On the Rescaled Hitting Time and Return Time Distributions to Asymptotically Small Sets

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Abstract

Consider a hyperbolic flow $\phi_t : M \to M$ on a smooth manifold M, and a sequence of open balls $(\Delta_n)_{n\in\mathbb{N}}$ with $\Delta_n \subset M$ and measure $m(\Delta_n) > 0$ but also satisfying $\lim_{n\to\infty} m(\Delta_n) = 0$. The expected time it takes for the flow to hit the set Δ_n, known as the hitting time, or the return time if the flow started in Δ_n, and each subsequent hit thereafter, is proportional to the measure $m(\Delta_n)$ of that set, provided the measure is ergodic.

In this thesis I study how the distribution of hitting times (and return times), rescaled by an appropriate sequence of constants, converges in the limit. I show conditions under which a Poisson limit law holds by considering the hitting time distributions of an associated discrete dynamical system.

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Declaration

I declare that this thesis is a presentation of original work and I am the sole author. This work has not previously been presented for an award at this, or any other, University. All sources are acknowledged as References.

Chapter 1

Introduction

One of the classic problems in dynamical systems and ergodic theory is the question of how long it takes to reach or return to a particular state, or equivalently how long does it take for a particle to reach a particular region of interest? Classic results on this due to Poincare and Kac and can be found in most ergodic theory textbooks (eg [15]). In theorem 1.1 and theorem 1.2 below, they suggest conditions under which orbits will take a finite time to reach a given region and Kac gives the expected amount of time to hit a region. The natural next step is to understand how long before the second, third, and subsequent times the particle reaches our region of interest. In this thesis I will consider this problem, in particular focussing on how the distribution of these times is affected when the target region is reduced to a set of measure zero.

Consider a dynamical system $(T,\Omega,\mu,\mathscr{B})$, with transformation $T:\Omega\to\Omega$ and μ a T-invariant probability measure.

Theorem 1.1 (Poincare's Recurrence Theorem)**.** *If μ is T-invariant and $A\in\mathscr{B}$ with $\mu(A)>0$ then for μ-a.e. $x\in A$ the orbit $\{T^n x\}_{n\in\mathbb{N}}$ intersects A infinitely often.*

Using this theorem, given a set $A\in\mathscr{B}$ then for $x\in A$ we can define the return time to A by

$$\eta_A(x) := \inf\{k>0 : T^k x\in A\}.$$

If $\mu(A)>0$ then $\eta_A(x)$ will almost surely be finite by Poincare's Recurrence Theorem.

Theorem 1.2 (Kac's Theorem)**.** *If μ is ergodic then for $A \in \mathscr{B}$ with $\mu(A) > 0$ and $x \in A$*

$$\mathbb{E}[\eta_A] = \frac{1}{\mu(A)} \int_A \eta_A(x)\, d\mu = \frac{1}{\mu(A)}.$$

In view of this theorem, given an ergodic system it is reasonable to consider return times to sets of positive measure. The expected time needed to return to a set of positive measure is proportional to the inverse of the measure of that set.

Definition 1.1. Given $x \in \Omega$ and $A \subset \Omega$ such that $\mu(A) > 0$, a *hitting time* for x to the set A is a measurable function $X : \Omega \to \mathbb{R}$ satisfying

$$X(x) = \inf\{i \in \mathbb{N} : T^i x \in A\}.$$

It is worth noting that the difference between a hitting time and a return time is the location of the initial point x; for a hitting time $x \in \Omega$ but for a return time $x \in A$. Although these two ideas are related, properties of one do not always imply properties of the other, see for example irrational rotations of the circle as discussed in the paper [6]. By assuming ergodicity however, Poincare's recurrence theorem extends to the full space. That is to say that the hitting times to a set of positive measure will be finite almost surely for an ergodic tranformation.

Throughout this thesis we are interested in what happens in the limiting case where sequence of target sets have measures tending to zero. Kac's theorem clearly indicates that it is not reasonable to expect that such a time might be finite, so we will consider the limiting case for a sequence of sets with positive measure, where the measure converges to zero. We now introduce some notation to describe this setting.

Consider a sequence of sets $A_n \in \mathscr{B}$ such that $\mu(A_n) > 0$ and

$$\lim_{n\to\infty} \mu(A_n) = 0.$$

for $x \in \Omega$ denote the first hitting time to A_n by

$$r_n^{(1)}(x) := \inf\{i \in \mathbb{N} : T^i x \in A_n\},$$

and subsequent hitting times are defined inductively as

$$r_n^{(k)}(x) := \inf\{i > r_n^{(k-1)} : T^i x \in A_n\}.$$

It is expected that these hitting times will be finite for μ-almost every $x \in \Omega$ by Theorems 1.1 and 1.2, but when one considers the hitting times in the limit as n tends to infinity Kac's Theorem implies that the hitting times and subsequent return times will become infinitely large. In order to tackle this problem and understand the limit law, the process must be rescaled.

This motivates the introduction of the point process of rescaled hitting times

$$X_n(x) = \sum_{k\in\mathbb{N}} \delta_{r_n^{(k)}(x)c_n}$$

where δ denotes the Dirac point mass and $(c_n)_{n\in\mathbb{N}}$ is a sequence of positive real numbers.

In this form X_n simply records the times at which a process hits the set A_n by assigning a point mass at each hitting time. By representing the process in this way it is possible to understand how the times are distributed. For the general hitting time problem the aim now is to find a suitable scale, c_n, such that X_n converges in distribution, and to then find the process to which it converges.

The main results in this thesis will be for a continuous dynamical system $(\phi_t, \Omega, m, \mathscr{B})$ with a continuous flow $(\phi_t : t \in \mathbb{R})$ and an invariant probability measure m. Consider a sequence of open sets $(U_n)_{n\in\mathbb{N}} \subset \mathscr{B}$ with measure $0 < m(U_n) < m(\Omega)$ but again

$$\lim_{n\to\infty} m(U_n) = 0.$$

For $\omega \in \Omega$ define the first hitting time to U_n by

$$\tau_n^{(1)}(\omega) = \inf\{t > 0 : \phi_t(\omega) \in U_n\},$$

and inductively define the subsequent hitting times

$$\tau_n^{(k)}(\omega) = \inf\left\{ t > \tau_n^{(k-1)} : \phi_t(\omega) \in U_n \text{ and } \exists s \in (\tau_n^{(k-1)}, t) \text{ such that } \phi_s(\omega) \notin U_n \right\}.$$

The hitting time process will be denoted

$$Z_n(\omega) = \sum_{k\in\mathbb{N}} \delta_{\tau_n^{(k)}(\omega)c_n}.$$

In this thesis I will give conditions on the flow ϕ_t and on Ω, along with suitable rescaling, and show that under such conditions the hitting time process (and the return time process) converges in distribution to a Poisson point process. I will do this by making use of known results in discrete systems introduced by Chazottes, Coelho and Collet [4, 5], Hirata [11], and Pitskel [14], and relating these to continuous systems, particularly making use of results by Bowen and Ruelle [1, 2].

Lemmas 3.2, 3.4, 3.14, 3.15, 3.16, and 3.17 along with Theorems 2.6, 3.10, 3.11, 3.13, 3.18, 3.19, and 3.21 are original results in this thesis.

First I will introduce notations and results from probability and ergodic theory.

1.1 Ergodic Theory

Here we will recall some theory and notation from ergodic theory which will be useful throughout this thesis.

Definition 1.2. Consider a dynamical system $(T, \Omega, \mu, \mathscr{B})$, with measurable transformation $T : \Omega \to \Omega$, and probability measure μ.

1. T is *measure preserving* if $\mu(T^{-1}(A)) = \mu(A)$ for any $A \in \mathscr{B}$.
2. T is *ergodic* if whenever $T^{-1}(A) = A$ then $\mu(A) = 0$ or 1.

Noting for example that Kac's theorem only applies to ergodic measures, these properties are therefore key to understanding recurrence times.

The following ergodic theorems are some of the main results in ergodic theory, and proofs of which can be found in [15]. Before stating these theorems some definitions are required.

Definition 1.3. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions on the measure space $(\Omega, \mathscr{B}, \mu)$. Then

1. f_n converges (μ-)almost surely to f if

$$\mu\left\{x \in \Omega : \lim_{n\to\infty} f_n(x) = f(x)\right\} = \mu(\Omega).$$

2. a function $f : \Omega \to \mathbb{R}$ is said to be L^p or $f \in L^p(\Omega, \mathscr{B}, \mu)$ for $1 \leq p < \infty$ if the integral

$$\int_\Omega |f(x)|^p \, d\mu(x)$$

exists and is finite.

3. f_n converges in L^p, for $1 \leq p < \infty$, to f if

$$\lim_{n\to\infty} \int |f_n(x) - f(x)|^p \, d\mu(x) = 0.$$

Theorem 1.3 (The Birkhoff Ergodic Theorem)**.** *Let $T : \Omega \to \Omega$ be an ergodic measure preserving transformation for the probability measure μ and let $f : \Omega \to \mathbb{R}$ be measurable and integrable. That is to say that the integral, $\int |f| \, d\mu$, exists and is finite. Then for μ-almost every $x \in \Omega$ the ergodic averages converge in the limit, that is*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x) = \int_\Omega f(x) \, d\mu(x).$$

Theorem 1.4 (The Von Neumann Ergodic Theorem)**.** *Let $T : \Omega \to \Omega$ be an ergodic measure preserving transformation for the probability measure μ and let $f \in L^2(\Omega, \mathscr{B}, \mu)$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x) = \int_\Omega f(x) \, d\mu(x)$$

where the convergence is in L^2.

A proof for both of these ergodic theorems can be found in [15]. It is worth noting that later in this thesis we will need similar versions of these theorems, where n is replaced by an increasing sequence (r_n) which diverges to infinity in the limit. In which case it follows that both theorems still hold and

$$\lim_{n\to\infty} \frac{1}{r_n} \sum_{i=0}^{r_n-1} f(T^i x) = \int_\Omega f(x) \, d\mu(x)$$

μ-almost surely or when the convergence is L^2, by noting that this is a subsequence of a convergent sequence.

Equilibrium Measures and Pressure

Definition 1.4 (Conditional Measures)**.** Let $(\Omega, \mu, \mathscr{B})$ be a measure space. Then for measurable sets A and B, with $\mu(B) > 0$, the measure of A conditioned on B is written as

$$\mu(A|B) = \frac{\mu(A \cap B)}{\mu(B)}.$$

It is also possible to define conditional measures with respect to sub-σ-algebras. Let $\mathscr{A}$ be a sub-σ-algebra of $\mathscr{B}$ then the conditional measure of $B \in \mathscr{B}$ given $\mathscr{A}$ is the function

$$\mu(B|\mathscr{A}) = \mathbb{E}(\chi_B|\mathscr{A}),$$

which is to say that it is a $\mathscr{A}$-measurable random variable satisfying

$$\int_A \mu(B|\mathscr{A})\, d\mu = \mu(A)$$

for each $A \in \mathscr{A}$.

Definition 1.5 (Entropy)**.**

1. Given a countable measurable partition, $\alpha = \{A_1, A_2, \ldots,\}$, of Ω, that is $A_i \in \mathscr{B}$ and $\bigcup_i A_i = \Omega$ but $A_i \cap A_j = \emptyset$ up to a set of μ-measure zero, then the *conditional entropy* of α given $\mathscr{A}$ is defined as
$$H(\alpha|\mathscr{A}) := -\sum_{A \in \alpha} \mu(A|\mathscr{A}) \log \mu(A|\mathscr{A}).$$

2. If $T : \Omega \to \Omega$ is measure preserving transformation then the *entropy of the partition* α *relative to the transformation* is defined as
$$h(T, \alpha) := \lim_{n \to \infty} H(\alpha| \bigvee_{i=1}^{n-1} T^{-i}\alpha)$$
where $\bigvee_{i=1}^{n-1} T^{-i}\alpha$ is a common refinement of the partitions $\alpha\, T^{-1}\alpha, \ldots, T^{-n+1}\alpha$.

3. The *measure theoretic entropy* of $T : \Omega \to \Omega$ is defined as
$$h(\mu) = h_\mu(T) := \sup\{h(T, \alpha) : H(\alpha) \text{ is finite}\}.$$

Definition 1.6 (Equilibrium States)**.** Given a potential $f : \Omega \to \mathbb{R}$, an *equilibrium measure* (or equilibrium state), where it exists, is a measure $\mu = \mu_f$ which realises the supremum

$$\begin{aligned} P(f) &:= \sup\left\{ h(m) + \int f\,dm \;:\; m \text{ is a } T\text{-invariant probability measure} \right\} \\ &= h(\mu) + \int f\,d\mu, \end{aligned}$$

where P is the pressure function and $h(m)$ is the measure-theoretic entropy of the system (T, m).

The pressure of a dynamical system and equilibrium measures are useful tools when attempting to understand how the system can behave when restricting the system to a smaller subsystem (see [4]). They are also useful for their relationship with the transfer operator, introduced in Chapter 2, and the spectral properties that emerge.

1.2 A Motivating Example: A Self-Similar Model

In 1999, Floriani and Lima [8] constructed a suspension flow which they described as a self-similar system. The model was developed in order to demonstrate properties of turbulence seen in fluid dynamics. In particular the structure gives rise to interesting flow patterns where a particle will remain in small pockets for long periods before potentially moving quickly between different areas until it reaches another small pocket. We will construct the model used and discuss some of the results, and why this example is of interest.

Starting with the interval $I = [0,1)$ and the doubling map $T : I \to I : x \mapsto 2x \mod 1$, with a T-invariant measure μ define a sequence of sets by the following fractal construction: $A_0 = \left[\frac{1}{4}, \frac{3}{4}\right]$ is the centre half of I. A_1 is then the union of the centre halves of each disjoint interval of $I \setminus A_0$, that is $A_1 = \left[\frac{1}{16}, \frac{3}{16}\right] \cup \left[\frac{13}{16}, \frac{15}{16}\right]$. A_2 is then the union of the centre halves of each of the four remaining intervals in $I \setminus (A_0 \cup A_1)$ and A_n is the union of the centre halves of each of the 2^n remaining intervals in $I \setminus \bigcup_{k=0}^{n-1} A_n$. Continuing as such, it should be clear that $I = \bigcup_{n=0}^{\infty} A_n$. See Fig 1.1.

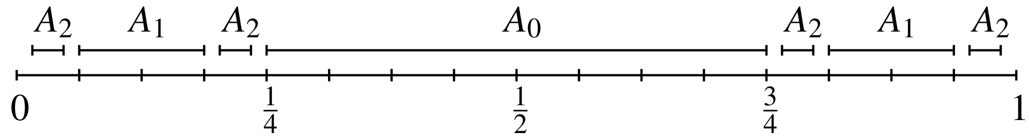

Figure 1.1: Showing the fractal construction of the sets A_0, A_1, A_2 in the unit interval.

Now create a suspended space above the interval by using a height function $\gamma : I \to I$ given by $\gamma(x) := \lambda^n$ if $x \in A_n$ for some $\lambda > 1$ and call the blocks $A_n \times [0, \lambda^n] = \Delta_n$. The value of λ should be chosen appropriately so that $\int \gamma d\mu < \infty$ (in particular if μ is lebesgue then $1 < \lambda < 2$). The suspended space is defined by

$$\Omega := \{\omega = (x,y) : x \in I, 0 \le y < \gamma(x)\} = \bigcup_{n=0}^{\infty} \Delta_n,$$

(see Fig 1.2) and define a new measure on Ω

$$\nu := \frac{\mu \times \text{Leb}}{\int \gamma d\mu}.$$

The suspension flow is then given by

$$S_t(\omega) = S_t(x,y) = \left(T^{\eta(t)}x, y + t - \sum_{i=0}^{\eta(t)-1} \gamma(T^i x) \right)$$

where $\eta(t)$ is the unique natural number which satisfies

$$0 \le y + t - \sum_{i=0}^{\eta(t)-1} \gamma(T^i x) < \gamma(T^{\eta(t)}x).$$

This construction gives the model a self-similar 'island within island' structure which closely mimics the effects of turbulence. Since the particle moves at a constant unit speed, once a particle reaches a set Δ_n for a large n it will remain in Δ_n for a disproportionate time while it travels the full height of Δ_n. However for a large n the width of Δ_n will be small so the probability of reaching such a pocket will be small. Kac's Theorem implies that the time taken for the base map T to reach $A_n \subset I$ will be proportional to

$$\frac{1}{\mu(A_n)}.$$

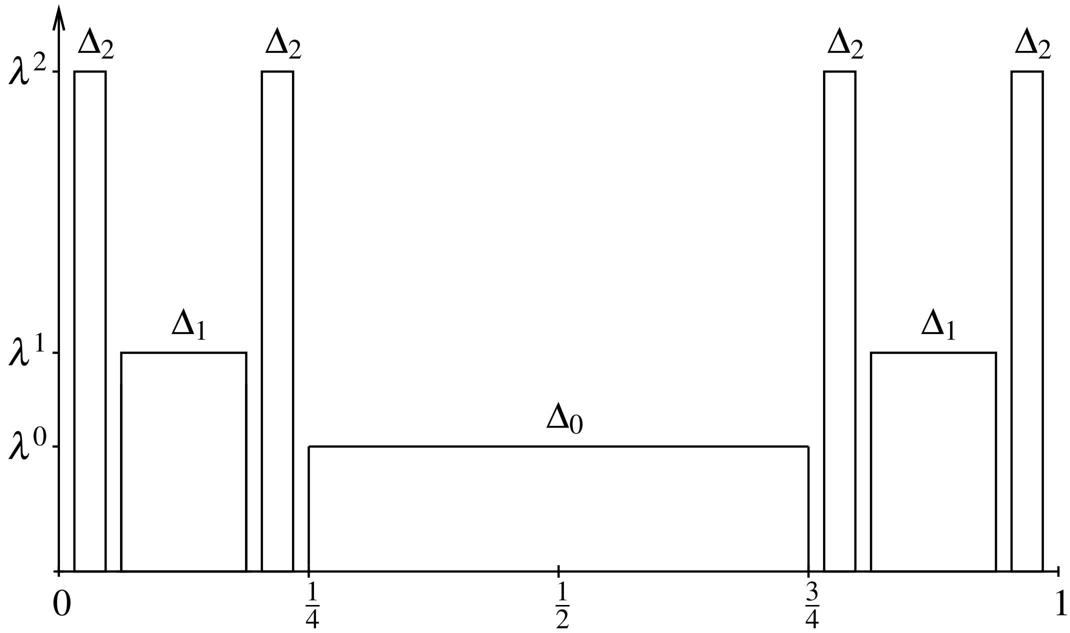

Figure 1.2: Showing the construction of the suspended space.

Floriani and Lima [8] considered the distribution of the first return times for the flow, given by

$$\mathscr{F}_n(t) = \frac{\nu\left\{(x,y) \in \Delta_n : \tau_n^{(1)} > t\right\}}{\nu(\Delta_n)},$$

and the first return times for the discrete system on the base, given by

$$F_n(t) = \frac{\mu\left\{x \in A_n : \tau_n^{(1)} > t\right\}}{\mu(A_n)}.$$

By considering rescaling factors $c_n^+ \propto \lambda^n$ and $c_n^- \propto \lambda^{n-1}$, Floriani and Lima showed bounds for these distributions satifying

$$F_n^-(t) \leq F_n(t) \leq F_n^+(t)$$

and similarly

$$\mathscr{F}_n^-(t) \leq \mathscr{F}_n(t) \leq \mathscr{F}_n^+(t).$$

For both cases

$$F_n^-(t),\ \mathscr{F}_n^-(t) \approx \left(\frac{t}{c_n^-}\right)^{-\alpha} \qquad \text{for } t \gg \lambda^n,$$

and

$$F_n^+(t),\ \mathscr{F}_n^+(t) \approx \left(\frac{t}{c_n^+}\right)^{-\alpha} \qquad \text{for } t \geq c_n^+ \gg \lambda^n,$$

where $\alpha = \frac{\log 2}{\log \lambda}$. The rescaling constants are given explicitly as

$$c_n^- = \lambda^{n-1}$$
$$c_n^+ = \frac{2(\lambda+1)}{\lambda-1}(2\lambda)^n\,(3\Gamma(\alpha+1))^{1/\alpha}$$

where Γ is the gamma function.

Remark 1.5. The notation $f(t) \approx g(t)$ is taken here to mean f and g have the same limiting behaviour and in particular

$$\lim_{t\to\infty} \frac{f(t)}{g(t)} = c \in \mathbb{R}_{>0}.$$

The notation $t \gg s$ means $t \geq cs$ for some constant $c \in \mathbb{R}$. Here it is needed as there are terms that will converge to constants for large t, but the main result is how these functions behave in the limit so these are omitted.

Here we say $c_n \propto C_n$ if the limit

$$\lim_{n\to\infty} \frac{c_n}{C_n}$$

exists and is finite.

Floriani and Lima suggested that their bounds indicate a polynomial law for the first return time distribution for any finite $n \in \mathbb{N}$, since the limiting behaviour, but do not rule out an exponential law.

This motivates the work to investigate the distribution of the rescaled hitting and return time processes of this system and other similar systems. This is especially due to the relationship between suspended flows and axiom A flows on manifolds, which will be discussed later in the thesis.

I will show in this thesis that for any $n \in \mathbb{N}$ and in the limiting case where n diverges to infinity, the distribution of return times follows a poisson law and that the first return time is indeed exponential.

1.3 Point Processes and Probabilistic Methods

I will now introduce some useful concepts from probability theory and statistics, and discuss briefly how these can be used in hitting time problems. First define a probability space $(\Omega, \mathbb{P}, \mathscr{B})$ where the probability, $\mathbb{P}$, is a probability measure.

Definition 1.7.
1. A random variable $X = X(\omega)$ is a measurable function $X : \Omega \to \mathbb{R}$.
2. A random vector $X = X(\omega) = (X_1, \ldots, X_n)$ is a vector consisting of random variables X_i.

Notice that a hitting time is a random variable since it is a measurable funtion $t : \Omega \to \mathbb{R}$. The idea of a random variable can be extended to the idea of a random element, something that takes on the same role as a random variable yet does not necessarily take on real values.

Definition 1.8. Given measurable spaces $(\Omega, \mathscr{B})$ and $(\Omega', \mathscr{B}')$ a *random element* taking values in Ω' is a measurable function $X : \Omega \to \Omega'$.

It is worth noting that random variables and random vectors are examples of random elements.

The objects we are to study are the times at which each orbit will hit the desired sets. These can be denoted as sets of events for example as

$$\{X_i \ : \ X_i \ \text{ is the } \ i\text{ -th hitting time } \}.$$

To understand the distribution of these times it is desirable to be able to count how many of these points are within an arbitrary time interval. This can be achieved by utilising the Dirac point measure, δ_x, on $\mathbb{R}$ which puts unit mass at $x \in \mathbb{R}$. That is for $A \in \mathscr{B}$)

$$\delta_x(A) := \begin{cases} 1 & x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Then describe the process as a random measure which puts a unit mass at each hitting time on the real numbers. That is to say let

$$X = \sum_{i \in \mathbb{N}} \delta_{X_i}.$$

This means that for an open interval $B \in \mathbb{R}$, $X(B)$ simply counts the number of hitting times that occur within the interval B. This motivates the following formulation of a point process.

Definition 1.9 (Point Process)**.** Consider a measurable spaces $(\Omega, \mu, \mathscr{B})$ and a sequence of random variables $(X_i)_{i \in \mathbb{N}}$ defined on this space. A *point process* is a σ-finite measure on $(0, \infty)$ with discrete support and can be written

$$X = X(\omega) = \sum_{i=1}^{\infty} \delta_{X_i}$$

where δ_x is the Dirac point measure with its mass at $x \in \mathbb{R}$.

We're particularly interested in point processes as hitting processes, that is where the X_i are in fact hitting times.

Given a sequence of sets $A_n \subset \Omega$ the hitting times to A_n can be written as a point process, that is as a point process where the random variables are the hitting times. So let the random variable $r_n^{(k)} = X_k : \Omega \to \mathbb{R}$ be the k-th hitting time to A_n, then for each n use

$$r_n = \sum_{k=1}^{\infty} \delta_{r_n^{(k)}}$$

to describe the hitting time process to the set A_n.

One particular point process of interest is the Poisson point processes on the real numbers. This process is characterised by the distribution of the intervals between consecutive random variables. Given random variables $X_i : \Omega \to \mathbb{R}$ the interval (or difference) between each consectutive pair given by $X_{i+1} - X_i$ is exponentially distributed and is independent of the index i, the size of any other interval $X_{j+1} - X_j$, and the value X_i.

Definition 1.10 (Poisson Point Process)**.** A *Poisson point process* of rate λ is a point process

$$X = \sum_{i=1}^{\infty} \delta_{X_i}$$

where

1. $X(B_1), \ldots, X(B_n)$ are independent for disjoint subsets $B_1, \ldots, B_n \subset \mathbb{R}$

2. For any bounded subset $B \subset \mathbb{R}$ the random variable $X(B) : \Omega \to \mathbb{R}$ has a Poisson distribution of rate $\lambda \,\text{Leb}\{B\}$, that is to say that the distribution is given by
$$\mathbb{P}(X(B) = k) = e^{-\lambda \,\text{Leb}\{B\}} \frac{(\lambda \,\text{Leb}\{B\})^k}{k!},$$
where 'Leb' is used to denote the Lebesgue measure on $\mathbb{R}$.

Convergence of Point Processes

There are many types of convergence when dealing with random variables, but here we are mostly interested in convergence in distribution, also known as convergence in law. This corresponds to a convergence of distribution functions.

Definition 1.11 (Distribution Functions)**.** Consider a measureable space $(\Omega, \mathscr{B})$.

1. For a random variable $X : \Omega \to \mathbb{R}$, a *distribution function* is a function $F : \mathbb{R} \to \mathbb{R}$ given by
$$F_X(t) = \mathbb{P}(X < t) = \mu\{\omega \in \Omega | X(\omega) < t\}.$$

2. For a random vector $X = (X_1, X_2, \ldots, X_n) : \Omega \to \mathbb{R}^n$ a *distribution function* is a function $F : \mathbb{R}^n \to \mathbb{R}$ given by
$$F_X(t_1, \ldots, t_n) = \mathbb{P}(X_i < t_i \;\forall i).$$

3. A sequence of random variables $(X_n)_{n\in\mathbb{N}}$ converges in distribution to X if the sequence of respective distribution functions, $(F_n)_{n\in\mathbb{N}}$, converge pointwise to $F : \mathbb{R}^n \to \mathbb{R}$, the distribution function for X, for all continuity points $x \in \mathbb{R}^n$.

The following theorem gives a useful understanding of the idea of convergence in distribution, and is a standard result from probability theory.

Theorem 1.6. *A sequence of random variables X_n converges in distribution if and only if for all continuous functions $h : \mathbb{R} \to \mathbb{R}$ with compact support the expectations $\mathbb{E}(h(X_n))$ converge to $\mathbb{E}(h(X))$ in the usual sense of convergence.*

This motivates the following generalisation of convergence in law (distribution) for a point process.

Definition 1.12. A point process

$$X_n = \sum_{k=1}^{N} \delta_{X_n^{(k)}}$$

converges in distribution to X if and only if for all continuous $g : \mathbb{R} \to \mathbb{R}$ with compact support the random variables given by

$$N_n(g)(\omega) = \int_0^\infty g(t) dX_n(\omega)(t) = \sum_{k=1}^{N} g(X_n^{(k)}(\omega))$$

converge in distribution, as $n \to \infty$, to

$$N(g)(\omega) = \int_0^\infty g(t)\, dX(\omega)(t).$$

A point process X_n is said to have a *Poisson limit law* (or converge in distribution to a Poisson process) if X is a Poisson point process.

This gives a simple object to study when considering the convergence in distribution of random processes, and so for the rest of this thesis we will be considering these objects and how they behave in the limit. The goal is to find Poisson limit laws for the systems under consideration.

Chapter 2

Discrete Dynamics

In this chapter we will discuss some previous results within the discrete dynamical setting. We will also make use of and develop ideas appearing in [4] to build theory around the self-similar model introduced earlier in section 1.2.

2.1 Markov Chains and Subshifts of Finite Type

Let $V = \{1, \dots, \ell\}$ be a finite set of symbols and let $A = \{A_{ij}\}$ be an $\ell \times \ell$ matrix of zeros and ones, that is $A_{ij} \in \{0, 1\}$. Define the phase space

$$\Sigma_A := \left\{ \omega = (\omega_0, \omega_1, \omega_2, \dots) \in V^{\mathbb{N}} : A(\omega_i, \omega_{i+1}) = 1 \quad \forall i \in \mathbb{N} \right\},$$

and associate with it the shift map $\sigma : \Sigma_A \to \Sigma_A$ defined by

$$\sigma(\omega_0, \omega_1, \omega_2, \dots) = (\omega_1, \omega_2, \omega_3, \dots).$$

This is the one-sided subshift of finite type. By considering sequences which are inifite in both directions, that is sequences of the form $(\dots, \omega_{-1}, \omega_0, \omega_1 \dots)$, along with the same shift map σ, we obtain the two-sided subshift of finite type.

Definition 2.1. (Markov Measures). Let $P = \{P_{ij}\}$ be an irreducible $\ell \times \ell$ stochastic matrix, that is any irreducible matrix such that $P_{i,j} \geq 0$ for all $1 \leq i, j \leq \ell$ and the rows sum to one, i.e.

$$\sum_{j=1}^{\ell} P_{ij} = 1,$$

and $P_{ij} = 0$ if and only if $A_{ij} = 0$. Then let $p = (p_1, \dots, p_\ell)$, satisfying $\sum_j p_j = 1$, be the left eigenvector which has eigenvalue 1. A *Markov measure* μ on Σ_A, with stochastic matrix P, is defined on cylinder sets by

$$\mu\left([\omega_{-m}, \dots, \omega_n]\right) := p_{\omega_{-m}} P_{\omega_{-m}\omega_{-m+1}} \dots P_{\omega_{n-1}\omega_n}.$$

A subshift of finite type with a Markov measure is called a *Markov Chain*.

Limit Laws for Open Balls

In 1991, Pitskel [14] studied the hitting time process to a sequence of open balls for mixing Markov chains, in the space of two sided sequences, and found a Poisson limit law.

Definition 2.2 (Mixing)**.** The system $(\Omega, \mathscr{B}, \mu, T)$ is called mixing, or strong mixing, if for any $A, B \in \mathscr{B}$

$$\lim_{n\to\infty} \mu(A \cap T^{-n}B) = \mu(A)\mu(B).$$

Let Ω be the space of allowable two sided sequences, σ the shift map, and μ a σ-invariant, mixing Markov measure. Then consider cylinder sets of the form

$$A_n = A_n(\omega^*) = \{\omega \in \Omega : \omega_i = \omega_i^*, -n \le i \le n\}$$

for an arbitrary, and fixed ω_i^*. Pitskel looked at the hitting times to these cylinder sets. So let $r_n^{(k)} = r_n^{(k)}(\omega)$ denote the k-th hitting time to A_n, and call the process

$$r_n(\omega) = \sum_{k=1}^{\infty} \delta_{c_n r_n^{(k)}}$$

where (c_n) is a rescaling sequence. Pitskel [14] showed the following theorem for such Markov chains.

Theorem 2.1. *Let A_n be a cylinder set in Ω, and $\lambda > 0$ then the hitting time process rescaled by $c_n = \lambda\mu(A_n)$ converges to a Poisson point process of rate λ.*

Axiom A Diffeomorphisms

In 1993, Hirata [11] produced a Poisson limit law for discrete axiom A diffeomorphisms. This is closely linked to the new work appearing in this thesis in that we obtain a result for axiom A flows in continuous dynamical systems.

Let M be a compact C^∞ Riemannian manifold and $f : M \to M$ be a C^1 diffeomorphism. The notion of axiom A was first introduced by Smale in [18]. We first require the definitions of hyperbolicity and non-wandering sets.

Definition 2.3 (Hyperbolic Sets)**.** A closed f-invariant set $\Lambda \subset M$ is *hyperbolic* if the tangent bundle restricted to Λ can be written $T_\Lambda M = E^s \oplus E^s$ where E^s, E^u are f-invariant subbundles and there are constants $c > 0$ and $\lambda \in (0,1)$ such that

1. $\|Df^k(z)\| \leq c\lambda^k \|z\|$ for every $z \in E^s$, $k \in \mathbb{N}$.
2. $\|Df^{-k}(z)\| \leq c\lambda^k \|z\|$ for every $z \in E^u$, $k \in \mathbb{N}$.

Definition 2.4 (Basic Hyperbolic Sets)**.** A closed invariant set Λ is a *basic hyperbolic set* if the following are satisfied:

1. Λ is hyperbolic.
2. The periodic orbits of $f|_\Lambda$ are dense in Λ.
3. For any open sets $U, V \subset \Lambda$, there is an integer $n \in \mathbb{N}$ satisfying
$$(f|_\Lambda)^n(U) \cap f|_\Lambda(V) \neq \emptyset$$
4. There is an open set $U \supset \Lambda$ with
$$\Lambda = \bigcap_{k \in \mathbb{Z}} f^k U.$$

Definition 2.5 (Non-Wandering Sets)**.** The *non-wandering set* $\Omega = \Omega_f \subset M$ is given by

$$\Omega_f := \left\{ z \in M : \text{ for every open } V \ni z,\ k_0 > 0\ \exists\, k > k_0 \text{ with } f^k(V) \cap V \neq \emptyset \right\}.$$

Definition 2.6. A diffeomorphism, $f : M \to M$, is said to satisfy *axiom A* if its non-wandering set Ω_f is a hyperbolic set.

Smale's spectral decomposition theorem states that the non-wandering set for an axiom A diffeomorphism is a disjoint union of a finite number of basic sets. And that without loss of generality it can be assumed that the diffeomorphism f is mixing.

As an example, consider the Smale horseshoe map.The horseshoe map $f : S \to S$ is a diffeomorphism defined on $S \subset \mathbb{R}^2$ into itself. The region S is a square capped by two semi-disks. f is defined through the composition of three transformations: First the square is contracted along the vertical direction by a factor $a < \frac{1}{2}$. The caps are contracted so as to remain semi-disks attached to the resulting rectangle. $a < \frac{1}{2}$ so that there will be a gap between the branches of the horseshoe. Next the rectangle is stretched horizontally by a factor of $\frac{1}{a}$. Finally the resulting strip is folded into a horseshoe-shape and placed back into S.

The interesting part of the dynamics is the image of the square into itself. Once that part is defined, the map can be extended to a diffeomorphism by defining its action on the caps. The caps are made to contract and eventually map inside one of the caps. The extension of f to the caps adds a fixed point to the non-wandering set of the map. To keep the class of horseshoe maps simple, the curved region of the horseshoe should not map back into the square.

In this example the non-wandering set will be the limiting matrix of points which when mapped are contained in each iteration of $f^n(S)$. The horseshoe map is one-to-one, when restricted to the non-wandering set.

Hirata [11] considered an axiom A diffeomorphism, $f : M \to M$, with non-wandering set $\Omega = \Omega_f$, and assumed that $f|_\Omega$ is mixing. Let $u : \Omega \to \mathbb{R}$ be Lipschitz and consider the associated unique equilibrium state $\mu = \mu_u$, that is the unique measure μ which realises the supremum

$$\begin{aligned} P(u) &= \sup\left\{ h(m) + \int u\,dm \ : \ m \text{ is a } T\text{-invariant probability measure} \right\} \\ &= h(\mu) + \int u\,d\mu, \end{aligned}$$

where P is the pressure function and $h(m)$ is the measure-theoretic entropy of the system (T,m). For a fixed $z \in \Omega$, consider the neighbourhoods of radius ε, denoted

$\{U_\varepsilon(z)\}$. Hirata showed the following Poisson limit law.

Theorem 2.2. *For μ-a.e. $z \in \Omega$ the return time process to $U_\varepsilon(z)$, rescaled by $\mu(U_\varepsilon)$, converges in distribution to a Poisson point process of rate 1 as $\varepsilon \to 0$.*

In order to show this Hirata looked at Markov partitions and in particular subshifts of finite type and produced the following result. This was achieved by showing that normalised waiting times, for each successive return, are mutually independent and are exponentially distributed in the limit. The return times were also shown to satisfy a Poisson limit law.

Theorem 2.3. *For μ-a.e. $z \in \Sigma_A$ the return time process to $U_\varepsilon(z)$, rescaled by $\mu(U_\varepsilon(z))$, converges to a Poisson point process of rate 1 as $n \to \infty$.*

Poisson Laws for Repeated Events

In 2009, Chazottes, Coelho and Collet [4] gave sufficient conditions for a Poisson limit law for hitting time processes. In particular they considered the structure of the target sets A_n.

Chazottes, Coelho and Collet considered the problem of how long until the same event is repeated n times in a row, and then considered the distribution of the waiting times between each of these sequences of repeat events. For example consider how long until a series of coin flips produces n heads in a row. They showed a Poisson limit law exists as n diverges to infinity.

Consider a subshift of finite type, (Σ_A, σ), for an irreducible and aperiodic matrix of allowable paths A and let $\Delta \subset \Sigma_A$ be a set of positive measure. Construct A_n to be the set of points that are restricted to Δ for the first n iterates of T. That is

$$A_n = \Delta \cap \sigma^{-1}\Delta \cap \cdots \cap \sigma^{(-n)}\Delta.$$

In order to obtain a Poisson limit law Chazottes, Coelho and Collet showed that the process of hitting times to A_n satisfies two particular properties. These two properties are enough to show a Poisson limit law for a general hitting time process to a sequence of measurable sets (Δ^n), with positive meaasure satisfying

$$\lim_{n\to\infty} \mu(\Delta^n) = 0.$$

Property 1. The following limit exists:

$$C_m = \lim_{n\to\infty} c_n \sum_{\substack{0=q_0<\cdots<q_{m-1} \\ q_s-q_{s-1}\leq \frac{n}{m}}} \mathbb{E}\left(\prod_{s=0}^{m-1} \chi_{\Delta^n}\circ T^{q_s}\right)$$

where c_n is taken to be $\mu(\Delta^n)$.

Property 1 is used to acheive a distribution of points in the point process that fits that of a Poisson process in the limit, and is controlled sufficiently by the sequence c_n. By considering all possible sequences satisfying the condition that $q_s - q_{s-1} \leq \frac{n}{m}$ and summing the expectation that all points lie in the target set gives a sufficient estimate for the return time and hitting time for the process to Δ_n.

Property 2. There exist $K_m > 0$ and $0 < \gamma < 1$ such that for every $0 = j_0 < j_1 < \cdots < j_m$ satisfying $j_s - j_{s-1} \leq \frac{n}{m}$ we have for sufficiently large n,

$$\left|\mathbb{E}\left(\prod_{s=0}^{m} \chi_{\Delta^n}\circ T^{j_s}\cdot \chi_B \circ T^{r+j_m}\right) - \mathbb{E}\left(\prod_{s=0}^{m} \chi_{\Delta^n}\circ T^{j_s}\right)\mu(B)\right| \leq K_m \gamma^{r+j_m}\mu(B)$$

for every $r > 0$, and for every $B \in \mathscr{B}$.

Property 2 is often called the ‘ decay of correlations’ and gives a strong enough condition for the dependence of the waiting times between each event to diminish so that in the limit the waiting times are independent.

In [4] the following theorem is established.

Theorem 2.4. *Let (Δ^n) be a sequence of sets in Σ such that $\lim_{n\to\infty}\mu(\Delta^n) = 0$ and (c_n) be a sequence of positive real numbers. If (Δ^n) and (c_n) satisfy properties 1 and 2 then thehitting time process rescaled by c_n converges to a Poisson point process of rate 1.*

They then apply this result to their construction, which satisfies properties 1 and 2, and conclude the following theorem.

Theorem 2.5. *The hitting time and return time process to A_n, rescaled by e^{-nP_Δ}, converges in law to a Poisson point process with rate* 1.

Here P_Δ is the pressure restricted to Δ. That is to say that if for some potential $f : \Omega \to \mathbb{R}$ the pressure is defined as

$$P(f) = \sup\left\{ h(m) + \int f\, dm \;:\; m \text{ is a } T\text{-invariant probability measure} \right\}$$

where $h(m)$ is the measure-theoretic entropy of the system (T,m), then the potential restricted to $\Delta \subset \Omega$ is given by

$$P_\Delta(f) = \sup\left\{ h_m(T|\Delta + \int_\Delta f dm \;:\; m \text{ is a } T\text{-invariant probability measure} \right\}.$$

2.2 Hitting Times for the Doubling Map with the Self-Similar Model

Recall the motivating example introduced by Floriani and Lima (see section 1.2). We shall be interested in the action of the doubling map on the base, $I = [0,1)$, and make use of the self-similar set structure to show that the hitting time process associated with this construction satisfies properties 1 and 2 and therefore show a Poisson limit law.

Recall the construction of A_n. $A_0 = \left[\frac{1}{4}, \frac{3}{4}\right]$ is the centre half of I. A_1 is then the union of the centre halves of each disjoint interval of $I \setminus A_0$, that is $A_1 = \left[\frac{1}{16}, \frac{3}{16}\right] \cup \left[\frac{13}{16}, \frac{15}{16}\right]$. A_2 is then the union of the centre halves of each of the four remaining intervals in $I \setminus (A_0 \cup A_1)$ and A_n is the union of the centre halves of each of the 2^n remaining intervals in $I \setminus \bigcup_{k=0}^{n-1} A_n$. See Fig 1.1 in section 1.2.

The remainder of this section will be dedicated to proving the following theorem.

Theorem 2.6. *The hitting time and return time process to A_n, rescaled by e^{-nP_Δ}, where P_Δ is the pressure function restricted to Δ converges in law to a Poisson point process with rate* 1.

The following lemma follows directly from the construction of the sets A_n.

Lemma 2.7. *For the system described above, the sets A_n can be written*

$$A_n = \Delta \cap T^{-2}\Delta \cap \cdots \cap T^{-2n+2}\Delta \cap T^{-2n}\Delta^c,$$

where $\Delta = A_0^c = [0, \frac{1}{4}) \cup (\frac{3}{4}, 1]$.

Proof. First let $n = 0$ and consider

$$A_0 = \Delta^c = [\tfrac{1}{4}, \tfrac{3}{4}].$$

Then for general $n \in \mathbb{N}$ assume the theorem holds and I will show it holds for $n+1$. A_{n+1} is the union of the centre halves of each disjoint interval of $I \setminus (A_0 \cup \cdots \cup A_n)$. But this is the same as the union of the centre halves of

$$\begin{aligned} B_n &= I \setminus (\Delta^c \cup \cdots \cup (\Delta \cap T^{-2}\Delta \ldots T^{-2n}\Delta^c) \\ &= \Delta \cap (\Delta \cup T^{-2}\Delta^c) \cap \cdots \cap (\Delta \cup \cdots \cup T^{-2n}\Delta^c) \\ &= \Delta \cup T^{-2}\Delta \cup \cdots \cup T^{-2n+2}\Delta \cup T^{-2n}\Delta^c. \end{aligned}$$

Therefore

$$\begin{aligned} A_{n+1} &= A_1^c \cap \cdots \cap A_n^c \cap T^{-2}B_n \\ &= \bigcap_{i=0}^{n} (\Delta^c \cup T^{-2}\Delta^c \cup \cdots \cup T^{-2i}\Delta) \cap (T^{-2}\Delta \cup T^{-4}\Delta \cup \cdots \cup T^{-2n-2}\Delta^c) \\ &= \Delta \cap T^{-2}\Delta \cap \cdots \cap T^{-2n}\Delta \cap T^{-2(n+1)}\Delta^c. \end{aligned}$$

□

It follows from this lemma that $T^2(A_n) = A_{n-1}$.

Compare lemma 2.7 to the construction in [4], which shows that for an ergodic transformation and a sequence of sets of the form

$$\Delta^n = \Delta \cap T^{-1}\Delta \cap \cdots \cap T^{-n}\Delta$$

the limit distribution of return times is indeed Poisson for a scaling sequence given by $c_n = e^{nP_\Delta}$. P_Δ is the pressure function restricted to the region Δ. We will use a similar method to show that for the sets

$$A_n = \Delta \cap T^{-2}\Delta \cap \cdots \cap T^{-2n+2}\Delta \cap T^{-2n}\Delta^c$$

there is a Poisson limit law.

Let $\Sigma = \{0,1\}^{\mathbb{N}}$ be the space of one-sided sequences on the symbols 0, and 1, and let $\sigma : \Sigma \to \Sigma$ be the shift map. That is $\sigma(\omega_n)_{n\in\mathbb{N}} = (\omega_{n+1})_{n\in\mathbb{N}}$. There is an injective map $\pi : I \to \Sigma$ satisfying

$$\pi \circ T = \sigma \circ \pi.$$

Explicitly this maps a point $x \in I$ to a sequence $\omega = (\omega_n) \in \Sigma$ as follows. If $x \in \Delta$ then $\omega_0 = 1$, otherwise $\omega_0 = 0$. Then for each $n \in \mathbb{N}$ let

$$\omega_n := \begin{cases} 1 & \text{if } T^n x \in \Delta, \\ 0 & otherwise. \end{cases}$$

In the space Σ one can describe sets called cylinder sets. These are usually denoted with square brackets and are defined as follows:

$$[a_i, \ldots, a_{i+m}]_i := \{\omega \in \Sigma : \omega_i = a_i, \ldots, \omega_{i+m} = a_{i+m}\}.$$

These cylinder sets form a basis of open sets for the topology of Σ, and can also be shown to be closed in Σ. By using cylinder sets, the problem of the orbit of a point in I hitting A_n can be reformulated in terms of the shift space, since $x \in A_n$ is equivalent to

$$\omega \in [1]_0 \cap [1]_2 \cap [1]_4 \cap \cdots \cap [1]_{2(n-2)} \cap [0]_{2n}.$$

In order for

$$\omega \in [1]_0 \cap [1]_2 \cap [1]_4 \cap \cdots \cap [1]_{2(n-2)} \cap [0]_{2n}$$

there is only a dependency on the first n even entries of the sequence ω, and the odd entries are entirely independent of whether or not

$$\omega \in [1]_0 \cap [1]_2 \cap [1]_4 \cap \cdots \cap [1]_{2(n-2)} \cap [0]_{2n}.$$

This means that the problem under consideration comes down to studying the system given by the map T^2.

Lemma 2.8. *Let μ be the Lebesgue measure on I, and consider the transformation $R: I \to I$ given by $R(x) = T^2(x) = 4x \mod 1$. Then consider also the hitting times to A_n for a point $x \in I$, with respect to R, given by*

$$\phi_n^{(1)}(x) = \inf\{i > 0 : R^i x \in A_n\},$$
$$\phi_n^{(k)}(x) = \inf\{i > \phi_n^{(k-1)}(x) : R^i x \in A_n\}$$

and the hitting times for the point $Tx \in [0,1)$ given by

$$\psi_n^{(1)}(x) = \inf\{i \geq 0 : R^i(Tx) \in A_n\},$$
$$\psi_n^{(k)}(x) = \inf\{i > \psi_n^{(k-1)}(x) : R^i(Tx) \in A_n\}.$$

Assume there exists a sequence c_n such that the point processes

$$X_n(\phi)(x) = \sum_{k \in \mathbb{N}} \delta_{\phi_n^{(k)}(x) c_n}$$

and

$$X_n(\psi)(x) = \sum_{k \in \mathbb{N}} \delta_{\psi_n^{(k)}(x) c_n}$$

both converge in distribution to a Poisson point process of rate 1, as n tends to infinity. Then the full point process, given by

$$X_n(x) = \sum_{k \in \mathbb{N}} \delta_{\tau_n^{(k)}(x) c_n},$$

also converges in distribution to a Poisson point process of rate 1.

Proof. Since μ is the lebesgue measure on I, with transformation $T: x \mapsto 2x \mod 1$, μ is a bernoulli measure with respect to T and T^2. This can be seen by considering a partition of I, $\{[0, \frac{1}{2}), [\frac{1}{2}, 1)\}$ and $\{[0, \frac{1}{4}), [\frac{1}{4}, \frac{1}{2}), [\frac{1}{2}, \frac{3}{4}), [\frac{3}{4}, 1)\}$ respectively. It therefore follows that the two point processes are independent and the full point process can be written as the sum

$$\begin{aligned} X_n(x) &= \sum_{k \in \mathbb{N}} \delta_{\tau_n^{(k)}(x) c_n} \\ &= \sum_{k \in \mathbb{N}} \delta_{2\phi_n^{(k)}(x) c_n} + \sum_{k \in \mathbb{N}} \delta_{(2\psi_n^{(k)}(x)+1) c_n} \\ &= X_n(2\phi)(x) + X_n(2\psi + 1)(x) \end{aligned}$$

Now recall that a point process converges in distribution to a Poisson process if and only if the random variable $N_n(g)(x)$ converges in distribution for every continous $g:\mathbb{R}\to\mathbb{R}$ with compact support. So let g be such a function and consider this as a random variable.

$$\begin{aligned} N_n(g)(x) &= \int_0^\infty g(t)\,dX_n(x)(t) \\ &= \sum_{k\in\mathbb{N}} g(t)\chi_{[\tau_n^{(k)}(x)c_n]}(t) \\ &= \sum_{k\in\mathbb{N}} g(\tau_n^{(k)}c_n) \\ &= \sum_{k\in\mathbb{N}} g(2\phi_n^{(k)}c_n) + \sum_{k\in\mathbb{N}} g((2\psi_n^{(k)}+1)c_n) \end{aligned}$$

$X_n(\phi)$ and $X_n(\psi)$ both converge in distribution to a Poisson process of rate one, and $2\phi_n^{(k)}$ and $(2\psi_n^{(k)}+1)$ are independent since ϕ_n and ψ_n are dependent on alternate points in the orbit of $(T^ix)_{i\in\mathbb{N}}$, and the measure μ is Bernoulli. Therefore the two random variables in the last line both converge in distribution to Poisson processes of rate $\frac{1}{2}$. And so it follows that $N_n(g)(x)$ converges in distribution to a Poisson process of rate 1.

□

With these ideas now consider, for a general ergodic measure μ and transformation T, the sets Δ^n defined by

$$\Delta^n = \Delta\cap T^{-1}\Delta\cap\cdots\cap T^{-n+1}\Delta\cap T^{-n}\Delta^c.$$

As explained earlier in order to show a Poisson limit law it is sufficient to check that two properties are satisfied. Recall property 1:

Property 1. The following limit exists:

$$C_m = \lim_{n\to\infty} c_n \sum_{\substack{0=q_0<\cdots<q_{m-1} \\ q_s-q_{s-1}\leq\frac{n}{m}}} \mathbb{E}(\prod_{s=0}^{m-1}\chi_{\Delta^n}\circ T^{q_s})$$

where c_n is taken to be $\mu(\Delta^n)$.

To see that this property is satisfied first observe that for $m=1$

$$C_1 = \lim_{n\to\infty}\mu(\Delta^n)^{-1}\mathbb{E}(\chi_{\Delta^n}) = \lim_{n\to\infty}\frac{\mu(\Delta^n)}{\mu(\Delta^n)} = 1.$$

For $m > 1$, I argue that $C_m = 0$: The sum is over a sequence of points, (q_s), which satisfy $q_s - q_{s-1} \leq \frac{n}{m} < n$. Therefore if $T^{q_s}x \in \Delta^n$ then it cannot be the case that $T^{q_{s-1}}x \in \Delta^n$, since

$$\Delta^n = \Delta \cap T^{-1}\Delta \cap \cdots \cap T^{-n+1}\Delta \cap T^{-n}\Delta^c.$$

So it then follows that the product is given by

$$\prod_{s=0}^{m-1} \chi_{\Delta^n} \circ T^{q_s} = 0,$$

which has expectation equal to zero, and hence C_m is zero, whenever $m > 1$.

This proves property 1 is satisfied, so now consider property 2.

Property 2. There exist $K_m > 0$ and $0 < \gamma < 1$ such that for every $0 = j_0 < j_1 < \cdots < j_m$ satisfying $j_s - j_{s-1} \leq \frac{n}{m}$ we have for sufficiently large n,

$$\left| \mathbb{E}\left(\prod_{s=0}^{m} \chi_{\Delta^n} \circ T^{j_s} \cdot \chi_B \circ T^{r+j_m} \right) - \mathbb{E}\left(\prod_{s=0}^{m} \chi_{\Delta^n} \circ T^{j_s} \right) \mu(B) \right| \leq K_m \gamma^{r+j_m} \mu(B)$$

for every $r > 0$, and for every $B \in \mathscr{B}$.

This property is not as easy to check. To begin with we introduce and recall notions from symbolic dynamics and thermodynamic formalism.

Let $V = \{1, \ldots, \ell\}$ be a finite alphabet. A will denote an irreducible and aperiodic $\ell \times \ell$ transition matrix, with entries either a 0 or a 1, indicating allowable transitions between vertices of a directed graph. Define the space of one-sided allowable paths in the graph by

$$\Sigma_A = \{x = (x_n) \in V^{\mathbb{N}} : A(x_{i-1}, x_i) = 1 \, \forall i \geq 1\}.$$

Σ_A is compact and metrisable with Tychonov product topology. Let $T : \Sigma_A \to \Sigma_A$ be the shift map, given by $T(x)_n = x_{n+1}$.

For $\varphi \in C(\Sigma_A)$, let $\mathrm{var}_n(\varphi) = \sup\{|\varphi(x) - \varphi(y)| : x_i = y_i, i \leq n\}$, and given $0 < \theta < 1$, define

$$|\varphi|_\theta := \sup \left\{ \frac{\mathrm{var}_n(\varphi)}{\theta^n} \right\}.$$

Then the space

$$\mathscr{F}_\theta := \{\varphi \in C(\Sigma_A) : |\varphi|_\theta < \infty\}$$

is a Banach space with norm $\|\varphi\|_\theta := \|\varphi\|_\infty + |\varphi|_\theta$.

Given a potential $\varphi \in \mathscr{F}_\theta$, let $\mathscr{L}_\varphi$ be the transfer operator on $\mathscr{F}_\theta$ defined by

$$(\mathscr{L}_\varphi \psi)(x) = \sum_{Ty=x} e^{\varphi(y)} \psi(y).$$

This operator $\mathscr{L}_\varphi$ has a maximum positive eigenvalue $e^{P(\varphi)}$ which is simple and isolated and the rest of the eigenvalues lie in a disc around the origin of radius strictly less than $e^{P(\varphi)}$. The value $P(\varphi)$ is the pressure of φ which satisfies

$$\begin{aligned} P(\varphi) &= \sup\left\{ h(m) + \int \varphi \, dm \ : \ m \text{ is a } T\text{-invariant probability measure} \right\} \\ &= h(\mu) + \int \varphi \, d\mu, \end{aligned}$$

where $h(m)$ denotes the measure-theoretic entropy of the system (T, m), and μ is the equilibrium state corresponding to φ which maximises the supremum above.

The positive function given by

$$\omega := \lim_{n\to\infty} e^{-nP(\varphi)} \mathscr{L}_\varphi^n(\mathbf{1}),$$

where $\mathbf{1}$ denotes the constant function equal to 1, can be shown to be an eigenfunction of $\mathscr{L}_\varphi$ corresponding to the eigenvalue $e^{P(\varphi)}$. Normalising φ by replacing it with $\varphi' = \varphi - P(\varphi) + \log(\omega) - \log(\omega \circ T)$ gives $\mathscr{L}_{\varphi'}(\mathbf{1}) = \mathbf{1}$ and $P(\varphi') = 0$. Furthermore φ and φ' have the same equilibrium state. So assume without loss of generality that φ is normalised. Thus $\mathscr{L}_\varphi$ satisfies

$$\int \varphi_1 \cdot (\varphi_2 \circ T) \, d\mu = \int \mathscr{L}_\varphi(\varphi_1) \cdot \varphi_2 \, d\mu$$

for any $\varphi, \varphi_1, \varphi_2 \in C(\Sigma_A)$.

Now consider a sub-alphabet $\Delta \subset V$ such that $\Delta \neq V$, and consider the associated closed T-invariant subset $\Sigma_\Delta \subseteq \Sigma_A$ defined by

$$\Sigma_\Delta = \{x \in \Sigma_A : x_i \in \Delta \ \forall i \geq 0\}.$$

I will only consider the case where Σ_Δ is an irreducible and aperiodic subshift of finite type. This is to say that the matrix A restricted to Δ defines a matrix A_Δ which is irreducible and aperiodic.

Let φ_Δ denote the restriction of φ to the subsystem Σ_Δ, and P_Δ be the pressure of φ_Δ with respect to (Σ_Δ, T). If φ is normalised, then $P_\varphi = 0$ and so $P_\Delta < 0$. Let μ_Δ denote the equilibrium state of φ_Δ. Let ω_Δ be the strictly positive, Hölder continuous function on Σ_Δ given by

$$\omega_\Delta = \lim_{n\to\infty} e^{-nP_\Delta} \mathscr{L}^n_{\varphi_\Delta}(\mathbf{1}).$$

Now define the restricted transfer operator $\mathscr{L}_\Delta$, acting on the space of Hölder continuous functions $\mathscr{F}_\theta$, by

$$\mathscr{L}_\Delta \varphi = \mathscr{L}(\varphi \cdot \chi_\Delta)$$

and consider the subset of Σ_A

$$\mathscr{Y}_\Delta = \{x \in \Sigma_A : \exists b \in \Delta^c,\, A(b, x_0) = 1\}.$$

Since A is irreducible and aperiodic in V, $\mathscr{Y}_\Delta$ is a non-empty finite union of cylinder sets of Σ_A, and in particular $\mu(\mathscr{Y}_\Delta) > 0$.

To continue we recall the following result from [4].

Proposition 2.9. *There exists a Hölder continuous function h_Δ defined on Σ_Δ such that*

$$\mathscr{L}_\Delta(h_\Delta) = e^{P_\Delta} h_\Delta$$

and $h_\Delta|_{\Sigma_\Delta} \equiv \omega_\Delta$. h_Δ is strictly positive on $\mathscr{Y}_\Delta$ and is zero on the complement $\mathscr{Y}^c_\Delta$. Moreover

$$\|e^{-nP_\Delta} \mathscr{L}^n_\Delta(\varphi) - h_\Delta \int_{\Sigma_\Delta} \varphi \, d\mu\|_\infty \underset{n\to\infty}{\longrightarrow} 0$$

for all $\varphi \in C(\Sigma_\Delta)$.

I am now ready that to show that property 2 is satisfied. I will do this via the following two lemmas.

Lemma 2.10. *Using the notation used above*

$$\lim_{n\to\infty} e^{-nP_\Delta} \mu(\Delta^n) = \int_{\Delta^c} h_\Delta \, d\mu.$$

Proof. Recall that

$$\Delta^n = \Delta \cap T^{-1}\Delta \cap \cdots \cap T^{-n+1}\Delta \cap T^{-n}\Delta^c.$$

Then it follows that

$$\begin{aligned}
\mu(\Delta^n) &= \int \chi_{\Delta^n}\, d\mu \\
&= \int \chi_\Delta \cdot (\chi_{\Delta^{n-1}} \circ T)\, d\mu \\
&= \int \mathscr{L}_\Delta(\mathbf{1}) \cdot \chi_{\Delta^{n-1}}\, d\mu \\
&= \int \mathscr{L}_\Delta^n(\mathbf{1}) \cdot \chi_{\Delta^0}\, d\mu
\end{aligned}$$

where $\Delta^0 := \Delta^c$. Thus by Proposition 2.9, we have that

$$e^{-nP_\Delta}\mu(\Delta^n) = \int h_\Delta \cdot \chi_{\Delta^c}\, d\mu + e^{-nP_\Delta} o(e^{nP_\Delta})$$

and therefore

$$\lim_{n\to\infty} e^{-nP_\Delta}\mu(\Delta^n) = \int_{\Delta^c} h_\Delta\, d\mu.$$

□

Lemma 2.11. *There exists $K > 0$ and $0 < \gamma < 1$ such that*

$$|\mathbb{E}(\chi_{\Delta^s} \cdot \chi_B \circ T^{s+r}) - \mu(\Delta^s)\mu(B)| \le K\gamma^r e^{sP_\Delta} \mu(B)$$

for every $s, r > 0$ and for every Borel set $B \subseteq \Sigma_\Delta$.

Proof. First note that from the properties of the transfer operator

$$\begin{aligned}
\mathbb{E}(\chi_{\Delta^s} \cdot \chi_B \circ T^{s+r}) &= \int \chi_{\Delta^s} \cdot \chi_B(T^{s+r})\, d\mu \\
&= \int \chi_B \cdot \mathscr{L}_\varphi^r(\mathscr{L}_\Delta^s(\mathbf{1}))\, d\mu \\
&= \mathbb{E}(\chi_B \cdot \mathscr{L}_\varphi^r(\mathscr{L}_\Delta^s(\mathbf{1})).
\end{aligned}$$

The spectral properties of $\mathscr{L}_\varphi$ imply the existence of $0 < \gamma < 1$ and $K > 0$ such that for all $k > 0$

$$\|\mathscr{L}_\varphi^k \omega\|_\infty \le K\gamma^k \|\omega\|_\theta,$$

whenever $\omega \in \mathscr{F}$ and $\int \omega\, d\mu = 0$. Since $e^{-sP_\Delta}\mathscr{L}_\Delta^s\mathbf{1}$ has uniformly bounded Hölder norm, taking

$$\omega = \omega_s = e^{-sP_\Delta}(\mathscr{L}_\Delta^s\mathbf{1} - \mu(\Delta^s)),$$

then

$$\int \omega\, d\mu = e^{-sP_\Delta} \int \mathscr{L}_\Delta^s(\mathbf{1}) - \mu(\Delta^s)\, d\mu = e^{-sP_\Delta}(\mu(\Delta^s) - \mu(\Delta^s)) = 0,$$

and there exists $K' > 0$ independent of r, s such that

$$|\mathbb{E}(\chi_B \cdot \mathscr{L}_\varphi^r \omega_s)| \le K' \gamma^r \mu(B)$$

for all $r, s > 0$. The result now follows. □

With lemmas 2.10 and 2.11 property 2 is now satisfied. The following theorem now follows immediately by theorem 2.4.

Theorem 2.12. *The hitting time and return time process of Δ^n, rescaled by e^{nP_Δ}, converges in law to a Poisson point process with rate* 1.

Theorem 2.6 now follows as an immediate corollary.

In the example of Floriani and Lima, I showed that the structure of the dynamics was such that it could be separated into two (sub-)sequences by considering alternate points. This means that in order to study the distribution of hitting and return times in this system it is enough to consider, without loss of generality, the map $T : x \mapsto 4x \mod 1$. This can be described using the double iteration of the shift map on the space of sequences of 0s and 1s with the map $\sigma_2 = \sigma^2 : \Sigma \to \Sigma$ where $\Sigma = \{x = (x_i) : (x_{2i}, x_{2i+1}) \in \{00, 01, 10, 11\} \forall i \in \mathbb{N}\}$ and associating cylinders $[00], [01], [10], [11]$ with the intervals $[0, \frac{1}{4}), [\frac{1}{4}, \frac{1}{2}), [\frac{1}{2}, \frac{3}{4}), [\frac{3}{4}, 1)$ respectively. This has a normalised potential of $-\log 4$ (noting that this is different from the normalised potential of the doubling map which is $-\log 2$), and this can be shown by calculating

$$\begin{aligned} P(-\log 4) &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{\sigma_2^n x = x} \exp(-n \log 4) \\ &= \lim_{n\to\infty} \frac{1}{n} \log(2^{2n} \cdot 2^{-2n}) \\ &= 0. \end{aligned}$$

The sub-alphabet for Δ will be the set of pairs $\{01, 10\}$. The restriction to Δ corresponds to the set of sequences $\Sigma_\Delta = \{x = (x_i)_\mathbb{N} : x_{2i} \neq x_{2i+1}\}$, and is therefore aperiodic and irreducible. Now calculating the pressure of the normalised potential restricted to Δ gives

$$\begin{aligned} P_\Delta(-\log 4) &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{\sigma_2^n x = x, x\in\Sigma_\Delta} \exp(-n\log 4) \\ &= \lim_{n\to\infty} \frac{1}{n} \log(2^n \cdot 2^{-2n}) \\ &= -\log 2. \end{aligned}$$

Therefore the rescaling constants, as given in Theorem 2.6, can be calculated as

$$e^{nP_\Delta} = \exp(-n\log 2) = 2^{-n}.$$

Chapter 3

Continuous Dynamics

In this chapter we will relate the results from discrete dynamics and probability theory to a continuous setting, in order to obtain sufficient conditions under which a Poisson limit law exists.

3.1 Manifolds and Axiom A Flows

As stated in the previous chapter, axiom A was first introduced by Smale [18]. We previously described axiom A diffeomorphisms; we now need an equivalent formulation for axiom A flows, and so we develop similar notions of hyperbolic sets for flows.

Let M be a compact Riemannian Manifold, that is a smooth manifold with a Riemannian metric, and let $\phi_t : M \to M$ be a differentiable flow on M, that is a one parameter family of diffeomorphisms such that $\phi_{t+s} = \phi_t \phi_s$.

Definition 3.1 (Hyperbolic Sets)**.** A closed (ϕ_t)-invariant set $\Lambda \subset M$ containing no fixed points is called *hyperbolic* if for every $x \in \Lambda$ the tangent space can be written $T_x M = E \oplus E^s \oplus E^s$ where E, E^s, E^u are $(D\phi_t)$-invariant and continuous subspaces and there are constants $c > 0$ and $\lambda \in (0,1)$ such that

1. E is the one dimensional subbundle tangent to the flow ϕ_t.
2. $\|D\phi_t(z)\| \leq ce^{-\lambda t}\|z\|$ for every $z \in E^s$, $t \geq 0$.
3. $\|D\phi_{-t}(z)\| \leq ce^{-\lambda t}\|z\|$ for every $z \in E^u$, $t \geq 0$.

It is possible to choose appropriate $t_0 > 0$ and $\lambda > 0$ for a given hyperbolic set $\Lambda \subset M$ such that the above conditions still hold with $c = 1$ when $t \geq t_0$. Assume that $t_0 \leq 1$ since this can be achieved by rescaling $t \to t' = \frac{t}{t_0}$.

Definition 3.2 (Basic Hyperbolic Sets)**.** A closed invariant set Λ is called *basic hyperbolic* if the following are satisfied:

1. Λ contains no fixed points and is hyperbolic.
2. The periodic orbits of $\phi_t|_\Lambda$ are dense in Λ.
3. $\phi_t|_\Lambda$ is a topologically transitive flow.
4. There is an open set $U \supset \Lambda$ with

$$\Lambda = \bigcap_{t \in \mathbb{R}} \phi_t U.$$

Definition 3.3 (Non-Wandering Sets)**.** The *non-wandering set* $\Omega = \Omega_\phi \subset M$ is given by

$$\Omega_\phi = \{z \in M : \text{ for every open } V \ni x,\ t_0 > 0\ \exists t > t_0 \text{ with } \phi_t(V) \cap V \neq \emptyset\}.$$

Definition 3.4. A flow, $\phi_t : M \to M$, is said to satisfy *Axiom A* if its non-wandering set Ω_ϕ is the disjoint union of a hyperbolic set with a finite number of hyperblic fixed points.

Smale's spectral decomposition theorem states that this hyperbolic set is a disjoint union of a finite number of basic sets.

In the remainder of this thesis we will only consider continuous dynamical systems $(\Omega_\phi, \phi_t, m, \mathscr{B})$ with a probability measure m on Ω_ϕ and flows $\phi_t : \Omega_\phi \to \Omega_\phi$ satisfying axiom A.

3.2 Suspension Flows and Axiom A Flows

Some of the ergodic theory and dynamics of axiom A flows have been studied by Bowen and Ruelle (see [1, 2]), who made the link to suspended flows. We will first define and construct a general suspended flow.

Given a measure preserving discrete dynamical system $(X, T, \mathscr{B}, \mu)$ and a measurable function $\gamma : X \to (0, \infty)$ with $\int \gamma d\mu < \infty$, define a new space, the suspended space

$$\Omega = \Omega_\gamma := \{\omega = (x, y) \in X \times \mathbb{R} : 0 \le y \le \gamma(x)\},$$

identifying points $(x, \gamma(x))$ with $(Tx, 0)$. Associate with this the product σ-algebra, and define a probability measure on Ω as the normalised product measure of μ and the Lebesgue measure

$$d\nu_\gamma = d\nu := \frac{d\mu \times d\,\mathrm{Leb}}{\int \gamma d\mu}.$$

Define a flow $S_t = S_{t,\gamma} : \Omega_\gamma \to \Omega_\gamma$ on the new measure space (Ω_γ, ν) by

$$S_t(\omega) = S_t(x, y) = \left(T^{\eta(t)} x,\, y + t - \sum_{i=0}^{\eta(t)-1} \gamma(T^i x) \right),$$

where $\eta(t)$ is the unique natural number which satisfies

$$0 \le y + t - \sum_{i=0}^{\eta(t)-1} \gamma(T^i x) < \gamma(T^{\eta(t)} x).$$

See Figure 3.1

Definition 3.5 (Suspension Flow)**.** (Ω_γ, ν) defined as above is a *suspended space* with respect to γ and S_t is a *suspension flow*.

We will only consider suspended flows over $X = \Sigma = \Sigma_A$, that is the space of two-sided sequences, and $T = \sigma = \sigma_A$ is the shift map, so that (Σ_A, σ_A) is a subshift of finite type.

The link between suspension flows and axiom A flows has been investigated by Bowen and Ruelle (see [1, 2]) and this relationship is described in the the theorem below.

Theorem 3.1. *For* $\gamma : \Sigma \to (0, \infty)$ *let*

$$var_n(\gamma) := \sup\{|\gamma(x) - \gamma(y)| : x, y \in \Sigma, x_i = y_i \;\; \forall |i| \le n\}$$

and

$$\mathscr{F} := \{\gamma \in C(\Sigma) : \exists b > 0,\; \alpha \in (0,1) \text{ so that } var_n\gamma \le b\alpha^n \;\; \forall n \ge 0\},$$

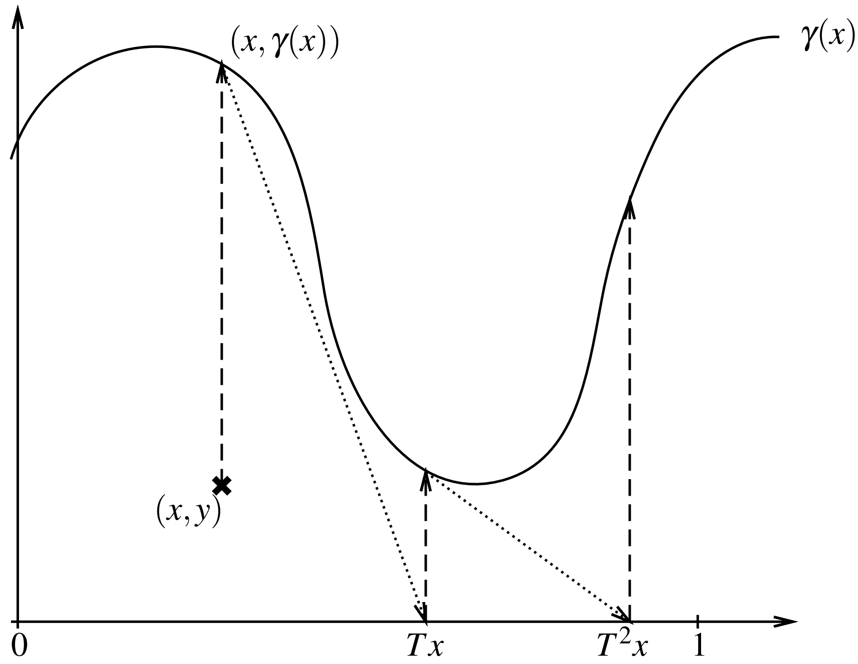

Figure 3.1: Showing the motion of a suspension flow for a height function γ. The flow travels vertically upwards until $y=\gamma(x)$, at which point it jumps back to $y=0$, and shifts along to Tx.

where $C(\Sigma)$ is the set of continuous functions on Σ, taking values in $\mathbb{R}$. Let ϕ_t be an axiom A flow on (M,ν) and Λ a basic hyperbolic set for ϕ_t. Then there is a subshift of finite type (Σ_A,σ_A) and a positive $\gamma\in\mathscr{F}$ and a continuous surjection $\rho:\Omega_\gamma\to\Lambda$ so that $\rho(S_t(\omega))=\phi_t(\rho(\omega))$.

A proof of this theorem can be found in [2]. This theorem along with Smale's decomposition theorem [18] means that for any axiom A flow there is a suspension flow which shares many of its properties. In particular note that if $\tau(z)$ is a hitting time of $z\in\Lambda$ to some open set $A\subset\Lambda$ with positive measure then there exists a (small) $\varepsilon>0$ such that $\phi_t(z)\in A$ for $t\in(\tau,\tau+\varepsilon)$ and $\phi_t(z)\notin A$ for $t\in(\tau-\varepsilon,\tau)$. But

$$\rho^{-1}(\phi_t(z))=S_t(\rho^{-1}z),$$

and $\rho^{-1}(\phi_t(z))\in\rho^{-1}(A)$ if and only if $\phi_t(z)\in A$. Therefore $\tau(z)\equiv\tau(\omega)$ when $z=\rho(\omega)$. So in order to study the hitting and return times of ϕ_t to a set A it is enough to look at the hitting and return times of the corresponding suspension flow, S_t, to the set $\rho^{-1}A$.

3.3 Hitting times in Suspended Flows

Recall that a suspended flow over a measure preserving discrete dynamical system $(X,T,\mathscr{B},\mu)$ with a measurable height function $\gamma : X \to (0,\infty)$ such that $\int \gamma d\mu < \infty$ is a flow on the suspended space

$$\Omega = \Omega_\gamma := \{\omega = (x,y) \in X \times \mathbb{R} : \ 0 \le y \le \gamma(x)\},$$

where the points $(x,\gamma(x))$ are identified with $(Tx,0)$. Associate with this the product σ-algebra, and define a probability measure on Ω by

$$d\nu_\gamma = d\nu = \frac{d\mu \times d\,\mathrm{Leb}}{\int \gamma d\mu}.$$

The suspension flow $(S_t : \Omega_\gamma \to \Omega_\gamma : \ t \in \mathbb{R})$ is given by

$$S_t(\omega) = S_t(x,y) = \left(T^{\eta(t)}x,\, y+t-\sum_{i=0}^{\eta(t)-1} \gamma(T^i x) \right)$$

where $\eta(t)$ is the unique natural number which satisfies

$$0 \le y+t-\sum_{i=0}^{\eta(t)-1} \gamma(T^i x) < \gamma(T^{\eta(t)}x).$$

Consider the projections $\pi : \Omega \to X$ such that $\pi(x,y) = x$ and $\pi' : \Omega \to \mathbb{R}$ the projection to the vertical axis such that $\pi'(x,y) = y$.

Let $(\Delta_n \subset \Omega)$ be a sequence of open balls with positive measure such that $\lim_{n\to\infty} \nu(\Delta_n) = 0$. Observe that $\mu(A_n) > 0$ since otherwise

$$\nu(\Delta_n) = \frac{\mu \times \mathrm{Leb}}{\int \gamma d\mu}(\Delta_n) \le \frac{\mu(\pi(\Delta_n) \times \mathrm{Leb}(\pi'(\Delta_n))}{\int \gamma d\mu} = 0.$$

Now consider the the hitting times, $\tau_n^{(k)}$, to Δ_n. Notice that from the construction of suspended flows these hitting times can be related to the hitting times, $r_n^{(k)}$, of the system $(X,T,\mathscr{B},\mu)$ at the base of the flow, to the projected sets $A_n = \pi(\Delta_n)$. This is formalised in the following lemma.

Lemma 3.2. *Let $(X, T, \mathscr{B}, \mu)$ be a T-invariant system with probability measure μ, and let $(\Omega_\gamma, S_t, \mathscr{B}, \nu)$ be the associated suspension flow with height $\gamma : X \to (0, \infty)$ uniformly bounded away from 0. Then consider a sequence of open balls $\Delta_n \subset \Omega_\gamma$ with positive measure such that $\lim_{n\to\infty} \nu(\Delta_n) = 0$. Let $\tau_n^{(k)}$ be the k-th hitting time for the suspension flow to hit Δ_n and let $r_n^{(k)}$ be the k-th hitting time for the transformation $T : X \to X$ to hit the projected sets $A_n = \pi(\Delta_n)$. Then*

$$\tau_n^{(k)} = r_n^{(k)} Y_n^{(k)}$$

where

$$Y_n^{(k)}(\omega) = Y_n^{(k)}(x,y) = \frac{1}{r_n^{(k)}} \left(\sum_{i=0}^{r_n^{(k)}-1} \gamma(T^i x) - y + h(T^{r_n^{(k)}} x) \right)$$

and $h : X \to \mathbb{R}$ is a positive measurable function with $0 \le h(x) \le \gamma(x)$.

Remark 3.3. The sequence of balls $\Delta_n \subset \Omega_\gamma$ are such that the measure converges to zero. In this setting of a suspended space there are two possible conditions that would give this convergence, convergence in measure of the projected sets to X given by $A_n = \pi(\Delta_n)$, or convergence in measure of the projected sets to $\mathbb{R}$ given by π'. It will be shown later that convergence in measure of the projection to X is associated to the $r_n^{(k)}$ term, and therefore it is sufficient to only consider convergence of $\mu(A_n)$.

Proof. $\tau_n^{(k)}(\omega)$ is the time it takes for the flow S_t to hit some particular set Δ_n of positive measure, starting at the point $\omega = (x,y)$. At the time $t = \tau_n^{(k)}(\omega)$

$$S_{\tau_n^{(k)}}(x,y) = \left(T^{\eta(\tau_n^{(k)})} x \, , \, y + \tau_n^{(k)} - \sum_{i=0}^{\eta(\tau_n^{(k)})-1} \gamma(T^i x) \right),$$

where $\eta(\tau_n^{(k)})$ is the unique natural number which satisfies

$$0 \le \; y + \tau_n^{(k)} - \sum_{i=0}^{\eta(\tau_n^{(k)})-1} \gamma(T^i x) \; < \; \gamma(T^{\eta(\tau_n^{(k)})} x).$$

We will show that $\eta(\tau_n^{(k)}) = r_n^{(k)}$

Each term in the sum coincides with the flow reaching the top of the suspension once, which is to say that the x-coordinate is translated to Tx for each term in the sum. In order for the flow to hit Δ_n for the k-th time, it will therefore have to continue until the x-coordinate hits the projected set $A_n = \pi(\Delta_n) \subset X$ for the

k-th time. After the first translation (that is the time to reach the point (Tx, 0)) S has travelled a distance of $\gamma(x) - y$, the height of γ over the first point minus the starting position. At subsequent translations the distance adds on the height of $\gamma(x)$ for each x coordinate visited so that after $r_n^{(k)}$ translations S has flowed a distance of

$$\gamma(x) - y + \gamma(Tx) + \cdots + \gamma(T^{r_n^{(k)}-1}x).$$

This must be at most the distance required to hit Δ_n for the k-th time. Since S_t travels at unit speed,

$$\tau_n^{(k)} \geq \left(\sum_{i=0}^{r_n^{(k)}-1} \gamma(T^i x) \right) - y.$$

Rearranging gives the left hand inequality

$$0 \leq y + \tau_n^{(k)} - \sum_{i=0}^{r_n^{(k)}-1} \gamma(T^i x).$$

To get the right hand inequality observe that S_t must hit Δ_n before the next translate. Therefore

$$\tau_n^{(k)} < \gamma(x) - y + \gamma(Tx) + \cdots + \gamma(T^{r_n^{(k)}} x).$$

Rearranging this inequality gives

$$y + \tau_n^{(k)} - \sum_{i=0}^{r_n^{(k)}-1} \gamma(T^i x) \; < \; \gamma(T^{r_n^{(k)}} x).$$

It follows that there is some function $h : X \to \mathbb{R}$ which satisfies

$$\tau_n^{(k)} = \sum_{i=0}^{r_n^{(k)}-1} \gamma(T^i x) - y + h(T^{r_n^{(k)}} x)$$

where $0 \leq h(x) < \gamma(x)$ for every $x \in X$. Define $h(x)$ to be the height required for S to hit Δ_n from $(x, 0)$ if $x \in A_n$ or 0 otherwise, that is to say

$$h(x) = \begin{cases} \tau_n^{(1)}(x,0) & x \in A_n \\ 0 & \text{otherwise.} \end{cases}$$

The measurability of h follows from the measurability of τ. □

We have now separated the hitting times into two random variables. The reason for this, as shown in the following lemma, is because the $Y_n^{(k)}$ terms converge almost surely to a constant.

Lemma 3.4. *$Y_n^{(k)}$ converges μ-almost surely to $\int \gamma d\mu$ as $n \to \infty$.*

Proof. This can be shown by rearranging and considering three terms separately, by writing

$$Y_n^{(k)} = \frac{1}{r_n^{(k)}} \sum_{i=0}^{r_n^{(k)}-1} \gamma(T^i x) - \frac{y}{r_n^{(k)}} + \frac{h(T^{r_n^{(k)}} x)}{r_n^{(k)}}$$

where $h(x)$ is the height required for S to hit Δ_n from $(x,0)$ if $x \in A_n$ or 0 otherwise, that is to say

$$h(x) = \begin{cases} \tau_n^{(1)}(x,0) & x \in A_n \\ 0 & \text{otherwise.} \end{cases}$$

For the second term, we recognise that $r_n^{(k)} \to \infty$ almost surely as $n \to \infty$ since the measure of the target set converges to zero and so the hitting times diverge according to Kac's theorem. As y is a constant here it follows that

$$\frac{y}{r_n^{(k)}} \to 0$$

μ-almost surely as $n \to \infty$.

For the third term it's already known that $0 \leq h(T^{r_n^{(k)}} x) < \gamma(T^{r_n^{(k)}} x)$. Since $\gamma \in L^1$ it follows that by the Birkhoff Ergodic theorem

$$\frac{1}{r_n^{(k)}} \sum_{k=0}^{r_n^{(k)}} \gamma(T^i x) \to \int \gamma d\mu.$$

So consider

$$\begin{aligned} \frac{1}{r_n^{(k)}} h(T^{r_n^{(k)}} x) &< \frac{1}{r_n^{(k)}} \gamma(T^{r_n^{(k)}} x) \\ &= \frac{1}{r_n^{(k)}} \sum_{k=0}^{r_n^{(k)}} \gamma(T^i x) - \frac{1}{(r_n^{(k)} - 1)} \sum_{k=0}^{r_n^{(k)}-1} \gamma(T^i x) \end{aligned}$$

Taking a limit as $n \to \infty$, and noting that $r_n^{(k)}$ is a subsequence of $(n)_{n \in \mathbb{N}}$, then both terms in the final expression converge to the integral of γ by the Birkhoff ergodic theorem. In particular, on a set of full measure,

$$0 \leq \lim_{n\to\infty} \frac{1}{r_n^{(k)}} h(T^{r_n^{(k)}} x) \leq \int \gamma d\mu - \int \gamma d\mu = 0,$$

and so $\frac{1}{r_n^{(k)}} h(T^{r_n^{(k)}} x) \to 0$ almost surely.

For the first term we use a generalised version of the Birkhoff ergodic theorem and notice that $r_n^{(k)}$ diverges almost surely so the averages must converge almost surely to the integral due to the ergodic properties of μ. That is

$$\frac{1}{r_n^{(k)}} \left(\sum_{i=0}^{r_n^{(k)}-1} \gamma(T^i x) - y \right) \to \int \gamma d\mu$$

almost surely as $n \to \infty$.

The result now follows. □

Remark 3.5. Here γ does not need to be continuous for the previous two lemmas to hold, but needs only be integrable in order to use the Birkhoff ergodic theorem in the proof.

Slutsky's Theorem and the Continuous Mapping Theorem

We now consider a theorem of Slutsky [9] and give a proof, along with some general and relevant remarks. We also consider related ideas including the continuous mapping theorem and how these ideas might be applied to the hitting time problem being considered in this thesis.

Theorem 3.6 (Continuous Mapping Theorem)**.** *Let Z_n be a d-dimensional random vector which converges in distribution to the random vector Z. If $g : \mathbb{R}^d \to \mathbb{R}$ is a continuous map with compact support then $g(Z_n) \to g(Z)$ in distribution.*

A proof of the continuous mapping theorem can be found in [13]. I now give a proof of Slutsky's theorem, in its usual form, using the continuous mapping theorem.

Theorem 3.7 (Slutsky's Theorem)**.** *Let (X_n) be a sequence of random variables that converge in distribution to X, and (Y_n) another sequence of random variables that converge in probability to a constant $c \in \mathbb{R}$. Then the product X_nY_n converges in distribution to the product Xc.*

Remark 3.8.

1. Convergence in probability of Y_n to c is defined as the convergence

$$\lim_{n\to\infty} \mathbb{P}(|Y_n - c| < \varepsilon) = 0$$

and is implied by convergence almost surely, so convergence of the variable Y_n almost surely to a constant is enough for the result to hold.

2. It is important that Y_n converges to a constant, and it is not generally the case that this theorem holds when Y_n converges to an arbitrary random variable. As a counterexample consider $\Omega = \{0,1\}^{\mathbb{N}}$. This is the one-sided full shift on two symbols. Consider the probability measure $\mu = (\frac{1}{2},\frac{1}{2})^{\mathbb{N}}$. Now for $\omega = (\omega_1, \omega_2, \dots)$ define a sequence of random variables by $X_n : \Omega \to \mathbb{R}$ by

$$X_n(\omega) = \begin{cases} 0 & \text{if } \omega_1 = 0, \quad n = 2k \\ 1 & \text{if } \omega_1 = 1, \quad n = 2k \\ 1 & \text{if } \omega_1 = 0, \quad n = 2k+1 \\ 0 & \text{if } \omega_1 = 1, \quad n = 2k+1 \end{cases}$$

Then both X_{2k} and X_{2k-1} converge in distribution to $X = X_0$, since they all have the same distribution function. However note that $X_{2k} \times X_{2k-1}$ is identically equal to the constant random variable 0, and this does not converge to $X_0 \times X_0 = X_0$ as $k \to \infty$.

Whilst proofs of this theorem can be found elsewhere (e.g. see [9]) we will consider a new proof as we will use it to generalise Slutsky's theorem later in the thesis.

Proof. We will prove this theorem using the continuous mapping theorem (theorem 3.6). The function that will be needed is $g(x,y) = xy$ and the random vector is (X_n, Y_n). The remainder of this proof is now devoted to showing that (X_n, Y_n) converges in distribution to (X, c), since from this the continuous mapping theorem can be used to complete the proof.

For any bounded and continuous function with compact support $h : \mathbb{R}^2 \to \mathbb{R}$ consider the difference

$$\begin{aligned} |\mathbb{E}[h(X_n,Y_n)] - &\mathbb{E}[h(X,c)]| \\ &= |\mathbb{E}[h(X_n,Y_n)] - \mathbb{E}[h(X_n,c)] + \mathbb{E}[h(X_n,c)] - \mathbb{E}[h(X,c)]| \\ &\leq |\mathbb{E}[h(X_n,Y_n)] - \mathbb{E}[h(X_n,c)]| + |\mathbb{E}[h(X_n,c)] - \mathbb{E}[h(X,c)]| \end{aligned}$$

Define $g(x) := h(x,c)$, which is continuous and has compact support from the compact support of h. Then the second term converges to zero as n tends to infinity since X_n converges to X in distribution.

The first term is harder and uses the fact that Y_n converge in probability. That is to say that for any $\varepsilon > 0$, $\mathbb{P}(|Y_n - c| > \varepsilon) \to 0$ as $n \to \infty$. Using this notation, for any $\varepsilon > 0$ it follows that

$$\begin{aligned}
&|\mathbb{E}[h(X_n,Y_n)] - \mathbb{E}[h(X_n,c)]| \\
&\qquad = \left| \int h(X_n,Y_n) - h(X_n,c)\, d\mathbb{P} \right| \\
&\qquad = \left| \int_{|Y_n - c| > \varepsilon} h(X_n,Y_n) - h(X_n,c)\, d\mathbb{P} \right. \\
&\qquad\qquad\qquad \left. + \int_{|Y_n - c| \le \varepsilon} h(X_n,Y_n) - h(X_n,c)\, d\mathbb{P} \right| \\
&\qquad \le \left| \int_{|Y_n - c| > \varepsilon} h(X_n,Y_n) - h(X_n,c)\, d\mathbb{P} \right| \\
&\qquad\qquad\qquad + \left| \int_{|Y_n - c| \le \varepsilon} h(X_n,Y_n) - h(X_n,c)\, d\mathbb{P} \right| \\
&\qquad \le \sup_{\{x,y : |y-c| \le \varepsilon\}} \{|h(x,y) - h(x,c)|\}\, \mathbb{P}(|Y_n - c| \le \varepsilon) \\
&\qquad\qquad\qquad + 2\|h\|_\infty \mathbb{P}(|Y_n - c| > \varepsilon).
\end{aligned}$$

Observe that h is continuous with compact support. In particular h is uniformly continuous, which is to say that for any $\eta > 0$ there is a $\delta = \delta_\eta > 0$ such that $d_1((x_1,y_1),(x_2,y_2)) < \delta \implies d_2(h(x_1,y_1),h(x_2,y_2)) < \eta$, where d_1 and d_2 are the metrics for $\mathbb{R}^2$ and $\mathbb{R}$ respectively. So there is a $\delta > 0$ such that $d_1((x,y),(x,c)) < \delta \implies d_2(h(x,y),h(x,c)) < \eta$. But the distance $d_1((x,y),(x,c)) = |y - c|$, so since in the above calculation ε was arbitrary, pick $\varepsilon < \delta_\eta$. Now for any $\eta > 0$ there exists ε such that

$$|\mathbb{E}[h(X_n,Y_n)] - \mathbb{E}[h(X_n,c)]| \le \eta + 2\|h\|_\infty \mathbb{P}(|Y_n - c| > \varepsilon).$$

Y_n converges in probability so in particular there exists some $N \in \mathbb{N}$ such that for $n > N$

$$|\mathbb{E}[h(X_n,Y_n)] - \mathbb{E}[h(X_n,c)]| \le \eta + \eta = 2\eta.$$

But η is arbitrary it follows that

$$\lim_{n\to\infty} \mathbb{E}[h(X_n,Y_n) - h(X_n,c)] = 0.$$

□

Remark 3.9. In the statement of the continuous mapping theorem the condition of convergence in distribution of $Z_n \to Z$ can be changed for convergence in probability (or convergence almost surely) to give $g(Z_n) \to g(Z)$ in probability (or almost surely respectively).

We are interested with the convergence in distribution of the processes given by

$$\tau_n(\omega) = \sum_{k \in \mathbb{N}} \delta_{\tau_n^{(k)}(\omega) c_n}$$

to the Poisson point process which will be denoted

$$\tau(\omega) = \sum_{k \in \mathbb{N}} \delta_{\tau^{(k)}(\omega)}.$$

It is enough to check that the integral

$$\int g \, d\tau_n$$

converges to

$$\int g \, d\tau$$

for any continuous $g : \mathbb{R} \to \mathbb{R}$ with compact support. We will now attempt to construct some theory generalising Slutsky's theorem and the continuous mapping theorem with this motivation, first by considering a point process with one point, and extending this to a more general point process.

A General Slutsky's Theorem for One Point

Consider again a sequence of random variables (X_n) which converge in distribution to X and another sequence of random variables (Y_n) which converge to a constant $c \in \mathbb{R}$ almost everywhere, which in particular means that Y_n converges to c in probabililty. Define a point process $\tau_n(\omega) = \delta_{X_n(\omega)Y_n(\omega)}$. This process converges in distribution to $\tau = \delta_{Xc}$ if for any continuous function with compact support, $g : \mathbb{R} \to \mathbb{R}$,

$$\int g \, d\tau_n = g(X_n Y_n) \to g(Xc) = \int g \, d\tau.$$

It is enough to check this is true for arbitrary indicator functions $g = \chi_{[a,b]}$ for $a < b$. Making use of Slutsky's Theorem above

$$\mathbb{P}(\chi_{[a,b]}(X_nY_n) \leq t) = \begin{cases} \mathbb{P}(X_nY_n \in [a,b]) & \text{if } t = 1 \\ 1 - \mathbb{P}(X_nY_n \in (a,b)) & \text{if } t = 0 \end{cases}$$

$$\to \begin{cases} \mathbb{P}(Xc \in [a,b]) & \text{if } t = 1 \\ 1 - \mathbb{P}(Xc \in [a,b]) & \text{if } t = 0 \end{cases}$$

$$= \mathbb{P}\left(\chi_{[a,b]}(X_nc) \leq t\right).$$

So τ_n converges in distribution to τ as n tends to infininty.

A General Slutsky's Theorem for Two Points

This time consider two sequences of random variables $X_n^{(1)}$ and $X_n^{(2)}$ which converge in distribution to $X^{(1)}$ and $X^{(2)}$ respectively and sequences $Y_n^{(1)}$ and $Y_n^{(2)}$ which converge almost everywhere to $Y^{(1)}$ and $Y^{(2)}$ respectively. Assume that $X^{(1)}$ and the difference $(X^{(2)} - X^{(1)})$ are independent and that $Y^{(1)} = Y^{(2)} = c$ is a constant. We are concerned with the convergence of the process given by $\tau_n(\omega) = \delta_{X_n^{(1)}Y_n^{(1)}} + \delta_{X_n^{(2)}Y_n^{(2)}}$.

To see that this converges in distribution it is enough to show that for any continuous function with compact support, $g : \mathbb{R} \to \mathbb{R}$,

$$\int g \, d\tau_n = g(X_n^{(1)}Y_n^{(1)}) + g(X_n^{(2)}Y_n^{(2)})$$

converges to

$$\int g \, d\tau.$$

The goal is to use the continuous mapping theorem to show this, using the function $G : \mathbb{R}^4 \to \mathbb{R}$ given by

$$G(u^{(1)}, u^{(2)}, v^{(1)}, v^{(2)}) = g(u^{(1)}v^{(1)}) + g((u^{(2)} - u^{(1)})v^{(2)})$$

and substituting $X_n^{(1)} = u^{(1)}$, $(X_n^{(2)} - X_n^{(1)}) = u^{(2)}$, $Y_n^{(1)} = v^{(1)}$, and $Y_n^{(2)} = v^{(2)}$. For ease of notation let $Z_n^{(1)} = X_n^{(1)}$, $Z^{(1)} = X^{(1)}$ and similarly $Z_n^{(2)} = (X_n^{(2)} - X_n^{(1)})$ and

$Z^{(2)} = (X^{(2)} - X^{(1)})$. We must show that the random vector $(Z_n^{(1)}, Z_n^{(2)}, Y_n^{(1)}, Y_n^{(2)})$ does indeed converge to $(Z^{(1)}, Z^{(2)}, c, c)$.

Assume that $(Z_n^{(1)}, Z_n^{(2)})$ converges in distribution to $(Z^{(1)}, Z^{(2)})$. Now we can adapt the proof of Slutsky's Theorem to show the convergence of $(Z_n^{(1)}, Z_n^{(2)}, Y_n^{(1)}, Y_n^{(2)})$.

Let $Z_n = (Z_n^{(1)}, Z_n^{(2)})$, $Y_n = Y_n^{(1)}$, and similarly $Z = (Z^{(1)}, Z^{(2)})$ and $Y = Y^{(1)} = c$. For any bounded and continuous function with compact support $h : \mathbb{R}^3 \to \mathbb{R}$ consider the difference

$$
\begin{aligned}
&|\mathbb{E}[h(Z_n, Y_n)] - \mathbb{E}[h(Z, c)]| \\
&\qquad = |\mathbb{E}[h(Z_n, Y_n)] - \mathbb{E}[h(Z_n, c)] + \mathbb{E}[h(Z_n, c)] - \mathbb{E}[h(Z, c)]| \\
&\qquad \leq |\mathbb{E}[h(Z_n, Y_n)] - \mathbb{E}[h(Z_n, c)]| + |\mathbb{E}[h(Z_n, c)] - \mathbb{E}[h(Z, c)]|
\end{aligned}
$$

Define $g(z) = h(z, c)$ which is continuous and has compact support from h. Therefore by the continuous mapping theorem the second term converges to zero as n tends to infinity since Z_n converges to Z in distribution.

The first term is less difficult and uses the fact that Y_n converges in probability, which is to say that for any $\varepsilon > 0$, $\mathbb{P}(|Y_n - c| > \varepsilon) \to 0$ as $n \to \infty$. Using this notation, for any $\varepsilon > 0$ it follows that

$$
\begin{aligned}
&|\mathbb{E}[h(Z_n, Y_n)] - \mathbb{E}[h(Z_n, c)]| \\
&\quad = \left| \int h(Z_n, Y_n) - h(Z_n, c)\, d\mathbb{P} \right| \\
&\quad = \left| \int_{|Y_n - c| > \varepsilon} h(Z_n, Y_n) - h(Z_n, c)\, d\mathbb{P} + \int_{|Y_n - c| \leq \varepsilon} h(Z_n, Y_n) - h(Z_n, c)\, d\mathbb{P} \right| \\
&\quad \leq \left| \int_{|Y_n - c| > \varepsilon} h(Z_n, Y_n) - h(Z_n, c)\, d\mathbb{P} \right| + \left| \int_{|Y_n - c| \leq \varepsilon} h(Z_n, Y_n) - h(Z_n, c)\, d\mathbb{P} \right| \\
&\quad \leq \sup_{\{z, y : |y - c| \leq \varepsilon\}} \{|h(z, y) - h(z, c)|\}\, \mathbb{P}(|Y_n - c| \leq \varepsilon) + 2\|h\|_\infty \mathbb{P}(|Y_n - c| > \varepsilon).
\end{aligned}
$$

Observe that h is continuous with compact support, in particular this means that h is uniformly continuous, which is to say that for any $\eta > 0$ there is a $\delta = \delta_\eta > 0$ such that $d_1((x_1, y_1), (x_2, y_2)) < \delta \implies d_2(h(x_1, y_1), h(x_2, y_2)) < \eta$, where d_1 and d_2 are the metrics for $\mathbb{R}^2$ and $\mathbb{R}$ respectively. So there is a $\delta > 0$ such that $d_1((x,y),(x,c)) < \delta \implies d_2(h(x,y),h(x,c)) < \eta$. But the distance $d_1((x,y),(x,c)) = |y-c|$, so since in the above calculation ε was arbitrary, pick $\varepsilon < \delta_\eta$. For any $\eta > 0$ there exists $\varepsilon > 0$ such that

$$|\mathbb{E}[h(X_n,Y_n)] - \mathbb{E}[h(X_n,c)]| \leq \eta + 2\|h\|_\infty \mathbb{P}(|Y_n - c| > \varepsilon).$$

Y_n converges in probability so there exists some $N \in \mathbb{N}$ such that for $n > N$

$$|\mathbb{E}[h(X_n,Y_n)] - \mathbb{E}[h(X_n,c)]| \leq \eta + \eta = 2\eta.$$

But η is arbitrary so it follows that

$$\lim_{n\to\infty} \mathbb{E}[h(X_n,Y_n) - h(X_n,c)] = 0.$$

It follows that (Z_n,Y_n) converges in distribution to (Z,Y).

Now let $Z_n = (Z_n^{(1)},Z_n^{(2)},Y_n^{(1)})$ and $Y_n = Y_n^{(2)}$ and repeat the calculation, to get that $(Z_n^{(1)},Z_n^{(2)},Y_n^{(1)},Y_n^{(2)})$ converges in distribution to $(Z^{(1)},Z^{(2)},c,c)$. Then by the continuous mapping theorem $G(Z_n^{(1)},Z_n^{(2)},Y_n^{(1)},Y_n^{(2)})$ converges in distribution to $G(Z^{(1)},Z^{(2)},c,c)$ which is to say the process τ_n converges in distribution to τ.

A General Slutsky's Theorem for Multiple Points

This time now consider m sequences of random variables $X_n^{(1)},X_n^{(2)},\ldots,X_n^{(m)}$ which converge in distribution to $X^{(1)},X^{(2)},\ldots,X^{(m)}$ respectively and sequences $Y_n^{(1)},Y_n^{(2)},\ldots,Y_n^{(m)}$ which converge almost surely to $Y^{(1)},Y^{(2)},\ldots,Y^{(m)}$ respectively. Assume that $X_n^{(1)}$ and the difference $(X_n^{(2)} - X_n^{(1)})$ are asymptotically independent and that for $i = 2,\ldots,m-1$ each pair consecutive intervals, $(X_n^{(i)} - X_n^{(i-1)})$ and $(X_n^{i+1} - X_n^{(i)})$, are pairwise asymptotically independent, and that $Y^{(1)} = Y^{(2)} = c \in \mathbb{R}$. Here two variables, Z_n^1, Z_n^2 are considered asymptotically independent if they converge in distribution to Z^1 and Z^2 respectively, and Z^1 and Z^2 are independent. We are concerned with the convergence of the process given by

$$\tau_n(\omega) = \sum_{i=1}^{m} \delta_{X_n^{(i)} Y_n^{(i)}}.$$

To see that this converges in distribution it is again enough to show that for any continuous function with compact support, $g : \mathbb{R} \to \mathbb{R}$,

$$\int g\, d\tau_n = \sum_{i=1}^{k} g(X_n^{(i)} Y_n^{(i)})$$

converges to

$$\int g\, d\tau.$$

This can be shown using the continuous mapping theorem for the continuous function $G : \mathbb{R}^{2m} \to \mathbb{R}$ given by

$$G(u^{(1)}, \dots, u^{(m)}, v^{(1)}, \dots, v^{(m)}) = g(u^{(1)} v^{(1)}) + \sum_{i=2}^{m} g((u^{(i)} - u^{(i-1)}) v^{(i)})$$

and substituting $X_n^{(1)} = u^{(1)}$, $(X_n^{(i)} - X_n^{(i-1)}) = u^{(i)}$, $Y_n^{(i)} = v^{(i)}$ for $i \geq 2$.

For ease of notation let $Z_n^{(1)} = X_n^{(1)}$, $Z^{(1)} = X^{(1)}$ and similarly $Z_n^{(i)} = (X_n^{(i)} - X_n^{(i-1)})$ and $Z^{(i)} = (X^{(i)} - X^{(i-1)})$ for $i \geq 2$. We therefore need to show that the random vector $(Z_n^{(1)}, \dots, Z_n^{(m)}, Y_n^{(1)}, \dots, Y_n^{(m)})$ does indeed converge to $(Z^{(1)}, \dots, Z^{(m)}, c, \dots, c)$. Assume also that $(Z_n^{(1)}, \dots, Z_n^{(m)})$ converges in distribution to $(Z^{(1)}, \dots, Z^{(m)})$.

Let $Z_n = (Z_n^{(1)}, \dots, Z_n^{(m)})$, $Y_n = Y_n^{(1)}$, and similarly $Z = (Z^{(1)}, \dots, Z^{(m)})$ and $Y = Y^{(1)} = c$. For any bounded and continuous function with compact support $h : \mathbb{R}^{m+1} \to \mathbb{R}$ consider the difference

$$\begin{aligned}
&|\mathbb{E}[h(Z_n, Y_n)] - \mathbb{E}[h(Z, c)]| \\
&\qquad = |\mathbb{E}[h(Z_n, Y_n)] - \mathbb{E}[h(Z_n, c)] + \mathbb{E}[h(Z_n, c)] - \mathbb{E}[h(Z, c)]| \\
&\qquad \leq |\mathbb{E}[h(Z_n, Y_n)] - \mathbb{E}[h(Z_n, c)]| + |\mathbb{E}[h(Z_n, c)] - \mathbb{E}[h(Z, c)]|
\end{aligned}$$

Defining $g(z) := h(z, c)$, which is continuous and has compact support from h.Then it follows from the continuous mapping theorem that second term converges to zero as n tends to infinity since Z_n converges to Z in distribution.

For the first term we will use the fact that Y_n converge in probability, which is to say that for any $\varepsilon > 0$, $\mathbb{P}(|Y_n - c| > \varepsilon) \to 0$ as $n \to \infty$. Using this notation, for any $\varepsilon > 0$

$$\begin{aligned}
&|\mathbb{E}[h(Z_n,Y_n)] - \mathbb{E}[h(Z_n,c)]| \\
&\qquad = \left| \int h(Z_n,Y_n) - h(Z_n,c)\, d\mathbb{P} \right| \\
&\qquad = \left| \int_{|Y_n-c|>\varepsilon} h(Z_n,Y_n) - h(Z_n,c)\, d\mathbb{P} + \int_{|Y_n-c|\leq\varepsilon} h(Z_n,Y_n) - h(Z_n,c)\, d\mathbb{P} \right| \\
&\qquad \leq \left| \int_{|Y_n-c|>\varepsilon} h(Z_n,Y_n) - h(Z_n,c)\, d\mathbb{P} \right| + \left| \int_{|Y_n-c|\leq\varepsilon} h(Z_n,Y_n) - h(Z_n,c)\, d\mathbb{P} \right| \\
&\qquad \leq \sup_{\{z,y:|y-c|\leq\varepsilon\}} \{|h(z,y) - h(z,c)|\}\, \mathbb{P}(|Y_n - c| \leq \varepsilon) + 2\|h\|_\infty \mathbb{P}(|Y_n - c| > \varepsilon).
\end{aligned}$$

Observe that h is continuous with compact support. In particular this means that h is uniformly continuous, which is to say that for any $\eta > 0$ there is a $\delta = \delta_\eta > 0$ such that $d_1((x_1,y_1),(x_2,y_2)) < \delta \implies d_2(h(x_1,y_1),h(x_2,y_2)) < \eta$, where d_1 and d_2 are the metrics for $\mathbb{R}^2$ and $\mathbb{R}$ respectively. So there is a $\delta > 0$ such that $d_1((x,y),(x,c)) < \delta \implies d_2(h(x,y),h(x,c)) < \eta$. But the distance $d_1((x,y),(x,c)) = |y - c|$, so since in the above calculation ε was arbitrary, pick $\varepsilon < \delta_\eta$. Therefore for any $\eta > 0$ there exists $\varepsilon > 0$ such that

$$|\mathbb{E}[h(X_n,Y_n)] - \mathbb{E}[h(X_n,c)]| \leq \eta + 2\|h\|_\infty \mathbb{P}(|Y_n - c| > \varepsilon).$$

Y_n converges in probability so there exists some $N \in \mathbb{N}$ such that for $n > N$

$$|\mathbb{E}[h(X_n,Y_n)] - \mathbb{E}[h(X_n,c)]| \leq \eta + \eta = 2\eta.$$

But η is arbitrary so it follows that

$$\lim_{n\to\infty} \mathbb{E}[h(X_n,Y_n) - h(X_n,c)] = 0.$$

Therefore (Z_n,Y_n) converges in distribution to (Z,Y).

Now let $Z_n = (Z_n^{(1)},\ldots,Z_n^{(m)},Y_n^{(1)})$ and $Y_n = Y_n^{(2)}$ and repeat the calculation, to get that $(Z_n^{(1)},\ldots Z_n^{(m)},Y_n^{(1)},Y_n^{(2)})$ converges in distribution to $(Z^{(1)},Z^{(2)},c,c)$ and continue for $Y_n = Y_n^3$ then $Y_n = Y_n^4$ etc. until we have that $(Z_n^{(1)},\ldots,Z_n^{(m)},Y_n^{(1)},\ldots,Y_n^{(m)})$ converges in distribution to $(Z^{(1)},\ldots,Z^{(m)},c,\ldots,c)$.

Then by the continuous mapping theorem $G(Z_n^{(1)}, \ldots, Z_n^{(m)},\ Y_n^{(1)}, \ldots, Y_n^{(m)})$ converges in distribution to $G(Z^{(1)}, \ldots, Z^{(2)}, c, \ldots, c)$ which is to say that the process τ_n converges in distribution to τ.

This proves the finite, m-dimensional case, where

$$\tau_n(\omega) = \sum_{i=1}^{m} \delta_{X_n^{(i)} Y_n^{(i)}}.$$

But for convergence in distribution for the infinite dimensional situation it is enough to prove convergence for every finite m. This can be seen by recalling the definition of convergence in distribution of a point process: τ_n converges in distribution to τ if and only if for all continuous $g : \mathbb{R} \to \mathbb{R}$ with compact support the random variables

$$\sum_{i=1}^{\infty} g(X_n^{(i)} Y_n^{(i)})$$

converge in distribution to

$$\sum_{i=1}^{\infty} g(X^{(i)} Y^{(i)}).$$

Theorem 3.10. *Let the process given by*

$$X_n = \sum_{i=1}^{\infty} \delta_{X_n^{(i)}}$$

converge in distribution, as $n \to \infty$, to

$$X = \sum_{i=1}^{\infty} \delta_{X^{(i)}},$$

where X is a Poisson point process of rate 1. If $Y_n^{(i)}$ converges μ-almost surely to a constant $c \in \mathbb{R}$ for all $i \in \mathbb{N}$ then the process

$$\tau_n = \sum_{i=1}^{\infty} \delta_{Y_n^{(i)} X_n^{(i)}}$$

converges in distribution, as $n \to \infty$, to

$$\tau = \sum_{i=1}^{\infty} \delta_{cX^{(i)}},$$

which is a Poisson point process of rate $\frac{1}{c}$.

3.3.1 Poisson Law for Suspended Flows

Consider a discrete dynamical system $(X, T, \mathscr{B}, \mu)$ with T-invariant probability measure μ, and the associated suspended space $(\Omega_\gamma, S_t, \mathscr{B}, \nu)$ for a height function $\gamma : X \to \mathbb{R}$. Let $\Delta_n \subset \Omega_\gamma$ be a sequence of open sets with positive measure satisfying $\lim_{n\to\infty} \nu(\Delta_n) = 0$. Recall the k-th hitting time for the action of S_t to hit Δ_n is denoted $\tau_n^{(k)}$ and the k-th hitting time for the action of T to hit the projected sets $A_n = \pi(\Delta_n)$ is denoted $r_n^{(k)}$.

Theorem 3.11. *Assume there is a positive sequence of real numbers (c_n) such that the hitting time process given by*

$$r_n(x) = \sum_{k\in\mathbb{N}} \delta_{r_n^{(k)}(x)c_n}$$

converges in distribution to a Poisson point process of rate 1. Then it follows that the process of rescaled hitting times

$$\tau_n(\omega) = \sum_{k\in\mathbb{N}} \delta_{\tau_n^{(k)} c_n}$$

converges in distribution to a Poisson point process of rate $\frac{1}{\int \gamma d\mu}$.

Proof. First check the criteria for Theorem 3.10 are satisfied. Lemma 3.2 states that $\tau_n^{(k)} = Y_n^{(k)} r_n^{(k)}$ and by lemma 3.4, $Y_n^{(k)}$ converges almost surely to $\int \gamma d\mu$ as $n \to \infty$, for any $k \in \mathbb{N}$. It therefore follows that $\tau_n(x,y)$ converges in distribution (with respect to the measure μ) to a point process of rate 1, which will be denoted τ.

For any continuous function $g : \mathbb{R} \to \mathbb{R}$ with compact support, consider the sequence of random variables given by

$$N_n(g)(\omega) = N_n(g)(x,y) = \int_0^\infty g(t)\, d\left(\sum_{k=0}^{\infty} \delta_{\tau_n^k(\omega)c_n}\right).$$

Theorem 3.10 states that

$$\int_\Sigma N_n(g)(x,y)\, d\mu \to \int_\Sigma N(g)(x,y)\, d\mu$$

where

$$N = N(\omega) = \int_0^\infty g(t)\, d\left(\sum_{k=0}^{\infty} \delta_{\tau^k(\omega)}\right).$$

However to show convergence in distribution with respect to the full measure given by

$$\nu = \frac{\mu \times \text{Leb}}{\int \gamma d\mu}$$

we need to show convergence of the integral

$$\int_\Omega N_n(g)(\omega)\, d\nu(\omega).$$

Consider the integral

$$\begin{aligned}
\int \gamma d\mu \int_\Omega N_n(g)(\omega)\, d\nu(\omega) &= \int_\Omega N_n(g)(\omega)\, d\mu(x) \otimes d\,\text{Leb}(y) \\
&= \int_\Sigma \left(\int_0^{\gamma(x)} N_n(g)(x,y)\, d\,\text{Leb}(y) \right) d\mu(x).
\end{aligned}$$

The internal integrals can be bounded above and below. First consider a lower bound.

Up to a set of μ-measure zero, the number of hits will eventually be minimised by considering a starting point $(x,0)$, as this requires the flow to travel the maximum distance before making the first hit, and then each subsequent hit. That is to say that

$$\gamma(x) N_n(g)(x,0) \leq \int_0^{\gamma(x)} N_n(g)(x,y)\, d\,\text{Leb}(y)$$

except on a set of μ-measure equal to zero. Therefore it follows that

$$\int_\Sigma \gamma(x) N_n(g)(x,0)\, d\mu(x) \leq \int_\Sigma \left(\int_0^{\gamma(x)} N_n(g)(x,y)\, d\,\text{Leb}(y) \right) d\mu(x).$$

Similarly for an upper bound up to a set of μ-measure zero, the number of hits will eventually be maximised by considering the highest starting point, $(x,\gamma(x)) = (Tx,0)$, as this requires the flow to travel the minimum distance before making the first hit, and each subsequent hit after. That is to say that

$$\int_0^{\gamma(x)} N_n(g)(x,y)\, d\,\text{Leb}(y) \leq \gamma(x) N_n(g)(x,\gamma(x)) = \gamma(x) N_n(g)(Tx,0)$$

except on a set of μ-measure equal to zero. Therefore it follows that

$$\int_\Sigma \left(\int_0^{\gamma(x)} N_n(g)(x,y)\, d\,\text{Leb}(y) \right) d\mu(x) \leq \int_\Sigma \gamma(x) N_n(g)(Tx,0)\, d\mu(x).$$

To understand the convergence of

$$\int_\Sigma \left(\int_0^{\gamma(x)} N_n(g)(x,y)\, d\,\mathrm{Leb}(y) \right) d\mu(x)$$

it is therefore enough to understand the convergence of the bounds

$$\int_\Sigma \gamma(x) N_n(g)(x,0)\, d\mu(x)$$

and

$$\int_\Sigma \gamma(x) N_n(g)(Tx,0)\, d\mu(x).$$

Consider the lower bound (given by the first integral). The two random variables given by $g(\tau_n^{(k)}(\cdot,0)c_n)$ and γ are asymptotically independent for any $k \in \mathbb{N}$; consider

$$g(\tau_n^{(k)}(x,0)c_n) = g\left(c_n \left(\sum_{i=1}^{r_n^{(k)}-1} \gamma(T^i x) + h(T^{r_n^{(k)}} x) \right) \right).$$

As g is a continuous function with compact support, this converges μ-almost surely, in the limit as n tends to infinity, to the integral

$$g\left(r^{(k)}(x) \int_\Sigma \gamma\, d\mu \right)$$

which is independent of $\gamma(x)$. Therefore

$$\begin{aligned}
\lim_{n\to\infty} \int_\Sigma \gamma(x) N_n(g)(x,0)\, d\mu(x) &= \lim_{n\to\infty} \int_\Sigma \gamma(x) \left(\int_0^\infty g(t)\, d\left(\sum_{k=1}^\infty \delta_{\tau_n^k c_n} \right) \right) d\mu(x) \\
&= \lim_{n\to\infty} \int_\Sigma \gamma(x) \left(\sum_{k=1}^\infty g(\tau_n^k(x,0)c_n) \right) d\mu(x) \\
&= \lim_{n\to\infty} \sum_{k=1}^\infty \int_\Sigma \gamma(x) g(\tau_n^k(x,0)c_n)\, d\mu(x).
\end{aligned}$$

Here we have made use of the monotone convergence theorem. Since g is continuous with compact support the sequence given by

$$\left(\sum_{k=1}^j g(\tau_n^k c_n) \right)_{j\in\mathbb{N}}$$

must eventually be monotonic (for a fixed n), yet also finite as $\tau_n^{(k)}$ diverges to infinity as k diverges. So the infinite sum and the integral may be exchanged.

Now observe that

$$\int_\Sigma \gamma(x) g(\tau_n^k(x,0)c_n)\,d\mu(x) \le \int \gamma(x)\,d\mu \|g\|_\infty.$$

By the dominated convergence theorem

$$\begin{aligned}
\lim_{n\to\infty}\int_\Sigma \gamma(x) N_n(g)(x,0)\,d\mu(x) &= \lim_{n\to\infty}\sum_{k=1}^\infty \int_\Sigma \gamma(x) g(\tau_n^{(k)}(x,0)c_n)\,d\mu(x)\\
&= \sum_{k=1}^\infty \lim_{n\to\infty}\int_\Sigma \gamma(x) g(\tau_n^{(k)}(x,0)c_n)\,d\mu(x)\\
&= \sum_{k=1}^\infty \int_\Sigma \gamma(x)\,d\mu \int_\Sigma g(\tau^{(k)}(x,0))\,d\mu(x)\\
&= \int \gamma d\mu \int N(g)(x)\,d\mu.
\end{aligned}$$

Similarly for the upper bound (given by the second integral), noting again that since $g(\tau_n^{(k)}(\cdot,0)c_n)$ and γ are asymptotically independent for any $k\in\mathbb{N}$

$$\begin{aligned}
\lim_{n\to\infty}\int_\Sigma \gamma(x) N_n(g)(Tx,0)\,d\mu(x) &= \lim_{n\to\infty}\int_\Sigma \gamma(x)\left(\int_0^\infty g(t)\,d\left(\sum_{k=1}^\infty \delta_{\tau_n^{(k)}c_n}\right)\right)d\mu(x)\\
&= \lim_{n\to\infty}\int_\Sigma \gamma(x)\left(\sum_{k=1}^\infty g(\tau_n^{(k)}(Tx,0)c_n)\right)d\mu(x)\\
&= \lim_{n\to\infty}\sum_{k=1}^\infty \int_\Sigma \gamma(x) g(\tau_n^{(k)}(Tx,0)c_n)\,d\mu(x)\\
&= \lim_{n\to\infty}\sum_{k=1}^\infty \int_\Sigma \gamma(x) g(\tau_n^{(k)}(Tx,0)c_n)\,d\mu(x)\\
&= \sum_{k=1}^\infty \lim_{n\to\infty}\int_\Sigma \gamma(x) g(\tau_n^{(k)}(Tx,0)c_n)\,d\mu(x)\\
&= \sum_{k=1}^\infty \int_\Sigma \gamma(x)\,d\mu \int_\Sigma g(\tau^{(k)}(Tx,0))\,d\mu(x)\\
&= \int \gamma d\mu \int N(g)(x)\,d\mu.
\end{aligned}$$

The upper and lower bound both converge to the same limit, therefore

$$\lim_{n\to\infty}\int_\Sigma \left(\int_0^{\gamma(x)} N_n(g)(x,y)\,d\,\mathrm{Leb}(y)\right)d\mu(x) = \int_\Sigma \gamma(x)\,d\mu(x)\int_\Sigma N(g)(x,0)\,d\mu(x),$$

and τ_n converges in ν-distribution to a Poisson process of rate $\frac{1}{\int \gamma d\mu}$. □

Corollary 3.12. *Consider an ergodic subshift of finite type, (Σ_A, σ), equiped with a suspended flow $S_t : \Omega_\gamma \to \Omega_\gamma$ for some $\gamma \in \mathscr{F}$, and let $\Delta_n \subset \Omega_\gamma$ be a sequence of open balls with measure $\nu(\Delta_n) > 0$ and $\lim_{n\to\infty} \nu(\Delta_n) = 0$. Then there is a sequence of positive real numbers $(c_n)_{n\in\mathbb{N}}$ such that the process of hitting times given by*

$$\tau_n(\omega) = \sum_{k\in\mathbb{N}} \delta_{\tau_n^{(k)}(\omega)c_n}$$

converges in distribution to a Poisson point process of rate 1.

This follows as a direct corollary from Theorem 3.11, and Pitskel's result (theorem 2.1), which gave a Poisson limit law for open balls in a Markov chain.

These results will extend naturally to give a Poisson limit law for the hitting time processes to a sequence of more general open sets Δ_n and not just open balls, so long as the hitting time process of the discrete system given by r_n converges to a Poisson process.

3.3.2 Poisson Law for Axiom A Flows

By applying Theorem 3.11 to the developed theory of axiom A flows, the following theorem holds.

Theorem 3.13. *Let $\phi_t : \Lambda \to \Lambda$ be an axiom A flow on a basic hyperbolic set, Λ, and $\{\Delta_n\}_n$ be a sequence of open sets with positive measure such that $\lim_{n\to\infty} m(\Delta_n) = 0$. Then the hitting time process to Δ_n given by*

$$\tau_n = \sum_{k=1}^{\infty} \delta_{c_n \tau_n^{(k)}}$$

converges in distribution to a Poisson point process of rate 1, where $c_n = \mu(\Delta_n) \int \gamma d\mu$.

Proof. The distribution of the rescaled hitting times in an axiom A flow is the same as the distribution of rescaled hitting times in the corresponding suspension flow. Open sets in the original space will also correspond to open sets in the suspended space. It therefore follows from Theorems 2.1 and 2.2 that there is a rescaling sequence c_n such that the rescaled hitting time process, of the action on the base of the suspension flow, converges in distribution to a Poisson point process of rate 1. Therefore by Theorem 3.11 there is a rescaling sequence such that the rescaled hitting time process of the full flow converges in distribution to a Poisson point process of rate 1. Therefore, for the same rescaling sequence, the rescaled hitting time process in the axiom A flow converges to a Poisson point process of rate 1. $\square$

3.4 Return Times

So far I have been looking at hitting times for flows, and I will now show how similar methods can be used to understand the distributions of the return times, proving a theorem on the rescaled limit distribution of return times in a similar setting. In general one would expect to find the return time process and hitting time process have similar limiting distributions, especially if the system has sufficient mixing properties, and I will show that the rescaled return time distribution is the same in the limit as that of hitting times, provided that similar conditions are satisfied.

Start by defining the return times. Consider a sequence of measurable open sets $\Delta_n \subset \Omega$ such that $\nu(\Delta_n) > 0$ and

$$\lim_{n\to\infty} \nu(\Delta_n) = 0.$$

For $\omega \in \Delta_n$ denote the first return time to Δ_n by

$$\tau_n^{(1)}(\omega) := \inf\{t > 0 : \phi_t(\omega) \in \Delta_n \text{ and } \exists s < t \text{ such that } \phi_s(\omega) \notin \Delta_n\},$$

and subsequent return times are defined inductively as

$$\tau_n^{(k)}(\omega) := \inf\{t > \tau_n^{(k-1)} : \phi_t(\omega) \in \Delta_n \text{ and } \exists s \in (\tau_n^{(k-1)}, t) \text{ such that } \phi_s(\omega) \notin \Delta_n\}.$$

We use the exact same notation here as for hitting times, as the definitions are identical, with the only exception being that return times are restricted to the target set. Because of this many of the useful observations made for hitting times still hold true for return times.

One of the main differences of studying the distribution of return times and the distribution of hitting times however is that because of the restriction to the target sets the distributions must be measured using conditional probability measures.

Denote by ν_n the conditional measure of ν given the condition Δ_n. That is to say for a measurable $\Delta \subset \Omega$

$$\nu_n(\Delta) := \nu(\Delta|\Delta_n) = \frac{\nu(\Delta \cap \Delta_n)}{\nu(\Delta_n)}.$$

3.4.1 Return Times for Suspension Flows

Consider a measure preserving discrete dynamical system, $(X, T, \mathscr{B}, \mu)$, with a measurable height function $\gamma : X \to (0, \infty)$ such that $\int \gamma d\mu < \infty$. The suspended space is given by

$$\Omega = \Omega_\gamma := \{\omega = (x,y) \in X \times \mathbb{R} : 0 \leq y \leq \gamma(x)\},$$

where the points $(x, \gamma(x))$ are identified with $(Tx, 0)$. Associate with this the product σ-algebra, and define a probability measure on Ω by

$$d\nu_\gamma = d\nu = \frac{d\mu \times d\,\mathrm{Leb}}{\int \gamma d\mu}.$$

The suspension flow $(S_t : \Omega_\gamma \to \Omega_\gamma : t \in \mathbb{R})$ is given by

$$S_t(\omega) = S_t(x,y) := \left(T^{\eta(t)}x, \, y + t - \sum_{i=0}^{\eta(t)-1} \gamma(T^i x) \right)$$

where $\eta(t)$ is the unique natural number which satisfies

$$0 \leq y + t - \sum_{i=0}^{\eta(t)-1} \gamma(T^i x) < \gamma(T^{\eta(t)} x).$$

Recall also the projections $\pi : \Omega \to X$ given $\pi(x,y) = x$ and $\pi' : \Omega \to \mathbb{R}$ given by $\pi'(x,y) = y$.

Consider a sequence of open balls $(\Delta_n \subset \Omega)$ with positive measure such that $\lim_{n\to\infty} \nu(\Delta_n) = 0$. Denote $\pi(\Delta_n) = A_n$ and observe that $\mu(A_n) > 0$ since otherwise

$$\nu(\Delta_n) = \frac{\mu \times \mathrm{Leb}}{\int \gamma d\mu}(\Delta_n) \leq \frac{\mu(\pi(\Delta_n) \times \mathrm{Leb}(\pi'(\Delta_n))}{\int \gamma d\mu} = 0.$$

Now consider the the return times $\tau_n^{(k)}$ to these sets. As shown with the hitting times, these return times can be related to the return times, $r_n^{(k)}$, of the system $(X, T, \mathscr{B}, \mu)$ at the base of the flow, to the projected sets $A_n = \pi(\Delta_n)$.

Lemma 3.14. *Let $(X, T, \mathscr{B}, \mu)$ be an invariant dynamical system with probability measure μ, and let $(\Omega_\gamma, S_t, \mathscr{B}, \nu)$ be the associated suspension flow with height given by the L^1 integrable function $\gamma : X \to \mathbb{R}$. Then consider a sequence of open balls $\Delta_n \subset \Omega_\gamma$ with positive measure such that $\lim_{n\to\infty} \nu(\Delta_n) = 0$. Let $\tau_n^{(k)}$ be the k-th return time for the suspension flow to hit Δ_n and let $r_n^{(k)}$ be the k-th return time for the transformation $T : X \to X$ to hit the projected sets $A_n = \pi(\Delta_n)$. Then*

$$\tau_n^{(k)} = r_n^{(k)} Y_n^{(k)}$$

where

$$Y_n^{(k)}(\omega) = Y_n^{(k)}(x,y) = \frac{1}{r_n^{(k)}} \left(\sum_{i=0}^{r_n^{(k)}-1} \gamma(T^i x) - y + h(T^{r_n^{(k)}} x) \right)$$

and $h : X \to \mathbb{R}$ is a positive measurable function with $0 \le h(x) \le \gamma$.

Proof. $\tau_n^{(k)}(\omega)$ is the time it takes for the flow S_t to hit some particular set Δ_n of positive measure, starting at the point $\omega = (x,y)$. At the time $t = \tau_n^{(k)}(\omega)$

$$S_{\tau_n^{(k)}}(x,y) = \left(T^{\eta(\tau_n^{(k)})} x \, , \, y + \tau_n^{(k)} - \sum_{i=0}^{\eta(\tau_n^{(k)})-1} \gamma(T^i x) \right),$$

where $\eta(\tau_n^{(k)})$ is the unique natural number which satisfies

$$0 \le \; y + \tau_n^{(k)} - \sum_{i=0}^{\eta(\tau_n^{(k)})-1} \gamma(T^i x) \; < \; \gamma(T^{\eta(\tau_n^{(k)})} x).$$

We will show that $\eta(\tau_n^{(k)}) = r_n^{(k)}$

Each term in the sum coincides with the flow reaching the top of the suspension once, which is to say that the x-coordinate is translated to Tx for each term in the sum. In order for the flow to hit Δ_n for the k-th time, it will therefore have to continue until the x-coordinate hits the projected set $A_n = \pi(\Delta_n) \in X$ for the k-th time. After the first translation (that is the time to reach the point (Tx, 0)) S has travelled a distance of $\gamma(x) - y$, the height of γ over the first point minus the starting position, and at subsequent translations the distance just adds on the height of γ each x coordinate visited so that after $r_n^{(k)}$ translations S has flowed a distance of

$$\gamma(x) - y + \gamma(Tx) + \cdots + \gamma(T^{r_n^{(k)}-1} x).$$

This must be at most the distance required to hit Δ_n for the k-th time. Since S_t travels at unit speed,

$$\tau_n^{(k)} \geq \left(\sum_{i=0}^{r_n^{(k)}-1} \gamma(T^i x) \right) - y.$$

Rearranging gives the left hand inequlity

$$0 \leq \ y + \tau_n^{(k)} - \sum_{i=0}^{r_n^{(k)}-1} \gamma(T^i x).$$

To get the right hand inequality observe that S_t must hit Δ_n before the next translate. So

$$\tau_n^{(k)} < \gamma(x) - y + \gamma(Tx) + \cdots + \gamma(T^{r_n^{(k)}} x).$$

Rearranging gives

$$y + \tau_n^{(k)} - \sum_{i=0}^{r_n^{(k)}-1} \gamma(T^i x) \ < \ \gamma(T^{r_n^{(k)}} x).$$

It follows that there is some function $h : X \to \mathbb{R}$ which satisfies

$$\tau_n^{(k)} = \sum_{i=0}^{r_n^{(k)}-1} \gamma(T^i x) - y + h(T^{r_n^{(k)}} x)$$

where $0 \leq h(x) < \gamma(x)$ for every $x \in X$. Define $h(x)$ to be the height required for S to hit Δ_n from $(x,0)$ if $x \in A_n$ or 0 otherwise, that is to say

$$h(x) = \begin{cases} \tau_n^{(1)}(x,0) & x \in A_n \\ 0 & \text{otherwise.} \end{cases}$$

The measurability of h follows from the measurability of τ. □

Lemma 3.15. *$Y_n^{(k)}$ converges μ-almost surely to $\int \gamma d\mu$ as $n \to \infty$.*

Proof. This can be shown by rearranging and considering three terms separately, by writing

$$Y_n^{(k)} = \frac{1}{r_n^{(k)}} \sum_{i=0}^{r_n^{(k)}-1} \gamma(T^i x) - \frac{y}{r_n^{(k)}} + \frac{h(T^{r_n^{(k)}} x)}{r_n^{(k)}}$$

where $h(x)$ is the height required for S to hit Δ_n from $(x,0)$ if $x \in A_n$ or 0 otherwise, that is to say

$$h(x) = \begin{cases} \tau_n^{(1)}(x,0) & x \in A_n \\ 0 & \text{otherwise.} \end{cases}$$

For the first term use a generalised version of the Birkhoff ergodic theorem observing that since $r_n^{(k)}$ diverges almost surely then the averages must converge almost surely to the integral due to the ergodic properties of μ. That is

$$\frac{1}{r_n^{(k)}}\left(\sum_{i=0}^{r_n^{(k)}-1}\gamma(T^i x)\right)\to\int\gamma d\mu$$

almost surely as $n\to\infty$.

For the second term recognise that $r_n^{(k)}\to\infty$ almost surely as $n\to\infty$ since the measure of the target set converges to 0 and so the hitting times diverge according to Kac's theorem. As y is a constant here so it follows that

$$\frac{y}{r_n^{(k)}}\to 0$$

μ-almost surely as $n\to\infty$.

For the third term it's already known that $0\leq h(T^{r_n^{(k)}}x)<\gamma(T^{r_n^{(k)}}x)$. Since $\gamma\in L^1$ it follows that by the Birkhoff Ergodic theorem

$$\frac{1}{r_n^{(k)}}\sum_{k=0}^{r_n^{(k)}}\gamma(T^i x)\to\int\gamma d\mu.$$

So consider

$$\begin{aligned}\frac{1}{r_n^{(k)}}\,h(T^{r_n^{(k)}}x)&<\frac{1}{r_n^{(k)}}\,\gamma(T^{r_n^{(k)}}x)\\&=\frac{1}{r_n^{(k)}}\sum_{k=0}^{r_n^{(k)}}\gamma(T^i x)-\frac{1}{(r_n^{(k)}-1)}\sum_{k=0}^{r_n^{(k)}-1}\gamma(T^i x)\end{aligned}$$

Taking a limit as $n\to\infty$, and noting that $r_n^{(k)}$ is a subsequence of $(n)_{n\in\mathbb{N}}$, then both terms in the final expression converge to the integral of γ by the Birkhoff ergodic theorem. In particular, on a set of full measure,

$$0\leq\lim_{n\to\infty}\frac{1}{r_n^{(k)}}\,h(T^{r_n^{(k)}}x)\;\leq\;\int\gamma d\mu\;-\;\int\gamma d\mu\;=0,$$

and so $\frac{1}{r_n^{(k)}}h(T^{r_n^{(k)}}x)\to 0$ almost surely.

The result now follows. □

This is where the immediate similarities with hitting times end, as in order to construct a general version of Slutky's Theorem which can be used here, a slightly different proof will be needed. The lemmas below will be required in that proof.

Lemma 3.16. *If $\gamma \in L^2(X, \mathscr{B}, \mu)$ then $Y_n^{(k)}$ converges in L^2 to $\int \gamma d\mu$ as $n \to \infty$.*

Proof. Again this can be shown by rearranging and considering three terms separately, by writing

$$Y_n^{(k)} = \frac{1}{r_n^{(k)}} \sum_{i=0}^{r_n^{(k)}-1} \gamma(T^i x) - \frac{y}{r_n^{(k)}} + \frac{h_n(T^{r_n^{(k)}} x)}{r_n^{(k)}}$$

where $h(x)$ is the height required for S to hit Δ_n from $(x, 0)$ if $x \in A_n$ or 0 otherwise, that is to say

$$h_n(x) = \begin{cases} \tau_n^{(1)}(x, 0) & x \in A_n \\ 0 & \text{otherwise.} \end{cases}$$

For the first term use a generalised version of the Von Neumann ergodic theorem observing that since $r_n^{(k)}$ diverge almost surely then the averages must converge in L^2 to the integral due to the ergodic properties of μ. That is

$$\frac{1}{r_n^{(k)}} \left(\sum_{i=0}^{r_n^{(k)}-1} \gamma(T^i x) \right) \to \int \gamma d\mu$$

in L^2 as $n \to \infty$.

For the second term note that since $\gamma \in L^2$ and $y \leq \gamma(x)$ the integral of y^2 exists and is finite and

$$\begin{aligned} \int \left| \tfrac{1}{r_n^{(k)}} y \right|^2 d\mu &\leq \int \left| \tfrac{1}{n} y \right|^2 d\mu \\ &\leq \frac{1}{n} \int |y|^2 d\mu, \end{aligned}$$

and this final term converges to zero as $n \to \infty$.

For the final term, again note that since γ is L^2 and $0 \leq h_n(x) < \gamma(x)$ then $\frac{1}{r_n^{(k)}} h_n$ is also L^2. Noting that h_n is zero outside of A_n the integral can be restricted to the set A_n and making use of the upper bound $h < \gamma$ and that both $h, \gamma \geq 0$ then

$$\begin{aligned}
\int \left| \tfrac{1}{r_n^{(k)}} h_n \right|^2 d\mu &\leq \int_{A_n} \left(\tfrac{1}{r_n^{(k)}} \gamma \right)^2 d\mu \\
&\leq \int_{A_n} \gamma^2 \, d\mu \\
&\leq \mu(A_n) \int \gamma^2 \, d\mu .
\end{aligned}$$

But $\lim_{n\to\infty} \mu(A_n) = 0$ and so this converges to zero.

The result now follows. □

Lemma 3.17. *If $Y_n^{(k)}$ converges in L^2 to a constant $c \in \mathbb{R}$ then the probability conditioned on A_n converges. That is to say*

$$\lim_{n\to\infty} \mathbb{P}_n \left(\left| Y_n^{(k)} - c \right| > \varepsilon \right) = 0.$$

Proof. Let $\varepsilon > 0$ and use the Markov inequality to calculate

$$\begin{aligned}
\mathbb{P}_n \left(\left| Y_n^{(k)} - c \right| > \varepsilon \right) &= \frac{\mathbb{P}\left(\chi_{A_n} \cdot \left| Y_n^{(k)} - c \right| > \varepsilon \right)}{\mathbb{P}(A_n)} \\
&\leq \frac{1}{\varepsilon \mathbb{P}(A_n)} \int \chi_{A_n} \cdot \left| Y_n^{(k)} - c \right| d\mu \\
&\leq \frac{\mathbb{P}(A_n)}{\varepsilon \mathbb{P}(A_n)} \int \left| Y_n^{(k)} - c \right| d\mu \\
&= \frac{1}{\varepsilon} \int \left| Y_n^{(k)} - c \right| d\mu .
\end{aligned}$$

But ε was fixed and so noting that $Y_n^{(k)}$ converges in L^2 to c, and therefore in L^1 to c, then taking a limit as $n \to \infty$ the result now follows. □

Theorem 3.18. *Let $A_n \subset X$ be a sequence of subsets such that $\mu(A_n) > 0$ and $\lim_{n\to\infty} \mu(A_n) = 0$. Let $X_n^{(i)}$ be a sequence of random variables defined on A_n such that the processes given by*

$$X_n = \sum_{i=1}^{\infty} \delta_{X_n^{(i)}}$$

converge in distribution, as $n \to \infty$, to

$$X = \sum_{i=1}^{\infty} \delta_{X^{(i)}},$$

a Poisson point process of rate 1. Let $Y_n^{(i)}$ be another sequence of random variables on A_n and $c \in \mathbb{R}$ be such that for any $\varepsilon > 0$ the limit

$$\lim_{n\to\infty} \mathbb{P}_n \left(|Y_n^{(i)} - c| > \varepsilon \right) = 0.$$

Then the process given by

$$\tau_n = \sum_{i=1}^{\infty} \delta_{Y_n^{(i)} X_n^{(i)}}$$

converges in distribution, as $n \to \infty$, to

$$\tau = \sum_{i=1}^{\infty} \delta_{cX^{(i)}},$$

which is a Poisson point process of rate $\frac{1}{c}$.

Proof. Consider m sequences of random variables $X_n^{(1)}, X_n^{(2)}, \ldots, X_n^{(m)}$ which converge in distribution to $X^{(1)}, X^{(2)}, \ldots, X^{(m)}$ respectively and sequences $Y_n^{(1)}, Y_n^{(2)}, \ldots, Y_n^{(m)}$ which converge almost everywhere to $Y^{(1)}, Y^{(2)}, \ldots, Y^{(m)}$ respectively. As X_n converges to a poisson process, $X_n^{(1)}$ and the difference $(X_n^{(2)} - X_n^{(1)})$ are asymptotically independent and for each $i = 2, \ldots, m-1$ the pair of differences $(X_n^{(i)} - X_n^{(i-1)})$ and $(X_n^{i+1} - X_n^{(i)})$ are pairwise asymptotically independent. Let $Y^{(i)} = c \in \mathbb{R}$. We are concerned with the convergence of the process given by $\tau_n(\omega) = \sum_{i=1}^{m} \delta_{X_n^{(i)} Y_n^{(i)}}$.

To see that this converges in distribution it is sufficient to show that for any continuous function, $g : \mathbb{R} \to \mathbb{R}$, with compact support

$$\int g \, d\tau_n = \sum_{i=1}^{m} g(X_n^{(i)} Y_n^{(i)}),$$

converges in distribution to

$$\int g \, d\tau = \sum_{i=1}^{m} g(X^{(i)} Y^{(i)}).$$

Recall theorem 3.6, the continuous mapping theorem: if Z_n is a d-dimensional random vector which converges in distribution to the random vector Z and $g : \mathbb{R}^d \to \mathbb{R}$ is a continuous map with compact supoprt then $g(Z_n) \to g(Z)$ in distribution.

Let $G : \mathbb{R}^{2m} \to \mathbb{R}$ given by

$$G(u^{(1)}, \ldots, u^{(m)}, v^{(1)}, \ldots, v^{(m)}) = g(u^{(1)} v^{(1)}) + \sum_{i=2}^{m} g((u^{(i)} - u^{(i-1)}) v^{(i)})$$

and substituting $X_n^{(1)} = u^{(1)}$, $(X_n^{(i)} - X_n^{(i-1)}) = u^{(i)}$, $Y_n^{(i)} = v^{(i)}$ for $i \geq 2$. For ease of notation write $Z_n^{(1)} = X_n^{(1)}$, $Z^{(1)} = X^{(1)}$ and similarly $Z_n^{(i)} = (X_n^{(i)} - X_n^{(i-1)})$ and $Z^{(i)} = (X^{(i)} - X^{(i-1)})$ for $i \geq 2$. We will show that the random vector $(Z_n^{(1)}, \ldots, Z_n^{(m)}, Y_n^{(1)}, \ldots, Y_n^{(m)})$ does indeed converge to $(Z^{(1)}, \ldots, Z^{(m)}, c, \ldots, c)$. Note that $(Z_n^{(1)}, \ldots, Z_n^{(m)})$ converges in distribution to $(Z^{(1)}, \ldots, Z^{(m)})$.

Let $Z_n = (Z_n^{(1)}, \ldots, Z_n^{(m)})$, $Y_n = Y_n^{(1)}$, and similarly $Z = (Z^{(1)}, \ldots, Z^{(m)})$ and $Y = Y^{(1)} = c$. Let $\mathbb{E}_n$ denote the conditional expectation, that is the expectation with respect to the measure $\mu_n = \mu(\cdot|A_n)$. For any bounded and continuous function with compact support $h : \mathbb{R}^{m+1} \to \mathbb{R}$ consider the difference given by

$$\begin{aligned} |\mathbb{E}_n[h(Z_n, Y_n)] - \mathbb{E}_n[h(Z, c)]| &= |\mathbb{E}_n[h(Z_n, Y_n)] - \mathbb{E}_n[h(Z_n, c)] \\ &\quad + \mathbb{E}_n[h(Z_n, c)] - \mathbb{E}_n[h(Z, c)]| \\ &\leq |\mathbb{E}_n[h(Z_n, Y_n)] - \mathbb{E}_n[h(Z_n, c)]| \\ &\quad + |\mathbb{E}_n[h(Z_n, c)] - \mathbb{E}_n[h(Z, c)]|. \end{aligned}$$

Define $g(z) := h(z, c)$, which is continuous and has compact support from h. Then by the continuous mapping theorem the second term converges to zero as n tends to infinity since Z_n converges to Z in distribution.

For the first term, note the assumption that for any $\varepsilon > 0$,

$$\mathbb{P}_n(|\, Y_n - c \,| > \varepsilon) \to 0$$

as $n \to \infty$. Using this notation, for any $\varepsilon > 0$

$$\begin{aligned} &|\mathbb{E}_n[h(Z_n, Y_n)] - \mathbb{E}_n[h(Z_n, c)]| \\ &\quad = \left| \int h(Z_n, Y_n) - h(Z_n, c)\, d\mathbb{P}_n \right| \\ &\quad = \left| \int_{|Y_n - c| > \varepsilon} h(Z_n, Y_n) - h(Z_n, c)\, d\mathbb{P}_n + \int_{|Y_n - c| \leq \varepsilon} h(Z_n, Y_n) - h(Z_n, c)\, d\mathbb{P}_n \right| \\ &\quad \leq \left| \int_{|Y_n - c| > \varepsilon} h(Z_n, Y_n) - h(Z_n, c)\, d\mathbb{P}_n \right| \\ &\qquad\qquad + \left| \int_{|Y_n - c| \leq \varepsilon} h(Z_n, Y_n) - h(Z_n, c)\, d\mathbb{P}_n \right| \\ &\quad \leq 2\|h\|_\infty \mathbb{P}_n(|Y_n - c| > \varepsilon) \\ &\qquad\qquad + \sup_{\{z, y : |y - c| \leq \varepsilon\}} \{|h(z, y) - h(z, c)|\}\, \mathbb{P}_n(|Y_n - c| \leq \varepsilon). \end{aligned}$$

Observe that h is continuous with compact support. In particular h is uniformly continuous, which is to say that for any $\eta > 0$ there is a $\delta = \delta_\eta > 0$ such that

$d_1((x_1,y_1),(x_2,y_2)) < \delta \implies d_2(h(x_1,y_1),h(x_2,y_2)) < \eta$, where d_1 and d_2 are the metrics for $\mathbb{R}^2$ and $\mathbb{R}$ respectively. So there is a $\delta > 0$ such that $d_1((x,y),(x,c)) < \delta \implies d_2(h(x,y),h(x,c)) < \eta$. But the distance $d_1((x,y),(x,c)) = |y-c|$. In the above calculation ε was arbitrary, so pick $\varepsilon < \delta_\eta$. Therefore for any $\eta > 0$ there exists $\varepsilon > 0$ such that

$$|\mathbb{E}_n[h(X_n,Y_n)] - \mathbb{E}_n[h(X_n,c)]| \leq 2\|h\|_\infty \mathbb{P}_n(|Y_n - c| > \varepsilon) + \eta.$$

Since

$$\lim_{n\to\infty} \mathbb{P}_n(|Y_n - c| > \varepsilon) = 0$$

there exists some $N \in \mathbb{N}$ such that for $n > N$

$$|\mathbb{E}_n[h(X_n,Y_n)] - \mathbb{E}_n[h(X_n,c)]| \leq \eta + \eta = 2\eta.$$

But η is arbitrary so it follows that

$$\lim_{n\to\infty} \mathbb{E}_n[h(X_n,Y_n) - h(X_n,c)] = 0.$$

Therefore (Z_n,Y_n) converges in distribution to (Z,Y).

Now let $Z_n = (Z_n^{(1)},\ldots,Z_n^{(m)},Y_n^{(1)})$ and $Y_n = Y_n^{(2)}$ and repeat the calculation, to get that $(Z_n^{(1)},\ldots,Z_n^{(m)},Y_n^{(1)},Y_n^{(2)})$ converges in distribution to $(Z^{(1)},\ldots,Z^{(m)},c,c)$ and continue for $Y_n = Y_n^3$ then $Y_n = Y_n^4$ etc. until we have that $(Z_n^{(1)},\ldots,Z_n^{(m)},Y_n^{(1)},\ldots,Y_n^{(m)})$ converges in distribution to $(Z^{(1)},\ldots,Z^{(m)},c,\ldots,c)$. Then by the continuous mapping theorem

$$G(Z_n^{(1)},\ldots,Z_n^{(m)},\ Y_n^{(1)},\ldots,Y_n^{(m)})$$

converges in distribution to

$$G(Z^{(1)},\ldots,Z^{(m)},\ c,\ldots,c)$$

which is to say that the process τ_n converges in distribution to τ.

This proves the finite, m-dimensional case, where

$$\tau_n(\omega) = \sum_{i=1}^{m} \delta_{X_n^{(i)} Y_n^{(i)}}.$$

But for convergence in distribution for the infinite dimensional situation it is enough to prove convergence for every finite m. This can be seen by recalling the definition of convergence in distribution of a point process: τ_n converges in distribution to τ if and only if for all continuous $g : \mathbb{R} \to \mathbb{R}$ with compact support the random variables

$$\sum_{i=1}^{\infty} g(X_n^{(i)} Y_n^{(i)})$$

converge in distribution to

$$\sum_{i=1}^{\infty} g(X^{(i)} Y^{(i)}).$$

Therefore the result follows. □

This shows a Poisson limit law, with respect to the measure μ, for the return time process rescaled by $\mu(A_n)$. Now we will prove a full result for the flow with respect to the product measure given by

$$\nu = \frac{\mu \times \text{Leb}}{\int \gamma d\mu}.$$

Theorem 3.19. *Assume there is a positive sequence of real numbers (c_n) such that the return time processes given by*

$$r_n(x) = \sum_{k \in \mathbb{N}} \delta_{r_n^{(k)}(x) c_n}$$

converge in distribution to a Poisson point process of rate 1. Then processes of rescaled return times

$$\tau_n(\omega) = \sum_{k \in \mathbb{N}} \delta_{\tau_n^{(k)} c_n}$$

converges in distribution to a Poisson point process of rate $\frac{1}{\int \gamma d\mu}$.

Proof. Lemma 3.14 states that $\tau_n^{(k)} = Y_n^{(k)} r_n^{(k)}$, and by lemma 3.15 $Y_n^{(k)}$ converges almost surely to $\int \gamma d\mu$ as $n \to \infty$, for any $k \in \mathbb{N}$. It therefore follows from Theorem 3.18 that $\tau_n(x,y)$ converges in distribution (with respect to the measure μ) to a point process of rate $\frac{1}{\int \gamma d\mu}$, which will be denoted τ.

For any continuous function $g : \mathbb{R} \to \mathbb{R}$ with compact support, consider the sequence of random variables given by

$$N_n(g)(\omega) = N_n(g)(x,y) = \int_0^{\infty} g(t)\, d\left(\sum_{k=0}^{\infty} \delta_{\tau_n^k(\omega) c_n}\right).$$

Theorem 3.18 states that

$$\int_{A_n} N_n(g)(x,y)\, d\mu_n \to \int_A N(g)(x,y)\, d\mu^*$$

where

$$N = N(\omega) = \int_0^\infty g(t)\, d\left(\sum_{k=0}^{\infty} \delta_{\tau^k(\omega)} \right),$$

and $(N(g), \mu^*)$ describe a Poisson point process of rate $\frac{1}{\int \gamma d\mu}$. To show convergence in distribution with respect to the full measure given by

$$\nu = \frac{\mu \times \mathrm{Leb}}{\int \gamma d\mu}$$

we will show the convergence of the integral

$$\int_\Omega N_n(g)(\omega)\, d\nu_n(\omega).$$

Note that for continuous γ an open ball Δ_n with respect to a product metric will be a square, if it has sufficiently small diameter, given by $\Delta_n = \pi(\Delta_n) \times \pi'(\Delta_n) = A_n \times (a_n, b_n)$, and so the conditional measure is given by

$$\begin{aligned}
\nu_n &= \frac{\nu(\Delta_n \cap \cdot)}{\nu(\Delta_n)} \\
&= \frac{\mu\left(\pi(\Delta_n \cap \cdot)\right) \times \mathrm{Leb}\left(\pi'(\Delta_n \cap \cdot)\right)}{\mu(\pi(\Delta_n)) \times \mathrm{Leb}(\pi'(\Delta_n))} \frac{\int \gamma d\mu}{\int \gamma d\mu} \\
&= \mu_n \times \mathrm{Leb}_n\,.
\end{aligned}$$

Then consider

$$\begin{aligned}
\int_\Omega N_n(g)(\omega)\, d\nu_n(\omega) &= \int_\Omega N_n(g)(\omega)\, d\mu_n(x) \otimes d\,\mathrm{Leb}_n(y) \\
&= \int_\Sigma \left(\int_0^{\gamma(x)} N_n(g)(x,y)\, d\,\mathrm{Leb}_n(y) \right) d\mu_n(x).
\end{aligned}$$

This double integral can be bounded above and below. First consider a lower bound by restricting the Lebesgue integral to only the points $y \in \pi'(\Delta_n) = (a_n, b_n)$, and noting that the number of hits will eventually be minimised by starting at the highest possible y value as the flow will need to travel the maximum distance for each hit. That is to say that

$$\begin{aligned}
&\int_\Sigma \left(\int_0^{\gamma} N_n(g)(x,y)\, d\,\mathrm{Leb}_n(y) \right) d\mu_n(x) \\
&\qquad = \int_\Sigma \left(\frac{1}{\mathrm{Leb}(\pi'(\Delta_n))} \int_{a_n}^{b_n} N_n(g)(x,y)\, d\,\mathrm{Leb}(y) \right) d\mu_n(x) \\
&\qquad \geq \int_\Sigma \frac{b_n - a_n}{b_n - a_n} N_n(g)(Tx, 0)\, d\mu_n \\
&\qquad = \int_\Sigma N_n(g)(Tx, 0)\, d\mu_n.
\end{aligned}$$

But it's already known that this converges to $\int N(g)\,d\mu^*$.

Similarly for an upper bound, the number of hits will be maximised by starting at the lowest possible y value as it will hit Δ_n immediately, before continuing to the subsequent hits. Therefore

$$\begin{aligned}
&\int_\Sigma \left(\int_0^\gamma N_n(g)(x,y)\,d\,\mathrm{Leb}_n(y) \right) d\mu_n(x) \\
&\qquad = \int_\Sigma \left(\frac{1}{\mathrm{Leb}(\pi'(\Delta_n))} \int_{a_n}^{b_n} N_n(g)(x,y)\,d\,\mathrm{Leb}(y) \right) d\mu_n(x) \\
&\qquad \leq \int_\Sigma \frac{b_n - a_n}{b_n - a_n} N_n(g)(x,0)\,d\mu_n \\
&\qquad = \int_\Sigma N_n(g)(x,0)\,d\mu_n.
\end{aligned}$$

Again it is already known that this converges to $\int N(g)\,d\mu^*$.

The upper and lower bound both converge to the same limit, therefore

$$\lim_{n\to\infty} \int_\Sigma \left(\int_0^{\gamma(x)} N_n(g)(x,y)\,d\,\mathrm{Leb}_n(y) \right) d\mu_n(x) = \int_\Sigma N(g)(x,0)\,d\mu^*(x),$$

and so τ_n converges in ν distribution to a Poisson process of rate $\frac{1}{\int \gamma d\mu}$. □

By applying this result with that of Pitskel (theorem 2.1, [14]) which gave a Poisson limit law for open balls in a Markov chain we obtain the following corollary.

Corollary 3.20. *Consider an ergodic subshift of finite type, (Σ_A, σ, ν), equiped with a suspended flow S_t on Ω_γ for some $\gamma \in \mathscr{F}$. Let $\Delta_n \subset \Omega_\gamma$ be a sequence of open balls with measure $\nu(\Delta_n) > 0$ and $\lim_{n\to\infty} \nu(\Delta_n) = 0$. If $\gamma \in L^2(\Sigma_A, \sigma)$ then there is a sequence of positive real numbers $(c_n)_{n\in\mathbb{N}}$ such that the process of return times given by*

$$\tau_n(\omega) = \sum_{k\in\mathbb{N}} \delta_{\tau_n^{(k)}(\omega)c_n}$$

converges in distribution to a Poisson point process of rate 1.

These results will extend naturally to give a Poisson limit law for the return time processes to a sequence of any open sets Δ_n and not just open balls, so long as the return time process of the discrete system given by r_n also converges to a Poisson process.

3.5 Hitting times to the Self-Similar Model

Recall the self-similar model (see [8] and section 1.2), creating a suspended flow over the unit interval, starting with the interval $I = [0,1)$ and the doubling map $T : I \to I : x \mapsto 2x \mod 1$, with a T-invariant measure μ. Define a sequence of sets by the following construction: $A_0 = \left[\frac{1}{4}, \frac{3}{4}\right]$ is the centre half of I. A_1 is then the union of the centre halves of each disjoint interval of $I \setminus A_0$, that is $A_1 = \left[\frac{1}{16}, \frac{3}{16}\right] \cup \left[\frac{13}{16}, \frac{15}{16}\right]$. A_2 is then the union of the centre halves of each of the four remaining intervals in $I \setminus (A_0 \cup A_1)$ and A_n is the union of the centre halves of each of the 2^n remaining intervals in $I \setminus \bigcup_{k=0}^{n-1} A_n$. Continuing as such, it should be clear that $I = \bigcup_{n=0}^{\infty} A_n$. See Fig 3.2.

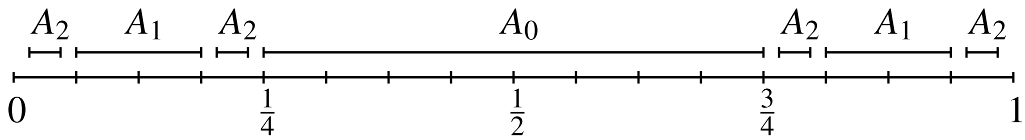

Figure 3.2: Showing the fractal construction of the sets A_0,A_1,A_2 in the unit interval.

For the suspended space use a height function $\gamma : I \to I$ given by $\gamma(x) = \lambda^n$ if $x \in A_n$ for some $\lambda > 1$ and call the blocks $A_n \times [0, \lambda^n] = \Delta_n$. The value of λ should be chosen appropriately so that $\int \gamma d\mu < \infty$. The suspended space is defined by

$$\Omega := \{\omega = (x,y) : x \in I, 0 \leq y < \gamma(x)\} = \bigcup_{n=0}^{\infty} \Delta_n,$$

(see Fig 3.3) and define a new measure on Ω by

$$\nu := \frac{\mu \times \text{Leb}}{\int \gamma d\mu}.$$

The suspension flow is then given by

$$S_t(\omega) = S_t(x,y) := \left(T^{n(t)} x, y + t - \sum_{i=0}^{\eta(t)-1} \gamma(T^i x) \right)$$

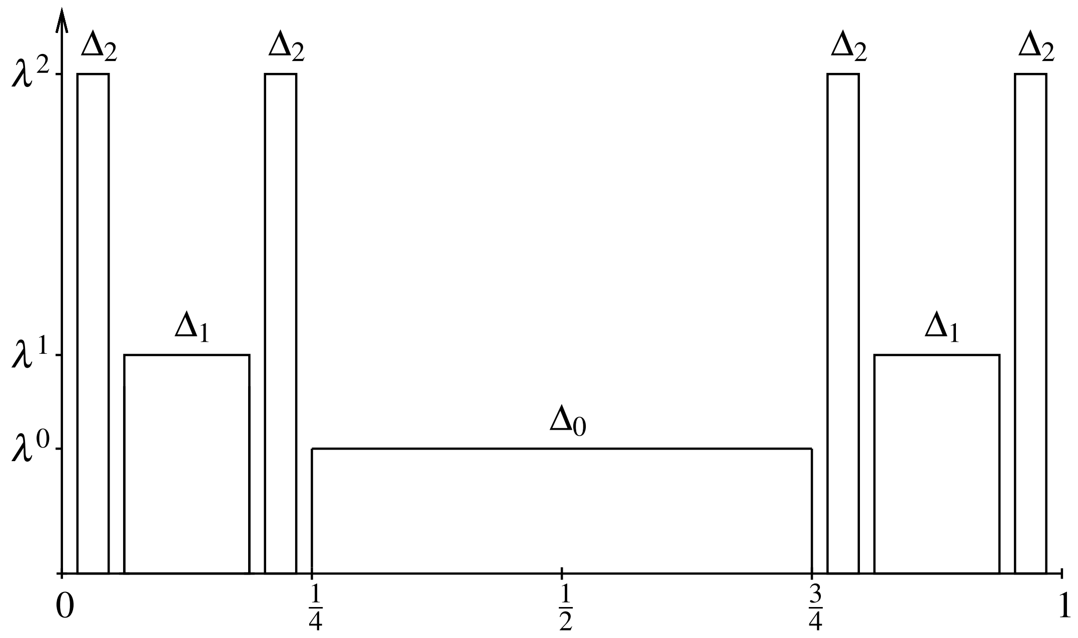

Figure 3.3: Showing the construction of the suspended space.

where $\eta(t)$ is the unique natural number which satisfies

$$0 \leq y + t - \sum_{i=0}^{\eta(t)-1} \gamma(T^i x) < \gamma(T^{\eta(t)} x).$$

Let $\pi : \Omega \to I$ be the projection given by $\pi(x,y) = x$. Recall theorem 2.6, that the hitting and return time process of the doubling map, $T : I \to I$, equipped with a T-invariant measure, to the sets $A_n = \pi(\Delta_n)$, rescaled by e^{-nP_Δ}, converges in law to a Poisson point process with rate 1.

In order to consider the hitting time process of the full suspension flow of the system, the problem is very similar to that handled in the previous sections. The aim is to use the convergence of the base process to get a convergence of the full process. The principle difference here is that instead of having a continuous height function, the height funtion is only integrable and is discontinuous on a set of measure zero.

As before it is possible to write the hitting times as two components, as shown in Lemma 3.2, as

$$\tau_n^{(k)}(\omega) = X_n^{(k)}(x) Y_n^{(k)}(\omega)$$

where $X_n^{(k)}$ is the hitting times in the base system, and

$$Y_n^{(k)}(\omega) = Y_n^{(k)}(x,y) = \frac{1}{X_n^{(k)}} \left(\sum_{i=0}^{X_n^{(k)}-1} \gamma(T^i x) - y + h(T^{X_n^{(k)}} x) \right)$$

with $h : X \to \mathbb{R}$ a positive measurable function such that $0 \leq h(x) \leq \gamma$. Recall the modified version of Slutsky's Theorem, which is given in Theorem 3.10, which states that since $Y_n^{(k)}$ converges in probability to a constant $\int \gamma d\mu$, then τ_n converges in law with respect to the measure μ.

The proof of the final theorem requires more care now that the height function γ can take infinite values.

Theorem 3.21. *The hitting time process for the self similar model, given by*

$$\tau_n(\omega) = \sum_{k \in \mathbb{N}} \delta_{\tau_n^{(k)}(\omega) c_n}$$

converges to a Poisson point process of rate 1, where $c_n = e^{-nP_\Delta} \int \gamma d\mu$.

Proof. First check the criteria for Theorem 3.10 are satisfied. Lemma 3.2 states that $\tau_n^{(k)} = Y_n^{(k)} X_n^{(k)}$, and by lemma 3.4, and noting the Birkhoff ergodic Theorem applies for any integrable γ, $Y_n^{(k)}$ converges almost surely to $\int \gamma d\mu$ as $n \to \infty$, for any $k \in \mathbb{N}$. It therefore follows that $\tau_n(x,y)$ converges in distribution (with respect to the measure μ) to a point process of rate 1, which will be denoted τ.

Now, for any continuous function $g : \mathbb{R} \to \mathbb{R}$ with compact support, consider the sequence of random variables given by

$$N_n(g)(\omega) = N_n(g)(x,y) = \int_0^\infty g(t)\, d \left(\sum_{k=0}^{\infty} \delta_{\tau_n^k(\omega) c_n} \right).$$

Theorem 3.10 states that

$$\int_\Sigma N_n(g)(x,y)\, d\mu \to \int_\Sigma N(g)(x,y)\, d\mu$$

where

$$N = N(\omega) = \int_0^\infty g(t)\, d \left(\sum_{k=0}^{\infty} \delta_{\tau^k(\omega)} \right).$$

To show convergence in distribution with respect to the full measure

$$\nu = \frac{\mu \times \mathrm{Leb}}{\int \gamma d\mu}$$

it is sufficient to show convergence of the integral

$$\int_\Omega N_n(g)(\omega)\, d\nu(\omega).$$

Consider

$$\begin{aligned}\int \gamma d\mu \int_\Omega N_n(g)(\omega)\, d\nu(\omega) &= \int_\Omega N_n(g)(\omega)\, d\mu(x) \otimes d\,\mathrm{Leb}(y) \\ &= \int_\Sigma \left(\int_0^{\gamma(x)} N_n(g)(x,y)\, d\,\mathrm{Leb}(y) \right) d\mu(x).\end{aligned}$$

The internal integrals can be bounded above and below. First consider a lower bound: The bounds are clear in the case γ is finite, however they are not where γ is infinite. Fortunately the set of discontinuities for γ has measure zero and so the infinite values of γ can be ignored as it is still integrable. Therefore, except on a set of μ-measure zero, the number of hits will be minimised by considering a starting point, $(x,0)$. That is

$$\int_\Sigma \gamma(x) N_n(g)(x,0)\, d\mu(x) \leq \int_\Sigma \left(\int_0^{\gamma(x)} N_n(g)(x,y)\, d\,\mathrm{Leb}(y) \right) d\mu(x).$$

Similarly, except on a set of μ-measure zero, the number of hits will be maximised by considering the highest starting point, $(x,\gamma(x)) = (Tx,0)$, so

$$\int_\Sigma \left(\int_0^{\gamma(x)} N_n(g)(x,y)\, d\,\mathrm{Leb}(y) \right) d\mu(x) \leq \int_\Sigma \gamma(x) N_n(g)(Tx,0)\, d\mu(x).$$

To understand the convergence of

$$\int_\Sigma \left(\int_0^{\gamma(x)} N_n(g)(x,y)\, d\,\mathrm{Leb}(y) \right) d\mu(x)$$

it is therefore enough to understand the convergence of

$$\int_\Sigma \gamma(x) N_n(g)(x,0)\, d\mu(x)$$

and

$$\int_\Sigma \gamma(x) N_n(g)(Tx,0)\, d\mu(x).$$

Consider the first integral. The two random variables are asymptotically independent as the time interval for counting hits (weighted by g) will expand, and therefore by the same arguement used in the proof of Theorem 3.11, using Lemma A.1, the value of the height function at the starting point and the number of hits are independent in the limit. Therefore

$$\begin{aligned}\lim_{n\to\infty}\int_\Sigma \gamma(x)N_n(g)(x,0)\,d\mu(x) &= \lim_{n\to\infty}\int_\Sigma \gamma(x)\,d\mu(x)\lim_{n\to\infty}\int_\Sigma N_n(g)(x,0)\,d\mu(x)\\ &= \int_\Sigma \gamma(x)\,d\mu(x)\int_\Sigma N(g)(x,0)\,d\mu(x).\end{aligned}$$

Similarly the same result will hold for the upper bound and

$$\begin{aligned}\lim_{n\to\infty}\int_\Sigma \gamma(x)N_n(g)(Tx,0)\,d\mu(x) &= \int_\Sigma \gamma(x)\,d\mu(x)\int_\Sigma N(g)(Tx,0)\,d\mu(x)\\ &= \int_\Sigma \gamma(x)\,d\mu(x)\int_\Sigma N(g)(x,0)\,d\mu(x).\end{aligned}$$

The upper and lower bound both converge to the same limit, therefore

$$\lim_{n\to\infty}\int_\Sigma\left(\int_0^{\gamma(x)} N_n(g)(x,y)\,d\mathrm{Leb}(y)\right)d\mu(x) = \int_\Sigma \gamma(x)\,d\mu(x)\int_\Sigma N(g)(x,0)\,d\mu(x),$$

and so τ_n converges in ν-distribution to a Poisson process of rate 1. □

Therefore we have found a suitable rescaling sequence for a Poisson limit law for the hitting time processes to the sets given in this self-similar model.

Chapter 4

Conclusion

This thesis has focussed on the limiting behaviour of hitting and then return time processes of flows to a sequence of sets with asymptotically small measure. Chapter 1 introduced a motivating example and notations being used in this work, before exploring some of the known results in discrete systems.

Chapter 3 created the link between continuous and discrete systems by considering suspension flows and axiom A flows: flows which could be identified with suspended flows making use of results by Bowen and Ruelle. Using this link I showed that for axiom A flows, if the rescaled hitting time process for an associated discrete system had a Poisson limit law of rate 1 then it would follow that the rescaled hitting time process for the flow would also have a Poisson limit law, with rate

$$\frac{1}{\int \gamma d\mu},$$

the integral of the height funtion for the suspended space.

I then showed, using similar techniques, that if the height function γ that describes the suspended space is L^2 then the rescaled return time process for the flow has a Poisson limit law if the rescaled return time process also has a Poisson limit law. The extra condition that $\gamma \in L^2$ was used to ensure that the conditional probabilities behaved in such a way as to give a property analagous to convergence in probability. A natural next question is whether L^2 is a necessary condition for this property to hold and whether it is necessary for the Poisson limit law to hold.

By making use of previous results for Poisson limit laws of hitting time processes and return time processes in discrete dynamics, the new results in chapter 3 can be extended to give Poisson limit laws for hitting and return time process for a wide variety of axiom A flows.

Theorem 2.6 gave a Poisson limit law for the hitting time process of the discrete system as described in Floriani and Lima's self-similar model. By then applying similar working I showed that the hitting time process for the continuous suspension flow in this model also had a Poisson limit law. This expanded on the results produced in the original paper to give an explicit limit law, and found suitable rescaling constants.

The main results in this thesis have focussed only on axiom A flows so that they can be identified with suspended flows and hence described using discrete systems. Next steps and further work could include looking for similar limit laws for hitting and return time processes for a wider class of flows, and whether axiom A is a necessary condition on flows to achieve a Poisson limit law.

Other work to note in the area is that of Rousseau [17], which studies in particular recurrence rates of Anosov flows, linking the Poincare recurrence rate with the dimension of the local measure.

Appendices

Here we will prove an asymptotic independence of two random variables, which is used in the proofs of theorem 3.11, theorem 3.19 and theorem 3.21. Asymptotic independence is independence in the limit between sequences of pairs of random variables where each finite pair may not be independent.

Let $g : \mathbb{R} \to \mathbb{R}$ be continuous with compact support and using the notation used throughout this thesis let

$$N_n(g)(\omega) = N_n(g)(x,y) = \int_0^\infty g(t)\, d\left(\sum_{k=0}^{\infty} \delta_{\tau_n^{(k)}(\omega)c_n}\right)(t) = \sum_{k=0}^{\infty} g(\tau_n^{(k)}(\omega)c_n).$$

Lemma A.1. *The random variables $g(\tau_n^{(k)}(\cdot,0)c_n)$ and $\gamma : \Sigma \to \mathbb{R}$ are asymptotically independent for any $k \in \mathbb{N}$. That is to say that if F_n and F_γ are the cumulative distribution functions of $g(\tau_n^{(k)}(\cdot,0)c_n)$ and γ respectively and $F_{n,\gamma}$ the joint distribution function then*

$$\lim_{n\to\infty} F_{n,\gamma} = \lim_{n\to\infty} F_n F_\gamma.$$

Proof. For $a,b > 0$ we will show that the conditional probability

$$\mathbb{P}\left(g(\tau_n^{(k)}(x,0)c_n) \in (0,a) | \gamma(x) < b\right)$$

converges as n tends to infinity to

$$\mathbb{P}\left(g\left(\tau^{(k)}(x,0)\right) \in (0,a)\right).$$

It follows that

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{P}\left(g(\tau_n^{(k)}(x,0)c_n) \in (0,a) | \gamma(x) < b\right) \\
&\quad = \lim_{n\to\infty} \mathbb{P}\left(c_n \left(\sum_{i=1}^{r_n^{(k)}-1} \gamma(T^i x) + h(T^{r_n^{(k)}} x)\right) \in g^{-1}(0,a) | \gamma(x) < b\right) \\
&\quad = \lim_{n\to\infty} \mathbb{P}\left(c_n r_n^{(k)} \left(\frac{1}{r_n^{(k)}} \sum_{i=1}^{r_n^{(k)}-1} \gamma(T^i x)\right) \in g^{-1}(0,a) | \gamma(x) < b\right) \\
&\quad = \mathbb{P}\left(r^{(k)} \int \gamma\, d\mu \in g^{-1}(0,a) | \gamma(x) < b\right) \\
&\quad = \mathbb{P}\left(r^{(k)} \int \gamma\, d\mu \in g^{-1}(0,a)\right).
\end{aligned}$$

For the last equality notice that $\int \gamma d\mu$ is a constant and that $r^{(k)}$ does not depend on γ.

$\square$

Bibliography

[1] R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms*, Lecture notes in Mathematics, Vol. 470, Springer-Verlag (1975)

[2] R. Bowen, D. Ruelle, *Ergodic Theory of Axiom A Flows*, Inventiones math. 29, 181-202 (1975)

[3] L. Breiman, *Probability*, Addison-Wesley Series in Statistics (1968)

[4] J. R. Chazottes, Z. Coelho, P. Collet, *Poisson processes for subsystems of finite type in symbolic dynamics*, Stochastics and Dynamics Journal 9(3):393-422 (2009)

[5] Z. Coelho, *Asymptotic laws for symbolic dynamical systems*, in Topics in Symbolic Dynamics and Applications, Eds. F. Blanchard, A. Maass and A. Nogueira, LMS Lecture Notes Series 279, Cambridge Univ. Press, (2000) 123-165

[6] Z. Coelho, *The loss of tightness of time distributions for homeomorphisms of the circle*. in Transactions of the American Mathematical Society, Vol. 356, No. 11, (2004) p.4427-4445

[7] D. Daley, D. Vere-Jones, *An Introduction to the Theory of Point Processes* Springer Series in Statistics, Springer-Verlag (1988)

[8] E. Floriani, R. Lima, *Statistics of Return Times in a Self-Similar Model*, American Institute of Physics: Chaos 9, 715-721 (1999)

[9] A. Gut, *Probability: A Graduate Course*, Springer Texts in Statistics, Springer (2005)

[10] B. Hasselblatt, A. Katok, *Introduction to the Modern Theory of Dynamical Systems*, Cambridge Univ. Press (1995)

[11] M. Hirata, *Poisson law for Axiom A diffeomorphisms*, Ergodic Theory and Dynamical Systems, 13, pp 533-556 (1993)

[12] G. Keller, *Equilibrium States in Ergodic Theory*, LMSST42, Cambridge Univ. Press (1998)

[13] H.B Mann, A. Wald, *On stochastic limit and order relationships*, The Annals of Mathematical Statistics 14 (3): 217-226 (1943)

[14] B. Pitskel, *Poisson limit law for Markov chains*, Ergodic Theory and Dynamical Systems, 11, pp 501-513 (1991)

[15] M. Pollicott, M. Yuri, *Dynamical Systems and Ergodic Theory*, LMSST40, Cambridge Univ. Press (1998)

[16] R.-D. Reiss, *A Course on Point Processes*, Springer Series in Statistics, Springer-Verlag (1993)

[17] J. Rousseau, *Recurrence Rates for Observations of Flows*, Ergodic Theory and Dynamical Systems, 32, pp1727-1751 (2012)

[18] S. Smale, *Differentiable Dynamical Systems*, Bull. A.M.S. 73, 747-817 (1967)