A Thesis

entitled

APPLICATIONS OF THE THEORY OF INTERNAL STRESSES IN CRYSTALS

by

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Summary

The study of the stress concentrations which exist at notches and cracks is of considerable importance in the understanding of the fracture of metals. The way in which these concentrations of stress are relieved by plastic deformation is not readily understood in terms of the classical theory of the elastic plastic solid since analytical solutions are obtained only in the simplest situations.

In this work a simple model of the relaxation process is considered, in which the crack and yielded regions are represented in terms of linear dislocation arrays. Alternatively the medium may be considered everywhere elastic and the cracks and yielded regions represented by arcs across which the stress is prescribed and relative displacements are permitted.

First the relaxation from a sharp isolated crack in an infinite medium is treated for conditions of plane strain and antiplane strain. In antiplane strain this provides a model of the relaxation round a surface notch in a semi infinite medium. Simple expressions are obtained for the relation between the yield stress, the applied stress, the relative displacement in the crack tips and the extent of the plastic zones.

The effect of free surfaces or of neighbouring cracks is considered by expanding the analysis to consider an infinite periodic coplanar array of identical cracks. It is

shown that the free surface causes plastic zones to spread more rapidly with increasing stress. The displacements for a given length of plastic zone are then reduced. If a critical displacement criterion is adopted for the initiation of fracture at a notch, then neglecting the effect of the free surface is shown to err on the safe side.

The effect of workhardening is also considered. An integral equation is obtained for the displacements and this is inverted numerically.

Finally a model of a tensile crack is treated in which the plastic zones from a single tip are represented by two linear arrays of dislocations inclined symmetrically to the plane of the crack. The applied tension is normal to the crack. Again this problem is treated numerically and preliminary calculations have been carried out to obtain the important relationships.

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Chapter I

Introduction

1.1 Modes of Fracture

It is well known that a metal subjected to high stresses will break across some surface passing through the material. Examination of these fracture surfaces reveals that fracture occurs in one of several modes which are not mutually exclusive. Fracture may accur as the metal breaks along planes of low energy in the crystalline grains, revealing bright facets in the surface. These planes are the cleavage planes of the crystal and we refer to this mode as cleavage fracture. Cleavage fracture is normally accompanied by only small amounts of plastic deformation and does not require high energies for its operation.

However a ductile material may fracture only after extensive plastic deformation which extends over some surface. The metal slides apart over this surface and the fracture has a dull, mat, or fibrous appearance. In this mode, ductile fracture, a considerable amount of energy is absorbed. There are a number of mechanisms by which ductile fracture may propogate. A single crystal may slide apart over a slip plane. In a polycrystal, pulled in tension, necking will occur and one or more holes will open in the middle. The stresses become concentrated on planes inclined at 45° to the tensile axis and the metal will fail by plastic sliding over these planes giving the familiar cup and cone fracture. The fracture process in the centre is a kind of internal necking starting from pre-existing holes, or inclusions around which holes may form.[1 2] These holes join up and often the fracture surface has a dimpled appearance where necking is initiated at many such inclusions.[3]

In a third mode the fracture surface may follow the grain boundaries. This is called intergranular fracture, and it can arise when there are metallurgical weaknesses in the boundary. Such weaknesses may, for example, be due to the presence of precipitates in or near the grain boundaries, or to the migration of vacancies or foreign atoms[4].

Failure may also occur by a process known as fatigue. This process does not easily classify into one of the above groupings. Fracture occurs after prolonged application of low cyclic stresses. The measurable plastic deformation is normally small but many reversals of plastic strain occur and the absolute sum of the strain increments will be large.

Failure is by cracking, one or more cracks being initiated in the specimen, and initially the crack growth is slow.

Cleavage fractures normally move rapidly through the structure but slow moving cleavage cracks are observed[5]. Cleavage fracture is frequently referred to as brittle fracture but this term should only be used when such fractures absorb little energy and move rapidly with small plastic deformation.

It is usual, but not general, for a cleavage fracture to be brittle in this sense. On the other hand ductile fracture is normally slow moving but here again catastrophic ductile failure is not unknown.

The problems of fracture have been extensively investigated and accounts of the work are given by Parker (1959)[6], Biggs (1960)[7]

1.2 Theoretical Strength

Early workers assumed that cleavage fracture would occur when the tensile stress exceeded some critical value. This critical stress has been calculated for the ideal lattice by considering the energy equations. (Polanyi (1921)[8], Orowan (1949)[4]. A rough estimate of the fracture strength may be obtained by the following simple argument.

Suppose that the material obeys Hooke's Law up to the fracture strength σ_m. Then the strain energy per unit volume of material is

$$\sigma_m^2/2E \qquad 1.2.1$$

where E is Youngs Modulus

The energy per unit area contained between two neighbouring planes of atoms a distance 'a' apart is then

$$a\,\sigma_m^2/2E \qquad 1.2.2$$

These planes will separate if this energy is equal to the energy of the free surfaces created. This will then give as the fracture stress

$$\sigma_m \sim \sqrt{E\gamma/a} \qquad 1.2.3$$

where γ is the energy per unit area of free surface. The relation 1.2.3 predicts a fracture stress which exceeds the observed strength by a factor ranging from 20 - 1,000. Detailed calculations have been carried out and stresses of the same high order as those given by relation 1.2.3 are predicted[9, 10]. This calculated value is generally known as the theoretical strength. Faced with this discrepancy between observed and calculated values of the fracture stress, two possibilities presented themselves. Either the calculations were wrong, or there is, in a real material, some mechanism which produces concentrations of stress of the order of the theoretical strength. Weaknesses such as micro-cracks will produce such internal stress, and cracks had been considered by Griffith (1920)[11] working on a related problem.

Similar calculations of the shear stress by Frenkel (1926)[12] also predicted a strength in excess of the observed value. Further it was shown by Taylor et al (1925)[13] that the energy expended in deforming a metal plastically does not all reappear as heat and the metal work hardens. These observations could not be explained retaining the assumption that metals are perfect crystals. Consequently Taylor (1928)[14] suggested that the dislocation was a suitable mechanism to raise the stress and elastic energy and to propogate slip. The dislocation concept had been suggested earlier[15] [16] and has since been developed[17] [18] [19] [20] [21]. A detailed account of the theory may be found in one of several books on the subject[22] [23].

1.3 Extension of a Completely Brittle Crack

Griffith (1920)[11] pointed out that micro cracks or surface scratches may be points of weakness in a material which cause dangerous concentrations of stress even when the applied stresses are within the elastic range. He was led to this conclusion by the fact that the incidence of rupture by fatigue can be reduced by polishing the surface of the specimen.

Further, some theoretical work suggested that surface grooves could increase the stresses and strains by a factor ranging from two to six and that this factor was not dependent upon the absolute size of the grooves.[24] The calculations were verified by experiments with wires containing spiral scratches. Applying these results to the existing fracture criteria however did not produce physically sensible conclusions.

Thus Griffith was prompted to investigate the behaviour of cracks and to formulate a new fracture criterion. A crack was considered as the limiting case of an elliptical hole and using the stresses derived by Inglis (1913)[25] an energy condition for the extension of the crack was obtained.

For the plane strain problem Griffith gave as the fracture stress :

$$\sigma = 2\sqrt{\{\mu\gamma/\pi\nu c\}} \qquad 1.3.1$$

and for plane strain :

$$\sigma = \sqrt{\{2E\gamma/\pi\nu c\}} \qquad 1.3.2$$

where μ is the modulus of shear

ν poissons ratio

2c the crack length

In a note to this paper Griffith draws attention to an error in the calculation. The correct result for plane strain should be :

$$\sigma = 2\sqrt{\{\gamma\mu/\pi(1-\nu)c\}} \qquad 1.3.3$$

The Griffith relationship has long been fundamental in the theory of fracture and has been derived subsequently by many workers. Sack (1946)[26] has made a similar calculation for a penny shaped crack and finds that this leads to the relation :

$$\sigma = \sqrt{\{\pi E\gamma/2c\ (1-\nu^2)\}} \qquad 1.3.4$$

Considering directly the stress at the crack tip, and setting this equal to the theoretical fracture stress over a region of atomic dimensions, Orowan (1949)[4] found the general agreement with the above relations.

Although the Griffith theory appears to explain the behaviour of glass with some success, its application to metals is limited. The theory predicts that cracks several mm. in length would be stable and it is difficult to see how such cracks could escape detection in specimens which have exhibited brittle behaviour. Before cleavage fracture can occur in a metal it is therefore necessary to nucleate a crack of a suitable size. This being so it is possible that the fracture strength of the material is related to the stresses required to nucleate a crack. These stresses will depend upon the mechanism of nucleation and several mechanisms have been proposed.

1.4 Formation of Cracks

Zener (1948)[27] suggested that a crack may form under the stress concentrated at the front of a row of dislocations piled up at a precipitate and Stroh (1954)[28] (1955)[29] has considered in detail a number of similar models.

In the first paper he considers the number of dislocations in a pile up required to produce, over some plane, a tensile stress sufficient to cause fracture. In the second he considers the number of dislocations required to force the leading dislocations together and form a wedge.

According to Stroh, the plane most favourably oriented for fracture is inclined at about 70° to the active slip plane and the potential energy of the medium will be reduced as this plane separates provided that the shear stress in the slip plane exceeds σ_s where

$$\sigma_s^2 = 3\pi\gamma\mu/8(1-\nu)L \qquad 1.4.1$$

In this equation L is the length of the pile up. This relationship does not contain the crack length and therefore the crack will grow at least over that length for which the approximations are valid. Now the number of dislocations in a pile up is given by[30]

$$L = \mu bn/\pi(1-\nu)\gamma \qquad 1.4.2$$

In this b is the Burgers vector of a dislocation and is of the order of the atomic distance. It then follows that the number of dislocations required to initiate a crack is given by

$$n\sigma_s b = 3/8\,\pi^2\gamma \qquad 1.4.3$$

In the second model Stroh considers the alternative condition for fracture that the distance between the two leading dislocations should be less than the atomic spacing. This will be so if the applied stress exceeds σ_s where

$$\sigma_s = 3.67\ \mu/4\ \pi\ n\ (1 - \nu) \qquad 1.4.4$$

and this agrees with equation 1.4.3 if we make use of the empirical relation :

$$Gb/\gamma \sim 8 \qquad 1.4.5$$

A crack nucleated by these mechanisms in a specimen subjected to an applied simple tension will propogate through the specimen since it is shown by Stroh (1957)[31] that under these conditions the number of dislocations required to initiate a crack exceeds the number of dislocations required for its propogation. The calculation assumes however that the energy per unit area of free surface is the same in both processes and this need not be the case as we shall see later.

Another important mechanism was postulated by Cottrell (1958)[32] in which a wedge crack forms on a (001) cleavage plane as certain dislocations on intersecting (101) $(10\bar{1})$ plane meet and coalesce. It is shown that this interaction reduces the elastic energy and it is supposed that this crack will grow as more dislocations move into the wedge. In principle this growth is not dependent upon the applied stress and will continue until the crack length is of the same order as the length of the slip lines. The crack will then propogate only if the applied stresses are suitable and this will not always be the case.

Burr and Thompson (1962)[33] have suggested a similar mechanism for zinc involving dislocations on the pyramidal and basal planes. Bell and Cahn (1948)[34] have observed cracks in zinc specimens pulled in tension but believe these cracks to have been formed by intersecting twins. However, it is suggested by Burr and Thompson that their dislocation mechanism and not twinning was the more probable mechanism of cracking in these experiments.

There is, nevertheless, considerable experimental evidence to suggest that cracks do form at intersecting twins[35-39]. Edmundson (1961)[38] working with iron single crystals below -145°C reports that no major cracks were detected which were not associated with a crack forming twin intersection.

Other mechanisms of crack formation are also discussed, Stroh (1958)[40] shows that in zinc the relative displacement of two parts of a twin boundary may lead to cracking in the basal plane. Further, Fisher (1955)[41] points out that vacancies are created when an edge dislocation cuts a screw which intersects in the slip plane. It is suggested that a crack may form if there are several such intersections.

All these mechanisms require at least localised plastic flow for their operation and Low[42] has confirmed that yield normally preceeds fracture. He showed that the yield stress in compression coincides numerically with the fracture stress in tension over a wide range of temperatureand other variables. To this extent all these models of crack initiation are satisfactory.

However, the Cottrell mechanism shows that the microcracks may be initiated rather easily during plastic deformation and it is supposed that they will only propogate if the applied stresses are suitable. If this is so one would expect to find stationary cracks in a yielded material and such observations have been made. Stationary cracks have been observed at the Luders front [43] and non propogating cracks in yielded regions but not in elastic regions which have been similarly stressed [44]. These observations are not consistent with a fracture theory in which crack nucleation is more difficult than crack propagation and this is an obstacle to theories based on the dislocation pile up mechanisms for crack initiation.

Another factor which operates against the pile up theory is that the macroscopic fracture criterion in such a theory depends upon the shear stress required to make dislocations coalesce. This implies that the fracture stress is independent of the hydrostatic stress, however, there is evidence to suggest [45] that hydrostatic stresses have a pronounced effect on ductile brittle behaviour and a theory which does not predict this cannot be entirely satisfactory.

1.5 The Propagation of Microcracks

It thus appears more reasonable to suppose that cracks are initiated easily during plastic deformation by a Cottrell type mechanism, twinning or some other process. Consequently the governing factor in brittle fracture will be the propagation of cracks and it is to be expected that hydrostatic stresses effect this process. The theory of crack propogation has been investigated by several workers and various modifications have been proposed [31] [32] [47-50]. Cottrell (1958)[32] considers the propagation of the wedge crack formed by his mechanism. He assumes the crack is subjected to normal tensile stresses and adopts a procedure similar to that used earlier by Stroh (1957)[31]. The potential energy of the system is written in the form :

$$W = 2\gamma \left[c_1 \ln \frac{4R}{c} + c - \frac{c^2}{2c_2} - 2c\sqrt{\frac{c_1}{c_2}} \right] \qquad 1.5.1$$

where

$$c_1 = \frac{\mu n^2 b^2}{8\pi(1-\nu)\gamma} \qquad c_2 = \frac{8\mu\gamma}{\pi(1-\nu)p} \qquad 1.5.2$$

p being the applied normal tension and R the effective radius of the stress field.

The equilibrium condition for the crack length is given by $\partial w/\partial c = 0$ and this is satisfied by the roots of the equation

$$c^2 - [1-2\left(\frac{c_1}{c_2}\right)^{1/2}]\, c\, c_2 + c_1 c_2 = 0 \qquad 1.5.3$$

If $p = 0$ then $c = c_1$. The quantity c_1 is therefore the length of the crack produced purely by dislocation interactions. Alternatively if $p > 0$ and $n = 0$, then the crack is a Griffith crack and c is equal either to c_2 or to zero. The quantity c_2 is therefore the unstable Griffith size. Further equation 1.5.3 has either no real roots or it has two real roots in which case the smaller gives the size of the stable crack and the larger the size of an unstable crack. By the usual analysis it follows that there will be no roots if :

$$\sqrt{(c_1/c_2)} > 1/4 \qquad 1.5.4$$

that is if :

$$p\, n\, b > 2\gamma \qquad 1.5.5$$

If this relationship is satisfied, there is no equilibrium length for the crack and it will spread catastrophically through the specimen. Otherwise the crack will grow until it reaches the stable size. Since cracks form at the yield stress a criterion for brittle behaviour will be obtained if the stress in relation 1.5.5 is set equal to the yield stress. The lower yield stress in mild steel is given by the well known relation :

$$\sigma_y = \sigma_i + Kd^{1/2} \qquad 1.5.6$$

in which σ_i is a stress opposing dislocation movement in the slip plane (friction stress), 2d is the grain diameter and K is a constant. To obtain nb in terms of the stresses the usual assumption is made :

$$nb = \alpha_1(\sigma - \sigma_i)\ d/\mu \qquad 1.5.6$$

where $\alpha_1 \sim 1$ is a constant. At the yield stress $\sigma = \sigma_y$ and from equation 1.5.6 it follows that

$$nb = \alpha_1 \quad Kd^{1/2} /\mu \qquad 1.5.7$$

The resolved shear stress may be obtained from the tensile stress using the general relation :

$$\sigma = \alpha_2\ p/2 \qquad 1.5.8$$

In the case of a simple tension $\alpha_2 = 1$ but in the presence of hydrostatic stresses this would not be the case. As before at the yield stress equation 1.5.6 gives :

$$p = 2(\sigma_i + Kd^{-1/2}) /\alpha_2 \qquad 1.5.9$$

Writing $\beta = \alpha_2/\alpha_1$ it then follows from 1.5.5, 1.5.8 and 1.5.9 that the condition for brittle failure of a material is :

$$\sigma_y \quad Kd^{1/2} > \beta\ \mu\ \gamma \qquad 1.5.10$$

$$\text{or } (\sigma_i \quad d^{1/2} + K)\ K > \beta\ \mu\ \gamma \qquad 1.5.11$$

If this condition is not satisfied cracks will form at the yield stress but will not propagate and the failure will be ductile. Thus there is a transition from ductile to brittle behaviour at the point where relations 1.5.10 or 1.5.11 are taken as equalities. The hydrostatic stresses enter this relation through the factor β. Increases in σ_i, d and K will tend to make the material brittle while increases in β, μ and γ will tend to make the material ductile.

The factor β for uniaxial tension is of the order unity but at the root of a notch the plastic constraint factor[50] causes a change so that $\beta \sim 3$. Therefore $\sigma_y K$ has to be reduced to preserve ductile behaviour. Estimates of the effective surface energy can be made from relations 1.5.10 and 1.5.11 by finding the parameters at which the material just becomes brittle. Using typical values for iron, Cottrell shows that the equations require a value of γ about 10 times the real surface energy. This high effective surface energy is interpreted as plastic work associated with the formation of the surface, for example tearing at river lines and grain boundaries[51] [52]. This idea that γ in the fracture criterion should take account of plastic work had been considered earlier by Orowan (1950)[54] Irwin (1949)[53] who give the fracture stress to be :

$$\sigma \sim (E\rho/c)^{1/2} \quad \sigma \sim (E\,\rho/c)^{1/2} \qquad 1.5.12$$

where ρ is the plastic work associated with the formation of the fracture surfaces rather than the true surface energy which is assumed negligible.

In all these theories the mathematical analysis contains no explicit discussion of the plastic relaxation which one would expect to find at the tip of a crack in a ductile material. There is evidence that high stresses are relieved in this way[54 55]. This plastic deformation may not affect these equations in only a simple way and the problem should be considered in more detail. Goodier and Field[56] have estimated the plastic work in these relaxed regions ahead of the crack and used this in relation 1.5.12. This procedure is suspect since the potential energy is derived assuming no such relaxation.

1.6 Objects of the Work

It is apparent that a full understanding of fracture requires some knowledge of the plastic deformation occuring near cracks and other similar inhomogeneities which act as sources of internal stress in metals. The analysis may be carried out using the classical macroscopic theory of the elastic plastic solid, but, except in very special cases extensive numerical work is required to exhibit the relations between the essential physical parameters involved.

An alternative approach is to simplify the model of a relaxing crack so that some important relations between these parameters may be determined analytically. Professor Cottrell has suggested a suitable model in which the plastic regions and the crack itself are represented by linear dislocation arrays.

This and related models of cracks and notches are treated in this work. In the analysis of these models, quantities will be referred to axes taken so that the crack lies in the plane $x_2 = 0$ with the crack tips parallel to the x_3 axis. These axes will be called standard axes. The material will be regarded as a perfectly elastic medium, relative displacements being permitted only across certain arcs in the x_1 , x_2 plane. These arcs, which will be called displacement arcs, will be used to represent the cracks and plastic regions. Yielded regions will be represented by arcs along which the stress is set equal to the yield stress and the cracks will be represented by arcs which are stress free. These arcs will be called the plastic arc and free arc, respectively. The elastic equations will then be solved subject to the condition that there are no singularities. The relative displacements may be represented by a continuous distribution of dislocations along the displacement arcs which may then be thought of as slip lines. Thus models may be treated as boundary value problems in an elastic medium or as dislocation problems.

The analysis will normally be carried out for plane strain shear and antiplane strain. By "Plane Strain Shear" it is to be understood that, referred to standard axes, the applied stress P at infinity and the stress σ_1 in the plastic arc are constant σ_{12} stresses, the displacements having the form $u_1(x_1, x_2)$ $u_2(x_1, x_2)$. Similarly by "Antiplane Strain" it is to be understood that the applied stress P at infinity and the stress σ_1 in the plastic arc are σ_{23} stresses, the only displacement being u_3 (x_1 , x_2).

Initially there is a discusion of the single crack in an infinite medium. The case of an infinite collinear sequence of cracks is discussed and in the antiplane strain case there are certain lines of symmetry perpendicular to the plane of the cracks which are stress free. The material may thus be divided along these lines to give models of a notched bar or a crack in the centre of a bar.

The Discussion considers, initially, only the non work-hardening material but in the final chapters the extension to include work hardening is considered for the single crack in an infinite medium.

Also a model is considered in which relaxation takes place along arcs not collinear with the crack. This is discussed only in the case of a single crack in an infinite material subjected to applied tensile stresses normal to the crack.

Finally some indication is given as to the manner in which these results may be applied to discuss theories of fracture.

Selected Work Related to the Relaxation Problem

In this chapter a discussion is given of some other theories of the plastic crack and of certain mathematical techniques which are to be used. However, the classical macroscopic theory of the elastic plastic solid is not used in this work and no attempt is made here to develop the theory. For an account of the classical theory reference may be made to Prager and Hodge (1951),[59] Hill (1956),[57] Thomas (1961),[58] or Johnson and Mellor (1962).[61] Further there exists an extensive bibliography compiled by Hodge (1958).[60] The only work involving the macroscopic theory to be included is a treatment of a long notch under conditions of antiplane strain. This treatment , which is due to Hult and McClintock (1957)[62] adequately demonstrates the complexity of classical methods applied to these problems. However, certain results correspond closely to the predictions based on simplified models.

Dugdale (1960)[63] considers a model of a crack based on a treatment of elliptical holes in an elastic medium developed by Muskhelishvili (1949).[67] This is a model of the type suggested in the previous section and as might be expected leads to conclusions similar to the results obtained from the Cottrell dislocation model.

Further consideration is given to the work of Leibfried (1951)[64] on the equilibrium of linear dislocation arrays, the dislocation density being derived in terms of an integral equation. This problem has also been considered by Head and Louat (1955)[65] who invert the integral equation using a general formula developed by Muskhelishvili (1946).[66] The derivation of this formula is given in the final section, the treatment being essentially that given by Muskhelishvili (1946)[66] (1949)[67] but including some simplification due to the writer. It will be seen later that this integral equation arises in other related theories and the method of inversion is fundamental to the work.

2.2 The Hult and McClintock Theory of a Notch

Hult and McClintock (1957)[62] have considered the plastic relaxation at the tip of a long sharp notch of angle θ and depth c in a semi infinite solid, fig (1). Standard axes are taken so that the material lies in the half space $x_1 \geq 0$ and the notch runs parallel to the x_3 axis. The system is symmetrical about the plane $x_2 = 0$. The analysis is carried out for the case of antiplane strain.

Let the stress components be σ_{ij}, the strain components ϵ_{ij} and the displacements u_i, where $ij = 1, 2, 3$. P is the applied stress at infinity which in this case is a uniform shear σ_{23}. Their analysis treates the problem as the limiting case of a longditudinal notch in a large cylinder subjected to torsion and follows the treatment of Prager and Hodge (1951).[59]

The general displacements throughout the medium are assumed to have the form :

$$\left.\begin{aligned} u_1 &= u_2 = 0 \\ u_3 &= (P/\mu)x_2 + \omega(x_1, x_2) \end{aligned}\right\} \qquad 2.2.1$$

where $\omega(x_1, x_2)$ is a warping function. The only stresses are σ_{23} and σ_{13} and therefore the only equation of equilibrium to be satisfied may be satisfied identically if the stresses are derived from a stress potential ϕ such that :

$$\partial \phi / \; x_1 = \sigma_{23} \qquad 2.2.2$$

$$\partial \phi / \; x_2 = - \sigma_{13} \qquad 2.2.3$$

$$\nabla^2 \phi = 0 \qquad 2.2.4$$

This is valid both in the elastic and the plastic regions. The criterion for yield is taken to be a maximum shear stress criterion :

$$\sigma_{13}^2 + \sigma_{23}^2 = k^2 \qquad 2.2.5$$

that is :

$$| \operatorname{grad} \phi | = K \qquad 2.2.6$$

Now ϕ, being a function of the two variables x_1 and x_2, may be represented by a three dimensional surface. The yield criterion 2.2.6 implies that this surface will be a surface of constant slope in the yielded region. Then the stress at any point, on the plane $x_3 = 0$, in the plastic region is normal to the direction of maximum slope of the ϕ surface.

Prager and Hodge show that ϕ is in general constant over any stress free surface, in which case the direction of maximum slope is perpendicular to that surface. In the case when the free surface bounding a plastic region is an arc of a circle, then the directions of maximum slope of the ϕ surface are radial lines. In the limit at a sharp corner the directions of maximum slope are straight lines radiating from that corner.

In polar co-ordinates (r, α) - with the origin at the notch tip and $\alpha = 0$ along the x_1 axis - the radius of the elastic plastic boundary is a function of α denoted by $R(\alpha)$. It is shown that in the elastic region :

$$\epsilon_{\alpha 3} \doteq R(\alpha)\, k/r\mu \qquad 2.2.7$$

The equation giving $R(\alpha)$ does not have a simple form even for the special case of $\theta = 0$. However the special value $R(0)$ may be expressed in terms of an integral and for the special case $\theta = 0$ it is easily shown that this reduces to an expression involving the complete elliptic integral of the second kind $E\,[\pi/2, Z]$.

Setting $\lambda = P/K$ then :

$$R(0) + c = [2c(1 + \lambda^2)/\pi(1- \lambda^2)]E[\pi/2, 2\lambda/(1 + \lambda^2)] \qquad 2.2.8$$

The procedure by which $R(0)$ is obtained involves an elaborate conformal mapping which is not given in the paper by Hult and McClintock. These mappings have been investigated by the author and the complete analysis is given in Appendix III to this work.

In the case of small λ a simple relationship is obtained, namely :

$$R(\alpha) = c\lambda^2 \cos \alpha \qquad 2.2.9$$

which is a circle of diameter $c\lambda^2$. From this theory which analyses only the most simple form of stress field it is apparent that only in the case of small applied stress does the elastic plastic boundary assume a simple form. Conditions of small applied stress are not the most useful for constructing a fracture theory.

Anything more complex than this involves the use of numerical techniques. Koskinen (1961)[20] has extended the above method to finite bodies and his results which are presented graphically will be compared with some results of this work in a later section.

2.3 The Dugdale Model of a Crack

Dugdale (1960)[63] proposed a model in which a relaxed crack is considered as the limiting case of an elliptical hole in an infinite elastic medium. Parts of the surface of the hole are stress free and collapse to form the crack. The remainder of the surface is subjected to a normal tensile stress Y, equal to the yield stress, and collapses to form the yielded region. The crack is of length 2c and the plastic regions are each of the length s. The stress field around elliptical holes, dying away at infinity is considered by Muskhelishvili (1949)[67].

This stress field is superimposed upon the stress field due to a crack of length 2(c + s) in an elastic medium subjected to a uniform tensile stress at infinity, the tensile axis being normal to the crack. This problem may also be considered as the limit of an ellipse and again the elliptical hole is treated by Muskhelishvili. The condition necessary to eliminate the stress singularity at the extremities of the ellipse is then derived. The following quantities are defined:

$$\left.\begin{aligned} x_1 &= a \cosh \alpha \\ \ell &= a \cos \beta \end{aligned}\right\} \qquad 2.3.1$$

Stresses are found in terms of a series about the point $x_1 = a$, that is $\alpha = 0$. For the case of the loaded elliptical hole the leading term in the series for the σ_{22} stress is :

$$-2Y\beta / \pi \alpha$$

For the case of the unloaded slit in a medium under an applied tensile stress P the corresponding expression is :

$$P/\alpha$$

The condition for finite stress is derived from the condition that the coefficient of $1/\alpha$ in the resulting series for σ_{22} should be zero. That is $P - (2Y\beta/\pi) = 0$. This expression reduces to the form :

$$c/a = \cos (\pi P/2\,Y) \qquad 2.3.2$$

Dugdale has conducted some experiments for internal and edge slits in tensile specimens and obtains substantial agreement with these predictions. These results are almost identical to those obtained by the present author using a dislocation model for the single crack and were published after the work using the dislocation model had begun.

2.4. The Equilibrium of a Continuous Distribution of Dislocations

Calculations using discrete dislocations are complicated and frequently a simplified theory is used. The actual distribution of dislocations each of Burgers vector b is replaced by a mathematical distribution of elementary dislocations each of Burgers vector δb. This mathematical distribution is chosen to give the same relative displacement over any sufficiently large region. The number of such dislocations is assumed to tend to infinity and the Burgers vector δb is assumed to tend to zero. Then any small region of the slip plane δx_1 containing x_1 encloses $N(x_1)\ \delta x_1$ elementary dislocations. The relative displacement over x_1 is given by

$$N(x_1)\ \delta b\ \delta x_1 \qquad 2.4.1$$

$N(x_1)$ is the density of elementary dislocations at x_1.
Writing :

$$D(x_1) = N(x_1)\ \delta b/b \qquad 2.4.2$$

the relative displacement over δx_1 is given by

$$D(x_1)b\ \ \delta x_1 \qquad 2.4.3$$

Thus $D(x_1)$ may be considered as the density, at x_1, of dislocations having Burgers vector b.
Take axes with their origin at the dislocation itself, and, in the case of edge dislocations set x_1 parallel to the Burgers vector. Then the stresses due to an edge dislocation are :

$$\sigma_{11} = -Ax_2(3x_1^2 + x_2^2)\ r^4 \qquad 2.4.4$$

$$\sigma_{22} = A\, x_2(x_1^2 - x_2^2)/r^4 \qquad 2.4.5$$

$$\sigma_{12} = A\, x_1(x_1^2 - x_2^2)/r^4 \qquad 2.4.6$$

where

$$A = \mu b/\, 2\pi\ (1 - \nu) \qquad 2.4.7$$

$$r^2 = x_1^2 + x_2^2 \qquad 2.4.8$$

The stresses due to a screw dislocation, with respect to polar coordinates (r,α), are :

$$\sigma_{r3} = A/r \qquad 2.4.9$$

$$A = \mu b/2\pi \qquad 2.4.10$$

In these relations μ is the shear modulus and ν Poissons ratio. The stress due to a similarly oriented dislocation at $x_1 = x_1'$ is thus given by

$$\sigma(x_1) = A/(x_1 - x_1') \qquad 2.4.11$$

where

$$\left.\begin{array}{l}\sigma \text{ is a } \sigma_{23} \text{ stress, } A = \mu b/2\pi \text{ for screws}\\ \sigma \text{ is a } \sigma_{12} \text{ stress, } A = \mu b/2\pi(1-\nu) \text{ for edges}\end{array}\right\} \qquad 2.4.12$$

The stress contribution due to a distribution $D(x_1')$ of dislocations in the small region $\delta x'$ is similarly :

$$\delta\sigma(x_1) = AD(x_1')\ \delta x_1' / (x_1 - x_1') \qquad 2.4.13$$

The stress due to a distribution over any large finite region L of the x_1 axis is :

$$\sigma(x_1) = \int_L [AD(x_1')/(x_1 - x_1')]dx_1' \qquad 2.4.14$$

Now let the resultant stress applied to the dislocation array be $p(x_1)$. This function gives the appropriate stress component and is positive if it moves a positive dislocation in the positive direction. Further $p(x_1)$ includes any resistance stress or friction. In order that the dislocations be in equilibrium under the applied stress $p(x_1)$ it is necessary that $p(x_1) + \sigma(x_1) = 0$ on L; that is the stress at any point of the dislocation is zero. The dislocations must adjust their positions so that :

$$A\int_L [D(x_1')/(x_1' - x_1)]dx_1' = p(x_1) \qquad 2.4.15$$

Leibfried (1951)[64] assumes that all the dislocations are of the same sign. Then for n dislocations, L = [-a,a] and uniform stress $p(x) = P$ applied to the dislocations. The dislocation density is shown to be :

$$D(x_1) = (n - Px_1/A)/\pi\sqrt{(a^2 - x_1^2)} \qquad 2.4.16$$

In general $D(x)$ will be infinite at $x = \pm a$ but solutions with $D(x)$ vanishing at one of these points may be obtained by suitable choice of n and a formula for a dislocation pile up at a rigid barrier is obtained. The distribution in two regions symmetrically placed about the origin (-a,-b) and (+b, +a) is then considered. If the dislocations in the first region are of opposite sign to the dislocations in the second, and the density is zero at $x_1 = \pm b$, then setting b = 0 a formula is obtained for a single region enclosing dislocations of both signs. The formula 2.4.16 is thus shown to apply in general. The dislocation density for a distribution of period $\ell/2$ of such pairs is obtained by interchanging the order of integration and summation and evaluating the infinite sum in the integrand. The relation is :

$$p(x_1) = (\pi/\ell) \int_b^a [AD(x_1') \sin (x_1'/\ell)/\{\cos(\pi x_1'/\ell)$$

$$- \cos(\pi x_1/\ell)\}] \, d x_1' \qquad 2.4.17$$

2.5 Inversion of the Singular Integral Equation.

It has been shown that in an array of dislocations, distributed continuously over a single slip line, the condition for zero stress at each point of the array is given by equation 2.4.15. In the case of a periodic array the condition is given by equation 2.4.17. These relations can all be reduced to the general form :

$$(1/\pi i) \int_L \{G(t)/t-t_o\}dt = \phi(t_o) \qquad 2.5.1$$

where the integral is taken over L, L being any set of n non intersecting arcs in the complex plane. On L, G(t) is an unknown complex function of a complex variable and $\phi(t)$ is a given complex function.

Singular integral equations of this type have been studied in detail by Muskhelishvili (1946)[66] and a method for finding a suitable G(t) is given. Part of the proof is summarised, Muskhelishvili (1949).[67]

In order to gain an understanding of the physical implications of the method, the underlying concepts of the analysis have been assembled here in a concise form.

For this purpose a simple procedure has been devised by which equations 2.5.4 and 2.5.8 are obtained from 2.5.1 and 2.5.2. This procedure is not strictly rigorous and is not that given by Muskhelishvili.

While the general treatment considers arbitrary arcs we shall assume such arcs to be segments of the real axis.

Preliminary formulae

Let F(z) be any complex function sectionally holomorphic in the complex plane outside L and let $F^+(t)$ and $F^-(t)$ be the limiting values of F(z) as $z \rightarrow t$ from the positive and negative sides of L respectively. The positive side of L lies to the left as one moves along the arc in the positive direction. Now consider the problem of finding F(z) satisfying, on L, the relation :

$$F^+(t_o) - F^-(t_o) = G(t_o) \qquad 2.5.2$$

where $G(t_o)$ is assumed to be known. A holomorphic F(z) may be found using the Cauchy theorem. If Γ is taken to be a rectangle containing L, then for z outside Γ :

$$F(z) = -(1/2\pi i) \int_\Gamma \{F(\zeta)/\zeta - z\} d\zeta \qquad 2.5.3$$

In the limiting case as Γ closes round L equations 2.5.3 and 2.5.2 give

$$F(z) = (1/2\pi i) \int_L \{G(t)/\ t-z\}dt \qquad 2.5.4$$

This function will be holomorphic outside L since Cauchy's theorem applies only for holomorphic functions. If F(z) is required with poles of orders $m_1, m_2, \ldots m_\ell$, m at the points $z_1, z_2, \ldots z_\ell$, ∞ not belonging to L then the function :

$$R(z) = \sum_{\ell=1}^{\ell} \sum_{j=0}^{i} a_{ij}\,(z-z_i)^{-m_j} + \sum_{i=0}^{m} a_i z^i \qquad 2.5.5$$

may be added to the R.H.S. of 2.5.4 and 2.5.2 will still hold since R(z) will be continuous over L and its positive and negative boundary values will be equal. Then the most general function satisfying 2.5.2 is :

$$F(z) = (1/2\pi i) \int_L \{G(t)/t-z\}dt + R(z) \qquad 2.5.6$$

where R(z) is a function continuous over L and having a finite number of poles. This is a most important general result which will be required later.

Reformulating the Problem

Now suppose that $G(t_o)$ is unknown and is a solution of 2.5.1. [Then $G(t_o)$ is directly related to the boundary value of the derivative of the stresses due to the dislocations and it follows that $F(z)$ will also be related to the derivative of the stresses throughout the material. If the stress is to have no singularities outside L then this condition must be imposed on $F(z)$]. $F(z)$ is taken to be sectionally holomorphic and zero at infinity. It is given by equation 2.5.4. Taking the limiting case of $F(t_o + i\delta) + F(t_o - i\delta)$ as $\delta \to 0$ gives the relation :

$$F^{+}(t_o) + F^{-}(t_o) = (1/\pi i) \int_L \{G(t)/t-t_o\} dt \qquad 2.5.7$$

and substituting from 2.5.1 the relation 2.5.7 becomes :

$$F^{+}(t_o) + F^{-}(t_o) = \phi(t_o) \qquad 2.5.8$$

The problem can now be reformulated as the problem of trying to find a holomorphic function satisfying 2.5.8. To do this a function is introduced holomorphic in the finite plane outside L and with the property $\chi^{+}/\chi^{-} = -1$. Further, functions $f = F/\chi$ and $g = \phi/\chi^{+}$ are defined. Relation 2.5.8 then becomes $f^{+} - f^{-} = g$ which is in the form 2.5.2

and thus has a solution in the form 2.5.6 since g is known.

Following Muskhelishvili equation 2.5.8 is now written in the form

$$F^{+}(t_o) - sF^{-}(t_o) = \phi(t_o) \qquad 2.5.9$$

and the following definition is made :

$$\chi_p(z) \equiv \prod_{k=1}^{n} (z-a_k)^{-\gamma} (z-b_k)^{\gamma-1} P_p(z) \qquad 2.5.10$$

Here a_k b_k are the end points of the kth arc comprising L and $P_p(z)$ is a polynomial of degree p with zeros on L. This polynomial is introduced in order to remove certain stress singularities on L and the form of $P_p(z)$ will be considered later.

Let z be any point on a circle centre a_k and radius r, then :

$$z - a_k = re^{i\theta} \qquad 2.5.11$$

As z moves around a_k from the negative to the positive side of L it follows that

$$[(t-a_k)^{-\gamma}]^{+} = e^{-2\pi i}\,[(t-a_k)^{-\gamma}]^{-} \qquad 2.5.12$$

Similarly it can be shown that :

$$[\,(t-b_k)^{\gamma-1}]^{+} = e^{2\pi i}\,[(t-b_k)^{\gamma-1}]^{-} \qquad 2.5.13$$

and it then follows that

$$\chi_p^{+}(t) = e^{2\pi i\gamma}\chi_p^{-}(t) \qquad 2.5.14$$

Now $\chi_{(z)}$ is holomorphic outside L except perhaps at infinity and also has the property

$$\chi_p^{+}(t)\,/\chi_p^{-}(t) = s \qquad 2.5.15$$

provided that :

$$\gamma = \log s/2\pi \qquad 2.5.16$$

The equation 2.5.10 therefore gives a $\chi_{(z)}$ of the required type.

Now define the following new functions :

$$f(z) = F(z)\,/\,\chi_p(z) \qquad 2.5.17$$

$$g(t_o) = \phi(t_o)\,/\,\chi_p^{+}(t_o) \qquad 2.5.18$$

The general solution

The equation 2.5.9 then becomes :

$$f^{+}(t_o) - f^{-}(t_o) = g(t_o) \qquad 2.5.20$$

in which $g(t_o)$ is a known function. The general solution to 2.5.20 is given by 2.5.6.

Now $\chi_p(z)$ will be constructed with the property that in the finite plane, all zeros of $\chi_p(z)$ belong to L. Further F(z) is required holomorphic everywhere outside L. Thus it follows from relation 2.5.17 that f(z) is holomorphic outside L except perhaps at infinity. This then implies that in the solution of 2.5.20 for f(z) the function R(z) in the general form 2.5.6 may be no more than a polynomial say $Q_m(z)$ of degree m.

From 2.5.17 and 2.5.20 it then follows that :

$$F(z) = \{\chi_p(z)/2\pi i\}\int_L \{\phi(t)/\chi^{+}(t)(t-z)\}dt + \chi_p(z)Q_m(z) \qquad 2.5.21$$

To obtain 2.5.8 from 2.5.9 it is necessary to set $s = -1$ and then from 2.5.16 it follows that

$$\gamma = 1/2 \qquad 2.5.22$$

Now $F(z)$ is not in general bounded at the end points of L but a solution bounded at a given set of ends $c_1 \ldots c_p$ may be obtained taking the polynomial $P_p(z)$ in equation 2.5.10 to be :

$$P_p(z) = \prod_{k=1}^{p} (z - c_k) \qquad 2.5.23$$

It is now convenient to define :

$$R_1 \equiv \prod_{j=o}^{p} (z-c_j) \qquad R_2 = \prod_{j=p+1}^{2n} (z-c_j) \qquad 2.5.24$$

where $c_1 \ldots c_p$ are the end points of L at which $F(z)$ is required bounded, $F(z)$ being unbounded at the remaining end points.

Setting $\gamma = 1/2$ it then follows from 2.5.24 and 2.5.23 that

$$\chi_p(z) = \sqrt{R_1(z)/R_2(z)} \qquad 2.5.25$$

and from 2.5.15 it follows that :

$$\overset{+}{\chi}_p(t_o) = -\overset{-}{\chi}_p(t_o) = \sqrt{R_1(t_o)/R_2(t_o)} \qquad 2.5.26$$

Then from equations 2.5.2, 2.5.21, 2.5.25 and 2.5.26 it follows that

$$G(t_o) = \frac{1}{\pi i}\left(\frac{R_1(t_o)}{R_2(t_o)}\right)^{1/2}\int\left(\frac{R_2(t)}{R_1(t)}\right)^{1/2}\frac{\phi(t)dt}{t-t_o}$$

$$+\left(\frac{R_1(t_o)}{R_2(t_o)}\right)^{1/2} Qm\,(t_o) \qquad 2.5.27$$

is the general solution to 2.5.1 where $Q_m(z)$ is an arbitrary polynomial of degree m and R_1, R_2 are given by equation 2.5.24.

Condition for finite stress

It remains only to examine F(z) at infinity and find the conditions for there to be no poles, since all unwanted poles have already been removed from the finite plane.

Expanding the integral in 2.5.21 in powers of t/z gives the relation :

$$F(z) = X_p(z)\left[\sum_{k=1}^{\infty}\{A_k/Z^k\} + Q_m(z)\right] \qquad 2.5.28$$

where

$$A_k = (1/2\pi i)\int_L \left\{-t^{k-1}\ \phi(t)/\chi_p^+(t)\right\} dt \qquad 2.5.29$$

Now let $O(r)$ denote a function which at infinity has the form :

$$O(r) = \sum_{i=-\infty}^{r} B_i z^i \qquad 2.5.30$$

Now from 2.5.10 it can be seen that :

$$\chi_p(z) = O(p-n) \qquad 2.5.31$$

and substituting into 2.5.28 it follows that :

$$F(z) = \sum_{k=1}^{\infty} O(p-n-k) + O(p-n+m) \qquad 2.5.32$$

If $F(z)$ is to be holomorphic at infinity it follows from 2.5.32 that :

$$\left.\begin{array}{lll} p-n-k & < & 0 \\ p-n+m & < & 0 \end{array}\right\} \qquad 2.5.33$$

These conditions may be stated as follows :

If n is the number of segments of L and p is the number of end points at which $F(z)$ is required to be bounded then :

$$\left.\begin{array}{l} \text{for } \; p \geq n, \quad Q_m(z) = 0, \quad A_k = 0 \text{ for } k = 1, 2, \ldots p-n \\ \text{and for } \; p<n, \quad m < n-p \end{array}\right\} \qquad 2.5.34$$

Summary

Thus the solution to 2.5.1 bounded at p of the $2n$ end points of L is given by 2.5.27 in which R_1 R_2 are defined by 2.5.24 and Q_m is an arbitrary polynomial of degree m, provided that the conditions 2.5.34 are satisfied in which A_k is defined by 2.5.29

Chapter III

The Isolated Crack

At the suggestion of Professor Cottrell the following model of the plastic relaxation round the tip of a sharp crack has been examined. Both the crack itself and the plastic regions are represented by linear dislocation arrays. The crack, or free arc, corresponds to a region of the slip plane in which there is no resistance to dislocation movement, and the plastic arc corresponds to a region in which the movement of dislocations will be opposed by a stress whose maximum value is equal to the yield stress. Provided that the applied stress increases monatonically and the dislocations move outward from the crack tips the resistance stress in the plastic arcs will oppose the applied stress and in fact be equal to the yield stress at each point.

The equilibrium of dislocation arrays has been discussed in Chapter 2, section 4. It is clear that one may discuss simultaneously the cases of screws and edges having their Burgers vector in the x_1 direction. These cases correspond to anti-plane strain and plane strain shear respectively.

With respect to standard axes having their origin at the centre of the crack, the crack lies in the region $|x_1| < c$, and the plastic arcs lie in the region $c < |x_1| < a$.

The magnitude of the applied stress will be P and the magnitude of the resistance stress will be σ_1. These stresses will be taken positive if they tend to move positive dislocations in the positive x_1 direction and the negative if they move positive dislocations in the negative x_1 direction. For edge dislocations P and σ_1 are σ_{12} stresses and

$$A = \mu b/2\pi\ (1 - \nu) \qquad 3.1.1$$

while for screw dislocations P and σ_1 are σ_{23} stresses and

$$A = \mu b/2\pi \qquad 3.1.2$$

In these relations μ is the shear modulus, ν Poisson's ratio and b the Burgers vector. The condition for equilibrium is then given by equation 2.4.15 in which

$$p(x_1) = \begin{cases} P & |x_1| < c \\ P - \sigma_1 & |x_1| > c \end{cases} \qquad 3.1.3$$

That is the dislocation density $D(x_1)$ is given by :

$$p(x_1) + A \int_L [D(x_1')/(x_1 - x_1')]dx' = 0 \qquad 3.1.4$$

Now if the following correspondence is set up :

$$\left.\begin{aligned} \varphi(x_1) &= P(x_1) \\ G(x_1) &= A\,\Pi i D(x_1) \end{aligned}\right\} \qquad 3.1.5$$

then the integral equations 3.1.4 and 2.5.1 are identical. However before proceeding to the solution given in 2.5 it is necessary to consider the validity of the procedure in this case since ϕ is a step function.

3.2 Inversion Procedure

In Chapter II, section 5, no attention is given to considerations of validity of the method. This is however considered by Muskhelishvili (1946). It is necessary, only, that the Cauchy principal valves of the integrals exist at each stage and a sufficient condition for this to be so is that the given function ϕ should satisfy the condition of the form :

$$|\phi(x_1) - \phi(x_1')| \leq K\,|x - x'|^{\mu} \qquad 3.2.1$$

where K, μ are positive constants.

It is clear that the step function $\sigma(x)$ defined by 3.1.3 does not satisfy this condition. It is therefore necessary to justify the use of these formulae when dealing with step functions. A brief argument is put forward here to show how step functions may be included. First approximate to the step function in the form :

$$p'(x_1) = \left.\begin{matrix} p(x_1), & |x \pm c| \geq \delta \\ P - (\sigma/2) - (|x| - c)\,\sigma/2\delta, & |x-c| \leq \delta \end{matrix}\right\} \qquad 3.2.2$$

see fig. (7).

This function will satisfy 3.2.1 for any finite δ and the required step function is the limiting value of a sequence of these functions as $\delta \to 0$. The required solution to the integral equations is then the limit of the sequence of solutions as $\delta \to 0$. Now suppose L is the single segment of the x_1 axis $|x_1| \leq a$, then from 3.1.4, 2.5.27 and 2.5.29 it is clear that in general the solution and its existance condition will include terms of the form :

$$S' = \int_L K(x,x')\, P'(x')\, dx' \qquad 3.2.3$$

where $K(x,x')$ is some function depending upon the conditions of the problem which may have only one singularity of the form $(1/\{x-x'\})$.

Suppose now that the dislocation density is required at x then choose $\delta < \delta_0$ where δ_0 is chosen so that

$$|x - c| > 2\delta_0 \quad > 2\delta \qquad 3.2.4$$

where c is the crack tip nearest to x.

Then for

$$|x' - c| < \delta \qquad 3.2.5$$

it follows that :

$$|x - x'| > |x - c| - |x' - c| > \delta_0 \qquad 3.2.6$$

Let the range of integration L be divided into subranges

$$\left.\begin{array}{l} |x - c| < \delta \\ |x - c| \geq \delta \end{array}\right\} \qquad 3.2.7$$

Denote by $L - \ell$ the union of the three ranges where both of $|x \pm c| > \delta$ then :

$$S' = \int_{\ell} K(x,x')p'(x')dx + \int_{L-\ell} K(x,x')p'(x')dx \qquad 3.2.8$$

Now ℓ is the union of the subranges where one of $|x \pm c| < \delta$ and in this range the integrand is bounded since the singularity has been removed to a sufficient distance by suitable choice of δ_0. It follows therefore that the first integral in 3.2.8 tends to zero as $\delta \to 0$ so that :

$$S' = \lim_{\delta \to 0} \int_{L-\ell} K(x\, x')p'(x')dx \qquad 3.2.9$$

Now S is defined by :

$$S = \int_{L} K(x\, x')\ p\ (x')dx' \qquad 3.2.10$$

From 3.2.2 it may be seen that $p'(x) = p(x)$ for $x \in L-\ell$ and substituting this relation into 3.2.9 S' is by definition equal to 3.2.10. This result is in fact equivalent to inverting the step function directly using relations 2.2.27 and 2.2.29. In this thesis this limiting process is implicit and will not be referred to directly. Analysis will be carried out as though the arguments of section 2.5 were valid for the step functions themselves.

3.3 Analysis for the single crack

In considering the solution to equation 3.1.4 it is first necessary to investigate the conditions for that solution to be bounded at the appropriate point. The dislocation density must be bounded at both the points $x_1 = +a$ and $x_1 = -a$, thus there is one crack and two bounded ends. In this case $p = 2$ and $n = 1$ so that the problem falls within the compass of the first of the conditions 2.5.34. There is one condition for a solution corresponding to $A_1 = 0$. Now from 2.5.24 it is clear that

$$\left.\begin{aligned} R_1^2(x_1) &= a^2 - x_1^2 \\ R_2^2 &= 1 \end{aligned}\right\} \qquad 3.3.1$$

and so the condition for a bounded solution becomes :

$$\int_{-a}^{a} [p(x_1) / \{\sqrt{(a^2 - x^2)}\}]\, dx_1 = 0 \qquad 3.3.2$$

It is convenient to make the following definitions :

$$R(x,y) = \{\sqrt{[(a^2 - x^2)/(a^2 - y^2)]}\} \{1/ x-y\} \qquad 3.3.3$$

$$H(a,x,y) = \cosh^{-1}\{|[(a^2 - x^2)/a(x - y)]+[x/a]|\}$$

3.3.4

The latter holding for all x,y contained in the interval $[-a,a]$ of the x_1 axis. The solution from 2.5.27 is then :

$$D(x_1) = (1/\pi^2 A) \int_{-a}^{a} p(y) \quad R(x_1,y)\, dy \qquad 3.3.5$$

Substituting for $p(x_1)$ from equation 3.1.3 the equation 3.3.2 gives rise to a relation for the extent of the plastic arcs :

$$c/a = \sin^{-1} (\pi/2)(1 - P/\sigma_1) = \cos (\pi/2)(P/\sigma_1) \qquad 3.3.6$$

This equation derives directly from the condition that the stresses are bounded at infinity and gives the relationship between the crack length and the length of the plastic zones. In the analysis of section 2.5 it is stated at the outset that solutions vanishing at infinity are required. To avoid poles at the ends of the arcs a multiplying factor is introduced which effectively removes the poles to infinity. To this extent the conditions 3.3.2 is equivalent to the condition that the stress shall be bounded at the ends of the plastic region, which is used by some other workers. (Smith, private communication, Dugdale, 1960).

As expected for large values of σ_1 or small applied stress P, c/a is of the order unity, that is there is only a small relaxation. Again in the case when $P \sim \sigma_1$ then $c/a \sim 0$ which implies that 'a' is large the plastic zones spreading great distances from the crack tip.

This relationship (3.3.6) for the spread of plasticity is compared numerically with the similar relationship obtained by McClintock and agreement is within 5% (fig. 9) In the case of small applied stress McClintock obtains a simpler equation for the elastic plastic boundary in the form :

$$r_\alpha = c(P/\sigma)^2 \cos \alpha \qquad 3.3.7$$

from which it follows that :

$$a/c = (r_0 + c)/c = 1 + (P/\sigma_1)^2 \qquad 3.3.8$$

Using the small angle approximation in 3.3.6 the current theory gives :

$$a/c = 1 + \pi^2/8\ (P/\sigma_1)^2 \qquad 3.3.9$$

The expression for the dislocation density 3.3.5 may be evaluated easily in terms of the integrals $I(\alpha,\beta)$ defined in Appendix I by relation A1.1.3 and evaluated in A1.4. The solution is :

$$D(x_1) = \{\sigma_1/\pi^2 A\}\{H(a,c,x) - H(a, -c, x)\} \qquad 3.3.10$$

where the function H is defined by 3.3.4 and its properties are considered in A1.2 and A1.3. The dislocation density is shown on fig (8) for the case when $(a/c) = 2$.

3.4 The Displacement Function

Now let $\Phi(x_1)$ be the relative displacement of the crack or plastic region. Then $\Phi(x_1)$ will be found by integrating the dislocation density from x_1 to the tip of the crack, and multiplying by the Burgers vector. This involves the integral $I_0(\alpha,\beta)$ defined in Appendix I by A1.1.4 and evaluated in A1.6. From A1.6.4 it follows that the relative displacement is :

$$\Phi(x_1) = (b\sigma_1/\pi^2 A)[(x_1 + c)H(a,-c,x_1) - (x_1-c)H(a,c,x_1)] \qquad 3.4.1$$

Finally for the displacement at c the second term vanishes by relation A1.5.3 so that :

$$\Phi(c) = (2cb\sigma_1/\pi^2 A) \ H(a,c,-c) \qquad 3.4.2$$

The displacement $\Phi(c)$ is shown in fig (15), (16).

It is of interest to examine the limiting value of this displacement as σ_1 becomes large. This limit is expected to be the displacement function for an unrelaxed crack which may be obtained from the results of Leibfried in 2.4. Taking standard axes with origin at the crack centre the dislocation density representing a crack of length 2c deduced from section 2.4 is :

$$(Px_1/\pi A)/\sqrt{(c^2 - x_1^2)} \qquad 3.4.3$$

and the corresponding displacement function is :

$$(Pb/\pi A)\sqrt{(c^2 - x_1^2)} \qquad 3.4.4$$

Now in considering the limit of $\Phi(x)$ given by 3.4.1 it is to be noted that for large σ_1, $a \rightarrow c$ so that the functions $H \rightarrow 0$. Thus Φ is the product of a large and a small function. It is necessary to express Φ as the quotient of two small functions so that the required limit follows as the limit of the quotient of the derivatives as $\sigma_1 \rightarrow \infty$.

Thus we write :

$$\Phi(x) = [(x_1+c)H(a,+c,-x_1)-(x_1-c)H(a,c,x)] / [\pi^2 A/\sigma_1 b] \qquad 3.4.5$$

in which the negative sign in the first H function has been transferred from c to x using A1.3.2. Now the numerator is a function of :

$$r = (c/a) = \cos\ [(\pi/2)(P/\sigma_1)] \qquad 3.4.6$$

and the derivative of the H function with respect to r is given in Appendix I by A1.3.11.

It now follows after some algebraic manipulation that :

$$\lim_{\sigma_1 \to \infty} \Phi(x) =$$

$$\lim_{\sigma_1 \to \infty} \left[(\pi P/2\sigma_1^2) \sin (\pi P/2\sigma_1)\right]\left[-2a\sqrt{\{(a^2 - x^2)/(a^2 - c^2)\}}\right] \Big/ \left[-\pi^2 A/\sigma_1^2 b\right] \qquad 3.4.7$$

Now it follows from 3.3.6 that

$$\sqrt{(a^2 - c^2)} = a \sin (\pi P/2\sigma_1) \qquad 3.4.8$$

and substituting this into 3.4.7 gives as the limit :

$$(Pb/\pi A)\sqrt{(c^2 - x^2)} \qquad 3.4.9$$

This is the result expected and thus it may be concluded that the displacement function is to this extent well behaved.

3.5 Potential Energy

The energy stored in the material surrounding a plastically relaxed crack will now be determined using the results of the previous section. Let Γ be some surface drawn in the material enclosing the crack and plastic zones. Make a cut along that part of $x_2 = 0$ where $x_1 > -a$. Let the two surfaces of this cut be denoted by s for $|x_1| \leq a$ and let the two surfaces joining s and Γ be denoted by γ. It is assumed throughout that the material behaves as though continuous across γ, that is that equal and opposite tractions are applied to the two faces to prevent relative displacement of the surfaces.

Denote by σ_{ij}^{I} , u_i^{I} and $\mathcal{E}^{I}$ the uniform elastic field and energy produced by a uniform positive applied stress of magnitude P together with the constant positive tractions of magnitude P on the surface s. Also denote by σ_{ij}^{II} , u_i^{II} and $\mathcal{E}^{II}$ the elastic field and energy obtained when tractions $-p(x)$ are applied along s only; $p(x)$ is given by 3.1.3.

If Γ is a circle with its centre at the origin then both u_i^{II} and σ_{ij}^{II} will be vanishingly small on this surface for sufficiently large Γ . The superposition of the elastic systems I and II when $\Gamma \rightarrow \infty$ is equivalent to a relaxed crack in an infinite medium with an applied stress at infinity of magnitude P and a yield stress σ_1. In this system the elastic field and energy will be denoted by σ_{ij}, u_i and $\mathcal{E}$.

Now $\mathcal{E}$ the total energy of the complete system is given by :

$$\begin{aligned} \mathcal{E} &= \mathcal{E}^{I} + \mathcal{E}^{II} + \int_{\Gamma+s+\gamma} \sigma_{ij}^{II} u_j^{I} \, ds_j \\ &= \mathcal{E}^{I} + \mathcal{E}^{II} + \int_{\Gamma+s+\gamma} \sigma_{ij}^{I} \, u_j^{II} \, ds_j \end{aligned} \qquad 3.5.1$$

In this relation the integrals over the two surfaces of γ are equal and opposite since the relative displacements over γ are zero identically. These integrals thus cancel and will be ignored.

As the perturbation (System II) is introduced into system I the work done by the external forces is :

$$F = \int_{\Gamma} \sigma_{ij}^{I} u_j^{II} \, ds_i = \int_{\Gamma+s} \sigma_{ij}^{II} u_j^{I} \, ds_j - \int_{s} \sigma_{ij}^{I} u_j^{II} \, ds_i \qquad 3.5.2$$

This follows by equating the integrals in 3.5.1.

Now define a quantity W by :

$$W = -\int_s \sigma_{ij} u_j \, ds_j \qquad 3.5.3$$

Now since this system of stress and displacement is obtained by superimposing system I and II and since there are no relative displacements over s in system I it follows that

$$W = -\int_s \left\{ \sigma_{ij}^{I} + \sigma_{ij}^{II} \right\} u_j^{II} \, ds_j = -2\mathcal{E} - \int_s \sigma_{ij}^{I} u_{ij}^{II} \, ds_j \qquad 3.5.4$$

It will be shown later that W is related to the plastic work of the system.

Now as the crack extends a distance δc the increment of energy released δv is the work done by the external forces reduced by the increase in the total elastic energy. This increment of energy supplies any plastic work or surface energy accompanying the extension. From 3.5.1 and 3.5.2 this is :

$$\delta v = \delta c \left[\partial(F - \mathcal{E}) \, \partial c \right]$$

$$= \delta c \left[-\partial/\partial c \left(\mathcal{E}^{I} + \mathcal{E}^{II} + \int_s \sigma_{ij}^{I} u_j^{II} \, ds_i \right) \right] \qquad 3.5.5$$

Now $\mathcal{E}^{I}$ is independent of c so that substituting W from 3.5.4 into 3.5.5 it follows that

$$\delta v = \delta c . \, \partial/\partial c \, (\mathcal{E}^{II} + W) \qquad 3.5.6$$

3.6 Plastic Work

In this model plastic work is to be interpreted as the work done against the stress in the plastic arc as the material on one side is displaced with respect to the material on the other. The amount of energy which has been lost as plastic work will not be a unique function of the state of the system but will depend on the path taken through the stress variables and the length c of the crack. The <u>increment</u> of plastic work is the quantity of physical interest and this will be determined using the results of section 3.4.

This increment is not the derivative of any unique function and must be determined directly. Physically, this is because the plastic work done in a particular region cannot subsequently be transferred to some other region. Now, suppose that before the crack extends a known equilibrium state of stress and strain exists within the material. It is assumed that as the crack extends the bonds break in the tip after which the displacements adjust themselves to a new state of equilibrium.

If there are displacements as the bonds break, that is as the bonding stresses reduce to zero, the work done as these displacements take place against the stresses is work expended in the formation of new surface. Therefore, the assumption does not introduce any error since the surface energy is to be treated separately.

There are certain important relationships concerning the nature of the incremental plastic work as the crack extends. In deriving these it is convenient to introduce the notation :

$$K_1\int_\alpha^\beta \gamma\,(x)dx + K_2\int_\gamma^\delta \gamma(x)dx = \left[K_1\int_\alpha^\beta + K_2\int_\gamma^\delta\right]\gamma\,(x)dx \qquad 3.6.1$$

where $\alpha\ \beta\ \gamma\ \delta$ are arbitrary limits and K_1, K_2 are arbitrary functions constant with respect to x . Now define a quantity $W(c)$ by the relationship :

$$W(c) = \left[\int_{-a}^{-c} + \int_{c}^{a}\right]\sigma_1\ \Phi\ (x)\ dx \qquad 3.6.2$$

in which $\Phi\ (x)$ is the even function giving the relative displacement of the single crack and is evaluated at 3.4.1. This quantity is the quantity W of the previous section and may be interpreted as the plastic work done as an elastic medium containing a crack of length c is gradually loaded at infinity to a stress of magnitude P while the crack length c remains constant. This plastic work is naturally a function of c.

Now let δW be the increment of plastic work done as the crack extends a small distance δc from c to c'. Let $\Phi(x)$ and a be the displacement and the length of the plastic zones for the crack of length c, and let $\Phi'(x)$ and a' be the corresponding quantities for a crack of length c'. Then :

$$\delta W = \left[\int_{-a'}^{-c'} + \int_{c'}^{a'} \right] \sigma_1 \left(\Phi'(x) - \Phi(x) \right) dx \qquad 3.6.3$$

in which it is assumed that

$$\Phi(x) = 0 \quad \text{when} \quad |x| > a$$

Now it is easily shown that :

$$\delta W = W(c') - W(c) - \left[\int_{-c}^{-c'} + \int_{c'}^{c} \right] \sigma_1 \, \Phi(x) \, dx \qquad 3.6.4$$

and in the case where δc is small, using the fact that $\Phi(c) = \Phi(-c)$, this relationship may be reduced to :

$$\delta W = \left\{ \left[\partial W(c) / \partial c \right] + 2\sigma_1 \, \Phi(c) \right\} \delta c \qquad 3.6.5$$

Although this result has been demonstrated only for the case when Φ is an even function symmetrical about the crack centre it holds in other cases.

In general the increment of plastic work lost as a crack extends is the product of the displacement in the tip the resistance stress σ_1 and the increment of extension. By increment lost it is understood to refer to that quantity in excess of $\partial W/\partial c$.

In the particular case under consideration the quantity $W(c)$ may be found by substituting $\Phi(c)$ from 3.4.1 into the definition 3.5.3 and integration using the relation A1.7.4

$$W(c) = (b\sigma_1/\pi^2 A)\left[\pi Pc\sqrt{\{a^2 - c^2\}} - 4\sigma_1 c^2 H(a,c,-c)\right] \quad 3.6.6$$

Substituting into 3.6.5 it follows that

$$\delta W = \delta c(2b\sigma_1/\pi^2 A)\,[\pi P\sqrt{\{a^2 - c^2\}} - 2\sigma_1 c H(a,c,-c)] \quad 3.6.7$$

Now in the limit for large σ_1 it is permissible to approximate to the natural logarithm and to the cosine by using the first term in the series expansions. Thus from 3.3.6 :

$$H(a,c,-c) = \ln(a/c) = (\pi^2 / 8)(P / \sigma_1)^2 \qquad 3.6.8$$

Similarly it follows that

$$\sqrt{(a^2 - c^2)} = a(\pi/2)(P/\sigma_1) \qquad 3.6.9$$

Substituting these two approximations into 3.6.6 and 3.6.7 it follows that

$$\lim_{\sigma_1 \to \infty} W(c) = 0 \qquad 3.6.10$$

and

$$\lim_{\sigma_1 \to \infty} \delta W = \delta c \quad (bP^2 c/2A) \qquad 3.6.11$$

The implications of these relationships will be considered in section 8.

3.7 The Elastic Energy δ^{II}

The only term remaining in the relation 3.5.6 for the potential energy is the term δ^{II}. The second term being related to the plastic work.

The general expression for δ^{II} is :

$$\mathcal{E}^{II} = (1/2) \int_{S+\Gamma} \sigma_{ij}^{II} \quad u_i^{II} \quad ds_j \qquad 3.7.1$$

Now along the surface S the non zero components of σ_{ij} and u_i are given by :

$$\left.\begin{aligned} \sigma_{ij}^{II} &= -P, \quad |x_1|, \quad < \quad c \\ &= \sigma_1 - P, \quad c < |x_1| < a \end{aligned}\right\} \qquad 3.7.2$$

$$u_i^{II} = \Phi\ (x_1)\ /2 \qquad 3.7.3$$

Further on Γ both u_i^{II} and σ_i^{II} are vanishingly small for sufficiently large Γ, thus, taking into account the symmetry of Φ within the ranges of integration, it follows from 3.7.1, 3.7.2, and 3.7.3 that :

$$\mathcal{E}^{II} = (b\sigma_1 / \pi^2 A)\ [(\sigma_1 - P) \int_{-a}^{a} - \sigma_1 \int_{-c}^{c}](c-x)H(a,c,x)dx$$

3.7.4

Now the relationship involves integrals of the type $I_1\ (\alpha,\beta)$ defined by A1.1.4 and evaluated in A1.7. Using the relationship 3.3.6 for $\sin^{-1}(a/c)$

it follows that :

$$\mathcal{E}^{\text{II}} = (b\sigma_1^{\,2}/\pi^2 A)2c^2 H(a,c,-c) \qquad 3.7.5$$

Now it is apparent from 3.4.1 that :

$$\Phi\,(c) = (2b\sigma_1/\pi^2 A)\,c\,H(a,-c,c) \qquad 3.7.6$$

and substituting this relation into the expression for $\mathcal{E}^{\text{II}}$ it follows that

$$\mathcal{E}^{\text{II}} = \sigma_1\, c\, \Phi\,(c) \qquad 3.7.7$$

Further differentiating 3.7.5 and substituting for $\Phi\,(c)$ it is clear that

$$\partial\, \mathcal{E}^{\text{II}}/\partial c = 2\sigma_1\, \Phi\,(c) \qquad 3.7.8$$

This latter result is equivalent to the second term in the expression 3.6.5 for δW and this equivalence has some fundamental consequences which will be discussed in the next section.

Although the expression for δW may be generalised the author has found no way of proving relation 3.7.8

as a general result.

Consider now the limiting value of $\mathcal{E}^{II}$ for large σ_1. From relation 3.6.8 it follows that :

$$\mathcal{E}^{II} = b\ P^2\ c^2 / 4A \qquad 3.7.9$$

One would expect this to be the elastic energy $\mathcal{E}^{II}$ of the unrelaxed crack. The displacement function for the unrelaxed case is given by 3.4.8 and substituting this relationship into 3.7.1 it follows at once that

$$\mathcal{E}^{II} = 1/2\ (bP^2 / \pi\ A) \int_{-c}^{c} \sqrt{(c^2 - x_1^2)}\ dx \qquad 3.7.10$$

$$= bP^2 c^2 / 4A$$

Comparing with 3.7.9 it is clear that this component of elastic energy conforms to the expected behaviour.

3.8 The Energy Propagation Criterion

In general a physical process is prevented from taking place if the process requires an increase of energy in excess of that available. Thus some energy condition often forms a necessary condition for a physical process but this need not be a sufficient condition. In the previous sections the incremental changes of energy associated with the crack extension have been evaluated and thus the energy condition for the crack extension may be examined.

If S is the energy per unit area of the crack surface then for the crack to extend it is necessary that :

$$\delta W + 4S\,\delta c < \delta V \qquad 3.8.1$$

that is the plastic work and energy of new surface should be less than the potential energy released. Now the expression for $\partial\delta^{\Pi}/\partial c$ is given at 3.7.8 and substituting this into the expression 3.5.6 for δV it follows that

$$\delta V = \delta c\,[2\sigma_1\,\Phi(c) + \partial W/\partial c] \qquad 3.8.2$$

Now this is equivalent to the expression for δW given at 3.6.5 . The relation 3.8.1 for the crack to extend then becomes

$$4S\,\delta c < 0 \qquad 3.8.3$$

and this is clearly a statement that all crack lengths are stable. It is stated in section 3.6. that the form of the relation 3.6.5 for δW is general and does not depend upon the particular configuration of cracks or displacements. On the other hand the most general relation for δV is given at 3.5.6 and the expression 3.7.7 for ε^{Π} which gives $\delta V = \delta W$ depends upon the substitution of a particular form of u_i^{Π}. From section 3.7 it is clear that this form of ε^{Π} which leads to relation 3.8.2 follows as a direct consequence of the relation 3.3.6 which gives the ratio c/a in terms of the ratio of the applied stress and the yield stress. That is $\delta W = \delta V$ follows as a direct consequence of the state of equilibrium. This result is not altogether unexpected. However, it is pointed out in 3.7 that the expression 3.7.8 for $\partial \varepsilon^{\Pi} / \partial c$ is not easily generalised, and it is not clear that this form holds for any other equilibrium state.

Since the completely brittle crack may be regarded as the limiting case of the relaxing crack as $\sigma_1 \rightarrow \infty$ the question is raised as to why an energy criterion for fracture has ever been obtained if no such criterion exists in the general problem.

Now it is clear from 3.6.10 that in the limit $W(c)$ is identically zero and then the relation 3.5.6 for δV in the brittle case, obtained by substituting for ε^{Π}, from 3.6.9 is :

$$\delta V = b\, P^2\, c/2A \cdot \delta c \qquad 3.8.4$$

Thus it is apparent that δV conforms to the expected behaviour. To obtain the Griffith theory δV must now be set equal to the energy of new surface. There is no plastic work in this problem although the quantity δW does not in fact tend to zero but has the limit $2\,\Phi(c)\,\sigma_1\,\epsilon$. This δW clearly does not exist in the brittle case and the fact that the limit of δW is finite is due to a reversal of the limiting processes.

For the brittle crack any finite δc does not extend into a region which has been plastically deformed and therefore this is true in the limit. That is, the density of plastic work lost over δc is zero as $\delta c \rightarrow 0$. However, for finite σ_1 a sufficiently small δc will be contained entirely within plastic regions so that the density of plastic work lost in the region contained by δc tends to $\sigma_1\,\Phi\,(c)$. This is true for all σ_1 and therefore it is true in the limit as $\sigma_1 \rightarrow \infty$. The extension of a crack in a completely brittle material may not be considered as the limiting case of a crack in a ductile material as $\sigma_1 \rightarrow \infty$.

In a physical system the increment by which a crack extends may not tend to zero but must be some small finite quantity. These limiting processes may then be interpreted as follows. If σ_1 is so large that the increment of extension is greater than the plastic zones, then this is equivalent to letting $\sigma_1 \to \infty$ before differentiating and a brittle crack theory is appropriate. If the increment of extension is small compared with the plastic zones then this is equivalent to differentiating before letting $\sigma_1 \to \infty$ and a relaxing theory is appropriate.

Chapter IV

Periodic Array of Cracks

In practical problems we often have to deal with a crack or notch which is not small in comparison with the dimensions of the body. In plane strain the extension of the previous analysis to a finite body requires extensive numerical work. In the case of antiplane strain a model of a notch in a semi infinite body may be obtained by cutting the infinite body containing a crack along the plane of symmetry perpendicular to the crack. However, this model still cannot explain the behaviour as the plasticity meets other defects or extends large distances across a finite specimen.

In order to understand these problems it is necessary to consider a system of several cracks explicitly. The most simple case is an infinite array of equally spaced identical coplanar cracks, since the use of Fourier series provides a convenient method of solution. In this case there are two different classes of planes of symmetry, namely the planes through the crack centres and the planes between the cracks. Both of these planes are stress free in the case of antiplane strain and so the infinite body may be cut in several ways to give a bar with one or two notched surfaces or a bar with one or more cracks along the cross section.

4.2 The Equations for a Periodic Array of Cracks

Consider an array of equally spaced cracks of length 2c. Let the distance between the crack centres be 2ℓ and let 2a be the total length of the displacement arcs. The length of the plastic arcs is then a-c. If standard axes are taken with their origin at the centre of the crack, (fig. (10)), then along $x_2 = 0$ the dislocation density is an odd function of x_1 and the required stress and displacements are even functions of x_1. Initially, however, this problem will not be treated in terms of the dislocation theory.† Let there be a positive uniform applied stress at infinity of magnitude P and let there be a prescribed positive stress in the plastic arc of magnitude σ_1. These stresses will be σ_{12} stresses for plane strain shear, σ_{11} stresses for plane strain tension and σ_{23} stresses for antiplane strain.

Now Appendix II contains a discussion of certain elastic problems in the half space $x_2 \geq 0$. In these

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† Smith has treated this problem using dislocation theory (Bilby, Cottrell, Smith and Swinden : to be published Proc. Roy. Soc.)

problems the state of stress and displacement is specified on $x_2 = 0$ and dies away as $x_2 \to \infty$.
The problems are set up in terms of two general functions $\sigma(x_1)$ and $u(x_1)$. For plane strain shear :

$$\left.\begin{aligned} \sigma(x_1) &= \sigma_{12} \\ u(x_1) &= \mu\, u_1 / (1-\nu) \end{aligned}\right\} \quad 4.2.1$$

where σ_{12} and u_1 are components of stress and displacement on the boundary $x_2 = 0$.
For plane strain tension :

$$\left.\begin{aligned} \sigma(x_1) &= \sigma_{11} \\ u(x_1) &= \mu u_2 / (1-\nu) \end{aligned}\right\} \quad 4.2.2$$

where σ_{11} and u_1 are the components of stress and displacement on $x_2 = 0$.
For antiplane strain :

$$\left.\begin{aligned} \sigma(x_1) &= \sigma_{23} \\ u(x_1) &= \mu\, u_3 \end{aligned}\right\} \quad 4.2.3$$

where σ_{23} and u_3 are the components of stress and displacement on $x_2 = 0$.

Define a function of q such that :

$$q(\pi x_1/\ell)\pi/\ell = \begin{cases} -P, & 2n\ell - c < x < 2n\ell + c \\ \sigma_1 - P, & \begin{cases} 2n\ell - a < x < 2n\ell - c \\ 2n\ell + c < x < 2n\ell + a \end{cases} \end{cases}$$

$$n = -\infty \ldots\ldots -1,0,1,2 \ldots\ldots \infty$$

4.2.4

That is q is defined on the displacement arcs such that $q = -P$ in the crack and $q = \sigma_1 - P$ on the plastic arc.

Now consider the half space elastic systems in which :

$$\sigma(x_1) = q\,(\pi x_1/\ell)\pi/\ell, \quad 2n\ell - c < x < 2n\ell + c$$

4.2.5

$$u(x_1) = 0 \qquad 2n\ell + c < x_1 < 2(n+1)\,\ell - c$$

4.2.6

The required models of a systems of cracks may be obtained by superimposing the above system on to the appropriate uniformly stressed half space in which the magnitude of the stress is P. Since the boundary stresses are even and periodic, a suitable general form is given by equations A2.6.12 and A2.6.13 setting $c_n \equiv 0.$ $n \geq 0.$ Thus the stresses are given by a Fourier cosine series. The boundary condition need then be stated only over the region $(-\ell, \ell)$. Now let :

$$\left.\begin{aligned} \theta &= \pi x/\ell \\ \alpha &= \pi a/\ell \\ \gamma &= \pi c/\ell \end{aligned}\right\} \qquad 4.2.7$$

$$U(\theta) = u(x) \qquad 4.2.8$$

Then from 4.2.5, 4.2.6, A2.6.12 and A2.6.13 the problem may be formulated by the following equations :

$$u(x) = \sum_{n=o}^{\infty} B_n \cos(n\theta) = 0, \quad c < |x| < \ell \qquad 4.2.9$$

$$(\ell/\pi)\,\sigma(x) = \sum_{n=o}^{\infty} nB_n \cos(n\theta) = q(\theta), \quad |x| < \ell \qquad 4.2.10$$

Define :

$$V(\theta) = -U'(\theta) = \sum_{n=1}^{\infty} n\,B_n \sin(n\theta) \qquad 4.2.11$$

Using the Fourier inversion theorem it follows from equation 4.2.11 that

$$nB_n = (1/\pi)\int_{-\pi}^{\pi} V(\theta)\sin(n\theta)\,d\theta \qquad 4.2.12$$

$$bD(x) = -d/dx \quad \Phi(x) \quad = +(b/\ell A) \ V(\theta) \qquad 4.2.16$$

From 4.2.7, $\zeta = \pi x/\ell$ and from 4.2.4 it follows that the stress applied to the dislocations is :

$$p(x) \quad = \quad -q(x) \ \pi/\ell \qquad 4.2.17$$

Substituting relations 4.2.16 and 4.2.17 into 4.2.14 the relation 2.4.17 , given by Leibfried, follows at once. It should be noted however that using the dislocation theory to set up the problem only the cases of plane strain shear and antiplane strain are encompassed in a natural way. Regarding the crack in terms of the half space elastic problem, plane strain tension is also included. However it is possible to represent a crack under tension by a distribution of dislocations which adjust their positions by a formal climbing process. One must, however, be careful regarding the interpretation of σ_1 . This problem will be considered in Chapter V.

4.3 The general solution

Making the substitutions :

$$\left.\begin{array}{lll} y & = & \cos\ \theta \\ y' & = & \cos\ \zeta \end{array}\right\} \qquad 4.3.1$$

The boundary condition 4.2.9 may now be used to restrict the range of integration from $[-\pi,\pi]$ to $[-\alpha,\alpha]$. Substituting this expression for nB_n into the relation 4.2.10 it follows that

$$\sum_{n=0}^{\infty} (1/\pi) \int_{-\alpha}^{\alpha} F(\zeta) \sin (n\zeta) \, d\zeta \cos (n\theta) = q(\theta) \qquad 4.2.13$$

Inverting the order of integration and summation and using the relation A1.9.9 for the evaluation of the series it follows that

$$1/2\pi \int_{-\alpha}^{\alpha} \left[\{V(\zeta) \sin \zeta\}/\{\cos \theta - \cos \zeta\} \right] d\zeta = +q(\theta) \qquad 4.2.14$$

Leibfried (1951) has obtained a similar relation in terms of the dislocation density $D(x)$ which is given at 2.4.17. Taking note of the fact that the displacement in the half space is half the relative displacement, it follows that the relative displacement is given by

$$\Phi(x) = u(x) \, b/ \pi A \qquad 4.2.15$$

where A is a function of the elastic constants defined at 2.4.12. The following relation between $D(x)$ and $V(\theta)$ is obtained from 4.2.11 :

$$\left.\begin{aligned} M &= \cos \alpha \\ N &= \cos \gamma \end{aligned}\right\} \qquad 4.3.2$$

$$\phi (y) = q (\theta) \qquad 4.3.3$$

$$G(y') = - iV(\zeta) \qquad 4.3.4$$

the equation 4.2.14 reduces to the general form 2.5.1 with the range of integration $L = (1 , M)$. Using equation 4.2.4 for $q(\theta)$ the condition for a solution (2.5.34) becomes :

$$\left[(- P\ell/\pi) \int_1^N + (\{\sigma_1 - P\} \ell/\pi) \int_N^M \right] (\{y' - M\}\{1 - y'\})^{-1/2} dy' = 0$$

4.3.5

that is $\cos (\pi\{\sigma_1 - P\}/\sigma_1) = (2N - 1 - M)/(1 - M)$ which reduces after some manipulation to :

$$\cos (\pi P/ 2 \sigma_1) = \sin (\gamma/2) / \sin (\alpha/2)$$

4.3.6

For small a/ℓ one may take the first term in the series expansion of sine as being equal to the sine itself and it is at once clear that the relation 4.3.6 reduces to the corresponding relation 3.3.6 , for the case of a crack in an infinite medium.

The form of the function $G(y)$ is given by relation 2.5.27 ; substitute for ϕ from 4.3.3, then, from 4.3.4, $V(y)$ is

$$iG(y) = (1/\pi)\surd(1-y)(y-m) \int_M^1 [q(\cos^{-1} y)/(y'-y)\surd\{(1-y')(y'-M)\}]dy' \qquad 4.3.7$$

These integrals are evaluated in the Appendix I and using equation A1.8.8, 4.3.7 reduces to :

$$iG(y) = (\sigma_1 \ell/\pi^2) \cosh^{-1} [|(\{M + 1 - 2N\}-2\{M-N\}\{1-N\}/\{y -N\})/(M-1)|] \qquad 4.3.8$$

From the definitions 4.2.6, 4.2.7 and 4.3.1, it is at once apparent that

$$u(x) = -\int_{\pi a/\ell}^{\pi x/\ell} iG(\cos \zeta)\, d\zeta \qquad 4.3.9$$

An analytical expression for $u(x)$ is not easily determined and recourse to numerical techniques is necessary. Calculations of $u(c)$ have been made using the Mercury Computer, the programme being written in the Manchester Autocode, Brucker et al (1961).[77] The specification of this programme is given in Appendix IV.

4.4 General yield

By general yield it will be understood that $a = \ell$; that is the plastic zones from neighbouring cracks just meet. Even in this special case it is difficult to evaluate the function $u(x)$, given at 4.3.9, by analytical methods. However in this case the stress is known at all points of the boundary and the function $u(c)$ at general yield may be determined directly from equation 4.2.10.

The following simple argument leads to the relation for the applied stress. Over any strip bounded by the surfaces $x_1 = n\ell$ the net force applied to the surface $x_2 = 0$ is zero since the medium is in equilibrium. Now the region intersecting the crack is of length c and subjected to a stress $-P$ and so the force per unit thickness of the material, applied over this region is $-Pc$. Similarly the force applied over the remainder of the surface $x_2 = 0$ is $(\sigma_1 - P)(\ell - c)$, since, in the case of general yield, all this surface is an intersection with the plastic arc. Thus the equation of equilibrium is :

$$(\sigma_1 - P)(\ell - c) - Pc = 0 \qquad 4.4.1$$

which reduces to :

$$P/\sigma_1 = (\ell - c)/\ell \qquad 4.4.2$$

Now to be consistent this must coincide with the limiting value of the relation 4.3.6. Setting $\alpha = \pi$ in that relation this correspondence follows at once. The displacements may be obtained directly from the form of the functions $u(x)$ and $\sigma(x)$ given at 4.2.4 and 4.2.5. Again the quantities θ, α, γ defined at 4.2.6 - 4.2.8 are used in the analysis.

From 4.2.10 at general yield the quantity nB_n may be obtained using the Fourier inversion theorem :

$$nB_n = (2/\pi)\int_0^{\pi} q(\theta)\cos(n\,\theta)\,d\theta \qquad 4.4.3$$

Substituting these quantities into the expression (4.2.9) for $u(x)$ then :

$$u(x) = B_o + \sum_{n=1}^{\infty} (2/n\pi)\int_0^{\pi} q(\zeta)\cos(n\zeta)d\zeta\cos n\theta \qquad 4.4.4$$

Inverting the order of integration and summation a sum is obtained which is evaluated in the Appendix I, section 10. From this it follows that

$$u(x) = B_o + (1/\pi)\int_{-\pi}^{\pi} q(\zeta)\ln[\sin(\{\zeta + \theta\}/2)]\,d\zeta \qquad 4.4.5$$

The term B_o is chosen so that the displacement at $x = \ell$ is zero. In terms of the Lobatchefsky function :

$$L(\theta) = \int_0^{\theta} \ell n (\cos \zeta) \, d\zeta \qquad 4.4.6$$

The expression for $u(x)$ is :

$$u(x) = -(2\sigma_1 \ell/\pi^2)\{L([\pi/2] - \gamma) + 2L(\gamma/2) - L(\pi/2)\} \qquad 4.4.7$$

The Lobatchefsky function has been tabulated by Tomontoaya, (Ryshek and Gradstein, 1957).[71]

4.5 Comparison with other models at general yield

The displacement is obtained from $u(x)$ from the set of relations 4.2.1 - 4.2.3, To facilitate comparison with the isolated crack the displacement is expressed in terms of the constant A defined at 2.4.12 (that is $A = \mu b/2\pi$ for antiplane strain and $A = \mu b/2\pi(1 - \nu)$ for plane strain). The relation for the relative displacement is thus :

$$\Phi(x) = u(x) b / \pi A \qquad 4.5.1$$

It should be noted that the relative displacement is twice the displacement on the boundary of the half space. Values of $\Phi(c)/\ell$ are given in Table I for conditions of general yield for the isolated crack theory (col i), the periodic array theory col (iii) and a double crack theory col (ii), due to Smith (to be published).

By general yield in the case of the isolated crack it is to be understood that the plastic zones have spread to a prescribed distance h from the centre of the crack. The double crack theory due to Smith is an analysis of two identical coplanar cracks the centres of which are separated by a distance 2ℓ. Conditions of general yield in this case will mean that plasticity has just spread between the two cracks and $\Phi(c)$ will denote the relative displacement at an inner tip.

It can be seen from Table I that the theory for an infinite array of cracks and the theory for two cracks predict essentially the same tip displacements; the difference being of the order of one per cent. The isolated crack theory, as one might expect, leads to greater differences of the order ten per cent. The displacements from the isolated crack are larger than those from multiple crack theories. This is because in the latter, dislocations in neighbouring cracks exert forces of attraction, whereas the only interaction forces on the dislocations in the plastic arc near a single crack are those due to the dislocations representing the isolated crack itself. Thus higher densities are to be expected. The extent of the plastic zones for conditions other than general yield are given in Table II. Values are given for the isolated crack, row (i), the array of cracks, row (ii), and for a classical model of a notched bar of finite thickness, row (iii). The latter results are calculated numerically by Koskinen (1961)[70] using an extension of the analysis given by Hult and McClintock (1957)[62] for the semi infinite medium. The calculations have been carried out varying the stress, the notch angle and the notch depth. The values in Table II are those for a zero notch angle. It is clear from these diagrams that the thickness of these plastic zones is dependent upon the notch angle but the extent of the zones does not appear to have such a dependence.

As is to be expected these lengths agree closely with the lengths calculated from equation 4.3.6 while the length of the zones from an isolated crack are shorter. For conditions of general yield (i.e. when $a = \ell$) comparisons are made graphically in fig(11). Shown here are the results for an isolated crack, those for a double crack and those for the array of cracks. Variations of curve 1 show the effect of workhardening 1A and relaxation along 2 slip system 1B.

4.6 Displacements in the tip

For the periodic problem the displacements in the tip have been calculated for a mesh of values of a/ℓ and c/ℓ and the variation of c/ℓ with a/ℓ is shown in fig. (13) for fixed displacement in the tip. These contours are determined by linear interpolation between the points of the mesh. The values on $a/\ell = 1$ agree with the equation 4.4.8? as expected. A similar contour graph has been constructed for a number of constant values of P/σ_1 fig. (12) A contour graph of some importance is fig. (14) which shows the variation of displacement with stress for a number of fixed values of c/a. These curves extend to the point of general yield and then terminate. The envelope of these curves will be called the yield envelope.

In what follows the term "Dangerous Crack" will mean a crack which will spread catastrophically under appropriate load and a "Safe Crack" will mean a crack for which no load will cause a catastrophic spread. It has been suggested, Cottrell (1960)[72], that a suitable criterion for the catastrophic spread of a crack in this model is that the displacement in the tip should exceed some critical value. Then a crack will be dangerous only if this displacement may be accommodated by the plastic zones before they spread completely through the material.

Now in fig. (14) a line may be constructed parallel to the stress axis corresponding to this critical displacement. For a given structure c/ℓ the point at which the contour intersects this critical displacement line gives the stress which catastrophic failure occurs. Fig (15) shows the displacement at general yield plotted against the structure size c/ℓ. Here again a horizontal critical displacement line may be constructed. The points of the curves which lie above this line are points at which catastrophic fracture may occur. The range of c/ℓ which are dangerous in the sense that the crack may spread catastrophically form a neighbourhood of a point $c/\ell \sim 1/3$. For smaller values of c/ℓ the displacements are small being proportional to c since the behaviour approximates to the isolated crack while for larger c/a the plastic zones are comparatively short and cannot accommodate a large displacement. That is, the material yields before the displacements have become large. Also shown in fig. (15) are displacements for fixed stresses in the periodic problem and for the isolated crack (broken line). In the latter case these contours are straight lines since the displacement is proportional to c and these contours are terminated at the point where $a = \ell$. The contours for the periodic problem are not straight and at the origin their tangents lie along the corresponding contours for the single crack. This is because the behaviour for a vanishing crack length approximates to the behaviour of the isolated crack.

A further curve, fig. (16) again shows the displacements for a range of c/ℓ. Whereas fig. (15) measures displacements in units of ℓ, fig. (16) gives displacements in units of c. Again some stress contours are shown. For small c/ℓ these contours are straight lines parallel to the axis c/ℓ. That is to say the displacement is proportional to c for small c/ℓ; again a line may be drawn corresponding to a critical displacement. A similar diagram was originally constructed by Smith and he suggests the following interpretation. The value of c/ℓ at which the critical displacement line intersects the curve is a transition point. For larger c/ℓ the structure is safe while for smaller c/ℓ the structure is unsafe.

The essential difference between figs. (15) and (16) is that from fig. (15) one may determine the dangerous crack size in a given size of specimen while from fig. (16) one may determine the size of the dangerous specimen given the crack size.

Chapter V.

An Isolated Crack in an Infinite Workhardening Material

It is well known that as a real material deforms plastically the stress necessary to produce further deformation increases. So far the analysis has taken no account of this workhardening effect. In order to obtain a more realistic picture a model is considered in this chapter in which the resistance to dislocation motion along the plastic arc is varied in proportion to the relative displacement. This proportionality may be varied along the plastic arc. The resistance is then given in the form :

$$\sigma_i' = \sigma_i + K(x_1)\,\Phi(x_1) \qquad 5.1.1$$

Now suppose that there is an isolated crack of length c in an infinite medium. Let the applied stress be P and the length of the plastic zones a-c. Representing the system by a linear array of dislocations, as in Chapter 3, the stress on the dislocations is :

$$p'(x_1) = \begin{cases} P, & |x_1| < c \\ P - \sigma_i', & c < |x_1| < c| \end{cases} \qquad 5.1.2$$

The method is as follows. Following the procedure of chapter 3 the singular integral equation for the dislocation density may be inverted. After some manipulation an integral equation is obtained in which the displacement occurs inside and outside the integral sign.

In this equation the kernel is singular and there is no obvious analytical solution. Further this singularity must be removed before numerical techniques may be applied. This may be done by evaluating the integral through the singularity by a modification of the trapezium rule. The problem is then reduced to a simple matrix equation.

5.2 The derivation of the Integral Equation

In this chapter no reference will be made to points off the x_1 axis and the suffix will be dropped from the coordinates. It is understood that all distances are measured along the x_1 axis.

Let $D(x)$ be the dislocation density, $\Phi(x)$ the relative displacement, $R(x,y)$ the quantity defined by A1.1.1 and $H(a,x,y)$ the quantity defined by A1.1.2. That is :

$$R(x,y) = \{\surd[(a^2 - x^2)/(a^2 - y^2)]\} \{1/(x - y)\}$$

$$H(a,x,y) = \cosh^{-1}\{|[(a^2 - x^2)/a(x - y)]+[x/a]|\}$$

The equations will be set up in terms of a function $s(x)$ related to the relative displacement by :

$$\Phi(x) = S(x)\ (\sigma_1\ b/\pi^2\ A) \qquad 5.2.1$$

where A and b have their usual meanings. A quantity $p(x)$ is defined in terms of the applied stress P and the initial resistance stress.

$$p(x) = \begin{cases} P & |x| < c \\ P - \sigma_1 & c < |x| \leq a \end{cases} \qquad 5.2.2$$

If $K(x)$ is defined such that

$$K(x) = 0 \quad |x| < c \qquad 5.2.3$$

then from 5.1.1 and 5.1.2 it follows that the stress on the dislocations is given by :

$$p'(x) = p(x) - K(x)\,\Phi(x) \qquad 5.2.4$$

The analysis follows exactly the analysis of chapter 3 replacing by $p'(x)$ the $p(x)$ of that chapter. The equation 3.3.5 for the dislocation density is then :

$$D(x') = (1/\pi^2 A)\left[\int_{-a}^{a} R(x',y)\,\{p(y) - K(y)\,\Phi(y)\}dy\right]$$

5.2.5

Now the relative displacement may be written :

$$(\sigma_1 b/\pi^2 A)\,S(x) = \Phi(x) = \int_{x}^{a} bD(x')\,dx' \qquad 5.2.6$$

and $D(x')$ may be substituted into 5.2.6 from 5.2.5. This substitution gives rise to an integral of the form :

$$\sigma_1 \quad S_0(x) = \int_x^a dx' \int_{-a}^{a} R(x',y)\, p\, (y)\, dy \qquad 5.2.7$$

This $S_0(x)$ is analogous to $(\pi^2 A/b)\,\Phi(x)$ in chapter 3 and it follows from 3.4.1 that :

$$S_0(x) = (c-x)\, H\, (a,c,x) + (c+x)\, H(a,-c,x) \qquad 5.2.8$$

Substituting into 5.2.6 from 5.2.5 and 5.2.7 and then making use of 5.2.3 and the symmetry of $R(x,y)$ to simplify the relation, it follows that :

$$\Phi(x) = (\sigma_1 \;\; b/\pi^2 A)\, S_0\, (x)$$

$$- \int_c^a \{bK(y)\Phi(y)/\pi^2 A\}\, dy \int_x^a \{R(x',y) + R(x',-y)\}\, dx' \qquad 5.2.9$$

The quantity $K(x)$ may be thought of as the hardening at a point x of the plastic arc for unit relative displacement across the arc. However, the quantity of physical interest is w the hardening per unit plastic strain and it is therefore necessary to determine a relationship between these quantities. That is, a relation between a relative displacement and a strain. To obtain such a relation it is necessary to introduce a gauge width and this we allow to vary along the plastic arc. Suppose therefore that the gauge width is given by $\alpha f(x)$ where α is some thickness and $f(x)$ some numerical function. Then it follows that

$$K(x) = w \alpha f(x) \tag{5.2.10}$$

Now one may substitute into 5.2.9 for $K(x)$ and $\Phi(x)$ using 5.2.10 and 5.2.1 respectively. It is then convenient to make the definition :

$$K(a,x,y) = [H(a,x,y) + H(a,x,-y) - 2\sqrt{\{(a^2 - x^2)/(a^2 - y^2)\}}] \tag{5.2.11}$$

Thus it follows that

$$S(x) = S_0(x) - (wb/\pi^2 A\alpha) \int_c^a K(a,x,y)[S(y)/f(y)]dy \tag{5.2.12}$$

This integral equation for $S(x)$ has a singularity in the kernel due to the H function when $x = y$ and due to $1/\sqrt{(a^2 - y^2)}$ at $y = a$. However the integral of both of these functions exists and is finite. The integral may be split up into a number of small regions within which $S(y)/f(y)$ may be represented to a suitable degree of accuracy by some polynomial. The coefficients of the polynomial will involve the values of the unknown function $S(y)$ at the ends of each interval and thus a set of simultaneous equations is obtained for these values of $S(y)$. This set of simultaneous equations involves integrals of $K(a,x,y)$ and will be linear if and only if the polynomial is a straight line. The straight line through the values of $[\, S(y)/f\,(y)]$ at the ends of the intervals has been used in this work since this is more accurate than a line parallel to the axis having some mean value.

5.3 Reduction to Matrix Form

Let the range of integration be divided into m equal intervals and make the following definitions :

$$h = (a-c)/m \qquad 5.3.1$$

$$x_j = c + jh \quad y_i = c + ih \qquad 5.3.2$$

$$f_i = f(y_i) \qquad 5.3.3$$

$$S_j = S(x_j) \qquad S_i = S(y_i) \qquad 5.3.4$$

$$S'_j = S_o(x_j) \qquad 5.3.5$$

$$X = 1/\lambda = wb/\pi^2 \; A\,\alpha \qquad 5.3.6$$

The equation 5.2.12 may then be written in the form :

$$\lambda\, S(x) = \lambda S_o(x) - \sum_{i=1}^{m} \int_{y_{i-1}}^{y_i} K(a,y,x)\, \{S(y) / f(y)\}\, dy \qquad 5.3.7$$

Now it is required to find a simple approximation to the function $\{S(y) / f(y)\}$. To this end it is convenient to define the quantities :

$$\alpha_{i+1} = S_i / f_i + y_i (S_i/f_i - S_{i+1}/ f_{i+1})/h \qquad 5.3.8$$

$$\beta_{i+1} = (S_i / f_i - S_{i+1} / f_{i+1})/h \qquad 5.3.9$$

and then one may write :

$$S(y) / f(y) \sim \alpha_i - \beta_i y \quad \text{for } y_{i-1} < y < y_i$$

5.3.10

In 5.3.10 the R.H.S. is a straight line and is equal to the L.H.S. for $y = y_{i-1}$ and for $y = y_i$. The integrals on the R.H.S. of 5.3.7 may then be evaluated approximately in terms of the integrals :

$$V_{ji} = \int_{y_{i-1}}^{y_i} K(a,y,x)dy \qquad 5.3.11$$

$$U_{ji} = \int_{y_{i-1}}^{y_i} yK(a,y,x)\, dy \qquad 5.3.12$$

Substituting these relations into equation 5.3.7 the following set of simultaneous equations is obtained :

$$\lambda S_j \sim \lambda S'_j - \sum_{i=1}^{m} \left[\alpha_i V_{ji} - \beta_i U_{ji} \right] \qquad 5.3.13$$

The α_i , β_i are given in terms of S_i by 5.3.8 and 5.3.9. Then one may collect the terms S_i on the R.H.S. of 5.3.13 and by setting :

$$V_{j,o} = V_{j,n+1} = U_{j,n+1} = U_{j,o} = 0 \qquad 5.3.14$$

the general coefficient of S_i in the equation for S_j is A_{ji} and is given by the relation :

$$f_i A_{ji} = \left\{ V_{j,i+1} y_{i+1} - U_{j,i+1} + U_{ji} - V_{ji} y_{i+1} \right\} / h$$

5.3.15

finally one obtains the following matrix relation :

$$S_i = \lambda \left[\lambda \delta_{ji} + A_{ji} \right]^{-1} S'_j \qquad 5.3.16$$

in which i and j run from o to m.

These equations have been solved using the Mercury Digital computer. The indexing system used here has been chosen so that the quantities V_{ji} U_{ji} are calculated into the locations V_j U_j in the computing store. This facilitates translation of the mathematics into the language of the Auto Code. The indexing system is thus dictated by the computer programme.

Nothing has been said about the form of the function f(y). In preparing the problem for the computer the form of f (y) must be written into the programme and a separate tape is required for each variation of f(y). Only one version of the programme is given in the Appendix, namely for f(y) in the form of a parabola.

The method, essentially based on the trapezium rule, neglects terms of order h^2. For the case m = 14 this should give an error of the order one half of one per cent. Comparing with a calculation at c/a = 3/8, M = 28 shows the error to be better than 5%. The following table shows values of the tip displacements using root strain, mean strain and parabolic strain methods (these methods to be defined later).

M = 14	M = 18	difference
5486	5602	2.32 %
7306	7297	0.13 %
6476	6728	4.20 %

5.4 Constant Gauge Width

Initially numerical calculations were carried out assuming $f(y) = 1$ in which case $K(x)$ is a constant given by :

$$K = w/a \qquad 5.4.1$$

K is the hardening for a unit displacement at any point on the plastic arc and α is the width of the plastic region. In choosing a value for α two factors are to be considered.

If it is to be supposed that fracture occurs when a given strain is reached in the plastic region then the maximum strain is of importance. This maximum will occur in the root. Thus one may suppose that the most suitable choice of width for the plastic region will be the diameter of the root of the notch. If one assumes a notch approximately one inch long with a radius of 0.01 inches in the root the ratio (diameter/crack length) will be 0.02 and thus we must take

$$\alpha = 0.02\ c \qquad 5.4.2$$

The solutions using this value of α will be called root strain solutions.

Alternatively one may argue that since the elastic plastic boundary meets the surfaces of the crack at right angles (Hult and McClintock 1957)[CL] then the distance over which large strains exist is small and a value of α chosen to give a correct root strain would over-emphasise the amount of hardening. Thus it might appear more reasonable to take a value of α giving some mean thickness of the plastic region. However a simple relationship of this kind is difficult to obtain theoretically since one must appeal to classical elastic plastic theories for the most simple treatment. Calculations are carried out with $10^3\ wb/\pi^2 A$ having values 0.4, 2 and 10.

The displacements in the root are presented graphically in fig (18) for the root strain method. The mean strain calculation shows little significant variation from the non workhardening case and over most of the range it is not possible to separate these curves on a graph.

Throughout the range of c/a negative displacements are calculated at points distant from the crack tip using the root strain method with $w = 10^{-2}\ \pi^2\ A/b$. This arises since the model assumes that there is always a forward stress on the dislocations greater than or equal to σ_1. However if the hardening rate is high the forward stress on the dislocations distant from the crack tip may fall as the crack tip hardens. In the physical situation the resistance would adjust itself to balance the forward stress but in the model such an adjustment is not permitted. The resistance is always assumed to have its maximum value σ_1' and this gives rise to a back stress. These negative displacements have appeared also when the plastic zones are large (a ~ 80 c) again using the root strain method of calculation.

It has been pointed out that a root strain method would tend to over-emphasise the work hardening and it is not surprising that such a model should be unsatisfactory. However the mean strain method clearly under-estimates the effect of work hardening and this probably arises since the description of conditions at the tip is inadequate.

5.5 Parabolic Gauge Width

In order to resolve these difficulties the gauge width has been varied along the plastic arc. Hult and McClintock (1957)[64] have shown that the elastic plastic boundary meets the crack surface at right angles. Thus a gauge width is proposed which is small at the tip and increases rapidly to some mean value.

Preliminary calculations have been carried out assuming the gauge width to increase in the form of a parabola. The apex of the parabola is just inside the crack tip so that there is a finite width 0.02 c at the crack tip itself and at a distance c from the crack tip the width is 0.6 c. This gives a strain at the crack tip of the required magnitude and at points removed from the tip a gauge width of the order suggested by Koskinen[70]'s graphs.

The parabola underestimates the workhardening at the end of the plastic arc removed from the crack tip. Since the displacements here are small the workhardening will be small and thus it is unlikely that underestimating this quantity will cause large errors.

Again values of $10^3 \, wb/\pi^2 A$ equal to 0.4, 2 and 10 have been used and the results of the calculations are shown graphically. The displacements are given in fig. (19) and the strain in the crack tip in fig. (20). No negative displacements have appeared in these calculations.

The ratio of the stresses is plotted as a function of c/a in fig. (11) (curve 1A) and may be compared with the other theories. For high stresses the length of the plastic zones is reduced by a factor ranging from 2 to 3.

Chapter VI

Generalisation of the Model

In the preceeding work the relaxed crack is represented by a set of arcs and the relative displacements across these arcs are determined. In chapter 3 the displacements are represented formally by a continuous distribution of dislocations the density of which is found in terms of a singular integral equation. In chapter 4 the material is divided along a surface intersecting the arcs and the problem considered in terms of an elastic half space with certain boundary conditions.

In this latter approach it is shown that at least the periodic case may be reduced to the solution of the same singular integral equation. Suitable general forms of stress and displacement are obtained in Appendix II for plane strain, normal tension and shear, and for anti-plane strain. These general relations for all three cases may be represented as Fourier series (equations A2.6.12, A2.6.13) and as Fourier integrals (equations A2.6.14, A2.6.15). The integral form is also obtained by Sneddon (1951). The boundary conditions thus reduce to a pair of dual integral equations or a pair of trigonometric series. Sneddon reduces the integral equations to a general form which is considered by Busbridge (1938)[67].

Since the stresses due to a dislocation (equations 2.4.4 - 2.4.10 are derived using the classical linear elastic theory one would expect the dual integral equations and the singular integral equation to be essentially the same.

6.2 The Equivalence of Dual Integral and Singular Integral Equations.

The equivalence of these two methods has been demonstrated by Smith (private communication) for even displacements functions. The following is a more general analysis. The form of the stress and displacements on the boundary of the half space $x_2 = 0$ of the elastic half space reduce to general forms given by A2.6.14, A2.6.15 and A2.6.1. These are :

$$u(x) = 2/\pi \int_{\infty}^{\infty} [\{B(\zeta) \cos(\zeta x) - c(\zeta) \sin(\zeta x)\}/\zeta] d\zeta \qquad 6.2.1$$

$$\sigma(x) = 2/\pi \int_{\infty}^{\infty} [B(\zeta) \cos(\zeta x) - c(\zeta) \sin(\zeta x)] d\zeta \qquad 6.2.2$$

The relative displacement, being twice the displacement in the half space, is given by relation 4.5.1 to be $\Phi(x) = u(x)b/\pi A$. The dislocation density then follows from the relation :

$$b\, D(x) = -d/dx\, \Phi(x) = -u'(x)b / \pi A \qquad 6.2.3$$

where $u'(x)$ denotes the derivative of $u(x)$.

Now assume that the stress and displacements are in fact given by the equations 6.2.1 and 6.2.2 and then consider the integral :

$$I = \int_L AD(x')/(x - x')\, dx' \qquad 6.2.4$$

Since the dislocation density is zero outside L this integral may be extended over the whole of the real axis and substituting from 6.2.3 it follows that :

$$I = -(1/\pi) \int_{-\infty}^{\infty} [u'(x')/(x - x')]dx' \qquad 6.2.5$$

Differentiate 6.2.1 to obtain $u'(x)$ and substitute this into 6.2.5. to obtain :

$$I = (2/\pi^2) \int_{\infty}^{\infty} dx'/(x-x') \int_{-\infty}^{\infty} [B(\zeta) \sin(\zeta x) + c(\zeta) \cos(\zeta x)]d\zeta$$

6.2.6

Now :

$$\int_{-\infty}^{\infty} [e^{i\zeta x'}/(x-x')]dx' = i\pi e^{i\zeta x} \qquad 6.2.7$$

So that inverting the order of integration in 6.2.6 the inner integral may be evaluated using two formulae obtained by taking the real and imaginary parts of 6.2.7. Thus :

$$I = (2/\pi) \int_{\infty}^{\infty} [B(\zeta) \cos(\zeta x) - c(\zeta) \sin(\zeta x)] d\zeta \qquad 6.2.8$$

But this is the expression for $\sigma(x)$, the stress due to the displacements given by equation 6.2.2 . Therefore, from 6.2.8 , 6.2.2 , and 6.2.4 it follows that :

$$\int_L A[D(x')/(x-x')] dx' = \sigma(x) \qquad 6.2.9$$

It is thus shown that given the forms 6.2.1 , 6.2.2 for stress and displacement the stress and dislocation density satisfy 6.2.9. The converse result is also true and a minor modification of the preceding analysis will demonstrate this.

Suppose that the relation 6.2.9 between the dislocation density and stress is given, then assume that $u(x)$ may be given in the form 6.2.1. It remains to show that the stress is then given by 6.2.2. Now define I by 6.2.4 and note that from 6.2.9

$$\sigma(x) = I \qquad 6.2.10$$

Relation 6.2.8 may be derived using exactly the same analysis and substituting for I from 6.2.10 the required relation 6.2.2 follows. It is therefore shown that the dual integral equations and the singular integral equation are equivalent methods of analysis.

6.3 A Dislocation Model of the Tensile Crack

It is apparent from the previous section that the tensile problem which hitherto has been set up only in terms of the half space problem, may always be solved in terms of a singular integral equation. This being so, what is the nature of the quantity $D(x)$ in this case.

It has been suggested, Friedel (1959)[73] that a crack may be represented formally by a distribution of "climbing" dislocations. In this case the Burgers vector is normal to the displacement arcs representing the relaxed crack, so that it is necessary to know the tensile stress exerted by an edge dislocation at points along a line normal to the Burgers vector. Taking axes at the dislocation with the x_1 direction parallel to the Burgers vector it follows from 2.4.5 that

$$\sigma_{22} = A/x_2 \qquad 6.3.1$$

where $A = \mu b/2\pi(1-\nu)$ is the quantity defined at 2.4.12 for edge dislocations. Now with respect to standard axes the tensile stress σ_{11} at a point x due to a dislocation at x_1' is

$$\sigma_{11} = A/(x_1 - x_1')$$

It then follows that if $\sigma(x_1)$ is the distribution of tensile stress over the surface $x_2 = 0$, due to a continuous distribution $D(x)$ of such edge dislocations in the region L of $x_2 = 0$, then :

$$\sigma(x_1) = A \int_L [D(x_1')/(x_1 - x_1')\, dx_1' \qquad 6.3.2$$

Relation 6.3.2. may be regarded as giving the state of stress on $x_2 = 0$ due to a set of displacements specified on $x_2 = 0$. This state of stress dies away for large x_2. Now let the body be subjected to a uniform positive tensile stress P at infinity. The resultant stress on L is then given by $P + \sigma(x_1)$. If the yield stress in tension is equal to σ_1 it is required that the resultant stress shall be σ_1 in the plastic arc and zero along the crack.

Define a quantity $P(x_1)$ by :

$$p(x_1) = \begin{cases} P - \sigma_1 & \text{plastic arc} \\ P & \text{free arc} \end{cases} \qquad 6.3.3$$

then the requirement is $p(x_1) + \sigma(x_1) = 0$. The quantity

$p(x_1)$ so defined is analogous to the stress on a dislocation in cases of shear. In terms of the quantity $D(x_1)$ the requirement is

$$A \int_L [D(x_1) / (x - x')] \, dx' + p(x) = 0 \qquad 6.3.4$$

This equation is identical to 2.4.15 or 3.1.5. An interpretation of σ_1 in terms of resistance to dislocation motion is not possible in this case. Instead σ_1 is interpreted directly as a tensile yield stress. The representation of the plastic arc by a set of "climbing" dislocations in this way is thus purely formal.

It follows that the tensile crack may be treated by any analysis suitable to the shear cases. Further, any two treatements based on linear elasticity should lead to the same conclusions. For these reasons Dugdale (1960)[63] obtains relations for an isolated tensile crack which are analogous to those obtained here for the shear case.

The solution for a tensile crack in an elastic material may also be obtained assuming no plastic zones and unbounded dislocation densities at the crack tips. Such an analysis may be used to determine Griffith relations.

6.4 Non-Collinear Dislocation Arrays

It is clear that a model of a crack under tension which allows dislocations in the plastic zones to move under shear stresses requires that relaxation take place along arcs other than the x_1 axis. As a next step a model is considered in which relaxation from each tip takes place along two planes symmetrically placed about the x_1 axis. Such a system of relaxation would resemble more closely certain other theoretical models, Southwell and Allen (1949)[74] Green (1953)[75], and also some experimental observations, Green and Hundy (1956)[76] Knott and Cottrell (1963).[78]

It is shown that relaxation takes place along curved arcs extending symmetrically from the root of the notch. Thus the crack will be represented by a distribution of climbing dislocations along a straight arc. The plastic zones extending from a single tip will be represented by a distribution dislocations gliding along two straight arcs inclined at angles $\pm\alpha$ to the plane of the crack fig. (21). The medium is subjected to an applied tensile stress P and the resistance in the slip lines representing the plastic zones is σ_i (N.B. σ_i is used in this section for the resistance to dislocation motion not σ_1, since this quantity may not now be considered as the tensile yield stress)

The equations for the equilibrium of such a distribution of dislocations are exceedingly complicated and have no obvious analytical solution.

6.5 Numerical Analysis of the Tensile Crack

By "Axes Taken at the Dislocation" it will be understood that the dislocation is an edge dislocation, the origin is at the dislocation and the Burgers vector lies in the x_1 direction. These axes are rotated from the standard axes through an angle θ say.

The analysis for the tensile crack is based on the set of equations 6.5.1 - 6.5.4 giving the stresses due to an edge dislocation with respect to axes taken at the dislocation.

$$P_{11} = - Ax_2(3x_1^2 + x_2^2) / r^4 \qquad 6.5.1$$

$$P_{22} = A \, x_2(x_1^2 - x_2^2)/ r^4 \qquad 6.5.2$$

$$P_{12} = A \, x_1(x_1^2 - x_2^2)/r^4 \qquad 6.5.3$$

$$r^2 = x_1^2 + x_2^2 \qquad 6.5.4$$

It will be necessary to rotate the stress matrix to and from the standard axes and the general formulae are :

$$\sigma_{11}' = \sigma_{11} \cos^2\theta + \sigma_{22} \sin^2\theta + 2\sigma_{12} \cos\theta \sin\theta \qquad 6.5.5$$

$$\sigma_{22}' = \sigma_{11} \sin^2\theta + \sigma_{22} \cos^2\theta - 2\sigma_{12} \cos\theta \sin\theta \qquad 6.5.6$$

$$\sigma_{12}' = \sigma_{12} (\cos^2\theta - \sin^2\theta) + (\sigma_{22} - \sigma_{11}) \cos\theta \sin\theta \qquad 6.5.7$$

The model is clearly symmetrical about the x_1 and x_2 axes; thus for any dislocation on the plastic arc in the first quadrant, at (x_1 , x_2) say, there will be dislocations at $(-x_1, x_2)$ $(-x_1, -x_2)$ and $(+x_1, -x_2)$. These four sets will be denoted by A, B, C and D respectively. If the Burgers vector in the first quadrant lies along the plastic arc directed away from the crack tip then this is true in the third quadrant while in the second and fourth quadrant the Burgers vector lies along the plastic arc directed towards the crack tip. Consider now the dislocations representing the crack itself. A dislocation at $(x_1, 0)$ say with $x_1 > 0$ has its Burgers vector lying in the positive x_2 direction. For every such dislocation there is a dislocation at $(-x_1, 0)$ with its Burgers vector in the negative x_2 direction. These sets will be called E and F respectively.

If the plastic arcs subtend an angle 2α at the crack tip the rotations to axes at the dislocation will be (fig. 20) :

$$\left.\begin{aligned} \theta^A &= +\alpha \\ \theta^B &= -\alpha \end{aligned}\right\} \quad \left.\begin{aligned} \theta^C &= \alpha - \pi \\ \theta^D &= \pi - \alpha \end{aligned}\right\} \quad \left.\begin{aligned} \theta^E &= \pi/2 \\ \theta^F &= -\pi/2 \end{aligned}\right\} \qquad 6.5.8$$

A programme has been written to determine the dislocation equilibrium; again using the Manchester Mercury Autocode.

This programme is built round a routine which will evaluate the stress matrix at any point due to an edge dislocation at any other point. The coordinates of these points and the angle θ between the Burgers vector of the dislocation and the standard axis x_1 must be specified as programme parameters. The stress matrix is calculated with respect to the axes taken at the dislocation and rotated to standard axes; equations 6.5.1 - 6.5.7 are used for these calculations.

Now the dislocations may be grouped in twos or fours for dislocations in the crack or plastic zones respectively. The above arithmetical routine is therefore built into a logical routine which groups the dislocations. Given α the inclination to the axis x_1 of the plastic arc in the first quadrant, this routine will evaluate the stress matrix at any point due to a dislocation in the first quadrant and all its symmetrical images. Again the matrix is given with respect to standard axes. This routine may be modified so that the stresses due to the dislocation in the first quadrant are omitted if the point at which the stresses are required coincides with that dislocation.

Now from the symmetry it is clear that if the dislocations in the first quadrant are in equilibrium then so are dislocations in the other quadrants. Thus the generating routine need only ask for stress in the first quadrant and need only specify the positions of the dislocations in the first quadrant. The general problem then is to determine the distribution of dislocations which will give specified stresses at specified points.

It is assumed that the dislocation density over a small region may be represented by a mean dislocation strength, concentrated at some point in that region. These concentrations will be denoted by D_j . Points on the plastic arcs are identified by the suffix i .

The programme generates a matrix A_{ij} with elements a_{ij} . If i is a point in the crack then the element a_{ij} is the normal tensile stress at the point i due to the dislocation concentration D_j and its images. If i is a point of the plastic arc then a_{ij} is the shear stress along the arc at the point i due to the dislocation concentration D_j and its images. It is understood that if the dislocations D_j are at the point i then a_{ij} is the stress due to image dislocations only.

Now at points i in the crack the dislocation distribution must balance the applied tensile stress P and at points i along the plastic arc the dislocations must balance the resistance σ_i and the resolved shear stress kP . Thus :

$$\left.\begin{array}{llll} a_{ij} \; D_j & = & -P & \text{point } i \text{ in the crack} \\ a_{ij} \; D_j & = & \sigma_i - kP & \text{point } i \text{ in the plastic arc} \end{array}\right\} \qquad 6.4.9$$

The programme then solves this set of simultaneous equations. A description of the programme has been given here since the method of solving the problem is entirely contained in the numerical analysis. The programme is not simply a method of evaluating a formula. A flow diagram is also given at fig (17).

Now in the analytical procedures there is a unique ratio P/σ which gives a bounded stress at the edge of the plastic zone and one would expect this to be the case in the present problem. Since a bounded stress must imply that the dislocation density at the tip is zero this affords a method of evaluating P/σ. First the problem is solved for $P = 0$ $\sigma_i = 1$ and then for $P = 1$, $\sigma_i = 0$, then the solutions are added in that ratio which removes the dislocation density at the tip.

It now remains to decide the positions of the dislocation concentrations and the points at which the stresses are to be balanced. Let the intervals over which the dislocations are concentrated be such that their projections onto the x_1 axis are equal. The dislocation concentrations and the points at which the stresses are balanced may then be placed at either end of the interval or at the centre (i.e. some intermediate point).

To place concentrations at the left of the intervals would lead to a zero determinant since the first dislocation would be anihilated by its image. To balance stresses at the right hand side of the intervals would specify the stress at the edge of the plastic zone, but effectively the removal of this stress is used to determine P/σ_1 and therefore it should remain unspecified. These two possibilities are therefore rejected.

To place the dislocation concentrations at the right of the interval effectively sets a zero concentration at the origin since there are no dislocations at the origin this increases the information in the equations.

Balancing the stress at the left hand side of the interval balances the stress at the origin which effectively reduces the number of points at which the stress is specified. This follows from the symmetry since specifying a stress in the half space $x_1 \geq 0$ effectively specifies a stress at the image point, in the half space $x_1 \leq 0$. The origin being its own image would be included twice. The dislocation concentrations are therefore placed at the right hand side of the intervals and the stresses are balanced at the centres of the intervals. This procedure tends to maximise the information contained in the equations and stable solutions have been obtained. Preliminary calculations using other procedures reveal some instability.

6.6 Some Numerical Relationships for the Tensile Crack

In the analysis it is assumed that the plastic arcs are straight and subtend an angle 2α at the crack tip. This is an idealisation since the plastic zones at a real crack are curved. The model will help to show qualitatively the effect of relaxation on several slip systems. Calculations have been carried out assuming $\alpha = \pi/2$.

Let a be the distance between the crack centre and the projection onto the x_1 axis of the tip of the plastic arc. The programme requires as data the number p of intervals in the distance a and the number q of those intervals in the distance c . The following results are based on p = 32 and c raising in steps of 4 from 4 to 28. A comparison is made for p = 16, c = 6 and for p = 32, c = 12. The differences are found to be of the order 5% .

The results are shown graphically in fig. (11) and fig. (22). In fig. (11) the relation between c/a and the stress ratio is shown for a variety of problems.

Now since the plastic arcs form an angle 45° with the tensile axes the yield stress σ_1 is obtained from the resistance to dislocation motion by setting $\sigma_1 = 2\sigma_i$

Making this substitution the ratio c/a is shown in fig. (11) curve 1B as a function of P/σ_1. The curve lies very close to curve 1 itself and coincides for small c/a. The single plane theory therefore gives a good estimate of the projection of the plastic zones onto the plane of the crack.

In fig. (22) curve B the displacements $\Phi(c)$ are given in terms of σ_i . Here $\Phi(c)$ is the relative shear displacement over a single plastic arc at the crack tip. In order to obtain the relative normal displacement u_2 at the crack tip it is necessary to add $\Phi(c)$ over both arcs. That is to multiply $\Phi(c)$ by $\sqrt{2}$. In order to obtain this displacement in terms of $\pi^2 A/\sigma_1 b$ it is necessary to make a further correction to the curve and divide by 2. Curve A shows the relative tip displacement as calculated by the single plane method and here $\sigma_i = \sigma_1$ and the curve remains fixed.

Curve A in fig (22) shows also the shear displacement at the tip of a crack in shear relaxing along one plane. This may be compared directly with the shear displacement over a plastic arc at the tip of a tensile crack. For large plastic zones these are comparable but greater differences are observed for smaller plastic zones.

In fig (22A) the curves have been adjusted according to the above proceedure and direct comparisons may be made between the relative u_2 displacements in the crack tip. predicted by the two theories

Chapter VII

Discussion

7.1 Introduction

The work of this thesis has been carried out in order to obtain simple theoretical quantitative relationships describing some aspects of the plastic behaviour of notches and cracks. Owing to the drastic simplifications, the models do not give certain physical quantities which are normally measured. In particular there is no plastic strain in these models, there is only plastic displacement.

However Cottrell (1960)[72] has suggested that a brittle fracture may be initiated when the displacement in the root or tip exceeds some critical value. This is the criterion adopted here to relate this work to the theory of fracture of macroscopic structures: sections 4.5 and 4.6.

It has been shown that the theoretical relationships for c/a and the displacement in the crack tip, which have been derived in this work for the isolated crack† are in reasonable agreement with experimental observations: Knott and Cottrell (1963)[78] and Tetelman (1963)[79]. When $P/\sigma_1 > 0.95$ experimental values of a/c are lower than those predicted by the theories of Chapters III and IV.

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† These results have been published, Bilby, Cottrell Swinden (1963) Proc.Roy. Soc. A272 304

This probably arises since these theories do not take account of hardening effects.

7.2 Work Hardening

A model has been considered in which the hardening is directly related to the relative displacement. Here it is necessary to introduce a gauge width in order to determine the work hardening law from the experimental values which give the hardening in terms of the plastic strain, not the relative displacement. Two slightly different procedures have been adopted in order to relate the hardening to the relative displacement. Both of these show the behaviour expected. The displacements are reduced and for high stresses the length of the plastic zones are also reduced. In the physical system, it is possible that the forward stress on a dislocation may fall below the maximum resistance stress σ_1 and again this effect is not considered.

Furthermore Professor Cottrell has pointed out (private communication) that one effect of work hardening is to spread the slip onto other systems . This effect is not considered explicitly in this model.

A model which takes account of these factors would be more reliable particularly at high stresses.

7.3 Non-Coplanar Relaxation

Preliminary work has been carried out on a model of a tensile crack in which the plastic deformation takes place along planes which are inclined to the plane of the crack. The displacement in the crack tip is shown to be less than in that predicted by the single plane model.

The extent of the plastic zones is determined in terms of the length a of their projection onto the plane of the crack, for a given stress ratio the ratio c/a is almost identical to the ratio c/a from the single plane model. This might have been expected since figs. (10) and (11) suggest that this relation is not very sensitive to the method of analysis.

7.4 Future Work

It is pointed out in chapter 6 that the analysis in the non-coplanar model has a tendency to become unstable. This obviously requires investigation in order to determine the causes and the extent to which the model may give reliable predictions. Then extension of the technique should be possible, to obtain a more realistic picture of work hardening and also to investigate the effect of curved plastic zones.

The single plane model of the crack in a work hardening material may be modified to relate the hardening to the dislocatin density rather than to the strain.

A fundamental resitriction imposed upon most theoretical work is that the shape of the boundary remains fixed. This causes difficulties in the interpretation of effects at sharp notches, for example, since such notches cannot remain sharp during plastic deformation. The non-coplanar model does in principle allow such movement but it would be fruitful to compare this with a classical theory.

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Notation

The following symbols throughout the work have the meanings given here. Other symbols have the meaning given in the particular sections in which they occur.

c	crack half length or notch depth.
s	length of plastic zone.
a	$c + s$.
ℓ	distance between the centre of two adjacent cracks. (N.B. Not Section 3.2).
x_i	coordinates.
σ_{ij}	stress.
ϵ_{ij}	strain.
u_i	displacement.
$\sigma(x_1)$	stress function giving general stress on $x_2 = 0$.
$p(x_1), q(\pi x_1/\ell)$	stress function giving prescribed stress on $x_2 = 0$.
$U(\pi x_1/\ell)$, $u(x)$	displacement function giving displacement on $x_2 = 0$.
$\Phi(x_1)$	relative displacement.
P	magnitude of the applied stress.
σ_1	magnitude of yield stress and resistance to dislocation motion if the yield stress in shear is inferred.
E	Young's modulus.
μ	shear modulus.
ν	Poissons ratio.

b Burgers vector,

$D(x_i)$ dislocation density,

$$A \begin{cases} \mu b/2\pi & \text{screws or antiplane strain} \\ 2b/2\pi(1-\nu) & \text{edges or plane strain,} \end{cases}$$

$$R(x,y) = [\sqrt{\{a^2 - x^2)/(a^2 - y^2)\}}]/[1/(x - y)]$$

$$H(a,x,y) = \cosh^{-1}\{|[(a^2 - x^2)/a(x - y)] + [x/a]|\}$$

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List of Diagrams

Table I

Displacements at the Crack Tip

$\frac{c}{a}$	$\frac{1-c/a}{1+c/a}$	$(\pi^2 A/\sigma_1 b)$ Isolated Crack	$(\Phi(c)/c)$ Double Crack	Infinite Array
0.2	2/3	3.22	2.91	2.9
0.33	1/2	2.20	1.91	1.84
0.5	1/3	1.39	1.15	1.17
0.6	1/4	1.02	0.82	0.83
0.714	1/6	0.67	0.52	0.53
0.818	1/10	0.40	0.30	0.30
0.905	1/20	0.20	0.14	0.14

Table II

Extent of Plastic Zones

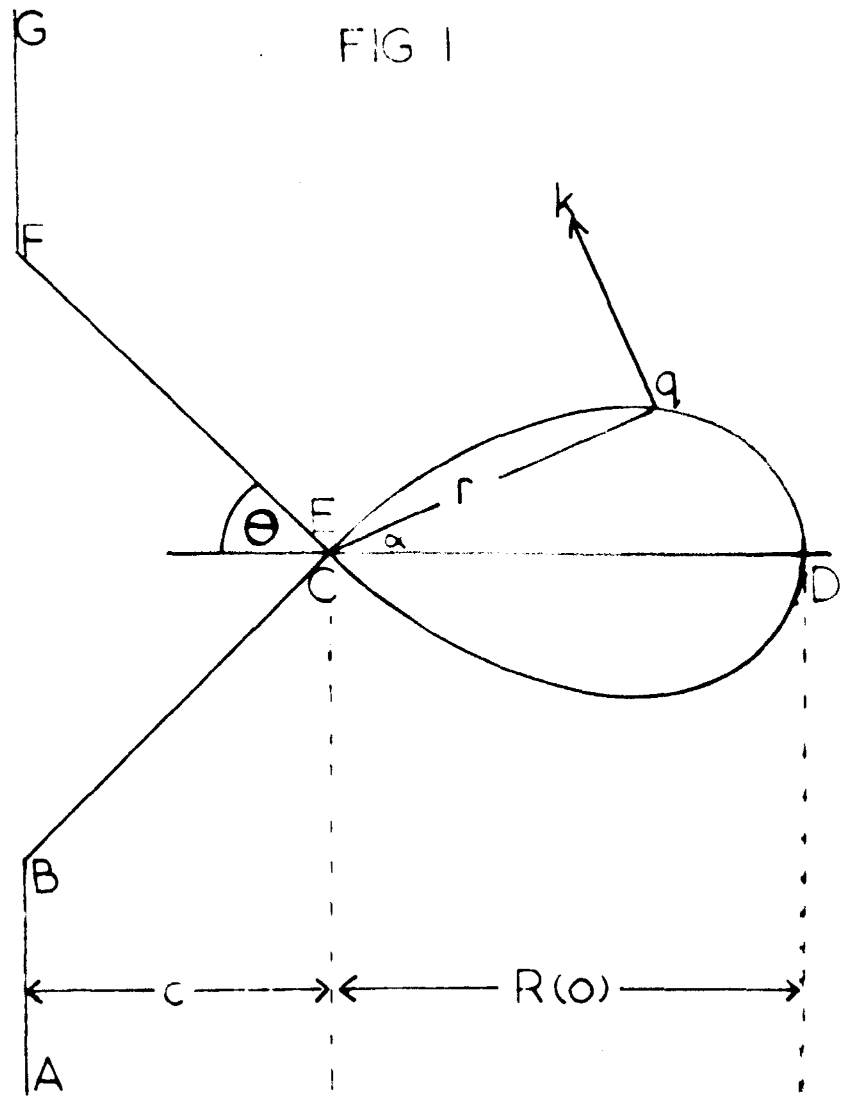
$\frac{h}{c}$	Model	$\sigma/\sigma_1(1 - c/h)$							
		0.97	0.91	0.83	0.78	0.61	0.57	0.38	0.36
4	Isolated Crack	3.26	-	1.81	-	1.33	-	1.10	-
4	Infinite Array	2.46	-	1.92	-	1.36	-	1.12	-
4	Koskinen	2.98	-	1.89	-	1.33	-	1.10	-
2	Isolated Crack	-	1.34	-	1.22	-	1.11	-	1.04
2	Infinite Array	-	1.52	-	1.28	-	1.15	-	1.05
2	Koskinen	-	1.54	-	1.28	-	1.15	-	1.04
$\frac{4}{3}$	Isolated Crack	1.08	-	1.05	-	1.03	-	1.01	-
$\frac{4}{3}$	Infinite Array	1.25	-	1.12	-	1.06	-	1.03	-
$\frac{4}{3}$	Koskinen	1.25	-	1.13	-	1.06	-	1.03	-

Figure 1

A Plastically Deforming Notch in Antiplane Strain

(following Hult and McClintock)

FIG 1



Conformal Mappings in the Hult and McClintock Analysis.

Fig. (2) Stress Space (σ plane)

Fig. (3) ζ plane

Fig. (4) η plane

Fig. (5) τ plane

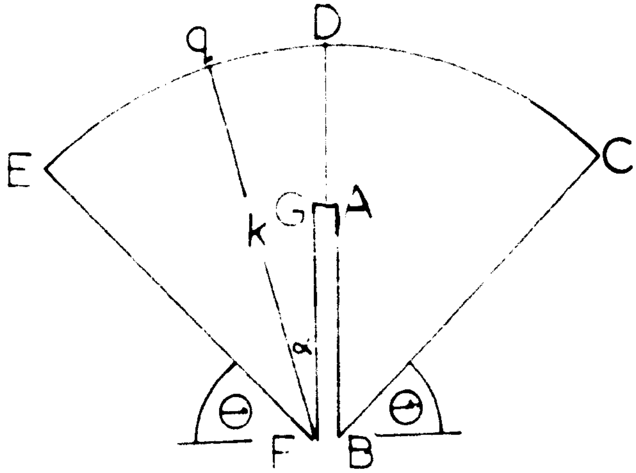
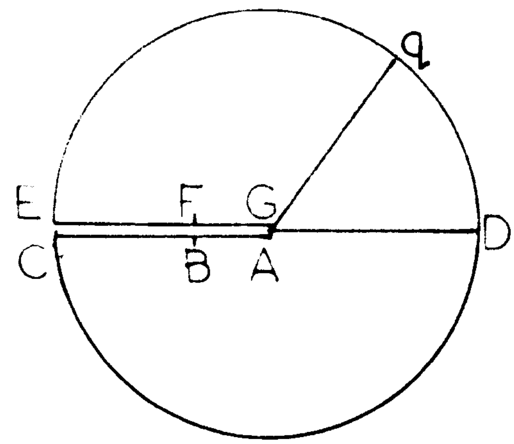


FIG 2

FIG 3



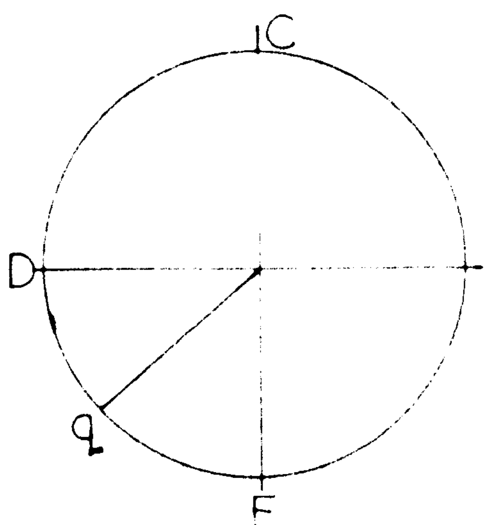


FIG 4

FIG 5

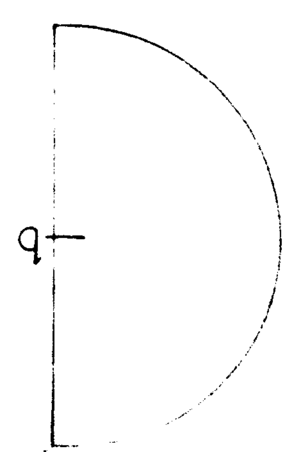


Figure 6

Crack Subjected to a Normal Tension Relaxing Plastically at the Tips. (Following Dugdale).

crack length 2c

extent of relaxation s

yield stress y

FIG 6

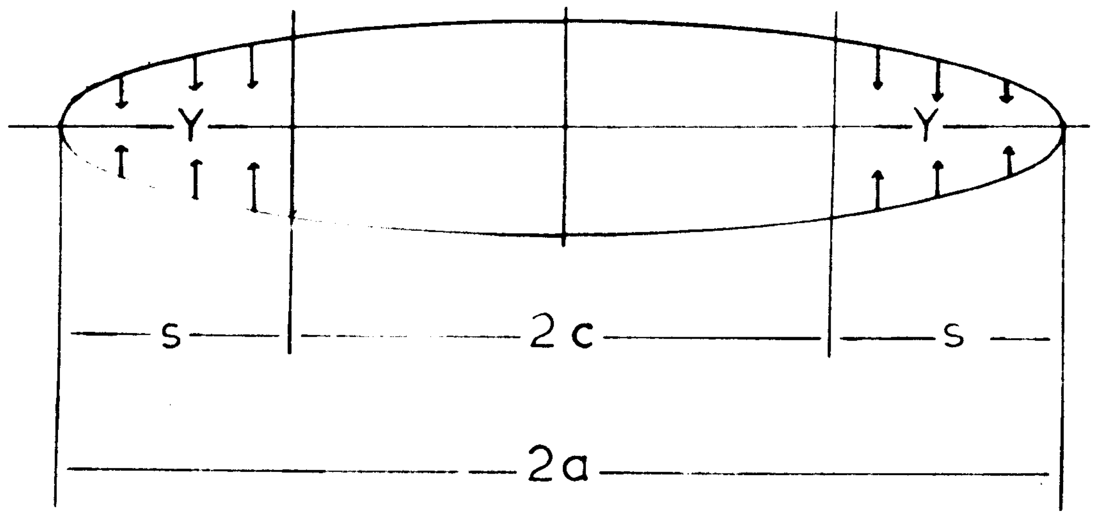


Figure 7

A Member of the Sequence of Stress Functions Having Limit $p(x_1)$.

FIG 7

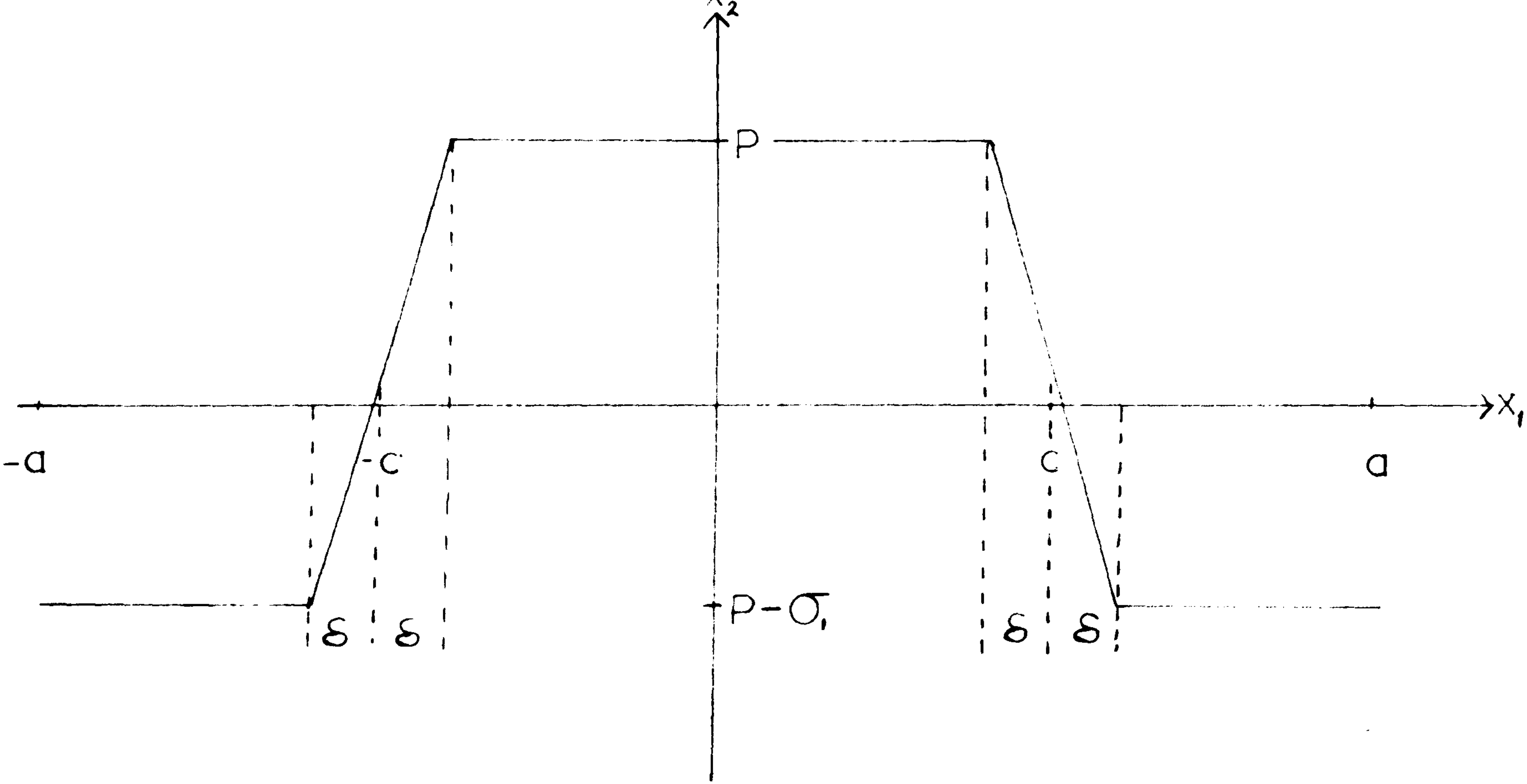


Figure 8

Dislocation Model of a Plastically Relaxing Shear Crack.

---

showing the distribution of dislocations along a sheared slit $|x_1| < c$ and its associated yield zones $c < |x_1| < a$.

FIG 8

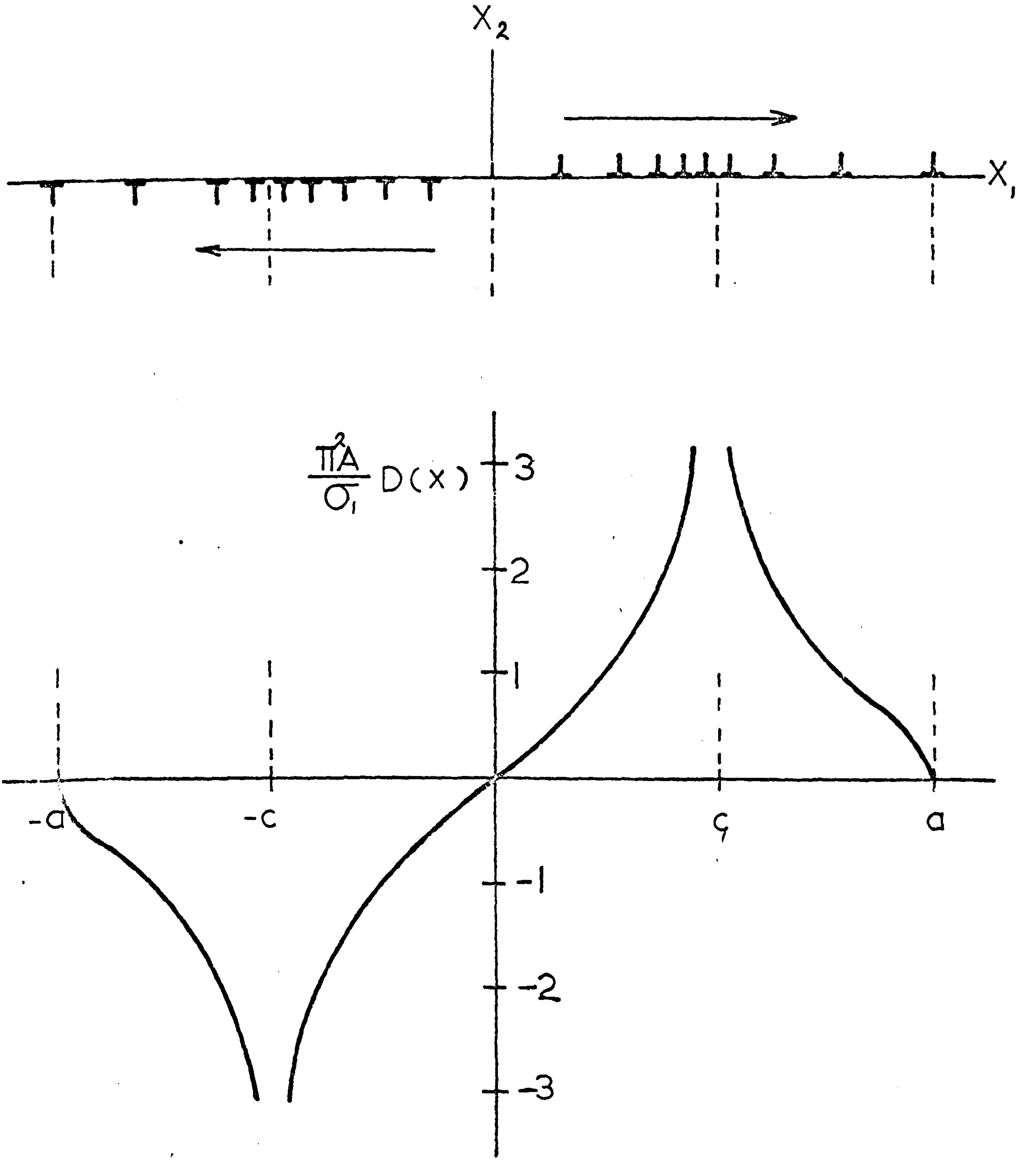


Figure 9

The Extent of Plastic Zones at an Isolated Crack as a Function of Applied Shear Stress.

Curve A according to the Dislocation model.

Curve B according to the Hult McClintock model.

FIG 9

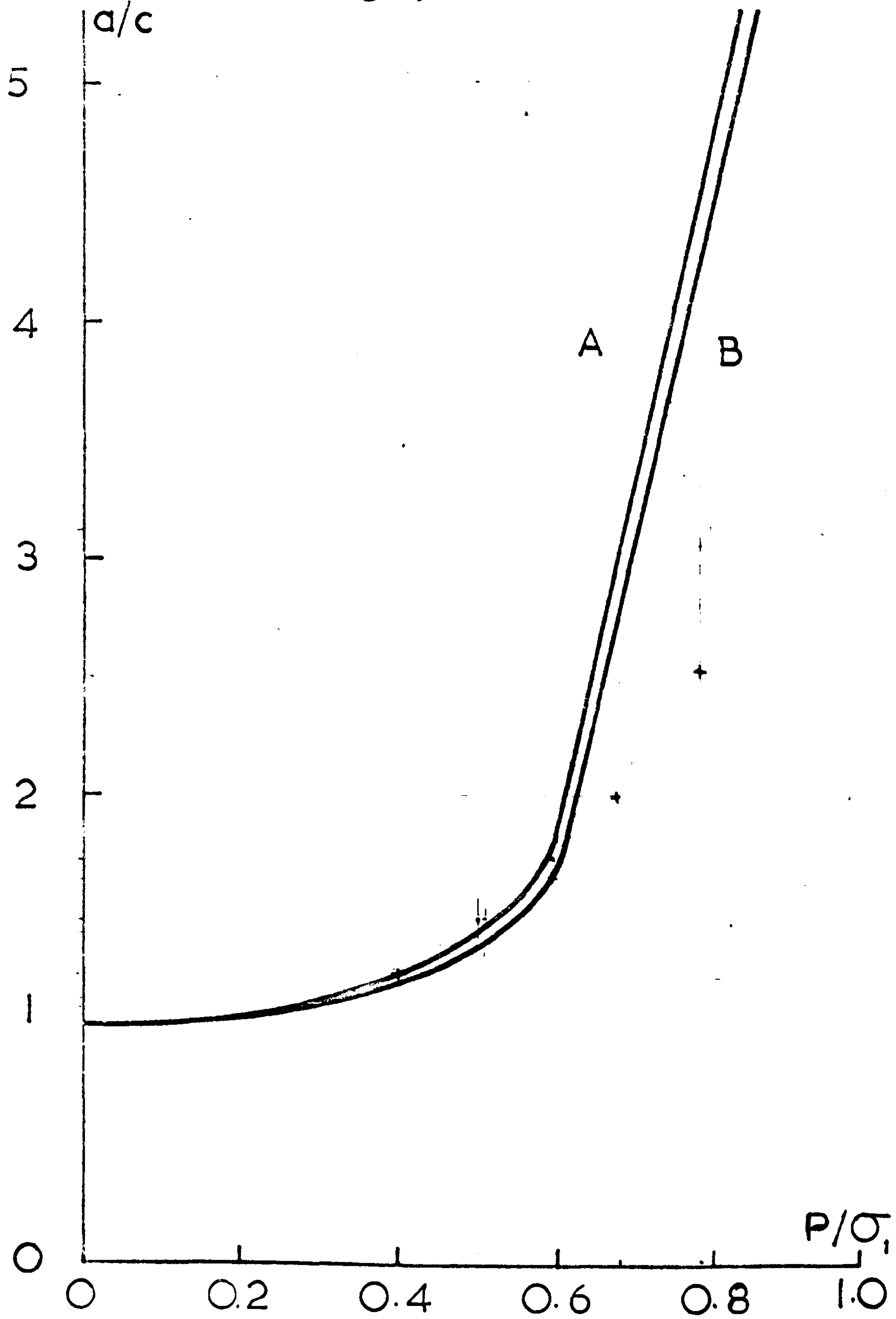


Figure 10

An Infinite Periodic Array of Coplanar Cracks Relaxing Plastically Under an Applied Stress

---

crack length $2c$

extent of relaxation $a-c$

period 2ℓ

FIG 10

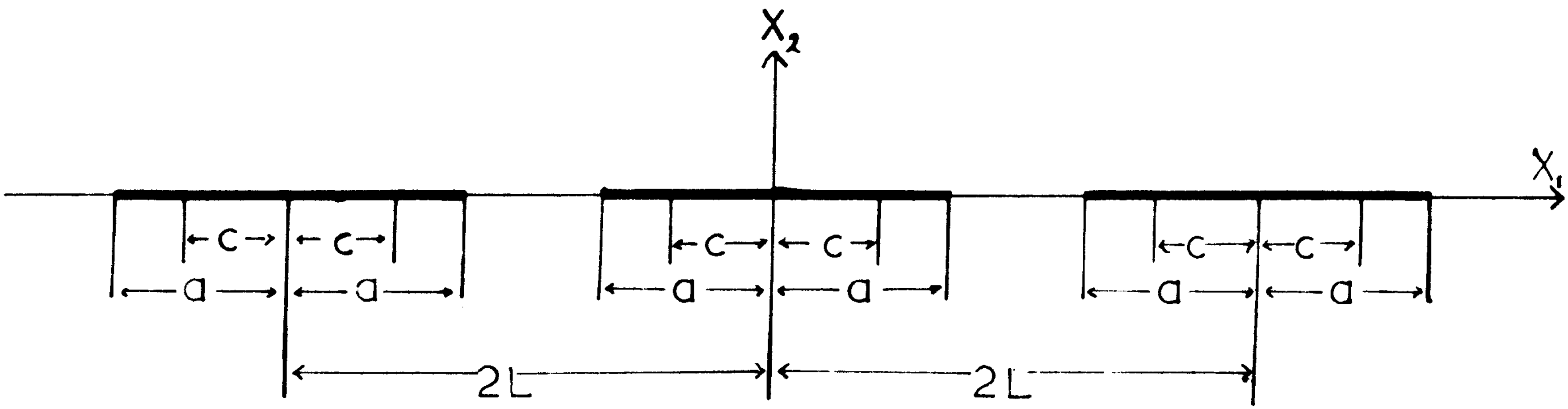


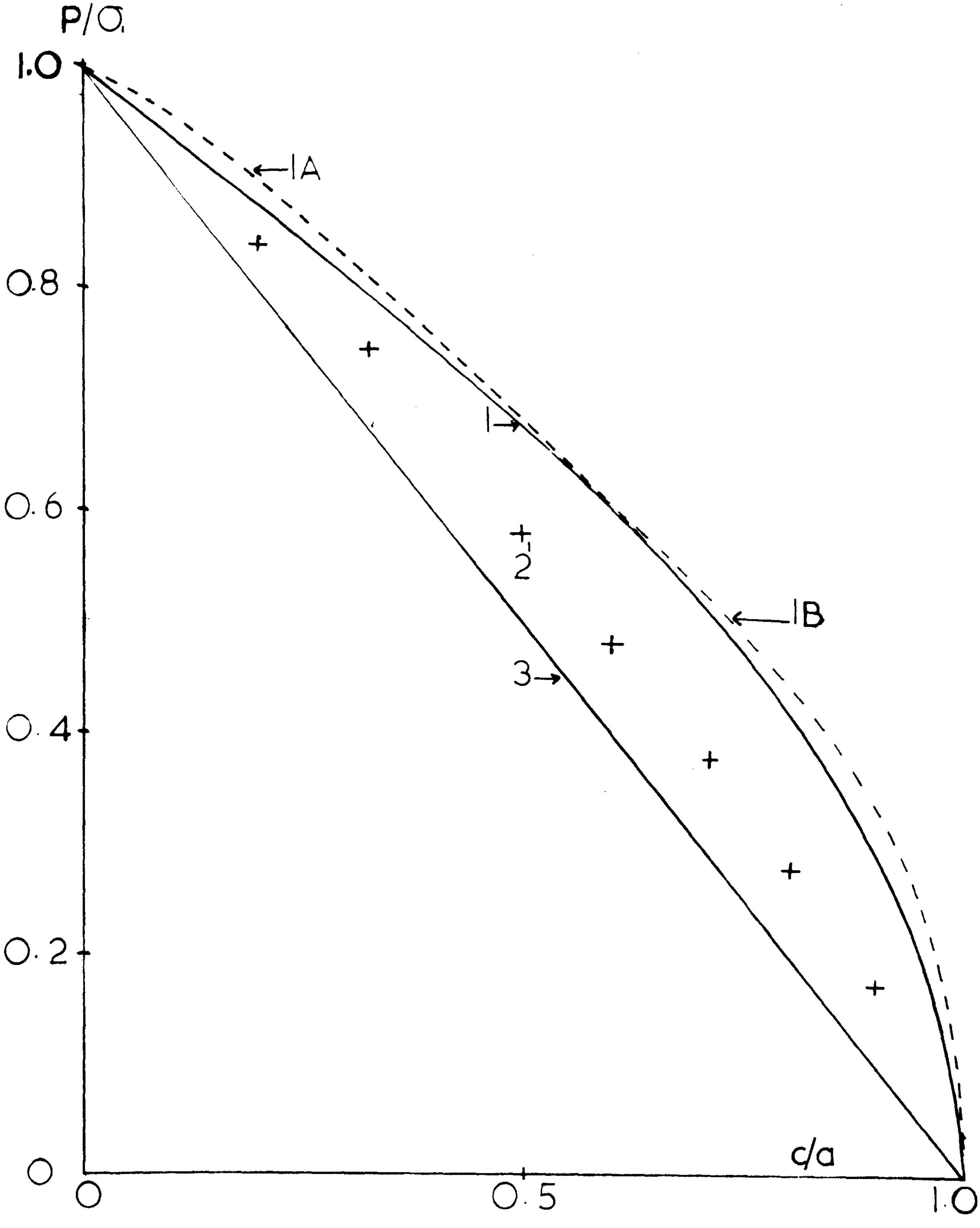
Figure 11

The Stress at General Yield According to the Various Theories.

P	applied stress
σ_1	yield stress
2c	crack length
a-c	length of plastic zone
	(at general yield $a = \ell$ where 2ℓ is the distance between the centres of the cracks or, in isolated crack theories, a has some prescribed value

---

curve	1	isolated crack theory
	1A	variation caused by workhardening
	1B	variation in tensile case when relaxation is along 2 inclined planes.
curve	2	double crack theory of Smith
curve	3	periodic array theory.



P/σ
1.0
0.8
0.6
0.4
0.2
0
1A
1
2
1B
3
c/a
0
0.5
1.0

Figure 12

Extent of Plastic Zones From a Crack of Length $2c$ in a Uniform Array of Period 2ℓ

---

showing the relationship between a/ℓ and c/ℓ for contours of fixed P/σ_1

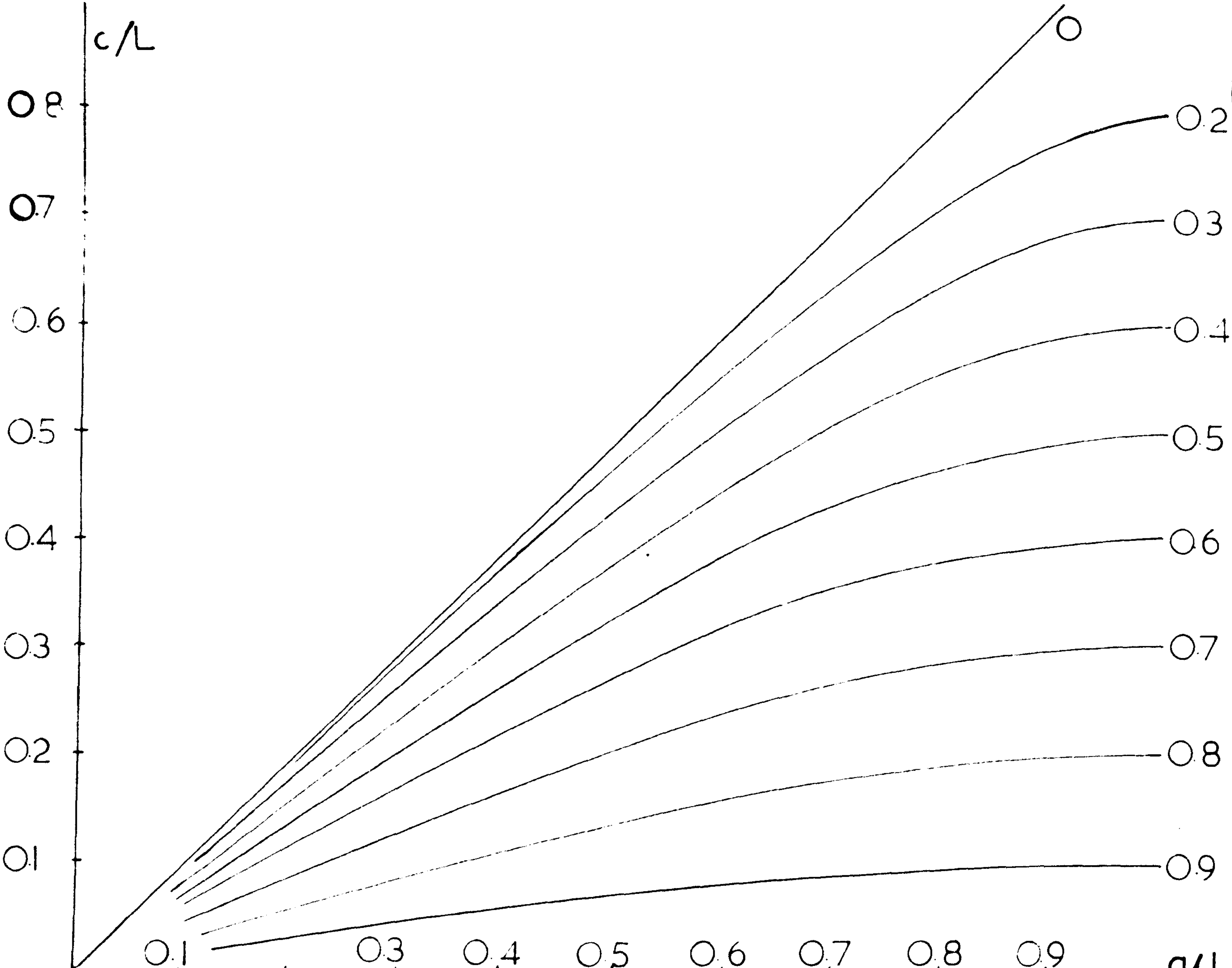
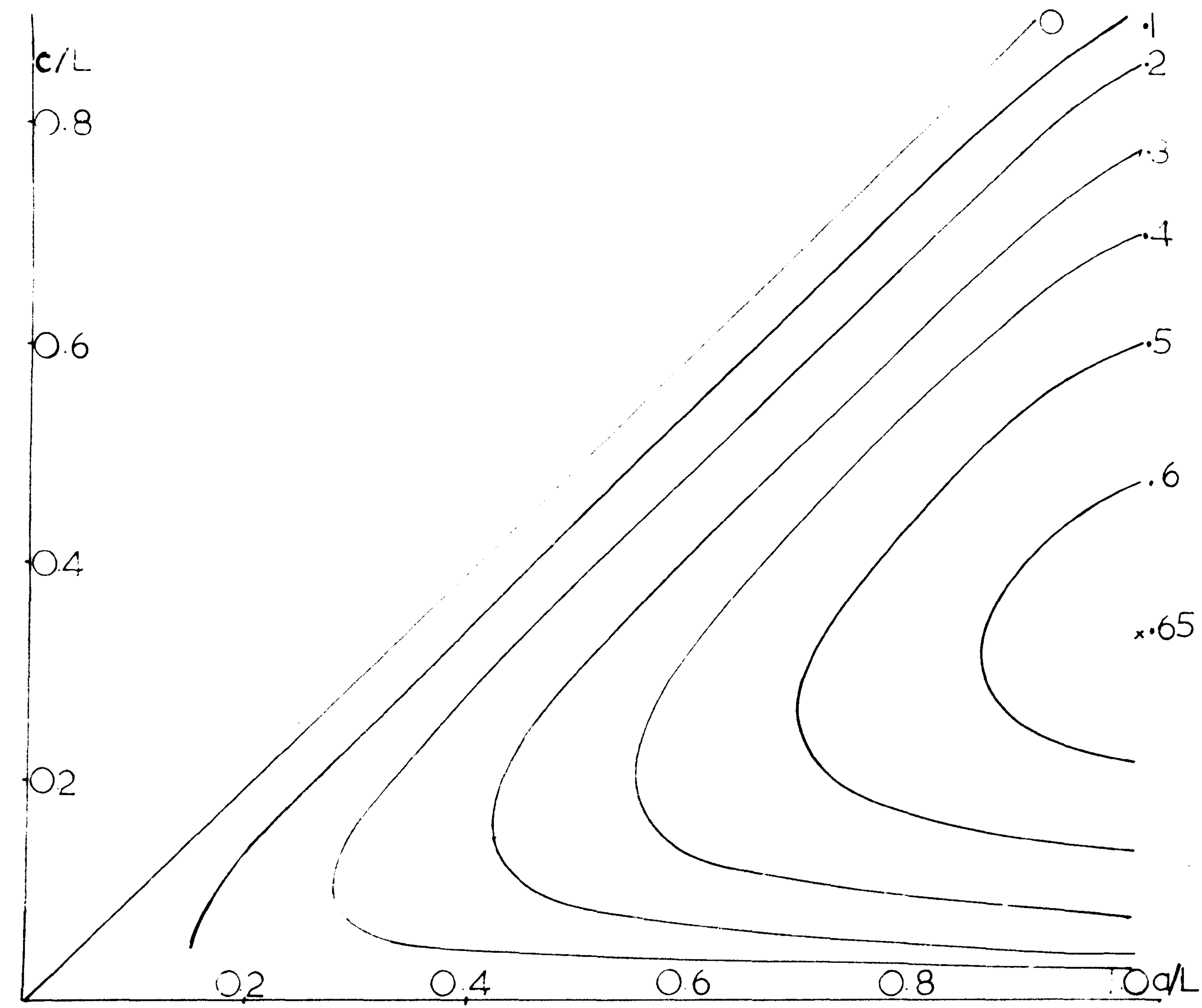

c/L
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
P/σ₁
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
0.1
0.3
0.4
0.5
0.6
0.7
0.8
0.9
a/L

Figure 13

The Relative Displacement at the Tip of a Crack in a Uniform Periodic Array Shown as a Function of a/ℓ and c/ℓ

---

showing the relation between a/ℓ and c/ℓ for contours of fixed $\frac{\pi^2 A}{\sigma_1 b} \frac{\Phi(c)}{\ell}$



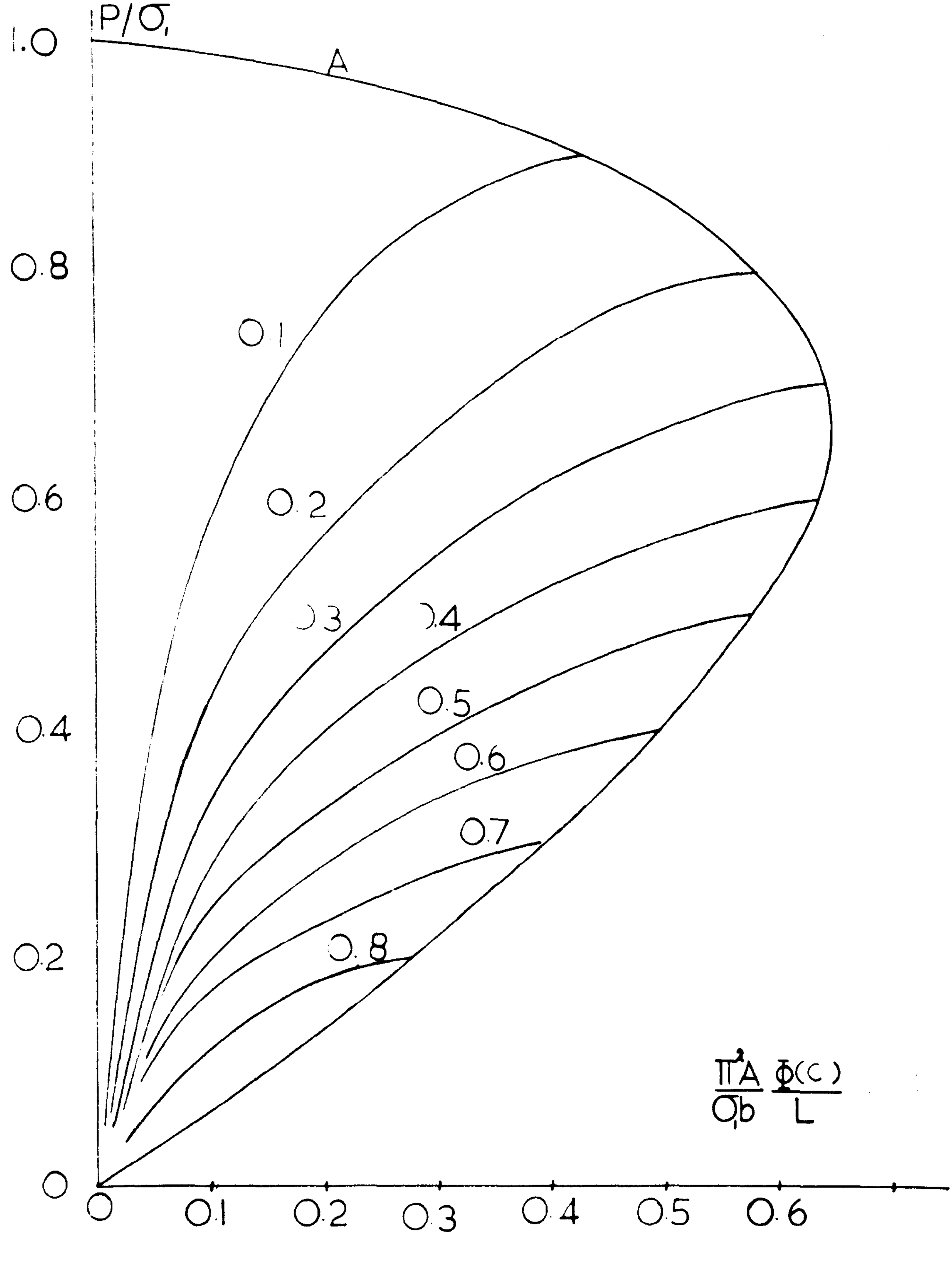
c/L
0.8
0.6
0.4
0.2
0.2
0.4
0.6
0.8
1.0
a/L
0
·1
·2
·3
·4
·5
·6
·65

Figure 14

Relation between Stress and Relative Displace at the Tip of a Crack of Length 2c in a Uniform Array of Period 2ℓ.

---

showing the stress P/σ_1 to produce a displacement $(\pi^2 A/\sigma_1 b)\Phi(c)/\ell$ for contours of fixed c/ℓ. The contours terminate on the yield envelope A.



P/σ₁
1.0
0.8
0.6
0.4
0.2
0
A
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
$\frac{\pi^2 A}{\sigma_1 b} \frac{\Phi(c)}{L}$
0
0.1
0.2
0.3
0.4
0.5
0.6

Figure 15

Relative Displacement $(\Phi(c)/\ell$ at the Tip of a Crack at General Yield, (and for certain stresses), according to the Isolated Crack Theory and the Periodic Array Theory.

$(\pi^2 A/\sigma_1 b)\ \Phi(c)/\ell$ as a function of c/ℓ

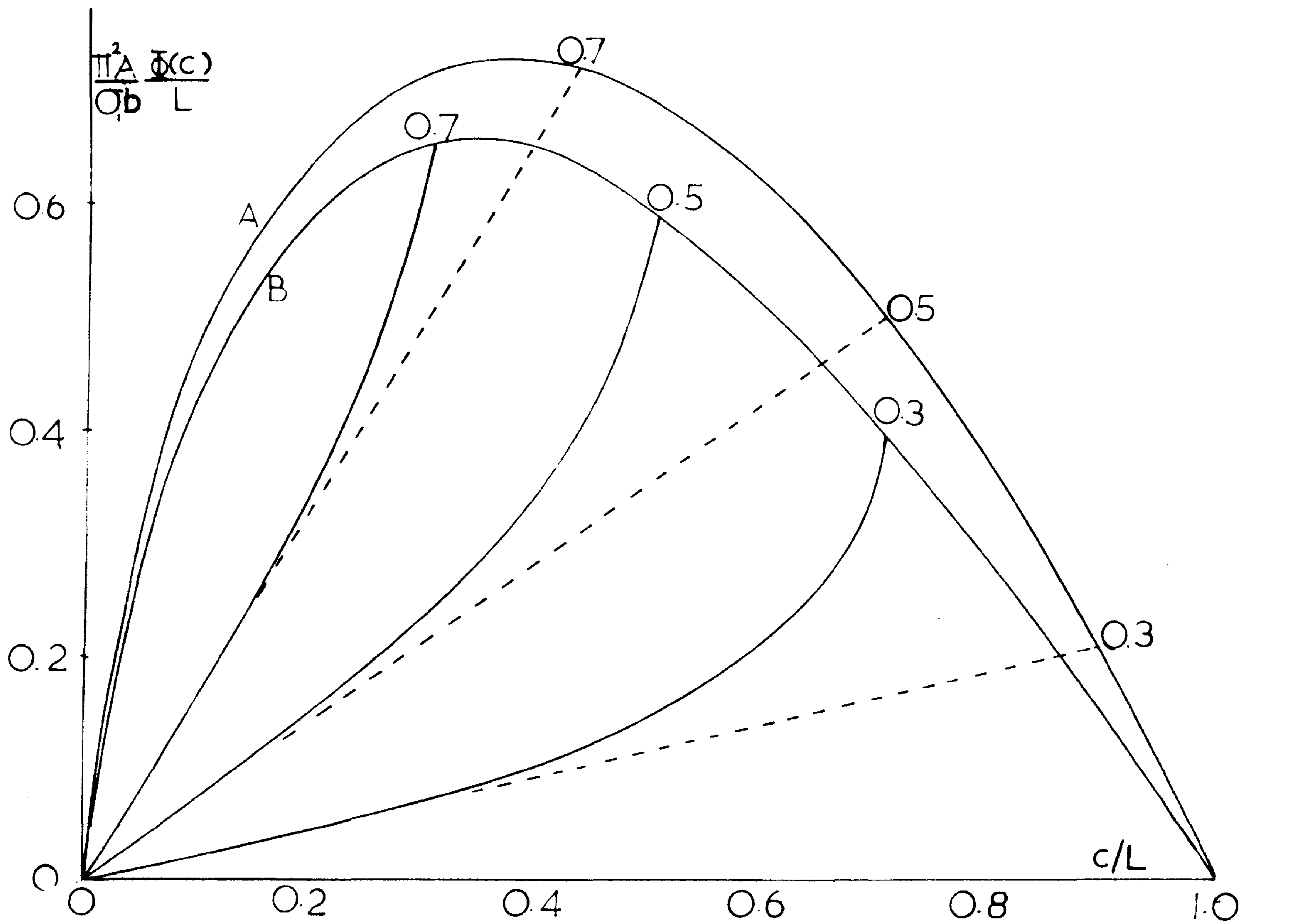
curve A - isolated crack theory

curve B - infinite array theory

$(\pi^2 A/\sigma_1 b)\ \Phi(c)/\ell$ as a function of c/ℓ for three values of the applied stress P/σ_1 equal to 0.7, 0.5, 0.]

Broken line - isolated crack theory

Full line - infinite array theory.



$\frac{\pi^2 A}{\sigma_1 b} \frac{\Phi(c)}{L}$
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1.0
c/L
A
B
0.7
0.7
0.5
0.5
0.3
0.3

Figure 16.

Relative Displacement $(\Phi(c)/c)$ at the Tip of a Crack at General Yield, (and for certain stresses), according to the Isolated Crack Theory and the Periodic Array Theory.

$(\pi^2 A/\sigma_1 b)\ \Phi(c)/\ell$ as a function of c/ℓ

curve A - isolated crack theory

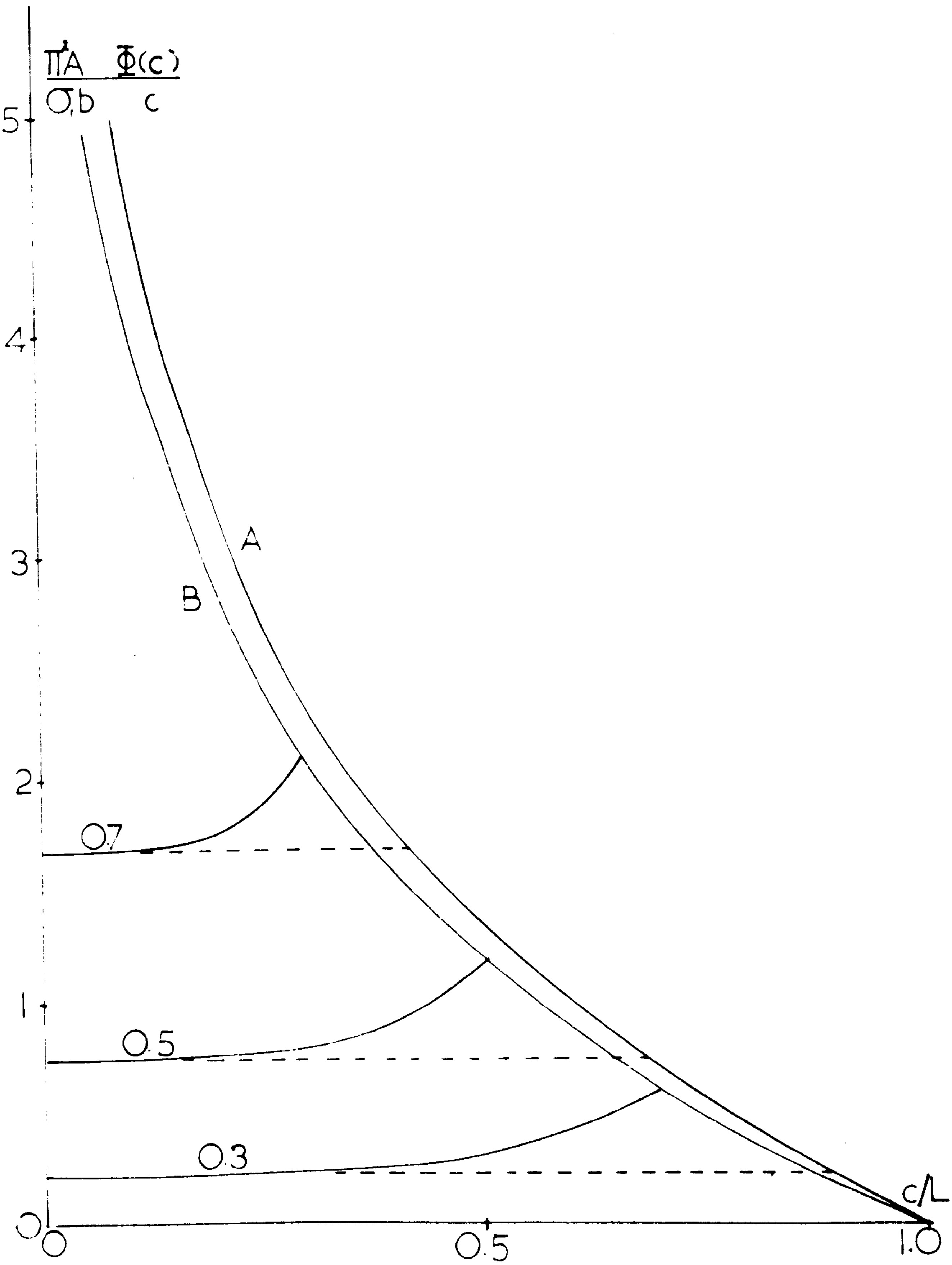
curve B - infinite array theory

(N.B. This section of the figure follows a suggestion of E. Smith. Bilby, Cottrell, Smith and Swinden 1964).

$(\pi^2 A/\sigma_1 b)\ \Phi(c)/\ell$ as a function of c/ℓ for three values of P/σ_1 equal to 0.7, 0.5 and 0.3.

Broken line - isolated crack theory

Full line - infinite array theory



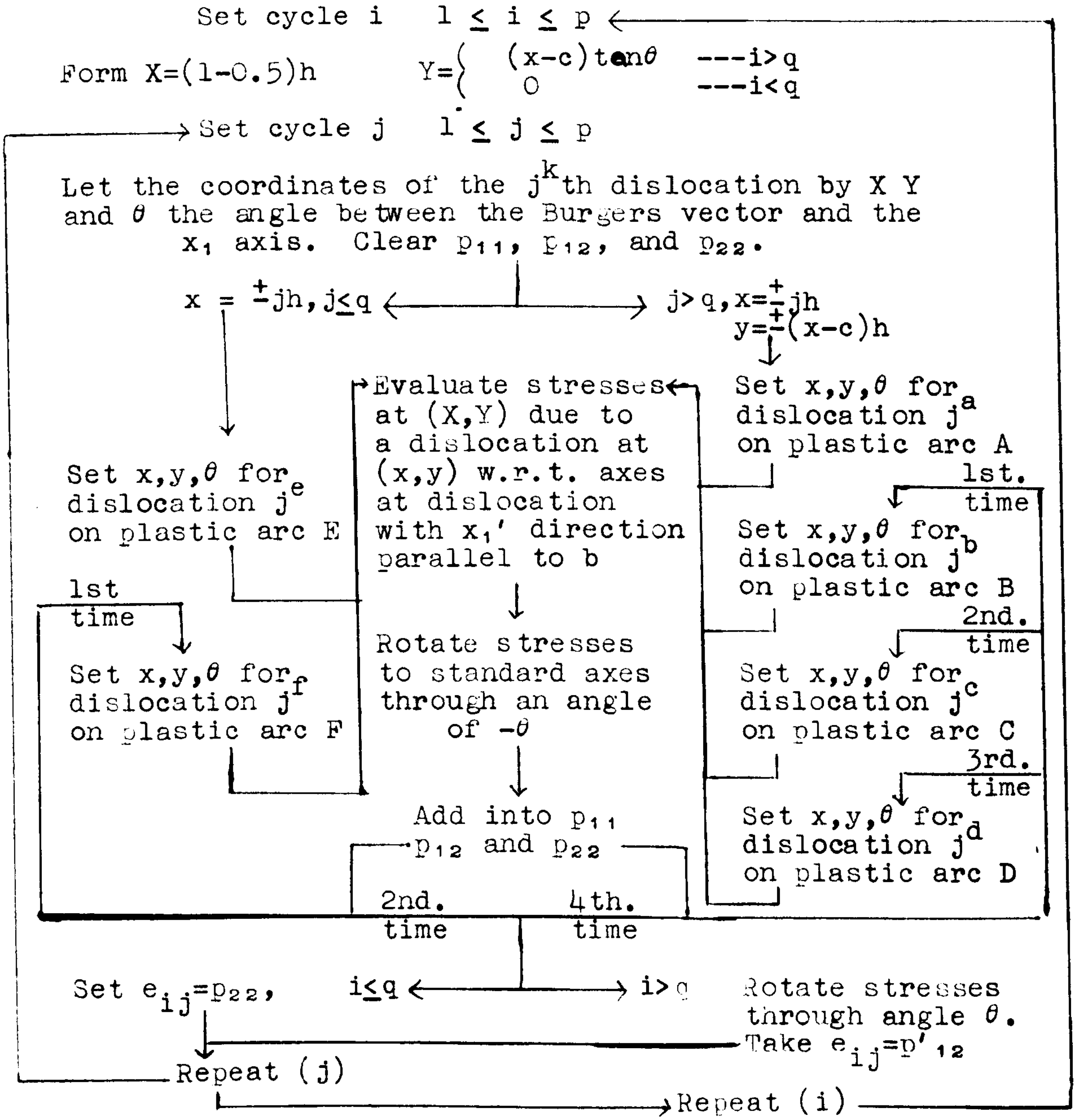
$\frac{\pi^2 A}{\sigma_1 b} \frac{\Phi(c)}{c}$
5
4
3
2
1
0
A
B
0.7
0.5
0.3
c/L
0
0.5
1.0

Figure 17

Flow diagram for the numerical analysis of a tensile crack relaxing along inclined planes.

Let the total number of intervals be p, the intervals of the crack q and the width of the intervals h.

Form the p x 2 stress matrix S
Col.1 Resistance stresses, Col. 2 Applied stresses



Form $D = E^{-1}S$ (where $E = e_{ij}$) and print optionally.

Form stress ratio, dislocation density and displacement at c.

Figure 18

Relative Displacements at the Tip of a Crack in an Infinite Workhardening Material Based on the Root Strain Method.

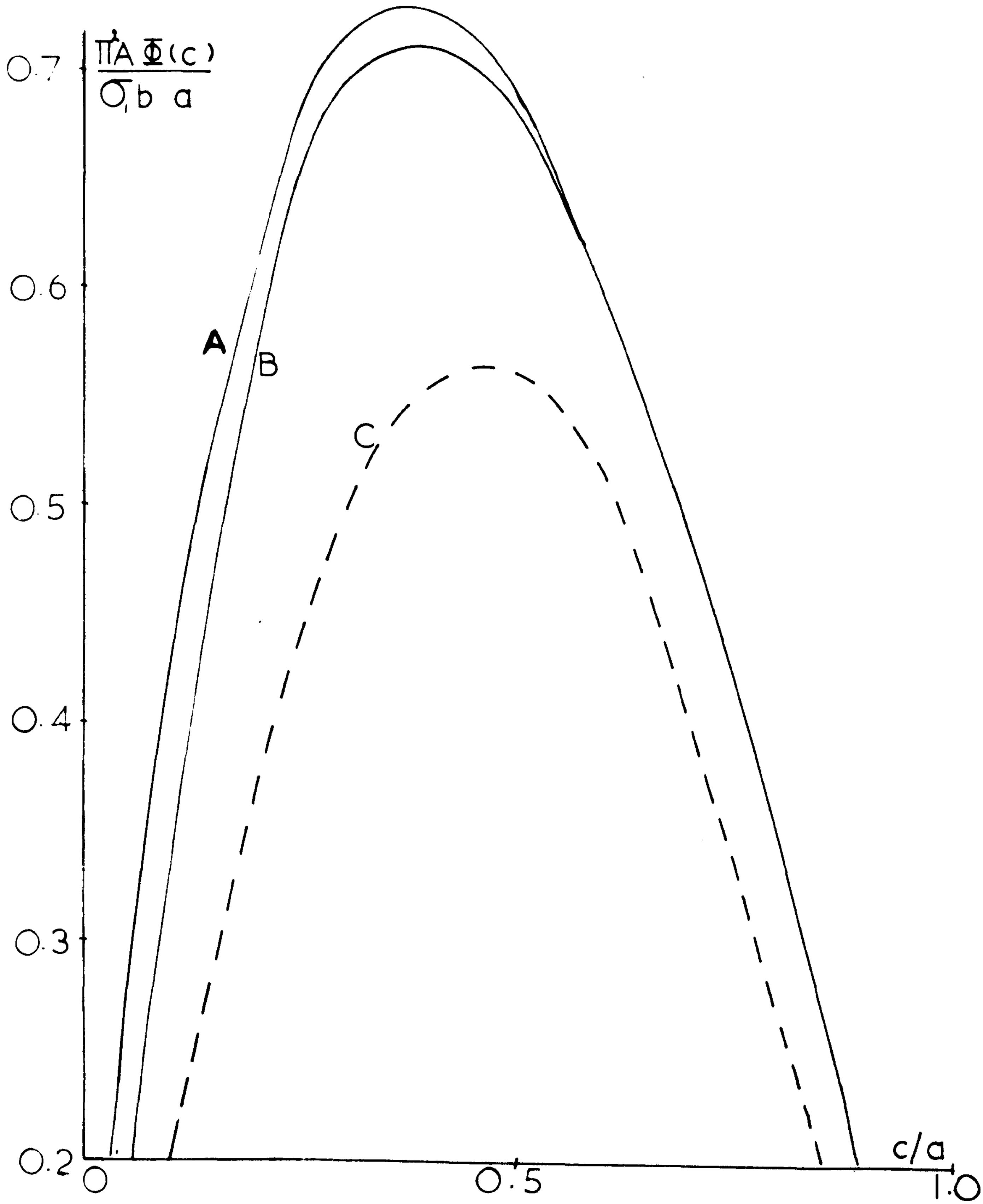
---

curve A - $X = wb/\pi^2 A = 0.4 \times 10^{-3}$

curve B - $= 2.0 \times 10^{-3}$

curve C - $= 10.0 \times 10^{-3}$

Broken line indicates negative displacements arise at points on the plastic arc removed from the crack tip.



$\frac{\pi A \Phi(c)}{\sigma_1 b\ a}$
0.7
0.6
0.5
0.4
0.3
0.2
A
B
C
0
0.5
1.0
c/a

Figure 19

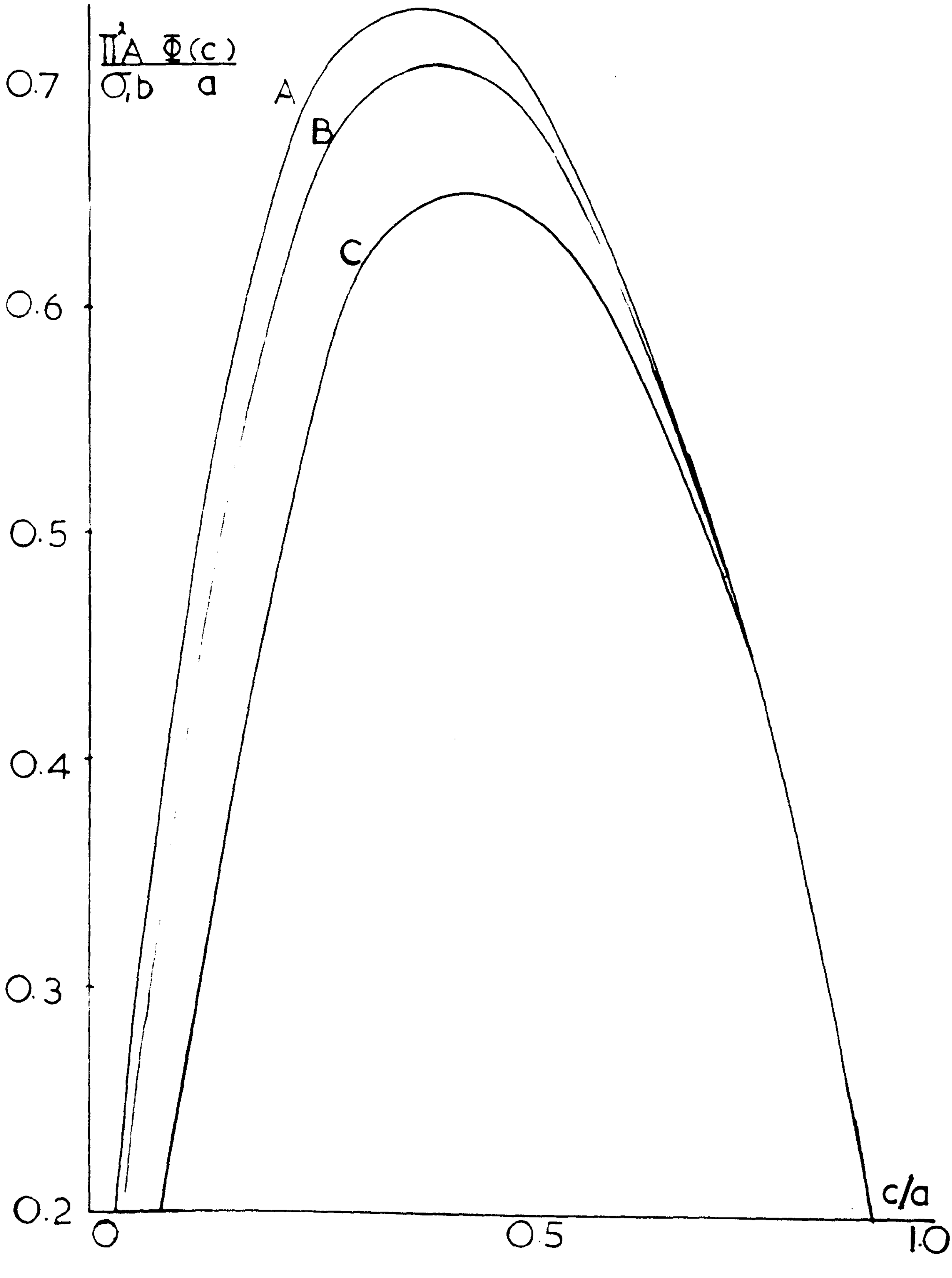
Relative Displacements at the Tip of a Crack in an Infinite Workhardening Material, Based on the Parabolic Method.

---

curve A $\quad X = wb/\pi^2 A = 0.4 \times 10^{-3}$

curve B $\quad = 2.0 \times 10^{-3}$

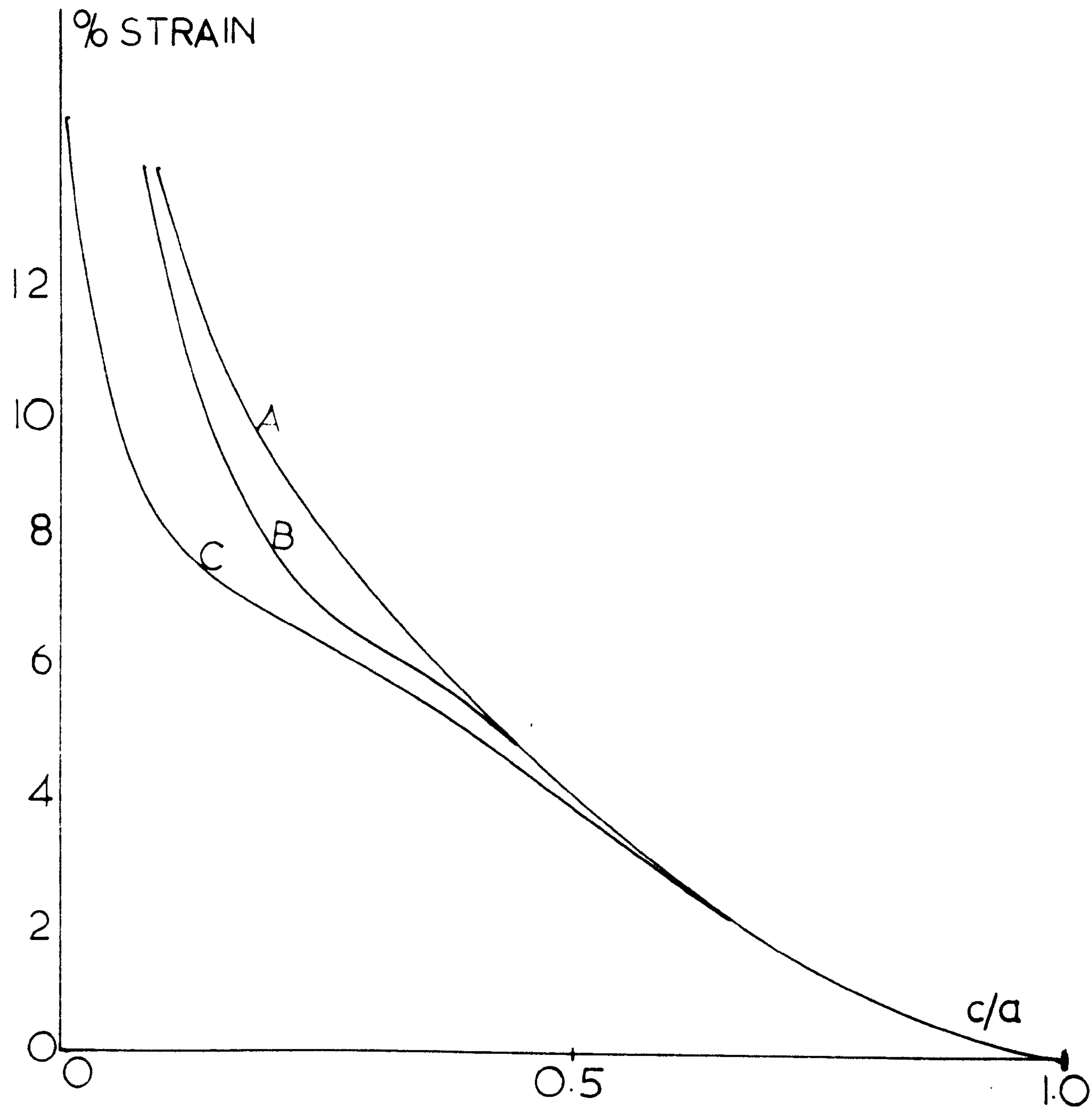
curve C $\quad = 10.0 \times 10^{-3}$



$\frac{\pi^2 A}{\sigma_1 b} \frac{\Phi(c)}{a}$
0.7
0.6
0.5
0.4
0.3
0.2
A
B
C
0
0.5
1.0
c/a

Figure 20

Strain at the Tip of a Crack in an Infinite Workhardening Material Based on the Parabolic Method.



% STRAIN
12
10
8
6
4
2
0
A
B
C
c/a
0
0.5
1.0

Figure 21

Orientation of Dislocations Representing an Isolated Crack Relaxing Along Inclined Planes.

---

intermediate axes taken at the dislocation are shown with the arrow pointing in the positive x_1' direction, that being the direction of the Burgers vector b.

crack length 2c

Projection of the plastic arcs on to the plane of the crack a-c

Plastic arcs meet in an angle 2α equal to one right angle in the analysis used.

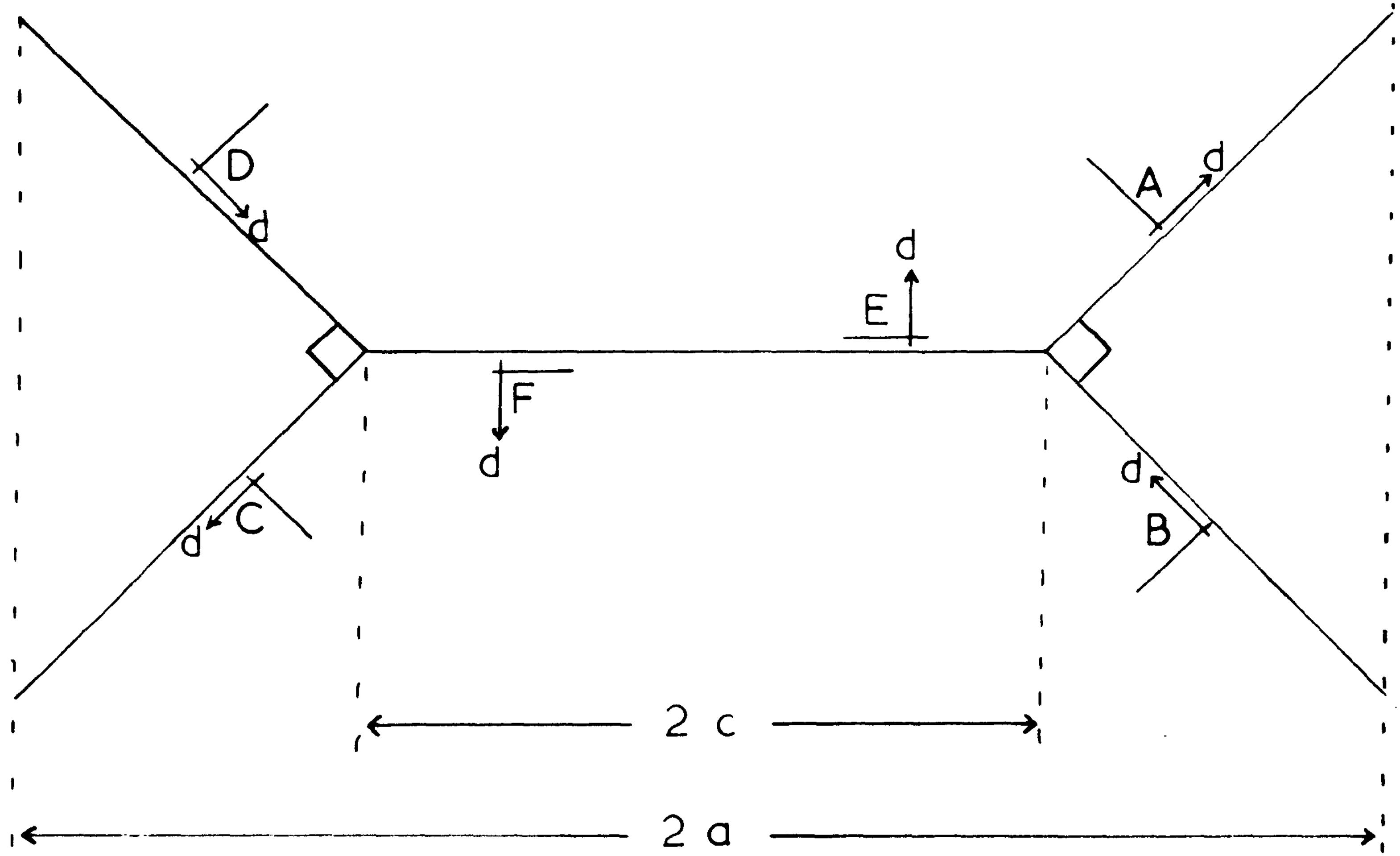

D
d
C
d
F
d
E
d
A
d
B
d
2 c
2 a

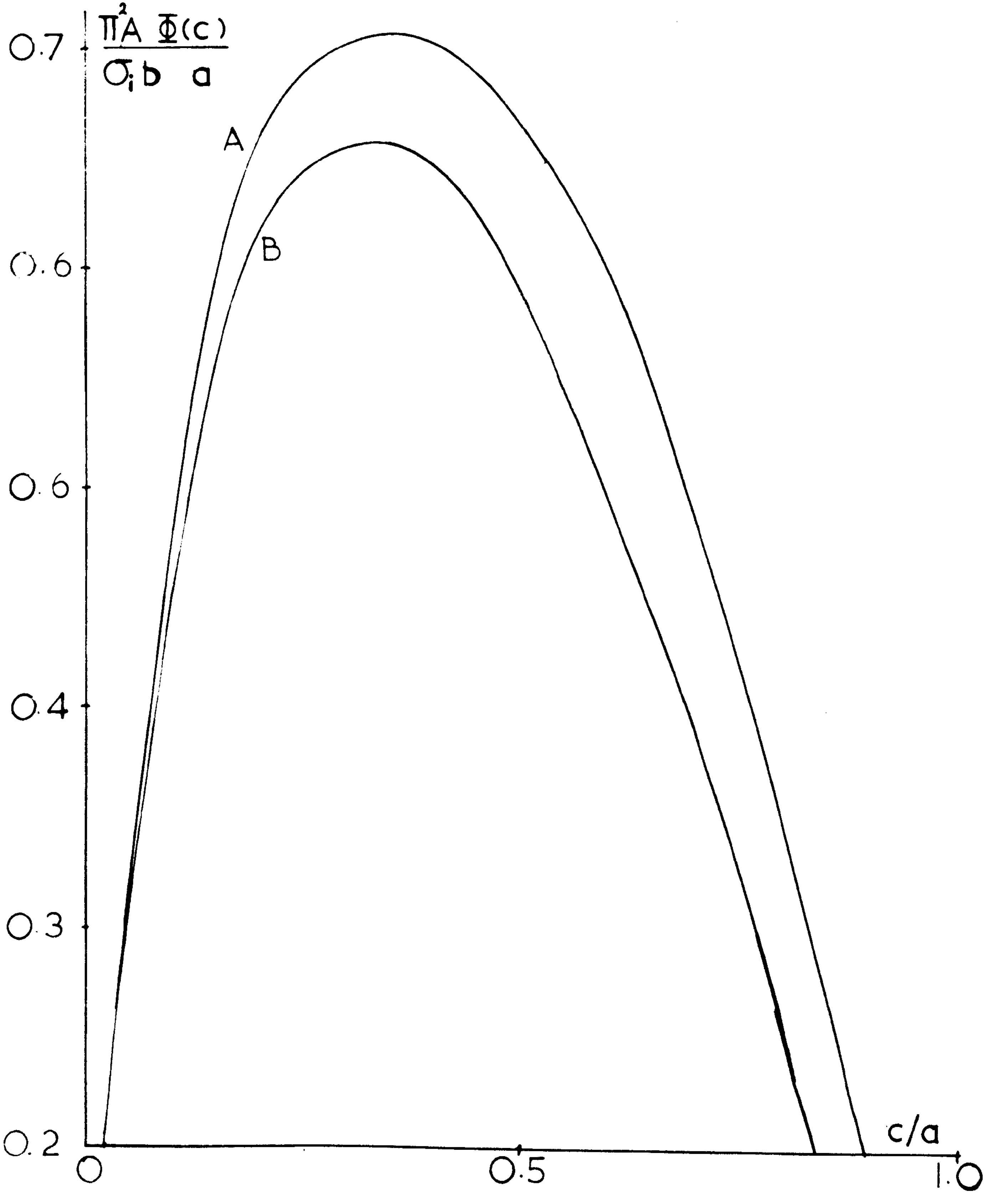
Figure 22

Displacement at the Tip of an Isolated Crack Relaxing Along Inclined Planes.

---

curve A — Displacements for single plane model in terms of σ_i the yield stress in the plastic zones. Shear and tensile case gives displacements of the same magnitude.

curve B — Shear displacement at the crack tip across a single plastic arc, in terms of the resistance to dislocation motion.



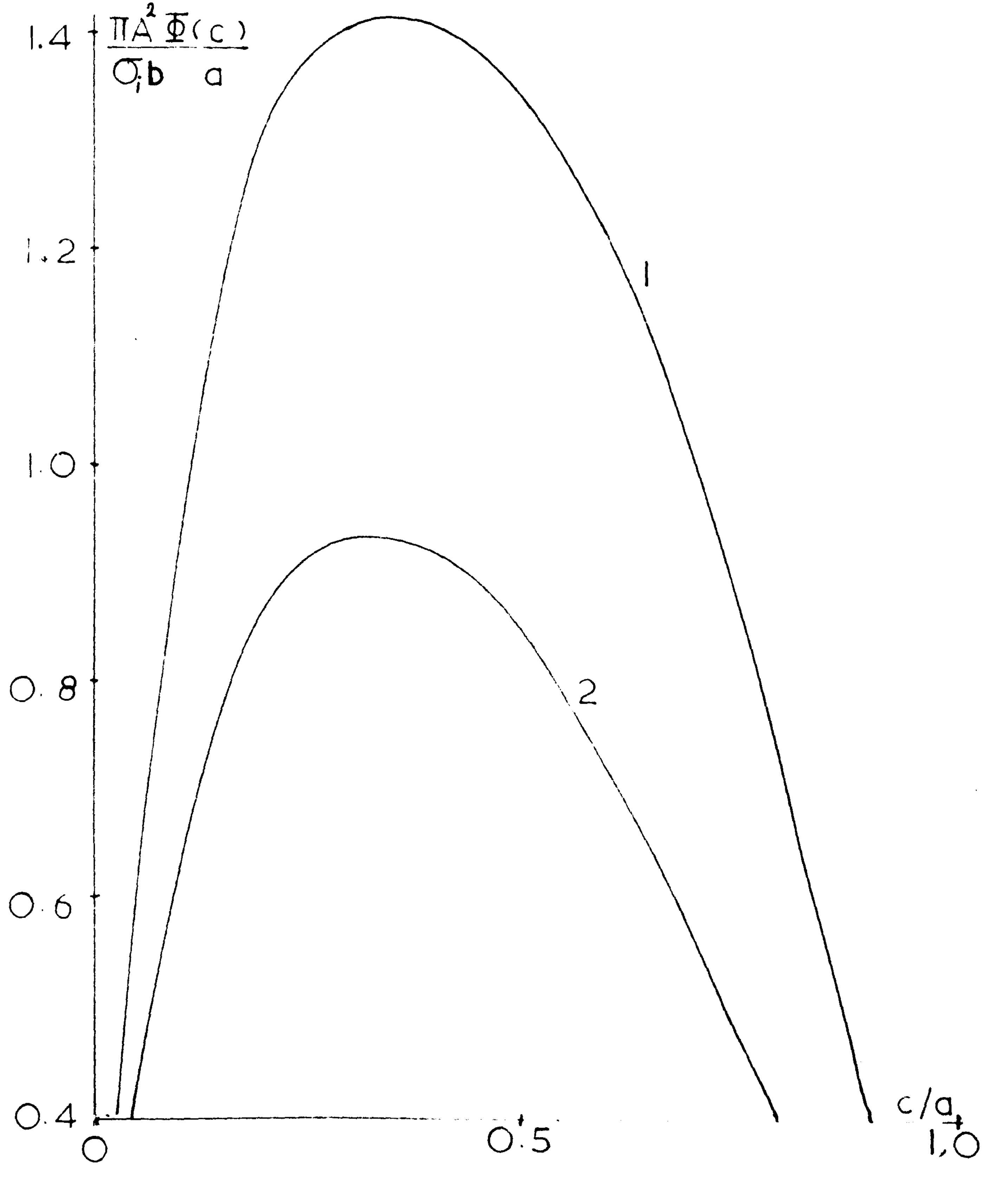
$\frac{\pi^2 A}{\sigma_i b} \frac{\Phi(c)}{a}$
0.7
0.6
0.6
0.4
0.3
0.2
A
B
c/a
0
0.5
1.0

Figure 22A

Displacement at the Tip of an Isolated Crack Relaxing Along Inclined Planes.

---

curve 1	The relative normal displacements in terms of σ_i where $2\sigma_i = \sigma_1$ = yield stress in tension.
curve 2	The relative normal displacement at the crack tip in terms of σ_i the resistance to dislocation motion.



$\frac{\pi A^2 \Phi(c)}{\sigma_i b \quad a}$
1.4
1.2
1.0
0.8
0.6
0.4
1
2
c/a
0
0.5
1.0

Appendix I

Evaluation of Integrals and Sums

A1.1 Preliminary definitions.

In this work it has been found necessary to define the following functions for y within the range $(-a,a)$ of the real axis :

$$R(x,y) = \left\{\sqrt{[(a^2 - x^2)/(a^2 - y^2)]}\right\}\left\{1/(x - y)\right\} \qquad \text{A1.1.1}$$

$$H(a,x,y) = \cosh^{-1}\left\{\left|[(a^2 - x^2)/a(x - y)]+[x/a]\right|\right\} \qquad \text{A1.1.2}$$

Further defined are the integrals :

$$I(\alpha,\beta) = \int_{\alpha}^{\beta} R(xy)dy \qquad \text{A1.1.3}$$

$$I_j(\alpha,\beta) = \int_{\alpha}^{\beta} (x-c)^j \, H(a,c,x)dx \qquad \text{A1.1.4}$$

Here again $\alpha < \beta$ are both contained in the range $(-a,a)$ of the real axis. The detailed steps in the evaluation of the Cauchy principle values of these integrals are given in this appendix rather than in the main text.

It is first necessary to consider the meaning to be given to the inverse hyperbolic cosine when the argument is negative. Since these functions arise from integrals a suitable working definitions is :

$$\cosh^{-1}(h) = \int_1^h \sqrt{(\nu^2-1)}\, d\nu \qquad \text{A1.1.5}$$

This is extended to include negative values as follows :

$$\cosh^{-1}(-h) = \int_1^{-h} \sqrt{(\nu^2-1)}\, d\nu \qquad \text{A1.1.6}$$

The integrand in the second term is imaginary and must be divided by (i). Then writing $u = -\nu$ in the first term gives :

$$\cosh^{-1}(-h) = -\int_1^h \sqrt{(u^2-1)}\, du - i\int_1^{-1} \sqrt{(1-\nu^2)}\, d\nu$$

$$= -\cosh^{-1}(+h) + \pi i \qquad \text{A1.1.7}$$

A1.2 Definition and properties of the function h(a,x,y)

In these integrals h is in general a function of three variables which has the form :

$$h(a,x,y) = [(a^2 - x^2)/a(x - y)] + [x/a] \qquad \text{A1.2.1}$$

where a>0, x and y being unrestricted. This function has some useful symmetry in the variables x and y. It follows from the relation

$$(a^2 - xy)/a(x-y) = [(a^2 - x^2)/a(x - y)] + [x/a]$$

$$= -[(a^2 - y^2)/a(y - x)] - [y/a] \qquad \text{A1.2.2}$$

that

$$h(a,x,y) = -h(a,y,x) = h(a,-y, -x) \qquad \text{A1.2.3}$$

Since the function $H = \cosh^{-1}(h)$ is required, it is necessary to consider the ranges of x and y for which $|h| > 1$.

Supposing x fixed and y variable, it is seen at once that :

$$h(a, x, -a) = +1 \qquad \text{A1.2.4}$$

$$h(a, x, +a) = -1 \qquad \text{A1.2.5}$$

Differentiation with respect to y gives :

$$\partial/\partial y[h(a,x,y)] = (a^2 - x^2)/a(x - y)^2 \qquad \text{A1.2.6}$$

h is therefore a monotonic function of y which is increasing for $|x|<a$, decreasing for $|x|>a$ and a constant $h \equiv 1$ for $x = a$. It follows therefore that h changes sign through all points where h is either zero or infinite. These points are :

$$\left.\begin{array}{ll} y = x & h = \infty \\ y = a^2/x & h = 0 \end{array}\right\} \qquad \text{A1.2.7}$$

Further it can be seen from A1.2.1 that :

$$h \to x/a \quad \text{as } |y| \to \infty \qquad \text{A1.2.8}$$

In view of relation A1.2.6 it is necessary to consider two separate cases $|x| < a$ and $|x| > a$.

In the first case h is monotonic increasing with y and changes sign through infinity in the range $|y| \leq a$ at the point $y = x$. Therefore relations A1.2.4, A1.2.5 and A1.2.8 give:

$$\begin{array}{ll} y < -a, & x/a < h < 1 \\ -a < y < x, & 1 < h \\ x < y < a, & h < -1 \\ a < y, & -1 < h < x/a \end{array} \qquad \text{A1.2.9}$$

In the second case h is monotonic decreasing with y and changes sign through zero in the range $|y| \geq a$ at the point $y = a^2 / x$. Similarly it can be shown that for $x < -a$:

$$\begin{aligned} y &< x, & h &< x/a < -1 \\ x &< y < -a, & h &> 1 \\ -a &< y < a, & -1 &< h < 1 \\ a &< y, & 1 &< h \end{aligned} \qquad \text{A1.2.10}$$

and for $x > a$:

$$\begin{aligned} y &< -a, & x/a &> h > 1 \\ -a &< y < a, & -1 &< h < 1 \\ a &< y < x, & 1 &< h \\ x &< y, & 1 &< x/a < h \end{aligned} \qquad \text{A1.2.11}$$

Finally it follows from A1.2.9, A1.2.10 and A1.2.11 that for $|h| \geq 1$ it is necessary and sufficient that x and y both belong to the same closed set of points ζ of the real axis, where the sets are $\zeta \leq |a|$ and $\zeta \geq |a|$.

A1.3 Definition and derivatives of the function H(a,x,y)

Now the following definition is made :

$$H(a,x,y) \equiv \cosh^{-1}(|h|) \qquad \text{A1.3.1}$$

where $|x| \leq a$, and $|y| \leq a$ and h is defined by A1.2.1 . The conclusion of the previous section shows that H so defined is a real function.

From A1.2.3 it follows that :

$$H(a,x,y) = H(a,y,x) = H(a,-y,-x) \qquad \text{A1.3.2}$$

To evaluate the derivative of H with respect to y it is necessary to consider two cases, namely h positive and h negative. Now $h > 0$, A1.2.9 implies $y < x$, that is $(x-y) \geq 0$. In this case :

$$\partial/\partial y\ [H(a,x,y)] = \partial/\partial y\ [\cosh^{-1}\{h(a,x,y)\}]$$

$$= [\{h^2(a,x,y) - 1\}^{-1/2}]\ [\partial/\partial y\{h(a,x,y)\}] \qquad \text{A1.3.3}$$

Substituting in this relation from A1.2.1, A1.2.2 and A1.2.6 gives :

$$\partial/\partial y\ [H(a,x,y)] = [\ (a^2 - xy) - (x-y)^2 a^2\ ]^{-1/2} [(a^2 - x^2)/(x-y)]$$

A1.3.4

In this last step it should be noted that $x - y$ is positive and taking this factor into the square root does not imply a change of sign.

Similarly for $h < 0$ relation A1.2.9 implies that $y > x$ or $(x - y) \leq 0$.

In this case :

$$\partial/\partial y\ [H(a,x,y)] = \partial/\partial y[\cosh^{-1}\{-h(a,x,y)\}]$$

$$= [\{h(a,x,y)-1\}^{-1/2}][-\partial/\partial y\{h(a,x,y)\}] \qquad \text{A1.3.5}$$

Substituting as before from A1.2.1, A1.2.2 and A1.2.6 gives :

$$\partial/\partial y[H(a,x,y)] = -[\{(a^2 - xy)/a(x-y)\}^2 -1]^{-1/2}[(a^2-x^2)/a(x-y)^2]$$

$$= +[(a^2-xy)^2-a^2(x-y)^2]^{-1/2}[(a^2 - x^2)/(x-y)] \qquad \text{A1.3.6}$$

Again in this last step it should be noted that $x - y$ is negative and taking this factor into the square root does require a correction of sign.

Thus both cases give rise to the same equation A1.3.4 and A1.3.6. Rearranging the expression inside the square root these relations reduce to the general formula:

$$\partial/\partial y[H(a,x,y)] = [(a^2-x^2)(a^2- y^2)]^{-1/2}[(a^2- x^2)/(x - y)]$$

$$= [(a^2 - x^2)/(a^2-y^2)]^{1/2}/\ (x - y) \qquad \text{A1.3.7}$$

The x derivative follows from A1.3.7 and A1.2.3 or it may be calculated directly as above.

In the problems discussed in this work the ratio c/a is normally fixed by the ratio of the stresses and thus a derivative with respect to stress will first require a derivative with respect to this ratio. To evaluate this derivative it is convenient to define :

$$r = c/a \qquad \text{A1.3.8}$$

and then the quantity h defined by A1.2.1 may be written in the form :

$$h(a,c,x) = \left\{[(1/r) - r]/[1-(x/c)]\right\} - \left\{r\right\} \qquad \text{A1.3.9}$$

Now H is defined by A1.3.1 and from this it follows that

$$\partial/\partial r \; H(a,c,x) = \left\{1-[-(1/r)^2-1]/[1-(x/c)]\right\}/(h^2-1) \qquad \text{A1.3.10}$$

Substituting for h from A1.2.1 and for r from A1.3.8 it can be shown, after some algebraic manipulation, that

$$\partial/\partial r \; H(a,c,x) = -(a/c)(a^2 + cx)/[(a^2 - c^2)(a^2 - x^2)] \qquad \text{A1.3.11}$$

A1.4 To evaluate the integral $I(\alpha\beta)$

Now from A1.1.1 and A1.1.3 :

$$I(\alpha\beta) \equiv (a^2 - x^2)^{1/2} \int_{\alpha}^{\beta} (a^2 - y^2)^{-1/2} (x - y)^{-1} dy \qquad \text{A1.4.1}$$

Consider first the indefinite integral :

$$I = (a^2 - x^2)^{1/2} \int (a^2 - y^2)^{-1/2} (x - y)^{-1} dy \qquad \text{A1.4.2}$$

and make the substitution :

$$u = (y - x)^{-1} \qquad \text{A1.4.3}$$

Then it follows that :

$$I = (a^2 - x^2)^{1/2} \int \mathrm{sgn}(u)[u^2 a^2 - (1 + xu)^2]^{-1/2} du \qquad \text{A1.4.4}$$

where sgn (u) is equal to (+1) or (-1) according as u is positive or negative. Multiplying numerator and denominator by $(a^2 - x^2)^{1/2}$ gives :

$$I = (a^2 - x^2)\int \text{sgn}\ (u) \left[(u[a^2 - x^2] - x)^2 - a^2\right]^{-1/2}$$

$$= \text{sgn}\ (y - x)\ \cosh^{-1}\left([(a^2 - x^2)/a(y-x)] - [x/a]\right) \qquad \text{A1.4.5}$$

Substituting from A1.2.1, A1.4.5 becomes :

$$I = \text{sgn}\ (y - x_0)\ \cosh^{-1}[-h\ (a,x,y)] \qquad \text{A1.4.6}$$

In considering the definite integral it is necessary to divide the range of integration into subranges such that 'u' does not change sign in any subrange. The integral A1.4.1 must therefore be considered in three separate cases. Case 1 in which $\beta - x < 0$. This condition implies that $y - x < 0$ throughout the range of integration and no sub-division is necessary. Therefore it follows from A1.4.6 that :

$$I(\alpha,\beta) = -\cosh^{-1}[-h(a,x,\beta)] + \cosh^{-1}[-h(a,x\ \alpha)] \qquad \text{A1.4.7}$$

Using equations A1.2.3 and A1.1.7 this becomes :

$$I(\alpha,\beta) = +\cosh^{-1}[-h(a,\beta,x)] - \cosh^{-1}[-h(a,\alpha,x)] \qquad \text{A1.4.8}$$

In this relation the arguments are positive. This follows from A1.2.9 which shows that $h(a,y,x)<0$ when $y - x < 0$.

Case 2 in which $\alpha - x > 0$. This condition implies that $y - x > 0$ throughout the range of integration and again no subdivision is necessary. It follows from A1.4.6 that :

$$I(\alpha,B) = + \cosh^{-1}[-h(a,x,\beta)] - \cosh^{-1}[-h(a,x,\beta)] \qquad \text{A1.4.9}$$

Using relation A1.2.3 this becomes :

$$I(\alpha,\beta) = + \cosh^{-1}[a,\beta,x)] - \cosh^{-1}[h(a,\alpha,x)] \qquad \text{A1.4.10}$$

Again in this relation the arguments are positive. This follows, as before, from A1.2.9 which shows that $h(a,y,x) > 0$ when $y - x > 0$.

<u>Case 3</u> in which $\alpha - x < 0$ and $\beta - x > 0$. This implies that $y - x$ changes sign from negative to positive within the range of integration and the range must be divided so that :

$$I(\alpha,\beta) = \lim_{\delta \to 0} [(\alpha,x - \delta) + I(x + \delta,\beta)] \qquad \text{A1.4.11}$$

Using the results A1.4.8 and A1.4.10 this becomes :

$$I(\alpha,\beta) = \lim \Big[+\cosh^{-1}[-h(a,x-\delta,x)] - \cosh^{-1}[-h(a,\alpha,x)] + \cosh^{-1}[h(a,\beta,x)] - \cosh^{-1}[h(a,x+\delta,x)] \Big] \qquad \text{A1.4.12}$$

Now writing :

$$L \equiv \lim_{\delta\to 0}\left[\cosh^{-1}[-h(a,x-\delta,x)] - \cosh^{-1}[h(a,x+\delta,x)]\right] \qquad \text{A1.4.13}$$

and using relations A1.2.3 and A1.2.1 it follows that :

$$L = \lim_{\delta\to 0}\left[\cosh^{-1}[\{(a^2-x^2)/a\delta\}+\{x/a\}]-\cosh^{-1}[\{(a^2-x^2)/a\delta-\{x/a\}]\right] \qquad \text{A1.4.14}$$

Expanding in a Taylor series about the point $[(a^2 - x^2)/a\delta]$ it can be demonstrated that $L = 0$. It then follows from A1.4.12 that :

$$I(\alpha,\beta) = +\cosh[h(a,\beta,x)] - \cosh^{-1}[-h(a,\alpha,x)] \qquad \text{A1.4.15}$$

The arguments in this relation are positive since it has been derived from results obtained earlier in which the arguments were arranged to be positive.

Finally relations A1.4.8, A1.4.10 and A1.4.15 may be summarised in the single relation :

$$I(\alpha,\beta) = H(a,\beta,x) - H(a,\alpha,x) \qquad \text{A1.4.16}$$

where H is defined by relation A1.3.1

A1.5 To evaluate the limit $L = \text{Lim}(c-x)\ H(a,c,x)$

This can be written in the form :

$$L = \lim_{x \to c} H(a,c,x)/(c - x)^{-1} \qquad \text{A1.5.1}$$

The numerator and denominator both tend to infinity and it follows that the limit is equal to the limit of the first derivatives. Thus from A1.3.7 :

$$L = \text{Lim}\ [(a^2 - c^2)/(a^2 - x^2)]^{1/2}\ (c - x) \qquad \text{A1.5.2}$$

From which it follows at once that :

$$L = 0 \qquad \text{A1.5.3}$$

Al.6 To evaluate the integral $I_0(\alpha,\beta)$

Now :

$$I_0(\alpha,\beta) \equiv \int_\alpha^\beta H(a,c,x)\,dx \qquad \text{Al.6.1}$$

Let I_0 be the indefinite integral:

$$I_0 = \int H(a,c,x)dx \qquad \text{Al.6.2}$$

Using Al.3.7 and integrating by parts Al.6.2 becomes

$$I_0 = (x-c)H(a,c,x) + \int[(a^2-c^2)/(a^2 - x^2)]^{1/2} dx$$

$$= (x-c)\, H(a,c,x) + (a^2 - c^2)^{1/2} \sin^{-1}(x/a) \qquad \text{Al.6.3}$$

Suppose now that $(c-x)$ does not change sign but is always either positive or negative throughout the range of integration, then from Al.6.3 :

$$I_0(\alpha,\beta) = \Big[(x-c)H(a,c,x) + (a^2 - c^2)^{1/2} \sin^{-1}(x/a)\Big]_\alpha^\beta$$

Al.6.4

In the case where $(c - x)$ changes sign, that is where c is contained within the range of integration it is necessary to write :

$$I_0(\alpha,\beta) = I_0(\alpha,\beta) + I_0(\alpha,\beta) \qquad \text{A1.6.5}$$

Then using the relations A1.6.4 and A1.5.3 to evaluate A1.6.5 it is shown that A1.6.4 applies also in this case. Thus A1.6.4 is the completely general relation.

A1.7 To evaluate the integral $I_1(\alpha,\beta)$

$$\text{Now } I_1(\alpha,\beta) = \int_\alpha^\beta (x - c)\, H\,(a,c,x)dx \qquad \text{A1.7.1}$$

Using A1.6.3 and integrating by parts A1.7.1 becomes :

$$I_1(\alpha,\beta) = \Big[(x-c)\Big((x-c)H(a,c,x) + (a^2 - c^2)^{1/2}\sin^{-1}(x/a)\Big)\Big]_\alpha^\beta$$

$$- \int_\alpha^\beta \Big[(x-c)H(a,c,x) + (a^2 - c^2)^{1/2}\sin\,(x/a)\Big]\,dx \qquad \text{A1.7.2}$$

$$\text{Now } \int_\alpha^\beta \sin^{-1}(x/a)dx = [x\,\sin^{-1}(x/a)]_\alpha^\beta - \int_\alpha^\beta x(a^2 - x^2)^{-1/2}dx$$

$$= [x\sin(x/a) + (a^2 - x^2)^{1/2}]_\alpha^\beta \qquad \text{A1.7.3}$$

It then follows from A1.7.1, A1.7.2 and A1.7.3 that

$$2I(\alpha,\beta) = \Big[(x-c)^2 H(a,c,x) - c(a^2-c^2)^{1/2}\sin^{-1}(x/a) - (a^2-x^2)^{1/2}(a^2-c^2)^{1/2}\Big]_{\alpha}^{\beta} \qquad \text{A1.7.4}$$

A1.8 To evaluate the nonsymmetric integral

Define :

$$I_{\alpha}^{\beta} \equiv \Big[(b-x)(x-a)\Big]^{1/2} \int_{\alpha}^{\beta} \Big[(b-y)(y-a)\Big]^{-1/2} (y-x)^{-1}\,dy \qquad \text{A1.8.1}$$

where $\alpha < \beta$ belong to the interval $[a,b]$ and x also belongs to this interval. Using the method of completion of squares this becomes :

$$I_{\alpha}^{\beta} = \Big[(b-a)^2/4 - (x-\{b+a\}/2)^2\Big]^{1/2} \int_{\alpha}^{\beta} \Big[[(b-a)^2/4-(y-(b+a)/2)^2\Big]^{-1/2} (y-x)^{-1}\,dy \qquad \text{A1.8.2}$$

Now writing :

$$\left.\begin{aligned}(b+a)/2 &= \theta \\ (b-a)/2 &= \phi\end{aligned}\right\} \qquad \text{A1.8.3}$$

$$\left.\begin{aligned}y-\theta &= r \\ x-\theta &= s\end{aligned}\right\} \qquad \text{A1.8.4}$$

relation A1.8.2 can be written in a symmetric form as follows :

$$I_{\alpha}^{\beta} = [\phi^2 - s^2]^{1/2} \int_{\alpha-\theta}^{\beta-\theta} [\phi^2 - r^2]^{-1/2} (r-s)^{-1} dr \qquad \text{A1.8.5}$$

where $(\alpha - \theta) < (\beta - \theta)$ belong to the interval $[-\phi, \phi]$ and s also belongs to this interval. Equation A1.2.5 is now in the general form of A1.4.1 and the general solution A1.4.16 may be applied. Thus

$$I_{\alpha}^{\beta} = H(\phi, \beta-\theta, s) - H(\phi, \alpha-\theta, s) \qquad \text{A1.8.6}$$

Substituting for θ ϕ and s from A1.8.3 and A1.8.4 and then using A1.3.1 and A1.2.1 gives :

$$H(\phi,\beta-\theta,s) = H(\{b-a\}/2,\beta-\{b+a\}/2,x-\{b+a\}/2)$$

$$=\cosh^{-1}\left(|[(2\beta-b-a)/(b-a)]-[2(\beta-a)(\beta-b)/(\beta-x)(b-a)]|\right) \quad \text{A1.8.7}$$

$H(\phi,\alpha-\theta,s)$ is found similarly and the value of I_{α}^{β} is found by substitution in A1.8.6.

A1.9 To evaluate the sum S_1

$$S_1 \equiv \sum_{n=0}^{\infty} \sin(n\theta)\cos(n\phi) \quad \text{A1.9.1}$$

Now :

$$\left.\begin{aligned} \exp(in\phi) &= \cos(n\phi) + i\sin(n\phi) \\ \exp(in\theta) &= \cos(n\theta) + i\sin(n\theta) \end{aligned}\right\} \quad \text{A1.9.2}$$

Thus it follows that S_1 is the imaginary part of some complex function $\overline{S}_1$ where :

$$\overline{S}_1 \equiv 1/2\sum_{n=0}^{\infty} \exp(in\theta)[\exp(in\phi) + \exp(-in\phi)] \quad \text{A1.9.3}$$

This may be written :

$$\bar{S}_1 = 1/2 \sum_{n=0}^{\infty} \left[\exp^n(i[\theta + \phi]) + \exp^n(i[\theta - \phi]) \right] \qquad A1.9.4$$

Using the relation :

$$\sum_{n=0}^{\infty} z^n = (1 - z)^{-1} \qquad A1.9.5$$

A1.9.4 reduces to :

$$\bar{S}_1 = 1/2 \left[\{1 - \exp i(\theta + \phi)\}^{-1} + \{1 - \exp i\,(\theta - \phi)\}^{-1} \right]$$

A1.9.6

Introducing a common denominator and dividing numerator and denominator by exp(i θ), it follows that :

$$\bar{S}_1 = 1/2 \left[2\exp(-i\theta) - \exp(-i\phi) - \exp(i\phi) \right] / \left[\exp(-i\theta) - \exp(i\phi) - \exp(-i\phi) + \exp(i\theta) \right] \qquad A1.9.7$$

Now since $\exp(i\theta) = \cos\theta + i\sin\theta$ this becomes

$$\bar{s}_1 = 1/2\,[\cos\theta - i\sin\theta - \cos\phi]/[\cos\theta - \cos\phi] \quad \text{A1.9.8}$$

Then taking the imaginary part it follows from A1.9.2 that :

$$S = -[\sin\theta]/2\,[\cos\theta - \cos\phi] \quad \text{A1.9.9}$$

A1.10 To evaluate the sum S_2

$$S_2 \equiv \sum_{n=1}^{\infty} (1/n)\cos(n\phi)\cos(n\theta) \quad \text{A1.10.1}$$

Adopting the procedure of the previous section, S_2 is the real part of a complex function :

$$\bar{S}_2 \equiv 1/2\sum_{n=1}^{\infty} (1/n)\exp(in\phi)[\exp(in\theta) + \exp(-in\theta)] \quad \text{A1.10.2}$$

This may be written :

$$\overline{S}_2 = 1/2 \sum_{n=1}^{\infty} \left[(1/n) \exp^n(i[\phi+\theta]) + (1/n) \exp^n(i[\phi-\theta]) \right] \quad \text{A1.10.3}$$

Using the relation

$$\sum_{n=o}^{\infty} z^n/n = \ell n\ (1-z) \quad \text{A1.10.4}$$

in which ℓn is the natural logarithm it follows that:

$$\overline{S}_2 = 1/2 \left[\ell n(1- \exp\{i[\theta+\phi]\}) + \ell n\ (1 - \exp\{i[\theta-\phi]\}) \right]$$

A1.10.5

This may be written :

$$\overline{S}_2 = 1/2 \Big[\ell n\Big(\exp[i(\theta+\phi)/2]\Big)\Big(\exp[-i\ (\theta+\phi)/2] - \exp[+i(\theta+\phi)/2]\Big)$$

$$+ \ell n\Big(\exp[i(\theta-\phi)/2]\Big)\Big(\exp[-i(\theta-\phi)/2] - \exp[+i(\theta-\phi)/2]\Big)\Big]$$

$$= 1/2 \Big[i\theta + 2\ell n(i) + \ell n\Big(\sin[(\theta+\phi)/2]\Big) + \ell n\Big(\sin[(\theta-\phi)/2]\Big)\Big]$$

A1.10.6

Now taking the real part it follows that :

$$S_2 = 1/2\left[\ell n(\sin[\{\theta+\phi\}/2]) + \ell n(\sin[\{\theta-\phi\}/2])\right] \qquad \text{A1.10.7}$$

Appendix II

A2.1 The half space problem in plane strain

Choose standard axis (x_1, x_2) and consider displacements $u_1(x_1, x_2)$ and $u_2(x_1, x_2)$ in the half space $x_2 \geq 0$. These displacements are required to satisfy the simultaneous differential equations :

$$(\lambda + \mu)\, \partial\Delta/\partial x_i \; + \; \mu\nabla^2 u_i \; = \; 0 \qquad \text{A2.1.1}$$

$$i = 1,2$$

where :

$$\Delta \; = \; \partial u_1/\partial x_1 + \partial u_2/\partial x_2 \qquad \text{A2.1.2}$$

is the dilatation and λ, μ are Lame's elastic constants. The strains are :

$$\gamma_{11} = \partial u_1/\partial x_1, \; \gamma_{22} = \partial u_2/\partial x_2, \; \gamma_{12} = (\partial u_1/\partial x_2 + \partial u_2/\partial x_1)/2 \qquad \text{A2.1.3}$$

The relations between the stresses and strains give :

$$E\partial u_1/\partial x_1 = (1-\nu^2)\sigma_{11} - \nu(1 + \nu)\,\sigma_{22} \qquad \text{A2.1.4}$$

$$E\partial u_2/\partial x_2 = (1-\nu^2)\sigma_{22} - \nu(1 + \nu)\,\sigma_{11} \qquad \text{A2.1.5}$$

Where ν is Poisson's ratio and E is Young's modulus

It follows that the stresses are given by :

$$(1+\nu)(1-2\nu)\sigma_{11} = E(1-\nu)\partial u_1/\partial x_1 + E\nu\partial u_2/\partial x_2 \qquad \text{A2.1.6}$$

$$(1+\nu)(1-2\nu)\sigma_{22} = E(1-\nu)\partial u_2/\partial x_2 + E\nu\partial u_1/\partial x_1 \qquad \text{A2.1.7}$$

A2.2 Displacements for which the σ_{12} stress is zero

Suitable displacements satisfying the equations A2.1.1 and dying away as $x_2 \to \infty$ are given by the relations :

$$\pi\mu u_1\zeta = [\zeta x_2-(1-2\nu)]\exp(-\zeta x_2)[B(\zeta)\sin(\zeta x_1)+C(\zeta)\cos(\zeta x_1)]$$

A2.2.1

$$\pi\mu u_2\zeta = [\zeta x_2+2(1-\nu)]\exp(-\zeta x_2)[B(\zeta)\cos(\zeta x_1)-C(\zeta)\sin\zeta x_1]$$

A2.2.2

ζ is any constant and these relations represent different systems for each distinct value of ζ. From equations A2.1.6 and A2.1.7 these displacements give rise to stresses :

$$\pi\sigma_{22} = -2(\zeta x_2 + 1) \exp(-\zeta x_2)[B(\zeta)\cos(\zeta x_1)-C(\zeta)\sin(\zeta x_1)]$$

A2.2.3

$$\pi\sigma_{11} = 2(\zeta x_2 - 2)\ \exp(-\zeta x_2)[B(\zeta)\cos(\zeta x_1) - C(\zeta)\sin(\zeta x_1)] \qquad \text{A2.2.4}$$

and from the relation :

$$\sigma_{12} = \mu(\partial u_1/\partial x_2 + \partial u_2/\partial x_1) \qquad \text{A2.2.5}$$

it follows that :

$$\pi\sigma_{12} = -2\zeta x_2 \exp(-\zeta x_2)[B(\zeta)\sin(\zeta x_1) + C(\zeta)\cos(\zeta x_1)] \qquad \text{A2.2.6}$$

On the boundary of the half space $x_2 \geq 0$, that is on $x_2 = 0$, relation A2.2.6 shows that the shear stress vanishes and so the equations represent a system on which only normal tensile stresses are applied to the boundary $x_2 = 0$.

A2.3 Displacements for which the σ_{22} stress is zero

Adopting the procesure of the previous section :

$$\pi\mu u_1\zeta = [\zeta x_2 - 2(1-\nu)]\exp(-\zeta x_2)[B(\zeta)\cos(\zeta x_1) - C(\zeta)\sin(\zeta x_1)] \qquad \text{A2.3.1}$$

$$\pi\mu u_2\zeta = [\zeta x_2 + 1 - 2\nu]\ \exp(-\zeta x_2)[-B(\zeta)\sin(\zeta x_1) + C(\zeta)\cos(\zeta x_1)] \qquad \text{A2.3.2}$$

are suitable displacements giving rise to stresses :

$$\pi\sigma_{22} = +2\zeta x_2 \exp(-\zeta x_2)[B(\zeta)\sin(\zeta x_1) + C(\zeta) \cos(\zeta x_1)] \quad \text{A2.3.3}$$

$$\pi\sigma_{11} = -2(\zeta x_2 - 2) \exp(-\zeta x_2)[B(\zeta)\sin(\zeta x_1) + C(\zeta)\cos(\zeta x_1)] \quad \text{A2.3.4}$$

$$\pi\sigma_{12} = 2(1-\zeta x_2) \exp(-\zeta x_2)[B(\zeta)\cos(\zeta x_1) - C(\zeta)\sin(\zeta x_1)] \quad \text{A2.3.5}$$

Here equation A2.3.3 shows that the normal stress on $x_2 = 0$ is zero and so the equations represent a system in which only shear stresses are applied to the boundary.

A2.4 The stress function

In the theory of plane strain it is known that the stresses may be derived from a function χ satisfying the byharmonic equation :

$$\nabla^4 \chi = 0 \quad \text{A2.4.1}$$

using the relations : $\partial^2\chi/\partial x_1^2 = \sigma_{22}$ A2.4.2

$$\partial^2\chi/\partial x_2^2 = \sigma_{11} \quad \text{A2.4.3}$$

$$\partial^2\chi/\partial x_1 \partial x_2 = \sigma_{12} \quad \text{A2.4.4}$$

A suitable function giving rise to the stresses A2.2.3, A2.2.4 and A2.2.6 is :

$$\pi\zeta^2\chi = 2(1+\zeta x_2)\exp(-\zeta x_2)[B(\zeta)\cos(\zeta x_1)-C(\zeta)\sin(\zeta x_1)] \qquad \text{A2.4.5}$$

and a stress function giving rise to the stresses A2.3.3, A2.3.4 and A2.3.5 is :

$$\pi\zeta^2\chi = -2\zeta x_2\exp(-\zeta x_2)[B(\zeta)\sin(\zeta x_1) + C(\zeta)\cos(\zeta x_1)] \qquad \text{A2.4.6}$$

A2.5 Antiplane Strain

In this system, referred to standard axes, the only displacement is of the form $u_3(x_1,x_2)$ and is normal to the x_1,x_2 plane. From the relations for compatibility of displacements it is required that :

$$\nabla^2 u = 0 \qquad \text{A2.5.1}$$

The relations :

$$\sigma_{ij} = \mu(\partial u_i/\partial x_j + \partial u_j/\partial x_i) \qquad \text{A2.5.2}$$

give rise to the stresses :

$$\sigma_{13} = \mu \partial u_3/\partial x_1 \qquad \text{A2.5.3}$$

$$\sigma_{23} = \mu \partial u_3/\partial x_2 \qquad \text{A2.5.4}$$

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{12} = 0 \qquad \text{A2.5.5}$$

A suitable displacement satisfying A2.5.1 and dying away as $x_2 \rightarrow \infty$ is :

$$\pi\zeta\,\mu\,u_3 = 2\exp(-\zeta x_2)[B(\zeta)\cos(\zeta x_1) - C(\zeta)\sin(\zeta x_1)] \qquad \text{A2.5.6}$$

and from A2.5.3, A2.5.4 the stresses are :

$$\pi\sigma_{23} = -2\exp(-\zeta x_2)[B(\zeta)\cos(\zeta x_1) - C(\zeta)\sin(\zeta x_1)] \qquad \text{A2.5.7}$$

$$\pi\sigma_{13} = +2\exp(-\zeta x_2)[B(\zeta)\sin(\zeta x_1) + C(\zeta)\cos(\zeta x_1)] \qquad \text{A2.5.8}$$

A2.6 Generalised Equations

In each case superimposing the stress systems obtained for various values of ζ it is possible to obtain a stress field satisfying certain conditions of stress and displacement on the boundary and dying away as $x_2 \rightarrow \infty$. Superimposition may be carried out by means of Fourier sums or integrals and simultaneous equations are obtained involving one stress and one displacement component. It will now be shown that these equations take the same form in each of the three systems discussed.

Make the following definitions :

$$A(\zeta,x_1) \equiv [B(\zeta) \cos(\zeta x_1) - C(\zeta) \sin(\zeta x_1)] \qquad \text{A2.6.1}$$

$$u \equiv (2/\pi)\, A(\zeta,x_1)/\zeta \qquad \text{A2.6.2}$$

$$\sigma \equiv (2/\pi)\, A(\zeta x_1) \qquad \text{A2.6.3}$$

For a system in which σ_{12} is zero on the boundary $x_2 = 0$ set :

$$\sigma_{22} = -\sigma \qquad \text{A2.6.4}$$

$$u_2 = u(1-\nu)/\mu \qquad \text{A2.6.5}$$

Then on $x_2 = 0$ equations A2.2.2 and A2.2.3 reduce to equations A2.6.2 and A2.6.3 respectively.

Similarly for a system in which σ_{22} is zero on $x_2 = 0$ Set :

$$\sigma_{12} = -\sigma \qquad \text{A2.6.6}$$

$$u_1 = +u(1-\nu)/\mu \qquad \text{A2.6.7}$$

Then on $x_2 = 0$ equations A2.3.1 and A2.3.5 reduce to relations A2.6.2 and A2.6.3.

Again in the case of antiplane strain set :

$$u_3 = u/\mu \qquad \text{A2.6.8}$$

$$\sigma_{23} = -\sigma \qquad \text{A2.6.9}$$

Then equations A2.5.6 and A2.5.7 reduce to A2.6.2 and A2.6.3.

To obtain a Fourier series let $B(\zeta)$ $C(\zeta)$ be defined only for:

$$\zeta = \pi n/\ell \qquad \text{A2.6.10}$$

and set :

$$\left.\begin{aligned} B(\zeta) &= n\pi^2 B_n/2\ell \\ C(\zeta) &= n\pi^2 B_n/2\ell \end{aligned}\right\} \quad n = 0,1,2, \text{ etc.} \qquad \text{A2.6.11}$$

where B_n and C_n are constants. Then from A2.6.2 and A2.6.3 :

$$u = \sum_{n=o}^{\infty} [B_n \cos(\pi n x_1/\ell) - C_n \sin(\pi n x_1/\ell)] \qquad \text{A2.6.12}$$

$$\sigma = (\pi/\ell) \sum_{n=o}^{\infty} n\, [B_n \cos(\pi n x_1/\ell) - C_n \sin(\pi n x_1/\ell)] \qquad \text{A2.6.13}$$

Integration with respect to ζ to obtain Fourier integrals gives :

$$u = (2/\pi) \int_{-\infty}^{\infty} [\, A(\zeta, x_1)/\zeta \,]\, d\zeta \qquad \text{A2.6.14}$$

$$\sigma = (2/\pi) \int_{-\infty}^{\infty} A(\zeta, x_1)\, d\zeta \qquad \text{A2.6.15}$$

These latter equations also follow directly from the theory of Fourier Transforms (Sneddon 1951).

Appendix III

Hult McClintock Theory of a Relaxed Notch

Hult and McClintock (1957)[62] have discussed the plastic relaxation at the root of a sharp notch of depth c and angle θ under conditions of anti plane strain. The analysis used to determine the extent of the plastic zones is not given in detail in their paper although the general procedure is indicated.

Following this procedure the complete analysis has been developed here and in principle the equation of the elastic plastic boundary may be determined. Analytic solutions for the boundary would be unduly complex and recourse to numerical techniques is necessary, except in a few special cases.

Now $\omega(x_1,x_2)$ is a working function, $\phi(x_1,x_2)$ a stress potential and P the applied σ_{23} stress at infinity. The equations of stress (2.2.2, 2.2.3 and 2.2.4 of the main text) may be written in the form :

$$\sigma_{13} = \mu\partial\omega/\partial x_1 = \partial(\phi + Px_1)/\partial x_2 \qquad \text{A3.1.1}$$

$$\sigma_{23} - P = \mu\partial\omega/\partial x_2 = -\partial(\phi + Px_1)/\partial x_1 \qquad \text{A3.1.2}$$

A complex stress vector is defined by :

$$\sigma = \sigma_{13} + i\sigma_{23} \qquad \text{A3.1.3}$$

and $q = R \exp(i\alpha)$ is an arbitrary point of the elastic plastic boundary. In the elastic region ϕ satisfies the Laplace equation, but this domain is unknown so that solutions for ϕ are not easily available. The equations in the elastic region are therefore transformed into equations in stress space.

A3.2 Transformation to stress space

Make the following definitions

$$z = x_1 + i\, x_2 \qquad \text{A3.2.1}$$

$$\bar{\sigma}' = \sigma_{13} - i\,(\sigma_{23} - P) \qquad \text{A3.2.2}$$

$$W = \mu\omega + i(\phi + Px_1) \qquad \text{A3.2.3}$$

From the first part of equation A3.1.1 and the second part of A3.1.2 it then follows that

$$\bar{\sigma}' = \partial W / \partial x_1 \qquad \text{A3.2.4}$$

The second parts of equations A3.1.1 and A3.1.2 are Cauchy Riemann equations in $(\mu\omega)$ and $(\phi + Px_1)$. It follows that these are conjugate harmonic functions and consequently W is an analytic function of Z. Therefore A3.2.4 may be written :

$$\bar{\sigma}' = \partial W/\partial Z \qquad \text{A3.2.5}$$

Also Z must be an analytic function of $\bar{\sigma}'$ and therefore the derivative of some other analytic function $A'(\bar{\sigma}')$ defined for values of stress in the elastic region :

$$Z = \partial A'(\bar{\sigma}')/\partial\bar{\sigma}' \qquad \text{A3.2.6}$$

Moving the origin of the σ space to the point (O,+P)

$$\bar{\sigma} = \bar{\sigma}' - iP = \sigma_{13} - i\sigma_{23} \qquad \text{A3.2.7}$$

equation A3.2.6 becomes :

$$Z = \partial A(\bar{\sigma})/\partial\bar{\sigma} \qquad \text{A3.2.8}$$

A3.2.7 shows that $\bar{\sigma}$ is the conjugate of the complex stress vector defined by A3.1.3.

Letting ψ be the imaginary part of A :

$$x_1 = \partial\psi/\partial\sigma_{23} \qquad \text{A3.2.9}$$

$$x_2 = -\partial\psi/\partial\sigma_{13} \qquad \text{A3.2.10}$$

$$\nabla^2 \psi = 0 \qquad \text{A3.2.11}$$

These equations in ψ are of the same form as the stress equations in ϕ (2.2.2, 2.2.3 and 2.2.4 of the main text.)

Now ψ is a function of stress defined for stresses in the elastic regions and in what follows ψ will be seen to have a rather simple set of boundary values which is not the case for the real part of A.

In the elastic region the only points at which $|\sigma| = k$ are on the elastic plastic boundary of the Z space (fig I). Using polar co-ordinates with the origin at the notch tip, the Radius R of the point q = (R,α) of the boundary is given by

$$R^2 = [x_1^2 + x_2^2]_{|\sigma| = k} \qquad \text{A3.2.12}$$

From A3.2.9 and A3.2.10 it is clear that

$$R = [\ |\text{grad}\ (\psi)|\ ]_{|P| = k} \qquad \text{A3.2.13}$$

This equation in σ space is again similar to an equation of the Z space, namely the yield criterion

$$k = [\ |\text{grad}\ (\phi)|\ ]_{|Z| = R} \qquad \text{A3.2.14}$$

A3.3 Method of solution

Consider the map of the elastic region of z space into the stress space (fig I, fig 2). Along a free surface the traction is zero. In the stress space this implies that, with the relative orientations of the diagrams, the complex stress vector in σ space must lie parallel to the free surface of the z space, since the ratio of the stress components is fixed.

Now in the yielded region the complex stress vector has a constant magnitude k and, again with the relative orientations of the diagram, a direction in σ space perpendicular to the radius vector in z space. The continuity conditions then imply that this relation holds on the elastic plastic boundary, so that the map of the point $q = (R,\alpha)$, on the edge of the elastic region in z space, is the point $q = (k, \{\pi/2\} + \alpha \})$ of σ space. The elastic region of fig (1) maps onto the enclosed region of fig (2), since outside this region the magnitude of the complex stress vector may exceed k.

Furthermore, since the complex stress vector is perpendicular to the radius in the plastic region, the stress <u>increment</u> between two neighbouring points will lie parallel to this vector. Therefore, continuity also implies that in the neighbourhood of the elastic plastic boundary, in the elastic region, as in the plastic region, the stress <u>increment</u> lies parallel to the radius vector.

The boundary values of ψ are obtained by integrating :

$$d\psi = [\partial\psi/\partial\sigma_{13}]d\sigma_{13} + [\partial\psi/\partial\sigma_{23}]d\sigma_{23}$$

$$= x_2 d\sigma_{13} + x_1 d\sigma_{23} \qquad \text{A3.3.1}$$

Now the continuous arc BCDEF in fig (2) corresponds either to free surface or to plastic boundary in fig (1) and in both cases the stress increments in σ space have been shown to be parallel to the radius vector in z space. Therefore ψ is constant over this arc since $x_1/x_2 = d\sigma_{23}/d\sigma_{13}$. Setting $\psi = 0$ over BCDEF it is at once apparent that over AB and FG :

$$\psi = -c\sigma_{23} \qquad \text{A3.3.2}$$

since $d\sigma_{13} = 0$ and from fig (1), x_1 has the constant value c over these arcs.

Denoting the real part of σ by $\Re\sigma$, then in stress space the boundary values of ψ take the form :

$$\left.\begin{aligned} \psi &= i\sigma & \Re\sigma &= 0 \\ \psi &= 0 & \Re\sigma &\neq 0 \end{aligned}\right\} \qquad \text{A3.3.3}$$

Now it is required to find grad (ψ) at the elastic plastic boundary. To this end suppose that the elastic region of σ space is mapped on to a semi-circle in τ space say, with polar co-ordinates

$$(t,\gamma) \text{ where } 0 \leq t \leq 1$$

$$-\pi/2 \leq \gamma \leq \pi/2 \qquad \text{A3.3.4}$$

and in such a way that a selected point q on the elastic plastic boundary maps on to the origin. Then a ψ satisfying the Laplace equation in τ space is :

$$\psi(t,\gamma) = \sum_{\nu=1}^{\infty} \left[\psi'_{\nu} \cos(\nu\gamma) + \psi''_{\nu} \sin(\nu\gamma) \right] t^{\nu} \qquad \text{A3.3.5}$$

Using the Fourier Theorem :

$$\psi'_{\nu} = (2/\pi) \int_{-\pi/2}^{\pi/2} \psi(1,\gamma) \cos(\nu\gamma) d\gamma \qquad \text{A3.3.6}$$

Since ψ is constant on the boundary near q it is constant in the neighbourhood of the origin in τ space in the direction $\gamma = \pi/2$. Therefore grad (ψ) in τ space is equal to $(\partial\psi/\partial t)_{\gamma=0}$ and from A3.3.5

$$\text{grad}_{\tau}\,(\psi) = (\partial\psi/\partial t)_{\substack{\gamma = 0 \\ t = 0}} = \left(\sum_{\nu=1}^{\infty} \nu\psi'_{\nu}\, t^{\nu-1}\right)_{t=0} = \psi'_1 \qquad \text{A3.3.7}$$

Thus

$$\text{grad}\,(\psi)_{t=0} = (2/\pi) \int_{-\pi/2}^{\pi/2} \psi(1,\gamma)\cos\gamma\, d\gamma \qquad \text{A3.3.8}$$

To determine grad (ψ) in τ space it is necessary to know $\psi(1,\gamma)$ on the boundary and this requires knowledge of the mapping function.

A3.4 The Conformal transformations

The following series of conformal mappings are compiled from a dictionary (Kober 1952).[80] Initially the σ space is mapped on to a circle in ζ space split along the x_1 axis where $x_1 \leq 0$. To do this σ space is rotated through an angle $-\pi/2$ and the arms EF, BC are then rotated to lie along the x_1 axis. Finally a bilinear transform shifts the coincident points GA to the origin.

Define : $\lambda = P/k$ A3.4.1

$a = 2\pi / (\pi - \theta)$ A3.4.2

then the transformation from σ space to ζ space is

$$\zeta = [(-i\sigma/k)^{a} - \lambda^{a}] \; / \; [1 - (-i\lambda\sigma/k)^{a}] \qquad \text{A3.4.3}$$

the derivative is

$$\partial\zeta/\partial\sigma = - [ia\{1 - \lambda^{2a}\}]/[k\{1-(i\lambda a/k)^{a}\}^{2}] \qquad \text{A3.4.4}$$

and the inverse is

$$\sigma = ik[\{\lambda^{a} + \zeta\}/\{\lambda^{a}\zeta + 1\}]^{1/a} \qquad \text{A3.4.5}$$

The divided circle is opened into a semicircle and then transformed into a complete circle in η space. The transformation from ζ space to η space is :

$$\eta = [\zeta - 1 + 2\sqrt{\zeta}]/[\zeta - 1 - 2\sqrt{\zeta}] \qquad \text{A3.4.6}$$

the derivative is :

$$\partial\eta/\partial\zeta = -2(\zeta + 1) \; / \; \sqrt{\zeta}(\zeta - 1 - 2\sqrt{\zeta})^{2} \qquad \text{A3.4.7}$$

and the inverse is :

$$\zeta = [\eta + 1 - \sqrt{\{2(\eta^2 + 1)\}}\ ]^2 \ / \ [\eta - 1]^2 \qquad \text{A3.4.8}$$

The η space is then rotated through an angle ρ so that the point q lies at (+1,0) and the circle is transformed back into a semicircle. Setting :

$$\eta' = \eta \exp(-i\rho) \qquad \text{A3.4.9}$$

These transformations are expressed by :

$$\tau = \left[\eta' + 1 - \sqrt{[2(\eta'^2 + 1)]}\right] \Big/ \left[\eta' - 1\right] \qquad \text{A3.4.10}$$

the derivative is :

$$d\tau/d\eta = \exp(-i\rho)\Big(\{2(\eta' + 1)/\sqrt{[2(\eta'^2 + 1)]}\} - 2/(\eta'-1)^2$$

A3.4.11

and the inverse is :

$$\eta = \exp(-i\rho)\ [\tau^2 - 1 + 2\tau]/\ [\tau^2 - 1 - 2\tau] \qquad \text{A3.4.12}$$

In general the rotation ρ is not easily determined. It is first necessary to find the map of the point q in σ space on to the η space. This is found by setting

$\sigma = -ik \exp(i\alpha)$ and combining equations A3.4.3 and A 3.4.6. The argument of the point so found is minus the rotation ρ. However if q is taken at the special point D then it follows at once that :

$$\rho = \pi \qquad \text{A3.4.13}$$

A3.5 To Determine R(o)

In order to determine grad (ψ) at the origin of τ space from equation A3.3.8 it is necessary to know $\psi(1,\gamma)$ for :

$$\tau = e^{i\gamma} \quad -\pi/2 \leq \gamma \leq \pi/2 \qquad \text{A3.5.1}$$

Now ψ is known in terms of σ from relation A3.3.3 and so using the transformations of the previous section the values of σ corresponding to the points τ in relation A3.5.1 must be found. From A3.4.12, A3.4.8 and A3.5.1 the relation between ζ and γ for $\rho = \pi$ is :

$$\zeta = \left[\left\{ -4 \exp(i\gamma) + \sqrt{\{4[\exp(2i\gamma)-1]^2 + 16 \exp 2i\gamma\}} \right\} \Big/ -2\{\exp(2i\gamma) - 1\} \right]^2 \qquad \text{A3.5.2}$$

$$= -(-\cos\gamma)/(1 + \cos\gamma) \qquad \text{A3.5.3}$$

This particularly simple form is only obtained for $\rho = \pi$. Substituting into A3.4.5 gives :

$$\sigma = ik\left[\left\{\lambda^a(1+\cos\gamma) - (1-\cos\gamma)\right\}\Big/\left\{(1+\cos\gamma) - \lambda^a(1-\cos\gamma)\right\}\right]^{1/a}$$

A3.5.4

By A3.3.3 ψ is zero at points where σ is not purely imaginary, that is at points where the expression in square brackets in relation A3.4.5 is not real and positive. These are the points at which :

$$|\cos\gamma| > (1 - \lambda^a)/(\lambda^a + 1) \qquad \text{A3.5.5}$$

that is :

$$|\gamma| > \sin^{-1} b \qquad \text{A3.5.6}$$

$$\text{where } b = (2\lambda^{a/2}/\lambda^a + 1) \qquad \text{A3.5.7}$$

The range of the integral in A3.3.8 may thus be restricted from $(-\pi/2, +\pi/2)$ to $(-\sin^{-1} b, + \sin^{-1} b)$.

Then from relations A3.3.3, A3.3.8, A3.5.4 and A3.5.6 it follows that :

$$\mathrm{grad}(\psi)_{t=o}=(2ck/\pi)\int_{-\sin^{-1}b}^{\sin^{-1}b}[\{\lambda^{a}-1+(\lambda^{a}+1)\cos\gamma\}/\{1-\lambda+(\lambda^{a}+1)\cos\gamma\}]\cos\gamma d\gamma$$

A3.5.8

Using the substitution :

$$xb = \sin\gamma$$

A3.5.9

and setting :

$$g(\lambda x)=\left[\sqrt{\{(\lambda^{a}+1)^{2}-4\lambda^{a}x^{2}\}}-1+\lambda^{a}\right]/\left[\sqrt{\{(\lambda^{a}+1)^{2}-4\lambda^{a}x^{2}\}}-1+\lambda^{a}\right]$$

A3.5.10

it follows that :

$$\mathrm{grad}(\psi)_{t=o} = (18ck/\pi)(\lambda^{a/2}/\lambda^{a}+1)\int_{o}^{1}[g(\lambda x)]^{1/a}dx$$

A3.5.11

The gradient of ψ in σ space is then obtained by multiplying by the factor :

$$(d\tau/d\sigma)_{(0,1)}=(\partial\tau/\partial\eta)_{(-1,0)}(\partial\eta/\partial\zeta)_{(1,0)}(\partial\zeta/\partial\sigma)_{(0,1)}$$

A3.5.12

Now from A3.4.11 differentiating numerator and denominator

$$(\partial\omega/\partial\eta)_{(-1,0)} = 1/4 \qquad \text{A3.5.13}$$

from A3.4.7 :

$$(\partial\eta/\partial\zeta)_{(1,0)} = -1 \qquad \text{A3.5.14}$$

and from A3.4.4 :

$$(\partial\zeta/\partial\sigma)_{(0,1)} = ia[1+\lambda^{a}]/k[1-\lambda^{a}] \qquad \text{A3.5.15}$$

so that the radius of the point D in Z space is :

$$R = (ca/\pi)(2\lambda^{a/2}/1-\lambda^{a})\int_{0}^{1}[g(\lambda x)]^{1/a}\,dx \qquad \text{A3.5.16}$$

Which is equivalent to the formula given by Hult and McClintock.

Appendix IV.

Programmes

This appendix contains the specification and transcript of the programmes used for certain of the numerical calculations in this work. All programmes are written in the Manchester Mercury Autocode (Brucker 1961)[77] and have been run only on the Mercury computer at Sheffield.

4.1 Periodic Array of Cracks

The programme calculates stresses or displacements for a triangular mesh of values of c/L and a/L where $2c$ is the crack length, $a-c$ the length of the plastic zones and $2L$ the period of the array. The points of the mesh are separated by a distance $h = L/\ell$ where ℓ is an integer. The range of integration is divided into $4i$ intervals, where i is an integer and the integration is carried out using Simpson's rule.

Order of Operation

(i) Read two unsigned integers from a data tape in the order

ℓ

i

(ii) Set values of c/L ranging from

$L - 2h$ to h in steps of $(-h)$.

(iii) Set values of a/L ranging from $(L-h)$ to $(c/L + h)$ in steps of $(-h)$.

(iv) Halt.

(v) Calculate displacements $(\pi^2 A/\sigma_1 h)\ \Phi\ (c)/L$ at all c and a if handkeys are equal to 3

or

Calculate stress ratio P/σ at all c and a if handkeys are equal to 4.

(vi) Return to (iv)

N.B. Output is in the form of a table in which fixed c/L form the rows and fixed a/L the columns. The page is 5 numbers wide and thus the rows occupy several lines of printing. Separate rows are separated by double spacing. The values of c/L and a/L for which calculations are made are printed down the side and across the bottom respectively.

Formulae

The integration uses Simpson's rule. Over sufficiently small regions this approximates to the integrand by means of a parabola. Such an approximation may not be used over any region containing c/L since the integrand is singular. However the singularity is known to be logarithmic and it is possible to use the following approximation to the integrant over the intervals $[c, \quad c + 2h]$

$$[F + (F_1 - F_2) \log h/\log 2] + [(F_2 - F_1)/\log 2] \log (x - c)$$

where F_1 and F_2 are the known values of the integrand at $c + h$ and $c + 2h$ respectively. The integral over this region is then :

$$F_2 \quad + \quad (F_1 - F_2)/ \ \log 2$$

A4.2 Displacements from a Thin Crack in a Work Hardening Material'

The behaviour of the programme is governed by a system of labels in Chapter 3. These labels may be fed into the machine either on tape (label (2)) or via the hand keys (label (0)). Entry to the programme is at label (12) of Chapter 3. The general sequence of behaviour at each label is given below. The following numbers may be read from a data tape :

P unsigned integer ; label number

M unsigned even integer : the number of intervals in the integration process,

C unsigned rational number : ratio c/a

X unsigned rational number : work hardening constant $wb/\pi^2 A$

Order of Operation

Labels in Chapter 3

1) (i) Read M and C and print 2, M and C preceded by 2 inches of blank tape.

(ii) Evaluate the (M + 1) x (M + 1) matrix A_{ji} for $c/a = C$.

(iii) Print the elements a_{ji}.

(iv) Store the elements a_{ji}.

(v) Print → * * * * * * preceded and followed by blank tape. (N.B. The output from this stage is suitable for re-input as data).

(vi) Go to label 10) with M, C and A_{ji} set.

2) (i) Read M, C and the (M + 1) x (M + 1) matrix A_{ji}. (N.B. That is the tape produced in the previous routine).

(ii) Pass control to label 6) with M, C, and A_{ji} set.

6) (i) The constants M, C, and A_{ji} must be set when control reaches label 6).

(ii) Print the title and the programme constants M, C preceded by captions.

(iii) Calculate, store and print the (M + 1) vector S'_j [displacements for non workhardening case].

(iv) Read and print X.

(v) Evaluate, print and store the (M + 1) vector S_j [The displacements in the form $(\pi^2 A/\sigma_1 b)(\Phi(x)/a)$] corresponding to the work hardening constant X.

(vi) Calculate the stress ratio and print preceded by 3 intermediate numbers.
[N.B. S_j is destroyed M, C and A_{ji} are preserved].

(vii) Halt. Read handkeys.

(viii) If handkeys are equal to 16 repeat from (iv) otherwise go to label 10) with M, C and A_{ji} set.

10) (i) Halt. Read handkeys.

(ii) Jump to label set on handkeys.

11) (i) Read M and C.

(ii) Go to label 10).
N.B. This may be used on a data tape following re entry by H.S. 9 and 1 I.T.B. when the matrix A_{ij} is set. Control is then passed to label 10) with M, C and A_{ij} set.

12) (i) Read P.

(ii) Pass control to label P.

Formulae used

In calculating the Matrix A_{ji} it is necessary to evaluate the integrals V_{ji} and (5.3.11 and 5.3.12 respectively). These calculations are based on the following relations :

$$2\int yH(1,y,x)dy=(y^2-x^2)H(1,y,x)+\sqrt{(1-x^2)}\{x \sin^{-1} y-\sqrt{(1-y^2)}\}$$

$$\int H(1,y,x)dy= (y-x)H(1,y,x)+\sqrt{(1-x^2)} \sin^{-1} y$$

These follow simply, integrating by parts and using A1.3.7 for the derivative of H.

<u>A Typical Operating Procedure.</u>

Suppose the data tape is shown in the form :

1, M, C, X_1, X_2, X_3, 1, M, C, X_1, X_2, X_3, etc.

If * denotes manual operations the procedure is then :

Read 1, M, C: calculate and print A_{ji} : Halt.

* Set handkeys to 6 : pass Halt.

Calculate S'_j : Read X_1 : calculate S_j and P/σ_i : Halt.

* Set handkeys to 16 : pass Halt.

Read X_2 : calculate S_j and P/σ_1 : Halt,

* Set handkeys to 16 : pass Halt.

Read X_3 : Calculate S_j and P/σ_1 : Halt.

* Set handkeys to 12 : pass Halt,

Halt,

* Set handkeys to 12 : pass Halt,

Read 1, M, C : calculate and print A_{ji} : Halt;

etc.

Alternatively one might have data of the form :

2, M, C, A_{ji}, X_1, X_2, X_3, 2, M, C, A_{ji}, etc.

The procedure would be as above. However, after the first label one may omit all subsequent labels and set these via the handswitches in the final operation of the cycle, instead of label 12). Further any combination of these two forms may be used.

A4.3 Thin Crack Relaxing along Two Planes.

$x_1 = a$ is the projection of the tip of the plastic arc onto the x_1 axis. The half length of the crack is unity. The range [0, a] is divided into P intervals and the range [0, 1] is divided into Q intervals. The plastic arcs are inclined at an angle α radians to the plane of the crack.

P, Q are integers, α a real number.

Order of Operation

(i) Print Title.

(ii) Read data in the order P Q α.

(iii)† Calculate the dislocation concentrations under stresses $P = 1, \sigma = 0$ and $P = 0, \sigma = 1$.

(iv) If, and only if, handkeys are set equal to 16 print the dislocation concentrations calculated in (iii).

(v) Print c/a, stress ratio and the dislocation column vector. [The density in the form $(A/\sigma_1 b)D(x)$]

(v)* Print the relative displacement at c across a single plastic arc. [Displacements in the form $(A/\sigma_1 b)(\Phi(c)/a)$]

(vi) Halt,

(vi) Pass halt return to (i),

Formulae.

The programme uses only standard formulae for the stress due to a dislocation and the rotation of the stress matrix.

Footnote :

† Owing to a programme error the stresses are set negative and all dislocations have the wrong sign.

* There is a programme error in this section of the calculation. These quantities have been determined using a desk calculating machine.

A4.4 Transcript

Periodic array of cracks.

```
TITLE
8080 SWINDEN.ARC.C.C/4

CHAPTER1
F→100

1)P = M(-1)1
C = PQ
SPACE
PRINT(C)0,1
T = 5
N = P+1
Q = L(-1)N
A = QC
H = A-C
C' = C
J = 4!
H = H/J
U = Aπ
V = Cπ
W = Hπ
A = ψCOS(U)
C = ψCOS(V)
X = A-1
Y = C-1
D = X-2Y
D = D/X
E = Y-X
E = 2EY/X
K = 1(1)J
X = ψCOS(KW+V)
X = X-C
X = D-E/X
Y = XX-1
JUMP3,0>Y
Y = ψSQRT(Y)
3)Y = X+Y
Y = ψMOD(Y)
Y = ψLOG(Y)
FK = ψMOD(Y)
REPEAT
```

```
Z = H/3
Y = 0
K = J(-2)4
X = F(K-2)+4F(K-1)+FK
Y = Y+ZX
REPEAT
X = F1-F2
X = XF+F2
Y = Y+2HX
PRINT(Y)0,3
C = C'
T = T-1
JUMP4,T≠0
NEWLINE
T = 0(1)10
SPACE
REPEAT
T = 5
4)REPEAT
NEWLINE
NEWLINE
REPEAT
ACROSS7/6

CLOSE
```

```
CHAPTER2
VARIABLES1

1)NEWLINE
NEWLINE
NEWLINE
CAPTION
                              RATIO OF STRESSES
NEWLINE
NEWLINE
CAPTION
TABLE GIVING VALUES FOR (C/L) VERTICALLY AND FOR (A/L) HORIZONTALLY
X = 1-0.5/π
NEWLINE
NEWLINE
NEWLINE
P = M(1)
C = PQ
SPACE
PRINT(C)0,1
C = 0.5πC
C = ψSIN(C)
N = P+1
T = 5
Q = L(-1)N
A = 0.5Qπ
A = ψSIN(A)

X = C/A
Y = 1-XX
Y = ψSQRT(Y)
X = ψARCTAN(X,Y)
X = 2X/π
PRINT(X)0,3
T = T-1
JUMP4,T≠0
NEWLINE
T = 0(1)10
SPACE
REPEAT
T = 5
4)REPEAT
NEWLINE
NEWLINE
REPEAT
ACROSS7/0

CLOSE
```

```
803
CHAPTER0
F→100

1)F = [illegible]LOG(2.0)
F = 1/F
2)READ(L)
READ(I)
C = 1/L
L = L-1
M = L-1
JUMP8
3)P = 0(1)6
NEWLINE
REPEAT
P = 0(1)8
SPACE
REPEAT
CAPTION
ARRAY OF CRACKS. DISPLACEMENTS AT THE TIPS
NEWLINE
NEWLINE
CAPTION
TABLE GIVING VALUES FOR (C/L) VERTICALLY AND FOR (A/L) HORIZONTALLY
NEWLINE
NEWLINE
NEWLINE
NEWLINE
ACROSS1/1
7)T = 0(1)10
SPACE
REPEAT
T = 5
K = L(-1)0
X = KC
PRINT(X)0,1
SPACE
SPACE
T = T-1
JUMP5,T≠0
NEWLINE
T = 0(1)10
SPACE
REPEAT
T = 5
5)REPEAT
8)HALT
HANDKEYS(J)
N) = J)
JUMP(N)
4)ACROSS1/2

CLOSE
→
```

A4.5 Transcript

Displacements from a thin crack in a workhardening material

```
TITLE
8080 SWINDEN. DWM.A.A/1

TITLE
DISPLACEMENTS FROM A THIN CRACK IN A WORKHARDENING MATERIAL

CHAPTER 1

U → 111
V → 111
W → 111
G → 111
π → 30

1)G = 1/H
J = 0(1)M
U = C+JH
V = 1-UU
JUMP2,0≥V
V = ψSQRT(V)
2)W = ψARCTAN(U,V)
I = 0(1)M
VI = 0
UI = 0
Y = C+IH
π0 = 1-YY
JUMP3,0≥π0
π1 = ψSQRT(π0)
3)π2 = ψARCTAN(π1,Y)
X = -U
T = I-J
4)X = -X
π4 = Y-X
π3 = 0
JUMP5*,T=0
π3 = π0/π4+Y
5)π5 = π3π3-1
T = 1
JUMP6,0≥π5
π5 = ψSQRT(π5)
6)π3 = ψMOD(π3+π5)
π3 = ψLOG(π3)
π3 = ψMOD(π3)
π5 = YY-XX
UI = 0.5π5π3+UI
VI = π4π3+VI
JUMP4,X>0
UI = Vπ2+UI
REPEAT
```

```
I = M(-1)1
UI = UI-U(I-1)
VI = VI-V(I-1)
REPEAT
U0 = 0
V0 = 0
V(M+1) = 0
U(M+1) = 0
I = 0(1)M
Y = I-1
Y = C+HY
X = Y+H+H
VI = XV(I+1)-U(I+1)+UI-YVI
VI = GVI
Y = IH
Y = 899Y/C+1
Y = ψSQRT(Y)
VI = VI/Y
NEWLINE
PRINT(VI)0,6
REPEAT
NEWLINE
N = M+1
A' ≃ NN
X ≃ JN
Z ≃ JN+A'
ψ7(Z)V0,N
ψ7(X)V0,N
REPEAT
I = 0(1)20
PUNCH(0)
REPEAT
PUNCH(20)
I = 0(1)6
PUNCH(31)
REPEAT
I = 0(1)6
PUNCH(0)
REPEAT
NEWLINE
NEWLINE
ACROSS10/3

ψSQRT
ψLOG
ψARCTAN
CLOSE
```

```
CHAPTER2
VARIABLES1

3)NEWLINE
CAPTION
C =
PRINT(C)0,5
CAPTION
M =
PRINT(M)3,0
NEWLINE
I = 0(1)M
X = C+IH
Y = 1-CC
U = C-X
V = C+X
JUMP1,I=0
Z = Y/U+C
A = ZZ-1
JUMP5,0>A
A = ψSQRT(A)
5)Z = Z+A
Z = ψMOD(Z)
Z = ψLOG(Z)
Z = ψMOD(Z)
1)π1 = UZ
Z = Y/V+C
A = ZZ-1
JUMP6,0>A
A = ψSQRT(A)
6)Z = Z+A
Z = ψMOD(Z)
Z = ψLOG(Z)
Z = ψMOD(Z)
π1 = π1+VZ
VI = π1
NEWLINE
PRINT(π1)0,5
REPEAT

N = M+1
A ≃ N
B ≃ AA
C' ≃ 2B
2)READ(X)
NEWLINE
NEWLINE
NEWLINE
CAPTION
VAR. X =
PRINT(X)0,5
NEWLINE
X = 1/X
```

```
ψ7(C')Vo,N
B = ψ19(o,X,A)
C' = ψ28(B,N,1)
ψ6(C')Uo,N
I = o(1)M
NEWLINE
UI = XUI
PRINT(UI)o,5
Y = IH
Y = 899Y/C+1
Y = ψSQRT(Y)
UI = UI/Y
REPEAT
NEWLINE

π30= o
J = M(-2)2
P = J
π2 = PH+C
π1 = π2-H
πo = π1-H
π3 = πo+π1
π4 = π1+π2
π5 = π2+πo
π6 = 2HH
π7 = U(P-2)-2U(P-1)+UP
π7 = π7/π6
π8 = U(P-2)π4-2U(P-1)π5+UPπ3
π8 = π8/π6
π9 = U(P-2)π1π2-2U(P-1)πoπ2+UPπoπ1
π9 = π9/π6

π3 = 1-πoπo
π4 = 1-π2π2
π3 = ψSQRT(π3)
JUMP4,o≥π4
π4 = ψSQRT(π4)

4) (π10,π11) = ψLOG(π3,πo)
(π12,π13) = ψLOG(π4,π2)
π14 = πoπ3-π2π4+π13-π11
π30 = π30+π7π14/2-π8π3+π8π4+π9π13-π9π11
REPEAT
```

```
NEWLINE
PRINT(π30)0,5
NEWLINE
π20 = 1-CC
π20 = ψSQRT(π20)
π20 = ψARCTAN(π20,C)
PRINT(π20)0,5
NEWLINE
π20 = -π30/K+π20
PRINT(π20)0,5
NEWLINE
Y = 0.5π
π20 = Y-π20
π20 = π20/Y
PRINT(π20)0,5
NEWLINE
NEWLINE
NEWLINE
HALT
HANDKEYS(J)
JUMP2,J = 16
ACROSS10/3

ψSQRT
ψLOG
*ψARCTAN

CLOSE
```

```
CHAPTER3
VARIABLES1

11)READ(M)
READ(C)
H = 1-C
H = H/M
JUMP10
12)READ(P)
N) = P)
JUMP(N)
1)READ(M)
READ(C)
H 8 1-C
H = H/M
P = 2
I = 0(1)50
PUNCH(0)
REPEAT
PRINT(P)1,0
PRINT(M)1,0
PRINT(C)0,5
ACROSS1/1
2)READ(M)
READ(C)
H = 1-C
H = H/M
U = M+1
U = UU
ψ10(0,U)
6)CAPTION
DISPLACEMENTS FROM A THIN CRACK IN A WORKHARDENING MATERIAL
ACROSS3/2

10)HALT
HANDKEYS(J)
N) = J)
JUMP(N)

CLOSE

*********

CHAPTER0
VARIABLES1
ACROSS12/3
CLOSE
→***************
1
```

A4.6 Transcript

Thin crack relaxing along two planes

```
TITLE
8080SWINDEN.TCH.D.C/C

CHAPTER1
A→200
Z→30

1)CAPTION
THIN CRACK RELAXING ALONG TWO PLANES
READ(P)
NEWLINE
READ(Q)
F = Q/P
READ(A)
B = ψSIN(A)
A = ψCOS(A)
B' = B/A
H = 1/Q
C = PH
D = P
E = D+D
K = P+P
I = 1(1)P
J = 2I
L = 2I-1
AL = 0
AJ = 1
JUMP2,Q>I
AL = -1
AJ = AB
2)REPEAT
ψ7(0)A1,K
DOWN1/2
```

```
HANDKEYS(J)
JUMP3, J=16
I = 1(1)P
J = 2I
L = 2I-1
NEWLINE
PRINT(AJ)0,3
PRINT(AL)0,3
REPEAT
3)U = -A(K-1)/AK
NEWLINE
NEWLINE
CAPTION
U =
PRINT(U)0,3
CAPTION
C/A =
PRINT(F)0,3
CAPTION
STRESS RATIO =
PRINT(U)0,3
NEWLINE
NEWLINE
CAPTION
D(IH) =    FOR H
PRINT(H)0,3
CAPTION
     0(1)
PRINT(P)2,0

NEWLINE
Y = 0
I = 2(2)K
AI = A(I-1)+UAI
X = AI/H
NEWLINE
PRINT(X)0,5
Y = Y+AI
REPEAT

NEWLINE
NEWLINE
CAPTION
C(C) =
PRINT(Y)0,5
NEWLINE
NEWLINE
HALT
JUMP1

CLOSE
```

```
5)Z1 = 0
A' = 1
Z2 = 0
Z3 = 1
N) = 6)
JUMP7
6)Z0 = -X
Z1 = 0
Z3 = -1
N) = 8)
JUMP7

7)Z11 = Z4-Z0
Z10 = Z5-Z1
Z9 = Z10Z3+Z11Z2
Z10 = Z10Z2-Z11Z3
Z11 = Z10Z10
Z12 = Z9Z9+Z11
Z12 = 1/Z12
Z11 = -2Z11Z12Z12+Z12
Z12 = 2Z12+Z11
Z9 = Z9Z11
Z12 = -Z10Z12
Z11 = Z10Z11
Z13 = Z2Z2
Z14 = Z3Z3
Z15 = Z2Z3
Z10 = 2Z15
W' = Z12Z13+Z11Z14-Z10Z9+W'
Z' = Z12Z14+Z11Z13+Z10Z9+Z'
Z13 = Z13-Z14
Z10 = Z12-Z11
Y' = Z9Z13+Z10Z15+Y'
JUMP(N)
8)AJ = Z'A'
JUMP10,Q>1
U = AA-BB
V = AB
Z = Z'-W'
AJ = UY'+ZV
10)REPEAT
X = IP+P
ψ7(X)A1,P
REPEAT
o = ψ28(E,P,2)
ψ6(o)A1,K
UP

CLOSE

CHAPTER0
ACROSS1/1
CLOSE
```