Lachlan Non-Splitting Pairs and High Computably Enumerable Turing Degrees

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Abstract

A given c.e. degree $\mathbf{a} > \mathbf{0}$ has a non-trivial splitting into c.e. degrees $\mathbf{v}$ and $\mathbf{w}$ if $\mathbf{a} = \mathbf{v} \vee \mathbf{w}$ and $\mathbf{v} \mid \mathbf{w}$. A Lachlan Non-Splitting Pair is a pair of c.e. degrees $\langle \mathbf{a}, \mathbf{d} \rangle$ such that $\mathbf{a} > \mathbf{d}$ and there is no non-trivial splitting of $\mathbf{a}$ into c.e. degrees $\mathbf{w}$ and $\mathbf{v}$ with $\mathbf{w} > \mathbf{d}$ and $\mathbf{v} > \mathbf{d}$. Lachlan [Lachlan, 1976] showed that such a pair exists by proving the Lachlan Non-Splitting Theorem. This theorem is remarkable for its discovery of the $\mathbf{0}'''$-priority method, and became known as the 'Monster' due to its significant complexity.

Harrington, Shore and Slaman subsequently tried to explain Lachlan's methods in more intuitive and comprehensible terms in a number of unpublished notes. Leonhardi [Leonhardi, 1997] then published a short account of the Lachlan Non-Splitting Theorem based on these notes and generalised the theorem in a different direction.

In their work on the separation of the jump class $High$ from the jump class Low_2, Shore and Slaman [Shore and Slaman, 1993] also conjectured that every $high$ c.e. degree strictly bounds a Lachlan Non-Splitting Pair, a fact which could be used to separate the two jump classes. While this separation was eventually achieved through the notion of a Slaman Triple, the conjecture itself remained an open question. Cooper, Yi and Li [Cooper et al., 2002] also defined the notion of a c.e. Robinson degree as one which does not strictly bound the base $\mathbf{d}$ of a Lachlan Non-Splitting Pair $\langle \mathbf{a}, \mathbf{d} \rangle$, and sought to understand the relationship of this notion to the High/Low Hierarchy.

In this dissertation we make the following two contributions. Firstly we show that a counter-example can be found to show that the account of the Lachlan Non-Splitting Theorem given by Leonhardi [Leonhardi, 1997] fails to satisfy its requirements. By rectifying the construction, we give a complete, correct and intuitive account of the Lachlan Non-Splitting Theorem. Secondly we show that the high permitting method developed by Shore and Slaman [Shore and Slaman, 1993] can be combined with the construction of the Lachlan Non-Splitting Theorem just described to prove that every $high$ c.e. degree strictly bounds a Lachlan Non-Splitting Pair. From this it follows that the existence of a Lachlan Non-Splitting Pair can be used to separate the jump classes $High$ and Low_2, that the distribution of Lachlan Non-Splitting Pairs with respect to these jump classes mirrors the one for Slaman Triples, and that there is no $high$ c.e. Robinson degree.

To my wife Clarissa Bondin.

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During the first discussion Prof. Slaman suggested that I prove a strong form of the negation of the original theorem instead, which meant showing that every $high$ c.e. degree strictly bounds a Lachlan Non-Splitting Pair. This required modifying the Lachlan 'Monster' theorem and his encouragment was very significant for me to overcome the dread which I felt at this stage.

During the second discussion, Prof. Slaman helped me focus on the fact that the central obstacle to applying the Slaman-Shore high permitting method to the Lachlan Non-Splitting Theorem was related to preserving the honesty of certain computations. He also made a number of suggestions on how one could approach this problem. It turned out that one of his remarks about modifying the requirements was key to ensure that honesty could be preserved.

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Contents

List of figures

Chapter 1

Introduction

In this chapter we shall introduce the necessary background to the problems which will be treated in this dissertation. In Section 1.1 we make the necessary definitions and introduce the notation which will be used in the rest of this work. In Section 1.2 we shall review the relevant literature. Finally in Section 1.3 we introduce the priority method, which we shall be using to structure the constructions in this dissertation.

1.1 Preliminaries

In this section we shall define the concepts related to Turing computability, relative computability and the High/Low hierarchy which will be used in this work. For an introduction to computability theory see [Soare, 1987],[Cooper, 2004] or [Soare, 2016].

1.1.1 Turing Computability

The notion of Turing computability is based on the concept of a Turing machine. A *Turing machine* M consists of a finite non-empty set of *states* Q, a set of *symbols* S containing a special blank symbol $B \in S$, a two way infinite *working tape* divided into cells each of which contains a symbol from S, a read-write *head* which sees one cell from the tape at time, and a partial function $\delta : Q \times S \to Q \times (S \cup \{R, L\})$ known as the *Turing program* of the machine. One of the states

$q_0 \in Q$ is designated as the *initial state*, and another state $q_h \in Q$ is designated as the *halting state*.

The machine goes through a sequence of stages t. During each stage it will find itself in some state $q \in Q$ and the read-write head will read the symbol $s \in S$ which is written on the present cell. The machine will then determine the value of $\delta(q, s)$, obtaining a tuple (q', x). This causes the machine to change its state to q' and to take one of the following actions depending on the value of x. If $x \in S$, we have that the read-write head writes the symbol x on the present cell. On the other hand the read-write head will move one cell to the left if $x = L$ and one cell to the right if $x = R$. After taking any one of the above actions, the machine will go to the next stage.

Initially the machine finds itself in the initial state q_0 and its read-write head is in some given starting cell. The machine receives a natural number x as an *input* in the form of x consecutive '1' symbols written on the working tape starting from the starting cell, and with all other cells being filled with the blank symbol B. If at some stage the machine reaches the halting state q_h, the machine M will stop and its *output* y will be the number of '1' symbols written on the tape. Note that since the machine can only have made a finite number of moves before reaching the halting state, the portion of the tape which needs to be examined to determine the output is bounded.

The Turing program δ of a Turing machine M (henceforth program) may be represented as a finite set of quadruples. Through the use of Gödel numbering it is possible to list all programs and assign a natural number e to each program. We denote the *e*th program on this list by P_e and say that e is the index of the program. We shall use the notation Φ_e to denote the function computed by P_e and say that Φ_e is a *partial computable* (p.c.) function. This listing also gives us a *standard numbering* of all the p.c. functions.

Given a p.c. function Φ_e we shall write $\Phi_e(x) \downarrow$ if the machine with program P_e eventually converges by reaching its halting state when given input x. If the machine gives output y when it converges, we write $\Phi_e(x) \downarrow= y$. If the machine converges in s steps, we write $\Phi_{e,s}(x) \downarrow$, and $\Phi_{e,s}(x) \downarrow= y$. if it converges with output y. On the other hand if the machine diverges in s steps, we shall write $\Phi_{e,s}(x) \uparrow$.

The Turing machine model allows us to define the notion of Turing computability as follows. We shall say that a function f is *Turing computable* (henceforth *computable*) if there exists a Turing machine M such that if $f(x) = y$ and M starts with x written on its working tape, then M reaches the state q_h after finitely many stages and the number of '1' symbols written on the working tape is y.

Since every set A can be associated with its *characteristic function* $\chi_A(x)$ (defined as $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \notin A$), we can say that a set A is computable if χ_A is computable. In these cases we shall write $A = \Phi_e$, where e is the index of the program P_e used by the machine which computes $\chi_A(x)$. We shall also say that a set A is computably enumerable (c.e) if its elements can be effectively listed. This notion is formally identified with A being the range of some p.c. function Φ_e.

1.1.2 Relative Computability

The notion of relative computability is based on the notion of an oracle Turing machine. We base our account of this notion on [Cooper, 2004]. An *oracle Turing machine O* is similar to a Turing machine except for the following modifications. In addition to the working tape found in a Turing machine, the oracle Turing machine also has a one way read only infinite *oracle tape*. This tape shall have the characteristic function of some set A written on it, which acts as an 'oracle' for the machine.

The function δ, will now be called an *oracle Turing program* and is of type $Q \times S \to Q \times (S \cup \{R, L\} \cup Q)$. During a given stage the oracle machine will determine the value of $\delta(q, s)$ as usual, obtaining the tuple (q', x). However this time if the oracle machine sees that $x \in Q$, it will instead query the oracle. The machine does this by counting the number n of '1' symbols on its working tape, and reading the oracle tape at the *n*th cell to determine whether $n \in A$ or not. If $n \in A$, the machine goes to state x, while if $n \notin A$, the machine goes to state q'. Either way the working tape is unaffected. The *use* of an oracle machine during the course of a computation is defined as being the largest natural number whose membership in A has been ascertained in this way.

The oracle Turing program δ of an oracle Turing machine O (henceforth oracle program) may also be represented as a finite set of quadruples. Once again it is possible to list all oracle programs and assign a natural number e to each oracle program. We denote the eth oracle program on this list by $\tilde{P}_e$ and say that e is the index of the oracle program. We shall use the notation Φ_e^A to denote the function computed by $\tilde{P}_e$ when it is given A as an oracle and say that Φ_e^A is a *partial computable* (p.c.) *functional*. We shall often abbreviate this to *functional*, making it explicit when it is total. This listing also gives us a standard numbering of all the p.c. functionals.

Given a p.c. functional Φ_e^A we shall write $\Phi_e^A(x) \downarrow$ if the oracle machine with program $\tilde{P}_e$ eventually converges by reaching its halting state when given input x and oracle A. If the oracle machine gives output y when it converges, we shall write that $\Phi_e^A(x) \downarrow= y$. If u is the largest number whose membership in A was queried by the oracle machine during this computation, we shall say that u is the use of the computation. If the oracle machine converges in s steps, we shall denote this by $\Phi_{e,s}^A(x) \downarrow$. If the output of this computation is y we shall write that $\Phi_{e,s}^A(x) \downarrow= y$, and if the use of this computation is u, we shall write that $\phi_{e,s}(x) = u$. On the other hand if oracle machine diverges in s steps, we shall denote this by $\Phi_{e,s}^A(x) \uparrow$.

We shall say that a function f is *Turing computable in* A (henceforth *computable in* A, or *computable with the help of* A) if there exists an oracle Turing machine O such that if $f(x) = y$ and O has x written on its working tape and A written on its oracle tape, then O reaches state q_h after finitely many stages and the number of '1' symbols written on the tape is y.

A set B is said to be *computable in* A if χ_B is computable in A. In these cases we shall write $B = \Phi_e^A$, where e is the index of the program $\tilde{P}_e$ used by the oracle machine which computes χ_B when χ_A is written on its oracle tape. We shall also say that a set B is *computably enumerable in* A (henceforth *c.e in* A, or *c.e. with the help of* A) if its elements can be effectively listed in A. This notion is formally identified with B being the range of some p.c. functional Φ_e^A.

Frequently we shall need to approximate c.e. sets or p.c. functionals. This need arises from the fact that these objects will be given to us, or will be constructed by us, on a stage by stage basis.

Given some c.e. set A, we shall denote the approximation to A at stage s by A_s. It shall also be convenient to use the notation $A \upharpoonright n$ to denote the characteristic function of A restricted up to argument n.

On the other hand following [Lachlan, 1980], we can think of a p.c. functional Φ^A as a c.e. set of axioms of the form $\Gamma^{\sigma}(x) = y$, where σ is an initial segment of some characteristic function whose length is equal to the use of the computation. This allows the use of the Lachlan notation $\Gamma^A[s](x)$, which corresponds to the computation performed by the finite functional defined by the set of axioms enumerated before stage s, when this is given the approximation A_s as an oracle and is run for s stages on the value x. Sometimes we shall write $\phi[s](x)$ in order to denote the use of this computation. The Lachlan notation also extends to functionals with more than one oracle in the expected way.

Turing Reducibility

The notion of computability in an oracle can be used to define the notion of Turing reducibility as follows. If a set B is computable in A, we shall say that B is *Turing reducible* to A, written $B \leq_T A$. We shall say that the functional Φ_e^A giving $B = \Phi_e^A$ is a *reduction* of B to A. The sets A and B are said to be *Turing equivalent*, written $A \equiv_T B$ iff $A \leq_T B$ and $B \leq_T A$. If $A \leq_T B$ but $B \not\leq_T A$, we shall write that $A <_T B$.

Turing Degrees

Turing reducibility can then be used to define the notion of a Turing degree as follows. The *Turing degree* (henceforth degree) of a set A, written $deg(A)$ consists of all the sets B which are Turing equivalent to A, or $\{B \mid B \equiv_T A\}$. We shall use the boldface letter $\mathbf{a}$ to denote $deg(A)$. The class of all such degrees is denoted by $\mathcal{D}$. The degrees $\mathcal{D}$ form a partially ordered set $(\mathcal{D}, \leq)$, where $deg(B) \leq deg(A)$ iff $B \leq_T A$. If $deg(B) \leq deg(A)$ but $deg(A) \not\leq deg(B)$, we shall write $deg(B) < deg(A)$. If $deg(B) \not\leq deg(A)$ and $deg(A) \not\leq deg(B)$, the degrees are incomparable and we shall write $deg(A) \mid deg(B)$.

The join of two degrees $deg(A) \vee deg(B)$ shall denote the least upper bound of $deg(A)$ and $deg(B)$. The least upper bound of $deg(A)$ and $deg(B)$ is equal to $deg(A \oplus B)$, where $A \oplus B = \{2x \mid x \in A\} \cup \{2x+1 \mid x \in B\}$ and $\oplus$ is called the computable join of the two sets. The degrees $\mathcal{D}$ also form an upper semi-lattice $(\mathcal{D}, \leq, \vee)$ where $\vee$ gives the supremum of two degrees, but for which an infimum does not always exist. There is also a least degree $\mathbf{0}$, which contains all of the computable sets.

If A is a set, we shall also denote the *Turing jump* of A by A' and define it as $A' = \{x \mid \Phi_x^A(x) \downarrow\}$. The *n*th jump of A is written as $A^{(n)}$ and is defined as follows: $A^{(0)} = A$ and $A^{(n+1)} = (A^{(n)})'$. Note that $A^{(1)} = A'$. Two important properties of the Turing jump are that $A <_T A'$ and that $A \leq_T B \Rightarrow A' \leq_T B'$. If $A \in \mathbf{a}$ then we shall write that $\mathbf{a}' = deg(A')$. Hence we have that the Turing jump of a degree is strictly greater than the original degree, that is $\mathbf{a} < \mathbf{a}'$. We also have that the Turing jump also preserves the order $\leq$, giving that $\mathbf{a} \leq \mathbf{b} \Rightarrow \mathbf{a}' \leq \mathbf{b}'$.

Computably Enumerable Turing Degrees

In this dissertation we shall be concerned with the computably enumerable Turing degrees which are defined as follows. A Turing degree $\mathbf{a}$ is a *computably enumerable Turing degree* (henceforth *c.e. degree*) if there is a c.e. set A such that $\mathbf{a} = deg(A)$. We shall denote the class of c.e. degrees by $\mathcal{C}$. The c.e. degrees also form an upper semi-lattice $(\mathcal{C}, \leq, \vee)$, with the ordering $\leq$ and join $\vee$ being the same for $\mathcal{D}$. As with the Turing degrees, the c.e. degrees are not a lattice because an infinimum does not always exist. Since every computable set is also computably enumerable, we have that $\mathbf{0}$ is also the least c.e. degree. In addition, the c.e. set $K = \{x \mid \Phi_x(x) \downarrow\}$ is *complete*, in the sense that $A \leq_T K$ for every c.e. set A. It follows that there is a largest c.e. degree $\mathbf{0}' = deg(K)$.

1.1.3 High and Low Hierarchy

The *High/Low Hierarchy* can be used to categorise the Turing degrees below $\mathbf{0}'$ according to their jumps. Intuitively a degree $\mathbf{a}$ below $\mathbf{0}'$ is $high$ if its jump is as high as possible, that is if $\mathbf{a}' = \mathbf{0}''$. On the other hand a degree $\mathbf{a}$ below $\mathbf{0}'$ is low if its jump is as low as possible, that is if $\mathbf{a}' = \mathbf{0}'$. Note that since every c.e. Turing degree is also a Turing degree this hierarchy can also be used to stratify the c.e. Turing degrees.

The concept described above can be generalised in a straightforward manner to consider iterated Turing jumps as well. For every natural number n, we define the jump classes $High_n$ as $\{\mathbf{a} \leq$

$\mathbf{0'} \mid \mathbf{a^{(n)}} = \mathbf{0^{(n+1)}}\}$ and Low_n as $\{\mathbf{a} \leq \mathbf{0'} \mid \mathbf{a^{(n)}} = \mathbf{0^{(n)}}\}$. If $deg(A) \in High_n$, we shall say that A and $deg(A)$ are $high_n$. Similarly if $deg(A) \in Low_n$, we shall say that A and $deg(A)$ are low_n. In the case where $n = 1$, we omit the subscript and refer to the jump classes $High$ and Low and to $high$ and low sets and degrees respectively.

1.2 Literature Review

In this section we shall review the literature relevant to the problems treated in this dissertation.

In Section 1.2.1 we trace the developments which led to the structure of the c.e. degrees becoming an object of research. We also outline a number of definitions and results relating to the structure of the c.e. degrees which are relevant for the rest of the literature review.

Once this background has been covered, we review the known splitting and non-splitting theorems for the structure of the c.e. degrees in Section 1.2.2. Of the theorems covered, the Lachlan Non-Splitting Theorem is central to this dissertation. We show how the original account of the Lachlan Non-Splitting Theorem was of considerable difficulty and how this led Leonhardi to publish a more intuitive account of the theorem based on the unpublished notes of Harrington, Slaman and Soare. In Chapter 2 we shall show that Leonhardi's construction fails to satisfy the requirements of the Lachlan Non-Splitting Theorem, and present a rectified construction.

The Lachlan Non-Splitting Theorem constructs an object known as a Lachlan Non-Splitting Pair. In Section 1.2.3 we review the problem of separating the jump classes $High$ and Low_2 in the structure of c.e. degrees. Shore and Slaman conjectured that the notion of strictly bounding a Lachlan Non-Splitting Pair could be used for this purpose, but then separated the jump classes using the simpler notion of a Slaman Triple. In Chapter 3 we shall prove that every $high$ c.e. degree strictly bounds a Lachlan Non-Splitting pair, thus showing that the conjecture holds.

Research on the separation of jump classes led to questions about the distribution of various objects in the structure of the c.e. degrees. In Section 1.2.4 we review the known results about the distribution of Lachlan Non-Splitting Pairs and Slaman Triples in the c.e. degrees. We also review the known results about the distribution of Robinson degrees which are relevant for failing to bound the base of a Lachlan Non-Splitting Pair. This allows us to situate the result of Chapter 3 in its broader context and to obtain its corollaries.

In Section 1.2.5 we shall conclude this chapter by summarising the contributions of this dissertation in light of the prior discussion and make explicit our claims of novel work.

1.2.1 Background

We shall start by describing the relevant developments which led to the class of the c.e. degrees being discovered and becoming an object of research. For a more detailed history of degree theory see [Ambos-Spies and Fejer, 2014].

Turing introduced the notion of a Turing machine as a mathematical model of effective calculability in [Turing, 1936]. Using this notion, Turing defined the concept of a computable function as one computable by a Turing machine. In this paper Turing also defined the notion of a Universal machine, which was able to simulate any other Turing machine given its description. The paper is also important for showing the existence of uncomputable problems.

Kleene introduced the notion of computably enumerable set as the range of a computable function in [Kleene, 1936]. Post had in fact anticipated this notion in the form of generated sets, but this work was not submitted for publication until 1941 and did not appear in print until 1965 in [Post, 1965]. In his paper Kleene also showed that the set $K = \{x \mid \Phi_x(x)\downarrow\}$ is computably enumerable, but not computable.

Turing introduced relativised computation through the notion of an oracle Turing machine in [Turing, 1939]. The oracle machine used by Turing had access to a fixed oracle corresponding to the set of well-formed formulas $A(n)$ of the conversion calculus of Church with the property of being 'dual', which means that the formula is convertible to '2' for all integers n. Turing then defined a set to be 'number-theoretic' if it could be computed by such an oracle Turing machine. Although this introduced the notion of a set being Turing reducible to a fixed oracle, it did not define the notion of Turing reducibility between two sets A and B.

Post introduced the notion of degree of unsolvability in [Post, 1944]. Given some specific reducibility, sets A and B were regarded as having the same degree of unsolvability if they were reducible to one other. Thus if A is reducible to B but B is not reducible to A, it would mean that A has a lower degree of unsolvability and B has a higher degree of unsolvability. If neither is reducible to the other, the sets have incomparable degrees of unsolvability. Post also defined the strong reducibilities known as one-one reducibility ($\leq_1$), many-one reducibility ($\leq_m$), bounded truth table reducibility ($\leq_{btt}$) and truth table reducibility ($\leq_{tt}$), but gave only an informal definition of Turing reducibility ($\leq_T$). Formal definitions of Turing reducibility were given by Kleene in terms of general recursive functions in [Kleene, 1943], by Post in terms of canonical sets in [Post, 1948] and again by Kleene in terms of oracle Turing machines in [Kleene, 1952].

In [Post, 1944] the author also showed that K was complete under Turing reducibility, in the sense that every c.e. set was Turing reducible to K. This paper also introduced what became known as Post's Problem and Post's Program. Post's Problem asked whether it was possible to find a non-computable incomplete c.e. set. On the other hand Post's Program formulated the way in which this was to be achieved, namely by defining some structural property of sets, proving that there were c.e. sets satisfying this property and finally showing that such sets were neither computable nor complete. Post was able to resolve Post's problem for the aformentioned strong reducibilities, but not for Turing reducibility.

Kleene and Post abstracted from the notion of degree of unsolvability introduced in [Post, 1944] by defining a Turing degree as an equivalence class of sets which were Turing reducible to one another in [Kleene and Post, 1954]. This allowed Kleene and Post to show that the class of Turing degrees $\mathcal{D}$ formed an upper semi-lattice. It was also shown that the Turing degrees do not form a lattice because the infimum of two degrees does not always exist. This paper also gave a notion of the Turing jump which was well-defined on the Turing degrees. The technique of breaking down the conditions which need to be satisfied by a set being constructed into infinitely many requirements was also introduced in this paper. Through the use of the finite extension method it was also shown that for every non-computable Turing degree there is a Turing degree which is incomparable with the former. However these degrees did not have to be c.e. degrees.

Post's problem was resolved independently by Friedberg in [Friedberg, 1957] and by Mučnik in [Mučnik, 1956], who constructed two incomparable c.e. degrees $\mathbf{a}$ and $\mathbf{b}$ through a finite injury priority argument.

Theorem 1.2.1. *(Friedberg-Mučnik Theorem). There exist c.e. degrees* $\mathbf{a}$ *and* $\mathbf{b}$ *such that* $\mathbf{a} \mid \mathbf{b}$.

This result by Friedberg and Mučnik led to a detailed investigation of the structure of the class of c.e. degrees $\mathcal{C}$, although much of this research did not in fact follow Post's Program. The following two theorems would become relevant to the development of splitting and non-splitting theorems for the c.e. degrees.

Sacks proved the Density Theorem in [Sacks, 1964] which showed that between every two c.e. degrees, one can always find a third.

Theorem 1.2.2. *(Density Theorem). Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be c.e. degrees, with* $\mathbf{a} > \mathbf{b}$. *Then there exists a c.e. degree* $\mathbf{c}$ *such that* $\mathbf{a} > \mathbf{c} > \mathbf{b}$.

The existence of a minimal pair of c.e. degrees was shown independently by Lachlan in [Lachlan, 1966] and Yates in [Yates, 1966]. A minimal pair consists of two c.e. degrees $\mathbf{a_0}$ and $\mathbf{a_1}$ whose greatest lower bound is $\mathbf{0}$.

Definition 1.2.3. *(Minimal Pair Theorem). There exist c.e. degrees* $\mathbf{a_0}$ *and* $\mathbf{a_1}$ *such that the following conditions hold.*

1. $\mathbf{a_0} > \mathbf{0}$.
2. $\mathbf{a_1} > \mathbf{0}$.
3. *For all c.e. degrees* $\mathbf{b}$, $\mathbf{b} \leq \mathbf{a_0} \ \wedge \ \mathbf{b} \leq \mathbf{a_1} \Rightarrow \mathbf{b} = \mathbf{0}$.

1.2.2 Splitting and Non-Splitting Theorems

We shall now review the known splitting and non-splitting results for the c.e. degrees. Given some non-computable c.e. degree $\mathbf{a}$, one often wants to find a *non-trivial splitting* of $\mathbf{a}$ into c.e. degrees $\mathbf{u}$ and $\mathbf{v}$.

Definition 1.2.4. *A c.e. degree* $\mathbf{a} > \mathbf{0}$ *has a non-trivial splitting into c.e. degrees* $\mathbf{u}$ *and* $\mathbf{v}$ *if the following conditions hold.*

1. $\mathbf{a} = \mathbf{u} \vee \mathbf{v}$.

2. $\mathbf{u} \mid \mathbf{v}$.

From the two conditions above, it also follows that $\mathbf{0} < \mathbf{u} < \mathbf{a}$ and $\mathbf{0} < \mathbf{v} < \mathbf{a}$.

Splitting and Non-Splitting theorems determine under which conditions such splittings can be found. We shall now review the results present in this field.

Friedberg proved the first splitting result in [Friedberg, 1958], which was related to the c.e. sets and not to the c.e. degrees. In this result Friedberg showed that if A is some non-computable c.e. set, then it can be split by finding non-computable c.e. sets U and V such that A forms their disjoint union $U \sqcup V$.

Theorem 1.2.5. *(Friedberg Splitting Theorem). Let $A >_T \emptyset$ be a c.e. set. Then there exist c.e. sets U and V such that:*

1. $U >_T \emptyset$ *and* $V >_T \emptyset$.

2. $A = U \sqcup V$.

Friedberg's result did not yield a corollary related to the non-trivial splitting of c.e. degrees. For although one can infer that $\mathbf{a} = \mathbf{u} \vee \mathbf{v}$ from the theorem, it is not possible to conclude that $\mathbf{u} \mid \mathbf{v}$.

Sacks improved on Friedberg's result in [Sacks, 1963] by showing that the c.e. sets U and V could be made low and to avoid the upper cone of any non-computable set C.

Theorem 1.2.6. *(Sacks Splitting Theorem). Let A and C be c.e. sets and let $C >_T \emptyset$. Then there exist c.e. sets U and V such that:*

1. $A = U \sqcup V$.

2. U *and* V *are* low.

3. $C \not\leq_T U$ *and* $C \not\leq_T V$.

The upper cone avoidance allows one to obtain a corollary about the non-trivial splitting of c.e. degrees. This states that given a non-computable c.e. degree $\mathbf{a}$ it is possible to find non-computable, *incomparable* and low c.e. degrees $\mathbf{u}$ and $\mathbf{v}$ which join to $\mathbf{a}$. The corollary is obtained by applying the Sacks Splitting Theorem to a c.e. set $A \in \mathbf{a}$ with $C = A$, giving low c.e. sets U and V such that $A = U \sqcup V$, which also gives that $A \equiv_T U \oplus V$.

From this fact, we can conclude that U and V are non-computable c.e. sets. For suppose, without loss of generality that V was a computable set. Then we would have that $V \leq_T U$, and thus that $A \equiv_T U$. But this would mean that $A \leq_T U$, contradicting the fact that $A \not\leq_T U$ through the choice of $C = A$. Similarly, we have that U and V are of incomparable degree. For suppose, once again without loss of generality, that $V \leq_T U$. It would then follow that $A \equiv_T U$. This would mean that $A \leq_T U$, contradicting the fact that $A \not\leq_T U$ through the choice of $C = A$.

By taking $\mathbf{u} = deg(U)$ and $\mathbf{v} = deg(V)$ it follows that $\mathbf{a} = \mathbf{u} \vee \mathbf{v}$ and $\mathbf{u} \mid \mathbf{v}$, where $\mathbf{u}$ and $\mathbf{v}$ are low non-computable c.e. degrees.

Corollary 1.2.7. *(Sacks Splitting Theorem). Let* $\mathbf{a} > \mathbf{0}$ *be a c.e. degree. Then there exist c.e. degrees* $\mathbf{u}$ *and* $\mathbf{v}$ *such that:*

1. $\mathbf{a} = \mathbf{u} \vee \mathbf{v}$.
2. $\mathbf{a} = \mathbf{u} \mid \mathbf{v}$.
3. $\mathbf{u}$ *and* $\mathbf{v}$ *are* low.

The Sacks Splitting Theorem can thus be used to establish the fact that given some non-computable c.e. degree $\mathbf{a}$, it is always possible to find a non-trivial splitting of $\mathbf{a}$ above $\mathbf{0}$.

Lachlan combined the techniques for building a minimal pair with those used to prove Sacks Splitting Theorem in [Lachlan, 1980] to show that for every non-computable c.e. degree $\mathbf{a}$, one could find c.e. degrees $\mathbf{u}$ and $\mathbf{v}$ which form a non-trivial splitting of $\mathbf{a}$ and have a non-computable c.e. degree $\mathbf{d}$ as their meet. This result is significant since infinima do not always exist for the c.e. degrees.

Theorem 1.2.8. *(Lachlan Splitting Theorem). Let* $\mathbf{a} > \mathbf{0}$ *be a c.e. degree. Then there exist c.e. degrees* $\mathbf{u}$ *and* $\mathbf{v}$ *and a c.e. degree* $\mathbf{d} > \mathbf{0}$ *such that:*

1. $\mathbf{a} = \mathbf{u} \vee \mathbf{v}$.
2. $\mathbf{u} \mid \mathbf{v}$.

3. $\mathbf{d} = \mathbf{u} \wedge \mathbf{v}$.

Robinson improved on Sacks' result in [Robinson, 1971] by showing that given some non-computable c.e. degree $\mathbf{a}$, and some low non-computable c.e. degree $\mathbf{c} < \mathbf{a}$, it is always possible to find low c.e. degrees $\mathbf{u}$ and $\mathbf{v}$ which form a non-trivial splitting of $\mathbf{a}$ above $\mathbf{c}$.

Theorem 1.2.9. *(Robinson Splitting Theorem). Let* $\mathbf{a}$ *and* $\mathbf{c}$ *be c.e. degrees, where* $\mathbf{c} > \mathbf{0}$*,* $\mathbf{a} > \mathbf{c}$ *and* $\mathbf{c}$ *is* low*. Then there exist c.e. degrees* $\mathbf{u}$ *and* $\mathbf{v}$ *such that:*

1. $\mathbf{a} = \mathbf{u} \vee \mathbf{v}$.
2. $\mathbf{u} \mid \mathbf{v}$.
3. $\mathbf{u}$ *and* $\mathbf{v}$ *are* low.
4. $\mathbf{u} > \mathbf{c}$ *and* $\mathbf{v} > \mathbf{c}$.

Shore and Slaman showed that the Robinson Splitting Theorem could be extended to the low_2 c.e. degrees in [Shore and Slaman, 1990]. Shore and Slaman indicate that this result had been independently discovered by Harrington and by Brickford and Mills, but had never been published.

Theorem 1.2.10. *(Shore and Slaman, Harrington, Brickford and Mills). Let* $\mathbf{a}$ *and* $\mathbf{c}$ *be c.e. degrees, where* $\mathbf{c} > \mathbf{0}$*,* $\mathbf{a} > \mathbf{c}$ *and* $\mathbf{c}$ *is* low_2*. Then there exist c.e. degrees* $\mathbf{u}$ *and* $\mathbf{v}$ *such that:*

1. $\mathbf{a} = \mathbf{u} \vee \mathbf{v}$.
2. $\mathbf{u} \mid \mathbf{v}$.
3. $\mathbf{u} > \mathbf{c}$ *and* $\mathbf{v} > \mathbf{c}$.

Given some c.e. degree $\mathbf{a}$, it is thus always possible to find a non-trivial splitting of $\mathbf{a}$ over any low_2 c.e. degree $\mathbf{c}$ strictly below $\mathbf{a}$.

On the other hand Lachlan showed that the the Sacks Density Theorem and the Sacks Splitting Theorem could not be combined, thus giving the first Non-Splitting Theorem in [Lachlan, 1976]. In this theorem Lachlan constructed non-computable c.e. degrees $\mathbf{a}$ and $\mathbf{d}$, with $\mathbf{a} > \mathbf{d}$ such that it was not possible to find a non-trivial splitting of $\mathbf{a}$ over $\mathbf{d}$. The pair $\langle \mathbf{a}, \mathbf{d} \rangle$ is called a *Lachlan Non-Splitting Pair*, where $\mathbf{a}$ is called a Lachlan Non-Splitting Top, and $\mathbf{d}$ is called a Lachlan Non-Splitting Base.

Theorem 1.2.11. *(Lachlan Non-Splitting Theorem). There is a pair of c.e. degrees* $\langle \mathbf{a}, \mathbf{d} \rangle$*, with* $\mathbf{d} > \mathbf{0}$ *and* $\mathbf{a} > \mathbf{d}$ *such that there is no non-trivial splitting of* $\mathbf{a}$ *into c.e. degrees* $\mathbf{u}$ *and* $\mathbf{v}$ *with* $\mathbf{u} > \mathbf{d}$ *and* $\mathbf{v} > \mathbf{d}$.

The Lachlan Non-Splitting Theorem introduced the $\mathbf{0}'''$-priority method for the first time, and is known as the Monster Theorem due to its great complexity. Prior to presenting the verification of his construction, Lachlan added the following note:

> This part of the paper is very complex. Its complexity stems from the way in which the construction was discovered. From very crude beginnings the final format of the construction was achieved only after a series of modifications each designed to eliminate a flaw found in the previous attempt. This process of evolution yielded only a cloudy intuition as to why the construction should work.

This cloudy intuition into why the construction should work motivated several researchers to study the construction in greater depth. Describing what happened in the aftermath of Lachlan's paper, Leonhardi states the following in [Leonhardi, 1997]:

> The proof was so complicated that the theorem became informally known as the 'Monster Theorem.' Eventually, notes by Harrington, Slaman, and Soare were circulated (but never published) which attempted to explain Lachlan's methods in more intuitive and comprehensible terms.

Harrington had also improved upon Lachlan's result in his notes [Harrington, 1980]. In fact he showed that the top of the Lachlan Non-Splitting Pair $\mathbf{a}$ could be made equal to $\mathbf{0}'$. This result, known as the Harrington Non-Splitting Theorem, proved the existence of Harrington Non-Splitting Bases $\mathbf{d}$ over which every non-trivial splitting of $\mathbf{0}'$ fails.

Theorem 1.2.12. *(Harrington Non-Splitting Theorem). There is a c.e. degree* $\mathbf{d} > \mathbf{0}$*, such that there is no non-trivial splitting of* $\mathbf{0}'$ *into c.e. degrees* $\mathbf{u}$ *and* $\mathbf{v}$ *such that* $\mathbf{u} > \mathbf{d}$ *and* $\mathbf{v} > \mathbf{d}$.

Leonhardi also gave a different account of the Lachlan Non-Splitting Theorem, and then proceeded to generalise this theorem in a different direction in [Leonhardi, 1997]. Leonhardi explains that this new account of the Lachlan Non-Splitting Theorem was derived from the aformentioned unpublished notes:

> Many of the ideas presented in this section, along with some terminology and notation, have been taken from the notes of Harrington, Slaman, and in particular from the notes of Soare on Lachlan's theorem. As with all such borrowings and adaptations, much credit should go to these authors, who were themselves inspired by the powerful and ingenious ideas presented in Lachlan's paper.

Leonhardi's published account of the Lachlan Non-Splitting Theorem shed considerable light on the techniques which need to be employed in order to prove Lachlan's Non-Splitting Theorem.

Regrettably this account, which is 8 pages in length, omits a number of important details when sketching out the construction, and contains no proper verification. It is in fact possible to find counter-examples to show that the construction as sketched by Leonhardi does not satisfy the requirements of the Lachlan Non-Splitting Theorem.

We remedy this situation in Chapter 2 of this dissertation, where we shall present a complete, correct and intuitive account of the Lachlan Non-Splitting Theorem.

1.2.3 Separation of Jump Classes High and Low

One important problem related to Lachlan Non-Splitting Pairs is the question of separating the jump class $High$ from the jump class Low. By separating the jump class $High$ from the jump class Low we mean finding some first order formula of degree theory $\theta(\mathbf{x})$ in the language of partial orders such that the formula is true whenever the c.e. degree $\mathbf{x}$ is $high$ and false when the c.e. degree $\mathbf{x}$ is low.

Shore and Slaman considered the problem of separating the jump class $High$ from the jump class Low in [Shore and Slaman, 1990, Shore and Slaman, 1993]. In [Shore and Slaman, 1990] they identified the existence of a Lachlan Non-Splitting Pair strictly below a given c.e. degree $\mathbf{x}$ as one possible way of separating the $high$ c.e. degrees from the low c.e. degrees.

Indeed, the following corollary of the Robinson Splitting Theorem would provide the first part of this result:

Corollary 1.2.13. *(Robinson Splitting Theorem). Let* $\mathbf{l}$ *be a low c.e. degree. Then there is no Lachlan Non-Splitting Pair* $\langle \mathbf{a}, \mathbf{d} \rangle$ *strictly below* $\mathbf{l}$*.*

Proof. Consider the low c.e. degree $\mathbf{l}$. By the definition of lowness and the fact the the Turing jump preserves the ordering between c.e. degrees, it follows that every c.e. degree below $\mathbf{l}$ is also low. Suppose that one chooses c.e. degrees $\mathbf{a}$ and $\mathbf{d}$ strictly below $\mathbf{l}$ with $\mathbf{a} > \mathbf{d}$. Then by the Robinson Splitting Theorem we have that it is possible to find a non-trivial splitting of $\mathbf{a}$ over $\mathbf{d}$. It follows that $\mathbf{l}$ cannot strictly bound some Lachlan Non-Splitting Pair $\langle \mathbf{a}, \mathbf{d} \rangle$. □

By applying an analogous argument to the result obtained by Shore and Slaman, Harrington and Brickford and Mills one obtains that a low_2 c.e. degree cannot strictly bound a Lachlan Non-Splitting Pair.

On the other hand, the second part of the result for separating the jump class $High$ from the jump class Low would have to be provided by proving that every $high$ c.e. degree $\mathbf{h}$ strictly bounds a Lachlan Non-Splitting Pair. Shore and Slaman announced that this was the case in [Shore and Slaman, 1990] and [Shore and Slaman, 1993], mentioning that this result would appear in a forthcoming paper, but this result was never published.

Slaman has confirmed that no proof of this fact exists [Slaman, 2015], and that Shore and himself had in fact singled out the notion of a *Slaman Triple* of c.e. degrees as a more tractable one for separating the jump class $High$ from the jump class Low, especially in the light of their new technique for high permitting introduced in [Shore and Slaman, 1993].

It is thus the case that the existence of a Lachlan Non-Splitting Pair strictly below every $high$ c.e degree remains a conjecture.

Conjecture 1.2.14. (Shore-Slaman Conjecture). *Let* $\mathbf{h}$ *be a* $high$ *c.e. degree. Then* $\mathbf{h}$ *strictly bounds a Lachlan Non-Splitting Pair.*

We shall prove that this conjecture holds in Chapter 3 of this Dissertation. The following unproven result will then follow from our theorem.

Corollary 1.2.15. (Separation of Jump Classes $High$ and Low_2). *The property of strictly bounding a Lachlan Non-Splitting Pair separates the jump classes $High$ and Low_2.*

Since every low c.e. degree is also a low_2 c.e. degree, we have that the property of strictly bounding a Lachlan Non-Splitting Pair separates the jump class $High$ from the jump class Low as a special case of the above.

1.2.4 Distribution of Various Degrees

Shore and Slaman's proposal for separating the jump classes $High$ and Low by asking whether a given c.e. degree strictly bounds some degree theoretical object such as a Lachlan Non-Splitting Pair or a Slaman Triple led to the question of how these objects were distributed in the c.e. degrees, and especially of their relation to c.e. degrees from various jump classes.

A related question involves the construction of c.e. degrees which do not strictly bound Lachlan Non-Splitting Bases or Harrington Non-Splitting Bases, and which thus strictly bound only well-behaved c.e. degrees $\mathbf{x}$ over which it is always possible to find a non-trivial splitting of any c.e. degree $\mathbf{a} > \mathbf{x}$ or of $\mathbf{0}'$ respectively.

We start by reviewing the proven and conjectured relationships between Lachlan Non-Splitting Pairs, Slaman Triples and c.e. degrees from various jump classes. It is interesting to compare the distribution of Lachlan Non-Splitting Pairs and Slaman Triples. For although the construction for Slaman Triples exhibits a resemblance to the construction for Lachlan Non-Splitting Pairs, the latter presents several features which complicate the situation. We shall then review the results for the Robinson degrees, defined as being those c.e. degrees which avoid all Lachlan Non-Splitting Bases in their lower cone.

Lachlan Non-Splitting Pairs

We start by restating the notion of a Lachlan Non-Splitting Pair [Lachlan, 1976].

Definition 1.2.16. *(Lachlan Non-Splitting Theorem). A Lachlan Non-Splitting Pair is a pair of c.e. degrees* $\langle \mathbf{a}, \mathbf{d} \rangle$*, with* $\mathbf{d} > \mathbf{0}$ *and* $\mathbf{a} > \mathbf{d}$ *such that there is no non-trivial splitting of* $\mathbf{a}$ *into c.e. degrees* $\mathbf{u}$ *and* $\mathbf{v}$ *with* $\mathbf{u} > \mathbf{d}$ *and* $\mathbf{v} > \mathbf{d}$.

The c.e. degree $\mathbf{a}$ *is called a Lachlan Non-Splitting Top, the c.e. degree* $\mathbf{d}$ *is called a Lachlan Non-Splitting Base, and if* $\mathbf{a} = \mathbf{0}'$*, the c.e. degree* $\mathbf{d}$ *is called a Harrington Non-Splitting Base.*

The known results and conjectures regarding the relationship of Lachlan Non-Splitting Pairs to various jump classes are as follows. Shore and Slaman have shown that low_2 c.e. degrees cannot strictly bound a Lachlan Non-Splitting Pair in [Shore and Slaman, 1990]. In addition Shore and Slaman have claimed that there is a low_3 c.e. degree which strictly bounds a Lachlan Non-Splitting Pair in [Shore and Slaman, 1993], but have not given a proof of this result. On the other hand, Cooper, Li and Yi have shown that there is a *non-*low_2 c.e. degree which does not strictly bound a Lachlan Non-Splitting Base in [Cooper et al., 2002]. Hence this degree cannot strictly bound a Lachlan Non-Splitting Pair either. Shore and Slaman have claimed that every $high$ c.e degree strictly bounds a Lachlan Non-Splitting Pair in [Shore and Slaman, 1990, Shore and Slaman, 1993] but have not given a proof of this result, as discussed in the previous section.

Slaman Triples

We shall now define the notion of a Slaman Triple [Shore and Slaman, 1993].

Definition 1.2.17. *A Slaman Triple is a triple of c.e. degrees* $\langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle$ *which obeys the following properties:*

1. $\mathbf{a} > \mathbf{0}$.
2. $\mathbf{c} \not\leq \mathbf{b}$.
3. *If* $\mathbf{d}$ *is a c.e. degree such that* $\mathbf{0} < \mathbf{d} < \mathbf{a}$*, we have that* $\mathbf{c} \leq \mathbf{d} \vee \mathbf{b}$.

The known results and conjectures regarding the relationship of Slaman Triples to various jump classes are as follows. Shore and Slaman have shown that low_2 c.e. degrees cannot strictly bound a

Slaman Triple in [Shore and Slaman, 1990]. In addition, Shore and Slaman have claimed that there is a low_3 c.e. degree which strictly bounds a Slaman Triple in [Shore and Slaman, 1993], but have not given a proof of this result. Shore and Slaman used a new high permitting method introduced in [Shore and Slaman, 1993] in order to show that every $high$ c.e. degree strictly bounds a Slaman Triple. In this way, Shore and Slaman have successfully separated the jump class $High$ from the jump class Low_2, and hence also from the jump class Low. On the other hand, Leonhardi has shown that there is a $high_2$ c.e. degree which does not strictly bound a Slaman Triple in [Leonhardi, 1996].

Robinson Degrees

We shall now define the notion of a Robinson degree [Cooper et al., 2002]. A Robinson degree is a c.e. degree $\mathbf{r}$ which does not strictly bound any Lachlan Non-Splitting Base. In particular this also means that a Robinson degree does not strictly bound any Lachlan Non-Splitting Pair.

Definition 1.2.18. *A Robinson degree is a c.e. degree* $\mathbf{r}$ *such that there is no Lachlan Non-Splitting Pair* $\langle \mathbf{a}, \mathbf{d} \rangle$ *with* $\mathbf{d} < \mathbf{r}$.

The known results relating to Robinson degrees are the following. From the Robinson Splitting Theorem it follows that no low c.e. degree strictly bounds a Lachlan Non-Splitting Base, which makes every low c.e. degree a Robinson degree. The result of Shore and Slaman, Harrington and Brickford and Mills showing that no low_2 c.e. degree bounds a Lachlan Non-Splitting Pair [Shore and Slaman, 1990] is not strong enough to show that no low_2 c.e. degree strictly bounds a Lachlan Non-Splitting Base. Cooper, Li and Yi have shown that there is a *non*-low_2 c.e. degree which does not strictly bound a Lachlan Non-Splitting Base in [Cooper et al., 2002], but have not determined the exact jump class of this degree. It is not known whether there is a $high$ Robinson degree, although from the Shore and Slaman Conjecture it would follow that no $high$ c.e. degree is a Robinson Degree.

Corollary 1.2.19. *(Shore and Slaman Conjecture). There is no* $high$ *c.e. Robinson degree.*

1.2.5 Contributions

We shall now summarise the contributions of this dissertation and the corollaries following from them, and make the novel elements explicit.

In Chapter 2 of this dissertation, we shall present a complete, correct and intuitive account of the Lachlan Non-Splitting Theorem. Our presentation will follow the outline of the Lachlan Non-Splitting Theorem given by Leonhardi in [Leonhardi, 1996], broadening this account into a detailed exposition of this construction. However there will also be a number of divergences and novel elements which are not present in Leonhardi's account, which we shall summarise as follows. These are the notion of R-Synchronisation (Section 2.6), the use of open and close stages (Section 2.7), the counterexample showing that Leonhardi's account fails to satisfy the requirements of the Lachlan Non-Splitting Theorem (Section 2.7), the notion of self-repair for $\mathcal{R}$ strategies which overcomes this difficulty (Section 2.8), the notion of fairness (Section 2.9), explicit strategies for the general case involving the satisfaction of many $\mathcal{R}$ and $\mathcal{S}$ requirements simultaneously (Section 2.9), a schema for generating the priority tree in the general case (Section 2.9) as well as the entire verification of the construction (Section 2.9).

In Chapter 3 of this dissertation we give an entirely novel result by showing that every $high$ c.e. degree strictly bounds a Lachlan Non-Splitting Pair, thus settling the Shore-Slaman conjecture. In order to achieve this, we shall permit the construction of the Lachlan Non-Splitting Theorem strictly below any given $high$ c.e. degree. Our theorem will show how to apply the high permitting method of Shore and Slaman given in [Shore and Slaman, 1993] to the Lachlan Non-Splitting construction given in Chapter 2 so as to obtain the desired result.

From this result it will follow that the property of strictly bounding a Lachlan Non-Splitting Pair can be used to separate the jump classes $High$ and Low_2, and hence also Low. This would mean that Lachlan Non-Splitting Pairs are distributed in a way analogous to that of Slaman Triples with respect to the jump classes $High$ and Low_2. In addition it will also follow that there is no $high$ Robinson degree. In particular, the *non-*low_2 Robinson degree built by Cooper, Li and Yi in [Cooper et al., 2002] cannot be $high$.

1.3 The Priority Method

In this section we shall outline the priority method, which will be used to organise the constructions found in Chapters 2 and 3. For further details about $\mathbf{0}'$ (finite injury), $\mathbf{0}''$ (infinite injury) and $\mathbf{0}'''$ priority arguments, see [Soare, 1987].

1.3.1 The Priority Ordering

The first step in the use of the priority method is to impose a strict total order $<_p$ on the infinite set of requirements which will need to be satisfied. This is called a *priority ordering* and results in the following sequence of requirements:

$$\mathcal{R}_1 <_p \mathcal{R}_2 <_p \mathcal{R}_3 <_p \dots$$

We shall then say that $\mathcal{R}_n$ is of higher priority than $\mathcal{R}_{n+1}$ and that the latter is of lower priority than the former. It is also possible to have requirements of different kinds. When this is the case the requirements will be interleaved in some way and will be ordered by the same priority ordering.

1.3.2 The Strategies

Every requirement will be satisfied through the action of one or more strategies. A *strategy* is a finite program with some (possibly infinite) set of outcomes Λ which attempts to satisfy one requirement. The outcomes Λ of the strategy are ordered by some strict total order $<_\Lambda$. If $a <_\Lambda b$ we shall say that the outcome a lies to the left of outcome b, and that b lies to the right of outcome a. We shall use Greek letters such as γ to denote strategies.

The construction will deem certain strategies to be *accessible* during every stage s. If a strategy γ becomes accessible at stage s it will execute its finite program, and this will leave it in some particular state. Each such state will correspond to an outcome and the strategy will choose the corresponding outcome $O_s(\gamma)$ at stage s.

Over the course of the construction the strategy may reach a limit outcome, or else it may reach some outcomes infinitely often. The order $a <_\Lambda b$ is usually defined so that if the strategy chooses outcome a infinitely often, then b cannot be the outcome which causes the strategy to satisfy the requirement. Hence it is possible to define the *true outcome* of the strategy as the limit outcome if it exists, and as the leftmost outcome which is visited infinitely often by the strategy otherwise.

1.3.3 The Priority Tree

Strategies are organised into a *priority tree*, which will be specific to each construction. A priority tree is a tree such that the nodes correspond to strategies and the outgoing edges of each node correspond to the outcomes of that strategy.

More formally, let $\tilde{\Lambda}$ be the set of all possible outcomes of the strategies which will be used in the construction. Let $\tilde{\Lambda}^{\omega}$ be the set of finite sequences of elements from $\tilde{\Lambda}$. The priority tree T of the construction will be a subset of $\tilde{\Lambda}^{\omega}$. We shall say that p is a path in the priority T tree if $p \in T$. Hence a path p consists of a sequence of outcomes.

Since every node of the priority tree corresponds to a strategy, we have that every path p in T also corresponds to some strategy γ. This strategy is the one obtained by starting at the root node of the priority tree and following all the edges corresponding to the sequence of outcomes p. The length of the path $|p|$ is obtained by counting the elements in the sequence, or in other words the number of edges on the path. The notation $p \restriction n$ will be used to denote the initial segment of p of length n. The expression $p \restriction n = \gamma$ shall be used as shorthand to denote that the strategy corresponding to $p \restriction n$ is γ. The notation $p(n)$ will denote the outcome of the strategy γ which corresponds to $p \restriction n$. The notation $[T]$ will be used to denote the set of infinite paths through T, where p is said to be an infinite path through T if $p \restriction n \in T$ for every n.

The notation $p\,\hat{}\,o$ shall be used to denote the path of length $|p| + 1$ obtained by appending the outcome o to the end of the sequence of outcomes of p. If $\gamma = p \restriction n$, we shall use the notation $\gamma \frown o$ to identify the edge with outcome o of the strategy γ. Given some path p of length $m > n$, and the fact that $\gamma = p \restriction n$, we shall use the notation γ^{+} to denote the successor strategy of γ on the path p.

Given paths p and p', we shall use the notation $p \subset p'$ to denote that p is an initial segment of p', and $p \subseteq p'$ to denote that p is either an initial segment of p' or equal to p'. The notation $p \supset p'$ and

$p \supseteq p'$ is then straightforward. We shall also write $p <_L p'$ if the path p lies to the left of the path p', defined as $(\exists\, a, b \in \tilde{\Lambda})(\exists\, p'' \in T)[p''\hat{\ }a \subseteq p \,\wedge\, p''\hat{\ }b \subseteq p' \,\wedge\, a <_\Lambda b]$.

Since every path p corresponds to a strategy γ, we can interpret $\gamma \subset \gamma'$ as meaning that the strategy γ is above γ' on the priority tree. Similarly $\gamma \subseteq \gamma'$ will mean that γ is above or equal to γ', $\gamma \supset \gamma'$ will mean that γ is below γ' and $\gamma \supseteq \gamma'$ will mean that the strategy γ is below or equal to γ'. Finally $\gamma <_L \gamma'$ shall mean that the strategy γ lies to the left of γ'.

1.3.4 The Construction

At any given stage s, the construction will construct a *current path* δ_s of length s through the priority tree. Strategies which lie on the current path will be declared to be *accessible* at stage s.

To build the current path, the construction goes through a sequence of substages $t \leq s$. During substage 0, we define δ_s as being the empty path of length 0, and declare the strategy lying at the end of this path to be accessible.

During substage $t + 1$, the construction takes the strategy γ lying at the end of the path δ_s which was defined during the previous substage. It then computes the outcome $O_s(\gamma)$ of the strategy γ at stage s. Furthermore it redefines δ_s by extending the path defined during the previous substage with this new outcome. Finally, the strategy at the end of the new path is declared to be *accessible* as before.

One this process concludes, the construction has finished building the current path δ_s. It will thus *initialise* all strategies lying to the right of the current path. This will cause these strategies to cancel all work done so far and restart their execution from the beginning.

Finally we define the *true path* f as follows. For every n, $f(n)$ is the true outcome of the strategy $f \restriction n$. If the construction is correct, the true path will be infinite and will correspond to the leftmost path which is visited by the construction infinitely often. By this we mean that there will be infinitely many stages s such that $\delta_s \subset f$. If a strategy γ lies on the true path, there will be some stage s_0 such that for all stages $t > s_0$ we have that $\delta_t \not<_L \gamma$, or in other words that the current path does not lie to the left of γ at stage t. Hence, the strategy γ will not be initialised by the construction after some stage s_0.

1.3.5 The Verification

The construction has to be verified in order to show that all requirements are satisfied. During the verification it is shown that for every requirement there is some strategy on the true path which will attempt and succeed in satisfying the requirement.

Chapter 2

The Lachlan Non-Splitting Theorem

2.1 Preliminaries for the Theorem

In this chapter we shall be proving the following theorem.

Theorem 2.1.1. *(Lachlan Non-Splitting Theorem [Lachlan, 1976]). There exist c.e. degrees* $\mathbf{a}$ *and* $\mathbf{d}$ *such that* $\mathbf{d} < \mathbf{a}$*, and there is no non-trivial splitting of* $\mathbf{a}$ *into c.e. degrees* $\mathbf{u}$ *and* $\mathbf{v}$ *such that* $\mathbf{d} < \mathbf{u}$ *and* $\mathbf{d} < \mathbf{v}$.

2.1.1 The Requirements

In order to prove the theorem we shall build two sets A and D, satisfying certain requirements, and then take $\mathbf{a} = deg(A \oplus D)$ and $\mathbf{d} = deg(D)$ to be the top and the base of the Lachlan Non-Splitting pair which we are required to construct.

Since $D \leq_T A \oplus D$, we have that $\mathbf{d} \leq \mathbf{a}$. To show that $\mathbf{d} < \mathbf{a}$, it is sufficient to prove that $\mathbf{a} \not\leq \mathbf{d}$. Hence we shall need to satisfy the following requirement.

$$\mathcal{S} : A \not\leq_T D$$

In order to show that there is no non-trivial splitting of $\mathbf{a}$ into c.e. degrees $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{d} < \mathbf{u}$ and $\mathbf{d} < \mathbf{v}$, we shall need to satisfy the requirement $\mathcal{R}_{(U,V)}$ for every pair of c.e. sets U and V.

$$\mathcal{R}_{(U,V)} : [A \leq_T U \oplus V] \Rightarrow [A \leq_T U \oplus D \ \vee \ A \leq_T V \oplus D]$$

We now show that satisfying these requirements is enough to prove the theorem.

Lemma 2.1.2. *If the requirement $\mathcal{S}$ is satisfied and the requirement $\mathcal{R}_{(U,V)}$ is satisfied for every pair of c.e. sets U and V, there is no non-trivial splitting of $\mathbf{a}$ into c.e. degrees $\mathbf{a_0}$ and $\mathbf{a_1}$ such that $\mathbf{d} < \mathbf{a_0}$ and $\mathbf{d} < \mathbf{a_1}$.*

Proof. Suppose for contradiction that $\mathbf{a}$ has a non-trivial splitting into c.e. degrees $\mathbf{a_0}$ and $\mathbf{a_1}$ such that $\mathbf{d} < \mathbf{a_0}$ and $\mathbf{d} < \mathbf{a_1}$. Let $\mathbf{a_0} = deg(A_0)$ and $\mathbf{a_1} = deg(A_1)$, where A_0 and A_1 are c.e. sets. Then by the definition of a non-trivial splitting we have that $\mathbf{a} = \mathbf{a_0} \vee \mathbf{a_1}$. Hence we have that $A \oplus D \equiv_T A_0 \oplus A_1$. From this it follows that $A \oplus D \leq_T A_0 \oplus A_1$ and hence that $A \leq_T A_0 \oplus A_1$. Thus from the fact that $\mathcal{R}_{(A_0,A_1)}$ holds we can conclude that $A \leq_T A_0 \oplus D$ or that $A \leq_T A_1 \oplus D$. Now without loss of generality, suppose that $A \leq_T A_0 \oplus D$ is the case. Then we have that $A \oplus D \leq_T A_0 \oplus D$. On the other hand, since $\mathbf{d} < \mathbf{a_0}$ we have that $D \leq_T A_0$, and thus that $A_0 \oplus D \leq_T A_0$. By transitivity we obtain that $A \oplus D \leq_T A_0$. In addition from $\mathbf{a} = \mathbf{a_0} \vee \mathbf{a_1}$, we can conclude that $A \oplus D \equiv_T A_0 \oplus A_1$ and therefore we have that $A_0 \oplus A_1 \leq_T A \oplus D$. By transitivity we have that $A_0 \oplus A_1 \leq_T A_0$ and hence that $A_1 \leq_T A_0$. But this means that $\mathbf{a_1} \leq \mathbf{a_0}$ and hence that $\mathbf{a_0} \nmid \mathbf{a_1}$. It follows that $\mathbf{a_0}$ and $\mathbf{a_1}$ cannot be a non-trivial splitting of $\mathbf{a}$, which gives the required contradiction. □

2.1.2 Implementation of the Requirements

In order for strategies to be able to satisfy the above requirements we will need to break them down into a simpler form. Let (Θ) be a standard listing of all p.c. functionals, and let (Φ, U, V) be a standard listing of all triples such that Φ is a p.c. functional, and U and V are c.e. sets.

Then the requirement $\mathcal{S}$ can be broken down into infinitely many requirements of the following form.

$$\mathcal{S}_{(\Theta)} : \Theta^D \neq A$$

On the other hand each requirement $\mathcal{R}_{(U,V)}$ can be broken down into infinitely many requirements of the following form.

$$\mathcal{R}_{(\Phi,U,V)} : [\Phi^{U,V} = A] \Rightarrow [\Gamma^{U,D} = A \ \vee \ \Gamma^{V,D} = A]$$

In this case the functionals $\Gamma^{U,D}$ or $\Gamma^{V,D}$ would need to be built by any strategy attempting to satisfy such a requirement.

2.1.3 Further Simplification of the Requirements

The requirements can be further simplified as follows. Consider a requirement of the form $\mathcal{R}_{(U,V)}$, and suppose that $A \leq_T U \oplus V$, so that it does not hold trivially.

Then in order to satisfy $\mathcal{R}_{(U,V)}$ it is sufficient to show that a requirement of the form $\mathcal{R}_{(\Phi,U,V)}$ holds for some p.c. functional Φ. This is because building a functional $\Gamma^{U,D} = A$ or a functional $\Gamma^{V,D} = A$ is sufficient to show that $A \leq_T U \oplus D$ or $A \leq_T V \oplus D$ respectively, thus satisfying $\mathcal{R}_{(U,V)}$.

Finally if a functional Φ satisfying $\Phi^{U,V} = A$ exists, there must also be a functional Ψ which gives $\Psi^{U,V} = A$ and which satisfies the following properties:

(1) $(\forall s)(\forall x)(\forall y)\ [x < y \Rightarrow \psi_s(x) < \psi_s(y)]$.

(2) $(\forall x)(\forall s)(\forall s')\ [s < s' \Rightarrow \psi_s(x) \leq \psi_{s'}(x)]$.

Thus it follows that in order to satisfy a requirement $\mathcal{R}_{(U,V)}$, it suffices to attempt to satisfy only those requirements $\mathcal{R}_{(\Phi,U,V)}$ such that the functional $\Phi^{U,V}$ satisfies properties corresponding to (1) and (2) as specified above.

2.1.4 Further Remarks

We shall now make a number of preliminary remarks on the construction which will be used to satisfy the requirements.

Priority Ordering of the Requirements

Earlier in this section we said that in order to prove the Lachlan Non-Splitting Theorem, we needed to satisfy infinitely many requirements of the form $\mathcal{S}_{(\Theta)}$ where (Θ) is a standard listing of p.c.

functionals, and infinitely many requirements of the form $\mathcal{R}_{(\Phi,U,V)}$, where (Φ, U, V) is a standard listing of all triples such that Φ is a p.c. functional and U and V are c.e. sets.

By making the index underlying each list of requirements explicit we can express the infinite list of requirements $\mathcal{S}_{(\Theta)}$ as $\mathcal{S}_i$, and the infinite list of requirements $\mathcal{R}_{(\Phi,U,V)}$ as $\mathcal{R}_i$, where i is a natural number. It is then possible to define the *priority ordering* $<_p$ on the set of all requirements, which is the strict total ordering below.

$$\mathcal{R}_1 <_p \mathcal{S}_1 <_p \mathcal{R}_2 <_p \mathcal{S}_2 <_p \dots$$

Strategies $\mathcal{R}$ and $\mathcal{S}$

Different strategies will be needed for the two kinds of requirement. Strategies which will attempt to satisfy a requirement $\mathcal{R}_i$ will be called $\mathcal{R}$-strategies, while strategies which will attempt to satisfy a requirement $\mathcal{S}_i$ will be called $\mathcal{S}$-strategies. In general we shall use the notation $\alpha, \alpha', \dots$ to denote $\mathcal{S}$-strategies, and $\beta, \beta', \dots$ to denote $\mathcal{R}$-strategies. We shall use the notation $\gamma, \gamma', \dots$ when there is no need to distinguish between the two types of strategy.

In order to avoid ambiguity for situations involving multiple $\mathcal{R}$ strategies, if an $\mathcal{R}$ strategy β builds a functional $\Gamma^{U,D}$, we shall sometimes denote this functional by $\Gamma_\beta^{U,D}$. Similarly, if an $\mathcal{R}$ strategy β builds a functional $\Gamma^{V,D}$ we willl sometimes denote the functional by $\Gamma_\beta^{V,D}$.

Witness, Threshold and Use Sets

During the course of its execution an $\mathcal{R}$ strategy may need to choose *uses* from some set, whilst an $\mathcal{S}$ strategy may need to choose *witnesses* from some set and *thresholds* from another. These uses, witnesses and thresholds shall sometimes be referred to as *parameters*. Different $\mathcal{S}$ strategies and different $\mathcal{R}$ strategies will have access to separate sets from which to choose these parameters.

In order to ensure that every set is disjoint from all others, we shall computably partition the set of natural numbers $\mathbb{N}$ into infinitely many infinite subsets, which we can then assign to the strategies as required.

We shall also totally order the $\mathcal{R}$ strategies on the priority tree, indexing them as β_e for some natural number e. Similarly we will totally order the $\mathcal{S}$ strategies on the priority tree, indexing them as α_e for some natural number e. In this manner we will be able to denote the set of uses of an $\mathcal{R}$ strategy β_e by U^e. Similarly, we will denote the set of witnesses of an $\mathcal{S}$ strategy α_e by W^e, and the set of thresholds of an $\mathcal{S}$ startegy α_e by V^e.

Expansionary Stages

An $\mathcal{R}$ strategy attempting to satisfy a requirement $\mathcal{R}_{(\Phi,U,V)}$ will need to observe the agreement between the functional $\Phi^{U,V}$ and the set A in order to decide what to do at any given stage s. We define some useful notions related to this process below. Let β be an $\mathcal{R}$ strategy.

The *length of agreement* between the functional $\Phi^{U,V}$ and the set A at stage s is denoted by $l_s(\Phi^{U,V}, A)$ and is defined as $\max\{x \mid (\forall y < x)[\,\Phi^{U,V}[s](y) = A_s(y)\,]\}$.

A stage s is called an $\mathcal{R}$-*expansionary stage* if $l_s(\Phi^{U,V}, A) > l_t(\Phi^{U,V}, A)$ for all stages $t < s$. This notion of expansionary stage is a global notion defined over all stages s. Thus if there are multiple $\mathcal{R}$ strategies β which are attempting to satisfy the same requirement, and an $\mathcal{R}$-expansionary stage occurs, this will be noted by all such strategies. Note that if there are only finitely many $\mathcal{R}$-expansionary stages, the $\mathcal{R}$ requirement is satisfied trivially.

Given a strategy β, a stage s is called a β-*expansionary stage* if s is a β-stage and $l_s(\Phi^{U,V}, A) > l_t(\Phi^{U,V}, A)$ for all β-stages t with $t < s$. This notion of expansionary stage is a local notion which is specific to the strategy β. When using this notion, β is only measuring the length of agreement at those stages at which it is accessible, and deciding whether there was an expansionary stage based on the length of agreement at these stages. Note that if there are infinitely many $\mathcal{R}$-expansionary stages, and β sees only finitely many of them despite being accessible infinitely often, it follows that $\Phi^{U,V}(x)\uparrow$ at some element x and that the $\mathcal{R}$ requirement is also satisfied trivially.

2.1.5 Stages and Substages

The construction shall proceed through a number of stages as described in section 1.3.4. During each stage s, it will construct a current path of finite length through the priority tree. This path is built during a number of substages, with a new strategy on this path becoming accessible and executing during each substage. Since each strategy executes during its own substage, we shall instruct the strategy to'end the substage' once it is done executing. This will allow the next strategy on the current path to execute, if there is any. When a strategy is instructed to end the substage, we will also indicate from which step it should resume execution when it becomes next accessible during the appropriate substage of some stage $s' > s$.

Note that if a requirement can be satisfied through a single strategy, the priority tree will consist of that strategy only. In these cases, the construction goes through a number of stages as usual, but each stage will only have one substage, during which the strategy becomes accessible. This will be the case in sections 2.2 and 2.3, where we consider how to satisfy an $\mathcal{R}$ and an $\mathcal{S}$ requirement in isolation respectively. On the other hand, any atempt to satisfy multiple requirements simultaneously will require the use of multiple strategies laid upon a priority tree, and in these cases each stage will have multiple substages.

2.2 One $\mathcal{R}$ Requirement

In this section we shall consider how to satisfy one $\mathcal{R}$ requirement in isolation.

There are two ways in which an $\mathcal{R}$ requirement can be satisfied. If the equality $\Phi^{U,V} = A$ does not hold, the requirement is satisfied trivially. On the other hand, if the equality $\Phi^{U,V} = A$ holds, it is sufficient to build a functional $\Gamma^{U,D}$ such that $\Gamma^{U,D} = A$ holds. Alternatively, it is also possible to build a functional $\Gamma^{V,D}$ such that $\Gamma^{V,D} = A$ holds.

If an $\mathcal{R}$ strategy is going to build a functional $\Gamma^{U,D}$, we shall say that it is *following a* Γ*-strategy*. On the other hand, if the strategy is going to build a functional $\Gamma^{V,D}$, we shall say that it is *following a* $\hat{\Gamma}$*-strategy*.

In order to satisfy one $\mathcal{R}$ requirement we shall use an $\mathcal{R}$ strategy β following a Γ-strategy. This strategy will be able to choose parameters from a set of uses U^e.

The strategy then works as follows. If the current stage s is not a β-expansionary stage, the strategy will do nothing. On the other hand, if stage s is a β-expansionary stage, the strategy will define the axiom $\Gamma^{U,D}[s](x) = A_s(x)$ for every $x < l_s(\Phi^{U,V}, A)$.

When defining $\Gamma^{U,D}[s](x)$, the strategy also needs to choose a use $\gamma_s(x)$. This use is the least element in the set of uses U^e satisfying certain conditions. The first condition is that use chosen has to be at least as large as any use which has previously been chosen for the element x. The second condition is that the use chosen must be greater than the use which is presently assigned to all elements which are smaller than x.

2.2.1 The $\mathcal{R}$ Strategy

We give the formal definition of the $\mathcal{R}$ strategy below.

The $\mathcal{R}$ Strategy

This strategy has a set of uses U^e, and follows a Γ-strategy.

(1) (Check for expansionary stage). Is stage s a β-expansionary stage?

   (a) (Yes) Go to step (2).

   (b) (No) End the substage and resume from step (1).

(2) (Define the functional). For every $x < l_s(\Phi^{U,V}, A)$ such that $\Gamma^{U,D}[s](x) \uparrow$, define the axiom $\Gamma^{U,D}[s](x) = A_s(x)$, and choose the corresponding use $\gamma_s(x)$ to be the least element in the set of uses U^e obeying the following conditions (if it exists):

   (a) $\gamma_s(x) \geq \gamma_t(x)$ for all $t < s$.

   (b) $\gamma_s(x) > \gamma_s(y)$ for all $y < x$.

   (c) $\gamma_s(x)$ is greater than the stage at which β was last initialised.

   End the substage and resume from step (1).

We shall now consider the outcomes of the $\mathcal{R}$ strategy and define the outcome which the strategy selects at stage s.

Outcome of the $\mathcal{R}$ Strategy

The $\mathcal{R}$ strategy has two outcomes, the infinitary outcome i and the finitary outcome f. These outcomes are ordered as follows: $i <_\Lambda f$. At stage s, the strategy decides which outcome to visit by performing the following case analysis:

(I) *Waiting for expansionary stage.* The strategy ends its substage at step (1). Then the present stage s is not a β-expansionary stage. The $\mathcal{R}$ strategy visits its f outcome.

(II) *Expansionary stage.* The strategy ends its substage at step (2). Then the present stage s is a β-expansionary stage. The $\mathcal{R}$ strategy visits its i outcome.

We are now in a position to analyse whether the $\mathcal{R}$ strategy satisfies the $\mathcal{R}$ requirement or not.

2.2.2 Satisfaction of Requirement

We perform a case analysis based on the leftmost outcome visited infinitely often by the $\mathcal{R}$ strategy.

[f] Suppose that f is the leftmost outcome to be visited by the $\mathcal{R}$ strategy. Then this means that there have only been finitely many β-expansionary stages, and that the strategy has satisfied its requirement trivially.

[i] Suppose that i is the leftmost outcome to be visited by the $\mathcal{R}$ strategy. Then this means that there have been infinitely many β-expansionary stages. But during each such β-expansionary stage, the strategy has defined its functional to agree with A according to the new length of agreement. Hence we have that $\Gamma^{U,V} = A$, and the requirement is satisfied.

2.3 One $\mathcal{S}$ Requirement

In this section we shall consider how to satisfy one $\mathcal{S}$ requirement in isolation.

In order to satisfy an $\mathcal{S}$ requirement, one needs to ensure that $\Theta^D \neq A$. This can be achieved through an $\mathcal{S}$ strategy α which follows the Friedberg-Mučnik procedure.

The strategy α will have an infinite set of witnesses W^e. It will start by choosing the least unused witness w from this set and wait for a stage s such that $\Theta^D[s](w) \downarrow = 0$. When this becomes the case, the strategy will diagonalise by enumerating w into A and restraining $D \restriction \theta_s(w)$ to prevent the computation from changing value.

We give the formal definition of the $\mathcal{S}$ strategy below.

2.3.1 The $\mathcal{S}$ Strategy

The $\mathcal{S}$ Strategy

The strategy has a set of witnesses W^e. It chooses one parameter, the witness w.

(1) (Select the witness). Choose a witness w. The value selected for this witness is the least value in W^e such that:

  (a) w is greater than the stage at which α was last initialised.

(2) (Wait for convergence). Is $\Theta^D[s](w) \downarrow = 0$?

  (a) (Yes) Go to step (3).

  (b) (No) End this substage. Resume from step (2).

(3) (Diagonalise). Enumerate w into A, restrain $D \restriction \theta_s(w)$. Go to step (4).

(4) (Successful Diagonalisation). End this substage and resume from step (4).

We shall now consider the outcomes of the $\mathcal{S}$ strategy and define the outcome which the strategy selects at stage s.

Outcome of the $\mathcal{S}$ Strategy

The $\mathcal{S}$ strategy has two outcomes, the diagonalisation outcome d and the wait outcome w. These outcomes are ordered as follows: $d <_\Lambda w$. At stage s, the strategy decides which outcome to visit by performing the following case analysis:

(I) *Waiting for computation.* The strategy ends its substage at step (2)(b). Then we have that $\Theta^D[s](w) \uparrow$, or that $\Theta^D[s](w) \downarrow= 1$. The strategy visits its w outcome.

(II) *Successful Diagonalisation.* The strategy ends its substage at step (4). Then we have that the strategy has diagonalised successfully by enumerating w into A and restraining $D \upharpoonright \theta_s(w)$. The strategy visits its d outcome.

We are now in a position to analyse whether the $\mathcal{S}$ strategy satisfies the $\mathcal{S}$ requirement or not.

2.3.2 Satisfaction of Requirement

We perform a case analysis based on the leftmost outcome visited infinitely often by the $\mathcal{S}$.

[w] Suppose w is the leftmost outcome to be visited infinitely often. Then one of two things must be the case. Either we have that $\Theta^D(w) \uparrow$, in which case $\Theta^D(w) \neq A(w)$ and the requirement is satisfied, or else we have that $\Theta^D(w) \downarrow= 1$. But in this case, the strategy never enumerates w into A. Thus in this case we have that $\Theta^D(w) \neq A(w)$ as well and the requirement is also satisfied.

[d] Suppose d is the leftmost outcome to be visited infinitely often. Then it must be the case that $\Theta^D(w) \downarrow= 0$. But in this case, we have that w has been enumerated into A and that the computation cannot change its value due to the restraint imposed on D. Hence we have that $\Theta^D(w) \neq A(w)$, and the requirement is satisfied.

2.4 $\mathcal{S}$ Below $\mathcal{R}$ - First Approximation

In this section we shall consider how to satisfy two requirements $\mathcal{R}$ and $\mathcal{S}$, with $\mathcal{R}$ being of higher priority than $\mathcal{S}$.

The simultaneous satisfaction of these two requirements will only be possible through the interaction of multiple $\mathcal{R}$ and $\mathcal{S}$ strategies which are of a specific form and are organised in a specific way on a priority tree.

The form of the $\mathcal{R}$ and $\mathcal{S}$ strategies which are necessary shall not be given immediately, but will be arrived at through a sequence of approximations. During each step of the approximation, a provisional version of the $\mathcal{R}$ and $\mathcal{S}$ strategies will be introduced. A number of strategies of this form will then be organised into a priority tree in an attempt to satisfy the requirements.

When the present arrangement fails to satisfy the requirements in some specific way, we move to a new step of the approximation, introducing a new version of the $\mathcal{R}$ and $\mathcal{S}$ strategies which is able to circumvent this problem. This process continues until one arrives at a sufficiently sophisticated form of $\mathcal{R}$ and $\mathcal{S}$ strategies.

We shall now consider the first approximation for satisfying the two requirements $\mathcal{R}$ and $\mathcal{S}$. As we have already seen, an $\mathcal{R}$ requirement in isolation can be satisfied through an $\mathcal{R}$ strategy β of the form found in Section 2.3. In order to deal with an additional $\mathcal{S}$ requirement, we shall require an $\mathcal{S}$ strategy α which is capable of taking the β strategy above it into consideration.

In what follows we shall assume that the $\mathcal{R}$ requirement is not satisfied trivially, and that thus β has to actually build its functional in order for the requirement to be satisfied. This in turn means that β sees infinitely many β-expansionary stages. If this were not the case, the $\mathcal{S}$ requirement could be satisfied by an $\mathcal{S}$ strategy which ignores the existence of β. Such an $\mathcal{S}$ strategy could thus be of the form found in Section 2.3.

2.4.1 The $\mathcal{S}$ Strategy

We shall begin by considering the use of an $\mathcal{S}$ strategy α of the form found in Section 2.3. Such a strategy α would attempt to diagonalise by enumerating its witness w into A and by restraining $D \upharpoonright \theta(w)$.

However the strategy β will now also be affected by this action, since the enumeration of w into A causes the occurrence of the disagreements $\Phi^{U,V}(w) \neq A(w)$ and $\Gamma^{U,D}(w) \neq A(w)$. Now, since there are infinitely many β-expansionary stages, the disagreement $\Phi^{U,V}(w) \neq A(w)$ must be removed. This can happen either through a $U \upharpoonright \phi(w)$ change, or through a $V \upharpoonright \phi(w)$ change.

Consider the situation where $\phi(w) < \gamma(w)$. In this case, a $U \upharpoonright \phi(w)$ change also undefines $\Gamma^{U,D}(w)$, allowing the $\mathcal{R}$ strategy to redefine $\Gamma^{U,D}(w)$ so as to agree with the new value of $A(w)$. In this case we shall say that $\Gamma^{U,D}(w)$ is *honest* with respect to $\Phi^{U,V}(w)$, or more simply that $\Gamma^{U,D}(w)$ is honest.

Since this situation is a beneficial one, the α strategy should try to ensure that $\Gamma^{U,D}(w)$ is honest before enumerating w into A. In this way the disagreement $\Gamma^{U,D}(w) \neq A(w)$ can be corrected automatically in the case of a U change. The α strategy can achieve this by enumerating $\gamma(w)$ into D in order to undefine $\Gamma^{U,D}(w)$. This allows the β strategy to redefine $\Gamma^{U,D}(w)$ with an increased use $\gamma(w)$, which will eventually become greater than or equal to $\phi(w)$. We refer to this process as *honestification*.

One should note that it is also possible for $\Phi^{U,V}(w)$ to become undefined through a $V \upharpoonright \phi(w)$ change. In this case $\phi(w)$ could increase once the functional is defined again. If this results in $\phi(w) > \gamma(w)$ the α strategy would be required to perform honestification once again.

On the other hand, if the α strategy sees that $\Phi^{U,V}(w)$ is honest, it will proceed to *open a gap*. The strategy will enumerate w into A and restrain $D \upharpoonright \theta(w)$. Opening a gap creates a disagreement $\Phi^{U,V}(w) \neq A(w)$ for the strategy β. This will be repaired during the next β-expansionary stage, either through a $U \upharpoonright \phi(w)$ change, or through a $V \upharpoonright \phi(w)$ change.

After the β-expansionary stage leads to one or both of these changes, the α strategy will *close a gap*. If there has been a $U \upharpoonright \phi(w)$ change, then we have that the disagreement $\Gamma^{U,D}(w) \neq A(w)$

has been removed without any need to change D. Hence the α strategy has *diagonalised*, and is able to protect this diagonalisation by preserving its restraint on D.

On the other hand, if there has been no $U \upharpoonright \phi(w)$ change, we have that there has been a $V \upharpoonright \phi(w)$ change. In this case the disagreement $\Phi^{U,V}(w) \neq A(w)$ has been removed, whilst the disagreement $\Gamma^{U,D}(w) \neq A(w)$ persists. This disagreement must thus be repaired through some other means.

The α strategy can repair this disagreement by enumerating some $x \leq \gamma(w)$ into D, thus undefining $\Gamma^{U,D}(w)$. However, if $\theta(w) \geq \gamma(w)$, such an action would destroy the computation $\Theta^D(w)$, and the diagonalisation would fail, meaning that a new witness would have to be chosen and that the α strategy would have to start over.

To choose the x to enumerate into D we proceed as follows. Let the strategy α have an infinite set of thresholds V^e and an infinite set of witnesses W^e. The strategy will select two parameters, a threshold v from V^e and a witness w from W^e. When selecting these parameters, it will seek to satisfy the following parameter choice constraint:

$$v < \gamma(v) < w < \gamma(w)$$

If the strategy α does not see a U change when closing a gap, it will enumerate $\gamma(v)$ into D in order to undefine $\Gamma^{U,D}(w)$. We refer this action as *capricious destruction*. Note that whilst the witness w needs to change each time a gap is opened, the threshold v remains fixed. This fact will be important when defining *work intervals*, which will be discussed at a later stage.

In our description of the strategy we postponed the mention of two important facts.

The first is that when the α strategy opens a gap by enumerating the witness w into A and restraining $D \upharpoonright \theta(w)$, it shall impose a constraint on the β strategy to choose $\gamma(w)$ to be greater than $\theta(w)$ if it becomes undefined. In this way, if a U change occurs whilst closing a gap, not only is the disagreement $\Gamma^{U,D}(w) \neq A(w)$ removed without changing D, but the computation is also protected from any enumeration of $\gamma(x)$ with $x \geq w$ into D.

The second is that the α strategy will wait until the functional built by β is in a suitable state before proceeding. Suppose that α has defined some parameter. Then we shall require α to check whether the functional is defined and equal to A at every element less than or equal to the parameter before proceeding.

On the one hand, this makes sure that the operations carried out by α and the work intervals which will be introduced at a later stage will be well-defined. On the other hand, it will stop α from taking any action involving any parameter until any disagreements between the functional and A have been repaired below that parameter, ensuring that actions affecting the functional built by β take place in an orderly fashion.

The aforementioned checks will take place through the use of a background task which the strategy α performs whenever it is accessible, before resuming from where it had left. Note that in order for this measure to apply consistently, α will also need to make the same checks prior to choosing any parameter.

We now proceed to formalise the $\mathcal{S}$ strategy.

The $\mathcal{S}$ Strategy

The strategy has a set of witnesses W^e and a set of thresholds V^e. It chooses two parameters, the witness w and the threshold v. It lies below one $\mathcal{R}$ strategy β which follows a Γ-strategy.

(*) (Background Step) Perform this step at the beginning of every substage during which the strategy is accessible. If the following conditions are met resume from the step last indicated by the strategy, or resume from step (1) if no such step exists. Otherwise end the substage.

  (a) If v is defined, $(\forall_{n \leq v})(\Gamma^{U,D}[s](n) \downarrow = A_s(n))$.

  (b) If w is defined, $(\forall_{n \leq w})(\Gamma^{U,D}[s](n) \downarrow = A_s(n))$.

(1) (Select the thresholds). Choose a threshold v. The value chosen for this threshold is the least value in V^e such that:

  (a) v is greater than the stage at which α was last initialised.

(2) (Select the witness). Choose a witness w. The value chosen for this witness is the least value in W^e such that:

(a) $(\forall_{n \leq w})(\Gamma_s^{U,D}(n) \downarrow= A_s(n))$.

(b) $w > \gamma_s(v)$.

(c) w is greater than the stage at which α was last initialised.

If a witness satisfying these conditions cannot be found, end this substage. Resume from step (1). Otherwise go to step (3).

(3) (Wait for convergence). Is $\Theta^D[s](w) \downarrow= 0$?

(a) (Yes) Go to step (4).

(b) (No) End this substage. Resume from step (3).

(4) (Honestification). Is $\phi_s(w) > \gamma_s(w)$?

(a) (Yes) Enumerate $\gamma(w)$ into D. End this substage and resume from step (3).

(b) (No) Go to step (5).

(5) (Gap open) Constrain the strategy β to choose uses $\gamma_{s'}(w) > \theta_s(w)$ at all stages $s' > s$. Enumerate w into A and restrain $D \upharpoonright \theta_s(w)$. Cancel the witness w and end this substage. Resume from step (6).

(6) (Gap close) Let t be the stage at which the strategy last opened a gap by ending its substage at step (5). Let t' be the the least $\mathcal{R}$-expansionary stage greater than t. Is it the case that $U_t \upharpoonright \phi_t(w) \neq U_{t'} \upharpoonright \phi_t(w)$?

(a) (Yes) Go to step (8).

(b) (No) Go to step (7).

(7) (Capricious destruction) Enumerate $\gamma_s(v)$ into D. End this substage and resume from step (1).

(8) (Successful diagonalisation). End this substage. Resume from step (8).

We shall now consider the outcomes of the $\mathcal{S}$ strategy and define the outcome which the strategy selects at stage s.

Outcome of the $\mathcal{S}$ Strategy

The $\mathcal{S}$ strategy has four outcomes, the diagonalisation outcome d, the gap outcome g, the honestification outcome h and the wait outcome w, with $d <_\Lambda g <_\Lambda h <_\Lambda w$. At stage s, the strategy decides which outcome to visit by performing the following case analysis:

(I) *Waiting for parameters.* The strategy ends its substage at step (*) or at step (2). Then the functional of the active strategy β fails to be defined and equal to A up to some parameter, or α has failed to choose some parameter. The strategy visits its w outcome and imposes the work interval $(0, \infty)$.

(II) *Waiting for computation.* The strategy ends its substage at step (3). Then we have that $\Theta^D[s](w) \uparrow$, or $\Theta^D[s](w) \downarrow = 1$. The strategy visits its w outcome.

(III) *Honestification.* The strategy ends its substage at step (4)(a). Then we have that $\Theta^D[s](w) \downarrow = 0$ and that $\gamma_s(w) < \phi_s(w)$. The strategy visits its h outcome.

(IV) *Opening a gap.* The strategy ends its substage at step (5). Then we have that $\Theta_s^{D_s}(w) \downarrow = 0$, $\phi_s(w) \leq \gamma_s(w)$ and that w has been enumerated into A. The strategy has opened a gap. The strategy visits its g outcome.

(V) *Closing a gap - capricious destruction.* The strategy ends its substage at step (7). Then we have that the strategy has last opened a gap at stage t and that $U_t \upharpoonright \phi_t(w) = U_s \upharpoonright \phi_t(w)$. The strategy is closing a gap. The strategy visits its g outcome.

(VI) *Closing a gap - successful diagonalisation.* The strategy ends its substage at step (8). Then we have that the strategy has last opened a gap at stage t and that $U_t \upharpoonright \phi_t(w) \neq U_s \upharpoonright \phi_t(w)$. The strategy has diagonalised successfully. The strategy visits its d outcome.

(VII) *Stopped.* The strategy ends its substage at step (8), and step (8) has been visited since the strategy was last initialised. Then we have that the strategy has already diagonalised successfully. The strategy visits its d outcome.

2.4.2 The $\mathcal{R}$ Strategy

A more sophisticated $\mathcal{R}$ strategy β will be needed to deal with the actions of the $\mathcal{S}$ strategy lying below it. The $\mathcal{R}$ strategy β introduced in this section will differ from the one found in Section 2.2 in three respects.

Firstly, the β strategy needs to consider the fact that it is now possible for the functional $\Gamma^{U,D}$ to disagree with the set A at some element. The reason for such a disagreement is that the strategy α could have enumerated its witness w into A by opening a gap during a stage at which $\Gamma^{U,D}(w)\downarrow$. Whilst this disagreement persists, the β strategy cannot continue building its functional. Instead it will have to wait until the strategy α closes its gap so that the disagreement is removed.

Secondly, when defining its functional, β will not choose elements from the set U^e which have already been enumerated into the set D. This means that whenever α enumerates some use $\gamma(x)$ into D during honestification or capricious destruction, the β strategy will have to choose an increased use when redefining its functional at the element x.

Thirdly, when defining its functional, the strategy β has to observe any constraints which may have been imposed on it by lower priority $\mathcal{S}$ strategies which have opened a gap. This means that whenever α opens a gap by enumerating its witness w into A, the β strategy will have to choose a use greater than $\theta(w)$ when redefining its functional at the element w.

We formalise the $\mathcal{R}$ strategy below.

The $\mathcal{R}$ Strategy

The strategy has a set of uses U^e, and follows a Γ-strategy.

(1) (Check for expansionary stage). Is stage s a β-expansionary stage?

- (a) (Yes) Go to step (2).
- (b) (No) End the substage and resume from step (1).

(2) (Check for disagreement). Is there an m such that $\Gamma^{U,D}[s](m) \neq A_s(m)$?

- (a) (Yes) End the substage and resume from step (1).

(b) (No) Go to step (3).

(3) (Define the functional). For every $x < l_s(\Phi^{U,V}, A)$ such that $\Gamma^{U,D}[s](x) \uparrow$, define the axiom $\Gamma^{U,D}[s](x) = A_s(x)$, and choose the corresponding use $\gamma_s(x)$ to be the least element in the set of uses U^e obeying the following conditions:

(a) $\gamma_s(x) \geq \gamma_t(x)$ for all $t < s$.

(b) $\gamma_s(x) > \gamma_s(y)$ for all $y < x$.

(c) $\gamma_s(x) \notin D$.

(d) $\gamma_s(x) > y$, where y is a constraint imposed by some $\mathcal{S}$ strategy below β.

(e) $\gamma_s(x)$ is greater than the stage at which β was last initialised.

End the substage and resume from step (1).

We shall now consider the outcomes of the $\mathcal{R}$ strategy and define the outcome which the strategy selects at stage s.

Outcome of the $\mathcal{R}$ Strategy

The $\mathcal{R}$ strategy has two outcomes, the infinitary outcome i and the finitary outcome f. These outcomes are ordered as follows: $i <_\Lambda f$. At stage s, the strategy decides which outcome to visit by performing the following case analysis:

(I) *Waiting for expansionary stage.* The strategy ends its substage at step (1). Then the present stage s is not an expansionary stage. The $\mathcal{R}$ strategy visits its f outcome.

(II) *Expansionary stage.* The strategy ends its substage at step (2) or step (3). Then the present stage s is an expansionary stage. The $\mathcal{R}$ strategy visits its i outcome.

2.4.3 Organisation of Priority Tree

We shall now make our first attempt at organising a priority tree to satisfy an $\mathcal{S}$ requirement below an $\mathcal{R}$ requirement. The following discussion will refer to the priority tree shown in Figure 2.1. The following notation will be used on the priority tree.

- β^U will denote an $\mathcal{R}$ strategy (from Section 2.4) which is following a Γ-strategy.
- α^U will denote an $\mathcal{S}$ strategy (from Section 2.4) which needs to take into consideration one $\mathcal{R}$ strategy following a Γ-strategy above it.
- α will denote an $\mathcal{S}$ strategy (from Section 2.3) which does not need to take into consideration any $\mathcal{R}$ strategy above it.

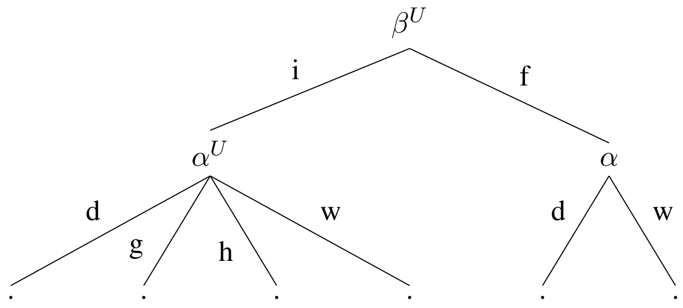

Figure 2.1: Priority tree for $\mathcal{S}$ below $\mathcal{R}$, first approximation

We shall start by attempting to satisfy the highest priority requirement, which happens to be $\mathcal{R}$. We assign an $\mathcal{R}$ strategy β^U of the form found in Section 2.4 for this purpose.

Now, the strategy β^U has two outcomes, i and f. The i outcome means that the strategy β^U sees infinitely many expansionary stages, and needs to build its functional in order to satisfy the $\mathcal{R}$ requirement. On the other hand, the f outcome means that β^U sees only finitely many expansionary stages. This would imply that the $\mathcal{R}$ requirement is satisfied trivially.

Below the f outcome of the β^U strategy, we have that the $\mathcal{S}$ requirement is the highest priority unsatisfied requirement. In addition the $\mathcal{R}$ requirement is satisfied trivially, with or without the action of the β^U strategy. Hence the $\mathcal{S}$ requirement can be satisfied through an $\mathcal{S}$ strategy α which ignores β^U. In this case, α can be of the form found in Section 2.3. The strategy α will then satisfy the $\mathcal{S}$ requirement, following the satisfaction of requirements analysis given in the same section.

Below the i outcome of the β^U strategy, we have that the $\mathcal{S}$ requirement is the highest priority unsatisfied requirement. But in this case, the $\mathcal{R}$ requirement is not satisfied trivially, and we require the strategy β^U to actually build its functional in order for the $\mathcal{R}$ requirement to be satisfied. This means that we need to use an $\mathcal{S}$ strategy which takes the $\mathcal{R}$ strategy above it into consideration.

We shall thus use an $\mathcal{S}$ strategy α^U of the form found in Section 2.4.

We are now in a position to analyse whether the strategies β^U and α^U acting together are sufficient to satisfy the $\mathcal{R}$ and $\mathcal{S}$ requirements on this branch of the priority tree.

2.4.4 Satisfaction of Requirements

We perform a case analysis based on the leftmost outcome visited infinitely often by the strategy α^U.

$\boxed{w}$ Suppose w is the leftmost outcome to be visited infinitely often. Then we either have that $\Theta^D(w) \uparrow$ or that $\Theta^D(w) \downarrow= 1$ for some witness w. In the first case, we have that $\Theta^D(w) \neq A(w)$ trivially. In the second case we have that the strategy never enumerates the witness w into A, and thus that $\Theta^D(w) \neq A(w)$ as well. Hence the $\mathcal{S}$ requirement is satisfied. In addition the strategy β^U is able to construct its functional $\Gamma^{U,V}$ without interference after some stage, meaning that the $\mathcal{R}$ requirement is satisfied as well.

$\boxed{h}$ Suppose h is the leftmost outcome to be visited infinitely often. Then there must be some witness w such that $\Gamma^{U,D}(w)$ is dishonest infinitely often. Each time $\Gamma^{U,D}(w)$ is dishonest, α^U performs honestification by enumerating $\gamma(w)$ into D. This undefines $\Gamma^{U,D}(w)$, leading the strategy β^U to redefine the functional by increasing the use $\gamma(w)$.

Since $\phi(w) > \gamma(w)$ infinitely often, honestification must take place infinitely often, causing $\lim_s \gamma_s(w) \to \infty$. On the other hand, the increase in $\gamma(w)$ must eventually cause $\gamma(w)$ to become greater than or equal to $\phi(w)$. Hence for $\Gamma^{U,D}(w)$ to be dishonest infinitely often, it follows that $\lim_s \phi_s(w) \to \infty$ must be the case as well. Hence we have that $\Gamma^{U,D}(w) \uparrow$, but also that $\Phi^{U,V}(w) \uparrow$, which means that the $\mathcal{R}$ requirement is satisfied trivially.

On the other hand the α^U strategy does not enumerate its witness w into A. Hence it does not manage to diagonalise against the computation $\Theta^D(w) \downarrow= 0$ and the $\mathcal{S}$ requirement remains unsatisfied. We shall thus require a *backup strategy* in order to satisfy this requirement, which will be discussed in the next section. It is important to note that despite the failure to satisfy both requirements, progress has been made through the satisfaction of the $\mathcal{R}$ requirement.

g Suppose g is the leftmost outcome to be visited infinitely often. Then there must be infinitely many witnesses w such that α^U opens a gap at some stage s by enumerating w into A and restraining $D \upharpoonright \theta_s(w)$. This is then followed by an expansionary stage at some stage $t > s$. Since a $U \upharpoonright \phi_s(w)$ change must fail to occur by stage t, we have that a $V \upharpoonright \phi_t(w)$ change must have occurred instead.

This means that the disagreement $\Gamma^{U,D}(w) \neq A(w)$ persists and that the α^U strategy has to perform capricious destruction by enumerating $\gamma(v)$ into D in order to undefine $\Gamma^{U,D}(w)$. It follows that $\Gamma^{U,D}(v)$ is undefined infinitely often, and that thus $\Gamma^{U,D}(v) \uparrow$. In addition, if $\gamma(v) \leq \theta(w)$, capricious destruction also destroys the computation $\Theta^D(w)$, forcing the α^U strategy to start anew. Since the $\mathcal{S}$ strategy never manages to diagonalise successfully, we have that the $\mathcal{S}$ requirement remains unsatisfied. Once again we shall need a backup strategy in order to satisfy the $\mathcal{S}$ requirement.

On the other hand, the β^U strategy has also failed to satisfy its requirement, because $\Gamma^{U,D}(v) \uparrow$. We thus also require a backup strategy in order to satisfy the $\mathcal{R}$ requirement. There seems to be a problem here in that no progress has been made in terms of satisfying at least one of the $\mathcal{R}$ and $\mathcal{S}$ requirements. However each time the strategy α^U fails to witness a $U \upharpoonright \phi(w)$ change, a $V \upharpoonright \phi(w)$ change occurs instead. It will be possible for a backup $\mathcal{R}$ strategy to use these changes in order to construct a functional $\Gamma^{V,D}$, as will be discussed in the next section.

d Suppose d is the leftmost outcome of the α^U strategy to be visited infinitely often. Then there must be a witness w such that α^U opens a gap at some stage s by enumerating w into A and restraining $D \upharpoonright \theta_s(w)$. This is then followed by an expansionary stage at some stage $t > s$, with a $U \upharpoonright \phi_s(w)$ change occurring by stage t. The U change undefines $\Gamma^{U,D}(w)$, allowing the β^U strategy to repair the disagreement without changing the set D.

The α^U strategy has diagonalised successfully, whilst protecting its computation $\Theta^D(w)$. Hence the $\mathcal{S}$ requirement has been satisfied. In addition after stage t the β^U strategy remains able to construct its functional $\Gamma^{U,D}$ without further interference, meaning that the $\mathcal{R}$ requirement is satisfied as well.

From the above discussion one can conclude that the g and h outcomes of the α^U strategy leave one or more requirements unsatisfied. This will require us to pass to a second approximation which is able to deal with this situation.

2.5 $\mathcal{S}$ Below $\mathcal{R}$ - Second Approximation

In this section we shall make a second attempt to satisfy an $\mathcal{S}$ requirement below an $\mathcal{R}$ requirement. In order to achieve this we shall need to present the second approximation to the $\mathcal{S}$ and $\mathcal{R}$ strategies. A number of concepts need to be introduced before one can proceed.

2.5.1 Backup Strategies

In Section 2.4, we have seen that some of the outcomes of the $\mathcal{S}$ strategy α leave one or both of the requirements unsatisfied. In general, when a strategy leaves a requirement unsatisfied, one resorts to a *backup strategy* which attempts to satisfy the requirement once again. Such a backup strategy has the added advantage of lying below the outcome which has led to the requirement remaining unsatisfied. Hence it is aware of why the original strategy has failed to satisfy the requirement in question. This represents additional information which can help the backup strategy to satisfy the requirement.

Note that such backup strategies will not work in the same way as the strategies which have left certain requirements unsatisfied. This is because they need to exploit the additional information which is available to them by virtue of being below the problematic outcomes. In addition these backup strategies now have to contend with the fact that they are operating below other $\mathcal{S}$ and $\mathcal{R}$ strategies, and this has to be taken into consideration.

In order to deal with unsatisfied requirements below an outcome of a given strategy, one proceeds as follows. One first lists the requirements which are left unsatisfied below a given outcome in priority order. Then one chooses the highest priority unsatisfied requirement and assigns a backup strategy of the appropriate kind to satisfy it below that outcome. This process can then be repeated below outcomes of the backup strategy itself, until every requirement has been satisfied.

2.5.2 Work Intervals

The $\mathcal{S}$ strategy α in Section 2.4 enumerates elements into the sets A and D. Strategies operating below one of the outcomes of α risk being injured by the actions being taken by α itself. Work intervals can be used to protect lower priority strategies from being injured.

A work interval is an interval of natural numbers $(a, b) = \{x \in \mathbb{N} \mid a < x < b\}$ which is associated to an outcome of an $\mathcal{S}$ strategy. In fact, every outcome of an $\mathcal{S}$ strategy will be assigned a work interval. $\mathcal{R}$ or $\mathcal{S}$ strategies lying below an outcome of an $\mathcal{S}$ strategy will work inside the associated work interval in the following manner.

An $\mathcal{R}$ strategy operating inside a work interval will choose uses lying inside the work interval. On the other hand $\mathcal{S}$ strategies operating inside a work interval will choose thresholds and witnesses inside the work interval. In addition an $\mathcal{S}$ strategy will only trust a computation $\Theta^D[s](w)$ if its use $\theta_s(w)$ is inside the work interval.

Suppose that an $\mathcal{S}$ strategy has a work interval (a, b) associated to one of its outcomes. Then when visiting this outcome, the strategy will only enumerate elements into the sets A or D which are greater than or equal to b. This will ensure that actions taken by the $\mathcal{S}$ strategy when visiting that outcome do not injure strategies located below that outcome.

In addition the upperbound b shall either be absent, or else will increase during visits to the outcome. In this way the work interval will increase in length, allowing strategies located below the outcome more space where to operate.

The work intervals associated to the different outcomes of the $\mathcal{S}$ strategy will also be ordered in a specific way, so that strategies operating below one outcome, do not interfere with strategies operating below an outcome to its left. This ordering between work intervals arises out of the ordering between parameters enforced by the $\mathcal{S}$ strategy.

Different forms of $\mathcal{S}$ strategy will give rise to different kinds of work interval. These will be made explicit when defining the outcomes of that particular form of $\mathcal{S}$ strategy.

To give an example of their use, we shall explain the work intervals which one can expect to find below the $\mathcal{S}$ strategy α of the form found in Section 2.4. This strategy has four outcomes d, g, h

and w, and a work interval has to be defined for each.

Consider the outcome w of the $\mathcal{S}$ strategy. When visiting this outcome, the $\mathcal{S}$ strategy does not enumerate any elements into the set D or A. Hence the strategy can impose the interval $(0, \infty)$ below this outcome. Clearly this does not place any constraint on lower priority strategies.

Consider now the outcome h of the $\mathcal{S}$ strategy. When visiting this outcome, the $\mathcal{S}$ may enumerate $\gamma(w)$ into the set D. In order to protect strategies below the h outcome from this enumeration into D, the strategy shall impose the work interval $(w, \gamma(w))$ below this outcome.

The next outcome to consider is the outcome g of the $\mathcal{S}$ strategy. When visiting this outcome, the $\mathcal{S}$ strategy may enumerated its witness w into the set A, and may also enumerate $\gamma(v)$ into D. Strategies below the g outcome of the $\mathcal{S}$ strategy will be protected from enumeration into the set D through the imposition of the work interval $(v, \gamma(v))$ below this outcome. Since the ordering between parameters ensures that $\gamma(v) < w$, we also have that strategies below the g outcome are protected from the enumeration of w into A by the same work interval.

The final outcome which needs to be considered is the outcome d of the $\mathcal{S}$ strategy. When visiting this outcome, the $\mathcal{S}$ strategy does not enumerate any elements into the set D or A. The strategy imposes the interval (s_1, ∞) below the d outcome, where s_1 is the stage at which the strategy has diagonalised successfully and visited the outcome d for the first time. This work interval stops all strategies below the d outcome of the $\mathcal{S}$ strategy from interfering with any computation which took place before stage s_1.

There is one final operation which is necessary to ensure that strategies operating below one outcome do not interfere with strategies operating below an outcome lying to its left. Whenever a strategy is accessible, it will initialise every strategy lying to its right. When a strategy which has been initialised selects its parameters, it will select parameters which are greater than the stage at which it is initialised. This is the reason behind the fact that no work interval is needed below the w outcome.

2.5.3 Active Strategies

In general, there will be many strategies attempting to satisfy a given requirement. However a given strategy γ might not be concerned with every higher priority strategy γ' located above it.

Since γ is located below a certain outcome of γ', it might be able to tell that the latter will fail to satisfy its requirement, or that the requirement would be satisfied trivially regardless of its actions.

If one of the above conditions is the case, γ' will be inactive for γ. Otherwise, we shall say that γ' is active for γ. A strategy γ only needs to take a higher priority strategy γ' into consideration if γ' is active for γ. Otherwise γ will be able to ignore its existence.

2.5.4 $\hat{\Gamma}$-Strategies

In Section 2.4 we introduced $\mathcal{R}$ strategies which were only capable of following Γ-strategies. In our second approximation, $\mathcal{R}$ strategies will either follow a Γ-strategy or a $\hat{\Gamma}$-strategy. This means that they will either build a functional $\Gamma^{U,D}$ or a functional $\Gamma^{V,D}$. The kind of strategy followed will depend on the position of the strategy on the priority tree.

2.5.5 Switching

In Section 2.4 we have also seen that visits to the g outcome of $\mathcal{S}$ strategies during capricious destruction are related to the occurrence of a V change. The first step towards exploiting these V-changes is to have strategies which are able to follow a $\hat{\Gamma}$-strategy and build a functional $\Gamma^{V,D}$. We shall say that the g outcome of the $\mathcal{S}$ strategy causes a *switch* in the manner of satisfying the $\mathcal{R}$ requirement. Such a switch causes $\mathcal{R}$ strategies below the g outcome of the $\mathcal{S}$ strategy to follow a $\hat{\Gamma}$-strategy instead.

2.5.6 Open Stages and Close Stages

It shall be convenient for a given strategy α to keep track of what strategies lying above it are doing during any given stage. In particular, a strategy will want to distinguish between stages during which higher priority strategies are all enumerating elements into D, and stages during which no higher priority strategies are enumerating elements into D. The first kind of stage occurs when higher priority strategies with g outcomes above α are performing capricious destruction and higher priority strategies with h outcomes are honestifying, and is called a close stage. The second kind of stage occurs when higher priority strategies with g outcomes above α are opening gaps by enumerating elements into A and higher priority strategies with h outcomes above α are not enumerating any elements into D. These definitions are useful to synchronise the actions of lower priority strategies with those of higher priority strategies as will be discussed in the following section.

2.5.7 S-Synchronisation

$\mathcal{S}$ strategies enumerate elements into A and restrain D when opening a gap, and enumerate elements into D when closing a gap through capricious destruction or when honestifying. Thus in a situation with multiple $\mathcal{S}$ strategies, it is possible for different $\mathcal{S}$ strategies to be enumerating and restraining the set D at the same time. Whilst work intervals protect lower priority strategies which are restraining D from the actions of higher priority strategies which are enumerating into D, the reverse is not the case.

To prevent such a situation from occurring we will require an $\mathcal{S}$ strategy α which is located below the g or h outcome of a higher priority $\mathcal{S}$ strategy α' to S-Synchronise with it.

In this case, the lower priority strategy will wait for the higher priority strategies to enumerate elements into D before enumerating elements of its own into D. In other words it will wait for a close stage before performing capricious destruction or honestification. This will stop the $\mathcal{S}$ strategy α from injuring any restraint which may have been imposed by some higher priority strategy α' on D.

In addition to this, the lower priority strategy will only enumerate elements into A if higher priority strategies with g outcomes above α have also enumerated elements into A, and if higher priority strategies with h outcomes above α have not enumerated elements into D. In other words, the lower priority strategy will wait for an open stage before opening a gap. When combined with work intervals, $\hat{\Gamma}$-strategies, and switching this will provide the second step towards exploiting V-changes to satisfy $\mathcal{R}$ requirements.

For suppose that at stage t the strategy α enumerates w into A and some strategy α' enumerates w' into A. This means that by the least $\mathcal{R}$-expansionary stage t' with $t' > t$, the disagreements between $\Phi^{U,V}$ and A at w and w' will have been removed. This is due to the occurence of a $U \restriction \phi_t(x)$ change or a $V \restriction \phi_t(x)$ change (or both) by stage t', where $x = \min\{w, w'\}$.

We shall arrange the priority tree so that whenever the strategy α' cannot use a $V \restriction \phi_t(x)$ change to diagonalise, the strategy α will be able to use the $V \restriction \phi_t(x)$ change to diagonalise instead. In addition this V change will undefine the functional being built by the active $\mathcal{R}$ strategy above α without the latter having to enumerate any elements into D.

We have now developed all the concepts required in order to discuss the second approximation to the $\mathcal{R}$ and $\mathcal{S}$ strategies.

2.5.8 The $\mathcal{R}$ Strategy

The main difference between the first approximation to the $\mathcal{R}$ strategy and the second approximation is that the latter can now either follow a Γ-strategy or a $\hat{\Gamma}$-strategy, depending on its position on the priority tree.

The $\mathcal{R}$ Strategy

This strategy has a set of uses U^e, and can either follow a Γ strategy of a $\hat{\Gamma}$-strategy. The strategy operates inside a work interval (a, b).

(1) (Check for expansionary stage). Is stage s a β-expansionary stage?

   (a) (Yes) Go to step (2).

   (b) (No) End the substage and resume from step (1).

(2) (Check for disagreement). Is there an m such that $\Gamma^{U,D}[s](m) \neq A_s(m)$? (or $\Gamma^{V,D}[s](m)$ resp.).

(a) (Yes) End the substage and resume from step (1).

(b) (No) Go to step (3).

(3) (Define the functional). For every $x < l_s(\Phi^{U,V}, A)$ such that $\Gamma^{U,D}[s](x) \uparrow$ (or $\Gamma^{V,D}[s](x)$ resp.), define the axiom $\Gamma^{U,D}[s](x) = A_s(x)$ (or $\Gamma^{V,D}[s](x)$ resp.), and choose the corresponding use $\gamma_s(x)$ to be the least element in the set of uses U^e obeying the following conditions (if it exists):

(a) $\gamma_s(x) \geq \gamma_t(x)$ for all $t < s$.

(b) $\gamma_s(x) > \gamma_s(y)$ for all $y < x$.

(c) $a < \gamma_s(x) < b$.

(d) $\gamma_s(x) \notin D$.

(e) $\gamma_s(x) > y$, where y is a constraint imposed by some $\mathcal{S}$ strategy below β.

(f) $\gamma_s(x)$ is greater than the stage at which β was last initialised.

End the substage and resume from step (1).

We shall now consider the outcomes of the $\mathcal{R}$ strategy and define the outcome which the strategy selects at stage s.

Outcomes of the $\mathcal{R}$ Strategy

The $\mathcal{R}$ strategy has two outcomes, the infinitary outcome i and the finitary outcome f. These outcomes are ordered as follows: $i <_\Lambda f$. At stage s, the strategy decides which outcome to visit by performing the following case analysis:

(I) *Waiting for expansionary stage.* The strategy ends its substage at step (1). Then the present stage s is not a β-expansionary stage. The $\mathcal{R}$ strategy visits its f outcome.

(II) *Expansionary stage.* The strategy ends its substage at step (2) or step (3). Then the present stage s is a β-expansionary stage. The $\mathcal{R}$ strategy visits its i outcome.

We now proceed to describe and formalise the second approximation to the $\mathcal{S}$ strategy.

2.5.9 The $\mathcal{S}$ Strategy

The $\mathcal{S}$ strategy α has to take into consideration one $\mathcal{R}$ strategy β lying above it, which can now be either following a Γ-strategy or a $\hat{\Gamma}$-strategy. It will thus choose a threshold v and a witness w as before, this time making sure that these parameters lie inside any work interval which has been imposed on it. It will then wait for $\Theta^D[s](w) \downarrow = 0$ to be the case, trusting this computation only if the use $\theta_s(w)$ is also inside the work interval.

The strategy α will then check whether $\Gamma^{U,D}(w)$ (or $\Gamma^{V,D}(w)$ resp.) is dishonest. If $\phi_s(w) > \gamma_s(w)$, the strategy needs to honestify $\Gamma^{U,D}(w)$ by enumerating $\gamma_s(w)$ into D.

Before honestifying, α must check whether stage s is a close stage as a result of S-Synchronisation. If this is the case, α will honestify by enumerating $\gamma_s(w)$ into D. If this is not the case, α will not enumerate $\gamma_s(w)$ into D, but will still behave as if it had honestified, by visiting the corresponding outcome. Without such a measure one would have a situation in which strategies lying below the h outcome of the α strategy would be accessible only during close stages.

If $\Gamma^{U,D}(w)$ is honest, the strategy α will want to open a gap. However prior to opening a gap α has to check whether stage s is an open stage as a result of S-Synchronisation.

If this is the case α will open a gap by enumerating its witness w into A and restraining $D \upharpoonright \theta_s(w)$. Otherwise the strategy will not open a gap and will wait instead.

Note that if s is not an open stage, the strategy will have to go through all the previous steps once again before making another attempt at opening a gap. In particular the strategy has to check whether $\Gamma^{U,D}(w)$ is still honest, as in the meantime $\phi(w)$ might have increased as a result of a U or a V change.

Once α opens a gap at some stage t, it creates a disagreement between $A(w)$ and $\Phi^{U,V}(w)$. Now, in order for α to become accessible again, there must have been some β-expansionary stage. This means that an $\mathcal{R}$ -expansionary stage must have taken place at some stage t', and that the disagreement must have been removed through a $U \upharpoonright \phi_t(w)$ change, or through a $V \upharpoonright \phi_t(w)$ change.

Now when α becomes accessible once again at some stage $s \geq t'$, it will proceed to close the gap by determining whether a $U \upharpoonright \phi_t(w)$ change has occurred between stage t and stage t'.

If β is following a Γ-strategy and a $U \upharpoonright \phi_t(w)$ change has occurred, then α has diagonalised successfully. Otherwise a $V \upharpoonright \phi_t(w)$ change must have occurred and α needs to perform capricious destruction.

However, prior to performing capricious destruction, the strategy α has to check whether stage s is a close stage as a result of S-Synchronisation. If this is the case, α enumerates $\gamma_s(v)$ into D. Otherwise it waits for such a stage to occur before performing capricious destruction.

On the other hand, if β is following a $\hat{\Gamma}$-strategy, α needs a $V \upharpoonright \phi(w)$ change to ensure that the functional being built by β is repaired without it having to enumerate elements into the set D. But the priority tree will be organised in a manner such that α will automatically obtain this change as a result of some other higher priority $\mathcal{S}$ strategy above it failing to obtain a $U \upharpoonright \phi_t(w)$ change.

Hence if the β strategy is following a $\hat{\Gamma}$-strategy, we have that the α strategy diagonalises successfully during the same stage it enumerates its witness into the set A, because the occurrence of the V change required to repair the functional built by β is guaranteed. This means that α does not need to check whether such a change has occurred, and that it does not need to perform any capricious destruction step.

We give the formal definition of the $\mathcal{S}$ strategy below.

The $\mathcal{S}$ Strategy

The strategy has a set of witnesses W^e and a set of thresholds V^e. It chooses two parameters, the witness w and the threshold v. It lies below one active $\mathcal{R}$ strategy β, which may either be following a Γ-strategy or a $\hat{\Gamma}$-strategy. The strategy operates inside a work interval (a, b).

(*) (Background Step) Perform this step at the beginning of every substage during which the strategy is accessible. If the following conditions are met resume from the step last indicated by the strategy, or resume from step (1) if no such step exists. Otherwise end the substage.

(a) If v is defined, $(\forall_{n \leq v})(\Gamma^{U,D}[s](n) \downarrow = A_s(n))$ (or $\Gamma^{V,D}[s](n)$ resp.).

(b) If w is defined, $(\forall_{n\leq w})(\Gamma^{U,D}[s](n)\downarrow= A_s(n))$ (or $\Gamma^{V,D}[s](n)$ resp.).

(c) If w is defined, $a < \theta_s(w) < b$.

(1) (Select the thresholds). If no threshold v corresponding to β is defined, choose a threshold v. The value selected for this threshold is the least value in V^e such that:

(a) $(\forall_{n\leq v})(\Gamma^{U,D}[s](n)\downarrow= A_s(n))$ (or $\Gamma^{V,D}[s](n)$ resp.).

(b) $a < v < b$.

(c) v is greater than the last stage at which α was last initialised.

If thresholds satisfying these conditions cannot be found, end this substage. Resume from step (1). Otherwise go to step (2).

(2) (Select the witness). Choose a witness w. The value selected for this witness is the least value in W^e such that:

(a) $(\forall_{n\leq w})(\Gamma^{U,D}[s](n)\downarrow= A_s(n))$ (or $\Gamma^{V,D}[s](n)$ resp.).

(b) $a < w < b$.

(c) $\Theta^D(w)\downarrow$

(d) $a < \theta_s(w) < b$.

(e) $w > \gamma_s(v)$.

(f) w is greater than the last stage at which α was last initialised.

If a witness satisfying these conditions cannot be found, end this substage. Resume from step (1). Otherwise go to step (3).

(3) (Wait for convergence). Is $\Theta^D[s](w)\downarrow= 0$?

(a) (Yes) Go to step (4).

(b) (No) End this substage. Resume from step (3).

(4) (Honestification). Is $\phi_s(w) > \gamma_s(w)$?

(a) (Yes) Is s a close stage?

(i) (Yes) Enumerate $\gamma_s(w)$ into D. End this substage and resume from step (3).

(ii) (No) End this substage and resume from step (3).

(b) (No) Go to step (5).

(5) (Gap open) Is stage s an open-stage?

(a) (Yes) Constrain β to choose uses $\gamma_{s'}(w) > \theta_s(w)$ at all stages $s' > s$. Enumerate w into A and restrain $D \upharpoonright \theta_s(w)$. Cancel the witness w. End this substage. Resume from step (6).

(b) (No) End this substage and resume from step (3).

(6) (Gap close) Is stage s a close-stage?

(a) (Yes) Let t be the stage at which the strategy last opened a gap by ending its substage at step (5), and let t' be the least $\mathcal{R}$-expansionary stage greater than t. If β is following a Γ-strategy and $U_t \upharpoonright \phi_t(w) = U_{t'} \upharpoonright \phi_t(w)$, go to step (7). If β is following a Γ-strategy and $U_t \upharpoonright \phi_t(w) \neq U_{t'} \upharpoonright \phi_t(w)$, go to step (8). If β is following a $\hat{\Gamma}$-strategy, go to step (8).

(b) (No) End this substage and resume from step (6).

(7) (Capricious destruction) Enumerate $\gamma_s(v)$ into D. End this substage and resume from step (2).

(8) (Successful diagonalisation). End this substage and resume from step (8).

Outcomes of $\mathcal{S}$ Strategy

The $\mathcal{S}$ strategy α has between three and four outcomes. The first three outcomes will be the diagonalisation outcome d, the honestification outcome h and the wait outcome w. The fourth or capricious destruction outcome g, occurs only if the active $\mathcal{R}$ strategy β lying above α is following a Γ-strategy. These outcomes, when present, are ordered as follows: $d <_\Lambda g <_\Lambda h <_\Lambda w$. At stage s, the strategy decides which outcome to visit by performing the following case analysis:

(I) *Waiting for parameters.* The strategy ends its substage at the (Background Step), step (1) or step (2). Then the functional of the active strategy β fails to be defined and be equal to A up to some parameter, or α has failed to choose some parameter. The strategy visits its w outcome and imposes the work interval $(0, \infty)$.

(II) *Waiting for computation.* The strategy ends its substage at step (3). Then we have that $\Theta^D[s](w) \uparrow$, or $\Theta^D[s](w) \downarrow= 1$. The strategy visits its w outcome and imposes the work interval $(0, \infty)$.

(III) *Honestification.* The strategy ends its substage at step (4)(a)(i). Then we have that $\Theta^D[s](w) \downarrow= 0$, $\phi_s(w) > \gamma_s(w)$ and that s is a close stage. The strategy visits its h outcome and imposes the work interval $(w, \gamma_s(w))$.

(IV) *Honestification - waiting for close stage.* The strategy ends its substage at step (4)(a)(ii). Then we have that $\Theta^D[s](w) \downarrow= 0$, and that $\phi_s(w) > \gamma_s(w)$, but s is not a close stage. The strategy visits its h outcome and imposes the work interval $(w, \gamma_s(w))$.

(V) *Opening a gap - g outcomes exist.* The strategy ends its substage at step (5)(a). Then we have that $\Theta^D[s](w) \downarrow= 0$, $\phi_s(w) \leq \gamma_s(w)$ and that w has been enumerated into A. In addition the strategy β is following a Γ-strategy. The strategy has opened a gap. The strategy visits its g outcome and imposes the work interval $(v, \gamma_s(v))$.

(VI) *Opening a gap - no g outcomes.* The strategy ends its substage at step (5)(a). Then we have that $\Theta^D[s](w) \downarrow= 0$, $\phi_s(w) \leq \gamma_s(w)$ and that w has been enumerated into A. In addition the strategy β is following a $\hat{\Gamma}$-strategy. Then we have that the strategy has diagonalised successfully. The strategy visits its d outcome and imposes the work interval (s_1, ∞), where s_1 is equal to the present stage s.

(VII) *Opening a gap - waiting for open stage.* The strategy ends its substage at step (5)(b). Then we have that $\Theta^D[s](w) \downarrow= 0$ and that $\phi_s(w) \leq \gamma_s(w)$ but s is not an open stage. The strategy visits its w outcome and imposes the work interval $(0, \infty)$.

(VIII) *Closing a gap - waiting for close stage.* The strategy ends its substage at step (6)(b). Then we have that the strategy has opened a gap at stage t, that there has been no close stage between t and s and that s is not a close stage. The strategy visits its w outcome and imposes the work interval $(0, \infty)$.

(IX) *Closing a gap - capricious destruction.* The strategy ends its substage at step (7). Then we have that the strategy has opened a gap at stage t, that there has been no close stage between t and s and that s is a close stage. In addition $U_{1,t} \upharpoonright \phi_{1,t}(w) = U_{1,t'} \upharpoonright \phi_{1,t}(w)$, where t' is the least $\mathcal{R}_1$-expansionary stage greater than t. The strategy is closing a gap. The strategy visits its g outcome and imposes the work interval $(v, \gamma_s(v))$.

(X) *Closing a gap - successful diagonalisation.* The strategy ends its substage at step (8). Then we have that the strategy has opened a gap at stage t, that there has been no close stage between t and s and that s is a close stage. In addition $U_{1,t} \upharpoonright \phi_{1,t}(w) \neq U_{1,t'} \upharpoonright \phi_{1,t}(w)$, where t' is the least $\mathcal{R}$-expansionary stage greater than t. The strategy has diagonalised successfully. The strategy visits its d outcome and imposes the work interval (s_1, ∞) where s_1 is the stage at which the strategy has diagonalised successfully and visited its outcome d.

(XI) *Stopped.* strategy ends its substage at step (8), and step (8) has been visited since the strategy was last initialised. Then we have that the strategy has already diagonalised successfully. The strategy visits its d outcome and imposes the work interval (s_1, ∞), where s_1 is the stage at which the strategy has first diagonalised successfully and visited its outcome d.

The next step is to present the priority tree.

2.5.10 Organisation of Priority Tree

We shall now organise a priority tree in order to make our second attempt at satisfying an $\mathcal{S}$ requirement below an $\mathcal{R}$ requirement. The following discussion will refer to the priority tree shown in Figure 2.2. The following notation will be used on the priority tree.

- β^U will denote an $\mathcal{R}$ strategy (from Section 2.5) which is following a Γ-strategy.
- β^V will denote an $\mathcal{R}$ strategy (from Section 2.5) which is following a $\hat{\Gamma}$-strategy.
- α^U will denote an $\mathcal{S}$ strategy (from Section 2.5) which needs to take into consideration one $\mathcal{R}$ strategy following a Γ-strategy above it.
- α^V will denote an $\mathcal{S}$ strategy (from Section 2.5) which needs to take into consideration one $\mathcal{R}$ strategy following a $\hat{\Gamma}$-strategy above it.

- α will denote an $\mathcal{S}$ strategy (from Section 2.3) which does not need to take into consideration any $\mathcal{R}$ strategy above it.

In our discussion we shall omit the parts of the priority tree occurring below the f outcomes of $\mathcal{R}$ strategies. The f outcome of an $\mathcal{R}$ strategy results in the trivial satisfaction of the $\mathcal{R}$ requirement, allowing the $\mathcal{S}$ requirement to be satisfied by an $\mathcal{S}$ strategy of the form found in Section 2.3.

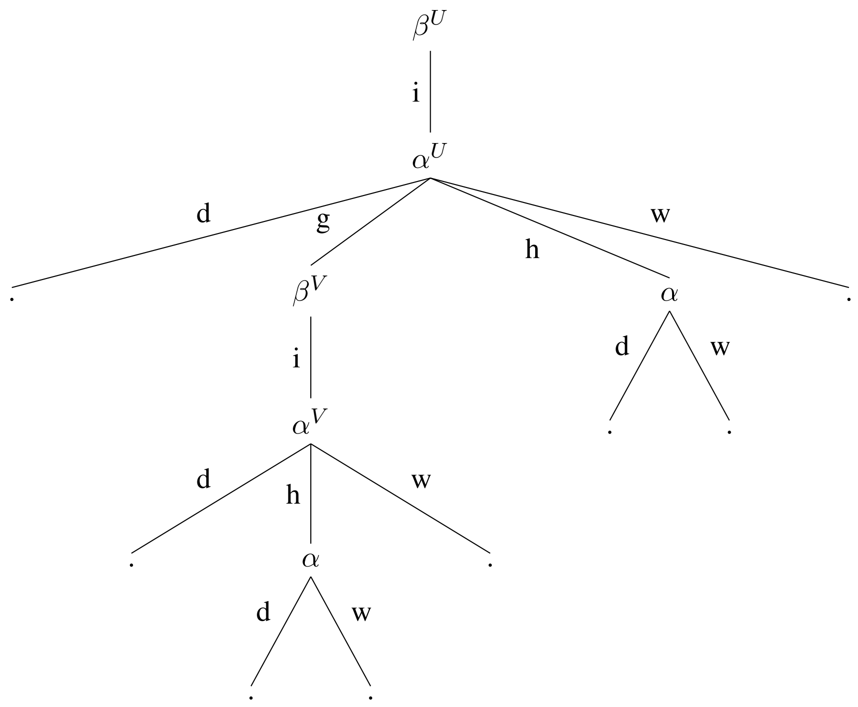

Figure 2.2: Priority tree for $\mathcal{S}$ below $\mathcal{R}$, second approximation

In order to build the priority tree we start by attempting to satisfy $\mathcal{R}$, which is the highest priority requirement. For this purpose, we shall use an $\mathcal{R}$ strategy β^U of the form found in Section 2.5. This strategy will be following a Γ-strategy. In the absence of interference from other strategies, this strategy will build its functional $\Gamma^{U,D}$ and will satisfy the $\mathcal{R}$ requirement.

Hence, the highest priority unsatisfied requirement below the i outcome of the strategy β^U is now the $\mathcal{S}$ requirement. To satisfy this requirement, we shall use an $\mathcal{S}$ strategy α^U of the form found in Section 2.5. This strategy considers the $\mathcal{R}$ strategy β^U to be active and thus needs to take it into consideration.

As we have already seen, the d and w outcomes of the strategy α^U result in the $\mathcal{S}$ requirement being satisfied without the satisfaction of the $\mathcal{R}$ requirement being compromised.

On the other hand, we have seen that the h outcome of the strategy α^U makes $\Phi^{U,V}$ partial, resulting in the trivial satisfaction of the $\mathcal{R}$ requirement. However this outcome also leaves the $\mathcal{S}$ requirement unsatisfied since α^U never enumerates a witness into A.

The trivial satisfaction of the $\mathcal{R}$ requirement however means that an $\mathcal{S}$ strategy α of the form found in Section 2.3 can be used below the h outcome in order to satisfy the $\mathcal{S}$ requirement. This strategy does not regard the strategy β^U as being active and is thus able to ignore it. The strategy α will then satisfy the $\mathcal{S}$ requirement regardless of whether it has a d or w outcome.

Finally, consider the g outcome of the α^U strategy. In this case we have seen that capricious destruction makes $\Gamma^{U,D}$ partial, leaving the $\mathcal{R}$ requirement unsatisfied. In addition the $\mathcal{S}$ requirement is left unsatisfied because α^U fails to see a U change each time it closes a gap, thus failing to diagonalise. We have also seen that this failure to obtain a U change, results in a V change occurring instead.

Thus we have that the highest priority unsatisfied requirement below the g outcome of the α^U strategy is once again the $\mathcal{R}$ requirement. However as we have seen earlier, the g outcome causes a switch in the manner of satisfying the $\mathcal{R}$ requirement. This switch means that we shall now assign an $\mathcal{R}$ strategy β^V of the form found in Section 2.5 to satisfy the $\mathcal{R}$ requirement. This strategy will be following a $\hat{\Gamma}$-strategy instead.

In the absence of interference from other strategies, the strategy β^V will build a functional $\Gamma^{V,D}$ and will satisfy the $\mathcal{R}$ requirement. This means that below the i outcome of the β^V strategy, the highest priority unsatisfied requirement is now the $\mathcal{S}$ requirement.

In order to satisfy the $\mathcal{S}$ requirement, we shall use an $\mathcal{S}$ strategy α^V of the form found in 2.5. This strategy regards β^U as having failed to satisfy its requirement and thus as inactive. Instead it believes that β^V is working to satisfy the $\mathcal{R}$ requirement, and thus treats β^V as an active strategy. Since β^V is following a $\hat{\Gamma}$-strategy, the α^V strategy will have only three outcomes, which are d, w and the honestification outcome h related to the functional $\Gamma^{V,D}$ being built by β^V.

Once again, the d and w outcomes of the α^V strategy result in the $\mathcal{S}$ requirement being satisfied, without the satisfaction of the $\mathcal{R}$ requirement by the β^V strategy being compromised.

On the other hand the h outcome of the α^V strategy makes $\Phi^{U,V}$ partial, resulting in the trivial satisfaction of the $\mathcal{R}$ requirement. This outcome also leaves the $\mathcal{S}$ requirement unsatisfied, because α^V never gets to enumerate a witness into A.

However the trivial satisfaction of the $\mathcal{R}$ requirement means that an $\mathcal{S}$ strategy α of the form found in Section 2.3 can be used below the h outcome in order to satisfy the $\mathcal{S}$ requirement. This strategy does not regard the strategy β^V as being active and is thus able to ignore it. The strategy α will then satisfy the $\mathcal{S}$ requirement regardless of whether it has a d or w outcome.

2.5.11 Satisfaction of Requirements

We shall now examine the satisfaction of the $\mathcal{R}$ and $\mathcal{S}$ requirements through the second approximation presented in this section. For this purpose it shall be sufficient to consider the leftmost outcome visited infinitely often by the strategy α^V.

$\boxed{\text{w}}$ Suppose that w is the leftmost outcome to be visited infinitely often by α^V. Then there is some stage s_0 beyond which the strategy α does not visit any outcome to the left of w. It also follows that α^V must have chosen some witness w' such that one of the following three cases holds.

In the first case one has that $\Theta^D(w') \uparrow$, in which case the $\mathcal{S}$ requirement is satisfied trivially.

In the second case, one has that $\Theta^D(w') \downarrow= 1$, which means that $\lim_s \theta_s(w')$ must be finite. Since the upper bound of any work interval increases without bound, there must exist some stage $t > s_0$ such that $a < \theta_s(w') < b$ for all stages $s \geq t$. This means that the strategy will hold the computation $\Theta^D(w') \downarrow= 1$ forever and that it will not enumerate w' into A. It follows that $\Theta^D(w') \neq A(w')$ and that the $\mathcal{S}$ requirement is satisfied once again.

In the third case, we have that $\Theta^D(w') \downarrow= 0$. Once again we must have that $\lim_s \theta_s(w')$ must be finite and that there is some stage $t > s_0$ such that for all $s \geq t$ we have that $a < \theta_s(w') < b$. Now at such stages s, we have must have that $\Gamma^{U,D}[s](w')$ is honest, or otherwise the strategy would visit the outcome h which is to the left of w, giving a contradiction.

In addition after stage t the strategy must visit its outcome w until it sees a close stage and is able to open a gap. If the strategy never sees a close stage, then we would have that $\Theta^D(w') = A(w')$ and the $\mathcal{S}$ requirement would remain unsatisfied. On the other hand if a close stage occurs the strategy would open a gap and visit the g outcome. Since this is to the left of w, this would give a contradiction.

We claim that it is possible to ensure that α^V sees infinitely many open and close stages, and that therefore this case cannot occur. However we postpone addressing this issue until Section 2.9.

Note that in each of the above three cases, we have that α^V does not interfere with β^V in the construction of its functional, and that the uses chosen to build this functional are protected from α^U through the work interval it imposes. Hence we have that the $\mathcal{R}$ requirement is satisfied as well.

$\boxed{\text{h}}$ Suppose h is the leftmost outcome to be visited infinitely often. Then there is some stage s_0 beyond which the strategy α does not visit any outcome to the left of h. It follows that after stage s_0 the strategy α^V must have chosen some witness w' such that $\Gamma^{V,D}(w')$ is dishonest infinitely often.

If at stage s we have that $\Gamma^{V,D}[s](w')$ is dishonest and s is a close stage, α^V performs honestification by enumerating $\gamma_s(w')$ into D and visiting outcome h. This undefines $\Gamma^{V,D}(w')$, leading the strategy β^V to redefine the functional by increasing the use $\gamma(w')$.

On the other hand if at stage s we have that $\Gamma^{V,D}[s](w')$ is dishonest and s is an open stage, α^V does not enumerate $\gamma_s(w')$ into D but still visits outcome h. Once again we claim that it is possible to ensure that α^V sees infinitely many open and close stages. Thus we have that α^V enumerates $\gamma_s(w')$ into D infinitely often.

This means that $\Gamma^{V,D}(w') \uparrow$. On the other hand, since $\Gamma^{V,D}(w')$ becomes dishonest infinitely often, it follows that $\lim_s \phi_s(w') \to \infty$, and hence we have that $\Phi^{U,V}(w') \uparrow$ as well. Hence the $\mathcal{R}$ requirement is satisfied trivially.

However the α^V strategy does not enumerate its witness w' into A and thus does not diagonalise. This means that the $\mathcal{S}$ requirement remains unsatisfied. This will cause another $\mathcal{S}$

strategy to occur below the h outcome of the strategy α^V. This $\mathcal{S}$ strategy does not consider the strategy β^V to be active, since the $\mathcal{R}$ requirement has been satisfied trivially. Therefore the $\mathcal{S}$ strategy α is of the form found in Section 2.3, and will now satisfy the $\mathcal{S}$ requirement independently of its outcome.

$\boxed{d}$ Suppose d is the leftmost outcome of the $\mathcal{S}$ strategy to be visited infinitely often. Then it follows that α^V must have opened a gap and diagonalised successfully.

In order for α^V to have opened a gap, there must have been an open stage t such that α^V was accessible at t and $\Theta^D[t](w') \downarrow= 0$. In addition $\Gamma^{V,D}[t](w')$ must have been honest. As a result, α^V must have enumerated w' into A, opening a gap at stage t.

Now t is an open stage, and α^V is accessible at stage t. This means that α^U must have been accessible at stage t, and that it has visited its g outcome by opening a gap. Hence, α^U must also have enumerated its witness w into A at stage t.

Since α^V is located inside the work interval $(v, \gamma_t(v))$ imposed by α^U, it must be the case that $v < w' < \gamma_t(v)$. In addition the ordering between the parameters of α^U ensures that $\gamma_t(v) < w$. It follows that $w' < w$.

The enumeration of the witnesses w and w' into A at stage t mean that the functional $\Phi^{U,V}$, the functional $\Gamma^{U,D}$ built by β^U, and the functional $\Gamma^{V,D}$ built by β^V now disagree with the set A at w and w'.

Now in order for the α^U strategy to be accessible again, β^U must have visited its i outcome and seen a β^U-expansionary stage. Hence there must have been some least stage $t' > t$ such that an $\mathcal{R}$ -expansionary stage has taken place. The $\mathcal{R}$ -expansionary stage removes the disagreement between $\Phi^{U,V}$ and A. This means that a $U \restriction \phi_t(w')$ change, or a $V \restriction \phi_t(w')$ change must have taken place between stages t and t'.

When α^U is accessible again at some close stage $s > t'$, it will attempt to close a gap. If a $U \restriction \phi_t(w')$ change has taken place, α^U would diagonalise successfully, and α^V would never be accessible again, which is a contradiction.

This means that a $V \restriction \phi_t(w')$ change must have occurred. In this case, α^U will close its gap by visiting its outcome g and performing capricious destruction. The strategy will then enumerate

$\gamma_s(v)$ into D (note that $\gamma_s(v) = \gamma_t(v)$). Whilst α^U undefines $\Gamma^{U,D}(w')$, this will not affect β^V or α^V in any way since these strategies are located inside the work interval $(v, \gamma_s(v))$.

Now in order for the α^V strategy to be accessible again, β^V must have visited its i outcome and seen a β^V-expansionary stage. We have already determined that the least $\mathcal{R}$ -expansionary stage greater than stage t occurs at stage t'. It is the changes which have occurred between t and t' which are considered to determine whether a strategy has diagonalised successfully.

Since α^V has ensured that $\Gamma^{V,D}[t](w')$ is honest prior to enumerating the witness into A, we have that the $V \upharpoonright \phi_t(w')$ change which took place between the stages t and t' is sufficient to undefine $\Gamma^{V,D}(w')$ as well.

In addition, the expansionary stage seen by β^V makes the strategy α^V accessible once again. Since β^V is the only active $\mathcal{R}$ strategy for α^V, and since the disagreement of the functional built by β^V has been removed without needing to enumerate any element into D, the α^V strategy has diagonalised successfully whilst preserving its restraint on D. Hence we have that α^V satisfies the $\mathcal{S}$ requirement.

In addition β^V is able to keep building its functional without interference after α^V has diagonalised successfully, and hence the $\mathcal{R}$ requirement is satisfied as required.

2.6 $\mathcal{S}$ Below $\mathcal{R}_2$ Below $\mathcal{R}_1$ - First Approximation

We now consider how to satisfy two $\mathcal{R}$ requirements and one $\mathcal{S}$ requirement. The $\mathcal{R}_1$ requirement will have the highest priority, followed by the $\mathcal{R}_2$ requirement and the $\mathcal{S}$ requirement.

In our first approximation we shall use an $\mathcal{R}$ strategy β_1 to satisfy the requirement $\mathcal{R}_1$, an $\mathcal{R}$ strategy β_2 to satisfy the requirement $\mathcal{R}_2$, and an $\mathcal{S}$ strategy α to satisfy the requirement $\mathcal{S}$. We shall assume that the $\mathcal{R}$ requirements $\mathcal{R}_1$ and $\mathcal{R}_2$ are not satisfied trivially.

Thus β_1 will see infinitely many β_1-expansionary stages and β_2 will see infinitely many β_2-expansionary stages. This means that these strategies will have to build a functional each in order to satisfy their corresponding requirement. In this first approximation, β_1 will be following a Γ-strategy and will build the functional $\Gamma_1^{U_1,D}$. Similarly β_2 will be following a Γ-strategy and will build the functional $\Gamma_2^{U_2,D}$.

Since the β_1 strategy is associated to the requirement of highest priority, it does not need to take any other strategy into consideration. On the other hand, the strategy β_2 considers the strategy β_1 to be active, and has to take it into consideration. Similarly, the strategy α considers β_1 and β_2 to be active, and has to take both into consideration.

We now introduce a few concepts which will be necessary in order to proceed.

2.6.1 R-Synchronisation

When an $\mathcal{R}$ strategy such as β_2 has an active $\mathcal{R}$ strategy β_1 above it, it will *R-Synchronise* with it in the following way. Whenever β_2 needs to define $\Gamma_2^{U_2,D}(x)$ and choose a use $\gamma_2(x)$, it will ensure that $\gamma_2(x) > \gamma_1(x)$. Note that this means that β_2 might not be able to define its functional at x until β_1 has also done so.

There are two reasons for synchronising the $\mathcal{R}$ strategies in this way.

The first is to ensure that enumeration of elements into the set D which makes the functional $\Gamma_1^{U_1,D}$ partial, also makes the functional $\Gamma_2^{U_2,D}$ partial. This ensures that $\mathcal{R}$ requirements are satisfied in order, in the sense that a lower priority $\mathcal{R}$ requirement will remain unsatisfied whilst higher priority $\mathcal{R}$ requirements remain unsatisfied.

The second is to ensure that if an $\mathcal{S}$ strategy cannot be injured by the enumeration of uses $\gamma_1(x)$ into the set D, then it is also safe from uses $\gamma_2(x)$ which are enumerated into the set D.

2.6.2 Gap Opening Convention

In order to diagonalise successfully, the $\mathcal{S}$ strategy α will need to enumerate its witness w into A and restrain $D \upharpoonright \theta(w)$. But this time, it needs to be concerned about the effect which the enumeration of w into A has on the functionals $\Gamma_1^{U_1,D}$ and $\Gamma_2^{U_2,D}$.

In particular it will need to make sure that $\Gamma_1^{U_1,D}(w)$ and $\Gamma_2^{U_2,D}(w)$ are honest before opening a gap. Following this, it will need to ensure that there has been both a $U_1 \upharpoonright \phi_1(w)$ change and a $U_2 \upharpoonright \phi_2(w)$ change when closing the gap, so that the disagreement which both functionals have with the set A at w can be removed without enumerating elements into D and running the risk of destroying the restraint on D in the process.

We thus have a situation where any one of $\Gamma_1^{U_1,D}(w)$ and $\Gamma_2^{U_2,D}(w)$ can fail to be honest prior to opening a gap, as well as a situation where any one of the sets U_1 and U_2 can fail to change when closing a gap. To reflect this fact α will need, besides the d and w outcomes, two honestification outcomes h_1 and h_2, as well as two capricious destruction outcomes g_1 and g_2.

Prior to opening a gap, the strategy α will visit the outcome h_1 when $\Gamma_1^{U_1,D}(w)$ fails to be honest, and will visit the outcome h_2 when $\Gamma_1^{U_1,D}(w)$ is honest but $\Gamma_2^{U_2,D}(w)$ is not. Similarly, when closing a gap, α will visit the outcome g_1 when a U_1 change has failed to occur, and will visit g_2 when a U_1 change has occurred, but a U_2 change has failed to occur.

The fact that there are now two outcomes g_1 and g_2 raises the question of which one should be visited when α is opening a gap. If α has never closed a gap before, we shall adopt the convention that the outcome with the least index (in this case g_1) should be visited. On the other hand, if α has closed a gap before, we choose the outcome which α has visited last when closing the gap.

The reason for adopting this convention is the following. Suppose that the outcome g_n is the leftmost outcome visited infinitely often by the strategy α. Then this means that α closes the gap unsuccessfully on g_n infinitely often. By having α open a gap on g_n each time it closes a gap on g_n, we ensure that the strategies below the outcome g_n see α opening a gap infinitely often. This arrangement guarantees that strategies below g_n would then be accessible during infinitely many open stages, just as they would be accessible during infinitely many close stages.

2.6.3 The $\mathcal{R}$ Strategy

The $\mathcal{R}$ strategy β which will be introduced in this section will be following a Γ-strategy. It will need to take at most one other $\mathcal{R}$ strategy β' above it into consideration, which will also be following a Γ-strategy. We formalise the $\mathcal{R}$ strategy below.

The $\mathcal{R}$ Strategy

This strategy has a set of uses U^e, and follows a Γ strategy. It lies below at most one higher priority $\mathcal{R}$ strategy β' which is following a Γ-strategy. The strategy operates inside a work interval (a, b).

(1) (Check for expansionary stage). Is stage s a β-expansionary stage?

   (a) (Yes) Go to step (2).

   (b) (No) End the substage and resume from step (1).

(2) (Check for disagreement). Is there an m such that $\Gamma^{U,D}[s](m) \neq A_s(m)$?

   (a) (Yes) End the substage and resume from step (1).

   (b) (No) Go to step (3).

(3) (Define the functional). For every $x < l_s(\Phi^{U,V}, A)$ such that $\Gamma^{U,D}[s](x) \uparrow$, define the axiom $\Gamma^{U,D}[s](x) = A_s(x)$, and choose the corresponding use $\gamma_s(x)$ to be the least element in the set of uses U^e obeying the following conditions (if it exists):

   (a) $\gamma_s(x) \geq \gamma_t(x)$ for all $t < s$.

   (b) $\gamma_s(x) > \gamma_s(y)$ for all $y < x$.

(c) $a < \gamma_s(x) < b$.

(d) $\gamma_s(x) \notin D$.

(e) $\gamma_s(x) > \gamma'_s(x)$, where $\gamma'(x)$ is the use of the functional being built by β' at the element x.

(f) $\gamma_s(x) > y$, where y is a constraint imposed by some $\mathcal{S}$ strategy below β.

(g) $\gamma_s(x)$ is greater than the stage at which β was last initialised.

End the substage and resume from step (1).

We shall now consider the outcomes of the $\mathcal{R}$ strategy and define the outcome which the strategy selects at stage s.

Outcomes of the $\mathcal{R}$ Strategy

The $\mathcal{R}$ strategy has two outcomes, the infinitary outcome i and the finitary outcome f. These outcomes are ordered as follows: $i <_\Lambda f$. At stage s, the strategy decides which outcome to visit by performing the following case analysis:

(I) *Waiting for expansionary stage.* The strategy ends its substage at step (1). Then the present stage s is not a β-expansionary stage. The $\mathcal{R}$ strategy visits its f outcome.

(II) *Expansionary stage.* The strategy ends its substage at step (2) or step (3). Then the present stage s is a β-expansionary stage. The $\mathcal{R}$ strategy visits its i outcome.

2.6.4 The $\mathcal{S}$ Strategy

The $\mathcal{S}$ strategy α which we shall introduce in this section needs to take two $\mathcal{R}$ strategies β_1 and β_2 into consideration. The strategies β_1 and β_2 will be following a Γ-strategy, and will build the functionals $\Gamma_1^{U_1,D}$ and $\Gamma_2^{U_2,D}$ respectively.

The strategy α will choose a witness w and wait for a stage t such that $\Theta^D[t](w) \downarrow= 0$. It will then check whether both $\Gamma_1^{U_1,D}[t](w)$ and $\Gamma_2^{U_2,D}[t](w)$ are honest.

If $\phi_{1,t}(w) > \gamma_{1,t}(w)$, then we have that $\Gamma_1^{U_1,D}[t](w)$ is dishonest and thus needs to be honestified. Prior to honestifying, α must check whether stage t is a close stage.

If this is the case, α will perform honestification for β_1 by enumerating $\gamma_{1,t}(w)$ into D. This undefines $\Gamma_1^{U_1,D}[t](w)$, allowing the strategy β_1 to redefine it by choosing a larger use. If t is not a close stage, α will not enumerate $\gamma_{1,t}(w)$ into D. However the strategy will still behave as if it had honestified, by visiting the corresponding outcome.

On the other hand, if $\phi_{1,t}(w) \leq \gamma_{1,t}(w)$, but $\phi_{2,t}(w) > \gamma_{2,t}(w)$, we have that $\Gamma_1^{U_1,D}[t](w)$ is honest, but that $\Gamma_2^{U_2,D}[t](w)$ is dishonest and thus needs to be honestified.

If this is the case, α will perform honestification for β_2 by enumerating $\gamma_{2,t}(w)$ into D. This undefines $\Gamma_2^{U_2,D}[t](w)$, allowing the strategy β_2 to redefine it by choosing a larger use. If t is not a close stage, α will not enumerate $\gamma_{2,t}(w)$ into D. However the strategy will still behave as if it had honestified, by visiting the corresponding outcome.

If both $\Gamma_1^{U_1,D}[t](w)$ and $\Gamma_2^{U_2,D}[t](w)$ are honest, the strategy α will try to open a gap.

Prior to opening a gap, the strategy will check whether stage t is an open stage. If this is not the case, then the strategy will wait instead. When it becomes accessible again it will have to start over before making another attempt at opening a gap.

In particular the strategy has to check whether $\Gamma_1^{U_1,D}(w)$ and $\Gamma_2^{U_2,D}(w)$ are still honest, as in the meantime $\phi_1(w)$ might have increased as a result of a U_1 or a V_1 change, and $\phi_2(w)$ might have increased as a result of a U_2 or a V_2 change.

On the other hand, if t is an open stage, the strategy opens a gap by enumerating w into A and restraining $D \upharpoonright \theta_t(w)$. This creates the disagreements $\Phi_1^{U_1,V_1}(w) \neq A(w)$, $\Phi_2^{U_2,V_2}(w) \neq A(w)$, $\Gamma_1^{U_1,D}(w) \neq A(w)$ and $\Gamma_2^{U_2,D}(w) \neq A(w)$.

Now, in order for α to become accessible again, there must have been some stage which is both a β_1-expansionary stage and a β_2-expansionary stage. For this to be the case there must have been some least stage $t_1 > t$ such that t_1 is an $\mathcal{R}_1$-expansionary stage and some least stage $t_2 > t$ such that t_2 is an $\mathcal{R}_2$-expansionary stage.

This means that the disagreement between $\Phi_1^{U_1,V_1}(w)$ and $A(w)$ is removed by stage t_1. This takes place either through a $U_1 \upharpoonright \phi_{1,t}(w)$ change or through a $V_1 \upharpoonright \phi_{1,t}(w)$ change between the stages t and t_1.

Similarly, the disagreement between $\Phi_2^{U_2,V_2}(w)$ and $A(w)$ is removed by stage t_2. This takes place either through a $U_2 \upharpoonright \phi_{2,t}(w)$ change or through a $V_2 \upharpoonright \phi_{2,t}(w)$ change between the stages t and t_2.

But despite these changes it is possible for $\Gamma_1^{U_1,D}(w)$ or $\Gamma_2^{U_2,D}(w)$ to still be in a state of disagreement with $A(w)$.

When α becomes accessible again at stage s, it will attempt to close its gap. Prior to closing the gap, the strategy will check whether stage s is a close stage. If this is not the case the strategy will wait. It will then try to close the gap when it becomes accessible once more. On the other hand if s is a close stage, the strategy can proceed to close its gap.

Since α needs to take two strategies into consideration, we shall need two thresholds. The threshold v_1 will correspond to the strategy β_1 and the threshold v_2 will correspond to the strategy β_2. The strategy α will seek to preserve the following ordering between parameters.

$$v_2 < \gamma_2(v_2) < v_1 < \gamma_1(v_1) < w < \gamma_1(w) < \gamma_2(w)$$

While the strategy α is able to choose the thresholds v_1 and v_2 and the witness w, it has no control over the uses selected by the β_1 and β_2 strategies. Hence α might have to redefine the threshold v_1 if $\gamma_2(v_2)$ becomes too large.

On the other hand, if $\gamma_2(v_2)$ or $\gamma_1(v_1)$ increases due to capricious destruction, the witness w must have already been enumerated into the set A, and thus presents no problems. Also note that $\gamma_1(w) < \gamma_2(w)$ due to the R-Synchronisation of the $\mathcal{R}$ strategies.

The strategy α can now proceed to check how the gap has been closed at stage s. Suppose that there has not been a $U_1 \upharpoonright \phi_{1,t}(w)$ change between stages t and t_1. Then a $V_1 \upharpoonright \phi_{1,t}(w)$ change must have occurred between stages t and t_1.

Therefore we have that the disagreement between $\Gamma_1^{U_1,D}(w)$ and $A(w)$ persists, and we undefine this disagreement through the enumeration of $\gamma_{1,s}(v_1)$ into D, where $\gamma_{1,s}(v_1) = \gamma_{1,t}(v_1)$. This is called capricious destruction for β_1.

On the other hand, suppose that there has been a $U_1 \upharpoonright \phi_{1,t}(w)$ change between stages t and t_2. Then the disagreement between $\Gamma_1^{U_1,D}(w)$ and $A(w)$ has been removed. However, it may now be the case that there has been no $U_2 \upharpoonright \phi_{2,t}(w)$ change between stages t and t_2.

Hence a $V_2 \upharpoonright \phi_{2,t}(w)$ change must have occurred between stages t and t_2. Therefore we have that the disagreement between $\Gamma_2^{U_2,D}(w)$ and $A(w)$ persists, and we undefine the disagreement through the enumeration of $\gamma_{2,s}(v_2)$ into D, where $\gamma_{2,s}(v_2) = \gamma_{2,t}(v_2)$. This is called capricious destruction for β_2.

Finally suppose that there has been both a $U_1 \upharpoonright \phi_{1,t}(w)$ change and a $U_2 \upharpoonright \phi_{2,t}(w)$ change. In this case both disagreements have been removed without the need to enumerate any element into D. In addition, the α strategy has managed to diagonalise and to protect its computation by preserving its restraint on the set D, so it can now stop.

We formalise the $\mathcal{S}$ strategy below.

The $\mathcal{S}$ Strategy

The strategy has a set of witnesses W^e and a set of thresholds V^e. It chooses three parameters, the witness w and two thresholds v_1 and v_2. It also lies below the sequence of active $\mathcal{R}$ strategies (β_1, β_2). Each strategy β_i in this sequence follows a Γ-strategy. The strategy operates inside a work interval (a, b).

(*) (Background Step) Perform this step at the beginning of every substage during which the strategy is accessible. If the following conditions are met resume from the step last indicated by the strategy, or resume from step (1) if no such step exists. Otherwise end the substage.

- (a) For all $1 \le i \le 2$, if v_i is defined, we have that $(\forall_{n \le v_i})(\Gamma_i^{U_i,D}[s](n) \downarrow = A_s(n))$ holds.
- (b) If w is defined, for all $1 \le i \le 2$ we have that $(\forall_{n \le w})(\Gamma_i^{U_i,D}[s](n) \downarrow = A_s(n))$ holds.
- (c) If w is defined, $a < \theta_s(w) < b$.

(1) (Select the thresholds). Let β_i be a strategy in the sequence (β_1, β_2). If no threshold v_i corresponding to β_i is defined, choose a threshold v_i. The value selected for this threshold is the least value in V^e such that:

   (a) $(\forall_{n \leq v_i})(\Gamma_{i,s}^{U_i,D}(n) \downarrow = A_s(n))$.

   (b) $a < v_i < b$.

   (c) $(\forall_{i<j\leq 2})(v_i > \gamma_{j,s}(v_j))$.

   (d) v_i is greater than the last stage at which α was last initialised.

   If thresholds satisfying these conditions cannot be found, end this substage. Resume from step (1). Otherwise go to step (2).

(2) (Select the witness). Choose a witness w. The value selected for this witness is the least value in W^e such that:

   (a) For all $1 \leq i \leq 2$, we have that $(\forall_{n\leq w})(\Gamma_{i,s}^{U_i,D}(n) \downarrow = A_s(n))$ holds.

   (b) $a < w < b$.

   (c) $\Theta^D(w) \downarrow$

   (d) $a < \theta_s(w) < b$.

   (e) $(\forall_{1\leq j\leq 2})(w > \gamma_{j,s}(v_j))$.

   (f) w is greater than the last stage at which α was last initialised.

   If a witness satisfying these conditions cannot be found, end this substage. Resume from step (1). Otherwise go to step (3).

(3) (Wait for convergence). Is $\Theta^D[s](w) \downarrow = 0$?,

   (a) (Yes) Go to step (4).

   (b) (No) Otherwise end this substage. Resume from step (3).

(4) (Honestification for β_1). Is $\phi_{1,s}(w) > \gamma_{1,s}(w)$?

   (a) (Yes) Is s a close stage?

      (i) (Yes) Enumerate $\gamma_{1,s}(w)$ into D. Go to step (3).

      (ii) (No) End this substage and resume from step (3).

(b) (No) Go to step (5).

(5) (Honestification for β_2). Is $\phi_{2,s}(w) > \gamma_{2,s}(w)$?

(a) (Yes) Is s a close stage?

(i) (Yes) Enumerate $\gamma_{2,s}(w)$ into D. Go to step (3).

(ii) (No) End this substage and resume from step (3).

(b) (No) Go to step (6).

(6) (Gap open) Is stage s an open stage?

(a) (Yes) Constrain each strategy β_i in the sequence (β_1, β_2) to choose uses $\gamma_{i,s'}(w) > \theta_s(w)$ at all stages $s' > s$. Enumerate w into A and restrain $D \restriction \theta_s(w)$. Cancel the witness w and end this substage. Resume from step (7).

(b) (No) End this substage and resume from step (3).

(7) (Gap close) Is stage s a close stage?

(a) (Yes) Let t be the stage at which the strategy last opened a gap by ending its substage at step (6). Let t_1 be the least $\mathcal{R}_1$-expansionary stage greater than t, and let t_2 be the least $\mathcal{R}_2$-expansionary stage greater than t. Let β_i be the least strategy in the sequence (β_1, β_2) such that $U_{i,t} \restriction \phi_t(w) = U_{i,t_i} \restriction \phi_t(w)$. If $\beta_i = \beta_1$ go to step (8), whilst if $\beta_i = \beta_2$, go to step (9). If there is no such β_i, go to step (10).

(b) (No) End this substage and resume from step (7).

(8) (Capricious destruction for β_1) Enumerate $\gamma_{1,s}(v_1)$ into D. End this substage and resume from step (1).

(9) (Capricious destruction for β_2) Enumerate $\gamma_{2,s}(v_2)$ into D. End this substage and resume from step (1).

(10) (Successful diagonalisation). End this substage and resume from step (10).

We shall now consider the outcomes of the $\mathcal{S}$ strategy and define the outcome which the $\mathcal{S}$ strategy selects at stage s.

Outcomes of the S Strategy

The α strategy has six outcomes, the diagonalisation outcome d, the capricious destruction for β_2 outcome g_2, the capricious destruction for β_1 outcome g_1, the honestification for β_1 outcome h_1, the honestification for β_2 outcome h_2 and the wait outcome w. These outcomes are ordered as follows: $d <_\Lambda g_2 <_\Lambda g_1 <_\Lambda h_1 <_\Lambda h_2 <_\Lambda w$. Note that the g outcomes are ordered in descending order, while the h outcomes are ordered in ascending order. At stage s, the strategy decides which outcome to visit by performing the following case analysis:

(I) *Waiting for parameters.* The strategy ends its substage at the (Background Step), step (1) or step (2). Then the functional of the active strategy β fails to be defined and equal to A up to some parameter, or α has failed to choose some parameter. The strategy visits its w outcome and imposes the work interval $(0, \infty)$.

(II) *Waiting for computation.* The strategy ends its substage at step (3). Then we have that $\Theta^D[s](w) \uparrow$, or $\Theta^D[s](w) \downarrow = 1$. The strategy visits its w outcome and imposes the work interval $(0, \infty)$.

(III) *Honestification for β_1.* The strategy ends its substage at step (4)(a)(i). Then we have that $\Theta^D[s](w) \downarrow = 0$, $\phi_{1,s}(w) > \gamma_{1,s}(w)$ and that s is a close stage. The strategy visits its h_1 outcome and imposes the work interval $(w, \gamma_{1,s}(w))$.

(IV) *Honestification for β_2.* The strategy ends its substage at step (5)(a)(i). Then we have that $\Theta^D[s](w) \downarrow = 0$, $\phi_{1,s}(w) \leq \gamma_{1,s}(w)$, $\phi_{2,s}(w) > \gamma_{2,s}(w)$ and that s is a close stage. The strategy visits its h_2 outcome and imposes the work interval $(w, \gamma_{2,s}(w))$.

(V) *Honestification for β_1 - waiting for close stage.* The strategy ends its substage at step (4)(a)(ii). Then we have that $\Theta^D[s](w) \downarrow = 0$, and that $\phi_{1,s}(w) > \gamma_{1,s}(w)$, but s is not a close stage. The strategy visits its h_1 outcome and imposes the work interval $(w, \gamma_{1,s}(w))$.

(VI) *Honestification for β_2 - waiting for close stage.* The strategy ends its substage at step (5)(a)(ii). Then we have that $\Theta^D[s](w) \downarrow = 0$, $\phi_{1,s}(w) \leq \gamma_{1,s}(w)$ and that $\phi_{2,s}(w) > \gamma_{2,s}(w)$ but s is not a close stage. The strategy visits its h_2 outcome and imposes the work interval $(w, \gamma_{2,s}(w))$.

(VII) *Opening a gap.* The strategy ends its substage at step (6)(a). Then we have that $\Theta^D[s](w) \downarrow = 0$,

$\phi_{1,s}(w) \leq \gamma_{1,s}(w)$, $\phi_{2,s}(w) \leq \gamma_{2,s}(w)$ and that w has been enumerated into A. The strategy has opened a gap. If the strategy has never closed a gap, the strategy visits the outcome g_1. Otherwise, the strategy visits the outcome on which a gap was last closed. The work interval $(v_i, \gamma_i(v_i))$ is imposed, where i is the index of the g_i outcome which has been visited last.

(VIII) *Opening a gap - waiting for open stage.* The strategy ends its substage at step (6)(b). Then we have that $\Theta^D[s](w) \downarrow= 0$, $\phi_{1,s}(w) \leq \gamma_{1,s}(w)$, $\phi_{2,s}(w) \leq \gamma_{2,s}(w)$ but s is not an open stage. The strategy visits its w outcome and imposes the work interval $(0, \infty)$.

(IX) *Closing a gap - waiting for close stage.* The strategy ends its substage at step (7)(b). Then the strategy has opened a gap at stage t, there has been no close stage between t and s and s is not a close stage. The strategy visits its w outcome and imposes the work interval $(0, \infty)$.

(X) *Closing a gap - capricious destruction for β_1.* The strategy ends its substage at step (8). Then we have that the strategy has opened a gap at stage t, that there has been no close stage between t and s and that s is a close stage. In addition $U_{1,t} \upharpoonright \phi_{1,t}(w) = U_{1,t_1} \upharpoonright \phi_{1,t}(w)$, where t_1 is the least $\mathcal{R}_1$-expansionary stage greater than t. The strategy is closing a gap. The strategy visits its g_1 outcome and imposes the work interval $(v_1, \gamma_{1,s}(v_1))$.

(XI) *Closing a gap - capricious destruction for β_2.* The strategy ends its substage at step (9). Then we have that the strategy has opened a gap at stage t, that there has been no close stage between t and s and that s is a close stage. In addition $U_{1,t} \upharpoonright \phi_{1,t}(w) \neq U_{1,t_1} \upharpoonright \phi_{1,t}(w)$ and $U_{2,t} \upharpoonright \phi_{2,t}(w) = U_{2,t_2} \upharpoonright \phi_{2,t}(w)$, where t_1 is the least $\mathcal{R}_1$-expansionary stage greater than t and t_2 is the least $\mathcal{R}_2$-expansionary stage greater than t. The strategy is closing a gap. The strategy visits its g_2 outcome and imposes the work interval $(v_2, \gamma_{2,s}(v_2))$.

(XII) *Closing a gap - successful diagonalisation.* Then we have that the strategy has opened a gap at stage t, that there has been no close stage between t and s and that s is a close stage. In addition $U_{1,t} \upharpoonright \phi_{1,t}(w) \neq U_{1,t_1} \upharpoonright \phi_{1,t}(w)$ and $U_{2,t} \upharpoonright \phi_{2,t}(w) \neq U_{2,t_2} \upharpoonright \phi_{2,t}(w)$, where t_1 is the least $\mathcal{R}_1$-expansionary stage greater than t and t_2 is the least $\mathcal{R}_2$-expansionary stage greater than t. The strategy has diagonalised successfully. The strategy visits its d outcome and imposes the work interval (s_1, ∞), where s_1 is equal to the present stage s.

(XIII) *Stopped.* The strategy ends its substage at step (10), and step (10) has been visited since the strategy was last initialised. Then we have that the strategy has already diagonalised successfully. The strategy visits its d outcome and imposes the work interval (s_1, ∞) , where s_1 is the stage at which the strategy has first diagonalised successfully and visited its outcome d.

2.6.5 Organisation of Priority Tree

We shall now make our first attempt at organising a priority tree to satisfy the $\mathcal{S}$ requirement below the $\mathcal{R}_2$ requirement below the $\mathcal{R}_1$ requirement. The following discussion will refer to the priority tree shown in Figure 2.3. The following notation will be used on the priority tree.

- $\beta_1^{U_1}$ will denote an $\mathcal{R}$ strategy (from Section 2.6) which is following a Γ-strategy.
- $\beta_2^{U_2}$ will denote an $\mathcal{R}$ strategy (from Section 2.6) which is following a Γ-strategy.
- α^{U_1,U_2} will denote an $\mathcal{S}$ strategy (from Section 2.6) which needs to take into consideration two $\mathcal{R}$ strategies above it, with each following a Γ-strategy.

Note that we shall omit the parts of the tree occurring below the f outcomes of the $\mathcal{R}_1$ and $\mathcal{R}_2$ strategies. The f outcome of $\mathcal{R}$ strategies results in the trivial satisfaction of the corresponding requirement. Hence, the $\mathcal{S}$ strategy would either be concerned with just one active $\mathcal{R}$ strategy above it, or none at all. This would then give rise to one of the cases already treated in previous sections.

In order to construct the tree, we start by attempting to satisfy the highest priority requirement, which is $\mathcal{R}_1$. We assign an $\mathcal{R}$ strategy $\beta_1^{U_1}$ of the form found in Section 2.6 for this purpose. In the absence of interference from other strategies, this strategy will build its functional $\Gamma_1^{U_1,D}$ and will satisfy the $\mathcal{R}_1$ requirement.

Below the i outcome of the $\beta_1^{U_1}$ strategy, we have that the $\mathcal{R}_2$ requirement is the highest priority unsatisfied requirement. We assign an $\mathcal{R}$ strategy $\beta_2^{U_2}$ of the form found in Section 2.6 for this purpose. Now, since the $\mathcal{R}_1$ requirement is not satisfied trivially, $\beta_2^{U_2}$ regards $\beta_1^{U_1}$ as an active strategy and must take it into consideration. This means that $\beta_2^{U_2}$ will R-Synchronise with $\beta_1^{U_1}$. In the absence of interference from other strategies, this strategy will build its functional $\Gamma_2^{U_2,D}$ and will satisfy the $\mathcal{R}_2$ requirement.

Below the i outcome of the $\beta_2^{U_2}$ strategy, we have that the $\mathcal{S}$ requirement is the highest priority unsatisfied requirement. Hence we assign an $\mathcal{S}$ strategy α^{U_1,U_2} of the form found in Section 2.6 to satisfy this requirement. The strategy α^{U_1,U_2} regards $\beta_1^{U_1}$ and $\beta_2^{U_2}$ as active strategies, and therefore must take both into consideration. This means that the strategy α^{U_1,U_2} needs to have the six outcomes, d, g_2, g_1, h_1, h_2 and w.

We are now in a position to analyse whether the strategies β_1, β_2 and α collectively satisfy the requirements $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{S}$.

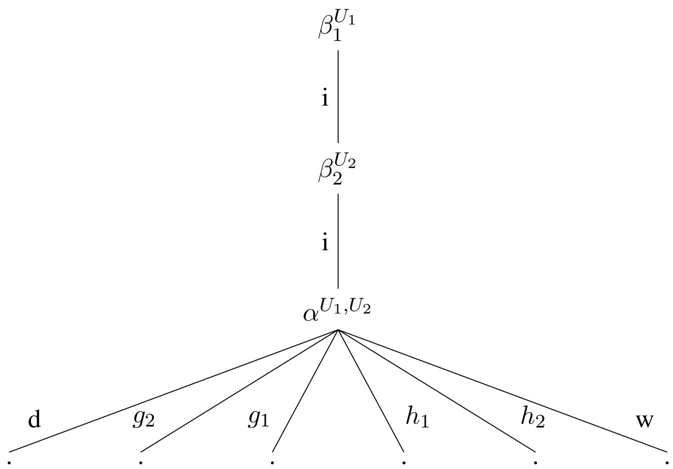

Figure 2.3: Priority tree for $\mathcal{S}$ below $\mathcal{R}_2$ below $\mathcal{R}_1$, first approximation

2.6.6 Satisfaction of Requirements

We perform a case analysis based on the leftmost outcome visited infinitely often by the strategy α^{U_1,U_2}.

[w] Suppose that w is the leftmost outcome visited infinitely often by α^{U_1,U_2}. Then there is some stage s_0 beyond which the strategy does not visit any outcome to the left of w. Hence the strategy chooses some witness w and holds it forever. It follows that one of the following cases holds.

If $\Theta^D(w)\uparrow$, we have that $\Theta^D(w) \neq A(w)$ trivially, and the $\mathcal{S}$ requirement is satisfied. On the other hand $\Theta^D(w)\downarrow = 1$, there has to be some stage at which this computation converges and remains convergent forever. Hence we have that the strategy holds its witness w forever and never enumerates it into A. It follows that $\Theta^D(w) \neq A(w)$ and that the $\mathcal{S}$ requirement is satisfied. Finally we have that $\Theta^D(w)\downarrow = 0$ cannot be the case if there are infinitely many open and close stages. For if this were the case, the strategy would either honestify or open a gap, thus visiting an outcome to the left of w which is a contradiction.

In addition, this means that after some stage, α^{U_1,U_2} no longer interferes with the strategies β_1 or β_2. Hence, the latter two strategies are able to build their functionals and satisfy the corresponding $\mathcal{R}_1$ and $\mathcal{R}_2$ requirements.

$\boxed{h_1}$ Suppose that h_1 is the leftmost outcome visited infinitely often by α^{U_1,U_2}. Then the strategy has chosen a witness w such that $\Gamma_1^{U_1,D}(w)$ fails to be honest infinitely often. In addition there might be infinitely many stages such that $\Gamma_1^{U_1,D}(w)$ is honest, but $\Gamma_2^{U_2,D}(w)$ is not honest. However it will never be the case that both $\Gamma_1^{U_1,D}(w)$ and $\Gamma_2^{U_2,D}(w)$ are honest.

The infinite honestification of β_1 makes $\Gamma_1^{U_1,D}(w)\uparrow$. But this also makes $\Phi_1^{U_1,V_1}(w)\uparrow$. Hence, the $\mathcal{R}_1$ requirement is satisfied. However, the R-Synchronisation of the $\mathcal{R}$ strategies will make $\Gamma_2^{U_2,D}(w)\uparrow$ as well, leaving the $\mathcal{R}_2$ requirement unsatisfied. Finally, the $\mathcal{S}$ strategy fails to diagonalise, and thus the $\mathcal{S}$ requirement remains unsatisfied.

$\boxed{h_2}$ Suppose that h_2 is the leftmost outcome visited infinitely often by α^{U_1,U_2}. Then the strategy has chosen a witness w such that $\Gamma_2^{U_2,D}(w)$ fails to be honest infinitely often. The infinite honestification of β_2 makes $\Gamma_2^{U_2,D}(w)\uparrow$, but we also have that $\Phi_2^{U_2,V_2}(w)\uparrow$ as well. Hence, the $\mathcal{R}_2$ requirement is satisfied.

On the other hand, the strategy β_1 will be able to build the functional $\Gamma_1^{U_1,D}$. This is the case because whenever the functional is undefined at some element, β_1 will simply redefine the functional at this element using its old use. Hence, the requirement $\mathcal{R}_1$ is satisfied.

Finally, the $\mathcal{S}$ strategy does not manage to diagonalise, and thus the $\mathcal{S}$ requirement remains unsatisfied.

$\boxed{g_1}$ Suppose that g_1 is the leftmost outcome visited infinitely often by α^{U_1,U_2}. Then the strategy will open infinitely many gaps which fail to obtain the required U_1 change. Infinite capricious destruction of β_1 makes $\Gamma_1^{U_1,D}(v_1)\uparrow$. Hence, the $\mathcal{R}_1$ requirement is not satisfied. Similarly, the R-Synchronisation of $\mathcal{R}$ strategies makes $\Gamma_2^{U_2,D}(v_1)\uparrow$ as well leaving the $\mathcal{R}_2$ requirement unsatisfied. Finally, the $\mathcal{S}$ strategy does not manage to diagonalise, and thus the $\mathcal{S}$ requirement remains unsatisfied. However progress in satisfying the requirements is made in the form of a V_1 change being obtained each time capricious destruction of β_1 takes place.

$\boxed{g_2}$ Suppose that g_2 is the leftmost outcome visited infinitely often by α^{U_1,U_2}. Then the strategy opens infinitely many gaps. During infinitely many of these gaps, it will obtain a U_1 change but fail to obtain a U_2 change. The infinite capricious destruction of β_2 makes $\Gamma_2^{U_2,D}(v_2)\uparrow$, leaving the $\mathcal{R}_2$ requirement unsatisfied.

However whilst β_2 is R-Synchronised with β_1, the reverse is not true. Hence if the enumeration of $\gamma_2(v_2)$ into D undefines the functional $\Gamma_1^{U_1,D}$ at some element, the β_1 strategy can redefine the functional at that element using its old use. Moreover as $\gamma_2(v_2)$ increases, it forces v_1 to be redefined.

Hence whenever the outcome g_1 which lies to the right of g_2 is visited, neither $\Gamma_1^{U_1,D}$, nor $\Gamma_2^{U_2,D}$ can be made partial at some fixed v_1. Hence we have that the strategy β_1 is able to build its functional and satisfy the $\mathcal{R}_1$ requirement, whilst the $\mathcal{R}_2$ requirement remains unsatisfied. Finally, the strategy α does not manage to diagonalise, and therefore the requirement $\mathcal{S}$ remains unsatisfied as well.

$\boxed{\text{d}}$ Suppose that d is the leftmost outcome visited infinitely often by α^{U_1,U_2}. In this case, the α^{U_1,U_2} strategy must have chosen a witness w and seen that $\Gamma_1^{U_1,D}(w)$ and $\Gamma_2^{U_2,D}(w)$ were honest. Following this the strategy must have opened a gap by enumerating w into A and restraining $D \upharpoonright \theta(w)$, and closed the gap successfully by obtaining the required U_1 and U_2 change.

Thus the α^{U_1,U_2} strategy diagonalises successfully and protects its computation $\Theta^D(w)$, and the $\mathcal{S}$ requirement is satisfied. In addition, this means that after some stage, α no longer interferes with the strategies β_1 or β_2. Hence, the latter two strategies are able to build their functionals and satisfy the corresponding $\mathcal{R}_1$ and $\mathcal{R}_2$ requirements as well.

From the above discussion one can conclude that the outcomes g_2, g_1, h_1 and h_2 of the α^{U_1,U_2} strategy leave one or more requirements unsatisfied. This will require us to pass to a second approximation which is able to deal with this situation.

2.7 $\mathcal{S}$ Below $\mathcal{R}_2$ Below $\mathcal{R}_1$ - Second Approximation

In this section we shall make a second attempt at satisfying the requirements $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{S}$. We shall therefore introduce the second approximation to the $\mathcal{S}$ and $\mathcal{R}$ strategies needed to achieve this goal. We start by introducing a number of concepts.

2.7.1 $\hat{\Gamma}$-Strategies

In the previous section we attempted to satisfy the requirement $\mathcal{R}_1$ by using an $\mathcal{R}$ strategy β_1 which followed a Γ-strategy and built the functional $\Gamma_1^{U_1,D}$. Similarly, we attempted to satisfy the requirement $\mathcal{R}_2$ by using an $\mathcal{R}$ strategy β_2 which followed a Γ-strategy and built the functional $\Gamma_2^{U_2,D}$.

In order to proceed $\mathcal{R}$ strategies of greater generality shall be required. Such strategies will be capable of following either a Γ-strategy or a $\hat{\Gamma}$-strategy. Thus an $\mathcal{R}$ strategy might now attempt to satisfy the $\mathcal{R}_1$ requirement by building a functional $\Gamma_1^{V_1,D}$. Similarly an $\mathcal{R}$ strategy might attempt to satisfy the $\mathcal{R}_2$ requirement by building a functional $\Gamma_2^{V_2,D}$. Whether an $\mathcal{R}$ strategy follows a Γ-strategy or a $\hat{\Gamma}$-strategy is determined by its position on the priority tree.

This however means that an $\mathcal{S}$ strategy α located below two active $\mathcal{R}$ strategies β_1 and β_2 needs to be able to deal with the fact that one or both of these strategies may be following a $\hat{\Gamma}$-strategy.

Note that if the strategy β_1 is following a Γ-strategy, the strategy α will have g_1 as an outcome. On the other hand if the strategy β_1 is following a $\hat{\Gamma}$-strategy, this outcome will be absent. In this case we would have already attempted to satisfy the $\mathcal{R}_1$ requirement by building functionals $\Gamma_1^{U_1,D}$ and $\Gamma_1^{V_1,D}$, and the priority tree would have been arranged so that the latter attempt would have succeeded.

Similarly, if the strategy β_2 is following a Γ-strategy, the strategy α will have g_2 as an outcome. On the other hand if the strategy β_2 is following a $\hat{\Gamma}$-strategy, this outcome will be absent. In this case we would have already attempted to satisfy the $\mathcal{R}_2$ requirement by building functionals $\Gamma_2^{U_2,D}$ and $\Gamma_2^{V_2,D}$, and the priority tree would have been arranged so that the latter attempt would have succeeded.

2.7.2 Switching

In the previous section we have seen that whenever an $\mathcal{S}$ strategy α performs capricious destruction for β_1 by visiting its g_1 outcome, a V_1 change must occur. Similarly, when an $\mathcal{S}$ strategy α performs capricious destruction for β_2 by visiting its g_2 outcome, a V_2 change must occur.

These V_1 and V_2 changes can be used to satisfy $\mathcal{R}$ requirements which were left unsatisfied by α. Clearly in order for an $\mathcal{R}$ strategy to make use of these changes, it has to be following a $\hat{\Gamma}$-strategy. Hence it must be building a functional $\Gamma_1^{V_1,D}$ or a functional $\Gamma_2^{V_2,D}$ respectively.

The outcome g_1 of an $\mathcal{S}$ strategy α will cause the $\mathcal{R}_1$ requirement to switch the manner in which it is satisfied. Any $\mathcal{R}$ strategy below the g_1 outcome which is attempting to satisfy the requirement $\mathcal{R}_1$ will now follow a $\hat{\Gamma}$-strategy. Similarly, the outcome g_2 of an $\mathcal{S}$ strategy α will cause the $\mathcal{R}_2$ requirement to switch the manner in which it is satisfied. Any $\mathcal{R}$ strategy below the g_2 outcome which is attempting to satisfy the requirement $\mathcal{R}_2$ will now follow a $\hat{\Gamma}$-strategy.

Thus the outcomes g_1 and g_2 of an $\mathcal{S}$ strategy have two important effects. On the one hand they are associated to the occurence of V_1 and V_2 changes, whilst on the other they cause the appearance of $\mathcal{R}$ strategies following a $\hat{\Gamma}$-strategy which are able to make use of such changes.

2.7.3 S-Synchronisation

Satisfying all the requirements will require multiple $\mathcal{S}$ strategies along certain paths of the priority tree. As in Section 2.5, an $\mathcal{S}$ strategy α has to S-Synchronise with a higher priority $\mathcal{S}$ strategy α' whenever α lies below the g_1 or the g_2 outcome of the α' strategy.

Hence before the $\mathcal{S}$ strategy α opens a gap by visiting a g_1 or g_2 outcome, it will check whether the present stage is an open stage, and wait otherwise. Similarly, before the $\mathcal{S}$ strategy α closes a gap by visiting a g_1 or g_2 outcome, it will check whether the present stage is a close stage, and wait otherwise.

We shall now formalise the $\mathcal{R}$ strategy required in this section.

2.7.4 The $\mathcal{R}$ Strategy

The main difference from the $\mathcal{R}$ strategy found in Section 2.5 is that it can now follow either a Γ-strategy or a $\hat{\Gamma}$-strategy.

The $\mathcal{R}$ Strategy

This strategy has a set of uses U^e, and follows either a Γ-strategy or a $\hat{\Gamma}$-strategy. It lies below at most one higher priority $\mathcal{R}$ strategy β'. The strategy β' follows either a Γ-strategy or a $\hat{\Gamma}$-strategy. The strategy operates inside a work interval (a, b).

(1) (Check for expansionary stage). Is stage s a β-expansionary stage?

  (a) (Yes) Go to step (2).

  (b) (No) End the substage and resume from step (1).

(2) (Check for disagreement). Is there an m such that $\Gamma^{U,D}[s](m) \neq A_s(m)$? (or $\Gamma^{V,D}[s](m)$ resp.).

  (a) (Yes) End the substage and resume from step (1).

  (b) (No) Go to step (3).

(3) (Define the functional). For every $x < l_s(\Phi^{U,V}, A)$ such that $\Gamma^{U,D}[s](x) \uparrow$ (or $\Gamma^{V,D}[s](x)$ resp.), define the axiom $\Gamma^{U,D}[s](x) = A_s(x)$ (or $\Gamma^{V,D}[s](x)$ resp.), and choose the corresponding use $\gamma_s(x)$ to be the least element in the set of uses U^e obeying the following conditions (if it exists):

  (a) $\gamma_s(x) \geq \gamma_t(x)$ for all $t < s$.

  (b) $\gamma_s(x) > \gamma_s(y)$ for all $y < x$.

  (c) $a < \gamma_s(x) < b$.

  (d) $\gamma_s(x) \notin D$.

  (e) $\gamma_s(x) > \gamma'_s(x)$, where $\gamma'(x)$ is the use of the functional being built by β' at the element x.

  (f) $\gamma_s(x) > y$, where y is a constraint imposed by some $\mathcal{S}$ strategy below β.

(g) $\gamma_s(x)$ is greater than the stage at which β was last initialised.

End the substage and resume from step (1).

We shall now consider the outcomes of the $\mathcal{R}$ strategy and define the outcome which the $\mathcal{R}$ strategy selects at stage s.

Outcomes of the $\mathcal{R}$ Strategy

The $\mathcal{R}$ strategy has two outcomes, the infinitary outcome i and the finitary outcome f. These outcomes are ordered as follows: $i <_\Lambda f$. At stage s, the strategy decides which outcome to visit by performing the following case analysis:

(I) *Waiting for expansionary stage.* The strategy ends its substage at step (1). Then the present stage s is not a β-expansionary stage. The $\mathcal{R}$ strategy visits its f outcome.

(II) *Expansionary stage.* The strategy ends its substage at step (2) or step (3). Then the present stage s is a β-expansionary stage. The $\mathcal{R}$ strategy visits its i outcome.

We shall now formalise the $\mathcal{S}$ strategy required in this section.

2.7.5 The $\mathcal{S}$ Strategy

The main difference from the $\mathcal{S}$ strategy found in Section 2.6 is that the $\mathcal{S}$ strategy α now has to deal with two $\mathcal{R}$ strategies β_1 and β_2 above it, each of which can be following a Γ-strategy or a $\hat{\Gamma}$-strategy.

The $\mathcal{S}$ strategy α will go through the usual steps of choosing a witness w, waiting for the computation to converge, checking the functionals built by the strategies β_1 and β_2 for honesty and then opening a gap.

Now, since β_1 and β_2 can now be following either a Γ-strategy or a $\hat{\Gamma}$-strategy, it follows that α has to wait for the appropriate combination of changes in the corresponding sets. Obtaining the correct combination of changes results in the functionals built by β_1 and β_2 becoming undefined at their least point of disagreement with A. In addition, the strategy does not have to change D, resulting in a successful diagonalisation.

Suppose that α opens a gap at stage t. Suppose also that the least $\mathcal{R}_1$-expansionary stage greater than t is t_1, while the least $\mathcal{R}_2$-expansionary stage greater than t is t_2. Then the changes which the strategy α has to wait for are determined as follows:

- If β_1 is following a Γ-strategy and β_2 is following a Γ-strategy, the strategy α will wait for a $U_1 \upharpoonright \phi_t(w)$ change between stages t and t_1 and for a $U_2 \upharpoonright \phi_t(w)$ change between stages t and t_2.
- If β_1 is following a Γ-strategy and β_2 is following a $\hat{\Gamma}$-strategy, the strategy α will wait for a $U_1 \upharpoonright \phi_t(w)$ change between stages t and t_1 and for a $V_2 \upharpoonright \phi_t(w)$ change between stages t and t_2.
- If β_1 is following a $\hat{\Gamma}$-strategy and β_2 is following a Γ-strategy, the strategy α will wait for a $V_1 \upharpoonright \phi_t(w)$ change between stages t and t_1 and for a $U_2 \upharpoonright \phi_t(w)$ change between stages t and t_2.
- If β_1 is following a $\hat{\Gamma}$-strategy and β_2 is following a $\hat{\Gamma}$-strategy, the strategy will wait for a $V_1 \upharpoonright \phi_t(w)$ change between stages t and t_1 and for a $V_2 \upharpoonright \phi_t(w)$ change between stages t and t_2.

Note that in practice, the $\mathcal{S}$ strategy α only needs to check whether it has obtained the changes related to those strategies which are following a Γ-strategy. It is not necessary for α to check whether V_1 or V_2 changes have taken place, because a strategy needing such a permission will be located below a g_1 or g_2 outcome which guarantees that such a change has occurred.

In general when depicting some $\mathcal{S}$ strategy α located below two active $\mathcal{R}$ strategies β_1 and β_2, we shall use a notation which helps us identify those sets whose change is necessary for the strategy α to diagonalise successfully when closing a gap. We shall write α^{U_1,U_2} when β_1 and β_2 both follow a Γ-strategy, and α^{V_1,V_2} when β_1 and β_2 both follow a $\hat{\Gamma}$-strategy. When β_1 follows a Γ-strategy and β_2 follows a $\hat{\Gamma}$-strategy, we write α^{U_1,V_2}, whilst when β_1 follows a $\hat{\Gamma}$-strategy and β_2 follows a Γ-strategy, we shall write α^{V_1,U_2}.

We shall now formalise the $\mathcal{S}$ strategy needed in this section.

The $\mathcal{S}$ Strategy

The strategy has a set of witnesses W^e and a set of thresholds V^e. It chooses three parameters, the witness w and two thresholds v_1 and v_2. It also lies below the sequence of active $\mathcal{R}$ strategies (β_1, β_2). Each strategy β_i in this sequence follows either a Γ-strategy or a $\hat{\Gamma}$-strategy. The strategy operates inside a work interval (a, b).

(*) (Background Step) Perform this step at the beginning of every substage during which the strategy is accessible. If the following conditions are met resume from the step last indicated by the strategy, or resume from step (1) if no such step exists. Otherwise end the substage.

  (a) For all $1 \leq i \leq 2$, if v_i is defined, we have that $(\forall_{n \leq v_i})(\Gamma_i^{U_i,D}[s](n) \downarrow = A_s(n))$ holds ($\Gamma^{V,D}$ resp).

  (b) If w is defined, for all $1 \leq i \leq 2$ we have that $(\forall_{n \leq w})(\Gamma_i^{U_i,D}[s](n) \downarrow = A_s(n))$ holds ($\Gamma^{V,D}$ resp).

  (c) If w is defined, $a < \theta_s(w) < b$.

(1) (Select the thresholds). Let β_i be a strategy in the sequence (β_1, β_2). If no threshold v_i corresponding to β_i is defined, choose a threshold v_i. The value selected for this threshold is the least value in V^e such that:

  (a) $(\forall_{n \leq v_i})(\Gamma_i^{U_i,D}[s](n) \downarrow = A_s(n))$ ($\Gamma^{V,D}$ resp).

  (b) $a < v_i < b$.

  (c) $(\forall_{i < j \leq 2})(v_i > \gamma_{j,s}(v_j))$.

  (d) v_i is greater than the last stage at which α was last initialised.

  If thresholds satisfying these conditions cannot be found, end this substage. Resume from step (1). Otherwise go to step (2).

(2) (Select the witness). Choose a witness w. The value selected for this witness is the least value in W^e such that:

(a) For all $1 \leq i \leq 2$, we have that $(\forall_{n \leq w})(\Gamma^{U_i,D}_{i,s}(n) \downarrow= A_s(n))$ holds ($\Gamma^{V,D}$ resp).

(b) $a < w < b$.

(c) $\Theta^D(w) \downarrow$

(d) $a < \theta_s(w) < b$.

(e) $(\forall_{1 \leq j \leq 2})(w > \gamma_{j,s}(v_j))$.

(f) w is greater than the last stage at which α was last initialised.

If a witness satisfying these conditions cannot be found, end this substage. Resume from step (1). Otherwise go to step (3).

(3) (Wait for convergence). Is $\Theta^D[s](w) \downarrow= 0$?

(a) (Yes) Go to step (4).

(b) (No) Otherwise end this substage. Resume from step (3).

(4) (Honestification for β_1). Is $\phi_{1,s}(w) > \gamma_{1,s}(w)$?

(a) (Yes) Is s a close stage?

(i) (Yes) Enumerate $\gamma_{1,s}(w)$ into D. Go to step (3).

(ii) (No) Go to step (3).

(b) (No) End this substage and resume from step (5).

(5) (Honestification for β_2). Is $\phi_{2,s}(w) > \gamma_{2,s}(w)$?

(a) (Yes) Is s a close stage?

(i) (Yes) Enumerate $\gamma_{2,s}(w)$ into D. Go to step (3).

(ii) (No) Go to step (3).

(b) (No) End this substage and resume from step (6).

(6) (Gap open) Is stage s an open stage?

(a) (Yes) Constrain each strategy β_i in the sequence (β_1, β_2) to choose uses $\gamma_{i,s'}(w) > \theta_s(w)$ at all stages $s' > s$. Enumerate w into A and restrain $D \restriction \theta_s(w)$. Cancel the witness w and end this substage. Resume from step (7).

(b) (No) End this substage and resume from step (3).

(7) (Gap close) Is stage s a close stage?

(a) (Yes) Let t be the stage at which the strategy last opened a gap by ending its substage at step (6). Let t_1 be the least $\mathcal{R}_1$-expansionary stage greater than t, and let t_2 be the least $\mathcal{R}_2$-expansionary stage greater than t. Let β_i be the least strategy in the sequence (β_1, β_2) such that $U_{i,t} \restriction \phi_t(w) = U_{i,t_i} \restriction \phi_t(w)$. If $\beta_i = \beta_1$ go to step (8), whilst if $\beta_i = \beta_2$, go to step (9). If there is no such β_i, go to step (10).

(b) (No) End this substage and resume from step (7).

(8) (Capricious destruction for β_1) Enumerate $\gamma_{1,s}(v_1)$ into D. End this substage and resume from step (1).

(9) (Capricious destruction for β_2) Enumerate $\gamma_{2,s}(v_2)$ into D. End this substage and resume from step (1).

(10) (Successful diagonalisation). End this substage and resume from step (10).

We shall now consider the outcomes of the $\mathcal{S}$ strategy and define the outcome which the strategy selects at stage s.

Outcome of the $\mathcal{S}$ Strategy

The $\mathcal{S}$ strategy α may have up to six outcomes. These will include the diagonalisation outcome d, the honestification for β_1 outcome h_1, the honestification for β_2 outcome h_2 and the wait outcome w. In addition if β_1 is following a Γ-strategy, α will have a g_1 ouctome, whilst if β_2 is following a Γ-strategy, α will have a g_2 outcome. These outcomes, when present, are ordered as follows: $d <_\Lambda g_2 <_\Lambda g_1 <_\Lambda h_1 <_\Lambda h_2 <_\Lambda w$. Once again note that the g_i outcomes are ordered in descending order, while the h_i outcomes are ordered in ascending order. At stage s, the strategy will decide which outcome to visit by performing the following case analysis:

(I) *Waiting for parameters.* The strategy ends its substage at the (Background Step), step (1) or step (2). Then the functional built by strategy β_1 or that built by strategy β_2 fails to be defined and equal to A up to some parameter, or α has failed to choose some parameter. The strategy visits its w outcome and imposes the work interval $(0, \infty)$.

(II) *Waiting for computation.* The strategy ends its substage at step (3). Then we have that $\Theta^D[s](w) \uparrow$, or $\Theta^D[s](w) \downarrow= 1$. The strategy visits its w outcome and imposes the work interval $(0, \infty)$.

(III) *Honestification for β_1.* The strategy ends its substage at step (4)(a)(i). Then we have that $\Theta^D[s](w) \downarrow= 0$, $\phi_{1,s}(w) > \gamma_{1,s}(w)$ and that s is a close stage. The strategy visits its h_1 outcome and imposes the work interval $(w, \gamma_{1,s}(w))$.

(IV) *Honestification for β_2.* The strategy ends its substage at step (5)(a)(i). Then we have that $\Theta^D[s](w) \downarrow= 0$, $\phi_{1,s}(w) \leq \gamma_{1,s}(w)$, $\phi_{2,s}(w) > \gamma_{2,s}(w)$ and that s is a close stage. The strategy visits its h_2 outcome and imposes the work interval $(w, \gamma_{2,s}(w))$.

(V) *Honestification for β_1 - waiting for close stage.* The strategy ends its substage at step (4)(a)(ii). Then we have that $\Theta^D[s](w) \downarrow= 0$, and that $\phi_{1,s}(w) > \gamma_{1,s}(w)$, but s is not a close stage. The strategy visits its h_1 outcome and imposes the work interval $(w, \gamma_{1,s}(w))$.

(VI) *Honestification for β_2 - waiting for close stage.* The strategy ends its substage at step (5)(a)(ii). Then we have that $\Theta^D[s](w) \downarrow= 0$, $\phi_{1,s}(w) \leq \gamma_{1,s}(w)$ and that $\phi_{2,s}(w) > \gamma_{2,s}(w)$ but s is not a close stage. The strategy visits its h_2 outcome and imposes the work interval $(w, \gamma_{2,s}(w))$.

(VII) *Opening a gap - g outcomes exist.* The strategy ends its substage at step (6)(a). Then we have that $\Theta^D[s](w) \downarrow= 0$, $\phi_{1,s}(w) \leq \gamma_{1,s}(w)$, $\phi_{2,s}(w) \leq \gamma_{2,s}(w)$ and that w has been enumerated into A. In addition at least one of β_1 and β_2 are following a Γ-strategy. The strategy has opened a gap. If the strategy has never closed a gap, the strategy visits the outcome g_i, where g_i is the rightmost g outcome of the α strategy. Otherwise, the strategy visits the outcome on which a gap was last closed. The work interval $(v_i, \gamma_i(v_i))$ is imposed, where i is the index of the g_i outcome which has been visited last.

(VIII) *Opening a gap - no g outcomes.* The strategy ends its substage at step (6)(a). Then we have that $\Theta^D[s](w) \downarrow= 0$, $\phi_{1,s}(w) \leq \gamma_{1,s}(w)$, $\phi_{2,s}(w) \leq \gamma_{2,s}(w)$ and that w has been enumerated into A. Both of β_1 and β_2 are following a $\hat{\Gamma}$-strategy. Then we have that the strategy has diagonalised successfully. The strategy visits its d outcome and imposes the work interval (s_1, ∞), where s_1 is equal to the present stage s.

(IX) *Opening a gap - waiting for open stage.* The strategy ends its substage at step (6)(b). Then we have that $\Theta^D[s](w) \downarrow= 0$, $\phi_{1,s}(w) \leq \gamma_{1,s}(w)$, $\phi_{2,s}(w) \leq \gamma_{2,s}(w)$ but s is not an open stage. The strategy visits its w outcome and imposes the work interval $(0, \infty)$.

(X) *Closing a gap - waiting for close stage.* The strategy ends its substage at step (7)(b). Then the strategy has opened a gap at stage t, there has been no close stage between t and s and s is not a close stage. The strategy visits its w outcome and imposes the work interval $(0, \infty)$.

(XI) *Closing a gap - capricious destruction for β_1.* The strategy ends its substage at step (8). Then we have that the strategy has opened a gap at stage t, that there has been no close stage between t and s and that s is a close stage. In addition, β_1 is following a Γ-strategy, and $U_{1,t} \upharpoonright \phi_{1,t}(w) = U_{1,t_1} \upharpoonright \phi_{1,t}(w)$, where t_1 is the least $\mathcal{R}_1$-expansionary stage greater than t. The strategy is closing a gap. The strategy visits its g_1 outcome and imposes the work interval $(v_1, \gamma_{1,s}(v_1))$.

(XII) *Closing a gap - capricious destruction for β_2.* The strategy ends its substage at step (9). Then we have that the strategy has opened a gap at stage t, that there has been no close stage between t and s and that s is a close stage. In addition, β_2 is following a Γ-strategy, $U_{1,t} \upharpoonright \phi_{1,t}(w) \neq U_{1,t_1} \upharpoonright \phi_{1,t}(w)$ and $U_{2,t} \upharpoonright \phi_{2,t}(w) = U_{2,t_2} \upharpoonright \phi_{2,t}(w)$, where t_1 is the least $\mathcal{R}_1$-expansionary stage greater than t and t_2 is the least $\mathcal{R}_2$-expansionary stage greater than t. The strategy is closing a gap. The strategy visits its g_2 outcome and imposes the work interval $(v_2, \gamma_{2,s}(v_2))$.

(XIII) *Closing a gap - successful diagonalisation.* Then we have that the strategy has opened a gap at stage t, that there has been no close stage between t and s and that s is a close stage. In addition $U_{i,t} \upharpoonright \phi_{i,t} \neq U_{i,t_i} \upharpoonright \phi_{i,t}$ whenever β_i is following a Γ-strategy. The strategy has diagonalised successfully. The strategy visits its d outcome and imposes the work interval (s_1, ∞), where s_1 is equal to the present stage s.

(XIV) *Stopped.* strategy ends its substage at step (10), and step (10) has been visited since the strategy was last initialised. Then we have that the strategy has already diagonalised successfully. The strategy visits its d outcome and imposes the work interval (s_1, ∞) , where s_1 is the stage at which the strategy has first diagonalised successfully and visited its outcome d.

We are now in a position to take the next step and expand the priority tree below the strategy α^{U_1,U_2}.

2.7.6 Organisation of Priority Tree

We shall now make our second attempt at organising a priority tree to satisfy the $\mathcal{S}$ requirement below the $\mathcal{R}_2$ requirement below the $\mathcal{R}_1$ requirement. The following discussion will refer to the priority tree shown in figure 2.4. For the time being we shall only expand the priority tree below the g_2 outcome of α^{U_1,U_2}. This will be sufficient to expose a difficulty in satisfying the requirements which can only be addressed through strategies of greater sophistication. We shall omit the f outcomes of $\mathcal{R}$ strategies as usual. The following notation will be used on the priority tree.

- $\beta_1^{U_1}$ will denote an $\mathcal{R}$ strategy (from Section 2.7) which is following a Γ-strategy.
- $\beta_2^{U_2}$ will denote an $\mathcal{R}$ strategy (from Section 2.7) which is following a Γ-strategy.
- α^{U_1,U_2} will denote an $\mathcal{S}$ strategy (from Section 2.7) which needs to take into consideration two $\mathcal{R}$ strategies above it, with each following a Γ-strategy.
- $\beta_2^{V_2}$ will denote an $\mathcal{R}$ strategy (from Section 2.7) which is following a $\hat{\Gamma}$-strategy.
- α^{U_1,V_2} will denote an $\mathcal{S}$ strategy (from Section 2.7) which needs to take into consideration two $\mathcal{R}$ strategies above it, with the first following a Γ-strategy and the second following a $\hat{\Gamma}$-strategy.

In order to construct the tree below the g_2 outcome of α^{U_1,U_2}, we recall that this outcome does not interfere with the attempts of the strategy β_1 to satisfy the requirement $\mathcal{R}_1$. On the other hand, the actions of α^{U_1,U_2} will make the functional built by β_2 partial, leaving the requirement $\mathcal{R}_2$ unsatisfied. Finally, α^{U_1,U_2} fails to satisfy the $\mathcal{S}$ requirement, because it never diagonalises successfully.

This means that the highest priority requirement which is left unsatisfied below the g_2 outcome of α^{U_1,U_2} is now $\mathcal{R}_2$. Now, the g_2 outcome causes a switch in the manner of satisfying the requirement $\mathcal{R}_2$. Hence in order to satisfy the $\mathcal{R}_2$ requirement we shall use an $\mathcal{R}$ strategy $\beta_2^{V_2}$ which follows a $\hat{\Gamma}$-strategy.

The highest priority requirement which is left unsatisfied below the i outcome of the $\mathcal{R}$ strategy $\beta_2^{V_2}$ is now the $\mathcal{S}$ requirement. The requirement can be satisfied by an $\mathcal{S}$ strategy α^{U_1,V_2} of the form found in Section 2.7.

This strategy considers the strategies $\beta_1^{U_1}$ and $\beta_2^{V_2}$ as being active and needs to take them into consideration. The strategy α^{U_1,V_2} will have the mandatory outcomes d, h_1, h_2 and w. It will also have a g_1 outcome, since $\beta_1^{U_1}$ is following a Γ-strategy, but no g_2 outcome, since $\beta_2^{V_2}$ is following a $\hat{\Gamma}$-strategy.

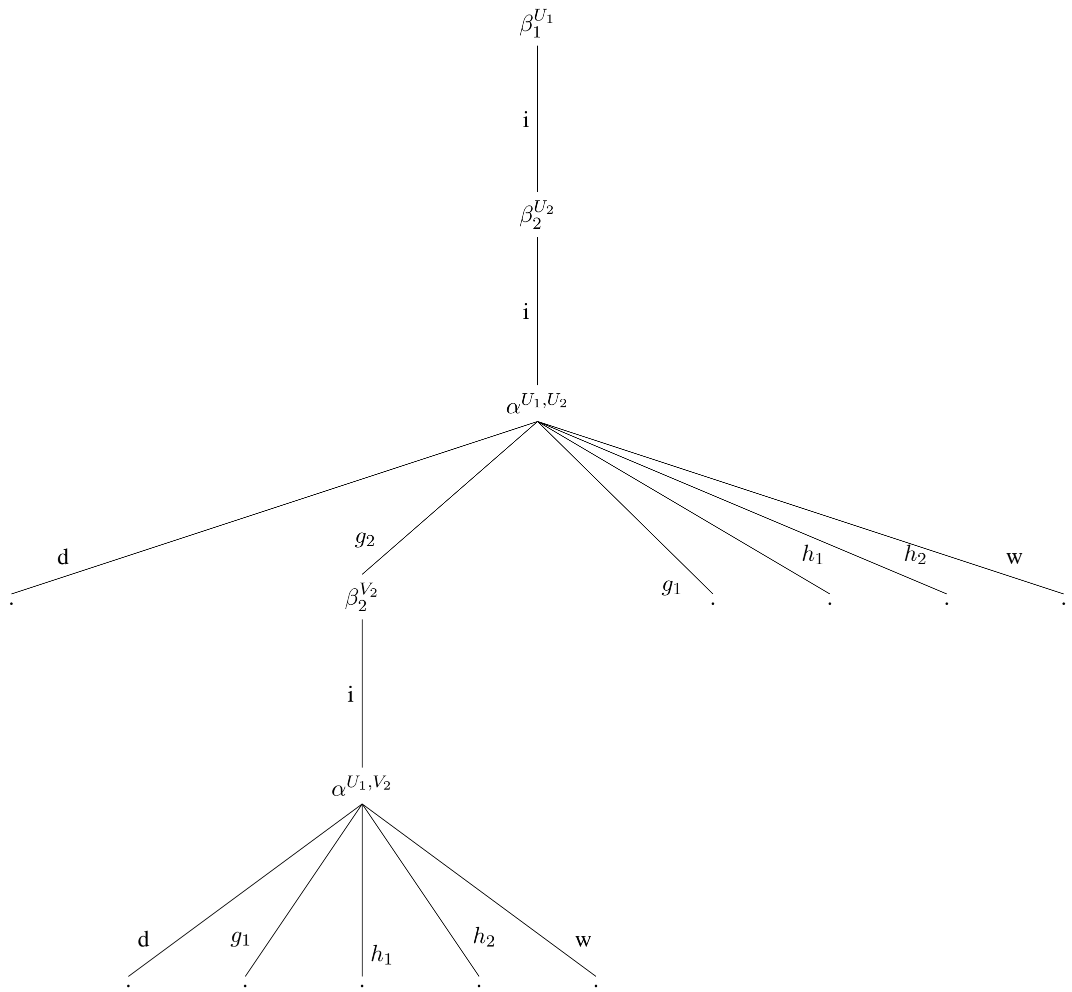

Figure 2.4: Priority tree for $\mathcal{S}$ below $\mathcal{R}_2$ below $\mathcal{R}_1$, second approximation

2.7.7 Satisfaction of Requirements

We shall now bring to the fore a problem with the present strategies. We consider the situation in which the leftmost outcome which is visited infinitely often by the strategy α^{U_1,V_2} is w.

$\boxed{w}$ Suppose w is the leftmost outcome to be visited infinitely often by α^{U_1,V_2}. Then there is a stage s_0 after which outcomes to the left of w are not accessible.

Suppose that at some stage $t \leq s_0$, α^{U_1,V_2} opens a gap by enumerating its witness w' into the set A. In order for this to be the case, α^{U_1,U_2} must have visited its g_2 outcome at t, and stage t must have been an open stage for α^{U_1,V_2}. For the latter to be the case α^{U_1,U_2} must also have opened a gap at stage t and enumerated a witness w into A.

Now, since α^{U_1,V_2} is located inside the work interval $(v_2, \gamma_2(v_2))$ imposed by α^{U_1,U_2}, it must be the case that $v_2 < w' < \gamma_2(v_2)$. In addition the ordering between the parameters of α^{U_1,U_2} ensures that $\gamma_2(v_2) < w$. It follows that $w' < w$.

Since w' has been enumerated into A, we have that the strategy $\beta_1^{U_1}$ sees that $\Phi_1^{U_1,V_1}(w')$ and $\Gamma_1^{U_1,D}(w')$ disagree with $A(w')$. Similarly the strategy $\beta_2^{U_2}$ sees that $\Phi_2^{U_2,V_2}(w')$ and $\Gamma_2^{U_2,D}(w')$ disagree with $A(w')$.

Now suppose that α^{U_1,U_2} becomes accessible again and closes the gap at stage s. Then there must have been an $\mathcal{R}_1$-expansionary stage at some least stage t_1 such that $t < t_1 < s$ and an $\mathcal{R}_2$-expansionary stage at some least stage t_2 such that $t < t_2 < s$.

Hence by the time α^{U_1,U_2} becomes accessible again the disagreement between $\Phi_1^{U_1,V_1}(w')$, $\Phi_2^{U_2,V_2}(w')$ and $A(w')$ must have been removed. Suppose that α^{U_1,U_2} closes its gap at stage s by visiting the outcome g_1. Then we have that $\Gamma_1^{U_1,D}(w')$ still disagrees with $A(w')$, and this disagreement can only be removed if $\gamma_1(w')$ or some smaller element enters D.

Now, since α^{U_1,V_2} is located below the g_2 outcome of the strategy α^{U_1,U_2}, it follows that the witness w' which it has enumerated when opening a gap is located within the work interval $(v_2, \gamma_2(v_2))$. On the other hand, the strategy α^{U_1,U_2} closes its gap by visiting the outcome g_1, which is associated to the work interval $(v_1, \gamma_1(v_1))$.

This means that we have that $v_2 < w' < \gamma_2(v_2) < v_1 < \gamma_1(v_1)$, and thus that $w' < v_1$. Now repairing the disagreement between $\Gamma_1^{U_1,D}(w')$ and $A(w')$ requires enumerating $\gamma_1(w')$ or some smaller element into D. But α^{U_1,U_2} enumerates $\gamma_1(v_1)$ into D, where $\gamma_1(w') < \gamma_1(v_1)$.

It follows that α^{U_1,U_2} has failed to repair the disagreement between $\Gamma_1^{U_1,D}(w')$ and $A(w')$ through this capricious destruction step.

Suppose now that after closing the gap at stage s, α^{U_1,U_2} is accessible once again at a stage greater than s_0. This means that after stage s, α^{U_1,U_2} can only ever visit outcome w.

In addition, every strategy below the w outcome of the α^{U_1,U_2} strategy is initialised at stage s, and can only choose parameters which are greater than stage s itself. Hence no element up to $\gamma_1(w')$ can ever enter D again.

Thus the $\mathcal{R}_1$ requirement will remain unsatisfied forever, because the functional built by the strategy $\beta_1^{U_1}$ remains in a state of disagreement.

2.7.8 Counterexample to the Leonhardi Account

Leonhardi's account of the Lachlan Non-Splitting Theorem [Leonhardi, 1996] differs from the one given in this dissertation in that the h_i outcomes of $\mathcal{S}$ strategies are ordered in descending order from left to right, whilst our account orders them in ascending order from left to right. This yields the priority tree shown in Figure 2.5. Since the priority tree is organised in the same manner as ours below the g_1 outcome of the strategy α^{U_1,U_2}, it follows that the counterexample introduced in the previous section also applies to the Leonhardi's account of the Lachlan Non-Splitting Theorem.

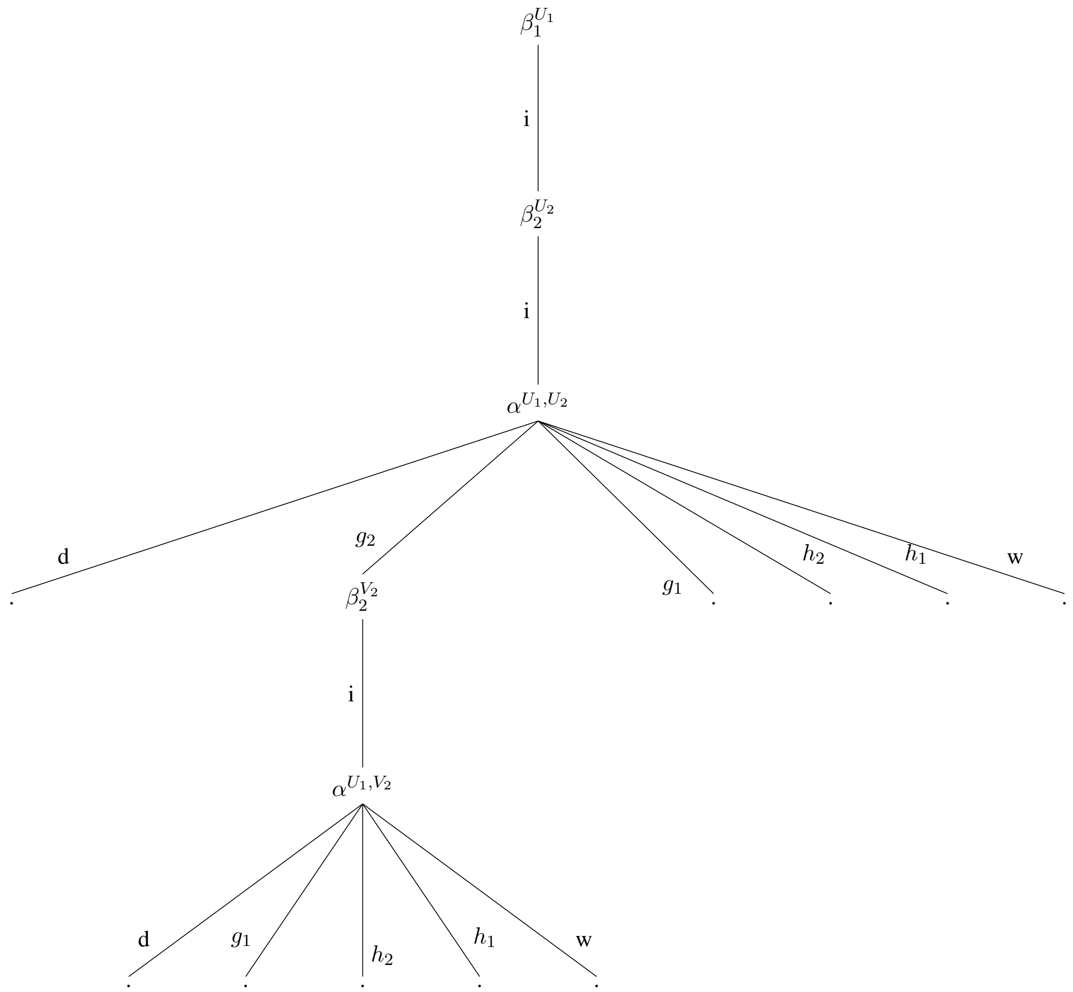

Figure 2.5: Priority tree for $\mathcal{S}$ below $\mathcal{R}_2$ below $\mathcal{R}_1$, Leonhardi's Account

2.8 $\mathcal{S}$ Below $\mathcal{R}_2$ Below $\mathcal{R}_1$ - Third Approximation

In this section we shall make the third and final attempt at satisfying the requirements $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{S}$. We shall therefore introduce the third approximation to the $\mathcal{S}$ and $\mathcal{R}$ strategies needed to achieve this goal. We start by introducing a number of concepts.

2.8.1 Self-repair of $\mathcal{R}$ Strategies

In Section 2.7 we have shown a situation in which the strategy α^{U_1,U_2} opened a gap by visiting its outcome g_2. This led to the strategy α^{U_1,V_2} being able to open a gap with a small witness w' inside the work interval below the g_2 outcome at the same stage. This enumeration consequently caused a disagreement between $\Gamma_1^{U_1,D}(w')$ and $A(w')$.

But when α^{U_1,U_2} closed the gap by visiting its g_1 outcome, it was unable to repair the disagreement. This happened because the enumeration of $\gamma_1(w)$ into D was too large to correct this disagreement, which needed to be at least as small $\gamma_1(w')$. Moreover, the strategy α^{U_1,U_2} has no control on the outcome it visits when it closes the gap, because this is controlled by the changes which fail to happen in the sets U_1 and U_2.

Since the disagreement between $\Gamma_1^{U_1,D}(w')$ and $A(w')$ cannot be repaired through the action of any $\mathcal{S}$ strategy below β_1, we shall allow the $\mathcal{R}$ strategy to self-repair its disagreements instead. When an $\mathcal{R}$ strategy sees a disagreement between $\Gamma^{U_1,D}(x)$ and $A(x)$, it will proceed to undefine its functional at this element by enumerating $\gamma(x)$ into the set D. The strategy will then be able to redefine its functional at a later stage so that it agrees with the set A once again.

Note that this turns the $\mathcal{R}$ strategies into positive strategies. The enumeration of elements into D aimed at repairing disagreements created by the enumeration of small witnesses can now result in the injury of other lower and higher priority $\mathcal{S}$ and $\mathcal{R}$ strategies. In particular, $\mathcal{S}$ strategies can have their restraint on D injured and their computation destroyed, and $\mathcal{R}$ strategies run the risk of having the functional they are building become partial at some element. The proof that this is not the case will be postponed until Section 2.10.

We have already determined that $\mathcal{S}$ strategies wishing to perform capricious destruction or honestification steps by enumerating elements into D should only do so during close stages. This stops the $\mathcal{S}$ strategy in question from interfering with higher priority strategies wishing to open a gap and restrain the set D. We shall adopt the same measure for $\mathcal{R}$ strategies performing self-repair, by allowing them to enumerate elements into D only during close stages.

2.8.2 Restarting

In order to be able to present the entire priority tree for $\mathcal{S}$ below $\mathcal{R}_2$ below $\mathcal{R}_1$, we shall need to introduce the concept of a *restart*.

Whenever an $\mathcal{S}$ strategy has a g_1 or h_1 outcome which is associated to the requirement $\mathcal{R}_1$, all switches related to the requirement $\mathcal{R}_2$ will be canceled. Note that the g_1 and h_1 outcomes render the requirement $\mathcal{R}_2$ unsatisfied through R-Synchronisation. The restart means that the next $\mathcal{R}$ strategy which attempts to satisfy the requirement $\mathcal{R}_2$ below the g_1 outcome must follow a Γ-strategy.

In general we have that whenever an $\mathcal{S}$ strategy has a g_n or h_n outcome associated with a requirement $\mathcal{R}_n$, this outcome will cancel all switches related to requirements $\mathcal{R}_m$ with $m > n$. These requirements will become unsatisfied and the $\mathcal{R}$ strategies which will be assigned to satisfy them once again will follow a Γ-strategy. Thus in these cases, progress made towards satisfying lower priority $\mathcal{R}$ requirements is discarded and the process is restarted once again.

We shall now formalise the $\mathcal{R}$ strategy required in this section.

2.8.3 The $\mathcal{R}$ Strategy

The main difference from the $\mathcal{R}$ strategy found in the previous section is that it can now follow either a Γ-strategy or a $\hat{\Gamma}$-strategy.

The $\mathcal{R}$ Strategy

This strategy has a set of uses U^e, and follows either a Γ-strategy or a $\hat{\Gamma}$-strategy. It lies below at most one higher priority $\mathcal{R}$ strategy β'. The strategy β' follows either a Γ-strategy or a $\hat{\Gamma}$-strategy. The strategy operates inside a work interval (a, b).

(1) (Check for expansionary stage). Is stage s a β-expansionary stage?

- (a) (Yes) Go to step (2).
- (b) (No) End the substage and resume from step (1).

(2) (Repair or define the functional). Is $\Gamma^{U,D}[s](m) \neq A_s(m)$ for some m? (or $\Gamma^{V,D}[s](m)$ resp.).

  (a) (Yes) Is stage s a close-stage?

    (i) (Yes) Enumerate $\gamma_s(m)$ into D. End the substage and resume from step (1).

    (ii) (No) End this substage and resume from step (1).

  (b) (No) (Define the functional). For every $x < l_s(\Phi^{U,V}, A)$ such that $\Gamma^{U,D}[s](x) \uparrow$ (or $\Gamma^{V,D}[s](x)$ resp.), define the axiom $\Gamma^{U,D}[s](x) = A_s(x)$ (or $\Gamma^{V,D}[s](x)$ resp.), and choose the corresponding use $\gamma_s(x)$ to be the least element in the set of uses U^e obeying the following conditions (if it exists):

    (i) $\gamma_s(x) \geq \gamma_t(x)$ for all $t < s$.

    (ii) $\gamma_s(x) > \gamma_s(y)$ for all $y < x$.

    (iii) $a < \gamma_s(x) < b$.

    (iv) $\gamma_s(x) \notin D$.

    (v) $\gamma_s(x) > \gamma'_s(x)$, where $\gamma'(x)$ is the use of the functional being built by β' at the element x.

    (vi) $\gamma_s(x) > y$, where y is a constraint imposed by some $\mathcal{S}$ strategy below β.

    (vii) $\gamma_s(x)$ is greater than the stage at which β was last initialised.

  End the substage and resume from step (1).

We shall now consider the outcomes of the $\mathcal{R}$ strategy and define the outcome which the $\mathcal{R}$ strategy selects at stage s.

Outcomes of the $\mathcal{R}$ Strategy

The $\mathcal{R}$ strategy has two outcomes, the infinitary outcome i and the finitary outcome f. These outcomes are ordered as follows: $i <_\Lambda f$. At stage s, the strategy decides which outcome to visit by performing the following case analysis:

(I) *Waiting for expansionary stage.* The strategy ends its substage at step (1). Then the present stage s is not a β-expansionary stage. The $\mathcal{R}$ strategy visits its f outcome.

(II) *Expansionary stage.* The strategy ends its substage at step (2)(a)(i) or step (2)(a)(ii) or step (2)(b). Then the present stage s is a β-expansionary stage. The $\mathcal{R}$ strategy visits its i outcome.

2.8.4 The $\mathcal{S}$ Strategy

The $\mathcal{S}$ strategy which we shall use in this section is identical to the one found in Section 2.7.

2.8.5 Organisation of the Priority Tree

We shall now make our final attempt at organising a priority tree to satisfy an $\mathcal{S}$ requirement below an $\mathcal{R}_2$ requirement below an $\mathcal{R}_1$ requirement. The following discussion will refer to the priority tree shown in Figure 2.6.

We shall build the full priority tree below the strategy α^{U_1,U_2} from right to left. As we descend down the tree, the strategies which are required become simpler and can be reused from previous sections.

Similarly when moving from the right to the left of the tree some structures will also be repeated and can be reused, simplifying the presentation of the priority tree.

In Figure 2.6 some branches of the tree terminate in a strategy rather than in the symbol '.'. This is an indication that the tree needs to be expanded below this strategy, according to a case which has been seen previously.

We shall also omit those parts of the tree occurring below the f outcomes of $\mathcal{R}$ strategies as usual.

The following notation will be used on the priority tree.

- $\beta_1^{U_1}$ will denote an $\mathcal{R}$ strategy (from Section 2.8) attempting to satisfy the $\mathcal{R}_1$ requirement by following a Γ-strategy.
- $\beta_2^{U_2}$ will denote an $\mathcal{R}$ strategy (from Section 2.8) attempting to satisfy the $\mathcal{R}_2$ requirement by following a Γ-strategy.
- $\beta_1^{V_1}$ will denote an $\mathcal{R}$ strategy (from Section 2.8) attempting to satisfy the $\mathcal{R}_1$ requirement by following a $\hat{\Gamma}$-strategy.

- $\beta_2^{V_2}$ will denote an $\mathcal{R}$ strategy (from Section 2.8) attempting to satisfy the $\mathcal{R}_2$ requirement by following a $\hat{\Gamma}$-strategy
- α^{U_1} will denote an $\mathcal{S}$ strategy (from Section 2.5) attempting to satisfy the $\mathcal{S}$ requirement. This strategy needs to take into consideration one $\mathcal{R}$ strategy which is attempting to satisfy the requirement $\mathcal{R}_1$ by following a Γ-strategy.
- α^{U_2} will denote an $\mathcal{S}$ strategy (from Section 2.5) attempting to satisfy the $\mathcal{S}$ requirement. This strategy needs to take into consideration one $\mathcal{R}$ strategy which is attempting to satisfy the requirement $\mathcal{R}_2$ by following a Γ-strategy.
- α^{V_1} will denote an $\mathcal{S}$ strategy (from Section 2.5) attempting to satisfy the $\mathcal{S}$ requirement. This strategy needs to take into consideration one $\mathcal{R}$ strategy which is attempting to satisfy the requirement $\mathcal{R}_1$ by following a $\hat{\Gamma}$-strategy.
- α^{V_2} will denote an $\mathcal{S}$ strategy (from Section 2.5) attempting to satisfy the $\mathcal{S}$ requirement. This strategy needs to take into consideration one $\mathcal{R}$ strategy which is attempting to satisfy the requirement $\mathcal{R}_2$ by following a $\hat{\Gamma}$-strategy.
- α^{U_1,U_2} will denote an $\mathcal{S}$ strategy (from Section 2.7) attempting to satisfy the $\mathcal{S}$ requirement. This strategy needs to take into consideration two $\mathcal{R}$ strategies which are attempting to satisfy the requirements $\mathcal{R}_1$ and $\mathcal{R}_2$ by following Γ-strategies.
- α^{U_1,V_1} will denote an $\mathcal{S}$ strategy (from Section 2.7) attempting to satisfy the $\mathcal{S}$ requirement. This strategy needs to take into consideration two $\mathcal{R}$ strategies which are attempting to satisfy the requirements $\mathcal{R}_1$ and $\mathcal{R}_2$ by following a Γ-strategy and a $\hat{\Gamma}$-strategy respectively.
- α^{V_1,U_2} will denote an $\mathcal{S}$ strategy (from Section 2.7) attempting to satisfy the $\mathcal{S}$ requirement. This strategy needs to take into consideration two $\mathcal{R}$ strategies which are attempting to satisfy the requirements $\mathcal{R}_1$ and $\mathcal{R}_2$ by following a $\hat{\Gamma}$-strategy and a Γ-strategy respectively.
- α^{V_1,V_2} will denote an $\mathcal{S}$ strategy (from Section 2.7) attempting to satisfy the $\mathcal{S}$ requirement. This strategy needs to take into consideration two $\mathcal{R}$ strategies which are attempting to satisfy the requirements $\mathcal{R}_1$ and $\mathcal{R}_2$ by following Γ-strategies.

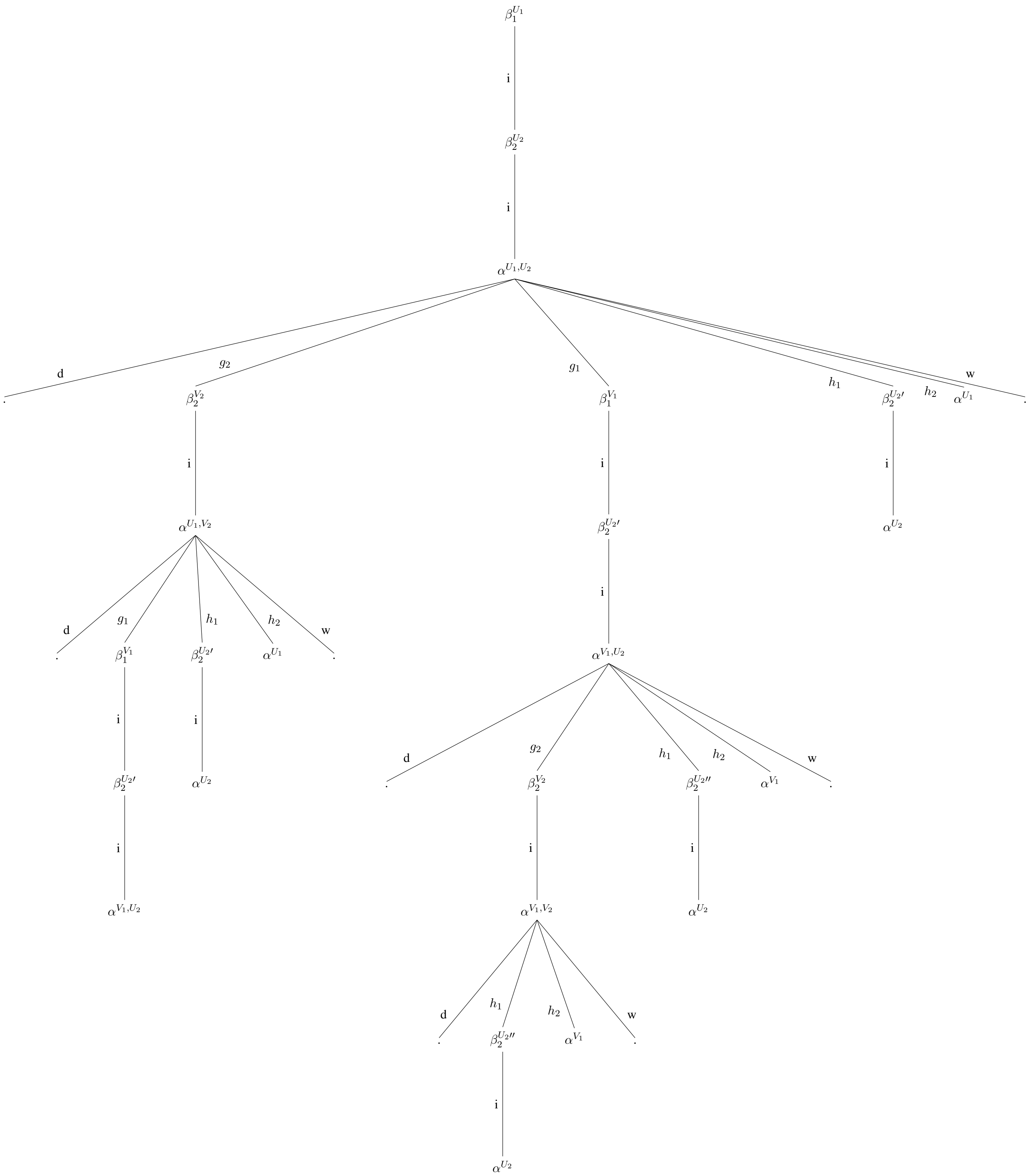

Figure 2.6: Priority tree for $\mathcal{S}$ below $\mathcal{R}_2$ below $\mathcal{R}_1$, third approximation.

(1) Below the w outcome of the α^{U_1,U_2} strategy we have that the $\mathcal{S}$, $\mathcal{R}_2$ and $\mathcal{R}_1$ requirements are satisfied.

(2) Below the h_2 outcome of the α^{U_1,U_2} strategy we have that the $\mathcal{R}_2$ requirement is satisfied trivially, whilst the $\mathcal{R}_1$ requirement will be satisfied by the strategy $\beta_1^{U_1}$ in the absence of interference by other strategies. On the other hand, the $\mathcal{S}$ requirement remains unsatisfied. Therefore we add an $\mathcal{S}$ strategy α^{U_1} which considers only the $\mathcal{R}$ strategy $\beta_1^{U_1}$ above it as active, which is a situation which has already been encountered before.

(3) Below the h_1 outcome of the α^{U_1,U_2} strategy we have that the $\mathcal{R}_1$ requirement is satisfied trivially, whilst the $\mathcal{R}_2$ requirement fails to be satisfied due to R-Synchronisation. Similarly the $\mathcal{S}$ requirement remains unsatisfied. The h_1 outcome of α^{U_1,U_2} causes a restart, meaning that in order to satisfy the $\mathcal{R}_2$ requirement we use an $\mathcal{R}$ strategy $\beta_2^{U_2\prime}$ which follows a Γ-strategy. In order to satisfy the $\mathcal{S}$ requirement, we use an $\mathcal{S}$ strategy α^{U_2} which considers only the $\mathcal{R}$ strategy $\beta_2^{U_2\prime}$ above it as active, which is a situation which has already been encountered before.

(4) Below the g_1 outcome of the α^{U_1,U_2} strategy we have that the $\mathcal{R}_1$ requirement is unsatisfied, and that the $\mathcal{R}_2$ requirement fails to be satisfied due to R-Synchronisation. Similarly the $\mathcal{S}$ requirement remains unsatisfied. The g_1 outcome causes a switch in the manner of satisfying $\mathcal{R}_1$ and a restart in the manner of satisfying $\mathcal{R}_2$. This means that in order to satisfy the requirement $\mathcal{R}_1$, we shall use a strategy $\beta_1^{V_1}$ following a $\hat{\Gamma}$-strategy, whilst to satisfy the requirement $\mathcal{R}_2$ we shall use a strategy $\beta_2^{U_2\prime}$ following a Γ-strategy. Finally, in order to satisfy the $\mathcal{S}$ strategy we shall use a strategy α^{V_1,U_2} which considers $\beta_1^{V_1}$ and $\beta_2^{U_2\prime}$ as active strategies.

We now consider the priority tree below the α^{V_1,U_2} strategy.

(4.1) Below the w outcome of the α^{V_1,U_2} strategy we have that the $\mathcal{S}$, $\mathcal{R}_2$ and $\mathcal{R}_1$ requirements are satisfied.

(4.2) Below the h_2 outcome of the α^{V_1,U_2} strategy we have that the $\mathcal{R}_2$ requirement is satisfied trivially, whilst the $\mathcal{R}_1$ requirement will be satisfied by the strategy $\beta_1^{V_1}$ in the absence of interference by other strategies. On the other hand, the $\mathcal{S}$ requirement remains unsatisfied. Therefore we add an $\mathcal{S}$ strategy α^{V_1} which considers only the $\mathcal{R}$ strategy $\beta_1^{V_1}$ above it as active, which is a situation which has already been encountered before.

(4.3) Below the h_1 outcome of the α^{V_1,U_2} strategy we have that the $\mathcal{R}_1$ requirement is satisfied trivially, whilst the $\mathcal{R}_2$ requirement fails to be satisfied due to R-Synchronisation. Similarly the $\mathcal{S}$ requirement remains unsatisfied. The h_1 outcome of α^{V_1,U_2} causes a restart in the manner of satisfying the requirement $\mathcal{R}_2$. Thus in order to satisfy the $\mathcal{R}_2$ requirement we use an $\mathcal{R}$ strategy $\beta_2^{U_2''}$ which follows a Γ-strategy. In order to satisfy the $\mathcal{S}$ requirement, we use an $\mathcal{S}$ strategy α^{U_2} which considers only the $\mathcal{R}$ strategy $\beta_2^{U_2''}$ above it as active, which is a situation which has already been encountered before.

(4.4) Below the g_2 outcome of the α^{V_1,U_2} strategy we have that the $\mathcal{R}_2$ requirement is unsatisfied, whilst the $\mathcal{R}_1$ requirement will be satisfied by the strategy $\beta_1^{V_1}$ in the absence of interference by other strategies. Similarly the $\mathcal{S}$ requirement remains unsatisfied. Now the g_2 outcome of the strategy α^{V_1,U_2} causes a switch in the manner of satisfying the $\mathcal{R}_2$ requirement. This means that in order to satisfy the requirement $\mathcal{R}_2$, we shall use a strategy $\beta_2^{V_2}$ which follows a $\hat{\Gamma}$-strategy. On the other hand to satisfy the $\mathcal{S}$ strategy we shall use a strategy α^{V_1,V_2} which considers $\beta_1^{V_1}$ and $\beta_2^{V_2}$ as active strategies.

We now consider the priority tree below the α^{V_1,V_2} strategy.

(4.4.1) Below the w outcome of the α^{V_1,V_2} strategy we have that the $\mathcal{S}$, $\mathcal{R}_2$ and $\mathcal{R}_1$ requirements are satisfied.

(4.4.2) Below the h_2 outcome of the α^{V_1,V_2} strategy we have that the $\mathcal{R}_2$ requirement is satisfied trivially, whilst the $\mathcal{R}_1$ requirement will be satisfied by the strategy $\beta_1^{V_1}$ in the absence of interference by other strategies. On the other hand, the $\mathcal{S}$ requirement remains unsatisfied. Therefore we add an $\mathcal{S}$ strategy α^{V_1} which considers only the $\mathcal{R}$ strategy $\beta_1^{V_1}$ above it as active, which is a situation which has already been encountered before.

(4.4.3) Below the h_1 outcome of the α^{V_1,V_2} strategy we have that the $\mathcal{R}_1$ requirement is satisfied trivially, whilst the $\mathcal{R}_2$ requirement fails to be satisfied due to R-Synchronisation. Similarly the $\mathcal{S}$ requirement remains unsatisfied. The h_1 outcome of α^{V_1,V_2} causes a restart, meaning that in order to satisfy the $\mathcal{R}_2$ requirement we use an $\mathcal{R}$ strategy $\beta_2^{U_2\prime\prime}$ which follows a Γ-strategy. In order to satisfy the $\mathcal{S}$ requirement, we use an $\mathcal{S}$ strategy α^{U_2} which considers only the $\mathcal{R}$ strategy $\beta_2^{U_2\prime\prime}$ above it as active, which is a situation which has already been encountered before.

(4.4.4) Below the d outcome of the α^{U_1,V_2} strategy we have that the $\mathcal{S}$, $\mathcal{R}_2$ and $\mathcal{R}_1$ requirements are satisfied.

(4.5) Below the w outcome of the α^{V_1,U_2} strategy we have that the $\mathcal{S}$, $\mathcal{R}_2$ and $\mathcal{R}_1$ requirements are satisfied.

(5) Below the g_2 outcome of the α^{U_1,U_2} strategy we have that the $\mathcal{R}_2$ requirement is unsatisfied, whilst the $\mathcal{R}_1$ requirement will be satisfied by the strategy $\beta_1^{U_1}$ in the absence of interference by other strategies. Similarly the $\mathcal{S}$ requirement remains unsatisfied. Now, the g_2 outcome of the α^{U_1,U_2} strategy causes a switch in the manner of satisfying the $\mathcal{R}_2$ requirement. This means that in order to satisfy the requirement $\mathcal{R}_2$, we shall use a strategy $\beta_2^{V_2}$ which follows a $\hat{\Gamma}$-strategy. On the other hand to satisfy the $\mathcal{S}$ strategy we shall use a strategy α^{U_1,V_2} which considers $\beta_1^{U_1}$ and $\beta_2^{V_2}$ as active strategies.

We now consider the priority tree below the α^{U_1,V_2} strategy.

(5.1) Below the w outcome of the α^{U_1,V_2} strategy we have that the $\mathcal{S}$, $\mathcal{R}_2$ and $\mathcal{R}_1$ requirements are satisfied.

(5.2) Below the h_2 outcome of the α^{U_1,V_2} strategy we have that the $\mathcal{R}_2$ requirement is satisfied trivially, whilst the $\mathcal{R}_1$ requirement will be satisfied by the strategy $\beta_1^{U_1}$ in the absence of interference by other strategies. On the other hand, the $\mathcal{S}$ requirement remains unsatisfied. Therefore we add an $\mathcal{S}$ strategy α^{U_1} which considers only the $\mathcal{R}$ strategy $\beta_1^{U_1}$ above it as active, which is a situation which has already been encountered before.

(5.3) Below the h_1 outcome of the α^{U_1,V_2} strategy we have that the $\mathcal{R}_1$ requirement is satisfied trivially, whilst the $\mathcal{R}_2$ requirement fails to be satisfied due to R-Synchronisation. Similarly the $\mathcal{S}$ requirement remains unsatisfied. The h_1 outcome of α^{U_1,V_2} causes a restart in the manner of satisfying the $\mathcal{R}_2$ requirement. This means that in order to satisfy the $\mathcal{R}_2$ requirement we use an $\mathcal{R}$ strategy $\beta_2^{U_2\prime}$ which follows a Γ-strategy. In order to satisfy the $\mathcal{S}$ requirement, we use an $\mathcal{S}$ strategy α^{U_2} which considers only the $\mathcal{R}$ strategy $\beta_2^{U_2\prime}$ above it as active, which is a situation which has already been encountered before.

(5.4) Below the g_1 outcome of the α^{U_1,V_2} strategy we have that the $\mathcal{R}_1$ requirement is unsatisfied, and that the $\mathcal{R}_2$ requirement fails to be satisfied due to R-Synchronisation. Similarly the $\mathcal{S}$ requirement remains unsatisfied. The g_1 outcome causes a switch in the manner of satisfying $\mathcal{R}_1$ and a restart in the manner of satisfying $\mathcal{R}_2$. This means that in order to satisfy the requirement $\mathcal{R}_1$, we shall use a strategy $\beta_1^{V_1}$ following a $\hat{\Gamma}$-strategy, whilst to satisfy the requirement $\mathcal{R}_2$ we shall use a strategy $\beta_2^{U_2\prime}$ following a Γ-strategy. Finally, in order to satisfy the $\mathcal{S}$ strategy we shall use a strategy α^{V_1,U_2} which considers $\beta_1^{V_1}$ and $\beta_2^{U_2\prime}$ as active strategies. This is a situation which has already been encountered before on the priority tree.

(5.5) Below the d outcome of the α^{U_1,V_2} strategy we have that the $\mathcal{S}$, $\mathcal{R}_2$ and $\mathcal{R}_1$ requirements are satisfied.

(6) Below the d outcome of the α^{U_1,U_2} strategy we have that the $\mathcal{S}$, $\mathcal{R}_2$ and $\mathcal{R}_1$ requirements are satisfied.

2.8.6 Satisfaction of Requirements

We shall now examine the satisfaction of the $\mathcal{S}$, $\mathcal{R}_2$ and $\mathcal{R}_1$ requirements, showing how they are finally satisfied by the priority tree in Figure 2.6. Many of the branches terminate in strategies representing a subtree which has already been shown to succeed in satisfying these requirements. Hence, we do not need to analyse whether these branches lead to the satisfaction of these requirements.

We shall instead focus on the most complicated branch which goes from α^{U_1,U_2} to α^{V_1,V_2}, as shown in Figure 2.7.

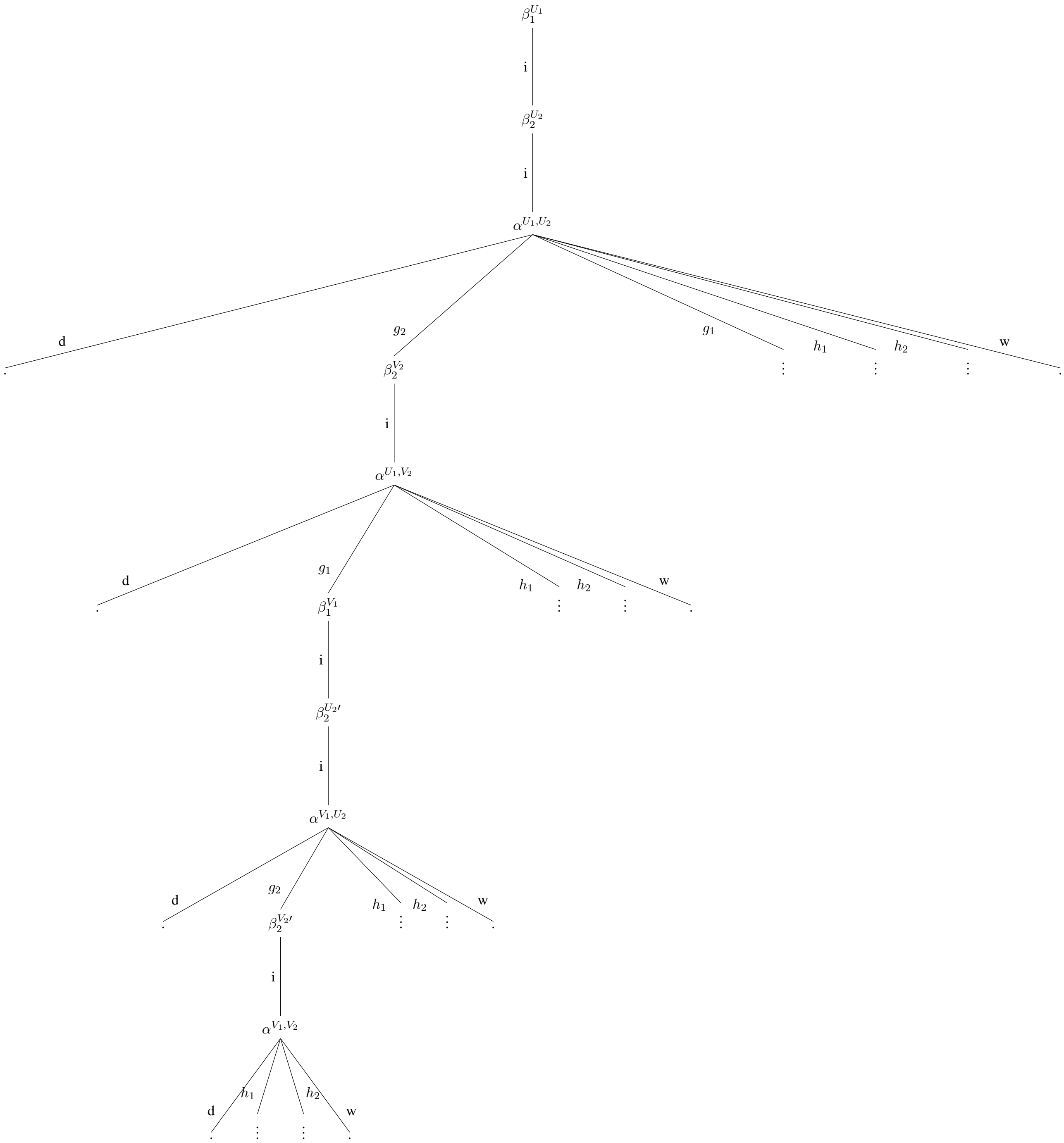

Figure 2.7: Priority tree for $\mathcal{S}$ below $\mathcal{R}_2$ below $\mathcal{R}_1$ - detail of the g_2 outcome of α^{U_1,U_2}.

For the purpose of our exposition, we shall consider only the situation where the d outcome is the leftmost outcome visited infinitely often by the strategy α^{V_1,V_2}, which reveals the operation of the most complex part of the construction.

Suppose that the strategy α^{V_1,V_2} and every other $\mathcal{S}$ strategy above it attempts to diagonalise during the same stage. Then the analysis below shows how each combination of changes in the sets U_1 and V_1 by the least $\mathcal{R}_1$-expansionary stage greater than s, and changes in the sets U_2 and V_2 by the least $\mathcal{R}_2$-expansionary stage greater than s causes one of the $\mathcal{S}$ strategies to succeed. It also shows that if α^{V_1,V_2} becomes accessible again after stage s, it must be the result of every $\mathcal{S}$ strategy above it failing to diagonalise, and that this result causes it to diagonalise instead, leading to the satisfaction of the $\mathcal{S}$ requirement.

$\boxed{\text{d}}$ Suppose d is the leftmost outcome of the strategy to be visited infinitely often. Then α^{V_1,V_2} must have opened a gap and diagonalised successfully as a result.

In order for α^{V_1,V_2} to open a gap at some stage t, α^{V_1,V_2} must have been accessible at stage t. This can only have been the case if the strategies α^{U_1,U_2}, α^{U_1,V_2} and α^{V_1,U_2} have also been accessible at stage t.

In this case α^{U_1,U_2} has opened a gap by visiting its outcome g_2, α^{U_1,V_2} has opened a gap by visiting its outcome g_1, and α^{V_1,U_2} has opened a gap by visiting outcome g_2.

When opening a gap, α^{U_1,U_2} enumerates its witness w, α^{U_1,V_2} enumerates its witness w', α^{V_1,U_2} enumerates its witness w'' and α^{V_1,V_2} enumerates its witness w'''.

Now, each $\mathcal{S}$ strategy on the priority tree is located inside the work interval imposed by the $\mathcal{S}$ strategy immediately above it. This means that the ordering $w''' < w'' < w' < w$ must hold between the witnesses. Hence w''' is now the least point of disagreement between the set A and the functionals built by $\mathcal{R}$ strategies lying on the path leading from the root of the tree to the strategy α^{V_1,V_2}.

In order for the strategy α^{U_1,U_2} to become accessible again after stage t, there must have been some least stage $t_1 > t$ such that t_1 is an $\mathcal{R}_1$-expansionary stage. In addition there must have been some least stage $t_2 > t$ such that t_2 is an $\mathcal{R}_2$-expansionary stage.

Hence we have that the disagreement between $\Phi_1^{U_1,V_1}$ and A has been repaired through some $U_1 \upharpoonright \phi_t(w''')$ or $V_1 \upharpoonright \phi_t(w''')$ change between stages t and t_1. Similarly, the disagreement between $\Phi_2^{U_2,V_2}$ and A has been repaired through a $U_2 \upharpoonright \phi_t(w''')$ or a $V_2 \upharpoonright \phi(w''')$ change between stages t and t_2.

Now, the functional $\Phi_1^{U_1,V_1}$ is shared amongst all strategies attempting to satisfy the $\mathcal{R}_1$ requirement and the functional $\Phi_2^{U_2,V_2}$ is shared amongst all strategies attempting to satisfy the $\mathcal{R}_2$ requirement. Hence t_1 and t_2 are the stages by which the strategies α^{U_1,U_2}, α^{U_1,V_2}, α^{V_1,U_2} and α^{V_1,V_2} will have to check whether U_1 or U_2 changes have occurred in response to the gaps which they have opened.

In addition, if an $\mathcal{R}$ strategy β which is trying to satisfy some requirement $\mathcal{R}_i$ sees an $\mathcal{R}_i$-expansionary stage at some given stage s, every $\mathcal{R}$ strategy β' attempting to satisfy the requirement $\mathcal{R}_i$ will see an $\mathcal{R}_i$-expansionary stage.

Suppose that α^{U_1,U_2} is accessible and is ready to close its gap at stage t'. Then it must check whether there has been the required U_1 change between t and t_1 and the required U_2 change between t and t_2.

If both changes have taken place, α^{U_1,U_2} would have diagonalised, making α^{U_1,V_2} inaccessible, which is impossible. On the other hand, we could have a U_1 change but no U_2 change (with a V_2 change instead), leading α^{U_1,U_2} to visit its outcome g_2. This makes the strategy α^{U_1,V_2} accessible at stage t'. The strategy α^{U_1,V_2} will then close its gap, and note that the U_1 and V_2 changes allow it to diagonalise successfully. But this is impossible once again.

Hence it follows that at stage t_1, a U_1 change must have failed to occur (with a V_1 change occurring instead). This leads α^{U_1,U_2} to close its gap by visiting outcome g_1. Note that although the strategies α^{U_1,V_2}, α^{V_1,U_2} and α^{V_1,V_2} are inaccessible at stage t', they remain in a suspended state waiting to close their gap. Eventually, α^{U_1,U_2} must visit its g_2 outcome again, and this can only happen if α^{U_1,U_2} has closed a gap on g_2 which it had opened at some stage greater than t'.

Suppose that α^{U_1,U_2} visits its g_2 outcome again at stage t''. Then α^{U_1,V_2} becomes accessible and closes its gap. When closing its gap, α^{U_1,V_2} notices that U_1 has failed to change at stage t_1, and thus closes its gap by visiting its outcome g_1.

This makes α^{V_1,U_2} accessible. This strategy knows that the failure of U_1 to change by stage t_1 guarantees the occurrence of a V_1 change by stage t_1. Hence, the strategy needs to check whether there has been a U_2 change by stage t_2. If this is the case, the strategy would diagonalise successfully, which is impossible. Hence, there must have been a V_2 change, and α^{V_1,U_2} closes by visiting the outcome g_2.

This makes the strategy α^{V_1,V_2} accessible. When the strategy closes its gap it must be the case that V_1 change has taken place by stage t_1 and that a V_2 changes has taken place by stage t_2. Hence the strategy must have been correct in believing that it would diagonalise successfully. The requirement $\mathcal{S}$ is thus satisfied by the strategy α^{V_1,V_2}.

On the other hand the $V_1 \upharpoonright \phi_{1,t}(w''')$ and $V_2 \upharpoonright \phi_{2,t}(w''')$ changes ensure that the functionals built by the strategies $\beta_2^{U_2\prime}$, and $\beta_2^{V_2\prime}$ no longer disagree with the set A at w'''. This follows from the fact that the α^{V_1,V_2} strategy ensures that the functionals built by the strategies $\beta_2^{U_2\prime}$, and $\beta_2^{V_2\prime}$ are honest at w''' before opening the gap.

It follows that the strategies $\beta_2^{U_2\prime}$ and $\beta_2^{V_2\prime}$ can keep building their functionals without further interference, satisfying the requirements $\mathcal{R}_1$ and $\mathcal{R}_2$.

2.9 The General Case

We shall now consider the general case, where we are required to satisfy many $\mathcal{R}$ and many $\mathcal{S}$ requirements simultaneously. In order to deal with many $\mathcal{R}$ and many $\mathcal{S}$ requirements, we shall generalise the approach found in Section 2.8, which we have used to successfully satisfy an $\mathcal{S}$ requirement below two $\mathcal{R}$ requirements.

We shall start by considering a straightforward generalisation of the $\mathcal{S}$ and $\mathcal{R}$ strategies which allows them to be used in a context involving many $\mathcal{R}$ and $\mathcal{S}$ requirements. Then we shall return to the problem of ensuring that the general $\mathcal{R}$ and $\mathcal{S}$ strategies are fair, in the sense of preserving infinitely many open and close stages for strategies located below them.

Following this the general $\mathcal{R}$ and $\mathcal{S}$ strategies will be formalised. We then proceed to consider the general priority tree, and formalise the way in which it is generated. We conclude by formalising the construction which decides which strategies are accessible on the priority tree at each stage. The verification, which proves that every requirement can be satisfied by the resulting system will be postponed to Section 2.10.

2.9.1 Generalising the Strategies

The general $\mathcal{R}$ strategy β must be able to follow either a Γ-strategy or a $\hat{\Gamma}$-strategy. It also needs to consider the fact that there might be n active $\mathcal{R}$ strategies lying above it. This means that when building its functional it will need to R-Synchronise with all of these strategies.

The general $\mathcal{S}$ strategy α needs to consider the fact that there might be a sequence of m active $\mathcal{R}$ strategies $(\beta_1, \ldots, \beta_m)$ above it. Hence it will need to choose a threshold v_i for each one of these strategies in addition to its witness w. This will also give rise to m different h_i outcomes. If m' of these m strategies are following a Γ-strategy, we will also have m' different g_i outcomes.

The strategy α will first choose a witness w and wait to see a computation $\Theta^D(w)\downarrow= 0$, visiting its outcome w until this is the case. Once this has become the case, the α strategy will determine whether there is some least active $\mathcal{R}$ strategy β_i above it whose functional is dishonest at w. If this is the case it will honestify the functional it is building and visit the corresponding h_i outcome.

When closing a gap, the strategy will consider each active $\mathcal{R}$ strategy above it which is following a Γ-strategy. It will then determine whether there is a least strategy β_i amongst these for which U_i has failed to change. If there is such an i, then the strategy will perform capricious destruction by visiting its g_i outcome. Otherwise it will diagonalise successfully and visit its d outcome.

The notions of S-Synchronisation, switching and restarting will now apply in their general form. Thus if an $\mathcal{S}$ strategy α lies below the g_i outcome of some $\mathcal{S}$ strategy α', it will need to S-Synchronise with it. In addition the g_i outcomes of an $\mathcal{S}$ strategy α cause a switch in the manner of satisfying the requirement $\mathcal{R}_i$. Satisfying such a requirement will now require an $\mathcal{R}$ strategy to follow a $\hat{\Gamma}$-strategy.

Finally, the g_i and h_i outcomes of an $\mathcal{S}$ strategy α cause a restart, canceling all switches affecting $\mathcal{R}$ requirements of priority lower than $\mathcal{R}_i$. Satisfying such a requirement will now require an $\mathcal{R}$ strategy to follow a Γ-strategy.

2.9.2 Fairness

Any $\mathcal{S}$ strategy α on the priority tree with a g_i outcome needs to see infinitely many open and close stages in order to function correctly. If the strategy is starved of one kind of stage it could be forced to wait forever, stuck at outcome w with a computation $\Theta^D(w)\downarrow= 0$ which is not suitable for satisfying the corresponding $\mathcal{S}$ requirement.

In order to avoid such a situation, we need to ensure that every $\mathcal{R}$ and $\mathcal{S}$ strategy passes infinitely many open and close stages down the leftmost outcome which they visit infinitely often, whichever this might be. We do this by assuming that the strategy in question sees infinitely many open and close stages, and then by showing that an infinite subset of each is preserved down the leftmost outcome which the strategy visits infinitely often.

The $\mathcal{S}$ strategy α shall ensure that this is the case in the following way.

Suppose that the outcome d or the outcome w is the leftmost outcome visited infinitely often by the strategy α. Then we have that the strategy eventually stops changing outcome and thus preserves every open and close stage it sees, passing it down the outcome in question.

Suppose that the outcome g_i is the leftmost outcome visited infinitely often by the strategy α. Then whenever the strategy closes a gap by visiting the outcome g_i, it will open the next gap by also visiting the outcome g_i. This is the approach adopted in previous sections and is sufficient to preserve infinitely many open and close stages and to pass them down the g_i outcome of the strategy.

Suppose that the outcome h_i is the leftmost outcome visited infinitely often by the strategy α. Whenever the functional built by some strategy above α is dishonest, the strategy will perform honestification in two parts.

During Part I the strategy will wait for a close stage and visit its outcome h_i. During this visit it will enumerate the required element into D in order to honestify the functional. During Part II the strategy will wait for an open stage and also visit its outcome h_i. However this time the strategy will not enumerate any element into D and will thus have no effect on the functional. This will ensure that the $\mathcal{S}$ strategy preserves infinitely many open and close stages and passes them down its h_i outcome.

On the other hand, the situation with the $\mathcal{R}$ strategy β is more complex. This is because the $\mathcal{R}$ strategy β chooses whether to visit its i or its f outcome depending on whether stage s is a β-expansionary stage or not. However whether s is a β-expansionary stage or not is completely outside of the strategy's control.

It might be the case that every β-expansionary stage coincides with an open stage, and that every stage which is not a β-expansionary stage coincides with a close stage (or vice versa). In such a scenario the strategy would visit its i outcome during open stages only and its f outcome during close stages only (or vice versa). It would thus fail to preserve infinitely many open stages and infinitely many close stages and to pass them down the leftmost outcome visited infinitely often by the strategy.

This difficulty can be resolved in the following way. Let us assume that there are infinitely many $\mathcal{R}$ -expansionary stages. At any given point in time β will want to pass an open stage or a close stage down its i outcome. The β strategy will first wait for a β-expansionary stage s. Then it will check whether that stage is of the required kind (say, an open stage). If this is the case, the

β-expansionary stage has coincided with a required stage. Therefore we can build the functional and visit the i outcome as usual, ensuring that an open stage has been passed down this outcome. We then set the required stage to be a close stage and repeat the process.

On the other hand, suppose that the β-expansionary stage occurs during a close stage. Then the β-expansionary stage and the open stage have not coincided. The strategy will then wait for an open stage s' such that the length of agreement between $\Phi^{U,V}$ and A is at least as large as the one witnessed during stage s. If this stage occurs, then we build the functional as usual according to the length of agreement, visit the i outcome and pass an open stage down this outcome as required.

If this stage never occurs, then the only explanation is that the functional keeps becoming undefined at some element which is less than the length of agreement seen at stage s. Hence, although there are infinitely many $\mathcal{R}$ -expansionary stages, the length of agreement between $\Phi^{U,V}$ and A is not infinite in the limit. This means that while the strategy visits only its f outcome after some stage, this is in effect the correct outcome to visit.

Note that in the situation where there are only finitely many β-expansionary stages, or where there is only finite agreement in the limit despite there being infinitely many $\mathcal{R}$ -expansionary stages, we have that the strategy visits only its f outcome. In this case, every open stage and close stage seen by the strategy, will be passed down its f outcome, as required.

2.9.3 The $\mathcal{R}$ Strategy

We shall now formally define the general $\mathcal{R}$ strategy.

The $\mathcal{R}$ strategy

This strategy has a set of uses U^e, and follows either a Γ-strategy or a $\hat{\Gamma}$-strategy. It lies below a sequence of n active of $\mathcal{R}$ strategies $(\beta_1, \ldots, \beta_n)$. Each strategy β_i in the sequence follows either a Γ-strategy or a $\hat{\Gamma}$-strategy. The strategy operates inside a work interval (a, b).

(1) (Initialise). Set the *required stage* to be an open-stage and go to step (2).

(2) (Check for expansionary stage). Is stage s a β-expansionary stage?

(a) (Yes) Go to step (3).

(b) (No) End the substage and resume from step (2).

(3) (Wait for the required stage). Is the *required stage* of the same kind as stage s, and is $l_s(A, \Phi^{U,V}) \geq l_t(A, \Phi^{U,V})$, where t is the last stage during which β last visited step (2)?

(a) (Yes) If the *required stage* is currently an open-stage, set the *required stage* to be a close-stage. On the other hand, if the *required stage* is currently a close-stage, set the *required stage* to be an open-stage. Go to step (4).

(b) (No) End the substage and resume from step (3).

(4) (Repair or define the functional). Is there an m such that $\Gamma^{U,D}[s](m) \neq A_s(m)$? (or $\Gamma^{V,D}[s](m)$ resp.).

(a) (Yes) Is stage s a close-stage?

(i) (Yes) Enumerate $\gamma_s(m)$ into D. End the substage and resume from step (2).

(ii) (No) End this substage and resume from step (2).

(b) (No) For every $x < l_s(A, \Phi^{U,V})$ such that $\Gamma^{U,D}[s](x) \uparrow$ (or $\Gamma^{V,D}[s](x)$ resp.), define the axiom $\Gamma^{U,D}[s](x) = A_s(x)$ (or $\Gamma^{V,D}[s](x)$ resp.), and choose the corresponding use $\gamma_s(x)$ to be the least element in the set of uses U^e obeying the following conditions (if it exists):

(i) $\gamma_s(x) \geq \gamma_t(x)$ for all $t < s$.

(ii) $\gamma_s(x) > \gamma_s(y)$ for all $y < x$.

(iii) $a < \gamma_s(x) < b$.

(iv) $\gamma_s(x) \notin D$.

(v) $\gamma_s(x) > \gamma_{j,s}(x)$, for every active $\mathcal{R}$ strategy $\beta_j \subset \beta$.

(vi) $\gamma_s(x) > y$, where y is a constraint imposed by some $\mathcal{S}$ strategy below β.

(vii) $\gamma_s(x)$ is greater than the last stage at which β was initialised.

End the substage and resume from step (2).

We shall now consider the outcomes of the $\mathcal{R}$ strategy and define the outcome which the strategy selects at stage s.

Outcome of the $\mathcal{R}$ strategy

The $\mathcal{R}$ strategy β has two outcomes, the infinitary outcome i and the finitary outcome f. These outcomes are ordered as follows: $i <_\Lambda f$. The outcome which is visited by the strategy β at stage s, and which we denote by $O_s(\beta)$ is chosen through the following case analysis:

(I) *Waiting for expansionary stage.* The strategy ends its substage at step (2)(b). The strategy is waiting for an expansionary stage. Then the $\mathcal{R}$ strategy visits its f outcome.

(II) *Waiting for a required stage.* The strategy ends its substage at step (3)(b). The strategy is waiting for a required stage which preserves the previous length of agreement. Then the $\mathcal{R}$ strategy visits its f outcome.

(III) *Expansion preserving required stage.* The strategy ends its substage at step (4)(a)(i), (4)(a)(ii) or (4)(b). The strategy has seen a required stage which preserves the previous length of agreement. Then the $\mathcal{R}$ strategy visits its i outcome.

2.9.4 The $\mathcal{S}$ Strategy

We shall now formally define the general $\mathcal{S}$ strategy.

The $\mathcal{S}$ strategy

The strategy has a set of witnesses W^e and a set of thresholds V^e. It chooses $m+1$ parameters, the witness w and m thresholds $v_1, \ldots, v_m$. It lies below the sequence of m active $\mathcal{R}$ strategies $(\beta_1, \ldots, \beta_m)$, each of which may either be following a Γ-strategy or a $\hat{\Gamma}$-strategy. The strategy operates inside a work interval (a, b).

(*) (Background Step) Perform this step at the beginning of every substage during which the strategy is accessible. If the following conditions are met resume from the step last indicated by the strategy, or resume from step (1) if no such step exists. Otherwise end the substage.

(a) For all $1 \leq i \leq m$, if v_i is defined, we have that $(\forall_{n \leq v_i})(\Gamma_i^{U_i,D}[s](n) \downarrow = A_s(n))$ holds ($\Gamma^{V,D}$ resp).

(b) If w is defined, for all $1 \leq i \leq m$ we have that $(\forall_{n \leq w})(\Gamma_i^{U_i,D}[s](n) \downarrow = A_s(n))$ holds ($\Gamma^{V,D}$ resp).

(c) If w is defined, $a < \theta_s(w) < b$.

(1) (Select the thresholds). Let β_i be a strategy in the sequence $(\beta_1, \ldots, \beta_m)$. If no threshold v_i corresponding to β_i is defined, choose a threshold v_i. The value selected for this threshold is the least value in V^e such that:

(a) $(\forall_{n \leq v_i})(\Gamma_i^{U_i,D}[s](n) \downarrow = A_s(n))$ ($\Gamma^{V,D}$ resp).

(b) $a < v_i < b$.

(c) $(\forall_{i < j \leq m})(v_i > \gamma_{j,s}(v_j))$.

(d) v_i is greater than the last stage at which α was initialised.

If thresholds satisfying these conditions cannot be found, end this substage. Resume from step (1). Otherwise go to step (2).

(2) (Select the witness). Choose a witness w. The value selected for this witness is the least value in W^e such that:

(a) For all $1 \leq i \leq m$, we have that $(\forall_{n \leq w})(\Gamma_{i,s}^{U_i,D}(n) \downarrow = A_s(n))$ holds ($\Gamma^{V,D}$ resp).

(b) $a < w < b$.

(c) $\Theta^D[s](w) \downarrow$

(d) $a < \theta_s(w) < b$.

(e) $(\forall_{1 \leq j \leq m})(w > \gamma_{j,s}(v_j))$.

(f) w is greater than the last stage at which α was initialised.

If a witness satisfying these conditions cannot be found, end this substage. Resume from step (1). Otherwise go to step (3).

(3) (Wait for convergence). Is $\Theta^D[s](w) \downarrow= 0$?

    (a) (Yes) Go to step (4).

    (b) (No) Otherwise end this substage. Resume from step (3).

(4) (Honestification, Part I). Is there some strategy β_i in the sequence $(\beta_1, \ldots, \beta_m)$ for which $\phi_{i,s}(w) > \gamma_{i,s}(w)$?

    (a) (Yes) Is stage s a close-stage?

        (i) (Yes) Let β_i be the least strategy for which $\phi_{i,s}(w) > \gamma_{i,s}(w)$. Enumerate $\gamma_{i,s}(w)$ into D. End this substage and resume from step (5).

        (ii) (No) End this substage and resume from step (3).

    (b) (No) Go to step (6).

(5) (Honestification, Part II). Is stage s an open-stage?

    (a) (Yes) End this substage and resume from step (3).

    (b) (No) End this substage and resume from step (5).

(6) (Gap open) Is stage s an open-stage?

    (a) (Yes) Constrain each strategy β_i in the sequence $(\beta_1, \ldots, \beta_m)$ to choose uses $\gamma_{i,s'}(w) > \theta_s(w)$ at all stages $s' > s$. Enumerate w into A and restrain $D \upharpoonright \theta_s(w)$. Cancel the witness w. End this substage. Resume from step (7).

    (b) (No) End this substage and resume from step (3).

(7) (Gap close) Is stage s a close-stage?

    (a) (Yes) Let t be the stage at which the strategy last opened a gap by ending its substage at step (6)(a). Let t_i be the least $\mathcal{R}_i$-expansionary stage greater than t for all i such that $1 \leq i \leq m$. Is there some strategy β_i in the sequence $(\beta_1, \ldots, \beta_m)$ such that β_i is following a Γ-strategy and such that $U_{i,t} \upharpoonright \phi_t(w) = U_{i,t_i} \upharpoonright \phi_t(w)$?

        (i) (Yes) Go to step (8).

        (ii) (No) Go to step (9).

    (b) (No) End the substage and resume from step (7).

(8) (Capricious destruction) Let β_i be the least strategy satisfying the condition in step (7a). Enumerate $\gamma_{i,s}(v_i)$ into D. Cancel all thresholds v_n with $n < i$. End this substage and resume from step (1).

(9) (Successful diagonalisation). End this substage and resume from step (9).

We shall now consider the outcomes of the $\mathcal{S}$ strategy and define the outcome which the $\mathcal{R}$ strategy selects at stage s.

Outcome of the $\mathcal{S}$ strategy

The $\mathcal{S}$ strategy α may have up to $2m+2$ outcomes, where m is the number of $\mathcal{R}$ strategies above α. These will include the diagonalisation outcome d, the wait outcome w and m honestification outcomes $h_1, \ldots, h_m$. If β_i for $i \in \{1 \ldots m\}$ is following a Γ-strategy, then g_i will also be an outcome of α. These outcomes, when present, are ordered in the following way: $d <_\Lambda g_m <_\Lambda \ldots <_\Lambda g_1 <_\Lambda h_1 \ldots <_\Lambda h_m <_\Lambda w$. Once again note that the g_i outcomes are ordered in descending order, while the h_i outcomes are ordered in ascending order. The outcome which is visited by the strategy α at stage s, and which we denote by $O_s(\alpha)$ is chosen through the following case analysis:

(I) *Waiting for parameters.* The strategy ends its substage at the (Background Step), step (1) or step (2). The functional built by some strategy β_i fails to be defined and equal to A up to some parameter, or α has failed to choose some parameter. The strategy visits its w outcome and imposes the work interval $(0, \infty)$.

(II) *Waiting for computation.* The strategy ends its substage at step (3). Then we have that $\Theta^D[s](w) \uparrow$, or $\Theta^D[s](w) \downarrow = 1$. The strategy visits its w outcome and imposes the work interval $(0, \infty)$.

(III) *Honestification, Part I for β_i.* The strategy ends its substage at step (4)(a)(i). Then we have that $\Theta^D[s](w) \downarrow = 0$, β_i is the least strategy such that $\phi_{i,s}(w) > \gamma_{i,s}(w)$, and that s is a close stage. The strategy visits its h_i outcome and imposes the work interval $(w, \gamma_{i,s}(w))$.

(IV) *Honestification, Part I for β_i - waiting for close stage.* The strategy ends its substage at step (4)(a)(ii). Then we have that $\Theta^D[s](w) \downarrow= 0$, β_i is the least strategy such that $\phi_{i,s}(w) > \gamma_{i,s}(w)$, but s is not a close stage. The strategy visits its w outcome and imposes the work interval $(0, \infty)$.

(V) *Honestification, Part II for β_i.* The strategy ends its substage at step (5)(a). Then the strategy has performed Part I honestification for β_i at some stage t, and has not performed Part II honestification for β_i between stages t and s. In addition stage s is an open stage. The strategy visits its h_i outcome and imposes the work interval $(w, \gamma_{i,s}(w))$.

(VI) *Honestification, Part II for β_i - waiting for open stage.* The strategy ends its substage at step (5)(b). Then the strategy has performed Part I honestification for β_i at some stage t, and has not performed Part II honestification for β_i between stages t and s. In addition stage s is an open stage. The strategy visits its w outcome and imposes the work interval $(0, \infty)$.

(VII) *Opening a gap - g outcomes exist.* The strategy ends its substage at step (6)(a). Then we have that $\Theta^D[s](w) \downarrow= 0$, $\phi_{i,s}(w) \leq \gamma_{i,s}(w)$ for every strategy β_i, and that s is an open stage. In addition at least one strategy β_i is following a Γ-strategy. The strategy has opened a gap. If the strategy has never closed a gap, the strategy visits the outcome g_j, where g_j is the rightmost g outcome of the α strategy. Otherwise, the strategy visits the outcome on which a gap was last closed. The work interval $(v_j, \gamma_j(v_j))$ is imposed.

(VIII) *Opening a gap - no g outcomes.* The strategy ends its substage at step (6)(a). Then we have that $\Theta^D[s](w) \downarrow= 0$, $\phi_{i,s}(w) \leq \gamma_{i,s}(w)$ for every strategy β_i, and that s is an open stage. Every strategy β_i is following a $\hat{\Gamma}$-strategy. Then we have that the strategy has diagonalised successfully. The strategy visits its d outcome and imposes the work interval (s_1, ∞), where s_1 is equal to the present stage s.

(IX) *Opening a gap - waiting for open stage.* The strategy ends its substage at step (6)(b). Then we have that $\Theta^D[s](w) \downarrow= 0$, $\phi_{i,s}(w) \leq \gamma_{i,s}(w)$ for every strategy β_i, but s is not an open stage. The strategy visits its w outcome and imposes the work interval $(0, \infty)$.

(X) *Closing a gap - waiting for close stage.* The strategy ends its substage at step (7)(b). Then the strategy has last opened a gap at stage t, but s is not a close stage. The strategy visits its w

outcome and imposes the work interval $(0, \infty)$.

(XI) *Closing a gap - capricious destruction for β_i.* The strategy ends its substage at step (8). Then we have that the strategy has last opened a gap at stage t, that β_i is the least strategy such that β_i is following a Γ-strategy, that t_i is the least stage greater than t such that there has been an $\mathcal{R}_i$-expansionary stage, and that $U_{i,t}(m) \upharpoonright \phi_t(w) = U_{i,t_i}(m) \upharpoonright \phi_t(w)$. In addition s is a close stage. The strategy is closing a gap by performing capricious destruction on β_i. The strategy visits its g_i outcome and imposes the work interval $(v_i, \gamma_{i,s}(v_i))$.

(XII) *Closing a gap - successful diagonalisation.* The strategy ends its substage at step (9). Then we have that there is no strategy β_i such that β_i is following a Γ-strategy, that t_i is the least stage greater than t such that β_i has seen an $\mathcal{R}_i$-expansionary stage, and that $U_{i,t}(m) \upharpoonright \phi_t(w) = U_{i,t_i}(m) \upharpoonright \phi_t(w)$. In addition stage s is a close stage. The strategy has diagonalised successfully. The strategy visits its d outcome and imposes the work interval (s_1, ∞), where s_1 is equal to the present stage s.

(XIII) *Stopped.* The strategy ends its substage at step (9), and step (9) has been visited since the strategy was last initialised. Then we have that the strategy has already diagonalised successfully. The strategy visits its d outcome and imposes the work interval (s_1, ∞), where s_1 is the stage at which the strategy has first diagonalised successfully and visited its outcome d.

2.9.5 Organisation of the Priority Tree

We shall now describe formally how the priority tree T is organised in general. The layout of strategies on the tree is not uniform and will be different for each path through the tree. This reflects the fact that a requirement is not in general satisfied by one strategy but as the result of action on the part of multiple strategies, and that different outcomes will necessitate the use of different backup strategies.

To build the priority tree T we first assign the following priority ordering to the requirements:

$$\mathcal{R}_0 <_p \mathcal{S}_0 <_p \mathcal{R}_1 <_p \mathcal{S}_1 <_p \dots$$

Every node on the tree will be assigned a label, corresponding to the highest priority requirement which is unsatisfied at that node. Each such node will be assigned a strategy of the appropriate kind in order to satisfy this requirement. Hence, nodes which are labeled $\mathcal{R}_i$ will be assigned an $\mathcal{R}$ strategy, while nodes which are labeled $\mathcal{S}_i$ will be assigned an $\mathcal{S}$ strategy.

An $\mathcal{R}$ strategy β assigned to a node labeled $\mathcal{R}_i$ will have two edges leaving it. These are labeled i and f with $i <_\Lambda f$, and represent the outcomes of the strategy.

The situation with an $\mathcal{S}$ strategy α assigned to a node labeled $\mathcal{S}_i$ is more complex. In order to define the edges leaving this strategy, we shall first need to introduce a few auxiliary concepts.

Firstly we shall define the concept of a strategy being *restarted*. Let $\overrightarrow{t}$ be a finite path. We shall say that a strategy γ assigned to a node labeled $\mathcal{R}_i$ on $\overrightarrow{t}$ is *restarted* on $\overrightarrow{t}$ if there is some $\mathcal{S}$ strategy α below γ, such that α has an edge labeled h_j or g_j with $j < i$ on $\overrightarrow{t}$.

In this case we say that the strategy α has caused a *restart* in the manner of satisfying the requirements $\mathcal{R}_i$ with $i > j$, thus canceling all switches affecting this requirement.

On the other hand a strategy γ assigned to a node labeled $\mathcal{S}_i$ on $\overrightarrow{t}$ is *restarted* on $\overrightarrow{t}$ if there is some $\mathcal{S}$ strategy α below γ, such that α has an edge labeled h_j or g_j for $j \leq i$ on $\overrightarrow{t}$.

Secondly we define the concept of a strategy following a Γ-strategy or a $\hat{\Gamma}$-strategy. Let $\overrightarrow{t}$ be a finite path. We shall say that a strategy β assigned to a node labeled $\mathcal{R}_i$ on $\overrightarrow{t}$ is *following a* $\hat{\Gamma}$-strategy if:

- There is some strategy β' which is assigned to a node labeled $\mathcal{R}_i$ above β.
- β' is the greatest strategy (under $\subset$) which is assigned to a node labeled $\mathcal{R}_i$ above β.
- There is some $\mathcal{S}$ strategy α assigned to a node between β' and β with the edge leaving α on $\overrightarrow{t}$ being labeled g_i.
- The strategy β' is not restarted on $\overrightarrow{t}$.

In this case we say that the strategy α has caused a *switch* in the manner of satisfying the requirement $\mathcal{R}_i$, forcing the β strategy to start following a $\hat{\Gamma}$-strategy. If an $\mathcal{R}$ strategy β lying on a finite path $\overrightarrow{t}$ is not *following a* $\hat{\Gamma}$-strategy, then we shall say that β is *following a* Γ-strategy.

Thirdly we need to define the concept of an *active* strategy. Let $\overrightarrow{t}$ be a finite path. We shall say that an $\mathcal{R}$ strategy β assigned to a node labeled $\mathcal{R}_j$ on $\overrightarrow{t}$ is *active* on $\overrightarrow{t}$ if:

- The edge leaving β on $\overrightarrow{t}$ is labeled i.
- There is no $\mathcal{S}$ strategy α assigned to a node below β with the edge leaving α on $\overrightarrow{t}$ being labeled h_j or g_j.
- The strategy β is not restarted on $\overrightarrow{t}$.

With the benefit of these concepts we can now define the edges which leave an $\mathcal{S}$ strategy α assigned to a node labeled $\mathcal{S}_i$. Two of the edges leaving α will be labeled d and w. In addition there will be an edge labeled h_n for every $\mathcal{R}$ strategy β obeying the following conditions:

- β is above α.
- β is assigned to a node labeled $\mathcal{R}_n$.
- β is active on the path terminating at α.

as well as an edge labeled g_n for every $\mathcal{R}$ strategy β obeying the following conditions:

- β is above α.
- β is assigned to a node labeled $\mathcal{R}_n$.
- β is active on the path terminating at α.
- β follows a Γ-strategy.

The edge labeled d will be the leftmost edge leaving α, followed by the edges labeled g_n arranged in descending order from left to right, followed by the edges labeled h_i arranged in ascending order from left to right and ending with the edge labeled w as the rightmost edge.

On any given finite path, there will be at most one strategy γ which is managing to satisfy a particular requirement, while all the other strategies associated to the requirement will be failing to do so. A strategy which is satisfying a particular requirement over a given finite path is said to *represent* the requirement on that path. We shall formalise this notion below.

Given a finite path $\overrightarrow{t}$, we say that an $\mathcal{S}$ strategy α *represents* a requirement $\mathcal{S}_i$ on $\overrightarrow{t}$ if all of the following are the case:

- α is on $\overrightarrow{t}$.
- α is assigned to a node labeled $\mathcal{S}_i$.
- The edge leaving α on $\overrightarrow{t}$ is labeled d or w.
- α is not restarted on $\overrightarrow{t}$.

Given a finite path $\overrightarrow{t}$, we say that an $\mathcal{R}$ strategy β *represents* a requirement $\mathcal{R}_i$ on $\overrightarrow{t}$ if all of the following are the case:

- β is on $\overrightarrow{t}$.
- β is assigned to a node labeled $\mathcal{R}_i$.
- β is not restarted on $\overrightarrow{t}$.

and in addition, *at least one* of the following is also the case:

- The edge leaving β on $\overrightarrow{t}$ is labeled f.
- There is some $\mathcal{S}$ strategy α below β on $\overrightarrow{t}$, such that the edge leaving α on $\overrightarrow{t}$ is labeled h_i.
- The strategy β is *active* on $\overrightarrow{t}$.

Note that although a strategy may represent some requirement on $\overrightarrow{t}$, it may stop doing so on some $\overrightarrow{t'}$ extending $\overrightarrow{t}$.

In addition, we shall say that a strategy represents some requirement on an infinite path p if it represents the requirement on $p \upharpoonright m$ for every natural number m.

The priority tree T can then be defined by induction as follows. Given any finite path $\overrightarrow{t}$ in T ending with an unlabeled node, we label the node with the highest priority requirement which is unrepresented on $\overrightarrow{t}$. We then assign a strategy of the appropriate kind to the node in order to satisfy this requirement. Following this we determine the edges leaving the strategy as detailed above, extending the tree with several new unlabeled nodes over which we can then repeat the same procedure.

We shall now formalise the construction which decides which strategies on the priority tree are accessible at each stage.

2.9.6 The Construction

During each stage s, we will generate a *current path* δ_s in T of length s by recursion, consisting of the edges visited by the construction during stage s. Whilst generating δ_s in this manner, we implicitly obtain the strategies lying on this path, which we declare to be accessible. The current path δ_s is generated by going through a sequence of substages $t \leq s$. During each substage, one chooses the last strategy on the path and computes its outcome at stage s, which corresponds to the next edge on the path.

The following procedure is followed to generate δ_s.

1. *Base case* ($t = 0$). Let $\delta_s = \emptyset$. Declare $\delta_s \upharpoonright 0$ to be *accessible*.

2. *Recursive case* ($t + 1$).

   If $t + 1 > s$, the path δ_s has been constructed. Go to the next stage $s + 1$.

   Otherwise consider the strategy $\gamma = \delta_s \upharpoonright t$ (that is, the strategy lying at the end of the path built during the last substage).

   Execute the strategy γ and compute its outcome $O_s(\gamma)$ at stage s.

   Otherwise γ goes to the next substage at stage s. Let $\delta_s = \delta_s \upharpoonright t \,^\frown O_s(\gamma)$. Declare $\delta_s \upharpoonright t{+}1$ to be *accessible*. Go to the next substage.

If a strategy γ is accessible at stage s, we shall say that s is a γ-*stage*.

Whenever a strategy γ is declared to be accessible, every strategy to its right is *initialised*. Strategies which have been initialised must start again from step (1). In addition initialised $\mathcal{S}$ strategies have all their parameters (witnesses and thresholds) canceled, while initialised $\mathcal{R}$ strategies have their functional destroyed and need to rebuild it anew.

2.10 Verification

2.10.1 Definitions

The true path f is defined as follows. The edge $f(n)$ is determined by identifying the strategy γ lying at the terminal node of the true path of length n and choosing the edge which the strategy converges to if it exists, or the leftmost edge which the strategy visits infinitely often otherwise. Note that the strategy lying at the terminal node of the true path of length 0 is the strategy located at the root node of the priority tree.

Definition 2.10.1. *(True Path). The true path f is defined by induction as follows. Let $f \upharpoonright n = \gamma$. If there are only finitely many γ-stages, $f(n)$ is undefined. Otherwise:*

(a) If γ is an $\mathcal{R}$ strategy,

$$f(n) = \begin{cases} i & \textit{if } \liminf_s O_s(\gamma) = i \\ f & \textit{if } \lim_s O_s(\gamma) = f \end{cases}$$

(b) If γ is an $\mathcal{S}$ strategy,

$$f(n) = \begin{cases} w & \textit{if } \lim_s O_s(\gamma) = w \\ g_j & \textit{if } \liminf_s O_s(\gamma) = g_j \\ h_k & \textit{if } \liminf_s O_s(\gamma) = h_k \\ d & \textit{if } \lim_s O_s(\gamma) = d \end{cases}$$

where:

$j \in \{n : \mathbb{N} \mid (\exists \beta \subset \gamma)[\beta$ *is labeled* $\mathcal{R}_n \;\wedge\; \beta$ *is active for* $\gamma \;\wedge\; \beta$ *follows a* Γ*-strategy*$]\}$.

$k \in \{n : \mathbb{N} \mid (\exists \beta \subset \gamma)[\beta$ *is labeled* $\mathcal{R}_n \;\wedge\; \beta$ *is active for* $\gamma]\}$.

$s \in \mathbb{N}_\gamma$*, where* $\mathbb{N}_\gamma$ *is the set of* γ*-stages.*

We shall now define the concepts of a γ-open stage and a γ-close stage for a strategy γ. A stage s is a γ-open stage if every $\mathcal{S}$ strategy α with outcome g_n above γ is opening a gap at s, and every $\mathcal{S}$ strategy α with outcome h_n above γ is performing Part II honestification at s. In addition any functional built by an $\mathcal{R}$ strategy above γ cannot disagree with the set A at any element at which it is defined during the stage under consideration. On the other hand a stage$s$ is a γ-close stage if every $\mathcal{S}$ strategy α with outcome g_n above γ is closing a gap at s, and every $\mathcal{S}$ strategy α with outcome h_n above γ is performing Part I honestification at s.

Definition 2.10.2. *(Open-Stages and Close-Stages) Let γ be an $\mathcal{R}$ or $\mathcal{S}$ strategy.*

A stage s is a γ-open stage if it satisfies conditions (O1)-(O3).

(1) *(Condition O1). γ is accessible at s.*

(2) *(Condition O2). If α is an $\mathcal{S}$ strategy with an edge with outcome g_n above γ, then α has chosen case (VII) when computing its outcome during stage s. Similarly, if α is an $\mathcal{S}$ strategy with an edge with outcome h_n above γ, then α has chosen case (V) when computing its outcome during stage s.*

(3) *(Condition O3). If β is an $\mathcal{R}$ strategy above γ building a functional $\Gamma^{U,D}$, there is no element m such that $A_s(m) \neq \Gamma^{U,D}[s](m)$ ($\Gamma^{V,D} resp$).*

A stage t is a γ-close stage for γ if it satisfies conditions (C1)-(C2).

(1) *(Condition C1). γ is accessible at t.*

(2) *(Condition O2). If α is an $\mathcal{S}$ strategy with an edge with outcome g_n above γ, then α has chosen case (XI) when computing its outcome during stage s. Similarly, if α is an $\mathcal{S}$ strategy with an edge with outcome h_n above γ, then α has chosen case (III) when computing its outcome during stage s.*

2.10.2 Representation Lemma

The Representation Lemma shows that given any requirement and any infinite path through the priority tree, there will eventually be a strategy which represents that requirement on the infinite path.

Lemma 2.10.3. *(Representation Lemma). Let Q be a requirement, and let p be an infinite path. Then there is some strategy γ and some natural number v such that:*

1. *γ represents Q on $p \upharpoonright v$.*

2. *For every $m \geq v$, γ represents Q on $p \upharpoonright m$.*

Proof. We proceed by strong induction on the list of requirements $\{Q_i\}_{i \in \mathbb{N}}$. Suppose, as the inductive hypothesis that for every requirement Q_i with $i \leq n$, there is some strategy γ_i and some natural number v_i such that γ_i represents Q_i on $p \upharpoonright v_i$, and such that for every $m \geq v_i$ we have that γ_i represents Q_i on $p \upharpoonright m$.

We shall prove that for requirement Q_{n+1} there is some strategy $\gamma_{n+1} \subset p$ and some natural number v_{n+1} such that γ_{n+1} represents Q_{n+1} on $p \upharpoonright v_{n+1}$, and that for every $m > v_{n+1}$ we have that γ_{n+1} represents Q_{n+1} on $p \upharpoonright m$.

From the inductive hypothesis it follows that for every requirement Q_i with $i \leq n$, there is some strategy γ_i and some *least* natural number v'_i such that γ_i represents Q_i on $p \upharpoonright v'_i$, and such that for every $m \geq v'_i$ we have that γ_i represents Q_i on $p \upharpoonright m$.

Let $u = \max_{i \leq n}\{v'_i\}$.

Then u must be the least natural number such that for every requirement Q_i with $i \leq n$ we have that γ_i represents Q_i on $p \upharpoonright u$ and such that for every $m \geq u$ we have that γ_j represents Q_i on $p \upharpoonright m$.

For if it were otherwise, there would exist some least natural number $u' < u$ such that for every requirement Q_i with $i \leq n$ we have that γ_i represents Q_i on $p \upharpoonright u'$, and that for every $m \geq u'$ we have that γ_i represents Q_i on $p \upharpoonright m$. But this contradicts the fact that there is some requirement Q_j with $j < n$ such that $v'_j = u$ is the least natural number such that γ_j represents Q_j on $p \upharpoonright v'_j$, and that for every $m \geq v'_j$ we have that γ_j represents Q_j on $p \upharpoonright m$.

Now, either Q_{n+1} is already represented by some strategy γ_{n+1} on $p \upharpoonright u$, or else this is not the case. We shall show that this is not the case by contradiction. Suppose that γ_{n+1} already represents Q_{n+1} on $p \upharpoonright u$. Then there must be some requirement Q_i with $i \leq n$, and some strategy γ_i with $\gamma_{n+1} \subset \gamma_i$ such that γ_i represents Q_i on $p \upharpoonright u$.

For suppose that this was not the case. Then γ_{n+1} can only represent Q_{n+1} on $p \upharpoonright u$ in one of the following ways. If Q_{n+1} is an $\mathcal{S}$ requirement, γ_{n+1} must have outcome d or outcome w on $p \upharpoonright u$. On the other hand, if Q_{n+1} is an $\mathcal{R}$ requirement, γ_{n+1} must either have outcome f on $p \upharpoonright u$, or have outcome i on $p \upharpoonright u$ along with one of the following conditions; either there is an $\mathcal{S}$ strategy below γ_{n+1} with an h outcome on $p \upharpoonright u$ associated to the requirement, or there is no $\mathcal{S}$ strategy with an h or g outcome on $p \upharpoonright u$ associated to the requirement.

In all of the above cases, we have that there is no effect on strategies γ_i for $i \leq n$ with respect to them representing requirement Q_i on $p \upharpoonright u$. It follows that every strategy γ_i for $i \leq n$ represents its requirement Q_i on $p \upharpoonright |\gamma_{n+1}|$, where $|\gamma_{n+1}| < u$. But this contradicts the fact that u is the least natural number such that for every requirement Q_i with $i \leq n$ we have that γ_i represents Q_i on $p \upharpoonright u$ and such that for every $m \geq u$ we have that γ_j represents Q_i on $p \upharpoonright m$.

Now, in order for the strategy γ_{n+1} labeled Q_{n+1} to appear on the tree at $p \upharpoonright |\gamma_{n+1}|$, it must be the case that Q_{n+1} is the highest priority unrepresented requirement on $p \upharpoonright |\gamma_{n+1}|$. Hence, the requirement Q_i must have been represented on $p \upharpoonright |\gamma_{n+1}|$.

But in order for the strategy γ_i labeled Q_i to appear on the tree at $p \upharpoonright |\gamma_i|$, it must be the case that Q_i is the highest priority unrepresented requirement on $p \upharpoonright |\gamma_i|$. Hence the requirement Q_i must have become unrepresented due to the strategy γ_i being restarted by the outcome of some strategy lying between $p \upharpoonright |\gamma_{n+1}|$ and $p \upharpoonright |\gamma_i|$. Since $i < n$, the outcome of this strategy must also have restarted the startegy γ_{n+1} labeled Q_{n+1}. It follows that Q_{n+1} is not represented by γ_{n+1} on $p \upharpoonright u$, which gives a contradiction.

Since every requirement Q_i with $i \leq n$ is represented by some γ_i on $p \upharpoonright u$, we have that the strategy at $p \upharpoonright u$, must be labeled Q_{n+1}, since this is the highest priority requirement which is not represented on $p \upharpoonright u$. Let this strategy be denoted by γ.

We are now in a position to show that there exists some strategy γ_{n+1} and some number v_{n+1} such that γ_{n+1} represents the requirement Q_{n+1} on $p \upharpoonright v_{n+1}$, and for all $m > v_{n+1}$, γ_{n+1} represents the requirement Q_{n+1} on $p \upharpoonright v_{n+1}$. In order to do so we split our analysis into two cases, depending on whether Q_{n+1} is an $\mathcal{R}$ or an $\mathcal{S}$ requirement.

Case 1: Q_{n+1} is an $\mathcal{R}$ requirement.

(1) If the outcome of γ labeled Q_{n+1} on p is f, we have that γ represents Q_{n+1} on $p \upharpoonright u + 1$. Now, the only way for γ to stop representing Q_{n+1} on extensions of $p \upharpoonright u + 1$ is for some $\mathcal{S}$ strategy α on this extension to have outcome g_j or h_j with $j \leq n$.

It is not possible for α to have outcome g_j with $j \leq n$ or outcome h_j with $j < n$, because this would cause some requirement Q_i with $i \leq n$ to become unrepresented on an extension of $p \upharpoonright u$. On the other hand it is possible for α to have outcome h_n, because this ensures that Q_n remains represented on extensions of $p \upharpoonright u$.

If α has outcome h_n, then we have that γ does not represent Q_{n+1} on $p \upharpoonright |\alpha|+1$, because the requirement Q_{n+1} has been restarted. Since every requirement Q_i with $i \leq n$ is represented on $p \upharpoonright m$ for all $m \geq u$, it follows that the strategy γ' at $p \upharpoonright |\alpha| + 1$ must be labeled $\mathcal{R}_{n+1}$.

If the outcome of γ' on p is f, then we have that γ' represents Q_{n+1} on $p \upharpoonright m$ for all $m \geq |\gamma'| + 1$. Note that there can now be no strategy α' below γ' with outcome h_n, because there is no strategy labeled $\mathcal{R}_n$ above α' which is active. Hence γ' now represents Q_{n+1} on all $p \upharpoonright m$ with $m \geq |\gamma'| + 1$.

On the other hand if the outcome of γ' on p is i, the analysis becomes identical to that found in (2)(i) for the strategy γ' with outcome i on p.

(2) If the outcome of γ labeled Q_{n+1} on p is i, we have that γ represents Q_{n+1} on $p \upharpoonright u + 1$. Now, the only way for γ to stop representing Q_{n+1} on extensions of $p \upharpoonright u + 1$ is for some $\mathcal{S}$ strategy α on this extension to have outcome g_j with $j \leq n + 1$ or outcome h_j with $j < n + 1$ on the same extension.

Now, if α has outcome g_j with $j \leq n$, it would make some requirement Q_i with $i \leq n$ unrepresented on the extension, which is a contradiction. Hence in this case, the only outcome possible is g_{n+1}.

Similarly if α has outcome h_j with $j < n$, it would make some requirement Q_i with $i \leq n$ unrepresented on the extension, which is a contradiction. Hence in this case, the only outcome possible is h_n.

(i) If α has outcome h_n, then we have that γ does not represent Q_{n+1} on $p \upharpoonright |\alpha| + 1$, because the requirement Q_{n+1} has been restarted. Since every requirement Q_i with $i \leq n$ is represented on $p \upharpoonright m$ for all $m \geq u$, it follows that the strategy γ' at $p \upharpoonright |\alpha|+1$ must be labeled $\mathcal{R}_{n+1}$.

If the outcome of γ' on p is f, then we have that γ' represents Q_{n+1} on $p \upharpoonright m$ for all

$m \geq |\gamma'| + 1$. Note that there can now be no strategy α' below γ' with outcome h_n. This arises from the fact that α has outcome h_n, which stops any strategy β labeled $\mathcal{R}_n$ which is active for α from being active for α'. In addition, the h_n outcome of α ensures that β represents $\mathcal{R}_n$ on $p \upharpoonright m'$ for all $m' > |\beta|$, meaning that this requirement will not appear again on the priority tree. Similarly, there can now be no strategy α' below γ' with outcome g_{n+1}, because the f outcome of γ' stops this strategy, which is labeled $\mathcal{R}_{n+1}$, from being active for α'. In addition the f outcome of γ' ensures that γ' represents $\mathcal{R}_{n+1}$ on $p \upharpoonright m'$ for all $m' > |\gamma'| + 1$, meaning that this requirement will not appear again on the priority tree.

If the outcome of γ' on p is i, then we have that γ' represents Q_{n+1} on $p \upharpoonright |\gamma'| + 1$. The only way in which γ' can stop representing Q_{n+1} on extensions of this path is for there to be a strategy α' below γ' with outcome g_{n+1}. Note that there can now be no strategy α' below γ' with outcome h_n. This arises from the fact that α has outcome h_n, which stops any strategy β labeled $\mathcal{R}_n$ which is active for α from being active for α'. In addition, the h_n outcome of α ensures that β represents $\mathcal{R}_n$ on $p \upharpoonright m'$ for all $m' > |\beta|$, meaning that this requirement will not appear again on the priority tree.

If the outcome of α' on p is g_{n+1}, we have that γ' does not represent Q_{n+1} on $p \upharpoonright |\alpha'| + 1$. Since every requirement Q_i with $i \leq n$ is represented on $p \upharpoonright m$ for all $m \geq u$, it follows that the strategy γ'' at $p \upharpoonright |\alpha| + 1$ must be labeled $\mathcal{R}_{n+1}$. Since the outcome g_{n+1} causes a switch in the manner of satisfying the requirement Q_{n+1}, we have that the strategy γ'' is following a $\hat{\Gamma}$-strategy.

If the outcome of γ'' on p is f, we have that γ'' represents Q_{n+1} on $p \upharpoonright m$ for all $m \geq |\gamma''| + 1$ through an argument identical to the one for γ'. On the other hand if the outcome of γ'' on p is i, we have that γ'' represents Q_{n+1} on $p \upharpoonright m$ for all $m \geq |\gamma''| + 1$. This can be the case for one of two reasons. The first is that the strategy γ'' is active at all extensions of $p \upharpoonright |\gamma''| + 1$, because there can now be no strategy α'' with outcomes g_{n+1} or h_n below γ''. The second is that there is some strategy α'' with outcome h_{n+1} on p below γ''. This is sufficient for α'' represent Q_{n+1} on all extensions of $|\gamma''| + 1$.

(ii) If α has outcome g_{n+1}, we have that γ does not represent Q_{n+1} on $p \upharpoonright |\alpha| + 1$. Since every requirement Q_i with $i \leq n$ is represented on $p \upharpoonright m$ for all $m \geq u$, it follows that the strategy γ' at $p \upharpoonright |\alpha| + 1$ must be labeled $\mathcal{R}_{n+1}$. Since the outcome g_{n+1} causes a switch in the manner of satisfying the requirement Q_{n+1}, we have that the strategy γ' is following a $\hat{\Gamma}$-strategy.

If the outcome of γ' on p is f, our analysis becomes identical to the one in (1). On the other hand if the outcome of γ' on p is i, we have that γ' represents Q_{n+1} on $p \upharpoonright |\gamma'| + 1$. The only way in which γ' can stop representing Q_{n+1} on extensions of this path is for there to be a strategy α' below γ' with outcome h_n. Note that now the strategy α' can have no outcome g_{n+1}, because γ' is already following a Γ-strategy. If such a strategy α' with outcome h_n exists, then our analysis becomes identical to the one in (2)(i).

Case 2: Q_{n+1} is an $\mathcal{S}$ requirement.

In this case the strategy γ at $p \upharpoonright u$ is an $\mathcal{S}$ strategy. In order to determine its outcomes we proceed as follows.

First of all we note that by the inductive hypothesis we have that any $\mathcal{R}$ requirement Q_r with $r \leq n$ is represented by some strategy γ_r on $p \upharpoonright u$ and on $p \upharpoonright m$ for all $m > u$. In order for this to be the case, one of the following conditions must hold.

(1) γ_r has outcome f on p. In addition γ_r is not restarted on p.

(2) γ_r has outcome i on p. In addition, there is some $\mathcal{S}$ strategy α below γ_r having outcome h_r on p. Finally γ_r is not restarted on p.

(3) γ_r has outcome i on p. In addition, γ_r must be active on p. This means that there is no $\mathcal{S}$ strategy α below γ_r having outcome h_r or g_r on p. Finally γ_r is not restarted on p.

We claim that in each of these three cases the $\mathcal{S}$ strategy γ does not have h_r or g_r as an outcome.

In the first case we have that γ_r does not have outcome i on p. Hence γ_r is not active for γ and the latter cannot have h_r or g_r as an outcome.

In the second case we have that there is some $\mathcal{S}$ strategy α between γ_r and γ with outcome h_r. Hence γ_r is not active for γ and the γ cannot have h_r or g_r as an outcome.

In the third case we have that there cannot be any $\mathcal{S}$ strategy α below γ_r with outcomes h_r or g_r on p. Since α can be γ, we have that γ cannot have h_r or g_r as an outcome.

In addition the same argument guarantees that there can be no $\mathcal{S}$ strategy α' below γ with outcomes h_r or g_r on p. Hence it is not possible for the strategy α to be restarted either.

From this it follows that γ can only have outcomes d or w on p, and that γ is never restarted on p. This means that that γ represents Q_{n+1} on $p \upharpoonright |\gamma| + 1$, and that γ represents Q_{n+1} on $p \upharpoonright m$, for all $m > |\gamma| + 1$ as required. □

2.10.3 Leftmost Path Lemma

The Leftmost Path Lemma shows that if a strategy lies on the true path, the current path generated by the construction at each stage can only lie to its left finitely often. This will mean that if a strategy lies on the true path, there will be some greatest stage s_0 such that the strategy is never initialised at stages $s > s_0$.

Lemma 2.10.4. *(Leftmost Path Lemma). Let f be the true path. If $f \upharpoonright n$ is defined, there are only finitely many stages s such that $\delta_s <_L \gamma$, where $\gamma = f \upharpoonright n$.*

Proof. We prove the lemma by induction on n.

For the Base Case $n = 0$ we consider $f \upharpoonright 0$. In this case we have that $f \upharpoonright 0$ is defined and is the strategy γ_0 located at the root of the priority tree. Hence there is no stage s such that $\delta_s <_L \gamma_0$.

For the Inductive Case we assume that the lemma holds for $n = k$. Thus we have that if $f \upharpoonright k$ is defined, there are only finitely many stages s such that $\delta_s <_L \gamma_k$, where $\gamma_k = f \upharpoonright k$.

We then prove that the lemma holds for $n = k + 1$. Thus we need to show that if $f \upharpoonright k + 1$ is defined, there are only finitely many stages s such that $\delta_s <_L \gamma_{k+1}$, where $\gamma_{k+1} = f \upharpoonright k + 1$.

Now if $f \upharpoonright k + 1$ is not defined, the lemma holds trivially.

Otherwise suppose that $f \upharpoonright k + 1$ is defined. This can only be the case if $f \upharpoonright k$ is defined and if $f(k)$ is defined.

Since $f \restriction k$ is defined, by the Inductive Hypothesis we have that there is some stage t such that $\delta_{t'} \not<_L \gamma_k$ for all $t' > t$. In addition, since $f(k)$ is defined, it must either be the case that $f(k) = \lim_s O_s(\gamma_k)$ or that $f(k) = \liminf_s O_s(\gamma_k)$, where $\gamma_k = f \restriction k$.

Hence there must be some stage $u > t$ such that $O_{u'}(\gamma_k) \not<_L f(k)$ for all $u' > u$. Since $f \restriction k+1 = f \restriction k\hat{\ }f(k)$, we have that $\delta_s \not< \gamma_{n+1}$ for all $s > u$. □

2.10.4 Infinite True Path Lemma

The Infinite True Path Lemma shows that the true path is infinite in length.

Lemma 2.10.5. *(Infinite True Path Lemma). Let f be the true path. Then $f(n)$ is defined for every n.*

Proof. To prove that $f(n)$ is defined for some n, we need to show two things. The first is that there are infinitely many γ_n-stages, where $\gamma_n = f \restriction n$. The second is that either $\lim_s O_s(\gamma_n)$ or $\liminf_s O_s(\gamma_n)$ exists.

We first prove a preliminary result by showing that for any strategy γ, we have that $\lim_s O_s(\gamma)$ or $\liminf_s O_s(\gamma)$ always exists.

Suppose that γ is an $\mathcal{R}$ strategy.

If γ does not visit its i outcome infinitely often there is some stage t such that γ must always visit its f outcome after t. In this case $\lim_s O_s(\gamma)$ exists and is equal to f. It follows that $f(n)$ is defined and is equal to f.

On the other hand it could be the case that γ visits its i outcome infinitely often. Since i is to the left of the outcome f, we have that $\liminf_s O_s(\gamma)$ exists and is equal to i. It follows that $f(n)$ is defined and is equal to i.

Suppose now that γ is an $\mathcal{S}$ strategy.

If γ visits its d outcome at some stage t, it will stop acting, visiting its d outcome at all subsequent stages. Hence, $\lim_s O_s(\gamma)$ exists, and is equal to d. It follows that $f(n)$ is defined and is equal to d.

If γ never visits its d outcome, it could be that it visits outcomes g_i for $i < k$ finitely often, but outcome g_k infinitely often. Since g_k is to the left of all other remaining outcomes, we have that $\liminf_s O_s(\gamma)$ exists, and is equal to g_k. It follows that $f(n)$ is defined and is equal to g_k.

If γ never visits its d outcome, and visits its g_i outcomes only finitely often, it could be the case that γ visits outcomes h_i for $i < k$ finitely often, but outcome h_k infinitely often. Since h_k is to the left of all other remaining outcomes, we have that $\liminf_s O_s(\gamma)$ exists, and is equal to h_k. It follows that $f(n)$ is defined and is equal to h_k.

If γ never visits its d outcome, and visits its g_i and h_i outcomes only finitely often, then γ must visit its w outcome at all subsequent stages. Hence $\lim_s O_s(\gamma)$ exists, and is equal to w. It follows that $f(n)$ is defined and is equal to w.

We are now in a position to prove the lemma by Induction on n.

For the Base Case $n = 0$ we need to show that $f(0)$ is defined. Consider $f \restriction 0$, then we have that the strategy $\gamma_0 = f \restriction 0$ is located at the root of the priority tree. This means that the strategy is accessible at every stage. Hence there are infinitely many γ_0-stages. By the preliminary result we also have that either $\lim_s O_s(\gamma_0)$ or $\liminf_s O_s(\gamma_0)$ exists. This means that $f(0)$ is defined as required.

For the Inductive Case we assume that the lemma holds for $n = k$. Then we have that $f(k)$ is defined. This means that there are infinitely many γ_k-stages, where $\gamma_k = f \restriction k$. We also have that either $\lim_s O_s(\gamma_k)$ or $\liminf_s O_s(\gamma_k)$ exists.

We then prove that the lemma holds for $n = k + 1$. This means that we need to prove that $f(k + 1)$ is defined. This requires showing that there are infinitely many γ_{k+1}-stages, where $\gamma_{k+1} = f \restriction k + 1$, as well as showing that either $\lim_s O_s(\gamma_{k+1})$ or $\liminf_s O_s(\gamma_{k+1})$ exists.

In order to show that there are infinitely many γ_{k+1}-stages we proceed as follows. By The Inductive Hypothesis we have that there are infinitely many γ_k-stages. This means that γ_k is accessible at infinitely many stages.

If $\lim_s O_s(\gamma_k)$ exists there must be some stage s such that γ_k visits $f(k)$ at every stage $s' > s$. Hence γ_{k+1} is accessible at infinitely many stages as well, which means that there are infinitely many γ_{k+1}-stages as required. If $\lim_s O_s(\gamma_k)$ does not exist but $\liminf_s O_s(\gamma_k)$ exists, it must be the case that γ_k visits $f(k)$ at infinitely many stages. Hence γ_{k+1} is accessible at infinitely many stages as well, which means that there are infinitely many γ_{k+1}-stages as required.

Finally, by the preliminary result we also have that either $\lim_s O_s(\gamma_{k+1})$ or $\liminf_s O_s(\gamma_{k+1})$ exists. This means that $f(k+1)$ is defined as required. $\square$

2.10.5 Restraint Lemma

The Restraint Lemma shows that the restraint imposed on any strategy on the true path by strategies located above it or to its left drops to some constant value during infinitely many of the stages at which it is accessible.

Lemma 2.10.6. *(Restraint Lemma). Let γ be an $\mathcal{R}$ strategy or an $\mathcal{S}$ strategy on the true path f. Let $r(\alpha, s)$ be the restraint imposed on γ by a strategy α lying to the left of or above γ at stage s. Then $\liminf\limits_{s \in \mathbb{N}_\gamma} r(\alpha, s)$ exists, where $\mathbb{N}_\gamma$ is the set of γ-stages.*

Proof. Consider the strategy γ on the true path f. Since γ is on the true path, by the *Leftmost Path Lemma* there is a stage s_0 such that no strategies to the left of γ are accessible at stages $s > s_0$. This means that the restraint imposed by strategies to the left of γ becomes constant during such stages. Hence we only need to consider restraints imposed by strategies above γ.

Now, $\mathcal{R}$ strategies located above γ do not impose any restraint on lower priority strategies. Therefore we only need to consider $\mathcal{S}$ strategies α located above γ. When determining the restraints which are imposed on γ, we are only interested in the restraints imposed during those stages at which γ is accessible. We thus consider the contribution of each such strategy α to the total restraint imposed on γ, by considering the outcome of α on the true path.

First of all we note that if the outcome of α on the true path is w or h_n for some n then α imposes no restraint on γ at stages at which γ is accessible. If the outcome of α on the true path is d, then α imposes the constant restraint $\theta(w)$ on γ at stages during which γ is accessible.

We now need to consider the situation which occurs when the strategies α have outcomes g_n for some n on the true path. Such strategies impose a restraint of $\theta(w)$ when they open a gap, and lower it to zero when they close a gap. We shall show that there are infinitely many stages such that γ is accessible, and such that the total restraint imposed by strategies α above γ with g_n outcomes at these stages is 0.

Let α^* be the greatest $\mathcal{S}$ strategy such that $\alpha^* \frown g_n \subset \gamma$ for some n. Since $\alpha^* \frown g_n$ is on the true path, α^* must be accessible and visit its g_n outcome infinitely often. If α^* opens a gap by visiting g_n, then it will be able to visit g_n again only once α^* closes some gap on g_n again. In addition, after α^* closes this gap by visiting g_n, it will open the next gap at g_n as well. Note that when α^* opens a gap, it will impose the restraint $\theta(w)$ on γ, and when it closes a gap, it will lower this restraint to 0. Now, due to S-Synchronisation α^* will only visit the g_n outcome whilst closing a gap if every higher priority $\mathcal{S}$ strategy on f is also closing a gap during the same stage. But this has to eventually occur, as otherwise the g_n outcome of the α^* strategy would not be on the true path. Thus, when α^* finally closes a gap by visiting its g_n outcome, it not only lowers its restraint down to 0, but does so during a stage when all higher priority $\mathcal{S}$ strategies have also lowered their restraints down to 0.

Hence we have that there are infinitely many stages such that the total restraint imposed by all $\mathcal{S}$ strategies with g_n outcomes which are of higher priority than γ is 0. During these stages the total restraint imposed on γ is thus the sum of the constant restraint imposed by strategies to the left of γ after stage s_0 and the constant restraint imposed by strategies above γ with d outcomes. Hence a $\liminf$ restraint on α exists, as required. □

2.10.6 Synchronisation Lemma

The Synchronisation Lemma shows that if a strategy is on the true path it is accessible during infinitely many open stages and infinitely many close stages.

Lemma 2.10.7. *(Synchronisation Lemma). Let γ be an $\mathcal{R}$ or $\mathcal{S}$ strategy on the true path. Then there are infinitely many open-stages for γ and infinitely many close-stages for γ.*

Proof. We prove this lemma by induction on the length of the true path.

For the base case consider the top strategy of the tree γ. Since γ is the top strategy of the tree, it must be accessible at every stage. Since γ has no other $\mathcal{R}$ or $\mathcal{S}$ strategy above it, it follows that every stage satisfies conditions (O1)-(O3) and (C1)-(C2) for γ. Hence, γ is accessible during infinitely many open-stages and close-stages.

For the inductive case we show that if there are infinitely many open-stages and infinitely many close-stages for a given strategy γ on the true path, then there are also infinitely many open-stages and infinitely many close-stages for its successor γ^+ on the true path. In order to prove this fact we shall have to show that γ preserves infinitely many open-stages and infinitely many close-stages down its outcome on the true path.

We split our analysis into two cases, depending on whether γ is an $\mathcal{R}$ strategy or an $\mathcal{S}$ strategy.

(1) *γ is an $\mathcal{R}$ strategy.* Suppose γ is accessible at infinitely many open-stages and at infinitely many close-stages. We split the analysis into a further two cases, depending on whether the $\mathcal{R}$ strategy has outcome i or outcome f on the true path.

  (a) *γ has outcome f on the true path.* By the *Leftmost Path Lemma* there is a stage s_0 after which γ visits its f outcome every time it is accessible. Suppose that γ is accessible during some open-stage $s > s_0$. Then we have that γ^+ is also accessible at stage s. Now, since s is an open-stage and γ is an $\mathcal{R}$ strategy, it follows that every $\mathcal{S}$ strategy above γ^+ with a g_n or h_n outcome for some n, is still opening a gap or performing Part II honestification at stage s. Finally, since s is an open-stage and the outcome of γ on the true path is f, it follows that γ^+ does not have any active strategy whose functional disagrees with the set A above it. Hence stage s satisfies conditions (O1)-(O3) for γ^+ as well.

  Suppose now that γ is accessible during a close-stage $t > s_0$. Then we have that γ^+ is also accessible at stage t. In addition, since t is a close-stage and γ is an $\mathcal{R}$ strategy, it follows that every $\mathcal{S}$ strategy above γ^+ with a g_n or h_n outcome for some n, is still closing a gap or performing Part I honestification at stage t. Hence stage t satisfies conditions (C1)-(C2) for γ^+ as well.

  (b) *γ has outcome i on the true path.* By the *Leftmost Path Lemma* γ visits its i outcome infinitely often. Now, for γ to visit its i outcome infinitely often, it must be the case that γ ends its substage at step (4)(a)(i), (4)(a)(ii) or (4)(b) infinitely many times. In order for γ to have reached step (4), it must have first passed from step (2)(a), and then from step (3)(a) respectively. By passing from step (2)(a), γ must have witnessed a γ-expansionary stage, whilst by passing from step (3)(a), the strategy must have witnessed a required stage and the preservation of the previous γ-expansionary stage. In addition, the visit to step (3)(a) flips the required stage. All these facts mean that γ witnesses infinitely many cycles leading from step (2)(a) to step (3)(a) to step (4), alternately witnessing an open-stage and a close-stage during these cycles.

In order to obtain an open-stage for γ^+ one proceeds as follows. First one waits for γ to have an open-stage as the required stage prior to beginning the cycle leading from step (2)(a) to step (4). Next, one waits for the strategy γ to pass through step (2)(a) and witness a γ-expansionary stage, and then to pass through step (3)(a) to witness a required stage which also preserves the previous γ-expansionary stage. The strategy will then pass through step (4) at some stage s and will check whether the $A_s(m) \neq \Gamma_{i,s}^{U_s,V_s}(m)$ for some m.

If there is no such disagreement, we have that γ visits its i outcome, and that the γ^+ strategy is now accessible. In addition, since s is an open-stage and γ is an $\mathcal{R}$ strategy, it follows that every $\mathcal{S}$ strategy above γ^+ with a g_n or h_n outcome for some n, is still opening a gap or performing Part II honestification at stage s. Hence we have that stage s satisfies conditions (O1)-(O3) for γ^+ as well.

On the other hand, if there is a disagreement, we have that γ visits its i outcome. While this makes γ^+ accessible, it fails to make it accessible at a stage satisfying conditions (O1)-(O3), because (O3) is not satisfied. Now, since s is an open-stage, it follows that the strategy must pass through step (4)(a)(ii), thus leaving the functional unrepaired.

One should note however that by passing from step (3)(a), the required stage was flipped to a close-stage. In order for the next cycle from step (2)(a) to step (4) to take place, such a close-stage must be witnessed during an intermediate step (3)(a). Upon reaching step (4) at stage s' however, γ will find that there is still some m such that $A_{s'}(m) \neq \Gamma_{i,s'}^{U_{s'},V_{s'}}(m)$.

Now, since s is a close-stage, the strategy will pass from step (4)(a)(i) and enumerate $\gamma_{i,s}(m)$ into D in order to undefine the functional at the point of disagreement. In addition, when visiting step (3)(a), the strategy must have flipped the required stage to an open-stage. Although γ^+ is accessible again, the disagreement will be removed only at the next stage, meaning that stage s fails to satisfy condition (O3) again for γ^+.

In order for the next cycle from step (2)(a) to step (4) to take place, such an open-stage must be witnessed during an intermediate step (3)(a). Upon reaching step (4) at stage s'', γ will note that there is now no disagreement between $A_{s''}$ and $\Gamma_{i,s''}^{U_{s''},V_{s''}}$. Hence, γ passes through step (4b), defining the functional up to the $l_{s''}(A, \Phi^{U,V})$.

Now since γ visits the outcome i when it passes through step (4b), we have that γ^+ is accessible at stage s''. Since s'' is an open-stage for γ, and γ is an $\mathcal{R}$ strategy, we have that every $\mathcal{S}$ strategy above γ^+ with a g_n or h_n outcome for some n, is still opening a gap or performing Part II honestification at stage s''. Finally, we know that γ witnesses no disagreement between the set $A_{s''}$ and $\Gamma_{i,s''}^{U_{s''},V_{s''}}$. Hence, s'' satisfies conditions (O1)-(O3) for γ^+, and is thus an open-stage for γ^+ as well.

We now show that γ preserves infinitely many close-stages, which therefore become close-stages for γ^+. In order to obtain a close stage for γ^+, we proceed as follows. First one waits for γ to have a close-stage as the required stage prior to beginning the cycle leading from step (2)(a) to step (4). Next, one waits for the strategy γ to pass through step (2)(a) and witness an γ-expansionary stage, and then to pass through step (3)(a) to witness a required stage which also preserves the previous γ-expansionary stage. The strategy will then pass through step (4) at stage t and will check whether the $A_t(m) \neq \Gamma_{i,t}^{U_t,V_t}(m)$ for some m.

Now irrespectively of whether γ finds a disagreement or not at step (4), it will visit its i outcome. Hence we have that γ^+ is accessible at stage t. In addition t is a close-stage for γ, and γ is an $\mathcal{R}$ strategy, we have that every $\mathcal{S}$ strategy above γ^+ with a g_n or h_n outcome for some n, is still closing a gap or performing Part I honestification at stage t. Hence, t satisfies conditions (C1)-(C2) for γ^+, and is thus a close-stage for γ^+ as well.

(2) *γ is an $\mathcal{S}$ strategy.* Suppose γ is accessible at infinitely many open-stages and at infinitely many close-stages. We split the analysis into four cases, depending on whether the $\mathcal{S}$

strategy has outcome w, h_n for some n, g_n for some n or d on the true path.

(a) *γ has outcome w on the true path.* By the *Leftmost Path Lemma* there is some stage s_0 after which γ no longer visits outcomes lying to the left of w. Hence γ must always visit outcome w after stage s_0.

This means that whenever γ is accessible at an open stage s, γ^+ is also accessible. In addition, since γ has w as an outcome on the true path and s is an open-stage, it follows that every $\mathcal{S}$ strategy above γ^+ with a g_n or h_n outcome for some n, is still opening a gap or performing Part II honestification at stage s. Finally, since s is an open-stage and γ is an $\mathcal{S}$ strategy, it is still the case that every $\mathcal{R}$ strategy above γ^+ witnesses no disagreement between A and its functional. Hence, s satisfies conditions (O1)-(O3) for γ^+, and is thus an open-stage for γ^+ as well.

Similarly, whenever γ is accessible at a close-stage t, γ^+ is also accessible. In addition, since γ has w as an outcome on the true path and t is an open-stage, it follows that every $\mathcal{S}$ strategy above γ^+ with a g_n or h_n outcome for some n, is still closing a gap or performing Part I honestification at stage t. Hence, t satisfies conditions (C1)-(C2) for γ^+, and is thus a close-stage for γ^+ as well.

(b) *γ has outcome h_n for some n on the true path.* By the *Leftmost Path Lemma* we have that there is some stage s_0 after which γ no longer visits outcomes lying to the left of h_n. In addition we have that the outcome h_n is visited infinitely often.

Now, for the outcome h_n to be visited at some stage s, γ must have seen a close-stage s and performed Part I Honestification. The following visit to h_n takes place when γ sees an open stage t, at which point it performs Part II Honestification. This cycle is then repeated upon the next visit to the h_n outcome.

Note that at a stage such as t, γ^+ is accessible. In addition, γ is performing Part II honestification. This means that every $\mathcal{S}$ strategy above γ^+ with a g_n or h_n outcome for some n, is still opening a gap or performing Part II honestification at stage s. Finally, since t is an open-stage and γ is an $\mathcal{S}$ strategy, it is still the case that every $\mathcal{R}$ strategy above γ^+ witnesses no disagreement between A and its functional. Hence, s satisfies conditions (O1)-(O3) for γ^+, and is thus an open-stage for γ^+ as well.

On the other hand, at a stage such as s, γ^+ is accessible. In addition, γ is performing Part I honestification. This means that every every $\mathcal{S}$ strategy above γ^+ with a g_n or h_n outcome for some n, is still closing a gap or performing Part II honestification at stage s. Hence, s satisfies conditions (C1)-(C2) for γ^+, and is thus a close-stage for γ^+ as well.

(c) *γ has outcome g_n for some n on the true path.* By the *Leftmost Path Lemma* we have that there is some stage s_0 after which γ no longer visits outcomes lying to the left of g_n. In addition we have that the outcome g_n is visited infinitely often.

Now for γ to visit outcome g_n, it must have first opened some gap on this outcome. To open such a gap, γ must have first waited for an open-stage s to appear, at which point it will have visited the outcome g_n. The following visit to g_n will occur once γ is ready to close a gap on the outcome g_n. To close the gap in this manner, γ must have waited for a close-stage t to appear, at which point it will visit g_n again. This cycle is then repeated upon the next visit to the outcome g_n.

Note that at a stage such as s, γ^+ is accessible. In addition, γ is opening a gap. This means that every $\mathcal{S}$ strategy above γ^+ with a g_n or h_n outcome for some n, is still opening a gap or performing Part II honestification at stage s. Finally, since s is an open-stage and γ is an $\mathcal{S}$ strategy, it is still the case that every $\mathcal{R}$ strategy above γ^+ witnesses no disagreement between A and its functional. Hence, s satisfies conditions (O1)-(O3) for γ^+, and is thus an open-stage for γ^+ as well.

On the other hand, at a stage such as t, γ^+ is accessible. In addition, γ is closing a gap. This means that every every $\mathcal{S}$ strategy above γ^+ with a g_n or h_n outcome for some n, is still closing a gap or performing Part I honestification at stage s. Hence, s satisfies conditions (C1)-(C2) for γ^+, and is thus a close-stage for γ^+ as well.

(d) *Suppose γ has outcome d on the true path.* By the *Leftmost Path Lemma* the γ strategy must visit its d outcome infinitely often. But once γ visits its d outcome, it stops and visits its d outcome at all subsequent stages. This means that from this point onwards whenever γ is accessible at an open stage s, γ^+ is also accessible. In addition, since γ has d as an outcome on the true path and s is an open-stage, it follows that every

$\mathcal{S}$ strategy above γ^+ with a g_n or h_n outcome for some n, is still opening a gap or performing Part II honestification at stage s. Finally, since s is an open-stage and γ is an $\mathcal{S}$ strategy, it is still the case that every $\mathcal{R}$ strategy above γ^+ witnesses no disagreement between A and its functional. Hence, s satisfies conditions (O1)-(O3) for γ^+, and is thus an open-stage for γ^+ as well.

Similarly, whenever γ is accessible at a close-stage t, γ is also accessible. In addition, since γ has d as an outcome on the true path and s is a close-stage, it follows that every $\mathcal{S}$ strategy above γ^+ with a g_n or h_n outcome for some n, is still closing a gap or performing Part I honestification at stage s. Hence, s satisfies conditions (C1)-(C2) for γ^+, and is thus an open-stage for γ^+ as well. □

2.10.7 Injury Lemma for $\mathcal{R}$ Strategies

We now show that if an $\mathcal{R}$ strategy represents an $\mathcal{R}$ requirement on the true path by being active on the true path, we have that the functional built by the strategy will be equal to the set A. To prove this fact, we first show that if the strategy redefines the functional whenever it becomes undefined, there will be some stage after which the functional will never become undefined again.

Lemma 2.10.8. *(Injury Lemma for $\mathcal{R}$ Strategies) Let $\mathcal{R}_i$ be an $\mathcal{R}$ requirement and let β be a strategy which represents $\mathcal{R}_i$ on the true path f. If β has outcome i and is active on the true path f then the following hold for every element x.*

(a) If there are infinitely many stages s such that $\Gamma_\beta^{U,D}[s](x)\downarrow$, there is some stage u such that for all $u' \geq u$, $\Gamma_\beta^{U,D}[u'](x)\downarrow$ ($\Gamma^{V,D} resp.$)

(b) $A(x) = \Gamma_\beta^{U,D}(x)$ ($\Gamma^{V,D} resp.$)

Proof. Lemma 2.10.8*, Part (a).* Suppose that there are infinitely many stages s such that $\Gamma_\beta^{U,D}[s](x)\downarrow$. We shall show that whenever some strategy enumerates an element into D, one of the following must be the case.

(i) $\Gamma_\beta^{U,D}(x)$ is not undefined.

(ii) $\Gamma_{\beta}^{U,D}(x)$ is undefined, but no constraint to increase $\gamma(x)$ is imposed, meaning that the strategy β can redefine the functional at x by choosing the same use $\gamma(x)$. Since there are only finitely many elements less than $\gamma(x)$, there will be a stage after which $\Gamma_{\beta}^{U,D}(x)$ cannot be undefined again.

(iii) $\Gamma_{\beta}^{U,D}(x)$ is undefined, and a constraint to increase $\gamma(x)$ is imposed. In these cases the constraint will only be imposed by some strategy if it holds some parameter which is less than or equal to x. It will then suffice to prove that such a strategy cannot hold this parameter forever. When combined with the fact that there are only finitely many parameters less than or equal to x, it follows that only finitely many strategies can impose such constraints, and that they can do so only finitely many times. Hence there will be a stage after which $\Gamma_{\beta}^{U,D}(x)$ cannot be undefined again.

We shall now examine the behaviour of all possible strategies which could enumerate elements into D and show that their behaviour falls into one of the above three cases. In the argument below, β is the strategy building $\Gamma_{\beta}^{U,D}$ (or $\Gamma_{\beta}^{V,D}$), β' is some $\mathcal{R}$ strategy on the priority tree, and α is some $\mathcal{S}$ strategy on the priority tree.

(1) We start by considering the situation where β itself enumerates some $\gamma_{\beta}(w)$ into D in order to repair a disagreement between $A(w)$ and $\Gamma_{\beta}^{U,D}(w)$. If $w > x$, β cannot undefine $\Gamma_{\beta}^{U,D}(x)$ by enumerating $\gamma_{\beta}(w)$ into D. Therefore one only needs to consider the situation where $w \leq x$. When β enumerates $\gamma_{\beta}(w)$ into D, it will also undefine $\Gamma_{\beta}^{U,D}(x)$. Since the use of $\Gamma_{\beta}^{U,D}(w)$ has to increase when the functional is redefined, this could lead to the use of $\Gamma_{\beta}^{U,D}(x)$ to increase as well. Now, since there are only finitely many elements $w \leq x$, it follows that such a disagreement can only arise and be repaired finitely many times. Thus it follows that β can only undefine $\Gamma_{\beta}^{U,D}(x)$ finitely many times, as required.

We now consider $\mathcal{R}$ strategies β' lying to the left, below, above or to the right of β, and the effect which they have on β when they enumerate elements into D.

(2) Consider the situation in which β' is to the left of β. Then by the *Leftmost Path Lemma* there is some stage s_0 such that β' is not accessible after stage s_0. This means that after stage s_0, β' cannot enumerate any elements into D, and hence does not undefine $\Gamma_{\beta}^{U,D}(x)$,

as required.

(3) Consider the situation in which β' is below β. In this case it is possible for β' to enumerate some element less than $\gamma_\beta(x)$ into D, thus undefining $\Gamma_\beta^{U,D}(x)$. However, such an action does not constrain β to choose a new use at x. Hence β will simply redefine $\Gamma_\beta^{U,D}(x)$ using its old use. This means that β' cannot cause the use of the $\Gamma_\beta^{U,D}(x)$ to increase, as required.

(4) Consider the situation in which β' is above β. Suppose that β' enumerates some $\gamma_{\beta'}(w)$ into D in order to repair a disagreement between $A(w)$ and $\Gamma_{\beta'}^{U,D}(w)$. Now it is either the case that β' is active for β, or else that β' is inactive for β.

Suppose that β' is active for β. Then we have that β is R-Synchronised with β', and that thus $\gamma_{\beta'}(x) < \gamma_\beta(x)$ for every x.

If $w > x$, the enumeration of $\gamma_{\beta'}(x)$ may undefine $\Gamma_\beta^{U,D}(w)$, but the strategy β can simply redefine its $\Gamma_\beta^{U,D}(w)$ using its old use $\gamma_{\beta'}(w)$. This means that β' cannot cause the use of the $\Gamma_\beta^{U,D}(x)$ to increase, as required.

On the other hand it could be the case that $w \leq x$. In this situation β' enumerates $\gamma_{\beta'}(w)$ into D, which also undefines $\Gamma_{\beta'}^{U,D}(x)$. In addition since β is R-synchronised with β', $\Gamma_\beta^{U,D}(x)$ will also be undefined.

Now, since $\gamma_{\beta'}(w)$ has been enumerated into D, it follows that the use of $\Gamma_{\beta'}^{U,D}(w)$ must increase when the functional is redefined at this value. Since this can lead to the use of $\Gamma_{\beta'}^{U,D}(x)$ to increase as well, R-Synchronisation can also cause an increase in the use $\Gamma_\beta^{U,D}(x)$.

Now, since there are finitely many elements $w \leq x$, it follows that such a disagreement can only arise and be repaired by β' finitely many times. Thus it follows that β' can only undefine $\Gamma_{\beta'}^{U,D}(x)$ finitely many times.

Once the use of $\Gamma_{\beta'}^{U,D}(x)$ stops changing, it follows that β can redefine $\Gamma_\beta^{U,D}(x)$ using its old use whenever this is undefined by β', whilst still observing R-Synchronisation. Since the use of $\Gamma_\beta^{U,D}(x)$ remains constant, there can only be finitely many elements below it. Hence β' can only undefine $\Gamma_\beta^{U,D}(x)$ finitely many times, as required.

Suppose that β' is inactive for β. In this case, β' enumerates $\gamma_\beta(w)$ into D, it may happen to undefine $\Gamma_{\beta}^{U,D}(x)$. But in this case there is no constraint stopping β from redefining $\Gamma_{\beta}^{U,D}(x)$ using its old use. Thus it follows that β' can only undefine $\Gamma_{\beta}^{U,D}(x)$ finitely many times, as required.

(5) Consider the situation in which β' is to the right of β. During stages at which β is accessible, β' is initialised. Once β' is accessible once again, it will rebuild its functional by selecting uses which are greater than any use assigned to some functional being built by a strategy appearing to the left of β'. It follows that if β' enumerates some $\gamma_{\beta'}(w)$ into D, it will not be able to undefine $\Gamma_{\beta}^{U,D}(x)$ at any element.

We proceed by considering $\mathcal{S}$ strategies α lying to the left, below, above or to the right of β, and the effect which they have on β when they enumerate elements into D.

(6) Consider the situation in which α is to the left of β. Then by the *Leftmost Path Lemma* there is some stage s_0 such that α is not accessible after stage s_0. This means that after stage s_0, α cannot enumerate any elements into D, and hence does not undefine $\Gamma_{\beta}^{U,D}(x)$, as required.

(7) Consider the situation in which α is below β, and α is on the true path. By assumption we have that β has outcome i, is active on the true path, and represents the requirement on the true path.

Since β is labeled $\mathcal{R}_i$, we have that that α cannot have outcomes h_j or g_j for $j \leq i$ on the true path. Hence the true outcome of α could only be one of d, g_j for $j > i$, h_j for $j > i$ or w. It is important to recall that while the g_j outcomes are ordered in descending order, the h_j outcomes are ordered in ascending order.

The strategy α will have a sequence of active $\mathcal{R}$ strategies $(\beta_1, \ldots, \beta_n)$ located above it, with β_m being labeled $\mathcal{R}_m$. The strategy β_m corresponds to the outcomes g_m and h_m of the strategy α. Note that the strategy β labeled $\mathcal{R}_i$ being analysed in this lemma corresponds to the strategy β_i in the aforementioned sequence.

We now examine the effect of the strategy α on the strategy β through a case analysis based on the true outcome of the α strategy.

(a) If the outcome of α on the true path is d, then α stops acting after it has successfully closed its gap for the first time. Hence, α can only enumerate finitely many elements into D and can only undefine $\Gamma^{U,D}_{\beta_i}(x)$ finitely often.

(b) If the outcome of α on the true path is g_j, with $j > i$, we have that $w \to \infty$ because infinitely many witnesses are enumerated into A during open stages, and that $\gamma_{\beta_j}(v_j) \to \infty$ because capricious destruction takes place infinitely often during close stages. In order to preserve the ordering between parameters, it follows that $v_n \to \infty$ for every $n < j$. Let s_0 be the stage after which strategies to the left of α are inaccessible, as by the *Leftmost Path Lemma*. Then there must be a stage $s_1 > s_0$ such that both the witness w and thresholds v_n with $n < j$ become greater than x.

Now, if α visits outcome g_j, it will enumerate $\gamma_{\beta_j}(v_j)$ into D. This may undefine $\Gamma^{U,D}_{\beta_i}(x)$. Now, since $\gamma_{\beta_j}(v_j)$ has been enumerated into D, it will have to increase. On the other hand, the strategy β_i can redefine $\Gamma^{U,D}_{\beta_i}(x)$ using is previous use, because this has not been enumerated into D. Since $\gamma_{\beta_j}(v_j) \to \infty$, whilst $\gamma_{\beta_i}(x)$ remains constant, it follows that α can only undefine $\Gamma^{U,D}_{\beta_i}(x)$ finitely often as required.

If α visits outcome g_n, with $n > i$, it will enumerate $\gamma_{\beta_n}(v_n)$ into D. By R-Synchronisation, $\gamma_{\beta_i}(v_n) < \gamma_{\beta_n}(v_n)$. After stage s_1, we also have that $x < v_n$, which by transitivity gives that $\gamma_{\beta_i}(x) < \gamma_{\beta_n}(v_n)$. Hence $\Gamma^{U,D}_{\beta_i}(x)$ cannot be undefined when α visits this outcome.

If α visits outcome g_i, it will enumerate $\gamma_{\beta_i}(v_i)$ into D. But after stage s_1, we have that $x < v_i$, which gives that $\gamma_{\beta_i}(x) < \gamma_{\beta_i}(v_i)$. Hence $\Gamma^{U,D}_{\beta_i}(x)$ cannot be undefined when α visits this outcome.

If α visits outcome g_n, with $n < i$, it will enumerate $\gamma_{\beta_n}(v_n)$ into D. Now, for any m such that $n < m \leq i$ it is possible that $\gamma_{\beta_n}(v_n) < \gamma_{\beta_m}(x)$, leading to $\Gamma^{U,D}_{\beta_m}(x)$ becoming undefined. However, whilst $\gamma_{\beta_n}(v_n)$ has entered D, $\gamma_{\beta_m}(x)$ has not. Hence each of these strategies is able to redefine its functional by choosing its old use. Since the value of $\gamma_{\beta_m}(x)$ for all m such that $n < m < i$ remains constant, we have that the value of $\gamma_{\beta_i}(x)$ does not increase through R-Synchronisation either. It follows that the functional $\Gamma^{U,D}_{\beta_i}(x)$ is undefined only finitely often.

The situation when visiting outcomes h_n for some n is similar.

If α visits outcome h_n, with $n > i$, it will enumerate $\gamma_{\beta_n}(w)$ into D. By R-

Synchronisation, $\gamma_{\beta_i}(w) < \gamma_{\beta_n}(w)$. After stage s_1, we also have that $x < w$, which by transitivity gives that $\gamma_{\beta_i}(x) < \gamma_{\beta_n}(w)$. Hence $\Gamma^{U,D}_{\beta_i}(x)$ cannot be undefined when α visits this outcome.

If α visits outcome h_i, it will enumerate $\gamma_{\beta_i}(w)$ into D. But after stage s_1, we have that $x < w$ giving that $\gamma_{\beta_i}(x) < \gamma_{\beta_i}(w)$. Hence $\Gamma^{U,D}_{\beta_i}(x)$ cannot be undefined when α visits this outcome.

If α visits outcome h_n, with $n < i$, it will enumerate $\gamma_{\beta_n}(w)$ into D. Now, for any m such that $n < m \leq i$ it is possible that $\gamma_{\beta_n}(w) < \gamma_{\beta_m}(x)$, leading to $\Gamma^{U,D}_{\beta_m}(x)$ becoming undefined. However, whilst $\gamma_{\beta_n}(w)$ has entered D, $\gamma_{\beta_m}(x)$ has not. Hence each of these strategies is able to redefine its functional by choosing its old use. Since the value of $\gamma_{\beta_m}(x)$ for all m such that $n < m < i$ remains constant, we have that the value of $\gamma_{\beta_i}(x)$ does not increase through R-Synchronisation either. It follows that the functional $\Gamma^{U,D}_{\beta_i}(x)$ is undefined only finitely often.

Finally if α visits the outcome w no element is enumerated into D. Hence α cannot undefine $\Gamma^{U,D}_{\beta_i}(x)$ after this stage.

(c) If the outcome of α on the true path is h_j with $j > i$, we have that after stage s_1 the strategy α chooses a witness w and holds it forever.

Now if α visits any outcome h_n with $n > i$, we have that it will enumerate $\gamma_{\beta_n}(w)$ into D. This may undefine $\Gamma^{U,D}_{\beta_i}(x)$. However whilst $\gamma_{\beta_n}(w)$ has entered D, $\gamma_{\beta_i}(x)$ has not. Hence the strategy β_i is able to redefine its functional by choosing its old use. Since $\gamma_{\beta_i}(x)$ remains constant while $\gamma_{\beta_n}(w)$ increases, it follows that $\Gamma^{U,D}_{\beta_i}(x)$ can only be undefined finitely many times.

On the other hand if α visits the outcome w no element is enumerated into D. Hence α cannot undefine $\Gamma^{U,D}_{\beta_i}(x)$ after this stage.

(d) Finally if the outcome of α on the true path is w, it must be the case that α no longer enumerates any element into D after stage s_1. Hence α cannot undefine $\Gamma^{U,D}_{\beta_i}(x)$ after this stage.

(8) Consider the situation in which α is below β, but this time, α is *to the right* of the true path. This means that α must be initialised infinitely many times. Each time α is initialised, it has

to choose new parameters p, which eventually must increase to become greater than x.

Once this has occurred α can only enumerate uses $\gamma_{\beta'}(p)$ for some $p > a$ for some β' active for α. Now, suppose $\Gamma_{\beta}^{U,D}(x)$ is undefined by this enumeration. In this case α cannot impose a constraint on β which forces it to choose a greater use when it defines $\Gamma_{\beta}^{U,D}(x)$ again, because such a constraint can now only be imposed at some parameter p greater than x.

Consider now strategies β' which are of higher priority than β and which are active for α. In this case we have that β does not increase its use due to a R-Synchronisation with a higher priority strategy β'. The reason for this is that the enumeration of $\gamma_{\beta'}(p)$ into D does not undefine $\Gamma_{\beta'}^{U,D}(x)$. Hence if $\Gamma_{\beta}^{U,D}(x)$ is undefined by this enumeration, β can simply redefine its functional using its old use. Since the use of $\Gamma_{\beta}^{U,D}(x)$ remains constant, there can only be finitely many elements below it. Hence $\Gamma_{\beta}^{U,D}(x)$ can only be undefined finitely many times, as required.

(9) Consider the situation in which α is above β. Depending on the outcome of α on the true path, α will be operating inside some work interval. If the outcome of α on the true path is d, then α must have already stopped acting for β to be accessible. Hence α does not enumerate any element into D.

If the outcome of α on the true path is g_n for some n, then we have that β is operating inside the work interval $(v_n, \gamma_n(v_n))$, and that it builds its functional by choosing uses inside this work interval. Now, if α visits g_n, it will enumerate $\gamma_n(v_n)$ into D. But β has built its functional by choosing uses which are all less than $\gamma_n(v_n)$. This means that α cannot undefine the functional built by β at any element. In addition, due to the ordering of the parameters chosen by the α strategy, outcomes to the right of g_n can only cause the enumeration of elements which are greater than $\gamma_n(v_n)$ into D, leaving the functional built by β unaffected. It follows that $\Gamma_{\beta}^{U,D}(x)$ can only be undefined finitely often by α.

If the outcome of α on the true path is h_n for some n, then we have that β is operating inside the work interval $(w, \gamma_n(w))$, and that it builds its functional by choosing uses inside this work interval. Now, if α visits h_n, it will enumerate $\gamma_n(w)$ into D. But β has built its functional by choosing uses which are all less than $\gamma_n(w)$. This means that α cannot undefine the functional built by β at any element. In addition, due to the ordering of the parameters chosen by the α strategy, outcomes to the right of h_n can only cause the enumeration of elements which are greater than $\gamma_n(w)$ into D, leaving the functional built by β unaffected.

Finally if the outcome of α on the true path is w, it must be the case that α no longer enumerates any element into D after stage s_1. Hence α will not affect the functional being built by β after stage s.

(10) Consider the situation in which α is to the right of β. The strategy α has a number of active $\mathcal{R}$ strategies β' above it. Each of these strategies chooses uses from its set of uses in order to build its functional. When α visits some g_n or h_n outcome, it will enumerate some use belonging to the functional built by one of the strategies β'.

Now, if β' is to the right of β, then β' is initialised whenever β is accessible. Once β' is accessible again, it rebuilds it functional by choosing uses which are greater than those of β. Hence, if α were to enumerate a use belonging to the functional built by β', this use would be too large to undefine the functional built by β at some element.

On the other hand, it could be that β' is above β. This means that β would be synchronised with β', and that $\gamma_\beta(x)$ has to be greater than $\gamma_{\beta'}(x)$ for every x. Now, if α undefines $\Gamma^{U,D}_{\beta'}(p)$ for some parameter $p \leq x$, then it will also undefine $\Gamma^{U,D}_{\beta'}(x)$ and $\Gamma^{U,D}_{\beta}(x)$. Since the enumeration by α into D can constrain β' to redefine $\Gamma^{U,D}_{\beta'}(p)$ using an increased use, this can also cause an increase in the use of $\Gamma^{U,D}_{\beta'}(x)$ and $\Gamma^{U,D}_{\beta}(x)$ when the appropriate strategies redefine their functionals.

However we know that when β is accessible, α is initialised. This means that all of its thresholds and witnesses have to be chosen again and thus increase. But because there are only finitely many elements which are less than x, there must be a stage such that all of the parameters chosen by α are greater than x. Hence, no R-Synchronisation constraint will be imposed on β to increase the use of $\Gamma^{U,D}_{\beta}(x)$ if it needs to be redefined. It follows that eventually α cannot undefine $\Gamma^{U,D}_{\beta}(x)$ anymore by enumerating elements into D. □

Proof. Lemma 2.10.8*, Part (b).* We shall now show that for all x we have that $A(x) = \Gamma^{U,D}_{\beta}(x)$

($\Gamma_\beta^{V,D}$ resp.)

We prove the above fact by Strong Induction. Suppose that for every $y < x$, there exists some stage t_y such that for all $s \geq t_y$, $\Gamma_\beta^{U,D}[s](y) = \Gamma_\beta^{U,D}[t_y](y) = A(y)$. Then we prove that for x there exists some stage t such that for all $s \geq t$, $\Gamma_\beta^{U,D}[s](x) = \Gamma_\beta^{U,D}[t](x) = A(x)$.

Let $y' = \max\{t_y \mid 1 \leq y < x\}$. Then $y' > s_0$, where s_0 is the least stage after which strategies to the left of β become inaccessible, and after which β cannot be initialised. For if this was not the case, β would be initialised at or after y', resulting in the functional $\Gamma_\beta^{U,D}[s](y)$ becoming undefined at all elements and contradicting the inductive hypothesis.

We now perform a case analysis, depending on whether $A(x) = 0$ or $A(x) = 1$.

(A) $A(x) = 0$. Suppose that $\Gamma_\beta^{U,D}[u](x) \uparrow$ for some stage $u \geq y'$.

Then since β has outcome i on the true path, we have that this outcome will be visited infinitely often. For this to be the case, β must reach reach step (4) infinitely often. Suppose that β reaches step (4) at stage $u_1 > u$. By the inductive hypothesis, the strategy β will then see that there is no disagreement between the functional and the set A at stage u_1. It will therefore go to step (4)(b) and define the computation $\Gamma_\beta^{U,D}[u_1](x)$ to be equal to $A_{u_1}(x) = A(x)$.

(B) $A(x) = 1$. Suppose that $\Gamma_\beta^{U,D}[u](x) \uparrow$ for some stage $u \geq y'$.

Then since β has outcome i on the true path, we have that this outcome will be visited infinitely often. For this to be the case, β must reach reach step (4) infinitely often. Suppose that β reaches step (4) at stage $u_1 > u$. By the inductive hypothesis, the strategy β will then see that there is no disagreement between the functional and the set A at stage u_1. It will therefore go to step (4)(b) and define the computation $\Gamma_\beta^{U,D}[u_1](x)$ to be equal to $A_{u_1}(x)$.

Hence if $A_{u_1}(x) = 1$, we have that the strategy defines $\Gamma_\beta^{U,D}[u_1](x)$ to be equal to $A(x)$.

On the other hand, it could be the case that $A_{u_1}(x) = 0$. Then we have that $\Gamma_\beta^{U,D}[u_1](x)$ is equal to $A_{u_1}(x)$. However, there must be some least stage $u_2 > u_1$ such that $A_{u_2}(x) = 1$. This means that a disagreement will arise between $\Gamma_\beta^{U,D}[u_2](x)$ and $A_{u_2}(x)$.

Now suppose that the strategy reaches step (4) again at stage $u_3 > u_2$. Then the strategy

β will see that there is a disagreement between the functional and the set A. If u_3 is not a close stage, we have that the strategy β reaches step (4) again at some close stage $u_4 > u_3$ by the *Synchronisation Lemma*.

Therefore at stage u_4 the strategy β will go to step (4)(a)(i) and enumerate $\gamma_{\beta,u_4}(x)$ into D, undefining $\Gamma_\beta^{U,D}(x)$.

Now, suppose that the strategy reaches step (4) again at stage $u_5 > u_4$. Then by the inductive hypothesis, the strategy β will see that there is no disagreement between the functional and the set A. Hence the strategy will go to step (4)(b) and define the computation $\Gamma_\beta^{U,D}[u_5](x)$ to be equal to $A(x)$.

Hence we have that if the functional is undefined at some stage $u \geq y'$, the strategy will eventually redefine it to be equal to $A(x)$. But then by *Lemma 3.8.11, Part (a)*, we have that there is some stage v such that for all $v' > v$, $\Gamma_\beta^{U,D}[v'](x) \downarrow$. It follows that there exists some stage t such that for all $s \geq t$, $\Gamma_\beta^{U,D}[s](x) = \Gamma_\beta^{U,D}[t](x) = A(x)$, as required.

□

2.10.8 Injury Lemma for $\mathcal{S}$ Strategies

The Injury Lemma for $\mathcal{S}$ Strategies describes those $\mathcal{S}$ strategies which represent an $\mathcal{S}$ requirement on the true path and which have outcome d on the true path. It shows that in such cases there is some stage t such that the strategy enumerates a witness w into A by opening a gap at stage t and restraining $D \upharpoonright \theta_t(w)$. In addition the strategy closes the gap successfully when it becomes accessible again by visiting outcome d. Finally and most importantly, no elements smaller than or equal to $\theta_t(w)$ enter D after stage t, meaning that the diagonalisation is preserved.

Lemma 2.10.9. *(Injury Lemma for $\mathcal{S}$ Strategies). Let α be a strategy which represents the requirement $\mathcal{S}_i$ on the true path f. Suppose that α has outcome d on the true path. Then there exists a stage t such that:*

(1) α imposes a restraint $D \upharpoonright \theta_t(w)$ for some witness w, and such that no element $x \leq \theta_t(w)$ enters the set D at or after stage t.

(2) *t is the least stage such that:*

   (a) *$t > s_0$, where s_0 is the stage such that no strategy to the left of α is accessible at stages $s > s_0$, as by the Leftmost Path Lemma.*

   (b) *α opens a gap at t.*

   (c) *α closes the gap opened at t successfully, visiting its d outcome.*

Proof. Let t be the stage defined in condition (2), that is the least stage such that $t > s_0$, α opens a gap at stage t, and α subsequently closes this gap by diagonalising successfully. We show that this stage t satisfies condition (1), that is α is no longer injured at and after stage t.

We have to consider possible injury to α at and after stage t coming from other $\mathcal{S}$ and $\mathcal{R}$ strategies on the tree. We start by examining possible injury coming from $\mathcal{S}$ strategies α' located to the left, below, above and to the right of α.

(1) Suppose that $\alpha' <_L \alpha$. Then α' is no longer accessible after stage s_0, and therefore cannot injure α.

(2) Suppose that $\alpha \subset \alpha'$. Then α' is not able to injure α, because α imposes the restraint $D \upharpoonright \theta(w)$ on α' at stage t.

(3) Suppose that $\alpha' \subset \alpha$. Consider the outcomes of α' on the true path.

   (a) Suppose that the outcome of α' on the true path is w. Then we have that α' does not enumerate elements into the set D after stage s_0.

   (b) Suppose that the outcome of α' on the true path is h_n for some n. Firstly we note that after stage s_0 outcomes to the left of h_n are not accessible. Secondly, the strategy α' imposes the work interval $(w', \gamma_n(w'))$ below its h_n outcome. Since α is operating inside this work interval, we have that $w' < w < \theta(w) < \gamma_n(w')$. Hence, if α' visits the outcome h_n and enumerates $\gamma_n(w')$ into D, it cannot injure α. Thirdly, if α' visits an h outcome to the right of h_n, it will only be able to enumerate elements greater than $\gamma_n(w')$ into D, thanks to the ordering between the upper bounds of work intervals. On the other hand if α' visits its w outcome, it does not enumerate any element into D.

   (c) Suppose that the outcome of α' on the true path is g_n for some n. Firstly we note that after stage s_0 outcomes to the left of g_n are not accessible. Secondly the strategy

α' imposes the work interval $(v'_n, \gamma(v'_n))$ below its g_n outcome. Since α is operating inside this work interval, we have that $v'_n < w < \theta(w) < \gamma_n(v'_n)$. Hence, if α' visits the outcome g_n and enumerates $\gamma_n(v'_n)$ into D, it cannot injure α. Thirdly, if α' visits a g or h outcome to the right of g_n, it will only be able to enumerate elements greater than $\gamma_n(v'_n)$ into D, thanks to the ordering between the upper bounds of work intervals. On the other hand if α' visits its w outcome, it does not enumerate any element into D.

(d) Suppose that the outcome of α' on the true path is d. At stage t, the node α is accessible. Since α is on the true path, this means that at this stage α' has visited outcome d. But once an $\mathcal{S}$ strategy visits its d outcome, it will stop and always visit its d outcome whenever it is accessible. The only exception to this would be if the α' strategy were initialised. But after stage s_0, strategies to the left of α are not accessible. Since $\alpha' \subset \alpha$, strategies to the left of α' are also inaccessible after s_0. Hence we can conclude that whenever α' is accessible at or after stage t, it will visit its outcome d and thus will not enumerate any element into the set D.

(4) Suppose that $\alpha <_L \alpha'$. When α opens the gap at stage t, every node α' to the right of α is initialised. Hence when α' becomes accessible again after stage t, it will first of all choose thresholds and witnesses which are greater than any parameters chosen by α, since this lies to its left. In particular any parameters chosen by α' have to be greater than w, the witness chosen by α.

Now, if α' becomes accessible after stage t it could enumerate some element of the form $\gamma_\beta(v_\beta)$ or $\gamma_\beta(w')$ into D, where the uses belong to the functional Γ_β being built by some strategy β which is active for α'. We have to consider two situations, one where $\beta \subset \alpha$, and the other when $\alpha <_L \beta$.

If $\beta \subset \alpha$, then we have that when α opened the gap at stage t, it has placed a constraint upon β to pick uses $\gamma_\beta(x) > \theta(w)$ for all $x \geq w$. Since β is above α', it has to be accessible before α' is accessible. Now, when β becomes accessible, it will either be undefined at w, or else it will undefine the functional at w itself by enumerating $\gamma_\beta(w)$ into D. The strategy α', on the other hand, will wait until it can choose uses belonging to the functional built by β for each of its thresholds and its witness. Suppose that α' succeeds in choosing uses from the functional built by β at any of its parameters, which as we have seen are greater than w. In order for this to have been possible, β must have redefined its functional at these elements, this time following the constraint imposed on it by α, and choosing uses greater than $\theta(w)$. Then if α' ever enumerates any one of these uses into D, we have that it cannot injure α.

On the other hand, if $\alpha <_L \beta$, we have that β is initialised when α opens its gap at stage t. Since β is above α', it has to be accessible before α' is accessible. The strategy β will redefine its functional during β-expansionary stages, choosing uses which are greater than any convergent computation lying to its left. Thus β will choose uses $\gamma_\beta(x) > \theta(w)$ for every x. The strategy α', waits until it can choose uses belonging to the functional built by β for each of its thresholds and its witness. Thus if α' ever succeeds in choosing uses from the functional built by β at any of its parameters and proceeds to enumerate such a use into D, we have that it cannot injure α.

We now examine the possible injury coming from $\mathcal{R}$ strategies β located to the left, below, above and to the right of α.

(1) Suppose that $\beta <_L \alpha$. Then β is no longer accessible after stage s_0, and therefore cannot injure α.

(2) Suppose that $\alpha \subset \beta$. Then β is not able to injure α, because α places the restraint $D \upharpoonright \theta(w)$ on β.

(3) Suppose that $\beta \subset \alpha$. We shall show separately that β cannot injure α at stage t, and that β cannot injure α at stages after t.

   (a) In order to show that β cannot injure α at stage t, we consider whether β is active or inactive for α.

      (i) Suppose that β is an active strategy for α. When α opens its gap at stage t, it will enumerate its witness w into A. Then this enumeration of w into A could have injured the functional being built by β, creating a disagreement $A(w) \neq \Gamma_\beta^{U,D}(w)$. Now when β is accessible again, it must have seen the required U change. This must have been the case because for α to diagonalise successfully and visit its d outcome at stage s, every higher priority $\mathcal{R}$ strategy β must have seen the required changes to repair its functional. Therefore in this case, β does not enumerate any element into D to correct its functional at w, and thus cannot injure α.

(ii) Suppose that β labeled $\mathcal{R}_i$ is inactive for α. When α opens its gap at stage t, it will enumerate its witness w into A. Then this enumeration of w into A could have injured the functional being built by β, creating a disagreement $A(w) \neq \Gamma_{\beta}^{U,D}(w)$. Now, for β to be inactive for α, there must exist some β' labeled $\mathcal{R}_j$ and some strategy α' such that we have $\beta' \subseteq \beta \subset \alpha' \frown o \subset \alpha$, where $o = g_j$ or $o = h_j$, for some $j \leq i$. We shall show that each of these two cases leads to the conclusion that β cannot injure α if it enumerates $\gamma_\beta(w)$ into D.

Before proving each of these two cases we shall need to show that β is synchronised with β'. We start from the observation that for α' to have any one of these two outcomes, β' has to be active for α'. Now, assume for contradiction that β was not synchronised with β'. This means that there would be some intervening strategy α'' between β' and β with outcome g_n or h_n, with $n \leq j$. But this would make β' inactive for β, and hence for α', which is a contradiction.

We shall now show that α cannot be injured by β if α' has outcome $o = g_j$, for some $j \leq i$. In this case, α' imposes a work interval $(v'_j, \gamma_{\beta'}(v'_j))$ on α. Since α has to select its witness w inside the work interval, and will only trust computations $\theta(w)$ which are in the work interval, we have that $v'_j < w < \theta(w) < \gamma_{\beta'}(v'_j)$. Now, since we have determined that β is synchronised with β', we have that $\gamma_{\beta'}(v'_j) < \gamma_\beta(v'_j)$. Since $v'_j < w$, it follows that $\gamma_\beta(v'_j) < \gamma_\beta(w)$. Hence β cannot injure α when trying to repair its disagreement at w.

We shall now show that α cannot be injured by β if α' has outcome $o = h_j$, for some $j \leq i$. In this case, α' imposes a work interval $(w', \gamma_{\beta'}(w'))$ on α. Since α has to select its witness w inside the work interval, and will only trust computations $\theta(w)$ which are in the work interval, we have that $w' < w < \theta(w) < \gamma_{\beta'}(w')$. Since we determined that β is synchronised with β', we have that $\gamma_{\beta'}(w') < \gamma_\beta(w')$. Now, since $w' < w$, it follows that $\gamma_\beta(w') < \gamma_\beta(w)$.

Once again, β cannot injure α when trying to repair its disagreement at w.

(b) In order to show that β cannot injure α after stage t, we proceed as follows. Suppose that α opens its gap at stage t. We start by showing that whenever some strategy α' enumerates some witness w' into A after stage t, we have that $w' > w$.

(i) Suppose that $\alpha' <_L \alpha$. Then α' is not accessible and does not enumerate any witness after stage s_0.

(ii) Suppose that $\alpha \subset \alpha'$. In this case, α' can only be accessible at some stage if α is also accessible at the same stage. Now, when α opens its gap at stage t, it will subsequently close this gap successfully at stage s by visiting its outcome d. Henceforth, α will stop, and always visit its d outcome whenever it is accessible. The only exception to this would occur if α were initialised, but strategies to the left of α are not accessible after stage s_0. Hence, if α' is accessible at or after stage t, it must be the case that α' is below the d outcome of α. From this we can conclude that α' must be working inside the work interval (s_1, ∞) imposed by α, where s_1 is the stage at which α has diagonalised successfully. This means that if α' picks some witness w', it must be the case that $w < \theta(w) < s_1 < w'$, as required.

(iii) Suppose that $\alpha' \subset \alpha$. If $\alpha' \frown g_n \subset \alpha$ for some n, then α picks its witness w inside the work interval specified by α'. In addition, α only opens a gap when α' opens a gap, because α is synchronised with α'. Since strategies opening a gap enumerate their witness into A, and then proceed to pick another one once they close the gap, it follows that the witness w picked by α is always smaller than the witness w' picked by α'. Hence we have that $w' > w$ as required.

(iv) Suppose that $\alpha <_L \alpha'$. When α opens the gap at stage t, every strategy α' to the right of α is initialised. When α' becomes accessible again, it will choose thresholds and witnesses which are larger than any parameters chosen by $\mathcal{S}$ strategies located to the left of α'. Since this includes α, the parameters of α' will thus be greater than those of α. Hence we have that $w' > w$ as required.

From the above argument we can conclude that after stage t, the functional built by any strategy $\beta \subset \alpha$ can only witness a disagreement at some $w' > w$. We shall now show that if β needs to enumerate $\gamma_\beta(w')$ into D in order to repair its functional at some such witness w', this would not injure α. We split our analysis in two cases, depending on whether β is an active or an inactive strategy for α.

Suppose β is an active strategy for α. When α opens its gap at stage t, it enumerates its witness w into A, and constrains every active $\mathcal{R}$ strategy $\beta \subset \alpha$ to choose uses $\gamma(x)$ such that $\theta(w) < \gamma(x)$ for every $w \leq x$. Now, when β becomes accessible again after stage t, it must have obtained the required changes to undefine its functional at w. Hence we have that that for all $x \geq w$, β must now choose uses obeying the constraint created by α. Suppose that some α' now enumerates some $w' > w$ into A, and that β tries to repair its functional by enumerating $\gamma_\beta(w')$ into D. But then we have that $\gamma_\beta(w') > \theta(w)$, meaning that β cannot injure α.

On the other hand suppose that β is an inactive strategy for α. In (2) we have already determined that if β enumerates $\gamma_\beta(w)$ into D, it cannot injure α. Suppose that some α' enumerates some $w' > w$ into A, and that β needs to repair its functional by enumerating $\gamma_\beta(w')$ into D. But since $w < w'$, we have that $\gamma_\beta(w) < \gamma_\beta(w')$, meaning that β cannot injure α.

(4) Consider the situation in which $\alpha <_L \beta$. When α opens the gap at stage t, every β strategy to the left of α is initialised. When such a strategy β rebuilds its functional, it will choose uses $\gamma_\beta(x)$ which are greater than the use of any converging computation belonging to strategies located to the left of β. Since this includes α, we have that β will define $\theta(w) < \gamma_\beta(x)$ for every x. This means that β cannot enumerate any use into D which would injure α. □

2.10.9 Truth of Outcome Theorem

The Truth of Outcome Theorem shows that every requirement is satisfied by the strategy which represents it on the true path.

Theorem 2.10.10. *(Truth of Outcome Theorem). Let f be the true path, and let Q be a requirement. Then there exists a strategy γ which satisfies Q on the true path f.*

Proof. We start by considering the case where Q is a requirement $\mathcal{R}_i$.

Consider the true path f. By the *Representation Lemma* there is some node β which represents $\mathcal{R}_i$ on f. For β to represent $\mathcal{R}_i$ on f, one of the following must be the case.

- β is labeled $\mathcal{R}_i$ has outcome f on the true path.
- β is labeled $\mathcal{R}_i$ and there is some α below β with outcome h_i on the true path.
- β is labeled $\mathcal{R}_i$, has outcome i on the true path, and there is no α below β with outcome h_j for $j < i$ or with outcome g_j for $j \leq i$ on the true path.

We consider these three cases in turn.

(1) Suppose that β has outcome f on the true path. Then by the *Leftmost Path Lemma* there is some stage s_0 such that β no longer witnesses any β-expansionary stage at stages $s > s_0$. This means that $l(\Phi_i^{U,V}, A)$ is finite, and that Q is satisfied trivially.

(2) Suppose that β is labeled $\mathcal{R}_i$ and there is some α below β with outcome h_i on the true path. Then α chooses some witness w for which $\phi_i(w) \to \infty$. Hence $\Phi_i^{U,V}(w) \uparrow$, and Q is satisfied trivially.

(3) Suppose that β is labeled $\mathcal{R}_i$, has outcome i on the true path, and there is no β below α with outcome h_j for $j < i$ or with outcome g_j for $j \leq i$ on the true path. In this case, β will build the functional $\Gamma_\beta^{U,D}$ in order to ensure its equality with the set A. By the *Injury Lemma for $\mathcal{R}$ Strategies* we have that for all x, $A(x) = \Gamma_\beta^{U,D}(x)$. Hence β satisfies its requirement Q.

We now consider the case where Q is a requirement $\mathcal{S}_i$.

Consider the true path f. By the *Representation Lemma* there is some strategy α which represents $\mathcal{S}_i$ on f. For α to represent $\mathcal{S}_i$ on f, we must either have that $\alpha \frown d \subset f$ or that $\alpha \frown w \subset f$.

(1) Suppose that $\alpha \frown d \subset f$. By the *Leftmost Path Lemma* there is some stage s_0 such that for all stages $s \geq s_0$ no strategy to the left of α is accessible. Since $\alpha \frown d \subset f$, by the *Injury Lemma for $\mathcal{S}$ Strategies* we have that there is some least stage $t > s_0$ such that α opens a gap at stage t, and subsequently closes it successfully. In addition, α is not injured at or after stage t. Now, for α to have opened the gap at stage t, it must have seen some computation $\Theta_t^{D_t}(w) \downarrow= 0$, and restrained $D \restriction \theta_t(w)$, for some witness w. Since no strategy injures α at or after t, and since α is never initialised again after s_0, we have that this computation is

preserved forever. In addition, α will enumerate its witness w into A. Hence we have that $A(w) \neq \Theta^D(w)$, and the requirement Q is satisfied.

(2) Suppose that $\alpha \frown w \subset f$. By the *Leftmost Path Lemma* there is some stage s_0 such that for all stages $s > s_0$ no strategy to the left of α is accessible. Since $\alpha \frown w \subset f$, the w outcome of α is the leftmost outcome which is visited infinitely often. This means that there is some stage $s_1 > s_0$ such that α does not move to the left of outcome w. Since α has no outcome lying to the right of w, this means that α has to visit outcome w every time it is accessible after stage s_1, holding its chosen witness w forever.

We shall show that α satisfies the requirement Q by splitting our analysis into two cases.

(a) $lim_s\ \theta_s(w) \to \infty$. In this case we have that $\Theta^D(w) \uparrow$ and that the requirement is satisfied trivially.

(b) $lim_s\ \theta_s(w)$ *is finite.* We consider the following four cases.

(i) Suppose that there is some stage t such that for all $s \geq t$ we have that $\Theta_s^{D_s}(w) \uparrow$. Then we have that $\Theta^D(w) \uparrow$ and the requirement Q is satisfied trivially.

(ii) Suppose that there is some stage t such that for all $s \geq t$ we have that $\Theta_s^{D_s}(w) \downarrow= 1$. In this case α never enumerates w into A. Hence we have that $\Theta^D(w) \neq A(w)$ and that the requirement Q is satisfied trivially.

(iii) Suppose that there is some stage t such that for all $s \geq t$ we have that $\Theta_s^{D_s}(w) \downarrow= 0$. In this case there must be some stage after t such that α tries to honestify or to open a gap, meaning that α visits an outcome to the left of w, which is a contradiction.

(iv) Suppose that there is some stage t such that for all $s \geq t$ we have that $\Theta_s^{D_s}(w) \downarrow= 0$, but α does not trust this computation at stage s because $\theta_s(w)$ is not within the work interval (a, b) imposed on it. But the upper bound of the work interval is either absent or it moves off to infinity. This means that this condition cannot hold for all stages $s \geq t$ and that we have a contradiction.

Hence we have that $A(w) \neq \Theta^D(w)$, and the requirement Q is satisfied. □

Chapter 3

High Permitting of Lachlan Non-Splitting Pairs

In this Chapter we shall use the high permitting method of Shore and Slaman to modify the construction of the Lachlan Non-Spliting Theorem from Chapter 2. This will allow us to prove that a Lachlan Non-Splitting Pair can be found strictly below every $high$ c.e. degree.

We start by describing the concept of high permitting in Section 3.1, before applying the high permitting of Shore and Slaman to the construction of the Lachlan Non-Splitting Theorem in the subsequent sections.

3.1 The High Permitting Method

Suppose that a construction is building some c.e. set X so as to satisfy some requirements, and that one of these requirements involves showing that $X \leq_T A$ for some given c.e. set A. This requirement can be satisfied by making use of some *permitting method.* If the method succeeds, the set A is said to be permitting X [Soare, 1987].

Similarly, a *high permitting* method is employed to build some c.e. set X so as to satisfy a requirement $X \leq_T H$ for some given $high$ c.e. set H. Such a method works by making use of the highness of the set H. Since this will be the only property used by the method, it follows that the set X can be built to be below or equal to any $high$ c.e. set.

Now suppose that the construction is building the c.e. set X so as to show the existence of some c.e. degree $\mathbf{x} = deg(X)$ satisfying certain requirements. Since for every $high$ c.e. degree h there is some $high$ c.e. set H such that $\mathbf{h} = deg(H)$, it follows that every $high$ c.e. degree $\mathbf{h}$ bounds a c.e. degree $\mathbf{x}$ satisfying these requirements.

We shall now review the two main high permitting methods. In Section 3.1.1 we describe a high permitting theorem based on a theorem of Martin, while in Section 3.1.2 we describe the high permitting method discovered by Shore and Slaman.

3.1.1 Martin High Permitting

The first high permitting method to be considered rests on the characterisation of $high$ c.e. sets given by a theorem of Martin in [Martin, 1966]. Before stating this theorem, we shall need to make a few definitions. We shall say that a total function f *dominates* a total function g if $(\exists y)(\forall x > y)[f(x) > g(x)]$. We shall also say that f is *dominant* if it dominates every total computable function. Then Martin's theorem says the following.

Theorem 3.1.1. (High Domination Theorem). *A c.e. set H is $high$ if and only if there is a dominant function $f \leq_T H$.*

We shall now describe how to build $X \leq_T H$ by making use of the dominant function f witnessing the highness of the set H (although the actual implementation will vary from construction to construction).

Since $f \leq_T H$, there must exist some total functional Ψ^H such that $f = \Psi^H$. In order to show that $X \leq_T H$, one builds a total functional Φ^H such that $X = \Phi^H$ in the following way. If at stage $s+1$ we have that $\Psi^H[s+1](x) \downarrow$ while $\Phi^H[s](x) \uparrow$ for some element x, we define $\Phi^H[s+1](x) = X_{s+1}(x)$, choosing a use $\phi[s+1](x) \geq \psi[s+1](x)$.

In addition we shall require strategies to refrain from enumerating an element x into the set X at stage $s+1$ if $\Phi^H[s+1](x) \downarrow$, so as to avoid creating an inequality between the two. On the other hand it will still be possible for strategies to enumerate x into X at stage $s+1$ if $\Phi^H[s+1](x)\uparrow$.

While the above restriction is sufficient to show that $X \leq_T H$, it creates a problem in that a strategy might be blocked from enumerating elements into the set X. This could in turn stop it from satisfying some other requirement. One way to approach this problem is for the strategy to build a total computable function g which f would then have to dominate. For instance if the strategy was not able to enumerate the element x into X at stage t, it could define $g_t(x)$ to be greater than $\Psi^H[t](x)$. While H may be able to ignore the value of $g(x)$, it will not be able to ignore the value of $g(x')$ for infinitely many elements x', or else f would not dominate g and H would not be $high$. Therefore one of two things have to be the case.

The first is that there is some stage u such that for all stages $u' > u$, if the strategy wishes to enumerate an element x into X at stage u', we have that $\Phi^H[u'](x)\uparrow$. This would mean that H stops blocking the strategy from enumerating witnesses into X after stage u, resulting in g being defined at only finitely many elements and f not having to dominate it.

The second is that there are infinitely many elements x such that H blocks the strategy from enumerating x into X at some stage u because $\Phi^H[u](x)\downarrow$. In this case the strategy can build g to be a total computable function which f has to dominate. Since $f=\Psi^H$, we have that an $H \restriction \psi_u(x)$ change must take place at some stage $u' > u$ for almost every x. This results in $\Psi^H[u'](x)\uparrow$. But since the construction of Φ^H ensures that $\phi[u'](x) > \psi[u'](x)$, we have that $\Phi^H[u'](x)\uparrow$ as well, allowing the strategy to enumerate x into X at stage u'.

3.1.2 Shore and Slaman High Permitting

The second method to be considered is the high permitting method of Shore and Slaman given in [Shore and Slaman, 1993]. We proceed to give a detailed exposition of this method, which will be used in Chapter 3.

Suppose that one starts with an existing construction which builds some c.e. degree $\mathbf{x}$ satisfying certain requirements. Then one can use this method to modify the construction and build this c.e. degree below any given $high$ c.e. degree $\mathbf{h}$, that is such that $\mathbf{x} \leq \mathbf{h}$.

The method is a general one and can be used to modify constructions which make use of Π_2^0 strategies on a priority tree. It also has the advantage of isolating the permitting problem to a question of whether events which take place during the course of the construction can be synchronised with the guesses which will be performed by the modified strategies regarding their occurrence.

It is needless to say that the c.e. degrees in question will be permitted only if the method succeeds. However it can be shown that if the method succeeds, the stronger statement $\mathbf{x} < \mathbf{h}$ can also be proved. This fact shall be made explicit in Section 3.1.2 once the entire method has been described.

We shall now proceed to describe the concepts required for the use of this method as well as the steps of the method itself.

Π_2^0 Strategies

A Π_2^0 strategy γ is a finite program with some (possibly infinite) set of outcomes Λ and a total order $<_\Lambda$ on this set. Π_2^0 strategies are characterised by the fact that they will either reach a limit outcome, or else will reach some leftmost outcome infinitely often. This outcome will be the true outcome of a Π_2^0 strategy.

Whether a given outcome of a Π_2^0 strategy is the true outcome can also be characterised in terms of certain events taking place or failing to take place infinitely often during the course of the construction. Given a Π_2^0 strategy, it shall be possible to find a sequence of Π_2^0 sentences related to the occurrence of these events, such that each sentence is stronger than the one preceding it, and such that each consistent sequence of truth values is tied to a certain outcome being the true outcome. This also means that for each outcome of a Π_2^0 strategy there will be a sentence consisting of a conjunction of finitely many Π_2^0 and Σ_2^0 sentences which specifies which conditions must hold for this outcome to be the true outcome.

We shall refer to each Π_2^0 sentence as a Π_2^0 question, whose answer will be ‘Yes’ if the truth value of the Π_2^0 sentence is true and ‘No’ if the truth value of the Π_2^0 sentence is false.

Limit Computability of Π^0_2 Questions in H

We shall now consider how the truth value of a Π^0_2 question can be determined in terms of a limit computation in H. This can be achieved through the following two observations.

The first observation is that since H is a high c.e. set we have that $\emptyset''$ is limit computable in H. This means that there exists a total functional Ψ^H witnessing the highness of H such that:

$$x \in \emptyset'' \Leftrightarrow \lim_{t\to\infty} \Psi^H(x,t) = 1$$

and

$$x \notin \emptyset'' \Leftrightarrow \lim_{t\to\infty} \Psi^H(x,t) = 0.$$

The second observation is that there is a uniform reduction of Π^0_2 sentences into $\emptyset''$. This means that there is some computable function f such that if Q is a Π^0_2 sentence we have that:

$$Q \Leftrightarrow (f(Q)) \notin \emptyset'')$$

By combining these two observations we have that the truth value of any Π^0_2 sentence Q is limit computable in H:

$$Q \Leftrightarrow \lim_{t\to\infty} \Psi^H(f(Q),t) = 0$$

From this it follows that the Π^0_2 question corresponding to a given Π^0_2 sentence will have a 'Yes' answer if and only if $\lim_{t\to\infty} \Psi^H(f(Q),t) = 0$ and a 'No' answer if and only if $\lim_{t\to\infty} \Psi^H(f(Q),t) = 1$.

Apparent Limit and Apparent Use

A computable construction is not able to compute the value of an expression such as $\lim_{t\to\infty} \Psi^H(f(Q),t)$ in order to determine the answer to a Π^2_0 question Q. This will instead have to be approximated at every stage s.

We shall suppress the notation $f(Q)$ for the moment and discuss how the value of an expression $\lim_{t\to\infty} \Psi^H(x,t)$ can be approximated for any x at each stage s through the concepts of an *apparent limit* and an *apparent use*. Before proceeding, we shall need the following preliminary definition of the *hat functional* $\hat{\Psi}^H(x,t)$. This functional will be computed from $\Psi^H[s](x,t)$ by any strategy γ wishing to approximate the value of $\lim_{t\to\infty} \Psi^H(x,t)$ at stage s.

Definition 3.1.2. *(Hat Functional). Let γ be a strategy. Given a stage s, let $s' < s$ be the greatest stage such that γ has been accessible at s'. If s' does not exist, we have that $\hat{\Psi}^H[s](x,t) \uparrow$ for every x and t. Otherwise $\hat{\Psi}^H[s](x,t)$ is defined as follows for every x and t:*

$$\hat{\Psi}^H[s](x,t) = \begin{cases} \Psi^H[s](x,t) \text{ if } (\forall t' \leq t)[\Psi^H[s](x,t') \downarrow = \Psi^H[s'](x,t')]; \\ \text{undefined otherwise}. \end{cases}$$

If $\hat{\Psi}^H[s](x,t) \downarrow$, we have that $\hat{\psi}[s](x,t) = \psi[s](x,t)$.

An important property of the hat functional is that if $H_s \upharpoonright \psi[s](x,t) = H \upharpoonright \psi[s](x,t)$ and $\hat{\Psi}^H[s](x,t) \downarrow$, we have that $\hat{\Psi}^H[s](x,t) = \Psi^H(x,t)$.

Given the above definition, a strategy can approximate $\lim_{t\to\infty} \Psi^H(x,t)$ at stage s by computing the finite sequence of values $\hat{\Psi}^H[s](x,t)$ for every $t < s$. From this sequence it will be able to extract the apparent limit and the apparent use for $\lim_{t\to\infty} \Psi^H(x,t)$ at stage s.

In order to define the notions of apperant limit and apparent use, we shall first need to define the number $tmax_s$ for every stage s. This corresponds to the largest value of t for which $\hat{\Psi}^H[s](x,t)$ is defined .

Definition 3.1.3. *($tmax_s$). The number $tmax_s < s$ be the greatest natural number such that $\hat{\Psi}^H[s](x, tmax_s) \downarrow$*

The apparent limit will then be the value of $\hat{\Psi}^H[s](x,t)$ at the largest t such that the functional is defined.

Definition 3.1.4. *(Apparent Limit). The apparent limit at stage s is the value of $\hat{\Psi}^H[s](x, tmax_s)$.*

On the other hand the apparent use will be the length of the initial segment of H_s which was necessary to establish the apparent limit. Suppose that at stage s there is some u such that for all $u' > u$ we either have that $\hat{\Psi}^H[s](x,u') \uparrow$ or that $\hat{\Psi}^H[s](x,u') = \hat{\Psi}^H[s](x,u)$. Then the apparent limit must have already been achieved at $\hat{\Psi}^H[s](x,u)$. Therefore as our apparent use we shall choose the maximum of all the uses required by computations up to and including $\hat{\Psi}^H[s](x,u)$, since these were the computations necessary to achieve the apparent limit.

Definition 3.1.5. *(Apparent Use). Let $u \leq tmax_s$ be the least number such that $\hat{\Psi}^H[s](x,v)$ is constant for all $v \in [u, tmax_s]$. The apparent use at stage s is the maximum of the uses of $\hat{\Psi}^H[s](x,t)$ for every $t \leq u$.*

Modification of the Π^0_2 Strategies

We have already seen that if H is $high$ and Q is a Π^0_2 question, we have that its answer is determined by the value of $\lim_{t\to\infty} \Psi^H(f(Q),t)$.

Now a Π^0_2 strategy as found in the original construction chooses its outcome at stage s depending on the events which have occurred up to stage s. In addition we have also said that the true outcome of such a Π^0_2 strategy can be determined by the answers to a particular sequence of Π^0_2 questions.

The concepts of apparent limit and apparent use can therefore be used to approximate what the answer to each such question is at stage s. This allows the Π^0_2 strategy to be modified so that the outcome selected by the strategy at stage s is determined by the approximations to the answers of these questions.

Each Π^0_2 strategy γ will be modified into a modified strategy γ^* as follows.

Let $Q_1, \ldots, Q_n$ be the sequence of Π^0_2 questions whose answers determine the true outcome of the strategy. At any stage s, one can calculate the apparent limit o_i and apparent use σ_i for $\lim_{t\to\infty} \Psi^H(f(Q_i),t)$ for every $1 \leq i \leq n$. From the sequence of apparent limits $\langle o_1, \ldots, o_n\rangle$ it is then possible to construct the corresponding sequence of 'Yes' or 'No' answers, which we shall denote by o. We shall also determine the apparent use of greatest length, which we shall denote by σ.

The set of outcomes Λ of the modified strategy will then correspond to all the tuples of the form $\langle o, \sigma\rangle$ described above. Each such edge will be accessed by the strategy for the first time at some stage s.

Note that the modified set of outcomes is thus infinite. At any given stage t, the modified strategy will also use the procedure described above in order to choose the outcome $\langle o, \sigma \rangle$ to visit at that stage.

We shall now describe the ordering between the outcomes of the modified strategy.

To achieve this we shall need to define a finite function $f_{x,\langle o,\sigma \rangle}(t)$ for every natural number x and outcome of the strategy $\langle o, \sigma \rangle$. This function is defined over the set $\{0, \dots, |\sigma|\}$, but does not need to be total. It will give us the value of $\hat{\Psi}^{\sigma}_{s}(x, t)$, as long as this is defined for all $t' \leq t$, where s is the least stage at which the strategy visits the outcome $\langle o, \sigma \rangle$.

$$f_{x,\langle o,\sigma \rangle}(t) = \begin{cases} y \text{ if } (\forall t' \leq t)[\, \hat{\Psi}^{\sigma}_{s}(x, t') \downarrow] \ \wedge \ \hat{\Psi}^{\sigma}_{s}(x, t) = y; \\ \text{undefined otherwise.} \end{cases}$$

By considering the largest element in the domain of $f_{x,\langle o,\sigma \rangle}(t)$, we obtain the largest value of t for which the value of $\hat{\Psi}^{\sigma}_{s}(x, t)$ exists.

Now let $\langle o, \sigma \rangle$ and $\langle o', \sigma' \rangle$ be outcomes of the modified strategy. We shall say that $\langle o', \sigma' \rangle <_{\Lambda} \langle o, \sigma \rangle$ if one of the following conditions holds.

(1) There exists some least x such that $\sigma'(y) = \sigma(y)$ for all $y < x$, and $\sigma'(x) = 1$ and $\sigma(x) = 0$.

(2) $\sigma' \subset \sigma$ and there exists some greatest $b \in dom(f_{x,\langle o' \sigma' \rangle})$ and some $t^* \in dom(f_{x,\langle o,\sigma \rangle})$ such that $t^* > b$ and $f_{x,\langle o',\sigma' \rangle}(b) \neq f_{x,\langle o,\sigma \rangle}(t^*)$.

We now make a number of observations.

Firstly, for the modified strategy to visit outcome $\langle o, \sigma \rangle$ at stage s, it must be the case that $\sigma \subset H_s$.

Secondly, suppose that the modified strategy visits outcome $\langle o, \sigma \rangle$ during some stage s, and that it visits an outcome $\langle o', \sigma' \rangle$ to its left when it is accessible again at some least stage $s' > s$. Then one of the following two things must have taken place.

(a) Suppose that there is some x such that σ' and σ agree for all $y < x$, and $\sigma'(x) = 1$ and $\sigma(x) = 0$. Then an $H \upharpoonright |\sigma|$ change must have occurred at some stage u such that $s' < u \leq s$.

(b) Suppose that $\sigma' \subset \sigma$. Then the apparent use σ' of $\lim_{t\to\infty} \hat{\Psi}^H(x,t)$ at stage s' is shorter in length than the apparent use σ' of $\lim_{t\to\infty} \hat{\Psi}^H(x,t)$ at stage s. This means that every computation $\hat{\Psi}^H[s](x,i)$ with $i \leq tmax_s$ and use $\hat{\psi}_s(x,i) > |\sigma|$ has become undefined at stage s'. Let $v = \min\{\hat{\psi}_s(x,i) \mid i \leq tmax_s \ \wedge \ \hat{\psi}_s(x,i) > |\sigma|\}$. Then it must be the case that an H change has occured at some element y with $v \leq y \leq |\sigma|$.

Thirdly, suppose that the modified strategy visits the outcome $\langle o', \sigma' \rangle$ during some stage s' and that it visits an outcome $\langle o, \sigma \rangle$ to its right when it is accessible again at some least stage $s > s'$. Then by case (2) we must have that $\hat{\Psi}(x,t)$ experiences a 'mind-change' when the computation is given greater resources in the form of a longer oracle σ and of a longer span of time in the form of s stages.

By mind-change we mean the following. Suppose that b is the largest value of t such that the value of $\hat{\Psi}(x,t)$ can be computed in s' stages and queries up to σ'. Then some $t^* > b$ is the largest value of t such that the value of $\hat{\Psi}(x,t)$ can be computed in s stages and queries up to σ, and in addition $\hat{\Psi}[s](x,t^*)$ does not agree with the value of $\hat{\Psi}[s'](x,b)$.

Fourthly, we have that $<_\Lambda$ is a strict total order. We prove that this is the case by showing that $<_\Lambda$ is irreflexive, anti-symmettrical, transitive and obeys the trichotomy law.

(1) We show that $<_\Lambda$ is irreflexive, that is that for any edge $\langle o, \sigma \rangle$ of the strategy γ, we have that $\langle o, \sigma \rangle \not<_\lambda \langle o, \sigma \rangle$.

    Suppose for contradiction that this is not the case. Then we have that $\langle o, \sigma \rangle <_\lambda \langle o, \sigma \rangle$. If this holds by case (1) of the ordering, we would have that there is some least element x such that $\sigma(x) = 0$ and $\sigma(x) = 1$, which gives a contradiction since $\sigma = \sigma$. On the other hand if this holds by case (2) of the ordering, we have that $\sigma \subset \sigma$, which also gives a contradiction, since $\sigma = \sigma$. Hence we have that $<_\Lambda$ is irreflexive, as required.

(2) We show that $<_\Lambda$ is anti-symmetric, that is that for any two edges $\langle o, \sigma \rangle$ and $\langle o', \sigma' \rangle$ of the strategy γ, we have that $\langle o, \sigma \rangle <_\lambda \langle o', \sigma' \rangle \Rightarrow \langle o', \sigma' \rangle \not<_\lambda \langle o, \sigma \rangle$.

    Suppose that $\langle o, \sigma \rangle <_\lambda \langle o', \sigma' \rangle$.

    If $\langle o, \sigma \rangle <_\lambda \langle o', \sigma' \rangle$ through case (1) of the ordering, we have that there is some least x such that σ and σ' agree for all $y < x$, and that $\sigma(x) = 1$ and $\sigma'(x) = 0$ (and hence that $\sigma|\sigma'$).

Now suppose for contradiction that $\langle o', \sigma' \rangle <_\lambda \langle o, \sigma \rangle$. If this was the case through case (1) of the ordering, we would have that there is some least x' such that σ and σ' agree for all $y < x'$, and that $\sigma'(x') = 1$ and $\sigma(x') = 0$. But this contradicts the fact that there is some least x such that σ and σ' agree for all $y < x$, and that $\sigma(x) = 1$ and $\sigma'(x) = 0$. Otherwise if this was the case through case (2) of the ordering, we have that $\sigma' \subset \sigma$. But this contradicts the fact that $\sigma \mid \sigma'$.

On the other hand it could be the case that $\langle o, \sigma \rangle <_\lambda \langle o', \sigma' \rangle$ through case (2) of the ordering. In particular, this would mean that $\sigma \subset \sigma'$.

Now suppose for contradiction that $\langle o', \sigma' \rangle <_\lambda \langle o, \sigma \rangle$. If this was the case through case (1) of the ordering, we would have that there is some least x' such that σ and σ' agree for all $y < x'$, and that $\sigma'(x') = 1$ and $\sigma(x') = 0$. But this would mean that $\sigma \mid \sigma'$, when we have already concluded that $\sigma \subset \sigma'$. Otherwise if this was the case through case (2) of the ordering, we have that $\sigma' \subset \sigma$. But this contradicts the fact that $\sigma \subset \sigma'$.

(3) We show that $<_\Lambda$ is transitive, that is that for any three edges $\langle o, \sigma \rangle$, $\langle o', \sigma' \rangle$ and $\langle o'', \sigma'' \rangle$ of the strategy γ, we have that if $\langle o, \sigma \rangle <_\lambda \langle o', \sigma' \rangle$ and $\langle o', \sigma' \rangle <_\lambda \langle o'', \sigma'' \rangle$, then $\langle o, \sigma \rangle <_\lambda \langle o'', \sigma'' \rangle$.

Suppose that $\langle o, \sigma \rangle <_\lambda \langle o', \sigma' \rangle$ and $\langle o', \sigma' \rangle <_\lambda \langle o'', \sigma'' \rangle$.

(a) Consider the situation where $\langle o, \sigma \rangle <_\lambda \langle o', \sigma' \rangle$ through case (1) of the ordering, and $\langle o', \sigma' \rangle <_\lambda \langle o'', \sigma'' \rangle$ through case (1) of the ordering. Then there is some least x such that σ and σ' agree for all $y < x$, and that $\sigma(x) = 1$ and $\sigma'(x) = 0$. We also have that there is some least x' such that σ' and σ'' agree for all $y' < x'$, and that $\sigma'(x') = 1$ and $\sigma''(x') = 0$. But this means that σ and σ'' agree for all $y < x$ and that $\sigma(x) = 1$ and $\sigma''(x) = 0$. Hence we have that $\langle o, \sigma \rangle <_\lambda \langle o'', \sigma'' \rangle$ through case (1) of the ordering.

(b) Consider the situation where $\langle o, \sigma \rangle <_\lambda \langle o', \sigma' \rangle$ through case (1) of the ordering, and $\langle o', \sigma' \rangle <_\lambda \langle o'', \sigma'' \rangle$ through case (2) of the ordering. Then there is some least x such that σ and σ' agree for all $y < x$, and that $\sigma(x) = 1$ and $\sigma'(x) = 0$. We also have that $\sigma' \subset \sigma''$. But this means that σ and σ'' agree for all $y < x$ and that $\sigma(x) = 1$ and

$\sigma''(x) = 0$. Hence we have that $\langle o, \sigma\rangle <_\lambda \langle o'', \sigma''\rangle$ through case (1) of the ordering.

(c) Consider the situation where $\langle o, \sigma\rangle <_\lambda \langle o', \sigma'\rangle$ through case (2) of the ordering, and $\langle o', \sigma'\rangle <_\lambda \langle o'', \sigma''\rangle$ through case (1) of the ordering. Then we have that σ' and σ'' agree for all $y < x$ and that $\sigma'(x) = 1$ and $\sigma''(x) = 0$. We also have that $\sigma \subset \sigma'$ and that there exists some greatest $b \in dom(f_{x,\langle o,\sigma\rangle})$ and some $t^* \in dom(f_{x,\langle o',\sigma'\rangle})$ such that $t^* > b$ and $f_{x,\langle o,\sigma\rangle}(b) \neq f_{x,\langle o',\sigma'\rangle}(t^*)$.

Now since $\sigma \subset \sigma'$, it could be the case that $|\sigma'| \geq x$. If this is the case, we have that σ and σ'' agree for all $y < x$ and that $\sigma(x) = 1$ and $\sigma''(x) = 0$. Hence we have that $\langle o, \sigma'\rangle <_\lambda \langle o'', \sigma''\rangle$ by case (1) of the ordering.

Otherwise we have that $\sigma \subset \sigma''$.

Since $\langle o, \sigma\rangle$ is an outcome of the strategy, we have that it is first accessed by the strategy at some stage s, when the strategy computes the apparent limit o and the apparent use σ of $\lim_{t\to\infty} \hat{\Psi}^H(x, t)$ at stage s. Since the apparent limit exists at stage s, we must have that there is at least one computation $\hat{\Psi}^H[s](x, 0)\downarrow$, giving that $dom(f_{x,\langle o,\sigma\rangle})$ is not empty.

Similarly, from the fact that $\langle o'', \sigma''\rangle$ is an outcome of the strategy, we have that this outcome is first accessed by the strategy at some stage s'', when the strategy computes the apparent limit o'' and the apparent use σ'' of $\lim_{t\to\infty} \hat{\Psi}^H(x, t)$ at stage s''. Since the apparent limit exists at stage s'', we must have that there is at least one computation $\hat{\Psi}^H[s''](x, 0)\downarrow$, giving that $dom(f_{x,\langle o'',\sigma''\rangle})$ is not empty.

Consider the greatest $b \in dom(f_{x,\langle o,\sigma\rangle})$.

If $s < s''$, we must have that $\hat{\Psi}^\sigma[s](x, i)\downarrow= \hat{\Psi}^{\sigma''}[s''](x, i)$ for every $i \leq b$, due to the fact that $\sigma \subset \sigma''$. On the other hand, if $s'' < s$, we have that $\hat{\Psi}^{\sigma''}[s''](x, i)\downarrow= \hat{\Psi}^\sigma[s](x, i)$ for every $i \leq b$, where $\hat{\psi}(x, i) \leq |\sigma|$ for all $i \leq b$.

Now, if there exists some $t^* > b$ such that $t^* \in dom(f_{x,\langle o'',\sigma''\rangle})$ and $f_{x,\langle o,\sigma\rangle}(b) \neq f_{x,\langle o'',\sigma''\rangle}(t^*)$, then it must be the case that $\langle o, \sigma\rangle <_\Lambda \langle o'', \sigma''\rangle$ by case (2) of the ordering.

On the other hand, if there does not exist some $t^* > b$ such that $t^* \in dom(f_{x,\langle o'',\sigma''\rangle})$ and $f_{x,\langle o,\sigma\rangle}(b) \neq f_{x,\langle o'',\sigma''\rangle}(t^*)$, then it must be the case that the apparent limit and apparent use of $\lim_{t\to\infty} \hat{\Psi}^H(x,t)$ at stage s'' is equal to the apparent limit and apparent use of $\lim_{t\to\infty} \hat{\Psi}^H(x,t)$ at stage s. In this case, we have that $\langle o, \sigma\rangle = \langle o'', \sigma''\rangle$. This means that the antecedent of the implication $\langle o,\sigma\rangle <_\lambda \langle o',\sigma'\rangle \wedge \langle o',\sigma'\rangle <_\lambda \langle o'',\sigma''\rangle \Rightarrow \langle o,\sigma\rangle <_\lambda \langle o'',\sigma''\rangle$ is of the form $\langle o,\sigma\rangle <_\lambda \langle o',\sigma'\rangle \wedge \langle o',\sigma'\rangle <_\lambda \langle o,\sigma\rangle$, which is false by antisymmetry, leading to the trivial satisfaction of this case.

(d) Consider the situation where $\langle o,\sigma\rangle <_\lambda \langle o',\sigma'\rangle$ through case (2) of the ordering, and $\langle o',\sigma'\rangle <_\lambda \langle o'',\sigma''\rangle$ through case (2) of the ordering. Then we have that $\sigma \subset \sigma'$ and that there exists some greatest $b \in dom(f_{x,\langle o,\sigma\rangle})$ and some $t^* \in dom(f_{x,\langle o',\sigma'\rangle})$ such that $t^* > b$ and $f_{x,\langle o,\sigma\rangle}(b) \neq f_{x,\langle o',\sigma'\rangle}(t^*)$. We also have that $\sigma' \subset \sigma''$ and that there exists some greatest $b' \in dom(f_{x,\langle o',\sigma'\rangle})$ and some $t^*{*} \in dom(f_{x,\langle o'',\sigma''\rangle})$ such that $t^{**} > b'$ and $f_{x,\langle o',\sigma'\rangle}(b') \neq f_{x,\langle o'',\sigma''\rangle}(t^{**})$.

Since $\langle o,\sigma\rangle$ is an outcome of the strategy, we have that it is first accessed by the strategy at some stage s, when the strategy computes the apparent limit o and the apparent use σ of $\lim_{t\to\infty} \hat{\Psi}^H(x,t)$ at stage s. Similarly, we have that $\langle o',\sigma'\rangle$ is first accessed by the strategy at some stage s', when it computes the apparent limit o' and the apparent use σ' of $\lim_{t\to\infty} \hat{\Psi}^H(x,t)$ at stage s'. Finally, we have that $\langle o'',\sigma''\rangle$ is first accessed by the strategy at some stage s'', when it computes the apparent limit o'' and the apparent use σ'' of $\lim_{t\to\infty} \hat{\Psi}^H(x,t)$ at stage s''.

Now, if $s < s'$, we must have that $\hat{\Psi}^\sigma[s](x,i)\downarrow = \hat{\Psi}^{\sigma'}[s'](x,i)$ for every $i \leq b$, due to the fact that $\sigma \subset \sigma'$. On the other hand, if $s' < s$, we have that $\hat{\Psi}^{\sigma'}[s'](x,i)\downarrow = \hat{\Psi}^\sigma[s](x,i)$ for every $i \leq b$, where $\hat{\psi}(x,i) \leq |\sigma|$ for all $i \leq b$.

Similarly, if $s' < s''$, we must have that $\hat{\Psi}^{\sigma'}[s](x,i)\downarrow = \hat{\Psi}^{\sigma''}[s'](x,i)$ for every $i \leq b'$, due to the fact that $\sigma' \subset \sigma''$. On the other hand, if $s'' < s'$, we have that $\hat{\Psi}^{\sigma''}[s''](x,i)\downarrow = \hat{\Psi}^{\sigma'}[s'](x,i)$ for every $i \leq b'$, where $\hat{\psi}(x,i) \leq |\sigma'|$ for all $i \leq b'$.

Now, since b' is the greatest element in $dom(f_{x,\langle o',\sigma'\rangle})$, we must have that $b' \geq t^*$. Hence, we have that $\hat{\Psi}^{\sigma''}[s''](x,t^*) = \hat{\Psi}^{\sigma'}[s'](x,t^*)$.

It follows that $t^* \in dom(f_{x,\langle o'',\sigma''\rangle})$, $t^* > b$ and $f_{x,\langle o,\sigma\rangle}(b) \neq f_{x,\langle o'',\sigma''\rangle}(t^*)$. Hence we have that $\langle o,\sigma\rangle <_L \langle o'',\sigma''\rangle$ by case (2) of the ordering, as required.

(4) We show that $<_\Lambda$ obeys the trichotomy law, that is that for any two edges $\langle o,\sigma\rangle$ and $\langle o',\sigma'\rangle$

of the strategy γ, we have that one of the following is the case; $\langle o, \sigma \rangle <_\lambda \langle o', \sigma' \rangle$, $\langle o', \sigma' \rangle <_\lambda \langle o, \sigma \rangle$ or $\langle o, \sigma \rangle = \langle o', \sigma' \rangle$.

Consider two edges $\langle o, \sigma \rangle$ and $\langle o', \sigma' \rangle$.

Suppose that $\sigma \mid \sigma'$. Then there must be some least x such that σ and σ' agree for all $y < x$, and $\sigma(x) = 1$ and $\sigma'(x) = 0$, or $\sigma'(x) = 1$ and $\sigma(x) = 0$. In the first case, we have that $\langle o, \sigma \rangle <_\Lambda \langle o', \sigma' \rangle$ by case (1) of the ordering, while in the second case we have that $\langle o', \sigma' \rangle <_\Lambda \langle o, \sigma \rangle$ by case (1) of the ordering.

On the other hand, it could be the case that $\sigma \subset \sigma'$ or that $\sigma' \subset \sigma$. Without loss of generality, suppose that $\sigma \subset \sigma'$.

Since $\langle o, \sigma \rangle$ is an outcome of the strategy, we have that it is first accessed by the strategy at some stage s, when the strategy computes the apparent limit o and the apparent use σ of $\lim_{t \to \infty} \hat{\Psi}^H(x, t)$ at stage s. Since the apparent limit exists at stage s, we must have that there is at least one computation $\hat{\Psi}^H[s](x, 0) \downarrow$, giving that $dom(f_{x,\langle o, \sigma \rangle})$ is not empty.

Similarly, from the fact that $\langle o', \sigma' \rangle$ is an outcome of the strategy, we have that this outcome is first accessed by the strategy at some stage s', when the strategy computes the apparent limit o and the apparent use σ of $\lim_{t \to \infty} \hat{\Psi}^H(x, t)$ at stage s'. Since the apparent limit exists at stage s', we must have that there is at least one computation $\hat{\Psi}^H[s'](x, 0) \downarrow$, giving that $dom(f_{x,\langle o', \sigma' \rangle})$ is not empty.

Consider the greatest $b \in dom(f_{x,\langle o, \sigma \rangle})$.

If $s < s'$, we must have that $\Phi^\sigma[s](x, i) \downarrow = \Phi^{\sigma'}[s'](x, i)$ for every $i \leq b$, due to the fact that $\sigma \subset \sigma'$. On the other hand, if $s' < s$, we have that $\hat{\Phi}^\sigma[s'](x, i) \downarrow = \hat{\Phi}^{\sigma'}[s](x, i)$ for every $i \leq b$, where $\hat{\phi}(x, i) \leq |\sigma|$ for all $i \leq b$.

Now, if there exists some $t^* > b$ such that $t^* \in dom(f_{x,\langle o', \sigma' \rangle})$ and $f_{x,\langle o, \sigma \rangle}(b) \neq f_{x,\langle o', \sigma' \rangle}(t^*)$, then it must be the case that $\langle o, \sigma \rangle <_\Lambda \langle o', \sigma' \rangle$ by case (2) of the ordering.

On the other hand, if there does not exist some $t^* > b$ such that $t^* \in dom(f_{x,\langle o', \sigma' \rangle})$ and $f_{x,\langle o, \sigma \rangle}(b) \neq f_{x,\langle o', \sigma' \rangle}(t^*)$, then it must be the case that the apparent limit and apparent use of $\lim_{t \to \infty} \hat{\Psi}^H(x, t)$ at stage s' is equal to the apparent limit and apparent use of $\lim_{t \to \infty} \hat{\Psi}^H(x, t)$ at stage s. In this case we have that $\langle o, \sigma \rangle = \langle o', \sigma' \rangle$.

Behaviour of the Modified Π^0_2 Strategies

Recall that the behaviour of a modified strategy depends on some finite number of Π^0_2 questions $Q_1, \ldots, Q_n$. The answer to each of the questions Q_i for every $1 \leq i \leq n$ is approximated at each stage s by computing an apparent limit o_i and an apparent use σ_i for $\lim_{t\to\infty} \Psi^H(f(Q_i), t)$ at stage s. The strategy then calculates the corresponding sequence of answers o and the apparent use of greatest length σ so as to choose its outcome $\langle o, \sigma \rangle$ at stage s.

In this section we shall show that this results in the modified strategy choosing some leftmost outcome infinitely often and that this outcome corresponds to the answers of the questions $Q_1, \ldots, Q_n$.

We start by considering the situation where the strategy depends only on one Π^0_2 question Q_1. In this case the answer to the question is determined by the value of $\lim_{t\to\infty} \Psi^H(f(Q_1), t)$.

In the following lemma we show that the existence of $\lim_{t\to\infty} \Psi^H(f(Q_1), t)$ guarantees that the apparent limit o_1 reaches the value $\lim_{t\to\infty} \Psi^H(f(Q_1), t)$ infinitely often, and that in each case the apparent use σ_1 returns to the same value σ^*. This means that the strategy visits the outcome $\langle o^*, \sigma^* \rangle$ infinitely often, where o^* is the answer of Q_1.

In addition to the above we show that there is some stage after which the apparent use σ_1 can only be an extension of σ^*. This means that there is a stage after which no outcome to the left of $\langle o^*, \sigma^* \rangle$ is visited by the strategy.

In the following lemma we shall suppress the notation $f(Q)$, denoting it by x for readability.

Lemma 3.1.6. *Suppose that $\lim_{t\to\infty} \Psi^H(x, t)$ exists. Then there is some least number u and some stage q such that the following conditions hold for infinitely many stages s.*

(1) The apparent limit at stage s is equal to $\hat{\Psi}^H[q](x, u)$ and the apparent use at stage s is equal to $\max\{\hat{\psi}_q(x, w) \mid w \leq u\}$.

(2) $\hat{\Psi}^H[q](x, u) = \lim_{t\to\infty} \Psi^H(x, t)$.

(3) $H_q \upharpoonright \max\{\hat{\psi}_q(x, w) \mid w \leq u\} = H \upharpoonright \max\{\hat{\psi}_q(x, w) \mid w \leq u\}$.

Proof. In order for $\lim_{t\to\infty} \Psi^H(x,t)$ to exist, there must be some least u such that for every $v \geq u$ we have that $\Psi^H(x,u) = \Psi^H(x,v)$. Let q be the least stage such that for every $w \leq u$ we have that $\Psi^H[q](x,w) \downarrow$, and such that $H_q \restriction \max\{\psi_q(x,w) \mid w \leq u\} \subset H$. The existence of this stage follows from the totality of Ψ^H. Then it is the case that for every $s \geq q$, $\Psi^H[s](x,u) = \Psi^H[q](x,u) = \lim_{t\to\infty} \Psi^H(x,t)$, and that for every $w \leq u$, $\psi_s(x,w) = \psi_q(x,w)$.

Now consider any stage $s \geq q$. Then one of the following cases must hold.

(a) For every v such that $u < v \leq tmax_s$, the computation $\hat{\Psi}^H[s](x,v)$ is equal to $\hat{\Psi}^H[q](x,u)$. Then we have that the apparent limit is $\hat{\Psi}^H[q](x,u)$ and that the apparent use is $\max\{\hat{\psi}_q(x,w) \mid w \leq u\}$, as required.

(b) There is some least v such that $u < v \leq tmax_s$ and such that the computation $\hat{\Psi}^H[s](x,v)$ is not equal to $\hat{\Psi}^H[q](x,u)$. Then it must be the case that $H_s \restriction \hat{\psi}_s(x,v) \not\subset H$. For otherwise u is not the least number such that $\Psi^H(x,u) = \Psi^H(x,v)$ for every $v \geq u$.

Hence if (b) is the case, there must be some stage $s' > s$ such that $H_{s'}(y) \neq H_s(y)$ for some y such that $\hat{\psi}_s(x,u) < y \leq \hat{\psi}_s(x,v)$. This means that at stage s' we have that u is the least number such that $\hat{\Psi}^H[s'](x,u)$ has value equal to $\lim_{t\to\infty} \Psi^H(x,t)$, and such that $\hat{\Psi}^H[s'](x,v')$ is constant for all $v' \in [u, tmax_{s'}]$.

It follows that the apparent limit is $\hat{\Psi}^H[s'](x,u) = \hat{\Psi}^H[q](x,u)$, and that the apparent use is $\max\{\hat{\psi}_q(x,w) \mid w \leq u\}$, as required for condition (1). In addition since $\hat{\Psi}^H[q](x,u) = \lim_{t\to\infty} \Psi^H(x,t)$, we have that $\hat{\Psi}^H[s'](x,u) = \lim_{t\to\infty} \Psi^H(x,t)$ as required for condition (2).

The above argument can be repeated to obtain infinitely many stages such that the apparent limit for $\lim_{t\to\infty} \Psi^H(x,t)$ is equal to $\hat{\Psi}^H[q](x,u)$ and the apparent use for $\lim_{t\to\infty} \Psi^H(x,t)$ is equal to $\max\{\hat{\psi}_q(x,w) \mid w \leq u\}$. Thus conditions (1) and (2) of the lemma hold as required.

In addition we have already determined that $H_q \restriction \max\{\hat{\psi}_q(x,w) \mid w \leq u\} \subset H$. This means that we have that $H_q \restriction \max\{\hat{\psi}_q(x,w) \mid w \leq u\} = H \restriction \max\{\hat{\psi}_q(x,w) \mid w \leq u\}$ and that condition (3) is met as well. □

We now proceed to consider the situation where the strategy depends on some finite number of Π^0_2 questions $Q_1, \ldots, Q_n$. In this case the truth value of each question Q_i is determined by the value of $\lim_{t\to\infty} \Psi^H(f(Q_i), t)$ for every $1 \leq i \leq n$

By Lemma 3.1.6, we have that for each of the limits $\lim_{t\to\infty} \Psi^H(f(Q_i), t)$ there are infinitely many stages s such that the apparent limit o_i attains the value $\lim_{t\to\infty} \Psi^H(f(Q_i), t)$ and such that the apparent use σ_i returns to some value σ_i^*.

We shall now prove the *Collation Lemma* by showing that there are infinitely many stages s such that all of the apparent limits o_i attain the value of their corresponding limit $\lim_{t\to\infty} \Psi^H(f(Q_i), t)$ simultaneously. In addition the maximum of all the apparent uses σ shall return to some value σ^* at these stages. This means that the strategy visits some outcome $\langle o^*, \sigma^* \rangle$ infinitely often, where o^* is the sequence of answers to the sequence of questions $Q_1, \ldots, Q_n$.

Furthermore we shall also show that there is some stage after which the maximum of the apparent uses can only be an extension of σ^*. Hence we have that there is a stage after which no outcome to the left of $\langle o^*, \sigma^* \rangle$ is visited by the strategy.

In the following lemma we shall suppress the notation $f(Q_i)$, denoting it by x_i for readability.

Lemma 3.1.7. *(Collation Lemma). Suppose that* $\lim_{t\to\infty} \Psi^H(x_i, t)$ *exists for every* $1 \leq i \leq n$*. Then there are infinitely many stages* s*, as well as a least number* u_i *and a stage* q_i *for all* $1 \leq i \leq n$*, such that the following conditions hold.*

(1) The apparent limit for $\lim_{t\to\infty} \Psi^H(x_i, t)$ *at stage* s *is equal to* $\hat{\Psi}^H[q_i](x_i, u_i)$ *and the apparent use for* $\lim_{t\to\infty} \Psi^H(x_i, t)$ *at stage* s *is equal to* $\max\{\hat{\psi}_{q_i}(x_i, w) \mid w \leq u_i\}$ *for every* $1 \leq i \leq n$*.*

(2) $\hat{\Psi}^H[q_i](x_i, u_i) = \lim_{t\to\infty} \Psi^H(x_i, t)$ *for every* $1 \leq i \leq n$*.*

(3) Let $z_i = \max\{\hat{\psi}_{q_i}(x_i, w) \mid w \leq u_i\}$ *for every* $1 \leq i \leq n$*, and let* $z = \max\{z_i \mid 1 \leq i \leq n\}$*. Then there is a stage* q' *such that* $H_{q'} \upharpoonright z = H \upharpoonright z$*.*

Proof. By Lemma 3.1.6 we have that for every $\lim_{t\to\infty} \Psi^H(x_i, t)$ for every $1 \leq i \leq n$, there exists some least number u_i and stage q_i such that there are infinitely many stages such that the apparent limit for $\lim_{t\to\infty} \Psi^H(x_i, t)$ is equal to $\hat{\Psi}^H[q_i](x_i, u_i)$ and the apparent use for $\lim_{t\to\infty} \Psi^H(x_i, t)$ is equal to $\max\{\hat{\psi}_{q_i}(x_i, w) \mid w \leq u_i\}$.

Let $q' = \max\{q_1, \ldots, q_n\}$. Consider some stage $s > q'$, and let $tmax_{i,s} < s$ be the greatest number such that $\hat{\Psi}^H[s](x_i, tmax_{i,s}) \downarrow$, for every $1 \leq i \leq n$.

Then for every $\lim_{t \to \infty} \Psi^H(x_i, t)$ such that $1 \leq i \leq n$ we have that case (a) or case (b) below must hold for the corresponding apparent limit and apparent use at stage s.

(a) For every v_i such that $u_i < v_i \leq tmax_{i,s}$, the computation $\hat{\Psi}^H[s](x_i, v_i)$ is equal to $\hat{\Psi}^H[q_i](x_i, u_i)$. Then we have that s is a stage such that the apparent limit for $\lim_{t \to \infty} \Psi^H(x_i, t)$ is equal to $\hat{\Psi}^H[q_i](x_i, u_i)$ and the apparent use for $\lim_{t \to \infty} \Psi^H(x_i, t)$ is equal to $\max\{\hat{\psi}_{q_i}(x_i, w) \mid w \leq u_i\}$ for every $1 \leq i \leq n$.

(b) There is some least v_i such that $u_i < v_i \leq tmax_{i,s}$ and such that the computation $\hat{\Psi}^H[s](x_i, v_i)$ is not equal to $\hat{\Psi}^H[q_i](x_i, u_i)$. Then there must be some stage $s' > s$ such that $H_s \upharpoonright \hat{\psi}_s(q_i, v_i) \neq H_{s'} \upharpoonright \hat{\psi}_s(q_i, v_i)$. For if this were not the case, u_i would not be the least number such that $\hat{\Psi}^H(x, u_i) = \hat{\Psi}^H(x, v)$ for every $v \geq u_i$.

If case (a) holds for every $\lim_{t \to \infty} \Psi^H(x_i, t)$ such that $1 \leq i \leq n$, then we have that stage s satisfies conditions (1) and (2).

Otherwise case (b) must hold for some $\lim_{t \to \infty} \Psi^H(x_i, t)$ such that $1 \leq i \leq n$.

Now when the $H \upharpoonright \hat{\psi}_s(q_i, v_i)$ change occurs at stage s', we either have that case (a) holds for every $\lim_{t \to \infty} \Psi^H(x_i, t)$ such that $1 \leq i \leq n$, in which case stage s' satisfies conditions (1) and (2) or else that case (b) holds once again.

In the latter case there is some $\lim_{t \to \infty} \Psi^H(x_j, t)$ for $1 \leq j \leq n$ and $j \neq i$ for which there is some least $v_j > u_j$ such that $\hat{\Psi}^H[s'](x_j, v_j) \neq \hat{\Psi}^H[q_j](x_j, u_j)$. But in order to have $\hat{\Psi}^H[s'](x_j, v_j) \downarrow$ it follows that $\hat{\psi}_{s'}(x_j, v_j) < \hat{\psi}_s(x_i, v_i)$.

Hence the repeated occurrence of case (2) constructs a strictly increasing sequence of stages $s_m > s$ with a corresponding strictly decreasing sequence of uses u_m such that there must be an $H \upharpoonright u_m$ change at stage s_m.

But by Lemma 3.1.6 we have that q_1 is a stage such that $\hat{\Psi}^H[q_1](x_1, u_1) \downarrow$ and such that $\hat{\psi}_{q_1}(x_1, u_1) \subset H$. This means that case (2) cannot cause an $H \upharpoonright \hat{\psi}_{q_1}(x_1, u_1)$ change. Hence the sequence of stages s_m corresponding to the decreasing sequence of uses u_m constructed by the occurrence of case (2) must be finite. Let s^* be the greatest stage in the sequence of stages constructed by the occurence of case (2). Then case (1) holds at stage s^* and conditions (1) and

(2) are satisfied at stage s^*.

The above argument can be repeated to obtain infinitely many stages such that the apparent limit for $\lim_{t\to\infty} \Psi^H(x_i, t)$ is equal to $\hat{\Psi}^H[q_i](x_i, u_i)$ and the apparent use for $\lim_{t\to\infty} \Psi^H(x_i, t)$ is equal to $\max\{\hat{\psi}_{q_i}(x_i, w) \mid w \leq u_i\}$ for every $1 \leq i \leq n$. Thus conditions (1) and (2) of the lemma hold as required.

In addition by Lemma 3.1.6 we have that $H_{q_i} \restriction \max\{\hat{\psi}_{q_i}(x, w) \mid w \leq u_i\} = H \restriction \max\{\hat{\psi}_{q_i}(x, w) \mid w \leq u_i\}$ for every $1 \leq i \leq n$. Let $z_i = \max\{\hat{\psi}_{q_i}(x, w) \mid w \leq u_i\}$ for every $1 \leq i \leq n$, and let $z = \max\{z_i \mid 1 \leq i \leq n\}$. Since $q' > q_i$ for every $1 \leq i \leq n$ we have that $H_{q'} \restriction z = H \restriction z$ as required, meaning that condition (3) is satisfied as well. $\square$

Synchronisation of Guesses and Events

The original strategy takes action and visits an outcome depending on the events which have taken place at stage s, or at some previous stage. On the other hand, the modified strategy will guess which outcome to visit at each stage as described in the previous section. This means that the outcome visited by the modified strategy at stage s bears no relation to the actual events which have taken place at stage s, or at some previous stage.

However in order for the construction to work properly, the modified strategies should only take the action associated to visiting some outcome if the underlying event on which the action depends has happened at stage s. In practice this cannot be guaranteed, so there has to be a way of preserving the event until the modified strategy visits an outcome which is able to make use of such an event. This is performed by using an *attachment procedure*.

During every stage s, the strategy will first execute the attachment procedure, which checks which events have occurred at stage s, and attaches the event to an edge which is able to make use of that event. The modified strategy then proceeds to guess the edge which is to be visited as before. Now, if the modified strategy visits an edge which has an event attached to it, the strategy will take corresponding action, whilst otherwise it will terminate the stage.

The main problem in implementing this form of synchronisation is to make sure that events which have been attached to an edge at stage s remain valid until the edge is actually visited by the strategy at some stage $s' > s$.

Making sure that this holds is non-trivial and might require changes to the Π^0_2 questions on which the modified strategy is basing its guesses. This would then affect the attachment procedure once again.

It is interesting to observe that the inability to preserve certain events is what causes the high permitting method to fail for certain constructions.

Inaccessibility on the Left of the True Path

The ordering between the outcomes allows us to show that there is some set Y of strategies γ^* and edges $\gamma^* \frown \langle o, \sigma \rangle$ on the priority tree which contains every node and edge on the true path as a subset, and for which it is possible to computably enumerate in H stages s such that no node or edge to the left of γ^* or $\gamma^* \frown \langle o, \sigma \rangle$ is accessible after stage s.

Lemma 3.1.8. *There is a set Y of strategies γ^* and edges $\gamma^* \frown \langle o, \sigma \rangle$ such that:*

(1) If γ^* is on the true path, then there is some stage s such that γ^* is enumerated into Y.

(2) If $\gamma^* \frown \langle o, \sigma \rangle$ is on the true path, then there is some stage s such that $\gamma^* \frown \langle o, \sigma \rangle$ is enumerated into Y.

(3) If γ^* is enumerated into Y at stage s, then no node to the left of γ^* is accessible after stage s.

(4) If $\gamma^* \frown \langle o, \sigma \rangle$ is enumerated into Y at stage s, then no edge to the left of $\gamma^* \frown \langle o, \sigma \rangle$ is accessible after stage s.

(5) Y is c.e. in H.

Proof. The procedure to computably enumerate the set Y in H will be the following.

Stage 0: Enumerate γ^*_0 into the set Y, where γ^*_0 is the root strategy of the priority tree.

Stage $s+1$: Consider every strategy γ^* on the priority tree which has been accessible at some stage $t < s + 1$. If every edge on the path leading to γ^* has been enumerated into the set Y, consider every edge $\gamma^* \frown \langle o, \sigma \rangle$ which has been accessible at some stage $u < s+1$. If the edge $\gamma^* \frown \langle o, \sigma \rangle$ has not yet been enumerated into Y, ask the oracle H whether $\sigma \subset H$. If this is the case determine whether $H_s \supseteq \sigma$. If this is also the case enumerate the edge $\gamma^* \frown \langle o, \sigma \rangle$ into Y, and enumerate the strategy $(\gamma^*)^+$ into Y, where $(\gamma^*)^+$ is the successor strategy of γ^* along the edge $\gamma^* \frown \langle o, \sigma \rangle$.

We shall now show that the set Y as constructed above satisfies properties (1)-(5).

(1) We prove this statement by induction on the length of the true path f.

For our base case we consider the strategy $\gamma_0^* = f \upharpoonright 0$. In this case we have that the strategy γ_0^* is enumerated into the set Y during stage 0 of the construction, as required.

For our inductive case we assume that there exists some stage s_n such that the strategy γ_n^* is enumerated into the set Y by the construction, where $\gamma_n^* = f \upharpoonright n$.

We then prove that there exists some stage s_{n+1} such that the strategy γ_{n+1}^* is enumerated into the set Y by the construction, where $\gamma_{n+1}^* = f \upharpoonright n+1$.

Consider the edge $\gamma_n^* \frown \langle o, \sigma \rangle$ lying on the true path. Since the strategy γ_n^* has been enumerated into the set Y, it must be the case that every edge on the path leading to the strategy γ_n^* must also have been enumerated into the set Y. Now, since the edge $\gamma_n^* \frown \langle o, \sigma \rangle$ is on the true path, there must be a stage such that it becomes accessible. In addition, since the edge is on the true path, we have that $\sigma \subset H$. Hence there must be some stage s_{n+1} such that $H_{s_{n+1}} \supseteq \sigma$. From this it follows that at stage s_{n+1} the construction enumerates the edge $\gamma_n^* \frown \langle o, \sigma \rangle$ and the strategy γ_{n+1}^* into the set Y as required.

(2) This statement follows directly from the proof of (1).

(3) Suppose that the strategy γ^* has been enumerated into Y at stage s. Then for each edge $\gamma' \frown \langle o, \sigma \rangle$ on the path leading to the strategy γ^* there must exist some stage s' such that the edge $\gamma' \frown \langle o, \sigma \rangle$ has been enumerated into Y. In order for this to be the case, we must have that $\sigma \subset H$.

Now, consider any edge $\gamma' \frown \langle o', \sigma' \rangle$ lying to the left of $\gamma' \frown \langle o, \sigma \rangle$.

If $\sigma' \mid \sigma$, we have that there is some x such that $\sigma'(x) = 1$ and $\sigma(x) = 0$. Hence in order for the edge $\gamma' \frown \langle o', \sigma' \rangle$ to become accessible at some stage $t > s$ it must be the case that

$H_t(x) = 1$. But this is impossible, since we have determined that $\sigma(x) = 0$ and $\sigma \subset H$.

On the other hand we could have that $\sigma' \subset \sigma$ and that there exists some $b \in dom(f_{x,\langle o',\sigma'\rangle})$ and some $t^* \in dom(f_{x,\langle o,\sigma\rangle})$ such that $t^* > b$ and $f_{x,\langle o',\sigma'\rangle}(b) \neq f_{x,\langle o,\sigma\rangle}(t^*)$. Hence in order for the edge $\gamma' \frown \langle o', \sigma'\rangle$ to become accessible at some stage $t > s$ there must be some x such that $|\sigma'| < x \leq |\sigma|$ such that $\sigma(x) = 0$ and $H_t(x) = 1$. But this is impossible, since we have already determined that $\sigma \subset H$.

Hence we have that the edge $\gamma' \frown \langle o', \sigma'\rangle$ is inaccessible at every stage $t > s$.

Now if γ'' lies to the left of γ^*, it must be the case that γ'' lies below some edge of the form $\gamma' \frown \langle o, \sigma\rangle$ as described above. Hence we have that γ'' is inaccessible at every stage $t > s$, as required.

(4) This statement follows directly from the proof of (3).

(5) This statement is true by our construction of the set Y. □

Showing $X \leq_T H$

In order to determine whether a given set X is computable with the help of H, we need to be able to determine whether $x \in X$ or not after some finite number of steps and in finitely many queries to H.

The construction will often be organised in such a way that it is possible to computably determine which strategy on the priority tree is able to enumerate the element x into the set X. In order for the modified strategy to make use of this element, it will first have to be attached to some edge of the appropriate kind.

Once the strategy which can potentially enumerate x into X has been identified, one can set in motion both the construction and the enumeration of the set Y with the help of H, which is the object of Lemma 3.1.8.

At this point one waits and observes the modified strategy until one of the following events occurs.

If the strategy enumerates x into X at some stage, we clearly have that $x \in X$.

Otherwise suppose that $x \notin X$. Then we need to be able to tell that this is the case in finitely many stages and in finitely many queries to H.

Suppose that x is attached to some edge $\gamma \frown \langle o, \sigma \rangle$ lying to the right of the true path. Then this edge is eventually initialised, resulting in x being discarded, allowing us to determine that $x \notin X$.

Suppose that x is attached to some edge $\gamma \frown \langle o, \sigma \rangle$ lying to the left of the true path. Then there is some stage s such that some strategy γ' or edge $\gamma' \frown \langle o, \sigma \rangle$ on the true path lying to the right of $\gamma \frown \langle o, \sigma \rangle$ will be enumerated into Y. This means that no strategy or edge to the left of $\gamma' \frown \langle o', \sigma' \rangle$ will be accessible after stage s. It follows that the edge $\gamma \frown \langle o, \sigma \rangle$ is now also inaccessible, and that thus x cannot be enumerated into X. Hence we have that $x \notin X$.

Suppose that x is attached to some edge $\gamma \frown \langle o, \sigma \rangle$ lying on the true path. Then this edge is visited infinitely often and it should be possible to tell from the action associated to the edge itself whether x will be enumerated into X or not.

Finally suppose that x is never attached to any edge. Then it should be possible to discover this fact in the following way. Suppose that $\gamma \frown \langle o, \sigma \rangle$ is the edge lying on the true path. Then this edge will be enumerated into the set Y at some stage t, meaning that no edge to the left of $\gamma \frown \langle o, \sigma \rangle$ is accessible after stage t.

Now, whenever the strategy γ visits this edge at some stage s, it will initialise every edge lying to its right, forcing the latter to discard any attached witness. In addition, a restraint will be imposed on edges lying to the right of $\gamma \frown \langle o, \sigma \rangle$ such that only elements which are greater s can now be attached to these edges.

One can then simply wait for $\gamma \frown \langle o, \sigma \rangle$ to become accessible again at some stage $t' > t$ such that we also have that $t' > x$. It then becomes impossible for x to be attached to any edge of the γ strategy, meaning that x cannot be enumerated into X either. Hence we have that $x \notin X$.

Showing $\mathbf{x} < \mathbf{h}$

Suppose that we have a construction which builds some c.e. set X and that the high permitting method succeeds in giving a correct modified construction. Then for any *high* c.e. degree $\mathbf{h}$ it is possible to choose a c.e. set H such that $\mathbf{h} = deg(H)$ and build X such that $X \leq_T H$. From this it follows that it is possible to build a c.e. degree $\mathbf{x} = deg(X)$ such that $\mathbf{x} \leq \mathbf{h}$.

However, for any $high$ c.e. degree $\mathbf{h}$, it is also possible to build X such that $\mathbf{x} = deg(X)$ and $\mathbf{x} < \mathbf{h}$. This can be carried out as follows.

Miller has shown that below every $high$ c.e. degree, there exists a $high$ minimal pair of c.e. degrees in [Miller, 1981].

Theorem 3.1.9. *(Miller). Let* $\mathbf{h}$ *be a* $high$ *c.e. degree. Then there exist* $high$ *c.e. degrees* $\mathbf{h_0}$ *and* $\mathbf{h_1}$ *such that* $\mathbf{h_0} \leq \mathbf{h}$, $\mathbf{h_1} \leq \mathbf{h}$ *and* $\mathbf{h_0}$ *and* $\mathbf{h_1}$ *form a minimal pair.*

Before proceeding we note that $\mathbf{h_0} < \mathbf{h}$. For if $\mathbf{h_0} = \mathbf{h}$, we would have that $\mathbf{h_1} \leq \mathbf{h_0}$. Since $\mathbf{h_1} \leq \mathbf{h_1}$, by the minimal pair property we would have that $\mathbf{h_1} = \mathbf{0}$. But this is not possible because $\mathbf{h_1} > \mathbf{0}$ by the definition of a minimal pair.

Hence given a $high$ c.e. degree $\mathbf{h}$, one can first build a $high$ minimal pair of c.e. degrees $\mathbf{h_0}$ and $\mathbf{h_1}$ below $\mathbf{h}$. The high permitting method is then used to build the c.e. set X such that $\mathbf{x} = deg(X)$ and $\mathbf{x} \leq \mathbf{h_0}$. Since $\mathbf{h_0} < \mathbf{h}$, we can then conclude that $\mathbf{x} < \mathbf{h}$ by transitivity, as required.

We are now in a position to apply the high permitting method of Shore and Slaman to the construction of the Lachlan Non-Spliting theorem presented in Chapter 2.

3.2 Preliminaries for the Theorem

In this section we shall show that a Lachlan Non-Splitting Pair can be found strictly below every $high$ c.e. degree.

Theorem 3.2.1. *(Every High c.e. Degree Bounds a Lachlan Non-Splitting Pair). For every high c.e. degree* $\mathbf{h}$ *there exist c.e. degrees* $\mathbf{a}$ *and* $\mathbf{d}$ *such that* $\mathbf{a} < \mathbf{h}$*,* $\mathbf{d} < \mathbf{a}$*, and there is no non-trivial splitting of* $\mathbf{a}$ *into c.e. degrees* $\mathbf{u}$ *and* $\mathbf{v}$ *such that* $\mathbf{d} < \mathbf{u}$ *and* $\mathbf{d} < \mathbf{v}$.

In order to prove the theorem we shall build two sets A and D, which satisfy certain 'Non-Splitting' requirements, causing $\mathbf{a} = deg(A \oplus D)$ and $\mathbf{d} = deg(D)$ to form the top and the base of a Lachlan Non-Splitting Pair. In order to show that the Lachlan Non-Splitting Pair $\langle \mathbf{a}, \mathbf{d} \rangle$ lies strictly below the given high c.e. degree $\mathbf{h}$, we proceed as follows. We first use Miller's Theorem (Theorem 3.1.9) to obtain a high c.e. degree $\mathbf{h_0} < \mathbf{h}$, where $\mathbf{h_0} = deg(H_0)$, and then use Shore and Slaman's high permitting technique to build $A \leq_T H_0$ and $D \leq_T H_0$. From this it follows that $A \oplus D \leq_T H_0$, and thus that $A \oplus D <_T H_0$. Thus we shall have that $\mathbf{a} < \mathbf{h}$ and $\mathbf{d} < \mathbf{h}$ as required.

3.2.1 The Non-Splitting Requirements

Since $D \leq_T A \oplus D$, we have that $\mathbf{d} \leq \mathbf{a}$. To show that $\mathbf{d} < \mathbf{a}$, it is sufficient to prove that $\mathbf{a} \not\leq \mathbf{d}$. Hence we shall need to satisfy the following requirement.

$$\mathcal{S} : A \not\leq_T D$$

In order to show that there is no non-trivial splitting of $\mathbf{a}$ into c.e. degrees $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{d} < \mathbf{u}$ and $\mathbf{d} < \mathbf{v}$, we shall need to satisfy a requirement $\mathcal{R}_{(U,V)}$ for every pair of c.e. sets U and V. These requirements are a weakened form of the corresponding requirements for the Lachlan Non-Splitting Theorem, and will also allow us to show that $\mathbf{a} < \mathbf{h}$. We postpone the argument for weakening the requirements to Section 3.2.5.

$$\begin{aligned} \mathcal{R}_{(U,V)} \quad : \quad & [A \leq_T U \oplus V \ \wedge \ U \leq_T A \oplus D \ \wedge \ V \leq_T A \oplus D] \Rightarrow \\ & [A \leq_T U \oplus D \ \vee \ A \leq_T V \oplus D] \end{aligned}$$

We now show that satisfying these requirements is enough to prove the theorem.

Lemma 3.2.2. *If the requirement $\mathcal{S}$ is satisfied and the requirement $\mathcal{R}_{(U,V)}$ is satisfied for every pair of c.e. sets U and V, there is no non-trivial splitting of $\mathbf{a}$ into c.e. degrees $\mathbf{a_0}$ and $\mathbf{a_1}$ such that $\mathbf{d} < \mathbf{a_0}$ and $\mathbf{d} < \mathbf{a_1}$.*

Proof. Suppose for contradiction that there is a non-trivial splitting of $\mathbf{a}$ into c.e. degrees $\mathbf{a_0}$ and $\mathbf{a_1}$ such that $\mathbf{d} < \mathbf{a_0}$ and $\mathbf{d} < \mathbf{a_1}$. Let $\mathbf{a_0} = deg(A_0)$ and $\mathbf{a_1} = deg(A_1)$, where A_0 and A_1 are c.e. sets. Then by the definition of a non-trivial splitting we have that $\mathbf{a} = \mathbf{a_0} \vee \mathbf{a_1}$, which means that $A \oplus D \equiv_T A_0 \oplus A_1$. In addition we have that $D <_T A_0$ and $D <_T A_1$.

We shall now derive the following three facts - $A \leq_T A_0 \oplus A_1$, $A_0 \leq_T A \oplus D$ and $A_1 \leq_T A \oplus D$. In order to derive $A \leq_T A_0 \oplus A_1$ we proceed as follows. Since $A \oplus D \equiv_T A_0 \oplus A_1$ we have that $A \oplus D \leq_T A_0 \oplus A_1$. On the other hand from $D <_T A_0$ it follows that $D \leq_T A_0$. Hence we can conclude that $A \leq_T A_0 \oplus A_1$. In order to derive $A_0 \leq_T A \oplus D$ and $A_1 \leq_T A \oplus D$ we use the fact that $A_0 \oplus A_1 \equiv_T A \oplus D$. From this it follows that $A_0 \oplus A_1 \leq_T A \oplus D$. Thus we can conclude that $A_0 \leq_T A \oplus D$ and $A_1 \leq_T A \oplus D$.

Now since the requirement $\mathcal{R}_{(U,V)}$ holds for any pair of c.e. sets U and V, and since A_0 and A_1 are c.e. sets, we have that $\mathcal{R}_{(A_0,A_1)}$ is the case. In addition since we have shown that $A \leq_T A_0 \oplus A_1$, $A_0 \leq_T A \oplus D$ and $A_1 \leq_T A \oplus D$, we have that the antecedent of the implication is satisfied and hence we can conclude that $A \leq_T A_0 \oplus D$ or that $A \leq_T A_1 \oplus D$.

Now without loss of generality, suppose that $A \leq_T A_0 \oplus D$ is the case. Then we have that $A \oplus D \leq_T A_0 \oplus D$. Since we have that $D <_T A_0$ and thus that $D \leq_T A_0$, we can conclude that $A_0 \oplus D \leq_T A_0$. Then by transitivity we have that $A \oplus D \leq_T A_0$. On the other hand we also know that $A \oplus D \equiv_T A_0 \oplus A_1$ and hence that $A_0 \oplus A_1 \leq_T A \oplus D$. Hence we have that $A_0 \oplus A_1 \leq_T A_0$ once again by transitivity. From this it follows that $A_1 \leq_T A_0$. But this means that $\mathbf{a_1} \leq \mathbf{a_0}$ and hence that $\mathbf{a_0} \nmid \mathbf{a_1}$. It follows that $\mathbf{a_0}$ and $\mathbf{a_1}$ cannot be a non-trivial splitting of $\mathbf{a}$, which gives the required contradiction. □

3.2.2 Implementation of the Non-Splitting Requirements

In order for strategies to be able to satisfy the above requirements we will need to break them down into a simpler form. Let (Θ) be a standard listing of all p.c. functionals, and let $(\Phi_1, \Phi_2, \Phi_3, U, V)$ be a standard listing of all 5-tuples such that Φ_1, Φ_2 and Φ_3 are p.c. functionals, and U and V are c.e. sets.

Then the requirement $\mathcal{S}$ can be broken down into infinitely many requirements of the following form.

$$\mathcal{S}_{(\Theta)} : \Theta^D \neq A$$

On the other hand each requirement $\mathcal{R}_{(U,V)}$ can be broken down into infinitely many requirements of the following form.

$$\begin{aligned} \mathcal{R}_{(\Phi_1,\Phi_2,\Phi_3,U,V)} \quad : \quad & [\Phi_1^{U,V} = A \ \wedge \ \Phi_2^{A,D} = U \ \wedge \ \Phi_3^{A,D} = V] \Rightarrow \\ & [\Gamma^{U,D} = A \ \vee \ \Gamma^{V,D} = A] \end{aligned}$$

In this case the functionals $\Gamma^{U,D}$ or $\Gamma^{V,D}$ would need to be built by any strategy attempting to satisfy such a requirement.

3.2.3 Further Simplification of the Non-Splitting Requirements

We can further simplify the requirements as follows. Consider a requirement of the form $\mathcal{R}_{(U,V)}$ and suppose that $A \leq_T U \oplus V$, $U \leq_T A \oplus D$ and $V \leq_T A \oplus D$, so that the requirement is not satisfied trivially.

Then in order to satisfy $\mathcal{R}_{(U,V)}$ it is sufficient to show that a requirement of the form $\mathcal{R}_{(\Phi_1,\Phi_2,\Phi_3,U,V)}$ holds for some p.c. functionals Φ_1, Φ_2 and Φ_3. This is because building a functional $\Gamma^{U,D} = A$ or a functional $\Gamma^{V,D} = A$ is sufficient to show that $A \leq_T U \oplus D$ or $A \leq_T V \oplus D$ respectively, thus satisfying $\mathcal{R}_{(U,V)}$.

Finally if there are functionals Φ_1, Φ_2 and Φ_3 such that $\Phi_1^{U,V} = A$, $\Phi_2^{A,D} = U$ and $\Phi_3^{A,D} = V$, there must also exist functionals Ψ_1, Ψ_2 and Ψ_3 giving $\Psi_1^{U,V} = A$, $\Psi_2^{A,D} = U$ and $\Psi_3^{A,D} = V$ and satisfying the following properties for all $1 \leq i \leq 3$:

(1) $(\forall s)(\forall x)(\forall y)\ [x < y \Rightarrow \psi_i[s](x) < \psi_i[s](y)]$.

(2) $(\forall x)(\forall s)(\forall s')\ [s < s' \Rightarrow \psi_i[s](x) \leq \psi_i[s'](x)]$.

By combining the above two observations we have that in order to satisfy a requirement $\mathcal{R}_{(U,V)}$ it suffices to attempt to satisfy only those requirements $\mathcal{R}_{(\Phi_1,\Phi_2,\Phi_3,U,V)}$ for which the functionals $\Phi_1^{U,V}$, $\Phi_2^{A,D}$ and $\Phi_3^{A,D}$ have the properties corresponding to (1) and (2) as specified above.

3.2.4 The High Permitting Requirements

We have already seen that in order to prove that $\mathbf{a} < \mathbf{h}$ and that $\mathbf{d} < \mathbf{h}$, we have to show that $A \leq_T H_0$ and that $D \leq_T H_0$. In order to achieve this, we shall apply the high permitting method of Shore and Slaman described in Section 3.1 to the construction of the Lachlan Non-Splitting Theorem given in Chapter 2.

In order to do this we shall mirror the development of the strategies given in Chapter 2, modifying the strategies in each case by following the procedure in Section 3.1.2. When introducing the modified startegy we shall start by defining the sequence $Q_1, \ldots, Q_n$ of Π_2^0 questions which determines the true outcome of the strategy. This allows the calculation of the apparent limits and apparent uses at stage s for every $\lim_{t\to\infty} \Psi^{H_0}(f(Q_i), t)$ such that $1 \leq i \leq n$. The apparent limits and the apparent use of greatest length can then be used to determine the outcome of the strategy at stage s. In this manner we obtain both the set of outcomes of the modified strategy and the means for allowing the modified strategy to guess the outcome which it should visit at stage s. The priority tree shall then be modified a natural way to take into account the fact that the outcomes of the strategy have changed.

The fact that the outcome chosen by a modified strategy is now driven by the stagewise approximation to H_0 represents a lack of control on the part of the strategy, which introduces a variety of technical problems which need to be resolved. The most serious of these problems arise from the fact that the modified strategy will now be divided into the following two parts. The first is an attachment procedure which attaches elements having certain properties at stage s to edges of a specific form. The second is a procedure which visits an edge at stage s as determined by the approximation $H_{0,s}$, taking the action associated to this edge based on any element which is attached to it.

While the property satisfied by an element which is attached to an edge at stage s will hold at stage s, this may no longer be the case when the strategy actually visits this edge at some stage $s' > s$. While certain properties can be preserved through direct means, this will not be the case in general. In fact the major obstacle in applying the Shore-Slaman high permitting method to the Lachlan Non-Splitting Theorem lies in preserving the honesty of computations $\Gamma^{U,D}(w)$ (or $\Gamma^{V,D}(w)$ resp.) for a given witness w. This problem shall be treated in greater length in Section 3.2.5 below.

The verification of the construction has two main objectives. The first is to show that the various modifications which have been performed still allow the Non-Splitting requirements discussed above to be satisfied. The second is to prove that $A \leq H_0$ and that $D \leq H_0$. In the latter case the set H_0 returns in the guise of an oracle, helping us to computably enumerate in H_0 the edges lying to the left of the true path of the construction. By combining this with a suitably devised attachment procedure, we shall be able to determine whether $x \notin A$ or $x \notin D$ for any element x in finite time and in finitely many queries to H_0.

3.2.5 Honesty Preservation

We shall now discuss how weakening the requirements $\mathcal{R}_{(U,V)}$ with respect to the analogous requirements for the Lachlan Non-Splitting Theorem, allows us to preserve the honesty of computations $\Gamma^{U,D}(w)$ (or $\Gamma^{V,D}(w)$ resp.) for a given witness w.

Consider the requirements $\mathcal{R}_{(\Phi_1,\Phi_2,\Phi_3,U,V)}$, corresponding to $\mathcal{R}_{(U,V)}$. Since the premises of these requirements now contain the expressions $U = \Phi_2^{A,D}$ and $V = \Phi_3^{A,D}$ we are now able to exercise indirect control over the sets U and V through our control of the sets A and D.

For in order for the element x to enter the set U, an $A \upharpoonright \phi_2(x)$ or $D \upharpoonright \phi_2(x)$ change is required. Similarly, in order for the element x to enter the set V, an $A \upharpoonright \phi_3(x)$ or $D \upharpoonright \phi_3(x)$ change is required. For if x were to enter one of these sets without the corresponding change to one of the sets A or D, a disagreement $U(x) \neq \Phi_2^{A,D}(x)$ or $V(x) \neq \Phi_3^{A,D}(x)$ would be the case and the requirement $\mathcal{R}_{(\Phi_1,\Phi_2,\Phi_3,U,V)}$ would be satisfied trivially.

Now suppose that a computation $\Gamma^{U,D}(w)$ (or $\Gamma^{V,D}(w)$ resp.) is honest with respect to $\Phi_1^{U,V}(w)$ at some stage s. This will be the case if $\gamma[s](w) > \phi_1[s](w)$. Then we have that the computation can only become dishonest at some stage $t > s$ if there is some $U \upharpoonright \phi_1[u](w)$ change or a $V \upharpoonright \phi_1[u](w)$ change at some stage u such that $s < u < t$.

However a $U \upharpoonright \phi_1[u](w)$ change is only possible if the construction allows an $A \upharpoonright \phi_2[u'](\phi_1[u'](w))$ change or a $D \upharpoonright \phi_2[u'](\phi_1[u'](w))$ change at some stage u' such that $s < u' < u$. Similarly a $V \upharpoonright \phi_1[u](w)$ change is only possible if the construction allows an $A \upharpoonright \phi_3[u'](\phi_1[u'](w))$ change or a $D \upharpoonright \phi_3[u'](\phi_1[u'](w))$ change at some stage u' such that $s < u' < u$.

Hence the weakened requirements allow us stop a computation $\Gamma^{U,D}[s](w)$ (or $\Gamma^{V,D}[s](w)$ resp.) for a given witness w. which is honest from becoming dishonest at some stage $t > s$. Preserving the honesty of computations in this way will be central to resolving the synchronisation problem which is created when the high permitting method of Shore and Slaman is applied to the construction of the Lachlan Non-Splitting Theorem.

3.2.6 Further Remarks

We shall now make a number of preliminary remarks on the construction which will be used to satisfy the requirements.

Witness, Threshold and Use Sets

As in Section 2.1.4 of Chapter 2, we shall totally order the $\mathcal{S}$ strategies on the priority tree, indexing them as α_e for some natural number e. Similarly we shall totally order the $\mathcal{R}$ strategies on the priority tree, indexing them as β_e for some natural number e. We shall also computably partition the set of natural numbers $\mathbb{N}$ into infinitely many infinite subsets, assigning the sets V^e and W^e to the strategy α_e for use as its threshold set and witness set respectively.

However, an $\mathcal{R}$ strategy shall now have to build a functional $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}$ or $\Gamma^{V,D}_{\beta\frown\langle i,\sigma\rangle}$ for each one of its edges $\beta \frown \langle i, \sigma\rangle$, of which there can be infinitely many. Each one of these edges shall therefore need its own use set $U^{e,\beta^\frown\langle i,\sigma\rangle}$ from which to choose the uses to define the corresponding functional.

Initialisation and Reseting

During every stage s, the construction will attempt to construct a *current path* δ_s of length s through the priority tree in the usual manner. Strategies which lie on the current path are said to be *accessible* at stage s. One important difference from the construction of the Lachlan Non-Splitting Theorem is that the current path at stage s does not have to be of length s. This situation results from the fact that some strategy lying on the current path might tell the construction to terminate the path early by going to the next stage, thus cutting the current path short.

Once the construction has built the current path δ_s, it will *initialise* all strategies and all edges lying to the right of the current path. A strategy which has been initialised behaves as if had never been previously accessible. It will also cancel all work performed so far by setting all restraints to zero, setting the variable 'suspend' to its initial state, and initialising every one of its edges. When an edge has been initialised, the construction will detach any element which has been attached to the edge, undefine every work interval which has been defined for the edge, set every boundary associated to the work interval to zero and cancel any functional which is associated to the edge. The mode of the edge will also be set to its initial state.

During the course of the construction a strategy may impose a diagonalisation restraint or a downward restraint on a strategy lying below it. Since the latter strategy may have chosen certain elements which are incompatible with the restraint, we shall *reset* the startegy whenever some strategy lying above it increases one of these two restraints. A strategy which has been reset will count as having been initialised the moment the higher priority strategy increased its restraint, with the exception that the diagonalisation restraint imposed by the strategy being reset will not be set to zero.

Expansionary and Expansionary* Stages

An $\mathcal{R}$ strategy attempting to satisfy a requirement $\mathcal{R}_{(\Phi_1,\Phi_2,\Phi_3,U,V)}$ will need to observe the length of agreement between the set A and the functionals $\Phi_1^{U,V}$, $\Phi_2^{A,D}$ and $\Phi_3^{A,D}$. These are defined as follows.

Definition 3.2.3. *(Length of Agreement).*

- $l_s(\Phi_1^{U,V}, A)$ *is the length of agreement between the functional* $\Phi_1^{U,V}$ *and the set* A *at stage* s*, and is defined as follows:*
$$l_s(\Phi_1^{U,V}, A) = \max\{x \mid (\forall y < x)[\,\Phi_1^{U,V}[s](y) = A_s(y)\,]\}$$
- $l_s(\Phi_2^{A,D}, U)$ *is the length of agreement between the functional* $\Phi_2^{A,D}$ *and the set* U *at stage* s*, and is defined as follows:*
$$l_s(\Phi_2^{A,D}, U) = \max\{x \mid (\forall y < x)[\,\Phi_2^{A,D}[s](y) = U_s(y)\,]\}$$
- $l_s(\Phi_3^{A,D}, V)$ *is the length of agreement between the functional* $\Phi_3^{A,D}$ *and the set* V *at stage* s*, and is defined as follows:*
$$l_s(\Phi_3^{A,D}, V) = \max\{x \mid (\forall y < x)[\,\Phi_3^{A,D}[s](y) = V_s(y)\,]\}$$

If any one of these lengths of agreement stops increasing, we have that the requirement is satisfied trivially and that the strategy does not need to take further action. If on the other hand this is not the case, the strategy will need to build one of the functionals $\Gamma^{U,D}$ or $\Gamma^{V,D}$ in order to satisfy the requirement.

In the construction of the Lachlan Non-Splitting Theorem, we used the occurrence of an expansionary stage to signal that strategy should continue to build its functional. Since we are now monitoring not one but three lengths of agreement, the notion of an expansionary stage must now take all three into account. However since the various lengths of agreement might not increase simultaneously, we shall first measure whether $l(\Phi_1^{U,V}, A)$ has expanded and then measure whether $l(\Phi_2^{A,D}, U)$ and $l(\Phi_3^{A,D}, U)$ have expanded as measured only over those stages at which $l(\Phi_1^{U,V}, A)$ has expanded. In this manner if $l(\Phi_2^{A,D}, U)$ expands at some stage $t_2 < s$, $l(\Phi_3^{A,D}, V)$ expands at some stage $t_3 < s$, both lengths of agreement are preserved at stage s and

$l(\Phi_1^{U,V}, A)$ expands at stage s, we have that s signals that any disagreement between $\Phi_1^{U,V}$ and A has been removed and that at stage s it appears that the requirement will not be satisfied trivially. This shall correspond to the required notion of *$\mathcal{R}$-expansionary stage*, which is defined below.

Definition 3.2.4. *($\mathcal{R}$-expansionary stage). Let β be an $\mathcal{R}$ strategy. A stage s is an $\mathcal{R}$-expansionary stage if:*

(1) $(\forall s' < s)[\, l_s(\Phi_1^{U,V}, A) > l_{s'}(\Phi_1^{U,V}, A)\,]$.

(2) $(\forall s' < s)[\,(\forall s'' < s')[\, l_{s'}(\Phi_1^{U,V}, A) > l_{s''}(\Phi_1^{U,V}, A)\,] \Rightarrow$
$l_s(\Phi_2^{A,D}, U) > l_{s'}(\Phi_2^{A,D}, U)\,]$.

(3) $(\forall s' < s)[\,(\forall s'' < s')[\, l_{s'}(\Phi_1^{U,V}, A) > l_{s''}(\Phi_1^{U,V}, A)\,] \Rightarrow$
$l_s(\Phi_3^{A,D}, V) > l_{s'}(\Phi_3^{A,D}, V)\,]$.

In practice a strategy γ will only measure the various lengths of agreement at those stages during which it is accessible. In general we refer to the stages at which a given strategy γ is accessible as γ-stages.

Definition 3.2.5. *(γ-stage). Let γ be an $\mathcal{R}$ or $\mathcal{S}$ strategy. A stage s is said to be a γ-stage if γ is accessible at stage s. If s is a γ-stage, we shall denote this by γ-stage(s). In addition we denote the set of γ-stages $\{s : \mathbb{N} \mid \gamma\text{-}stage(s)\}$ by $\mathbb{N}_\gamma$.*

Given an $\mathcal{R}$ strategy β, we can then define the notion of a β-expansionary stage by measuring the expansion of the lengths of agreement at those stages at which β is accessible. A β-expansionary stage s thus signals to the strategy β that it should proceed to build its functional.

Definition 3.2.6. *(β-expansionary stage). Let β be an $\mathcal{R}$ strategy. A stage s is a β-expansionary stage if:*

(1) *s is a β-stage.*

(2) $(\forall s' < s)[\,\beta\text{-}stage(s') \Rightarrow l_s(\Phi_1^{U,V}, A) > l_{s'}(\Phi_1^{U,V}, A)\,]$.

(3) $(\forall s' < s)[\,\beta\text{-}stage(s') \wedge (\forall s'' < s')[\,\beta\text{-}stage(s'') \Rightarrow l_{s'}(\Phi_1^{U,V}, A) > l_{s''}(\Phi_1^{U,V}, A)\,] \Rightarrow$
$l_s(\Phi_2^{A,D}, U) > l_{s'}(\Phi_2^{A,D}, U)\,]$.

(4) $(\forall s' < s)[\,\beta\text{-}stage(s') \wedge (\forall s'' < s')[\,\beta\text{-}stage(s'') \Rightarrow l_{s'}(\Phi_1^{U,V}, A) > l_{s''}(\Phi_1^{U,V}, A)\,] \Rightarrow$
$l_s(\Phi_3^{A,D}, V) > l_{s'}(\Phi_3^{A,D}, V)\,]$.

In the previous section we also indicated that we shall be attempting to preserve the honesty of computations $\Gamma^{U,D}(x)$. In order to do this we shall define the *length of honesty preservation* to be the length of the initial segment of U and V which cannot change without a corresponding A or D change as described in the previous section, or without resulting in the requirement being satisfied trivially.

Definition 3.2.7. *(Length of Honesty Preservation).*
$h_s(\Phi_1^{U,V}, \Phi_2^{A,D}, \Phi_3^{A,D})$ *is the length of honesty preservation between the functional* $\Phi_1^{U,V}$ *and the set* A *at stage* s*, and is defined as follows:*

$$h_s(\Phi_1^{U,V}, \Phi_2^{A,D}, \Phi_3^{A,D}) = \max\{x \mid (\forall y < x)[\,\phi_2[s](\phi_1[s](y))\downarrow \wedge\, \phi_3[s](\phi_1[s](y))\downarrow\,]\,\}$$

$\mathcal{R}$ strategies implementing mechanisms for honesty preservation will also require expansions in the length of honesty preservation, as measured over those stages s at which $l_s(\Phi_1^{U,V}, A)$ has expanded. Infinitely many expansions of the length of honesty are required so as to ensure that the uses of longer initial segments of $\Phi_1^{U,V}$ will not change value. In this manner lower priority $\mathcal{S}$ strategies which need to take the $\mathcal{R}$ strategy into consideration are also guaranteed the honesty of longer initial segments of $\Gamma^{U,D}$. We expand the notion of an $\mathcal{R}$*-expansionary stage* to the one of an $\mathcal{R}$*-expansionary* stage* by including this constraint in the same manner as we have done with the other lengths of agreement.

Definition 3.2.8. *($\mathcal{R}$-expansionary* stage). Let β be an $\mathcal{R}$ strategy. A stage s is an $\mathcal{R}$-*expansionary stage if:*

(1) s *is an* $\mathcal{R}$*-expansionary stage.*

(2) $(\forall s' < s)[\,(\forall s'' < s')[\,l_{s'}(\Phi_1^{U,V}, A) > l_{s''}(\Phi_1^{U,V}, A)\,] \Rightarrow$
$h_s(\Phi_1^{U,V}, \Phi_2^{A,D}, \Phi_3^{A,D}) > h_{s'}(\Phi_1^{U,V}, \Phi_2^{A,D}, \Phi_3^{A,D})\,]$.

Given an $\mathcal{R}$ strategy β, we can then define the notion of a β-expansionary* stage by measuring the expansion of the lengths of agreement and the length of honesty preservation at those stages at which β is accessible.

Definition 3.2.9. *(β-expansionary* stage). Let β be an $\mathcal{R}$ strategy. A stage s is a β-*expansionary stage if:*

(1) s *is an* β*-expansionary stage.*

(2) $(\forall s' < s)[\,\beta\text{-}stage(s') \wedge (\forall s'' < s')[\,\beta\text{-}stage(s'') \Rightarrow l_{s'}(\Phi_1^{U,V}, A) > l_{s''}(\Phi_1^{U,V}, A)\,] \Rightarrow h_s(\Phi_1^{U,V}, \Phi_2^{A,D}, \Phi_3^{A,D}) > h_{s'}(\Phi_1^{U,V}, \Phi_2^{A,D}, \Phi_3^{A,D})\,]$.

The following lemma tells us that if the length of agreement between $\Phi_1^{U,V}$ and A as seen by an $\mathcal{R}$ strategy β attempting to satisfy a requirement $\mathcal{R}_{(\Phi_1,\Phi_2,\Phi_3,U,V)}$ is unbounded, but the length of honesty preservation as seen by the strategy β is finite, we have that the requirement is satisfied trivially. It follows that the notion of an expansionary* stage is an appropriate guide to when an $\mathcal{R}$ strategy with honesty preservation mechanisms should proceed to build a functional.

Lemma 3.2.10. *Let β be an $\mathcal{R}$ strategy attempting to satisfy a requirement $\mathcal{R}_{(\Phi_1,\Phi_2,\Phi_3,U,V)}$. If $\lim_{s\in\mathbb{N}_\beta} h_s(\Phi_1^{U,V}, \Phi_2^{A,D}, \Phi_3^{A,D})$ is finite, then the requirement $\mathcal{R}_{(\Phi_1,\Phi_2,\Phi_3,U,V)}$ is satisfied trivially.*

Proof. Suppose that $\lim_{s\in\mathbb{N}_\beta} l_s(\Phi_1^{U,V}, A)$ is unbounded and that $\lim_{s\in\mathbb{N}_\beta} h_s(\Phi_1^{U,V}, \Phi_2^{A,D}, \Phi_3^{A,D})$ is finite. From the latter it follows that there is some element x such that $\lim_{s\in\mathbb{N}_\beta} \phi_2[s](\phi_1[s](x))$ is unbounded or such that $\lim_{s\in\mathbb{N}_\beta} \phi_3[s](\phi_1[s](x))$ is unbounded.

In the first case we either have that $\lim_{s\in\mathbb{N}_\beta} \phi_1[s](x)$ is unbounded, or that $\lim_{s\in\mathbb{N}_\beta} \phi_1[s](x) = u$ for some finite u and $\lim_{s\in\mathbb{N}_\beta} \phi_2[s](u)$ is unbounded. Hence we either have that $\Phi_1^{U,V}(x)\uparrow$, or that $\Phi_2^{A,D}(u)\uparrow$. Similarly in the second case we either have that $\lim_{s\in\mathbb{N}_\beta} \phi_1[s](x)$ is unbounded, or that $\lim_s \phi_1[s](x) = u$ for some finite u, whilst $\lim_{s\in\mathbb{N}_\beta} \phi_3[s](u)$ is unbounded. Hence we either have that $\Phi_1^{U,V}(x)\uparrow$ or that $\Phi_3^{A,D}(u)\uparrow$.

Therefore we have that the requirement $\mathcal{R}_{(\Phi_1,\Phi_2,\Phi_3,U,V)}$ is satisfied trivially as required. □

Note that all $\mathcal{R}$ strategies used in this chapter which are defined prior to Section 3.7 do not contain the mechanisms for honesty preservation and are based on the notion of an expansionary stage. On the other hand the strategies defined in Section 3.7 have the mechanisms for honesty preservation and are therefore based on the notion of an expansionary* stage.

3.3 One $\mathcal{R}$ Requirement

In this section we shall show how one can satisfy one $\mathcal{R}$ requirement. This will require defining an $\mathcal{R}$ strategy to satisfy the $\mathcal{R}$ requirement. We start by defining the questions needed for the $\mathcal{R}$ strategy to determine its outcome at any given stage.

3.3.1 Questions for the $\mathcal{R}$ Strategy

The $\mathcal{R}$ strategy β will need to ask one question, which we denote by Q_1. The question asks whether the strategy β sees infinitely many β-expansionary stages:

(1) Are there infinitely many stages $q \in \mathbb{N}_\beta$ such that the following holds?

(i) $(\forall q' < q)[\beta\text{-stage}(q') \Rightarrow l_q(\Phi_1^{U,V}, A) > l_{q'}(\Phi_1^{U,V}, A)]$.

(ii) $(\forall q' < q)[\beta\text{-stage}(q') \wedge (\forall q'' < q')[\,\beta\text{-stage}(q'') \Rightarrow l_{q'}(\Phi_1^{U,V}, A) > l_{q''}(\Phi_1^{U,V}, A)\,] \Rightarrow l_q(\Phi_2^{A,D}, U) > l_{q'}(\Phi_2^{A,D}, U)]$.

(iii) $(\forall q' < q)[\beta\text{-stage}(q') \wedge (\forall q'' < q')[\,\beta\text{-stage}(q'') \Rightarrow l_{q'}(\Phi_1^{U,V}, A) > l_{q''}(\Phi_1^{U,V}, A)\,] \Rightarrow l_q(\Phi_3^{A,D}, V) > l_{q'}(\Phi_3^{A,D}, V)]$.

If the strategy is accessible at some stage s, it will guess the answer to Q_1 by computing the apparent limit o and apparent use σ for $\lim_{t\to\infty} \Psi^{H_0}(f(Q_1), t)$ at stage s. If the answer corresponding to o_1 is 'No', we denote the outcome by $\langle f, \sigma\rangle$. On the other hand, if the answer corresponding to o_1 is 'Yes', we denote the outcome by $\langle i, \sigma\rangle$.

We now describe the $\mathcal{R}$ strategy itself.

3.3.2 The $\mathcal{R}$ strategy

The $\mathcal{R}$ strategy β follows a Γ-strategy and has outcomes of the form $\langle i, \sigma\rangle$ and $\langle f, \sigma\rangle$. The strategy will build a different functional $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}$ below every edge $\beta \frown \langle i, \sigma\rangle$ leaving β. Each edge will also have a separate set of uses $U^{e,\beta\frown\langle i,\sigma\rangle}$ from which the strategy chooses uses when defining the functional associated to the edge $\beta \frown \langle i, \sigma\rangle$. Note that e is the index of the strategy β in the total ordering of the $\mathcal{R}$ strategies lying on the priority tree.

The strategy goes through the following steps at stage s.

Firstly, the strategy calculates a rightward restraint $r(\beta \frown \langle o', \sigma' \rangle, s)$ for every edge $\beta \frown \langle o', \sigma' \rangle$ which has been previously accessible. This restraint has to be observed by all edges lying to the right of $\beta \frown \langle o', \sigma' \rangle$.

The restraint itself is equal to the maximum of two values. These are the stage at which the edge was last visited and any β-expansionary stage which might be attached to the edge. The first constraint protects the functional $\Gamma^{U,D}_{\beta \frown \langle i, \sigma \rangle}$ associated to the edge from computations taking place to the right of the edge. This follows from the fact that this functional can only be defined when the strategy visits the edge. The second constraint protects the β-expansionary stage itself from computations taking place to the right of the edge. This will include protecting the functionals $\Phi^{U,V}_1$, $\Phi^{A,D}_2$ and $\Phi^{A,D}_3$ at those elements which are defined during the β-expansionary stage.

Secondly, the strategy performs its attachment procedure. The strategy will check whether the present stage is a β-expansionary stage, and if so it will attach this stage to the leftmost edge of the form $\beta \frown \langle i, \sigma' \rangle$ which has been previously accessible and which has no β-expansionary stage attached to it. The attachment of a β-expansionary stage to the edge $\beta \frown \langle i, \sigma' \rangle$ causes all β-expansionary stages attached to edges lying to its right to become detached.

Thirdly the strategy will impose an attachment restraint $a(\beta \frown \langle o', \sigma' \rangle, s)$ on every edge which has been previously accessible. This restraint will have to be obeyed by the edge $\beta \frown \langle o', \sigma' \rangle$ itself. If a β-expansionary stage has become attached to some edge $\beta \frown \langle o'', \sigma'' \rangle$ lying to the left of $\beta \frown \langle o', \sigma' \rangle$ during stage s, we set $a(\beta \frown \langle o', \sigma' \rangle, s)$ to be equal to s. Otherwise it will be equal to 0. This restraint protects any β-expansionary stage which has been attached to an edge during the present stage from computations taking place to the right of this edge, in the same manner as before.

Fourthly, the strategy chooses the edge $\beta \frown \langle o, \sigma \rangle$ to visit during the present stage. It then takes action according to the outcome of this edge.

If the outcome is $\langle f, \sigma \rangle$, we do nothing and proceed with the next substage.

If the outcome is $\langle i, \sigma \rangle$, and there is no β-expansionary stage attached to the edge, then we do not build the functional associated to the edge, and terminate the stage. On the other hand if the outcome is $\langle i, \sigma \rangle$, and there is a β-expansionary stage attached to the edge, the functional associated to the edge can be built based on this β-expansionary stage, and the strategy can proceed with the next substage.

When defining the functional, the uses which are chosen must obey certain constraints. In particular when choosing a use to define a functional at some given element x, the strategy makes sure that this use is at least as large as any use which has already been chosen for this element, and larger than the use which presently holds for all smaller elements. In addition the strategy makes sure that the use chosen is greater than any rightward restraint imposed by an edge lying to the left of the one being visited, and greater than the attachment restraint imposed on the edge.

Note that strictly speaking, the use of rightward restraints and attachment restraints is not necessary to satisfy one $\mathcal{R}$ requirement in isolation. We include these restraints in this section to introduce the fact that certain parts of the construction will need to be protected from computations taking place to their right. This provides an anticipation of the more complex restraints which need to be introduced in future sections, and avoids having to introduce all of these restraints at once.

We shall now formalise the modified $\mathcal{R}$ strategy.

The $\mathcal{R}$ Strategy

The strategy β will be following a Γ-strategy. Each edge of the form $\beta \frown \langle i, \sigma \rangle$ will have a functional $\Gamma^{U,D}_{\beta \frown \langle i, \sigma \rangle}$ associated to it, which will be built by the strategy when the strategy visits that edge. Each edge $\beta \frown \langle i, \sigma \rangle$ will also have its own set of uses $U^{e, \beta \frown \langle i, \sigma \rangle}$ which will be used when defining the respective functionals.

(1) Define the rightward restraint $r(\beta \frown \langle o', \sigma' \rangle, s)$ for every edge $\beta \frown \langle o', \sigma' \rangle$ which has been previously accessible as the least element x such that:

   (a) $x \geq t$, where t is some stage attached to $\beta \frown \langle o', \sigma' \rangle$.

   (b) $x \geq t$, where t is the last stage at which $\beta \frown \langle o', \sigma' \rangle$ has been accessible.

   Go to step (2).

(2) If stage s is a β-expansionary stage, and there is some edge $\beta \frown \langle i, \sigma' \rangle$ which has been accessible at some previous stage and which has no β-expansionary stage attached to it, attach s to the leftmost such edge.

If a β-expansionary stage s has been attached to some edge $\beta \frown \langle i, \sigma' \rangle$, consider every edge $\beta \frown \langle i, \sigma'' \rangle$ lying to the right of $\beta \frown \langle i, \sigma' \rangle$. If some β-expansionary stage s' is attached to $\beta \frown \langle i, \sigma'' \rangle$, detach the β-expansionary stage from the edge.

Go to step (3).

(3) Define the attachment procedure restraint $a(\beta \frown \langle o', \sigma' \rangle, s)$ for every edge $\beta \frown \langle o', \sigma' \rangle$ which has been previously accessible. If the strategy has not attached a β-expansionary stage to some edge $\beta \frown \langle o'', \sigma'' \rangle <_L \beta \frown \langle o', \sigma' \rangle$ at stage s, define $a(\beta \frown \langle o', \sigma' \rangle, s) = 0$. Otherwise define $a(\beta \frown \langle o', \sigma' \rangle, s) = s$.

Go to step (4).

(4) Calculate the outcome $\beta \frown \langle o, \sigma \rangle$ of the strategy at stage s. Take action by performing the following case analysis.

(a) $o = f$. Continue with the next substage.

(b) $o = i$.

(i) The edge $\beta \frown \langle i, \sigma \rangle$ has no β-expansionary stage attached to it. End stage s, and go to stage $s + 1$.

(ii) The edge $\beta \frown \langle i, \sigma \rangle$ has a β-expansionary stage attached to it. Detach the stage from the edge. For every $x < l_s(\Phi_1^{U,V}, A)$ such that $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}[s](x) \uparrow$, define the axiom $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}[s](x) = A_s(x)$, and choose the corresponding use $\gamma_{\beta\frown\langle i,\sigma\rangle}[s](x)$ to be the least element u in the set $U^{e,\beta\frown\langle i,\sigma\rangle}$ obeying the following conditions (if it exists):

(i) $u \geq \gamma_{\beta\frown\langle i,\sigma\rangle}[t](x)$ for all $t < s$.

(ii) $u > \gamma_{\beta\frown\langle i,\sigma\rangle}[s](y)$ for all $y < x$.

(iii) $u > r(\beta \frown \langle o', \sigma' \rangle, s)$ for every edge $\beta \frown \langle o', \sigma' \rangle$ lying to the left of $\beta \frown \langle o, \sigma \rangle$.

(iv) $u > a(\beta \frown \langle o', \sigma' \rangle)$.

We are now in a position to analyse whether the $\mathcal{R}$ strategy β satisfies the $\mathcal{R}$ requirement.

3.3.3 Satisfaction of Requirement

Consider the leftmost edge $\beta \frown \langle o, \sigma \rangle$ which is visited infinitely often by β. We perform the following case analysis depending on the outcome $\langle o, \sigma \rangle$.

[f] Suppose that the outcome is $\langle f, \sigma \rangle$. Then the answer to question Q_1 is 'No'. This means that there are only finitely many β-expansionary stages. Hence one or more of $\Phi_1^{U,V} \neq A$, $\Phi_2^{A,D} \neq U$ or $\Phi_3^{A,D} \neq V$ must be the case. It follows that the strategy satisfies its requirement trivially.

[i] Suppose that the outcome is $\langle i, \sigma \rangle$. Then the answer to question Q_1 is 'Yes'. This means that there are infinitely many β-expansionary stages.

Now since $\beta \frown \langle i, \sigma \rangle$ is the leftmost edge which is accessible infinitely often, we have that there is a stage s_0 after which no edge to its left is accessible. Hence only finitely many edges to the left of $\beta \frown \langle i, \sigma \rangle$ can have been accessible at stages $s < s_0$. Suppose that the edge $\beta \frown \langle i, \sigma \rangle$ does not have a β-expansionary stage attached at some stage $s_1 > s_0$. Since β-expansionary stages are attached to the leftmost edge of the form $\beta \frown \langle i, \sigma \rangle$ which has no element attached, and since there are infinitely many β-expansionary stages, it follows that a β-expansionary stage is eventually attached to $\beta \frown \langle i, \sigma \rangle$ at some stage $s_2 > s_1$.

But whenever $\beta \frown \langle i, \sigma \rangle$ is visited by the strategy, and a β-expansionary stage s is attached to this edge, we have that the strategy defines the functional $\Gamma^{U,D}_{\beta \frown \langle i, \sigma \rangle}$ to agree with the set A up to $l_s(\Phi_1^{U,V}, A) - 1$. Hence it is the case that $\Gamma^{U,D}_{\beta \frown \langle i, \sigma \rangle} = A$, and the requirement is satisfied.

3.4 One $\mathcal{S}$ Requirement

In this section we shall show how one can satisfy one $\mathcal{S}$ requirement. This will require defining an $\mathcal{S}$ strategy to satisfy the $\mathcal{S}$ requirement. We start by defining the questions needed for the $\mathcal{S}$ strategy to determine its outcome at any given stage.

3.4.1 Questions for the $\mathcal{S}$ Strategy

The $\mathcal{S}$ strategy α will need to ask one question, which we denote by Q_1. This question asks whether there are infinitely many witnesses w and stages s such that $\Theta^D[s](w)\downarrow= 0$. In addition, it also asks whether the length of agreement between the functional Θ^D and the set A expands infinitely often.

(1) Are there infinitely many witnesses $w \in W^e$ and stages $s \in \mathbb{N}_\alpha$ and $q \in \mathbb{N}_\alpha$ such that the following holds?

   (i) $\Theta^D[s](w)\downarrow= 0$.

   (ii) $(\forall q' < q)[\,\alpha\text{-stage}(q') \Rightarrow l_{q'}(\Theta^D, A) < l_q(\Theta^D, A)\,]$.

If the strategy is accessible at some stage s, it will guess the answer to Q_1 by computing the apparent limit o and apparent use σ for $\lim_{t\to\infty} \Psi^{H_0}(f(Q_1), t)$ at stage s. If the answer corresponding to o is 'No', we denote the outcome by $\langle w, \sigma\rangle$. On the other hand, if the answer corresponding to o is 'Yes', we denote the outcome by $\langle d, \sigma\rangle$.

We now describe the $\mathcal{S}$ strategy itself.

3.4.2 The $\mathcal{S}$ Strategy

The $\mathcal{S}$ strategy α has an infinite set of witnesses W^e, where e is the index of the strategy α in the total ordering of the $\mathcal{S}$ strategies lying on the priority tree. At any given stage s the strategy will be able to impose a restraint $R_{\alpha,s}$. Initially, we have that $R_{\alpha,0}$ is equal to 0. The strategy α shall use the fact that $R_{\alpha,s} > 0$ to signal that it has diagonalised. Once this restraint has been set, it will keep its value during subsequent stages. The strategy will have outcomes of the form $\langle d, \sigma \rangle$ and $\langle w, \sigma \rangle$.

The strategy goes through the following steps at stage s.

Firstly, the strategy determines whether it has enumerated some witness w' during the last stage t at which it was accessible (assuming it was accessible at least once before). If this is the case, the strategy has diagonalised. It will thus set the restraint $R_{\alpha,s} = \theta_t(w)$ so as to protect the use of the computation used in the diagonalisation.

Secondly, the strategy calculates a rightward restraint $r(\alpha \frown \langle o', \sigma' \rangle, s)$ for every edge $\alpha \frown \langle o', \sigma' \rangle$ which has been previously accessible. The restraint is equal to the maximum of two values. These are the stage at which this edge was last visited, and the use $\theta_t(w)$ of any computation $\Theta^D[t](w)$ such that w is a witness attached to the edge and t is the stage at which it was attached. Such a restraint will have to be obeyed by all edges lying to the right of $\alpha \frown \langle o', \sigma' \rangle$. In this way we shall have that the uses $\theta_s(w)$ for witnesses w attached to $\alpha \frown \langle o', \sigma' \rangle$ are protected from any computation taking place to the right of $\alpha \frown \langle o', \sigma' \rangle$.

Thirdly, the strategy performs its attachment procedure. If the strategy believes that it has already diagonalised ($R_{\alpha,s} > 0$), the attachment procedure is terminated and no further witnesses are attached.

Otherwise the attachment procedure will consider in turn every witness w in W^e which at stage s yields a computation $\Theta^D[s](w) \downarrow= 0$ and which has not been attached to an edge so far. The attachment procedure will be seeking to attach one of these witnesses to an edge, and will stop considering further witnesses once this has been achieved.

The strategy will try to attach the witness under consideration to the leftmost edge of the form $\alpha \frown \langle d, \sigma' \rangle$ which does not have a witness attached. Prior to attaching the witness to the edge it wil make sure that the witness is also greater than the restraint $r(\alpha \frown \langle d, \sigma' \rangle, s)$, as discussed before.

Fourthly, the strategy will impose an attachment restraint $a(\alpha \frown \langle o', \sigma' \rangle, s)$ on every edge which has been previously accessible. This restraint will have to be obeyed by the edge $\alpha \frown \langle o', \sigma' \rangle$ itself. If a witness w has become attached to some edge $\alpha \frown \langle o'', \sigma'' \rangle$ lying to the left of $\alpha \frown$

$\langle o', \sigma' \rangle$ during stage s, we set $a(\alpha \frown \langle o', \sigma' \rangle, s) = \theta_s(w)$. Otherwise it will be equal to 0. This restraint protects the use $\theta_s(w)$ from computations taking place to the right of this edge, in the same manner as before.

Fifthly, the strategy guesses the edge $\beta \frown \langle o, \sigma \rangle$ to visit during the present stage. It then takes action according to the outcome of this edge.

If the outcome is $\langle w, \sigma \rangle$, we do nothing and proceed with the next substage.

On the other hand if the outcome is $\langle d, \sigma \rangle$ and a witness is attached to the edge, we enumerate the witness w into A and end the stage. If the outcome is $\langle d, \sigma \rangle$ and there is no witness attached to the edge, we do nothing and end the stage.

Note that strictly speaking, the use of rightward restraints and attachment restraints is not necessary to satisfy one $\mathcal{S}$ requirement in isolation. We include these restraints in this section to introduce the fact that certain parts of the construction will need to be protected from computations taking place to their right. This provides an anticipation of the more complex restraints which need to be introduced in future sections, and avoids having to introduce all of these restraints at once.

We shall now formalise the modified $\mathcal{S}$ strategy.

The $\mathcal{S}$ Strategy

The strategy has a set of witnesses W^e and imposes a restraint $R_{\alpha,s}$ at each stage s. Initially we have that $R_{\alpha,0} = 0$.

(1) Let t be the stage at which α was last accessible. If such a stage does not exist, go to step (2). Otherwise, has α has enumerated some witness w into A at stage t?

   (a) (No) Do nothing.

   (b) (Yes) Set the restraint $R_{\alpha,s}$ to $\theta_t(w)$. Consider every edge $\alpha \frown \langle o', \sigma' \rangle$ of α which has been previously accessible. If some witness w' is attached to $\alpha \frown \langle o', \sigma' \rangle$, detach the witness.

   Go to step (2).

(2) Define the rightward restraint $r(\alpha \frown \langle o', \sigma' \rangle, s)$ for every edge $\alpha \frown \langle o', \sigma' \rangle$ which was previously accessible as the least element x such that:

(a) $x \geq \theta_t(w)$, where w is a witness attached to $\alpha \frown \langle o', \sigma' \rangle$ and t is the stage at which the witness was attached.

(b) $x \geq t$, where t is the last stage at which $\alpha \frown \langle o', \sigma' \rangle$ was last accessible.

Go to step (3).

(3) Consider the finite set of witnesses w in W^e such that $w < s$ and $\Theta^D[s](w) \downarrow = 0$ and such that w has not been attached to an edge at some stage $u < s$. Perform the following case analysis for every such witness in turn (under the order $<$), until one witness is attached successfully to an edge or until no more witnesses are available.

(a) Suppose $R_{\alpha,s} > 0$. Do nothing.

(b) Suppose $R_{\alpha,s} = 0$. If there is an edge $\alpha \frown \langle d, \sigma' \rangle$ such that:

(i) $\alpha \frown \langle d, \sigma' \rangle$ has been accessible during a previous stage.

(ii) $\alpha \frown \langle d, \sigma' \rangle$ has no witness attached to it.

(iii) $w > \sup\{r(\alpha \frown \langle o'', \sigma'' \rangle, s) \mid \alpha \frown \langle o'', \sigma'' \rangle <_L \alpha \frown \langle d, \sigma' \rangle \ \wedge$
$\alpha \frown \langle o'', \sigma'' \rangle$ has been previously accessible$\}$.

Attach w to the leftmost such $\alpha \frown \langle d, \sigma' \rangle$.

If a witness w has been attached to some edge $\alpha \frown \langle o', \sigma' \rangle$ at stage s, consider every edge $\alpha \frown \langle o'', \sigma'' \rangle$ lying to the right of $\alpha \frown \langle o', \sigma' \rangle$. If some witness w' is attached to $\alpha \frown \langle o'', \sigma'' \rangle$, detach the witness from the edge.

Go to step (4).

(4) Define the attachment procedure restraint $a(\alpha \frown \langle o', \sigma' \rangle, s)$ for every edge $\alpha \frown \langle o', \sigma' \rangle$ which has been previously accessible. If the strategy has not attached a witness w to some edge $\alpha \frown \langle o'', \sigma'' \rangle <_L \alpha \frown \langle o', \sigma' \rangle$ at stage s, define $a(\alpha \frown \langle o', \sigma' \rangle, s) = 0$. Otherwise define $a(\alpha \frown \langle o', \sigma' \rangle, s) = \theta_s(w)$.

Go to step (5).

(5) Calculate the outcome $\alpha \frown \langle o, \sigma \rangle$ of the strategy at stage s. Take action by performing the following case analysis.

   (a) $o = w$. Go to the next substage.

   (b) $o = d$.

      (i) $R_{\alpha,s} > 0$. End stage s, and go to stage $s + 1$.

      (ii) $R_{\alpha,s} = 0$, and the edge $\alpha \frown \langle d, \sigma \rangle$ has no witness attached to it. End stage s, and go to stage $s + 1$.

      (iii) $R_{\alpha,s} = 0$, and the edge $\alpha \frown \langle d, \sigma \rangle$ has a witness attached to it. Enumerate w into A. End stage s, and go to stage $s + 1$.

We are now in a position to analyse whether the $\mathcal{S}$ strategy α satisfies the $\mathcal{S}$ requirement or not.

3.4.3 Satisfaction of Requirement

Consider the leftmost edge $\alpha \frown \langle o, \sigma \rangle$ which is visited infinitely often by α. We perform the following case analysis depending on the outcome $\langle o, \sigma \rangle$.

[w] Suppose that the outcome is $\langle w, \sigma \rangle$.

Then the answer to question Q_1 must be 'No'.

If condition (i) of question Q_1 fails, we have that there are only finitely many witnesses w and stages s such that $\Theta^D[s](w) \downarrow= 0$. Then there must be some stage t' and some element x' such that for every $t > t'$ and $x > x'$, we have that $\Theta^D[t](x) \uparrow$ or $\Theta^D[t](x) \downarrow= 1$.

Now, if $\Theta^D(x) \uparrow$ for some x, we have that $\Theta^D(x) \neq A(x)$ and the requirement is satisfied. On the other hand, if $\Theta^D(x) \downarrow= 1$ for some x, we have that the strategy will never enumerate x into A. This means that $\Theta^D(x) \neq A(x)$ and that the requirement is also satisfied.

If condition (i) of question Q_1 holds but condition (ii) of question Q_1 fails, we have that there are infinitely many witnesses w and stages s such that $\Theta^D[s](w) \downarrow= 0$, but only finitely many stages q such that $(\forall q' < q)[l_{q'}(\Theta^D, A) < l_q(\Theta^D, A)]$. Then there must be some x such that $\Theta^D(x) \neq A(x)$, meaning that the requirement is satisfied.

It is important to note that one way for this outcome to be on the true path is for the strategy to diagonalise successfully.

d Suppose that the outcome is $\langle d, \sigma \rangle$.

Then the answer to question Q_1 must be 'Yes'.

This means that there are infinitely many witnesses w and stages s such that $\Theta^D[s](w) \downarrow = 0$, but also that there are infinitely many stages q such that $(\forall q' < q)[l_{q'}(\Theta^D, A) < l_q(\Theta^D, A)]$.

Now since $\alpha \frown \langle d, \sigma \rangle$ is the leftmost edge which is accessible infinitely often, we have that there is a stage s_0 after which no edge to its left is accessible. Hence only finitely many edges to the left of $\alpha \frown \langle d, \sigma \rangle$ can have been accessible at stages $s < s_0$. Suppose that the edge $\alpha \frown \langle d, \sigma \rangle$ does not have a witness attached at some stage $s_1 > s_0$. Since witnesses satisfying the conditions for attachment are attached to the leftmost edge of the form $\alpha \frown \langle d, \sigma \rangle$ which has no other witness attached, and since there are infinitely many such witnesses, it follows that a witness satisfying these conditions is eventually attached to $\alpha \frown \langle d, \sigma \rangle$ at some stage $s_2 > s_1$.

Now suppose that a witness w is attached to $\alpha \frown \langle d, \sigma \rangle$ at stage s_2. When α visits $\alpha \frown \langle d, \sigma \rangle$ at the least stage $s_3 > s_2$, the strategy will enumerate w into A. Once the strategy is accessible again at the least stage $s_4 > s_3$, it will determine that it has diagonalised. It will therefore set the restraint R_{α, s_4} to $\theta_{s_2}(w)$. Hence it must be the case that $\Theta^D(w) \neq A(w)$. However, this contradicts the fact that there are infinitely many stages q such that $(\forall q' < q)[l_{q'}(\Theta^D, A) < l_q(\Theta^D, A)]$.

From this contradiction, it follows that no edge with outcome $\langle d, \sigma \rangle$ can be on the true path. In fact, only an edge with outcome $\langle w, \sigma \rangle$ can be on the true path if the strategy diagonalises successfully. This is the only outcome for which Q_1 can have a negative answer, which in turn allows the length of agreement between Θ^D and A to be finite in length.

3.5 $\mathcal{S}$ Below $\mathcal{R}$

In this section we shall show how one can satisfy one $\mathcal{S}$ requirement below one $\mathcal{R}$ requirement.

Unlike the situation in the previous section, satisfying an $\mathcal{S}$ requirement below an $\mathcal{R}$ requirement will require multiple $\mathcal{S}$ and $\mathcal{R}$ strategies organised into a priority tree. Strategies located lower down the priority tree will be able to guess the outcome which strategies above them will be choosing. This additional information allows the use of a special case of the strategy used to satisfy a given requirement.

The most concise way of presenting all of the strategies which we will need to lay out on the priority tree is the following. We shall give an $\mathcal{R}$ strategy and an $\mathcal{S}$ strategy which is sufficiently comprehensive to describe any strategy lying on the priority tree. Depending on the strategy's location on the priority tree, different parts of the $\mathcal{R}$ and $\mathcal{S}$ strategy will then be executed. This will involve determining what kind of $\mathcal{R}$ and $\mathcal{S}$ strategies (if any) lie above the strategy under consideration.

As with the Lachlan Non-Splitting Theorem, a strategy γ will need to take into consideration those strategies γ' lying above it which it believes will satisfy their requirement. Such a strategy γ' is said to be active for γ. An $\mathcal{S}$ strategy γ' may also define a work interval on one of its edges. An $\mathcal{R}$ strategy γ lying below such an edge will have to choose uses which lie inside this work interval when defining its functionals. An $\mathcal{S}$ strategy γ lying below such an edge will have to choose witnesses and thresholds and uses which lie inside this work interval, and will only believe a computation $\Theta^D[s](w)$ if $\theta_s(w)$ lies inside the work interval.

We shall now discuss the concept of open and close modes related to the edges of the strategies used in this section.

3.5.1 Open and Close Modes

Given any strategy γ, each one of its edges will be in one of two *modes*. If γ is an $\mathcal{R}$ strategy, edges with outcomes $\langle f, \sigma\rangle$ or $\langle i, \sigma\rangle$ outcomes will either be in *open mode* or in *close mode*. If

γ is an $\mathcal{S}$ strategy, edges with $\langle w, \sigma\rangle$, $\langle g, \sigma\rangle$ or $\langle d, \sigma\rangle$ outcomes will either be in open mode or in close mode, while edges with an $\langle h, \sigma\rangle$ outcome will either be in *Part I mode* or in *Part II mode*.

Initially, edges start in open mode or in Part I mode, respectively. If the edge is initialised by the current path moving to its left, or reset by higher priority strategies increasing their restraints, the mode of an edge is also changed to open mode or to Part I mode, respectively.

If a strategy needs to enumerate some element into the set A when visiting an edge, it shall do so when the edge is in open mode. Similarly, if a strategy needs to enumerate some element into the set D when visiting an edge, it shall do so when the edge is in close mode (or Part I mode respectively). If the strategy manages to take its intended action when visiting an edge, it will change the mode of the edge to the opposite mode. The mode of the edge will also be changed if proceeding requires enumerating an element into the set associated with the opposite mode.

3.5.2 Open and Close Stages

A strategy will also attempt to synchronise its actions with actions of the same kind taken by higher priority strategies.

We shall say that a stage s is a γ-*open stage* if every edge lying on the path leading to γ is in open mode or Part II mode at stage s. Similarly we shall say that a stage s is a γ-*close stage* if every edge lying on the path leading to γ is in close mode or Part I mode at stage s.

A strategy γ visiting an edge which is in open mode or Part II mode, will only take the action associated to the mode of the edge if the present stage s is a γ-open stage. Similarly, a strategy γ visiting an edge which is in close mode or Part I mode, will only take the action associated to the mode of the edge if the present stage s is a γ-close stage. Either way if the mode of the edge does not match the required kind of stage, the strategy will terminate the stage early, and strategies below it will not be accessible during that stage.

There are three important consequences of this arrangement.

The first consequence is that during any given stage, either all accessible strategies will be able to enumerate elements into the set A, or else all accessible strategies will be able to enumerate elements into the set D. This allows us to associate actions which are incompatible with elements entering into D during the same stage with open modes (or Part II modes), and actions which are incompatible with elements entering into A during the same stage with close modes (or Part I modes).

The second consequence is the following. Suppose that there are infinitely many γ-open stages and infinitely many γ-close stages. Now, while a strategy γ may visit the edge lying on the true path during infinitely many stages, it may be the case that only finitely many of these are γ-open stages, or that only finitely many of these are γ-close stages. This is a result of the fact that strategies are now guessing the outcome which is to be visited during each stage. In practice we shall want to ensure that the existence of infinitely many γ-open stages and infinitely many γ-close stages guarantees that there are infinitely many γ^+-open stages and infinitely many γ^+-close stages, where γ^+ is the successor of γ on the true path. We shall postpone resolving this problem until section 3.7.1, when we shall address the question of 'fairness' in the context of satisfying multiple $\mathcal{R}$ and $\mathcal{S}$ requirements simultaneously.

The third consequence is that $\mathcal{S}$ *-Synchronisation* can be implemented in the following way. We shall only allow an $\mathcal{S}$ strategy γ to go to the next substage when visiting an edge with outcome $\langle g, \sigma \rangle$ which is in open mode if γ also enumerates a witness into A during the same stage. The same shall apply to the special case of strategies γ which enumerate elements into A when visiting edges with outcome $\langle d, \sigma \rangle$, and to strategies γ with outcome $\langle g_i, \sigma \rangle$ in more general settings. This solution implements $\mathcal{S}$ -Synchronisation due to the fact that if an $\mathcal{S}$ strategy γ is accessible during some stage s and visits an edge which allows it to enumerate a witness w into A, it must be the case that s is a γ-open stage. Hence the edge of any $\mathcal{S}$ strategy γ' on the path to γ must either be in open mode or Part II mode. It follows that every strategy γ' with an edge with outcome $\langle g, \sigma \rangle$ on the path to γ must be in open mode, and that γ can only be accessible if γ' has also enumerated a witness w' into A when visiting the edge, as required for S-Synchronisation.

In the above discussion we have omitted mentioning that there is an additional requirement for the present stage to qualify as a γ-open stage. The additional requirement states that if γ' is an $\mathcal{R}$ strategy above γ building a functional associated to an edge on the path to γ, then there can be no disagreement between the functional and the set A. In this manner, any $\mathcal{S}$ strategy γ below an $\mathcal{R}$ strategy γ' will refrain from enumerating elements into A when some functional being built by γ' for the edge lying above it already has a disagreement. This will allow γ' to remove the disagreement without γ introducing a new one during the same stage.

We are now in a position to define the questions needed for the modified $\mathcal{R}$ strategy.

3.5.3 Questions for the $\mathcal{R}$ Strategy

The $\mathcal{R}$ strategy β, will need to ask one question, which we denote by question Q_1. The question asks whether the strategy β sees infinitely many β-expansionary stages:

(1) Are there infinitely many $q \in \mathbb{N}_\beta$ such that the following holds?

(i) $(\forall q' < q)[\beta\text{-stage}(q') \Rightarrow l_q(\Phi_1^{U,V}, A) > l_{q'}(\Phi_1^{U,V}, A)]$.

(ii) $(\forall q' < q)[\beta\text{-stage}(q') \wedge (\forall q'' < q')[\,\beta\text{-stage}(q'') \Rightarrow l_{q'}(\Phi_1^{U,V}, A) > l_{q''}(\Phi_1^{U,V}, A)\,] \Rightarrow l_q(\Phi_2^{A,D}, U) > l_{q'}(\Phi_2^{A,D}, U)]$.

(iii) $(\forall q' < q)[\beta\text{-stage}(q') \wedge (\forall q'' < q')[\,\beta\text{-stage}(q'') \Rightarrow l_{q'}(\Phi_1^{U,V}, A) > l_{q''}(\Phi_1^{U,V}, A)\,] \Rightarrow l_q(\Phi_3^{A,D}, V) > l_{q'}(\Phi_3^{A,D}, V)]$.

If the strategy is accessible at some stage s, it will guess the answer to Q_1 by computing the apparent limit o and apparent use σ for $\lim_{t\to\infty} \Psi^{H_0}(f(Q_1), t)$ at stage s. If the answer corresponding to o is 'No', we denote the outcome by $\langle f, \sigma\rangle$. On the other hand, if the answer corresponding to o is 'Yes', we denote the outcome by $\langle i, \sigma\rangle$.

We now describe the $\mathcal{R}$ strategy itself.

3.5.4 The $\mathcal{R}$ Strategy

A $\mathcal{R}$ strategy β will either be following a Γ-strategy or a $\hat{\Gamma}$-strategy which will depend on its location on the priority tree. The outcomes of the strategy will be of the form $\langle i, \sigma\rangle$ and $\langle f, \sigma\rangle$. If the strategy is following a Γ-strategy, it will build a different functional $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}$ for every edge $\beta \frown \langle i, \sigma\rangle$ leaving β. Similarly if the strategy is following a $\hat{\Gamma}$-strategy, it will build a different functional $\Gamma^{V,D}_{\beta\frown\langle i,\sigma\rangle}$ for every edge $\beta \frown \langle i, \sigma\rangle$ leaving β. Each edge will have a separate set of uses $U^{e,\beta\frown\langle i,\sigma\rangle}$ from which the strategy will choose uses when defining the functional associated to the edge $\beta \frown \langle i, \sigma\rangle$. Note that e is the index of the strategy β in the total ordering of the $\mathcal{R}$ strategies lying on the priority tree.

The strategy goes through the following steps at stage s.

During its first step, the strategy β will calculate a rightward restraint $r(\beta \frown \langle o', \sigma'\rangle)$ for every edge $\beta \frown \langle o', \sigma'\rangle$ which has been previously accessible, exactly as in the previous section.

Similarly during its second step, the strategy β will perform its attachment procedure as in the previous section, attaching the stage s to a suitable edge if s is a β-expansionary stage.

During its third step the strategy β will calculate its attachment restraint $a(\beta \frown \langle o', \sigma'\rangle)$ for every edge $\beta \frown \langle o', \sigma'\rangle$ which has been previously accessible as in the previous section.

This is followed by its fourth step, where the strategy will calculate the edge $\beta \frown \langle o, \sigma\rangle$ to visit during the present stage.

Once the outcome has been determined, the strategy will perform its fifth step by calculating a downward restraint $d(\beta \frown \langle o', \sigma'\rangle, s)$. The downward restraint consists of three parts; the supremum of the rightward restraints imposed by edges lying to the left of $\beta \frown \langle o', \sigma'\rangle$, the attachment restraint imposed on any edge lying to the left of $\beta \frown \langle o', \sigma'\rangle$, and any previously computed downward restraint for the edge. In this way edges lying to the left of $\beta \frown \langle o', \sigma'\rangle$, any β-expansionary stages attached to them and any functionals associated to them will be protected from lower priority strategies.

During the final and sixth step, the strategy takes action depending on the outcome of the edge $\beta \frown \langle o, \sigma\rangle$.

Suppose that the outcome is $\langle f, \sigma\rangle$ and the edge is in open mode. If the present stage is not a β-open stage, we terminate the stage so as to wait for a β-open stage. Otherwise we go to the next substage and change the mode to close mode. On the other hand, suppose that the outcome is $\langle f, \sigma\rangle$ and the edge is in close mode. If the present stage is not a β-close stage, we terminate the stage so as to wait for a β-close stage. Otherwise we go to the next substage and change the mode back to open mode.

On the other hand, if the outcome is $\langle i, \sigma \rangle$, we have to consider the case where the edge is in open mode and the case where the edge is in close mode.

We start by considering the case where the edge is in open mode. In this case, we shall attempt to define the functional associated to the edge. We make the following observations.

Firstly, in order to define the functional, we first wait for a β-expansionary stage to be attached to the edge. Such a β-expansionary stage means that the strategy has determined that the length of agreement between $\Phi_1^{U,V}$ and A has increased, and that the strategy needs to increase the length of agreement between $\Gamma^{U,D}$ and A in response. Thus if no β-expansionary stage is attached to the edge, we terminate the stage and wait for a β-expansionary stage to become attached to the edge.

Secondly, the functional will only be defined if the present stage is a β-open stage. Since no elements are enumerated into D during a β-open stage, we avoid defining the functional and undefining it during the same stage. Thus if a β-expansionary stage is attached to the edge, but the present stage is not a β-open stage, we terminate the stage and wait for a β-open stage.

Thirdly, the functional cannot be defined if it presently disagrees with the set A at some element.

If there is no such disagreement, a β-expansionary stage is attached to the edge, and the present stage is a β-open stage, the strategy proceeds to define its functional based on the present length of agreement between $\Phi_1^{U,V}$ and A and detaches the β-expansionary stage from the edge. Whilst choosing uses to define the functional, the strategy will consider a number of additional constraints to those found in the previous section.

First, any uses chosen must be within any work interval imposed on the strategy. In this way the functional will be unaffected by higher priority strategies which enumerate elements into the sets A and D which are greater than or equal to the upper bound of the work interval. Second, elements which have already been enumerated into the set D cannot be chosen as uses. This is because β or some other strategy below β which takes the latter into consideration might need to enumerate this use into D. Third, if some lower priority strategy imposes a constraint on the functional to choose uses greater than a given element, the strategy will obey this constraint. Finally, suppose that the strategy needs to redefine the functional at some element x. If it cannot do so by choosing its previous use, the strategy will choose a use which is greater that every use which has been previously chosen by the strategy itself. This fourth condition is necessary to show that $D \leq_T H$ during the verification of the construction.

On the other hand, it could be the case that there is a disagreement between the functional and the set A. In this case we need the strategy to 'self-repair' and remove this disagreement by enumerating the use of the element into D. However elements can only be enumerated into D if the edge is in close mode. Thus if a β-expansionary stage is attached to the edge, the present stage is a β-open stage, but the functional disagrees with A at some element, we change the mode of the edge to close mode, and terminate the stage.

When the edge is in close mode, the strategy will determine whether the functional disagrees with the set A at some element, and if so will try to remove the disagreement.

We now consider the case where the edge is in close mode.

If there is no disagreement, but the present stage is not a β-close stage, we terminate the stage so as to wait for a β-close stage. Otherwise we go to the next substage and change the mode to open mode.

If there is a disagreement $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}(x) \neq A(x)$ at some element x, the strategy will first determine whether a β-expansionary stage is attached to the edge. Such a β-expansionary stage indicates that the length of agreement between $\Phi_1^{U,V}$ and A has increased. This means that any disagreement $\Phi_1^{U,V}(x) \neq A(x)$ which would have arisen as a result of x entering A at an earlier stage has been removed through a U or V change. Therefore the strategy needs to increase the length of agreement between $\Gamma^{U,D}$ and A in response. Thus if no β-expansionary stage is attached to the edge, we terminate the stage and wait for a β-expansionary stage to become attached to the edge.

If the edge is in close mode, there is a disagreement $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}(m) \neq A(m)$, and a β-expansionary stage is attached to the edge, the strategy will determine whether the present stage is a β-close stage. If this is not the case, we terminate the stage and wait for a β-close stage to occur. Otherwise the strategy is now in a position to enumerate $\gamma_{\beta\frown\langle i,\sigma\rangle}(m)$ into D so as to 'self-repair' and remove the disagreement. The stage is terminated so as not to allow lower priority strategies to introduce new disagreements, and the edge is changed to open mode. The strategy can now attempt to define its functional once again.

We shall now formalise the $\mathcal{R}$ strategy.

The $\mathcal{R}$ Strategy

The strategy β labeled $\mathcal{R}_i$ will either be following a Γ-strategy or a $\hat{\Gamma}$-strategy. Every edge $\beta \frown \langle i, \sigma \rangle$ has a functional $\Gamma^{U,D}_{\beta \frown \langle i,\sigma \rangle}$ (or $\Gamma^{V,D}_{\beta \frown \langle i,\sigma \rangle}$ resp.) associated to it, which the strategy will build when it visits that edge. Each edge $\beta \frown \langle i, \sigma \rangle$ will also have its own set of uses $U^{e,\beta \frown \langle i,\sigma \rangle}$ from which uses will be chosen when defining the respective functionals.

The strategy β may lie below a number of $\mathcal{R}$ strategies β'. Each such strategy β' imposes a downward restraint $d(\beta' \frown \langle o', \sigma' \rangle, s)$ on β at stage s, where $\beta' \frown \langle o', \sigma' \rangle$ is the edge of β' on the path leading to β.

The strategy β may also lie below a number of $\mathcal{S}$ strategies α'. Each such strategy α' imposes a downward restraint $d(\alpha' \frown \langle o', \sigma' \rangle, s)$ on β at stage s, where $\alpha' \frown \langle o', \sigma' \rangle$ is the edge of α' on the path leading to β. The strategy α' also imposes the diagonalisation restraint $R_{\alpha',s}$ on β at stage s. The strategy α' may also impose a work interval on β at stage s, depending on its outcome on the path leading to β. Finally, let $\alpha'' \subset \beta$ be the greatest $\mathcal{S}$ strategy (under $\subset$) which imposes a work interval on β. We shall denote the work interval imposed by α'' on β at stage s by (a_s, b_s).

(1) Define the rightward restraint $r(\beta \frown \langle o', \sigma' \rangle, s)$ for every edge $\beta \frown \langle o', \sigma' \rangle$ which was previously accessible as the least element x such that:

 (a) $x \geq t$ where t is some β-expansionary stage attached to $\beta \frown \langle o', \sigma' \rangle$.

 (b) $x \geq t$ where t is the last stage at which $\beta \frown \langle o', \sigma' \rangle$ was accessible.

 Go to step (2).

(2) If stage s is a β-expansionary stage, and there is some edge $\beta \frown \langle i, \sigma' \rangle$ which has been previously accessible and which has no β-expansionary stage attached to it, attach s to the leftmost such edge.

 If a β-expansionary stage s has been attached to some edge $\beta \frown \langle i, \sigma' \rangle$, consider every edge $\beta \frown \langle i, \sigma'' \rangle$ lying to the right of $\beta \frown \langle i, \sigma' \rangle$. If some β-expansionary stage s' is attached to $\beta \frown \langle i, \sigma'' \rangle$, detach the β-expansionary stage from the edge.

Go to step (3).

(3) Determine the edge $\beta \frown \langle o, \sigma \rangle$ which the strategy should visit at stage s.

Go to step (4).

(4) Define the attachment procedure restraint $a(\beta \frown \langle o', \sigma' \rangle, s)$ for every edge $\beta \frown \langle o', \sigma' \rangle$ which was previously accessible. If the strategy has not attached a β-expansionary stage s to some edge $\beta \frown \langle o'', \sigma'' \rangle <_L \beta \frown \langle o', \sigma' \rangle$ at stage s, define $a(\beta \frown \langle o', \sigma' \rangle, s) = 0$. Otherwise define $a(\beta \frown \langle o', \sigma' \rangle, s) = s$.

Also define the downward restraint $d(\beta \frown \langle o, \sigma \rangle, s)$ as the least element x such that:

(a) $x \geq \sup\{r(\beta \frown \langle o', \sigma' \rangle, s) \mid \beta \frown \langle o', \sigma' \rangle <_L \beta \frown \langle o, \sigma \rangle \ \wedge$
$\beta \frown \langle o', \sigma' \rangle$ has been previously accessible$\}$.

(b) $x \geq a(\beta \frown \langle o, \sigma \rangle, s)$.

(c) $x \geq d(\beta \frown \langle o, \sigma \rangle, t)$ for all $t < s$.

Go to step (5).

(5) Consider the edge $\beta \frown \langle o, \sigma \rangle$ being visited by the strategy at stage s. Take action according to the value of o through the following case analysis.

(a) $o = f$.

(i) $\beta \frown \langle f, \sigma \rangle$ is in open mode and s is not a β-open stage. End the stage s, and go to stage $s + 1$.

(ii) $\beta \frown \langle f, \sigma \rangle$ is in open mode and s is a β-open stage. Set the edge to close mode. Go to the next substage.

(iii) $\beta \frown \langle f, \sigma \rangle$ is in close mode and s is not a β-close stage. End the stage s, and go to stage $s + 1$.

(iv) $\beta \frown \langle f, \sigma \rangle$ is in close mode and s is a β-close stage. Set the edge to open mode. Go to the next substage.

(b) $o = i$.

(i) $\beta \frown \langle i, \sigma\rangle$ is in open mode and there is no β-expansionary stage attached to the edge. End the stage s, and go to stage $s+1$.

(ii) $\beta \frown \langle i, \sigma\rangle$ is in open mode and there is a β-expansionary stage attached to the edge and s is not a β-open stage. End the stage s, and go to stage $s+1$.

(iii) $\beta \frown \langle i, \sigma\rangle$ is in open mode and there is a β-expansionary stage attached to the edge and s is a β-open stage.

If there is some element m such that $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}[s](m) \neq A_s(m)$ (or $\Gamma^{V,D}_{\beta\frown\langle i,\sigma\rangle}[s](m)$ resp.), set the edge to close mode. End the stage s, and go to stage $s+1$.

Otherwise, detach the stage s from the edge. Consider every $x < l_s(\Phi_1^{U,V}, A)$ such that $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}[s](x) \uparrow$. Define the axiom $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}[s](x) = A_s(x)$. Consider the least element $u < s$ in $U^{e,\beta\frown\langle i,\sigma\rangle}$ (if it exists) such that:

(A) $u \geq \gamma_{\beta\frown\langle i,\sigma\rangle}[t](x)$ for all $t < s$.

(B) $u > \gamma_{\beta\frown\langle i,\sigma\rangle}[s](y)$ for all $y < x$.

(C) $u > \sup\{r(\beta \frown \langle o', \sigma'\rangle, s) \mid \beta \frown \langle o', \sigma'\rangle <_L \beta \frown \langle o, \sigma\rangle \ \wedge$
$\beta \frown \langle o', \sigma'\rangle$ has been previously accessible$\}$.

(D) $u > a(\beta \frown \langle i, \sigma\rangle, s)$.

(E) $u > R_{\alpha',s}$, for every $\mathcal{S}$ strategy $\alpha' \subset \beta$.

(F) $u > d(\beta' \frown \langle o', \sigma'\rangle, s)$, for every $\mathcal{R}$ strategy $\beta' \subset \beta$ with edge $\beta' \frown \langle o', \sigma'\rangle$ on the path leading to β.

(G) $u > d(\alpha' \frown \langle o', \sigma'\rangle, s)$, for every $\mathcal{S}$ strategy $\alpha' \subset \beta$ with edge $\alpha' \frown \langle o', \sigma'\rangle$ on the path leading to β.

(H) $a_s < u < b_s$.

(I) $u \notin D$.

(J) $u > y$, where y is a constraint imposed by some $\mathcal{S}$ strategy α below β.

(K) $u > t$, where t is the last stage at which the edge $\beta \frown \langle i, \sigma\rangle$ was last initialised.

If u does not exist, $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}(x)$ is not defined.

Otherwise let $t' < s$ be the greatest stage such that $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}[t'](x) \downarrow$, and let

u' be the greatest use which the strategy has chosen so far when defining its functional at some element.

If t' does not exist, define $\gamma_{\beta\frown\langle i,\sigma\rangle}[s](x) = u$.

If t' exists and $u > \gamma_{\beta\frown\langle i,\sigma\rangle}[t'](x)$, define $\gamma_{\beta\frown\langle i,\sigma\rangle}[s](x)$ to be the least element in $U^{e,\beta\frown\langle i,\sigma\rangle}$ which is greater than u'.

Otherwise define $\gamma_{\beta\frown\langle i,\sigma\rangle}[s](x) = \gamma_{\beta\frown\langle i,\sigma\rangle}[t'](x)$.

Set the edge to close mode. Go to the next substage.

($\Gamma^{V,D}$ resp.)

(iv) $\beta \frown \langle i,\sigma\rangle$ is in close mode and there is an element m such that $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}[s](m) \neq A_s(m)$ (or $\Gamma^{V,D}_{\beta\frown\langle i,\sigma\rangle}[s](m)$ resp.) and there is no β-expansionary stage attached to the edge. End the stage s, and go to stage $s+1$.

(v) $\beta \frown \langle i,\sigma\rangle$ is in close mode and there is an element m such that $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}[s](m) \neq A_s(m)$ (or $\Gamma^{V,D}_{\beta\frown\langle i,\sigma\rangle}[s](m)$ resp.) and there is a β-expansionary stage attached to the edge and s is not a β-close stage. End the stage s, and go to stage $s+1$.

(vi) $\beta \frown \langle i,\sigma\rangle$ is in close mode and there is an element m such that $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}[s](m) \neq A_s(m)$ (or $\Gamma^{V,D}_{\beta\frown\langle i,\sigma\rangle}[s](m)$ resp.) and there is a β-expansionary stage attached to the edge and s is a β-close stage. Enumerate $\gamma_{\beta\frown\langle i,\sigma\rangle}[s](m)$ into D. Set the edge to open mode. End the stage s and go to stage $s+1$.

(vii) $\beta \frown \langle i,\sigma\rangle$ is in close mode and there is no element m such that $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}[s](m) \neq A_s(m)$ (or $\Gamma^{V,D}_{\beta\frown\langle i,\sigma\rangle}[s](m)$ resp.) and s is not a β-close stage. End the stage s, and go to stage $s+1$.

(viii) $\beta \frown \langle i,\sigma\rangle$ is in close mode and there is no element m such that $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}[s](m) \neq A_s(m)$ (or $\Gamma^{V,D}_{\beta\frown\langle i,\sigma\rangle}[s](m)$ resp.) and s is a β-close stage. Set the edge to open mode. Go to the next substage.

We are now in a position to analyse whether the $\mathcal{R}$ strategy β satisfies the $\mathcal{R}$ requirement.

3.5.5 Analysis of Outcomes

Consider the leftmost outcome $\beta \frown \langle o, \sigma \rangle$ which is visited infinitely often by β. We perform the following case analysis depending on the outcome $\langle o, \sigma \rangle$.

[f] Suppose that the outcome is $\langle f, \sigma \rangle$. Then the answer to question Q_1 is 'No'. This means that there are only finitely many β-expansionary stages. Hence one or more of $\Phi_1^{U,V} \neq A$, $\Phi_2^{A,D} \neq U$ or $\Phi_3^{A,D} \neq V$ must be the case. It follows that the strategy satisfies its requirement trivially.

[i] Suppose that the outcome is $\langle i, \sigma \rangle$. Then the answer to question Q_1 is 'Yes'. This means that there are infinitely many β-expansionary stages.

Through an argument identical to the one found in the previous section, we have that if there is no β-expansionary stage attached to the edge $\beta \frown \langle o, \sigma \rangle$ at some stage $s_1 > s_0$, some β-expansionary stage will eventually be attached to the edge at some stage $s_2 > s_1$.

In addition we claim that it is possible to ensure that the edge $\beta \frown \langle i, \sigma \rangle$ is visited during infinitely many β-open stages and infinitely many β-close stages. We address this claim when we discuss *fairness* in section 3.7.1.

Suppose that there is some stage $s > s_0$ such that for all $t > s$, there is no element x giving a disagreement $\Gamma^{U,D}_{\beta \frown \langle i, \sigma \rangle}[t](x) \downarrow \neq A_t(x)$. Then if the edge is in open mode at such a stage t, we have that a β-expansionary stage is eventually attached to the edge. We also have that the edge eventually becomes accessible during some β-open stage. This allows the strategy to build the functional $\Gamma^{U,D}_{\beta \frown \langle i, \sigma \rangle}$ to agree with the set A according to the length of agreement witnessed at stage s and to change the mode of the edge to close mode. On the other hand, if the edge is in close mode at some stage $t > s$, we have that the edge eventually becomes accessible during some β-close stage, allowing the strategy to set the edge to open mode. It follows that the length of agreement between $\Gamma^{U,D}_{\beta \frown \langle i, \sigma \rangle}$ and A increases infinitely often. This means that $\Gamma^{U,D}_{\beta \frown \langle i, \sigma \rangle} = A$, and that the requirement is satisfied

On the other hand suppose that at some stage $s > s_0$, there is some disagreement $\Gamma^{U,D}_{\beta \frown \langle i, \sigma \rangle}[s](w) \neq A_s(w)$ for some witness w. Then we have that a β-expansionary stage is eventually attached to the edge. If the edge is in open mode, we also have that the edge is eventually visited during a β-open stage, which causes the mode of the edge to be changed to close mode. Once the edge is in close mode, we have that the edge is eventually visited during a β-close stage. This causes the strategy β to enumerate $\gamma_{\beta\frown\langle i,\sigma\rangle}(w)$ into D and remove the disagreement, change the mode of the edge to open mode and terminate the stage.

Now, if the strategy visits the edge again, it will terminate the stage unless the present stage is a β-open stage. Hence, no $\mathcal{S}$ strategy α below β is able to enumerate any element into A until this is the case. Once the strategy visits the edge during a β-open stage, it will now proceed to build the functional $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}$ to agree with the set A according to the length of agreement witnessed at stage s.

Hence we have that every disagreement results in the length of agreement between $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}$ and A increasing. Since there are infinitely many disagreements, and each one is removed in this manner, we have that $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle} = A$, and that the requirement is satisfied.

(The above also holds for $\Gamma^{V,D}$ respectively).

We shall now proceed to discuss the $\mathcal{S}$ strategy. We start by defining the questions which will be needed for this purpose.

3.5.6 Questions for the $\mathcal{S}$ Strategy

The $\mathcal{S}$ strategy α will need to ask a number of questions, which take the context of the strategy into consideration. In particular α may lie below some $\mathcal{R}$ strategy β which is active for α and which follows a Γ-strategy or a $\hat{\Gamma}$-strategy. In this case the strategy β will have an outgoing edge $\beta \frown \langle i, \sigma\rangle$, which lies on the path leading to the strategy α. In addition $\mathcal{S}$ strategies α' lying above α may impose a work interval on α. The work interval imposed at stage s by the greatest strategy α' (under $\subset$) above α is denoted by (a_s, b_s).

The strategy starts by asking question Q_1. This question asks whether there are infinitely many witnesses w and stages s such that w and $\theta_s(w)$ lie inside the work interval (a_s, b_s) at stage s and such that the computation $\Theta^D[s](w) \downarrow= 0$ holds. In addition the question also asks whether the length of agreement between the functional Θ^D and the set A expands infinitely often.

(1) Are there infinitely many $w \in W^e$, $s \in \mathbb{N}_\alpha$ and $q \in \mathbb{N}_\alpha$ such that the following hold?

(i) $\Theta^D[s](w) \downarrow = 0$.

(ii) $a_s < w < b_s$ (if a work interval is imposed on the strategy).

(iii) $a_s < \theta_s(w) < b_s$ (if a work interval is imposed on the strategy).

(iv) $(\forall q' < q)[\, \alpha\text{-stage}(q') \Rightarrow l_{q'}(\Theta^D, A) < l_q(\Theta^D, A)\,]$.

If β exists, the strategy proceeds by asking question Q_2. A positive answer to question Q_1 asserts that there are infinitely many witnesses w and stages s such that w and $\theta_s(w)$ lie inside the work interval (a_s, b_s) and such that the computation $\Theta^D[s](w) \downarrow = 0$ holds. Question Q_2 also asks whether infinitely many of these witnesses w and stages s give rise to computations $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}[s](w)$ (or $\Gamma^{V,D}_{\beta\frown\langle i,\sigma\rangle}[s](m)$ resp.) which are honest.

(2) Are there infinitely many $w \in W^e$, $s \in \mathbb{N}_\alpha$ and $q \in \mathbb{N}_\alpha$ such that the following hold?

(i) $\Theta^D[s](w) \downarrow = 0$.

(ii) $a_s < w < b_s$ (if a work interval is imposed on the strategy).

(iii) $a_s < \theta_s(w) < b_s$ (if a work interval is imposed on the strategy).

(iv) $(\forall q' < q)[\, \alpha\text{-stage}(q') \Rightarrow l_{q'}(\Theta^D, A) < l_q(\Theta^D, A)\,]$.

(v) $\phi_{1,s}(w) \leq \gamma_{\beta\frown\langle i,\sigma\rangle}[s](w)$.

Following this if β exists and is following a Γ-strategy, we proceed by asking question Q_3. A positive answer to question Q_1 and question Q_2 asserts that there are infinitely many witnesses w and stages s which give rise to computations $\Theta^D[s](w) \downarrow = 0$, and to honest computations $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}[s](w)$, where w and $\theta_s(w)$ lie inside (a_s, b_s). Question Q_3 also asks whether infinitely many of these witnesses w enter A at stage s. In addition it asks whether a $U \upharpoonright \phi_{1,s}(w)$ change occurs by the least $\mathcal{R}$ -expansionary stage $t > s$.

(3) Are there infinitely many $w \in W^e$, $s \in \mathbb{N}_\alpha$, $t \in \mathbb{N}$ and $q \in \mathbb{N}_\alpha$ such that the following hold?

(i) $\Theta^D[s](w) \downarrow = 0$.

(ii) $a_s < w < b_s$ (if a work interval is imposed on the strategy).

(iii) $a_s < \theta_s(w) < b_s$ (if a work interval is imposed on the strategy).

(iv) $(\forall q' < q)[\, \alpha\text{-stage}(q') \Rightarrow l_{q'}(\Theta^D, A) < l_q(\Theta^D, A)\,]$.

(v) $\phi_{1,s}(w) \leq \gamma_{\beta^\frown\langle i,\sigma\rangle}[s](w)$.

(vi) $A_s(w) = 0$.

(vii) $A_{s+1}(w) = 1$.

(viii) $t > s$.

(ix) $(\forall s < t' < t)[U_{t'} \upharpoonright \phi_{1,s}(w) = U_s \upharpoonright \phi_{1,s}(w)]$.

(x) $(\forall s < t' < t)[V_{t'} \upharpoonright \phi_{1,s}(w) = V_s \upharpoonright \phi_{1,s}(w)]$.

(xi) $U_t \upharpoonright \phi_{1,s}(w) \neq U_s \upharpoonright \phi_{1,s}(w)$.

If the strategy is accessible at some stage s, it will guess the answer to questions Q_1, Q_2 and Q_3 (where applicable). This is done by computing an apparent limit o_i and an apparent use σ_i for each $\lim_{t\to\infty} \Psi^{H_0}(f(Q_i), t)$ at stage s. Let σ be the apparent use of greatest length. The outcome visited by the strategy at stage s is calculated as follows.

- If the answer corresponding to o_1 is 'No', we denote the outcome by $\langle w, \sigma\rangle$.
- If β exists, the answer corresponding to o_1 is 'Yes' and the answer corresponding to o_2 is 'No', we denote the outcome by $\langle h, \sigma\rangle$.
- If β exists and is following a Γ-strategy, the answer corresponding to o_1 is 'Yes', the answer corresponding to o_2 is 'Yes', and the answer corresponding to o_3 is 'No', we denote the outcome by $\langle g, \sigma\rangle$.
- If β exists and is following a Γ-strategy, the answer corresponding to o_1 is 'Yes', the answer corresponding to o_2 is 'Yes', and the answer corresponding to o_3 is 'Yes', we denote the outcome by $\langle d, \sigma\rangle$.
- If β does not exist and the answer corresponding to o_1 is 'Yes', we denote the outcome by $\langle d, \sigma\rangle$.
- If β exists and follows a $\hat{\Gamma}$-strategy, the answer corresponding to o_1 is 'Yes', and the answer corresponding to o_2 is 'Yes', we denote the outcome by $\langle d, \sigma\rangle$.

We now formalise the $\mathcal{S}$ strategy.

3.5.7 The $\mathcal{S}$ Strategy

The $\mathcal{S}$ strategy α has an infinite set of witnesses W^e, and an infinite set of thresholds V^e, where e is the index of the strategy α in the total ordering of the $\mathcal{S}$ strategies lying on the priority tree. At any given stage s the strategy will also be able to impose a restraint $R_{\alpha,s}$ on all lower priority strategies. Initially, we have that $R_{\alpha,0}$ is equal to 0. The strategy α shall use the fact that $R_{\alpha,s} > 0$ to signal that it has diagonalised. Once this restraint has been set, it will keep its value during subsequent stages. The strategy may lie below at most one $\mathcal{R}$ strategy β which is active for α. The outcomes of the strategy will be of the form $\langle d, \sigma \rangle$, $\langle h, \sigma \rangle$ and $\langle w, \sigma \rangle$, while the outcome $\langle g, \sigma \rangle$ will be present if β exists and is following a Γ-strategy.

The strategy goes through the following steps at stage s.

During its first step, the strategy determines whether it has opened a gap by enumerating some witness w' into the set A during the last stage t at which it was accessible (assuming it was accessible at least once before, and that it has not been initialised in the meantime). Suppose that this has been the case, then we have to consider the following three scenarios.

Suppose that there is an $\mathcal{R}$ strategy $\beta \subset \alpha$ which is active for α and is following a Γ-strategy. Then we have that the strategy β has an edge $\beta \frown \langle i, \sigma \rangle$ on the path leading to α. In this case, the strategy α must determine the way in which the disagreement $\Phi^{U,V}(w') \neq A(w')$ which was introduced when the strategy enumerated w' into A at stage t was removed. It is important to note that the disagreement must have been removed due to the occurrence of an $\mathcal{R}$ expansionary stage at some least stage t' such that $t < t' \leq s$. If this were not the case, β would not have gone to the next substage whilst visiting the edge $\beta \frown \langle i, \sigma \rangle$ above α at stage s, and α would not have been accessible at stage s, giving a contradiction. Now if a $U \restriction \phi_1[t](w')$ change has occurred, the strategy has diagonalised and will set the restraint $R_{\alpha,s} = \theta_t(w)$ so as to protect the computation used in the diagonalisation, and go to the next step. If this is not the case, the diagonalisation attempt has failed and the strategy goes to the next step.

On the other hand suppose that there is an $\mathcal{R}$ strategy $\beta \subset \alpha$ which is active for α and is following a $\hat{\Gamma}$-strategy. Then we must have that the disagreement $\Phi^{U,V}(w') \neq A(w')$ was removed through a $V \restriction \phi_1[t](w')$ change due to the way in which the strategies are organised on the priority tree.

The strategy has diagonalised and will go to the next step.

Finally suppose there is no $\mathcal{R}$ strategy $\beta \subset \alpha$ which is active for α. Then we have that the strategy α is not concerned with the functionals built by any higher priority $\mathcal{R}$ strategy. Hence the enumeration of w' into A at stage t is sufficient for the strategy to diagonalise. The strategy has diagonalised and will go to the next step.

If the strategy has diagonalised, it will set the restraint $R_{\alpha,s} = \theta_t(w)$ to protect this diagonalisation. The strategy will also detach every witness attached to one of its edges, and will undefine every work interval defined for one its edges. This is due to the fact that once the strategy has diagonalised, its work is complete and it has no further use for these elements.

During its second step, the strategy α calculates a rightward restraint $r(\alpha \frown \langle o', \sigma' \rangle)$ for every edge $\alpha \frown \langle o', \sigma' \rangle$ which has been previously accessible, exactly as in the previous section.

During its third step, the strategy α will perform its attachment procedure.

If the strategy has already diagonalised ($R_{\alpha,s} > 0$), no further action needs to be taken and the attachment procedure will be terminated.

Otherwise the attachment procedure will consider in turn every witness w in W^e which at stage s yields a computation $\Theta^D[s](w) \downarrow = 0$ and which has not been attached to an edge so far. The attachment procedure will be seeking to attach one of these witnesses to an edge, and will stop considering further witnesses once this has been achieved.

In order to decide which edge the witness under consideration should be attached to, the strategy will first consider the kind of outcomes which the strategy has. These are determined by the position of the strategy on the priority tree.

There are three cases to consider.

In the first (and most general) case we have that the $\mathcal{S}$ strategy α lies below some $\mathcal{R}$ strategy β which is active for α and which follows a Γ-strategy. In this case, α has edges with outcomes $\langle d, \sigma \rangle$, $\langle g, \sigma \rangle$, $\langle h, \sigma \rangle$ and $\langle w, \sigma \rangle$. Hence the strategy will have to determine whether the computation $\Gamma^{U,D}_{\beta \frown \langle i,\sigma \rangle}[s](w)$ is honest, which will be the case if $\phi_1[s](w) \leq \gamma_{\beta \frown \langle i,\sigma \rangle}[s](w)$. The strategy will then attach the witness to the appropriate edge depending on the result.

If the witness w is honest at stage s, we attach it to the leftmost edge of the form $\alpha \frown \langle g, \sigma \rangle$ which does not have a witness attached and which has been previously accessible, as long as it satisfies the following constraints.

First, the witness w has to be greater than all restraints imposed on the edge. This includes the supremum of all rightward restraints imposed by edges lying to the left of $\alpha \frown \langle g, \sigma \rangle$, the downward restraints which might be imposed by some higher priority strategy on an edge leading to α and any diagonalisation restraint $R_{\alpha',s}$ which might be imposed by some $\mathcal{S}$ strategy α' lying above α. Second the witness w and the use $\theta_s(w)$ must lie inside any work interval imposed by some higher priority $\mathcal{S}$ strategy which might lie above it at stage s. This ensures that the computation $\Theta^D[s](w)$ is not affected when the higher priority strategy enumerates some element greater than or equal to the upper bound of the work interval into D. Third, the edge $\alpha \frown \langle g, \sigma \rangle$ has to be in open mode. In this way the witness is only attached to the edge if the edge needs to enumerate a witness into A. Fourth, a work interval for the edge $\alpha \frown \langle g, \sigma \rangle$ must be defined. Fifth, the witness must be greater than the upper bound of this work interval. Sixth, the witness must be greater than the last stage at which the edge was last initialised. In this way w cannot affect any computation which has previously taken place to the left of the edge. Finally w must be greater than any witness which has previously been attached to the edge. This will be necessary to show that $D \leq_T H$ during the verification of the construction.

On the other hand, if the witness w is dishonest at stage s, we attach it to the leftmost edge of the form $\alpha \frown \langle h, \sigma \rangle$ which does not have a witness attached and which has been previously accessible, as long as it satisfies a number of constraints.

First, the witness w has to be greater than all restraints imposed on the edge. This includes the supremum of all rightward restraints imposed by edges lying to the right of $\alpha \frown \langle h, \sigma \rangle$, the downward restraints which might be imposed by some higher priority strategy on an edge leading to α and any diagonalisation restraint $R_{\alpha',s}$ which might be imposed by some $\mathcal{S}$ strategy α' lying above α. Second, the use $\theta_s(w)$ must lie inside any work interval imposed by some higher priority $\mathcal{S}$ strategy which might lie above it at stage s. Third, the witness must be greater than the upper bound of any work interval defined for some edge lying to the left of $\alpha \frown \langle h, \sigma \rangle$. Since attaching the witness will also define a work interval for the edge, this constraint ensures that the proper ordering between work intervals is maintained. Fourth, the witness must be greater than the last stage at which the edge was last initialised. Fifth the witness must be greater than any witness which has previously been attached to the edge. This will be necessary to show that $D \leq_T H$ during the verification of the construction.

When a witness w is attached to an edge with outcome $\langle h, \sigma \rangle$ at stage s, the strategy defines a work interval $(w, \gamma_{\beta \frown \langle i, \sigma \rangle}[s](w))$ for the edge.

In the second case we consider the special case where the $\mathcal{S}$ strategy α lies below some $\mathcal{R}$ strategy β which is active for α but which is following a $\hat{\Gamma}$-strategy. Hence we have that α has edges with outcomes $\langle d, \sigma \rangle$, $\langle h, \sigma \rangle$ and $\langle w, \sigma \rangle$. This means that there are no edges with outcome $\langle g, \sigma \rangle$ to which the strategy can attach witnesses w giving honest computations $\Gamma^{V,D}_{\beta \frown \langle i, \sigma \rangle}[s](w)$. Hence the attachment procedure will instead attach these witnesses to the leftmost edge with outcome $\langle d, \sigma \rangle$ which has been visited previously and which presently has no witness attached. The witness will also have to satisfy an appropriate subset of the constraints described above for edges with $\langle g, \sigma \rangle$ outcomes.

In the third case we consider the special case where there is no active $\mathcal{R}$ strategy β above α. Hence we have that α has edges with outcomes $\langle d, \sigma \rangle$ and $\langle w, \sigma \rangle$. This means that α is not concerned with any $\mathcal{R}$ strategies lying above it. This means that the strategy does not need to discern between honest and dishonest witnesses. In addition there are no edges with outcome g to which the strategy can attach witnesses. Thus the strategy will attach any witness which gives a computation $\Theta^D[s](w) \downarrow = 0$ to the leftmost edge with outcome $\langle d, \sigma \rangle$ which has been visited previously and which presently has no witness attached. The witness will also have to satisfy an appropriate subset of the constraints described above for edges with $\langle g, \sigma \rangle$ outcomes.

The attachment procedure concludes by determining whether it has attached a witness w to some edge at stage s. If this is the case it will consider every edge lying to its right and detach any witness and undefine every work interval defined for the edge. It will also set the edge to open mode or to Part I mode as appropriate.

During its fourth step the strategy α will calculate the edge $\alpha \frown \langle o, \sigma \rangle$ to visit during the present stage. It then takes action according to the outcome of this edge.

This is followed by its fifth step, where the strategy α will calculate the attachment restraint $a(\alpha \frown \langle o', \sigma' \rangle, s)$ for every edge $\alpha \frown \langle o', \sigma' \rangle$ which has been previously accessible. If the strategy has attached a witness w to some edge $\alpha \frown \langle o'', \sigma'' \rangle$ lying to the left of $\alpha \frown \langle o', \sigma' \rangle$ at stage s, we set $a(\alpha \frown \langle o', \sigma' \rangle) = \theta_s(w)$. The restraint will have to be obeyed by the edge $\alpha \frown \langle o', \sigma' \rangle$ and protects the computation $\Theta^D[s](w) \downarrow = 0$ from any of the computations taking place to the right of $\alpha \frown \langle o'', \sigma'' \rangle$.

The strategy also calculates a downward restraint $d(\alpha \frown \langle o, \sigma \rangle, s)$ for the edge $\alpha \frown \langle o, \sigma \rangle$ which is visited by the strategy during stage s. The downward restraint consists of three parts; the supremum of the rightward restraints imposed by edges lying to the left of $\alpha \frown \langle o, \sigma \rangle$, the attachment restraint imposed on any edge lying to the left of $\alpha \frown \langle o, \sigma \rangle$, and any previously computed downward restraint for the edge. In this way edges lying to the left of $\alpha \frown \langle o, \sigma \rangle$, and the uses $\theta_t(w')$ of the witnesses w' attached to them at some stage $t \leq s$ will be protected from lower priority strategies.

During the final and fifth step, the strategy takes action depending on the outcome of the edge $\alpha \frown \langle o, \sigma \rangle$ which the strategy is visiting during stage s.

There are three cases to consider.

In the first (and most general) case we have that the $\mathcal{S}$ strategy α lies below some $\mathcal{R}$ strategy β which is active for α and which follows a Γ-strategy.

Suppose that the strategy visits an outcome of the form $\alpha \frown \langle w, \sigma \rangle$ and the edge is in open mode. If the present stage is not a β-open stage, we terminate the stage so as to wait for a β-open stage. Otherwise the strategy will count visiting the edge as having taken action successfully, changing the mode of the edge back to close mode and going to the next substage.

On the other hand, suppose that the outcome is $\langle w, \sigma \rangle$ and the edge is in close mode. If the present stage is not an α-close stage, we terminate the stage so as to wait for an α-close stage. Otherwise the strategy will count visiting the edge as having taken action successfully, changing the mode of the edge back to open mode and going to the next substage.

Suppose that the strategy visits an outcome of the form $\alpha \frown \langle g, \sigma \rangle$. If the strategy has already diagonalised, the stage is terminated. Otherwise we have that the edge is either in open mode or in close mode.

If the edge is in open mode, the strategy will first determine whether a work interval for the edge is defined. If this is not the case, the strategy will choose a threshold $v < s$ from V^e so as to define a work interval $(v, \gamma_{\beta\frown\langle i,\sigma\rangle}[s](v))$ for the edge. This threshold has to obey certain constraints.

First, the threshold v has to be greater than all restraints imposed on the edge. This includes the supremum of all rightward restraints imposed by edges lying to the left of $\alpha \frown \langle g, \sigma\rangle$, the attachment restraint imposed on $\alpha \frown \langle g, \sigma\rangle$, any diagonalisation restraint $R_{\alpha',s}$ which might be imposed by some $\mathcal{S}$ strategy α' lying above α and the downward restraints which might be imposed by higher priority strategies on an edge leading to α. Second, the threshold must lie inside any work interval imposed on α. Third, the threshold v must be greater than the upperbound of any work interval defined for some edge lying to the left of $\alpha \frown \langle g, \sigma\rangle$. Fourth, the threshold v must be greater than an witness attached to some edge lying to the left of $\alpha \frown \langle g, \sigma\rangle$. Fifth, the threshold must be greater than the last stage at which the edge was last initialised.

Once a work interval is defined for the edge, the strategy will determine whether a witness is attached to the edge. If this is not the case, the strategy will terminate the stage and wait for a witness to be attached. If a work interval is defined for the edge and a witness has been attached to the edge, the strategy will determine whether the witness still gives an honest computation, that is whether $\phi_{1,s}(w) \leq \gamma_{\beta\frown\langle i,\sigma\rangle}[s](w)$. If the witness has become dishonest since it was attached, the witness is detached from the edge. If a work interval is defined for the edge, a witness has been attached to the edge and the witness is honest, the strategy determines whether the present stage is an α-open stage. If this is not the case, the strategy will terminate the stage and wait for an α-open stage. If the strategy visits the edge, a work interval has been defined, a witness w has been attached, the witness is honest, and the present stage is an α-open stage, the strategy can finally take action and open a gap by enumerating the witness w into the set A. Since the strategy has taken action successfully, it changes the mode of the edge to close mode and goes to the next substage.

When the edge is in close mode, the strategy will determine whether the present stage is an α-close stage. If this is not the case, the strategy will terminate the stage and wait for an α-close stage. If the strategy visits the edge and the present stage is an α-close stage, the strategy will perform capricious destruction by enumerating the upper bound of the work interval of the edge $\gamma_{\beta \frown \langle i,\sigma \rangle}[s](v)$ into the set D. Since the strategy has taken action successfully, it changes the mode of the edge to open mode and goes to the next substage.

Suppose now that the strategy visits an outcome of the form $\alpha \frown \langle h, \sigma \rangle$. If the strategy has already diagonalised, the stage is terminated. Otherwise we have that the edge is either in Part I mode or in Part II mode.

If the edge is in Part I mode, the strategy will determine whether a witness is attached to the edge. If this is not the case, the strategy will terminate the stage and wait for a witness to be attached. If a witness w is attached to the edge during the present stage, the work interval $(w, \gamma_{\beta \frown \langle i,\sigma \rangle}[s](w))$ will be defined for the edge and the stage will be terminated. If a witness is attached to the edge and the work interval is defined for the edge, the strategy will determine whether the present stage is an α-close stage. If this is not the case, the strategy will terminate the stage and wait for an α-close stage. Otherwise, the strategy will determine whether the witness w attached to the edge is still dishonest. If this is not the case, the strategy will terminate the stage and wait until the witness becomes dishonest again. If a witness is attached to the edge and the work interval for the edge is defined, the present stage is an α-close stage, and the witness w is dishonest, it will perform honestification by enumerating $\gamma_{\beta \frown \langle i,\sigma \rangle}[s](w)$ into the set D. Since the strategy has taken action successfully, it changes the mode of the edge to Part II mode and goes to the next substage.

When the edge is in Part II mode, the strategy will determine whether the present stage is an α-open stage. If this is not the case, the strategy will terminate the stage and wait for an α-open stage. Otherwise, the strategy will take no action. This will count as the strategy having taken action successfully, changing the mode of the edge back to Part I mode and going to the next substage.

Suppose that the strategy visits an outcome of the form $\alpha \frown \langle d, \sigma \rangle$. There are three cases to consider.

In the first case we have that the $\mathcal{S}$ strategy α lies below some $\mathcal{R}$ strategy β which is active for α and which follows a Γ-strategy. Then we do nothing and terminate the stage.

In the second case we consider the special case where the $\mathcal{S}$ strategy α lies below some $\mathcal{R}$ strategy

β which is active for α but which is following a $\hat{\Gamma}$-strategy. In this case α has edges with outcomes $\langle d, \sigma\rangle$, $\langle h, \sigma\rangle$ and $\langle w, \sigma\rangle$. The actions taken when the strategy visits edges with outcomes $\langle h, \sigma\rangle$ and $\langle w, \sigma\rangle$ remain the same as described above. On the other hand the actions taken when the strategy visits an edge with outcome $\langle d, \sigma\rangle$ change so as to reflect the fact that witnesses are now enumerated into the set A by first being attached to an edge with outcome $\langle d, \sigma\rangle$.

Suppose that the strategy visits an outcome of the form $\alpha \frown \langle d, \sigma\rangle$ and that the edge has no witness attached. Then the strategy terminates the stage and waits for a witness to become attached. Once a witness becomes attached to the edge, the strategy will determine whether the witness still gives an honest computation, that is whether $\phi_1[s](w) \leq \gamma_{\beta\frown\langle i,\sigma\rangle}[s](w)$. If the computation has become dishonest since it was attached, the witness is detached from the edge. If a witness has been attached to the edge and the witness gives an honest computation, the strategy determines whether the present stage is an α-open stage. If this is not the case, the strategy will terminate the stage and wait for an α-open stage. If the strategy visits the edge, a witness w has been attached, the witness gives an honest computation, and the present stage is an α-open stage, the strategy can finally take action and open a gap by enumerating the witness w into the set A. The strategy terminates the stage.

In the third case we consider the special case where there is no active $\mathcal{R}$ strategy β above α. Hence we have that α has edges with outcomes $\langle d, \sigma\rangle$ and $\langle w, \sigma\rangle$. The actions taken when the strategy visits edges with outcome w remains the same as described above. On the other hand the actions taken when the strategy visits an edge with outcome $\langle d, \sigma\rangle$ change so as to reflect the fact that witnesses are now enumerated into the set A by first being attached to an edge with outcome $\langle d, \sigma\rangle$, and the fact that α ignores all higher priority $\mathcal{R}$ strategies, and is thus able to open a gap without the witness having to give honest computations.

Suppose that the strategy visits an outcome of the form $\alpha \frown \langle d, \sigma\rangle$ and that the edge has no witness attached. Then the strategy terminates the stage and waits for a witness to become attached. Once a witness becomes attached to the edge, the strategy determines whether the present stage is an α-open stage. If this is not the case, the strategy will terminate the stage and wait for an α-open stage. If the strategy visits the edge, a witness w has been attached and the present stage is an α-open stage, the strategy can finally take action and open a gap by enumerating the witness w

into the set A. The strategy terminates the stage.

We shall now formalise the $\mathcal{S}$ strategy.

The $\mathcal{S}$ Strategy

The $\mathcal{S}$ strategy α has a set of witnesses W^e and a set of thresholds V^e, and at every stage s is able to impose a restraint $R_{\alpha,s}$ on lower priority strategies. Initially we have that $R_{\alpha,0} = 0$, and if the strategy sets $R_{\alpha,s} > 0$ during some stage s, the restraint will maintain this value unless the strategy has been initialised.

The strategy α may lie below a number of $\mathcal{R}$ strategies β'. Each such strategy β' imposes a downward restraint $d(\beta' \frown \langle o', \sigma' \rangle, s)$ on α at stage s, where $\beta' \frown \langle o', \sigma' \rangle$ is the edge of β' on the path leading to α. One of these strategies, which we denote by β will be active for α. In this case we denote the edge of the strategy lying on the path leading to α by $\beta \frown \langle i, \sigma \rangle$. This strategy will either be following a Γ-strategy or a $\hat{\Gamma}$-strategy.

The strategy α may also lie below a number of $\mathcal{S}$ strategies α'. Each such strategy α' imposes a downward restraint $d(\alpha' \frown \langle o', \sigma' \rangle, s)$ on α at stage s, where $\alpha' \frown \langle o', \sigma' \rangle$ is the edge of α' on the path leading to α. The strategy α' also imposes the diagonalisation restraint $R_{\alpha',s}$ on α at stage s. The strategy α' may also impose a work interval on α at stage s, depending on its outcome on the path leading to α. Finally let $\alpha'' \subset \alpha$ be the greatest $\mathcal{S}$ strategy (under $\subset$) which imposes a work interval on α. We shall denote the work interval imposed by α'' on α at stage s by (a_s, b_s).

(1) Consider the last stage t at which α was accessible. If t does not exist, or the strategy α has been initialised at some stage t' such that $t < t' < s$, go to step (2).

If t exists, has α enumerated some witness w into A at stage t?

(a) (No) Go to step (2).

(b) (Yes) Is it the case that (i) there is no $\mathcal{R}$ strategy $\beta \subset \alpha$ which is active for α or (ii) there is some $\mathcal{R}$ strategy $\beta \subset \alpha$ which is active for α, and it is following a $\hat{\Gamma}$-strategy or (iii) there is some $\mathcal{R}$ strategy $\beta \subset \alpha$ which is active for α, and it is following a Γ-strategy and $U_t \upharpoonright \phi_1[t](w) \neq U_{t'} \upharpoonright \phi_1[t](w)$, where t' is the least $\mathcal{R}$-expansionary stage greater than t?

(i) (No) Go to step (2).

(ii) (Yes) Set the restraint $R_{\alpha,s}$ to $\theta_t(w)$. Consider every edge $\alpha \frown \langle o', \sigma' \rangle$ of α which has been previously accessible. If a work interval is defined for $\alpha \frown \langle o', \sigma' \rangle$, cancel the work interval. If some witness is attached to $\alpha \frown \langle o', \sigma' \rangle$, detach the witness. If o' is equal to d, g or w, set the edge to open mode. If o' is equal to h, set the edge to Part I mode. Go to step (2).

(2) Define the rightward restraint $r(\alpha \frown \langle o', \sigma' \rangle, s)$ for every edge $\alpha \frown \langle o', \sigma' \rangle$ which was previously accessible as the least element x such that:

(a) $x \geq \theta_t(w)$, where w is a witness attached to $\alpha \frown \langle o', \sigma' \rangle$ and t is the stage at which the witness was attached.

(b) $x \geq t$, where t is the last stage at which $\alpha \frown \langle o', \sigma' \rangle$ was last accessible.

Go to step (3).

(3) Consider the finite set of witnesses w in W^e such that $w < s$ and $\Theta^D[s](w) \downarrow = 0$ and such that w has not been attached to an edge at some stage $u < s$. Perform the following case analysis for every such witness in turn (under the order $<$), until one witness is attached successfully to an edge or until no more witnesses are available.

(a) Suppose that $R_{\alpha,s} > 0$. End stage s, and go to stage $s+1$.

(b) Suppose that $R_{\alpha,s} = 0$ and that no $\mathcal{R}$ strategy $\beta \subset \alpha$ is active for α. If there is an edge $\alpha \frown \langle d, \sigma' \rangle$ such that:

(i) $\alpha \frown \langle d, \sigma' \rangle$ has been accessible during a previous stage.

(ii) $\alpha \frown \langle d, \sigma' \rangle$ has no witness attached to it.

(iii) $w > \sup\{r(\alpha \frown \langle o'', \sigma'' \rangle, s) \mid \alpha \frown \langle o'', \sigma'' \rangle <_L \alpha \frown \langle d, \sigma' \rangle \ \wedge$ $\alpha \frown \langle o'', \sigma'' \rangle$ has been previously accessible$\}$.

(iv) $w > R_{\alpha',s}$ for every $\mathcal{S}$ strategy $\alpha' \subset \alpha$.

(v) $w > d(\beta' \frown \langle o', \sigma' \rangle, s)$, for every $\mathcal{R}$ strategy $\beta' \subset \alpha$ with edge $\beta' \frown \langle o', \sigma' \rangle$ on the path leading to α.

(vi) $w > d(\alpha' \frown \langle o', \sigma' \rangle, s)$, for every $\mathcal{S}$ strategy $\alpha' \subset \alpha$ with edge $\alpha' \frown \langle o', \sigma' \rangle$ on the path leading to α.

(vii) $a_s < w < b_s$.

(viii) $a_s < \theta_s(w) < b_s$.

(ix) w is greater than the upper bound of the work interval at stage s defined for any edge $\alpha \frown \langle o'', \sigma'' \rangle$ which was previously accessible and which lies to the left of $\alpha \frown \langle d, \sigma' \rangle$.

(x) $w > t$, where t is the last stage at which the edge $\alpha \frown \langle d, \sigma' \rangle$ was initialised.

(xi) $w > w'$, where w' is any witness which has been attached to $\alpha \frown \langle d, \sigma' \rangle$ at some stage $t < s$.

Then attach w to the leftmost such $\alpha \frown \langle d, \sigma' \rangle$.

(c) Suppose that $R_{\alpha,s} = 0$, that one $\mathcal{R}$ strategy $\beta \subset \alpha$ is active for α and that $\phi_1[s](w) > \gamma_{\beta \frown \langle i, \sigma \rangle}[s](w)$. If there is an edge $\alpha \frown \langle h, \sigma' \rangle$ such that:

(i) $\alpha \frown \langle h, \sigma' \rangle$ has been accessible during a previous stage.

(ii) $\alpha \frown \langle h, \sigma' \rangle$ has no witness attached to it.

(iii) $w > \sup\{r(\alpha \frown \langle o'', \sigma'' \rangle, s) \mid \alpha \frown \langle o'', \sigma'' \rangle <_L \alpha \frown \langle h, \sigma' \rangle \ \wedge$ $\alpha \frown \langle o'', \sigma'' \rangle$ has been previously accessible$\}$.

(iv) $w > R_{\alpha',s}$ for every $\mathcal{S}$ strategy $\alpha' \subset \alpha$.

(v) $w > d(\beta' \frown \langle o', \sigma' \rangle, s)$, for every $\mathcal{R}$ strategy $\beta' \subset \alpha$ with edge $\beta' \frown \langle o', \sigma' \rangle$ on the path leading to α.

(vi) $w > d(\alpha' \frown \langle o', \sigma' \rangle, s)$, for every $\mathcal{S}$ strategy $\alpha' \subset \alpha$ with edge $\alpha' \frown \langle o', \sigma' \rangle$ on the path leading to α.

(vii) $a_s < w < b_s$.

(viii) $a_s < \theta_s(w) < b_s$.

(ix) w is greater than the upper bound of the work interval at stage s defined for any edge $\alpha \frown \langle o'', \sigma'' \rangle$ which was previously accessible and which lies to the left of $\alpha \frown \langle h_i, \sigma' \rangle$.

(x) $w > t$, where t is the last stage at which the edge $\alpha \frown \langle h, \sigma' \rangle$ was initialised.

(xi) $w > w'$, where w' is any witness which has been attached to this edge at some stage $t < s$.

Attach w to the leftmost such $\alpha \frown \langle h, \sigma' \rangle$. Define the work interval of the edge $\alpha \frown \langle h, \sigma' \rangle$ to be $(w, \gamma_{\beta' \frown \langle i, \sigma' \rangle}[s](w))$.

(d) Suppose that $R_{\alpha,s} = 0$, that one $\mathcal{R}$ strategy $\beta \subset \alpha$ is active for α and follows a Γ-strategy and that $\phi_1[s](w) \leq \gamma_{\beta \frown \langle i, \sigma \rangle}[s](w)$. If there is an edge $\alpha \frown \langle g, \sigma' \rangle$ such that:

(i) $\alpha \frown \langle g, \sigma' \rangle$ has been accessible during a previous stage.

(ii) $\alpha \frown \langle g, \sigma' \rangle$ has no witness attached to it.

(iii) $w > \sup\{r(\alpha \frown \langle o'', \sigma'' \rangle, s) \mid \alpha \frown \langle o'', \sigma'' \rangle <_L \alpha \frown \langle g, \sigma' \rangle \ \wedge$ $\alpha \frown \langle o'', \sigma'' \rangle$ has been previously accessible$\}$.

(iv) $w > R_{\alpha',s}$ for every $\mathcal{S}$ strategy $\alpha' \subset \alpha$.

(v) $w > d(\beta' \frown \langle o', \sigma' \rangle, s)$, for every $\mathcal{R}$ strategy $\beta' \subset \alpha$ with edge $\beta' \frown \langle o', \sigma' \rangle$ on the path leading to α.

(vi) $w > d(\alpha' \frown \langle o', \sigma' \rangle, s)$, for every $\mathcal{S}$ strategy $\alpha' \subset \alpha$ with edge $\alpha' \frown \langle o', \sigma' \rangle$ on the path leading to α.

(vii) $a_s < w < b_s$.

(viii) $a_s < \theta_s(w) < b_s$.

(ix) $\alpha \frown \langle g, \sigma' \rangle$ is in open mode.

(x) The work interval for the edge $\alpha \frown \langle g, \sigma' \rangle$ is defined.

(xi) w is greater than the upper bound of the work interval for the edge $\alpha \frown \langle g, \sigma' \rangle$.

(xii) $w > t$, where t is the last stage at which the edge $\alpha \frown \langle g, \sigma' \rangle$ was initialised.

(xiii) $w > w'$, where w' is any witness which has been attached to this edge at some stage $t < s$.

Then attach w to the leftmost such $\alpha \frown \langle g, \sigma' \rangle$.

(e) Suppose that $R_{\alpha,s} = 0$, that one $\mathcal{R}$ strategy $\beta \subset \alpha$ is active for α and follows a

$\hat{\Gamma}$-strategy and that $\phi_1[s](w) \leq \gamma_{\beta' \frown \langle i, \sigma' \rangle}[s](w)$. If there is an edge $\alpha \frown \langle d, \sigma' \rangle$ such that:

(i) $\alpha \frown \langle d, \sigma' \rangle$ has been accessible during a previous stage.

(ii) $\alpha \frown \langle d, \sigma' \rangle$ has no witness attached to it.

(iii) $w > \sup\{r(\alpha \frown \langle o'', \sigma'' \rangle, s) \mid \alpha \frown \langle o'', \sigma'' \rangle <_L \alpha \frown \langle d, \sigma' \rangle \wedge$ $\alpha \frown \langle o'', \sigma'' \rangle$ has been previously accessible$\}$.

(iv) $w > R_{\alpha', s}$ for every $\mathcal{S}$ strategy $\alpha' \subset \alpha$.

(v) $w > d(\beta' \frown \langle o', \sigma' \rangle, s)$, for every $\mathcal{R}$ strategy $\beta' \subset \alpha$ with edge $\beta' \frown \langle o', \sigma' \rangle$ on the path leading to α.

(vi) $w > d(\alpha' \frown \langle o', \sigma' \rangle, s)$, for every $\mathcal{S}$ strategy $\alpha' \subset \alpha$ with edge $\alpha' \frown \langle o', \sigma' \rangle$ on the path leading to α.

(vii) $a_s < w < b_s$.

(viii) $a_s < \theta_s(w) < b_s$.

(ix) w is greater than the upper bound of the work interval at stage s defined for any edge $\alpha \frown \langle o'', \sigma'' \rangle$ which was previously accessible and which lies to the left of $\alpha \frown \langle d, \sigma' \rangle$.

(x) $w > t$, where t is the last stage at which the edge $\alpha \frown \langle d, \sigma' \rangle$ was initialised.

(xi) $w > w'$, where w' is any witness which has been attached to this edge at some stage $t < s$.

Then attach w to the leftmost such $\alpha \frown \langle d, \sigma' \rangle$.

If a witness w has been attached to some edge $\alpha \frown \langle o', \sigma' \rangle$, consider every edge $\alpha \frown \langle o'', \sigma'' \rangle$ lying to the right of $\alpha \frown \langle o', \sigma' \rangle$. If some witness w' is attached to $\alpha \frown \langle o'', \sigma'' \rangle$, detach the witness from the edge. If some work interval is defined for $\alpha \frown \langle o'', \sigma'' \rangle$, undefine the work interval. If o'' is equal to d, w or g, set the edge to open mode. If o'' is equal to h, set the edge to Part I mode.

Go to step (4).

(4) Determine the edge $\alpha \frown \langle o, \sigma \rangle$ which the strategy should visit at stage s.

Go to step (5).

(5) Define the attachment procedure restraint $a(\alpha \frown \langle o', \sigma' \rangle, s)$ for every edge $\alpha \frown \langle o', \sigma' \rangle$ which was previously accessible. If the strategy has not attached a witness w to some edge $\alpha \frown \langle o'', \sigma'' \rangle <_L \alpha \frown \langle o', \sigma' \rangle$ at stage s, define $a(\alpha \frown \langle o', \sigma' \rangle, s) = 0$. Otherwise define $a(\alpha \frown \langle o', \sigma' \rangle, s) = \theta_s(w)$.

Also define the downward restraint $d(\alpha \frown \langle o, \sigma \rangle, s)$ as the least element x such that:

(a) $x \geq \sup\{r(\alpha \frown \langle o', \sigma' \rangle, s) \mid \alpha \frown \langle o', \sigma' \rangle <_L \alpha \frown \langle o, \sigma \rangle \ \wedge$
$\alpha \frown \langle o', \sigma' \rangle$ has been previously accessible$\}$.

(b) $x \geq a(\alpha \frown \langle o, \sigma \rangle, s)$.

(c) $x \geq d(\alpha \frown \langle o, \sigma \rangle, t)$ for all $t < s$.

Go to step (6).

(6) Consider the edge $\alpha \frown \langle o, \sigma \rangle$ being visited by the strategy at stage s. Take action according to the value of o through the following case analysis.

(a) $o = w$.

(i) Suppose that $\alpha \frown \langle w, \sigma \rangle$ is in open mode, and s is not an α-open stage. End the stage s, and go to stage $s + 1$.

(ii) Suppose that $\alpha \frown \langle w, \sigma \rangle$ is in open mode, and s is an α-open stage. Set the edge to close mode. Go to the next substage.

(iii) Suppose that $\alpha \frown \langle w, \sigma \rangle$ is in close mode, and s is an α-close stage. End the stage s, and go to stage $s + 1$.

(iv) Suppose that $\alpha \frown \langle w, \sigma \rangle$ is in close mode, and s is an α-close stage. Set the edge to open mode. Go to the next substage.

(b) $o = g$.

(i) Suppose that $R_{\alpha,s} > 0$. End stage s, and go to stage $s + 1$.

(ii) Suppose that $R_{\alpha,s} = 0$, and the work interval for the edge $\alpha \frown \langle g, \sigma \rangle$ is undefined. If there is some least threshold $v < s$ in V^e such that:

(A) $v > \sup\{r(\alpha \frown \langle o', \sigma' \rangle, s) \mid \alpha \frown \langle o', \sigma' \rangle <_L \alpha \frown \langle g, \sigma \rangle \ \wedge$
$\alpha \frown \langle o', \sigma' \rangle$ has been previously accessible$\}$.

(B) $v > a(\alpha \frown \langle g, \sigma \rangle, s)$.

(C) $v > R_{\alpha', s}$ for every $\mathcal{S}$ strategy $\alpha' \subset \alpha$.

(D) $v > d(\beta' \frown \langle o', \sigma' \rangle, s)$, for every $\mathcal{R}$ strategy $\beta' \subset \alpha$ with edge $\beta' \frown \langle o', \sigma' \rangle$ on the path leading to α.

(E) $v > d(\alpha' \frown \langle o', \sigma' \rangle, s)$, for every $\mathcal{S}$ strategy $\alpha' \subset \alpha$ with edge $\alpha' \frown \langle o', \sigma' \rangle$ on the path leading to α.

(F) $a_s < v < b_s$.

(G) v is greater than the upper bound of a work interval defined for an edge $\alpha \frown \langle o', \sigma' \rangle$ lying to the left of $\alpha \frown \langle o, \sigma \rangle$.

(H) $v > t$, where t is the stage at which the edge $\alpha \frown \langle g, \sigma \rangle$ was last initialised.

Define the work interval of the edge $\alpha \frown \langle g, \sigma \rangle$ to be $(v, \gamma_{\beta \frown \langle i, \sigma \rangle}[s](v))$. End stage s, and go to stage $s + 1$.

(iii) Suppose that $R_{\alpha, s} = 0$, a work interval is defined for the edge $\langle g, \sigma \rangle$ and the edge is in open mode, but no witness w is attached to the edge. End stage s, and go to stage $s + 1$.

(iv) Suppose that $R_{\alpha, s} = 0$, a work interval is defined for the edge $\langle g, \sigma \rangle$, the edge is in open mode and a witness w is attached to the edge, but $\phi_1[s](w) > \gamma_{\beta \frown \langle i, \sigma \rangle}[s](w)$. Detach w from $\alpha \frown \langle g, \sigma \rangle$. End stage s, and go to stage $s + 1$.

(v) Suppose that $R_{\alpha, s} = 0$, a work interval is defined for the edge $\langle g, \sigma \rangle$, the edge is in open mode, a witness w is attached to the edge and $\phi_1[s](w) \leq \gamma_{\beta \frown \langle i, \sigma \rangle}[s](w)$, but s is not an α-open stage. End stage s, and go to stage $s + 1$.

(vi) Suppose that $R_{\alpha, s} = 0$, a work interval is defined for the edge $\langle g, \sigma \rangle$, the edge is in open mode, a witness w is attached to the edge, $\phi_1[s](w) \leq \gamma_{\beta \frown \langle i, \sigma \rangle}[s](w)$ and s is an α-open stage. Enumerate w into A. Set the edge $\langle g, \sigma \rangle$ to close mode. Go to the next substage.

(vii) Suppose that $R_{\alpha, s} = 0$, a work interval is defined for the edge $\langle g, \sigma \rangle$, the edge is in close mode and s is not an α-close stage. End stage s, and go to stage $s+1$.

(viii) Suppose that $R_{\alpha,s} = 0$, a work interval is defined for the edge $\langle g, \sigma \rangle$, the edge is in close mode and s is an α-close stage. Enumerate $\gamma_{\beta\frown\langle i,\sigma\rangle}[s](v)$ into D. Set the edge $\langle g, \sigma \rangle$ to open mode. Go to the next substage.

(c) $o = h$.

(i) Suppose that $R_{\alpha,s} > 0$. End stage s, and go to stage $s+1$.

(ii) Suppose that $R_{\alpha,s} = 0$, but the edge $\langle h, \sigma \rangle$ has no witness w attached to it. End stage s, and go to stage $s+1$.

(iii) Suppose that $R_{\alpha,s} = 0$, and the strategy has attached a witness w to the edge $\langle h, \sigma \rangle$ during this stage s. End stage s, and go to stage $s+1$.

(iv) Suppose that $R_{\alpha,s} = 0$, the work interval for the edge $\langle h, \sigma \rangle$ is defined and the edge $\langle h, \sigma \rangle$ is in Part I mode, but s is not an α-close stage. End stage s, and go to stage $s+1$.

(v) Suppose that $R_{\alpha,s} = 0$, the work interval for the edge $\langle h, \sigma \rangle$ is defined, the edge $\langle h, \sigma \rangle$ is in Part I mode, and s is an α-close stage, but $\phi_1[s](w) \leq \gamma_{\beta\frown\langle i,\sigma\rangle}[s](w)$. End stage s, and go to stage $s+1$.

(vi) Suppose that $R_{\alpha,s} = 0$, and the work interval for the edge $\langle h, \sigma \rangle$ is defined, the edge $\langle h, \sigma \rangle$ is in Part I mode, s is an α-close stage, and $\phi_1[s](w) > \gamma_{\beta\frown\langle i,\sigma\rangle}[s](w)$. Enumerate $\gamma_{\beta\frown\langle i,\sigma\rangle}[s](w)$ into D. Set the edge $\langle h, \sigma \rangle$ to Part II mode. Go to the next substage.

(vii) Suppose that $R_{\alpha,s} = 0$, the edge $\langle h, \sigma \rangle$ has a witness w attached to it, the edge is in Part II mode and s is not an α-open stage. End stage s, and go to stage $s+1$.

(viii) Suppose that $R_{\alpha,s} = 0$, the edge $\langle h, \sigma \rangle$ has a witness w attached to it, the edge is in Part II mode and s is an α-open stage. Set the edge $\langle h, \sigma \rangle$ to Part I mode. Go to the next substage.

(d) $o = d$.

(i) Suppose that $R_{\alpha,s} > 0$. End stage s, and go to stage $s+1$.

(ii) Suppose that $R_{\alpha,s} = 0$, that there is no $\mathcal{R}$ strategy $\beta \subset \alpha$ active for α, and that no witness w is attached to this edge. End stage s, and go to stage $s+1$.

(iii) Suppose that $R_{\alpha,s} = 0$, that there is no $\mathcal{R}$ strategy $\beta \subset \alpha$ active for α and that a witness w is attached to this edge, but that s is not an α-open stage. End stage s, and go to stage $s+1$.

(iv) Suppose that $R_{\alpha,s} = 0$, that there is no $\mathcal{R}$ strategy $\beta \subset \alpha$ active for α, that a witness w is attached to this edge, and that s is not an α-open stage. Enumerate w into A. End stage s, and go to stage $s+1$.

(v) Suppose that $R_{\alpha,s} = 0$, that there is one $\mathcal{R}$ strategy $\beta \subset \alpha$ active for α following a $\hat{\Gamma}$-strategy and that no witness w is attached to this edge. End stage s, and go to stage $s+1$.

(vi) Suppose that $R_{\alpha,s} = 0$, that there is one $\mathcal{R}$ strategy $\beta \subset \alpha$ active for α following a $\hat{\Gamma}$-strategy and that a witness w is attached to this edge, but $\phi_1[s](w) > \gamma_{\beta \frown \langle i,\sigma \rangle}[s](w)$. Detach the witness w from the edge. End stage s, and go to stage $s+1$.

(vii) Suppose that $R_{\alpha,s} = 0$, that there is one $\mathcal{R}$ strategy $\beta \subset \alpha$ active for α following a $\hat{\Gamma}$-strategy, that a witness w is attached to this edge and that $\phi_1[s](w) \leq \gamma_{\beta \frown \langle i,\sigma \rangle}[s](w)$, but s is not an α-open stage. End stage s, and go to stage $s+1$.

(viii) Suppose that $R_{\alpha,s} = 0$, that there is one $\mathcal{R}$ strategy $\beta \subset \alpha$ active for α following a $\hat{\Gamma}$-strategy, that a witness w is attached to this edge, that $\phi_1[s](w) \leq \gamma_{\beta \frown \langle i,\sigma \rangle}[s](w)$ and that s is an α-open stage. End stage s, and go to stage $s+1$. Enumerate w into A. End stage s, and go to stage $s+1$.

(viii) Suppose that $R_{\alpha,s} = 0$ and that there is one $\mathcal{R}$ strategy $\beta \subset \alpha$ active for α following a Γ-strategy. End stage s, and go to stage $s+1$.

3.5.8 Analysis of Outcomes

We shall now consider the effect of the $\mathcal{S}$ strategy α on the satisfaction of the requirements $\mathcal{R}$ and $\mathcal{S}$. When certain edges of the strategy α are on the true path, it may be the case that certain requirements are left unsatisfied. We shall show that the relationship between the outcome on the true path and the requirements which are left unsatisfied is the same as the one for the Lachlan Non-Splitting Theorem. The immediate consequence of this will be that the priority tree can be structured in an analogous way to the priority tree of the Lachlan Non-Splitting Theorem. This will allow us to perform a similar analysis in order to show that all requirements are actually satisfied at a later stage.

In order to analyse the effect of the $\mathcal{S}$ strategy α on the satisfaction of the requirements $\mathcal{R}$ and $\mathcal{S}$, we consider the leftmost edge $\alpha \frown \langle o, \sigma \rangle$ which is visited infinitely often by the strategy α. The following case analysis can then be made depending on the outcome $\langle o, \sigma \rangle$

$\boxed{\text{W}}$ Suppose that the outcome is $\langle w, \sigma \rangle$.

Then the answer to question Q_1 must be 'No'.

If condition (i) of question Q_1 fails, we have that there are only finitely many witnesses $w \in W^e$ and stages $s \in \mathbb{N}_\alpha$ such that $\Theta^D[s](w) \downarrow= 0$. Then there must be some stage $t \in \mathbb{N}_\alpha$ such that for every $t' \in \mathbb{N}_\alpha$ with $t' > t$ and every element $x \in W^e$, we have that $\Theta^D[t'](x) \uparrow$ or $\Theta^D[t'](x) \downarrow= 1$. Now, if $\Theta^D(x) \uparrow$ for some $x \in W^e$, we have that $\Theta^D(x) \neq A(x)$ and the $\mathcal{S}$ requirement is satisfied. On the other hand, if $\Theta^D(x) \downarrow= 1$ for some $x \in W^e$, we have that the strategy will never enumerate x into A. This means that $\Theta^D(x) \neq A(x)$ and that the $\mathcal{S}$ requirement is also satisfied.

If condition (i) of question Q_1 holds but condition (ii) of question Q_1 fails, we have that there are infinitely many witnesses $w \in W^e$ and stages $s \in \mathbb{N}_\alpha$ such that $\Theta^D[s](w) \downarrow= 0$, but $a_s < w < b_s$ for only finitely many of these witnesses and stages. This means that there is some stage $t \in \mathbb{N}_\alpha$ such that for all $t' \in \mathbb{N}_\alpha$ with $t' > t$ and every element $x \in W^e$ such that $a_{t'} < x < b_{t'}$ we have that $\Theta^D[t'](w) \downarrow= 1$ or that $\Theta^D[t'](w) \uparrow$. Now since the strategy imposing this work interval is on the true path, there is some stage s_0 such that for all $s' > s_0$ its edge on the true path is no longer initialised. Hence once this strategy defines its work interval at some least stage $u > s_0$, we have that $a_u = a_{u'}$ for all $u' \geq u$. In addition, the upper bound of the work interval will become unbounded. It follows that there is some element $x' \in W^e$ and some stage $p \in \mathbb{N}_\alpha$ such that for all stages $p' \in \mathbb{N}_\alpha$ with $p' > p$, $p' > t$ and $p' > u$ we have that $a_{p'} < x' < b_{p'}$ and that $\Theta^D[p'](x') \uparrow$ or $\Theta^D[p'](x') \downarrow= 1$. Now, if $\Theta^D(x') \uparrow$, we have that $\Theta^D(x') \neq A(x')$ and the $\mathcal{S}$ requirement is satisfied. On the other hand, if $\Theta^D(x') \downarrow= 1$, we have that the strategy will never enumerate x' into A. This means that $\Theta^D(x') \neq A(x')$ and that the $\mathcal{S}$ requirement is also satisfied.

If conditions (i) and (ii) of question Q_1 hold but condition (iii) of question Q_1 fails, we have that there are infinitely many witnesses $w \in W^e$ and stages $s \in \mathbb{N}_\alpha$ such that $\Theta^D[s](w) \downarrow= 0$ and $a_s < w < b_s$, but $a_s < \theta_s(w) < b_s$ for only finitely many of these witnesses and stages. This means that there is some stage $t \in \mathbb{N}_\alpha$ such that for all $t' \in \mathbb{N}_\alpha$ with $t' > t$ and every element $x \in W^e$ such that $a_{t'} < x < b_{t'}$ we have that $\theta_{t'}(x) > b_{t'}$. Now since the strategy imposing this work interval is on the true path, there is some stage s_0 such that for all $s' > s_0$ its edge on the true path is no longer initialised. Hence once this strategy defines its work interval at some least stage $u > s_0$, we have that $a_u = a_{u'}$ for all $u' \geq u$. In addition, the upper bound of the work interval will become unbounded. It follows that there is some element $x' \in W^e$ and some stage $p \in \mathbb{N}_\alpha$ such that for all stages $p' \in \mathbb{N}_\alpha$ with $p' > p$, $p' > t$ and $p' > u$ we have that $b_{p'} < \theta_{p'}(x')$. But since the upper bound of the work interval is unbounded, it must be the case that $\Theta^D(x') \uparrow$, which means that the $\mathcal{S}$ requirement is satisfied.

If conditions (i) and (ii) and (iii) of question Q_1 hold but condition (iv) of question Q_1 fails, we have that there are infinitely many witnesses $w \in W^e$ and stages $s \in \mathbb{N}_\alpha$ such that $\Theta^D[s](w) \downarrow= 0$, $a_s < w < b_s$ and $a_s < \theta_s(w) < b_s$. However there are only finitely many stages $q \in \mathbb{N}_\alpha$ such that $(\forall q' < q)[l_{q'}(\Theta^D, A) < l_q(\Theta^D, A)]$, where q' ranges over $\mathbb{N}_\alpha$. But in this case there must be some x such that $\Theta^D(x) \neq A(x)$, meaning that the $\mathcal{S}$ requirement is satisfied.

Hence if $\langle w, \sigma \rangle$ is the outcome of the edge lying on the true path we have that the $\mathcal{S}$ requirement is satisfied, whilst the $\mathcal{R}$ strategy can build its functional without interference after some stage, satisfying the $\mathcal{R}$ requirement as well. It is important to note that one way for this outcome to be on the true path is for the strategy to diagonalise successfully.

h Suppose that the outcome is $\langle h, \sigma \rangle$.

(This outcome is only present if there is some strategy β above α which is active for α).

In this case, we have that the answers to questions Q_1 and Q_2 guarantee that there are infinitely many witnesses $w \in W^e$ and stages $s \in \mathbb{N}_\alpha$ such that $\Theta^D[s](w) \downarrow = 0$, $a_s < w < b_s$ and $a_s < \theta_s(w) < b_s$. However only finitely many of these witnesses w will give honest computations $\Gamma^{U,D}_{\beta \frown \langle i,\sigma \rangle}[s](w)$. It follows that there is some stage t' such that for every $s > t'$, we have that the computations $\Gamma^{U,D}_{\beta \frown \langle i,\sigma \rangle}[s](w)$ are dishonest. Finally we also have that the length of agrement between Θ^D and A increases infinitely often.

Now since $\alpha \frown \langle h, \sigma \rangle$ is the leftmost edge which is accessible infinitely often, we have that there is a stage s_0 after which no edge to its left is accessible. Hence only finitely many edges to the left of $\alpha \frown \langle h, \sigma \rangle$ can have been accessible at stages $s < s_0$. Suppose that the edge $\alpha \frown \langle h, \sigma \rangle$ does not have a witness giving a dishonest computation attached at some stage $s_1 > s_0$. Since witnesses giving a dishonest computation are attached to the leftmost edge of the form $\alpha \frown \langle h, \sigma \rangle$ which has no other witness attached, and since there are infinitely many such witnesses, it follows that a witness satisfying these conditions is eventually attached to $\alpha \frown \langle h, \sigma \rangle$ at some stage $s_2 > s_1$.

In addition we claim that it is possible to ensure that the edge $\alpha \frown \langle h, \sigma \rangle$ is visited during infinitely many α-open stages and infinitely many α-close stages. We address this claim when we discuss *fairness* in section 3.7.1.

Hence, if the edge $\alpha \frown \langle h, \sigma \rangle$ does not have a witness giving a dishonest computation attached at some stage, it must be the case that such a witness will eventually be attached to the edge at some stage s. This defines a work interval $(w, \gamma_{\beta \frown \langle i,\sigma \rangle}[s](w))$ for the edge. If the edge is in Part I mode, we have that the strategy visits the edge during an α-close stage. This allows the strategy to honestify and to change the mode of the edge to Part II mode. If the edge is in Part II mode, we have that the strategy visits the edge during an α-open stage, and that it changes the mode of its edge back to Part I mode.

Now it could be the case that a witness w which was attached to $\alpha \frown \langle h, \sigma \rangle$ at stage s gave a dishonest computation $\Gamma^{U,D}_{\beta \frown \langle i,\sigma \rangle}[s](w)$, but then started giving an honest computation

$\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}[s'](w)$ at some stage $s' > s$ at which the strategy visited the edge once again. In this case we have that the strategy would be blocked from honestifying. However, we have already seen that there is some greatest stage t' such that $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}[t](w)$ is dishonest at every stage $t > t'$. Therefore the strategy will not be blocked from honestifying after stage t'.

Therefore the strategy will honestify during infinitely many stages u, by enumerating $\gamma_{\beta\frown\langle i,\sigma\rangle}[u](w)$ into D. Hence we have that $\lim_{q\to\infty} \gamma_{\beta\frown\langle i,\sigma\rangle}[q](w)$ is unbounded and that $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}(w) \uparrow$. In addition to this we also have that $\lim_{q\to\infty} \phi_1[q](w)$ is unbounded and that $\Phi^{U,V}_1(w) \uparrow$.

Hence if $\langle h, \sigma\rangle$ is the outcome of the edge lying on the true path we have that the $\mathcal{R}$ requirement is satisfied trivially. Finally the $\mathcal{S}$ requirement remains unsatisfied as well because the strategy does not diagonalise. For if this were the case, is would contradict the fact that the length of agrement between Θ^D and A increases infinitely often.

g Suppose that the outcome is $\langle g, \sigma\rangle$.

(This outcome is not present unless there is some strategy β above α which is active for α and which is following a Γ-strategy).

In this case, we have that the answers to questions Q_1, Q_2 and Q_3 guarantee that there are infinitely many witnesses $w \in W^e$ and stages $s \in \mathbb{N}_\alpha$ such that $\Theta^D[s](w) \downarrow= 0$, the computation $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}[s](w)$ is honest, $a_s < w < b_s$ and $a_s < \theta_s(w) < b_s$. Infinitely many of these witnesses w will also enter the set A at stage s, but only finitely many of them will cause a $U \upharpoonright \phi_1[s](w)$ change to occur by the least $\mathcal{R}$ -expansionary stage $t > s$. Finally we also have that the length of agrement between Θ^D and A increases infinitely often.

Now since $\alpha \frown \langle g, \sigma\rangle$ is the leftmost edge which is accessible infinitely often, we have that there is a stage s_0 after which no edge to its left is accessible. Hence only finitely many edges to the left of $\alpha \frown \langle g, \sigma\rangle$ can have been accessible at stages $s < s_0$. Suppose that the edge $\alpha \frown \langle g, \sigma\rangle$ does not have a witness giving a honest computation attached at some stage $s_1 > s_0$. Since witnesses giving an honest computation are attached to the leftmost edge of the form $\alpha \frown \langle g, \sigma\rangle$ which has no other witness attached, and since there are infinitely many such witnesses, it follows that a witness satisfying these conditions is eventually attached to

$\alpha \frown \langle g, \sigma \rangle$ at some stage $s_2 > s_1$.

In addition we claim that it is possible to ensure that the edge $\alpha \frown \langle g, \sigma \rangle$ is visited during infinitely many α-open stages and infinitely many α-close stages. We address this claim when we discuss *fairness* in section 3.7.1.

We also claim that if a witness w is attached to the edge $\alpha \frown \langle g, \sigma \rangle$ at some stage s and the witness gives an honest computation $\Gamma^{U,D}_{\beta \frown \langle i, \sigma \rangle}[s](w)$, it is possible to stop elements from entering A or $D \restriction \phi_2[s](\phi_1[s](w))$ and A or $D \restriction \phi_3[s](\phi_1[s](w))$ at some stage $s' \geq s$. In this way the honesty of the witness can be preserved until the strategy determines that it should enumerate it into the set A. We address this claim when we discuss *honesty preservation* in section 3.7.2.

Now if the edge $\alpha \frown \langle g, \sigma \rangle$ does not have a work interval defined at some stage $s > s_0$, the strategy will choose a threshold v and define a work interval $(v, \gamma_{\beta \frown \langle i, \sigma \rangle}[s](v))$ for the edge. Once the work interval has been defined, a witness w giving an honest computation will eventually be attached to $\alpha \frown \langle g, \sigma \rangle$. Then if the edge is in open mode, we have that the strategy eventually visits the edge during an α-open stage, enumerating w into A and changing the mode of the edge to close mode.

Now consider the least $\mathcal{R}$ -expansionary stage $t > s$, and suppose that $U \restriction \phi_{1,s}(w) \neq U \restriction \phi_{1,t}(w)$. Then once the strategy α becomes accessible again at some stage $u \geq t$ it diagonalises and sets the restraint $R_{\alpha,u} = \theta_s(w)$. But this would mean that $\Theta^D(w) \neq A(w)$, contradicting the fact that the length of agreement between $\Theta^D(w)$ and A expands infinitely often. It follows that $V \restriction \phi_{1,s}(w) \neq V \restriction \phi_{1,t}(w)$ instead and that α does not diagonalise.

Since the edge is in close mode, we have that the strategy eventually visits the edge during an α-close stage s', performing capricious destruction and enumerating $\gamma_{\beta \frown \langle i, \sigma \rangle}[s'](v)$ into D, while changing the mode of the edge to open mode again.

Since the strategy performs capricious destruction infinitely often, we have that $\lim_{q \to \infty} \gamma_{\beta \frown \langle i, \sigma \rangle}[q](v)$ is unbounded and that $\Gamma^{U,D}_{\beta \frown \langle i, \sigma \rangle}(v) \uparrow$.

Hence if $\langle g, \sigma \rangle$ is the outcome of the edge lying on the true path we have that the $\mathcal{R}$ requirement is not satisfied. This will require the next $\mathcal{R}$ strategy attempting to satisfy the requirement on the true path to switch to following a $\hat{\Gamma}$-strategy. Finally the $\mathcal{S}$ requirement remains unsatisfied as well because the strategy does not diagonalise. For if this were the case, is would contradict the fact that the length of agrement between Θ^D and A increases infinitely often.

$\boxed{\text{d}}$ Suppose that the outcome is $\langle d, \sigma \rangle$.

Three different cases have to be considered.

(a) Suppose that there is no $\mathcal{R}$ strategy β above α which is active for α.

In this case the strategy has no edge with outcome $\langle h, \sigma \rangle$ or $\langle g, \sigma \rangle$. Hence the analysis is identical to the one for the $\langle d, \sigma \rangle$ outcome of the $\mathcal{S}$ strategy found in Section 3.4.

(b) Suppose that there is some $\mathcal{R}$ strategy β above α which is active for α, but every such strategy follows a $\hat{\Gamma}$-strategy.

In this case the strategy has edges with outcome $\langle h, \sigma \rangle$, but no edges with outcome $\langle g, \sigma \rangle$. We also have that the answers to questions Q_1, Q_2 and Q_3 guarantee that there are infinitely many witnesses $w \in W^e$ and stages $s \in \mathbb{N}_\alpha$ such that $\Theta^D[s](w) \downarrow = 0$, the computation $\Gamma^{U,D}_{\beta \frown \langle i, \sigma \rangle}[s](x)$ is honest, $a_s < w < b_s$ and $a_s < \theta_s(w) < b_s$.

Now since $\alpha \frown \langle d, \sigma \rangle$ is the leftmost edge which is accessible infinitely often, we have that there is a stage s_0 after which no edge to its left is accessible. Hence only finitely many edges to the left of $\alpha \frown \langle d, \sigma \rangle$ can have been accessible at stages $s < s_0$. Suppose that the edge $\alpha \frown \langle d, \sigma \rangle$ does not have a witness giving a honest computation attached at some stage $s_1 > s_0$. Since witnesses giving an honest computation are attached to the leftmost edge of the form $\alpha \frown \langle d, \sigma \rangle$ which has no other witness attached, and since there are infinitely many such witnesses, it follows that a witness satisfying these conditions is eventually attached to $\alpha \frown \langle d, \sigma \rangle$ at some stage $s_2 > s_1$.

In addition we claim that it is possible to ensure that the edge $\alpha \frown \langle d, \sigma \rangle$ is visited during infinitely many α-open stages and infinitely many α-close stages. We address this claim when we discuss *fairness* in section 3.7.1.

We also claim that if a witness w is attached to the edge $\alpha \frown \langle d, \sigma \rangle$ at some stage s and the witness gives an honest computation $\Gamma^{U,D}_{\beta \frown \langle i, \sigma \rangle}[s](w)$, it is possible to stop elements from entering A or $D \restriction \phi_2[s](\phi_1[s](w))$ and A or $D \restriction \phi_3[s](\phi_1[s](w))$ at some stage $s' \geq s$. In this way the honesty of the witness can be preserved until the strategy determines that it should enumerate it into the set A. We address this claim when we discuss *honesty preservation* in section 3.7.2.

Now the strategy will eventually attach a witness w giving an honest computation to $\alpha \frown \langle d, \sigma \rangle$. If the edge is in open mode, we have that the strategy eventually visits the edge during an α-open stage, enumerating w into A at some stage u. Hence when the strategy becomes accessible again at some stage $s' \geq u$, we have that it diagonalises and sets $R_{\alpha,s'} = \theta_u(w)$. The strategy will in fact have diagonalised successfully because the priority tree will be arranged such that a $U \restriction \phi_1[u](w)$ change cannot take place at the least $\mathcal{R}$ expansionary stage $u' > u$ without α becoming inaccessible. Since α is on the true path, it will then follow that a $U \restriction \phi_1[u](w)$ change has instead taken place at the least $\mathcal{R}$ expansionary stage $u' > u$.

However since the strategy has diagonalised we have that $\Theta^D[s''](w) \neq A_{s''}(w)$ for all $s'' > s'$, which contradicts the fact that the length of agreement between $\Theta^D(w)$ and A expands infinitely often.

(c) Suppose that there is some $\mathcal{R}$ strategy β above α which is active for α, and which follows a Γ-strategy.

In this case the strategy has edges with outcome $\langle h, \sigma \rangle$ and edges with outcome $\langle g, \sigma \rangle$. We also have that the answers to questions Q_1, Q_2 and Q_3 guarantee that there are infinitely many witnesses $w \in W^e$ and stages $s \in \mathbb{N}_\alpha$ such that $\Theta^D[s](w) \downarrow= 0$, the computation $\Gamma^{U,D}_{\beta \frown \langle i,\sigma \rangle}[s](x)$ is honest, $a_s < w < b_s$ and $a_s < \theta_s(w) < b_s$. In addition infinitely many of these witnesses are enumerated into the set A at stage s and each of these causes a $U \restriction \phi_1[s](w)$ change to take place by the least $\mathcal{R}$ -expansionary stage $t > s$. Finally we have that the length of agreement between $\Theta^D(w)$ and A expands infinitely often.

Now, suppose that the strategy opens a gap by enumerating a witness w into A at some stage u, and that a $U \restriction \phi_1[u](w)$ change takes place at some least $\mathcal{R}$ expansionary stage $u' > u$. If the strategy becomes accessible again at some stage $s' \geq u'$, we have that it sets $R_{\alpha,s'} > \theta_u(w)$. But this would mean that $\Theta^D[s''](w) \neq A_{s''}(w)$ for all $s'' > s'$, which contradicts the fact that there are infinitely many stages such that the length of agreement between $\Theta^D(w)$ and A expands.

Since in each of the above three cases we have a contradiction, it follows that no edge with outcome $\langle d, \sigma \rangle$ can be on the true path. In fact, only edges with outcome $\langle w, \sigma \rangle$ can be on the true path if the strategy diagonalises successfully. This is the only outcome for which Q_1 can have a negative answer, which in turn allows the length of agreement between Θ^D and A to be finite in length.

3.5.9 Organisation of Priority Tree

We shall now organise a priority tree in order to satisfy an $\mathcal{S}$ requirement below an $\mathcal{R}$ requirement. The following notation shall be used when depicting the priority tree shown in figure 3.1. Note that when we say that some strategy γ takes into consideration another strategy γ', we mean that γ' is active for γ.

- β^U will denote an $\mathcal{R}$ strategy which is following a Γ-strategy.
- β^V will denote an $\mathcal{R}$ strategy which is following a $\hat{\Gamma}$-strategy.
- α^U will denote an $\mathcal{S}$ strategy which needs to take into consideration one $\mathcal{R}$ strategy following a Γ-strategy lying above it.
- α^V will denote an $\mathcal{S}$ strategy which needs to take into consideration one $\mathcal{R}$ strategy following a $\hat{\Gamma}$-strategy lying above it.
- α will denote an $\mathcal{S}$ strategy which does not need to take into consideration any $\mathcal{R}$ strategy lying above it.

When presenting the priority tree we omit the infinitely many edges of a given $\mathcal{R}$ strategy, focusing on just one edge with outcome $\langle i, \sigma \rangle$. Similarly, we omit the infinitely many edges of a given $\mathcal{S}$ strategy, focusing on just one edge with outcome $\langle w, \sigma \rangle$, one with outcome $\langle d, \sigma \rangle$, one with outcome $\langle h, \sigma \rangle$ and one with outcome $\langle g, \sigma \rangle$ (whenever the last two kinds of edge are present). To simplify our presentation we shall simply write σ to denote the use of any outcome being depicted. We also recall that the infinitely many edges of each strategy are ordered by the value of σ and not by the value of o. The outcomes will thus be depicted on the priority tree as being in no particular order.

We have already seen that the effect of the various outcomes of an $\mathcal{S}$ strategy on the satisfaction of the $\mathcal{R}$ and $\mathcal{S}$ requirements is similar to the effect of an $\mathcal{S}$ strategy in the original Lachlan Non-Splitting Theorem. Hence it is possible to build the priority tree in this section in an analogous way to the priority tree for the Lachlan Non-Splitting Theorem.

As is the case with the priority tree of the Lachlan Non-Splitting Theorem, the highest priority unsatisfied requirement at a given node can be determined though the analysis of the outcomes covered in the previous section. As before, the highest priority unsatisfied requirement at a given node causes a strategy of the corresponding kind to appear at that node. When an $\mathcal{S}$ strategy has an edge with outcome $\langle g, \sigma \rangle$ it will cause the corresponding $\mathcal{R}$ requirement to *switch* its mode of satisfaction from a Γ-strategy to a $\hat{\Gamma}$-strategy below the edge. This will cause any $\mathcal{R}$ strategy below the edge to follow a $\hat{\Gamma}$-strategy instead of a Γ-strategy. In addition, we have already seen that $\mathcal{S}$ strategies will follow *S-Synchronisation*. As in the construction of the Lachlan Non-Splitting Theorem, the combination of switching and S-Synchronisation will allow an $\mathcal{S}$ requirement below an $\mathcal{R}$ requirement to be satisfied when strategies are organised according to the priority tree in figure 3.1.

The resulting modified priority tree is shown below.

3.5.10 Satisfaction of Requirements

We shall now examine the simultaneous satisfaction of the $\mathcal{R}$ and $\mathcal{S}$ requirements by the strategies and priority tree found in this section. For our purposes it shall be sufficient to consider the most complex situation, which occurs when the strategy α^V is on the true path and the leftmost edge visited infinitely often by the strategy is $\alpha \frown \langle w, \sigma \rangle$.

[w] Suppose that the edge $\alpha^V \frown \langle w, \sigma \rangle$ is on the true path.

The analysis for showing that the $\mathcal{S}$ requirement is satisfied is identical to the one found in Section 3.5.8 for the case where the outcome $\langle w, \sigma \rangle$ of the strategy is on the true path.

We now consider the satisfaction of the $\mathcal{R}$ requirement by the strategy β^V.

If α^V never enumerates a witness w into the set A, then we have that it cannot interfere with

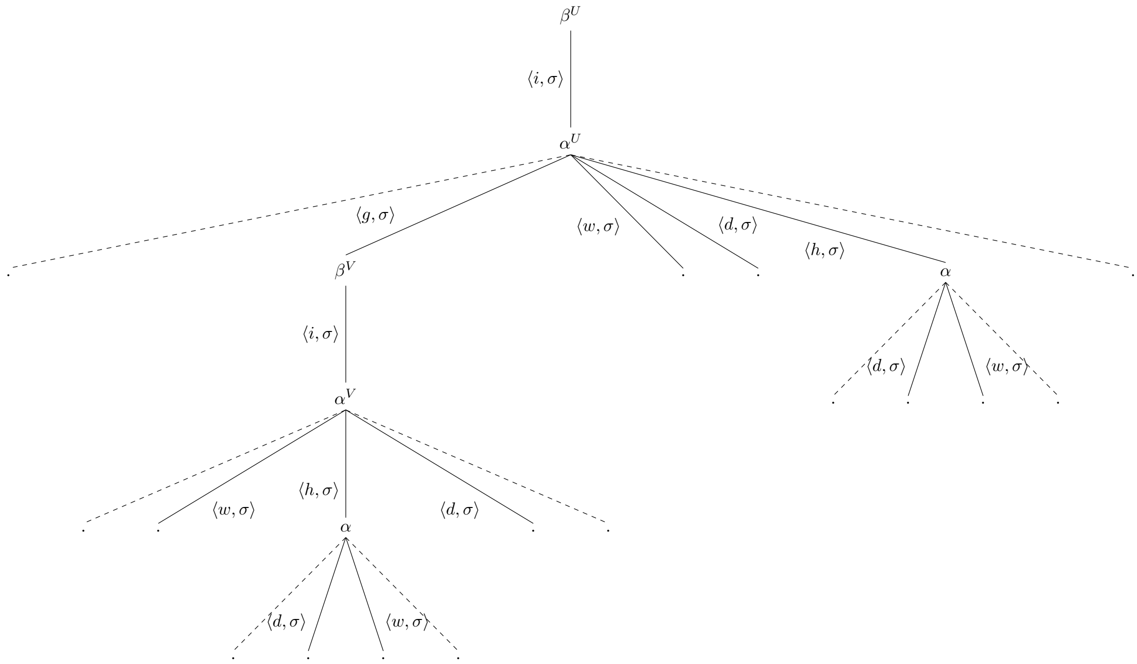

Figure 3.1: Priority tree for $\mathcal{S}$ below $\mathcal{R}$

β^V building $\Gamma^{V,D}_{\beta^V \frown \langle i,\sigma\rangle} = A$ and satisfying the $\mathcal{R}$ requirement. On the other hand, suppose that α^V has enumerated a witness w into the set A at stage u. Then by $\mathcal{S}$ -Synchronisation the strategy α^U must also have enumerated some witness w' into the set A at stage u.

Now at stage u, the strategy α^V lies inside the work interval $(v, \gamma_{\beta^U \frown \langle i,\sigma\rangle}[u](v))$ of the edge $\alpha^U \frown \langle g,\sigma\rangle$, and therefore both w and $\theta_u(w)$ must lie inside this work interval. On the other hand, α^U can only choose witnesses w' which are greater than the upper bound of its work interval. Hence we have that $w < w'$. The enumeration of w and w' into the set A at stage u therefore create least disagreements $\Gamma^{U,D}_{\beta^U \frown \langle i,\sigma\rangle}(w) \neq A(w)$ and $\Gamma^{V,D}_{\beta^V \frown \langle i,\sigma\rangle}(w) \neq A(w)$.

Now, in order for α^V to lie on the true path, it must become accessible again. But for this to be the case, the strategy β^U must have visited the edge $\beta^U \frown \langle i,\sigma\rangle$ and continued to the next substage at some least stage $u' > u$. Since $\Gamma^{U,D}_{\beta^U \frown \langle i,\sigma\rangle}[u'](w) \neq A_{u'}(w)$, β^U can only have gone to the next substage whilst visiting the edge if a β-expansionary stage u'' such that $u < u'' \leq u'$ has been attached to the edge.

We now make two considerations.

Firstly, in order for α^V to have enumerated w into A at stage u, the edge $\beta^U \frown \langle i, \sigma \rangle$ must have been in open mode at stage u, and the strategy must have gone to the next substage when visiting the edge at stage u. This means that the strategy must have changed the mode of the edge to close mode at stage u. Therefore when β^U visits the edge $\beta^U \frown \langle i, \sigma \rangle$ at stage u', it will see the disagreement $\Gamma^{U,D}_{\beta^U \frown \langle i,\sigma \rangle}[u'](w) \neq A_{u'}(w)$ and enumerate $\gamma_{\beta^U \frown \langle i,\sigma \rangle}[u'](v)$ into D so as to remove it. Now the uses of the functional must be non decreasing with respect to stages, and therefore we have that $\gamma_{\beta^U \frown \langle i,\sigma \rangle}[u'](v) \geq \gamma_{\beta^U \frown \langle i,\sigma \rangle}[u](v)$. Now both α^V and β^V lie inside the work interval of the edge $\alpha^U \frown \langle g, \sigma \rangle$ and have not been accessible since stage u. It follows that $\theta_u(w) < \gamma_{\beta^U \frown \langle i,\sigma \rangle}[u'](v)$, and that any use chosen in defining the functional $\Gamma^{V,D}_{\beta^V \frown \langle i,\sigma \rangle}$ at or prior to stage u must also be smaller than $\gamma_{\beta^U \frown \langle i,\sigma \rangle}[u'](v)$. Hence the enumeration of $\gamma_{\beta^U \frown \langle i,\sigma \rangle}[u'](v)$ into D has no effect on the strategies α^V and β^V.

Secondly, from the fact that a β-expansionary stage u'' has been attached to $\beta^U \frown \langle i\sigma \rangle$, we can infer that there has been some least $\mathcal{R}$ -expansionary stage t such that $u < t \leq u''$. If $U \upharpoonright \phi_{1,u}(w) \neq U \upharpoonright \phi_{1,t}(w)$, we have that α^U determines that it has diagonalised when it is accessible again at some least stage $u''' \geq u'$. This causes α^U to detach every witness attached to one of its edges, stop attaching witnesses and to set $R_{\alpha^U, u'''} > 0$. But the latter means that α^U will start terminating the stage whenever it visits the edge $\alpha^U \frown \langle g, \sigma \rangle$ at and after stage u''', making the strategy α^V inaccessible, which is a contradiction.

Hence it must be the case that $V \upharpoonright \phi_{1,u}(w) \neq V \upharpoonright \phi_{1,t}(w)$. From this it follows that $\Gamma^{V,D}_{\beta^V \frown \langle i,\sigma \rangle}(w)$ becomes undefined at stage t, meaning that it no longer disagrees with $A(w)$. Hence once the strategy β^V visits its edge $\beta^V \frown \langle i, \sigma \rangle$ again at some least stage $u'''' \geq u'$, there will be no need to repair a disagreement and the strategy will not enumerate $\gamma_{\beta^V \frown \langle i,\sigma \rangle}[u''''](v)$ into D. Thus when α^V becomes accessible again at some least stage $u''''' \geq u'''$, we have that the strategy α^V has diagonalised and preserved this diagonalisation. Therefore the $\mathcal{S}$ requirement is satisfied.

Similarly, the functional being built by the strategy β^V no longer disagrees with the set A, and the strategy β^V is now able to build $\Gamma^{V,D}_{\beta^V \frown \langle i,\sigma \rangle} = A$ without the interference of α^V, thus satisfying the $\mathcal{R}$ requirement.

3.6 $\mathcal{S}$ Below $\mathcal{R}_2$ Below $\mathcal{R}_1$

In this section we shall show how one can satisfy one $\mathcal{S}$ requirement below one $\mathcal{R}$ requirement labeled $\mathcal{R}_2$ below one $\mathcal{R}$ requirement labeled $\mathcal{R}_1$.

Similarly to the previous section, this will require the use of multiple $\mathcal{R}$ and $\mathcal{S}$ strategies organised into a priority tree. In this section we shall introduce $\mathcal{R}$ strategies which are able to take into consideration one other $\mathcal{R}$ strategy lying above them, and $\mathcal{S}$ strategies which are able to take into consideration two other $\mathcal{R}$ strategies lying above them. These strategies can then be used in conjunction with the simpler strategies found in Section 3.5 to satisfy the requirements.

We shall now consider the new $\mathcal{R}$ strategies which will be required. We start by defining the questions needed for the $\mathcal{R}$ strategy to determine its outcome at any given stage.

3.6.1 Questions for the $\mathcal{R}$ Strategy

The $\mathcal{R}$ strategy β, will need to ask one question, which we denote by Q_1. The question asks whether the strategy β sees infinitely many β-expansionary stages:

(1) Are there infinitely many $q \in \mathbb{N}_\beta$ such that the following holds?

(i) $(\forall q' < q)[\beta\text{-stage}(q') \Rightarrow l_q(\Phi_1^{U,V}, A) > l_{q'}(\Phi_1^{U,V}, A)]$.

(ii) $(\forall q' < q)[\beta\text{-stage}(q') \wedge (\forall q'' < q')[\,\beta\text{-stage}(q'') \Rightarrow l_{q'}(\Phi_1^{U,V}, A) > l_{q''}(\Phi_1^{U,V}, A)\,] \Rightarrow l_q(\Phi_2^{A,D}, U) > l_{q'}(\Phi_2^{A,D}, U)]$.

(iii) $(\forall q' < q)[\beta\text{-stage}(q') \wedge (\forall q'' < q')[\,\beta\text{-stage}(q'') \Rightarrow l_{q'}(\Phi_1^{U,V}, A) > l_{q''}(\Phi_1^{U,V}, A)\,] \Rightarrow l_q(\Phi_3^{A,D}, V) > l_{q'}(\Phi_3^{A,D}, V)]$.

If the strategy is accessible at some stage s, it will guess the answer to Q_1 by computing the apparent limit o and apparent use σ for $\lim_{t\to\infty} \Psi^{H_0}(f(Q_1), t)$ at stage s. If the answer corresponding to o is 'No', we denote the outcome by $\langle f, \sigma\rangle$. On the other hand, if the answer corresponding to o is 'Yes', we denote the outcome by $\langle i, \sigma\rangle$.

We now describe the $\mathcal{R}$ strategy itself.

3.6.2 The $\mathcal{R}$ Strategy

The $\mathcal{R}$ strategy β will work as follows. First of all, it can either be following a Γ-strategy or a $\hat{\Gamma}$-strategy which will depend on its location on the priority tree. The outcomes of the strategy will be of the form $\langle i, \sigma\rangle$ and $\langle f, \sigma\rangle$. If the strategy is following a Γ-strategy, it will build a different functional $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}$ below every edge $\beta \frown \langle i, \sigma\rangle$ leaving β. Similarly if the strategy is following a $\hat{\Gamma}$-strategy, it will build a different functional $\Gamma^{V,D}_{\beta\frown\langle i,\sigma\rangle}$ below every edge $\beta \frown \langle i, \sigma\rangle$ leaving β. On the other hand, the strategy will not build any functional below edges of the form $\beta \frown \langle f, \sigma\rangle$, irrespectively of whether it is following a Γ-strategy or a $\hat{\Gamma}$-strategy.

Each edge of the form $\beta \frown \langle i, \sigma\rangle$ will have a separate set of uses $U^{e,\beta\frown\langle i,\sigma\rangle}$ from which the strategy will choose uses when defining the functional associated to the edge $\beta \frown \langle i, \sigma\rangle$. Note that e is the index of the strategy β in the total ordering of the $\mathcal{R}$ strategies lying on the priority tree. The most important difference from the $\mathcal{R}$ strategy described in the previous section is that the $\mathcal{R}$ strategy β will now lie below at most one $\mathcal{R}$ strategy β_1 which is active for β, and which the latter must now take into consideration.

The strategy goes through the following steps at stage s.

During its first step, the strategy β will calculate a rightward restraint $r(\beta \frown \langle o', \sigma'\rangle)$ for every edge $\beta \frown \langle o', \sigma'\rangle$ which has been previously accessible, exactly as in the previous section.

Similarly during its second step, the strategy β will perform its attachment procedure as in the previous section, attaching the stage s to a suitable edge if s is a β-expansionary stage.

During its third step the strategy β will calculate its attachment restraint $a(\beta \frown \langle o', \sigma'\rangle)$ for every edge $\beta \frown \langle o', \sigma'\rangle$ which has been previously accessible as in the previous section.

This is followed by its fourth step, where the strategy will calculate the outcome $\beta \frown \langle o, \sigma\rangle$ to visit during the present stage.

Once the outcome has been determined, the strategy will perform its fifth step by calculating the downward restraint $d(\beta \frown \langle o, \sigma\rangle, s)$ for the edge $\beta \frown \langle o, \sigma\rangle$ as in the previous section.

During the final and sixth step, the strategy will take action depending on the outcome of the edge $\beta \frown \langle o, \sigma\rangle$. This will be mostly identical to the procedure found in the previous section, with one important exception. If there is an $\mathcal{R}$ strategy β_1 above β which is active for β, we have that β must now *R-Synchronise* with β_1. Thus if the strategy β_1 has an edge $\beta_1 \frown \langle i, \sigma_1 \rangle$ on the path leading to β, and β defines the functional associated to some edge $\beta \frown \langle i, \sigma \rangle$ at some element x at stage s by choosing some use $\gamma_{\beta \frown \langle i,\sigma \rangle}[s](x)$, we must have that $\gamma_{\beta \frown \langle i,\sigma \rangle}[s](x) > \gamma_{\beta_1 \frown \langle i,\sigma_1 \rangle}[s](x)$.

We shall now formalise the $\mathcal{R}$ strategy.

The $\mathcal{R}$ Strategy

The strategy β labeled $\mathcal{R}_i$ will either be following a Γ-strategy or a $\hat{\Gamma}$-strategy. Every edge $\beta \frown \langle i, \sigma \rangle$ has a functional $\Gamma^{U_i,D}_{\beta \frown \langle i,\sigma \rangle}$ (or $\Gamma^{V_i,D}_{\beta \frown \langle i,\sigma \rangle}$ resp.) associated to it, which the strategy will build when it visits that edge. Each edge $\beta \frown \langle i, \sigma \rangle$ will also have its own set of uses $U^{e,\beta \frown \langle i,\sigma \rangle}$ from which uses will be chosen when defining the respective functionals.

The strategy β lies below a number of $\mathcal{R}$ strategies β'. Each such strategy β' imposes a downward restraint $d(\beta' \frown \langle o', \sigma' \rangle, s)$ on β at stage s, where $\beta' \frown \langle o', \sigma' \rangle$ is the edge of β' on the path leading to β. One of these $\mathcal{R}$ strategies will be active for β. In this case we denote this $\mathcal{R}$ strategy by β_1 and its edge lying on the path leading to β by $\beta_1 \frown \langle i, \sigma_1 \rangle$. This strategy will either be following a Γ-strategy or a $\hat{\Gamma}$-strategy.

The strategy β may also lie below a number of $\mathcal{S}$ strategies α'. Each such strategy α' imposes a downward restraint $d(\alpha' \frown \langle o', \sigma' \rangle, s)$ on β at stage s, where $\alpha' \frown \langle o', \sigma' \rangle$ is the edge of α' on the path leading to β. The strategy α' also imposes the diagonalisation restraint $R_{\alpha',s}$ on β at stage s. The strategy α' may also impose a work interval on β at stage s, depending on its outcome on the path leading to β. Finally let $\alpha'' \subset \beta$ be the greatest $\mathcal{S}$ strategy (under $\subset$) which imposes a work interval on β. We shall denote the work interval imposed by α'' on β at stage s by (a_s, b_s).

(1) Define the rightward restraint $r(\beta \frown \langle o', \sigma' \rangle, s)$ for every edge $\beta \frown \langle o', \sigma' \rangle$ which was previously accessible as the least element x such that:

  (a) $x \geq t$ where t is some β-expansionary stage attached to $\beta \frown \langle o', \sigma' \rangle$.

  (b) $x \geq t$ where t is the last stage at which $\beta \frown \langle o', \sigma' \rangle$ was accessible.

Go to step (2).

(2) If stage s is a β-expansionary stage, and there is some edge $\beta \frown \langle i, \sigma' \rangle$ which has been previously accessible and which has no β-expansionary stage attached to it, attach s to the leftmost such edge.

If a β-expansionary stage s has been attached to some edge $\beta \frown \langle i, \sigma' \rangle$, consider every edge $\beta \frown \langle i, \sigma'' \rangle$ lying to the right of $\beta \frown \langle i, \sigma' \rangle$. If some β-expansionary stage s' is attached to $\beta \frown \langle i, \sigma'' \rangle$, detach the β-expansionary stage from the edge.

Go to step (3).

(3) Determine the edge $\beta \frown \langle o, \sigma \rangle$ which the strategy should visit at stage s.

Go to step (4).

(4) Define the attachment procedure restraint $a(\beta \frown \langle o', \sigma' \rangle, s)$ for every edge $\beta \frown \langle o', \sigma' \rangle$ which was previously accessible. If the strategy has not attached a β-expansionary stage s to some edge $\beta \frown \langle o'', \sigma'' \rangle <_L \beta \frown \langle o', \sigma' \rangle$ at stage s, define $a(\beta \frown \langle o', \sigma' \rangle, s) = 0$. Otherwise define $a(\beta \frown \langle o', \sigma' \rangle, s) = s$.

Also define the downward restraint $d(\beta \frown \langle o, \sigma \rangle, s)$ as the least element x such that:

(a) $x \geq \sup\{r(\beta \frown \langle o', \sigma' \rangle, s) \mid \beta \frown \langle o', \sigma' \rangle <_L \beta \frown \langle o, \sigma \rangle \ \wedge$ $\beta \frown \langle o', \sigma' \rangle$ has been previously accessible$\}$.

(b) $x \geq a(\beta \frown \langle o, \sigma \rangle, s)$.

(c) $x \geq d(\beta \frown \langle o, \sigma \rangle, t)$ for all $t < s$.

Go to step (5).

(5) Consider the edge $\beta \frown \langle o, \sigma \rangle$ being visited by the strategy at stage s. Take action according to the value of o through the following case analysis.

(a) $o = f$.

(i) $\beta \frown \langle f, \sigma \rangle$ is in open mode and s is not a β-open stage. End the stage s, and go to stage $s + 1$.

(ii) $\beta \frown \langle f, \sigma \rangle$ is in open mode and s is a β-open stage. Set the edge to close mode. Go to the next substage.

(iii) $\beta \frown \langle f, \sigma \rangle$ is in close mode and s is not a β-close stage. End the stage s, and go to stage $s+1$.

(iv) $\beta \frown \langle f, \sigma \rangle$ is in close mode and s is a β-close stage. Set the edge to open mode. Go to the next substage.

(b) $o = i$.

(i) $\beta \frown \langle i, \sigma \rangle$ is in open mode and there is no β-expansionary stage attached to the edge. End the stage s, and go to stage $s+1$.

(ii) $\beta \frown \langle i, \sigma \rangle$ is in open mode and there is a β-expansionary stage attached to the edge and s is not a β-open stage. End the stage s, and go to stage $s+1$.

(iii) $\beta \frown \langle i, \sigma \rangle$ is in open mode and there is a β-expansionary stage attached to the edge and s is a β-open stage.

If there is some element m such that $\Gamma^{U_i,D}_{\beta \frown \langle i,\sigma \rangle}[s](m) \neq A_s(m)$ (or $\Gamma^{V_i,D}_{\beta \frown \langle i,\sigma \rangle}[s](m)$ resp.), set the edge to close mode. End the stage s, and go to stage $s+1$.

Otherwise, detach the stage s from the edge. Consider every $x < l_s(\Phi_1^{U_i,V_i}, A)$ such that $\Gamma^{U_i,D}_{\beta \frown \langle i,\sigma \rangle}[s](x) \uparrow$. Define the axiom $\Gamma^{U_i,D}_{\beta \frown \langle i,\sigma \rangle}[s](x) = A_s(x)$. Consider the least element $u < s$ in $U^{e,\beta \frown \langle i,\sigma \rangle}$ such that:

(A) $u \geq \gamma_{\beta \frown \langle i,\sigma \rangle}[t](x)$ for all $t < s$.

(B) $u > \gamma_{\beta \frown \langle i,\sigma \rangle}[s](y)$ for all $y < x$.

(C) $u > \sup\{r(\beta \frown \langle o', \sigma' \rangle, s) \mid \beta \frown \langle o', \sigma' \rangle <_L \beta \frown \langle o, \sigma \rangle \ \wedge$ $\beta \frown \langle o', \sigma' \rangle$ has been previously accessible$\}$.

(D) $u > a(\beta \frown \langle i, \sigma \rangle, s)$.

(E) $u > R_{\alpha', s}$, for every $\mathcal{S}$ strategy $\alpha' \subset \beta$.

(F) $u > d(\beta' \frown \langle o', \sigma' \rangle, s)$, for every $\mathcal{R}$ strategy $\beta' \subset \beta$ with edge $\beta' \frown \langle o', \sigma' \rangle$ on the path leading to β.

(G) $u > d(\alpha' \frown \langle o', \sigma' \rangle, s)$, for every $\mathcal{S}$ strategy $\alpha' \subset \beta$ with edge $\alpha' \frown \langle o', \sigma' \rangle$ on the path leading to β.

(H) $a_s < u < b_s$.

(I) $u \notin D$.

(J) $u > y$, where y is a constraint imposed by some $\mathcal{S}$ strategy α below β.

(K) $u > t$, where t is the last stage at which the edge $\beta \frown \langle i, \sigma \rangle$ was last initialised.

(L) $u > \gamma_{\beta_1 \frown \langle i, o_1 \rangle}[s](x)$.

If u does not exist, $\Gamma^{U_i,D}_{\beta \frown \langle i,\sigma \rangle}(x)$ is not defined.

Otherwise let $t' < s$ be the greatest stage such that $\Gamma^{U_i,D}_{\beta \frown \langle i,\sigma \rangle}[t'](x) \downarrow$, and let u' be the greatest use which the strategy has chosen so far when defining its functional at some element.

If t' does not exist, define $\gamma_{\beta \frown \langle i,\sigma \rangle}[s](x) = u$.

If t' exists and $u > \gamma_{\beta \frown \langle i,\sigma \rangle}[t'](x)$, define $\gamma_{\beta \frown \langle i,\sigma \rangle}[s](x)$ to be the least element in $U^{e,\beta \frown \langle i,\sigma \rangle}$ which is greater than u'.

Otherwise define $\gamma_{\beta \frown \langle i,\sigma \rangle}[s](x) = \gamma_{\beta \frown \langle i,\sigma \rangle}[t'](x)$.

Set the edge to close mode. Go to the next substage.

($\Gamma^{V,D}$ *resp.*)

(iv) $\beta \frown \langle i, \sigma \rangle$ is in close mode and there is an element m such that $\Gamma^{U_i,D}_{\beta \frown \langle i,\sigma \rangle}[s](m) \neq A_s(m)$ (or $\Gamma^{V_i,D}_{\beta \frown \langle i,\sigma \rangle}[s](m)$ resp.) and there is no β-expansionary stage attached to the edge. End the stage s, and go to stage $s+1$.

(v) $\beta \frown \langle i, \sigma \rangle$ is in close mode and there is an element m such that $\Gamma^{U_i,D}_{\beta \frown \langle i,\sigma \rangle}[s](m) \neq A_s(m)$ (or $\Gamma^{V_i,D}_{\beta \frown \langle i,\sigma \rangle}[s](m)$ resp.) and there is a β-expansionary stage attached to the edge and s is not a β-close stage. End the stage s, and go to stage $s+1$.

(vi) $\beta \frown \langle i, \sigma \rangle$ is in close mode and there is an element m such that $\Gamma^{U_i,D}_{\beta \frown \langle i,\sigma \rangle}[s](m) \neq A_s(m)$ (or $\Gamma^{V_i,D}_{\beta \frown \langle i,\sigma \rangle}[s](m)$ resp.) and there is a β-expansionary stage attached to the edge and s is a β-close stage. Enumerate $\gamma_{\beta \frown \langle i,\sigma \rangle}[s](m)$ into D. Set the edge to open mode. End the stage s and go to stage $s+1$.

(vii) $\beta \frown \langle i, \sigma \rangle$ is in close mode and there is no element m such that $\Gamma^{U_i,D}_{\beta \frown \langle i,\sigma \rangle}[s](m) \neq A_s(m)$ (or $\Gamma^{V_i,D}_{\beta \frown \langle i,\sigma \rangle}[s](m)$ resp.) and s is not a β-close stage. End the stage s, and go to stage $s+1$.

(viii) $\beta \frown \langle i, \sigma \rangle$ is in close mode and there is no element m such that $\Gamma^{U_i,D}_{\beta \frown \langle i,\sigma \rangle}[s](m) \neq A_s(m)$ (or $\Gamma^{V_i,D}_{\beta \frown \langle i,\sigma \rangle}[s](m)$ resp.) and s is a β-close stage. Set the edge to open mode. Go to the next substage.

3.6.3 Analysis of Outcomes

The analysis is identical to the one found in Section 3.5.

We shall now consider the new $\mathcal{S}$ strategies which will be required. We start by defining the questions needed for the $\mathcal{S}$ strategy to determine its outcome at any given stage.

3.6.4 Questions for the $\mathcal{S}$ Strategy

The $\mathcal{S}$ strategy α will need to ask a number of questions, which take the context of the strategy into consideration. We assume that the strategy α lies below two $\mathcal{R}$ strategies β_i for $1 \leq i \leq 2$ which are active for α and which can follow a Γ-strategy or a $\hat{\Gamma}$-strategy. The strategy β_i will have an outgoing edge $\beta_i \frown \langle i, \sigma_i \rangle$, which lies on the path leading to the strategy α. In addition $\mathcal{S}$ strategies α' lying above α may impose a work interval on α. The work interval imposed at stage s by the greatest strategy α' (under $\subset$) above α is denoted by (a_s, b_s). In addition, we shall also use the notation $\phi_{i,1}[s](w)$ for $1 \leq i \leq 2$ to denote the use of the computation $\Phi^{U_i,V_i}_{i,1}[s](w)$.

The strategy starts by asking question Q_1. This question asks whether there are infinitely many witnesses w and stages s such that w and $\theta_s(w)$ lie inside the work interval (a_s, b_s) at stage s and such that the computation $\Theta^D[s](w) \downarrow = 0$ holds. In addition the question also asks whether the length of agreement between the functional Θ^D and the set A expands infinitely often.

(1) Are there infinitely many $w \in W^e$, $s \in \mathbb{N}_\alpha$ and $q \in \mathbb{N}_\alpha$ such that the following hold?

(i) $\Theta^D[s](w) \downarrow = 0$.

(ii) $a_s < w < b_s$ (if a work interval is imposed on the strategy).

(iii) $a_s < \theta_s(w) < b_s$ (if a work interval is imposed on the strategy).

(iv) $(\forall q' < q)[\,\alpha\text{-stage}(q') \Rightarrow l_{q'}(\Theta^D, A) < l_q(\Theta^D, A)\,]$.

The strategy then proceeds by asking question $Q_{2.1}$. A positive answer to question Q_1 asserts that there are infinitely many witnesses w and stages s such that w and $\theta_s(w)$ lie inside the work interval (a_s, b_s) and such that the computation $\Theta^D[s](w)\downarrow= 0$ holds. Question $Q_{2.1}$ asks whether infinitely many of these witnesses w and stages s give rise to computations $\Gamma^{U_1,D}_{\beta_1\frown\langle i,\sigma_1\rangle}[s](w)$ (or $\Gamma^{V_1,D}_{\beta_1\frown\langle i,\sigma_1\rangle}[s](w)$ resp.) which are honest.

(2.1) Are there infinitely many $w \in W^e$, $s \in \mathbb{N}_\alpha$ and $q \in \mathbb{N}_\alpha$ such that the following hold?

(i) $\Theta^D[s](w)\downarrow= 0$.

(ii) $a_s < w < b_s$ (if a work interval is imposed on the strategy).

(iii) $a_s < \theta_s(w) < b_s$ (if a work interval is imposed on the strategy).

(iv) $(\forall q' < q)[\,\alpha\text{-stage}(q') \Rightarrow l_{q'}(\Theta^D, A) < l_q(\Theta^D, A)\,]$.

(v) $\phi_{1,1}[s](w) \leq \gamma_{\beta_1\frown\langle i,\sigma_1\rangle}[s](w)$.

The strategy then proceeds by asking question $Q_{2.2}$. A positive answer to question Q_1 and question $Q_{2.1}$ asserts that there are infinitely many witnesses w and stages s such that w and $\theta_s(w)$ lie inside the work interval (a_s, b_s), the computation $\Theta^D[s](w)\downarrow= 0$ holds and the computation $\Gamma^{U_1,D}_{\beta_1\frown\langle i,\sigma_1\rangle}[s](w)$ (or $\Gamma^{V_1,D}_{\beta_1\frown\langle i,\sigma_1\rangle}[s](w)$ resp.) is honest. Question $Q_{2.2}$ asks whether infinitely many of these witnesses w and stages s give rise to computations $\Gamma^{U_2,D}_{\beta_2\frown\langle i,\sigma_2\rangle}[s](w)$ (or $\Gamma^{V_2,D}_{\beta_2\frown\langle i,\sigma_2\rangle}[s](w)$ resp.) which are also honest.

(2.2) Are there infinitely many $w \in W^e$, $s \in \mathbb{N}_\alpha$ and $q \in \mathbb{N}_\alpha$ such that the following hold?

(i) $\Theta^D[s](w)\downarrow= 0$.

(ii) $a_s < w < b_s$ (if a work interval is imposed on the strategy).

(iii) $a_s < \theta_s(w) < b_s$ (if a work interval is imposed on the strategy).

(iv) $(\forall q' < q)[\,\alpha\text{-stage}(q') \Rightarrow l_{q'}(\Theta^D, A) < l_q(\Theta^D, A)\,]$.

(v) $\phi_{1,1}[s](w) \leq \gamma_{\beta_1\frown\langle i,\sigma_1\rangle}[s](w)$.

(vi) $\phi_{2,1}[s](w) \leq \gamma_{\beta_2\frown\langle i,\sigma_2\rangle}[s](w)$.

Subsequently, if β_1 is following a Γ-strategy, we proceed by asking question $Q_{3.1}$. A positive answer to questions Q_1, $Q_{2.1}$ and $Q_{2.2}$ asserts that there are infinitely many witnesses w and stages s such that w and $\theta_s(w)$ lie inside the work interval (a_s, b_s), the computation $\Theta^D[s](w)\downarrow= 0$ holds and the computations $\Gamma^{U_1,D}_{\beta_1\frown\langle i,\sigma_1\rangle}[s](w)$ and $\Gamma^{U_2,D}_{\beta_2\frown\langle i,\sigma_2\rangle}[s](w)$ (or $\Gamma^{V_2,D}_{\beta_2\frown\langle i,\sigma_2\rangle}[s](w)$ resp.) are honest. Question $Q_{3.1}$ asks whether infinitely many of these witnesses w enter A at stage s, and whether a $U_1 \restriction \phi_{1,1}[s](w)$ change occurs by the least $\mathcal{R}_1$-expansionary stage $t_1 > s$.

(3.1) Are there infinitely many $w \in W^e$ $s \in \mathbb{N}_\alpha$, $t_1 \in \mathbb{N}$ and $q \in \mathbb{N}_\alpha$ such that the following hold?

(i) $\Theta^D[s](w)\downarrow= 0$.

(ii) $a_s < w < b_s$ (if a work interval is imposed on the strategy).

(iii) $a_s < \theta_s(w) < b_s$ (if a work interval is imposed on the strategy).

(iv) $(\forall q' < q)[\,\alpha\text{-stage}(q') \Rightarrow l_{q'}(\Theta^D, A) < l_q(\Theta^D, A)\,]$.

(v) $\phi_{1,1}[s](w) \leq \gamma_{\beta_1\frown\langle i,\sigma_1\rangle}[s](w)$.

(vi) $\phi_{2,1}[s](w) \leq \gamma_{\beta_2\frown\langle i,\sigma_2\rangle}[s](w)$.

(vii) $A_s(w) = 0$.

(viii) $A_{s+1}(w) = 1$.

(ix) $t_1 > s$.

(xi) $(\forall s < t' < t_1)[U_{1,t'} \restriction \phi_{1,1}[s](w) = U_{1,s} \restriction \phi_{1,1}[s](w)]$.

(xii) $(\forall s < t' < t_1)[V_{1,t'} \restriction \phi_{1,1}[s](w) = V_{1,s} \restriction \phi_{1,1}[s](w)]$.

(xiii) $U_{1,t_1} \restriction \phi_{1,1}[s](w) \neq U_{1,s} \restriction \phi_{1,1}[s](w)]$.

Finally if β_2 is following a Γ-strategy, we proceed by asking question $Q_{3.2}$. A positive answer to questions Q_1, $Q_{2.1}$ and $Q_{2.2}$ asserts that there are infinitely many witnesses w and stages s such that w and $\theta_s(w)$ lie inside the work interval (a_s, b_s), the computation $\Theta^D[s](w)\downarrow= 0$ holds and the computations $\Gamma^{U_1,D}_{\beta_1\frown\langle i,\sigma_1\rangle}[s](w)$ (or $\Gamma^{V_1,D}_{\beta_1\frown\langle i,\sigma_1\rangle}[s](w)$ resp.) and $\Gamma^{U_2,D}_{\beta_2\frown\langle i,\sigma_2\rangle}[s](w)$ are honest.

Now if question $Q_{3.1}$ is asked and answered positively we also have that infinitely many of these witnesses w enter A at stage s, and infinitely many of these cause a $U_1 \restriction \phi_{1,1}[s](w)$ change to occur by the least $\mathcal{R}_1$-expansionary stage $t_1 > s$. Question $Q_{3.2}$ would then ask whether infinitely many of the witnesses causing a $U_1 \restriction \phi_{1,1}[s](w)$ change also cause a $U_2 \restriction \phi_{2,1}[s](w)$ change to occur by the least $\mathcal{R}_2$-expansionary stage $t_2 > s$.

On the other hand, if question $Q_{3.1}$ is not asked, we have that question $Q_{3.2}$ asks whether infinitely many of the witnesses w enter A at stage s, and whether infinitely many of these cause a $U_2 \restriction \phi_{2,1}[s](w)$ change to occur by the least $\mathcal{R}_2$-expansionary stage $t_2 > s$.

(3.2) Are there infinitely many $w \in W^e$, $s \in \mathbb{N}_\alpha$, $t_1 \in \mathbb{N}, t_2 \in \mathbb{N}$ and $q \in \mathbb{N}_\alpha$ such that the following hold?

(i) $\Theta^D[s](w) \downarrow= 0$.

(ii) $a_s < w < b_s$ (if a work interval is imposed on the strategy).

(iii) $a_s < \theta_s(w) < b_s$ (if a work interval is imposed on the strategy).

(iv) $(\forall q' < q)[\,\alpha\text{-stage}(q') \Rightarrow l_{q'}(\Theta^D, A) < l_q(\Theta^D, A)\,]$.

(v) $\phi_{1,1}[s](w) \leq \gamma_{\beta_1 \frown \langle i,\sigma_1\rangle}[s](w)$.

(vi) $\phi_{2,1}[s](w) \leq \gamma_{\beta_2 \frown \langle i,\sigma_2\rangle}[s](w)$.

(vii) $A_s(w) = 0$.

(viii) $A_{s+1}(w) = 1$.

(ix) $t_1 > s$.

(x) $t_2 > s$.

(xi) β_1 exists and follows a Γ-strategy $\Rightarrow$ $(\forall s < t' < t_1)[U_{1,t'} \restriction \phi_{1,1}[s](w) = U_{1,s} \restriction \phi_{1,1}[s](w)]$.

(xii) β_1 exists and follows a Γ-strategy $\Rightarrow$ $(\forall s < t' < t_1)[V_{1,t'} \restriction \phi_{1,1}[s](w) = V_{1,s} \restriction \phi_{1,1}[s](w)]$.

(xiii) β_1 exists and follows a Γ-strategy $\Rightarrow U_{1,t_1} \restriction \phi_{1,1}[s](w) \neq U_{1,s} \restriction \phi_{1,1}[s](w)]$.

(xiv) $(\forall s < t' < t_2)[U_{2,t'} \restriction \phi_{2,1}[s](w) = U_{2,s} \restriction \phi_{2,1}[s](w)]$.

(xv) $(\forall s < t' < t_2)[V_{2,t'} \restriction \phi_{2,1}[s](w) = V_{2,s} \restriction \phi_{2,1}[s](w)]$.

(xvi) $U_{2,t_2} \restriction \phi_{2,1}[s](w) \neq U_{2,s} \restriction \phi_{2,1}[s](w)]$.

If the strategy is accessible at some stage s, it will guess the answer to questions Q_1, $Q_{2.1}$, $Q_{2.2}$, $Q_{3.1}$ and $Q_{3.2}$ (where applicable). This is done by computing an apparent limit o_i and an apparent use σ_i for each $\lim_{t\to\infty} \Psi^{H_0}(f(Q_i), t)$ at stage s. Let σ be the apparent use of greatest length. The outcome visited by the strategy at stage s is calculated as follows.

- If the answer corresponding to o_1 is 'No', we denote the outcome by $\langle w, \sigma\rangle$.
- If β_1 exists, the answer corresponding to o_1 is 'Yes', and the answer corresponding to $o_{2.1}$ is 'No', we denote the outcome by $\langle h_1, \sigma\rangle$.
- If β_1 and β_2 exist, the answer corresponding to o_1 is 'Yes', the answer corresponding to $o_{2.1}$ is 'Yes', and the answer corresponding to $o_{2.2}$ is 'No', we denote the outcome by $\langle h_2, \sigma\rangle$.
- If β_1 and β_2 exist, β_1 is following a Γ-strategy, the answer corresponding to o_1 is 'Yes', the answer corresponding to $o_{2.1}$ is 'Yes', the answer corresponding to $o_{2.2}$ is 'Yes', and the answer corresponding to $o_{3.1}$ is 'No', we denote the outcome by $\langle g_1, \sigma\rangle$.
- If β_1 and β_2 exist, β_1 is following a Γ-strategy, β_2 is following a Γ-strategy, the answer corresponding to o_1 is 'Yes', the answer corresponding to $o_{2.1}$ is 'Yes', the answer corresponding to $o_{2.2}$ is 'Yes', the answer corresponding to $o_{3.1}$ is 'Yes', and the answer corresponding to $o_{3.2}$ is 'No', we denote the outcome by $\langle g_2, \sigma\rangle$.
- If β_1 and β_2 exist, β_1 is following a Γ-strategy, β_2 is following a Γ-strategy, the answer corresponding to o_1 is 'Yes', the answer corresponding to $o_{2.1}$ is 'Yes', the answer corresponding to $o_{2.2}$ is 'Yes', the answer corresponding to $o_{3.1}$ is 'Yes', and the answer corresponding to $o_{3.2}$ is 'Yes', we denote the outcome by $\langle d, \sigma\rangle$.
- If β_1 and β_2 exist, β_1 is following a $\hat{\Gamma}$-strategy, β_2 is following a Γ-strategy, the answer corresponding to o_1 is 'Yes', the answer corresponding to $o_{2.1}$ is 'Yes', the answer corresponding to $o_{2.2}$ is 'Yes', and the answer corresponding to $o_{3.2}$ is 'No', we denote the outcome by $\langle g_2, \sigma\rangle$.
- If β_1 and β_2 exist, β_1 is following a $\hat{\Gamma}$-strategy, β_2 is following a Γ-strategy, the answer corresponding to o_1 is 'Yes', the answer corresponding to $o_{2.1}$ is 'Yes', the answer corresponding to $o_{2.2}$ is 'Yes', and the answer corresponding to $o_{3.2}$ is 'Yes', we denote the outcome by $\langle d, \sigma\rangle$.
- If β_1 and β_2 exist, β_1 is following a $\hat{\Gamma}$-strategy, β_2 is following a $\hat{\Gamma}$-strategy, the answer corresponding to o_1 is 'Yes', the answer corresponding to $o_{2.1}$ is 'Yes' and the answer

corresponding to $o_{2.2}$ is 'Yes', we denote the outcome by $\langle d, \sigma \rangle$.

We shall now proceed to discuss the $\mathcal{S}$ strategy.

3.6.5 The $\mathcal{S}$ Strategy

The $\mathcal{S}$ strategy α has an infinite set of witnesses W^e, and an infinite set of thresholds V^e, where e is the index of the strategy α in the total ordering of the $\mathcal{S}$ strategies lying on the priority tree. At any given stage s it will also be able to impose a restraint $R_{\alpha,s}$ on all lower priority strategies. Initially, we have that $R_{\alpha,0}$ is equal to 0. The strategy α shall use the fact that $R_{\alpha,s} > 0$ to signal that it has diagonalised. Once this restraint has been set, it will keep its value during subsequent stages.

The strategy α lies below two $\mathcal{R}$ strategies β_i for $1 \leq i \leq 2$ which are active for α and which can follow a Γ-strategy or a $\hat{\Gamma}$-strategy. In this case the strategy β_i will have an outgoing edge $\beta_i \frown \langle i, \sigma_i \rangle$, which lies on the path leading to the strategy α. The outcomes of the strategy will be of the form $\langle d, \sigma \rangle$, $\langle h_1, \sigma \rangle$, $\langle h_2, \sigma \rangle$ and $\langle w, \sigma \rangle$, while the outcome $\langle g_1, \sigma \rangle$ will be present if β_1 is following a Γ-strategy, and the outcome $\langle g_2, \sigma \rangle$ will be present if β_2 is following a Γ-strategy.

The strategy goes through the following steps at stage s.

During its first step, the strategy determines whether it has enumerated some witness w' into the set A during the last stage t at which it was accessible (assuming it was accessible at least once before, and that it has not been initialised in the meantime). Suppose that this has been the case.

Then the strategy needs to determine the way in which the disagreements $\Phi_{1,1}^{U_1,V_1}(w') \neq A(w')$ and $\Phi_{2,1}^{U_2,V_2}(w') \neq A(w')$ which were introduced when the strategy enumerated w' into A at stage t were removed.

In order to do this, the strategy determines two things. Firstly, whether a $U_1 \upharpoonright \phi_{1,1}[t](w')$ change has occurred between stage t and the least $\mathcal{R}_1$-expansionary stage $t_1 > t$. Secondly, whether a $U_2 \upharpoonright \phi_{2,1}[t](w')$ change has occurred between stage t and the least $\mathcal{R}_2$-expansionary stage $t_2 > t$. If both are the case, the strategy has diagonalised and the strategy sets $R_{\alpha,s} = \theta_t(w)$, so as to protect the use of the computation. Note that the first check is required only if the strategy β_1

exists and is active for α and is following a Γ-strategy, whilst the second check is required only if the strategy β_2 exists and is active for α and is also following a Γ-strategy.

During its second step, the strategy α will calculate a rightward restraint $r(\alpha \frown \langle o', \sigma' \rangle, s)$ for every edge $\alpha \frown \langle o', \sigma' \rangle$ which has been previously accessible, exactly as in the previous section.

During its third step, the strategy α will perform its attachment procedure.

If the strategy has already diagonalised ($R_{\alpha,s} > 0$), no further action needs to be taken and the attachment procedure will be terminated.

Otherwise the attachment procedure will consider in turn every witness w in W^e which at stage s yields a computation $\Theta^D[s](w) \downarrow = 0$ and which has not been attached to an edge so far. The attachment procedure will be seeking to attach one of these witnesses to an edge, and will stop considering further witnesses once this has been achieved.

In order to decide which edge the witness under consideration should be attached to, the strategy will first consider the kind of outcomes which the strategy has. These are determined by the position of the strategy on the priority tree.

There are three cases to consider.

In the first (and most general) case the $\mathcal{S}$ strategy α lies below two $\mathcal{R}$ strategies β_1 and β_2 which are active for α and which follow a Γ-strategy. In this case, α has edges with outcomes d, g_1, g_2, h_1, h_2 and w. Hence the strategy has to determine whether the computations $\Gamma^{U_1,D}_{\beta_1 \frown \langle i,\sigma_1 \rangle}[s](w)$ and $\Gamma^{U_1,D}_{\beta_2 \frown \langle i,\sigma_2 \rangle}[s](w)$ are honest. The first computation is honest if $\phi_{1,1}[s](w) \leq \gamma_{\beta_1 \frown \langle i,\sigma_1 \rangle}[s](w)$, while the second computation is honest if $\phi_{2,1}[s](w) \leq \gamma_{\beta_2 \frown \langle i,\sigma_2 \rangle}[s](w)$. The strategy will then attach the witness to the appropriate edge depending on the result.

In this case the attachment procedure will consider every witness w which at stage s yields a computation $\Theta^D[s](w) \downarrow = 0$. The strategy will then determine whether the witness w gives computations $\Gamma^{U_1,D}_{\beta_1 \frown \langle i,\sigma_1 \rangle}[s](w)$ (or $\Gamma^{V_1,D}_{\beta_1 \frown \langle i,\sigma_1 \rangle}[s](w)$ resp.) and $\Gamma^{U_2,D}_{\beta_2 \frown \langle i,\sigma_2 \rangle}[s](w)$ (or $\Gamma^{V_2,D}_{\beta_2 \frown \langle i,\sigma_2 \rangle}[s](w)$ resp.) which are honest.

If both of these computations are honest, the strategy attaches the witness w to the leftmost edge of the form $\alpha \frown \langle g_1, \sigma \rangle$ or $\alpha \frown \langle g_2, \sigma \rangle$ which does not have a witness attached. In either of these two cases this witness also has to satisfy a number of constraints, similar to the ones described in the previous section.

If the attachment procedure sees that the witness w gives a dishonest computation $\Gamma^{U_1,D}_{\beta_1\frown\langle i,\sigma_1\rangle}[s](w)$ (or $\Gamma^{V_1,D}_{\beta_1\frown\langle i,\sigma_1\rangle}[s](w)$ resp.), the strategy will attach the witness w to the leftmost edge of the form $\alpha \frown \langle h_1, \sigma\rangle$, subject to the witness satisfying a number of constraints, similar to the ones described in the previous section. The attachment of a witness to this edge also defines the work interval $(w, \gamma_{\beta_1\frown\langle i,\sigma_1\rangle}[s](w))$ for this edge.

On the other hand, if the witness w gives an honest computation $\Gamma^{U_1,D}_{\beta_1\frown\langle i,\sigma_1\rangle}[s](w)$ (or $\Gamma^{V_1,D}_{\beta_1\frown\langle i,\sigma_1\rangle}[s](w)$ resp.), and a dishonest computation $\Gamma^{U_2,D}_{\beta_2\frown\langle i,\sigma_2\rangle}[s](w)$ (or $\Gamma^{V_2,D}_{\beta_2\frown\langle i,\sigma_2\rangle}[s](w)$ resp.), the strategy will attach the witness w to the leftmost edge of the form $\alpha \frown \langle h_2, \sigma\rangle$, subject to the witness satisfying a number of constraints similar to the ones described in the previous section. The attachment of a witness to this edge also defines the work interval $(w, \gamma_{\beta_2\frown\langle i,\sigma_2\rangle}[s](w))$ for this edge.

We now consider two special cases of the $\mathcal{S}$ strategy which may appear lower down the priority tree in this section as backup $\mathcal{S}$ strategies, and which are not of a form covered in the previous section.

In the second case we consider the special case where one of the strategies β_1 or β_2 is following a $\hat{\Gamma}$-strategy. If β_1 is following a $\hat{\Gamma}$-strategy, the strategy α will have edges d, h_1, h_2, g_2 and d. The attachment procedure will therefore attach witnesses w giving honest computations $\Gamma^{V_1,D}_{\beta_1\frown\langle i,\sigma_1\rangle}[s](w)$ and $\Gamma^{U_2,D}_{\beta_2\frown\langle i,\sigma_2\rangle}[s](w)$ to edges with outcome g_2 satisfying the usual constraints. On the other hand if β_2 is following a $\hat{\Gamma}$-strategy, the strategy α will have edges d, h_1, h_2, g_1 and d. Hence the attachment procedure will attach witnesses w giving honest computations $\Gamma^{U_1,D}_{\beta_1\frown\langle i,\sigma_1\rangle}[s](w)$ and $\Gamma^{V_2,D}_{\beta_2\frown\langle i,\sigma_2\rangle}[s](w)$ to edges with outcome g_1 satisfying the usual constraints.

In the third case we consider the special case where both of the strategies β_1 and β_2 are following a $\hat{\Gamma}$-strategy. In this case there are no edges with outcomes g_1 or g_2 to which the strategy can attach witnesses w giving honest computations $\Gamma^{V_1,D}_{\beta_1\frown\langle i,\sigma_1\rangle}[s](w)$ and $\Gamma^{V_2,D}_{\beta_2\frown\langle i,\sigma_2\rangle}[s](w)$. Hence the attachment procedure will instead attach these witnesses to the leftmost edge with outcome d which has been visited previously and which presently has no witness attached. The witness will also have to satisfy an appropriate subset of the constraints described above for edges with g_1 or g_2 outcomes.

During its fourth step the strategy α calculates the edge $\alpha \frown \langle o, \sigma \rangle$ to visit during the present stage. It then takes action according to the outcome of this edge.

This is followed by its fifth step, where the strategy α will calculate the attachment restraints and the downward restraint as in the previous section.

During the final and sixth step, the strategy takes action depending on the outcome of the edge $\alpha \frown \langle o, \sigma \rangle$ visited by the strategy at stage s.

Suppose that the strategy visits an outcome of the form $\alpha \frown \langle w, \sigma \rangle$ and the edge is in open mode. If the present stage is not an α-open stage, we terminate the stage so as to wait for an α-open stage. Otherwise the strategy will count visiting the edge as having taken action successfully, changing the mode of the edge back to close mode and going to the next substage.

On the other hand, suppose that the outcome is w and the edge is in close mode. If the present stage is not an α-close stage, we terminate the stage so as to wait for an α-close stage. Otherwise the strategy will count visiting the edge as having taken action successfully, changing the mode of the edge back to open mode and going to the next substage.

Suppose that the strategy visits an outcome of the form $\alpha \frown \langle g_i, \sigma \rangle$ for some $1 \leq i \leq 2$. If the strategy has diagonalised as a result of enumerating some witness w' into A at some prior stage, the stage is terminated. Otherwise we have that the edge is either in open mode or in close mode.

If the edge is in open mode, the strategy will first determine whether a work interval for the edge is defined. If this is not the case, the strategy will choose a threshold v so as to define a work interval $(v, \gamma_{\beta_i \frown \langle i, \sigma_i \rangle}[s](v))$ for the edge. This threshold has to obey certain constraints as detailed in the previous section.

Once a work interval is defined for the edge, the strategy will determine whether a witness is attached to the edge. If this is not the case, the strategy will terminate the stage and wait for a witness to be attached. If a work interval is defined for the edge, a witness w is attached to the edge, the strategy will determine whether the witness still gives an honest computation, that is whether $\phi_{1,1}[s](w) \leq \gamma_{\beta_1 \frown \langle i,\sigma_1 \rangle}[s](w)$ and $\phi_{2,1}[s](w) \leq \gamma_{\beta_2 \frown \langle i,\sigma_2 \rangle}[s](w)$. If this is no longer the case, the witness is detached from the edge. If a work interval is defined for the edge, a witness has been attached to the edge and the witness gives an honest computation, the strategy determines whether the present stage is an α-open stage. If this is not the case, the strategy will terminate the stage and wait for an α-open stage. If the strategy visits the edge, a work interval has been defined, a witness w has been attached, the witness is honest, and the present stage is an α-open stage, the strategy can finally take action and open a gap by enumerating the witness w into the set A. Since the strategy has taken action successfully, it changes the mode of the edge to close mode and goes to the next substage.

If the edge is in close mode, the strategy will determine whether the present stage is an α-close stage. If this is not the case, the strategy will terminate the stage and wait for an α-close stage. If the strategy visits the edge and the present stage is an α-close stage, the strategy will perform capricious destruction for $\Gamma^{U_1,D}_{\beta_i \frown \langle i,\sigma_i \rangle}$ by enumerating the upper bound of the work interval of the edge $\gamma_{\beta_i \frown \langle i,\sigma_i \rangle}[s](v)$ into the set D. Since the strategy has taken action successfully, it changes the mode of the edge to open mode and goes to the next substage.

Suppose now that the strategy visits an outcome of the form $\alpha \frown \langle h_i, \sigma \rangle$ for some $1 \leq i \leq 2$. If the strategy has diagonalised as a result of enumerating some witness w' into A at some prior stage, the stage is terminated. Otherwise we have that the edge is either in Part I mode or in Part II mode.

If the edge is in Part I mode, the strategy will determine whether a witness is attached to the edge. If this is not the case, the strategy will terminate the stage and wait for a witness to be attached. If a witness w has been attached to the edge during the present stage, the work interval $(w, \gamma_{\beta \frown \langle i,\sigma \rangle}[s](w))$ is defined for the edge and the stage is terminated. If a work interval is defined for the edge and a witness has been attached to the edge, the strategy will determine whether the present stage is an α-close stage. If this is not the case, the strategy will terminate the stage and wait for an α-close stage. Otherwise, the strategy will determine whether the witness w attached to the edge still gives the appropriate dishonest computations.

Hence if the outcome is h_1, the strategy will determine whether $\Gamma^{U_1,D}_{\beta_1 \frown \langle i,\sigma_1 \rangle}[s](w)$ (or

$\Gamma^{V_1,D}_{\beta_1 \frown \langle i,\sigma_1\rangle}[s](w)$ resp.) is still dishonest at the present stage s. If this is not the case, the strategy will terminate the stage and wait for a stage until the computation becomes dishonest again.

On the other hand if the outcome is h_2, the strategy will determine whether the computation $\Gamma^{U_2,D}_{\beta_2 \frown \langle i,\sigma_2\rangle}[s](w)$ (or $\Gamma^{V_2,D}_{\beta_2 \frown \langle i,\sigma_2\rangle}[s](w)$ resp.) is still dishonest at the present stage s. If this is not the case, the strategy will terminate the stage and wait for a stage until the computation becomes dishonest again. It is important to note that in this case it is sufficient to ensure that the computation $\Gamma^{U_2,D}_{\beta_2 \frown \langle i,\sigma_2\rangle}[s](x)$ (or $\Gamma^{V_2,D}_{\beta_2 \frown \langle i,\sigma_2\rangle}[s](w)$ resp.) remains dishonest. The reason for this is that if the edge $\alpha \frown \langle h_2, \sigma\rangle$ is on the true path, the goal of the strategy will only be to honestify the functional $\Gamma^{U_2,D}_{\beta_2 \frown \langle i,\sigma_2\rangle}$ (or $\Gamma^{V_2,D}_{\beta_2 \frown \langle i,\sigma_2\rangle}$ resp.) so as to make $\Phi^{U_2,V_2}_{2,1}$ partial as a consequence.

If the strategy visits the edge, a work interval is defined, a witness has been attached to the edge, the present stage is an α-close stage, and the witness w gives a dishonest computation as described above, it will perform honestification for $\Gamma^{U_i,D}_{\beta_i \frown \langle i,\sigma_i\rangle}$ by enumerating $\gamma_{\beta_i \frown \langle i,\sigma_i\rangle}(w)$ into the set D. Since the strategy has taken action successfully, it changes the mode of the edge to Part II mode and goes to the next substage.

If the edge is in Part II mode, the strategy will determine whether the present stage is an α-open stage. If this is not the case, the strategy will terminate the stage and wait for an α-open stage. Otherwise, the strategy will take no action. This will count as the strategy having taken action successfully, changing the mode of the edge back to Part I mode and going to the next substage.

Finally suppose that the strategy visits an outcome of the form $\alpha \frown \langle d, \sigma\rangle$. We have to distinguish between two cases.

If at least one of the strategies β_1 and β_2 are following a Γ-strategy, we do nothing and terminate the stage. On the other hand, if both of the strategies β_1 and β_2 are following a $\hat{\Gamma}$-strategy, we have that α does not have edges of the form $\langle g_i, \sigma\rangle$ for some $1 \leq i \leq 2$, and that strategies can enumerate witnesses into A when visiting edges of the form $\langle d, \sigma\rangle$.

Thus suppose that the strategy visits an outcome of the form $\alpha \frown \langle d, \sigma\rangle$ and the edge has no witness attached. Then the strategy terminates the stage and waits for a witness to become attached. Once a witness becomes attached to the edge, the strategy will determine whether the witness still gives an honest computation, that is whether $\phi_{1,1}[s](w) \leq \gamma_{\beta_1 \frown \langle i,\sigma_1\rangle}[s](w)$ and

$\phi_{2,1}[s](w) \leq \gamma_{\beta_2 \frown \langle i,\sigma_2 \rangle}[s](w)$. If the computation has become dishonest since it was attached, the witness is detached from the edge. If a witness has been attached to the edge and the computation is honest, the strategy determines whether the present stage is an α-open stage. If this is not the case, the strategy will terminate the stage and wait for an α-open stage. If the strategy visits the edge, a witness w has been attached, the computation is honest, and the present stage is an α-open stage, the strategy can finally take action and open a gap by enumerating the witness w into the set A. The strategy terminates the stage.

We shall now formalise the $\mathcal{S}$ strategy.

The $\mathcal{S}$ Strategy

The strategy α has a set of witnesses W^e and a set of thresholds V^e, and at every stage s is able to impose a restraint $R_{\alpha,s}$ on lower priority strategies. Initially we have that $R_{\alpha,0} = 0$, and if the strategy sets $R_{\alpha,s} > 0$ during some stage s, the restraint will maintain this value unless the strategy has been initialised.

The strategy α lies below a number of $\mathcal{R}$ strategies β'. Each such strategy β' imposes a downward restraint $d(\beta' \frown \langle o', \sigma' \rangle, s)$ on α at stage s, where $\beta' \frown \langle o', \sigma' \rangle$ is the edge of β' on the path leading to α. Two of these $\mathcal{R}$ strategies, which we denote by β_i for $1 \leq i \leq 2$ will be active for α. Similarly the corresponding edges lying on the path leading to α will be denoted by $\beta_i \frown \langle i, \sigma_i \rangle$ for $1 \leq i \leq 2$. Each of these strategies may either be following a Γ-strategy or a $\hat{\Gamma}$-strategy.

The strategy α may also lie below a number of $\mathcal{S}$ strategies α'. Each such strategy α' imposes a downward restraint $d(\alpha' \frown \langle o', \sigma' \rangle, s)$ on α at stage s, where $\alpha' \frown \langle o', \sigma' \rangle$ is the edge of α' on the path leading to α. The strategy α' also imposes the diagonalisation restraint $R_{\alpha',s}$ on α at stage s. The strategy α' may also impose a work interval on α at stage s, depending on its outcome on the path leading to α. Let $\alpha'' \subset \beta$ be the greatest $\mathcal{S}$ strategy (under $\subset$) which imposes a work interval on α. We shall denote the work interval imposed by α'' on α at stage s by (a_s, b_s).

(1) Consider the last stage t at which α was accessible. If t does not exist, or the strategy α

has been initialised at some stage t' such that $t < t' < s$, go to step (2).

If t exists, has α enumerated some witness w into A at stage t?

(a) (No) Go to step (2).

(b) (Yes) Is it the case that for every $\mathcal{R}$ strategy $\beta_i \subset \alpha$ which is active for α, and is following a Γ-strategy we have that $U_t \upharpoonright \phi_{i,1}[t](w) \neq U_{t_i} \upharpoonright \phi_{i,1}[t](w)$, where t_i be the least $\mathcal{R}_i$-expansionary stage greater than t?

(i) (No) Go to step (2).

(ii) (Yes) Set the restraint $R_{\alpha,s}$ to $\theta_t(w)$. Consider every edge $\alpha \frown \langle o', \sigma' \rangle$ of α which has been previously accessible. If a work interval is defined for $\alpha \frown \langle o', \sigma' \rangle$, cancel the work interval. If some witness is attached to $\alpha \frown \langle o', \sigma' \rangle$, detach the witness. If o' is equal to d, g_1, g_2 or w, set the edge to open mode. If o' is equal to h_1 or h_2, set the edge to Part I mode. Go to step (2).

(2) Define the rightward restraint $r(\alpha \frown \langle o', \sigma' \rangle, s)$ for every edge $\alpha \frown \langle o', \sigma' \rangle$ which was previously accessible as the least element x such that:

(a) $x \geq \theta_t(w)$, where w is a witness attached to $\alpha \frown \langle o', \sigma' \rangle$ and t is the stage at which the witness was attached.

(b) $x \geq t$, where t is the last stage at which $\alpha \frown \langle o', \sigma' \rangle$ was last accessible.

Go to step (3).

(3) Consider the finite set of witnesses w in W^e such that $w < s$ and $\Theta^D[s](w) \downarrow= 0$ and such that w has not been attached to an edge at some stage $u < s$. Perform the following case analysis for every such witness in turn (under the order $<$), until one witness is attached successfully to an edge or until no more witnesses are available.

(a) Suppose that $R_{\alpha,s} > 0$. End stage s, and go to stage $s + 1$.

(b) Suppose that $R_{\alpha,s} = 0$ and that there is some least (under $\subset$) $\mathcal{R}$ strategy $\beta_i \subset \alpha$ which is active for α such that $\phi_{i,1}[s](w) > \gamma_{\beta_i \frown \langle i, \sigma_i \rangle}[s](w)$, where $1 \leq i \leq 2$. If there is an edge $\alpha \frown \langle h_i, \sigma' \rangle$ such that:

(i) $\alpha \frown \langle h_i, \sigma' \rangle$ has been accessible during a previous stage.

(ii) $\alpha \frown \langle h_i, \sigma' \rangle$ has no witness attached to it.

(iii) $w > \sup\{r(\alpha \frown \langle o'', \sigma'' \rangle, s) \mid \alpha \frown \langle o'', \sigma'' \rangle <_L \alpha \frown \langle h_i, \sigma' \rangle \ \wedge$ $\alpha \frown \langle o'', \sigma'' \rangle$ has been previously accessible$\}$.

(iv) $w > R_{\alpha',s}$ for every $\mathcal{S}$ strategy $\alpha' \subset \alpha$.

(v) $w > d(\beta' \frown \langle o', \sigma' \rangle, s)$, for every $\mathcal{R}$ strategy $\beta' \subset \alpha$ with edge $\beta' \frown \langle o', \sigma' \rangle$ on the path leading to α.

(vi) $w > d(\alpha' \frown \langle o', \sigma' \rangle, s)$, for every $\mathcal{S}$ strategy $\alpha' \subset \alpha$ with edge $\alpha' \frown \langle o', \sigma' \rangle$ on the path leading to α.

(vii) $a_s < w < b_s$.

(viii) $a_s < \theta_s(w) < b_s$.

(ix) w is greater than the upper bound of the work interval at stage s defined for any edge $\alpha \frown \langle o'', \sigma'' \rangle$ which was previously accessible and which lies to the left of $\alpha \frown \langle h_i, \sigma' \rangle$.

(x) $w > t$, where t is the last stage at which the edge $\alpha \frown \langle h_i, \sigma' \rangle$ was initialised.

(xi) $w > w'$, where w' is any witness which has been attached to this edge at some stage $t < s$.

Attach w to the leftmost such $\alpha \frown \langle h_i, \sigma' \rangle$. Define the work interval of the edge $\alpha \frown \langle h_i, \sigma' \rangle$ to be $(w, \gamma_{\beta_i \frown \langle i, \sigma_i \rangle}[s](w))$.

(c) Suppose that $R_{\alpha,s} = 0$ and that there is some $\mathcal{R}$ strategy which is active for α and is following a Γ-strategy, and that $\phi_{i,1}[s](w) \leq \gamma_{\beta_i \frown \langle i, \sigma_i \rangle}[s](w)$ for every $\mathcal{R}$ strategy $\beta_i \subset \alpha$ which is active for α, where $1 \leq i \leq 2$. If there is an edge $\alpha \frown \langle g_j, \sigma' \rangle$ with $1 \leq j \leq 2$ such that:

(i) $\alpha \frown \langle g_j, \sigma' \rangle$ has been accessible during a previous stage.

(ii) $\alpha \frown \langle g_j, \sigma' \rangle$ has no witness attached to it.

(iii) $w > \sup\{r(\alpha \frown \langle o'', \sigma'' \rangle, s) \mid \alpha \frown \langle o'', \sigma'' \rangle <_L \alpha \frown \langle g_j, \sigma' \rangle \ \wedge$ $\alpha \frown \langle o'', \sigma'' \rangle$ has been previously accessible$\}$.

(iv) $w > R_{\alpha',s}$ for every $\mathcal{S}$ strategy $\alpha' \subset \alpha$.

(v) $w > d(\beta' \frown \langle o', \sigma' \rangle, s)$, for every $\mathcal{R}$ strategy $\beta' \subset \alpha$ with edge $\beta' \frown \langle o', \sigma' \rangle$

on the path leading to α.

(vi) $w > d(\alpha' \frown \langle o', \sigma' \rangle, s)$, for every $\mathcal{S}$ strategy $\alpha' \subset \alpha$ with edge $\alpha' \frown \langle o', \sigma' \rangle$ on the path leading to α.

(vii) $a_s < w < b_s$.

(viii) $a_s < \theta_s(w) < b_s$.

(ix) $\alpha \frown \langle g_j, \sigma' \rangle$ is in open mode.

(x) The work interval for the edge $\alpha \frown \langle g_j, \sigma' \rangle$ is defined.

(xi) w is greater than the upper bound of the work interval for the edge $\alpha \frown \langle g_j, \sigma' \rangle$.

(xii) $w > t$, where t is the last stage at which the edge $\alpha \frown \langle g_j, \sigma' \rangle$ was initialised.

(xiii) $w > w'$, where w' is any witness which has been attached to this edge at some stage $t < s$.

Then attach w to the leftmost such $\alpha \frown \langle g_j, \sigma' \rangle$.

(d) Suppose that $R_{\alpha,s} = 0$ and that there is some $\mathcal{R}$ strategy which is active for α, and that every such strategy is following a $\hat{\Gamma}$-strategy, and that $\phi_{i,1}[s](w) \leq \gamma_{\beta_i \frown \langle i, \sigma_i \rangle}[s](w)$ for every $\mathcal{R}$ strategy $\beta_i \subset \alpha$ which is active for α, where $1 \leq i \leq 2$. If there is an edge $\alpha \frown \langle d, \sigma' \rangle$ such that:

(i) $\alpha \frown \langle d, \sigma' \rangle$ has been accessible during a previous stage.

(ii) $\alpha \frown \langle d, \sigma' \rangle$ has no witness attached to it.

(iii) $w > \sup\{r(\alpha \frown \langle o'', \sigma'' \rangle, s) \mid \alpha \frown \langle o'', \sigma'' \rangle <_L \alpha \frown \langle d, \sigma' \rangle \ \wedge$
$\alpha \frown \langle o'', \sigma'' \rangle$ has been previously accessible$\}$.

(iv) $w > R_{\alpha',s}$ for every $\mathcal{S}$ strategy $\alpha' \subset \alpha$.

(v) $w > d(\beta' \frown \langle o', \sigma' \rangle, s)$, for every $\mathcal{R}$ strategy $\beta' \subset \alpha$ with edge $\beta' \frown \langle o', \sigma' \rangle$ on the path leading to α.

(vi) $w > d(\alpha' \frown \langle o', \sigma' \rangle, s)$, for every $\mathcal{S}$ strategy $\alpha' \subset \alpha$ with edge $\alpha' \frown \langle o', \sigma' \rangle$ on the path leading to α.

(vii) $a_s < w < b_s$.

(viii) $a_s < \theta_s(w) < b_s$.

(ix) w is greater than the upper bound of the work interval at stage s defined for any edge $\alpha \frown \langle o'', \sigma'' \rangle$ which was previously accessible and which lies to the left of $\alpha \frown \langle d, \sigma' \rangle$.

(x) $w > t$, where t is the last stage at which the edge $\alpha \frown \langle d, \sigma' \rangle$ was initialised.

(xi) $w > w'$, where w' is any witness which has been attached to this edge at some stage $t < s$.

Then attach w to the leftmost such $\alpha \frown \langle d, \sigma' \rangle$.

If a witness w has been attached to some edge $\alpha \frown \langle o', \sigma' \rangle$, consider every edge $\alpha \frown \langle o'', \sigma'' \rangle$ lying to the right of $\alpha \frown \langle o', \sigma' \rangle$. If some witness w' is attached to $\alpha \frown \langle o'', \sigma'' \rangle$, detach the witness from the edge. If some work interval is defined for $\alpha \frown \langle o'', \sigma'' \rangle$, undefine the work interval. If o'' is equal to d, g_1, g_2 or w, set the edge to open mode. If o'' is equal to h_1 or h_2, set the edge to Part I mode.

Go to step (4).

(4) Determine the edge $\alpha \frown \langle o, \sigma \rangle$ which the strategy should visit at stage s.

Go to step (5).

(5) Define the attachment procedure restraint $a(\alpha \frown \langle o', \sigma' \rangle, s)$ for every edge $\alpha \frown \langle o', \sigma' \rangle$ which was previously accessible. If the strategy has not attached a witness w to some edge $\alpha \frown \langle o'', \sigma'' \rangle <_L \alpha \frown \langle o', \sigma' \rangle$ at stage s, define $a(\alpha \frown \langle o', \sigma' \rangle, s) = 0$. Otherwise define $a(\alpha \frown \langle o', \sigma' \rangle, s) = \theta_s(w)$.

Also define the downward restraint $d(\alpha \frown \langle o, \sigma \rangle, s)$ as the least element x such that:

(a) $x \geq \sup\{r(\alpha \frown \langle o', \sigma' \rangle, s) \mid \alpha \frown \langle o', \sigma' \rangle <_L \alpha \frown \langle o, \sigma \rangle \ \wedge$
$\alpha \frown \langle o', \sigma' \rangle$ has been previously accessible$\}$.

(b) $x \geq a(\alpha \frown \langle o, \sigma \rangle, s)$.

(c) $x \geq d(\alpha \frown \langle o, \sigma \rangle, t)$ for all $t < s$.

Go to step (6).

(6) Consider the edge $\alpha \frown \langle o, \sigma \rangle$ being visited by the strategy at stage s. Take action according to the value of o through the following case analysis.

(a) $o = w$.

(i) Suppose that $\alpha \frown \langle w, \sigma\rangle$ is in open mode, and s is not an α-open stage. End the stage s, and go to stage $s+1$.

(ii) Suppose that $\alpha \frown \langle w, \sigma\rangle$ is in open mode, and s is an α-open stage. Set the edge to close mode. Go to the next substage.

(iii) Suppose that $\alpha \frown \langle w, \sigma\rangle$ is in close mode, and s is not an α-close stage. End the stage s, and go to stage $s+1$.

(iv) Suppose that $\alpha \frown \langle w, \sigma\rangle$ is in close mode, and s is an α-close stage. Set the edge to open mode. Go to the next substage.

(b) $o = g_j$, for $1 \leq j \leq 2$.

(i) Suppose that $R_{\alpha,s} > 0$. End stage s, and go to stage $s+1$.

(ii) Suppose that $R_{\alpha,s} = 0$, and the work interval for the edge $\alpha \frown \langle g_j, \sigma\rangle$ is undefined. If there is some least threshold $v < s$ in V^e such that:

(A) $v > \sup\{r(\alpha \frown \langle o', \sigma'\rangle, s) \mid \alpha \frown \langle o', \sigma'\rangle <_L \alpha \frown \langle g_j, \sigma\rangle \;\wedge$ $\alpha \frown \langle o', \sigma'\rangle$ has been previously accessible$\}$.

(B) $v > a(\alpha \frown \langle g_j, \sigma\rangle, s)$.

(C) $v > R_{\alpha',s}$ for every $\mathcal{S}$ strategy $\alpha' \subset \alpha$.

(D) $v > d(\beta' \frown \langle o', \sigma'\rangle, s)$, for every $\mathcal{R}$ strategy $\beta' \subset \alpha$ with edge $\beta' \frown \langle o', \sigma'\rangle$ on the path leading to α.

(E) $v > d(\alpha' \frown \langle o', \sigma'\rangle, s)$, for every $\mathcal{S}$ strategy $\alpha' \subset \alpha$ with edge $\alpha' \frown \langle o', \sigma'\rangle$ on the path leading to α.

(F) $a_s < v < b_s$.

(G) v is greater than the upper bound of a work interval defined for an edge $\alpha \frown \langle o', \sigma'\rangle$ lying to the left of $\alpha \frown \langle o, \sigma\rangle$.

(H) $v > t$, where t is the stage at which the edge $\alpha \frown \langle g_j, \sigma\rangle$ was last initialised.

Define the work interval of the edge $\alpha \frown \langle g_j, \sigma\rangle$ to be $(v, \gamma_{\beta_j \frown \langle i, \sigma_j\rangle}[s](v))$. End stage s, and go to stage $s+1$.

(iii) Suppose that $R_{\alpha,s} = 0$, a work interval is defined for the edge $\langle g_j, \sigma\rangle$ and the edge is in open mode, but no witness w is attached to the edge. End stage s, and go to stage $s+1$.

(iv) Suppose that $R_{\alpha,s} = 0$, a work interval is defined for the edge $\langle g_j, \sigma\rangle$, the edge is in open mode and a witness w is attached to the edge, but $\phi_{i,1}[s](w) > \gamma_{\beta_i \frown \langle i,\sigma_i\rangle}[s](w)$ for some $\beta_i \subset \alpha$ which is active for α, where $1 \leq i \leq 2$. Detach w from $\alpha \frown \langle g_j, \sigma\rangle$. End stage s, and go to stage $s+1$.

(v) Suppose that $R_{\alpha,s} = 0$, a work interval is defined for the edge $\langle g_j, \sigma\rangle$, the edge is in open mode, a witness w is attached to the edge and $\phi_{i,1}[s](w) \leq \gamma_{\beta_i \frown \langle i,\sigma_i\rangle}[s](w)$ for every $\beta_i \subset \alpha$ which is active for α, where $1 \leq i \leq 2$, but s is not an α-open stage. End stage s, and go to stage $s+1$.

(vi) Suppose that $R_{\alpha,s} = 0$, a work interval is defined for the edge $\langle g_j, \sigma\rangle$, the edge is in open mode, a witness w is attached to the edge, $\phi_{i,1}[s](w) \leq \gamma_{\beta_i \frown \langle i,\sigma_i\rangle}[s](w)$ for every $\beta_i \subset \alpha$ which is active for α, where $1 \leq i \leq 2$ and s is an α-open stage. Enumerate w into A. Set the edge $\langle g_j, \sigma\rangle$ to close mode. Go to the next substage.

(vii) Suppose that $R_{\alpha,s} = 0$, a work interval is defined for the edge $\langle g_j, \sigma\rangle$, the edge is in close mode and s is not an α-close stage. End stage s, and go to stage $s+1$.

(viii) Suppose that $R_{\alpha,s} = 0$, a work interval is defined for the edge $\langle g_j, \sigma\rangle$, the edge is in close mode and s is an α-close stage. Enumerate $\gamma_{\beta_j \frown \langle i,\sigma_j\rangle}[s](v)$ into D. Set the edge $\langle g_j, \sigma\rangle$ to open mode. Go to the next substage.

(c) $o = h_j$, for $1 \leq j \leq 2$.

(i) Suppose that $R_{\alpha,s} > 0$. End stage s, and go to stage $s+1$.

(ii) Suppose that $R_{\alpha,s} = 0$, but the edge $\langle h_j, \sigma\rangle$ has no witness w attached to it. End stage s, and go to stage $s+1$.

(iii) Suppose that $R_{\alpha,s} = 0$, and the strategy has attached a witness w to the edge $\langle h_j, \sigma\rangle$ during this stage s. End stage s, and go to stage $s+1$.

(iv) Suppose that $R_{\alpha,s} = 0$, the work interval for the edge $\langle h_j, \sigma\rangle$ is defined and the edge $\langle h_j, \sigma\rangle$ is in Part I mode, but s is not an α-close stage. End stage s, and go to stage $s+1$.

(v) Suppose that $R_{\alpha,s} = 0$, the work interval for the edge $\langle h_j, \sigma\rangle$ is defined, the edge $\langle h_j, \sigma\rangle$ is in Part I mode, and s is an α-close stage, but $\phi_{j,1}[s](w) \leq \gamma_{\beta_j \frown \langle i,\sigma_j\rangle}[s](w)$. End stage s, and go to stage $s+1$.

(vi) Suppose that $R_{\alpha,s} = 0$, and the work interval for the edge $\langle h_j, \sigma\rangle$ is defined, the edge $\langle h_j, \sigma\rangle$ is in Part I mode, s is an α-close stage, and $\phi_{j,1}[s](w) > \gamma_{\beta_j \frown \langle i,\sigma_j\rangle}[s](w)$. Enumerate $\gamma_{\beta_j \frown \langle i,\sigma_j\rangle}[s](w)$ into D. Set the edge $\langle h_j, \sigma\rangle$ to Part II mode. Go to the next substage.

(vii) Suppose that $R_{\alpha,s} = 0$, the edge $\langle h_j, \sigma\rangle$ has a witness w attached to it, the edge is in Part II mode and s is not an α-open stage. End stage s, and go to stage $s+1$.

(viii) Suppose that $R_{\alpha,s} = 0$, the edge $\langle h_j, \sigma\rangle$ has a witness w attached to it, the edge is in Part II mode and s is an α-open stage. Set the edge $\langle h_j, \sigma\rangle$ to Part I mode. Go to the next substage.

(d) $o = d$.

(i) Suppose that $R_{\alpha,s} > 0$. End stage s, and go to stage $s+1$.

(ii) Suppose that $R_{\alpha,s} = 0$, that there is some $\mathcal{R}$ strategy $\beta_i \subset \alpha$ active for α where $1 \leq i \leq 2$, that every such strategy is following a $\hat{\Gamma}$-strategy and that no witness w is attached to this edge. End stage s, and go to stage $s+1$.

(iii) Suppose that $R_{\alpha,s} = 0$, that there is some $\mathcal{R}$ strategy $\beta_i \subset \alpha$ active for α where $1 \leq i \leq 2$, that every such strategy is following a $\hat{\Gamma}$-strategy and that a witness w is attached to this edge, but $\phi_{i,1}[s](w) > \gamma_{\beta_i \frown \langle i,\sigma_i\rangle}[s](w)$ for some $\beta_i \subset \alpha$ active for α. Detach the witness w from the edge. End stage s, and go to stage $s+1$.

(iv) Suppose that $R_{\alpha,s} = 0$, that there is some $\mathcal{R}$ strategy $\beta_i \subset \alpha$ active for α where $1 \leq i \leq 2$, that every such strategy is following a $\hat{\Gamma}$-strategy, that a witness w is attached to this edge and that $\phi_{i,1}[s](w) \leq \gamma_{\beta_i \frown \langle i,\sigma_i\rangle}[s](w)$ for every $\beta_i \subset \alpha$ active for α, but s is not an α-open stage. End stage s, and go to stage $s+1$.

(v) Suppose that $R_{\alpha,s} = 0$, that there is some $\mathcal{R}$ strategy $\beta_i \subset \alpha$ active for α where $1 \leq i \leq 2$, that every such strategy is following a $\hat{\Gamma}$-strategy, that a witness w is attached to this edge, that $\phi_{i,1}[s](w) \leq \gamma_{\beta_i \frown \langle i,\sigma_i \rangle}[s](w)$ for every $\beta_i \subset \alpha$ active for α and that s is an α-open stage. Enumerate w into A. End stage s, and go to stage $s+1$.

(vi) Suppose that $R_{\alpha,s} = 0$ and that there is some $\mathcal{R}$ strategy $\beta_i \subset \alpha$ active for α which is following a Γ-strategy, where $1 \leq i \leq 2$. End stage s, and go to stage $s+1$.

3.6.6 Analysis of Outcomes

We shall now consider the effect of the $\mathcal{S}$ strategy α on the satisfaction of the requirements $\mathcal{S}$ below $\mathcal{R}_2$ below $\mathcal{R}_1$. When certain edges of the strategy α are on the true path, it may be the case that certain requirements are left unsatisfied. We shall show that the relationship between the outcome on the true path and the requirements which are left unsatisfied is the same as the one for the Lachlan Non-Splitting Theorem. The immediate consequence of this will be that the priority tree for satisfying the requirements $\mathcal{S}$ below $\mathcal{R}_2$ below $\mathcal{R}_1$ can be structured in an analogous way to the priority tree of the Lachlan Non-Splitting Theorem as described in Chapter 2. This will allow us to perform a similar analysis in order to show that all requirements are actually satisfied at a later stage.

In order to analyse the effect of the $\mathcal{S}$ strategy α on the satisfaction of the requirements $\mathcal{R}$ and $\mathcal{S}$, we consider the leftmost edge $\alpha \frown \langle o, \sigma \rangle$ which is visited infinitely often by the strategy α. The following case analysis can then be made depending on the outcome $\langle o, \sigma \rangle$.

$\boxed{w}$ Suppose that the outcome is $\langle w, \sigma \rangle$.

The analysis for showing that the $\mathcal{S}$ requirement is satisfied is identical to the one found in Section 3.5.8 for the case where the outcome $\langle w, \sigma \rangle$ of the strategy is on the true path.

Hence if $\langle w, \sigma \rangle$ is the outcome of the edge lying on the true path we have that the $\mathcal{S}$ requirement is satisfied, whilst the $\mathcal{R}$ strategies β_1 and β_2 above α can build their functional without interference after some stage, satisfying the $\mathcal{R}$ requirements $\mathcal{R}_1$ and $\mathcal{R}_2$ respectively as well. It is important to note that one way for this outcome to be on the true path is for the strategy to diagonalise successfully.

$\boxed{h_1}$ Suppose that the outcome is $\langle h_1, \sigma\rangle$.

In this case, we have that the answers to questions Q_1 and $Q_{2.1}$ guarantee that there are infinitely many witnesses $w \in W^e$ and stages $s \in \mathbb{N}_\alpha$ such that $\Theta^D[s](w) \downarrow= 0$, $a_s < w < b_s$ and $a_s < \theta_s < b_s$. However only finitely many of these witnesses w will give honest computations $\Gamma^{U_1,D}_{\beta_1 \frown \langle i,\sigma_1\rangle}[s](w)$. It follows that there is some stage t' such that for every $s > t'$, we have that the computations $\Gamma^{U_1,D}_{\beta_1 \frown \langle i,\sigma_1\rangle}[s](w)$ are dishonest.

Now since $\alpha \frown \langle h_1, \sigma\rangle$ is the leftmost edge which is accessible infinitely often, we have that there is a stage s_0 after which no edge to its left is accessible. Hence only finitely many edges to the left of $\alpha \frown \langle h_1, \sigma\rangle$ can have been accessible at stages $s < s_0$. Suppose that the edge $\alpha \frown \langle h_1, \sigma\rangle$ does not have a witness attached at some stage $s_1 > s_0$. Since witnesses giving a dishonest computation $\Gamma^{U_1,D}_{\beta_1 \frown \langle i,\sigma_1\rangle}[s](w)$ (or $\Gamma^{V_1,D}_{\beta_1 \frown \langle i,\sigma_1\rangle}[s](w)$ resp.) are attached to the leftmost edge of the form $\alpha \frown \langle h_1, \sigma\rangle$ which has no other witness attached, and since there are infinitely many such witnesses, it follows that a witness satisfying these conditions is eventually attached to $\alpha \frown \langle h_1, \sigma\rangle$ at some stage $s_2 > s_1$.

In addition we claim that it is possible to ensure that the edge $\alpha \frown \langle h_1, \sigma\rangle$ is visited during infinitely many α-open stages and infinitely many α-close stages. We address this claim when we discuss *fairness* in Section 3.7.1.

Hence, if the edge $\alpha \frown \langle h_1, \sigma\rangle$ does not have a witness attached at some stage, it must be the case that a witness w giving a dishonest computation $\Gamma^{U_1,D}_{\beta_1 \frown \langle i,\sigma_1\rangle}[s](w)$ will eventually be attached to the edge at some stage s. This defines a work interval $(w, \gamma_{\beta_1 \frown \langle i,\sigma_1\rangle}[s](w))$ for the edge. Now if the edge is in Part I mode, we have that the strategy eventually visits the edge during an α-close stage, honestifying and changing the mode to Part II mode. If the edge is in Part II mode, we have that the strategy eventually visits the edge during an α-open stage, changing the mode of its edge back to Part I mode.

Now, it could be the case that a witness w which was attached to $\alpha \frown \langle h_1, \sigma\rangle$ at stage s gave a dishonest computation $\Gamma^{U_1,D}_{\beta_1 \frown \langle i,\sigma_1\rangle}[s](w)$ (or $\Gamma^{V_1,D}_{\beta_1 \frown \langle i,\sigma_1\rangle}[s](w)$ resp.), but that at some stage

$s' > s$ the computation $\Gamma^{U_1,D}_{\beta_1 \frown \langle i,\sigma_1\rangle}[s'](w)$ (or $\Gamma^{V_1,D}_{\beta_1 \frown \langle i,\sigma_1\rangle}[s'](w)$ resp.) becomes honest again. In this case the strategy would be blocked from honestifying. However, we have already seen that there is some greatest stage t' such that $\Gamma^{U_1,D}_{\beta_1 \frown \langle i,\sigma_1\rangle}[t](w)$ (or $\Gamma^{V_1,D}_{\beta_1 \frown \langle i,\sigma_1\rangle}[t](w)$ resp.) is dishonest at every stage $t > t'$. This means that the strategy will be able to honestify after stage t'.

Therefore the strategy will honestify during infinitely many stages s', by enumerating $\gamma_{\beta_1 \frown \langle i,\sigma_1\rangle}[s'](w)$ into D. As a result of this infinite honestification we have that both $\lim_{q\to\infty} \gamma_{\beta_1 \frown \langle i,\sigma_1\rangle}[q](w)$ and $\lim_{q\to\infty} \phi_{1,1}[q](w)$ are unbounded. Hence we have that $\Gamma^{U_1,D}_{\beta_1 \frown \langle i,\sigma_1\rangle}(w)\uparrow$ (or $\Gamma^{V_1,D}_{\beta_1 \frown \langle i,\sigma_1\rangle}(w)\uparrow$ resp.) and $\Phi^{U_1,D}_{1,1}(w)\uparrow$. On the other hand, the R-Synchronisation of β_2 with β_1, results in $\lim_{q\to\infty} \gamma_{\beta_2 \frown \langle i,\sigma_2\rangle}[q](w)$ also becoming unbounded, without any guarantee that this will be the case for $\lim_{q\to\infty} \phi_{2,1}[q](w)$.

Hence if h_1 is the outcome of the edge lying on the true path we have that the $\mathcal{R}_1$ requirement is satisfied trivially. On the other hand we have that the $\mathcal{R}_2$ requirement may remain unsatisfied. Finally the $\mathcal{S}$ requirement remains unsatisfied as well because the strategy does not diagonalise. For if this were the case, is would contradict the fact that the length of agrement between Θ^D and A increases infinitely often.

$\boxed{h_2}$ Suppose that the outcome is $\langle h_2, \sigma\rangle$.

In this case, we have that the answers to questions Q_1 and $Q_{2.1}$ and $Q_{2.2}$ guarantee that there are infinitely many witnesses $w \in W^e$ and stages $s \in \mathbb{N}_\alpha$ such that $\Theta^D[s](w)\downarrow= 0$, the computation $\Gamma^{U_1,D}_{\beta_1 \frown \langle i,\sigma_1\rangle}[s](w)$ (or $\Gamma^{V_1,D}_{\beta_1 \frown \langle i,\sigma_1\rangle}[s](w)$ resp.) is honest, $a_s < w < b_s$ and $a_s < \theta_s < b_s$. However only finitely many of these witnesses w and stages s will give computations $\Gamma^{U_2,D}_{\beta_2 \frown \langle i,\sigma_2\rangle}[s](w)$ (or $\Gamma^{V_2,D}_{\beta_2 \frown \langle i,\sigma_2\rangle}[s](w)$ resp.) which are honest. It follows that there is some stage t' such that for every $s > t'$, we have that the computations $\Gamma^{U_2,D}_{\beta_2 \frown \langle i,\sigma_2\rangle}[s](w)$ (or $\Gamma^{V_2,D}_{\beta_2 \frown \langle i,\sigma_2\rangle}$ resp.) are dishonest.

Now since $\alpha \frown \langle h_2, \sigma\rangle$ is the leftmost edge which is accessible infinitely often, we have that there is a stage s_0 after which no edge to its left is accessible. Hence only finitely many edges to the left of $\alpha \frown \langle h_2, \sigma\rangle$ can have been accessible at stages $s < s_0$. Suppose that the edge $\alpha \frown \langle h_2, \sigma\rangle$ does not have a witness attached at some stage $s_1 > s_0$. Since witnesses giving an honest computation $\Gamma^{U_1,D}_{\beta_1 \frown \langle i,\sigma_1\rangle}[s](w)$ and a dishonest computation $\Gamma^{U_2,D}_{\beta_2 \frown \langle i,\sigma_2\rangle}[s](w)$

(or $\Gamma^{V_2,D}_{\beta_2 \frown \langle i,\sigma_2\rangle}[s](w)$ resp.) are attached to the leftmost edge of the form $\alpha \frown \langle h_2, \sigma\rangle$ which has no other witness attached, and since there are infinitely many such witnesses, it follows that a witness satisfying these conditions is eventually attached to $\alpha \frown \langle h_2, \sigma\rangle$ at some stage $s_2 > s_1$.

In addition we claim that it is possible to ensure that the edge $\alpha \frown \langle h_2, \sigma\rangle$ is visited during infinitely many α-open stages and infinitely many α-close stages. We address this claim when we discuss *fairness* in Section 3.7.1.

Hence, if the edge $\alpha \frown \langle h_2, \sigma\rangle$ does not have a witness attached at some stage, it must be the case that a witness giving an honest computation $\Gamma^{U_1,D}_{\beta_1 \frown \langle i,\sigma_1\rangle}[s](w)$ and a dishonest computation $\Gamma^{U_2,D}_{\beta_2 \frown \langle i,\sigma_2\rangle}[s](w)$ (or $\Gamma^{V_2,D}_{\beta_2 \frown \langle i,\sigma_2\rangle}[s](w)$ resp.) will eventually be attached to the edge at some stage s. This defines a work interval $(w, \gamma_{\beta_2 \frown \langle i,\sigma_2\rangle}[s](w))$ for the edge. If the edge is in Part I mode, we have the strategy visits the edge during an α-close stage, honestifying and changing the mode to Part II mode. If the edge is in Part II mode, we have that the strategy visits the edge during an α-close stage, changing the mode of its edge back to Part I mode.

Now, it could be the case that a witness w which was attached to $\alpha \frown \langle h_2, \sigma\rangle$ at stage s gave a dishonest computation $\Gamma^{U_2,D}_{\beta_2 \frown \langle i,\sigma_2\rangle}[s](w)$ (or $\Gamma^{V_2,D}_{\beta_2 \frown \langle i,\sigma_2\rangle}[s](w)$ resp.), but that at some stage $s' > s$ the computation $\Gamma^{U_2,D}_{\beta_2 \frown \langle i,\sigma_2\rangle}[s'](w)$ (or $\Gamma^{V_2,D}_{\beta_2 \frown \langle i,\sigma_2\rangle}[s'](w)$ resp.) becomes honest again. In this case the strategy would be blocked from honestifying. However, we have already seen that there is some greatest stage t' such that $\Gamma^{U_2,D}_{\beta_2 \frown \langle i,\sigma_2\rangle}[t](w)$ (or $\Gamma^{V_2,D}_{\beta_2 \frown \langle i,\sigma_2\rangle}[t](w)$ resp.) is dishonest at every stage $t > t'$. Therefore the strategy will not be blocked from honestifying after stage t'.

Therefore the strategy will honestify during infinitely many stages s', by enumerating $\gamma_{\beta_2 \frown \langle i,\sigma_2\rangle}[s'](w)$ into D. As a result of this infinite honestification we have that both $\lim_{q\to\infty} \gamma_{\beta_2 \frown \langle i,\sigma_2\rangle}[q](w)$ and $\lim_{q\to\infty} \phi_{2,1}[q](w)$ are unbounded. Hence we have that $\Gamma^{U_2,D}_{\beta_2 \frown \langle i,\sigma_2\rangle}(w) \uparrow$ (or $\Gamma^{V_2,D}_{\beta_2 \frown \langle i,\sigma_2\rangle}(w) \uparrow$ resp.) and $\Phi^{U_2,D}_{2,1}(w) \uparrow$.

Hence if h_2 is the outcome of the edge lying on the true path we have that the $\mathcal{R}_2$ requirement is satisfied trivially. On the other hand we have that the strategy β_1 can continue to build the functional $\Gamma^{U_1,D}_{\beta_1 \frown \langle o,\sigma_1\rangle}(w)$ so as to ensure its agreement with the set A, and thus in the absence of further interference, the $\mathcal{R}_1$ requirement is satisfied. Finally the $\mathcal{S}$ requirement remains unsatisfied as well because the strategy does not diagonalise. For if this were the case, is would contradict the fact that the length of agrement between Θ^D and A increases infinitely often.

$\boxed{g_1}$ Suppose that the outcome is $\langle g_1, \sigma \rangle$.

In this case, we have that the answers to questions Q_1, $Q_{2.1}$, $Q_{2.2}$ and $Q_{3.1}$ guarantee that there are infinitely many witnesses $w \in W^e$ and stages $s \in \mathbb{N}_\alpha$ such that $\Theta^D[s](w) \downarrow = 0$, the computations $\Gamma^{U_1,D}_{\beta_1 \frown \langle i,\sigma_1 \rangle}[s](w)$ and $\Gamma^{U_2,D}_{\beta_2 \frown \langle i,\sigma_2 \rangle}[s](w)$ (or $\Gamma^{V_2,D}_{\beta_2 \frown \langle i,\sigma_2 \rangle}[s](w) resp.$) are honest, $a_s < w < b_s$ and $a_s < \theta_s < b_s$. In addition we have that infinitely many of these witnesses w enter the set A at stage s, but only finitely many of these witnesses cause a $U_1 \upharpoonright \phi_{1,1}[s](w)$ change between stage s and the least $\mathcal{R}_1$-expansionary stage $t_1 > s$. Finally we also have that the length of agreement between Θ^D and A expands infinitely often.

Now since $\alpha \frown \langle g_1, \sigma \rangle$ is the leftmost edge which is accessible infinitely often, we have that there is a stage s_0 after which no edge to its left is accessible. Hence only finitely many edges to the left of $\alpha \frown \langle g_1, \sigma \rangle$ can have been accessible at stages $s < s_0$. Suppose that the edge $\alpha \frown \langle g_1, \sigma \rangle$ does not have a witness attached at some stage $s_1 > s_0$. Since witnesses giving honest computations $\Gamma^{U_1,D}_{\beta_1 \frown \langle i,\sigma_1 \rangle}[s](w)$ and $\Gamma^{U_2,D}_{\beta_2 \frown \langle i,\sigma_2 \rangle}[s](w)$ (or $\Gamma^{V_2,D}_{\beta_2 \frown \langle i,\sigma_2 \rangle}[s](w)$ resp.) are attached to the leftmost edge of the form $\alpha \frown \langle g_i, \sigma \rangle$ for $1 \leq i \leq 2$ which has no other witness attached, and since there are infinitely many such witnesses, it follows that a witness satisfying these conditions is eventually attached to $\alpha \frown \langle g_1, \sigma \rangle$ at some stage $s_2 > s_1$.

In addition we claim that it is possible to ensure that the edge $\alpha \frown \langle g_1, \sigma \rangle$ is visited during infinitely many α-open stages and infinitely many α-close stages. We address this claim when we discuss *fairness* in Section 3.7.1.

Finally we claim that if a witness w is attached to the edge $\alpha \frown \langle g_1, \sigma \rangle$ at some stage s and the witness gives honest computations $\Gamma^{U_1,D}_{\beta_1 \frown \langle i,\sigma_1 \rangle}[s](w)$ and $\Gamma^{U_2,D}_{\beta_2 \frown \langle i,\sigma_2 \rangle}[s](w)$ (or $\Gamma^{V_2,D}_{\beta_2 \frown \langle i,\sigma_2 \rangle}[s](w)$ resp.), it is possible to stop elements from entering A or D up to $\phi_{1,2}[s](\phi_{1,1}[s](w))$ and $\phi_{1,3}[s](\phi_{1,1}[s](w))$, and up to $\phi_{2,2}[s](\phi_{2,1}[s](w))$ and $\phi_{2,3}[s](\phi_{2,1}[s](w))$ at some stage $s' \geq s$. In this way the honesty of the witness can be preserved until the strategy determines that it should enumerate it into the set A. We address this claim when we discuss *honesty preservation* in Section 3.7.2.

Now if the edge $\alpha \frown \langle g_1, \sigma \rangle$ does not have a work interval defined at some stage s, the strategy will choose a threshold v and define a work interval $(v, \gamma_{\beta_1 \frown \langle i,\sigma_1 \rangle}[s](v))$ for the edge. Once the strategy has taken this step, an honest witness w will eventually be attached to $\alpha \frown \langle g_1, \sigma \rangle$. Then if the edge is in open mode, we have that the strategy eventually visits the edge during an α-open stage, enumerating w into A and changing the mode of the edge to close mode.

Now consider the least $\mathcal{R}_1$-expansionary stage $t_1 > s$, and the least $\mathcal{R}_2$-expansionary stage $t_2 > s$. Since α has edges with outcome g_1, it must be the case that β_1 is following a Γ-strategy. Suppose that β_2 is following a Γ-strategy. Then we claim that it cannot be the case that $U_1 \upharpoonright \phi_{1,1}[s](w) \neq U_1 \upharpoonright \phi_{1,1}[t_1](w)$ and $U_2 \upharpoonright \phi_{2,1}[s](w) \neq U_2 \upharpoonright \phi_{2,1}[t_2](w)$. Similarly if β_2 is following a $\hat{\Gamma}$-strategy we claim that it cannot be the case that $U_1 \upharpoonright \phi_{1,1}[s](w) \neq U_1 \upharpoonright \phi_{1,1}[t_1](w)$ and $V_2 \upharpoonright \phi_{2,1}[s](w) \neq V_2 \upharpoonright \phi_{2,1}[t_2](w)$. For in either case the strategy α would diagonalise and impose the restraint $R_{\alpha,u} > 0$ when it becomes accessible again at some stage $u \geq t$. But this would mean that $\Theta^D(w) \neq A(w)$, contradicting the fact that the length of agreement between $\Theta^D(w)$ and A expands infinitely often.

Hence once the edge is in close mode, we have that the strategy eventually visits the edge during an α-close stage s', performing capricious destruction and enumerating $\gamma_{\beta_1 \frown \langle i,\sigma_1 \rangle}[s'](v)$ into D, while changing the mode of the edge to open mode again.

Since the strategy performs capricious destruction infinitely often, we have that $\lim_{q\to\infty} \gamma_{\beta_1 \frown \langle i,\sigma_1 \rangle}[q](v)$ is unbounded and that $\Gamma^{U_1,D}_{\beta_1 \frown \langle i,\sigma_1 \rangle}(v) \uparrow$. However this does not imply that $\Phi^{U_1,V_1}(v) \uparrow$ as well. On the other hand, the R-Synchronisation of β_2 with β_1, results in $\lim_{q\to\infty} \gamma_{\beta_2 \frown \langle o,\sigma_2 \rangle}[q](w)$ also becoming unbounded, meaning that $\Gamma^{U_2,D}_{\beta_2 \frown \langle o,\sigma_2 \rangle}(v) \uparrow$ (or $\Gamma^{V_2,D}_{\beta_2 \frown \langle o,\sigma_2 \rangle}$ resp.) without $\Phi^{U_2,V_2}(v) \uparrow$ having to be the case as well. Hence both the $\mathcal{R}_1$ and the $\mathcal{R}_2$ requirements remain unsatisfied, although the g_1 outcome will cause the next $\mathcal{R}$ strategy labeled $\mathcal{R}_1$ on the true path to switch to following a $\hat{\Gamma}$-strategy.

Finally the $\mathcal{S}$ requirement remains unsatisfied as well because the strategy does not diagonalise. For if this were the case, is would contradict the fact that the length of agrement between Θ^D and A increases infinitely often.

$\boxed{g_2}$ Suppose that the outcome is $\langle g_2, \sigma \rangle$.

Two different cases have to be considered.

(a) (β_1 *is following a* Γ-*strategy*).

In this case the answer to questions Q_1, $Q_{2.1}$, $Q_{2.2}$ and $Q_{3.1}$ and $Q_{3.2}$ guarantee that there are infinitely many witnesses w and stages s such that $\Theta^D[s](w)\downarrow= 0$, the computations $\Gamma^{U_1,D}_{\beta_1\frown\langle i,\sigma_1\rangle}[s](w)$ and $\Gamma^{U_2,D}_{\beta_2\frown\langle i,\sigma_2\rangle}[s](w)$ are honest, $a_s < w < b_s$ and $a_s < \theta_s < b_s$. We also have that infinitely many of these witnesses w enter the set A at stage s, and that infinitely many of these witnesses cause a $U_1 \upharpoonright \phi_{1,1}[s](w)$ change to occur between stage s and the least $\mathcal{R}_1$-expansionary stage $t_1 > s$. However, only finitely many of these witnesses cause a $U_2 \upharpoonright \phi_{2,1}[s](w)$ change to occur between stage s and the least $\mathcal{R}_2$-expansionary stage $t_2 > s$. Finally we also have that the length of agreement between Θ^D and A expands infinitely often.

Now since $\alpha \frown \langle g_2, \sigma\rangle$ is the leftmost edge which is accessible infinitely often, we have that there is a stage s_0 after which no edge to its left is accessible. Hence only finitely many edges to the left of $\alpha \frown \langle g_2, \sigma\rangle$ can have been accessible at stages $s < s_0$. Suppose that the edge $\alpha \frown \langle g_2, \sigma\rangle$ does not have a witness attached at some stage $s_1 > s_0$. Since witnesses giving honest computations $\Gamma^{U_1,D}_{\beta_1\frown\langle i,\sigma_1\rangle}[s](w)$ and $\Gamma^{U_2,D}_{\beta_2\frown\langle i,\sigma_2\rangle}[s](w)$ are attached to the leftmost edge of the form $\alpha \frown \langle g_i, \sigma\rangle$ for $1 \le i \le 2$ which has no other witness attached, and since there are infinitely many such witnesses, it follows that a witness satisfying these conditions is eventually attached to $\alpha \frown \langle g_1, \sigma\rangle$ at some stage $s_2 > s_1$.

In addition we claim that it is possible to ensure that the edge $\alpha \frown \langle g_2, \sigma\rangle$ is visited during infinitely many α-open stages and infinitely many α-close stages. We address this claim when we discuss *fairness* in Section 3.7.1.

Finally we claim that if a witness w is attached to the edge $\alpha \frown \langle g_2, \sigma\rangle$ at some stage s and the witness gives honest computations $\Gamma^{U_1,D}_{\beta_1\frown\langle i,\sigma_1\rangle}[s](w)$ and $\Gamma^{U_2,D}_{\beta_2\frown\langle i,\sigma_2\rangle}[s](w)$, it is possible to stop elements from entering A or D up to $\phi_{1,2}[s](\phi_{1,1}[s](w))$ and $\phi_{1,3}[s](\phi_{1,1}[s](w))$, and up to $\phi_{2,2}[s](\phi_{2,1}[s](w))$ and $\phi_{2,3}[s](\phi_{2,1}[s](w))$ at some stage $s' \ge s$. In this way the honesty of the witness can be preserved until the strategy determines that it should enumerate it into the set A. We address this claim when we discuss *honesty preservation* in Section 3.7.2.

Now if the edge $\alpha \frown \langle g_2, \sigma\rangle$ does not have a work interval defined at some stage s, the strategy will choose a threshold v and define a work interval $(v, \gamma_{\beta_2 \frown \langle i,\sigma_2\rangle}[s](v))$ for the edge. Once the strategy has taken this step, a witness w giving honest computations $\Gamma^{U_1,D}_{\beta_1 \frown \langle i,\sigma_1\rangle}[s](w)$ and $\Gamma^{U_2,D}_{\beta_2 \frown \langle i,\sigma_2\rangle}[s](w)$ will eventually be attached to $\alpha \frown \langle g_2, \sigma\rangle$. Then if the edge is in open mode, we have that the strategy eventually visits the edge during an α-open stage, enumerating w into A and changing the mode of the edge to close mode.

Consider the least $\mathcal{R}_1$-expansionary stage $t_1 > s$, and the least $\mathcal{R}_2$-expansionary stage $t_2 > s$. Since both β_1 and β_2 are following Γ-strategies, we claim that it cannot be the case that $U_1 \upharpoonright \phi_{1,1}[s](w) \neq U_1 \upharpoonright \phi_{1,1}[t_1](w)$ and $U_2 \upharpoonright \phi_{2,1}[s](w) \neq U_2 \upharpoonright \phi_{2,1}[t_2](w)$. Otherwise the strategy α would diagonalise successfully and impose the restraint $R_{\alpha,u} > 0$ when it becomes accessible again at some stage $u \geq t$. But this would mean that $\Theta^D(w) \neq A(w)$, contradicting the fact that the length of agreement between $\Theta^D(w)$ and A expands infinitely often.

Since the edge is now in close mode, we have that the strategy eventually visits the edge during an α-close stage s', performing capricious destruction and enumerating $\gamma_{\beta_2 \frown \langle i,\sigma_2\rangle}[s'](v)$ into D, while changing the mode of the edge to open mode again.

Now since the strategy performs capricious destruction infinitely often, we have that $\lim_{q\to\infty} \gamma_{\beta_2 \frown \langle i,\sigma_2\rangle}[q](v)$ is unbounded. This means that $\Gamma^{U_2,D}_{\beta_2 \frown \langle i,\sigma_2\rangle}(v) \uparrow$ is unbounded, without $\Phi^{U_2,V_2}(v) \uparrow$ having to be the case as well. Hence the $\mathcal{R}_2$ requirement is unsatisfied, and the next $\mathcal{R}$ strategy labeled $\mathcal{R}_2$ to appear on the priority tree will switch to following a $\hat{\Gamma}$-strategy. On the other hand we have that the strategy β_1 can continue to build the functional $\Gamma^{U_1,D}_{\beta_1 \frown \langle i,\sigma_1\rangle}(w)$ so as to ensure its agreement with the set A, and thus in the absence of further interference, the $\mathcal{R}_1$ requirement will be satisfied.

Finally the $\mathcal{S}$ requirement remains unsatisfied as well because the strategy does not diagonalise. For if this were the case, is would contradict the fact that the length of agrement between Θ^D and A increases infinitely often.

(b) (β_1 *is following a* $\hat{\Gamma}$*-strategy*).

Then the answer to questions Q_1, $Q_{2.1}$, $Q_{2.2}$ and $Q_{3.2}$ guarantee that there are infinitely many witnesses w and stages s such that $\Theta^D[s](w) \downarrow = 0$, the computations $\Gamma^{U_1,D}_{\beta_1 \frown \langle i,\sigma_1\rangle}[s](w)$ and $\Gamma^{U_2,D}_{\beta_2 \frown \langle i,\sigma_2\rangle}[s](w)$ are honest, $a_s < w < b_s$ and $a_s < \theta_s < b_s$.

In addition, infinitely many of these witnesses w and stages s give rise to computations $\Gamma^{V_1,D}_{\beta_1 \frown \langle i,\sigma_1 \rangle}[s](w)$ and $\Gamma^{U_2,D}_{\beta_2 \frown \langle i,\sigma_2 \rangle}[s](w)$ which are honest. We also have that infinitely many of these witnesses w enter the set A at stage s, but only finitely many of these witnesses cause a $U_2 \upharpoonright \phi_{2,1}[s](w)$ change between stage s and the least $\mathcal{R}_2$-expansionary stage $t_2 > s$. Finally we also have that the length of agreement between Θ^D and A increases infinitely often.

Now since $\alpha \frown \langle g_2, \sigma \rangle$ is the leftmost edge which is accessible infinitely often, we have that there is a stage s_0 after which no edge to its left is accessible. Hence only finitely many edges to the left of $\alpha \frown \langle g_2, \sigma \rangle$ can have been accessible at stages $s < s_0$. Suppose that the edge $\alpha \frown \langle g_2, \sigma \rangle$ does not have a witness attached at some stage $s_1 > s_0$. Since witnesses giving honest computations $\Gamma^{V_1,D}_{\beta_1 \frown \langle i,\sigma_1 \rangle}[s](w)$ and $\Gamma^{U_2,D}_{\beta_2 \frown \langle i,\sigma_2 \rangle}[s](w)$ are attached to the leftmost edge of the form $\alpha \frown \langle g_2, \sigma \rangle$ which has no other witness attached, and since there are infinitely many such witnesses, it follows that a witness satisfying these conditions is eventually attached to $\alpha \frown \langle g_2, \sigma \rangle$ at some stage $s_2 > s_1$.

In addition we claim that it is possible to ensure that the edge $\alpha \frown \langle g_2, \sigma \rangle$ is visited during infinitely many α-open stages and infinitely many α-close stages. We address this claim when we discuss *fairness* in Section 3.7.1.

Finally we claim that if a witness w is attached to the edge $\alpha \frown \langle g_2, \sigma \rangle$ at some stage s and the witness gives honest computations $\Gamma^{V_1,D}_{\beta_1 \frown \langle i,\sigma_1 \rangle}[s](w)$ and $\Gamma^{U_2,D}_{\beta_2 \frown \langle i,\sigma_2 \rangle}[s](w)$ it is possible to preserve its honesty as in the previous case.

Now if the edge $\alpha \frown \langle g_2, \sigma \rangle$ does not have a work interval defined at some stage s, the strategy will choose a threshold v and define a work interval $(v, \gamma_{\beta_2 \frown \langle i,\sigma_2 \rangle}[s](v))$ for the edge. Once the strategy has taken this step, an honest witness w will eventually be attached to $\alpha \frown \langle g_2, \sigma \rangle$. Then if the edge is in open mode, we have that the strategy eventually visits the edge during an α-open stage, enumerating w into A and changing the mode of the edge to close mode.

Consider the least $\mathcal{R}_1$-expansionary stage $t_1 > s$, and the least $\mathcal{R}_2$-expansionary stage $t_2 > s$. We claim that it cannot be the case that $V_1 \upharpoonright \phi_{1,1}[s](w) \neq V_1 \upharpoonright \phi_{1,1}[t_1](w)$ and $U_2 \upharpoonright \phi_{2,1}[s](w) \neq U_2 \upharpoonright \phi_{2,1}[t_2](w)$. For in this case the strategy α would diagonalise successfully and impose the restraint $R_{\alpha,u} > 0$ when it becomes accessible again at some stage $u \geq t$. But this would mean that $\Theta^D(w) \neq A(w)$, contradicting the fact that the length of agreement between $\Theta^D(w)$ and A increases infinitely often.

Hence once the edge is in close mode, we have that the strategy eventually visits the edge during an α-close stage s', performing capricious destruction and enumerating $\gamma_{\beta_2 \frown \langle i,\sigma_2 \rangle}[s'](v)$ into D, while changing the mode of the edge to open mode again.

Since the strategy performs capricious destruction infinitely often, we have that $\lim_{q\to\infty} \gamma_{\beta_2 \frown \langle i,\sigma_2 \rangle}[q](v)$ is unbounded. This means that $\Gamma^{U_2,D}_{\beta_2 \frown \langle i,\sigma_2 \rangle}(v) \uparrow$ is unbounded, without $\Phi^{U_2,V_2}(v) \uparrow$ having to be the case as well. Hence the $\mathcal{R}_2$ requirement is unsatisfied, and the next $\mathcal{R}$ strategy labeled $\mathcal{R}_2$ to appear on the priority tree will switch to following a $\hat{\Gamma}$-strategy. On the other hand we have that the strategy β_1 can continue to build the functional $\Gamma^{V_1,D}_{\beta_1 \frown \langle i,\sigma_1 \rangle}(w)$ so as to ensure its agreement with the set A, and thus in the absence of further interference, the $\mathcal{R}_1$ requirement will be satisfied.

Finally the $\mathcal{S}$ requirement remains unsatisfied as well because the strategy does not diagonalise. For if this were the case, is would contradict the fact that the length of agrement between Θ^D and A increases infinitely often.

[d] Suppose that the outcome is $\langle d, \sigma \rangle$.

Four different cases have to be considered.

(a) (*β_1 and β_2 are active for α and follow a Γ-strategy*).

In this case we have that the answers to the questions Q_1, $Q_{2.1}$, $Q_{2.2}$, $Q_{3.1}$ and $Q_{3.2}$ guarantee that there are infinitely many witnesses w and stages s such that $\Theta^D[s](w) \downarrow = 0$, the computations $\Gamma^{U_1,D}_{\beta_1 \frown \langle i,\sigma_1 \rangle}[s](w)$ and $\Gamma^{U_2,D}_{\beta_2 \frown \langle i,\sigma_2 \rangle}[s](w)$ are honest, $a_s < w < b_s$ and $a_s < \theta_s < b_s$. In addition we have that infinitely many of these witnesses w enter the set A at stage s, and infinitely many of these witnesses cause a $U_1 \upharpoonright \phi_{1,1}[s](w)$ change between stage s and the least $\mathcal{R}_1$-expansionary stage $t_1 > s$, and a $U_2 \upharpoonright \phi_{2,1}[s](w)$ change between stage s and the least $\mathcal{R}_2$-expansionary stage $t_2 > s$. Finally we have that the length of agreement between Θ^D and A expands infinitely often.

Now suppose that the strategy opens a gap by enumerating a witness w into A when visiting some edge of the form $\alpha \frown \langle g_i, \sigma \rangle$ for $1 \leq i \leq 2$ at some stage u, and that a

$U_1 \upharpoonright \phi_{1,u}(w)$ change takes place at some least $\mathcal{R}_1$ expansionary stage $u_1 > u$ and that a $U_2 \upharpoonright \phi_{2,u}(w)$ change takes place at some least $\mathcal{R}_2$ expansionary stage $u_2 > u$. If the strategy becomes accessible again at some stage $s' \geq u'$, we have that it sets $R_{\alpha,s'} > 0$. But this would mean that $\Theta^D[s''](w) \neq A_{s''}(w)$ for all $s'' > s'$, which contradicts the fact that the length of agreement between $\Theta^D(w)$ and A expands infinitely often.

(b) (β_1 *and* β_2 *are active for* α, β_1 *is following a* Γ-*strategy and* β_2 *is following a* $\hat{\Gamma}$-*strategy*).

In this case we have that the answers to the questions Q_1, $Q_{2.1}$, $Q_{2.2}$ and $Q_{3.1}$ guarantee that there are infinitely many witnesses w and stages s such that $\Theta^D[s](w) \downarrow= 0$, the computations $\Gamma^{U_1,D}_{\beta_1\frown\langle i,\sigma_1\rangle}[s](w)$ and $\Gamma^{V_2,D}_{\beta_2\frown\langle i,\sigma_2\rangle}[s](w)$ are honest, $a_s < w < b_s$ and $a_s < \theta_s < b_s$. In addition we also have that infinitely many of these witnesses w enter the set A at stage s, and infinitely many of these witnesses cause a $U_1 \upharpoonright \phi_{1,1}[s](w)$ change between stage s and the least $\mathcal{R}_1$-expansionary stage $t_1 > s$.

Now suppose that the strategy opens a gap by enumerating a witness w into A when visiting some edge of the form $\alpha \frown \langle d, \sigma\rangle$ at some stage u, and that a $U_1 \upharpoonright \phi_{1,u}(w)$ change takes place at some least $\mathcal{R}_1$ expansionary stage $u_1 > u$. If the strategy becomes accessible again at some stage $s' \geq u$, we have that it sets $R_{\alpha,s'} > 0$. The strategy will in fact have diagonalised successfully because the priority tree will be arranged such that a $U_2 \upharpoonright \phi_{2,u}(w)$ change could not have taken place at the least $\mathcal{R}_2$ expansionary stage $u_2 > u$ without α becoming inaccessible. Since α is on the true path, it will then follow that a $V_2 \upharpoonright \phi_{2,u}(w)$ change has instead taken place at the least $\mathcal{R}_2$ expansionary stage $u_2 > u$.

However since the strategy has diagonalised we have that $\Theta^D[s''](w) \neq A_{s''}(w)$ for all $s'' > s'$, which contradicts the fact that the length of agreement between $\Theta^D(w)$ and A expands infinitely often.

(c) (β_1 *and* β_2 *are active for* α, β_1 *is following a* $\hat{\Gamma}$-*strategy and* β_2 *is following a* Γ-*strategy*).

In this case we have that the answers to the questions Q_1, $Q_{2.1}$, $Q_{2.2}$ and $Q_{3.2}$ guarantee that there are infinitely many witnesses w and stages s such that $\Theta^D[s](w) \downarrow= 0$, the computations $\Gamma^{V_1,D}_{\beta_1\frown\langle i,\sigma_1\rangle}[s](w)$ and $\Gamma^{U_2,D}_{\beta_2\frown\langle i,\sigma_2\rangle}[s](w)$ are honest, $a_s < w < b_s$ and $a_s < \theta_s < b_s$. In addition we also have that infinitely many of these witnesses w enter the set A

at stage s, and infinitely many of these witnesses cause a $U_2 \upharpoonright \phi_{2,1}[s](w)$ change between stage s and the least $\mathcal{R}_2$-expansionary stage $t_2 > s$.

Now suppose that the strategy opens a gap by enumerating a witness w into A when visiting some edge of the form $\alpha \frown \langle d, \sigma \rangle$ at some stage u, and that a $U_2 \upharpoonright \phi_{2,u}(w)$ change takes place at some least $\mathcal{R}_2$ expansionary stage $u_2 > u$. If the strategy becomes accessible again at some stage $s' \geq u$, we have that it sets $R_{\alpha,s'} > 0$. The strategy will in fact have diagonalised successfully because the priority tree will be arranged such that a $U_1 \upharpoonright \phi_{1,u}(w)$ could not have taken place at the least $\mathcal{R}_1$ expansionary stage $u_1 > u$ without α becoming inaccessible. Since α is on the true path, it will then follow that a $V_1 \upharpoonright \phi_{1,u}(w)$ change has instead taken place at the least $\mathcal{R}$ expansionary stage $u_1 > u$.

However since the strategy has diagonalised we have that $\Theta^D[s''](w) \neq A_{s''}(w)$ for all $s'' > s'$, which contradicts the fact that the length of agreement between $\Theta^D(w)$ and A expands infinitely often.

(d) (β_1 *and* β_2 *are active for* α *and follow a* $\hat{\Gamma}$*-strategy*).

In this case we have that the answers to the questions Q_1, $Q_{2.1}$, $Q_{2.2}$ guarantee that there are infinitely many witnesses w and stages s such that $\Theta^D[s](w) \downarrow = 0$, the computations $\Gamma^{V_1,D}_{\beta_1 \frown \langle i,\sigma_1 \rangle}[s](w)$ and $\Gamma^{V_2,D}_{\beta_2 \frown \langle i,\sigma_2 \rangle}[s](w)$ are honest, $a_s < w < b_s$ and $a_s < \theta_s < b_s$.

Now since $\alpha \frown \langle d, \sigma \rangle$ is the leftmost edge which is accessible infinitely often, we have that there is a stage s_0 after which no edge to its left is accessible. Hence only finitely many edges to the left of $\alpha \frown \langle d, \sigma \rangle$ can have been accessible at stages $s < s_0$. Suppose that the edge $\alpha \frown \langle d, \sigma \rangle$ does not have a witness attached at some stage $s_1 > s_0$. Since witnesses giving honest computations $\Gamma^{V_1,D}_{\beta_1 \frown \langle i,\sigma_1 \rangle}[s](w)$ and $\Gamma^{V_2,D}_{\beta_2 \frown \langle i,\sigma_2 \rangle}[s](w)$ are attached to the leftmost edge of the form $\alpha \frown \langle d, \sigma \rangle$ which has no other witness attached, and since there are infinitely many such witnesses, it follows that a witness satisfying these conditions is eventually attached to $\alpha \frown \langle d, \sigma \rangle$ at some stage $s_2 > s_1$.

In addition we claim that it is possible to ensure that the edge $\alpha \frown \langle d, \sigma \rangle$ is visited during infinitely many α-open stages and infinitely many α-close stages. We address this claim when we discuss *fairness* in Section 3.7.1.

Finally we claim that if a witness w is attached to the edge $\alpha \frown \langle d, \sigma \rangle$ at some stage s and the witness gives honest computations $\Gamma_{\beta_1 \frown \langle i,\sigma_1 \rangle}[s]^{V_1,D}(w)$ and $\Gamma_{\beta_2 \frown \langle i,\sigma_2 \rangle}[s]^{V_2,D}(w)$, it is possible to stop elements from entering sets A or D up to $\phi_{1,2}[s](\phi_{1,1}[s](w))$ and $\phi_{1,3}[s](\phi_{1,1}[s](w))$, as well as up to $\phi_{2,2}[s](\phi_{2,1}[s](w))$ and $\phi_{2,3}[s](\phi_{2,1}[s](w))$. In this way the honesty of the witness can be preserved until the strategy determines that it should enumerate it into the set A. We address this claim when we discuss *honesty preservation* in Section 3.7.2.

Now the strategy will eventually attach an honest witness w attached to $\alpha \frown \langle d, \sigma \rangle$. Then if the edge is in open mode, we have that the strategy eventually visits the edge during an α-open stage, enumerating w into A at some stage u.

Now suppose that the strategy opens a gap by enumerating a witness w into A when visiting some edge of the form $\alpha \frown \langle d, \sigma \rangle$ at some stage u. If the strategy becomes accessible again at some stage $s' \geq u$, we have that it sets $R_{\alpha,s'} > 0$. The strategy will in fact have diagonalised successfully because the priority tree will be arranged so that α would become inaccessible if a $U_1 \upharpoonright \phi_{1,u}(w)$ change takes place at the least $\mathcal{R}_1$ expansionary stage $u_1 > u$, or if a $U_2 \upharpoonright \phi_{2,u}(w)$ change takes place at the least $\mathcal{R}_1$ expansionary stage $u_2 > u$. Since α is on the true path, it will then follow that a $V_1 \upharpoonright \phi_{1,u}(w)$ change has instead taken place at the least $\mathcal{R}_1$ expansionary stage $u_1 > u$ and that a $V_2 \upharpoonright \phi_{2,u}(w)$ change has instead taken place at the least $\mathcal{R}_2$ expansionary stage $u_2 > u$.

However since the strategy has diagonalised we have that $\Theta^D[s''](w) \neq A_{s''}(w)$ for all $s'' > s'$, which contradicts the fact that the length of agreement between $\Theta^D(w)$ and A expands infinitely often.

Since in each of the above three cases we have a contradiction, it follows that no edge with outcome $\langle d, \sigma \rangle$ can be on the true path. In fact, only edges with outcome $\langle w, \sigma \rangle$ can be on the true path if the strategy diagonalises successfully. This is the only outcome for which Q_1 can have a negative answer, which in turn allows the length of agreement between Θ^D and A to be finite in length.

3.6.7 Organisation of Priority Tree

We shall now organise a priority tree in order to satisfy an $\mathcal{S}$ requirement below the $\mathcal{R}_2$ requirement below the $\mathcal{R}_1$ requirement. The following notation shall be used when depicting the priority tree shown in Figure 3.2. Note that when we say that some strategy γ takes into consideration another strategy γ', we mean that γ' is active for γ.

- β^{U_1} will denote an $\mathcal{R}$ strategy labeled $\mathcal{R}_1$ which is following a Γ-strategy.
- β^{V_1} will denote an $\mathcal{R}$ strategy labeled $\mathcal{R}_1$ which is following a $\hat{\Gamma}$-strategy.
- β^{U_2} will denote an $\mathcal{R}$ strategy labeled $\mathcal{R}_2$ which is following a Γ-strategy.
- β^{V_2} will denote an $\mathcal{R}$ strategy labeled $\mathcal{R}_2$ which is following a $\hat{\Gamma}$-strategy.
- α^{U_1,U_2} will denote an $\mathcal{S}$ strategy which needs to take into consideration an $\mathcal{R}$ strategy labeled $\mathcal{R}_1$ and an $\mathcal{R}$ strategy labeled $\mathcal{R}_2$ above it, each of which are following a Γ-strategy.
- α^{U_1,V_2} will denote an $\mathcal{S}$ strategy which needs to take into consideration an $\mathcal{R}$ strategy labeled $\mathcal{R}_1$ and an $\mathcal{R}$ strategy labeled $\mathcal{R}_2$ above it, with the first following a Γ-strategy and the second following a $\hat{\Gamma}$-strategy.
- α^{V_1,U_2} will denote an $\mathcal{S}$ strategy which needs to take into consideration an $\mathcal{R}$ strategy labeled $\mathcal{R}_1$ and an $\mathcal{R}$ strategy labeled $\mathcal{R}_2$ above it, with the first following a $\hat{\Gamma}$-strategy and the second following a Γ-strategy.
- α^{V_1,V_2} will denote an $\mathcal{S}$ strategy which needs to take into consideration an $\mathcal{R}$ strategy labeled $\mathcal{R}_1$ and an $\mathcal{R}$ strategy labeled $\mathcal{R}_2$ above it, each of which are following a $\hat{\Gamma}$-strategy.

When multiple strategies of the same kind appear on the same branch of the priority tree, we shall use primed versions of the notation above to distinguish between them.

When presenting the priority tree we omit the infinitely many edges of a given $\mathcal{R}$ strategy, focusing on just one edge with outcome $\langle i, \sigma \rangle$. Similarly, we omit the infinitely many edges of a given $\mathcal{S}$ strategy, focusing on just one edge of each of the following kinds $\langle w, \sigma \rangle$, $\langle d, \sigma \rangle$, $\langle h_1, \sigma \rangle$, $\langle h_2, \sigma \rangle$, $\langle g_1, \sigma \rangle$ and $\langle g_2, \sigma \rangle$ (whenever the last two kinds of edge are present). To simplify our presentation we shall simply write σ to denote the use of any outcome being depicted. We also recall that the infinitely many edges of each strategy are ordered by the value of σ and not by the value of o. The outcomes will thus be depicted on the priority tree as being in no particular order.

We have already seen that the effect of the various outcomes of an $\mathcal{S}$ strategy on the satisfaction of the $\mathcal{R}$ and $\mathcal{S}$ requirements is similar to the effect of an $\mathcal{S}$ strategy in the original Lachlan Non-Splitting Theorem. Hence it is possible to build the priority tree in this section in an analogous way to the priority tree for the Lachlan Non-Splitting Theorem as described in Chapter 2.

As is the case with the priority tree of the Lachlan Non-Splitting Theorem, the highest priority unsatisfied requirement at a given node can be determined though the analysis of the outcomes covered in the previous section. As before, the highest priority unsatisfied requirement at a given node causes a strategy of the corresponding kind to appear at that node.

When an $\mathcal{S}$ strategy has an edge with outcome g_1 it will cause the $\mathcal{R}_1$ requirement to *switch* its mode of satisfaction from a Γ-strategy to a $\hat{\Gamma}$-strategy below the edge. This will cause any $\mathcal{R}$ strategy labeled $\mathcal{R}_1$ below the edge to follow a $\hat{\Gamma}$-strategy instead of a Γ-strategy. In addition it will cause a *restart* of the $\mathcal{R}_2$ by changing its mode of satisfaction to a Γ-strategy below the edge.

Similarly when an $\mathcal{S}$ strategy has an edge with outcome g_2 it will cause the $\mathcal{R}_2$ requirement to *switch* its mode of satisfaction from a Γ-strategy to a $\hat{\Gamma}$-strategy below the edge. This will cause any $\mathcal{R}$ strategy labeled $\mathcal{R}_2$ below the edge to follow a $\hat{\Gamma}$-strategy instead of a Γ-strategy.

In addition, we have already seen that $\mathcal{S}$ strategies will follow both *S-Synchronisation* and *R-Synchronisation*. As in the construction for the Lachlan Non-Splitting Theorem, the combination of switching and S-Synchronisation will allow the $\mathcal{S}$ requirement below the $\mathcal{R}_2$ requirement below the $\mathcal{R}_1$ requirement to be satisfied when strategies are organised according to the priority tree in Figure 3.2.

The resulting priority tree is shown below.

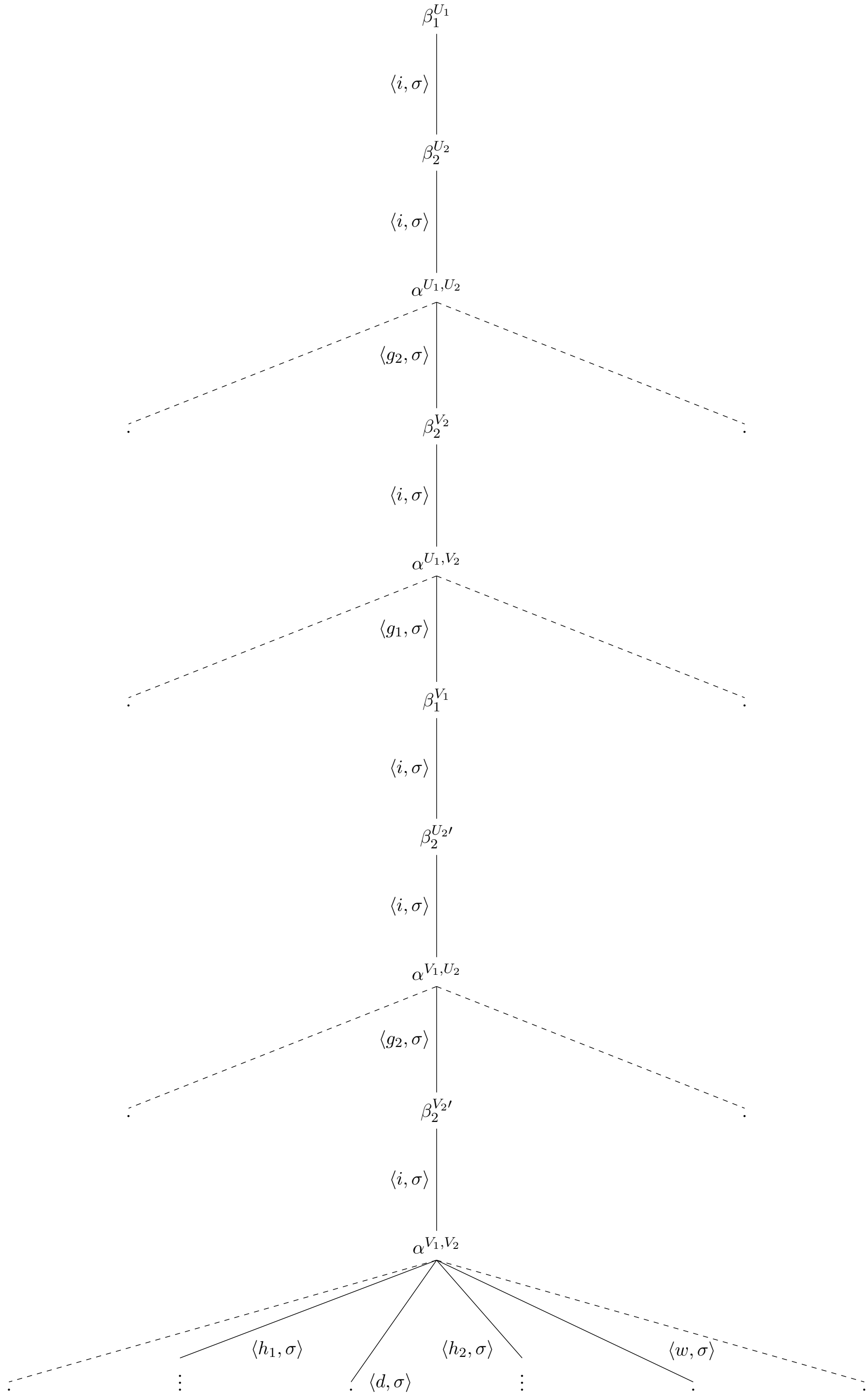

Figure 3.2: Priority tree for $\mathcal{S}$ below $\mathcal{R}_2$ below $\mathcal{R}_1$ - path leading to α^{V_1,V_2}.

3.6.8 Satisfaction of Requirements

We shall now examine the simultaneous satisfaction of an $\mathcal{S}$ requirement below an $\mathcal{R}_2$ requirement below an $\mathcal{R}_1$ requirement by the strategies and priority tree found in this section. For our purposes it shall be sufficient to consider the most complex situation, which occurs when the edge of the strategy α^{V_1,V_2} on the true path is of the form $\alpha^{V_1,V_2} \frown \langle w, \sigma\rangle$.

[w] The analysis for showing that the $\mathcal{S}$ requirement is satisfied is identical to the one found in Section 3.5.8 for the case where the outcome $\langle w, \sigma\rangle$ of the strategy is on the true path.

We now consider the satisfaction of the $\mathcal{R}_1$ requirement by the strategy $\beta^{V_1\prime}$ and the satisfaction of the $\mathcal{R}_2$ requirement by the strategy $\beta^{V_2\prime}$.

Suppose that α^{V_1,V_2} never enumerates a witness w into the set A. Then $\beta_1^{V_1}$ can build $\Gamma^{V_1,D}_{\beta_1^{V_1}\frown\langle i,\sigma\rangle} = A$ without interference, thus satisfying the $\mathcal{R}_1$ requirement. Similarly $\beta_2^{V_2\prime}$ can build $\Gamma^{V_2,D}_{\beta_2^{V_2\prime}\frown\langle i,\sigma\rangle} = A$ without interference, thus satisfying the $\mathcal{R}_2$ requirement.

On the other hand, suppose that α^{V_1,V_2} has enumerated a witness w into the set A at stage s. Then by $\mathcal{S}$-Synchronisation the strategies α^{U_1,U_2}, α^{U_1,V_2}, α^{V_1,U_2} must also have enumerated witnesses w', w'' and w''' into the set A during the same stage s. Since α^{V_1,V_2} lies inside the work interval imposed by the aforementioned strategies down the path leading to α^{V_1,V_2}, we have that w is smaller than either of w', w'' or w''', each of which is greater than the upper bound of the corresponding work interval. The enumeration of these elements then creates a least disagreement at the element w between the set A and every functional associated to an edge with outcome $\langle i, \sigma\rangle$ lying on the path leading to α^{V_1,V_2}.

Now, in order for α^{V_1,V_2} to lie on the true path, the strategy must eventually become accessible again. For this to be the case, the strategy $\beta_1^{U_1}$ must have visited its outcome $\beta_1^{U_1} \frown \langle i, \sigma\rangle$ and proceeded to the next substage. This means that some β_1-expansionary stage $s_1 > s$ must have been attached to this edge, which implies that some least $\mathcal{R}_1$-expansionary stage t_1 such that $s < t_1 \leq s_1$ must also have taken place. Similarly, the strategy $\beta_2^{V_2\prime}$ must have visited its outcome $\beta_2^{V_2\prime} \frown \langle i, \sigma\rangle$ and also proceeded to the next substage. This means that some expansionary stage $s_2 > s$ must have also been attached to this edge, which implies that some least $\mathcal{R}_2$-expansionary stage t_2 such that $s < t_2 \leq s_2$ must also have taken place.

Now, if $U_{1,s} \restriction \phi_{1,1}[s](w) \neq U_{1,t_1} \restriction \phi_{1,1}[s](w)$ and $U_{2,s} \restriction \phi_{2,1}[s](w) \neq U_{2,t_2} \restriction \phi_{2,1}[s](w)$, we have that α^{U_1,U_2} has diagonalised. Hence the strategy α^{U_1,U_2} would terminate the stage whenever it visits the edge $\alpha^{U_1,U_2} \frown \langle g_2, \sigma \rangle$, making the strategy α^{V_1,V_2} inaccessible, which is a contradiction.

Similarly, if $U_{1,s} \restriction \phi_{1,1}[s](w) \neq U_{1,t_1} \restriction \phi_{1,1}[s](w)$ and $U_{2,s} \restriction \phi_{2,1}[s](w) = U_{2,t_2} \restriction \phi_{2,1}[s](w)$, we have that α^{U_1,V_2} has diagonalised making α^{V_1,V_2} inaccessible. The same happens if $U_{1,s} \restriction \phi_{1,1}[s](w) = U_{1,t_1} \restriction \phi_{1,1}[s](w)$ and $U_{2,s} \restriction \phi_{2,1}[s](w) \neq U_{2,t_2} \restriction \phi_{2,1}[s](w)$, which causes α^{V_1,U_2} to diagonalise, making α^{V_1,V_2} inaccessible. Hence both of these situations also lead to a contradiction.

Thus it follows that it must be the case that $U_{1,s} \restriction \phi_{1,1}[s](w) = U_{1,t_1} \restriction \phi_{1,1}[s](w)$ and $U_{2,s} \restriction \phi_{2,1}[s](w) = U_{2,t_2} \restriction \phi_{2,1}[s](w)$. Hence we must have that $V_{1,s} \restriction \phi_{1,1}[s](w) \neq V_{1,t_1} \restriction \phi_{1,1}[s](w)$ and $V_{2,s} \restriction \phi_{2,1}[s](w) \neq V_{2,t_2} \restriction \phi_{2,1}[s](w)$. This means that the functionals $\Gamma^{V_1,D}_{\beta_1^{V_1} \frown \langle i,\sigma \rangle}$ and $\Gamma^{V_2,D}_{\beta_2^{V_2\prime} \frown \langle i,\sigma \rangle}$ no longer disagree with the set A at w, because they have been undefined at this element. Hence these functionals do not need to remove the disagreement by enumerating some element into D. Thus we have that the strategy α^{V_1,V_2} has not only diagonalised but also successfully preserved its diagonalisation.

Hence the strategy α^{V_1,V_2} has satisfied the $\mathcal{S}$ requirement, while the strategies $\beta_1^{V_1}$ and $\beta_2^{V_2\prime}$ can now build the corresponding functionals which lie on the true path to agree with the set A without interference. Hence we have that the requirements $\mathcal{R}_1$ and $\mathcal{R}_2$ are satisfied as well.

3.7 The General Case

In this section we shall show how one can satisfy many $\mathcal{R}$ and many $\mathcal{S}$ requirements simultaneously. In order to deal with many $\mathcal{R}$ and many $\mathcal{S}$ requirements, we shall generalise the approach found in Section 3.6, which we have used to successfully satisfy an $\mathcal{S}$ requirement below two $\mathcal{R}$ requirements. The formalised $\mathcal{R}$ and $\mathcal{S}$ strategies which we shall present in this section will be comprehensive enough to represent every possible strategy which shall be required on the priority tree for some given list of requirements. However prior to introducing these strategies we shall have to address two problems which we have so far postponed.

In Section 3.7.1 we consider the problem of fairness. Since $\mathcal{R}$ and $\mathcal{S}$ strategies γ are now guessing the outcome which they should visit at any given stage, it may be the case that the leftmost outcome chosen infinitely often by the strategy is only visited during finitely many γ-open or γ-close stages, which stops the strategy from working correctly. In order to resolve the problem we introduce a solution based on suspending the guessing until certain conditions are met.

In Section 3.7.2 we address the problem of preserving the honesty of the witnesses which an $\mathcal{S}$ strategy attaches to edges with an outcome g_i (if there is an $\mathcal{R}$ strategy following a Γ-strategy which is active for the $\mathcal{S}$ strategy) or to edges with an outcome d (if there is an $\mathcal{R}$ strategy active for the $\mathcal{S}$ strategy but no such strategy is following a Γ-strategy). This problem will be resolved by having $\mathcal{R}$ strategies take the length of honesty preservation into account when defining their functionals, by modifying the restraints imposed by $\mathcal{S}$ strategies and by introducing the notion of *H-Synchronisation*.

Following this, the general modified $\mathcal{R}$ strategy will be formalised in Sections 3.7.3 and 3.7.4. Similarly, the general modified $\mathcal{S}$ strategy will be formalised in Sections 3.7.5 and 3.7.6. We proceed to consider the general modified priority tree in Section 3.7.7, formalising the way in which it is generated. We conclude by formalising the modified construction in Section 3.7.8. The verification of the entire modified construction, which proves that every requirement can satisfied by the resulting system is postponed to Section 3.8.

3.7.1 Fairness

In previous sections we have seen that a strategy γ will guess the edge $\gamma \frown \langle o, \sigma \rangle$ which it will visit at any given stage. It is thus possible that the leftmost outcome chosen infinitely often by the strategy is only visited during finitely many γ-open or γ-close stages.

In this case the strategy γ will eventually become unable to take actions which require a certain kind of stage when visiting this edge. In addition it will also start terminating the stage early when visiting the edge, making all the strategies lying below it inaccessible. Both of these reasons can cause the construction to fail.

In order to avoid this situation, we shall proceed as follows. Suppose that at some stage s the strategy guesses that it should visit a given edge $\gamma \frown \langle o, \sigma \rangle$. In addition suppose that any elements which are required for the strategy to take the action associated with visiting that edge have already been attached to the edge.

If the edge $\gamma \frown \langle o, \sigma \rangle$ finds itself in some given mode, and stage s does not match the kind of stage required by this mode, the guessing will be suspended. This means that whenever the strategy becomes accessible again at some stage $t > s$, it will visit the edge $\gamma \frown \langle o, \sigma \rangle$ once again. This will keep taking place until one of the following two conditions are met.

Firstly it could be the case that stage t is of the kind required by the mode of the edge. Then the strategy is able to take the action associated with the mode of the edge. Having taken this action, the strategy will change the mode of the edge. It can also resume guessing the edges which it should visit when it becomes accessible.

Secondly it could be the case that if the guessing were not suspended at stage t, the strategy would have visited an edge $\gamma \frown \langle o', \sigma' \rangle$ lying to the left of $\gamma \frown \langle o, \sigma \rangle$. In this situation the guessing will no longer be suspended and the edge $\gamma \frown \langle o', \sigma' \rangle$ will be visited instead. We shall then go through the same procedure with edge $\gamma \frown \langle o', \sigma' \rangle$.

The reason for resuming the guessing in the second condition is that otherwise the strategy might only guess to visit the leftmost edge chosen infinitely often during those stages at which the guessing is suspended, leading the construction to fail.

It is important to note that since the strategy guesses to visit some leftmost edge infinitely often we have that access to this edge cannot be stopped through suspending the strategy's guessing. In addition once the strategy guesses to visit this edge, it will keep visiting the edge until a stage of the required kind has occurred, allowing the strategy to take the appropriate action.

3.7.2 Honesty Preservation

We start by describing the honesty preservation problem.

Consider an $\mathcal{S}$ strategy α lying below a number of $\mathcal{R}$ strategies β_i for $1 \leq i \leq m$ which are active for α, and let $\beta_i \frown \langle i, \sigma_i \rangle$ denote the edge of β_i on the path leading to α.

Then if there is some β_i is following a Γ-strategy we have that α will only attach a witness w to an edge of the form $\alpha \frown \langle g_j, \sigma \rangle$ at some stage t if the witness gives honest computations $\Gamma^{U_i,D}_{\beta \frown \langle i,\sigma \rangle}[t](w)$ (or $\Gamma^{V_i,D}_{\beta \frown \langle i,\sigma \rangle}[t](w)$ resp.) for every $1 \leq i \leq m$. On the other hand, if no β_i is following a Γ-strategy, we have that α will only attach a witness w to an edge of the form $\alpha \frown \langle d, \sigma \rangle$ if the witness gives honest computations $\Gamma^{U_i,D}_{\beta \frown \langle i,\sigma \rangle}[t](w)$ (or $\Gamma^{V_i,D}_{\beta \frown \langle i,\sigma \rangle}[t](w)$ resp.) for every $1 \leq i \leq m$.

Thus in both of these cases, the strategy α will determine whether $\phi_{i,1}[t](w) \leq \gamma_{\beta_i \frown \langle i,\sigma_i \rangle}[t](w)$ for every $1 \leq i \leq m$ prior to attaching w to one of the aforementioned edges at stage t.

Now suppose that the strategy α does attach a witness w to an edge of the form $\alpha \frown \langle g_j, \sigma \rangle$ or $\alpha \frown \langle d, \sigma \rangle$ as detailed above at some stage t. Then if the strategy visits the edge at some stage $s \geq t$ it will only be able to enumerate w into A if the witness is still honest, that is if $\phi_{i,1}[s](w) \leq \gamma_{\beta_i \frown \langle i,\sigma_i \rangle}[s](w)$ for all $1 \leq i \leq m$.

However, the honesty of the witness might not be preserved between stage t and stage s. This can be the case if at some stage t' such that $t < t' \leq s$ there is an $A \upharpoonright \phi_{i,2}[t'](\phi_{i,1}[t'](w))$ or $A \upharpoonright \phi_{i,3}[t'](\phi_{i,1}[t'](w))$ change for some for $1 \leq i \leq m$, or alternatively a $D \upharpoonright \phi_{i,2}[t''](\phi_{i,1}[t'](s))$ or $D \upharpoonright \phi_{i,3}[t'](\phi_{i,1}[t'](s))$ change for some $1 \leq i \leq m$. Thus the honesty of the computations will need to be preserved by ensuring that no strategy can enumerate such an element into A or D once the witness w has been attached to an edge lying on the true path.

In order to implement honesty preservation we shall need to introduce a number of new constraints, which will be considered below.

Length of Honesty Preservation Constraint

The $\mathcal{R}$ strategies shall be required to take the length of honesty preservation into account when defining their functionals. In particular at any given stage s, a strategy β labeled $\mathcal{R}_i$ will only define a functional $\Gamma^{U_i,D}_{\beta\frown\langle i,\sigma\rangle}$ (or $\Gamma^{V_i,D}_{\beta\frown\langle i,\sigma\rangle}$ resp.) associated to one of its edges $\beta \frown \langle i, \sigma\rangle$ at some element x if this lies within the length of honesty preservation, that is if $x < h_s(\Phi^{U_i,V_i}_{i,1}, \Phi^{A,D}_{i,2}, \Phi^{A,D}_{i,3})$. In this manner if the element x is chosen by some $\mathcal{S}$ strategy α below β as a witness giving certain honest computations, we have that α knows which elements need to be kept out of A and D in order to protect the honesty of these computations.

We shall also require that if the $\mathcal{R}$ strategy β labeled $\mathcal{R}_i$ defines $\Gamma^{U_i,D}_{\beta\frown\langle i,\sigma\rangle}[s](x)$ (or $\Gamma^{V_i,D}_{\beta\frown\langle i,\sigma\rangle}[s](x)$ resp.) during some stage s, it chooses a use $\gamma_{\beta\frown\langle i,\sigma\rangle}[s](x)$ which is greater than $\phi_{i,2}[s](\phi_{i,1}[s](x))$ and $\phi_{i,3}[s](\phi_{i,1}[s](x))$. This additional constraint shall be required to implement H-Synchronisation, as will be described further below.

Honesty Preserving Restraints

The $\mathcal{S}$ strategies shall also be required to observe additional constraints when defining their attachment restraints, rightward restraints and downward restraints.

We start by considering the attachment restraints imposed by an $\mathcal{S}$ strategy.

At the moment we have that if an $\mathcal{S}$ strategy α attaches a witness w giving honest computations to an edge $\alpha \frown \langle g_j, \sigma\rangle$ or $\alpha \frown \langle d, \sigma\rangle$ at some stage s, every witness attached to an edge lying to its right is detached, and any work interval defined for an edge lying to its right is undefined.

Now if the strategy α attaches a witness w giving honest computations to an edge of the form $\alpha \frown \langle g_j, \sigma\rangle$ or $\alpha \frown \langle d, \sigma\rangle$ at some stage t, we shall require the strategy to impose an attachment restraint $a(\alpha \frown \langle o', \sigma'\rangle, t)$ which is greater than $\phi_{i,2}[t](\phi_{i,1}[t](w))$ and $\phi_{i,3}[t](\phi_{i,1}[t](w))$ for all $1 \leq i \leq m$ on every edge $\alpha \frown \langle o', \sigma'\rangle$ lying to the right of $\alpha \frown \langle g_j, \sigma\rangle$ or $\alpha \frown \langle d, \sigma\rangle$.

In this way if α tries to choose a threshold v to define a work interval for the edge $\alpha \frown \langle o', \sigma'\rangle$ at stage t itself, we shall have that $v > \phi_{i,2}[s](\phi_{i,1}[s](w))$ and $v > \phi_{i,3}[s](\phi_{i,1}[s](w))$ for $1 \leq i \leq m$.

It follows that if at some stage $s > t$ the strategy enumerates $\gamma_{\beta_j \frown \langle i,\sigma_j\rangle}[s](v)$ into D for some $\mathcal{R}$ strategy β_j above α which is active for α and which follows a Γ-strategy, we have that $\gamma_{\beta_j \frown \langle i,\sigma_j\rangle}[s](v) > \phi_{i,2}[s](\phi_{i,1}[s](w))$ and that $\gamma_{\beta_j \frown \langle i,\sigma_j\rangle}[s](v) > \phi_{i,3}[s](\phi_{i,1}[s](w))$ for all $1 \leq i \leq m$. But since uses are non-decreasing with respect to stages, we also have that $\phi_{i,2}[s](\phi_{i,1}[s](w)) > \phi_{i,2}[t](\phi_{i,1}[s](w))$ and $\phi_{i,3}[s](\phi_{i,1}[s](w)) > \phi_{i,3}[t](\phi_{i,1}[s](w))$, meaning that the honesty of the computations cannot be compromised by the enumeration of $\gamma_{\beta_j \frown \langle i,\sigma_j\rangle}[s](v)$ into D, as required.

Next we shall consider the rightward restraints imposed by an $\mathcal{S}$ strategy.

When calculating restraint $r(\alpha \frown \langle o', \sigma'\rangle, s)$ for some edge $\alpha \frown \langle o', \sigma'\rangle$ at some stage s, we shall require the restraint to be greater than $\phi_{i,2}[t](\phi_{i,1}[t](w))$ and $\phi_{i,3}[t](\phi_{i,1}[t](w))$ for $1 \leq i \leq m$, where w is the witness attached to the edge and t is the stage at which the witness was attached.

Thus if a witness w' is attached to some edge $\alpha \frown \langle g_k, \sigma'\rangle$ lying to the right of $\alpha \frown \langle g_j, \sigma\rangle$ at some stage $t' > t$, we have that w' must be greater than the supremum of the rightward restraints imposed by edges lying to the left of $\alpha \frown \langle g_k, \sigma'\rangle$. The same holds if the witness w' is attached to an edge of the form $\alpha \frown \langle d, \sigma'\rangle$ lying to the right of $\alpha \frown \langle d, \sigma\rangle$ instead. This gives that $w' > \phi_{i,2}[t](\phi_{i,1}[t](w))$ and that $w' > \phi_{i,3}[t](\phi_{i,1}[t](w))$ for all $1 \leq i \leq m$. It follows that the honesty of w is not compromised if the strategy enumerates w' into the set A by visiting $\alpha \frown \langle g_k, \sigma'\rangle$ or $\alpha \frown \langle d, \sigma'\rangle$ at some stage $u > t'$.

Similarly if a witness w' is attached to some edge $\alpha \frown \langle h_k, \sigma'\rangle$ lying to the right of $\alpha \frown \langle g_j, \sigma\rangle$ at some stage $t' > t$, we have that $w' > \phi_{i,2}[t](\phi_{i,1}[t](w))$ and that $w' > \phi_{i,3}[t](\phi_{i,1}[t](w))$ for all $1 \leq i \leq m$. It follows that if the strategy enumerates $\gamma_{\beta_j \frown \langle i,\sigma_j\rangle}[u](w')$ into the set D at some stage $u > t'$, we have that $\gamma_{\beta_j \frown \langle i,\sigma_j\rangle}[u](w') > \phi_{i,2}[t](\phi_{i,1}[t](w'))$ and that $\gamma_{\beta_j \frown \langle i,\sigma_j\rangle}[u](w) > \phi_{i,3}[t](\phi_{i,1}[t](w))$ as required.

On the other hand if a threshold v is chosen to define a work interval for some edge $\alpha \frown \langle g_k, \sigma'\rangle$ lying to the right of $\alpha \frown \langle g_j, \sigma\rangle$ at some stage $t' > t$, we have that v must be greater than the supremum of the rightward restraints imposed by edges lying to the left of $\alpha \frown \langle g_k, \sigma'\rangle$. This gives that $v > \phi_{i,2}[t](\phi_{i,1}[t](w))$ and that $v > \phi_{i,3}[t](\phi_{i,1}[t](w))$ for all $1 \leq i \leq m$ as required.

It follows that if the strategy enumerates $\gamma_{\beta_j \frown \langle i,\sigma_j\rangle}[u](v)$ into the set D at some stage $u > t'$, we have that $\gamma_{\beta_j \frown \langle i,\sigma_j\rangle}[u](v) > \phi_{i,2}[t](\phi_{i,1}[t](w))$ and that $\gamma_{\beta_j \frown \langle i,\sigma_j\rangle}[u](v) > \phi_{i,3}[t](\phi_{i,1}[t](w))$ as required.

Finally we consider the downward restraints imposed by an $\mathcal{S}$ strategy.

In this case we have that the strategy α will impose a downward restraint $d(\alpha \frown \langle o', \sigma' \rangle, s)$ on the edge $\alpha \frown \langle o', \sigma' \rangle$ which is visited during the current stage s as before. Recall that this restraint is the maximum of the supremum of every rightward restraint $r(\alpha \frown \langle o'', \sigma'' \rangle, s)$ imposed by edges lying to the left of $\alpha \frown \langle o', \sigma' \rangle$, the attachment restraint $a(\alpha \frown \langle o', \sigma' \rangle, s)$ and any downward restraint previously computed for the edge. Since honesty preserving constraints have been added to the calculation of the rightward restraint, these will also be reflected in the following effects caused by the downward restraint.

Suppose that some witness w has been attached at some stage t to some edge $\alpha \frown \langle o'', \sigma'' \rangle$ which lies to the left of, or is equal to $\alpha \frown \langle o', \sigma' \rangle$. Then when an $\mathcal{S}$ strategy α' lying below the edge $\alpha \frown \langle o', \sigma' \rangle$ notes that the downward restraint imposed upon it has increased at some stage $t' \geq t$, it will detach all witnesses from its edges and undefine all work intervals associated to its edges. Hence if α' attaches some witness w' to one of its edges or chooses some threshold v to define a work interval for one of its edges at some stage $u \geq t'$, it will choose w' or v to be greater than $d(\alpha \frown \langle o', \sigma' \rangle, u)$. This means that both w' and v will be greater than $\phi_{i,2}[t](\phi_{i,1}[t](w))$ and $\phi_{i,3}[t](\phi_{i,1}[t](w))$ for all $1 \leq i \leq m$ as required.

Thus if α' visits some edge of the form $\alpha' \frown \langle g_k, \sigma \rangle$ or $\alpha' \frown \langle d, \sigma \rangle$ and enumerates an attached witness w' into A at some stage $u' \geq u$, we have that the honesty of the witness w cannot be compromised. On the other hand suppose that α' visits some edge of the form $\alpha' \frown \langle h_k, \sigma \rangle$ at some stage $u' > u$ which has a work interval $(w', \gamma_{\beta_k \frown \langle i,\sigma_k\rangle}[u'](w'))$ defined. Then if α' enumerates $\gamma_{\beta_k \frown \langle i,\sigma_k\rangle}[u'](w')$ into D at stage u', we have that the honesty of the witness w cannot be compromised either. Finally suppose that α' visits some edge of the form $\alpha' \frown \langle g_k, \sigma \rangle$ at some stage $u' > u$ which has a work interval $(v, \gamma_{\beta_k \frown \langle i,\sigma_k\rangle}[u'](v))$ defined. Then if α' enumerates $\gamma_{\beta_k \frown \langle i,\sigma_k\rangle}[u'](v)$ into D, we have that the honesty of the witness w cannot be compromised once again as required.

The situation is similar for an $\mathcal{R}$ strategy β' lying below the edge $\alpha \frown \langle o', \sigma' \rangle$. If β' determines that the downward restraint has increased at some stage $t' > t$, it will detach all β'-expansionary* stages (see Definition **??**) from its edges and cancel every functional associated to one of its edges. Furthermore at any stage $u \geq t'$, the strategy β' will only choose uses which are greater than $d(\alpha \frown \langle o', \sigma' \rangle, t')$. Hence we have that any use chosen will be greater than $\phi_{i,2}[t](\phi_{i,1}[t](w))$ and $\phi_{i,3}[t](\phi_{i,1}[t](w))$ for all $1 \leq i \leq m$. It follows that if at some stage $u' > u$ the strategy β' visits some edge of the form $\beta' \frown \langle i, \sigma' \rangle$ and enumerates some use into D, we have that the honesty of the witness w cannot be compromised as required.

H-Synchronisation

The last important notion which needs to be introduced is that of *H-Synchronisation*.

Consider an $\mathcal{S}$ strategy α lying below a number of $\mathcal{R}$ strategies β_i for $1 \leq i < m$ which are active for α, and let $\beta_i \frown \langle i, \sigma_i \rangle$ be the edge of β_i lying on the path to α.

In the present arrangement α will attach a witness w to an edge of the form $\alpha \frown \langle h_j, \sigma \rangle$ at stage s in order to define a work interval $(w, \gamma_{\beta_j \frown \langle i, \sigma_j \rangle}[s](w))$ for it. Similarly α will choose a threshold v at stage s in order to define a work interval $(v, \gamma_{\beta_j \frown \langle i, \sigma_j \rangle}[s](v))$ for edges of the form $\alpha \frown \langle g_j, \sigma \rangle$.

We shall modify this scheme by having the strategy α define a *boundary* n_s for every such edge during every stage s. Initially we have that $n_s = 0$, and the boundary will keep its value until it is changed by the strategy. On the other hand if the strategy is initialised or reset during some stage s', we shall have that $n_{s'} = 0$.

If the strategy α visits the edge $\alpha \frown \langle h_j, \sigma \rangle$ at some stage u and enumerates the upper bound of its work interval $\gamma_{\beta_j \frown \langle i, \sigma_j \rangle}[u](w)$ into D, we shall increment the corresponding boundary by defining $n_{u+1} = n_u + 1$. The strategy α will also constrain the strategy β_j to henceforth H-Synchronise with every strategy β_k such that $1 \leq k < j$ by choosing uses $\gamma_{\beta_j \frown \langle i, \sigma_j \rangle}[u'](w)$ to be greater than $\gamma_{\beta_k \frown \langle i, \sigma_k \rangle}[u'](w + (n_u + 1))$ for every $1 \leq k < j$ and during all stages $u' > u$.

Similarly if the strategy α visits the edge $\alpha \frown \langle g_j, \sigma \rangle$ at some stage u and enumerates the upper bound of its work interval $\gamma_{\beta_j \frown \langle i, \sigma_j \rangle}[u](v)$ into D, we shall increment the corresponding boundary by defining $n_{u+1} = n_u + 1$. The strategy α will also constrain the strategy β_j to henceforth choose uses $\gamma_{\beta_j \frown \langle i,\sigma_j\rangle}[u'](v)$ to be greater than $\gamma_{\beta_k \frown \langle i,\sigma_k\rangle}[u'](v + (n_u + 1))$ for every $1 \leq k < j$ and during all stages $u' > u$.

The function of the boundary at stage s is to create a subinterval $(w, w + n_s)$ inside the work interval $(w, \gamma_{\beta_j \frown \langle i,\sigma_j\rangle}[s](w))$ defined for edges $\alpha \frown \langle h_j, \sigma\rangle$, and a subinterval $(v, v + n_s)$ inside the work interval $(v, \gamma_{\beta_j \frown \langle i,\sigma_j\rangle}[s](v))$ defined for edges $\alpha \frown \langle g_j, \sigma\rangle$.

These subintervals can then be used to protect the honesty of the computations given by witnesses w' which are chosen by an $\mathcal{S}$ strategy α' lying below the edge $\alpha \frown \langle h_j, \sigma\rangle$ or $\alpha \frown \langle g_j, \sigma\rangle$. This is achieved in the following way.

Firstly if α' lies below the edge $\alpha \frown \langle h_j, \sigma\rangle$ it will only attach a witness w' to one of its edges at some stage t if $w < w' < w + n_t$. Similarly if α' lies below the edge $\alpha \frown \langle g_j, \sigma\rangle$ it will only attach a witness w' to one of its edges at some stage t if $v < w' < v + n_t$.

Secondly we observe that every strategy β_k such that $1 \leq k < j$ is not only active for α, but also potentially active for α'. Therefore we need to ensure that the honesty of the computations $\Gamma^{U_k,D}_{\beta_k \frown \langle i,\sigma_k\rangle}[t'](w')$ (or $\Gamma^{V_k,D}_{\beta_k \frown \langle i,\sigma_k\rangle}[t'](w')$ resp.) is preserved for every $1 \leq k < j$ during all stages $t' > t$, unless α' is initialised or reset.

Now if α visits the edge $\alpha \frown \langle h_j, \sigma\rangle$ at some stage $t' > t$ and enumerates the upper bound of its work interval $\gamma_{\beta_j \frown \langle i,\sigma_j\rangle}[t'](w)$ into D, we shall have that $\gamma_{\beta_j \frown \langle i,\sigma_j\rangle}[t'](w) > \gamma_{\beta_k \frown \langle i,\sigma_k\rangle}[t'](w + n_{t'})$ for every $1 \leq k < j$ by H-Synchronisation. We also have that $\gamma_{\beta_k \frown \langle i,\sigma_k\rangle}[t'](w + n_{t'}) > \gamma_{\beta_k \frown \langle i,\sigma_k\rangle}[t'](w')$ for all $1 \leq k < j$ because $w < w' < w + n_t \leq w + n_{t'}$. In addition $\gamma_{\beta_k \frown \langle i,\sigma_k\rangle}[t'](w') \geq \gamma_{\beta_k \frown \langle i,\sigma_k\rangle}[t](w')$ for every $1 \leq k < j$, because the strategy β_k chooses uses which are non-decreasing with respect to stages for any given element when defining its functionals. But $\gamma_{\beta_k \frown \langle i,\sigma_k\rangle}[t](w')$ is greater than $\phi_{k,2}[t](\phi_{k,1}[t](w'))$ and $\phi_{k,3}[t](\phi_{k,1}[t](w'))$ for every $1 \leq i < j$, which means that the honesty of the aforementioned functionals cannot be compromised by the enumeration of $\gamma_{\beta_j \frown \langle i,\sigma_j\rangle}[t'](w)$ into D.

Similarly if α visits the edge $\alpha \frown \langle g_j, \sigma\rangle$ at some stage $t' > t$ it may enumerate $\gamma_{\beta_j \frown \langle i,\sigma_j\rangle}[t'](v)$ into D, or else enumerate some witness w attached to the edge into A. Since we have that $w > \gamma_{\beta_j \frown \langle i,\sigma_j\rangle}[t'](v)$ it will suffice to consider the first case only. In this case we have that $\gamma_{\beta_j\frown\langle i,\sigma_j\rangle}[t'](v) > \gamma_{\beta_k\frown\langle i,\sigma_k\rangle}[t'](v+n_{t'})$ for every $1 \leq k < j$ by H-Synchronisation. Now $\gamma_{\beta_k\frown\langle i,\sigma_k\rangle}[t'](v+n_{t'}) > \gamma_{\beta_k\frown\langle i,\sigma_k\rangle}[t'](w')$ for every $1 \leq k < j$ because $v < w' < v+n_t \leq v+n_{t'}$. In addition $\gamma_{\beta_k\frown\langle i,\sigma_k\rangle}[t'](w') \geq \gamma_{\beta_k\frown\langle i,\sigma_k\rangle}[t](w')$ for every $1 \leq k < j$, because for any given element the strategy β_k chooses uses which are non-decreasing with respect to stages. But $\gamma_{\beta_k\frown\langle i,\sigma_k\rangle}[t](w')$ is greater than $\phi_{k,2}[t](\phi_{k,1}[t](w'))$ and $\phi_{k,3}[t](\phi_{k,1}[t](w'))$ for every $1 \leq i < j$, which means that the honesty of the aforementioned functionals cannot be compromised by the enumeration of $\gamma_{\beta_j\frown\langle i,\sigma_j\rangle}[t'](v)$ into D, or by the enumeration of some witness $w > \gamma_{\beta_j\frown\langle i,\sigma_j\rangle}[t'](v)$ into A.

In addition to the above we have that the honesty of computations $\Gamma^{U_k,D}_{\beta_k\frown\langle i,\sigma_k\rangle}[t'](w)$ (or $\Gamma^{V_k,D}_{\beta_k\frown\langle i,\sigma_k\rangle}[t'](w)$ resp.) for $\mathcal{R}$ strategies β_k lying above α' but below α must also be preserved at stages $t' > t$. In this case we claim that α cannot compromise the honesty of such computations by enumerating the upper bound $b_{t'}$ of its work interval into D, nor by enumerating some witness $w > b_{t'}$ into A at some stage $t' > t$. Once again it shall suffice to consider the first case only. Since β_k chooses uses which lie inside the work interval imposed by α when defining its functionals, and since the upper bound of this work interval can only increase, we must have that $\gamma_{\beta_k\frown\langle i,\sigma_k\rangle}[t'](w') < b_{t'}$. In addition $\gamma_{\beta_k\frown\langle i,\sigma_k\rangle}[t'](w') \geq \gamma_{\beta_k\frown\langle i,\sigma_k\rangle}[t](w')$, because for any given element the strategy β_k chooses uses which are non-decreasing with respect to stages. But $\gamma_{\beta_k\frown\langle i,\sigma_k\rangle}[t](w')$ is greater than $\phi_{k,2}[t](\phi_{k,1}[t](w'))$ and $\phi_{k,3}[t](\phi_{k,1}[t](w'))$, which means that the honesty of the aformentioned functionals cannot be compromised by the enumeration of $b_{t'}$ into D or by the enumeration of some witness $w > b_{t'}$ into A at some stage $t' > t$.

We are now in a position to define the questions needed for the general $\mathcal{R}$ strategy.

3.7.3 Questions for the $\mathcal{R}$ Strategy

The $\mathcal{R}$ strategy β, will need to ask one question, which we denote by Q_1. The question asks whether the strategy β sees infinitely many β-expansionary* stages:

(1) Are there infinitely many stages $q \in \mathbb{N}_\beta$ such that the following holds?

(i) $(\forall q' < q)[\beta\text{-stage}(q') \Rightarrow l_q(\Phi_1^{U,V}, A) > l_{q'}(\Phi_1^{U,V}, A)]$.

(ii) $(\forall q' < q)[\beta\text{-stage}(q') \wedge (\forall q'' < q')[\,\beta\text{-stage}(q'') \Rightarrow l_{q'}(\Phi_1^{U,V}, A) > l_{q''}(\Phi_1^{U,V}, A)\,] \Rightarrow l_q(\Phi_2^{A,D}, U) > l_{q'}(\Phi_2^{A,D}, U)]$.

(iii) $(\forall q' < q)[\beta\text{-stage}(q') \wedge (\forall q'' < q')[\,\beta\text{-stage}(q'') \Rightarrow l_{q'}(\Phi_1^{U,V}, A) > l_{q''}(\Phi_1^{U,V}, A)\,] \Rightarrow l_q(\Phi_3^{A,D}, V) > l_{q'}(\Phi_3^{A,D}, V)]$.

(iv) $(\forall q' < q)[\beta\text{-stage}(q') \wedge (\forall q'' < q')[\,\beta\text{-stage}(q'') \Rightarrow l_{q'}(\Phi_1^{U,V}, A) > l_{q''}(\Phi_1^{U,V}, A)\,] \Rightarrow h_q(\Phi_1^{U,V}, \Phi_2^{A,D}, \Phi_3^{A,D}) > h_{q'}(\Phi_1^{U,V}, \Phi_2^{A,D}, \Phi_3^{A,D})]$.

If the strategy is accessible at some stage s, it will guess the answer to Q_1 by computing the apparent limit o and apparent use σ for $\lim_{t \to \infty} \Psi^{H_0}(f(Q_1), t)$ at stage s. If the answer corresponding to o is 'No', we denote the outcome by $\langle f, \sigma \rangle$. On the other hand, if the answer corresponding to o is 'Yes', we denote the outcome by $\langle i, \sigma \rangle$.

We now describe the $\mathcal{R}$ strategy itself.

3.7.4 The $\mathcal{R}$ Strategy

The general $\mathcal{R}$ strategy β labeled $\mathcal{R}_i$ will either be following a Γ-strategy or a $\hat{\Gamma}$-strategy, depending on its location on the priority tree. The strategy will have outcomes of the form $\langle i, \sigma \rangle$ and $\langle f, \sigma \rangle$. If it is following a Γ-strategy, it will build a different functional $\Gamma^{U_i, D}_{\beta \frown \langle i, \sigma \rangle}$ below every edge $\beta \frown \langle i, \sigma \rangle$ leaving β. Similarly if the strategy is following a $\hat{\Gamma}$-strategy, it will build a different functional $\Gamma^{V_i, D}_{\beta \frown \langle i, \sigma \rangle}$ below every edge $\beta \frown \langle i, \sigma \rangle$ leaving β. Every edge has a separate set of uses $U^{e, \beta \frown \langle i, \sigma \rangle}$, from which the strategy will choose uses when defining the functional associated to the edge $\beta \frown \langle i, \sigma \rangle$. Note that e is the index of the strategy β in the total ordering of the $\mathcal{R}$ strategies lying on the priority tree. The strategy β may also lie below a number of $\mathcal{R}$ strategies β_i for $1 \leq i \leq m$ which are active for β, with which it will need to R-Synchronise.

In order to implement fairness the strategy will need to keep track of whether it should suspend the guessing at the present stage. For this purpose we shall use a Boolean variable *suspend*, which is initialised to the value *false*. When the variable *suspend* is *false*, the strategy calculates which edge to visit as normal. When the variable *suspend* is *true*, the strategy will also calculate the edge it should visit. However if this edge lies to the right of the edge which was visited when the strategy was last accessible, the strategy will visit the latter edge instead.

The strategy goes through the following steps at stage s.

During its first step, the strategy β will calculate a rightward restraint $r(\beta \frown \langle o', \sigma' \rangle, s)$ for every edge $\beta \frown \langle o', \sigma' \rangle$ which has been previously accessible, as in the previous section.

Similarly during its second step, the strategy β will perform its attachment procedure as in the previous section, attaching the stage s to a suitable edge if s is a β-expansionary* stage.

This is followed by its third step, where the strategy will calculate the edge $\beta \frown \langle o, \sigma \rangle$ which will be visited at stage s. The strategy will start by calculating the edge $\beta \frown \langle o_\lambda, \sigma_\lambda \rangle$ which should be visited at stage s as usual, but will then consider the value of the variable *suspend*. If *suspend* is *false* the edge $\beta \frown \langle o, \sigma \rangle$ is set to $\beta \frown \langle o_\lambda, \sigma_\lambda \rangle$. On the other hand if *suspend* is *true* the strategy determines the edge $\beta \frown \langle o', \sigma' \rangle$ which the strategy has visited when it was last accessible. Then if $\beta \frown \langle o', \sigma' \rangle$ is to the left of $\beta \frown \langle o_\lambda, \sigma_\lambda \rangle$, the edge $\beta \frown \langle o, \sigma \rangle$ is set to $\beta \frown \langle o', \sigma' \rangle$. Otherwise the edge $\beta \frown \langle o, \sigma \rangle$ is set to $\beta \frown \langle o_\lambda, \sigma_\lambda \rangle$.

Once the edge $\beta \frown \langle o, \sigma \rangle$ has been determined, the strategy will perform its fourth step by calculating the attachment restraint $a(\beta \frown \langle o', \sigma' \rangle, s)$ for every edge $\beta \frown \langle o', \sigma' \rangle$ which has been previously accessible, as in the previous section. The strategy also calculates the downward restraint $d(\beta \frown \langle o, \sigma \rangle, s)$, as in the previous section.

During the final and fifth step, the strategy will take action depending on the outcome of $\beta \frown \langle o, \sigma \rangle$. This is mostly identical to the procedure found in the previous section, with the following four exceptions.

The first exception is that the value of the variable *suspend* must now be taken into consideration.

Thus in the case where the outcome of the edge is f, the strategy will set *suspend* to *true* if the present stage does not match the mode of the edge. Otherwise, the strategy will set *suspend* to *false*.

In the case where the outcome of the edge is i we have the following.

If the edge is in β-open mode but has no β-expansionary* stage attached, the strategy will set *suspend* to *false*, since the edge is not yet ready for the strategy to take action when visiting it. On the other hand if the edge has a β-expansionary* stage attached but the present stage is not a β-open stage, we set *suspend* to *true* to wait for an appropriate stage. Once the edge has a β-expansionary* stage attached and the present stage is a β-open stage, the strategy can take action and sets *suspend* to *false*, independently of whether there is a disagreement between the functional associated to the edge and the set A.

If the edge is in close mode and there is a disagreement between the functional associated to the edge and the set A, but there is no β-expansionary* stage attached to the edge, the strategy will set *suspend* to *false*, since the edge is not yet ready for the strategy to take action when visiting it. On the other hand if the edge has a β-expansionary* stage attached but the present stage is not a β-close stage, we set *suspend* to *true* to wait for an appropriate stage. Once the edge has a β-expansionary* stage attached and the present stage is a β-close stage, the strategy can take action and sets *suspend* to *false*.

Finally if the edge is in close mode and there is no disagreement between the functional associated to the edge and the set A, but the present stage is not a β-close stage, we set *suspend* to *true* to wait for an appropriate stage. Once the present stage is a β-close stage, the strategy can take action and sets *suspend* to *false*.

The second exception is that the strategy β will only define a functional $\Gamma^{U_i,D}_{\beta\frown\langle i,\sigma\rangle}$ (or $\Gamma^{V_i,D}_{\beta\frown\langle i,\sigma\rangle}$ resp.) associated to one of its edges $\beta \frown \langle i,\sigma\rangle$ at some element x at stage s if it lies within the length of honesty preservation, that is if $x \leq h[s](\Phi^{U,V}_{i,1}, \Phi^{A,D}_{i,2}, \Phi^{A,D}_{i,3})$.

The third exception is that the strategy β must now R-Synchronise with every $\mathcal{R}$ strategy β_i for $1 \leq i \leq m$. Thus if the strategy β_i has an edge $\beta_i \frown \langle i,\sigma_i\rangle$ lying on the path leading to β, and β defines the functional $\Gamma^{U_i,D}_{\beta\frown\langle i,\sigma\rangle}[s](x)$ (or $\Gamma^{V_i,D}_{\beta\frown\langle i,\sigma\rangle}[s](x)$ resp.) associated to some edge $\beta \frown \langle i,\sigma\rangle$ by choosing some use $\gamma_{\beta\frown\langle i,\sigma\rangle}[s](x)$, we must have that $\gamma_{\beta\frown\langle i,\sigma\rangle}[s](x) > \gamma_{\beta_i\frown\langle i,\sigma_i\rangle}[s](x)$ for every $1 \leq i \leq m$.

The fourth exception is that the strategy β must now obey H-Synchronisation constraints imposed by $\mathcal{S}$ strategies α lying below some edge $\beta \frown \langle i,\sigma\rangle$ of the strategy β. These constraints are imposed when α visits some edge of the form $\alpha \frown \langle g_j,\sigma\rangle$ or $\alpha \frown \langle h_j,\sigma\rangle$ which has an associated boundary n. Thus whilst defining the functional $\Gamma^{U_i,D}_{\beta\frown\langle i,\sigma\rangle}(x)[s]$ (or $\Gamma^{V_i,D}_{\beta\frown\langle i,\sigma\rangle}(x)[s]$ resp.), we have that β could be constrained to choose uses $\gamma_{\beta\frown\langle i,\sigma\rangle}[s](x) > \gamma_{\beta_k\frown\langle i,\sigma_k\rangle}[s](x+n_s)$ for every β_k

such that $1 \leq k < j$.

We shall now formalise the modified $\mathcal{R}$ strategy.

The $\mathcal{R}$ Strategy

The strategy β labeled $\mathcal{R}_i$ will either be following a Γ-strategy or a $\hat{\Gamma}$-strategy. Every edge $\beta \frown \langle i, \sigma \rangle$ has a functional $\Gamma^{U_i,D}_{\beta \frown \langle i,\sigma \rangle}$ (or $\Gamma^{V_i,D}_{\beta \frown \langle i,\sigma \rangle}$ resp.) associated to it, which the strategy will build when it visits that edge. Each edge $\beta \frown \langle i, \sigma \rangle$ will also have its own set of uses $U^{e,\beta \frown \langle i,\sigma \rangle}$ from which uses will be chosen when defining the respective functionals.

The strategy β may lie below a number of $\mathcal{R}$ strategies β'. Each such strategy β' imposes a downward restraint $d(\beta' \frown \langle o', \sigma' \rangle, s)$ on β at stage s, where $\beta' \frown \langle o', \sigma' \rangle$ is the edge of β' on the path leading to β. A number of these $\mathcal{R}$ strategies may be active for β. We denote these $\mathcal{R}$ strategies by β_i for $1 \leq i \leq m$. Similarly the corresponding edges lying on the path leading to β will be denoted by $\beta_i \frown \langle i, \sigma_i \rangle$ for every $1 \leq i \leq m$. Each of these strategies may either be following a Γ-strategy or a $\hat{\Gamma}$-strategy.

The strategy β may also lie below a number of $\mathcal{S}$ strategies α'. Each such strategy α' imposes a downward restraint $d(\alpha' \frown \langle o', \sigma' \rangle, s)$ on β at stage s, where $\alpha' \frown \langle o', \sigma' \rangle$ is the edge of α' on the path leading to β. The strategy α' also imposes the diagonalisation restraint $R_{\alpha',s}$ on β at stage s. The strategy α' may also impose a work interval on β at stage s, depending on its outcome on the path leading to β. Finally let $\alpha'' \subset \beta$ be the greatest $\mathcal{S}$ strategy (under $\subset$) which imposes a work interval on β. We shall denote the work interval imposed by α'' on β at stage s by (a_s, b_s).

Finally the strategy β has a Boolean variable *suspend* which is initialised to the value *false*.

(1) Define the rightward restraint $r(\beta \frown \langle o', \sigma' \rangle, s)$ for every edge $\beta \frown \langle o', \sigma' \rangle$ which was previously accessible as the least element x such that:

  (a) $x \geq t$ where t is some β-expansionary* stage attached to $\beta \frown \langle o', \sigma' \rangle$.

  (b) $x \geq t$ where t is the last stage at which $\beta \frown \langle o', \sigma' \rangle$ was accessible.

  Go to step (2).

(2) If stage s is a β-expansionary* stage, and there is some edge $\beta \frown \langle i, \sigma' \rangle$ which has been previously accessible and which has no β-expansionary* stage attached to it, attach s to the leftmost such edge.

If a β-expansionary* stage s has been attached to some edge $\beta \frown \langle i, \sigma' \rangle$, consider every edge $\beta \frown \langle i, \sigma'' \rangle$ lying to the right of $\beta \frown \langle i, \sigma' \rangle$. If some β-expansionary* stage s' is attached to $\beta \frown \langle i, \sigma'' \rangle$, detach the β-expansionary* stage from the edge.

Go to step (3).

(3) Determine the edge $\beta \frown \langle o, \sigma \rangle$ which is to be visited during stage s as follows. Calculate the edge $\beta \frown \langle o_\lambda, \sigma_\lambda \rangle$ which the strategy should visit at stage s, and consider the value of the variable *suspend*.

(a) If *suspend* is *true*, let $\beta \frown \langle o', \sigma' \rangle$ be the edge which was accessible when the strategy was last visited at stage t. If $\beta \frown \langle o', \sigma' \rangle$ is to the left of $\beta \frown \langle o_\lambda, \sigma_\lambda \rangle$, let $\beta \frown \langle o, \sigma \rangle = \beta \frown \langle o', \sigma' \rangle$. Otherwise, let $\beta \frown \langle o, \sigma \rangle = \beta \frown \langle o_\lambda, \sigma_\lambda \rangle$.

(b) If *suspend* is *false*, let $\beta \frown \langle o, \sigma \rangle = \beta \frown \langle o_\lambda, \sigma_\lambda \rangle$.

Go to step (4).

(4) Define the attachment procedure restraint $a(\beta \frown \langle o', \sigma' \rangle, s)$ for every edge $\beta \frown \langle o', \sigma' \rangle$ which was previously accessible. If the strategy has not attached a β expansionary* stage s to some edge $\beta \frown \langle o'', \sigma'' \rangle <_L \beta \frown \langle o', \sigma' \rangle$ at stage s, define $a(\beta \frown \langle o', \sigma' \rangle, s) = 0$. Otherwise define $a(\beta \frown \langle o', \sigma' \rangle, s) = s$.

Also define the downward restraint $d(\beta \frown \langle o, \sigma \rangle, s)$ as the least element x such that:

(a) $x \geq \sup\{r(\beta \frown \langle o', \sigma' \rangle, s) \mid \beta \frown \langle o', \sigma' \rangle <_L \beta \frown \langle o, \sigma \rangle \ \wedge$
$\beta \frown \langle o', \sigma' \rangle$ has been previously accessible$\}$.

(b) $x \geq a(\beta \frown \langle o, \sigma \rangle, s)$.

(c) $x \geq d(\beta \frown \langle o, \sigma \rangle, t)$ for all $t < s$.

Go to step (5).

(5) Consider the edge $\beta \frown \langle o, \sigma\rangle$ being visited by the strategy at stage s. Take action according to the value of o through the following case analysis.

(a) $o = f$.

(i) $\beta \frown \langle f, \sigma\rangle$ is in open mode and s is not a β-open stage. Set *suspend* to *true*. End the stage s, and go to stage $s+1$.

(ii) $\beta \frown \langle f, \sigma\rangle$ is in open mode and s is a β-open stage. Set the edge to close mode, and set *suspend* to *false*. Go to the next substage.

(iii) $\beta \frown \langle f, \sigma\rangle$ is in close mode and s is not a β-close stage. Set *suspend* to *true*. End the stage s, and go to stage $s+1$.

(iv) $\beta \frown \langle f, \sigma\rangle$ is in close mode and s is a β-close stage. Set the edge to open mode, and set *suspend* to *false*. Go to the next substage.

(b) $o = i$.

(i) $\beta \frown \langle i, \sigma\rangle$ is in open mode and there is no β-expansionary* stage attached to the edge. Set *suspend* to *false*. End the stage s, and go to stage $s+1$.

(ii) $\beta \frown \langle i, \sigma\rangle$ is in open mode and there is a β-expansionary* stage attached to the edge and s is not a β-open stage. Set *suspend* to *true*. End the stage s, and go to stage $s+1$.

(iii) $\beta \frown \langle i, \sigma\rangle$ is in open mode and there is a β-expansionary* stage attached to the edge and s is a β-open stage.

If there is some element m such that $\Gamma^{U_i,D}_{\beta\frown\langle i,\sigma\rangle}[s](m) \neq A_s(m)$, set the edge to close mode and set *suspend* to *false*. End the stage s, and go to stage $s+1$.

Otherwise, detach the stage s from the edge. Consider every $x < l_s(\Phi^{U_i,V_i}_{i,1}, A)$ such that $\Gamma^{U_i,D}_{\beta\frown\langle i,\sigma\rangle}[s](x)\uparrow$. If $x < h_s(\Phi^{U_i,V_i}_{i,1}, \Phi^{A,D}_{i,2}, \Phi^{A,D}_{i,3})$, define the axiom $\Gamma^{U_i,D}_{\beta\frown\langle i,\sigma\rangle}[s](x) = A_s(x)$.

Consider the least $u < s$ such that:

(A) $u \geq \gamma_{\beta\frown\langle i,\sigma\rangle}[t](x)$ for all $t < s$.

(B) $u > \gamma_{\beta\frown\langle i,\sigma\rangle}[s](y)$ for all $y < x$.

(C) $u > \sup\{r(\beta \frown \langle o', \sigma'\rangle, s) \mid \beta \frown \langle o', \sigma'\rangle <_L \beta \frown \langle o, \sigma\rangle\ \wedge$

$\beta \frown \langle o', \sigma' \rangle$ has been previously accessible$\}$.

(D) $u > a(\beta \frown \langle i, \sigma \rangle, s)$.

(E) $u > R_{\alpha', s}$, for every $\mathcal{S}$ strategy $\alpha' \subset \beta$.

(F) $u > d(\beta' \frown \langle o', \sigma' \rangle, s)$, for every $\mathcal{R}$ strategy $\beta' \subset \beta$ with edge $\beta' \frown \langle o', \sigma' \rangle$ on the path leading to β.

(G) $u > d(\alpha' \frown \langle o', \sigma' \rangle, s)$, for every $\mathcal{S}$ strategy $\alpha' \subset \beta$ with edge $\alpha' \frown \langle o', \sigma' \rangle$ on the path leading to β.

(H) $a_s < u < b_s$.

(I) $u \notin D$.

(J) $u > y$, where y is a constraint imposed by some $\mathcal{S}$ strategy α below β.

(K) $u > t$, where t is the last stage at which the edge $\beta \frown \langle i, \sigma \rangle$ was last initialised.

(L) $u > \gamma_{\beta_j \frown \langle i, \sigma_j \rangle}[s](x)$, for all $1 \leq j \leq m$.

(M) $u > \phi_{i,2}[s](\phi_{i,1}[s](x))$.

(N) $u > \phi_{i,3}[s](\phi_{i,1}[s](x))$.

If u does not exist, $\Gamma^{U_i, D}_{\beta \frown \langle i, \sigma \rangle}(x)$ is not defined.

Otherwise let $t' < s$ be the greatest stage such that $\Gamma^{U_i, D}_{\beta \frown \langle i, \sigma \rangle}[t'](x) \downarrow$, and let u' be the greatest use which the strategy has chosen so far when defining its functional at some element.

If t' does not exist, define $\gamma_{\beta \frown \langle i, \sigma \rangle}[s](x) = u$.

If t' exists and $u > \gamma_{\beta \frown \langle i, \sigma \rangle}[t'](x)$, define $\gamma_{\beta \frown \langle i, \sigma \rangle}[s](x)$ to be the least element in $U^{e, \beta \frown \langle i, \sigma \rangle}$ which is greater than u'.

Otherwise define $\gamma_{\beta \frown \langle i, \sigma \rangle}[s](x) = \gamma_{\beta \frown \langle i, \sigma \rangle}[t'](x)$.

Set the edge to close mode and *suspend* to *false*. Go to the next substage.

($\Gamma^{V_i, D}$ resp.).

(iv) $\beta \frown \langle i, \sigma \rangle$ is in close mode and there is an element m such that $\Gamma^{U_i, D}_{\beta \frown \langle i, \sigma \rangle}[s](m) \neq A_s(m)$ (or $\Gamma^{V_i, D}_{\beta \frown \langle i, \sigma \rangle}[s](m)$ resp.) and there is no β-expansionary* stage attached to the edge. Set *suspend* to *false*. End the stage s, and go to stage $s+1$.

(v) $\beta \frown \langle i,\sigma\rangle$ is in close mode and there is an element m such that $\Gamma^{U_i,D}_{\beta\frown\langle i,\sigma\rangle}[s](m) \neq A_s(m)$ (or $\Gamma^{V_i,D}_{\beta\frown\langle i,\sigma\rangle}[s](m)$ resp.) and there is a β-expansionary* stage attached to the edge and s is not a β-close stage. Set *suspend* to *true*. End the stage s, and go to stage $s+1$.

(vi) $\beta \frown \langle i,\sigma\rangle$ is in close mode and there is an element m such that $\Gamma^{U_i,D}_{\beta\frown\langle i,\sigma\rangle}[s](m) \neq A_s(m)$ (or $\Gamma^{V_i,D}_{\beta\frown\langle i,\sigma\rangle}[s](m)$ resp.) and there is a β-expansionary* stage attached to the edge and s is a β-close stage. Enumerate $\gamma_{\beta\frown\langle i,\sigma\rangle}[s](m)$ into D. Set the edge to open mode. Set *suspend* to *false*. End the stage s and go to stage $s+1$.

(vii) $\beta \frown \langle i,\sigma\rangle$ is in close mode and there is no element m such that $\Gamma^{U_i,D}_{\beta\frown\langle i,\sigma\rangle}[s](m) \neq A_s(m)$ (or $\Gamma^{V_i,D}_{\beta\frown\langle i,\sigma\rangle}[s](m)$ resp.) and s is not a β-close stage. Set *suspend* to *true*. End the stage s, and go to stage $s+1$.

(viii) $\beta \frown \langle i,\sigma\rangle$ is in close mode and there is no element m such that $\Gamma^{U_i,D}_{\beta\frown\langle i,\sigma\rangle}[s](m) \neq A_s(m)$ (or $\Gamma^{V_i,D}_{\beta\frown\langle i,\sigma\rangle}[s](m)$ resp.) and s is a β-close stage. Set the edge to open mode. Set *suspend* to *false* and go to the next substage.

3.7.5 Questions for the $\mathcal{S}$ Strategy

The $\mathcal{S}$ strategy α will need to ask a number of questions, which take the context of the strategy into consideration. The strategy α may lie below a number of $\mathcal{R}$ strategies β_i for $1 \leq i \leq m$ which are active for α and which can follow a Γ-strategy or a $\hat{\Gamma}$-strategy. Every strategy β_i will have an outgoing edge $\beta_i \frown \langle i,\sigma_i\rangle$, which lies on the path leading to the strategy α. In addition $\mathcal{S}$ strategies α' lying above α may impose a work interval on α. The work interval imposed at stage s by the greatest strategy α' (under $\subset$) above α is denoted by (a_s, b_s), and the boundary imposed by this strategy on α at stage s is denoted by n_s.

The strategy starts by asking question Q_1. This question asks whether there are infinitely many witnesses w and stages s such that w and $\theta_s(w)$ lie inside the work interval (a_s, b_s), w lies inside the subinterval $(a_s, a_s + n_s)$, and the computation $\Theta^D[s](w)\downarrow= 0$ holds. In addition the question also asks whether the length of agreement between the functional Θ^D and the set A expands infinitely often.

(1) Are there infinitely many $w \in W^e$, $s \in \mathbb{N}_\alpha$ and $q \in \mathbb{N}_\alpha$ such that the following hold?

(i) $\Theta^D[s](w) \downarrow= 0$.

(ii) $a_s < w < b_s$ (if a work interval is imposed on the strategy).

(iii) $a_s < \theta_s(w) < b_s$ (if a work interval is imposed on the strategy).

(iv) $a_s < w < a_s + n_s$ (if a work interval is imposed on the strategy).

(v) $(\forall q' < q)[\,\alpha\text{-stage}(q') \Rightarrow l_{q'}(\Theta^D, A) < l_q(\Theta^D, A)\,]$.

If there is some $\mathcal{R}$ strategy $\beta_i \subset \alpha$ which is active for α, we ask question $Q_{2.i}$ for every $1 \leq i \leq m$. A positive answer to question Q_1 asserts that there are infinitely many witnesses w and stages s such that w and $\theta_s(w)$ lie inside the work interval (a_s, b_s), w lies inside the subinterval (a_s, a_s+n_s) and the computation $\Theta^D[s](w) \downarrow= 0$ holds. Question $Q_{2.i}$ asks whether infinitely many of these witnesses w and stages s give rise to computations $\Gamma^{U_j,D}_{\beta_j\frown\langle i,\sigma_j\rangle}[s](w)$ (or $\Gamma^{V_j,D}_{\beta_j\frown\langle i,\sigma_j\rangle}[s](w)$ resp.) which are honest for every $1 \leq j \leq i$.

(2.i) Are there infinitely many $w \in W^e$, $s \in \mathbb{N}_\alpha$ and $q \in \mathbb{N}_\alpha$ such that the following hold?

(i) $\Theta^D[s](w) \downarrow= 0$.

(ii) $a_s < w < b_s$ (if a work interval is imposed on the strategy).

(iii) $a_s < \theta_s(w) < b_s$ (if a work interval is imposed on the strategy).

(iv) $a_s < w < a_s + n_s$ (if a work interval is imposed on the strategy).

(v) $(\forall q' < q)[\,\alpha\text{-stage}(q') \Rightarrow l_{q'}(\Theta^D, A) < l_q(\Theta^D, A)\,]$.

(vi) $(\forall\, 1 \leq j \leq i)[\phi_{j,1}[s](w) \leq \gamma_{\beta_j\frown\langle i,\sigma_j\rangle}[s](w)]$.

If there is some $\mathcal{R}$ strategy $\beta_i \subset \alpha$ which is active for α and follows a Γ-strategy, we ask question $Q_{3.i}$ for every $1 \leq i \leq m$ such that β_i follows a Γ-strategy. A positive answer to question Q_1 and questions $Q_{2.i}$ for $1 \leq i \leq m$ asserts that there are infinitely many witnesses w and stages s such that w and $\theta_s(w)$ lie inside the work interval (a_s, b_s), w lies inside the subinterval $(a_s, a_s + n_s)$ the computation $\Theta^D[s](w) \downarrow= 0$ holds, and the computations $\Gamma^{U_i,D}_{\beta_i\frown\langle i,\sigma_i\rangle}[s](w)$ (or $\Gamma^{V_i,D}_{\beta_i\frown\langle i,\sigma_i\rangle}[s](w)$ resp.) are honest for every $1 \leq i \leq m$. Question $Q_{3.i}$ asks whether infinitely many of these witnesses w enter A at stage s. In addition it asks whether a $U_j \upharpoonright \phi_{j,1}[s](w)$ change has occurred by the least $\mathcal{R}_j$-expansionary* stage $t_j > s$ for every strategy β_i with $1 \leq j \leq i$ which is following a Γ-strategy.

(3.i) Are there infinitely many $w \in W^e$, $s \in \mathbb{N}_\alpha$, $t_1, \ldots t_j \in \mathbb{N}$ and $q \in \mathbb{N}_\alpha$ such that the following hold?

(i) $\Theta^D[s](w) \downarrow = 0$.

(ii) $a_s < w < b_s$ (if a work interval is imposed on the strategy).

(iii) $a_s < \theta_s(w) < b_s$ (if a work interval is imposed on the strategy).

(iv) $a_s < w < a_s + n_s$ (if a work interval is imposed on the strategy).

(v) $(\forall q' < q)[\,\alpha\text{-stage}(q') \Rightarrow l_{q'}(\Theta^D, A) < l_q(\Theta^D, A)\,]$.

(vi) $(\forall\, 1 \leq i \leq m)[\phi_{i,1}[s](w) \leq \gamma_{\beta_i \frown \langle i,\sigma_i\rangle}[s](w)]$.

(vii) $A_s(w) = 0$.

(viii) $A_{s+1}(w) = 1$.

(ix) $(\forall\, 1 \leq j \leq i)[t_j > s]$.

(x) $(\forall\, 1 \leq j < i)[\beta_j$ follows a Γ-strategy $\Rightarrow (\forall s < t' < t_j)[U_{j,t'} \upharpoonright \phi_{j,1}[s](w) = U_{j,s} \upharpoonright \phi_{j,1}[s](w)]]$.

(xi) $(\forall\, 1 \leq j < i)[\beta_j$ follows a Γ-strategy $\Rightarrow (\forall s < t' < t_j)[V_{j,t'} \upharpoonright \phi_{j,1}[s](w) = V_{j,s} \upharpoonright \phi_{j,1}[s](w)]]$.

(xii) $(\forall\, 1 \leq j < i)[\beta_j$ follows a Γ-strategy $\Rightarrow (\forall s < t' < t_j)[U_{j,t_j} \upharpoonright \phi_{j,1}[s](w) \neq U_s \upharpoonright \phi_{j,1}[s](w)]]$.

(xiii) $(\forall s < t' < t_i)[U_{i,t'} \upharpoonright \phi_{i,1}[s](w) = U_{i,s} \upharpoonright \phi_{i,1}[s](w)]$.

(xiv) $(\forall s < t' < t_i)[V_{i,t'} \upharpoonright \phi_{i,1}[s](w) = V_{i,s} \upharpoonright \phi_{i,1}[s](w)]$.

(xv) $U_{i,t_i} \upharpoonright \phi_{i,1}[s](w) \neq U_s \upharpoonright \phi_{i,1}[s](w)$.

If the strategy is accessible at some stage s, it will guess the answer to questions Q_1, $Q_{2.i}$ for every $1 \leq i \leq m$ and to questions $Q_{3.i}$ for every $1 \leq i \leq m$ (where is applicable). This is done by computing an apparent limit o_i and an apparent use σ_i for each $\lim_{t \to \infty} \Psi^{H_0}(f(Q_i), t)$ at stage s. Let σ be the apparent use of greatest length. The outcome visited by the strategy at stage s is calculated as follows.

- If the answer corresponding to o_1 is 'No', we denote the outcome by $\langle w, \sigma \rangle$.
- If the answer corresponding to o_1 is 'Yes', there is an $\mathcal{R}$ strategy $\beta_i \subset \alpha$ which is active for α, there is some $1 \leq j \leq m$ such that the answer corresponding to $o_{2.i}$ is 'Yes' for every $1 \leq i < j$, and the answer corresponding to $o_{2.j}$ is 'No', we denote the outcome by $\langle h_j, \sigma \rangle$.
- If the answer corresponding to o_1 is 'Yes', there is an $\mathcal{R}$ strategy $\beta_i \subset \alpha$ which is active for α, the answer corresponding to $o_{2.i}$ is 'Yes' for every $1 \leq i \leq m$, there is an $\mathcal{R}$ strategy $\beta_i \subset \alpha$ which is active for α and is following a Γ-strategy, there is some $1 \leq j \leq m$ such that the answer corresponding to $o_{3.i}$ is 'Yes' for every $1 \leq i < j$, and the answer corresponding to $o_{3.j}$ is 'No', we denote the outcome by $\langle g_j, \sigma \rangle$.
- If the answer corresponding to o_1 is 'Yes', there is an $\mathcal{R}$ strategy $\beta_i \subset \alpha$ which is active for α, the answer corresponding to $o_{2.i}$ is 'Yes' for every $1 \leq i \leq m$, there is an $\mathcal{R}$ strategy $\beta_i \subset \alpha$ which is active for α and is following a Γ-strategy, and the answer corresponding to $o_{3.i}$ is 'Yes' for every $1 \leq i \leq m$ such that β_i is following a Γ-strategy, we denote the outcome by $\langle d, \sigma \rangle$.
- If the answer corresponding to o_1 is 'Yes', and there is no strategy $\beta_i \subset \alpha$ which is active for α we denote the outcome by $\langle d, \sigma \rangle$.
- If the answer corresponding to o_1 is 'Yes', there is an $\mathcal{R}$ strategy $\beta_i \subset \alpha$ which is active for α, the answer corresponding to $o_{2.i}$ is 'Yes' for every $1 \leq i \leq m$, and there is no $\mathcal{R}$ strategy $\beta_i \subset \alpha$ which is active for α and is following a Γ-strategy, we denote the outcome by $\langle d, \sigma \rangle$.

We shall now proceed to discuss the modified $\mathcal{S}$ strategy.

3.7.6 The $\mathcal{S}$ Strategy

The general $\mathcal{S}$ strategy α has an infinite set of witnesses W^e, and an infinite set of thresholds V^e, where e is the index of the strategy α in the total ordering of the $\mathcal{S}$ strategies lying on the priority tree. At any given stage s it will also be able to impose a restraint $R_{\alpha,s}$ on all lower priority strategies. Initially, we have that $R_{\alpha,0}$ is equal to 0. The strategy α shall use the fact that $R_{\alpha,s} > 0$ to signal that it has diagonalised. Once this restraint has been set, it will keep its value during subsequent stages.

The strategy α may lie below a number of $\mathcal{R}$ strategies β_i for $1 \leq i \leq m$ which are active for α and which can follow a Γ-strategy or a $\hat{\Gamma}$-strategy. In this case the strategy β_i will have an outgoing edge $\beta_i \frown \langle i, \sigma_i \rangle$, which lies on the path leading to the strategy α. The outcomes of the strategy will be of the form $\langle d, \sigma \rangle$, $\langle h_i, \sigma \rangle$ for $1 \leq i \leq m$ (assuming some $\mathcal{R}$ strategy $\beta_i \subset \alpha$ active for α exists), $\langle g_i, \sigma \rangle$ for every $1 \leq i \leq m$ such that β_i follows a Γ-strategy (assuming some $\mathcal{R}$ strategy $\beta_i \subset \alpha$ active for α and following a Γ-strategy exists) and $\langle w, \sigma \rangle$.

In order to implement fairness the strategy will need to keep track of whether it should suspend the guessing at the present stage. For this purpose we shall use a Boolean variable *suspend*, which is initialised to the value *false*. When the variable *suspend* is *false*, the strategy calculates which edge to visit as normal. When the variable *suspend* is *true*, the strategy will also calculate the edge it should visit. However if this edge lies to the right of the edge which was visited when the strategy was last accessible, the strategy will visit the latter edge instead.

The strategy goes through the following steps at stage s.

During its first step, the strategy determines whether it has enumerated some witness w' into the set A during the last stage t at which it was accessible (assuming it was accessible at least once before, and that it has not been initialised in the meantime). Suppose that this has been the case.

Then the strategy needs to determine the way in which the disagreements $\Gamma^{U_i,D}_{\beta_i \frown \langle i,\sigma_i \rangle}(w') \neq A(w')$ (or $\Gamma^{V_i,D}_{\beta_i \frown \langle i,\sigma_i \rangle}(w')$ resp.) which were introduced when the strategy enumerated w' into A at stage t were removed for all $1 \leq i \leq m$.

In order to do this, the strategy will determine whether a $U_i \upharpoonright \phi_{i,1}[s](w)$ change has occurred between stage t and the least $\mathcal{R}_i$-expansionary* stage $t_i > t$ for every $1 \leq i \leq m$. If this is the case, the strategy has diagonalised and the strategy sets $R_{\alpha,t} = \theta_t(w)$ so as to protect the use of the computation. Since the strategy has diagonalised and does not need to take further action, it will detach every witness, undefine every work interval, set every boundary to zero, set the mode of every edge to its initial mode and set suspend to false.

During its second step, the strategy α will calculate a rightward restraint $r(\alpha \frown \langle o', \sigma' \rangle, s)$ for every edge $\alpha \frown \langle o', \sigma' \rangle$ which has been previously accessible, exactly as in the previous section. If a witness w' is attached to an edge $\alpha \frown \langle d, \sigma' \rangle$ or $\alpha \frown \langle g_i, \sigma' \rangle$ for $1 \leq i \leq m$, we shall require $r(\alpha \frown \langle o', \sigma' \rangle, s)$ to be greater than or equal to $\phi_{i,2}[t](\phi_{i,1}[t](w))$ and $\phi_{i,3}[t](\phi_{i,1}[t](w))$ for $1 \leq i \leq m$ where t is the stage at which the witness was attached to the edge. This is done so as to preserve the honesty of w'.

During its third step, the strategy α will perform its attachment procedure.

If the strategy has already diagonalised ($R_{\alpha,s} > 0$), no further action needs to be taken and the attachment procedure will be terminated.

Otherwise the attachment procedure will consider in turn every witness w in W^e which at stage s yields a computation $\Theta^D[s](w) \downarrow= 0$ and which has not been attached to an edge so far. The attachment procedure will be seeking to attach one of these witnesses to an edge, and will stop considering further witnesses once this has been achieved.

In order to decide which edge the witness under consideration should be attached to, the strategy will consider the kind of outcomes possessed by the strategy, and the honesty of the computations $\Gamma^{U_i,D}_{\beta_i \frown \langle i,\sigma_i \rangle}[s](w)$ (or $\Gamma^{V_i,D}_{\beta_i \frown \langle i,\sigma_i \rangle}[s](w)$ resp.) for all $1 \leq i \leq m$.

If no $\mathcal{R}$ strategy $\beta_i \subset \alpha$ is active for α, we have that the strategy has no edges with outcomes $\langle g_i, \sigma \rangle$ or $\langle h_i, \sigma \rangle$ for some $1 \leq i \leq m$. Hence we shall attach the witness w to the leftmost edge of the form $\alpha \frown \langle d, \sigma \rangle$ which does not have a witness attached, and which obeys the usual constraints.

If there is some $\mathcal{R}$ strategy $\beta_i \subset \alpha$ which is active for α, we have that the strategy has edges with outcomes $\langle h_i, \sigma \rangle$ for some $1 \leq i \leq m$, and may also have edges with outcomes $\langle g_i, \sigma \rangle$ for some $1 \leq i \leq m$.

In this case the strategy determines whether the computations $\Gamma^{U_i,D}_{\beta_i \frown \langle i,\sigma_i \rangle}[s](w)$ (or $\Gamma^{V_i,D}_{\beta_i \frown \langle i,\sigma_i \rangle}[s](w)$ resp.) are honest for every $1 \leq i \leq m$. If this is not the case, there is some least j such that the computation $\Gamma^{U_j,D}_{\beta_j \frown \langle i,\sigma_j \rangle}[s](w)$ (or $\Gamma^{V_j,D}_{\beta_j \frown \langle i,\sigma_j \rangle}[s](w)$ resp.) is dishonest. Hence we attach the witness w to the leftmost edge of the form $\alpha \frown \langle h_j, \sigma \rangle$ which does not have a witness attached, and which obeys the usual constraints.

On the other hand if the computations $\Gamma^{U_i,D}_{\beta_i \frown \langle i,\sigma_i \rangle}[s](w)$ (or $\Gamma^{V_i,D}_{\beta_i \frown \langle i,\sigma_i \rangle}[s](w)$ resp.) are honest for every $1 \leq i \leq m$, the strategy will determine whether there is some strategy $\beta_i \subset \alpha$ active for α

which is following a Γ-strategy.

If this is not the case, the strategy does not have any edges with outcome $\langle g_i, \sigma\rangle$. Hence we attach the witness w to the leftmost edge of the form $\alpha \frown \langle d, \sigma\rangle$ which does not have a witness attached, as long as it satisfies $a_s < w < a_s + n_s$ in addition to the usual constraints.

On the other hand if there is some strategy $\beta_i \subset \alpha$ which is active for α and is following a Γ-strategy, we have that the strategy has edges with outcome $\langle g_i, \sigma\rangle$. Hence it will attach the witness w to the leftmost edge of the form $\alpha \frown \langle g_i, \sigma\rangle$ for some $1 \leq i \leq m$ where β_i is following a Γ-strategy, as long as this edge does not have a witness attached and the witness satisfies $a_s < w < a_s + n_s$ in addition to the usual constraints. In this case we do not distinguish between g_i outcomes with different values of i when deciding the edge to which the witness should be attached.

Before proceeding we remark that edges with outcomes of the form $\langle w, \sigma\rangle$, $\langle d, \sigma\rangle$ or $\langle g_i, \sigma\rangle$ will either be in open mode or in close mode, while edges with outcomes of the form $\langle h_i, \sigma\rangle$ will either be in Part I mode or in Part II mode. Edges of the first form are initially in open mode, while edges of the second form are initially in Part I mode.

This is followed by the fourth step, where the strategy will calculate the edge $\alpha \frown \langle o, \sigma\rangle$ which will be visited at stage s. The strategy will start by calculating the edge $\alpha \frown \langle o_\lambda, \sigma_\lambda\rangle$ which should be visited at stage s as usual, but will then consider the value of the variable *suspend*. If *suspend* is *false* the edge $\alpha \frown \langle o, \sigma\rangle$ is set to $\alpha \frown \langle o_\lambda, \sigma_\lambda\rangle$. On the other hand if *suspend* is *true* the strategy determines the edge $\alpha \frown \langle o', \sigma'\rangle$ which the strategy has visited when it was last accessible. Then if $\alpha \frown \langle o', \sigma'\rangle$ is to the left of $\alpha \frown \langle o_\lambda, \sigma_\lambda\rangle$, the edge $\alpha \frown \langle o, \sigma\rangle$ is set to $\alpha \frown \langle o', \sigma'\rangle$. Otherwise the edge $\alpha \frown \langle o, \sigma\rangle$ is set to $\alpha \frown \langle o_\lambda, \sigma_\lambda\rangle$.

This is followed by its fifth step, where the strategy α will calculate its attachment restraint and downward restraint as in the previous section. If a witness w' has been attached to some edge with outcome $\langle g_i, \sigma'\rangle$ or $\langle d, \sigma'\rangle$ at stage s, and $\alpha \frown \langle o'', \sigma''\rangle$ lies to the right of this edge, we shall require $a(\alpha \frown \langle o'', \sigma''\rangle)$ to be greater than or equal to $\phi_{i,2}[t](\phi_{i,1}[t](w))$ and $\phi_{i,3}[t](\phi_{i,1}[t](w))$ for $1 \leq i \leq m$, where t is the stage at which the witness was attached to the edge. This is done so as to preserve the honesty of w'. On the other hand the downward restraint implicitly contains the required honesty preservation constraints and no further constraints need to be imposed in this case.

During the final and sixth step, the strategy takes action depending on the the outcome of the edge $\alpha \frown \langle o, \sigma \rangle$ being visited during stage s.

We start by considering the case where the $\mathcal{S}$ strategy α lies below some $\mathcal{R}$ strategy β_i which is active for α and which is following a Γ-strategy. In this case, the strategy α has edges with outcomes $\langle h_i, \sigma \rangle$ for $1 \leq i \leq m$ and $\langle g_i, \sigma \rangle$ for $1 \leq i \leq m$ such that β_i is following a Γ-strategy.

Suppose that the strategy visits an edge with outcome $\langle w, \sigma \rangle$ and that the edge is in open mode. If the present stage is not an α-open stage, we terminate the stage and set suspend to true so as to wait for an α-open stage. Otherwise the strategy will count visiting the edge as having taken action successfully, changing the mode of the edge back to close mode, setting suspend to false and going to the next substage.

On the other hand, suppose that the outcome is $\langle w, \sigma \rangle$ and the edge is in close mode. If the present stage is not an α-close stage, we terminate the stage and set suspend to true so as to wait for an α-close stage. Otherwise the strategy will count visiting the edge as having taken action successfully, changing the mode of the edge back to open mode, setting suspend to false and going to the next substage.

Suppose that the strategy visits an edge with outcome $\langle g_i, \sigma \rangle$. If the strategy has diagonalised as a result of enumerating some witness w' into A at some prior stage, the stage is terminated. Otherwise we have that the edge is either in open mode or in close mode.

If the edge in open mode, the strategy will first determine whether a work interval for the edge is defined. If this is not the case, the strategy will choose a threshold v so as to define a work interval $(v, \gamma_{\beta_i \frown \langle i, \sigma_i \rangle}[s](v))$ for the edge. This threshold has to obey certain constraints as detailed in the previous section. The variable suspend is set to false because the strategy is not yet ready to take action when visiting the edge.

Once a work interval is defined for the edge, the strategy will determine whether a witness is attached to the edge. If this is not the case, the strategy will terminate the stage and wait for a witness w giving honest computations $\Gamma^{U_i, D}_{\beta_i \frown \langle i, \sigma_i \rangle}[s](w)$ (or $\Gamma^{V_i, D}_{\beta_i \frown \langle i, \sigma_i \rangle}[s](w)$ resp.) to be attached.

The variable suspend is set to false because the strategy is not yet ready to take action when visiting the edge.

If a work interval is defined for the edge and a witness w has been attached to the edge, the strategy will determine whether the witness is still honest, that is whether $\phi_{i,1}[s](w) \leq \gamma_{\beta_i \frown \langle i,\sigma_i \rangle}(w)[s]$ for all $1 \leq i \leq m$. If this is no longer the case, the witness is detached from the edge. The variable suspend is set to false because the strategy is not yet ready to take action when visiting the edge.

If a work interval is defined for the edge, a witness has been attached to the edge and the witness gives honest computations, the strategy determines whether the present stage is an α-open stage. If this is not the case, the strategy will terminate the stage, set suspend to true and wait for an α-open stage. If the strategy visits the edge, a work interval has been defined, a witness w has been attached, the witness gives honest computations, and the present stage is an α-open stage, the strategy can finally take action and open a gap by enumerating the witness w into the set A. Since the strategy has taken action successfully, it changes the mode of the edge to close mode, sets suspend to false and goes to the next substage.

If the edge is in close mode, the strategy will determine whether the present stage is an α-close stage. If this is not the case, the strategy will terminate the stage, set suspend to true and wait for an α-close stage. If the strategy visits the edge and the present stage is an α-close stage, the strategy will perform capricious destruction for $\Gamma^{U_i,D}_{\beta_i \frown \langle i,\sigma_i \rangle}$ by enumerating the upper bound of the work interval of the edge $\gamma_{\beta_i \frown \langle i,\sigma_i \rangle}[s](v)$ into the set D. The strategy constrains β_i to choose uses $\gamma_{\beta_i \frown \langle i,\sigma_i \rangle}[t](v) > \gamma_{\beta_j \frown \langle i,\sigma_j \rangle}[t](v + n'_s)$ for all $1 \leq j < i$ and $t > s$, where n'_s is the boundary of the work interval at stage s, and also increments the boundary by one. Since the strategy has taken action successfully, it changes the mode of the edge to open mode, sets suspend to false and goes to the next substage.

Suppose now that the strategy visits an edge with outcome $\langle h_i, \sigma \rangle$. If the strategy has diagonalised as a result of enumerating some witness w' into A at some prior stage, the stage is terminated. Otherwise we have that the edge is either in Part I mode or in Part II mode.

If the edge in Part I mode, the strategy will determine whether a witness is attached to the edge. If this is not the case, the strategy will terminate the stage and wait for a witness to be attached.

The variable suspend is set to false as the strategy is not yet ready to take action when visiting the edge. If a witness w has been attached to the edge at the present stage, the work interval $(w, \gamma_{\beta' \frown \langle i,\sigma' \rangle}[s](w))$ is defined for the edge and the stage is terminated. The variable suspend is set to false as the strategy is not yet ready to take action when visiting the edge. If a witness w is attached to the edge and the work interval is defined for the edge, the strategy will determine whether the present stage is an α-close stage. If this is not the case, the strategy will terminate the stage, set suspend to true and wait for an α-close stage.

Otherwise, the strategy will determine whether the witness w is attached to the edge still gives a dishonest computation $\Gamma^{U_i,D}_{\beta_i \frown \langle i,\sigma_i \rangle}[s](w)$ (or $\Gamma^{V_i,D}_{\beta_i \frown \langle i,\sigma_i \rangle}[s](w)$ resp.) at the present stage s. If this is not the case, the strategy will terminate the stage and wait for a stage until the computation becomes dishonest again. Suspend is set to false as the strategy is not yet ready to take action.

If the strategy visits the edge and a witness which has been previously attached is still attached, the present stage is an α-close stage, and the witness w gives a dishonest computation $\Gamma^{U_i,D}_{\beta_i \frown \langle i,\sigma_i \rangle}[s](w)$ (or $\Gamma^{V_i,D}_{\beta_i \frown \langle i,\sigma_i \rangle}[s](w)$ resp.), the strategy will perform honestification for $\Gamma^{U_i,D}_{\beta_i \frown \langle i,\sigma_i \rangle}$ (or $\Gamma^{V_i,D}_{\beta_i \frown \langle i,\sigma_i \rangle}$ resp.), by enumerating the upper bound $\gamma_{\beta_i \frown \langle i,\sigma_i \rangle}[s](w)$ of the work interval defined for the edge into the set D. The strategy constrains β_i to choose uses $\gamma_{\beta_i \frown \langle i,\sigma_i \rangle}[t](w) > \gamma_{\beta_j \frown \langle i,\sigma_j \rangle}[t](w + n'_s)$ for every $1 \leq j < i$ and $t > s$, where n'_s is the boundary of the work interval at stage s. The strategy also increments the boundary of the work interval by one. Since the strategy has taken action successfully, it changes the mode of the edge to Part II mode, sets suspend to false and goes to the next substage.

If the edge is in Part II mode, the strategy will determine whether the present stage is an α-open stage. If this is not the case, the strategy will terminate the stage, set suspend to true and wait for an α-open stage. Otherwise, the strategy will take no action. This will count as the strategy having taken action successfully. The strategy thus changes the mode of the edge back to Part I mode, sets suspend to false and goes to the next substage.

Finally suppose that the strategy visits an edge with outcome $\langle d, \sigma \rangle$. The strategy first determines whether it has already diagonalised, and in this case terminates the stage.

Otherwise, we have to consider three cases.

If there is no $\mathcal{R}$ strategy $\beta_i \subset \alpha$ which is active for α, we have that witnesses are attached to the edges of the form $\alpha \frown \langle d, \sigma \rangle$, and that they can be enumerated into A when such edges are visited. Thus if the edge has a witness attached and is in open mode, but the present stage is not an α-open stage, the strategy will terminate the stage, set suspend to true and wait for an α-open stage. If the edge has a witness attached and is in open mode and the present stage is an α-open stage, it will enumerate this witness into A, set suspend to false and terminate the stage.

If there is some $\mathcal{R}$ strategy $\beta_i \subset \alpha$ which is active for α but no such strategy is following a $\hat{\Gamma}$-strategy we have that witnesses are attached to the edges of the form $\alpha \frown \langle d, \sigma \rangle$, and that they can be enumerated into A when the edge is visited at stage s, this time on condition that they give honest computations $\Gamma^{U_i,D}_{\beta_i \frown \langle i, \sigma_i \rangle}[s](w)$. Thus if the edge has a witness giving honest computations attached and is in open mode, but the present stage is not an α-open stage, it will terminate the stage, set suspend to true and wait for an α-open stage. If the edge has a witness giving honest computations attached and is in open mode and the present stage is an α-open stage, it will enumerate this witness into A, set suspend to false and terminate the stage.

Finally, if there is some strategy $\beta_i \subset \alpha$ which is active for α and which is following a Γ-strategy we have that the strategy α has edges of the form $\alpha \frown \langle g_i, \sigma \rangle$ for some $1 \leq i \leq m$, and that witnesses are attached to these edges, and not to edges of the form $\alpha \frown \langle d, \sigma \rangle$. Thus there is no action to take when the strategy visits an edge of the form $\alpha \frown \langle d, \sigma \rangle$, and the stage will be terminated when this happens.

The $\mathcal{S}$ Strategy

The strategy α has a set of witnesses W^e and a set of thresholds V^e, and at every stage s is able to impose a restraint $R_{\alpha,s}$ on lower priority strategies. Initially we have that $R_{\alpha,0} = 0$, and if the strategy sets $R_{\alpha,s} > 0$ during some stage s, the restraint will maintain this value unless the strategy has been initialised.

The strategy α may lie below a number of $\mathcal{R}$ strategies β'. Each such strategy β' imposes a downward restraint $d(\beta' \frown \langle o', \sigma' \rangle, s)$ on α at stage s, where $\beta' \frown \langle o', \sigma' \rangle$ is the edge of β' on the path leading to α. A number of these $\mathcal{R}$ strategies may be active for β. We denote these $\mathcal{R}$ strategies by β_i for $1 \le i \le m$. Similarly the corresponding edges lying on the path leading to β will be denoted by $\beta_i \frown \langle i, \sigma_i \rangle$ for $1 \le i \le m$. Each of these strategies may either be following a Γ-strategy or a $\hat{\Gamma}$-strategy.

The strategy α may also lie below a number of $\mathcal{S}$ strategies α'. Each such strategy α' imposes a downward restraint $d(\alpha' \frown \langle o', \sigma' \rangle, s)$ on α at stage s, where $\alpha' \frown \langle o', \sigma' \rangle$ is the edge of α' on the path leading to α. The strategy α' also imposes the diagonalisation restraint $R_{\alpha',s}$ on α at stage s. Finally the strategy α' may also impose a work interval on α at stage s, depending on its outcome on the path leading to α. The work interval imposed at stage s by the greatest strategy α' (under $\subset$) which lies above α and which does impose a work interval is denoted by (a_s, b_s). Since these work intervals are nested, it will be sufficient for α to observe this work interval during the course of its computation.

Finally the strategy α has a Boolean variable *suspend* which is initialised to the value *false*.

(1) Consider the last stage t at which α was accessible. If t does not exist, or the strategy α has been initialised at some stage t' such that $t < t' < s$, go to step (2).

If t exists, has α enumerated some witness w into A at stage t?

(a) (No) Go to step (2).

(b) (Yes) Is it the case that (i) there is no $\mathcal{R}$ strategy $\beta_i \subset \alpha$ which is active for α, or that (ii) for every $\mathcal{R}$ strategy $\beta_i \subset \alpha$ which is active for α, and is following a Γ-strategy, where $1 \le i \le m$ we have that $U_t \restriction \phi_{i,1}[t](w) \neq U_{t_i} \restriction \phi_{i,1}[t](w)$, where t_i be the least $\mathcal{R}_i$-expansionary stage greater than t?

(i) (No) Go to step (2).

(ii) (Yes) Set the restraint $R_{\alpha,s}$ to $\theta_t(w)$. Consider every edge $\alpha \frown \langle o', \sigma' \rangle$ of α which has already been accessed. If a work interval is defined for $\alpha \frown \langle o', \sigma' \rangle$, cancel the work interval and reset the boundary to 0. If some witness is attached to $\alpha \frown \langle o', \sigma' \rangle$, detach the witness. If o' is equal to d, w or g_i for some $1 \le i \le m$, set the edge to open mode. If o' is equal to h_i for some $1 \le i \le m$, set the edge to Part I mode. Set *suspend* to *false*. Go to step (2).

(2) Define the rightward restraint $r(\alpha \frown \langle o', \sigma' \rangle, s)$ for every edge $\alpha \frown \langle o', \sigma' \rangle$ which was previously accessible as the least element x such that:

(a) $x \geq \theta_t(w)$, where w is a witness attached to $\alpha \frown \langle o', \sigma' \rangle$ and t is the stage at which the witness was attached.

(b) $x \geq t$, where t is the last stage at which $\alpha \frown \langle o', \sigma' \rangle$ was last accessible.

(c) $x \geq \phi_{i,2}[t](\phi_{i,1}[t](w))$ for all $1 \leq i \leq m$, where w is a witness attached to $\alpha \frown \langle o', \sigma' \rangle$ and t is the stage at which the witness was attached.

(d) $x \geq \phi_{i,3}[t](\phi_{i,1}[t](w))$ for all $1 \leq i \leq m$, where w is a witness attached to $\alpha \frown \langle o', \sigma' \rangle$ and t is the stage at which the witness was attached.

Go to step (3).

(3) Consider the finite set of witnesses w in W^e such that $w < s$ and $\Theta^D[s](w) \downarrow= 0$ and such that w has not been attached to an edge at some stage $u < s$. Perform the following case analysis for every such witness in turn (under the order $<$), until one witness is attached successfully to an edge or until no more witnesses are available.

(a) Suppose that $R_{\alpha,s} > 0$. End stage s, and go to stage $s + 1$.

(b) Suppose that $R_{\alpha,s} = 0$ and that no $\mathcal{R}$ strategy $\beta_i \subset \alpha$ is active for α. If there is an edge $\alpha \frown \langle d, \sigma' \rangle$ such that:

(i) $\alpha \frown \langle d, \sigma' \rangle$ has been accessible during a previous stage.

(ii) $\alpha \frown \langle d, \sigma' \rangle$ has no witness attached to it.

(iii) $w > \sup\{r(\alpha \frown \langle o'', \sigma'' \rangle, s) \mid \alpha \frown \langle o'', \sigma'' \rangle <_L \alpha \frown \langle d, \sigma' \rangle \ \wedge$ $\alpha \frown \langle o'', \sigma'' \rangle$ has been previously accessible$\}$.

(iv) $w > R_{\alpha',s}$ for every $\mathcal{S}$ strategy $\alpha' \subset \alpha$.

(v) $w > d(\beta' \frown \langle o', \sigma' \rangle, s)$, for every $\mathcal{R}$ strategy $\beta' \subset \alpha$ with edge $\beta' \frown \langle o', \sigma' \rangle$ on the path leading to α.

(vi) $w > d(\alpha' \frown \langle o', \sigma' \rangle, s)$, for every $\mathcal{S}$ strategy $\alpha' \subset \alpha$ with edge $\alpha' \frown \langle o', \sigma' \rangle$ on the path leading to α.

(vii) $a_s < w < b_s$.

(viii) $a_s < \theta_s(w) < b_s$.

(ix) w is greater than the upper bound of the work interval at stage s defined for any edge $\alpha \frown \langle o'', \sigma'' \rangle$ which was previously accessible and which lies to the left of $\alpha \frown \langle d, \sigma' \rangle$.

(x) $w > t$, where t is the last stage at which the edge $\alpha \frown \langle d, \sigma' \rangle$ was initialised.

(xi) $w > w'$, where w' is any witness which has been attached to $\alpha \frown \langle d, \sigma' \rangle$ at some stage $t < s$.

Then attach w to the leftmost such $\alpha \frown \langle d, \sigma' \rangle$.

(c) Suppose that $R_{\alpha,s} = 0$ and that there is some least (under $\subset$) $\mathcal{R}$ strategy $\beta_i \subset \alpha$ which is active for α such that $\phi_{i,1}[s](w) > \gamma_{\beta_i \frown \langle i, \sigma_i \rangle}[s](w)$, where $1 \leq i \leq m$. If there is an edge $\alpha \frown \langle h_i, \sigma' \rangle$ such that:

(i) $\alpha \frown \langle h_i, \sigma' \rangle$ has been accessible during a previous stage.

(ii) $\alpha \frown \langle h_i, \sigma' \rangle$ has no witness attached to it.

(iii) $w > \sup\{r(\alpha \frown \langle o'', \sigma'' \rangle, s) \mid \alpha \frown \langle o'', \sigma'' \rangle <_L \alpha \frown \langle h_i, \sigma' \rangle \ \wedge$ $\alpha \frown \langle o'', \sigma'' \rangle$ has been previously accessible$\}$.

(iv) $w > R_{\alpha',s}$ for every $\mathcal{S}$ strategy $\alpha' \subset \alpha$.

(v) $w > d(\beta' \frown \langle o', \sigma' \rangle, s)$, for every $\mathcal{R}$ strategy $\beta' \subset \alpha$ with edge $\beta' \frown \langle o', \sigma' \rangle$ on the path leading to α.

(vi) $w > d(\alpha' \frown \langle o', \sigma' \rangle, s)$, for every $\mathcal{S}$ strategy $\alpha' \subset \alpha$ with edge $\alpha' \frown \langle o', \sigma' \rangle$ on the path leading to α.

(vii) $a_s < w < b_s$.

(viii) $a_s < \theta_s(w) < b_s$.

(ix) w is greater than the upper bound of the work interval at stage s defined for any edge $\alpha \frown \langle o'', \sigma'' \rangle$ which was previously accessible and which lies to the left of $\alpha \frown \langle h_i, \sigma' \rangle$.

(x) $w > t$, where t is the last stage at which the edge $\alpha \frown \langle h_i, \sigma' \rangle$ was initialised.

(xi) $w > w'$, where w' is any witness which has been attached to this edge at some stage $t < s$.

Consider the leftmost such edge $\alpha \frown \langle h_j, \sigma' \rangle$ and define the work interval of the edge to be $(w, \gamma_{\beta_j \frown \langle i,\sigma_j \rangle}[s](w))$ and its boundary to be 0.

(d) Suppose that $R_{\alpha,s} = 0$ and that there is some $\mathcal{R}$ strategy which is active for α and is following a Γ-strategy, and that $\phi_{i,1}[s](w) \leq \gamma_{\beta_i \frown \langle i,\sigma_i \rangle}[s](w)$ for every $\mathcal{R}$ strategy $\beta_i \subset \alpha$ which is active for α, where $1 \leq i \leq m$. If there is an edge $\alpha \frown \langle g_j, \sigma' \rangle$ with $1 \leq j \leq m$ such that:

(i) $\alpha \frown \langle g_j, \sigma' \rangle$ has been accessible during a previous stage.

(ii) $\alpha \frown \langle g_j, \sigma' \rangle$ has no witness attached to it.

(iii) $w > \sup\{r(\alpha \frown \langle o'', \sigma'' \rangle, s) \mid \alpha \frown \langle o'', \sigma'' \rangle <_L \alpha \frown \langle g_j, \sigma' \rangle \ \wedge$ $\alpha \frown \langle o'', \sigma'' \rangle$ has been previously accessible$\}$.

(iv) $w > R_{\alpha',s}$ for every $\mathcal{S}$ strategy $\alpha' \subset \alpha$.

(v) $w > d(\beta' \frown \langle o', \sigma' \rangle, s)$, for every $\mathcal{R}$ strategy $\beta' \subset \alpha$ with edge $\beta' \frown \langle o', \sigma' \rangle$ on the path leading to α.

(vi) $w > d(\alpha' \frown \langle o', \sigma' \rangle, s)$, for every $\mathcal{S}$ strategy $\alpha' \subset \alpha$ with edge $\alpha' \frown \langle o', \sigma' \rangle$ on the path leading to α.

(vii) $a_s < w < b_s$.

(viii) $a_s < \theta_s(w) < b_s$.

(ix) $a_s < w < a_s + n_s$.

(x) $\alpha \frown \langle g_j, \sigma' \rangle$ is in open mode.

(xi) The work interval of the edge $\alpha \frown \langle g_j, \sigma' \rangle$ is defined.

(xii) w is greater than the upper bound of the work interval for the edge $\alpha \frown \langle g_j, \sigma' \rangle$.

(xiii) $w > t$, where t is the last stage at which the edge $\alpha \frown \langle g_j, \sigma' \rangle$ was initialised.

(xiv) $w > w'$, where w' is any witness which has been attached to this edge at some stage $t < s$.

Then attach w to the leftmost such $\alpha \frown \langle g_j, \sigma' \rangle$.

(e) Suppose that $R_{\alpha,s} = 0$ and that there is some $\mathcal{R}$ strategy which is active for α, and that every such strategy is following a $\hat{\Gamma}$-strategy, and that $\phi_{i,1}[s](w) \leq \gamma_{\beta_i \frown \langle i,\sigma_i \rangle}[s](w)$ for every $\mathcal{R}$ strategy $\beta_i \subset \alpha$ which is active for α, where $1 \leq i \leq m$. If there is an edge $\alpha \frown \langle d, \sigma' \rangle$ such that:

(i) $\alpha \frown \langle d, \sigma' \rangle$ has been accessible during a previous stage.

(ii) $\alpha \frown \langle d, \sigma' \rangle$ has no witness attached to it.

(iii) $w > \sup\{r(\alpha \frown \langle o'', \sigma'' \rangle, s) \mid \alpha \frown \langle o'', \sigma'' \rangle <_L \alpha \frown \langle d, \sigma' \rangle \ \wedge$ $\alpha \frown \langle o'', \sigma'' \rangle$ has been previously accessible$\}$.

(iv) $w > R_{\alpha', s}$ for every $\mathcal{S}$ strategy $\alpha' \subset \alpha$.

(v) $w > d(\beta' \frown \langle o', \sigma' \rangle, s)$, for every $\mathcal{R}$ strategy $\beta' \subset \alpha$ with edge $\beta' \frown \langle o', \sigma' \rangle$ on the path leading to α.

(vi) $w > d(\alpha' \frown \langle o', \sigma' \rangle, s)$, for every $\mathcal{S}$ strategy $\alpha' \subset \alpha$ with edge $\alpha' \frown \langle o', \sigma' \rangle$ on the path leading to α.

(vii) $a_s < w < b_s$.

(viii) $a_s < \theta_s(w) < b_s$.

(ix) $a_s < w < a_s + n_s$.

(x) w is greater than the upper bound of the work interval at stage s defined for any edge $\alpha \frown \langle o'', \sigma'' \rangle$ which was previously accessible and which lies to the left of $\alpha \frown \langle d, \sigma' \rangle$.

(xi) $w > t$, where t is the last stage at which the edge $\alpha \frown \langle d, \sigma' \rangle$ was initialised.

(xii) $w > w'$, where w' is any witness which has been attached to this edge at some stage $t < s$.

Then attach w to the leftmost such $\alpha \frown \langle d, \sigma' \rangle$.

If a witness w has been attached to some edge $\alpha \frown \langle o', \sigma' \rangle$, consider every edge $\alpha \frown \langle o'', \sigma'' \rangle$ lying to the right of $\alpha \frown \langle o', \sigma' \rangle$. If some witness w' is attached to $\alpha \frown \langle o'', \sigma'' \rangle$, detach the witness from the edge. If some work interval is defined for $\alpha \frown \langle o'', \sigma'' \rangle$, undefine the work interval of the edge and reset the boundary to 0. If o'' is equal to d, w or g_i for some $1 \leq i \leq m$, set the edge to open mode. If o'' is equal to h_i for some $1 \leq i \leq m$, set the edge to Part I mode.

Go to step (4).

(4) Consider the value of the variable *suspend*.

(a) If *suspend* is *true*, let $\beta \frown \langle o', \sigma' \rangle$ be the edge which was accessible when the strategy was last visited at stage t. Determine the edge $\beta \frown \langle o, \sigma \rangle$ which the strategy should visit at stage s. If $\beta \frown \langle o', \sigma' \rangle$ is to the left of $\beta \frown \langle o, \sigma \rangle$, let $\beta \frown \langle o, \sigma \rangle = \beta \frown \langle o', \sigma' \rangle$.

(b) If *suspend* is *false*, determine the edge $\beta \frown \langle o, \sigma \rangle$ which the strategy should visit at stage s.

Go to step (5).

(5) Define the attachment procedure restraint $a(\alpha \frown \langle o', \sigma' \rangle, s)$ for every edge $\alpha \frown \langle o', \sigma' \rangle$ which was previously accessible. If the strategy has not attached a witness w to some edge $\alpha \frown \langle o'', \sigma'' \rangle <_L \alpha \frown \langle o', \sigma' \rangle$ at stage s, define $a(\alpha \frown \langle o', \sigma' \rangle, s) = 0$. Otherwise define $a(\alpha \frown \langle o', \sigma' \rangle, s)$ as the least element x such that:

(a) $x \geq \theta_s(w)$.

(b) $x \geq \phi_{i,2}[s](\phi_{i,1}[s](w))$ for all $1 \leq i \leq m$.

(c) $x \geq \phi_{i,3}[s](\phi_{i,1}[s](w))$ for all $1 \leq i \leq m$.

Also define the downward restraint $d(\alpha \frown \langle o, \sigma \rangle, s)$ as the least element x such that:

(a) $x \geq \sup\{r(\alpha \frown \langle o', \sigma' \rangle, s) \mid \alpha \frown \langle o', \sigma' \rangle <_L \alpha \frown \langle o, \sigma \rangle \ \wedge$ $\alpha \frown \langle o', \sigma' \rangle$ has been previously accessible$\}$.

(b) $x \geq a(\alpha \frown \langle o, \sigma \rangle, s)$.

(c) $x \geq d(\alpha \frown \langle o, \sigma \rangle, t)$ for all $t < s$.

Go to step (6).

(6) Take action according to the outcome of $\alpha \frown \langle o, \sigma \rangle$.

(a) $o = w$.

(i) Suppose that $\alpha \frown \langle w, \sigma \rangle$ is in open mode, but s is not an open stage. Set *suspend* to *true*. End the stage s, and go to stage $s + 1$.

(ii) Suppose that $\alpha \frown \langle w, \sigma \rangle$ is in open mode, and s is an open stage. Set the edge to close mode, and set *suspend* to *false*. Go to the next substage.

(iii) Suppose that $\alpha \frown \langle w, \sigma \rangle$ is in close mode, but s is not an α-close stage. Set *suspend* to *true*. End the stage s, and go to stage $s+1$.

(iv) Suppose that $\alpha \frown \langle w, \sigma \rangle$ is in close mode, and s is an α-close stage. Set the edge to open mode, and set *suspend* to *false*. Go to the next substage.

(b) $o = g_j$, for some $1 \leq j \leq m$.

(i) Suppose that $R_{\alpha,s} > 0$. Set *suspend* to *false*. End stage s, and go to stage $s+1$.

(ii) Suppose that $R_{\alpha,s} = 0$, and the work interval for the edge $\alpha \frown \langle g_j, \sigma \rangle$ is undefined. If there is some least threshold $v < s$ in V^e such that:

(A) $v > \sup\{r(\alpha \frown \langle o', \sigma' \rangle, s) \mid \alpha \frown \langle o', \sigma' \rangle <_L \alpha \frown \langle g_j, \sigma \rangle \ \wedge$ $\alpha \frown \langle o', \sigma' \rangle$ has been previously accessible$\}$.

(B) $v > a(\alpha \frown \langle g_j, \sigma \rangle, s)$.

(C) $v > R_{\alpha',s}$ for every $\mathcal{S}$ strategy $\alpha' \subset \alpha$.

(D) $v > d(\beta' \frown \langle o', \sigma' \rangle, s)$, for every $\mathcal{R}$ strategy $\beta' \subset \alpha$ with edge $\beta' \frown \langle o', \sigma' \rangle$ on the path leading to α.

(E) $v > d(\alpha' \frown \langle o', \sigma' \rangle, s)$, for every $\mathcal{S}$ strategy $\alpha' \subset \alpha$ with edge $\alpha' \frown \langle o', \sigma' \rangle$ on the path leading to α.

(F) $a_s < v < b_s$.

(G) v is greater than the upper bound of the work interval at stage s defined for any edge $\alpha \frown \langle o', \sigma' \rangle$ which was previously accessible and which lies to the left of $\alpha \frown \langle g_j, \sigma \rangle$.

(H) $v > t$, where t is the stage at which the edge $\alpha \frown \langle g_j, \sigma \rangle$ was last initialised.

Define the work interval of the edge $\alpha \frown \langle g_j, \sigma \rangle$ to be $(v, \gamma_{\beta_j \frown \langle i, \sigma_j \rangle}[s](v))$ and the boundary to be 0. Set *suspend* to *false*. End stage s, and go to stage $s+1$.

(iii) Suppose that $R_{\alpha,s} = 0$, a work interval is defined for the edge $\langle g_j, \sigma \rangle$ and the edge is in open mode, but no witness w is attached to the edge. Set *suspend* to *false*. End stage s, and go to stage $s+1$.

(iv) Suppose that $R_{\alpha,s} = 0$, a work interval is defined for the edge $\langle g_j, \sigma \rangle$, the edge is in open mode and a witness w is attached to the edge, but $\phi_{i,1}[s](w) > \gamma_{\beta_i \frown \langle i,\sigma_i \rangle}[s](w)$ for some $\beta_i \subset \alpha$ which is active for α, where $1 \leq i \leq m$. Detach w from $\alpha \frown \langle g_j, \sigma \rangle$. Set *suspend* to *false*. End stage s, and go to stage $s+1$.

(v) Suppose that $R_{\alpha,s} = 0$, a work interval is defined for the edge $\langle g_j, \sigma \rangle$, the edge is in open mode, a witness w is attached to the edge, and $\phi_{i,1}[s](w) \leq \gamma_{\beta_i \frown \langle i,\sigma_i \rangle}[s](w)$for every $\beta_i \subset \alpha$ which is active for α, where $1 \leq i \leq m$, but s is not an open stage. Set *suspend* to *true*. End stage s, and go to stage $s+1$.

(vi) Suppose that $R_{\alpha,s} = 0$, a work interval is defined for the edge $\langle g_j, \sigma \rangle$, the edge is in open mode, a witness w is attached to the edge, $\phi_{i,1}[s](w) \leq \gamma_{\beta_i \frown \langle i,\sigma_i \rangle}[s](w)$ for every $\beta_i \subset \alpha$ which is active for α, where $1 \leq i \leq m$ and s is an open-stage. Enumerate w into A. Set the edge $\langle g_j, \sigma \rangle$ to close mode. Set *suspend* to *false*. Go to the next substage.

(vii) Suppose that $R_{\alpha,s} = 0$, a work interval is defined for the edge $\langle g_j, \sigma \rangle$, the edge is in close mode and s is not an α-close stage. Set *suspend* to *true*. End stage s, and go to stage $s+1$.

(viii) Suppose that $R_{\alpha,s} = 0$, a work interval is defined for the edge $\langle g_j, \sigma \rangle$, the edge is in close mode and s is an α-close stage. Enumerate $\gamma_{\beta_j \frown \langle i,\sigma_j \rangle}[s](v)$ into D. Increment the boundary n' of the edge by 1. Constrain the strategy β_j to choose uses $\gamma_{\beta_j \frown \langle i,\sigma_j \rangle}[t](v) > \gamma_{\beta_k \frown \langle i,\sigma_k \rangle}[t](v+n')$ for all $k < j$ at stages $t > s$. Set the edge $\langle g_j, \sigma \rangle$ to open mode. Set *suspend* to *false*. Continue with the next substage.

(c) $o = h_j$, for some $1 \leq j \leq m$.

(i) Suppose that $R_{\alpha,s} > 0$. Set *suspend* to *false*. End stage s, and go to stage $s+1$.

(ii) Suppose that $R_{\alpha,s} = 0$, but the edge $\langle h_j, \sigma \rangle$ has no witness w attached to it. Set *suspend* to *false*. End stage s, and go to stage $s+1$.

(iii) Suppose that $R_{\alpha,s} = 0$, and the strategy has attached a witness w to the edge $\langle h_j, \sigma \rangle$ during this stage s. Set *suspend* to *false*. End stage s, and go to stage $s+1$.

(iv) Suppose that $R_{\alpha,s} = 0$, the work interval for the edge $\langle h_j, \sigma \rangle$ is defined and the edge $\langle h_j, \sigma \rangle$ is in Part I mode, but s is not an α-close stage. Set *suspend* to *true*. End stage s, and go to stage $s+1$.

(v) Suppose that $R_{\alpha,s} = 0$, the work interval for the edge $\langle h_j, \sigma \rangle$ is defined, the edge $\langle h_j, \sigma \rangle$ is in Part I mode, and s is an α-close stage, but $\phi_{j,1}[s](w) \leq \gamma_{\beta_j \frown \langle i, \sigma_j \rangle}[s](w)$. Set *suspend* to *false*. End stage s, and go to stage $s+1$.

(vi) Suppose that $R_{\alpha,s} = 0$, and the work interval for the edge $\langle h_j, \sigma \rangle$ is defined, the edge $\langle h_j, \sigma \rangle$ is in Part I mode, s is an α-close stage, and $\phi_{j,1}[s](w) > \gamma_{\beta_j \frown \langle i, \sigma_j \rangle}[s](w)$. Enumerate $\gamma_{\beta_j \frown \langle i, \sigma_j \rangle, s}(w)$ into D. Increment the boundary n' of the edge by 1. Constrain the strategy β_j to choose uses $\gamma_{\beta_j \frown \langle i, \sigma_j \rangle}[t](w) > \gamma_{\beta_k \frown \langle i, \sigma_k \rangle}[t](w+n')$ for all $k < j$ at stages $t > s$. Set the edge $\langle h_j, \sigma \rangle$ to Part II mode. Set *suspend* to *false*. Go to the next substage.

(vii) Suppose that $R_{\alpha,s} = 0$, and the edge $\langle h_j, \sigma \rangle$ has a witness w attached to it, the edge is in Part II mode and s is not an open-stage. Set *suspend* to *true*. End stage s, and go to stage $s+1$.

(viii) Suppose that $R_{\alpha,s} = 0$, and the edge $\langle h_j, \sigma \rangle$ has a witness w attached to it, the edge is in Part II mode and s is an open-stage. Set the edge $\langle h_j, \sigma \rangle$ to Part I mode. Set *suspend* to *false*. Go to the next substage.

(d) $o = d$.

(i) Suppose that $R_{\alpha,s} > 0$. Set *suspend* to *false*. End stage s, and go to stage $s+1$.

(ii) Suppose that $R_{\alpha,s} = 0$, that there is no $\mathcal{R}$ strategy $\beta_i \subset \alpha$ active for α, and that no witness w is attached to this edge. Set *suspend* to *false*. End stage s, and go to stage $s+1$.

(iii) Suppose that $R_{\alpha,s} = 0$, that there is no $\mathcal{R}$ strategy $\beta_i \subset \alpha$ active for α and that a witness w is attached to this edge, but that s is not an open stage. Set *suspend* to *true*. End stage s, and go to stage $s + 1$.

(iv) Suppose that $R_{\alpha,s} = 0$, that there is no $\mathcal{R}$ strategy $\beta_i \subset \alpha$ active for α, that a witness w is attached to this edge, and that s is an open stage. Enumerate w into A. Set *suspend* to *false*. End stage s, and go to stage $s + 1$.

(v) $R_{\alpha,s} = 0$. Suppose that $R_{\alpha,s} = 0$, that there is some $\mathcal{R}$ strategy $\beta_i \subset \alpha$ active for α where $1 \leq i \leq m$, that every such strategy is following a $\hat{\Gamma}$-strategy and that no witness w is attached to this edge. Set *suspend* to *false*. End stage s, and go to stage $s + 1$.

(vi) Suppose that $R_{\alpha,s} = 0$, that there is some $\mathcal{R}$ strategy $\beta_i \subset \alpha$ active for α where $1 \leq i \leq m$, that every such strategy is following a $\hat{\Gamma}$-strategy and that a witness w is attached to this edge, but $\phi_{i,1}[s](w) > \gamma_{\beta_i \frown \langle i,\sigma_i \rangle}[s](w)$ for some $\beta_i \subset \alpha$ active for α. Detach the witness w from the edge. Set *suspend* to *false*. End stage s, and go to stage $s + 1$.

(vii) Suppose that $R_{\alpha,s} = 0$, that there is some $\mathcal{R}$ strategy $\beta_i \subset \alpha$ active for α where $1 \leq i \leq m$, that every such strategy is following a $\hat{\Gamma}$-strategy, that a witness w is attached to this edge and that $\phi_{i,1}[s](w) \leq \gamma_{\beta_i \frown \langle i,\sigma_i \rangle}[s](w)$ for every $\beta_i \subset \alpha$ active for α, but s is not an open stage. Set *suspend* to *true*. End stage s, and go to stage $s + 1$.

(viii) Suppose that $R_{\alpha,s} = 0$, that there is some $\mathcal{R}$ strategy $\beta_i \subset \alpha$ active for α where $1 \leq i \leq m$, that every such strategy is following a $\hat{\Gamma}$-strategy, that a witness w is attached to this edge, that $\phi_{i,1}[s](w) \leq \gamma_{\beta_i \frown \langle i,\sigma_i \rangle}[s](w)$ for every $\beta_i \subset \alpha$ active for α and that s is an open stage. Enumerate w into A. Set *suspend* to *false*. End stage s, and go to stage $s + 1$.

(ix) Suppose that $R_{\alpha,s} = 0$ and that there is some $\mathcal{R}$ strategy $\beta_i \subset \alpha$ active for α which is following a Γ-strategy, where $1 \leq i \leq m$. Set *suspend* to *false*. End stage s, and go to stage $s + 1$.

3.7.7 Organisation of the Priority Tree

We shall now describe formally how the priority tree T is organised in general. The layout of the strategies on the priority tree is not uniform and will be different for each path through the tree. To build the priority tree T we first assign the following priority ordering to the requirements:

$$\mathcal{R}_0 <_p \mathcal{S}_0 <_p \mathcal{R}_1 <_p \mathcal{S}_1 <_p \ldots$$

Every node on the tree will be labeled with the highest priority requirement which is unsatisfied at that node. Each node will then be assigned a strategy of the appropriate kind in order to satisfy this requirement. Thus nodes which are labeled $\mathcal{R}_i$ for some $i \in \mathbb{N}$ will be assigned an $\mathcal{R}$ strategy, while nodes which are labeled $\mathcal{S}_i$ for some $i \in \mathbb{N}$ will be assigned an $\mathcal{S}$ strategy. The strategies are said to be labeled with the requirement which they are trying to satisfy.

The set of outcomes of an $\mathcal{R}$ strategy β labeled $\mathcal{R}_i$ can then be defined as the set of outcomes obtained by following the procedure in Section 3.7.3 at every stage s. The ordering $\beta \frown \langle o, \sigma \rangle <_L \beta \frown \langle o', \sigma' \rangle$ between any two edges of the strategy β will depend only on the apparent uses σ and σ', and is determined according to the conditions (1) and (2) of the ordering $<_L$ as defined in Section 3.1.2.

The situation for an $\mathcal{S}$ strategy α labeled $\mathcal{S}_i$ is more complex. In order to define the edges leaving this strategy, we shall first need to introduce a number of auxiliary concepts.

Firstly we define the concept of a strategy being *restarted*. Let p be a finite path.

A strategy $\gamma \subset p$ labeled $\mathcal{R}_i$ is *restarted* on p if there is some $\mathcal{S}$ strategy $\alpha \subset p$ such that $\gamma \subset \alpha$ and α has outcome $\alpha \frown \langle h_j, \sigma \rangle$ or $\alpha \frown \langle g_j, \sigma \rangle$ with $j < i$ on p.

A strategy $\gamma \subset p$ labeled $\mathcal{S}_i$ is *restarted on* p if there is some $\mathcal{S}$ strategy $\alpha \subset p$ such that $\gamma \subset \alpha$ and α has outcome $\alpha \frown \langle h_j, \sigma \rangle$ or $\alpha \frown \langle g_j, \sigma \rangle$ with $j \leq i$ on p.

Secondly we define the concept of a strategy following a Γ-strategy or a $\hat{\Gamma}$-strategy. Let p be a finite path. We shall say that a strategy β labeled $\mathcal{R}_i$ on p is *following a* $\hat{\Gamma}$-strategy if there is some greatest (under $\subset$) strategy $\beta' \subset p$ labeled $\mathcal{R}_i$ such that:

- $\beta' \subset \beta$.

- There exists some $\mathcal{S}$ strategy α such that $\beta' \subset \alpha \subset \beta$ and α has outcome $\alpha \frown \langle g_i, \sigma \rangle$ on p.
- β' is not restarted on β.

In this case we say that the strategy α has caused a *switch* in the manner of satisfying the requirement $\mathcal{R}_i$, forcing the β strategy to start following a $\hat{\Gamma}$-strategy. If an $\mathcal{R}$ strategy β lying on a finite path p is not *following a* $\hat{\Gamma}$*-strategy*, then we shall say that β is *following a* Γ*-strategy*.

Thirdly we define the concept of an *active* strategy. Let p be a finite path. We shall say that a strategy $\beta \subset p$ labeled $\mathcal{R}_j$ is *active on* p if:

- β has outcome $\langle i, \sigma \rangle$ on p.
- There is no $\mathcal{S}$ strategy $\alpha \subset p$ such that $\beta \subset \alpha$ and α has outcome $\langle h_i, \sigma \rangle$ or $\langle g_i, \sigma \rangle$ on p.
- β' is not restarted on p.

If γ is some $\mathcal{R}$ or $\mathcal{S}$ strategy and there is some $\mathcal{R}$ strategy $\beta \subset \gamma$ which is active on γ, we shall simply say that β is active for γ.

With these concepts having been defined, the set of outcomes of an $\mathcal{S}$ strategy α labeled $\mathcal{S}_i$ can then be defined as the set of outcomes obtained by following the procedure in Section 3.7.5 at every stage s. The ordering $\alpha \frown \langle o, \sigma \rangle <_L \alpha \frown \langle o', \sigma' \rangle$ between any two edges of the strategy α will depend only on the apparent uses σ and σ', and is determined according to the conditions (1) and (2) of the ordering $<_L$ as defined in Section 3.1.2.

On any finite path p, there will be at most one strategy γ which is managing to satisfy a given requirement, while all other strategies attempting to satisfy this requirement will be failing to do so. This leads us to the concept of a strategy representing a requirement on a path.

An $\mathcal{R}$ strategy β labeled $\mathcal{R}_i$ such that $\beta \subset p$ *represents* $\mathcal{R}_i$ *on* p if the following conditions hold:

- β has outcome $\langle f, \sigma \rangle$ on p.
- There exists an $\mathcal{S}$ strategy $\alpha \subset p$ such that $\beta \subset \alpha$ and α has outcome $\langle h_i, \sigma \rangle$ on p.
- β is active on p.

An $\mathcal{S}$ strategy α labeled $\mathcal{S}_i$ such that $\alpha \subset p$ *represents* $\mathcal{S}_i$ *on* p if the following conditions hold:

- α has outcome $\langle w, \sigma \rangle$ on p.
- α is not restarted on p.

Note that although a strategy may represent some requirement on p, it may stop doing so on some $p' \supset p$.

In addition, we shall say that a strategy represents some requirement on an infinite path p if it represents the requirement on $p \upharpoonright m$ for every natural number m.

The priority tree T is approximated as a sequence of trees T_s, where s is a natural number. During the stage $s = 0$, we define T_0 as the set containing only the empty path. During stage $s + 1$, we define a set P_{s+1} of extended paths as follows. Consider every finite path $p \in T_s$. Every such path p ends in an unlabeled node, which we label with the highest priority requirement which is not represented on p. A strategy of the appropriate kind is then assigned to the node, and an extended path p' is obtained by determining the outcome of the strategy at stage s and appending it to p. Let P_{s+1} be the set of all such extended paths obtained during stage $s + 1$. Then T_{s+1} is defined as $T_s \cup P$.

We shall now formalise the construction which decides which strategies on the priority tree are accessible during a given stage.

3.7.8 The Construction

During each stage s, we will generate a *current path* δ_s in T of length s by recursion, consisting of the edges visited by the construction during stage s. Whilst generating δ_s in this manner, we implicitly obtain the strategies lying on this path, which we declare to be accessible. The current path δ_s is generated by going through a sequence of substages $t \leq s$. During each substage, one chooses the last strategy on the path and computes its outcome at stage s. This outcome corresponds to the next edge on the path, unless the strategy terminates the stage early and goes to the next stage.

The following procedure is followed to generate δ_s.

1. *Base case* ($t = 0$). Let $\delta_s = \emptyset$. Declare $\delta_s \upharpoonright 0$ to be *accessible*.

2. *Recursive case* $(t+1)$.

   If $t+1 > s$, the path δ_s has been constructed. Go to the next stage $s+1$.

   Otherwise consider the strategy $\gamma = \delta_s \upharpoonright t$.

   Execute the strategy γ and compute its outcome $O_s(\gamma)$ at stage s.

   If γ goes to the next stage, the path δ_s has been constructed. Go to stage $s+1$.

   Otherwise γ goes to the next substage at stage s. Let $\delta_s = \delta_s \upharpoonright t \,^\frown O_s(\gamma)$. Declare $\delta_s \upharpoonright t+1$ to be *accessible*.

   If $\gamma' = \delta_s \upharpoonright t+1$ was accessible prior to stage s, let $u < s$ be the greatest stage at which γ' was accessible. Otherwise, let $u = 0$.

   The strategy γ' has been *reset* at stage s if one of the following is the case.

   (a) $R_{\alpha,s} > R_{\alpha,u}$ for some $\mathcal{S}$ strategy $\alpha = \delta_s \upharpoonright t'$ such that $t' < t+1$.
   (b) $d(\alpha \frown \langle o,\sigma\rangle, s) > d(\alpha \frown \langle o,\sigma\rangle, u)$ for some $\mathcal{S}$ strategy $\alpha = \delta_s \upharpoonright t'$ with $t' < t+1$ and edge $\alpha \frown \langle o,\sigma\rangle$ with $\alpha \frown \langle o,\sigma\rangle = \delta_s(t')$.
   (c) $d(\beta \frown \langle o,\sigma\rangle, s) > d(\beta \frown \langle o,\sigma\rangle, u)$ for some $\mathcal{R}$ strategy $\beta = \delta_s \upharpoonright t'$ with $t' < t+1$ and edge $\beta \frown \langle o,\sigma\rangle$ with $\beta \frown \langle o,\sigma\rangle = \delta_s(t')$.

   Go to the next substage.

If a strategy γ is accessible at stage s, we shall say that s is a γ-*stage*.

If γ is an $\mathcal{R}$ strategy and is reset at stage s, every stage attached to any one of its edges is detached, and every functional associated to one of its edges is canceled. Every edge of the form $\gamma \frown \langle f,\sigma\rangle$ or $\gamma \frown \langle i,\sigma\rangle$ is set to open mode, and the variable *suspend* is set to *false*.

If γ is an $\mathcal{S}$ strategy and is reset at stage s, every witness attached to one of its edges is detached, and every work interval and boundary associated to one of its edges is undefined. Every edge of the form $\gamma \frown \langle d,\sigma\rangle$, $\gamma \frown \langle g_i,\sigma\rangle$ or $\gamma \frown \langle w,\sigma\rangle$ is set to open mode, every edge of the form $\gamma \frown \langle h_i,\sigma\rangle$ is set to Part I mode and the variable *suspend* is set to *false*.

If a strategy γ is accessible at stage s, every strategy to its right is *initialised* at stage s.

If γ is an $\mathcal{R}$ strategy and is initialised at stage s, every stage attached to any one of its edges is detached, and every functional associated to one of its edges is canceled. Every edge of the form $\gamma \frown \langle f, \sigma \rangle$ or $\gamma \frown \langle i, \sigma \rangle$ is set to open mode, and the variable *suspend* is set to *false*. The strategy and its edges are regarded as not having been accessible prior to stage s.

If γ is an $\mathcal{S}$ strategy and is initialised at stage s, every witness attached to one of its edges is detached, and every work interval and boundary associated to one of its edges is undefined. Every edge of the form $\gamma \frown \langle d, \sigma \rangle$, $\gamma \frown \langle g_i, \sigma \rangle$ or $\gamma \frown \langle w, \sigma \rangle$ is set to open mode, every edge of the form $\gamma \frown \langle h_i, \sigma \rangle$ is set to Part I mode and the variable *suspend* is set to *false*. The strategy and its edges are regarded as not having been accessible prior to stage s and we have that $R_{\gamma,s} = 0$.

Whenever some edge $\gamma \frown \langle o, \sigma \rangle$ of the strategy γ is on the current path δ_s, we have that every edge $\gamma \frown \langle o', \sigma' \rangle$ with $\gamma \frown \langle o, \sigma \rangle <_L \gamma \frown \langle o', \sigma' \rangle$ is initialised. If γ is an $\mathcal{R}$ strategy, any stage attached to the edge is detached and every functional associated to the edge is canceled. If γ is an $\mathcal{S}$ strategy, any witness attached to the edge is detached and any work interval and boundary defined for the edge is undefined. If the edge is of the form $\gamma \frown \langle d, \sigma \rangle$, $\gamma \frown \langle g_i, \sigma \rangle$ or $\gamma \frown \langle w, \sigma \rangle$ the edge is set to open mode and if $\gamma \frown \langle h_i, \sigma \rangle$ the edge is set to Part I mode.

3.8 Verification

3.8.1 Definitions

The true path f is defined as follows. The edge $f(n)$ is determined by identifying the strategy γ lying at the terminal node of the true path of length n, and choosing the leftmost edge visited infinitely often by this strategy (provided the strategy is in fact accessible at infinitely many stages). Note that the strategy lying at the terminal node of the true path of length 0 is the strategy located at the root node of the priority tree, which is accessible during every stage.

Definition 3.8.1. *(True Path). The true path f is defined by induction as follows. Let $f \upharpoonright n = \gamma$. If there are finitely many γ-stages, $f(n)$ is undefined. Otherwise $f(n) = \liminf_s O_s(\gamma)$, where s ranges over γ-stages.*

Note that according to this definition, it is possible for the true path to be finite. In fact it could be the case that the terminal strategy on a true path of a certain length is not accessible infinitely often. This could result from the previous strategy going to the next substage only finitely often when visiting its edge on the true path.

We shall now define the concepts of a γ-open stage and a γ-close stage for a given strategy γ. A stage is a γ-open stage if the strategy is accessible during this stage and every edge with outcome d, g_n and w of an $\mathcal{S}$ strategy above γ is in open mode and every edge with outcome h_n of an $\mathcal{S}$ strategy above γ is in Part II mode. In addition no functional associated to an edge of an $\mathcal{R}$ strategy above γ disagrees with the set A. On the other hand a stage is a γ-close stage if γ is accessible during this stage and if every edge with outcome d, g_n and w of an $\mathcal{S}$ strategy above γ is in close mode and every edge with outcome h_n associated to an $\mathcal{S}$ strategy above γ is in Part I mode.

Definition 3.8.2. *(Open-Stages and Close-Stages) Let γ be an $\mathcal{R}$ or $\mathcal{S}$ strategy.*

A stage s is a γ-open stage if it satisfies conditions (O1)-(O3).

1. *(Condition O1). γ is accessible at s.*
2. *(Condition O2). If α is an $\mathcal{S}$ strategy with an edge $\alpha \frown \langle d, \sigma\rangle$, $\alpha \frown \langle w, \sigma\rangle$ or $\alpha \frown \langle g_n, \sigma\rangle$ for some n and σ above γ, then this edge is in open mode at stage s. On the other hand, if α is an $\mathcal{S}$ strategy with edge $\alpha \frown \langle h_n, \sigma\rangle$ for some n and σ above γ, then this edge is in Part II mode at stage s.*
3. *(Condition O3). If β is an $\mathcal{R}$ strategy with edge $\beta \frown \langle i, \sigma\rangle$ above γ for some σ, then there is no m such that $A_s(m) \neq \Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}[s](m)$ $(or \Gamma^{V,D}_{\beta\frown\langle i,\sigma\rangle}[s](m) resp.)$.*

A stage t is a γ-close stage for γ if it satisfies conditions (C1)-(C2).

1. *(Condition C1). γ is accessible at t.*
2. *(Condition C2). If α is an $\mathcal{S}$ strategy with edge $\alpha \frown \langle d, \sigma\rangle$, $\alpha \frown \langle w, \sigma\rangle$ or $\alpha \frown \langle g_n, \sigma\rangle$ for some n and σ above γ, then the edge is in close mode at stage t. On the other hand, if α is an $\mathcal{S}$ strategy with edge $\alpha \frown \langle h_n, \sigma\rangle$ for some n and σ above γ, then the edge is in Part I mode at stage t.*

3.8.2 Representation Lemma

The Representation Lemma shows that given any requirement and any infinite path through the priority tree, there will eventually be a strategy which represents that requirement on the infinite path.

Lemma 3.8.3. *(Representation Lemma). Let U be a requirement, and let p be an infinite path. Suppose that there is no $\mathcal{S}$ strategy α with edge $\alpha \frown \langle d, \sigma \rangle$ on p. Then for every requirement U there is some strategy γ and some natural number v such that:*

(i) *γ represents U on $p \upharpoonright v$.*

(ii) *For every $m \geq v$, γ represents U on $p \upharpoonright m$.*

Proof. In generating the priority tree, we are using almost the same procedure used to generate the priority tree for the Lachlan Non-Splitting Theorem in Section 2.9.5. While it is the case that each strategy γ now has infinitely many edges of the form $\gamma \frown \langle o, \sigma \rangle$, it is also the case that in building the tree below this edge it is only the outcome o of the edge which is considered. In fact, with one exception, building the priority tree below an edge $\gamma \frown \langle o, \sigma \rangle$ is identical to building the priority tree below $\gamma \frown o$ in the Lachlan Non-Splitting Theorem.

The exception which has been mentioned when building the priority tree is the following. Let γ labeled $\mathcal{S}_i$ have outcome $\gamma \frown \langle d, \sigma \rangle$ on some infinite path p through the priority tree, and let γ^+ be the successor strategy of γ on p. Then we have that γ does not represent the requirement $\mathcal{S}_i$ on γ^+. However note that the Representation Lemma (Lemma 3.8.3) has now been qualified so that its does not need to apply to infinite paths p with strategies such as γ lying on them. This will have no effect on the satisfaction of requirements by strategies lying on the true path, because edges of the form $\gamma \frown \langle d, \sigma \rangle$ can never lie on the true path.

Hence we have that the proof of Lemma 3.8.3 follows the proof of Lemma 2.10.3 and will not be repeated. □

3.8.3 Leftmost Path Lemma

The Leftmost Path Lemma shows that if a strategy lies on the true path, the current path generated by the construction at each stage can only lie to its left finitely often.

Lemma 3.8.4. *(Leftmost Path Lemma). Let f be the true path. If $f \restriction n$ is defined, there are only finitely many stages s such that $\delta_s <_L \gamma$, where $\gamma = f \restriction n$.*

Proof. We prove the lemma by Induction on n.

For the Base Case $n = 0$ we consider $f \restriction 0$. In this case we have that $f \restriction 0$ is defined and is the strategy γ_0 located at the root of the priority tree. Hence there is no stage s such that $\delta_s <_L \gamma_0$.

For the Inductive Case we assume that the lemma holds for $n = k$. Thus we have that if $f \restriction k$ is defined, there are only finitely many stages s such that $\delta_s <_L \gamma_k$, where $\gamma_k = f \restriction k$.

We then prove that the lemma holds for $n = k + 1$. Thus we need to show that if $f \restriction k + 1$ is defined, there are only finitely many stages s such that $\delta_s <_L \gamma_{k+1}$, where $\gamma_{k+1} = f \restriction k + 1$.

Now if $f \restriction k + 1$ is not defined, the lemma holds trivially.

Otherwise suppose that $f \restriction k + 1$ is defined. This can only be the case if $f \restriction k$ is defined and if $f(k)$ is defined.

Since $f \restriction k$ is defined, by the Inductive Hypothesis we have that there is some stage t such that $\delta_{t'} \not<_L \gamma_k$ for all $t' > t$. In addition, since $f(k)$ is defined, it must either be the case that $f(k) = \liminf_s O_s(\gamma_k)$, where $\gamma_k = f \restriction k$.

Hence there must be some stage $u > t$ such that $O_{u'}(\gamma_k) \not<_L f(k)$ for all $u' > u$. Since $f \restriction k + 1 = f \restriction k\hat{\ } f(k)$, we have that $\delta_s \not< \gamma_{n+1}$ for all $s > u$ as required. □

From the previous lemma it follows that for every strategy γ on the true path f there is some least stage s_0 such that $\gamma_s \not< \gamma$ for all stages $s > s_0$.

3.8.4 Attachment Procedure Lemma

Suppose that the attachment procedure of a strategy γ attaches certain elements to edges of a specific kind. The Attachment Procedure Lemma shows that if an edge lies on the true path and does not have an element attached at some given stage, an element will eventually be attached to the edge at some subsequent stage.

In the lemma, X is the set from which elements which need to be attached are chosen, $P(\gamma \frown \langle o', \sigma' \rangle, s)$ is a relation which tells us whether the edge $\gamma \frown \langle o', \sigma' \rangle$ is suitable to receive the elements in question at stage s and $R(x, s)$ is a relation which determines whether an element x is suitable to be attached to an edge for which $P(\gamma \frown \langle o', \sigma' \rangle, s)$ holds at stage s.

In practice, the set X will be a set of witnesses, thresholds, uses or β-expansionary* stages. The relation $R(x, s)$ will determine whether a witness w gives a computation $\Theta^D[s](w) \downarrow= 0$ and whether this is honest; or whether a given stage is a β-expansionary* stage. Finally the relation $P(\gamma \frown \langle o', \sigma' \rangle, s)$ will determine whether the edge has an outcome of a certain kind or whether a work interval is defined for an edge.

Lemma 3.8.5. *(Attachment Procedure Lemma). Let γ be some strategy, $\mathbb{E}$ be its set of edges $\gamma \frown \langle o', \sigma' \rangle$, $\mathbb{N}_\gamma$ be the set of γ-stages and $\gamma \frown \langle o, \sigma \rangle$ be the edge on the true path. In addition let X be some computable subset of $\mathbb{N}$, $R(x, s)$ be a computable relation over $\mathbb{N} \times \mathbb{N}_\gamma$ and $P(\gamma \frown \langle o', \sigma' \rangle, s)$ be a computable relation over $\mathbb{E} \times \mathbb{N}_\gamma$.*

Suppose that the following conditions hold.

- *(1) There are infinitely many elements $x \in X$ and γ-stages s such that $R(x, s)$ holds.*
- *(2) Let $x \in X$ be the least element such that $R(x, s)$ holds and such that the strategy γ has not attached x to an edge at some stage $t < s$. Suppose that if s is a γ-stage, the attachment procedure attaches x to the leftmost edge of the form $\gamma \frown \langle o', \sigma' \rangle$ such that:*
  - *(a) $\gamma \frown \langle o', \sigma' \rangle$ has been visited at some stage $t' < s$.*
  - *(b) $\gamma \frown \langle o', \sigma' \rangle$ does not have some $x' \in X$ attached at stage s.*
  - *(c) $P(\gamma \frown \langle o', \sigma' \rangle, s)$ holds.*

(3) Let $\gamma \frown \langle o', \sigma' \rangle$ be an edge in $\mathbb{E}$. If $\gamma \frown \langle o', \sigma' \rangle \leq_L \gamma \frown \langle o, \sigma \rangle$ and there exists a stage $s > s_0$ such that $P(\gamma \frown \langle o', \sigma' \rangle, s)$ holds, then $P(\gamma \frown \langle o', \sigma' \rangle, t)$ holds for every $t > s$.

(4) $P(\gamma \frown \langle o, \sigma \rangle, s_1)$ holds for some stage $s_1 > s_0$.

Then if $\gamma \frown \langle o, \sigma \rangle$ is accessible at stage s_1 and does not have a witness attached at stage s_1, there is some stage $s_2 > s_1$ such that some element $y \in X$ for which $R(y, s_2)$ holds will be attached to $\gamma \frown \langle o, \sigma \rangle$ at stage s_2.

Proof. Consider the stage $s_1 > s_0$ and suppose that there is no witness attached to the edge $\gamma \frown \langle o, \sigma \rangle$ at stage s_1.

Since the edge $\gamma \frown \langle o, \sigma \rangle$ lies on the true path, we have that strategies and edges to the left of $\gamma \frown \langle o, \sigma \rangle$ are inaccessible at stages $s > s_0$ by the *Leftmost Path Lemma* (Lemma 3.8.4).

Consider the finitely many edges $\gamma \frown \langle o', \sigma' \rangle$ lying to the left of the edge $\gamma \frown \langle o, \sigma \rangle$, which have been accessible at some stage $t < s_1$ and for which $P(\gamma \frown \langle o', \sigma' \rangle, s_1)$ holds.

Then by (3) we have that $P(\gamma \frown \langle o', \sigma' \rangle, t)$ holds for all $t > s_1$. In addition by (4) we have that $P(\gamma \frown \langle o', \sigma' \rangle, s_1)$ holds. Hence by (3) we also have that $P(\gamma \frown \langle o', \sigma' \rangle, t)$ holds for all $t > s_1$.

In addition since no strategy or edge to the left of $\gamma \frown \langle o, \sigma \rangle$ is accessible after stage s_0, we have that $\gamma \frown \langle o', \sigma' \rangle$ cannot be initialised after s_0. It is also the case that since $\gamma \frown \langle o', \sigma' \rangle$ is inaccessible after stage s_0 we have that no element can be detached by γ from $\gamma \frown \langle o, \sigma \rangle$ after stage s_0.

Now by (1) there are infinitely many elements $x \in X$ and infinitely many γ-stages s such that $R(x, s)$ holds. Therefore by (2) we must have that there is some stage $u > s_1$ such that a witness is attached to every edge of the form $\gamma \frown \langle o', \sigma' \rangle$ for which $P(\gamma \frown \langle o', \sigma' \rangle, u)$ holds.

Let $s_2 \geq u$ be the least stage such that $\gamma \frown \langle o, \sigma \rangle$ has been accessible at some stage $t \leq s_2$. Such a stage must exist because $\gamma \frown \langle o, \sigma \rangle$ is on the true path and is thus visited infinitely often.

Then by (2) the strategy γ attaches the least $y \in X$ such that $R(y, s_2)$ holds to the edge $\gamma \frown \langle o, \sigma \rangle$ as required. $\square$

3.8.5 Honesty Preservation Lemma

The Honesty Preservation Lemma shows that once an honest witness is attached to an edge lying on the true path, it cannot become dishonest prior to becoming enumerated into the set A.

Lemma 3.8.6. *(Honesty Preservation Lemma). Let α be an $\mathcal{S}$ strategy lying below some non-empty sequence of active $\mathcal{R}$ strategies $(\beta_1, \ldots, \beta_m)$, and let the edge $\alpha \frown \langle o, \sigma \rangle$ lie on the true path. Suppose that one of the following is the case.*

(1) Every strategy $\beta_i \in (\beta_1, \ldots, \beta_m)$ is following a $\hat{\Gamma}$-strategy, and the edge on the true path is of the form $\alpha \frown \langle d, \sigma \rangle$.

(2) Some strategy $\beta_i \in (\beta_1, \ldots, \beta_m)$ is following a Γ-strategy, and the edge on the true path is of the form $\alpha \frown \langle g_i, \sigma \rangle$, for some $1 \leq i \leq m$.

Suppose that a witness w is attached to $\alpha \frown \langle o, \sigma \rangle$ at stage s. Then if some element x is enumerated into the set A or D at some stage $t \geq s$, one of the following must be the case.

(a) $x > \phi_{i,2}[s](\phi_{i,1}[s](w))$ and $x > \phi_{i,3}[s](\phi_{i,1}[s](w))$ for all $1 \leq i \leq m$.

(b) w is enumerated into A at some stage $t' \leq t$.

(c) w is detached from the edge at some stage $u > s$ due to some $\mathcal{S}$ strategy $\alpha' \subset \alpha$ setting $R_{\alpha',t'} > R_{\alpha',s}$ for some t' such that $s < t' < u$.

Proof. We consider every possible way in which a strategy can enumerate an element $x \leq \phi_{i,2}[s](\phi_{i,1}[s](w))$ or $x \leq \phi_{i,3}[s](\phi_{i,1}[s](w))$ for some $1 \leq i \leq m$ into the set A or D at a stage $t \geq s$, and show that none of these can be the case.

(1) Consider the strategy α itself. Let the edge $\alpha \frown \langle o', \sigma' \rangle$ lie to the right of $\alpha \frown \langle o, \sigma \rangle$.

Then when the strategy α attaches the witness w to the edge $\alpha \frown \langle o, \sigma \rangle$ at stage s, we must have that every witness attached to edge $\alpha \frown \langle o', \sigma' \rangle$ is detached. Similarly, every work interval defined for the edge $\alpha \frown \langle o', \sigma' \rangle$ is undefined.

We now proceed by performing the following case analysis.

(1.1) Suppose that every $\mathcal{R}$ strategy $\beta_i \in (\beta_1, \ldots, \beta_m)$ above α which is active for α is following a $\hat{\Gamma}$-strategy. Then the strategy α must have attached w to an edge of the form $\alpha \frown \langle d, \sigma \rangle$ at stage s.

Since only one witness can be attached by the strategy α at any given stage, we have that no witness can be attached to an edge $\alpha \frown \langle d, \sigma' \rangle$ lying to the right of $\alpha \frown \langle d, \sigma \rangle$ at stage s.

On the other hand, suppose the strategy α attaches a witness w' to an edge $\alpha \frown \langle d, \sigma' \rangle$ lying to the right of $\alpha \frown \langle d, \sigma \rangle$ at some stage $u > s$. Then α must have chosen w' to be greater than $\sup\{r(\alpha \frown \langle o'', \sigma'' \rangle, s) \mid \alpha \frown \langle o'', \sigma'' \rangle <_L \alpha \frown \langle d, \sigma' \rangle \ \wedge \ \alpha \frown \langle o'', \sigma'' \rangle$ has been previously accessible$\}$. Since w is attached to $\alpha \frown \langle d, \sigma \rangle$ at stage s, and $\alpha \frown \langle d, \sigma \rangle <_L \alpha \frown \langle d, \sigma' \rangle$ we have that $w' > \phi_{i,2}[s](\phi_{i,1}[s](w))$ and $w' > \phi_{i,3}[s](\phi_{i,1}[s](w))$ for all $1 \leq i \leq m$ as required.

Similarly, suppose that the strategy α attaches some witness w' to an edge $\alpha \frown \langle h_i, \sigma' \rangle$ lying to the right of $\alpha \frown \langle d, \sigma \rangle$ at some stage $u > s$ so as to define a work interval $(w', \gamma_{\beta \frown \langle i, \sigma \rangle}[u](w'))$ for the edge at stage u. If the strategy α enumerates $\gamma_{\beta \frown \langle i, \sigma \rangle}[t](w')$ into D at some stage $t \geq u$, we must have that $\gamma_{\beta \frown \langle i, \sigma \rangle}[t](w') > w'$. Hence by the above it follows that $\gamma_{\beta \frown \langle i, \sigma \rangle}[t](w')) > \phi_{i,2}[s](\phi_{i,1}[s](w))$ and $\gamma_{\beta \frown \langle i, \sigma \rangle}[t](w') > \phi_{i,3}[s](\phi_{i,1}[s](w))$ for all $1 \leq i \leq m$ as required.

(1.2) Suppose that some $\mathcal{R}$ strategy $\beta_i \in (\beta_1, \ldots, \beta_m)$ above α which is active for α is following a Γ-strategy. Then the strategy α must have attached w to an edge of the form $\alpha \frown \langle g_i, \sigma \rangle$ for some $1 \leq i \leq m$ at stage s.

Since only one witness can be attached by the strategy α at any given stage, we have that no witness can be attached to an edge $\alpha \frown \langle g_j, \sigma' \rangle$ for some $1 \leq j \leq m$ lying to the right of $\alpha \frown \langle g_i, \sigma \rangle$ at stage s.

On the other hand, suppose the strategy α attaches a witness w' to an edge $\alpha \frown \langle g_j, \sigma' \rangle$ for some $1 \leq j \leq m$ lying to the right of $\alpha \frown \langle g_i, \sigma \rangle$ at some stage $u > s$. Then α must have chosen w' to be greater than $\sup\{r(\alpha \frown \langle o'', \sigma'' \rangle, s) \mid \alpha \frown \langle o'', \sigma'' \rangle <_L \alpha \frown \langle g_j, \sigma' \rangle \ \wedge \ \alpha \frown \langle o'', \sigma'' \rangle$ has been previously accessible$\}$. Since w is attached to $\alpha \frown \langle g_i, \sigma \rangle$ at stage s, and $\alpha \frown \langle g_i, \sigma \rangle <_L \alpha \frown \langle g_j, \sigma' \rangle$ we have that $w' > \phi_{i,2}[s](\phi_{i,1}[s](w))$ and $w' > \phi_{i,3}[s](\phi_{i,1}[s](w))$ for all $1 \leq i \leq m$ as required.

Similarly, suppose that the strategy α attaches some witness w' to an edge $\alpha \frown \langle h_j, \sigma' \rangle$ for some $1 \leq j \leq m$ lying to the right of $\alpha \frown \langle g_i, \sigma \rangle$ at some stage $u > s$ so as to define a work interval $(w', \gamma_{\beta \frown \langle i,\sigma \rangle}[u](w'))$ for the edge at stage u. If the strategy α enumerates $\gamma_{\beta \frown \langle i,\sigma \rangle}[t](w')$ into D at some stage $t \geq u$, we must have that $\gamma_{\beta \frown \langle i,\sigma \rangle}[t](w') > w'$. Hence by the above it follows that $\gamma_{\beta \frown \langle i,\sigma \rangle}[t](w')) > \phi_{i,2}[s](\phi_{i,1}[s](w))$ and $\gamma_{\beta \frown \langle i,\sigma \rangle}[t](w') > \phi_{i,3}[s](\phi_{i,1}[s](w))$ for all $1 \leq i \leq m$ as required.

Finally, suppose that the strategy α chooses some threshold v to define a work interval $(v, \gamma_{\beta \frown \langle i,\sigma \rangle}[u](v))$ for some edge $\alpha \frown \langle g_j, \sigma' \rangle$ for some $1 \leq j \leq m$ lying to the right of $\alpha \frown \langle g_i, \sigma \rangle$ at some stage $u \geq s$.

Then if $u = s$, we have that the strategy α must choose v to be greater than $a(\alpha \frown \langle g_j, \sigma' \rangle)$. But since α has attached the witness w to the edge $\alpha \frown \langle g_i, \sigma \rangle$ at stage s, and $\alpha \frown \langle g_i, \sigma \rangle <_L \alpha \frown \langle g_j, \sigma \rangle$ it must be the case that $v > \phi_{i,2}[s](\phi_{i,1}[s](w))$ and $v > \phi_{i,3}[s](\phi_{i,1}[s](w))$ for all $1 \leq i \leq m$.

On the other hand, if $u > s$, α must have chosen v to be greater than $\sup\{r(\alpha \frown \langle o'', \sigma'' \rangle, s) \mid \alpha \frown \langle o'', \sigma'' \rangle <_L \alpha \frown \langle g_j, \sigma' \rangle \wedge \alpha \frown \langle o'', \sigma'' \rangle$ has been previously accessible$\}$. Since v is attached to $\alpha \frown \langle g_i, \sigma \rangle$ at stage s, and $\alpha \frown \langle g_i, \sigma \rangle <_L \alpha \frown \langle g_j, \sigma' \rangle$ we also have that $v > \phi_{i,2}[s](\phi_{i,1}[s](w))$ and $v > \phi_{i,3}[s](\phi_{i,1}[s](w))$ for all $1 \leq i \leq m$.

Then if the strategy α enumerates $\gamma_{\beta \frown \langle i,\sigma \rangle}[t](v)$ into D at some stage $t \geq u$, we must have that $\gamma_{\beta \frown \langle i,\sigma \rangle}[t](v) > v$. Hence by the above it follows that $\gamma_{\beta \frown \langle i,\sigma \rangle}[t](v)) > \phi_{i,2}[s](\phi_{i,1}[s](w))$ and $\gamma_{\beta \frown \langle i,\sigma \rangle}[t](v) > \phi_{i,3}[s](\phi_{i,1}[s](w))$ for all $1 \leq i \leq m$ as required.

(2) Consider the strategy α' such that $\alpha \subset \alpha'$. Let $\alpha \frown \langle o, \sigma \rangle$ be the edge of α on the true path, and let $(\beta'_1, \ldots, \beta'_n)$ be the sequence of $\mathcal{R}$ strategies above α' which are active for α'.

We perform the following case analysis.

(2.1) Suppose that α' lies below $\alpha \frown \langle o, \sigma \rangle$.

In order for α' to enumerate some element into the set A or the set D at some stage $t \geq s$, it must first be the case that α' is accessible at stage t. But this means that the edge $\alpha \frown \langle o, \sigma \rangle$ must be of the form $\alpha \frown \langle g_i, \sigma \rangle$ for some $1 \leq i \leq m$. For α never goes to the next substage when visiting an edge of the form $\alpha \frown \langle d, \sigma \rangle$ meaning that α' could never become accessible.

We now perform the following case analysis.

(2.1.1) Suppose that every $\beta_i \in (\beta'_1, \ldots, \beta'_n)$ is following a $\hat{\Gamma}$-strategy.

Consider an edge of the form $\alpha' \frown \langle d, \sigma' \rangle$. Then we have that α' can enumerate a witness w' which was attached to the edge $\alpha' \frown \langle d, \sigma' \rangle$ into A when it visits the edge at stage t.

In order for this to be the case, we must have that the edge $\alpha' \frown \langle d, \sigma \rangle$ is in open mode at stage t and that stage t is an α'-open stage. But in order for t to be an α'-open stage, we must have that t is also an α-open stage.

Now the only way for α to go to the next substage during the α-open stage t is for α to be in open mode at stage t. But this means that α enumerates w into A at stage t when visiting the edge $\alpha \frown \langle g_i, \sigma \rangle$ and the lemma is satisfied trivially.

On the other hand consider an edge of the form $\alpha' \frown \langle h_j, \sigma' \rangle$ for some $\beta_j \in (\beta'_1, \ldots, \beta'_n)$, where the latter is the sequence of $\mathcal{R}$ strategies which are above α' and are active for α'.

Suppose that a witness w' is attached to the edge $\alpha' \frown \langle h_j, \sigma' \rangle$ and that α' enumerates $\gamma_{\beta_j \frown \langle i, \sigma \rangle}[t](w')$ into D at some stage $t \geq s$.

In order for this to be the case, the edge $\alpha' \frown \langle h_j, \sigma' \rangle$ must be in Part I mode at stage t, and t must be an α'-close stage.

But t can only be an α'-close stage if it is also an α-close stage. In addition for α to go to the next substage when visiting the edge $\alpha \frown \langle h_j, \sigma \rangle$ during an α-close stage, the edge $\alpha \frown \langle h_j, \sigma \rangle$ must be in close mode during stage t.

Now the edge $\alpha \frown \langle h_j, \sigma \rangle$ can only be in close mode at stage t if it was in open mode at some greatest α-open stage $t' < t$, and α also enumerated some witness z into A during stage t'.

But the strategy α does not attach another witness z' to the edge $\alpha \frown \langle h_j, \sigma \rangle$ if the edge is not in open mode. But we have already determined that $t' < t$ is the greatest stage such that the edge $\alpha \frown \langle h_j, \sigma \rangle$ is in open mode. Hence it follows that w cannot be attached to the edge $\alpha \frown \langle h_j, \sigma \rangle$ at stage t. Hence the lemma is satisfied trivially.

(2.1.2) Suppose that some $\beta_i \in (\beta'_1, \ldots, \beta'_n)$ is following a Γ-strategy.

Consider an edge of the form $\alpha' \frown \langle g_j, \sigma' \rangle$ for some $\beta_j \in (\beta'_1, \ldots, \beta'_n)$, where the latter is the sequence of $\mathcal{R}$ strategies which are above α' and are active for α'.

Suppose that a witness w' is attached to the edge $\alpha' \frown \langle g_j, \sigma' \rangle$ and that α' enumerates w' into A when visiting the edge at some stage $t \geq s$.

In order for this to be the case, we must have that the edge $\alpha' \frown \langle g_j, \sigma' \rangle$ is in open mode at stage t and that stage t is an α'-open stage. But in order for t to be an α'-open stage, we must have that t is also an α-open stage.

Now the only way for α to go to the next substage during the α-open stage t is for α to be in open mode at stage t. But this means that α enumerates w into A at stage t when visiting the edge $\alpha \frown \langle g_i, \sigma \rangle$ and the lemma is satisfied trivially.

In addition we also have that α' can enumerate elements into D by visiting some edge of the form $\alpha \frown \langle g_j, \sigma' \rangle$ for some $\beta_j \in (\beta'_1, \ldots, \beta'_n)$.

Suppose that a work interval $(v, \gamma_{\beta_j \frown \langle i, \sigma \rangle}[t](w'))$ is defined for an edge $\alpha' \frown \langle g_j, \sigma' \rangle$ at stage t, and that α' enumerates $\gamma_{\beta_j \frown \langle i, \sigma \rangle}[t](w')$ into D at stage t.

In order for this to be the case, the edge $\alpha' \frown \langle g_j, \sigma' \rangle$ must be in close mode at stage t, and t must be an α'-close stage.

Now t can only be an α'-close stage if it is also an α-close stage. In addition for α to go to the next substage when visiting the edge $\alpha \frown \langle g_i, \sigma \rangle$ during an α-close stage, the edge $\alpha \frown \langle g_i, \sigma \rangle$ must be in close mode during stage t.

However the edge $\alpha \frown \langle g_i, \sigma \rangle$ can only be in close mode at stage t if it was in open mode at some greatest α-open stage $t' < t$, and α also enumerated some witness z into A during stage t'.

But the strategy α does not attach another witness z' to the edge $\alpha \frown \langle g_i, \sigma \rangle$ if the edge is not in open mode. But we have already determined that $t' < t$ is the greatest stage such that the edge $\alpha \frown \langle g_i, \sigma \rangle$ is in open mode. Hence it follows that w cannot be attached to the edge $\alpha \frown \langle g_i, \sigma \rangle$ at stage t. Hence the lemma is satisfied trivially.

Finally we also have that α' can enumerate elements into D by visiting some edge of the form $\alpha \frown \langle h_j, \sigma' \rangle$ for some $\beta_j \in (\beta'_1, \ldots, \beta'_n)$.

Suppose that a witness w' is attached to the edge $\alpha' \frown \langle h_j, \sigma' \rangle$ and that α' enumerates $\gamma_{\beta_j \frown \langle i, \sigma \rangle}[t](w')$ into D at some stage $t \geq s$.

In order for this to be the case, the edge $\alpha' \frown \langle h_j, \sigma' \rangle$ must be in Part I mode at stage t, and t must be an α'-close stage.

But t can only be an α'-close stage if it is also an α-close stage. In addition for α to go to the next substage when visiting the edge $\alpha \frown \langle h_j, \sigma \rangle$ during an α-close stage, the edge $\alpha \frown \langle h_j, \sigma \rangle$ must be in close mode during stage t.

Now the edge $\alpha \frown \langle h_j, \sigma \rangle$ can only be in close mode at stage t if it was in α-open mode at some greatest α-open stage $t' < t$, and α also enumerated some witness z into A during stage t'.

But the strategy α does not attach another witness z' to the edge $\alpha \frown \langle h_j, \sigma \rangle$ if the edge is not in open mode. But we have already determined that $t' < t$ is the greatest stage such that the edge $\alpha \frown \langle h_j, \sigma \rangle$ is in open mode. Hence it follows that w cannot be attached to the edge $\alpha \frown \langle h_j, \sigma \rangle$ at stage t. Hence the lemma is satisfied trivially.

(2.2) Suppose that α' lies below some $\alpha \frown \langle o', \sigma' \rangle$ where $\alpha \frown \langle o', \sigma' \rangle$ lies to the right of $\alpha \frown \langle o, \sigma \rangle$.

In order for the strategy α to attach the witness w to the edge $\alpha \frown \langle o, \sigma \rangle$ at stage s, it must be the case that $\phi_{i,2}[s](\phi_{i,1}[s](w)) \downarrow$ and $\phi_{i,3}[s](\phi_{i,1}[s](w)) \downarrow$ for every strategy $\beta_i \in (\beta_1, \ldots, \beta_n)$. In addition, the strategy α must have been accessible at stage s.

Now in order for α' to be accessible at some least stage $u \geq s$ it must be the case that α has visited the edge $\alpha \frown \langle o', \sigma' \rangle$ at stage u. If w is no longer attached to the edge $\alpha \frown \langle o, \sigma \rangle$ at stage u, the lemma is satisfied trivially. Otherwise we have that α imposes the downward restraint $d(\alpha \frown \langle o', \sigma' \rangle, u)$ on strategies lying below the edge $\alpha \frown \langle o', \sigma' \rangle$.

Now α' is not accessible at any stage t such that $s < t < u$ and w has been attached to $\alpha \frown \langle o, \sigma \rangle$ at stage s. Suppose α' had been accessible at some greatest stage $s' < s$. Then when it becomes accessible at stage u it notes that $d(\alpha \frown \langle o', \sigma' \rangle, u) > d(\alpha \frown \langle o', \sigma' \rangle, s')$, where s' is the greatest stage less than s such that β was accessible. Hence at stage u we have that α' will detach every witness which was attached to one of its edges, and undefine every work interval which was defined for one of its edges. Otherwise we have that α' was never accessible prior to stage s, and that it has never attached a witness to some edge or defined a work interval for some edge.

Therefore if some witness w' is attached to $\alpha' \frown \langle o', \sigma' \rangle$ at some stage $u' \geq u$, we must have that $w' > \phi_{i,2}[s](\phi_{i,1}[s](w))$ and that $w' > \phi_{i,3}[s](\phi_{i,1}[s](w))$ for every strategy $\beta_i \in (\beta_1, \ldots, \beta_m)$. Similarly if the strategy chooses a threshold v to define a work interval for the edge $\alpha' \frown \langle o', \sigma' \rangle$ at some stage $u > s$ we have that $v > \phi_{i,2}[s](\phi_{i,1}[s](w))$ and that $v > \phi_{i,3}[s](\phi_{i,1}[s](w))$ for every strategy $\beta_i \in (\beta_1, \ldots, \beta_m)$.

Hence, if $\alpha' \frown \langle o', \sigma' \rangle$ has outcome d or g_j and α' enumerates a witness w' into the set A at some stage $t \geq u' \geq s$ we have that $w' > \phi_{i,2}[s](\phi_{i,1}[s](w))$ and that $w' > \phi_{i,2}[s](\phi_{i,1}[s](w))$ for every strategy $\beta_i \in (\beta_1, \ldots, \beta_m)$.

Similarly if $\alpha' \frown \langle o', \sigma' \rangle$ has outcome g_j and α' enumerates $\gamma_{\beta_j \frown \langle i, \sigma \rangle}[t](v)$ into the set D at some stage $t \geq u' \geq s$ we have that $\gamma_{\beta_j \frown \langle i, \sigma \rangle}[t](v) > \phi_{i,2}[s](\phi_{i,1}[s](w))$ and that $\gamma_{\beta_j \frown \langle i, \sigma \rangle}[t](v) > \phi_{i,2}[s](\phi_{i,1}[s](w))$ for every strategy $\beta_i \in (\beta_1, \ldots, \beta_m)$.

Finally if $\alpha' \frown \langle o', \sigma' \rangle$ has outcome h_j and α' enumerates $\gamma_{\beta_j \frown \langle i, \sigma \rangle}[t](w')$ into the set D at some stage $t \geq u' \geq s$ we have that $\gamma_{\beta_j \frown \langle i, \sigma \rangle}[t](w') > \phi_{i,2}[s](\phi_{i,1}[s](w))$ and that $\gamma_{\beta_j \frown \langle i, \sigma \rangle}[t](w') > \phi_{i,2}[s](\phi_{i,1}[s](w))$ for every strategy $\beta_i \in (\beta_1, \ldots, \beta_m)$ as required.

(3) Consider a strategy α' such that $\alpha' \subset \alpha$.

Let $(\beta'_1, \ldots, \beta'_n)$ be the sequence of $\mathcal{R}$ strategies above α' which are active for α'.

We perform the following case analysis.

(3.1) Suppose that every $\beta_i \in (\beta'_1, \ldots, \beta'_n)$ is following a $\hat{\Gamma}$-strategy.

In this case we have that α' can only enumerate elements into A by visiting some edge of the form $\alpha' \frown \langle d, \sigma' \rangle$. On the other hand, α' can enumerate elements into D by visiting some edge of the form $\alpha' \frown \langle h_i, \sigma' \rangle$ for some $\beta_i \in (\beta'_1, \ldots, \beta'_n)$. We perform the following case analysis.

(3.1.1) Consider an edge $\alpha' \frown \langle d, \sigma' \rangle$ lying above α.

If the strategy α' visits such an edge, it will never go to the next substage. Thus α is never be accessible and cannot attach the witness w to the edge $\alpha \frown \langle o, \sigma' \rangle$ at stage s, which is a contradiction.

(3.1.2) Consider an edge $\alpha' \frown \langle d, \sigma' \rangle$ lying to the right of α.

Suppose that α' enumerates a witness w' attached to the edge $\alpha' \frown \langle d, \sigma' \rangle$ into A at a stage $t \geq s$.

Now every $\beta_i \in (\beta'_1, \ldots, \beta'_n)$ is following a $\hat{\Gamma}$-strategy. Hence when α' becomes accessible again at some stage $t' > t$, we have that the functional built by each such β_i becomes undefined at w' due to a $V_i \upharpoonright \phi_{i,1}[t](w')$ change.

Hence we have that α' sets $R_{\alpha',t'} > 0$ at stage t' to preserve its successful diagonalisation.

But when α becomes accessible again at some stage $t'' > t$ it will determine that $\alpha' \subset \alpha$ has set its restraint $R_{\alpha',t''} > 0$.

Hence α will detach every witness attached to any of its edges, which includes the witness w attached to $\alpha \frown \langle o, \sigma \rangle$.

(3.1.3) Consider an edge $\alpha' \frown \langle h_i, \sigma' \rangle$ lying above α.

We have already seen that $(\beta'_1, \ldots, \beta'_n)$ is the sequence of $\mathcal{R}$ strategies which are above α' and active for α'.

Now since α lies below the edge $\alpha' \frown \langle h_i, \sigma' \rangle$, we have that no strategy $\beta'_j \in (\beta'_1, \ldots, \beta'_n)$ is active for α if $i \leq j \leq n$.

On the other hand it is possible for strategies $\beta'_j \in (\beta'_1, \ldots, \beta'_n)$ to be active for α if $1 \leq j < i$.

Suppose that α' enumerates $\gamma_{\beta_i \frown \langle i, \sigma \rangle}[t](w')$ into D at a stage $t > u$, where w' is a witness which is attached to the edge. Then we shall need to show that $\gamma_{\beta_i \frown \langle i, \sigma \rangle}[t](w') > \phi_{j,2}[t](\phi_{j,1}[t](w))$ and $\gamma_{\beta_i \frown \langle i, \sigma \rangle}[t](w') > \phi_{j,3}[t](\phi_{j,1}[t](w))$ for all $1 \leq j < i$.

To prove this we observe that in order for α to attach the witness w to the edge $\alpha \frown \langle o, \sigma \rangle$ at stage s, α must have been accessible at stage s. In addition we have that a work interval

$(w', \gamma_{\beta_i \frown \langle i,\sigma\rangle}(w')[u])$ must have been defined for the edge $\alpha' \frown \langle h_i, \sigma' \rangle$ at some stage $u < s$.

Hence it must be that case that $w' < w < w' + n_{\beta \frown \langle i,\sigma\rangle}[s]$. But this means that there is some stage $p < s$ such that a constraint to the effect that $\gamma_{\beta_i \frown \langle i,\sigma\rangle}(w')[p'] > \gamma_{\beta_j \frown \langle i,\sigma\rangle}(w' + n_{\beta_i \frown \langle i,\sigma\rangle}[s])[p']$ has been imposed on β_i for all stages $p' > p$ and $1 \leq j < i$.

In addition, $\gamma_{\beta_i \frown \langle i,\sigma\rangle}[p](w')$ must have been enumerated into D at stage p, undefining the functional $\Gamma^{U_i,D}_{\beta_i \frown \langle i,\sigma\rangle}(w')$. Since the functional is redefined in accordance with the constraints it follows that $\gamma_{\beta_i \frown \langle i,\sigma\rangle}(w')[t] > \gamma_{\beta_j \frown \langle i,\sigma\rangle}(w)[t]$ for all $1 \leq j < i$ and for all $t \geq s$.

Now suppose that α' enumerates $\gamma_{\beta_i \frown \langle i,\sigma\rangle}(w')[t]$ whilst visiting the edge $\alpha' \frown \langle h_i, \sigma' \rangle$ at some stage $t \geq s$. Then we have that $\gamma_{\beta_i \frown \langle i,\sigma\rangle}(w')[t] > \gamma_{\beta_j \frown \langle i,\sigma\rangle}(w)[t]$ for all $1 \leq j < i$. In addition $\gamma_{\beta_j \frown \langle i,\sigma\rangle}(w)[t] > \phi_{j,2}[t](\phi_{j,1}[t](w))$ and $\gamma_{\beta_j \frown \langle i,\sigma\rangle}(w)[t] > \phi_{j,3}[t](\phi_{j,1}[t](w))$ for all $1 \leq j < i$, as required.

In addition to the above we also have to consider the sequence of $\mathcal{R}$ strategies $(\beta''_1, \ldots, \beta''_n)$ which are active for α and lie below α'.

Suppose that α' enumerates $\gamma_{\beta_i \frown \langle i,\sigma\rangle}[t](w')$ into D at a stage $t > u$, where w' is a witness which is attached to the edge. Then we shall need to show that $\gamma_{\beta_i \frown \langle i,\sigma\rangle}[t](w') > \phi_{j,2}[t](\phi_{j,1}[t](w))$ and $\gamma_{\beta_i \frown \langle i,\sigma\rangle}[t](w') > \phi_{j,3}[t](\phi_{j,1}[t](w))$ for every $\beta''_j \in (\beta''_1, \ldots, \beta''_n)$.

To prove this we observe that every such β''_j lies below the edge $\alpha' \frown \langle h_i, \sigma' \rangle$ of the strategy. We have already determined that α' defines the work interval $(w', \gamma_{\beta_i \frown \langle i,\sigma\rangle}(w')[u])$ at some stage u. Then if at some stage $u' \geq u$ the strategy β''_j defines the functional associated with the edge $\beta''_j \frown \langle i, \sigma \rangle$ at the element w, we have that it must choose some use $w' < \gamma_{\beta_j \frown \langle i,\sigma\rangle}[u'](w) < \gamma_{\beta_i \frown \langle i,\sigma\rangle}[u'](w')$. Now suppose that α' enumerates $\gamma_{\beta_i \frown \langle i,\sigma\rangle}(w')[t]$ whilst visiting the edge $\alpha' \frown \langle h_i, \sigma' \rangle$ at some stage $t \geq s$. Then we have that $\gamma_{\beta_i \frown \langle i,\sigma\rangle}(w')[t] > \gamma_{\beta_j \frown \langle i,\sigma\rangle}(w)[t]$. But $\gamma_{\beta_j \frown \langle i,\sigma\rangle}[t](w) > \phi_{j,2}[t](\phi_{j,1}[t](w))$ and $\gamma_{\beta_j \frown \langle i,\sigma\rangle}[t](w) > \phi_{j,3}[t](\phi_{j,1}[t](w))$, as required.

(3.1.4) Consider an edge $\alpha' \frown \langle h_i, \sigma' \rangle$ lying to the right of $\alpha \frown \langle o, \sigma \rangle$.

In order for the strategy α to attach the witness w to the edge $\alpha \frown \langle o, \sigma \rangle$ at stage s, it must be the case that $\phi_{j,2}[s](\phi_{j,1}[s](w)) \downarrow$ and $\phi_{j,3}[s](\phi_{j,1}[s](w)) \downarrow$ for every strategy $\beta_j \in (\beta_1, \ldots, \beta_n)$. Hence we have that $\phi_{j,2}[s](\phi_{j,1}[s](w)) < s$ and $\phi_{j,3}[s](\phi_{j,1}[s](w)) < s$ for every such β_j. In addition, the strategy α must have been accessible at stage s.

But this means that the edge $\alpha' \frown \langle h_i, \sigma' \rangle$, which lies to the right of α is initialised at stage s. Hence any witness attached to the edge $\alpha' \frown \langle h_i, \sigma' \rangle$ is detached at stage s. Now, if some witness w' is attached to the edge $\alpha' \frown \langle h_i, \sigma' \rangle$ at some stage $u > s$, we must have that $w' > s$. But this also means that $\gamma_{\beta_i \frown \langle i,\sigma \rangle}(w') > s$.

Hence, if α' enumerates $\gamma_{\beta_i \frown \langle i,\sigma \rangle}(w')$ into the set D at some stage $t \geq s$, we have that $\gamma_{\beta_i \frown \langle i,\sigma \rangle}(w') > \phi_{j,2}[s](\phi_{j,1}[s](w))$ and that $\gamma_{\beta_i \frown \langle i,\sigma \rangle}(w') > \phi_{j,2}[s](\phi_{j,1}[s](w)) < s$ for every strategy $\beta_j \in (\beta_1, \ldots, \beta_n)$.

(3.2) Suppose that some $\beta'_j \in (\beta'_1, \ldots, \beta'_n)$ is following a Γ-strategy.

In this case we have that α' can only enumerate elements into A by visiting some edge of the form $\alpha' \frown \langle g_i, \sigma' \rangle$, for some $\beta'_i \in (\beta'_1, \ldots, \beta'_n)$ such that β'_i follows a Γ-strategy. On the other hand, α' can enumerate elements into D by visiting some edge of the form $\alpha' \frown \langle g_i, \sigma' \rangle$ for some $\beta'_i \in (\beta'_1, \ldots, \beta'_n)$ following a Γ-strategy, or else by visiting some edge of the form $\alpha' \frown \langle h_i, \sigma' \rangle$ for some $\beta'_i \in (\beta'_1, \ldots, \beta'_n)$. We perform the following case analysis.

(3.2.1) Consider an edge $\alpha' \frown \langle g_i, \sigma' \rangle$ lying above α.

Firstly, we consider the strategies $\beta'_j \in (\beta'_1, \ldots, \beta'_n)$. Since α lies below the edge $\alpha' \frown \langle g_i, \sigma' \rangle$, we have that no strategy β'_j for $i \leq j \leq m$ is active for α. On the other hand it is possible for strategies β'_j for $1 \leq j < i$ to also be active for α. Hence we shall need to show that if α' enumerates some witness w' into A at stage t, it must be the case that $w' > \phi_{j,2}[t](\phi_{j,1}[t](w))$ and $w' > \phi_{j,3}[t](\phi_{j,1}[t](w))$ for all $1 \leq j < i$.

To prove this we observe that in order for α to attach the witness w to the edge $\alpha \frown \langle o, \sigma \rangle$ at stage s, α must have been accessible at stage s. But this means that α' must also have been accessible at stage s and that a work interval $(v, \gamma_{\beta'_i \frown \langle i,\sigma \rangle}(v)[u])$ must have been defined for the edge $\alpha' \frown \langle g_i, \sigma' \rangle$ at some stage $u < s$.

But this means that $v < w < v + n_{\beta'_i \frown \langle i,\sigma \rangle}[s]$. Hence there is some stage $p < s$ such that a constraint to the effect that $\gamma_{\beta'_i \frown \langle i,\sigma\rangle}(v)[p'] > \gamma_{\beta'_j \frown \langle i,\sigma\rangle}(v + n_{\beta'_i \frown \langle i,\sigma\rangle}[s])[p']$ has been imposed on β'_i for all stages $p' > p$ and $1 \leq j < i$.

In addition, $\gamma_{\beta'_i \frown \langle i,\sigma\rangle}[p](v)$ is enumerated into D at stage p, undefining the functional $\Gamma^{U_i,D}_{\beta'_i \frown \langle i,\sigma\rangle}(v)$. Hence it follows that when the functional is redefined we have that $\gamma_{\beta'_i \frown \langle i,\sigma\rangle}(v)[t] > \gamma_{\beta'_j \frown \langle i,\sigma\rangle}(w)[t]$ for all $1 \leq j < i$ and for all $t \geq s$.

Now suppose that α' enumerates a witness w' attached to the edge $\alpha' \frown \langle g_i, \sigma'\rangle$ at some stage $t \geq s$. Then we have that $w' > \gamma_{\beta'_i \frown \langle i,\sigma\rangle}(v)[t]$. But $\gamma_{\beta'_i \frown \langle i,\sigma\rangle}(v)[t] > \gamma_{\beta'_j \frown \langle i,\sigma\rangle}(w)[t]$ for all $1 \leq j < i$. In addition $\gamma_{\beta'_j \frown \langle i,\sigma\rangle}(w)[t] > \phi_{j,2}[t](\phi_{j,1}[t](w))$ and $\gamma_{\beta'_j \frown \langle i,\sigma\rangle}(w)[t] > \phi_{j,3}[t](\phi_{j,1}[t](w))$ for all $1 \leq j < i$. Hence it follows that $w' > \phi_{j,2}[t](\phi_{j,1}[t](w))$ and $w' > \phi_{j,3}[t](\phi_{j,1}[t](w))$ for all $1 \leq j < i$ as required.

Secondly, we consider the sequence of $\mathcal{R}$ strategies $\beta''_j \in (\beta''_1, \ldots, \beta''_l)$ which are active for α and lie below α'. In this case we shall need to show that if α' enumerates some witness w' into A at stage t, it must be the case that $w' > \phi_{j,2}[t](\phi_{j,1}[t](w))$ and $w' > \phi_{j,3}[t](\phi_{j,1}[t](w))$ for every β''_j.

To prove this we observe that such a β''_j lies below the edge $\alpha' \frown \langle g_i, \sigma'\rangle$ of the strategy. We have already determined that α' defines the work interval $(v, \gamma_{\beta'_i \frown \langle i,\sigma\rangle}(v)[u])$ at some stage u. Then if at some stage $u' \geq u$ the strategy β''_j defines the functional associated with the edge leading to α at the element w, we have that it must choose some use $v < \gamma_{\beta''_j \frown \langle i,\sigma\rangle}[u'](w) < \gamma_{\beta'_i \frown \langle i,\sigma\rangle}[u'](v)$. It is also the case that $s > u'$ since α can only attach its witness w to $\alpha \frown \langle o, \sigma\rangle$ after the work interval is defined at stage u and after $\Gamma^{U,D}_{\beta''_j \frown \langle i,\sigma\rangle}(w)$ has been defined for every $\beta''_j \in (\beta''_1, \ldots, \beta''_l)$.

Now suppose that α' enumerates a witness w' attached to the edge $\alpha' \frown \langle g_i, \sigma'\rangle$ at some stage $t \geq s$. Then we have that $w' > \gamma_{\beta'_i \frown \langle i,\sigma\rangle}(v)[t]$. But $\gamma_{\beta'_i \frown \langle i,\sigma\rangle}(v)[t] > \gamma_{\beta''_j \frown \langle i,\sigma\rangle}(w)[t]$ for all $1 \leq j < i$. In addition $\gamma_{\beta''_j \frown \langle i,\sigma\rangle}(w)[t] > \phi_{j,2}[t](\phi_{j,1}[t](w))$ and $\gamma_{\beta''_j \frown \langle i,\sigma\rangle}(w)[t] > \phi_{j,3}[t](\phi_{j,1}[t](w))$ for all $1 \leq j < i$. Hence it follows that $w' > \phi_{j,2}[t](\phi_{j,1}[t](w))$ and $w' > \phi_{j,3}[t](\phi_{j,1}[t](w))$ for all $\beta''_j \in (\beta''_1, \ldots, \beta''_n)$ such that $1 \leq j < i$ as required.

We also need to consider the situation where α' enumerates some element into D when it visits some edge of the form $\alpha' \frown \langle g_i, \sigma'\rangle$ at some stage $t \geq s$.

In the third paragraph of case (3.2.1) we determined that in order for w to be attached to $\alpha \frown \langle o, \sigma \rangle$ at stage s, it must be the case that some work interval $(v, \gamma_{\beta'_i \frown \langle i,\sigma \rangle}(v))$ must have been defined for the edge $\alpha' \frown \langle g_i, \sigma' \rangle$ at some stage $u < s$.

We have also seen that for all $t \geq s$ we have that $\gamma_{\beta'_i \frown \langle i,\sigma \rangle}[t](v) > \gamma_{\beta'_j \frown \langle i,\sigma \rangle}[t](w)$ for all $\beta'_j \in (\beta'_1, \ldots, \beta'_n)$ with $1 \leq j < i$. Hence if α' enumerates $\gamma_{\beta'_i \frown \langle i,\sigma \rangle}[t](v)$ into D at some stage $t \geq s$ we have that $\gamma_{\beta'_i \frown \langle i,\sigma \rangle}[t](v) > \phi_2[s](\phi_1[s](w))$ and that $\gamma_{\beta'_i \frown \langle i,\sigma \rangle}[t](v) > \phi_3[s](\phi_1[s](w))$ as required.

Similarly we have also seen that for all $t \geq s$ we have that $\gamma_{\beta'_i \frown \langle i,\sigma \rangle}[t](v) > \gamma_{\beta''_j \frown \langle i,\sigma \rangle}[t](w)$ for all $\beta''_j \in (\beta''_1, \ldots, \beta''_l)$. Hence if α' enumerates $\gamma_{\beta'_i \frown \langle i,\sigma \rangle}[t](v)$ into D at some stage $t \geq s$ we have that $\gamma_{\beta'_i \frown \langle i,\sigma \rangle}[t](v) > \phi_2[s](\phi_1[s](w))$ and that $\gamma_{\beta'_i \frown \langle i,\sigma \rangle}[t](v) > \phi_3[s](\phi_1[s](w))$ as required.

(3.2.2) Consider an edge $\alpha' \frown \langle g_i, \sigma' \rangle$ lying to the right of $\alpha \frown \langle o, \sigma \rangle$.

Let $(\beta'_1, \ldots, \beta'_n)$ be the sequence of $\mathcal{R}$ strategies lying above α' and which are active for α'.

In order for the strategy α to attach the witness w to the edge $\alpha \frown \langle o, \sigma \rangle$ at stage s, it must be the case that $\phi_{j,2}[s](\phi_{j,1}[s](w)) \downarrow$ and $\phi_{j,3}[s](\phi_{j,1}[s](w)) \downarrow$ for every strategy $\beta_j \in (\beta_1, \ldots, \beta_n)$. Hence we have that $\phi_{j,2}[s](\phi_{j,1}[s](w)) < s$ and $\phi_{j,3}[s](\phi_{j,1}[s](w)) < s$ for every strategy $\beta_j \in (\beta_1, \ldots, \beta_n)$ which is active for α.

In addition the strategy α must have been accessible at stage s. But this means that the edge $\alpha' \frown \langle g_i, \sigma' \rangle$, which lies to the right of α is initialised at stage s. Hence any work interval defined for the edge $\alpha' \frown \langle g_i, \sigma' \rangle$ is undefined at stage s.

Now, if some work interval $(v, \gamma_{\beta'_i \frown \langle i,\sigma' \rangle}[u](v))$ is defined for the edge $\alpha' \frown \langle g_i, \sigma' \rangle$ at some stage $u > s$, we must have that $v > s$. But this also means that $\gamma_{\beta'_i \frown \langle i,\sigma' \rangle}[u](v) > s$. In addition if a witness w' is attached to the edge $\alpha' \frown \langle g_i, \sigma' \rangle$ at some stage $u' > u$ we have that $w' > \gamma_{\beta'_i \frown \langle i,\sigma' \rangle}[u'](v)$.

Therefore if α' enumerates w' into the set A at some stage $t > u'$, we have that $\gamma_{\beta'_i \frown \langle i,\sigma' \rangle}[t](v) > \phi_{j,2}[s](\phi_{j,1}[s](w))$ and that $\gamma_{\beta'_i \frown \langle i,\sigma' \rangle}[t](w') > \phi_{j,3}[s](\phi_{j,1}[s](w))$ for every strategy $\beta_j \in (\beta_1, \ldots, \beta_n)$.

Similarly if α' enumerates $\gamma_{\beta'_i \frown \langle i,\sigma' \rangle}[t'](v)$ into the set D at some $t' > u$, we have that

$\gamma_{\beta_i' \frown \langle i,\sigma' \rangle}[t'](v) > \phi_{j,2}[s](\phi_{j,1}[s](w))$ and that $\gamma_{\beta_i' \frown \langle i,\sigma' \rangle}[t'](v) > \phi_{j,3}[s](\phi_{j,1}[s](w))$ for every strategy $\beta_j \in (\beta_1, \ldots, \beta_n)$.

(3.2.3) Consider an edge $\alpha' \frown \langle h_i, \sigma' \rangle$ lying above α.

In this case the proof is the same as the one for case 3.1.3.

(3.2.4) Consider an edge $\alpha' \frown \langle h_i, \sigma' \rangle$ lying to the right of $\alpha \frown \langle o, \sigma \rangle$.

In this case the proof is the same as the one for case 3.1.4.

(4) Consider a strategy α' such that $\alpha <_L \alpha'$.

In order for the strategy α to attach the witness w to the edge $\alpha \frown \langle o, \sigma \rangle$ at stage s, it must be the case that $\phi_{i,2}[s](\phi_{i,1}[s](w)) \downarrow$ and $\phi_{i,3}[s](\phi_{i,1}[s](w)) \downarrow$ for every strategy $\beta_i \in (\beta_1, \ldots, \beta_n)$. Hence we have that $\phi_{i,2}[s](\phi_{j,1}[s](w)) < s$ and $\phi_{i,3}[s](\phi_{i,1}[s](w)) < s$ for every such β_i. In addition, the strategy α must have been accessible at stage s.

Let $\alpha' \frown \langle o', \sigma' \rangle$ be an edge of the strategy α'. Since the strategy α' is initialised at stage s we have that any witness attached to $\alpha' \frown \langle o', \sigma' \rangle$ is detached at stage s. Now, if some witness w' is attached to $\alpha' \frown \langle o', \sigma' \rangle$ at some stage $u > s$, we must have that $w' > s$. Similarly if the strategy chooses a threshold v to define a work interval for the edge $\alpha' \frown \langle o', \sigma' \rangle$ at some stage $u > s$ we have that $v > s$.

Hence, if $\alpha' \frown \langle o', \sigma' \rangle$ has outcome d or g_j and α' enumerates a witness w' into the set A at some stage $t \geq s$ we have that $w' > \phi_{i,2}[s](\phi_{i,1}[s](w))$ and that $w' > \phi_{i,2}[s](\phi_{i,1}[s](w))$ for every strategy $\beta_i \in (\beta_1, \ldots, \beta_m)$.

Similarly if $\alpha' \frown \langle o', \sigma' \rangle$ has outcome g_j and α' enumerates $\gamma_{\beta_j \frown \langle i,\sigma \rangle}[t](v)$ into the set D at some stage $t \geq s$ we have that $\gamma_{\beta_j \frown \langle i,\sigma \rangle}[t](v) > \phi_{i,2}[s](\phi_{i,1}[s](w))$ and that $\gamma_{\beta_j \frown \langle i,\sigma \rangle}[t](v) > \phi_{i,2}[s](\phi_{i,1}[s](w))$ for every strategy $\beta_i \in (\beta_1, \ldots, \beta_m)$.

Finally if $\alpha' \frown \langle o', \sigma' \rangle$ has outcome h_j and α' enumerates $\gamma_{\beta_j \frown \langle i,\sigma \rangle}[t](w')$ into the set D at some stage $t \geq s$ we have that $\gamma_{\beta_j \frown \langle i,\sigma \rangle}[t](w') > \phi_{i,2}[s](\phi_{i,1}[s](w))$ and that $\gamma_{\beta_j \frown \langle i,\sigma \rangle}[t](w') > \phi_{i,2}[s](\phi_{i,1}[s](w))$ for every strategy $\beta_i \in (\beta_1, \ldots, \beta_m)$ as required.

(5) Consider the strategy β such that $\alpha \subset \beta$.

Let $\alpha \frown \langle o, \sigma \rangle$ be the edge of α on the true path.

We perform the following case analysis.

(5.1) Suppose that β lies below $\alpha \frown \langle o, \sigma \rangle$.

In order for the strategy β to enumerate some element into the set D at some stage $t \geq s$, it must be the case that the strategy α visits its edge $\alpha \frown \langle o, \sigma \rangle$ at stage t and that it goes to the next substage.

Now α attaches the witness w to $\alpha \frown \langle o, \sigma \rangle$ at stage s. Hence we have that this edge is in open mode at stage s. Let $u \geq s$ be the least stage such that β is accessible at stage u. For this to be the case, u must be an α-open stage. But this means that α visits its edge $\alpha \frown \langle o, \sigma \rangle$ at an α-open stage while it is in open mode. Hence α enumerates w into A into stage u.

Hence we have that if β enumerates some element into D at some stage $t \geq s$, α has already enumerated w into A at some stage $t' \leq t$.

(5.2) Suppose that β lies below some $\alpha \frown \langle o', \sigma' \rangle$ where $\alpha \frown \langle o', \sigma' \rangle$ lies to the right of $\alpha \frown \langle o, \sigma \rangle$.

In order for the strategy α to attach the witness w to the edge $\alpha \frown \langle o, \sigma \rangle$ at stage s, it must be the case that $\phi_{i,2}[s](\phi_{i,1}[s](w)) \downarrow$ and $\phi_{i,3}[s](\phi_{i,1}[s](w)) \downarrow$ for every strategy $\beta_i \in (\beta_1, \ldots, \beta_n)$. In addition, the strategy α must have been accessible at stage s.

Now in order for β to be accessible at some least stage $u \geq s$ it must be the case that α has visited the edge $\alpha \frown \langle o', \sigma' \rangle$ at stage u. If w is no longer attached to the edge $\alpha \frown \langle o, \sigma \rangle$ at stage u, the lemma is satisfied trivially. Otherwise we have that α imposes the downward restraint $d(\alpha \frown \langle o', \sigma' \rangle, u)$ on strategies lying below the edge $\alpha \frown \langle o', \sigma' \rangle$.

Now β is not accessible at any stage t such that $s < t < u$ and w has been attached to $\alpha \frown \langle o, \sigma \rangle$ at stage s. Suppose β had been accessible at some greatest stage $s' < s$. Then when it becomes accessible at stage u it notes that $d(\alpha \frown \langle o', \sigma' \rangle, u) > d(\alpha \frown \langle o', \sigma' \rangle, s')$, where s' is the greatest stage less than s such that β was accessible. Hence at stage u we have that β will cancel every functional attached to each of its edges. Otherwise we have that β was never accessible prior to stage s, and that it has never defined any functional at any element.

Therefore if β defines the functional associated to an edge $\beta \frown \langle i, \sigma \rangle$ by choosing some use x at some stage $u' \geq u$, we must have that $x > \phi_{i,2}[s](\phi_{i,1}[s](w))$ and that $x > \phi_{i,3}[s](\phi_{i,1}[s](w))$ for every strategy $\beta_i \in (\beta_1, \ldots, \beta_m)$. Hence, if β enumerates the use x into the set D at some stage $t \geq u' \geq s$ we have that $x > \phi_{i,2}[s](\phi_{i,1}[s](w))$ and that $x > \phi_{i,2}[s](\phi_{i,1}[s](w))$ for every strategy $\beta_i \in (\beta_1, \ldots, \beta_m)$, as required.

(6) Consider a strategy β such that $\beta \subset \alpha$.

Let $\beta \frown \langle i, \sigma \rangle$ be the edge of the strategy β above α.

The strategy α attaches the witness w to the edge $\alpha \frown \langle o, \sigma \rangle$ at stage s. In order for α to attach this witness at stage s, we have that α has to be accessible at stage s.

But this means that β must visit the edge $\beta \frown \langle i, \sigma \rangle$ at stage s and go to the next substage. This can only be the case if β does not enumerate any element into D at stage s.

Now, in order for β to enumerate an element at some stage $t > s$ into the set D, it must be the case that some strategy α' below β has enumerated some element w' into the set A at some stage t' such that $s \leq t' < t$, creating a disagreement between $\Gamma^{U,D}_{\beta \frown \langle i, \sigma \rangle}(w')$ and $A(w')$. This would cause β to enumerate $\gamma_{\beta \frown \langle i, \sigma \rangle}[t](w')$ at stage t in order to undefine $\Gamma^{U,D}_{\beta \frown \langle i, \sigma \rangle}(w')$.

We perform the following case analysis.

(6.1) Suppose that α' is above α. Then by cases 3.1.1, 3.1.2, 3.2.1 and 3.2.2 we have that if α' enumerates a witness w' into A at stage t', it must be the case that $w' > \phi_{j,2}[s](\phi_{j,1}[s](w))$ and $w' > \phi_{j,3}[s](\phi_{j,1}[s](w))$ for every $\beta_j \in (\beta_1, \ldots, \beta_n)$.

But $\gamma_{\beta \frown \langle i, \sigma \rangle}[t](w') > w'$. Hence if β enumerates $\gamma_{\beta \frown \langle i, \sigma \rangle}[t](w')$ into the set D at stage t, it must be the case that $\gamma_{\beta \frown \langle i, \sigma \rangle}[t](w') > \phi_{j,2}[s](\phi_{j,1}[s](w))$ and $\gamma_{\beta \frown \langle i, \sigma \rangle}[t](w') > \phi_{j,3}[s](\phi_{j,1}[s](w))$ for every strategy $\beta_j \in (\beta_1, \ldots, \beta_n)$ as required.

(6.2) Suppose that α' is to the right of α.

In order for the strategy α to attach the witness w to the edge $\alpha \frown \langle o, \sigma \rangle$ at stage s, it must be the case that $\phi_{i,2}[s](\phi_{i,1}[s](w)) \downarrow$ and $\phi_{i,3}[s](\phi_{i,1}[s](w)) \downarrow$ for every strategy $\beta_i \in (\beta_1, \ldots, \beta_n)$. Hence we have that $\phi_{i,2}[s](\phi_{j,1}[s](w)) < s$ and $\phi_{i,3}[s](\phi_{i,1}[s](w)) < s$ for every such β_i. In addition, the strategy α must have been accessible at stage s.

Let $\alpha' \frown \langle o', \sigma' \rangle$ be an edge of the strategy α'. Since the strategy α' is initialised at stage s we have that any witness attached to $\alpha' \frown \langle o', \sigma' \rangle$ is detached at stage s. Now, if some witness w' is attached to $\alpha' \frown \langle o', \sigma' \rangle$ at some stage $u > s$, we must have that $w' > s$.

Hence, if $\alpha' \frown \langle o', \sigma' \rangle$ has outcome d or g_j and α' enumerates a witness w' into the set A at some stage $u' > s$ we have that $w' > \phi_{i,2}[s](\phi_{i,1}[s](w))$ and that $w' > \phi_{i,2}[s](\phi_{i,1}[s](w))$ for every strategy $\beta_i \in (\beta_1, \ldots, \beta_m)$.

But $\gamma_{\beta \frown \langle i,\sigma \rangle}[t](w') > w'$. Hence if β enumerates $\gamma_{\beta \frown \langle i,\sigma \rangle}[t](w')$ into the set D at stage t, it must be the case that $\gamma_{\beta \frown \langle i,\sigma \rangle}[t](w') > \phi_{i,2}[s](\phi_{i,1}[s](w))$ and $\gamma_{\beta \frown \langle i,\sigma \rangle}[t](w') > \phi_{i,3}[s](\phi_{i,1}[s](w))$ for every strategy $\beta_i \in (\beta_1, \ldots, \beta_n)$ as required.

(7) Consider a strategy β such that $\alpha <_L \beta$.

In order for the strategy β to attach the witness w to the edge $\alpha \frown \langle o, \sigma \rangle$ at stage s, it must be the case that $\phi_{i,2}[s](\phi_{i,1}[s](w)) \downarrow$ and $\phi_{i,3}[s](\phi_{i,1}[s](w)) \downarrow$ for every strategy $\beta_i \in (\beta_1, \ldots, \beta_n)$. Hence we have that $\phi_{i,2}[s](\phi_{j,1}[s](w)) < s$ and $\phi_{i,3}[s](\phi_{i,1}[s](w)) < s$ for every such β_i. In addition, the strategy α must have been accessible at stage s.

Let $\beta \frown \langle i, \sigma' \rangle$ be an edge of the strategy β. Since the strategy β is initialised at stage s, we have that the functional associated to $\beta \frown \langle i, \sigma' \rangle$ is canceled at stage s.

Hence if β visits the edge $\beta \frown \langle i, \sigma' \rangle$ at some stage $u > s$ and defines the corresponding functional at some element x, it must choose a use $\gamma_{\beta \frown \langle i,\sigma' \rangle}[u](x)$ which is greater than s.

This means that if the strategy β enumerates $\gamma_{\beta \frown \langle i,\sigma' \rangle}[t](x)$ into D at some stage $t > s$, we have that $\gamma_{\beta \frown \langle i,\sigma' \rangle}[t](x) > s$. It follows that $\gamma_{\beta \frown \langle i,\sigma' \rangle}[t](x) > \phi_{j,2}[s](\phi_{j,1}[s](w))$ and that $\gamma_{\beta \frown \langle i,\sigma' \rangle}[t](x) > \phi_{j,3}[s](\phi_{j,1}[s](w))$ for every $\beta_i \in (\beta_1, \ldots, \beta_m)$ as required. □

3.8.6 Synchronisation Lemma

The Synchronisation Lemma shows that if γ is a strategy on the true path and there are no edges with outcome d on the path leading to γ we have that γ is accessible during infinitely many γ-open stages and infinitely many γ-close stages.

Lemma 3.8.7. *(Synchronisation Lemma). Let f be the true path and let $\gamma = f \upharpoonright n$ for some n. If $f(m)$ is defined and not equal to $\langle d, \sigma \rangle$ for every $m < n$, we have that there are infinitely many γ-open stages and infinitely many γ-close stages.*

Proof. We prove this lemma by induction on n.

For the Base Case $n = 0$ we have that the antecedent holds trivially. Therefore we need to show that there are infinitely many γ_0-open stages and infinitely many γ_0-close stages, where $\gamma_0 = f \upharpoonright 0$.

Now the strategy γ_0 is located at the root of the priority tree. Hence it is accessible at every stage. In addition since γ_0 has no other $\mathcal{R}$ or $\mathcal{S}$ strategy above it, it follows that every stage satisfies conditions (O1)-(O3) and (C1)-(C2) for γ_0. Hence there are infinitely many γ-open stages and infinitely many γ-close stages as required.

For the Inductive Case we proceed as follows.

First we assume that the lemma holds for $n = k$ as our Inductive Hypothesis.

Let $\gamma_k = f \upharpoonright k$. Then we have that if $f(m)$ is defined and not equal to $\langle d, \sigma \rangle$ for every $m < k$, there are infinitely many γ_k-open stages and infinitely many γ_k-close stages.

We then prove that the lemma holds for $n = k + 1$.

Let $\gamma_{k+1} = f \upharpoonright k + 1$. Then we have to prove that if $f(m)$ is defined and not equal to $\langle d, \sigma \rangle$ for every $m < k + 1$, we have that there are infinitely many γ_{k+1}-open stages and infinitely many γ_{k+1}-close stages.

Assume that $f(m)$ is defined and not equal to $\langle d, \sigma \rangle$ for every $m < k+1$. Then we have that $f(m)$ is defined and not equal to $\langle d, \sigma \rangle$ for every $m < k$ as well. Hence by the Inductive Hypothesis we have that there are infinitely many γ_k-open stages and infinitely many γ_k-close stages.

We shall now show that there are infinitely many γ_{k+1}-open stages and infinitely many γ_{k+1}-close stages.

Let $\gamma_k \frown \langle o, \sigma \rangle$ be the edge lying on the true path. We shall show that the strategy γ_k visits the edge $\gamma_k \frown \langle o, \sigma \rangle$ at infinitely many γ_k-open stages and at infinitely many γ_k-close stages such that it goes to the next substage in each case. This makes γ_{k+1} accessible during infinitely many γ_{k+1}-open stages and infinitely many γ_{k+1}-close stages.

If γ_k is an $\mathcal{R}$ strategy we perform a case analysis depending on the outcome of edge $\gamma_k \frown \langle o, \sigma\rangle$.

(1) Suppose $\gamma_k \frown \langle f, \sigma\rangle$ is on the true path. By the *Leftmost Path Lemma* (Lemma 3.8.4) there is some stage s_0 after which edges to the left of $\gamma_k \frown \langle f, \sigma\rangle$ are inaccessible.

We start by noting the following fact. Suppose that γ_k visits the edge $\gamma_k \frown \langle f, \sigma\rangle$ at some stage t and sets the variable suspend to *true* at stage t. Let $t' > t$ be the least stage such that γ_k sets the variable suspend to *false* at stage t'. Then we have that γ_k visits the edge $\gamma_k \frown \langle f, \sigma\rangle$ at every γ_k-stage t'' such that $t < t'' \leq t'$.

Now suppose that at some stage $s_1 > s_0$, the strategy γ_k visits the edge $\gamma_k \frown \langle f, \sigma\rangle$, and that the edge is in open mode at stage s_1. If stage s_1 is not a γ_k-open stage the variable *suspend* is set to true at stage s_1. Hence the edge will be visited again by the strategy until it sets the variable *suspend* to false.

Since there are infinitely many γ_k-open stages we have that there is some stage $s_2 > s_1$ such that γ_k visits the edge $\gamma_k \frown \langle f, \sigma\rangle$ at stage s_2, the stage s_2 is an γ_k-open stage and $\gamma_k \frown \langle f, \sigma\rangle$ is in open mode at stage s_2.

Hence we have that the strategy γ_k visits the edge $\gamma_k \frown \langle f, \sigma\rangle$ at stage s_2 and goes to the next substage. Since the strategy γ_k does not build any functional whilst visiting the edge $\gamma_k \frown \langle f, \sigma\rangle$, we also have that conditions (O1)-(O3) are satisfied for γ_{k+1}, and that stage s_2 is a γ_{k+1}-open stage.

In addition the strategy will also change the mode of the edge to close mode and set the variable *suspend* to false at stage s_2. This means that the strategy is now free to visit any other edge.

Now, let s_3 be the least γ_k-stage greater than s_2 such that the strategy visits the edge $\gamma_k \frown \langle f, \sigma\rangle$. If stage s_3 is not a γ_k-close stage, the variable *suspend* is set to true at stage s_3. Hence the edge will be visited again by the strategy until it sets the variable *suspend* to false.

Since there are infinitely many γ_k-close stages we have that there is some stage $s_4 > s_3$

such that γ_k visits the edge $\gamma_k \frown \langle f, \sigma \rangle$ at stage s_4, the stage s_4 is a γ_k-close stage and $\gamma_k \frown \langle f, \sigma \rangle$ is in close mode at stage s_4.

Hence we have that the strategy γ_k visits the edge $\gamma_k \frown \langle f, \sigma \rangle$ at stage s_4 and goes to the next substage. This means that conditions (C1)-(C2) are now satisfied for γ_{k+1}, and that stage s_4 is a γ_{k+1}-close stage.

In addition the strategy will also change the mode of the edge to open mode and set the variable *suspend* to false at stage s_4. Hence the strategy γ_k is now able to visit other edges once again.

Now s_0 is the least stage such that no edge to the left of $\gamma_k \frown \langle f, \sigma \rangle$ is accessible after stage s_0. This also means that s_0 is the greatest stage such that the edge $\gamma_k \frown \langle f, \sigma \rangle$ is initialised.

When the edge is initialised at stage s_0, it is set to open mode. Therefore if u is the least γ_k-stage greater than s_0, we have that $\gamma_k \frown \langle f, \sigma \rangle$ must be in open mode at stage u. Therefore we can use our earlier argument to generate a γ_{k+1}-open stage and a γ_{k+1}-close stage. But the strategy γ_{k+1} returns to its original state once it generates the γ_{k+1}-close stage.

Therefore the argument can be repeated to show that there are infinitely many γ_{k+1}-open stages and infinitely many γ_{k+1}-close stages as required.

(2) Suppose $\gamma_k \frown \langle i, \sigma \rangle$ is on the true path. By the *Leftmost Path Lemma* (Lemma 3.8.4) there is some stage s_0 after which edges to the left of $\gamma_k \frown \langle i, \sigma \rangle$ are inaccessible.

We start by noting the following fact. Suppose that γ_k visits the edge $\gamma_k \frown \langle i, \sigma \rangle$ at some stage t and sets the variable *suspend* to true at stage t. Let $t' > t$ be the least stage such that γ_k sets the variable suspend to *false* at stage t'. Then we have that γ_k visits the edge $\gamma_k \frown \langle i, \sigma \rangle$ at every γ_k-stage t'' such that $t < t'' \leq t'$.

Suppose that γ_k visits its edge $\gamma \frown \langle i, \sigma \rangle$ at some stage $s_1 > s_0$. Let $\Gamma^{U,D}_{\gamma_k \frown \langle i, \sigma \rangle}$ be the functional which is associated to the edge $\gamma_k \frown \langle i, \sigma \rangle$.

We now perform a case analysis depending on whether the edge $\gamma_k \frown \langle i, \sigma \rangle$ is in close mode or in open mode at stage s_1.

(2.1) Suppose $\gamma_k \frown \langle i, \sigma \rangle$ is in close mode at stage s_1. We need to consider the following two cases.

(2.1.1) Suppose there is no m such that $\Gamma^{U,D}_{\gamma_k \frown \langle i,\sigma \rangle}[s_1](m) \neq A[s_1](m)$.

If there is no γ_k-expansionary* stage attached to $\gamma_k \frown \langle i, \sigma \rangle$ at stage s_1 we make the following observations.

The edge $\gamma_k \frown \langle i, \sigma \rangle$ is on the true path. Hence the answer to question Q_1 of the γ_k-strategy must be 'Yes' and there are infinitely many γ_k-expansionary* stages. In addition the strategy γ_k always attaches γ_k-expansionary* stages to the leftmost edge which has already been visited and which has no γ_k-expansionary* stage attached.

Therefore by the *Attachment Procedure Lemma* (Lemma 3.8.5) there must be some least γ_k-stage $s_2 > s_1$ such that a γ_k-expansionary* stage is attached to $\gamma_k \frown \langle i, \sigma \rangle$.

If s_2 is not a γ_k-close stage we have that the strategy γ_k sets *suspend* to true at stage s_2 and goes to the next stage. Hence the edge will be visited again by the strategy until it sets the variable *suspend* to false.

Since there are infinitely many γ_k-close stages we have that there is some stage $s_3 > s_2$ such that γ_k visits the edge $\gamma_k \frown \langle i, \sigma \rangle$ at stage s_3, the stage s_3 is a γ_k-close stage and $\gamma_k \frown \langle i, \sigma \rangle$ is in close mode at stage s_3.

Hence we have that the strategy γ_k visits the edge $\gamma_k \frown \langle i, \sigma \rangle$ at stage s_3 and goes to the next substage. This means that conditions (C1)-(C2) are satisfied for γ_{k+1}, and that stage s_3 is a γ_{k+1}-close stage.

Now at stage s_3 the strategy γ_k sets *suspend* to false and sets the mode of the edge to open mode and set the variable *suspend* to false at stage s_4. Hence the strategy γ_k is now able to visit other edges once again.

We also note that since the edge is in close mode for every stage t such that $s_1 \leq t \leq s_2$ we have that no $\mathcal{S}$ strategy α below γ is accessible during stage t. Hence no such strategy α can enumerate an element m' into the set A during stage t. This means that no new disagreement $\Gamma^{U,D}_{\gamma_k \frown \langle i,\sigma \rangle}(m') \neq A(m')$ can be created at stage t.

Now suppose that the strategy γ_k visits $\gamma_k \frown \langle i, \sigma \rangle$ again at some least stage $s_4 > s_3$. If there is no γ_k-expansionary* stage attached to $\gamma_k \frown \langle i, \sigma \rangle$ at stage s_4 then by the *Attachment Procedure Lemma* (Lemma 3.8.5) there must be some least γ_k-stage $s_5 > s_4$

such that a γ_k-expansionary* stage is attached to $\gamma_k \frown \langle i, \sigma \rangle$ as before.

If s_5 is not a γ_k-open stage we have that the strategy γ_k sets *suspend* to true at stage s_4 and goes to the next stage. Hence the edge will be visited again by the strategy until it sets the variable *suspend* to false.

Since there are infinitely many γ_k-open stages we have that there is some stage $s_6 > s_5$ such that γ_k visits the edge $\gamma_k \frown \langle i, \sigma \rangle$ at stage s_6, the stage s_6 is an γ_k-open stage and $\gamma_k \frown \langle i, \sigma \rangle$ is in open mode at stage s_6. Hence we have that the strategy γ_k goes to the next substage at stage s_6.

In addition the strategy does not go to the next substage for every stage t such that $s_5 \leq t \leq s_6$. Hence no $\mathcal{S}$ strategy α below γ is accessible during stage t. It follows that no such strategy α can enumerate an element m' into the set A during stage t, and that no new disagreement $\Gamma^{U,D}(m') \neq A(m')$ can be created at stage t.

Hence we have that conditions (O1)-(O3) are satisfied for γ_{k+1}, and that stage s_5 is a γ_{k+1}-open stage.

Also note that the strategy γ_k sets the edge to close mode and the variable *suspend* to false at stage s_6 allowing the strategy to visit other edges once again. This means that the strategy γ_k has returned to the state which it started from.

(2.1.2) Suppose there is some m such that $\Gamma^{U,D}_{\gamma_k \frown \langle i, \sigma \rangle}[s_1](m) \neq A[s_1](m)$.

If there is no γ_k-expansionary* stage attached to $\gamma_k \frown \langle i, \sigma \rangle$ at stage s_1 we make the following observations.

The edge $\gamma_k \frown \langle i, \sigma \rangle$ is on the true path. Hence the answer to question Q_1 of the γ_k-strategy must be 'Yes' and there are infinitely many γ_k-expansionary* stages. In addition the strategy γ_k always attaches γ_k-expansionary* stages to the leftmost edge which has already been visited and which has no γ_k-expansionary* stage attached.

Therefore by the *Attachment Procedure Lemma* (Lemma 3.8.5) there must be some least γ_k-stage $s_2 > s_1$ such that a γ_k-expansionary* stage is attached to $\gamma_k \frown \langle i, \sigma \rangle$.

If s_2 is not a γ_k-close stage we have that the strategy γ_k sets *suspend* to true at stage s_2 and goes to the next stage. Hence the edge will be visited again by the strategy until it sets the variable *suspend* to false.

Since there are infinitely many γ_k-close stages we have that there is some stage $s_3 > s_2$ such that γ_k visits the edge $\gamma_k \frown \langle i, \sigma \rangle$ at stage s_3, the stage s_3 is a γ_k-close stage and $\gamma_k \frown \langle i, \sigma \rangle$ is in close mode at stage s_3.

Therefore the strategy γ_k will enumerate $\gamma_{\gamma_k \frown \langle i, \sigma \rangle}[s_3](m)$ into the set D so as to undefine $\Gamma^{U,D}_{\gamma_k \frown \langle i, \sigma \rangle}(m)$. The strategy will also set the edge to open mode and set *suspend* to false and will terminate the stage.

Now since the edge is in close mode for every stage t such that $s_1 \leq t \leq s_3$, we have that no $\mathcal{S}$ strategy α below γ is accessible during stage t. Hence we have that no such strategy α can enumerate an element m' into the set A during stage t. This means that no new disagreement $\Gamma^{U,D}(m') \neq A(m')$ can be created at stage t.

Let s_4 be the least γ_k-stage greater than stage s_3. Then at stage s_4 the strategy γ_k finds itself in case 2.1.1. It follows that we can use the analysis found in that case to obtain a γ_{k+1}-close stage followed by a γ_{k+1}-open stage.

(2.2) $\gamma \frown \langle i, \sigma \rangle$ is in open mode at stage s_1.

(2.2.1) Suppose there is no m such that $\Gamma^{U,D}_{\gamma_k \frown \langle i, \sigma \rangle}[s_1](m) \neq A[s_1](m)$.

If there is no γ_k-expansionary* stage attached to $\gamma_k \frown \langle i, \sigma \rangle$ at stage s_1 we make the following observations.

The edge $\gamma_k \frown \langle i, \sigma \rangle$ is on the true path. Hence the answer to question Q_1 of the γ_k-strategy must be 'Yes' and there are infinitely many γ_k-expansionary* stages. In addition the strategy γ_k always attaches γ_k-expansionary* stages to the leftmost edge which has already been visited and which has no γ_k-expansionary* stage attached.

Therefore by the *Attachment Procedure Lemma* (Lemma 3.8.5) there must be some least γ_k-stage $s_2 > s_1$ such that a γ_k-expansionary* stage is attached to $\gamma_k \frown \langle i, \sigma \rangle$.

If s_2 is not an γ_k-open stage we have that the strategy γ_k sets *suspend* to true at stage s_2 and goes to the next stage. Hence the edge will be visited again by the strategy until it sets the variable *suspend* to false.

Since there are infinitely many γ_k-open stages we have that there is some stage $s_3 > s_2$ such that γ_k visits the edge $\gamma_k \frown \langle i, \sigma \rangle$ at stage s_3, the stage s_3 is an γ_k-open stage and $\gamma_k \frown \langle i, \sigma \rangle$ is in open mode at stage s_3.

Hence we have that the strategy γ_k visits the edge $\gamma_k \frown \langle i, \sigma \rangle$ at stage s_3. It will therefore set the edge to close mode and set *suspend* to false, allowing the strategy γ_k to visit other edges once again. The strategy will then end the stage.

Note that for every stage t such that $s_1 \leq t \leq s_3$, we have that no $\mathcal{S}$ strategy α below γ is accessible during stage t. This follows from the fact that during these stages there is either no γ_k-expansionary* stage attached to the edge, or that t is not a γ_k-open stage, or that γ_k notes that there is a disagreement between the functional and the set A.

Hence we have that no such strategy α can enumerate an element m' into the set A during stage t. This means that no new disagreement $\Gamma^{U,D}(m') \neq A(m')$ can be created at stage t. Therefore we can conclude that stage s_3 satisfies conditions (O1)-(O3) for γ_{k+1} and thus that s_3 is a γ_{k+1}-open stage.

Now, since strategies below γ_k are accessible at stages s_3 it could be the case that one of these strategies enumerates a witness m' into A at stage s_3, creating a disagreement $\Gamma^{U,D}_{\gamma_k \frown \langle i, \sigma \rangle}(m') \neq A(m')$.

If this is not the case let s_4 be the least γ_k-stage greater than s_3. Then at stage s_4 the strategy γ_k finds itself in case 2.1.1. It follows that we can use the analysis found in that case to obtain a γ_{k+1}-close stage as required, followed by yet another γ_{k+1}-open stage.

Otherwise if this is the case let s_4 be the least γ_k-stage greater than s_3. Then at stage s_4 the strategy γ_k finds itself in case 2.1.2. It follows that we can use the analysis found in that case to obtain a γ_{k+1}-close stage as required, followed by yet another γ_{k+1}-open stage.

(2.2.2) Suppose there is an m such that $\Gamma^{U,D}_{\gamma_k \frown \langle i, \sigma \rangle}[s_1](m) \neq A[s_1](m)$.

If there is no γ_k-expansionary* stage attached to $\gamma_k \frown \langle i, \sigma \rangle$ at stage s_1 we make the following observations.

The edge $\gamma_k \frown \langle i, \sigma \rangle$ is on the true path. Hence the answer to question Q_1 of the γ_k-strategy must be 'Yes' and there are infinitely many γ_k-expansionary* stages. In addition the strategy γ_k always attaches γ_k-expansionary* stages to the leftmost edge which has already been visited and which has no γ_k-expansionary* stage attached.

Therefore by the *Attachment Procedure Lemma* (Lemma 3.8.5) there must be some least γ_k-stage $s_2 > s_1$ such that a γ_k-expansionary* stage is attached to $\gamma_k \frown \langle i, \sigma \rangle$.

If s_2 is not a γ_k-open stage we have that the strategy γ_k sets *suspend* to true at stage s_2 and goes to the next stage. Hence the edge will be visited again by the strategy until it sets the variable *suspend* to false.

Since there are infinitely many γ_k-open stages we have that there is some stage $s_3 > s_2$ such that γ_k visits the edge $\gamma_k \frown \langle i, \sigma \rangle$ at stage s_3, the stage s_3 is a γ_k-open stage and $\gamma_k \frown \langle i, \sigma \rangle$ is in open mode at stage s_3.

Hence we have that the strategy γ_k visits the edge $\gamma_k \frown \langle i, \sigma \rangle$ at stage s_3 and notes the disagreement $\Gamma^{U,D}_{\gamma_k \frown \langle i, \sigma \rangle}[s_1](m) \neq A[s_1](m)$. It will therefore set the edge to close mode and set *suspend* to false, allowing the strategy γ_k to visit other edges once again. The strategy will then end the stage.

Let s_4 be the least γ_k-stage greater than s_3. Then at stage s_4 the strategy γ_k finds itself in case 2.1.2. It follows that we can use the analysis found in that case to obtain a γ_{k+1}-close stage followed by a γ_{k+1}-open stage.

If γ_k is an $\mathcal{S}$ strategy we proceed as follows.

Suppose that the edge $\gamma_k \frown \langle o, \sigma \rangle$ is on the true path. Then by the *Leftmost Path Lemma* (Lemma 3.8.4) there is some stage s_0 after which edges to the left of $\gamma_k \frown \langle o, \sigma \rangle$ are inaccessible.

We also note the following fact. Suppose that γ_k visits the edge $\gamma_k \frown \langle o, \sigma \rangle$ at some stage t and sets the variable *suspend* to *true* at stage t. Let $t' > t$ be the least stage such that γ_k sets the variable *suspend* to false at stage t'. Then we have that γ_k visits the edge $\gamma_k \frown \langle o, \sigma \rangle$ at every γ_k-stage t'' such that $t < t'' \leq t'$.

We then perform a case analysis depending on the outcome of edge $\gamma_k \frown \langle o, \sigma \rangle$.

(1) Suppose $\gamma \frown \langle w, \sigma \rangle$ is on the true path.

Consider the situation where the edge $\gamma \frown \langle w, \sigma \rangle$ is in open mode at stage s_1.

If s_1 is not a γ_k-open stage we have that the strategy γ_k sets *suspend* to true at stage s_1 and goes to the next stage. Hence the edge will be visited again by the strategy until it sets the variable *suspend* to false.

Since there are infinitely many γ_k-open stages we have that there is some stage $s_2 > s_1$ such that γ_k visits the edge $\gamma_k \frown \langle w, \sigma \rangle$ at stage s_2, the stage s_2 is a γ_k-open stage and $\gamma_k \frown \langle w, \sigma \rangle$ is in open mode at stage s_2.

Hence we have that the strategy γ_k visits the edge $\gamma_k \frown \langle w, \sigma \rangle$ at stage s_2.

Now s_2 is a γ_k-open stage which must satisfy condition (O3) for γ_k. Let β be any $\mathcal{R}$ strategy with an edge of the form $\beta \frown \langle i, \sigma' \rangle$ above γ_{k+1}, and let $\Gamma^{U,D}_{\beta \frown \langle i, \sigma' \rangle}$ be its corresponding functional. Then it must be the case that there is no element m such that $\Gamma^{U,D}_{\beta \frown \langle i, \sigma' \rangle}[s_2](m) \neq A_{s_2}(m)$.

But since γ_{k+1} is not an $\mathcal{R}$ strategy, we still have that there is no element m such that $\Gamma^{U,D}_{\beta \frown \langle i, \sigma' \rangle}[s_2](m) \neq A_{s_2}(m)$ for some $\mathcal{R}$ strategy β with an edge of the form $\beta \frown \langle i, \sigma' \rangle$ above γ_{k+1}. Therefore stage s_2 satisfies conditions (O1)-(O3) for γ_{k+1} and is a γ_{k+1}-open stage.

In addition to the above we have that at stage s_2, the strategy γ_k will set the edge to close mode and set *suspend* to false, allowing the strategy γ_k to visit other edges once again. The strategy will then go to the next substage.

Let s_3 be the greatest γ_k stage greater than s_2. If s_3 is not a γ_k-close stage we have that the strategy γ_k sets *suspend* to true at stage s_3 and goes to the next stage. Hence the edge will be visited again by the strategy until it sets the variable *suspend* to false.

Since there are infinitely many γ_k-open stages we have that there is some stage $s_4 > s_3$ such that γ_k visits the edge $\gamma_k \frown \langle w, \sigma \rangle$ at stage s_4, the stage s_4 is a γ_k-close stage and $\gamma_k \frown \langle w, \sigma \rangle$ is in close mode at stage s_4.

Hence we have that s_4 satisfies conditions (C1)-(C2) and that s_4 is a γ_{k+1}-close stage.

(2) Suppose $\gamma \frown \langle g_i, \sigma \rangle$ is on the true path for some $1 \leq i \leq m$.

If there is no work interval is defined for $\gamma_k \frown \langle g_i, \sigma \rangle$ at stage s_1 we make the following observations.

After stage s_0, we have that no strategy or edge to the left of γ_k is accessible at stages $s > s_0$. Hence at any such stage s, we have that there are only finitely many edges of the form $\gamma_k \frown \langle g_i, \sigma' \rangle$ lying to the left of $\gamma_k \frown \langle g_i, \sigma \rangle$ whose work interval is defined at stage s.

In addition, we have that the edge $\gamma \frown \langle g_i, \sigma \rangle$ is on the true path. Therefore it must be the case that the answer to question Q_1 is 'Yes' and that the answers to questions $Q_{2.j}$ for every $1 \leq j \leq m$ are also 'Yes'.

Let $(\beta_1, \ldots, \beta_n)$ be the sequence of active $\mathcal{R}$ strategies above γ_k. Also, let $\alpha^* \subset \gamma_k$ be the greatest (under $\subset$) $\mathcal{S}$ strategy which imposes a work interval on γ_k, (a_s, b_s) be the work interval it imposes on γ_k at stage s and n_s be the boundary of the work interval at stage s. Then the positive answer to question Q_1 guarantees there must be infinitely many witnesses w and stages s such that $\Theta^D[s](w) \downarrow= 0$, $a_s < w < b_s$, $a_s < \theta_s(w) < b_s$, $a_s < \theta_s(w) < a_s + n_s$ and such that the computations $\Gamma^{U,D}_{\beta_j \frown \langle i, \sigma \rangle}[s](w)$ are honest for every $1 \leq j \leq n$.

Now, the strategy γ_k will always attach witnesses of the above form to the leftmost edge of the form $\gamma_k \frown \langle g_i, \sigma' \rangle$ which has been already visited and whose work interval is already defined.

In addition we have that after stage s_0 no work interval defined for an edge lying to the left of $\gamma_k \frown \langle g_i, \sigma \rangle$ can be undefined, and that no witness attached to such an edge can be detached. This follows from the fact that no edge to the left of $\gamma_k \frown \langle g_i, \sigma \rangle$ is accessible after stage s_0.

Hence there must be some least γ_k-stage $s_2 > s_1$ such that γ_k visits the edge $\gamma_k \frown \langle g_i, \sigma \rangle$ at stage s_2 and such that every edge lying to the left of $\gamma_k \frown \langle g_i, \sigma \rangle$ whose work interval is defined at stage s_2 has a witness attached to it. This means that at stage s_2 the strategy γ_k will choose a threshold v and define the work interval $(v, \gamma_i[s_2](v))$ for the edge $\gamma_k \frown \langle g_i, \sigma \rangle$.

Now if there is no witness w attached to $\gamma_k \frown \langle g_i, \sigma \rangle$ at stage s_2 we have that there is some stage $s_3 > s_2$ such that a witness is attached to the edge. This follows from the fact that the work interval has now been defined for the edge, and the fact that we have already determined that every edge to the left of $\gamma_k \frown \langle g_i, \sigma \rangle$ which has a work interval defined, already has a witness attached.

Now consider the situation where the edge $\gamma_k \frown \langle g_i, \sigma \rangle$ is in open mode at stage s_3.

If s_3 is not a γ_k-open stage we have that the strategy γ_k sets *suspend* to true at stage s_3 and goes to the next stage. Hence the edge will be visited again by the strategy until it sets the variable *suspend* to false.

Since there are infinitely many γ_k-open stages we have that there is some stage $s_4 > s_3$ such that γ_k visits the edge $\gamma_k \frown \langle g_i, \sigma \rangle$ at stage s_4, the stage s_4 is a γ_k-open stage and $\gamma_k \frown \langle g_i, \sigma \rangle$ is in open mode at stage s_4.

Now s_4 is a γ_k-open stage which must satisfy condition (O3) for γ_k. Let β be any $\mathcal{R}$ strategy with an edge of the form $\beta \frown \langle i, \sigma' \rangle$ above γ_{k+1}, and let $\Gamma^{U,D}_{\beta \frown \langle i, \sigma' \rangle}$ be its corresponding functional. Then it must be the case that there is no element m such that $\Gamma^{U,D}_{\beta \frown \langle i, \sigma' \rangle}[s_4](m) \neq A_{s_4}(m)$.

But since γ_{k+1} is not an $\mathcal{R}$ strategy, we still have that there is no element m such that $\Gamma^{U,D}_{\beta \frown \langle i, \sigma' \rangle}[s_4](m) \neq A_{s_4}(m)$ for some $\mathcal{R}$ strategy β with an edge of the form $\beta \frown \langle i, \sigma' \rangle$ above γ_{k+1}. Therefore stage s_4 satisfies conditions (O1)-(O3) for γ_{k+1} and is a γ_{k+1}-open stage.

In addition to the above we have that at stage s_4, the strategy γ_k will set the edge to close mode and set *suspend* to false, allowing the strategy γ_k to visit other edges once again. The strategy will then go to the next substage.

Let s_5 be the greatest γ_k stage greater than s_4. If s_5 is not a γ_k-close stage we have that the strategy γ_k sets *suspend* to true at stage s_5 and goes to the next stage. Hence the edge will be visited again by the strategy until it sets the variable *suspend* to false.

Since there are infinitely many γ_k-open stages we have that there is some stage $s_6 > s_5$ such that γ_k visits the edge $\gamma_k \frown \langle g_i, \sigma \rangle$ at stage s_6, the stage s_6 is a γ_k-close stage and $\gamma_k \frown \langle g_i, \sigma \rangle$ is in close mode at stage s_6.

Hence we have that the strategy γ_k visits the edge $\gamma_k \frown \langle g_i, \sigma \rangle$ at stage s_6. Since stage s_6 satisfies conditions (C1)-(C2) for γ_{k+1} we have that s_6 is a γ_{k+1}-close stage.

In addition to the above we have that at stage s_6, the strategy γ_k will set the edge to open mode and set *suspend* to false, allowing the strategy γ_k to visit other edges once again. The strategy will then go to the next substage.

(3) Suppose $\gamma \frown \langle h_i, \sigma\rangle$ is on the true path for some $1 \leq i \leq m$.

If there is no witness w attached to $\gamma_k \frown \langle h_i, \sigma\rangle$ at stage s_1 we make the following observations.

The edge $\gamma_k \frown \langle h_i, \sigma\rangle$ is on the true path. Hence the answer to question Q_1 of the γ_k-strategy is 'Yes', the answer to questions $Q_{2.j}$ for all $j < i$ is 'Yes' and the answer to question $Q_{2.i}$ is 'No'.

Let $(\beta_1, \ldots, \beta_n)$ be the sequence of active $\mathcal{R}$ strategies above γ_k. Also, let $\alpha^* \subset \gamma_k$ be the greatest (under $\subset$) $\mathcal{S}$ strategy which imposes a work interval on γ_k, (a_s, b_s) be the work interval it imposes on γ_k at stage s and n_s be the boundary of the work interval at stage s. Then the positive answer to question Q_1 guarantees there must be infinitely many witnesses w and stages s such that $\Theta^D[s](w) \downarrow= 0$, $a_s < w < b_s$, $a_s < \theta_s(w) < b_s$, $a_s < \theta_s(w) < a_s + n_s$ and such that the computations $\Gamma^{U,D}_{\beta_j \frown \langle i,\sigma'\rangle}[s](w)$ are honest for every $j < i$. However, there must also be some stage t such that for all stages $s > t$ and every witness w, we have that the computation $\Gamma^{U,D}_{\beta_i}[s](w)$ is dishonest.

In addition the strategy γ_k always attaches a witness w giving rise to honest computations $\Gamma^{U,D}_{\beta_j \frown \langle i,\sigma'\rangle}[s](w)$ for every $j < i$ and a dishonest computation $\Gamma^{U,D}_{\beta_i \frown \langle i,\sigma'\rangle}[s](w)$ to the leftmost edge of the form $\gamma_k \frown \langle h_i, \sigma\rangle$ which has already been visited and which has no such witness attached.

Therefore by the *Attachment Procedure Lemma* (Lemma 3.8.5) there must be some least γ_k-stage $s_2 > s_1$ such that a witness w giving rise to honest computations $\Gamma^{U,D}_{\beta_j \frown \langle i,\sigma'\rangle}[s_2](w)$ for every $j < i$ and a dishonest computation $\Gamma^{U,D}_{\beta_i \frown \langle i,\sigma'\rangle}[s_2](w)$ is attached to the edge $\gamma_k \frown \langle h_i, \sigma\rangle$.

Now consider the situation where the edge $\gamma_k \frown \langle h_i, \sigma\rangle$ is in Part I mode at stage s_2.

If s_2 is not a γ_k-close stage we have that the strategy γ_k sets *suspend* to true at stage s_2 and goes to the next stage. Hence the edge will be visited again by the strategy until it sets the variable *suspend* to false.

Since there are infinitely many γ_k-close stages we have that there is some stage $s_3 > s_2$ such that γ_k visits the edge $\gamma_k \frown \langle h_i, \sigma \rangle$ at stage s_3, the stage s_3 is a γ_k-close stage and $\gamma_k \frown \langle h_i, \sigma \rangle$ is in Part I mode at stage s_3.

Hence we have that the strategy γ_k visits the edge $\gamma_k \frown \langle h_i, \sigma \rangle$ at stage s_3. Since stage s_3 satisfies conditions (C1)-(C2) for γ_{k+1} we have that s_3 is a γ_{k+1}-close stage.

In addition to the above we have that at stage s_3, the strategy γ_k will set the edge to Part II mode and set *suspend* to false, allowing the strategy γ_k to visit other edges once again. The strategy will then go to the next substage.

Let s_4 be the greatest γ_k stage greater than s_3. If s_4 is not a γ_k-open stage we have that the strategy γ_k sets *suspend* to true at stage s_4 and goes to the next stage. Hence the edge will be visited again by the strategy until it sets the variable *suspend* to false.

Since there are infinitely many γ_k-open stages we have that there is some stage $s_5 > s_4$ such that γ_k visits the edge $\gamma_k \frown \langle h_i, \sigma \rangle$ at stage s_5, the stage s_5 is a γ_k-open stage and $\gamma_k \frown \langle h_i, \sigma \rangle$ is in Part II mode at stage s_5.

Now s_5 is a γ_k-open stage which must satisfy condition (O3) for γ_k. Let β be any $\mathcal{R}$ strategy with an edge of the form $\beta \frown \langle i, \sigma' \rangle$ above γ_{k+1}, and let $\Gamma^{U,D}_{\beta \frown \langle i, \sigma' \rangle}$ be its corresponding functional. Then it must be the case that there is no element m such that $\Gamma^{U,D}_{\beta \frown \langle i, \sigma' \rangle}[s_5](m) \neq A_{s_5}(m)$.

But since γ_{k+1} is not an $\mathcal{R}$ strategy, we still have that there is no element m such that $\Gamma^{U,D}_{\beta \frown \langle i, \sigma' \rangle}[s_5](m) \neq A_{s_5}(m)$ for some $\mathcal{R}$ strategy β with an edge of the form $\beta \frown \langle i, \sigma' \rangle$ above γ_{k+1}. Therefore stage s_5 satisfies conditions (O1)-(O3) for γ_{k+1} and is a γ_{k+1}-open stage.

In addition to the above we have that at stage s_5, the strategy γ_k will set the edge to Part I mode and set *suspend* to false, allowing the strategy γ_k to visit other edges once again. The strategy will then go to the next substage.

(4) Suppose $\gamma \frown \langle d, \sigma \rangle$ is on the true path.

In the Inductive Case we assumed that $f(m)$ is defined and not equal to $\langle d, \sigma \rangle$ for every $m < k + 1$. Since the outcome of γ_k on the true path is $f(k)$, we have that $f(k)$ cannot be of the form $\langle d, \sigma \rangle$ which gives us a contradiction. Hence we do not need to consider this case. $\square$

3.8.7 Qualified Infinite True Path Lemma

We now show that the true path is infinite in length, as long as there are no strategies γ with outcomes of the form $\gamma \frown \langle d, \sigma \rangle$ on the true path. We shall be able to remove this qualification once we prove the *Pseudo Outcome Lemma* (Lemma 3.8.12). It will then follow that the true path is infinite in length.

Lemma 3.8.8. *(Infinite True Path Lemma). Let f be the true path. If $f(m)$ is defined and not equal to $\langle d, \sigma \rangle$ for every $m < n$, we have that $f(n)$ is defined as well.*

Proof. We start by noting that to prove that $f(n)$ is defined for some n, we need to show two things. The first is that there are infinitely many γ_n-stages, where $\gamma_n = f \upharpoonright n$. The second is that $\liminf_s O_s(\gamma_n)$ exists.

We now prove the lemma by induction on n.

For the Base Case $n = 0$ we have that the antecedent holds trivially. Therefore we need to prove that $f(0)$ is defined. Consider $f \upharpoonright 0$. Then we have that the strategy $\gamma_0 = f \upharpoonright 0$ is located at the root of the priority tree. This means that the strategy is accessible at every stage. Hence there are infinitely many γ_0-stages. In addition by the *Collation Lemma* (Lemma 3.1.7) we have that $\liminf_s O_s(\gamma_0)$ exists. This means that $f(0)$ is defined as required.

For the Inductive Case we proceed as follows.

As the Inductive Hypothesis we shall assume that the lemma holds for $n = k$ as the Inductive Hypothesis. Then we have that if $f(m)$ is defined and not equal to $\langle d, \sigma \rangle$ for every $m < k$, then $f(k)$ is defined as well.

We then prove that the lemma holds for $n = k + 1$. Hence we need to show that if $f(m)$ is defined and not equal to $\langle d, \sigma \rangle$ for every $m < k + 1$, we have that $f(k + 1)$ is defined as well.

Suppose that $f(m)$ is defined and not equal to $\langle d, \sigma \rangle$ for every $m < k + 1$. Then $f(m)$ is also defined and not equal to $\langle d, \sigma \rangle$ for every $m < k$. Hence by the Inductive Hypothesis we have that $f(k)$ is defined. This means that there are infinitely many γ_k stages and that $\liminf_s O_s(\gamma_k)$ exists.

We shall now prove that there are infinitely many γ_{k+1}-open stages and infinitely many γ_{k+1}-close stages.

Since $f(m)$ is defined and not equal to $\langle d, \sigma \rangle$ for every $m < k+1$, we have that γ_{k+1} is accessible at infinitely many γ_{k+1}-open stages and infinitely many γ_{k+1}-close stages by the *Synchronisation Lemma* (Lemma 3.8.7). Hence there are infinitely many γ_{k+1} stages as required. Furthermore by the *Collation Lemma* (Lemma 3.1.7) we have that $\liminf_s O_s(\gamma_{k+1})$ exists.

Therefore we have that $f(k+1)$ is defined, as required. $\square$

3.8.8 Restraint Lemma

In the following lemma we shall show that the sum of the restraints which are imposed on any edge lying on the true path will eventually converge to some finite number.

Lemma 3.8.9. *(Restraint Lemma). Let γ be a strategy with edge $\gamma \frown \langle o, \sigma \rangle$ on the true path, and let the total restraint $\hat{r}(\gamma \frown \langle o, \sigma \rangle, s)$ imposed on the edge at stage s be the sum of:*

(1) $\sup\{r(\gamma \frown \langle o', \sigma' \rangle, s) \mid \gamma \frown \langle o', \sigma' \rangle <_L \gamma \frown \langle o, \sigma \rangle \ \wedge \ \gamma \frown \langle o', \sigma' \rangle$ *has been previously accessible*$\}$.

(2) $a(\gamma \frown \langle o, \sigma \rangle, s)$.

(3) $\max\{d(\gamma_1 \frown \langle o_1, \sigma_1 \rangle, s) \ldots d(\gamma_m \frown \langle o_m, \sigma_m \rangle, s)\}$, *where each $\gamma_i \in (\gamma_1, \ldots, \gamma_m)$ is a strategy on the true path leading to γ, and $\gamma_i \frown \langle o_i, \sigma_i \rangle$ is its corresponding edge on the true path.*

(4) $\max\{R_{\alpha_1,s}, \ldots, R_{\alpha_n,s}\}$, *where $(\alpha_1, \ldots, \alpha_n)$ are the $\mathcal{S}$ strategies on the true path leading to γ.*

Then $\lim_s \hat{r}(\gamma \frown \langle o, \sigma \rangle, s)$ exists and is finite.

Proof. By the *Leftmost Path Lemma* (Lemma 3.8.4) there is a least stage s_0 such that no strategy and no edge to the left of $\gamma \frown \langle o, \sigma \rangle$ is accessible at stages $s > s_0$.

We consider the restraints imposed under conditions (1), (2) and (3) and (4) in turn.

(1) Consider the restraint imposed under condition (1).

We perform a case analysis depending on whether the strategy γ is an $\mathcal{R}$ strategy or an $\mathcal{S}$ strategy.

(1.1) Suppose γ is an $\mathcal{R}$ strategy.

We claim that there is some stage $s_1 > s_0$ such that for every stage $s > s_1$ we have that: $\sup\{r(\gamma \frown \langle o', \sigma' \rangle, s) \mid \gamma \frown \langle o', \sigma' \rangle <_L \gamma \frown \langle o, \sigma \rangle \ \wedge \ \gamma \frown \langle o', \sigma' \rangle$ has been previously accessible$\} = \sup\{r(\gamma \frown \langle o', \sigma' \rangle, s_1) \mid \gamma \frown \langle o', \sigma' \rangle <_L \gamma \frown \langle o, \sigma \rangle \ \wedge \ \gamma \frown \langle o', \sigma' \rangle$ has been previously accessible$\}$. Consider $\sup\{r(\gamma \frown \langle o', \sigma' \rangle, t) \mid \gamma \frown \langle o', \sigma' \rangle <_L \gamma \frown \langle o, \sigma \rangle \ \wedge \ \gamma \frown \langle o', \sigma' \rangle$ has been previously accessible$\}$ for any stage $t > s_0$. For each edge $\gamma \frown \langle o', \sigma' \rangle <_L \gamma \frown \langle o, \sigma \rangle$ we have that $r(\gamma \frown \langle o', \sigma' \rangle, t)$ is the least number which is greater than the following two constraints.

Firstly, $r(\gamma \frown \langle o', \sigma' \rangle, t)$ is greater than the greatest stage $t' < t$ such that the edge was accessible at stage t'. Since $\gamma \frown \langle o', \sigma' \rangle$ lies to the left of the true path, we have that the edge is inaccessible at stages $t'' > s_0$. Hence the first constraint contributes a constant value to the restraint $r(\gamma \frown \langle o', \sigma' \rangle, t'')$ at all stages $t'' > s_0$.

Secondly, $r(\gamma \frown \langle o', \sigma' \rangle, t)$ is greater than any γ-expansionary* stage attached to the edge $\gamma \frown \langle o', \sigma' \rangle$. Suppose that the edge $\gamma \frown \langle o', \sigma' \rangle$ has no γ-expansionary* stage attached to it at some stage $t' > s_0$. Then it is either the case that no γ-expansionary* stage will ever be attached, or that one will be attached at some stage $u \geq t'$.

In the latter case we have that the edge $\gamma \frown \langle o', \sigma' \rangle$ is inaccessible after stage s_0. Therefore we have that once attached, the γ-expansionary* stage is never detached from the edge. Similarly, the edge $\gamma \frown \langle o', \sigma' \rangle$ is never initialised at stages $s > s_0$. It follows that the second constraint contributes a constant value to the restraint $r(\gamma \frown \langle o', \sigma' \rangle, t)$ at all stages $s \geq u$.

Therefore given an edge $\gamma \frown \langle o', \sigma' \rangle$ lying to the left of $\gamma \frown \langle o, \sigma \rangle$, we have that $r(\gamma \frown \langle o', \sigma' \rangle, s) = r(\gamma \frown \langle o', \sigma' \rangle, u)$ for all $s > u$.

Since this argument holds for every edge $\gamma \frown \langle o', \sigma' \rangle$ lying to the left of $\gamma \frown \langle o, \sigma \rangle$, and only finitely many such edges are accessible at stages $s \leq s_0$, it follows that there is some stage s_1 such that for all $s > s_1$ we have that: $\sup\{r(\gamma \frown \langle o', \sigma' \rangle, s) \mid \gamma \frown \langle o', \sigma' \rangle <_L \gamma \frown \langle o, \sigma \rangle \ \wedge \ \gamma \frown \langle o', \sigma' \rangle$ has been previously accessible$\} = \sup\{r(\gamma \frown \langle o', \sigma' \rangle, s_1) \mid \gamma \frown \langle o', \sigma' \rangle <_L \gamma \frown \langle o, \sigma \rangle \ \wedge \ \gamma \frown \langle o', \sigma' \rangle$ has been previously accessible$\}$ as required.

(1.2) Suppose γ is an $\mathcal{S}$ strategy.

We claim that there is some stage $s_1 > s_0$ such that for every stage $s > s_1$ we have that: $\sup\{r(\gamma \frown \langle o', \sigma' \rangle, s) \mid \gamma \frown \langle o', \sigma' \rangle <_L \gamma \frown \langle o, \sigma \rangle \ \wedge \ \gamma \frown \langle o', \sigma' \rangle$ has been previously accessible$\} = \sup\{r(\gamma \frown \langle o', \sigma' \rangle, s_1) \mid \gamma \frown \langle o', \sigma' \rangle <_L \gamma \frown \langle o, \sigma \rangle \ \wedge \ \gamma \frown \langle o', \sigma' \rangle$ has been previously accessible$\}$.

To show this we proceed as follows. Consider $\sup\{r(\gamma \frown \langle o', \sigma' \rangle, t) \mid \gamma \frown \langle o', \sigma' \rangle <_L \gamma \frown \langle o, \sigma \rangle \ \wedge \ \gamma \frown \langle o', \sigma' \rangle$ has been previously accessible$\}$ for any $t > s_0$. For each edge $\gamma \frown \langle o', \sigma' \rangle <_L \gamma \frown \langle o, \sigma \rangle$ we have that $r(\gamma \frown \langle o', \sigma' \rangle, t)$ is the least number which is greater than the following four constraints.

Firstly, $r(\gamma \frown \langle o', \sigma' \rangle, t)$ is greater than $\theta_{t'}(w')$, where w' is any witness attached to the edge $\gamma \frown \langle o', \sigma' \rangle$ and t' is the stage at which the witness was attached to the edge. Suppose that the edge $\gamma \frown \langle o', \sigma' \rangle$ has no witness attached to it at some stage $t'' > s_0$. Then it is either the case that no witness w' will ever be attached, or that one will be attached at some stage $u \geq t''$.

In the latter case we have that the edge $\gamma \frown \langle o', \sigma' \rangle$ is inaccessible at stages $s > s_0$. Therefore w' is never detached from the edge once it is attached. Similarly, the edge $\gamma \frown \langle o', \sigma' \rangle$ is never initialised after stage s_0. It follows that the first constraint contributes a constant value to the restraint $r(\gamma \frown \langle o', \sigma' \rangle, s)$ at all stages $s > u$.

Secondly, $r(\gamma \frown \langle o', \sigma' \rangle, t)$ is greater than the greatest stage $t' < t$ such that the edge was accessible at stage t'. Since $\gamma \frown \langle o', \sigma' \rangle$ lies to the left of the true path, we have that the edge is inaccessible at stages $s > s_0$. Hence the second constraint contributes a constant value to the restraint $r(\gamma \frown \langle o', \sigma' \rangle, s)$ at all stages $s > s_0$.

Thirdly and fourthly, $r(\gamma \frown \langle o', \sigma' \rangle, t)$ is greater than $\phi_{i,2}[t'](\phi_{i,1}[t'](w'))$ and $\phi_{i,3}[t'](\phi_{i,1}[t'](w'))$ for every $\mathcal{R}$ strategy $\beta_i \subset \gamma$ labeled $\mathcal{R}_i$ active for γ and w' is some witness attached to the edge and t' is the stage at which the witness was attached to the edge. Suppose that the edge $\alpha \frown \langle o', \sigma' \rangle$ has no witness attached to it at some stage $t'' > s_0$. Then it is either the case that no witness w' will ever be attached, or that one will be attached at some stage u' with $u' > t''$.

In the latter case the edge $\gamma \frown \langle o', \sigma' \rangle$ is inaccessible at stages $s > s_0$. Therefore we have that w' is never detached from the edge. Similarly, the edge $\gamma \frown \langle o', \sigma' \rangle$ is never initialised after stage s_0. Since there are only finitely many such edges which are accessible at some stage $s \leq s_0$, it follows that the third and fourth constraints contribute a constant value to the restraint $r(\gamma \frown \langle o', \sigma' \rangle, s)$ at all stages $s > u'$.

Therefore given an edge $\gamma \frown \langle o', \sigma' \rangle$ lying to the left of $\gamma \frown \langle o, \sigma \rangle$, we have that there is a stage $v = \max\{u, u'\}$ such that for all $s > v$ we have that $r(\gamma \frown \langle o', \sigma' \rangle, s) = r(\gamma \frown \langle o', \sigma' \rangle, v)$.

Since this argument holds for every edge $\gamma \frown \langle o', \sigma' \rangle$ lying to the left of $\gamma \frown \langle o, \sigma \rangle$, and only finitely many such edges are accessible at stages $s \leq s_0$, it follows that there is some stage s_1 such that for all $s > s_1$ we have that: $\sup\{r(\gamma \frown \langle o', \sigma' \rangle, s) \mid \gamma \frown \langle o', \sigma' \rangle <_L \gamma \frown \langle o, \sigma \rangle \ \wedge \ \gamma \frown \langle o', \sigma' \rangle$ has been previously accessible$\} = \sup\{r(\gamma \frown \langle o', \sigma' \rangle, s_1) \mid \gamma \frown \langle o', \sigma' \rangle <_L \gamma \frown \langle o, \sigma \rangle \ \wedge \ \gamma \frown \langle o', \sigma' \rangle$ has been previously accessible$\}$ as required.

(2) Consider the restraint imposed under condition (2).

We claim that there is some stage $s_1 > s_0$ such that for every stage $s > s_1$ we have that $a(\gamma \frown \langle o, \sigma \rangle, s) = 0$. To show this we proceed as follows. We perform a case analysis depending on whether the strategy γ is an $\mathcal{R}$ strategy or an $\mathcal{S}$ strategy.

(2.1) Suppose γ is an $\mathcal{R}$ strategy.

Consider the restraint $a(\gamma \frown \langle o, \sigma \rangle, t)$ for any $t > s_0$. Then we have that $a(\gamma \frown \langle o, \sigma \rangle, t) > 0$ if some γ-expansionary* stage has been attached to some edge $\gamma \frown \langle i, \sigma' \rangle$ lying to the left of $\gamma \frown \langle o, \sigma \rangle$ at stage t.

Suppose that the edge $\gamma \frown \langle i, \sigma' \rangle$ has no γ-expansionary* stage attached to it at some stage $t' > s_0$. Then it is either the case that no γ-expansionary* stage will ever be attached, or that one will be attached at some stage $u \geq t'$.

In the latter case we have that the edge $\gamma \frown \langle i, \sigma' \rangle$ is inaccessible at stages $s > s_0$. It is also the case that the edge $\gamma \frown \langle i, \sigma' \rangle$ is never initialised after stage s_0. Hence we have that the γ-expansionary* stage which is attached to the edge at stage u is never detached by the strategy.

Since this argument holds for every edge $\gamma \frown \langle i, \sigma' \rangle$ lying to the left of $\gamma \frown \langle o, \sigma \rangle$, and only finitely many such edges are accessible at stages $s \leq s_0$, and the strategy γ attaches a γ-expansionary* stage only to those edges which have been previously accessible, it follows that there is some stage s_1 such that for all $s > s_1$ we have that $a(\gamma \frown \langle o, \sigma \rangle, s) = 0$, as required.

(2.2) Suppose γ is an $\mathcal{S}$ strategy.

Consider the restraint $a(\gamma \frown \langle o, \sigma \rangle, t)$ for any $t > s_0$.

If there is no $\mathcal{R}$ strategy $\beta \subset \gamma$ which is active for γ and is following a Γ-strategy, we have that $a(\gamma \frown \langle o, \sigma \rangle, t) > 0$ if some witness w has been attached to some edge $\gamma \frown \langle d, \sigma' \rangle$ lying to the left of $\gamma \frown \langle o, \sigma \rangle$ at stage t.

Consider every edge $\gamma \frown \langle d, \sigma' \rangle <_L \gamma \frown \langle o, \sigma \rangle$ which has been accessible at some stage $s \leq s_0$. If the edge $\gamma \frown \langle d, \sigma' \rangle$ has no witness attached to it at some stage $t' > s_0$, then it is either the case that no witness w' will ever be attached, or that one will be attached at some stage $u \geq t'$. In the latter case we have that the edge $\gamma \frown \langle d, \sigma' \rangle$ is inaccessible at stages $s > s_0$. It is also the case that the edge $\gamma \frown \langle d, \sigma' \rangle$ is never initialised after stage s_0. Hence we have that any witness which is attached to the edge at stage u is never detached by the strategy.

Since this argument holds for every edge $\gamma \frown \langle d, \sigma' \rangle$ lying to the left of $\gamma \frown \langle o, \sigma \rangle$, and only finitely many such edges are accessible at stages $s \leq s_0$, and the strategy γ attaches a witness only to those edges which have been previously accessible, it follows that there is some stage s_1 such that for all $s > s_1$ we have that $a(\gamma \frown \langle o, \sigma \rangle, s) = 0$ for all stages $s > s_1$, as required.

On the other hand, if there is some $\mathcal{R}$ strategy $\beta \subset \gamma$ which is active for γ and is following a

Γ-strategy, we have that $a(\gamma \frown \langle o, \sigma \rangle, t) > 0$ if some witness w has been attached to some edge $\gamma \frown \langle g_i, \sigma' \rangle$ lying to the left of $\gamma \frown \langle o, \sigma \rangle$ at stage t.

Consider every edge $\gamma \frown \langle g_i, \sigma' \rangle <_L \gamma \frown \langle o, \sigma \rangle$ which has been accessible at some stage $s \leq s_0$. If the edge $\gamma \frown \langle g_i, \sigma' \rangle$ has no witness attached to it at some stage $t' > s_0$, then it is either the case that no witness w' will ever be attached, or that one will be attached at some stage $u \geq t'$. In the latter case we have that the edge $\gamma \frown \langle g_i, \sigma' \rangle$ is inaccessible at stages $s > s_0$. It is also the case that the edge $\gamma \frown \langle g_i, \sigma' \rangle$ is never initialised after stage s_0. Hence we have that any witness which is attached to the edge at stage u is never detached by the strategy.

Since this argument holds for every edge $\gamma \frown \langle g_i, \sigma' \rangle$ lying to the left of $\gamma \frown \langle o, \sigma \rangle$, and only finitely many such edges are accessible at stages $s \leq s_0$, and the strategy γ attaches a witness only to those edges which have been previously accessible, it follows that there is some stage s_1 such that for all $s > s_1$ we have that $a(\gamma \frown \langle o, \sigma \rangle, s) = 0$ for all stages $s > s_1$, as required.

(3) Consider the restraint imposed under condition (3).

We claim that there is some stage $s_1 > s_0$ such that for every stage $s > s_1$ we have that $d(\gamma_i \frown \langle o_i, \sigma_i \rangle, s) = d(\gamma_i \frown \langle o_i, \sigma_i \rangle, s_1)$, for every $\gamma_i \subset \gamma$, where $\gamma_i \frown \langle o_i, \sigma_i \rangle$ lies on the path leading to γ.

Consider $d(\gamma' \frown \langle o', \sigma' \rangle, t)$ for some $\gamma' \subset \gamma$ with edge $\gamma' \frown \langle o', \sigma' \rangle$ lying on the true path for any $t \geq s_0$. Then we have that $d(\gamma' \frown \langle o', \sigma' \rangle, t)$ is the least number which is greater than the following two constraints.

Firstly, $d(\gamma' \frown \langle o', \sigma' \rangle, t)$ is greater than $\sup\{r(\gamma' \frown \langle o'', \sigma'' \rangle, t) \mid \gamma' \frown \langle o'', \sigma'' \rangle <_L \gamma' \frown \langle o', \sigma' \rangle \ \wedge \ \gamma' \frown \langle o'', \sigma'' \rangle \text{ has been previously accessible}\}$. But γ' and $\gamma' \frown \langle o', \sigma' \rangle$ are on the true path. Therefore by the proof for condition (1) of the lemma there must be some stage $u > s_0$ such that for all $s > u$ we have that: $\sup\{r(\gamma' \frown \langle o'', \sigma'' \rangle, s) \mid \gamma' \frown \langle o'', \sigma'' \rangle <_L \gamma' \frown \langle o', \sigma' \rangle \ \wedge \ \gamma' \frown \langle o'', \sigma'' \rangle \text{ has been previously accessible}\} = \sup\{r(\gamma' \frown \langle o'', \sigma'' \rangle, u) \mid \gamma' \frown \langle o'', \sigma'' \rangle <_L \gamma' \frown \langle o', \sigma' \rangle \ \wedge \ \gamma' \frown \langle o'', \sigma'' \rangle \text{ has been previously accessible}\}$.

Secondly, $d(\gamma' \frown \langle o', \sigma' \rangle, t)$ is greater than or equal to $a(\gamma' \frown \langle o', \sigma' \rangle, t)$. But γ' and $\gamma' \frown \langle o', \sigma' \rangle$ are on the true path. Therefore by the proof for condition (2) of the lemma there must be some stage $u' > s_0$ such that for all $s > u'$ we have that $a(\gamma' \frown \langle o', \sigma' \rangle, s) = a(\gamma' \frown \langle o', \sigma' \rangle, u') = 0$.

Therefore we have that there is a stage $v = \max\{u, u'\}$ such that for all $s > v$ we have that $r(\gamma' \frown \langle o', \sigma' \rangle, s) = r(\gamma \frown \langle o', \sigma' \rangle, v)$.

Since this argument can be applied to the finitely many strategies $\gamma_i \subset \gamma$, there must be a stage s_1 such that for every $s > s_1$ we have that $d(\gamma_i \frown \langle o_i, \sigma_i \rangle, s) = d(\gamma_i \frown \langle o_i, \sigma_i \rangle, s_1)$, for every $\gamma_i \subset \gamma$, where $\gamma_i \frown \langle o_i, \sigma_i \rangle$ lies on the path leading to γ.

(4) Consider the restraint imposed under condition (4).

We claim that there is some stage $s_1 > s_0$ such that for every $\mathcal{S}$ strategy $\alpha_i \subset \gamma$ and every stage $s > s_1$ we have that $R_{\alpha_i,s} = R_{\alpha_i,s_1}$.

Consider $R_{\alpha',t}$ for some $\mathcal{S}$ strategy $\alpha' \subset \gamma$ at any stage $t > s_0$. Since the strategy α' lies on the true path, we have that no strategy $\gamma' <_L \alpha'$ is accessible at stages $s > s_0$ and that the strategy α' cannot be initialised at stages $s > s_0$. Now if $R_{\alpha',t} = 0$ at some stage $t > s_0$, it is either the case that $R_{\alpha',t'} = 0$ for all $t' \geq t$, or else that there is some stage $u \geq t$ such that $R_{\alpha',u} > 0$. In the latter case we also have that $R_{\alpha',u'} = R_{\alpha',u}$ for all $u' > u$, because α' cannot be initialised at stages $s > s_0$.

Since there are only finitely many $\mathcal{S}$ strategies $\alpha_i \subset \gamma$, we have that there must be some stage $s_1 > s_0$ such that $R_{\alpha_i,s} = R_{\alpha_i,s_1}$ for every stage $s \geq s_1$ and every $\mathcal{S}$ strategy $\alpha_i \subset \gamma$, as required.

Since for each of the restraints imposed by conditions (1), (2) (3) and (4) we have found some stage s_1 such that the corresponding restraints become constant after stage s_1, we have that $\lim_s \hat{r}(\gamma \frown \langle o, \sigma \rangle, s)$ exists and is finite, as required. $\square$

3.8.9 Injury Lemma for $\mathcal{S}$ Strategies

The Injury Lemma for $\mathcal{S}$ Strategies shows that if an $\mathcal{S}$ strategy α lying on the true path enumerates a witness w into A at some stage t after it can no longer be initialised, and finds that it has diagonalised successfully when it becomes accessible again at some stage $t' > t$, we have that no element smaller than or equal to $\theta_t(w)$ can enter into D after stage t. Hence the diagonalisation of the $\mathcal{S}$ strategy is preserved.

Lemma 3.8.10. *(Injury Lemma for $\mathcal{S}$ Strategies) Let α be an $\mathcal{S}$ strategy on the true path f. Let s_0 be the least stage such that for all $s \geq s_0$, no strategy $\gamma <_L \alpha$ is accessible at s. Suppose that the following is the case:*

(i) There is some stage $t \geq s_0$ such that α enumerates a witness w into A.

(ii) $t' > t$ is the least stage such that α is accessible at t'.

(iii) $R_{\alpha,t'} > 0$.

Then no strategy γ enumerates some $x \leq R_{\alpha,t'}$ into D at some stage $s \geq t$.

Proof. Let t be the stage defined in condition (i), that is the least stage such that $t \geq s_0$ and α enumerates a witness w into A. By condition (ii), the strategy α will then diagonalise successfully during the next stage t' at which it is accessible again, and by condition (iii) we have that it sets the restraint $R_{\alpha,t'}$ on D to $\theta_t(w)$.

Now, if any strategy enumerates some element $x \leq \theta_t(w)$ into D at some stage $s \geq t$, it will injure and cancel this diagonalisation. Hence, we need to show that no $\mathcal{S}$ or $\mathcal{R}$ strategy on the priority tree enumerates such an x into D at some stage $s \geq t$.

We start by considering $\mathcal{S}$ strategies α' located to the left, below, above and to the right of α.

(1) Suppose that $\alpha' <_L \alpha$.

Then α' is no longer accessible after stage s_0. Since we have that $t \geq s_0$, it follows that α' cannot enumerate elements into the set D at some stage $s \geq t$.

(2) Suppose that $\alpha \subset \alpha'$.

Since t is an α-open stage we have that t cannot be an α'-close stage. Hence α' cannot enumerate any element into D at stage t. On the other hand when α becomes accessible again at some least stage $t' > t$, we have that α sets the restraint $R_{\alpha,t'}$ to $\theta_t(w)$.

Suppose that the strategy α' becomes accessible again at some least stage $u \geq t$. Then we have that α' detaches all its witnesses and cancels every work interval at stage u. In order for α' to attach a witness w' to one of its edges at some stage $u' \geq u$ it must be the case that $w' > R_{\alpha,t'}$. Similarly in order for α' to choose a threshold v so as to define a work interval for one of its edges at at some stage $u' \geq u$ it must be the case that $v > R_{\alpha,t'}$.

Now in order for the strategy α' to enumerate an element into D at some stage $u' \geq u$, one of two things must be the case.

The first is for α' to visit some edge of the form $\alpha' \frown \langle h_i, \sigma \rangle$ which already has some witness w attached. This causes α' to enumerate $\gamma_{i,u'}(w)$ into D. But in this case we have that $\gamma_{i,u'}(w) > w > R_{\alpha,t'}$ as required.

The second is for the strategy to visit some edge of the form $\alpha' \frown \langle g_i, \sigma \rangle$ whose work interval has already been defined. In this case the strategy will enumerate $\gamma_{i,u'}(v)$ into D, where v is the threshold and lower bound of the work interval. Once again we must then have that $\gamma_{i,u'}(v) > v > R_{\alpha,t'}$, as required.

It follows that α' cannot enumerate an element $x \leq \theta_t(w)$ at some stage $s \geq t$.

(3) Suppose that $\alpha' \subset \alpha$.

Consider the outcome of the edge $\alpha' \frown \langle o, \sigma \rangle$ lying on the true path.

(3.1) Suppose that the outcome is w.

If the strategy α' visits the edge $\alpha' \frown \langle w, \sigma \rangle$, it will not enumerate any element in the set D at some stage $s \geq t$.

On the other hand suppose the strategy α' visits some edge $\alpha' \frown \langle o', \sigma' \rangle$ lying to the right of the edge $\alpha' \frown \langle w, \sigma \rangle$. We claim that the strategy cannot enumerate some element $x \leq \theta_t(w)$ when visiting such an edge at some stage $s \geq t$.

To see why this is the case, consider the stage t at which the strategy α enumerates its witness w into the set A. Since α is accessible at stage t, we have that α' is also accessible at stage t. Hence we have that edges to the right of $\alpha' \frown \langle w, \sigma \rangle$ are initialised at stage t. This means that any witness attached to the edge $\alpha' \frown \langle o', \sigma' \rangle$, and any work interval defined for the edge $\alpha' \frown \langle o', \sigma' \rangle$ is canceled at stage t.

Therefore, if α' attaches a witness w' to the edge $\alpha' \frown \langle o', \sigma' \rangle$ at some stage $u > t$, we have that $w' > t$. Similarly if α' chooses a threshold v to define a work interval for the edge $\alpha' \frown \langle o', \sigma' \rangle$ at some stage $u > t$, we have that $v > t$.

In addition it is important to note that in order for α to have enumerated w into A at stage t, it must be the case that $\theta_t(w) < t$.

Now in order for the strategy α' to enumerate an element into D at some stage $u > t$, one of two things must be the case.

The first is for the strategy to visit some edge of the form $\alpha' \frown \langle h_i, \sigma' \rangle$ which has some witness w' attached, in which case it enumerates $\gamma_{i,u}(w')$ into D. In this case we have that $\gamma_{i,u}(w') > w' > \theta_t(w)$ as required.

The second is for the strategy to visit some edge of the form $\alpha' \frown \langle g_i, \sigma' \rangle$ whose work interval has already been defined. In this case the strategy will enumerate $\gamma_{i,u}(v)$ into D, where v is the threshold and lower bound of the work interval. Once again we must then have that $\gamma_{i,u}(v) > v > \theta_t(w)$, as required.

It follows that α' cannot enumerate an element $x \leq \theta_t(w)$ at some stage $s \geq t$.

(3.2) Suppose that the outcome is h_i for some i.

Consider the witness w enumerated by α into A at stage t. Then the strategy α must have attached w to one of its edges at some stage $u \leq t$. Now α works inside the work interval imposed by the edge $\alpha' \frown \langle h_i, \sigma \rangle$. This means that when α attached the witness w to its edge at stage u, the work interval for $\alpha' \frown \langle h_i, \sigma \rangle$ must have been defined. Let this work interval be $(w', \gamma_{i,u}(w'))$ for some witness w'. Then we have that α must have chosen w such that $w' < \theta(w) < \gamma_{i,u}(w')$.

Now suppose that α' visits the edge $\alpha' \frown \langle h_i, \sigma \rangle$ at some stage $s \geq t$, enumerating $\gamma_{i,s}(w')$ into D. Since the uses chosen when defining $\Gamma_i^{U,D}(w')$ are non-decreasing, we must have that $w' < \theta_t(w) < \gamma_{i,s}(w')$. Hence it follows that α' does not enumerate any $x \leq \theta_t(w)$ into D when visiting the edge $\alpha' \frown \langle h_i, \sigma \rangle$. at some stage $s \geq t$.

The argument for showing that α' does not enumerate any element $x \leq \theta_t(w)$ into D when visiting an edge $\alpha' \frown \langle o', \sigma' \rangle$ lying to the right of $\alpha' \frown \langle h_i, \sigma \rangle$ at some stage $s \geq t$ is similar to the one found in case 3.1.

(3.3) Suppose that the outcome is g_i for some i.

Consider the witness w enumerated by α into A at stage t. Then the strategy α must have attached w to one of its edges at some stage $u \leq t$. Now α works inside the work interval imposed by the edge $\alpha' \frown \langle g_i, \sigma \rangle$. This means that when α attached the witness w to its edge at stage u, the work interval for $\alpha' \frown \langle h_i, \sigma \rangle$ must have been defined. Let this work interval be $(v, \gamma_{i,u}(v))$ for some threshold v. Then we have that α must have chosen w such that $v < \theta(w) < \gamma_{i,u}(v)$.

Now suppose that α' visits the edge $\alpha' \frown \langle g_i, \sigma \rangle$ at some stage $s \geq t$, enumerating $\gamma_{i,s}(v)$ into D. Since the uses chosen when defining $\Gamma_i^{U,D}(v)$ are non-decreasing, we must have that $v < \theta_t(w) < \gamma_{i,s}(v)$. Hence it follows that α' does not enumerate any $x \leq \theta_t(w)$ into D when visiting the edge $\alpha' \frown \langle g_i, \sigma \rangle$ at some stage $s \geq t$.

The argument for showing that α' does not enumerate any element $x \leq \theta_t(w)$ into D when visiting an edge $\alpha' \frown \langle o', \sigma' \rangle$ lying to the right of $\alpha' \frown \langle g_i, \sigma \rangle$ at some stage $s \geq t$ is similar to the one found in case 3.1.

(3.4) Suppose that the outcome is d.

If the strategy α' visits the edge $\alpha' \frown \langle d, \sigma \rangle$, it will not enumerate any element in the set D at any stage $s \geq t$.

The argument for showing that α' does not enumerate any element $x \leq \theta_t(w)$ into D when visiting an edge $\alpha' \frown \langle o', \sigma' \rangle$ lying to the right of $\alpha' \frown \langle d, \sigma \rangle$ at some stage $s \geq t$ is similar to the one found in case 3.1.

(4) Suppose that $\alpha <_L \alpha'$.

When α enumerates its witness w into A at stage t, it initialises the strategy α' which lies to its right. Hence when α' becomes accessible again after stage t, we have that any witness attached to one of its edges, as well as any threshold chosen to define the work interval of any one of its edges, must now be greater than t. Then the argument for showing that α' does not enumerate any element smaller than $\theta_t(w)$ into D when visiting one of its edges at some stage $s \geq t$ is similar to the one found in case (a).

We now consider $\mathcal{R}$ strategies β located to the left, below, above and to the right of α.

(1) Suppose that $\beta <_L \alpha$.

Then β is no longer accessible after stage s_0. Since we have that $t \geq s_0$, it follows that β cannot enumerate elements into the set D at some stage $s \geq t$.

(2) Suppose that $\alpha \subset \beta$.

Since t is an α-open stage we have that t cannot be a β-close stage. Hence β cannot enumerate any element into D at stage t. On the other hand when α becomes accessible again at some least stage $t' > t$, we have that α sets the restraint $R_{\alpha,t'}$ to $\theta_t(w)$.

Therefore when the strategy β becomes accessible again at some least stage $u \geq t$ we have that β cancels every functional built by the strategy whilst visiting one of its edges. In addition in order for β to define one of its functionals at some stage $u' \geq u$, it must choose uses which are greater than the restraint $R_{\alpha,t'}$. Hence it follows that if the strategy β becomes accessible at some stage $s \geq t$ and enumerates some use into D, this element must be greater than $\theta_t(w)$, as required.

(3) Suppose that $\beta \subset \alpha$.

Let α lie below the edge $\beta \frown \langle i, \sigma \rangle$ of the strategy β, and let $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}$ (or $\Gamma^{V,D}_{\beta\frown\langle i,\sigma\rangle}$ resp.) be the functional corresponding to this edge.

Then we need to consider the following two cases. The first concerns the enumeration of elements into A occuring at stage t itself, which may cause a disagreement between $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}$ and A (or $\Gamma^{V,D}_{\beta\frown\langle i,\sigma\rangle}$ resp.). The second concerns the enumeration of elements into A occuring after stage t, which may also cause a disagreement between $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}$ and A (or $\Gamma^{V,D}_{\beta\frown\langle i,\sigma\rangle}$ resp.).

(3.1) When the strategy α enumerates the witness w into the set A at stage t, it is possible that $\mathcal{S}$ strategies α' with $\alpha \subset \alpha'$ also enumerate some witness $w' < w$ into A during stage t. Let w^* be the least witness enumerated into A at stage t. Then the enumeration of this witness could create a least disagreement disagreement between $A(w^*)$ and $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}(w^*)$. Hence it is possible for β to enumerate $\gamma_{\beta\frown\langle i,\sigma\rangle}(w^*) \leq \theta_t(w)$ into D at some stage $s > t$ in order to remove this disagreement.

Before showing that β cannot enumerate such an element into D at some stage $s > t$ we make two preliminary remarks.

The first is that edges $\beta \frown \langle i, \sigma' \rangle$ located to the right of $\beta \frown \langle i, \sigma \rangle$ are initialised at stage t. Hence every functional associated to such an edge is canceled at stage t. This also means that if β defines such a functional after stage t, it must choose uses which are greater than t. But since $\theta_t(w) < t$, it follows that β can never enumerate a use into D which is smaller or equal to $\theta_t(w)$ when visiting such an edge at some stage $s > t$. One should also note that β does not enumerate any use into D when visiting edges of the form $\beta \frown \langle f, \sigma' \rangle$, because no functional is associated to such an edge.

We shall now return to the original problem and show that it is not possible for β to enumerate $\gamma_{\beta \frown \langle i,\sigma \rangle}(w^*) \leq \theta_t(w)$ into D at some stage $s > t$ in order to remove the disagreement between $A(w^*)$ and $\Gamma^{U,D}_{\beta \frown \langle i,\sigma \rangle}(w^*)$ created at stage t.

We split the analysis into two cases, the first where β is active for α and the second where β is inactive for α.

(3.1.1) Suppose that β labeled $\mathcal{R}_i$ is active strategy for α.

In order for α to be accessible at stage t, the strategy β must have visited the edge $\beta \frown \langle i, \sigma \rangle$ and gone to the next substage. But this is only possible if a β-expansionary* stage was attached to the edge at stage t. In addition such a β-expansionary* stage must have become detached from the edge when the strategy β moved to its next substage.

Once α becomes accessible at stage t, it enumerates its witness w into A. Other $\mathcal{S}$ strategies α' with $\alpha \subset \alpha'$ could also enumerate some witness $w' < w$ into A during stage t. Let w^* be the least witness enumerated into A at stage t. Suppose that this creates a least disagreement between $A(w^*)$ and $\Gamma^{U,D}_{\beta \frown \langle i,\sigma \rangle}(w^*)$.

Now, in order for α to become accessible again at some least stage $t' > t$, it follows that the strategy β must have visited the edge $\beta \frown \langle i, \sigma \rangle$ and gone to the next substage. But this is only possible if a β-expansionary* stage p such that $t < p \leq t'$ is attached to the edge at stage t'.

Similarly suppose that β enumerates some use $\gamma_{\beta \frown \langle i,\sigma \rangle, s}(w^*)$ into D at a stage $t'' > t$ to remove the disagreement between $A(w^*)$ and $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}(w^*)$. This can only take place if the strategy β has visited the edge $\beta \frown \langle i,\sigma\rangle$ and a β-expansionary* stage q such that $t < q \leq t''$ is attached to the edge at stage t''.

But in both of these cases, the existence of a β-expansionary* stage p or q implies the existence of some least $\mathcal{R}_i$-expansionary* stage r such that $t < r \leq p$ and $t < r \leq q$. The existence of such an $\mathcal{R}_i$-expansionary* stage r means that a $U_i \upharpoonright \phi_{i,1}[t](w^*)$ or a $V_i \upharpoonright \phi_{i,1}[t](w^*)$ change has taken place at some stage u such that $t < u \leq r$.

Now, when the strategy α becomes accessible again at stage t' it sets the restraint $R_{\alpha,t}$ to $\theta_t(w)$.

If β is following a Γ-strategy, we have that a $U_i \upharpoonright \phi_{i,1}[t](w^*)$ change must have taken place at stage u, or otherwise α would not have set the restraint at stage t'. Hence $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}(w^*)$ must have become undefined at stage u.

On the other hand, if β is following a $\hat{\Gamma}$-strategy, we have that a $V_i \upharpoonright \phi_{i,1}[t](w^*)$ change must have taken place at stage u. Otherwise some $\mathcal{S}$ strategy $\alpha' \subset \alpha$ would have set its restraint $R_{\alpha',q} > 0$ at some stage $q < t'$. This would have caused α to become initialised at stage t' meaning that α would not set its restraint $R_{\alpha,t'} > 0$ at stage t'. Hence $\Gamma^{V,D}_{\beta\frown\langle i,\sigma\rangle}(w^*)$ must have become undefined at stage u once again.

But we have already determined that β can only enumerate $\gamma_{\beta\frown\langle i,\sigma\rangle,s}(w^*)$ into D if the disagreement between $A(w^*)$ and $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}(w^*)$ is present at some stage $r \geq u$. Since in both of the above cases $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}(w^*)$ becomes undefined at stage u, we have that β does not enumerate $\gamma_{\beta\frown\langle i,\sigma\rangle,s}(w^*)$ element into the set D at some stage $s > t$.

Note that since the $U_i \upharpoonright \phi_{i,1}[t](w^*)$ or $V_i \upharpoonright \phi_{i,1}[t](w^*)$ change which takes place at stage u undefines $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}(x)$ (or $\Gamma^{V,D}_{\beta\frown\langle i,\sigma\rangle}(x)$ resp.) for every $x \geq w^*$, it also follows that if $w' > w^*$ is some witness which was enumerated into A at stage t, we have that β does not enumerate $\gamma_{\beta\frown\langle i,\sigma\rangle,s}(w')$ into the set D at some stage $s > t$.

(3.1.2) Suppose that β labeled $\mathcal{R}_i$ is not an active strategy for α.

For the strategy β to be inactive for α, there must exist some $\mathcal{R}$ strategy β' labeled $\mathcal{R}_j$ and some $\mathcal{S}$ strategy α' such that we have $\beta' \subseteq \beta \subset \alpha' \subset \alpha$, and that α' has an edge of the form

$\alpha' \frown \langle g_j, \sigma \rangle$ or $\alpha' \frown \langle h_j, \sigma' \rangle$ lying on the true path, for some $j \leq i$.

Before proceeding we shall need to show that β is R-Synchronised with β'. We start from the observation that for α' to have an edge of the form $\alpha' \frown \langle g_j, \sigma \rangle$ or $\alpha' \frown \langle h_j, \sigma \rangle$ lying on the true path, β' has to be active for α'. Now, assume for contradiction that β was not R-Synchronised with β'. Then β' is not active for β. This means that there must be some intervening strategy α'' between β' and β with an edge of the form $\alpha'' \frown \langle g_k, \sigma \rangle$ or $\alpha'' \frown \langle h_k, \sigma \rangle$ lying on the true path, with $k \leq j$. This would make β' inactive for α', giving a contradiction.

Once α becomes accessible at stage t, it enumerates its witness w into A. Other strategies $\alpha''' \supset \alpha'$ could also enumerate some witness $w' < w$ into A during stage t. Let w^* be the least witness enumerated into A at stage t, and let $\alpha^* \supseteq \alpha$ be the strategy which has enumerated w^* into A at stage t. Suppose that this creates a disagreement between $A(w^*)$ and $\Gamma^{U,D}_{\beta \frown \langle i, \sigma \rangle}(w^*)$.

Suppose α' has an edge of the form $\alpha' \frown \langle g_j, \sigma \rangle$ lying on the true path, for some $j \leq i$.

In order for α to enumerate w into A at stage t, the strategy must have first attached w to one of its edges at some stage $u \leq t$. Similarly in order for α^* to enumerate w^* into A at stage t, the strategy must have first have attached w^* to one of its edges at some stage $u' \leq t$. These stages u and u' must be greater than s_0, or otherwise both α and α^* would be initialised after s_0 and these witnesses would be canceled, meaning that they would not be enumerated into A at stage t.

Now α can only attach its witness w to one of its edges at stage u if the work interval associated to the edge $\alpha' \frown \langle g_j, \sigma \rangle$ is defined at stage u. Similarly, α^* can only attach its witness w^* to one of its edges at stage u' if the work interval associated to the edge $\alpha' \frown \langle g_j, \sigma \rangle$ is defined at stage u'. Let $u_{min} = \min\{u, u'\}$ and $u_{max} = \max\{u, u'\}$.

Then we have that the work interval for the edge $\alpha' \frown \langle g_j, \sigma \rangle$ is defined at stage u_{min}, and is not initialised after stage u_{min}. Let the work interval defined at this stage be $(v, \gamma_{\beta' \frown \langle i, \sigma \rangle, u_{min}}(v))$ for some threshold v. Since for each element the use function of the functional $\Gamma^{U,D}_{\beta' \frown \langle i, \sigma \rangle}$ (or $\Gamma^{V,D}_{\beta' \frown \langle i, \sigma \rangle}$ resp.) is monotonically increasing over the set of stages, we must then have that $v < w < \theta_t(w) < \gamma_{\beta' \frown \langle i, \sigma \rangle, u_{max}}(v)$ and that $v < w^* <$

$\gamma_{\beta' \frown \langle i,\sigma \rangle, u_{max}}(v)$.

Now suppose that β visits the edge $\beta \frown \langle i, \sigma \rangle$ at some stage $s \geq t$, enumerating $\gamma_{\beta \frown \langle i,\sigma \rangle, s}(w^*)$ into D. Since for a given stage the use function of a functional is strictly increasing over the set of elements, we have that $\gamma_{\beta \frown \langle i,\sigma \rangle, s}(v) < \gamma_{\beta \frown \langle i,\sigma \rangle, s}(w^*)$. In addition since β is R-Synchronised with β', we have that $\gamma_{\beta' \frown \langle i,\sigma \rangle, s}(v) < \gamma_{\beta \frown \langle i,\sigma \rangle, s}(v)$. Finally we have that for each element the use function of a functional is monotonically increasing over the set of stages, giving that $\gamma_{\beta' \frown \langle i,\sigma \rangle, u_{max}}(v) \leq \gamma_{\beta' \frown \langle i,\sigma \rangle, s}(v)$. By transitivity, we thus have that $\theta_t(w) < \gamma_{\beta \frown \langle i,\sigma \rangle, s}(w)$.

Now $\gamma_{\beta \frown \langle i,\sigma \rangle, s}(x) > \gamma_{\beta \frown \langle i,\sigma \rangle, s}(w^*)$ for all $x > w^*$, it follows that if $w' > w^*$ is some witness which was enumerated into A at stage t, we have that β does not enumerate $\gamma_{\beta \frown \langle i,\sigma \rangle, s}(w')$ into the set D at some stage $s > t$ either.

Similarly suppose that α' has an edge of the form $\alpha' \frown \langle h_j, \sigma \rangle$ lying on the true path, for some $j \leq i$.

In order for α to enumerate w into A at stage t, the strategy must have first attached w to one of its edges at some stage $u \leq t$. Similarly in order for α^* to enumerate w^* into A at stage t, the strategy must have first have attached w^* to one of its edges at some stage $u' \leq t$. These stages u and u' must be greater than s_0, or otherwise both α and α^* would be initialised after s_0 and these witnesses would be canceled, meaning that they would not be enumerated into A at stage t.

Now α can only attach its witness w to one of its edges at stage u if the work interval associated to the edge $\alpha' \frown \langle h_j, \sigma \rangle$ is defined at stage u. Similarly, α^* can only attach its witness w^* to one of its edges at stage u' if the work interval associated to the edge $\alpha' \frown \langle h_j, \sigma \rangle$ is defined at stage u'. Let $u_{min} = \min\{u, u'\}$ and $u_{max} = \max\{u, u'\}$.

Then we have that the work interval for the edge $\alpha' \frown \langle h_j, \sigma \rangle$ is defined at stage u_{min}, and is not initialised after stage u_{min}. Let the work interval defined at this stage be $(w', \gamma_{\beta' \frown \langle i,\sigma \rangle, u_{min}}(w'))$ for some witness w'. Since for each element the use function of the functional $\Gamma^{U,D}_{\beta' \frown \langle i,\sigma \rangle}$ (or $\Gamma^{V,D}_{\beta' \frown \langle i,\sigma \rangle}$ resp.) is monotonically increasing over the set of stages, we must then have that $w' < w < \theta_t(w) < \gamma_{\beta' \frown \langle i,\sigma \rangle, u_{max}}(w')$ and that $w' < w^* < \gamma_{\beta' \frown \langle i,\sigma \rangle, u_{max}}(w')$.

Now suppose that β visits the edge $\beta \frown \langle i, \sigma \rangle$ at some stage $s \geq t$, enumerating $\gamma_{\beta \frown \langle i,\sigma \rangle, s}(w^*)$ into D. Since for a given stage the use function of a functional is strictly increasing over the set of elements, we have that $\gamma_{\beta \frown \langle i,\sigma \rangle, s}(w') < \gamma_{\beta \frown \langle i,\sigma \rangle, s}(w^*)$. In addition since β is R-Synchronised with β', we have that $\gamma_{\beta' \frown \langle i,\sigma \rangle, s}(w') < \gamma_{\beta \frown \langle i,\sigma \rangle, s}(w')$. Finally we have that for each element the use function of a functional is monotonically increasing over the set of stages, giving that $\gamma_{\beta' \frown \langle i,\sigma \rangle, u_{max}}(w') \leq \gamma_{\beta' \frown \langle i,\sigma \rangle, s}(w')$. By transitivity, we thus have that $\theta_t(w) < \gamma_{\beta \frown \langle i,\sigma \rangle, s}(w)$.

Now $\gamma_{\beta \frown \langle i,\sigma \rangle, s}(x) > \gamma_{\beta \frown \langle i,\sigma \rangle, s}(w^*)$ for all $x > w^*$, it follows that if $w'' > w^*$ is some witness which was enumerated into A at stage t, we have that β does not enumerate $\gamma_{\beta \frown \langle i,\sigma \rangle, s}(w'')$ into the set D at some stage $s > t$ either.

(3.2) When some strategy α' enumerates some witness w' into A at some stage $u > t$ it may create a disagreement between $\Gamma^{U,D}_{\beta \frown \langle i,\sigma \rangle}(w')$ and $A(w')$. Hence β could enumerate $\gamma_{\beta \frown \langle i,\sigma \rangle}[s](w')$ into D at some stage $s > u$ in order to remove this disagreement. We shall show that if α' enumerates a witness w' into A at some stage $u > t$, it must be the case that $w' > \theta_t(w)$. Hence for every stage $s > u$, we have that $\gamma_{\beta \frown \langle i,\sigma \rangle}[s](w') > w' > \theta_t(w)$ as required. We consider $\mathcal{S}$ strategies α' located to the left, below, above and to the right of α.

(3.2.1) Suppose that $\alpha' <_L \alpha$.

Then α' is no longer accessible after stage s_0. Since we have that $t' > t > s_0$, we cannot enumerate elements into the set D at some stage $s > t'$.

(3.2.2) Suppose that $\alpha \subset \alpha'$.

Then α' is only accessible if α is also accessible at the same stage. Now α enumerates the witness w into A at stage t. When α becomes accessible again at some least stage $t' > t$ it sets $R_{\alpha,t'}$ to $\theta_t(w)$. This initialises the strategy α'. Thus when the strategy α' becomes accessible again at some least stage $s \geq t'$, we have that α' detaches every witness attached to its edges. Furthermore at all stages $s'' > t$ we have that α' will only attach a witnesses w' to one of its edges if $w' > R_{\alpha,t'}$. Hence we have that every such $w' > \theta_t(w)$ as required.

(3.2.3) Suppose that $\alpha' \subset \alpha$.

There are two cases to consider.

In the first case, there is no active $\mathcal{R}$ strategy $\beta \subset \alpha'$ which is following a Γ-strategy. In this case, the strategy α' does not have any edges of the form $\alpha' \frown \langle g_i, \sigma\rangle$. Hence, the only way for the strategy α' to enumerate witnesses into the set A is for the witness to be attached to an edge of the form $\alpha' \frown \langle d, \sigma\rangle$.

Now, in order for α to be accessible at stage t, α' also has to be accessible at stage t. In addition, α' must have visited the edge $\alpha \frown \langle d, \sigma\rangle$ at stage t and gone to the next substage. But α can never go to the next substage when visiting the outcome $\alpha \frown \langle d, \sigma\rangle$. But this means that α cannot be accessible at stage t, which gives a contradiction.

Hence the edge $\alpha \frown \langle d, \sigma\rangle$ cannot lie on the true path. Therefore it follows that α' has some edge of the form $\alpha' \frown \langle w, \sigma\rangle$ or $\alpha' \frown \langle h_i, \sigma\rangle$ for some i on the true path. But this means that α' cannot enumerate any witness into A when visiting such an edge.

Let α' have edge $\alpha' \frown \langle o, \sigma\rangle$ on the true path. We now consider the situation concerning edges $\alpha' \frown \langle o, \sigma\rangle$ lying to the right of $\alpha' \frown \langle o, \sigma\rangle$. If α enumerates w into A at stage t, α' must also have been accessible at stage t. Therefore edges of the form $\alpha' \frown \langle o', \sigma'\rangle$ must be initialised at stage t. Hence we have that every witness w' which is attached to some edge lying to the right of $\alpha' \frown \langle o, \sigma\rangle$ must be greater than stage t. Since $\theta_t(w) < t$, we have that $w' > \theta_t(w)$, as required.

In the second case, there is some active $\mathcal{R}$ strategy $\beta \subset \alpha'$ which is following a Γ-strategy. In this case the only way for the strategy α' to enumerate witnesses into the set A is for the witness to be attached to an edge of the form $\alpha' \frown \langle g_i, \sigma\rangle$ for some i.

Suppose that α' has such an edge on the true path. In order for α to have enumerated witness w into A at stage t, α' must have first defined a work interval for its edge at some stage $u < t$. Let this work interval be $(v, \gamma_{\beta,u}(v))$ for some threshold v. Then α must have chosen the witness w such that $\theta_t(w)$ lies inside the work interval. Therefore we have that $v < \theta_t(w) < \gamma_{\beta,u}(v)$. But for α' to attach a witness w' to the edge $\alpha' \frown \langle g_i, \sigma\rangle$, it must be greater than $\gamma_{\beta,u}(v)$. Hence we have that $w' > \theta_t(w)$, as required.

On the other hand, suppose that α' does not have such an edge on the true path. Then α' cannot enumerate a witness w' into A when visiting this edge. On the other hand α' might enumerate some witness w' whilst visiting some edge which lies to the right of the true path.

Now when α enumerates w into A at stage t, α' must also have been accessible. This means that whenever α' attaches some witness w' to an edge lying to the right of the true path after stage t, it must be the case that $w' > t$. Since $\theta_t(w) < t$, we have that $w' > \theta_t(w)$, as required.

(3.2.4) Suppose that $\alpha <_L \alpha'$.

When α enumerates its witness w into A at stage t, every strategy to the right of α is initialised. This means that all witnesses are detached from the edges of α' at stage t. In addition when α' becomes accessible again, it will choose witnesses w' which are greater than stage t. Since $\theta_t(w) < t$ we have that $w' > \theta_t(w)$ as required.

(4) Suppose that $\alpha <_L \beta$.

When α enumerates its witness w into A at stage t, every strategy to the right of α is initialised. Hence every functional built by the β when visiting its edges is canceled at stage t. In addition when β chooses some use in order to define a functional associated to one of its edges, we have that these uses must be greater than stage t. Since $\theta_t(w) < t$ we have that β cannot enumerate an element $x \leq \theta_t(w)$ into D at some stage $s > t$. In addition since and β is not accessible at stage t, we have that β cannot enumerate an element $x \leq \theta_t(w)$ into D at some stage $s \geq t$ as required. □

3.8.10 Injury Lemma for $\mathcal{R}$ Strategies

We now show that if an $\mathcal{R}$ strategy represents an $\mathcal{R}$ requirement on the true path by being active on the true path, we have that the functional built by the strategy will be equal to the set A. To prove this fact, we first show that if the strategy redefines the functional whenever it becomes undefined, there will be some stage after which the functional will never become undefined again.

Lemma 3.8.11. *(Injury Lemma for $\mathcal{R}$ Strategies) Let β be an $\mathcal{R}$ strategy on the true path f. Suppose that β represents requirement $\mathcal{R}_i$ on f by being active on f. Then we have that the following holds for every x.*

(a) *If there are infinitely many stages s such that $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}[s](x)\downarrow$, there is some stage u such that for all $u' \geq u$, $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}[u'](x)\downarrow$ ($\Gamma^{V,D} resp.$)*

(b) $A(x) = \Gamma^{U,D}_{\beta^\frown\langle i,\sigma\rangle}(x)$ *(*$\Gamma^{V,D}_{\beta^\frown\langle i,\sigma\rangle}(x)$ *resp.)*

Proof. Lemma 3.8.11, Part (*a*). By the *Leftmost Path Lemma* (Lemma 3.8.4), we have that there exists some stage s_0 such that strategies and edges to the left of β are inaccessible after stage s_0. Suppose that there are infinitely many stages s such that $\Gamma^{U,D}_{\beta}[s](x)\downarrow$. Our approach will be to classify every strategy on the priority tree into one of the following four cases depending on its behaviour after stage s_0.

(i) The strategy cannot enumerate any element z into D after stage s_0.

(ii) The strategy can only enumerate elements $z > \gamma_{\beta^\frown\langle i,\sigma\rangle}[s](x)$ into D at stages $s > s_0$.

(iii) The strategy can enumerate elements $z \leq \gamma_{\beta^\frown\langle i,\sigma\rangle}[s](x)$ into D at stages $s > s_0$, but imposes no constraint $\gamma_{\beta^\frown\langle i,\sigma\rangle}[t](x) > \gamma_{\beta^\frown\langle i,\sigma\rangle}[s](x)$ for stages $t > s$.

    This means that the strategy β can redefine its functional by choosing the same use $\gamma_{\beta^\frown\langle i,\sigma\rangle}[s](x)$. Since there are only finitely many elements less than $\gamma_{\beta^\frown\langle i,\sigma\rangle}[s](x)$, there is some stage u such that for all $u' > u$ we have that the strategy cannot enumerate some $z' \leq \gamma_{\beta^\frown\langle i,\sigma\rangle}[s](x)$ into D at u'. It follows that the strategy can only enumerate elements $z \leq \gamma_{\beta^\frown\langle i,\sigma\rangle}[s](x)$ into D at some stage $s > s_0$ finitely often.

(iv) The strategy can enumerate elements $z \leq \gamma_{\beta^\frown\langle i,\sigma\rangle}[s](x)$ into D at stages $s > s_0$, and can impose a constraint $\gamma_{\beta^\frown\langle i,\sigma\rangle}[t](x) > \gamma_{\beta^\frown\langle i,\sigma\rangle}[s](x)$ for stages $t > s$.

    In this case we shall prove that the following two facts hold.

    Firstly that the strategy in question only enumerates such a z at some stage s if a work interval based on a witness $w \leq x$ or threshold $v \leq x$ is defined for one of its edges at stage s. Secondly that if such a work interval is defined at stage s, it must become undefined at some stage $s' > s$.

    Since there are only finitely many witnesses and thresholds less than x, there must be a stage u after which no such constraint can be imposed by the strategy. Thus after stage u, we can use the analysis in case (iii) to show that the strategy can only enumerate elements $z \leq \gamma_{\beta^\frown\langle i,\sigma\rangle}[u](x)$ into D at some stage $s > s_0$ finitely often.

Now each strategy classified into cases (iii) and (iv) can only undefine $\Gamma^{U,D}_{\beta^\frown\langle i,\sigma\rangle}(x)$ finitely often.

To ensure that $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}(x)$ is undefined finitely often one also has to show that only finitely many strategies will be classified into cases (iii) and (iv).

For a strategy to be classified into cases (iii) and (iv), it must be able to enumerate elements $z > \gamma_{\beta\frown\langle i,\sigma\rangle}[s](x)$ into D at stages $s > s_0$.

We first consider why only finitely many strategies can be classified into case (iv). In the analysis for case (iv) we have stated that a strategy can only enumerate an elements $z > \gamma_{\beta\frown\langle i,\sigma\rangle}[s](x)$ into D at stage s if a work interval based on a witness $w \leq x$ or a threshold $v \leq x$ is defined for one of the edges of the strategy. Since there are only finitely many such witnesses and thresholds, we have that only finitely many strategies are able to enumerate elements $z > \gamma_{\beta\frown\langle i,\sigma\rangle}[s](x)$ into D at stages $s > s_0$. Hence only finitely many strategies can be classified into case (iv).

We now consider why only finitely many strategies can be classified into case (iii). Since there are only finitely many case (iv) strategies, and since each strategy can only impose finitely many constraints, we have that there is some stage v such that the use of $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}(x)$ is no longer constrained to increase. Since there are only finitely many elements below $\gamma_{\beta\frown\langle i,\sigma\rangle}[u](x)$ it follows that there are only finitely many strategies which can enumerate elements $z \leq \gamma_{\beta\frown\langle i,\sigma\rangle}[s](x)$ into D at stages $s > s_0$. Hence only finitely many strategies can be classified into case (iii).

We now show that every strategy on the priority tree can be classified into cases (i)-(iv) above.

Consider the strategy β itself.

(1) The strategy β can enumerate $\gamma_{\beta\frown\langle i,\sigma\rangle}[s](w)$ into D at stage s in order to repair a disagreement between $A_s(w)$ and $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}[s](w)$.

If $w > x$ we have that $\gamma_{\beta\frown\langle i,\sigma\rangle}[s](w) > \gamma_{\beta\frown\langle i,\sigma\rangle}[s](x)$. Therefore β cannot undefine $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}(x)$ by enumerating $\gamma_{\beta\frown\langle i,\sigma\rangle}[s](w)$ into D at stage s.

If $w \leq x$ we have that $\gamma_{\beta\frown\langle i,\sigma\rangle}[s](x) \leq \gamma_{\beta\frown\langle i,\sigma\rangle}[s](w)$. Therefore if β enumerates $\gamma_{\beta\frown\langle i,\sigma\rangle}[s](w)$ into D, it also undefines $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}(x)$.

Now when β redefines $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}(x)$ at stage t, we have that $\gamma_{\beta\frown\langle i,\sigma\rangle}[t](w) > \gamma_{\beta\frown\langle i,\sigma\rangle}[s](w)$ because $\gamma_{\beta\frown\langle i,\sigma\rangle}[s](w)$ is now an element of D. But this means that it is possible for

$\gamma_{\beta\frown\langle i,\sigma\rangle}[t](x) > \gamma_{\beta\frown\langle i,\sigma\rangle}[s](x)$ to be the case as well.

Since there are only finitely many elements $w \leq x$, it follows that such a disagreement can only occur and be removed finitely often. Thus it follows that β can only undefine $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}(x)$ finitely many times.

Thus we can classify the strategy β into case (iii).

Consider $\mathcal{R}$ strategies β' lying to the left, below, above or to the right of β.

(2) Suppose that $\beta' <_L \beta$.

Then β' is no longer accessible after stage s_0. Hence after stage s_0 we have that β' cannot enumerate any element z into D, and thus cannot undefine $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}(x)$.

Thus we can classify the strategy β' into case (i).

(3) Suppose that $\beta \subset \beta'$.

Then β' can enumerate some element $z \leq \gamma_{\beta\frown\langle i,\sigma\rangle}[s](x)$ into D at some stage $s \geq s_0$, undefining $\Gamma^{U,D}_{\beta}(x)$. However β' does not constrain β to choose a use $\gamma_{\beta\frown\langle i,\sigma\rangle}[t](x) > \gamma_{\beta\frown\langle i,\sigma\rangle}[s](x)$ when redefining its functional at some stage $t > s$. Hence β will redefine $\Gamma^{U,D}_{\beta}(x)$ by choosing $\gamma_{\beta\frown\langle i,\sigma\rangle}[t](x)$ to be equal to $\gamma_{\beta\frown\langle i,\sigma\rangle}[s](x)$.

Thus we can classify the strategy β' into case (iii).

(4) Suppose that $\beta' \subset \beta$.

Let $\beta' \frown \langle i,\sigma'\rangle$ be the edge of β' on the true path f, and let $\beta' \frown \langle i,\sigma''\rangle$ be an edge lying to its right.

We start by showing that β' cannot enumerate some element $z \leq \gamma_{\beta\frown\langle i,\sigma\rangle}[s](x)$ when visiting the edge $\beta' \frown \langle i,\sigma''\rangle$ at some stage $s > s_0$.

Suppose that β defines $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}(x)$ at some stage t. Then we have that it must choose a use $\gamma_{\beta\frown\langle i,\sigma\rangle}(x) < t$. In addition since β is accessible at stage t, the strategy β' must also have been accessible at stage t, and must have visited the edge $\beta' \frown \langle i,\sigma'\rangle$.

This means that the edge $\beta' \frown \langle i,\sigma''\rangle$ must have been initialised at stage t and that the functional attached to it must have been canceled at stage t. Hence if β defines the functional $\Gamma^{U,D}_{\beta'\frown\langle i,\sigma''\rangle}$ at some stage $t' > t$, it must choose some use which is greater than t. It follows that β' cannot enumerate some element $z \leq \gamma_{\beta\frown\langle i,\sigma\rangle}[s](x)$ when visiting the edge $\beta' \frown \langle i, \sigma''\rangle$ at stage $s > s_0$.

We now show that β' cannot enumerate some element $z \leq \gamma_{\beta\frown\langle i,\sigma\rangle}[s](x)$ when visiting the edge $\beta' \frown \langle i, \sigma'\rangle$ at some stage $s > s_0$. We start by observing that β' can enumerate some $\gamma_{\beta'\frown\langle i,\sigma'\rangle}[s](w)$ into D in order to remove a disagreement between $A_s(w)$ and $\Gamma^{U,D}_{\beta'\frown\langle i,\sigma'\rangle}[s](w)$.

Now it is either the case that β' is active for β, or else that β' is inactive for β.

(4.1) Suppose that β' is active for β.

Then we have that β is R-Synchronised with β', and thus that $\gamma_{\beta'\frown\langle i,\sigma'\rangle}(x) < \gamma_{\beta\frown\langle i,\sigma\rangle}(x)$ for every x.

If $w > x$, the enumeration of $\gamma_{\beta'\frown\langle i,\sigma'\rangle}[s](w)$ into D at some stage s may undefine $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}(x)$. However when the strategy β redefines its functional $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}(x)$ at some stage $t > s$, it can choose the use $\gamma_{\beta\frown\langle i,\sigma\rangle}[t](x)$ to be equal to $\gamma_{\beta\frown\langle i,\sigma\rangle}[s](x)$ because R-Synchronisation only requires that $\gamma_{\beta\frown\langle i,\sigma\rangle}(x) > \gamma_{\beta'\frown\langle i,\sigma'\rangle}(x)$, and $\Gamma^{U,D}_{\beta'\frown\langle i,\sigma'\rangle}(x)$ has not been undefined at stage s.

If $w \leq x$, the enumeration of $\gamma_{\beta'\frown\langle i,\sigma'\rangle}[s](w)$ into D at some stage s may undefine $\Gamma^{U,D}_{\beta'\frown\langle i,\sigma'\rangle}(x)$. Since β is R-Synchronised with β' we have that $\gamma_{\beta\frown\langle i,\sigma\rangle}[s](x) > \gamma_{\beta'\frown\langle i,\sigma'\rangle}[s](x)$. This means that $\Gamma_{\beta\frown\langle i,\sigma\rangle}(x)$ will become undefined as well at stage s.

Since $\gamma_{\beta'\frown\langle i,\sigma'\rangle}[s](w)$ has been enumerated into D, we have that β' redefines the functional at stage t such that $\gamma_{\beta'\frown\langle i,\sigma'\rangle}[t](w) > \gamma_{\beta'\frown\langle i,\sigma'\rangle}[s](w)$. But this means that β' could choose $\gamma_{\beta'\frown\langle i,\sigma'\rangle}[t](x) > \gamma_{\beta'\frown\langle i,\sigma'\rangle}[s](x)$ as well. We also have that when the strategy β defines $\Gamma^{U,D}_{\beta\frown\langle i,\sigma'\rangle}(x)$ at stage $t' > t$ it must choose $\gamma_{\beta\frown\langle i,\sigma\rangle}[t'](x) > \gamma_{\beta'\frown\langle i,\sigma'\rangle}[t](x)$ by R-Synchronisation.

Now there are only finitely many elements $w \leq x$. This means that only finitely many disagreements of the form $A(w) \neq \Gamma^{U,D}_{\beta'\frown\langle i,\sigma'\rangle}(x)$ can occur for $w \leq x$. Thus there must be some stage u such that for all $u' \geq u$ β' can only enumerate $\gamma_{\beta'\frown\langle i,\sigma\rangle}[u'](x)$ into D for some $x' > w$.

If β' enumerates $\gamma_{\beta'\frown\langle i,\sigma\rangle}[u'](x')$ into D at some stage $u' \geq u$, it is possible that

$\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}(x)$ may also become undefined at stage u'. But this means that β can redefine the functional at stage $u'' > u'$ by choosing $\gamma_{\beta\frown\langle i,\sigma\rangle}[u''](x)$ to be equal to $\gamma_{\beta\frown\langle i,\sigma\rangle}[u](x)$ whilst still observing R-Synchronisation. This follows from the fact that R-Synchronisation requires only that $\gamma_{\beta\frown\langle i,\sigma\rangle}[u''](x) > \gamma_{\beta'\frown\langle i,\sigma'\rangle}[u''](x)$ and that $\Gamma^{U,D}_{\beta'\frown\langle i,\sigma'\rangle}(x)$ has not been undefined at stage u.

Thus we have that β' cannot cause the use of $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}(x)$ to increase after stage u. Hence there must be some stage v such that for all $v' \geq v$ we have that β' cannot enumerate some $z \leq \gamma_{\beta\frown\langle i,\sigma\rangle}[u](x)$ into D at stage v' either.

Thus we can classify the strategy β' into case (iii).

(4.2) Suppose that β' is inactive for β.

Then when β' enumerates $\gamma_{\beta'\frown\langle i,\sigma\rangle}[s](w)$ into D at some stage s, it may undefine $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}(x)$. But β' does not constrain β to choose a use $\gamma_{\beta\frown\langle i,\sigma\rangle}[t](x) > \gamma_{\beta\frown\langle i,\sigma\rangle}[s](x)$ when redefining its functional at some stage $t > s$. This follows from the fact that β is not R-Synchronised with β'.

Thus we can classify the strategy β' into case (iii).

(5) Suppose that $\beta <_L \beta'$.

If β defines $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}(x)$ at some stage t we have that it must choose a use $\gamma_{\beta\frown\langle i,\sigma\rangle}(x) < t$. In addition since β is accessible at stage t, the strategy β' must have been initialised at stage t. This means that every functional associated to one of the edges of β' is canceled at stage t. Hence if β defines a functional at some stage $t' > t$, it must choose some use which is greater than t. It follows that β' cannot enumerate some element $z \leq \gamma_{\beta\frown\langle i,\sigma\rangle}[s](x)$ when visiting one of its edges at some stage $s > s_0$.

Thus we can classify the strategy β' into case (ii).

We now consider the case where some $\mathcal{S}$ strategy α lying to the left, below, above or to the right of β enumerates some element into D.

(6) Suppose that $\alpha <_L \beta$.

Then α is no longer accessible after stage s_0. Hence after stage s_0 we have that α cannot enumerate any element z into D, and thus cannot undefine $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}(x)$.

Thus we can classify the strategy α into case (i).

(7) Suppose that $\beta \subset \alpha$, and that α is on the true path. The lemma assumes that the strategy β has an edge of the form $\beta \frown \langle i, \sigma\rangle$ on the true path, and that β represents the requirement $\mathcal{R}_i$ on the true path by being active on the true path.

Therefore, we have that the strategy α cannot have outcomes of the form $\alpha \frown \langle h_j, \sigma\rangle$ or $\alpha \frown \langle g_j, \sigma\rangle$ on the true path, for any $j \leq i$. Hence the edge leaving α which lies on the true path must be of the form $\alpha \frown \langle w, \sigma\rangle$, $\alpha \frown \langle h_j, \sigma\rangle$ for $j > i$ or $\alpha \frown \langle g_j, \sigma\rangle$ for $j > i$. This also means that the strategy β is active for α.

Now, the strategy α will have a sequence of active $\mathcal{R}$ strategies $(\beta_1, \ldots, \beta_n)$ above it, with $\beta_m \in (\beta_1, \ldots, \beta_n)$ being labeled $\mathcal{R}_m$. The strategy β_m corresponds to edges of the form $\alpha \frown \langle h_m, \sigma\rangle$ and (if β_m follows a Γ-strategy) edges of the form $\alpha \frown \langle g_m, \sigma\rangle$ of the strategy α. Let the strategy β labeled $\mathcal{R}_i$ correspond to the strategy β_i in the aforementioned sequence.

We shall now perform a case analysis based on the edge leaving α on the true path.

(7.1) Suppose that the edge on the true path is $\alpha \frown \langle w, \sigma\rangle$. Then the strategy α does not enumerate any element into D when it visits the edge.

Thus we can classify the strategy β' into case (i).

(7.2) Suppose that the edge on the true path is $\alpha \frown \langle g_j, \sigma\rangle$ for $j > i$. If α visits this edge at some stage s and a work interval $(v, \gamma_{\beta_j\frown\langle i,\sigma_j\rangle}[s](v))$ is defined for some threshold v, it may enumerate $\gamma_{\beta_j\frown\langle i,\sigma_j\rangle}[s](v)$ into D. This could make $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}(x)$ undefined. However since $j > i$ we have that α does not impose any constraint on β_i to increase its use for $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}(x)$. Hence we have that at stage $t > s$ the strategy β redefines $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}(x)$ by choosing $\gamma_{\beta\frown\langle i,\sigma\rangle}[t](x)$ to be equal to $\gamma_{\beta\frown\langle i,\sigma\rangle}[s](x)$. Since the use of $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}(x)$ does not increase, it follows that there is some stage u such that for all $u' \geq u$, β' can no longer undefine $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}(x)$ by enumerating $\gamma_{\beta_j\frown\langle i,\sigma_j\rangle}[u'](v)$ into D.

Thus we can classify the strategy β' into case (iii).

(7.3) Suppose that the edge on the true path is $\alpha \frown \langle h_j, \sigma\rangle$ for $j > i$. If α visits this edge at some stage s and a work interval $(w, \gamma_{\beta_j \frown \langle i, \sigma_j\rangle}[s](w))$ is defined for some witness w, it may enumerate $\gamma_{\beta_j \frown \langle i, \sigma_j\rangle}[s](w)$ into D. This could make $\Gamma^{U,D}_{\beta \frown \langle i,\sigma\rangle}(x)$ undefined. However since $j > i$ we have that α does not impose any constraint on β_i to increase its use for $\Gamma^{U,D}_{\beta \frown \langle i,\sigma\rangle}(x)$. Hence we have that at stage $t > s$ the strategy β redefines $\Gamma^{U,D}_{\beta \frown \langle i,\sigma\rangle}(x)$ by choosing $\gamma_{\beta \frown \langle i,\sigma\rangle}[t](x)$ to be equal to $\gamma_{\beta \frown \langle i,\sigma\rangle}[s](x)$. Since the use of $\Gamma^{U,D}_{\beta \frown \langle i,\sigma\rangle}(x)$ does not increase, it follows that there is some stage u such that for all $u' \geq u$, β' can no longer undefine $\Gamma^{U,D}_{\beta \frown \langle i,\sigma\rangle}(x)$ by enumerating $\gamma_{\beta_j \frown \langle i,\sigma_j\rangle}[u'](w)$ into D.

Thus we can classify the strategy β' into case (iii).

(7.4) Suppose that the edge on the true path is $\alpha \frown \langle d, \sigma\rangle$. By Lemma 3.8.12 we have that this cannot be the case, so this situation does not need to be considered.

Now, suppose that $\alpha \frown \langle o, \sigma\rangle$ is the edge on the true path. Consider an edge $\alpha \frown \langle o', \sigma'\rangle$ lying to the right of $\alpha \frown \langle o, \sigma\rangle$. If the edge $\alpha \frown \langle o, \sigma\rangle$ becomes accessible at stage t, we have that the edge $\alpha \frown \langle o', \sigma'\rangle$ is initialised at stage t. This means that any work interval defined for this edge is undefined at stage t. In addition, if $\alpha \frown \langle o', \sigma'\rangle$ is accessible at some stage $t' > t$, α will only choose witnesses w and thresholds v which are greater than stage t when defining a work interval for this edge.

Since the edge $\alpha \frown \langle o, \sigma\rangle$ is visited infinitely often, we must have that $\alpha \frown \langle o', \sigma'\rangle$ is initialised infinitely often. Hence there must be some stage q such that for all $q' \geq q$, α chooses only witnesses $w > x$ and thresholds $v > x$ when defining a work interval for the edge $\alpha \frown \langle o', \sigma'\rangle$.

Now, edges to the right of the true path can be of the form $\alpha \frown \langle d, \sigma\rangle$, $\alpha \frown \langle w, \sigma\rangle$, $\alpha \frown \langle g_j, \sigma\rangle$ for $1 \leq j \leq m$ or $\alpha \frown \langle h_j, \sigma\rangle$ for $1 \leq j \leq m$. When α visits edges of the form $\alpha \frown \langle d, \sigma\rangle$ or $\alpha \frown \langle w, \sigma\rangle$, it does not enumerate any element into D. On the other hand if α visits an edge of the form $\alpha \frown \langle g_j, \sigma\rangle$ during some stage s it can cause an element of the form $\gamma_{\beta_j \frown \langle g_j, \sigma_j\rangle}[s](v)$ to enter D. Similarly if α visits an edge of the form $\alpha \frown \langle h_j, \sigma\rangle$ during some stage s it can cause an element of the form $\gamma_{\beta_j \frown \langle g_j, \sigma_j\rangle}[s](w)$ to enter D.

Suppose that $\gamma_{\beta_j \frown \langle g_j, \sigma_j\rangle}[s](v)$ or $\gamma_{\beta_j \frown \langle g_j, \sigma_j\rangle}[s](w)$ has been enumerated into D at some stage $s \geq q$, meaning that w and v are both greater than x.

If $i < j$ we have that $\gamma_{\beta_i \frown \langle i,\sigma_i \rangle}[s](x) < \gamma_{\beta_j \frown \langle i,\sigma_j \rangle}[s](x)$ by R-Synchronisation. In addition $\gamma_{\beta_j \frown \langle i,\sigma_j \rangle}(x)$ is less than both $\gamma_{\beta_j \frown \langle g_j,\sigma_j \rangle}[s](v)$ and $\gamma_{\beta_j \frown \langle g_j,\sigma_j \rangle}[s](w)$. Hence we have that $\Gamma^{U,D}_{\beta_i \frown \langle i,\sigma_i \rangle}(x)$ cannot be undefined at stage s.

If $i = j$, we have that $\gamma_{\beta_i \frown \langle i,\sigma_i \rangle}[s](x)$ is less than both $\gamma_{\beta_i \frown \langle i,\sigma_i \rangle}[s](v)$ and $\gamma_{\beta_i \frown \langle i,\sigma_i \rangle}[s](w)$. Hence we have that $\Gamma^{U,D}_{\beta_i \frown \langle i,\sigma_i \rangle}(x)$ cannot b-e undefined at stage s.

If $j < i$, let k be the least natural number such that $j < k < m$ and the functional $\Gamma^{U,D}_{\beta_k \frown \langle i,\sigma_k \rangle}(x)$ is undefined at stage s. If there is no such k, or $i < k$, we have that $\Gamma^{U,D}_{\beta_i \frown \langle i,\sigma_i \rangle}(x)$ cannot be undefined at stage s.

On the other hand if k exists and $i \geq k$ we have that for every strategy $\beta_n \in (\beta_k, \ldots, \beta_m)$, $\gamma_{\beta_n \frown \langle i,\sigma_n \rangle}[s](x) > \gamma_{\beta_j \frown \langle i,\sigma_j \rangle}[s](x)$ due to R-Synchronisation. Hence, $\Gamma^{U,D}_{\beta_n \frown \langle i,\sigma_n \rangle}(x)$ is undefined at stage s.

In addition we have that for every strategy $\beta_{n'} \in (\beta_1, \ldots, \beta_{k-1})$, $\Gamma^{U,D}_{\beta_{n'} \frown \langle i,\sigma_{n'} \rangle}(x)$ remains defined at stage s. This means that every strategy $\beta_n \in (\beta_k, \ldots, \beta_m)$ can redefine $\Gamma^{U,D}_{\beta_n \frown \langle i,\sigma_n \rangle}(x)$ at some least stage $t_n > s$ by choosing a use $\gamma_{\beta_n \frown \langle i,\sigma_n \rangle}[t_n](x) = \gamma_{\beta_n \frown \langle i,\sigma_n \rangle}[s](x)$ and still satisfy all its R-Synchronisation constraints.

Since $i \geq k$ and the use of $\Gamma^{U,D}_{\beta_i \frown \langle i,\sigma_i \rangle}(x)$ does not increase there must be some stage u such that for all $u' \geq u$, α can no longer undefine $\Gamma^{U,D}_{\beta_i \frown \langle i,\sigma_i \rangle}(x)$ by enumerating elements into D when visiting edges $\alpha \frown \langle o', \sigma' \rangle$lying to the right of $\alpha \frown \langle o, \sigma \rangle$.

(8) Suppose that $\beta \subset \alpha$, and that α is to the right of the true path.

If β defines $\Gamma^{U,D}_{\beta \frown \langle i,\sigma \rangle}(x)$ at some stage t we have that it must choose a use $\gamma_{\beta \frown \langle i,\sigma \rangle}(x) < t$. In addition since the edge $\beta \frown \langle i, \sigma \rangle$ on the true path is accessible at stage t, the strategy α must have been initialised at stage t. This means that every work interval associated to one of the edges of α is canceled at stage t. We also have that if α chooses a witness w or a threshold v at some stage $t' > t$ it follows that $w > t$ and $v > t$.

Therefore if α visits an edge with work interval $(v, \gamma_{\beta' \frown \langle i,\sigma' \rangle}(v))$ at some stage $t' > t$ and enumerates $\gamma_{\beta' \frown \langle i,\sigma' \rangle}(v)$ into D for some strategy β', we must have that $\gamma_{\beta' \frown \langle i,\sigma' \rangle}(v) > t$. Similarly, if α visits an edge with work interval $(w, \gamma_{\beta' \frown \langle i,\sigma' \rangle}(w))$ at some stage $t' > t$ and enumerates $\gamma_{\beta' \frown \langle i,\sigma' \rangle}(w)$ into D for some strategy β', we must have that $\gamma_{\beta' \frown \langle i,\sigma' \rangle}(w) > t$. It follows that α cannot enumerate some element $z \leq \gamma_{\beta \frown \langle i,\sigma \rangle}[s](x)$ when visiting one of its edges at some stage $s > s_0$.

Thus we can classify the strategy α into case (ii).

(9) Suppose that $\alpha \subset \beta$. In order for the strategy α to enumerate some element into the set D, there must be some $\mathcal{R}$ strategy $\beta' \subset \alpha$ which is labeled $\mathcal{R}_i$ and is active for α, and α must visit some edge of the form $\alpha \frown \langle g_i, \sigma \rangle$ or $\alpha \frown \langle h_i, \sigma \rangle$ during some stage s at which the work interval of these edges is defined.

Suppose that an edge of the form $\alpha \frown \langle g_i, \sigma \rangle$ or $\alpha \frown \langle h_i, \sigma \rangle$ lies on the true path. Then at stage s we have that the strategy β lies inside a work interval of the form $(v, \gamma_{\beta' \frown \langle i,\sigma' \rangle}[s](v))$ where v is some threshold, whilst in the second it lies inside a work interval of the form $(w, \gamma_{\beta' \frown \langle i,\sigma' \rangle}[s](w))$ where w is some witness. In either case, whenever the strategy β defines a functional at some element whenever it visits an edge, it must choose uses which lie inside the appropriate work interval.

Now if α visits the edge $\alpha \frown \langle g_i, \sigma \rangle$ at some stage s and the work interval is defined the strategy enumerates $\gamma_{\beta' \frown \langle i,\sigma' \rangle}[s](v)$ into D. Similarly, if α visits the edge $\alpha \frown \langle h_i, \sigma \rangle$ at some stage s and the work interval is defined, the strategy enumerates $\gamma_{\beta' \frown \langle i,\sigma' \rangle}[s](w)$ into D. Since we have that these elements are greater than any use chosen by the strategy β, we have that the strategy α cannot undefine $\Gamma^{U,D}_{\beta \frown \langle i,\sigma \rangle}(x)$.

Now suppose $\alpha \frown \langle o, \sigma \rangle$ lies on the true path, and let $\alpha \frown \langle o', \sigma' \rangle$ be some edge lying to its right. If β is accessible at some stage t and defines $\Gamma^{U,D}_{\beta \frown \langle i,\sigma \rangle}(x)$, it must choose some use $\gamma_{\beta \frown \langle i,\sigma \rangle}[t](x) < t$.

In addition for β to be accessible at stage t, α must have visited the edge $\alpha \frown \langle o, \sigma \rangle$ at stage t. This means that the edge $\alpha \frown \langle o', \sigma' \rangle$ must have been initialised at stage t. Hence any work interval defined for this edge becomes undefined at stage t. In addition if α defines some work interval for the edge at some stage $t' > t$ we have that the work interval must be based on some threshold $v > t$ or witness $w > t$.

Therefore if α visits an edge with work interval $(v, \gamma_{\beta' \frown \langle i,\sigma' \rangle}(v))$ at some stage $s >$

t and enumerates $\gamma_{\beta' \frown \langle i,\sigma' \rangle}[s](v)$ into D for some strategy β', we must have that $\gamma_{\beta' \frown \langle i,\sigma' \rangle}[s](v) > \gamma_{\beta \frown \langle i,\sigma \rangle}[t](x)$. Similarly, if α visits an edge with work interval $(w, \gamma_{\beta' \frown \langle i,\sigma' \rangle}[s](w))$ at some stage $s > t$ and enumerates $\gamma_{\beta' \frown \langle i,\sigma' \rangle}[s](w)$ into D for some strategy β', we must have that $\gamma_{\beta' \frown \langle i,\sigma' \rangle}[s](w) > \gamma_{\beta \frown \langle i,\sigma \rangle}[t](w)$. It follows that α cannot enumerate some element $z \leq \gamma_{\beta \frown \langle i,\sigma \rangle}[s](x)$ when visiting an edge of the form $\alpha \frown \langle o', \sigma' \rangle$ at some stage $s > s_0$.

(10) Suppose that $\beta <_L \alpha$.

If β defines $\Gamma^{U,D}_{\beta \frown \langle i,\sigma \rangle}(x)$ at some stage t we have that it must choose a use $\gamma_{\beta \frown \langle i,\sigma \rangle}[t](x) < t$. In addition since β is accessible at stage t, the strategy α must have been initialised at stage t. This means that every work interval associated to one of the edges of α is canceled at stage t. Hence if α defines a work interval $(v, \gamma_{\beta' \frown \langle i,\sigma \rangle}[t'](v))$ or $(w, \gamma_{\beta' \frown \langle i,\sigma \rangle}[t'](w))$ at some stage $t' > t$ for some strategy β', it must be the case that $v > t$ and $w > t$ respectively. Hence it follows that $\gamma_{\beta \frown \langle i,\sigma \rangle}[t'](v) > \gamma_{\beta \frown \langle i,\sigma \rangle}[t](x)$ and that $\gamma_{\beta \frown \langle i,\sigma \rangle}[t'](v) > \gamma_{\beta \frown \langle i,\sigma \rangle}[t](x)$. It follows that α cannot enumerate some element $z \leq \gamma_{\beta \frown \langle i,\sigma \rangle}[s](x)$ when visiting one of its edges at some stage $s > s_0$. □

Proof. Lemma 3.8.11, Part (*b*). We shall now show that for all x we have that $A(x) = \Gamma^{U,D}_{\beta \frown \langle i,\sigma \rangle}(x)$ ($\Gamma^{V,D}_{\beta \frown \langle i,\sigma \rangle}$ resp.)

We prove the above fact by Strong Induction. Suppose that for every $y < x$, there exists some stage t_y such that for all $s \geq t_y$, $\Gamma^{U,D}_{\beta \frown \langle i,\sigma \rangle}[s](y) = \Gamma^{U,D}_{\beta \frown \langle i,\sigma \rangle}[t_y](y) = A(y)$. Then we prove that for x there exists some stage t such that for all $s \geq t$, $\Gamma^{U,D}_{\beta \frown \langle i,\sigma \rangle}[s](x) = \Gamma^{U,D}_{\beta \frown \langle i,\sigma \rangle}[t](x) = A(x)$.

Let $y' = \max\{t_y \mid 1 \leq y < x\}$. Then $y' > s_0$, where s_0 is the least stage after which strategies to the left of β become inaccessible, and after which β cannot be initialised. For if this was not the case, β would be initialised at or after y', resulting in the functional $\Gamma^{U,D}_{\beta \frown \langle i,\sigma \rangle}[s](y)$ becoming undefined at all elements and contradicting the inductive hypothesis.

We now perform a case analysis, depending on whether $A(x) = 0$ or $A(x) = 1$.

(A) $A(x) = 0$. Suppose that $\Gamma^{U,D}_{\beta \frown \langle i,\sigma \rangle}[u](x) \uparrow$ for some stage $u \geq y'$. We distinguish between two cases, depending on whether $\beta \frown \langle i, \sigma \rangle$ is in open mode or in close mode at stage u.

(i) $\beta \frown \langle i, \sigma \rangle$ is in open mode at stage u.

Then since β has outcome $\beta \frown \langle i, \sigma \rangle$ on the true path, we have that this outcome will be visited infinitely often. For this to be the case, β must reach reach step (5)(b) infinitely often. Suppose that the strategy reaches step (5)(b) at stage $u_1 > u$.

If the edge $\beta \frown \langle i, \sigma \rangle$ does not have a β-expansionary* stage attached at stage u_1, we have that there is some stage $u_2 > u_1$ such that a β-expansionary* stage is attached to this edge by the *Attachment Procedure Lemma* (Lemma 3.8.5).

Suppose that the strategy reaches step (5)(b) again at stage $u_3 > u_2$. Then if u_3 is not a β-open stage, we have that the edge is accessible during some β-open stage $u_4 > u_3$ by the *Synchronisation Lemma* (Lemma 3.8.7).

By the inductive hypothesis, the strategy β will then see that there is no disagreement between the functional and the set A, reach step (5)(b)(iii) and define the computation $\Gamma^{U,D}_{\beta \frown \langle i, \sigma \rangle}[u_4](x)$ to be equal to $A_{u_4}(x) = A(x)$.

(ii) $\beta \frown \langle i, \sigma \rangle$ is in close mode at stage u.

Then since β has outcome $\beta \frown \langle i, \sigma \rangle$ on the true path, we have that this outcome will be visited infinitely often. For this to be the case, β must reach reach step (5)(b) infinitely often.

Suppose that the strategy reaches step (5)(b) at stage $u_1 > u$. If there is no β-expansionary* stage attached to the edge, one will be attached at some stage $u_2 > u_1$ by the *Attachment Procedure Lemma* (Lemma 3.8.5).

If the strategy reaches step (5)(b) again at some stage $u_3 \geq u_2$ and finds that u_3 is not a β-close stage, we have that the strategy visits the edge at some β-close stage $u_4 > u_3$ by the *Synchronisation Lemma* (Lemma 3.8.7).

It will then go through step (5)(b)(viii) and set the edge $\beta \frown \langle i, \sigma \rangle$ to open mode. Case (A)(ii) of the proof then reduces to case (A)(i), which means that $\Gamma^{U,D}_{\beta \frown \langle i, \sigma \rangle}[u'](x)$ is defined as being equal to $A(x)$ at some stage $u' > u_4$ by the strategy β.

(B) $A(x) = 1$. Suppose that $\Gamma^{U,D}_{\beta \frown \langle i, \sigma \rangle}[u](x) \uparrow$ for some stage $u \geq y'$.

We distinguish between two cases, depending on whether $\beta \frown \langle i, \sigma \rangle$ is in open mode or in close mode at stage u.

(i) $\beta \frown \langle i, \sigma \rangle$ is in open mode at stage u.

Since β has outcome $\beta \frown \langle i, \sigma\rangle$ on the true path, we have that this outcome will be visited infinitely often. For this to be the case, β must reach reach step (5)(b) infinitely often. Suppose that the strategy reaches step (5)(b) at stage $u_1 > u$.

If the edge $\beta \frown \langle i, \sigma\rangle$ does not have a β-expansionary* stage attached at stage u_1, by the *Attachment Procedure Lemma* (Lemma 3.8.5) we have that there is some stage $u_2 > u_1$ such that a β-expansionary* stage is attached to this edge.

Suppose that the strategy reaches step (5)(b) again at stage $u_3 \geq u_2$. Then if u_3 is not a β-open stage, we have that the edge is accessible during some β-open stage $u_4 > u_3$ by the *Synchronisation Lemma* (Lemma 3.8.7).

By the inductive hypothesis, the strategy β will see that there is no disagreement between the functional and the set A, reach step (5)(b)(iii) and define the computation $\Gamma^{U,D}_{\beta \frown \langle i,\sigma\rangle}[u_4](x)$ to be equal to $A_{u_4}(x)$, changing the mode of the edge to close mode and detaching the β-expansionary* stage from the edge.

Hence if $A_{u_4}(x) = 1$, we have that the strategy β defines $\Gamma^{U,D}_{\beta \frown \langle i,\sigma\rangle}[u_4](x)$ to be equal to $A(x)$.

On the other hand, it could be the case that $A_{u_4}(x) = 0$. Then we have that $\Gamma^{U,D}_{\beta \frown \langle i,\sigma\rangle}[u_4](x)$ is equal to $A_{u_4}(x)$. However, there must be some least stage $u_5 > u_4$ such that $A_{u_5}(x) = 1$. This means that a disagreement will arise between $\Gamma^{U,D}_{\beta \frown \langle i,\sigma\rangle}[u_5](x)$ and $A_{u_5}(x)$.

Now suppose that the strategy reaches step (5)(b) again at stage $u_6 > u_5$. Then the strategy β will see the disagreement between the functional and the set A. If there is no β-expansionary* stage attached to the edge $\beta \frown \langle i, \sigma\rangle$ at stage u_6, we have that one will be attached to this edge at some stage $u_7 > u_6$ by the *Attachment Procedure Lemma* (Lemma 3.8.5).

Suppose that the strategy β reaches step (5)(b) again at some stage $u_8 \geq u_7$.

We now perform a case analysis depending on whether the edge $\beta \frown \langle i, \sigma\rangle$ is in close mode or in open mode at stage u_8.

(I) $\beta \frown \langle i, \sigma\rangle$ is in close mode at stage u_8.

If no β-expansionary* stage is attached to $\beta \frown \langle i, \sigma\rangle$ at stage u_8, we have that a

β-expansionary* stage will be attached to this edge at some stage $u_9 > u_8$ by the *Attachment Procedure Lemma* (Lemma 3.8.5).

Suppose that the strategy reaches step (5)(b) again at stage $u_{10} \geq s_9$. If u_{10} is not a β-close stage, we have that the strategy will visit the edge again during some close stage $u_{11} > u_{10}$ by the *Synchronisation Lemma* (Lemma 3.8.7).

The strategy will then pass through step (5)(b)(vi) and enumerate $\gamma_{\beta,u_{11}}(x)$ into D, undefining $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}(x)$, and changing the mode of the edge to open mode.

Suppose that the strategy reaches step (5)(b) again at stage $u_{12} > u_{11}$. If u_{12} is not a β-open stage, we have that the strategy will visit the edge again during some close stage $u_{13} > u_{12}$ by the *Synchronisation Lemma* (Lemma 3.8.7).

By the inductive hypothesis, the strategy β will then see that there is no disagreement between the functional and the set A, reach step (5)(b)(iii) and define the computation $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}[u_{13}](x)$ to be equal to $A_{u_{13}}(x) = A(x)$.

(II) $\beta \frown \langle i, \sigma\rangle$ is in open mode at stage s_8. Then if u_8 is not a β-open stage, we have that the strategy will visit the edge again during some open stage $u_9 > u_8$ by the *Synchronisation Lemma* (Lemma 3.8.7). The strategy will then pass through step (5)(b)(iii) and change the mode of the edge to close mode, and will detach the β-expansionary* stage from the edge.

Case (B)(i)(II) of the proof then reduces to case (B)(i)(I), which means that $\Gamma^{U,D}_{\beta\frown\langle i,\sigma\rangle}[u'](x)$ is defined as being equal to $A(x)$ at some stage $u' > u_9$ by the strategy β.

(ii) $\beta \frown \langle i, \sigma\rangle$ is in close mode at stage u.

Since β has outcome $\beta \frown \langle i, \sigma\rangle$ on the true path, we have that this outcome will be visited infinitely often. For this to be the case, β must reach reach step (5)(b) infinitely often. Suppose that the strategy reaches step (5)(b) at stage $u_1 > u$.

If $\beta \frown \langle i, \sigma\rangle$ is in close mode at stage u_1 and there is no β-expansionary* stage attached to the edge, one will be attached at some stage $u_2 > u_1$ by the the *Attachment Procedure Lemma* (Lemma 3.8.5).

If the strategy visits the edge again at some stage $u_3 \geq u_2$ and finds that u_3 is not a β-

close stage, we have that the strategy will visit the edge at some β-close stage $u_4 > u_3$. It will then go through step (5)(b)(viii) and set the edge to open mode. Case (B)(ii) then reduces to case (B)(i), which means that $\Gamma^{U,D}_{\beta \frown \langle i,\sigma \rangle}[u'](x)$ is defined as being equal to $A(x)$ at some stage $u' > u_4$.

Hence we have that if the functional is undefined at some stage $u \geq y'$, the strategy will eventually redefine it to be equal to $A(x)$. But then by Lemma 3.8.11, *Part (a)*, we must have that there is some stage v such that for all $v' > v$, $\Gamma^{U,D}_{\beta}[v'](x) \downarrow$. It follows that there exists some stage t such that for all $s \geq t$, $\Gamma^{U,D}_{\beta}[s](x) = \Gamma^{U,D}_{\beta}[t](x) = A(x)$, as required.

□

3.8.11 Pseudo Outcome Lemma

The Pseudo Outcome Lemma shows that there is in fact no edge of the form $\langle d, \sigma \rangle$ on the true path.

Lemma 3.8.12. *(Pseudo Outcome Lemma). Let α be an $\mathcal{S}$ strategy on the true path f. Then no edge of the form $\alpha \frown \langle d, \sigma \rangle$ is on the true path f.*

Proof. Assume for contradiction that an outcome of the form $\alpha \frown \langle d, \sigma \rangle$ lies on the true path.

We perform the following case analysis.

(1) (There is no active $\beta \subset \alpha$). In this case the strategy α asks only question Q_1, which must have a 'Yes' answer. Let $\alpha^* \subset \alpha$ be the greatest (under $\subset$) $\mathcal{S}$ strategy which imposes a work interval on α, (a_s, b_s) be the work interval it imposes on α at stage s and n_s be the boundary of the work interval at stage s. Then the positive answer to question Q_1 guarantees there must be infinitely many witnesses w and stages s such that $\Theta^D[s](w) \downarrow = 0$, $a_s < w < b_s$, $a_s < \theta_s(w) < b_s$, $a_s < \theta_s(w) < a_s + n_s$. In addition this answer ensures that $\lim_{q \to \infty} l_q(\Theta^D, A) = \infty$.

By the *Leftmost Path Lemma* (Lemma 3.8.4) there is some stage s_0 such that no strategy $\gamma <_L \alpha$ is accessible. By the *Restraint Lemma* (Lemma 3.8.9), we have that there is some stage $s_1 > s_0$ such that $R_{\alpha',s} = R_{\alpha',s_1}$ for every strategy $\alpha' \subset \alpha$ and every stage $s \geq s_1$. It follows that α is not initialised after stage s_1.

Now, the edge $\alpha \frown \langle d, \sigma \rangle$ is on the true path. Hence we have that $\liminf_s O(\alpha) = d$. Then by the Attachment Procedure Lemma (Lemma 3.8.5), we have that if no witness w is attached to the edge at some stage $t > s_1$, a witness will be attached to the edge at some stage $t' > t$. But this means that the strategy will enumerate infinitely many witnesses when visiting the edge throughout the construction.

Now suppose that α has enumerated some witness at stage $u > s_1$, and that α has become accessible again at some least stage $u' > u$. Since α is not initialised after stage s_1, it must be the case that α sets $R_{\alpha,u'}$ to $\theta_u(w)$. But by the *Injury Lemma for $\mathcal{S}$ Strategies* (Lemma 3.8.10), this means that no strategy γ can enumerate any $x \leq R_{\alpha,u'}$ at some stage $v \geq u$. In addition since $u' > u > s_1$, we have that $R_{\alpha,v} = R_{\alpha,u'}$ for all $v > u'$. Hence we must have that $\Theta^D(w) \neq A(w)$.

But this contradicts the fact that $\lim_{q \to \infty} l_q(\Theta^D, A) = \infty$. Hence it cannot be the case that the answer to question Q_1 is 'Yes'. It follows that the edge $\alpha \frown \langle d, \sigma \rangle$ cannot lie on the true path, as required.

(2) (Every active $\beta \subset \alpha$ follows a $\hat{\Gamma}$-strategy). In this case the strategy α lies below some sequence of active $\mathcal{R}$ strategies $(\beta_1, \ldots, \beta_n)$, each of which is following a $\hat{\Gamma}$-strategy. In this case the strategy asks question Q_1 and questions $Q_{2.i}$ for every $\beta_i \in (\beta_1, \ldots, \beta_n)$. Each of these questions which must have a 'Yes' answer.

Let $\alpha^* \subset \alpha$ be the greatest (under $\subset$) $\mathcal{S}$ strategy which imposes a work interval on α, (a_s, b_s) be the work interval it imposes on α at stage s and n_s be the boundary of the work interval at stage s. Then the positive answer to question Q_1 guarantees there must be infinitely many witnesses w and stages s such that $\Theta^D[s](w) \downarrow= 0$, $a_s < w < b_s$, $a_s < \theta_s(w) < b_s$, $a_s < \theta_s(w) < a_s + n_s$. In addition this answer ensures that $\lim_{q \to \infty} l_q(\Theta^D, A) = \infty$.

The positive answer to question $Q_{2.i}$ for every $\beta_i \in (\beta_1, \ldots, \beta_n)$, ensures that infinitely many of these witnesses w and stages s give rise to honest computations $\Gamma^{U_i,D}_{\beta_i \frown \langle i, \sigma_i \rangle}[s](w)$ for every $\beta_i \in (\beta_1, \ldots, \beta_n)$.

By the *Leftmost Path Lemma* (Lemma 3.8.4) there is some stage s_0 such that no strategy $\gamma <_L \alpha$ is accessible. By the *Restraint Lemma* (Lemma 3.8.9), we have that there is some stage $s_1 > s_0$ such that $R_{\alpha',s} = R_{\alpha',s_1}$ for every strategy $\alpha' \subset \alpha$ and every stage $s \geq s_1$. It follows that α is not initialised after stage s_1.

Now, the edge $\alpha \frown \langle d, \sigma \rangle$ is on the true path. Hence we have that $\liminf_s O(\alpha) = d$. Then by the Attachment Procedure Lemma (Lemma 3.8.5), we have that if no witness w is attached to the edge at some stage $t > s_1$, a witness w giving honest computations $\Gamma^{U_i,D}_{\beta \frown \langle i,\sigma \rangle}(w)$ for all $1 \leq i \leq n$ will be attached to the edge at some stage $t' > t$. By the *Honesty Preservation Lemma* (Lemma 3.8.6) we have that these computations remain honest at all stages $t'' \geq t'$. But this means that the strategy will enumerate infinitely many witnesses when visiting the edge throughout the construction.

Now suppose that α has enumerated some witness at stage $u > s_1$, and that α has become accessible again at some least stage $u' > u$. Since α is not initialised after stage s_1, it must be the case that α sets $R_{\alpha,u'}$ to $\theta_u(w)$. But by the *Injury Lemma for $\mathcal{S}$ Strategies* (Lemma 3.8.10), this means that no strategy γ can enumerate any $x \leq R_{\alpha,u'}$ at some stage $v \geq u$. In addition since $u' > u > s_1$, we have that $R_{\alpha,v} = R_{\alpha,u'}$ for all $v > u'$. Hence we must have that $\Theta^D(w) \neq A(w)$.

But this contradicts the fact that $\lim_{q \to \infty} l_q(\Theta^D, A) = \infty$. Hence it cannot be the case that the answer to question Q_1 is 'Yes'. It follows that the edge $\alpha \frown \langle d, \sigma \rangle$ cannot lie on the true path, as required.

(3) (Some active $\beta \subset \alpha$ follows a Γ-strategy). In this case the strategy α lies below some sequence of active $\mathcal{R}$ strategies $(\beta_1, \ldots, \beta_n)$ which may be following either a Γ-strategy or a $\hat{\Gamma}$-strategy. In addition, there is some subsequence of active $\mathcal{R}$ strategies $(\beta'_1, \ldots, \beta'_m)$ above α which are following a Γ-strategy.

In this case the strategy asks question Q_1, questions $Q_{2.i}$ for every $\beta_i \in (\beta_1, \ldots, \beta_n)$ and questions $Q_{3.j}$ for every $\beta'_j \in (\beta'_1, \ldots, \beta'_m)$ and each of these questions must have a 'Yes' answer.

Let $\alpha^* \subset \alpha$ be the greatest (under $\subset$) $\mathcal{S}$ strategy which imposes a work interval on α, (a_s, b_s) be the work interval it imposes on α at stage s and n_s be the boundary of the work interval at stage s. Then the positive answer to question Q_1 guarantees there must be infinitely many witnesses w and stages s such that $\Theta^D[s](w)\downarrow= 0$, $a_s < w < b_s$, $a_s < \theta_s(w) < b_s$, $a_s < \theta_s(w) < a_s + n_s$. In addition this answer ensures that $\lim_{q\to\infty} l_q(\Theta^D, A) = \infty$.

The positive answer to question $Q_{2.i}$ for every $\beta_i \in (\beta_1, \dots, \beta_n)$, ensures that infinitely many of these witnesses w and stages s give rise to honest computations $\Gamma^{U_i,D}_{\beta_i \frown \langle i,\sigma_i\rangle}[s](w)$ for every $\beta_i \in (\beta_1, \dots, \beta_n)$

The positive answer to question $Q_{3.j}$ for every $\beta'_j \in (\beta'_1, \dots, \beta'_m)$, guarantees that infinitely many of the witnesses w giving rise to honest computations $\Gamma^{U_i,D}_{\beta_i \frown \langle i,\sigma_i\rangle}[s](w)$ for every $\beta_i \in (\beta_1, \dots, \beta_n)$ are enumerated into A at stage s, and infinitely many of these cause a $U_j \upharpoonright \phi_{j,1}[s](w)$ change to occur by the least $\mathcal{R}_j$-expansionary* stage $t_j > s$, for every $\beta_j \in (\beta'_1, \dots, \beta'_m)$.

Now since there are active $\mathcal{R}$ strategies $\beta \subset \alpha$ following a Γ-strategy, we have that witnesses giving honest computations can only be enumerated into the set A by first becoming attached to edges of the form $\alpha \frown \langle g_j, \sigma\rangle$, for some $\beta_j \in (\beta'_1, \dots, \beta'_m)$. The 'Yes' answers to questions $Q_{3.j}$ for every $\beta'_j \in (\beta'_1, \dots, \beta'_m)$ guarantee that this takes place infinitely many times, and that infinitely many witnesses are enumerated into A.

By the *Leftmost Path Lemma* (Lemma 3.8.10) there is some stage s_0 such that no strategy $\gamma <_L \alpha$ is accessible. By the *Restraint Lemma* (Lemma 3.8.9), we have that there is some stage $s_1 > s_0$ such that $R_{\alpha',s} = R_{\alpha',s_1}$ for every strategy $\alpha' \subset \alpha$ and every stage $s \geq s_1$. It follows that α is not initialised after stage s_1.

Now suppose that α has enumerated some witness at stage $u > s_1$, and that α has become accessible again at some least stage $u' > u$. Since α is not initialised after stage s_1, it must be the case that α sets $R_{\alpha,u'}$ to $\theta_u(w)$. But by the *Injury Lemma for $\mathcal{S}$ Strategies* (Lemma 3.8.10), this means that no strategy γ can enumerate any $x \leq R_{\alpha,u'}$ at some stage $v \geq u$. In addition since $u' > u > s_1$, we have that $R_{\alpha,v} = R_{\alpha,u'}$ for all $v > u'$. Hence we must have that $\Theta^D(w) \neq A(w)$.

But this contradicts the fact that $\lim_{q\to\infty} l_q(\Theta^D, A) = \infty$. Hence it cannot be the case that the answer to question Q_1 is 'Yes'. It follows that the edge $\alpha \frown \langle d, \sigma\rangle$ cannot lie on the true path, as required. $\square$

Having determined that edges of the form $\langle d, \sigma\rangle$ cannot lie on the true path, we can now remove the qualification in the True Path Existence Lemma (Lemma 3.8.8) and conclude that the true path is infinite.

Corollary 3.8.13. *The true path f is infinite.*

3.8.12 Truth of Outcome Theorem

The Truth of Outcome Theorem shows that every requirement is satisfied by the strategy which represents it on the true path.

Theorem 3.8.14. *(Truth of Outcome Theorem). Let f be the true path and let U be a requirement. Then there exists a strategy γ on f which satisfies U.*

Proof. We start by considering the case where U is a requirement $\mathcal{R}_i$.

By the *Pseudo Outcome Lemma* (Lemma 3.8.12) we have that no edges of the form $\gamma \frown \langle d, \sigma\rangle$ lie on the true path f. By Corollary 3.8.13 we also have that the true path f is infinite. Hence by the *Representation Lemma* (Lemma 3.8.3) there is some $\mathcal{R}$ strategy β which represents the requirement $\mathcal{R}_i$ on the true path. For the strategy β to represent the requirement $\mathcal{R}_i$ on the true path, one of the following must be the case.

- β has an edge $\beta \frown \langle f, \sigma\rangle$ on the true path.
- β is labeled $\mathcal{R}_i$ and there is some $\mathcal{S}$ strategy α such that $\beta \subset \alpha$ and α has an edge $\alpha \frown \langle h_i, \sigma\rangle$ on the true path.
- β is labeled $\mathcal{R}_i$ and has an edge $\beta \frown \langle i, \sigma\rangle$ on the true path, and β is active on the true path.

We consider these three cases in turn.

(1) Suppose that β has an edge $\beta \frown \langle f, \sigma\rangle$ on the true path. Then the answer to question Q_1 of the strategy must be 'No'. Hence we have that there are only finitely many β-expansionary* stages. This means that $l(\Phi_i^{U_i,V_i}, A)$ is finite, and that $\mathcal{R}_i$ is satisfied trivially.

(2) Suppose that β is labeled $\mathcal{R}_i$ and there is some strategy α such that $\beta \supset \alpha$ and α has an edge $\alpha \frown \langle h_i, \sigma\rangle$ on the true path. Suppose that α lies below some sequence $(\beta_1, \ldots, \beta_n)$

of $\mathcal{R}$ strategies, with $\beta_i = \beta$. Then we have that the answer to question Q_1 of the strategy must be 'Yes', the answers to questions $Q_{2.j}$ with $j < i$ must be 'Yes', and the answer to question $Q_{2.i}$ must be 'No'.

Let $\alpha^* \subset \alpha$ be the greatest (under $\subset$) $\mathcal{S}$ strategy which imposes a work interval on α, (a_s, b_s) be the work interval it imposes on α at stage s and n_s be the boundary of the work interval at stage s. Then the positive answer to question Q_1 guarantees there must be infinitely many witnesses w and stages s such that $\Theta^D[s](w) \downarrow = 0$, $a_s < w < b_s$, $a_s < \theta_s(w) < b_s$, $a_s < \theta_s(w) < a_s + n_s$ and such that the computations $\Gamma^{U_j,D}_{\beta_i \frown \langle j,\sigma\rangle}[s](w)$ are honest for all $1 \leq j < i$. However only finitely many of these witnesses and stages give computations $\Gamma^{U_i,D}_{\beta_i \frown \langle i,\sigma\rangle}[s](w)$ which are honest.

Now, by the *Attachment Procedure Lemma* (Lemma 3.8.5) we have that a witness w which gives honest computations $\Gamma^{U_j,D}_{\beta_i \frown \langle j,\sigma\rangle}[s_1](w)$ for all $1 \leq j < i$ and a dishonest computation $\Gamma^{U_i,D}_{\beta_i \frown \langle i,\sigma\rangle}[s_1](w)$ is attached to the edge at stage s_1. Moreover since question $Q_{2.i}$ has a 'No' answer we have that there is some stage $s_2 \geq s_1$ such that the computation $\Gamma^{U_i,D}_{\beta_i \frown \langle i,\sigma\rangle}[s](w)$ is dishonest for every stage $s \geq s_2$.

Now since there are infinitely many stages such that α visits the edge during an α-close stage when it is in Part I mode, we have that α enumerates $\gamma_{\beta_i \frown \langle i,\sigma\rangle}(w)$ into D infinitely often. Since β_i cannot redefine $\Gamma^{U_i,D}_{\beta_i \frown \langle i,\sigma\rangle}(w)$ without choosing a larger use, we have that $\lim_t \gamma_{\beta_i \frown \langle i,\sigma\rangle}[t](w) \to \infty$. But since after stage s_2 we have that the computation $\Gamma^{U_i,D}_{\beta_i \frown \langle i,\sigma\rangle}(w)$ is always dishonest, it must be the case that $\lim_t \phi_{i,1}[t](w) \to \infty$ as well. Hence we have that $\Phi^{U_i,V_i}_{i,1}(w) \uparrow$, and $\mathcal{R}_i$ is satisfied trivially.

(3) Suppose that β is labeled $\mathcal{R}_i$, has an edge $\beta \frown \langle i,\sigma\rangle$ on the true path and β is active on the true path. This means that there is no $\mathcal{S}$ strategy α such that $\beta \subset \alpha$ and α has an edge $\alpha \frown \langle h_j,\sigma\rangle$ for $j < i$ or an edge $\alpha \frown \langle g_j,\sigma\rangle$ for $j \leq i$ on the true path. Then by the *Injury Lemma for $\mathcal{R}$ Strategies* (Lemma 3.8.10) we have that the strategy β is able to build a functional $\Gamma^{U,D}_{\beta \frown \langle i,\sigma\rangle}$ such that for all elements x, $\Gamma^{U_i,D}_{\beta \frown \langle i,\sigma\rangle}(x) = A(x)$. Hence we have that β satisfies the requirement $\mathcal{R}_i$.

We now consider the case where U is a requirement $\mathcal{S}_i$.

By the *Pseudo Outcome Lemma* (Lemma 3.8.12) we have that no edges of the form $\gamma \frown \langle d, \sigma \rangle$ lie on the true path f. By Corollary 3.8.13 we also have that the true path f is infinite. Hence by the *Representation Lemma* (Lemma 3.8.3) we have that there is some $\mathcal{S}$ strategy α on f which represents the requirement $\mathcal{S}_i$ on the true path. Let e be the index of the strategy α in the total ordering of all $\mathcal{S}$ strategies on the priority tree.

In order for the strategy α to represent the requirement $\mathcal{S}_i$ on the true path, we must have that α has an edge $\alpha \frown \langle w, \sigma \rangle$ on the true path. But this means that the answer to question Q_1 of the strategy α must be 'No'. Hence one or more of the conditions of question Q_1 have failed.

Let $\alpha^* \subset \alpha$ be the greatest (under $\subset$) $\mathcal{S}$ strategy which imposes a work interval on α, (a_s, b_s) be the work interval it imposes on α at stage s, n_s be the boundary of the work interval at stage s and $\alpha^* \frown \langle o', \sigma' \rangle$ be the edge of α^* lying on the true path.

If condition (i) of question Q_1 fails, we have that there are only finitely many witnesses $w \in W^e$ and stages $s \in \mathbb{N}_\alpha$ such that $\Theta^D[s](w) \downarrow= 0$. Then there must be some stage $t \in \mathbb{N}_\alpha$ such that for every $t' \in \mathbb{N}_\alpha$ with $t' > t$ and every element $x \in W^e$, we have that $\Theta^D[t'](x) \uparrow$ or $\Theta^D[t'](x) \downarrow= 1$. Now, if $\Theta^D(x) \uparrow$ for some $x \in W^e$, we have that $\Theta^D(x) \neq A(x)$ and the $\mathcal{S}$ requirement is satisfied. On the other hand, if $\Theta^D(x) \downarrow= 1$ for some $x \in W^e$, we have that the strategy will never enumerate x into A. This means that $\Theta^D(x) \neq A(x)$ and that the $\mathcal{S}$ requirement is also satisfied.

If condition (i) of question Q_1 holds but condition (ii) of question Q_1 fails, we have that there are infinitely many witnesses $w \in W^e$ and stages $s \in \mathbb{N}_\alpha$ such that $\Theta^D[s](w) \downarrow= 0$, but $a_s < w < b_s$ for only finitely many of these witnesses and stages. This means that there is some stage $t \in \mathbb{N}_\alpha$ such that for all $t' \in \mathbb{N}_\alpha$ with $t' > t$ and every element $x \in W^e$ such that $a_{t'} < x < b_{t'}$ we have that $\Theta^D[t'](x) \downarrow= 1$ or that $\Theta^D[t'](x) \uparrow$. Now since α is on the true path, there is some stage s_0 such that for all $s' > s_0$, the edge $\alpha^* \frown \langle o', \sigma \rangle$ is no longer initialised. Hence once α^* defines a work interval for the edge $\alpha^* \frown \langle o', \sigma' \rangle$ at some least stage $u > s_0$, we have that $a_u = a_{u'}$ for all $u' \geq u$. In addition, since α is accessible infinitely often, we have that α^* must have visited $\alpha^* \frown \langle o', \sigma \rangle$ during infinitely many close stages and gone to the next substage. It follows that the upper bound of the work interval is enumerated into the set D infinitely often, giving that

$\lim_{v \to \infty} b_v \to \infty$. Hence there is some element $x' \in W^e$ and some stage $p \in \mathbb{N}_\alpha$ such that for all stages $p' \in \mathbb{N}_\alpha$ with $p' > p$, $p' > t$ and $p' > u$ we have that $a_{p'} < x' < b_{p'}$ and that $\Theta^D[p'](x') \uparrow$ or $\Theta^D[p'](x') \downarrow = 1$. Now, if $\Theta^D(x') \uparrow$, we have that $\Theta^D(x') \neq A(x')$ and the $\mathcal{S}$ requirement is satisfied. On the other hand, if $\Theta^D(x') \downarrow = 1$, we have that the strategy will never enumerate x' into A. This means that $\Theta^D(x') \neq A(x')$ and that the $\mathcal{S}$ requirement is also satisfied.

If conditions (i) and (ii) of question Q_1 hold but condition (iii) of question Q_1 fails, we have that there are infinitely many witnesses $w \in W^e$ and stages $s \in \mathbb{N}_\alpha$ such that $\Theta^D[s](w) \downarrow = 0$ and $a_s < w < b_s$, but $a_s < \theta_s(w) < b_s$ for only finitely many of these witnesses and stages. This means that there is some stage $t \in \mathbb{N}_\alpha$ such that for all $t' \in \mathbb{N}_\alpha$ with $t' > t$ and every element $x \in W^e$ such that $a_{t'} < x < b_{t'}$ we have that $\theta_{t'}(x) > b_{t'}$. Now since α is on the true path, there is some stage s_0 such that for all $s' > s_0$, the edge $\alpha^* \frown \langle o', \sigma \rangle$ is no longer initialised. Hence once α^* defines a work interval for the edge $\alpha^* \frown \langle o', \sigma' \rangle$ at some least stage $u > s_0$, we have that $a_u = a_{u'}$ for all $u' \geq u$. In addition, since α is accessible infinitely often, we have that α^* must have visited $\alpha^* \frown \langle o', \sigma \rangle$ during infinitely many close stages and gone to the next substage. It follows that the upper bound of the work interval is enumerated into the set D infinitely often, giving that $\lim_{v \to \infty} b_v \to \infty$. Hence there is some element $x' \in W^e$ and some stage $p \in \mathbb{N}_\alpha$ such that for all stages $p' \in \mathbb{N}_\alpha$ with $p' > p$, $p' > t$ and $p' > u$ we have that $b_{p'} < \theta_{p'}(x')$. But since the upper bound of the work interval is unbounded, it must be the case that $\Theta^D(x') \uparrow$, which means that the $\mathcal{S}$ requirement is satisfied.

If conditions (i) and (ii) and (iii) of question Q_1 hold but condition (iv) of question Q_1 fails, we have that there are infinitely many witnesses $w \in W^e$ and stages $s \in \mathbb{N}_\alpha$ such that $\Theta^D[s](w) \downarrow = 0$ and $a_s < w < b_s$ and $a_s < \theta_s(w) < b_s$, but that $a_s < w < a_s + n_s$ for only finitely many of these witnesses and stages. This means that there is some stage $t \in \mathbb{N}_\alpha$ such that for all $t' \in \mathbb{N}_\alpha$ with $t' > t$ and every element $x \in W^e$ such that $a_{t'} < x < a_{t'} + n_{t'}$ we have that $\Theta^D[t'](x) \downarrow = 1$ or that $\Theta^D[t'](x) \uparrow$. Now since α is on the true path, there is some stage s_0 such that for all $s' > s_0$, the edge $\alpha^* \frown \langle o', \sigma \rangle$ is no longer initialised. Hence once α^* defines a work interval for the edge $\alpha^* \frown \langle o', \sigma' \rangle$ at some least stage $u > s_0$, we have that $a_u = a_{u'}$ for all $u' \geq u$. In addition since α is accessible infinitely often, we have that α^* must have visited $\alpha^* \frown \langle o', \sigma \rangle$ during infinitely many close stages and gone to the next substage. It follows that the boundary of the work interval is incremented infinitely often, giving that $\lim_{v \to \infty} n_v \to \infty$. Hence there is some element $x' \in W^e$ and some stage $p \in \mathbb{N}_\alpha$ such that for all stages $p' \in \mathbb{N}_\alpha$ with $p' > p$, $p' > t$ and $p' > u$ we have that $a_{p'} < x' < a_{p'} + n_{p'}$ and that $\Theta^D[p'](x') \uparrow$ or $\Theta^D[p'](x') \downarrow = 1$. Now, if $\Theta^D(x') \uparrow$, we have that $\Theta^D(x') \neq A(x')$ and the $\mathcal{S}$ requirement is satisfied. On the other hand, if $\Theta^D(x') \downarrow = 1$, we have that the strategy will never enumerate x' into A. This means that $\Theta^D(x') \neq A(x')$ and that the $\mathcal{S}$ requirement is also satisfied.

If conditions (i) and (ii) and (iii) and (iv) of question Q_1 hold but condition (v) of question Q_1 fails, we have that there are infinitely many witnesses $w \in W^e$ and stages $s \in \mathbb{N}_\alpha$ such that $\Theta^D[s](w) \downarrow = 0$, $a_s < w < b_s$, $a_s < \theta_s(w) < b_s$ and $a_s < w < a_s + n_s$. However there are only finitely many stages $q \in \mathbb{N}_\alpha$ such that $(\forall q' < q)[l_{q'}(\Theta^D, A) < l_q(\Theta^D, A)]$, where q' ranges over $\mathbb{N}_\alpha$. But in this case there must be some x such that $\Theta^D(x) \neq A(x)$, meaning that the $\mathcal{S}$ requirement is satisfied.

Either way we have that $A \neq \Theta^D$, and that the requirement $\mathcal{S}_i$ is satisfied. □

3.8.13 High Permitting Theorem

The High Permitting Theorem shows that the sets A and D lie below H_0.

Theorem 3.8.15. *(High Permitting Theorem). The following are the case.*

(I) $A \leq_T H_0$.

(II) $D \leq_T H_0$.

Proof. Theorem 3.8.15, Part (I). In order to show that A can be computed with the help of H_0 we shall proceed as follows. Given some w, we shall want to determine whether $w \in A$ or $w \notin A$ in some finite amount of time and in finitely many queries to H_0.

In order to do this we set in motion the construction from Section 3.7, whilst simultaneously enumerating the set Y in H_0 from Lemma 3.1.8. Since only $\mathcal{S}$ strategies can enumerate witnesses w into the set A, we also identify the $\mathcal{S}$ strategy α which has the witness w in its witness set W^e.

We then wait for a stage s such that one of the following cases occurs.

(1) The witness w is attached to an edge $\alpha \frown \langle o, \sigma \rangle$ at stage s.

If the edge is of the form $\alpha \frown \langle h_i, \sigma \rangle$, we have that the strategy will never enumerate the witness w into A whilst visiting the edge. Hence we have that $w \notin A$.

Otherwise we wait for a stage $t \geq s$ with the following properties:

(a) The strategy has attached the witness w to an edge of the form $\alpha \frown \langle g_i, \sigma \rangle$ or $\alpha \frown \langle d, \sigma \rangle$ at stage s, and enumerates the witness w into A when it visits this edge at stage t. Then we have that $w \in A$.

(b) Some strategy γ with $\gamma <_L \alpha$ has become accessible at stage t, or alternatively some edge $\gamma \frown \langle o', \sigma' \rangle$ with $\gamma \frown \langle o', \sigma' \rangle <_L \alpha \frown \langle o, \sigma \rangle$ becomes accessible at stage t. This means that the edge $\alpha \frown \langle o, \sigma \rangle$ is initialised at stage t and that the witness w is detached from the edge at stage t. Therefore α cannot enumerate w into A and we have that $w \notin A$.

(c) Some strategy γ with $\alpha <_L \gamma$ enters Y at stage t, or alternatively some edge $\gamma \frown \langle o', \sigma' \rangle$ with $\alpha \frown \langle o, \sigma \rangle <_L \gamma \frown \langle o', \sigma' \rangle$ enters Y at stage t. In this case we have that the edge $\alpha \frown \langle o, \sigma \rangle$ is no longer accessible at stages $t' > t$. Therefore α cannot enumerate w into A at stages $t' > t$. We also have that α cannot enumerate w into A at stage $t = s$ either, because otherwise case (a) would hold. Then we have that $w \notin A$.

(2) The strategy α visits an edge $\alpha \frown \langle o', \sigma' \rangle$ at stage s, some edge $\alpha \frown \langle o', \sigma' \rangle$ has entered Y at some stage $t < s$ and $s > w$. In addition one of the following conditions holds.

(a) $o' = w$.

(b) $o' = d$ and no $\mathcal{R}$ strategy β above α which is active for α is following a Γ-strategy. In addition we have that some witness $w' > w$ has been attached to the edge at some stage t' such that $t < t' \leq s$.

(c) $o' = d$ and there is some $\mathcal{R}$ strategy β above α which is active for α and is following a Γ-strategy.

(d) $o' = g_i$ and we have that some witness $w' > w$ has been attached to $\alpha \frown \langle g_i, \sigma' \rangle$ at some stage t' such that $t < t' \leq s$.

(e) $o' = h_i$ and we have that some witness w' has been attached to $\alpha \frown \langle h_i, \sigma' \rangle$ at some stage t' such that $t < t' \leq s$.

We now show that if case (2) holds, we have that $w \notin A$.

Suppose a stage s satisfying condition (2) exists. Then we have that edges $\alpha \frown \langle o'', \sigma'' \rangle$ lying to the left of $\alpha \frown \langle o', \sigma' \rangle$ are not accessible at stages $t' > t$. Since $s > t$ we have that if w is attached to some edge $\alpha \frown \langle o'', \sigma'' \rangle$ lying to the left of $\alpha \frown \langle o', \sigma' \rangle$ at some stage $s' > s$, the strategy α cannot enumerate it into the set A. Note that w cannot have become attached to such an edge at some stage $s' \leq s$, or we would have that case (1) holds instead of case (2).

In addition we have that edges $\alpha \frown \langle o'', \sigma'' \rangle$ lying to the right of $\alpha \frown \langle o', \sigma' \rangle$ are initialised at stage s. This means that only witnesses $w > s$ can be attached to an edge $\alpha \frown \langle o'', \sigma'' \rangle$ lying to the right of $\alpha \frown \langle o', \sigma' \rangle$ at stages $s' \geq s$.

Now consider the value o' of the edge $\alpha \frown \langle o', \sigma' \rangle$.

If $o' = w$, we have that no witness is ever attached to the edge. Hence the witness w cannot become attached to the edge.

If $o' = d$, and there is some $\mathcal{R}$ strategy β above α which is active for α and is following a Γ-strategy, we have that no witness is ever attached to the edge. Hence the witness w cannot become attached to the edge.

If $o' = d$, and there is no $\mathcal{R}$ strategy β above α which is active for α and is following a Γ-strategy, we have that some witness w' has been attached to the edge at some stage t' such that $t < t' \leq s$. It follows that only witnesses which are greater than w' can be attached to the edge at stages $s' \geq s$. Hence the witness w can no longer be attached to the edge.

If $o' = g_i$, we have that some witness w' has been attached to the edge at some stage t' such that $t < t' \leq s$. It follows that only witnesses which are greater than w' can be attached to the edge at stages $s' \geq s$. Hence the witness w can no longer be attached to the edge.

If $o' = h_i$, we have that a witness w' has been attached to the edge at some stage t' such that $t < t' \leq s$. Since the edge is never initialised at stages $s' > t$ we have that the witness w' is never detached from the edge and that the witness w cannot become attached to the edge.

Thus we can conclude that $w \notin A$ as required.

It follows that A is computable in H_0. □

Proof. Theorem 3.8.15, Part (II). In order to show that D can be computed with the help of H_0 we proceed as follows.

Given some use u, we shall want to determine whether $u \in D$ or $u \notin D$ in some finite amount of time and in finitely many queries to H_0. We set up the construction from Section 3.7, whilst simultaneously enumerating the set Y in H_0 from Lemma 3.1.8. We also identify the $\mathcal{R}$ strategy β labeled $\mathcal{R}_i$ and the edge $\beta \frown \langle i, \sigma \rangle$ such that $u \in U^{e, \beta \frown \langle o, \sigma \rangle}$.

We then wait for a stage s such that one of the following cases occurs.

(1.1) The strategy defines $\Gamma^{U_i, D}_{\beta \frown \langle i, \sigma \rangle}[s](z)$ for some z by choosing $\gamma_{\beta \frown \langle i, \sigma \rangle}[s](z) = u$ at stage s.

(1.2) Some strategy γ with $\beta <_L \gamma$ enters Y at stage s, or alternatively some edge $\gamma \frown \langle o', \sigma' \rangle$ with $\beta \frown \langle i, \sigma \rangle <_L \gamma \frown \langle o', \sigma' \rangle$ enters Y at stage s. In this case we have that the edge $\beta \frown \langle i, \sigma \rangle$ is no longer accessible at stages $s' > s$. Therefore β cannot define $\Gamma^{U_i, D}_{\beta \frown \langle i, \sigma \rangle}[s'](z)$ for some z by choosing $\gamma_{\beta \frown \langle i, \sigma \rangle}[s'](z) = u$ at stages $s' > s$.

(1.3) Some strategy γ with $\gamma <_L \beta$ has become accessible at some stage $s > u$, or alternatively some edge $\gamma \frown \langle o', \sigma' \rangle$ with $\gamma \frown \langle o', \sigma' \rangle <_L \beta \frown \langle i, \sigma \rangle$ becomes accessible at some stage $s > u$. This means that the edge $\beta \frown \langle i, \sigma \rangle$ is initialised at stage s and that the functional associated to the edge is canceled at stage s. In addition if β defines $\Gamma^{U_i, D}_{\beta \frown \langle i, \sigma \rangle}[s'](z)$ for some z at stages $s' > s$, it must choose a use which is greater than s, and hence greater than u.

(1.4) Some strategy γ with $\gamma \subset \beta$ has reset the strategy β and imposed a restraint on β which greater than u at stage s. This means that the functional associated to the edge $\beta \frown \langle i, \sigma \rangle$ is canceled at stage s. In addition if β defines $\Gamma^{U_i, D}_{\beta \frown \langle i, \sigma \rangle}[s'](z)$ for some z at stages $s' > s$, it must choose a use which is greater than this restraint, and hence greater than u.

(1.5) The strategy defines $\Gamma^{U_i, D}_{\beta \frown \langle i, \sigma \rangle}[s](z)$ for some z by choosing $\gamma_{\beta \frown \langle i, \sigma \rangle}[s](z) > u$ at stage s. Suppose that the strategy needs to define $\Gamma^{U_i, D}_{\beta \frown \langle i, \sigma \rangle}[s'](z')$ for some z' at some stage $s' > s$. Then the strategy will compute the least use which is compatible with conditions (5)(b)(iii)(A) - (5)(b)(iii)(L) of the $\mathcal{R}$ strategy as defined in section 3.7. If it finds that the least use can be equal to $\gamma_{\beta \frown \langle i, \sigma \rangle}[s](z')$ it will choose this use once again. Otherwise it will find that the least use compatible with these conditions has to be greater than $\gamma_{\beta \frown \langle i, \sigma \rangle}[s](z')$.

In this case it will choose the least use which is greater than all previous uses chosen by the strategy when defining the functional. But since the strategy has already chosen some use greater than u when defining $\Gamma^{U_i,D}_{\beta\frown\langle i,\sigma\rangle}[s](z)$, we have that the strategy can never choose u at some stage $s' > s$.

Now if stage s follows one of cases (1.2), (1.3), (1.4) or (1.5), we have that the use u can no longer be chosen by the strategy β when defining $\Gamma^{U_i,D}_{\beta\frown\langle i,\sigma\rangle}[s'](z)$ for some z at some stage $s' > s$. This means that no strategy can ever enumerate u into D after stage s. Hence we have that $u \notin D$.

On the other hand if stage s follows case (1.1) we proceed as follows. We identify the $\mathcal{S}$ strategy α which has z as a witness in its witness set W^e or which has z as a threshold in its threshold set V^e.

We then have to determine whether the strategy attaches witness z to an edge or whether the strategy chooses threshold z to define a work interval for some edge.

In order to determine whether α attaches witness z to some edge in finite time and in finitely many queries to H_0, we wait for a stage $t \geq s$ such that one of the following cases occurs.

(2.1) The strategy α attaches witness z to an edge $\alpha \frown \langle g_i, \sigma\rangle$ at stage t.

(2.2) The strategy α attaches witness z to an edge $\alpha \frown \langle d, \sigma\rangle$ at stage t.

(2.3) The strategy α attaches witness z to an edge $\alpha \frown \langle h_i, \sigma\rangle$ at stage t.

(2.4) The strategy α visits an edge $\alpha \frown \langle o', \sigma'\rangle$ at stage t, the edge $\alpha \frown \langle o', \sigma'\rangle$ has entered Y at some stage $t' < t$ and $t > z$. In addition one of the following conditions holds.

   (a) $o' = w$.

   (b) $o' = d$ and no $\mathcal{R}$ strategy β above α which is active for α is following a Γ-strategy. In addition we have that some witness $w' > z$ has been attached to the edge at some stage t'' such that $t' < t'' \leq t$.

   (c) $o' = d$ and there is some $\mathcal{R}$ strategy β above α which is active for α and is following a Γ-strategy.

   (d) $o' = g_i$ and we have that some witness $w' > z$ has been attached to $\alpha \frown \langle g_i, \sigma'\rangle$ at some stage t'' such that $t' < t'' \leq t$.

(e) $o' = h_i$ and we have that some witness w' has been attached to $\alpha \frown \langle h_i, \sigma' \rangle$ at some stage t'' such that $t' < t'' \leq t$.

On the other hand in order to determine whether α chooses threshold z to define a work interval for some edge in finite time and in finitely many queries to H_0, we wait for a stage $t \geq s$ such that one of the following things occurs.

(3.1) The strategy α chooses threshold z to define a work interval for some edge $\alpha \frown \langle g_i, \sigma \rangle$ at stage t.

(3.2) The strategy α visits an edge $\alpha \frown \langle o', \sigma' \rangle$ at stage t, edge $\alpha \frown \langle o', \sigma' \rangle$ has entered Y at some stage $t' < t$ and $t > z$. In addition one of the following conditions holds.

(a) $o' = w$.

(b) $o' = d$.

(c) $o' = g_i$ and we have that some threshold $v' > z$ has been used to define a work interval for $\alpha \frown \langle g_i, \sigma' \rangle$ at some stage t'' such that $t' < t'' \leq t$.

(d) $o' = h_i$.

The fact that the occurrence of clause (2.4) is sufficient to show that the witness z will not be attached to any edge is identical to the argument given in Part I of the proof. The fact that the occurrence of clause (3.2) is sufficient to show that the threshold z will not be chosen to define the work interval of any edge is a straightforward variation on the same argument. Hence if we have that case (2.4) holds for the witness z, or that case (3.2) holds for the threshold z it follows that $u \notin D$.

Otherwise we have that α has taken one of the following actions.

(4.1) The strategy α has chosen threshold z for some edge $\alpha \frown \langle g_i, \sigma \rangle$ at some stage $t \geq s$, defining a work interval for this edge.

(4.2) The strategy α has attached witness z to some edge $\alpha \frown \langle h_i, \sigma \rangle$ at some stage $t \geq s$, defining a work interval for this edge.

(4.3) The strategy α has attached witness z to some edge $\alpha \frown \langle g_i, \sigma \rangle$ at some stage $t \geq s$.

(4.4) The strategy α has attached witness z to some edge $\alpha \frown \langle d, \sigma \rangle$ at some stage $t \geq s$.

Suppose that one of cases (4.1) or (4.2) has taken place. Let $\alpha \frown \langle o, \sigma \rangle$ denote the edge for which the work interval has been defined, and let $(z, \gamma_{\beta' \frown \langle i, \sigma' \rangle})[t](z)$ be the work interval, where β' is some $\mathcal{R}$ strategy labeled $\mathcal{R}_j$ such that $\beta' \subset \alpha$, and $\gamma_{\beta' \frown \langle i, \sigma' \rangle}[t](z)$ is the use of the functional built by β' for the edge $\beta' \frown \langle i, \sigma' \rangle$.

Then if $\gamma_{\beta \frown \langle i, \sigma \rangle}(z) \neq u$, we have that $u \neq D$.

Otherwise we have that $\gamma_{\beta \frown \langle i, \sigma \rangle}(z) = u$, and wait for a stage $p \geq t$ such that one of the following takes place:

(5.1) The strategy α enumerates the upper bound $\gamma_{\beta \frown \langle i, \sigma \rangle}[t](z)$ of the work interval into D.

(5.2) Some strategy γ with $\alpha <_L \gamma$ enters Y at stage p, or alternatively some edge $\gamma \frown \langle o', \sigma' \rangle$ with $\alpha \frown \langle o, \sigma \rangle <_L \gamma \frown \langle o', \sigma' \rangle$ enters Y at stage p. In this case we have that the edge $\alpha \frown \langle o, \sigma \rangle$ is no longer accessible at stages $p' > p$.

(5.3) Some strategy γ with $\gamma <_L \alpha$ has become accessible at stage p, or alternatively some edge $\gamma \frown \langle o', \sigma' \rangle$ with $\gamma \frown \langle o', \sigma' \rangle <_L \alpha \frown \langle o, \sigma \rangle$ becomes accessible at stage p. This means that the edge $\alpha \frown \langle o, \sigma \rangle$ is initialised at stage p. Hence any work interval defined for the edge is undefined at stage p.

(5.4) Some strategy γ with $\gamma \subset \alpha$ has reset the strategy α at stage p. Hence any work interval defined for the edge is undefined at stage p.

Now if (5.1) is the case, we have that $u \in D$. Otherwise, if one of cases (5.2), (5.3) or (5.4) holds, we have that $u \notin D$.

Now suppose (4.3) or (4.4) holds. Let $\alpha \frown \langle o, \sigma \rangle$ denote the edge to which the witness z has been attached at stage t. In this case we wait for a stage $p \geq t$ such that one of the following takes place:

(6.1) The strategy α enumerates the witness z into A at stage p.

(6.2) Some strategy γ with $\alpha <_L \gamma$ enters Y at stage p, or alternatively some edge $\gamma \frown \langle o', \sigma' \rangle$ with $\alpha \frown \langle o, \sigma \rangle <_L \gamma \frown \langle o', \sigma' \rangle$ enters Y at stage p. In this case we have that the edge $\alpha \frown \langle o, \sigma \rangle$ is no longer accessible at stages $p' > p$.

(6.3) Some strategy γ with $\gamma <_L \alpha$ has become accessible at stage p, or alternatively some edge $\gamma \frown \langle o', \sigma' \rangle$ with $\gamma \frown \langle o', \sigma' \rangle <_L \alpha \frown \langle o, \sigma \rangle$ becomes accessible at stage p. This means that the edge $\alpha \frown \langle o, \sigma \rangle$ is initialised at stage p. Hence the witness z is detached from the edge at stage p.

(6.4) Some strategy γ with $\gamma \subset \alpha$ has reset the strategy α at stage p. Hence the witness z is detached from the edge at stage p.

(6.5) The strategy α visits the edge $\alpha \frown \langle o, \sigma \rangle$ at stage p and determines that the witness z is dishonest, thus discarding the witness z.

If one of cases (6.2), (6.3), (6.4) and (6.5) holds, we have that $u \notin D$.

Otherwise, we have that case (6.1) holds and that a disagreement now exists between $\Gamma^{U,D}_{\beta \frown \langle i, \sigma \rangle}(z)$ and $A(z)$.

Therefore we will wait for a stage $q > p$ such that one of the following takes place:

(7.1) The strategy β visits the edge $\beta \frown \langle i, \sigma \rangle$ at stage q, determines that $\Gamma^{U,D}_{\beta \frown \langle i, \sigma \rangle}[q](z) \neq A_q(z)$ and self-repairs by enumerating $\gamma_{\beta \frown \langle i, \sigma \rangle}(z)$ into D.

(7.2) Some strategy γ with $\beta <_L \gamma$ enters Y at stage p, or alternatively some edge $\gamma \frown \langle o', \sigma' \rangle$ with $\beta \frown \langle i, \sigma \rangle <_L \gamma \frown \langle o', \sigma' \rangle$ enters Y at stage q. In this case we have that the edge $\beta \frown \langle i, \sigma \rangle$ is no longer accessible at stages $q' > q$.

(7.3) Some strategy γ with $\gamma <_L \beta$ has become accessible at stage q, or alternatively some edge $\gamma \frown \langle o', \sigma' \rangle$ with $\gamma \frown \langle o', \sigma' \rangle <_L \beta \frown \langle i, \sigma \rangle$ becomes accessible at stage q. This means that the edge $\beta \frown \langle i, \sigma \rangle$ is initialised at stage q. Hence the functional $\Gamma^{U,D}_{\beta \frown \langle i, \sigma \rangle}$ is canceled at stage q.

(7.4) Some strategy γ with $\gamma \subset \beta$ has reset the strategy β at stage p. Hence the functional $\Gamma^{U,D}_{\beta \frown \langle i, \sigma \rangle}$ is canceled at stage q.

(7.5) The strategy β visits the edge $\beta \frown \langle i, \sigma \rangle$ at stage q and determines that $\Gamma^{U,D}_{\beta \frown \langle i, \sigma \rangle}[q](z) \uparrow$, or that $\gamma_{\beta \frown \langle i, \sigma \rangle}[q](z) \neq u$.

Then in case (7.1) we have that $u \in D$, while in cases (7.2), (7.3), (7.4) and (7.5) we have that $u \notin D$.

It follows that D is computable in H_0. □

Bibliography

[Ambos-Spies and Fejer, 2014] Ambos-Spies, K. and Fejer, P. A. (2014). Degrees of unsolvability. In Siekmann, J. H., editor, *Computational Logic*, volume 9 of *Handbook of the History of Logic*, pages 443–494. Elsevier.

[Cooper, 2004] Cooper, S. B. (2004). *Computability Theory*. Chapman and Hall/CRC.

[Cooper et al., 2002] Cooper, S. B., Li, A., and Yi, X. (2002). On the distribution of Lachlan nonsplitting bases. *Archive for Mathematical Logic*, 41(5):455–482.

[Friedberg, 1957] Friedberg, R. M. (1957). Two recursively enumerable sets of incomparable degrees of unsolvability. *Proceedings of the National Academy of Sciences of the United States of America*, 43(2):236–238.

[Friedberg, 1958] Friedberg, R. M. (1958). Three theorems on recursive enumeration. i. decomposition. ii. maximal set. iii. enumeration without duplication. *The Journal of Symbolic Logic*, 23(3):309–316.

[Harrington, 1980] Harrington, L. A. (1980). Understanding Lachlan's monster paper. Handwritten Notes.

[Kleene, 1936] Kleene, S. C. (1936). General recursive functions of natural numbers. *Mathematische Annalen*, 112:727–742.

[Kleene, 1943] Kleene, S. C. (1943). Recursive predicates and quantifiers. *Transactions of the American Mathematical Society*, 53(1):41–73.

[Kleene, 1952] Kleene, S. C. (1952). *Introduction to Metamathematics*. Bibliotheca Mathematica. Wolters-Noordhoff.

[Kleene and Post, 1954] Kleene, S. C. and Post, E. L. (1954). The upper semi-lattice of degrees of recursive unsolvability. *Annals of Mathematics*, 59(3):379–407.

[Lachlan, 1966] Lachlan, A. H. (1966). Lower bounds for pairs of recursively enumerable degrees. *Proceedings of the London Mathematical Society*, s3-16(1):537–569.

[Lachlan, 1976] Lachlan, A. H. (1976). A recursively enumerable degree which will not split over all lesser ones. *Annals of Mathematical Logic*, 9(4):307–365.

[Lachlan, 1980] Lachlan, A. H. (1980). Decomposition of recursively enumerable degrees. *Proceedings of the American Mathematical Society*, 79(4):629–634.

[Leonhardi, 1996] Leonhardi, S. D. (1996). Nonbounding and Slaman triples. *Annals of Pure and Applied Logic*, 79(2):139 – 163.

[Leonhardi, 1997] Leonhardi, S. D. (1997). Generalized nonsplitting in the recursively enumerable degrees. *The Journal of Symbolic Logic*, 62(2):397–437.

[Martin, 1966] Martin, D. A. (1966). Classes of recursively enumerable sets and degrees of unsolvability. *Zeitschrift fur mathematische Logik und Grundlagen der Mathematik*, 12(1):295–310.

[Miller, 1981] Miller, D. P. (1981). *The relationship between the structure and the degrees of recursively enumerable sets*. PhD thesis, The University of Chicago, Chicago, Illinois.

[Mučnik, 1956] Mučnik, A. A. (1956). On the unsolvability of the problem of reducibility in the theory of algorithms. *Doklady Akademii Nauk SSSR (Russian)*, 108(2):194–197.

[Post, 1944] Post, E. L. (1944). Recursively enumerable sets of positive integers and their decision problems. *Bulletin of the American Mathematical Society*, 50(5):284–316.

[Post, 1948] Post, E. L. (1948). Degrees of recursive unsolvability: preliminary report (abstract). *Bulletin of the American Mathematical Society*, 54(7):641–642.

[Post, 1965] Post, E. L. (1965). Absolutely unsolvable problems and relatively undecidable propositions - account of an anticipation. In Davis, M. D., editor, *The Undecidable. Basic Papers on Undecidable Propositions, Unsolvable Problems and Computable Functions*, pages 338–433. Raven Press, Hewlett, N.Y.

[Robinson, 1971] Robinson, R. W. (1971). Interpolation and embedding in the recursively enumerable degrees. *Annals of Mathematics*, 93(2):285–314.

[Sacks, 1963] Sacks, G. E. (1963). On the degrees less than 0'. *Annals of Mathematics*, 77(2):211–231.

[Sacks, 1964] Sacks, G. E. (1964). The recursively enumerable degrees are dense. *Annals of Mathematics*, 80(2):300–312.

[Shore and Slaman, 1990] Shore, R. A. and Slaman, T. A. (1990). Working below a low2 recursively enumerably degree. *Archive for Mathematical Logic*, 29(3):201–211.

[Shore and Slaman, 1993] Shore, R. A. and Slaman, T. A. (1993). Working below a high recursively enumerable degree. *The Journal of Symbolic Logic*, 58(3):824–859.

[Slaman, 2015] Slaman, T. A. (2015). Personal communication (Oct. 29, 2015).

[Soare, 1987] Soare, R. I. (1987). *Recursively Enumerable Sets and Degrees*. Springer-Verlag New York, Inc.

[Soare, 2016] Soare, R. I. (2016). *Turing Computability, Theory and Applications*. Springer-Verlag Berlin Heidelberg.

[Turing, 1936] Turing, A. M. (1936). On computable numbers, with an application to the Entscheidungsproblem. *Proceedings of the London Mathematical Society*, 2(42):230–265.

[Turing, 1939] Turing, A. M. (1939). Systems of logic based on ordinals. *Proceedings of the London Mathematical Society*, s2-45(1):161–228.

[Yates, 1966] Yates, C. E. M. (1966). A minimal pair of recursively enumerable degrees. *The Journal of Symbolic Logic*, 31(2):159–168.