Hypergeometric Equation and Differential-Difference Bispectrality

Abdul Muqeet Khalid

Submitted in accordance with the requirements for the degree of Doctor of Philosophy

The University of Leeds

School of Mathematics

June 2018

The candidate confirms that the work submitted is his own and that appropriate credit has been given where reference has been made to the work of others.

Acknowledgements

First of all, my appreciation goes to almighty God, whose blessings and guidance have given me the health, resilience and insight to complete this work.

I would like to thank my supervisor Dr Oleg Chalykh for his support, commitment and advice throughout the course of my research. Indeed, no amount of appreciation on my part can do justice to the wealth of expertise and knowledge that he imparted. I have learnt a lot from his work ethics, organisation and insights into the subject. I am deeply indebted to him for all the time, guidance and encouragement that he provided.

I want to mention my parents Dr Khalid Iqbal and Mrs. Mussarat Ruhi Khalid for always being there for me. This work was made possible through their endless support. They always made things easy for me, so I could work with total dedication. My affectionate thanks to them both.

Finally, I am grateful to the School of Mathematics, University of Leeds, for funding this project.

Abstract

The bispectral problem was posed by Duistermaat and Grünbaum in 1986. Since then, many interesting links of this problem with nonlinear integrable PDEs, algebraic geometry, orthogonal polynomials and special functions have been found. Bispectral operators of rank one are related to the KP equation and have been completely classified by G. Wilson. For rank greater than 1 some large families related to Bessel functions are known, although the classification problem remains open.

If one generalises the bispectral problem by allowing difference operators in the spectral variable, then this has a clear parallel with the three-term recurrence relation in the theory of orthogonal polynomials. This differential-difference version of the bispectral problem has also been studied extensively, more recently in the context of the exceptional orthogonal polynomials. However, the associated special functions have not been treated in such a way, until now.

In our work we make a step in that direction by constructing a large family of bispectral operators related to the hypergeometric equation. In this thesis, we will fully explain our construction.

Contents

Chapter 1

Introduction and Background

The aim of our research project is to construct bispectral extensions of the Jacobi polynomials and related hypergeometric functions. The idea is to start with a bispectral pair for hypergeometric function and construct new bispectral operators by applying suitable Darboux transformations. This work is a follow up to earlier work carried out by various authors including Bakalov, Horozov and Yakimov [1], Haine and Iliev [2], Grünbaum and Yakimov [3], Plamen Iliev's work on the Askey-Wilson polynomials [4] and others.

The study of bispectral equations was initiated by J. J. Duistermaat and F. A. Grünbaum when they considered the Schrödinger operators for which differential operators in the spectral parameter could be found [5]. Their problem was as follows: For which linear ordinary differential operators $L(x, \partial_x)$ do there exist eigenfunctions $\phi(x, \lambda)$ (which depend smoothly on x) which are simultaneously eigenfunctions of a differential operator $A(\lambda, \partial_\lambda)$ in the spectral parameter λ.

In other words, for such an operator $L(x, \partial_x)$, if ϕ satisfies:

$$L(x, \partial_x)\phi(x, \lambda) = f(\lambda)\phi(x, \lambda), \tag{1.1}$$

then there exists an operator $A\big(\lambda, \partial_\lambda\big)$ such that

$$A\big(\lambda, \partial_\lambda\big)\phi\big(x, \lambda\big) = g\big(x\big)\phi\big(x, \lambda\big). \tag{1.2}$$

The equations (1.1) and (1.2) are referred to as the bispectral equations. It was discovered that the Schrödinger operators for which this could be done were obtained from a few simple ones through finitely many rational Darboux transformations. The answer was as follows: the Schrödinger operators which satisfy the bispectral property are of the form ${\partial_x}^2 + V\big(x\big)$, where V is one of the following potentials:

- $V\big(x\big) = \alpha x + \beta$, where $\alpha, \beta \in \mathbb{C}$.
- $V\big(x\big) = c(x-a)^{-2} + b$, where $a, b, c \in \mathbb{C}$.
- V is obtained from $V = 0$ from finitely many rational Darboux transformations up to translation and scaling in x and V.
- V is obtained from $V = -1/4x^2$ from finitely many rational Darboux transformations up to translation and scaling in x and V.

Wilson [6] proposed classifying commutative algebras of bispectral ordinary differential operators. All operators in such an algebra would share a common eigenspace which would solve an eigenvalue problem in the spectral variable. The dimension of this joint eigenspace would be the greatest common divisor of the orders of the operators in the algebra and was called the rank of the algebra. Wilson classified all rank 1 bispectral algebras.

The question of complete classification of all bispectral ordinary differential operators is usually referred to as the bispectral problem. In full generality, it remains wide open, and the aforementioned results of [5] and [6] are essentially the only examples where a complete classification has been achieved. Most subsequent work has aimed at constructing interesting examples of bispectral operators and exploring their links to other areas. The bispectral problem has also been studied in a more general setting where one allows difference operators in place of differential operators.

One main idea used for constructing new examples of bispectral operators is based on the concept of Darboux transformations. It can be summarised as follows.

Suppose we have the bispectral pair

$$L(x, \partial_x)\phi(x, \lambda) = f(\lambda)\phi(x, \lambda),$$

$$A(\lambda, \partial_\lambda)\phi(x, \lambda) = g(x)\phi(x, \lambda).$$

We assume that L is independent of λ and A is independent of x. Let $h(t)$ be a polynomial in the variable t. Denote the operator $h(L)$ as $\mathcal{L}$. Suppose $\mathcal{L} = Q \circ P$ is a factorisation of $\mathcal{L}$, with P and Q being differential operators. Then interchanging the factors produces a Darboux transformation of $\mathcal{L}$.

$$\mathcal{L} = Q \circ P \quad \mapsto \quad \hat{\mathcal{L}} = P \circ Q.$$

Set $\psi = P\phi$. It automatically follows that $\hat{\mathcal{L}}\psi = h(f(\lambda))\psi$.

In general, $\hat{\mathcal{L}}$ will not be bispectral; certain conditions have to be imposed on P and Q in order to make ψ satisfy an eigenvalue problem in the spectral variable.

As an example, one can start with the following very simple bispectral pair

$$\partial_x\phi = \lambda\phi,$$

$$\partial_\lambda\phi = x\phi,$$

with $\phi = e^{\lambda x}$ being the common eigenfunction. In this case $L = \partial_x$ and $\mathcal{L} = h(\partial_x)$ is an arbitrary differential operator with constant coefficients. One of the results of [6] can formulated as follows (see also [7]): if $h(\partial_x)$ is factorised as $Q \circ P$ where P and Q have coefficients which are rational in x, then $\hat{\mathcal{L}} = P \circ Q$ would be bispectral. This led to large multi-parametric families of bispectral algebras of rank one, which can be organised in to the so-called adelic Grassmannian Gr^{ad} introduced in [6]. Namely, a rational factorisation

$$h(\partial_x) = Q \circ P$$

is determined by choosing $\ker P \subset \ker h(\partial_x)$. Writing

$$h(\partial_x) = \prod_i (\partial_x - \lambda_i)^{m_i}$$

with pairwise distinct λ_i and some $m_i \geq 1$, we have:

$$\ker h(\partial_x) = \bigoplus_i \operatorname{span}\Big\{ x^j e^{\lambda_i x} : 0 \leq j \leq m_j - 1 \Big\}.$$

It follows from [6] that to guarantee that P has rational coefficients, the subspace $W = \ker P$ should be of the form

$$W = \bigoplus_i W_i, \quad W_i \subset \operatorname{span}\Big\{ x^j e^{\lambda_i x} : 0 \leq j \leq m_i - 1 \Big\}.$$

The adelic Grassmannian Gr^{ad} can be defined as the set of all such W, with varying $h(t)$, with the restriction that $e^{\lambda_i x} \notin W_i$ for all i.

As a natural modification of the space Gr^{ad}, Haine and Iliev [8] start from the following bispectral pair:

$$\partial_x \phi = \lambda \phi,$$

$$T_\lambda \phi = e^x \phi,$$

where T_λ is a shift operator acting by $T_\lambda f(\lambda) = f(\lambda + 1)$ and $\phi = e^{\lambda x}$. In this case, we have a difference equation in the spectral variable, so this is usually referred to as differential-difference bispectrality. The appropriate Darboux transformations in this situation correspond to factorisations $h(\partial_x) = Q \circ P$, where P and Q have coefficients which are rational functions of e^x. All such factorisations were described in theorem 3.3 in [9] (see also [8] and [10] for related results). A space, parametrising all such factorisations, was introduced in [9] and called the trigonometric Grassmannian (see definition 3.4 in [9]).

Let us also mention a few other possibilities. One can start with

$$T_x\phi = e^{\lambda}\phi, \qquad T_\lambda\phi = e^{x}\phi \tag{1.3}$$

as in [11]. In this case, one can construct factorisations $\mathcal{L} = h(T_x) = Q \circ P$ such that $\hat{\mathcal{L}} = P \circ Q$ would have bispectral property if P and Q have coefficients which are rational in e^x.

Another idea is to replace T_x and T_λ in (1.3) by q-derivatines as in [12].

An important example [1] involves taking the Bessel operator

$$L_x = \partial_x^2 - \frac{c(c-1)}{x^2}$$

as a starting point. In this case, the initial bispectral pair is

$$L_x\phi = \left[\partial_x^2 - \frac{c(c-1)}{x^2}\right]\phi = \lambda\phi,$$

$$A_\lambda\phi = \left[\partial_\lambda^2 - \frac{c(c-1)}{\lambda^2}\right]\phi = x\phi.$$

(In fact, Bakalov, Horozov and Yakimov even considered in [1] a more general case of L_x being a higher-order analogue of the above L_x.)

One of their main results [1, Theorem 3.3] says that if $h(L_x)$ is factorised as $h(L_x) = Q \circ P$ where P and Q have rational coefficients and are invariant under the reflection $x \mapsto -x$, then $\hat{\mathcal{L}} = P \circ Q$ is bispectral. Furthermore, they give a complete classification of such factorisations, which can be viewed as a Bessel analogue of Wilson's adelic Grassmannian. More precisely, assuming that P is monic, it is uniquely determined by its kernel $W \subset \ker h(L_x)$.

The main result of [1] gives a complete description of all possible choices of W (see [1, (2.18) - (2.20)]). Note that the bispectral algebras resulting from that construction will be of rank greater than 1.

Taking an inspiration from [1], our goal is to construct new families of bispectral operators starting from the following initial bispectral pair:

$$L_x(x, \partial_x)\phi(x, \lambda) = \lambda^2\phi(x, \lambda), \tag{1.4}$$

$$A_\lambda(\lambda, T_\lambda)\phi(x, \lambda) = -4\sin^2\left(\frac{x}{2}\right)\phi(x, \lambda), \tag{1.5}$$

where

$$L_x(x, \partial_x) = -\partial_x^2 + u, \qquad u = \frac{g(g-1)}{4\sin\left(\frac{x}{2}\right)} + \frac{h(h-1)}{4\cos\left(\frac{x}{2}\right)},$$

and

$$A_\lambda(\lambda, T_\lambda) = A_+T_\lambda + A_0 + A_-T_\lambda^{-1},$$

$$A_\pm = \left(1 \pm \frac{g+h}{2\lambda}\right)\left(1 \pm \frac{g-h}{2\lambda \pm 1}\right) \quad \text{and} \quad A_0 = -A_+ - A_-.$$

L_x is known as the Darboux-Pöschl-Teller (DPT) operator, which is well known in the theory of Jacobi polynomials. A_λ is closely related to the three term recurrence relation for Jacobi polynomials. The equation $L_x\phi = \lambda^2\phi$ is essentially the celebrated hypergeometric equation up to a gauge transformation. This can be seen using the following relabelling of parameters:

$$a = \lambda + \frac{g+h}{2}, \quad b = -\lambda + \frac{g+h}{2}, \quad c = g + \frac{1}{2}, \quad z = \sin^2\frac{x}{2}.$$

Then $L_x = \mathfrak{g}^{-1}L_z\mathfrak{g}$, where $\mathfrak{g} = \sin^{-g}\left(\frac{x}{2}\right)\cos^{-h}\left(\frac{x}{2}\right)$ is our gauge function and L_z is the hypergeometric differential operator:

$$L_z(z, \partial_z) = z(1-z)\partial_z^2 + [c - (a+b+1)z]\partial_z - ab + \lambda^2.$$

In a manner similar to previous works, we construct new families of bispectral operators by employing suitable Darboux transformations following the framework of [1]: we consider arbitrary polynomials $h(L_x)$ and look for possible factorisations $h(L) = Q \circ P$.

It turns out that to ensure bispectrality of $\hat{\mathcal{L}} = P \circ Q$, we need that P and Q are invariant under $x \mapsto -x$ and have trigonometric coefficients (coefficients which are rational functions of e^{ix}). See theorem 3.11 for this.

In its turn, P is determined by its kernel $W \subset \ker h(L_x)$. To this end, we describe all possible spaces W that guarantee that P and Q will have those properties. See sections 4.4 and 5.4 for this.

Although our results have a similar flavour to those of [1], the methods therein do not generalise to our case easily. Therefore, some new ideas were needed. In particular, our crucial idea was to use the monodromy of the hypergeometric equation (section 4.1).

Let us explain how our results compare to some earlier work done in this area. First, some bispectral families of operators have been constructed from the Jacobi difference operator by Grünbaum and Yakimov in [3]. However, their construction required certain restrictions on the parameters g and h, namely, either g pr h (or both) had to be half-integers. In contrast, our analysis excludes such values of g and h (for instance, the expressions (4.8) are not well defined for $g \in \frac{1}{2}\mathbb{Z}$). It is possible that the bispectral families constructed in [3] are a limiting case of our families, however, this is difficult to tell because the authors of [3] work primarily with difference operators. It should be possible to extend our analysis to the cases when g or h is a half integer, but this is not straightforward.

Another related result is a construction of bispectral extensions of the Askey-Wilson (AW) operator by Iliev [4]. Since the DPT operator can be obtained from the AW operator in a certain limit, Iliev's result has an analogue for our case. However the Darboux transformations in [4] are only performed at special eigenvalues, while here, λ is allowed to be any complex number.

Finally let us mention a link to exceptional Jacobi polynomials. Bispectrality of exceptional Jacobi polynomials was established by Odake [13]. Their construction is based on Darboux transformations corresponding to the choice of W being spanned by a collection of Jacobi polynomials. This constitutes a very small subclass within our approach, as will become clear in our analysis (see section 5.5).

We now describe the structure of the thesis.

Chapter 2 is a review of known results that we will need later. In this chapter, we will recall Jacobi polynomials and their link to the DPT operator. We will derive (1.5) from the three term recurrence relation satisfied by Jacobi polynomials. We will also state the eigenfunctions of L_x in terms of the Gaussian hypergeometric function.

In chapter 3, we will look in to the conditions on P and Q which guarantee bispectrality. We will mostly follow the approach in [1] and [2] to derive those conditions and prove the main result in that chapter, theorem 3.11, which gives us bispectrality.

In chapter 4, we make a link between bispectrality and the monodromy of the hypergeometric equation. We will show in theorem 4.4 how the monodromy of hypergeometric equation ensures that monodromy invariant spaces $W = \ker P$ will produce P and Q with required properties. We will then go on to describe the structure of possible subspaces W.

We will show in chapter 5 that our monodromy invariant solution spaces can be seen as modules over the free algebra generated by analytic continuations around the poles of the hypergeometric equation. Using module theoretic arguments we will algebraically prove the structure of monodromy invariant spaces described in chapter 4.

Finally, in chapter 6, we will conclude by providing an account of some related work that we did for Jacobi and Hermite polynomials. All this is expected to lay the ground work for future investigations in to the bispectrality of Hermite differential operator as well as links to other areas in the theory of integrable systems.

Chapter 2

Jacobi Polynomials and Darboux-Pöschl-Teller Operator

We begin by recapping Jacobi polynomials, and the equations and identities that they satisfy. We extend these polynomials to infinite series in an natural way using hypergeometric function definition and re-express them as eigenfunctions of the Darboux-Pöschl-Teller operator, which is the subject of our investigation.

2.1 Jacobi Polynomials

Our starting point are the Jacobi polynomials $P_n^{\alpha,\beta}$ [14]. They are eigenfunctions of the Jacobi differential operator

$$L_J(z,\partial_z)P_n^{\alpha,\beta}(z) = N\big(n\big)P_n^{\alpha,\beta}\big(z\big), \tag{2.1}$$

where

$$L_J(z,\partial_z) = \big(1-z^2\big)\partial_z^2 + \big(\beta-\alpha-\big(\alpha+\beta+2\big)z\big)\partial_z,$$

$$N\big(n\big) = -n\big(n+\alpha+\beta+1\big),$$

and α, β and z are all complex numbers.

They are orthogonal with respect to the weight $\big(1-z\big)^{\alpha}\big(1-z\big)^{\beta}$:

$$\int_{-1}^{1}(1-z)^{\alpha}(1+z)^{\beta}P_n^{\alpha,\beta}(z)P_m^{\alpha,\beta}(z)dz = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+_1)n!}\delta_{nm}.$$

Equation (2.1) is a special case of the hypergeometric differential equation. Therefore its solutions can be expressed in terms of the Gaussian hypergeometric function ${}_2F_1$:

$$\frac{(\alpha+1)_n}{n!}{}_2F_1\bigg(-n, n+\alpha+\beta+1, 1+\alpha; \frac{1-z}{2}\bigg), \tag{2.2}$$

$$\frac{(\alpha+1)_n}{n!}{}_2F_1\bigg(-n-\alpha, n+\beta+1, 1-\alpha; \frac{z-1}{2}\bigg). \tag{2.3}$$

where

$${}_2F_1(a,b,c;z) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k\, k!}z^k.$$

Here, $(q)_n$ is the Pochhammer's symbol for rising factorial defined as follows:

$$(q)_0 = 1, \quad (q)_n = q(q+1)(q+2)...(q+n-1) = \prod_{i=0}^{n-1}(q+i).$$

Solution (2.2) is a Jacobi polynomial $P_n^{\alpha,\beta}\big(z\big)$ because the hypergeometric series terminates when a is a non-positive integer in ${}_2F_1(a,b,c;z)$.

The Jacobi differential operator in (2.1) is *bispectral*. This means that Jacobi polynomials are also eigenfunctions of a three term recurrence operator in the variable n:

$$A_J(n,T)P_n^{\alpha,\beta}(z) = zP_n^{\alpha,\beta}(z). \tag{2.4}$$

where

$$A_J(n,T) = A_+(n)T + A_0(n) + A_-(n)T^{-1}.$$

Here, T is the shift operator given by $n \mapsto n+1$ and the coefficients are:

$$\begin{aligned} A^+(n) &= \frac{(2n+2)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}, \\ A^0(n) &= \frac{(\beta^2-\alpha^2)}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}, \\ A^-(n) &= \frac{2(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}. \end{aligned} \tag{2.5}$$

We consider a more general form of the above equations in which the index n is not necessarily an integer. The resulting solutions would no longer be polynomials, but they satisfy similar equations. We call them Jacobi functions. These are as follows (see [15], [16] and [17]):

$$P_\epsilon^{\alpha,\beta}(n,z) = \frac{(\epsilon+\alpha+1)_n}{(\epsilon+1)_n} F(-(n+\epsilon), n+\epsilon+\alpha+\beta+1, \alpha+1; (1-z)/2). \tag{2.6}$$

Here $\epsilon, \alpha \notin \mathbb{Z}_{<0}$. The functions (2.6) satisfy the ordinary differential equation

$$L_J(z,\partial_z)P_\epsilon^{\alpha,\beta}(n,z) = N(n+\epsilon)P_\epsilon^{\alpha,\beta}(n,z)$$

as well as the difference operator

$$\begin{aligned}A_J(n+\epsilon,T) = &\frac{2(n+\epsilon+1)(n+\epsilon+\alpha+\beta+1)}{(2n+2\epsilon+\alpha+\beta+1)(2n+2\epsilon+\alpha+\beta+2)}T \\ &+ \frac{(\beta^2-\alpha^2)}{(2n+2\epsilon+\alpha+\beta)(2n+2\epsilon+\alpha+\beta+2)} \\ &+ \frac{2(n+\epsilon+\alpha)(n+\epsilon+\beta)}{(2n+2\epsilon+\alpha+\beta)(2n+2\epsilon+\alpha+\beta+1)}T^{-1}.\end{aligned}$$

Since Jacobi functions are a special case of Gaussian hypergeometric functions, we work with the hypergeometric differential operator. We do this because we later take advantage of monodromy of hypergeometric functions around $z = 0, 1$ and ∞.

We transform the hypergeometric equation into the Darboux-Pöschl-Teller equation because that is what we would like to construct bispectral extensions for.

2.2 Darboux-Pöschl-Teller Operator

The standard hypergeometric differential equation is

$$z(1-z)\frac{d^2w}{dz^2} + [c-(a+b+1)z]\frac{dw}{dz} - abw = 0. \tag{2.7}$$

This equation has regular singularities at $z = 0, 1$ and ∞. The Frobenius method gives a basis of solutions as series expansions around each of the singular points. These solutions are classically known and are expressed in terms of hypergeometric series.

The corresponding solutions to (2.7) near $z = 0$ are

$$_2F_1\Big(a, b, c; z\Big) \quad \text{and}$$

$$z^{1-c}(1-z)^{c-a-b}{}_2F_1\Big(1-a, 1-b, 2-c; z\Big). \tag{2.8}$$

The solutions to (2.7) near $z = 1$ are

$$ {}_2F_1\Big(a, b, a+b-c+1; 1-z\Big) \quad \text{and} $$

$$ z^{1-c}(1-z)^{c-a-b}{}_2F_1\Big(1-a, 1-b, c-a-b+1; 1-z\Big). \tag{2.9} $$

The solutions to (2.7) near $z = \infty$ are:

$$ (-4z)^{-a}{}_2F_1\Big(a, a-c+1, a-b+1; z^{-1}\Big) \quad \text{and} $$

$$ (-4z)^{-b}{}_2F_1\Big(b, b-c+1, b-a+1; z^{-1}\Big) \quad . \tag{2.10} $$

Jacobi polynomials are a special case of hypergeometric series. This can be seen from the following relabelling of parameters:

$$ a := -n, \quad b := n+\alpha+\beta+1, \quad c := \alpha+1, \quad z \mapsto 1-2z. $$

In working with the Darboux-Pöschl-Teller (DPT) equation, we are going to use properties of the hypergeometric equation. To see the link between hypergeometric differential equation and the DPT equation, we relabel our parameters as follows.

$$ a = \lambda + \frac{g+h}{2}, \quad b = -\lambda + \frac{g+h}{2}, \quad c = g + \frac{1}{2}. \tag{2.11} $$

Here, $g, h \in \mathbb{C}$. This re-parametrization turns equation (2.7) into

$$ L_z\big(z, \partial_z\big)\phi\big(\lambda, z\big) = \lambda^2\phi\big(\lambda, z\big), \tag{2.12} $$

where

$$ L_z\big(z, \partial_z\big) = z\big(z-1\big)\partial_z^2 + \Bigg[\bigg(\big(1+g+h\big)z - \frac{1}{2} + g\bigg)\Bigg]\partial_z + \Big(\frac{g+h}{2}\Big)^2. \tag{2.13} $$

We will use the Frobenius series solutions (2.10) near $z = \infty$ most frequently. With the re-parametrisation (2.11), these are given explicitly as

$$\phi_{\pm} = \phi\big(z, \pm\lambda\big) = (-4z)^{\mp\lambda - \frac{g+h}{2}} {}_2F_1\left(\frac{g+h}{2} \pm \lambda, \frac{1}{2} \pm \lambda - \frac{g-h}{2}, 1 \pm 2\lambda; \frac{1}{z}\right). \tag{2.14}$$

Other pairs of solutions will be used in special cases which will be described in later chapters.

We would like to construct bispectral extensions of Darboux-Pöschl-Teller (DPT) operator. To acquire the DPT operator from the operator (2.13), we substitute $z = \frac{1}{2} - \frac{1}{4}\big(e^{ix} + e^{-ix}\big) = \sin^2(x/2)$ to obtain

$$L\big(x, \partial_x\big) = -\partial_x^2 - \left((g-h)\cot\left(\frac{x}{2}\right) + 2h\cot x\right)\partial_x + \frac{(g+h)^2}{2}. \tag{2.15}$$

Eigenfunctions for this operator are found by substituting a series of the form [18]:

$$f = \sum_{\nu \geq 0} \Gamma_\nu(\mu) e^{ix(\mu+\nu)}. \tag{2.16}$$

This is known as Frobenius method, which allows us to find the coefficients Γ_ν recursively by setting $\Gamma_0 = 1$. It gives an equation for possible values of μ; this is known as the indicial equation. The solutions to indicial equation in this case are $\mu = \pm\lambda + (g+h)/2$. The resulting operator (2.15) has a first order derivative. To remove it and obtain a Schrödinger operator, a gauge transformation is required. This involves conjugating (2.15) with a gauge function g which removes the first order derivative. This process gives the Darboux-Pöschl-Teller (DPT) operator [19]:

$$L_x\big(x, \partial_x\big) := \mathrm{g}^{-1} \circ L(x, \partial_x) \circ \mathrm{g} = -\partial_x{}^2 + u, \qquad u = \frac{g(g-1)}{4\sin^2\left(\frac{x}{2}\right)} + \frac{h(h-1)}{4\cos^2\left(\frac{x}{2}\right)}. \tag{2.17}$$

The gauge function g is

$$\mathrm{g} = \sin^{-g}\left(\frac{x}{2}\right)\cos^{-h}\left(\frac{x}{2}\right). \tag{2.18}$$

If $f(z)$ is an eigenfunction to (2.13), then $\mathrm{g}^{-1}(x) f(\sin^2(x/2))$ would be an eigenfunction to (2.17).

The eigenfunctions to the operator (2.15) are found by substituting a series of the form (2.16). For (2.17), the series would be of the form

$$\sum_{\nu\geq 0}\tilde{\Gamma}_\nu(\lambda)e^{i(\pm\lambda+\nu)x}. \tag{2.19}$$

That is, the gauge function will get rid of the index $(g+h)/2$. This is because

$$\mathrm{g}^{-1} = \left(\sin\left(\frac{x}{2}\right)\right)^g\left(\cos\left(\frac{x}{2}\right)\right)^h$$

$$= \Big(\frac{e^{ix/2}-e^{-ix/2}}{2i}\Big)^g\Big(\frac{e^{ix/2}+e^{-ix/2}}{2}\Big)^h$$

$$= e^{-gix/2}\Big(\frac{e^{ix}-1}{2i}\Big)^g e^{-hix/2}\Big(\frac{e^{ix}+1}{2}\Big)^h \propto e^{-ix(\frac{g+h}{2})}+...$$

$$\Longrightarrow \mathrm{g}^{-1}f \propto e^{-ix(\frac{g+h}{2})}\sum_{\nu\geq 0}\Gamma_\nu(\lambda)e^{i(\pm\lambda+\frac{g+h}{2}+\nu)x} \propto \sum_{\nu\geq 0}\tilde{\Gamma}_\nu(\lambda)e^{i(\pm\lambda+\nu)x}.$$

There are 4 choices for pairs of g and h: replacing g by $1-g$ and/or replacing h by $1-h$ leaves (2.17) unchanged. Of course, making any such change would change the solution $\mathrm{g}^{-1}f$ as well as the original hypergeometric equation (2.12). So there are 4 separate hypergeometric equations which can be transformed in to (2.17) by conjugating with an appropriate gauge function. Each of these 4 equations has Jacobi functions as its special eigenfunctions. As a result the DPT operator has 4 families of elementary eigenfunctions.

The fact that DPT can have elementary eigenfunctions is of significance to bispectrality. These elementary functions arise when one of the following is a negative integer.

$$\lambda+\frac{g+h}{2},\quad -\lambda+\frac{g+h}{2},\ \lambda+\frac{g+1-h}{2},\quad -\lambda+\frac{g+1-h}{2}.$$

The emergence of elementary solutions coincides with the reducibility of the monodromy group for the hypergeometric equation (that is, existence of a solution which is invariant, up to a factor, under the monodromy transformations). This allows for more possibilities for creating bispectral extensions (see section 4.4.5).

The DPT operator (2.17) is bispectral. For example, its solutions near $z = 0$ can be written in terms of Jacobi polynomials when $\lambda = n + (g + h)/2$, $n \in \mathbb{N}$. Those solutions would therefore satisfy the three term recurrence relation (2.4). This would be true even for $n \notin \mathbb{N}$.

It is more useful to work with solutions near $z = \infty$ (2.14). We normalise those solutions with the following function for reasons which will become clear in section 4.2:

$$c(\lambda) = \frac{2^{2\lambda+g+h}\Gamma\big(\frac{1}{2}+g\big)\Gamma\big(-2\lambda\big)}{\Gamma\big(-\lambda+\frac{g+h}{2}\big)\Gamma\big(\frac{g-h+1}{2}-\lambda\big)}.$$

Set

$$\psi(x, \lambda) = c(\lambda)\mathrm{g}^{-1}(x)\phi\Big(\sin^2\frac{x}{2}, \lambda\Big), \tag{2.20}$$

where g is as in (2.18).

The solution $\psi(x, \lambda)$ satisfies the following three term recurrence relation,

$$A_\lambda\big(\lambda, T\big)\psi(x, \lambda) = -4\sin^2\left(\frac{x}{2}\right)\psi(x, \lambda),$$

where $A_\lambda\big(\lambda, T\big)$ is the difference operator:

$$\left(1+\frac{g+h}{2\lambda}\right)\left(1+\frac{g-h}{2\lambda+1}\right)(T-1)+\left(1-\frac{g+h}{2\lambda}\right)\left(1-\frac{g-h}{2\lambda-1}\right)(T^{-1}-1). \tag{2.21}$$

For general λ this would follow from contiguity relations on hypergeometric functions. In the special case that $h = 0$, this observation was made in [20]. It can also be viewed as a one-variable case of [21, Theorem 6.12]. Below, we provide a proof for this three term recurrence relation.

Theorem 2.1. *$\psi(x, \lambda)$ and $\psi(x, -\lambda)$ are both eigenfunctions of the difference operator $A_\lambda(\lambda, T)$. Specifically,*

$$A_\lambda(\lambda, T)\psi(x, \pm\lambda) = -4\sin^2\left(\frac{x}{2}\right)\psi(x, \pm\lambda).$$

Proof. The solutions of DPT equation ($\psi(x, \pm\lambda)$ and ${}_2F_1(\lambda - \frac{g+h}{2}, \lambda + \frac{g+h}{2}, g + \frac{1}{2}; z)$) are related. We have the connection formula for hypergeometric function [22]:

$$F(a, b, c; z) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} f_1 + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} f_2 \tag{2.22}$$

Here,

$$f_1 = e^{a\pi i} z^{-a} {}_2F_1\Big(a, a-c+1, a-b+1; z^{-1}\Big) \quad \text{and}$$

$$f_2 = e^{b\pi i} z^{-b} {}_2F_1\Big(b, b-c+1, b-a+1; z^{-1}\Big).$$

With the reparametrisation (2.11), the connection formula (2.22) becomes

$$F(x, \lambda) := \mathrm{g}^{-1} {}_2F_1\left(\lambda - \frac{g+h}{2}, \lambda + \frac{g+h}{2}, g + \frac{1}{2}; z\right) = \psi(x, \lambda) + \psi(x, -\lambda). \tag{2.23}$$

We will first show using the properties of Jacobi polynomials that the hypergeometric function $F(x, \lambda)$ on the left hand side is an eigenfunction of A_λ. We will then show that this property gets inherited by both $\psi(x, \pm\lambda)$ in the linear combination.

In the spirit of (2.6), allow n to be non-integer. Hypergeometric functions can be written in terms of Jacobi functions as mentioned before in (2.2).

$${}_2F_1\Big(-n, n+\alpha+\beta+1, \alpha+1; z\Big) = \frac{n!}{(\alpha+1)_n} P_n^{\alpha,\beta}\big(1-2z\big). \tag{2.24}$$

Furthermore the Jacobi polynomials satisfy the three term recurrence (2.4):

$$\Big[A^+(n)T + A^0(n) + A^-(n)T^{-1}\Big] P_n^{\alpha,\beta}\big(1-2z\big) = \big(1-2z\big) P_n^{\alpha,\beta}\big(1-2z\big)$$

$$\Longrightarrow \Big[2A^+(n)T + 2(A^0(n)-1) + 2A^-(n)T^{-1}\Big] P_n^{\alpha,\beta}\big(1-2z\big) = -4z P_n^{\alpha,\beta}\big(1-2z\big). \tag{2.25}$$

The coefficients A^+, A^0 and A^- are given in (2.5).

The two equations (2.24) and (2.25) show that ${}_2F_1(-n, n+\alpha+\beta+1, \alpha+1; z)$ is an eigenfunction of the difference operator

$$\frac{n!}{(\alpha+1)_n}\Big[2A^+(n)T + 2(A^0(n)-1) + 2A^-(n)T^{-1}\Big]\frac{(\alpha+1)_n}{n!} \tag{2.26}$$

with eigenvalue $-4z = -4\sin^2(x/2)$. By substituting

$$\alpha = g - \frac{1}{2}, \quad \beta = h - \frac{1}{2} \quad \text{and} \quad n = \lambda - \frac{g+h}{2} \tag{2.27}$$

we find that the operator (2.26) is in fact $A_\lambda(\lambda, T)$. Therefore

$$A_\lambda\big(\lambda, T\big)F(x,\lambda) = -4\sin^2\left(\frac{x}{2}\right)F(x,\lambda).$$

Now to show that $\psi(x, \pm\lambda)$ is an eigenfunction of $A_\lambda(\lambda, T)$, we map $x \mapsto x + 2\pi$. Recall that

$$\psi(x, \pm\lambda) = \sum_{n\geq 0}^{\infty} c(\lambda)\tilde{\Gamma}_n(\lambda)e^{i(\pm\lambda+n)x},$$

where $\tilde{\Gamma}_n(\lambda)$ are coefficients in Frobenius series solution for $\psi(x, \pm\lambda)$, see (2.19). As a result,

$$\psi(x+2\pi, \pm\lambda) = \sum_{n\geq 0}^{\infty} c(\lambda)\tilde{\Gamma}_n(\lambda)e^{i(\pm\lambda+n)(x+2\pi)} = e^{\pm 2\pi i\lambda}\psi(x, \pm\lambda).$$

$$F(x,\lambda) = \psi(x,\lambda) + \psi(x,-\lambda) \implies F(x+2\pi,\lambda) = \psi(x+2\pi,\lambda) + \psi(x+2\pi,-\lambda).$$

$$\implies F(x+2\pi,\lambda) = e^{2\pi i\lambda}\psi(x,\lambda) + e^{-2\pi i\lambda}\psi(x,-\lambda).$$

$$-4\sin^2\left(\frac{x+2\pi}{2}\right) = -4\Big(-\sin\left(\frac{x}{2}\right)\Big)^2 = -4\sin^2\left(\frac{x}{2}\right).$$

So $F(x+2\pi,\lambda)$ is an eigenfunction of $A_\lambda(\lambda, T)$. Also, because

$$T^m e^{-2\pi i\lambda} = e^{-2\pi i(\lambda+m)}\ T^m = e^{-2\pi i\lambda}e^{-2\pi i m}\ T^m = e^{-2\pi i\lambda}\ T^m$$

for all $m \in \mathbb{Z}$, it follows that

$$A_\lambda(\lambda, T)[e^{-2\pi i\lambda}F(x,\lambda)] = e^{-2\pi i\lambda}A(\lambda, T)F(x,\lambda) = -4z[e^{-2\pi i\lambda}F(x,\lambda)].$$

So we have found a basis for the eigenspace of $D_\lambda(\lambda, T)$:

$$\text{Ker}\left(A_\lambda(\lambda, T) + 4\sin^2\left(\frac{x}{2}\right)\right) = \text{span}\Big\{F(x+2\pi, \lambda), e^{-2\pi i\lambda}F(x, \lambda)\Big\}$$

$$= \text{span}\Big\{e^{2\pi i\lambda}\psi(x, \lambda) + e^{-2\pi i\lambda}\psi(x, -\lambda), e^{-2\pi i\lambda}\psi(x, \lambda) + e^{-2\pi i\lambda}\psi(x, -\lambda)\Big\}.$$

From this it follows that $\psi(x, \lambda)$ is an eigenfunction of $A_\lambda(\lambda, T)$.

$$\psi(x, \lambda) = \frac{F(x+2\pi, \lambda) - e^{-2\pi i\lambda}F(x, \lambda)}{2i\sin(2\pi\lambda)} \in \ker\Big(D_\lambda(\lambda, T) + 4\sin^2(x/2)\Big). \tag{2.28}$$

The operator $A_\lambda(\lambda, T)$ is invariant under the reflection $\lambda \mapsto -\lambda$. So with the particular normalisation (2.20), we would also have

$$A_\lambda\big(\lambda, T\big)\psi(x, -\lambda) = -4\sin^2\left(\frac{x}{2}\right)\psi(x, -\lambda).$$

■

The differential and difference operators in the above equations all have coefficients which are rational in terms of z, e^{ix} and λ. The coefficients of (2.17) are called trigonometric coefficients: these are coefficients which are rational in terms of e^{ix}.

In the next chapter, we prove that performing Darboux transformations with trigonometric coefficients on polynomials of (2.17) would gives us new bispectral operators.

Chapter 3

Bispectrality and Polynomial Darboux Transformations

In this chapter, we will define polynomial Darboux transformations. We prove that Darboux transformations with certain properties will lead to new operators which are bispectral. To that end, we will set up the notation and the framework for various structures we use. We will then move on to the proof itself.

3.1 Polynomial Darboux Transformations

Let us first explain the general scheme for constructing polynomial Darboux transformations, which are higher order analogues of the classical Darboux transformations.

Classically, a Darboux transformation of a differential operator L is given by expressing it as a product of two factors, and then interchanging their orders [23]. Since composing differential operators is not a commutative operation, the resulting operator is going to be different from the original one.

$$L = Q \circ P \quad \mapsto \quad \hat{L} = P \circ Q.$$

If ψ is in the kernel of L, then $\hat{\psi} = P\psi$ is in the kernel of $\hat{L}$.

Such a factorisation exists if and only if the kernel of P is a subspace of kernel of L. So we find the kernel of L, pick some subspace from it and construct P. For example, if we choose a space with the basis $\{f_1, f_2, ..., f_n\}$, then the operator P whose leading coefficient is 1 and whose kernel is equal to this space is found using the following:

$$P\varphi = \frac{Wr\{f_1, f_2, ...f_n, \varphi\}}{Wr\{f_1, f_2, ...f_n\}}, \tag{3.1}$$

where $Wr\{f_1, f_2, ...f_n\}$ is the Wronskian of the functions f_i. P is a differential operator of order n. The reason why this works is because normally, $L \circ P^{-1}$ would be a pseudo-differential operator of the form $L \circ P^{-1} = Q + R \circ P^{-1}$, where Q is a quotient and R is a remainder operator whose order is less than that of P. This implies that $L = Q \circ P + R$. If f is in the kernel of P, then since $\ker P$ is a subset of kernel of L, f is also in $\ker L$.

We get $Lf = Q \circ Pf + Rf = Q(Pf) + Rf = Q(0) + Rf = Rf = 0$. Since order of P is greater than the order of R and f is any function in the kernel of P, therefore the only way that $Rf = 0$ is if $R \equiv 0$.

To construct entire algebras of bispectral operators, we want to perform Darboux transformations on polynomials of the differential operator L_x. In the spirit of [1] and [9], we start off by selecting arbitrary values λ_1, λ_2, ... , λ_n. There would also be a multiplicity associated to each λ_l, called m_l.

This gives a polynomial differential operator of the form:

$$h\big(L_x\big) = \prod_{l=1}^{n} \big(L_x - {\lambda_l}^2\big)^{m_l}. \tag{3.2}$$

We would like first to describe the kernel of $h(L_x)$. This would be

$$\ker h\big(L_x\big) = \bigoplus_{l=1}^{n} {S_l}^{\pm}, \tag{3.3}$$

where

$${S_l}^{\pm} = \operatorname{span}\left\{ \frac{\partial^k}{\partial \lambda^k} \phi\big(\pm \lambda_l, x\big) : 0 \leq k \leq m_l - 1 \right\}.$$

Lemma 3.1. *For* $\lambda \neq 0$*, if* $\{\phi_+, \phi_-\}$ *is a basis for* $\ker(L_x - \lambda^2)$*, then*

$$\left\{ \frac{\partial^p}{\partial \lambda^p} \phi_\pm : 0 \leq p \leq m \right\} \tag{3.4}$$

is a basis for $\ker(L_x - \lambda^2)^{m+1}$.

Proof. First we will prove

$$(L_x - \lambda^2)^n \frac{\partial^p}{\partial \lambda^p} \phi_\pm = \sum_{i=n}^{2n} a_i \frac{p!}{(p-i)!} \frac{\partial^{p-i}}{\partial \lambda^{p-i}} \phi_\pm, \tag{3.5}$$

where a_i are some coefficients and $n \in \mathbb{N}$. We can do this by induction.

<u>Induction Base</u>: When $n = 0$,

$$\text{LHS} = (L_x - \lambda^2)^0 \frac{\partial^p}{\partial \lambda^p} \phi_\pm = \frac{\partial^p}{\partial \lambda^p} \phi_\pm.$$

$$\text{RHS} = a_0 \frac{p!}{(p-0)!} \frac{\partial^{p-0}}{\partial \lambda^{p-0}} \phi_\pm = a_0 \frac{\partial^p}{\partial \lambda^p} \phi_\pm, \text{ so } a_0 = 1.$$

The $n = 0$ case is highly trivial; there is more structure to the formula (3.5). To give the reader a better understanding of what the operator $(L_x - \lambda^2)^n$ is doing, we perform the above calculation for a couple more values of n. For $n = 1$,

$$\text{LHS} = (L_x - \lambda^2)^1 \frac{\partial^p}{\partial \lambda^p}\phi_\pm = 2\lambda p \frac{\partial^{p-1}}{\partial \lambda^{p-1}}\phi_\pm + p(p-1)\frac{\partial^{p-2}}{\partial \lambda^{p-2}}\phi_\pm. \tag{3.6}$$

$$\text{RHS} = \sum_{i=1}^{2} a_i \frac{p!}{(p-i)!}\frac{\partial^{p-i}}{\partial \lambda^{p-i}}\phi_\pm = a_1 p \frac{\partial^{p-1}}{\partial \lambda^{p-1}}\phi_\pm + a_2 p(p-1)\frac{\partial^{p-2}}{\partial \lambda^{p-2}}\phi_\pm. \tag{3.7}$$

So LHS = RHS with $a_1 = 2\lambda$ and $a_2 = 1$.

For $n = 2$,

$$\text{LHS} = (L_x - \lambda^2)^2 \frac{\partial^p}{\partial \lambda^p}\phi_\pm = 4\lambda^2 p(p-1)\frac{\partial^{p-2}}{\partial \lambda^{p-2}}\phi_\pm$$

$$+4\lambda p(p-1)\frac{\partial^{p-3}}{\partial \lambda^{p-3}}\phi_\pm + p(p-1)(p-2)(p-3)\frac{\partial^{p-4}}{\partial \lambda^{p-4}}\phi_\pm.$$

$$\text{RHS} = \sum_{i=2}^{4} a_i \frac{p!}{(p-i)!}\frac{\partial^{p-i}}{\partial \lambda^{p-i}}\phi_\pm = a_2 p(p-1)\frac{\partial^{p-2}}{\partial \lambda^{p-2}}\phi_\pm$$

$$+a_3 p(p-1)(p-2)\frac{\partial^{p-3}}{\partial \lambda^{p-3}}\phi_\pm + a_4 p(p-1)(p-2)(p-3)\frac{\partial^{p-4}}{\partial \lambda^{p-4}}\phi_\pm.$$

So for $n = 2$, LHS = RHS with $a_2 = 4\lambda^2$, $a_3 = 4\lambda$ and $a_4 = 1$.

Induction Assumption: Suppose that for some k,

$$(L_x - \lambda^2)^k \frac{\partial^p}{\partial \lambda^p}\phi_\pm = \sum_{i=k}^{2k} a_i \frac{p!}{(p-i)!}\frac{\partial^{p-i}}{\partial \lambda^{p-i}}\phi_\pm.$$

Induction Step: Advance k by 1.

$$(L_x - \lambda^2)^{k+1} \frac{\partial^p}{\partial \lambda^p}\phi_\pm = \sum_{i=k}^{2k} a_i \frac{p!}{(p-i)!}(L_x - \lambda^2)\frac{\partial^{p-i}}{\partial \lambda^{p-i}}\phi_\pm.$$

$$= \sum_{i=k}^{2k} a_i \frac{p!}{(p-i)!}\left[2\lambda(p-i)\frac{\partial^{p-i-1}}{\partial \lambda^{p-i-1}}\phi_\pm + (p-i)(p-i-1)\frac{\partial^{p-i-2}}{\partial \lambda^{p-i-2}}\phi_\pm\right]$$

$$= \sum_{i=k}^{2k} a_i 2\lambda \frac{p!}{(p-(i+1))!}\frac{\partial^{p-(i+1)}}{\partial \lambda^{p-(i+1)}}\phi_\pm + \sum_{i=k}^{2k} a_i \frac{p!}{(p-(i+2))!}\frac{\partial^{p-(i+2)}}{\partial \lambda^{p-(i+2)}}\phi_\pm$$

$$= \sum_{i=k+1}^{2k+1} a_{i-1} 2\lambda \frac{p!}{(p-i)!}\frac{\partial^{p-i}}{\partial \lambda^{p-i}}\phi_\pm + \sum_{i=k+2}^{2k+2} a_{i-2} \frac{p!}{(p-i)!}\frac{\partial^{p-i}}{\partial \lambda^{p-i}}\phi_\pm$$

$$= \sum_{i=k+1}^{2(k+1)} b_i \frac{p!}{(p-i)!} \frac{\partial^{p-i}}{\partial \lambda^{p-i}} \phi_\pm,$$

where $b_{k+1} = 2a_k\lambda$, $b_{2(k+1)} = a_{2k}$ and $b_i = 2\lambda a_{i-1} + a_{i-2}$ for $k+2 \leq i \leq 2k+1$.

Hence, by the principle of mathematical induction, the sum (3.5) is correct for $n \leq p$. Beyond that,

$$(L_x - \lambda^2)^{p+1} \frac{\partial^p}{\partial \lambda^p} \phi_\pm = \sum_{i=p+1}^{2(p+1)} a_i \frac{p!}{(p-i)!} \frac{\partial^{p-i}}{\partial \lambda^{p-i}} \phi_\pm = 0, \text{ because } \frac{p!}{(p-i)!} = 0$$

for $i \geq p$. This shows that $\partial_\lambda^p \phi_\pm \in \ker(L - \lambda^2)^{m+1}$ for $0 \leq p \leq m$.

The following method for establishing linear independence of the proposed basis functions was inspired by lemma 1.1 in [5]. For $n = p$, we get

$$(L_x - \lambda^2)^p \frac{\partial^p}{\partial \lambda^p} \phi_\pm = a_p p! \phi_\pm \propto \phi_\pm. \tag{3.8}$$

Consider the linear combination equation:

$$\sum_{p=0}^{m} \left[c_{+,p} \frac{\partial^p}{\partial \lambda^p} \phi_+ + c_{-,p} \frac{\partial^p}{\partial \lambda^p} \phi_- \right] = 0. \tag{3.9}$$

Apply $(L_x - \lambda^2)^m$ to (3.9) to get

$$c_{+,m}\phi_+ + c_{-,m}\phi_- = 0.$$

$$\implies c_{+,m} = c_{-,m} = 0.$$

This follows from the linear independence of $\{\phi_+, \phi_-\}$. Substitute $c_{\pm,m}$ back into (3.9) and apply $(L_x - \lambda^2)^{m-1}$ to get obtain $c_{\pm,m-1} = 0$.

Repeating this process iteratively gives us $c_{\pm,i} = 0$ for $0 \leq i \leq m$. Substituting these zero coefficients back into (3.9) shows us that for all p, $\partial_\lambda^p \phi_\pm$ are linearly independent.

Furthermore,

$$\dim\ker(L_x-\lambda^2)^{m+1} = \text{order}(L_x-\lambda^2)^{m+1} = 2(m+1)$$

and the set (3.4) contains $2(m+1)$ linearly independent functions. Thus the set (3.4) is a basis for the kernel of $(L_x-\lambda^2)^{m+1}$. ■

For completeness, the $\lambda = 0$ case is also treated in the following theorem.

Lemma 3.2. *Suppose* $\{\phi_+, \phi_-\}$ *is a basis for* $\ker L_x$. *Then*

$$\left\{\frac{\partial^{2p}}{\partial\lambda^{2p}}\phi_\pm : 0 \le p \le m\right\} \tag{3.10}$$

is a basis for $\ker L_x^{m+1}$.

Proof. For non-zero λ,

$$(L-\lambda^2)\frac{\partial^{2p}}{\partial\lambda^{2p}}\phi_\pm = 2\lambda(2p)\frac{\partial^{2p-1}}{\partial\lambda^{2p-1}}\phi_\pm + 2p(p-1)\frac{\partial^{2p-1}}{\partial\lambda^{2p-2}}\phi_\pm.$$

When we apply $(L_x-\lambda^2)$ repetitively, the only term which does not vanish by setting $\lambda = 0$ comes from the last term which is not already being multiplied by λ.

$$\implies (L_x-\lambda^2)^2\Big|_{\lambda=0}\frac{\partial^{2p}}{\partial\lambda^{2p}}\phi_\pm = \frac{(2p)!}{(2p-4)!}\frac{\partial^{2p-4}}{\partial\lambda^{2p-4}}\phi_\pm \text{ and so on.}$$

$$\implies (L_x-\lambda^2)^n\Big|_{\lambda=0}\frac{\partial^{2p}}{\partial\lambda^{2p}}\phi_\pm = \frac{(2p)!}{(2p-2n)!}\frac{\partial^{2p-2n}}{\partial\lambda^{2p-2n}}\phi_\pm, \text{ for } n \le p.$$

For $n = p$ and $n = p+1$ we would get

$$(L_x-\lambda^2)^p\Big|_{\lambda=0}\frac{\partial^{2p}}{\partial\lambda^{2p}}\phi_\pm = \frac{(2p)!}{(2p-2p)!}\frac{\partial^{2p-2p}}{\partial\lambda^{2p-2p}}\phi_\pm = \frac{(2p)!}{(0)!}\frac{\partial^0}{\partial\lambda^0}\phi_\pm = (2p)!\phi_\pm$$

and

$$(L_x-\lambda^2)^{p+1}\Big|_{\lambda=0}\frac{\partial^{2p}}{\partial\lambda^{2p}}\phi_\pm = 0 \implies \frac{\partial^{2p}}{\partial\lambda^{2p}}\phi_\pm \in \ker L^{m+1}.$$

Consider

$$\sum_{p=0}^{m} c_{+,2p}\frac{\partial^{2p}}{\partial\lambda^{2p}}\phi_{+} + c_{-,2p}\frac{\partial^{2p}}{\partial\lambda^{2p}}\phi_{-} = 0. \tag{3.11}$$

Apply L_x^m to (3.11) to get $c_{+,2m}\phi_+ + c_{-,2m}\phi_- = 0$. Linear independence of $\{\phi_+, \phi_-\}$ then gives us $c_{\pm,2m} = 0$.

Apply L_x^{m-i} inductively to get $c_{\pm,2i} = 0$ for $0 \le i \le m$. Therefore $\{\partial_\lambda^{2p}\phi_\pm : 0 \le p \le m\}$ is linearly independent. Additionally,

$$\dim\ker L_x^{m+1} = \mathrm{order} L_x^{m+1} = 2(m+1).$$

Therefore (3.10) is a basis for the kernel of L_x^{m+1}. ■

We want to perform Darboux transformations on (3.2) in such a way that the resulting operator is bispectral. This will happen if our Darboux factorisation takes a specific form. In this form, the factorisation will fit in to a general result which will ensure bispectrality. To explain all this, we first need to set up some notation.

3.2 Bispectral Triples and Darboux Transformations

In this section we prove that certain Darboux transformations will lead to bispectral operators. The constructions here were motivated by earlier works on the bispectral problem such as [1], [2], [3] and [24].

Let $\mathcal{B}$ be the algebra of differential operators generated by L_x and $-4\sin^2(x/2)$:

$$\mathcal{B} = \left\langle L_x, -4\sin^2\left(\frac{x}{2}\right)\right\rangle. \tag{3.12}$$

Similarly, let

$$\mathcal{B}' = \langle \lambda^2, A_\lambda \rangle \tag{3.13}$$

be the algebra of difference operators generated by λ^2 and A_λ (2.21) where λ is an unfixed variable. Define the following anti-isomorphism:

$$\mathfrak{b} : \mathcal{B} \to \mathcal{B}',$$

$$L_x \mapsto \lambda^2, \qquad -4\sin^2\left(\frac{x}{2}\right) \mapsto A_\lambda,$$

$$\text{and} \quad \mathfrak{b}(u \circ v) = \mathfrak{b}(v) \circ \mathfrak{b}(u). \tag{3.14}$$

We call this the bispectral anti-isomorphism; it encodes the bispectral property.

$$\mathfrak{b}\big(X\big)\psi\big(x,\lambda\big) = X\psi\big(x,\lambda\big) \quad \forall X \in \mathcal{B}, \tag{3.15}$$

where $\psi(x, \lambda)$ is given in (2.20).

Further to the algebras above, we also define two commutative subalgebras of functions:

$$\mathcal{K} = \left\langle -4\sin^2\left(\frac{x}{2}\right)\right\rangle \quad \text{and} \quad \mathcal{K}' = \langle \lambda^2 \rangle. \tag{3.16}$$

Finally, we would like to have the following sets as well.

$$\mathcal{K}^{-1}\mathcal{B} = \Big\{\Theta^{-1}V : \Theta \in \mathcal{K}, V \in \mathcal{B}\Big\},$$

$$\mathcal{B}\mathcal{K}^{-1} = \Big\{U\Gamma^{-1} : \Gamma \in \mathcal{K}, U \in \mathcal{B}\Big\}. \tag{3.17}$$

The next theorem from [24] gives us bispectrality of specific Darboux transformations; it tells us about the existence of a dual difference operator. We present it here along with its brief proof.

Theorem 3.3. *Let $h(L_x) \in \mathcal{B}$ be a constant coefficient polynomial in terms of L_x. Suppose it factorises as:*

$$h(L_x) = Q\big(x, \partial_x\big) \circ P\big(x, \partial_x\big) \tag{3.18}$$

in such a way that

$$Q\big(x, \partial_x\big) = U\Gamma^{-1} \in \mathcal{B}\mathcal{K}^{-1} \quad \textit{and} \quad P\big(x, \partial_x\big) = \Theta^{-1}V \in \mathcal{K}^{-1}\mathcal{B}, \tag{3.19}$$

then $\hat{L}_x = P \circ Q$ *is a bispectral operator.*

Specifically, defining $f = \mathfrak{b}\big[h(L_x)\big] = h\big(\mu^2\big)$ *and* $\hat{\psi} = P\psi$ *(where* ψ *is the eigenfunction of* $h(L_x)$*), we have the bispectral pair:*

$$\hat{L}_x \hat{\psi} = f \hat{\psi}, \tag{3.20}$$

$$\hat{A}_\mu \hat{\psi} = \Theta\Gamma \hat{\psi}, \tag{3.21}$$

where

$$\hat{A}_\mu = \mathfrak{b}(V)\mathfrak{b}(U)\frac{1}{f}. \tag{3.22}$$

Proof. Using (3.15), we obtain (3.20).

$$\hat{L}_x \hat{\psi} = P \circ Q \circ P \hat{\psi} = P \circ h(L_x)\psi = P \circ \mathfrak{b}(h(L_x))\psi = fP\psi = f\hat{\psi}.$$

From (3.19), we can re-write $\hat{\psi}$ as follows:

$$\hat{\psi} = P\psi = \Theta^{-1}V\psi = \Theta^{-1}\mathfrak{b}(V)\psi.$$

Substituting (3.19) into (3.18) and rearranging gives

$$Vh(L_x(x, \partial_x))^{-1}U = \Theta\Gamma.$$

$$\implies \mathfrak{b}(\Theta\Gamma) = \mathfrak{b}(Vh(L_x(x, \partial_x))^{-1}U) = \mathfrak{b}(U)\frac{1}{f}\mathfrak{b}(V).$$

Putting all of the above equations together gives

$$\Theta\Gamma\hat{\psi} = \Theta\Gamma\Theta^{-1}\mathfrak{b}(V)\psi = \Theta^{-1}\mathfrak{b}(V)\Theta\Gamma\psi$$

$$= \Theta^{-1}\mathfrak{b}(V)\mathfrak{b}(\Theta\Gamma)\psi = \Theta^{-1}\mathfrak{b}(V)\mathfrak{b}(U)f^{-1}\mathfrak{b}(V)\psi$$

$$= \mathfrak{b}(V)\mathfrak{b}(U)f^{-1}\Theta^{-1}\mathfrak{b}(V)\psi = \mathfrak{b}(V)\mathfrak{b}(U)f^{-1}P\psi$$

$$\mathfrak{b}(V)\mathfrak{b}(U)f^{-1}\hat{\psi} = \hat{A}_\mu\hat{\psi}.$$

■

This theorem would be applicable to polynomials in the DPT operator provided that the factors $Q(x, \partial_x)$ and $P(x, \partial_x)$ are in $\mathcal{B}\mathcal{K}^{-1}$ and $\mathcal{K}^{-1}\mathcal{B}$ respectively. It is not immediately clear why these operators will be of the required form; we prove this here by formulating a strategy similar to what was used in [3]. We are going to show that the conditions imposed on operators P and Q are equivalent to the following:

- P and Q have trigonometric coefficients.
- P and Q are invariant under the reflection $x \mapsto -x$.

So in other words, we are looking for differential operators P and Q which have trigonometric coefficients, and are I-invariant, where the involution I is the reflection $x \mapsto -x$.

$$I(f(x)) = f(-x).$$

If a function or an operator is invariant under the involution I, we will call it $\mathbb{Z}_2$-invariant. Our claim is that factorisations with the above-mentioned properties will give us bispectrality.

Note that all elements of $\mathcal{B}$ are $\mathbb{Z}_2$-invariant. This is because $\mathcal{B}$ is generated by $L_x(x, \partial_x)$ and $-4\sin^2(x/2)$, both of which are $\mathbb{Z}_2$-invariant:

$$I\Big(4\sin^2\left(\frac{x}{2}\right)\Big) = 4\sin^2\left(-\frac{x}{2}\right) = 4\Big(-\sin\left(\frac{x}{2}\right)\Big)^2 = 4\sin^2\left(\frac{x}{2}\right),$$

$$I(u) = \frac{g(g-1)}{I\Big(4\sin^2\left(\frac{x}{2}\right)\Big)} + \frac{h(h-1)}{I\Big(4\cos^2\left(\frac{x}{2}\right)\Big)} = u$$

$$\implies I(-\partial_x^2 + u) = -(I(\partial_x))^2 + I(u) = -(-\partial_x)^2 + u = -\partial_x^2 + u$$

Let $\mathcal{R}\{e^{ix}, \partial_x\}$ be the algebra of all the differential operators of the form

$$E = \sum_{n=0}^{m} h_n(e^{ix})\partial_x^n, \qquad \text{where } h_n(e^{ix}) \text{ are rational in } e^{ix}.$$

Let Δ^I be the subalgebra of $\mathcal{R}\{e^{ix}, \partial_x\}$ which only contains $\mathbb{Z}_2$-invariant operators:

$$\Delta^I = \{E \in \mathcal{R}\{e^{ix}, \partial_x\} : I(E) = E\}. \tag{3.23}$$

Let Δ^I_m be the subalgebra of Δ^I which contains operators of order up to and including $2m$.

$$\Delta^I_m = \left\{E \in \Delta^I : E = \sum_{j=0}^{2m} h_j(e^{ix})\partial_x^j\right\}. \tag{3.24}$$

The following inclusion is clear.

$$\mathcal{K} \subset \mathcal{B} \subset \mathcal{K}^{-1}\mathcal{B} \cup \mathcal{B}\mathcal{K}^{-1} \subset \Delta^I \subset \mathcal{R}\{e^{ix}, \partial_x\}. \tag{3.25}$$

Lemma 3.4. *The Laurent polynomials in e^{ix},*

$$p(e^{ix}) = \sum_{k=-m}^{n} c_k e^{ixk}, \quad m, n \in \mathbb{N},$$

are in the algebra $\mathcal{K}$ if and only if they are $\mathbb{Z}_2$-invariant.

Proof. If $p(e^{ix}) \in \mathcal{K}$, then $p(e^{ix}) = q(4\sin^2(x/2))$ where $q(y)$ is some other polynomial in y. Then

$$I\left(q\left(4\sin^2\left(\frac{x}{2}\right)\right)\right) = q\left(4\sin^2\left(\frac{-x}{2}\right)\right) = q\left(4\left(-\sin\left(\frac{x}{2}\right)\right)^2\right) = q\left(4\sin^2\left(\frac{x}{2}\right)\right),$$

so $I(p(e^{ix})) = p(e^{ix})$.

Conversely, if $I(p(e^{ix})) = p(e^{ix})$, then

$$p(e^{ix}) = \sum_{k=-m}^{n} c_k e^{ixk} \implies p(e^{-ix}) = \sum_{k=-m}^{n} c_k e^{-ixk} = \sum_{k=-n}^{m} c_{-k} e^{ixk}.$$

Now, since $p(e^{ix}) = p(e^{-ix})$, $p(e^{ix}) - p(e^{-ix}) = 0$.

$$\implies \sum_{k=-m}^{n} c_k e^{ixk} - \sum_{k=-n}^{m} c_{-k} e^{ixk} = 0.$$

Without loss of generality, let $n \geq m$. Set $c_k = 0$ for $-n \leq k < -m$. From this it follows that $c_{-k} = 0$ for $m < k \leq n$; this allows us to give the above sum a symmetric support $[-n, n]$.

$$\implies \sum_{k=-n}^{n} c_k e^{ixk} - \sum_{k=-n}^{n} c_{-k} e^{ixk} = 0.$$

$$\implies \sum_{k=-n}^{n} (c_k - c_{-k}) e^{ixk} = 0 \quad \forall x.$$

$$\implies \forall k, \quad c_k - c_{-k} = 0 \implies c_k = c_{-k}.$$

Since $c_k = 0$ for $-n \leq k < -m$, $c_{-k} = 0$ for $-n < k \leq -m$. Also, since $c_{-k} = 0$ for $m < k \leq n$, $c_k = 0$ for $m < k \leq n$. This means that $p(e^{ix})$ has a symmetric support $[-m, m]$:

$$p(e^{ix}) = \sum_{k=-m}^{m} c_k e^{ixk} = c_0 + \sum_{k=1}^{m} c_k (e^{ixk} + e^{-ixk})$$

$$= c_0 + \sum_{k=1}^{m} 2c_k \cos(kx) = c_0 + \sum_{k=1}^{m} 2c_k T_k(\cos x).$$

Here, $\cos(kx) = T_k(\cos x)$, where T_k is the kth Chebyshev polynomial.

$$\cos x = \cos(2 \times \frac{x}{2}) = 1 - 2\sin^2 \frac{x}{2} = 1 - \frac{1}{2}\Big(4\sin^2 \frac{x}{2}\Big).$$

Therefore,

$$p(e^{ix}) = c_0 + \sum_{k=1}^{m} 2c_k T_k\left(1 + \frac{1}{2}\left[-4\sin^2 \frac{x}{2}\right]\right) \in \mathcal{K}.$$

∎

I is the involution $x \mapsto -x$. So if a function $f(x)$ is $\mathbb{Z}_2$-invariant then it is an *even* function because $I(f(x)) = f(-x) = f(x)$. By lemma 3.4, the functions in $\mathcal{K}$ are even functions. We can say something similar to lemma 3.4 for *odd* functions.

Lemma 3.5. *If* $p(e^{ix})$ *is an odd Laurent polynomial in* e^{ix}*, that is, if* $I(p(e^{ix})) = p(e^{-ix}) = -p(e^{ix})$*, then* $p(e^{ix})$ *is some polynomial of* $\sin^2(x/2)$ *multiplied by a single instance of* $\sin x$*. So* $p(e^{ix})$ *is of the form:*

$$p(e^{ix}) = q\left(4\sin^2\left(\frac{x}{2}\right)\right) \times \sin x,$$

where $q(y)$ *is some polynomial.*

Proof. If $p(e^{-ix}) = -p(e^{ix})$, then

$$p(e^{ix}) = \sum_{k=-m}^{n} c_k e^{ixk} = c_{-m}e^{-ixm} + ... + c_n e^{ixn}$$

$$\implies p(e^{-ix}) = \sum_{k=-m}^{n} c_k e^{-ixk} = c_{-m}e^{ixm} + ... + c_n e^{-ixn} = \sum_{k=-n}^{m} c_{-k} e^{ixk}.$$

Now $p(e^{ix}) = -p(e^{-ix})$, which implies that $p(e^{ix}) + p(e^{-ix}) = 0$.

$$\implies \sum_{k=-m}^{n} c_k e^{ixk} + \sum_{k=-n}^{m} c_{-k} e^{ixk} = 0. \tag{3.26}$$

Without loss of generality, assume that $n \geq m$. Set $c_l = 0$ for $-n \leq l < -m$ and $c_{-l} = 0$ for $m < l \leq n$. The equation (3.26) becomes

$$\sum_{k=-n}^{n} c_k e^{ixk} + \sum_{k=-n}^{n} c_{-k} e^{ixk} = \sum_{k=-n}^{n} (c_k + c_{-k})e^{ixk} = 0 \quad \forall x.$$

$$\implies c_k + c_{-k} = 0 \quad \forall k, \implies c_k = -c_{-k}.$$

Note here that $c_0 = 0$ because $c_k = -c_{-k}$ gives us $c_0 = -c_{-0} \implies c_0 = -c_0 \implies 2c_0 = 0$. Furthermore, since $c_l = 0$ for $-n \leq l <$ and $c_{-l} = 0$ for $m < l \leq n$, $p(e^{ix})$ has a symmetric support $[-m, m]$:

$$p(e^{ix}) = \sum_{k=-m}^{m} c_k e^{ixk} = c_0 + \sum_{k=1}^{m} c_k e^{ixk} + \sum_{k=-m}^{-1} c_k e^{ixk}$$

$$= 0 + \sum_{k=1}^{m} c_k e^{ixk} + \sum_{k=1}^{m} c_{-k} e^{-ixk} = \sum_{k=1}^{m} \left(c_k e^{ixk} + c_{-k} e^{-ixk} \right)$$

$$= \sum_{k=1}^{m} \left(c_k e^{ixk} - c_k e^{-ixk} \right) \quad \text{(because } \ c_{-k} = -c_k)$$

$$= \sum_{k=1}^{m} c_k \left(e^{ixk} - e^{-ixk} \right) = \sum_{k=1}^{m} 2ic_k \left(\frac{e^{ixk} - e^{-ixk}}{2i} \right)$$

$$\implies p(e^{ix}) = \sum_{k=1}^{m} 2ic_k \sin(xk). \tag{3.27}$$

Here we have to use the following identity from François Viète (1540 - 1603),

$$\sin(xk) = \sum_{l=1}^{\infty} (-1)^l \binom{k}{2l+1} \cos^{k-2l-1}(x) \sin^{2l+1}(x). \tag{3.28}$$

Writing the sum to infinity is fine because the binomial coefficient is zero for $2l + 1 > k$. Substituting (3.28) into (3.27) gives us the desired result:

$$p(e^{ix}) = \left[\sum_{k=1}^{m} 2ic_k \sum_{l=1}^{\infty} (-1)^l \binom{k}{2l+1} \cos^{k-2l-1}(x) \left(1 - \cos^2 x \right)^l \right] \sin x.$$

■

The two lemmas above describe what happens if a Laurent *polynomial* is even or odd. We can go a step further by generalising these results to *rational functions* which are even or odd.

Lemma 3.6. *If $r(e^{ix})$ is an even function, rational in e^{ix}, then it is of the form*

$$r(e^{ix}) = \frac{f(\cos x)}{g(\cos x)},$$

where f and g are some polynomials and g is not the zero polynomial.

Proof. Since r is rational,

$$r(e^{ix}) = \frac{p(e^{ix})}{q(e^{ix})}, \tag{3.29}$$

where p and q are some Laurent polynomials in e^{ix}. Since r is even,

$$r(e^{ix}) = r(e^{-ix}) \implies \frac{p(e^{ix})}{q(e^{ix})} = \frac{p(e^{-ix})}{q(e^{-ix})} \implies p(e^{ix})q(e^{-ix}) = p(e^{-ix})q(e^{ix}).$$

So $p(e^{ix})q(e^{-ix})$ is an even ($\mathbb{Z}_2$-invariant) Laurent polynomial. By lemma 3.4, $p(e^{ix})q(e^{-ix}) = f(\cos x)$, where f is a polynomial.

$$\implies p(e^{ix}) = \frac{f(\cos x)}{q(e^{-ix})}. \tag{3.30}$$

Substitute (3.30) in to (3.29).

$$r(e^{ix}) = \frac{f(\cos x)}{q(e^{ix})q(e^{-ix})}, \tag{3.31}$$

The denominator $q(e^{ix})q(e^{-ix})$ is an even Laurent polynomial. By lemma 1 again, $q(e^{ix})q(e^{-ix}) = g(\cos x)$ for some polynomial g. Hence

$$r(e^{ix}) = \frac{f(\cos x)}{g(\cos x)}.$$

■

Lemma 3.7. *If $r(e^{ix})$ is an odd function, rational in e^{ix}, then it is of the form*

$$r(e^{ix}) = \frac{f(\cos x)}{g(\cos x)} \times \sin x,$$

where f and g are polynomials.

Proof. $r(e^{ix})$ and $\sin x$ are both odd functions. The quotient of these two odd functions is an even function, rational in e^{ix}. So by lemma 3.6,

$$\frac{r(e^{ix})}{\sin x} = \frac{f(\cos x)}{g(\cos x)} \implies r(e^{ix}) = \frac{f(\cos x)}{g(\cos x)} \sin x.$$

■

We need further technical results.

Lemma 3.8. *The operators* $\sin x \circ \partial_x$ *and* $\partial_x \circ \sin x$ *are both in* $\mathcal{B}$.

Proof. We note that L_x and $\cos x$ are in $\mathcal{B}$. Then

$$\sin x \circ \partial_x = \frac{1}{2}(L_x \circ \cos x - \cos x \circ L_x - \cos x) \in \mathcal{B}, \tag{3.32}$$

and

$$\partial_x \circ \sin x = \cos x + \sin x \circ \partial_x = \frac{1}{2}(L_x \circ \cos x - \cos x \circ L_x + \cos x) \in \mathcal{B}. \tag{3.33}$$

■

Lemma 3.9. *Let* $\Theta \in \mathcal{K}$ *and* $V \in \mathcal{B}$ *(so that* $\Theta^{-1}V \in \mathcal{K}^{-1}\mathcal{B}$*). Then* $\sin x \circ \partial_x \circ \Theta^{-1}V \in \mathcal{K}^{-1}\mathcal{B}$.

Proof. $\sin x \circ \partial_x$ acts on Θ^{-1} as follows:

$$\sin x \circ \partial_x \circ \Theta^{-1} = \sin x \Theta^{-1} \partial_x - \sin x \Theta^{-2} \Theta'.$$

Therefore:

$$\sin x \circ \partial_x \circ \Theta^{-1} V = \frac{1}{\Theta} \sin x \circ \partial_x \circ V - \frac{1}{\Theta^2} \sin x \circ \Theta' \circ V.$$

<u>First term</u>: By lemma 3.8, $\sin x \circ \partial_x \in \mathcal{B}$. $V \in \mathcal{B}$. So $\sin x \circ \partial_x \circ V \in \mathcal{B}$. $\Theta \in \mathcal{K}$. Hence

$$\frac{1}{\Theta} \sin x \circ \partial_x \circ V \in \mathcal{K}^{-1}\mathcal{B}.$$

<u>Second term</u>:

$$\Theta' = \frac{d\Theta}{dx} = \frac{d\Theta}{dy}\frac{dy}{dx}, \text{ where } y = 4\sin^2\left(\frac{x}{2}\right).$$

Θ is a polynomial in $4\sin^2(x/2) = y$, and so is $d\Theta/dy$, implying that $d\Theta/dy \in \mathcal{K}$.

$$\frac{dy}{dx} = 2\sin x \implies \Theta' = 2\sin x \frac{d\Theta}{dy}.$$

$$\implies \frac{1}{\Theta^2}\sin x\circ\Theta'\circ V = \frac{1}{\Theta^2}\sin x\times 2\sin x\frac{d\Theta}{dy}\circ V = \frac{1}{\Theta^2}2\sin^2 x\frac{d\Theta}{dy}\circ V.$$

Here, $2\sin^2 x \in \mathcal{K}\subset\mathcal{B}$, $d\Theta/dy \in \mathcal{K}\subset\mathcal{B}$ and $V\in\mathcal{B}$. Furthermore, $\Theta^2\in\mathcal{K}$. So the second term is also in $\mathcal{K}^{-1}\mathcal{B}$.

The conclusion is that $\Theta^{-1}V\in\mathcal{K}^{-1}\mathcal{B} \implies \sin x\partial_x\circ\Theta^{-1}V\in\mathcal{K}^{-1}\mathcal{B}$. ∎

Lemma 3.10. *$\forall m\in\mathbb{N}$, $\partial_x^{2m}\in\mathcal{K}^{-1}\mathcal{B}$.*

Proof. We use proof by induction.

Induction Base: Set $m=1$. $\partial_x^2 = \partial_x^2 - u + u = -(-\partial_x^2+u)+u = -L_x+u$. Here,

$$u = \frac{g(g-1)}{4\sin^2\frac{x}{2}} + \frac{h(h-1)}{4\cos^2\frac{x}{2}} = \frac{g(g-1)\cos^2\frac{x}{2}+h(h-1)\sin^2\frac{x}{2}}{4\sin^2\frac{x}{2}\cos^2\frac{x}{2}}.$$

Therefore,

$$-L_x+u = \frac{1}{4\sin^2\frac{x}{2}\cos^2\frac{x}{2}}\Bigg[-4\sin^2\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right)\circ L_x$$
$$+g(g-1)\cos^2\left(\frac{x}{2}\right)+h(h-1)\sin^2\left(\frac{x}{2}\right)\Bigg],$$

which is in $\mathcal{K}^{-1}\mathcal{B}$.

Induction Assumption: Suppose $\partial_x^{2k}\in\mathcal{K}^{-1}\mathcal{B}$ for some $k\in\mathbb{N}$. Let

$$\partial_x^{2k} = \frac{1}{\Theta_k}V_k$$

for some $\Theta_k\in\mathcal{K}$ and $V_k\in\mathcal{B}$.

Induction Step: Consider $\partial_x^{2(k+1)}$. By induction hypothesis,

$$\partial_x^{2(k+1)} = \partial_x^{2+2k} = \partial_x^2\circ\partial_x^{2k} = \partial_x^2\circ\frac{1}{\Theta_k}V_k.$$

$$\implies \partial_x^{2(k+1)}\phi = \partial_x^2\left(\frac{1}{\Theta_k}V_k\phi\right) = \partial_x\left[\partial_x\left(\frac{1}{\Theta_k}V_k\phi\right)\right]$$
$$= \partial_x\left[\frac{1}{\sin x}\times\sin x\partial_x\left(\frac{1}{\Theta_k}V_k\phi\right)\right]$$

$$= \left[\frac{\sin x\partial_x \circ \sin x\partial_x \circ \Theta_k^{-1} V_k}{\sin^2 x} - \frac{\cos x \circ \sin x\partial_x \circ \Theta_k^{-1} V_k}{\sin^2 x} \right] \phi.$$

First term: $\Theta_k^{-1} V_k \in \mathcal{K}^{-1}\mathcal{B}$. By lemma 3.9, $\sin x\partial_x \circ \Theta_k^{-1} V_k \in \mathcal{K}^{-1}\mathcal{B}$. By another application of lemma 3.9, $\sin x\partial_x \circ \sin x\partial_x \circ \Theta_k^{-1} V_k \in \mathcal{K}^{-1}\mathcal{B}$. So the first term is in $\mathcal{K}^{-1}\mathcal{B}$.

Second term: $\Theta_k^{-1} V_k \in \mathcal{K}^{-1}\mathcal{B}$. By lemma 3.9, $\sin x\partial_x \circ \Theta_k^{-1} V_k \in \mathcal{K}^{-1}\mathcal{B}$. $\cos x = 1 - (1/2)(4\sin^2(x/2)) \in \mathcal{K} \subset \mathcal{B}$. This implies that $\cos x \circ \sin x\partial_x \circ \Theta_k^{-1} V_k \in \mathcal{K}^{-1}\mathcal{B}$. Thus the second term is also in $\mathcal{K}^{-1}\mathcal{B}$.

Therefore we conclude that $\partial_x^{2k} \in \mathcal{K}^{-1}\mathcal{B} \implies \partial_x^{2(k+1)} \in \mathcal{K}^{-1}\mathcal{B}$. By the principle of mathematical induction, $\partial_x^{2m} \in \mathcal{K}^{-1}\mathcal{B}\ \forall m \in \mathbb{N}$. ■

3.3 Proof of Bispectrality

With all the structures established above, we prove the following.

Theorem 3.11. *The following 3 statements are equivalent:*

- (A) *Operator E is $\mathbb{Z}_2$-invariant, so $E \in \Delta^I$.*
- (B) *$E \in \mathcal{K}^{-1}\mathcal{B}$.*
- (C) *$E \in \mathcal{B}\mathcal{K}^{-1}$.*

Proof. (A) $\implies$ (B): We know that $E \in \Delta_m^I \subset \Delta^I$ for some $m \in \mathbb{N}$. So

$$E = k_{2m}(e^{ix})\partial_x^{2m} + k_{2m-1}(e^{ix})\partial_x^{2m-1} + ... + k_1(e^{ix})\partial_x + k_0(e^{ix})\text{id}$$

$$= k_{2m}(e^{ix})\partial_x^{2m} + k_{2m-1}(e^{ix})\partial_x^{2m-1} \bmod \Delta_{m-1}^I, \tag{3.34}$$

where k_i are rational functions. We will show that $E = \Theta_m^{-1} V_m \bmod \Delta_{m-1}^I$, where $\Theta_m \in \mathcal{K}$ and $V_m \in \mathcal{B}$. From this the proof follows by induction on m.

$$I(E) = I(k_{2m}(e^{ix}))I(\partial_x^{2m}) + I(k_{2m-1}(e^{ix}))I(\partial_x^{2m-1}) \bmod \Delta_{m-1}^I.$$

Here, $I(\partial_x^{2m-1}) = -\partial_x^{2m-1}$, $I(\partial_x^{2m}) = I(\partial_x^{2m})$ and $I(k_i(e^{ix})) = k_i(e^{-ix})$. So,

$$I(E) = k_{2m}(e^{-ix})\partial_x^{2m} - k_{2m-1}(e^{-ix})\partial_x^{2m-1} \mod \Delta_{m-1}^I. \tag{3.35}$$

We have that $I(E) = E$ for all x. Comparing the coefficients in (3.34) with the coefficients in (3.35) gives:

$k_{2m}(e^{-ix}) = k_{2m}(e^{ix}) \implies k_{2m}$ is <u>even</u> rational in e^{ix}. By lemma 3.6,

$$k_{2m}(e^{ix}) = \frac{f(\cos x)}{g(\cos x)},$$

where $f(\cos x)$ and $g(\cos x) \in \mathcal{K} \subset \mathcal{B}$.

By lemma 3.10, $\partial_x^{2m} \in \mathcal{K}^{-1}\mathcal{B}$. So $\partial_x^{2m} = \bar{\Theta}^{-1}\bar{V}$. So,

$$k_{2m}(e^{ix})\partial_x^{2m} = \frac{f(\cos x)}{g(\cos x)}\frac{1}{\bar{\Theta}}\bar{V} = \frac{1}{\bar{\Theta}g(\cos x)}f(\cos x)\bar{V} \in \mathcal{K}^{-1}\mathcal{B}. \tag{3.36}$$

$k_{2m-1}(e^{ix}) = -k_{2m-1}(e^{-ix})$, so $k_{2m-1}(e^{ix})$ is <u>odd</u> rational in e^{ix}. By lemma 3.7,

$$k_{2m-1}(e^{ix}) = \frac{h(\cos x)}{i(\cos x)}\sin x,$$

where $h(\cos x)$ and $i(\cos x) \in \mathcal{K} \subset \mathcal{B}$.

$$k_{2m-1}(e^{ix})\partial_x^{2m-1} = \frac{h(\cos x)}{i(\cos x)}\sin x \circ \partial_x \circ \partial_x^{2m-2}.$$

Here, by lemma 3.10, $\partial_x^{2m-2} \in \mathcal{K}^{-1}\mathcal{B}$. By lemma 3.9, $\sin x \circ \partial_x \circ \partial_x^{2m-2} \in \mathcal{K}^{-1}\mathcal{B}$. So $\sin x \circ \partial_x \circ \partial_x^{2m-2} = \underline{\Theta}^{-1}\underline{V}$.

$$\implies k_{2m-1}(e^{ix})\partial_x^{2m-1} = \frac{h(\cos x)}{i(\cos x)}\frac{1}{\underline{\Theta}}\underline{V} = \frac{1}{i(\cos x)\underline{\Theta}}h(\cos x)\underline{V} \in \mathcal{K}^{-1}\mathcal{B}. \tag{3.37}$$

From (3.36) and (3.37) we see that $E = \Theta_m^{-1}V_m \mod \Delta_{m-1}^I$. Iteratively we get $E \in \mathcal{K}^{-1}\mathcal{B}$.

<u>(B) $\implies$ (C)</u>: Let $E \in \mathcal{K}^{-1}\mathcal{B}$. Then $E = \Theta^{-1}V$.

To turn it in to the form $U\Gamma^{-1}$, we need to essentially pass Θ^{-1} through V to the other side. Note that

$$V \in \mathcal{B} = \left\langle L_x, 4\sin^2\left(\frac{x}{2}\right)\right\rangle = \text{span}\Big\{x_1 \circ x_2 \circ ... \circ x_n : x_i \in \Big\{L_x, 4\sin^2\left(\frac{x}{2}\right)\Big\}, n \in \mathbb{N}\Big\},$$

whereas

$$\Theta \in \mathcal{K} = \left\langle 4\sin^2\left(\frac{x}{2}\right)\right\rangle = \text{span}\Big\{\left(4\sin^2\left(\frac{x}{2}\right)^n : n \in \mathbb{N}\Big\}.$$

$$\implies E = \frac{1}{\Theta}V = \frac{1}{\Theta}\sum_i c_i x_{1,i} \circ x_{2,i} \circ ... \circ x_{n_i,i}, \quad \text{where} \quad x_{l,i} \in \Big\{L_x, 4\sin^2\left(\frac{x}{2}\right)\Big\}.$$

So we just need to know how Θ^{-1} passes through each x_i, whether it be $4\sin^2(x/2)$ or L_x. We must check that after Θ^{-1} has passed through each x_i, the result is of the form $U\Gamma^{-1}$.

<u>If $x_i = 4\sin^2(x/2)$</u>, then Θ^{-1} and $4\sin^2(x/2)$ simply commute because they are both functions. So $\Theta^{-1} \circ 4\sin^2(x/2) = 4\sin^2(x/2) \circ \Theta^{-1} \in \mathcal{B}\mathcal{K}^{-1}$. In other words, Θ^{-1} passes through $4\sin^2(x/2)$ unaffected.

<u>If $x_i = L_x$</u>, then Θ^{-1} and L_x do not commute, and Θ^{-1} passes through L_x in some non-trivial way. We determine this as shown below.

$\Theta^{-1} \circ L_x = \Theta^{-1} \circ (-\partial_x^2 + u)$ is a second order differential operator. So after Θ^{-1} has passed through, we still expect to get a second order ordinary differential operator:

$$\frac{1}{\Theta} \circ (-\partial_x^2 + u) = \partial_x^2 \circ h + \partial_x \circ i + j \tag{3.38}$$

where h, i and j are functions to be determined. Expanding both sides in (3.38) gives

$$-\frac{1}{\Theta}\partial_x^2 + \frac{u}{\Theta} = h\partial_x^2 + (2h' + i)\partial + (h'' + i' + j).$$

Compare the coefficients:

<u>∂_x^2 terms:</u> $\qquad h = -\Theta^{-1}$.

$\underline{\partial_x \text{ terms}}$: $\quad 2h' + i = 0,$

$$\implies i = -2h' = -2\frac{\Theta'}{\Theta^2}.$$

$\underline{\text{Non-derivative terms}}$: $\quad h'' + i' + j = u\Theta^{-1},$

$$\implies j = u\Theta^{-1} + h''.$$

$$h' = \frac{\Theta'}{\Theta^2} \implies h'' = \Theta'' \circ \frac{1}{\Theta^2} - 2\Theta'^2 \circ \frac{1}{\Theta^3}.$$

We had $\Theta' = 2\sin x \ d\Theta/dy$.

$$\implies \Theta'' = 2\cos\frac{d\Theta}{dy} + 4\sin^2 x\frac{d^2\Theta}{dy^2}.$$

Since $d\Theta/dy$ is a polynomial in y, $d^2\Theta/dy^2$ is still a polynomial in $y = 4\sin^2(x/2)$. So

$$\frac{d^2\Theta}{dy^2} \in \mathcal{K} \subset \mathcal{B}, \ \ \sin^2 x = 4\sin^2\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right) \in \mathcal{K} \text{ and } \cos x \in \mathcal{K}.$$

$$\implies \Theta'' \in \mathcal{K} \subset \mathcal{B} \implies \Theta'' \circ \frac{1}{\Theta^2} \in \mathcal{B}\mathcal{K}^{-1}.$$

Also,

$$\Theta' = 2\sin x\frac{d\Theta}{dy} \implies \Theta'^2 = 4\sin^2\left(\frac{d\Theta}{dy}\right)^2 \in \mathcal{K} \subset \mathcal{B}.$$

$$\implies 2\Theta'^2 \circ \frac{1}{\Theta^3} \in \mathcal{B}\mathcal{K}^{-1} \implies h'' \in \mathcal{B}\mathcal{K}^{-1}.$$

Putting all these in to the equation (3.38), we get

$$\frac{1}{\Theta}\circ L_x = \left[L\circ\Theta^2 - 4\partial_x\circ\sin x\circ\frac{d\Theta}{dy}\Theta + \left(2\cos x\frac{d\Theta}{dy} + 4\sin^2 x\frac{d^2\Theta}{dy^2}\right)\Theta - 4\sin^2 x\left(\frac{d\Theta}{dy}\right)^2\right]\circ\frac{1}{\Theta^3} \in \mathcal{B}\mathcal{K}^{-1}.$$

So $\Theta^{-1} \circ L_x$ is of the form $U\Gamma^{-1}$ for the above $U \in \mathcal{B}$ and $\Gamma = \Theta^3 \in \mathcal{K}$.

This Γ^{-1} then passes through $x_{i+1} \in \{L_x, 4\sin^2(x/2)\}$ in a similar way as described above.

Repeating the process, we get that $E \in \mathcal{B}\mathcal{K}^{-1}$.

$\underline{\text{(C)} \implies \text{(A)}}$: From inclusion (3.25), it follows that if $E \in \mathcal{B}\mathcal{K}^{-1}$, then $E \in \Delta^I$.

We have shown that (A) $\implies$ (B), (B) $\implies$ (C) and (C) $\implies$ (A). This completes the proof of the theorem. ■

Theorem 3.3 tells us that if a Darboux factorisation on $h(L_x) = Q(x, \partial_x) \circ P(x, \partial_x)$ takes the form (3.19), then the new operator would be bispectral. Theorem 3.11 gives us sufficient conditions on Q and P to ensure that they take the form (3.19). They are:

- P and Q are $\mathbb{Z}_2$-invariant.
- P and Q have coefficients rational in e^{ix}.

In the next chapter we show that monodromy invariant solution spaces would have the above properties.

Chapter 4

Possible Darboux Factorisations for DPT Operator

In the previous chapter we established the properties of Darboux factorisations which will ensure bispectrality. Here, we will show that monodromy invariant spaces have those properties. We will explain a method by which such solution spaces can be generated. We will go on to describe solution spaces that lead to new bispectral operators.

4.1 Monodromy Group of Hypergeometric Differential Equation

In the last chapter we saw that performing a Darboux transformation on an operator with the factorisation $h(L_x) = Q \circ P$ will produce a bispectral operator if

- P and Q have trigonometric coefficients.
- P and Q are $\mathbb{Z}_2$-invariant.

The operator $h(L_x)$ would have trigonometric coefficients and would be invariant in $x \mapsto -x$. Therefore if we find a factor P with those properties, then Q will automatically satisfy them as well. This is useful to note because it allows us to concentrate on finding classifications for P.

We propose that monodromy invariant subspaces of solutions will give us P with the above properties.

The monodromy group of a linear differential equation on the Riemann sphere is a linear representation of the fundamental group of the punctured Riemann sphere (where each puncture represents a singular point of the equation).

Riemann sphere [25] is a stereographic projection of the extended complex plane $\bar{\mathbb{C}}$, that is, the complex plane plus a point representing infinity, so $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. As a manifold, it is charted by two copies of $\mathbb{C}$:

$$\bar{\mathbb{C}} = C_0 \cup C_\infty.$$

If z and w are respective coordinates in C_0 and C_∞ then on the intersection of these charts we have $w = z^{-1}$.

Monodromy group describes analytic continuations of the solutions along loops in $\bar{\mathbb{C}}$ as linear maps (see for instance [23, section 15.91]). More precisely, one chooses a regular (non-singular) point z_0 for the given differential equation and considers its solutions in a small neighbourhood of z_0. By existence and uniqueness theorem for solutions of ordinary differential equations, the solution space will be a vector space over $\mathbb{C}$ of dimension equal to the order of the equation. The process of analytic continuations along a loop on punctured sphere which starts and ends at z_0 gives a linear transformation on the solution space. Composing analytic continuations along different paths corresponds to composing the corresponding linear transformations. If one chooses a basis of the solution space then the monodromy transformations can be represented by matrices.

Individual functions in $\ker h(L_x)$ are not preserved under analytic continuations around a singularity. However, it might be possible to have a subspace $W \subset \ker h(L_x)$ such that functions in W remain within W even after an analytic continuation. In this situation, the space W as a whole is preserved under monodromy transformations; we call such spaces monodromy invariant.

4.1.1 Link Between Monodromy and Rationality

We have to deal with the fact that the DPT operator (2.17) has infinitely many singularities. A workaround is to use the monodromy information of the hypergeometric differential equation (2.12) in z-variable. This is because that equation only has three regular singular points: $z = 0$, $z = 1$ and $z = \infty$. Once we have established that monodromy invariant subspaces give us bispectral Darboux transformations in z variable, we move back to the x-variable; the next lemma makes it certain that we can do that. z and x variables are related as follows:

$$z = \sin^2\left(\frac{x}{2}\right) = \frac{1}{2}(1 - \cos x) = \frac{1}{4}(2 - e^{ix} - e^{-ix}).$$

Lemma 4.1. *P has trigonometric coefficients and is $\mathbb{Z}_2$-invariant in x if and only if it has rational coefficients when written in the z variable.*

Proof. If P has rational coefficients in z variable, then it is of the form

$$P(z, \partial_z) = \sum_{i \geq 0} f_i(z) \Big(\frac{d}{dz}\Big)^i,$$

where f_i is a rational function and

$$\frac{d}{dz} = \frac{2}{\sin x} \frac{d}{dx}.$$

$$I(f_i(z)) = I(f_i(\sin^2(x/2))) = f_i(\sin^2(-x/2)) = f_i(\sin^2(x/2)) = f_i(z),$$

so f_i are $\mathbb{Z}_2$-invariant and are trigonometric in x.

$$I\Big(\frac{d}{dz}\Big)^i = \Big(I\Big(\frac{d}{dz}\Big)\Big)^i = \Big(I\Big(\frac{2}{\sin x}\frac{d}{dx}\Big)\Big)^i = \Big(\frac{2}{\sin(-x)}\frac{d}{d(-x)}\Big)^i$$

$$= \Big(\frac{2}{\sin x}\frac{d}{dx}\Big)^i = \Big(\frac{d}{dz}\Big)^i.$$

So d/dz is also I-invariant. Therefore in x, P would be a $\mathbb{Z}_2$-invariant operator with trigonometric coefficients.

On the other hand, if P is already $\mathbb{Z}_2$-invariant with trigonometric coefficients, then by theorem 3.11, it is in $\mathcal{K}^{-1}\mathcal{B}$ (3.17) where

$$\mathcal{B} = \Big\langle -\partial_x^2 + u, -4\sin^2\Big(\frac{x}{2}\Big)\Big\rangle \quad \text{and} \quad \mathcal{K} = \Big\langle -4\sin^2\Big(\frac{x}{2}\Big)\Big\rangle.$$

$$-\partial_x^2 + u = (2z - 2z^2)\frac{d^2}{dz^2} + \frac{1-2z}{2}\frac{d}{dz} + \frac{g(g-1)}{4z} + \frac{h(h-1)}{4(1-z)}. \tag{4.1}$$

$$-4\sin^2\Big(\frac{x}{2}\Big) = -4z. \tag{4.2}$$

So the algebra $\mathcal{B}$ is generated by $-4z$ and the differential operator (4.1) with rational coefficients in z variable. Differentiating a rational function gives another rational function. Therefore all the elements of $\mathcal{B}$ have rational coefficients when written in z variable. In particular, P will have rational coefficients in z. ∎

We need to use the following two theorems in order to prove that monodromy-invariant spaces give us bispectrality. The first one is the Riemann Removable Singularities theorem (theorem 4.1.1 in [26]).

Theorem 4.2 (Riemann Removable Singularities Theorem)**.** *Let* $f : D(P,r)\backslash P \to \mathbb{C}$ *be analytic and bounded, where* $D(P,r)$ *is a disk of radius* r *centred at point* P*. Then*

- $\lim_{z\to P} f(z)$ *exists;*
- *the function* $\hat{f} : D(P,r) \to \mathbb{C}$ *defined by*

$$\hat{f} = \begin{cases} f(z) & if \quad z \neq P \\ \lim_{l\to P} f(l) & if \quad z = P \end{cases}$$

is analytic.

Secondly, we need the following theorem (theorem 4.7.7 in [26]).

Theorem 4.3. *A function is meromorphic on extended complex plane* $\bar{\mathbb{C}}$ *if and only if it is rational.*

Using the above theorems, we can prove the following important result.

Theorem 4.4. *Suppose* P *is a monic differential operator with kernel* $W \subset ker\, h(L_z(z,\partial_z))$*, where* $L_z(z,\partial_z)$ *is given in* (2.13)*. Then* P *would be a differential operator with rational coefficients in* z *if and only if* W *is a monodromy invariant subspace.*

Proof. Let W be a monodromy invariant subspace of $\ker h(L(z,\partial_z))$. Then for all $f \in W$,

- f is multivalued.
- f is locally analytic (holomorphic) away from the singularities 0, 1 and ∞.
- f has moderate growth near the singularities.

The notion of moderate growth is as follows. Suppose a function f was holomorphic on a disk punctured at zero, where it had a singularity, and it was not monodromy invariant near 0. Then f is said to have *moderate growth* if on any sector of the disc, there exists an $\epsilon > 0$, and there also exist $c > 0$, $N \in \mathbb{Z}_+$ such that:

$$|f(z)| \leq c\frac{1}{|z|^N}, \tag{4.3}$$

where z is in the sector, and $|z| < \epsilon$.

Even though, individual functions in the space W are multivalued, the operator whose kernel is W is monodromy invariant. This is because W itself is a monodromy invariant space. So the coefficients of the operator are

- single valued in the domain $\bar{\mathbb{C}} - \{0, 1, \infty\}$,
- meromorphic away from 0, 1 and ∞, and
- have moderate growth.

Suppose k is a coefficient of the operator whose kernel is the space W. The equation (4.3) can be rearranged as $|k(z)||z|^N \leq c$. The above properties imply that this function, $|k(z)||z|^N$, satisfies theorem 4.2.

In the local neighbourhood of each singularity, $|k(z)||z|^N$ is analytic for some N. Away from the singularities 0, 1 and ∞, $|k(z)||z|^N$ is already analytic. So $k(z)$ is meromorphic across the entire extended complex plane $\bar{\mathbb{C}}$. By theorem 4.3, $k(z)$ would be a rational function.

Therefore, the coefficients of the operator P obtained from a monodromy invariant space must be rational in z. ■

Now that we know that P with a monodromy invariant kernel would have rational coefficients in z, we can revert to x variable and by lemma 4.1, P would be a $\mathbb{Z}_2$-invariant operator with trigonometric coefficients.

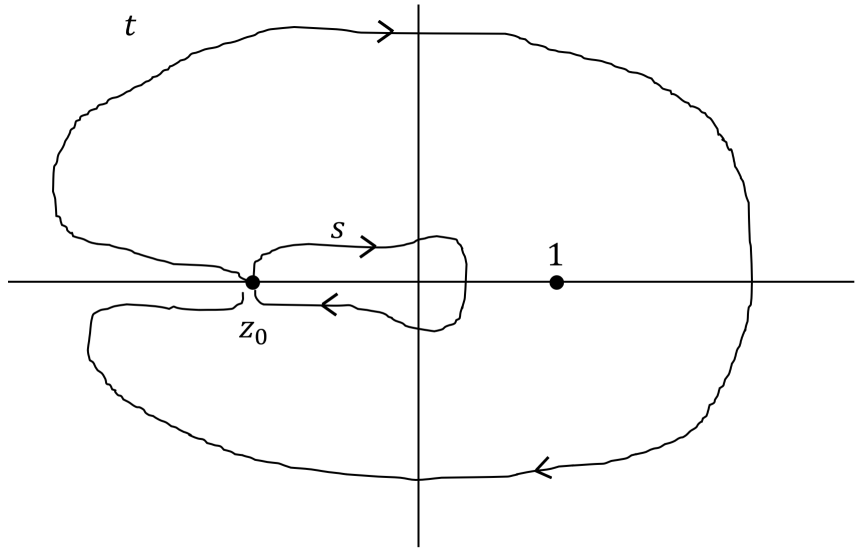
In our situation we need to look at the linear differential equation $h(L_z) = 0$. By theorem 4.4, to construct our factorisations, we need to ensure that we choose a subset of (3.3) as $\ker P$, such that it is an invariant subspace under monodromy transformations. To find possibilities for $\ker P$, we need to understand the monodromy group of hypergeometric differential equation.

4.1.2 Monodromy Matrices

For the two dimensional solution space of the hypergeometric equation, the monodromy representation is explicitly known. It can be found in, for instance, [23, section 15.93]. We will be closely following the notation used in [18]. Let us choose the basis of solutions (2.14):

$$\phi_\pm = \phi\big(z, \pm\lambda\big) = (-4z)^{\mp\lambda - \frac{g+h}{2}} {}_2F_1\left(\frac{g+h}{2} \pm \lambda, \frac{1}{2} \pm \lambda - \frac{g-h}{2}, 1 \pm 2\lambda; \frac{1}{z}\right). \tag{4.4}$$

Use the following mapping: $\ker(L_x - \lambda^2) \to \mathbb{C}^2, \phi_+ \mapsto (1,0)$ and $\phi_- \mapsto (0,1)$.



The above figure of the complex plane sets out the situation. z_0 is a point on the negative real axis. The loop s starts and ends at z_0 and goes around the singular point $z = 0$, whereas the loop t starts at z_0, goes around $z = \infty$ and comes back to z_0. Let Sol_{z_0} be the space of solutions of hypergeometric equation at the point $z = z_0$.

We denote by $\mathcal{M}_0$ and $\mathcal{M}_\infty$ the linear transformations on the solution space Sol_{z_0}, corresponding to the analytic continuations along s and t. When working with a specific basis of Sol_{z_0}, we may replace $\mathcal{M}_0$ and $\mathcal{M}_\infty$ by their matrices, corresponding to that basis. We will denote these matrices by M_0 and M_∞.

For the particular basis (4.4), the monodromy matrices for analytic continuations in loops around $z = 0$, $z = \infty$ and $z = 1$ are:

$$M_0 = C(\lambda) \begin{bmatrix} 1 & 0 \\ 0 & -e^{2\pi i g} \end{bmatrix} C(\lambda)^{-1}, \tag{4.5}$$

$$M_\infty = e^{\pi i (g+h)} \begin{bmatrix} e^{2\pi i \lambda} & 0 \\ 0 & e^{-2\pi i \lambda} \end{bmatrix}, \tag{4.6}$$

$$M_1 = M_\infty{}^{-1} M_0. \tag{4.7}$$

Here, the matrix $C(\lambda)$ is given by

$$C(\lambda) = \begin{bmatrix} c_+ & c'_+ \\ c_- & c'_- \end{bmatrix}.$$

The entries of $C(\lambda)$ are the following functions of λ:

$$c_+ := c(+\lambda) = \frac{2^{2\lambda+g+h}\Gamma\left(\frac{1}{2}+g\right)\Gamma\left(-2\lambda\right)}{\Gamma\left(-\lambda+\frac{g+h}{2}\right)\Gamma\left(\frac{g-h+1}{2}-\lambda\right)}, \quad c'_+ := c'(+\lambda) = \frac{2^{2+2\lambda-g-h}\Gamma\left(\frac{3}{2}-g\right)\Gamma\left(-2\lambda\right)}{\Gamma\left(1-\lambda-\frac{g+h}{2}\right)\Gamma\left(\frac{1-g+h}{2}-\lambda\right)}$$

$$c_- := c(-\lambda) = \frac{2^{-2\lambda+g+h}\Gamma\left(\frac{1}{2}+g\right)\Gamma\left(2\lambda\right)}{\Gamma\left(\lambda+\frac{g+h}{2}\right)\Gamma\left(\frac{g-h+1}{2}+\lambda\right)}, \quad c'_- := c'(-\lambda) = \frac{2^{2-2\lambda-g-h}\Gamma\left(\frac{3}{2}-g\right)\Gamma\left(2\lambda\right)}{\Gamma\left(1+\lambda-\frac{g+h}{2}\right)\Gamma\left(\frac{1-g+h}{2}+\lambda\right)} \tag{4.8}$$

The functions $c_\pm$ are the coefficients $c(\pm\lambda)$ in (2.20). $c'_\pm$ are not derivatives; they are just $c_\pm$ with a transformation in the parameters.

The above formulas are (4.9), (4.13) and (4.14) in [18]. They are valid for specifically chosen loops around $z = 0, 1$ and ∞. Our parameters are related to the ones in [18] as follows:

$$g = k_\alpha + k_\beta, \quad h = k_\alpha, \quad \rho = \frac{g+h}{2}.$$

In the basis $\{\phi_+, \phi_-\}$, the above matrices obviously map the solution space $S = \mathrm{span}\{\phi_+, \phi_-\}$ to itself. For a more general operator which is a polynomial in terms of L_x, our solution space would be bigger in the sense that there would be some combination of λ-derivatives. In other words it would take the form (3.3). Later on we will even take integer differences in λ's into account.

The solution space would have an even dimension bigger than 2 and so we would like to know what monodromy matrices for such a space would look like and which subspaces of that space are preserved by those matrices. For a space to be invariant under monodromy, it is sufficient for it to be invariant under transformations $\mathcal{M}_0$ and $\mathcal{M}_\infty$ because any monodromy transformation can be written as a linear combination of compositions of $\mathcal{M}_0$ and $\mathcal{M}_\infty$. This is true for $\mathcal{M}_1$ as well, which satisfies $\mathcal{M}_1 = {\mathcal{M}_\infty}^{-1}\mathcal{M}_0$.

More precisely, let $h(l)$ be a polynomial in terms of the variable l. Then any polynomial operator $h(L_x)$ can be written as a product of factors of the form $(L_x - \lambda^2)$.

$$h(L_x) = \prod_{r=1}^{n} \left(L_x - \lambda_r^2\right)^{m_r}, \tag{4.9}$$

where $\lambda_r \neq \lambda s$ if $r - s \notin \mathbb{Z}$. The kernel of (4.9) is

$$\ker h(L_x) = \bigoplus_{r=1}^{n} {S_r}^{\pm}, \tag{4.10}$$

where

$${S_r}^{\pm} = \mathrm{span}\left\{ \frac{\partial^k}{\partial \lambda^k} \phi\big(\pm \lambda_r, x\big) : 0 \leq k \leq m_r - 1 \right\}.$$

For a 2-dimensional space $S_1 = \mathrm{span}\{\phi_+, \phi_-\} = \ker\left(L_x - \lambda^2\right)$, the monodromy matrices are $2{\times}2$ matrices. For a bigger space $S_m = \ker\left(L_x - \lambda^2\right)^m = \mathrm{span}\{{\partial_\lambda}^k \phi_\pm : 0 \leq k \leq m-1\}$, the monodromy matrices would be of size $2m \times 2m$.

Suppose $\mathcal{M}$ is a general monodromy transformation over $S_1 := \ker(L - \lambda^2)$ with $\mathcal{M}\phi_+ = A\phi_+ + B\phi_-$ and $\mathcal{M}\phi_- = C\phi_+ + D\phi_-$. Then in matrix form, $\mathcal{M}$ is represented by the matrix M:

$$M = \begin{pmatrix} A & C \\ B & D \end{pmatrix}.$$

In the space $S_2 := \ker(L - \lambda^2)^2 = \text{span}\{\phi_+, \phi_-, \partial_\lambda\phi_+, \partial_\lambda\phi_-\}$, $\mathcal{M}[\partial_\lambda\phi_+] = \partial_\lambda[\mathcal{M}\phi_+] = \partial_\lambda[A\phi_+ + B\phi_-] = A\partial_\lambda\phi_+ + A'\phi_+ + B\partial_\lambda\phi_- + B'\phi_-$, where $A' = \partial_\lambda A$ and $B' = \partial_\lambda B$.

Similarly, $\mathcal{M}[\partial_\lambda\phi_-] = \partial_\lambda[\mathcal{M}\phi_-] = \partial_\lambda[C\phi_+ + D\phi_-] = C\partial_\lambda\phi_+ + C'\phi_+ + D\partial_\lambda\phi_- + D'\phi_-$. This gives a 4×4 monodromy matrix M over S_2 of the form:

$$\begin{bmatrix} A & C & A' & C' \\ B & D & B' & D' \\ 0 & 0 & A & C \\ 0 & 0 & B & D \end{bmatrix}$$

In the space $S_3 := \ker(L-\lambda^2)^3 = \text{span}\{\phi_+, \phi_-, \partial_\lambda\phi_+, \partial_\lambda\phi_-, {\partial_\lambda}^2\phi_+, {\partial_\lambda}^2\phi_-\}$, $\mathcal{M}[{\partial_\lambda}^2\phi_+] = {\partial_\lambda}^2[\mathcal{M}\phi_+] = {\partial_\lambda}^2[A\phi_+ + B\phi_-] = A{\partial_\lambda}^2\phi_+ + 2A'\partial_\lambda\phi_+ + A''\phi_+ + B{\partial_\lambda}^2\phi_- + 2B'\partial_\lambda\phi_- + B''\phi_-$.

Similarly, $\mathcal{M}[{\partial_\lambda}^2\phi_-] = {\partial_\lambda}^2[\mathcal{M}\phi_-] = {\partial_\lambda}^2[C\phi_+ + D\phi_-] = C{\partial_\lambda}^2\phi_+ + 2C'\partial_\lambda\phi_+ + C''\phi_+ + D{\partial_\lambda}^2\phi_- + 2D'\partial_\lambda\phi_- + D''\phi_-$. This gives a 6×6 monodromy matrix M over S_3 of the form:

$$\begin{bmatrix} A & C & A' & C' & A'' & C'' \\ B & D & B' & D' & B'' & D'' \\ 0 & 0 & A & C & 2A' & 2C' \\ 0 & 0 & B & D & 2B' & 2D' \\ 0 & 0 & 0 & 0 & A & C \\ 0 & 0 & 0 & 0 & B & D \end{bmatrix}$$

We note that the above matrices consists of 2×2 blocks which are either zeroes, or derivatives of the basic block for S_1, multiplied by some binomial coefficient. This suggests that for any k, the general Leibniz rule for differentiation can be used to determine the matrix M for S_k. Let $\mathbf{m}$ denote the 2×2 block:

$$\mathbf{m} := \begin{matrix} A & C \\ B & D \end{matrix}$$

Then, the 2×2 block in the qp position of the matrix M is given by

$$\big[M\big]_{qp} = \binom{p-1}{q-1} \left(\frac{\partial}{\partial \lambda}\right)^{p-q} \mathbf{m} \tag{4.11}$$

for $1 \leq p, q \leq k$.

Explicitly, (4.11) looks like this:

$$M = \begin{bmatrix} \mathbf{m} & \mathbf{m}' & \mathbf{m}'' & \mathbf{m}^{(3)} & \mathbf{m}^{(4)} & \dots & \mathbf{m}^{(k-1)} \\ \mathbf{0} & \mathbf{m} & \binom{2}{1}\mathbf{m}' & \binom{3}{1}\mathbf{m}'' & \binom{4}{1}\mathbf{m}^{(3)} & \dots & \binom{k-1}{1}\mathbf{m}^{(k-2)} \\ \mathbf{0} & \mathbf{0} & \mathbf{m} & \binom{3}{2}\mathbf{m}' & \binom{4}{2}\mathbf{m}'' & \dots & \binom{k-1}{2}\mathbf{m}^{(k-3)} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{m} & \binom{4}{3}\mathbf{m}' & \dots & \binom{k-1}{3}\mathbf{m}^{(k-4)} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{m} & \dots & \binom{k-1}{4}\mathbf{m}^{(k-5)} \\ \cdot & \cdot & & & & \cdot & \cdot \\ \cdot & \cdot & & & & \cdot & \cdot \\ \cdot & \cdot & & & & \cdot & \cdot \\ \cdot & \cdot & & & & & \binom{k-1}{k-2}\mathbf{m}' \\ \mathbf{0} & \mathbf{0} & \cdot & \cdot & \cdot & \cdot & \mathbf{m} \end{bmatrix}$$

This matrix would be multiplied with a $2k$-dimensional vector corresponding to some function in S_k.

So far, we have learnt that monodromy invariant subspaces give bispectral Darboux transformations thanks to theorem 4.4. We also know how analytic continuations of hypergeometric series work. We can now think about what kinds of subspaces we can use and how we might construct them.

Let us start with a simple example of the operator $h(L_x) = (L_x - \lambda^2)^2$. Then kernel of $h(L_x)$ is $\text{span}\{\phi_+, \phi_-, \partial_\lambda\phi_+, \partial_\lambda\phi_-\}$ by lemma 3.1. This a four dimensional space. If

$h(L_x) = Q \circ P$ is a non-trivial factorisation then dimension of $\ker P$ has to be 1, 2 or 3. Suppose we try to find a first order P with one-dimensional kernel $\mathrm{span}\{\phi_+\}$ without loss of generality.

Then for any λ which keeps the monodromy group irreducible, $\mathcal{M}_0\phi_+ = A\phi_+ + B\phi_-$ where $A, B \neq 0$. $\mathcal{M}_0\phi_+ \in \ker P$ because this space is preserved by $\mathcal{M}_0$. A and B are the left column entries of (4.5). So kernel of P must include $\phi_- = B^{-1}\mathcal{M}_0\phi_+ - B^{-1}A\phi_+$ and so it can not be a one-dimensional space. Therefore $\ker P = \mathrm{span}\{\phi_+, \phi_-\} = \ker(L_x - \lambda^2)$, which gives a trivial factorisation $h(L_x) = (L_x - \lambda^2) \circ (L_x - \lambda^2)$.

Similarly, if we try to take any kind of space with dimension 3, we find that it would actually have to be four dimensional space $\ker h(L_x)$. To illustrate this, take a starting basis function $f_1 := \alpha\partial_\lambda\phi_+ + \beta\partial_\lambda\phi_- + \gamma\phi_+ + \zeta\phi_- \in \ker P$. Then $\mathcal{M}_\infty f_1$ is proportional to

$$e^{2\pi i\lambda}(\alpha\partial_\lambda\phi_+ + (\gamma + 2\pi i\alpha)\phi_+) + e^{-2\pi i\lambda}(\beta\partial_\lambda\phi_- + (\zeta - 2\pi i\beta)\phi_-).$$

Then $f_2 := \left(\mathcal{M}_\infty - e^{2\pi i\lambda}\right) f_1 \in \ker P$ would be proportional to

$$2\pi i\alpha e^{2\pi i\lambda}\phi_+ + \left(e^{-2\pi i\lambda} - e^{2\pi i\lambda}\right)\beta\partial_\lambda\phi_- + \left(e^{-2\pi i\lambda}(\zeta - 2\pi i\beta) - e^{2\pi i\lambda}\zeta\right)\phi_-. \tag{4.12}$$

Another application of $\mathcal{M}_\infty - e^{2\pi i\lambda}$ gives

$$\begin{aligned} f_3 := &(e^{4\pi i\lambda} - 2 + e^{-4\pi i\lambda})\partial_\lambda\phi_- \\ &+ (4\pi i\beta - 2\zeta - 4\pi i e^{-4\pi i\lambda}\beta + \zeta(e^{4\pi i\lambda} + e^{-4\pi i\lambda}))\phi_- \in \ker P. \end{aligned} \tag{4.13}$$

Applying $\mathcal{M}_\infty - e^{-2\pi i\lambda}$ to f_3 produces

$$f_4 := -2\pi i(e^{4\pi i\lambda} - 2 + e^{-4\pi i\lambda})e^{-2\pi i\lambda}\phi_- \in \ker P. \tag{4.14}$$

Thus the monodromy invariance condition has resulted in the existence of three additional linearly independent functions (4.12), (4.13) and (4.14) in kernel of P. So $P = h(L_x)$ is the original fourth order operator and we are left with the trivial factorisation $h(L_x) =$

$1 \circ (L_x - \lambda^2)^2$.

In fact, this is true for higher multiplicities as well: if $h(L_x) = (L_x - \lambda^2)^n$, then the only monodromy invariant subspaces of $\ker h(L_x)$ are $\ker(L_x - \lambda^2)^m$, $0 \leq m \leq n$. This important property of our solution subspaces is called *uniseriality* and will be introduced and proved in the next chapter.

In order to get non-trivial factorisations, we need to mix different λ's. So take another simple example: $h(L_x) = (L_x - \lambda^2) \circ (L_x - (\lambda+1)^2)$ with $\ker h(L_x) = \operatorname{span}\{\phi_+(\lambda), \phi_-(\lambda), \phi_+(\lambda+1), \phi_-(\lambda+1)\}$. We have to take basis functions which sit across the two spaces $\ker(L_x - \lambda^2)$ and $\ker(L_x - (\lambda+1)^2)$. Take a basis function $f_1 := \alpha\phi_+(\lambda) + \beta\phi_+(\lambda+1)$. Abbreviate M_0 as

$$M_0 := \begin{bmatrix} A & C \\ B & D \end{bmatrix}.$$

Then

$$\mathcal{M}_0 f_1 = \alpha(A(\lambda)\phi_+(\lambda) + B(\lambda)\phi_-(\lambda)) + \beta(A(\lambda+1)\phi_+(\lambda+1) + B(\lambda+1)\phi_-(\lambda+1))$$

$$= (A(\lambda)\alpha\phi_+(\lambda) + A(\lambda+1)\beta\phi_+(\lambda+1)) + (B(\lambda)\alpha\phi_-(\lambda) + B(\lambda+1)\beta\phi_-(\lambda+1)).$$

If we renormalised our solutions so that $A(\lambda) = A(\lambda+1)$ and $B(\lambda) = B(\lambda+1)$ then we would get

$$f_2 := \frac{1}{B(\lambda)}(\mathcal{M}_0 - A(\lambda))f_1 \in \ker P.$$

So if we find an appropriate re-normalisation which makes the monodromy matrices periodic in λ, then we would find that the integer shift $\lambda \mapsto \lambda + 1$ actually does lead to a 2-dimensional non-trivial monodromy invariant solution space $\{f_1, f_2\}$. We will provide a complete treatment of this example at the end of this chapter, but before that, we need to develop a couple of observations further.

- First of all, there is the integer shift $\lambda \mapsto \lambda + 1$. Without integer shifts, we just get trivial factorisations. Integer shifts allow us to construct non-trivial factorisations. This is because in the specific normalisation $\psi(\lambda) = c(\lambda)\mathrm{g}^{-1}\phi(\lambda)$ (2.20), the monodromy matrices become periodic in λ. We explain this in section 4.2.

- Then there is the method with which we used monodromy invariance property to construct the new basis functions (4.12 - 4.14). This is turned in to a formal algorithm in section 4.3.

4.2 Periodicity of Monodromy Representation

The previous example illustrates that it would be useful to have bases in which the monodromy matrices would satisfy $M_0(\lambda) = M_0(\lambda+1)$ and $M_\infty(\lambda) = M_\infty(\lambda+1)$.

The monodromy matrices (4.5 - 4.7) were given with respect to the basis (4.4). Note that the matrices (4.5) and (4.7) are not periodic in λ. Recall the functions $c_\pm$ and $c'_\pm$ introduced in (4.8).

Define these new pairs of eigenfunctions:

$$\begin{cases} \psi_+ = c_+ \mathrm{g}^{-1}\phi_+ \\ \psi_- = c_- \mathrm{g}^{-1}\phi_- \end{cases} \quad \text{and} \quad \begin{cases} \psi'_+ = c'_+ \mathrm{g}^{-1}\phi_+ \\ \psi'_- = c'_- \mathrm{g}^{-1}\psi_- \end{cases}. \tag{4.15}$$

Note that here $\psi'_\pm$ are not derivatives of $\psi_\pm$ but represent an alternative basis of solutions.

Recall that g is the gauge function (2.18) viewed as a function of z. With respect to the basis $\{\psi_+, \psi_-\}$ the matrices (4.5) and (4.6) become:

$$M_0 = \frac{1}{c_+c'_- - c'_+c_-} \begin{bmatrix} c_+c'_- + e^{2\pi i g}c'_+c_- & -\left(1+e^{2\pi i g}\right)c'_+c_- \\ \left(1+e^{2\pi i g}\right)c_+c'_- & -e^{2\pi i g}c_+c'_- - c'_+c_- \end{bmatrix}, \tag{4.16}$$

$$M_\infty = e^{\pi i(g+h)} \begin{bmatrix} e^{2\pi i\lambda} & 0 \\ 0 & e^{-2\pi i\lambda} \end{bmatrix}. \tag{4.17}$$

We claim that the above matrices are periodic.

Lemma 4.5. *If none of* 2λ, $\lambda + \frac{g+h}{2}$, $-\lambda + \frac{g+h}{2}$, $\frac{1}{2}\left(1+g-h-2\lambda\right)$ *and* $\frac{1}{2}\left(1+g-h+2\lambda\right)$

is an integer, then with respect to the basis $\{\psi_+, \psi_-\}$, the monodromy matrices M_0 and M_∞ are invariant under the translation $\lambda \mapsto \lambda + 1$.

Proof. The periodicity of (4.17) is obvious. The periodicity of (4.16) requires a little more work: we need to use the Euler's reflection formula for Gamma function. Observe the two terms $c_+c'_-$ and c'_+c_- which appear repeatedly; we calculate those.

$$
\begin{aligned}
c_+c'_- &= \frac{2^{2\lambda+g+h}\Gamma\big(\frac{1}{2}+g\big)\Gamma\big(-2\lambda\big)}{\Gamma\big(-\lambda+\frac{g+h}{2}\big)\Gamma\big(\frac{g-h+1}{2}-\lambda\big)}\frac{2^{2-2\lambda-g-h}\Gamma\big(\frac{3}{2}-g\big)\Gamma\big(2\lambda\big)}{\Gamma\big(1+\lambda-\frac{g+h}{2}\big)\Gamma\big(\frac{1-g+h}{2}+\lambda\big)} \\
&= \frac{4\Gamma\big(\frac{1}{2}+g\big)\Gamma\big(\frac{3}{2}-g\big)\Gamma\big(2\lambda\big)\Gamma\big(-2\lambda\big)}{\Gamma\big(-\lambda+\frac{g+h}{2}\big)\Gamma\big(1+\lambda-\frac{g+h}{2}\big)\Gamma\big(\frac{1-g+h}{2}+\lambda\big)\Gamma\big(\frac{g-h+1}{2}-\lambda\big)}
\end{aligned}
$$

$$
= \frac{4}{\pi^2}\Gamma\Big(\frac{1}{2}+g\Big)\Gamma\Big(\frac{3}{2}-g\Big)\Gamma(2\lambda)\Gamma(-2\lambda)\sin\Big(\pi\Big(-\lambda+\frac{g+h}{2}\Big)\Big)\sin\Big(\pi\Big(\frac{1-g+h}{2}+\lambda\Big)\Big).
$$

In the step above, we used the Euler's reflection formula [15]:

$$
\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \qquad z \notin \mathbb{Z}.
$$

$$
\begin{aligned}
c'_+c_- &= \frac{2^{2+2\lambda-g-h}\Gamma\big(\frac{3}{2}-g\big)\Gamma\big(-2\lambda\big)}{\Gamma\big(1-\lambda-\frac{g+h}{2}\big)\Gamma\big(\frac{1-g+h}{2}-\lambda\big)}\frac{2^{-2\lambda+g+h}\Gamma\big(\frac{1}{2}+g\big)\Gamma\big(2\lambda\big)}{\Gamma\big(\lambda+\frac{g+h}{2}\big)\Gamma\big(\frac{g-h+1}{2}+\lambda\big)} \\
&= \frac{4\Gamma\big(\frac{1}{2}+g\big)\Gamma\big(\frac{3}{2}-g\big)\Gamma\big(2\lambda\big)\Gamma\big(-2\lambda\big)}{\Gamma\big(\lambda+\frac{g+h}{2}\big)\Gamma\big(1-\lambda-\frac{g+h}{2}\big)\Gamma\big(\frac{g-h+1}{2}+\lambda\big)\Gamma\big(\frac{1-g+h}{2}-\lambda\big)}
\end{aligned}
$$

$$
= \frac{4}{\pi^2}\Gamma\Big(\frac{1}{2}+g\Big)\Gamma\Big(\frac{3}{2}-g\Big)\Gamma(2\lambda)\Gamma(-2\lambda)\sin\Big(\pi\Big(\lambda+\frac{g+h}{2}\Big)\Big)\sin\Big(\pi\Big(\frac{1-g+h}{2}-\lambda\Big)\Big).
$$

Let

$$
\xi := \frac{4}{\pi^2}\Gamma\Big(\frac{1}{2}+g\Big)\Gamma\Big(\frac{3}{2}-g\Big)\Gamma(2\lambda)\Gamma(-2\lambda),
$$

so that

$$
c_+c'_- = \xi\sin\Big(\pi\Big(-\lambda+\frac{g+h}{2}\Big)\Big)\sin\Big(\pi\Big(\frac{1-g+h}{2}+\lambda\Big)\Big)
$$

and

$$c'_+ c_- = \xi \sin\left(\pi\left(\lambda + \frac{g+h}{2}\right)\right)\sin\left(\pi\left(\frac{1-g+h}{2} - \lambda\right)\right).$$

Consider the top right term.

$$(e^{2\pi ic}-1)\frac{c'_+ c_-}{c_+ c'_- - c'_+ c_-} = (e^{2\pi ic}-1)\xi \sin\left(\pi\left(\lambda + \frac{g+h}{2}\right)\right)\sin\left(\pi\left(\frac{1-g+h}{2} - \lambda\right)\right)$$

$$\times\frac{1}{\xi}\frac{1}{\sin\left(\pi\left(-\lambda+\frac{g+h}{2}\right)\right)\sin\left(\pi\left(\frac{1-g+h}{2}+\lambda\right)\right) - \sin\left(\pi\left(\lambda+\frac{g+h}{2}\right)\right)\sin\left(\pi\left(\frac{1-g+h}{2}-\lambda\right)\right)}$$

$$= \frac{(e^{2\pi ic}-1)\sin\left(\pi\left(\lambda+\frac{g+h}{2}\right)\right)\sin\left(\pi\left(\frac{1-g+h}{2}-\lambda\right)\right)}{\sin\left(\pi\left(-\lambda+\frac{g+h}{2}\right)\right)\sin\left(\pi\left(\frac{1-g+h}{2}+\lambda\right)\right) - \sin\left(\pi\left(\lambda+\frac{g+h}{2}\right)\right)\sin\left(\pi\left(\frac{1-g+h}{2}-\lambda\right)\right)}.$$

Here, the only terms with λ in them are of the form "$\sin(f \pm \pi\lambda)\sin(f' \mp \pi\lambda)$" where f and f' are some constants in terms of g and h. It is clear that such expressions are periodic under $\lambda \mapsto \lambda + 1$. Therefore the top right entry of (4.16) is periodic in λ. Similar calculations hold for the other three entries; this shows that (4.16) is also invariant under the mapping $\lambda \mapsto \lambda + 1$. ■

Remark: The above calculation holds as long as none of the following is an integer: 2λ, $\lambda + \frac{g+h}{2}$, $-\lambda + \frac{g+h}{2}$, $\frac{1}{2}\big(1+g-h-2\lambda\big)$ and $\frac{1}{2}\big(1+g-h+2\lambda\big) \notin \mathbb{Z}$. If $2\lambda \in \mathbb{Z}$, then we would have to work in a different basis as described in section 4.4.4. For the other special cases, we start getting poles in the entries of the monodromy matrices. As we run through the arithmetic progression in λ, we may eventually get simple poles. So the above calculation may only hold for a certain number of integer shifts in λ, after which we have to switch basis. Therefore in those cases the arithmetic progression in λ must be split in to two separate non-isomorphic progressions. This is explained in sections 4.4.5 - 4.4.6.

Remark: It should be stressed that the other basis $\{\psi'_+, \psi'_-\}$ is equally good and can be used instead of $\{\psi_+, \psi_-\}$. Its monodromy matrices are also periodic in λ; M_∞ is the same as the one for $\{\psi_+, \psi_-\}$ whereas

$$M_0 = \frac{1}{c_+c'_- - c'_+c_-} \begin{bmatrix} c_+c'_- + e^{2\pi i g}c'_+c_- & -\left(1+e^{2\pi i g}\right)c_+c'_- \\ \left(1+e^{2\pi i g}\right)c'_+c_- & -e^{2\pi i g}c_+c'_- - c'_+c_- \end{bmatrix}. \tag{4.18}$$

The functions $\{\psi'_+, \psi'_-\}$ will be useful in section 4.4.4.

Remark: Another way of seeing the periodicity of the monodromy is as follows. First note that parameters a, b and c of the hypergeometric equation under consideration are given by (2.11). Under the shift $\lambda \mapsto \lambda + 1$, they get shifted by integers. Therefore, we can use corollary 2.2.6 in [27]:

Proposition: *If the monodromy group for $F(a, b, c; z)$ is irreducible (that is, the space of solutions does not admit a non-trivial subspace which is invariant under monodromy transformations), then up to conjugation, it only depends on the values of a, b and c modulo $\mathbb{Z}$.*

Note that the monodromy group is irreducible exactly when one of a, b, $a-c$ or $b-c$ is an integer (see e.g. [27, Corollary 2.2.2]). The above proposition just tells us that the monodromy matrices can be made periodic by choosing suitable bases; it does not tell us how to choose such bases. In comparison, our proof above is constructive as it gives such bases explicitly.

4.3 Generating Subspaces

We look for spaces of solutions which are invariant under $\mathcal{M}_0$ and $\mathcal{M}_\infty$. In this section we describe how to construct bases for monodromy invariant spaces. For instance, in the example of $h(L_x) = (L_x - \lambda^2)^2$, we used monodromy invariance to construct a basis. We can generalise that method.

The most general form of $h\big(L_x\big)$ is

$$h(L_x) = \prod_{j=0}^{n} \left(L_x - \lambda_j^2\right)^{m_j}, \tag{4.19}$$

with some $\lambda_j \in \mathbb{C}$ and $m_j \in \mathbb{N}$. As we will see, our analysis will depend on the type of λ_j's involved. We classify the values of $\lambda = \lambda_j$ into the following categories:

- Type I λ: This is completely general, that is, it is not of the two types described below.
- Type II λ: $2\lambda \in \mathbb{Z}$. This leads to simple poles in some places in our equations that need to be treated with care.
- Type III λ: λ takes a value which makes one of the following an integer: $\lambda + \frac{g+h}{2}$, $\lambda - \frac{g+h}{2}$, $\frac{1}{2}\big(1+g-h-2\lambda\big)$ or $\frac{1}{2}\big(1+g-h+2\lambda\big)$.

The distinction between types I and II is mostly technical, whereas type III is substantially different because these are exactly the values of λ for which the monodromy of the equation (2.12) becomes reducible (see e.g. [27, Corollary 2.2.2]).

Motivated by the periodicity of monodromy discussed in section 4.2, we choose λ of type I and $h\big(L_x\big)$ of the form

$$h(L_x) = \prod_{j=0}^{n} \left(L_x - (\lambda - j)^2\right)^{m_j}. \tag{4.20}$$

A general element of $\ker h(L_x)$ is of the form

$$\phi = \sum_{j=0}^{n} \sum_{k=0}^{m_j-1} \left[a_{k,j}{}^{+} \partial_\lambda{}^{k} \psi_+\big(\lambda - j\big) + a_{k,j}{}^{-} \partial_\lambda{}^{k} \psi_-\big(\lambda - j\big) \right]. \tag{4.21}$$

A possible approach is to take a particular ϕ of this form, and generate a monodromy invariant subspace by acting on ϕ by all monodromy transformations. We start with $\mathcal{M}_\infty$, i.e. the analytic continuation around $z = \infty$. Note that $\mathcal{M}_\infty$ acts on $\psi_\pm(x, \lambda)$ in a simple way (4.17). As a result,

$$\mathcal{M}_\infty\phi = \sum_{j=0}^{n}\sum_{k=0}^{m_j-1}\left[a_{k,j}{}^{+}\mathcal{M}_\infty\Big(\partial_\lambda{}^k\psi_+(\lambda-j)\Big) + a_{k,j}{}^{-}\mathcal{M}_\infty\Big(\partial_\lambda{}^k\psi_-(\lambda-j)\Big)\right].$$

Here,

$\mathcal{M}_\infty\big(\partial_\lambda{}^k\psi_+(\lambda-j)\big) = e^{2\pi i(\lambda+\rho)}\big(\partial_\lambda+2\pi i\big)^k\psi_+(\lambda-j) = e^{2\pi i(\lambda+\rho)}\partial_\lambda{}^k\psi_+(\lambda-j)+$ lower order derivative terms, and also
$\mathcal{M}_\infty\big(\partial_\lambda{}^k\psi_-(\lambda-j)\big) = e^{2\pi i(-\lambda+\rho)}\big(\partial_\lambda-2\pi i\big)^k\psi_+(\lambda-j) = e^{2\pi i(-\lambda+\rho)}\partial_\lambda{}^k\psi_-(\lambda-j)+$ lower order derivative terms. $\rho=(g+h)/2$.

So the linear operation $\mathcal{M}_\infty - e^{2\pi i(\lambda+\rho)}I$ removes the highest order derivative of ψ_+ from (4.21) for each j and $\mathcal{M}_\infty - e^{2\pi i(-\lambda+\rho)}I$ removes the highest order derivative of ψ_- from (4.21) for each j. Either way, after repeatedly applying $\mathcal{M}_\infty - e^{2\pi i(-\lambda+\rho)}I$ to ϕ, we get:

$$\phi^{(+,1)} = \sum_{j=0}^{n}\sum_{k=0}^{m_j-1} b_{k,j}{}^{+}\psi_+(\lambda-j). \tag{4.22}$$

The point of this exercise is to demonstrate that all subspaces of (4.20) have elements which are linear combinations of derivatives of ψ_+ only. Applying $\mathcal{M}_\infty - e^{2\pi i(\lambda+\rho)}I$ to $\phi^{(+,1)}$ gives us $\phi^{(+,2)}$ which has highest order derivatives for each j removed. Apply $\mathcal{M}_\infty - e^{2\pi i(\lambda+\rho)}I$ recursively to obtain $\phi^{(+,3)}$, $\phi^{(+,4)}$ and so on. These $\phi^{(+,i)}$ would be half of the basis functions.

To get the other half of the basis functions, we observe that the matrix M_0 is not diagonal. Applying $\mathcal{M}_0$ to $\phi^{(+,1)}$ would produce a function which would be a mixture of derivatives of ψ_+ and ψ_-.

Use $\mathcal{M}_\infty - e^{2\pi i(\lambda+\rho)}I$ recursively on this new function to remove all the derivatives of ψ_+. Once all those derivatives are gone, the remaining function would be a linear combination of derivatives of ψ_-, similar to (4.22). We would call this function $\phi^{(-,1)}$. We would then go on to use $\mathcal{M}_\infty - e^{2\pi i(-\lambda+\rho)}I$ to get $\phi^{(-,2)}$, $\phi^{(-,3)}$, $\phi^{(-,4)}$ and so on.

The space generated by this basis satisfies the conditions required to ensure rationality of coefficients of P. In the next theorem, we illustrate the above process and prove the following points:

- For every basis function $\phi^{(+,i)}$ (or $\phi^{(-,i)}$) for any monodromy invariant subspace, there is a mirror function $\phi^{(-,i)}$ (or $\phi^{(+,i)}$) which has the same coefficients as $\phi^{(+,i)}$, and also has the same number of derivatives as $\phi^{(+,i)}$. The only difference is that all the ψ_+'s are replaced by ψ_-'s.
- For every monodromy invariant subspace of (3.3), there necessarily exists a basis of the form $\{\phi^{(\pm,i)}\}$ described above. This means that the procedure that we described above can produce all the subspaces with the required properties, and there are no subspaces which cannot be constructed using the above method.

Theorem 4.6. *Suppose λ is of type I. For the operator*

$$h(L_x) = \prod_{j=0}^{n} \left(L_x - (\lambda - j)^2\right)^{m_j},$$

the factorisation $h(L_x) = Q \circ P$ has trigonometric coefficients (that is, coefficients rational in terms of e^{ix}) if and only if kernel of P has a basis of the form:

$$\left\{ \sum_{j=0}^{n} \sum_{k=0}^{m_j-1} c_{kj} \binom{k}{l} \frac{\partial^{k-l}}{\partial \mu^{k-l}} \psi_\pm(\mu) \Big|_{\mu=\lambda-j}, \ l \in \mathbb{N} \right\}. \tag{4.23}$$

Proof. If kernel of P has a basis of the form (4.23), then for each basis function f, $\mathcal{M}_0 f \in \ker P$ and $\mathcal{M}_\infty f \in \ker P$. Therefore we must check this invariance.

$$\mathcal{M}_\infty \frac{\partial^k}{\partial \mu^k} \psi_+(\mu) \Big|_{\mu=\lambda-j} = e^{2\pi i \lambda} \sum_{p=0}^{k} \binom{k}{p} (2\pi i)^p \frac{\partial^{k-p}}{\partial \mu^{k-p}} \psi_+(\mu) \Big|_{\mu=\lambda-j}$$

and

$$\mathcal{M}_\infty \frac{\partial^k}{\partial \mu^k} \psi_-(\mu) \Big|_{\mu=\lambda-j} = e^{-2\pi i \lambda} \sum_{p=0}^{k} \binom{k}{p} (-2\pi i)^p \frac{\partial^{k-p}}{\partial \mu^{k-p}} \psi_-(\mu) \Big|_{\mu=\lambda-j}.$$

Applying $\mathcal{M}_\infty$ to a basis function of the form given in (4.23) gives

$$\mathcal{M}_\infty \sum_{j=0}^{n}\sum_{k=0}^{m_j-1} c_{kj}\binom{k}{l}\frac{\partial^{k-l}}{\partial\mu^{k-l}}\psi_\pm(\mu)\bigg|_{\mu=\lambda-j} = \sum_{j=0}^{n}\sum_{k=0}^{m_j-1} c_{kj}\binom{k}{l} M_\infty \frac{\partial^{k-l}}{\partial\mu^{k-l}}\psi_\pm(\mu)\bigg|_{\mu=\lambda-j}$$

$$= \sum_{j=0}^{n}\sum_{k=0}^{m_j-1} c_{kj}\binom{k}{l} e^{\pm 2\pi i\lambda}\sum_{p=0}^{k-l}\binom{k-l}{p}(\pm 2\pi i)^p\frac{\partial^{k-l-p}}{\partial\mu^{k-l-p}}\psi_\pm(\mu)\bigg|_{\mu=\lambda-j}. \tag{4.24}$$

The terms corresponding to each individual index p are:

$$\sum_{j=0}^{n}\sum_{k=0}^{m_j-1} c_{kj}\binom{k}{l} e^{\pm 2\pi i\lambda}\binom{k-l}{p}(\pm 2\pi i)^p\frac{\partial^{k-l-p}}{\partial\mu^{k-l-p}}\psi_\pm(\mu)\bigg|_{\mu=\lambda-j}.$$

Here,

$$\binom{k}{l}\binom{k-l}{p} = \frac{(l+p)!}{l!p!}\binom{k}{l+p}.$$

Therefore, the sum (4.24) can be reordered as

$$e^{\pm 2\pi i\lambda}\sum_{p\geq 0}\left[\frac{(l+p)!}{l!p!}(\pm 2\pi i)^p\right]\left[\sum_{j=0}^{n}\sum_{k=0}^{m_j-1} c_{kj}\binom{k}{l+p}\frac{\partial^{k-(l+p)}}{\partial\mu^{k-(l+p)}}\psi_\pm(\mu)\bigg|_{\mu=\lambda-j}\right]. \tag{4.25}$$

This is a linear combination in terms of the basis functions in (4.23). So the proposed kernel for P is $\mathcal{M}_\infty$-invariant. Now, let $\mathcal{M}_0$ denote the monodromy transformation corresponding to a loop around $z=0$. The action of $\mathcal{M}_0$ on $\ker(L-\mu^2)$ is given by (4.16). Let us write the corresponding matrix as

$$\begin{bmatrix} A & C \\ B & D \end{bmatrix}.$$

Note that since $\mu = \lambda - j$ is of type I, so B and C are non-zero because otherwise the 2×2 matrices M_0 and M_∞ would have an invariant one dimensional subspace in $\ker(L-\mu^2)$.

Now we have

$$\mathcal{M}_0\frac{\partial^k}{\partial\mu^k}\psi_+(\mu) = \frac{\partial^k}{\partial\mu^k}\Big(M_0\psi_+(\mu)\Big) = \frac{\partial^k}{\partial\mu^k}\Big(A\psi_+(\mu)+B\psi_-(\mu)\Big)$$

$$= \sum_{p=0}^{k}\binom{k}{p}\left[A^{(p)}\frac{\partial^{k-p}}{\partial\mu^{k-p}}\psi_+(\mu) + B^{(p)}\frac{\partial^{k-p}}{\partial\mu^{k-p}}\psi_-(\mu)\right],$$

and

$$\mathcal{M}_0 \frac{\partial^k}{\partial \mu^k} \psi_-(\mu) = \frac{\partial^k}{\partial \mu^k} \Big(M_0 \psi_-(\mu) \Big) = \frac{\partial^k}{\partial \mu^k} \Big(C\psi_+(\mu) + D\psi_-(\mu) \Big)$$

$$= \sum_{p=0}^{k} \binom{k}{p} \left[C^{(p)} \frac{\partial^{k-p}}{\partial \mu^{k-p}} \psi_+(\mu) + D^{(p)} \frac{\partial^{k-p}}{\partial \mu^{k-p}} \psi_-(\mu) \right],$$

where $A^{(p)} = \partial_\mu^p A$. With the equations above, we now apply $\mathcal{M}_0$ to a basis function.

$$\mathcal{M}_0 \sum_{j=0}^{n} \sum_{k=0}^{m_j-1} c_{kj} \binom{k}{l} \frac{\partial^{k-l}}{\partial \mu^{k-l}} \psi_+(\mu) = \sum_{j=0}^{n} \sum_{k=0}^{m_j-1} c_{kj} \binom{k}{l} \mathcal{M}_0 \frac{\partial^{k-l}}{\partial \mu^{k-l}} \psi_+(\mu)$$

$$= \sum_{j=0}^{n} \sum_{k=0}^{m_j-1} c_{kj} \binom{k}{l} \sum_{p=0}^{k-l} \binom{k-l}{p} \left[A^{(p)} \frac{\partial^{k-l-p}}{\partial \mu^{k-l-p}} \psi_+(\mu) + B^{(p)} \frac{\partial^{k-l-p}}{\partial \mu^{k-l-p}} \psi_-(\mu) \right]. \tag{4.26}$$

The terms corresponding to each individual index p are:

$$\sum_{j=0}^{n} \sum_{k=0}^{m_j-1} c_{kj} \binom{k}{l} \binom{k-l}{p} \left[A^{(p)} \frac{\partial^{k-l-p}}{\partial \mu^{k-l-p}} \psi_+(\mu) + B^{(p)} \frac{\partial^{k-l-p}}{\partial \mu^{k-l-p}} \psi_-(\mu) \right].$$

$$= \sum_{j=0}^{n} \sum_{k=0}^{m_j-1} c_{kj} \frac{(l+p)!}{l!p!} \binom{k}{l+p} \left[A^{(p)} \frac{\partial^{k-l-p}}{\partial \mu^{k-l-p}} \psi_+(\mu) + B^{(p)} \frac{\partial^{k-l-p}}{\partial \mu^{k-l-p}} \psi_-(\mu) \right].$$

Therefore, the sum (4.26) can be reordered as follows.

$$\sum_{p \geq 0} \left[\frac{(l+p)!}{l!p!} A^{(p)} \right] \left[\sum_{j=0}^{n} \sum_{k=0}^{m_j-1} c_{kj} \binom{k}{l+p} \frac{\partial^{k-(l+p)}}{\partial \mu^{k-(l+p)}} \psi_+(\mu) \right]$$

$$+ \sum_{p \geq 0} \left[\frac{(l+p)!}{l!p!} B^{(p)} \right] \left[\sum_{j=0}^{n} \sum_{k=0}^{m_j-1} c_{kj} \binom{k}{l+p} \frac{\partial^{k-(l+p)}}{\partial \mu^{k-(l+p)}} \psi_-(\mu) \right]. \tag{4.27}$$

Similarly,

$$M_0 \sum_{j=0}^{n} \sum_{k=0}^{m_j-1} c_{kj} \binom{k}{l} \frac{\partial^{k-l}}{\partial \mu^{k-l}} \psi_-(\mu) = \sum_{j=0}^{n} \sum_{k=0}^{m_j-1} c_{kj} \binom{k}{l} M_0 \frac{\partial^{k-l}}{\partial \mu^{k-l}} \psi_-(\mu)$$

$$= \sum_{j=0}^{n} \sum_{k=0}^{m_j-1} c_{kj} \binom{k}{l} \sum_{p=0}^{k-l} \binom{k-l}{p} \left[C^{(p)} \frac{\partial^{k-l-p}}{\partial \mu^{k-l-p}} \psi_+(\mu) + D^{(p)} \frac{\partial^{k-l-p}}{\partial \mu^{k-l-p}} \psi_-(\mu) \right].$$

$$= \sum_{p\geq 0}\left[\frac{(l+p)!}{l!p!}C^{(p)}\right]\left[\sum_{j=0}^{n}\sum_{k=0}^{m_j-1}c_{kj}\binom{k}{l+p}\frac{\partial^{k-(l+p)}}{\partial\mu^{k-(l+p)}}\psi_+(\mu)\right]$$

$$+ \sum_{p\geq 0}\left[\frac{(l+p)!}{l!p!}D^{(p)}\right]\left[\sum_{j=0}^{n}\sum_{k=0}^{m_j-1}c_{kj}\binom{k}{l+p}\frac{\partial^{k-(l+p)}}{\partial\mu^{k-(l+p)}}\psi_-(\mu)\right] \quad (4.28)$$

(4.27) and (4.28) are linear combinations of the basis functions in (4.23). Therefore, the proposed solution space is also invariant under the action of $\mathcal{M}_0$. The monodromy invariance of $\ker P$ implies that P is itself monodromy invariant and hence has trigonometric coefficients.

The converse statement asserts two things:

- Kernel of P breaks down into a direct sum $W_+ \oplus W_-$; W_+ only contains terms involving ψ_+ and W_- only contains terms involving ψ_-.
- Kernel of P is symmetric. So for every element in W_+, there is an analogous element in W_- with the same coefficients but with ψ_+ replaced with ψ_-.

Ker $P = W_+ \oplus W_-$:

Assume that the factorisation $h(L_x) = Q \circ P$ has trigonometric coefficients. Since $\forall l \in \mathbb{Z}, e^{i(x+2\pi l)} = e^{ix+2\pi il} = e^{ix}e^{2\pi il} = e^{ix}$, P is invariant under the transformation $x \mapsto x+2\pi l$. So the kernel of P is invariant under $x \mapsto x + 2\pi l$. This transformation is represented by $\mathcal{M}_0$, so $\mathcal{M}_0(\ker P) \subset \ker P$. Recall that

$$\ker h(L_x) = \text{span}\left\{\frac{\partial^k\psi_\pm}{\partial\mu^k}\bigg|_{\mu=\lambda-j} : 0 \leq j \leq n, 0 \leq k \leq m_j - 1\right\}$$

and $\ker P \subset \ker h(L_x)$. Take any $\phi \in \ker P$. Then, because it is in $\ker h(L_x)$, it must be of the form

$$\phi(x) = \sum_{j=0}^{n}\sum_{k=0}^{k_1} a_{kj}^{+}\frac{\partial^k\psi_+(\mu)}{\partial\mu^k}\bigg|_{\mu=\lambda-j} + \sum_{j=0}^{n}\sum_{k=0}^{k_2} a_{kj}^{-}\frac{\partial^k\psi_-(\mu)}{\partial\mu^k}\bigg|_{\mu=\lambda-j}$$

where $k_1, k_2 \in \mathbb{N}$ and we assume without loss of generality that the leading coefficient

$a_{k_1 0} \neq 0$. $\mathcal{M}_0\phi(x) = \phi(x+2\pi) \in \ker P$.

$$\mathcal{M}_\infty\phi(x) = \sum_{j=0}^{n}\sum_{k=0}^{k_1} a_{kj}^{+}\mathcal{M}_\infty \frac{\partial^k \psi_+(\mu)}{\partial\mu^k}\bigg|_{\mu=\lambda-j} + \sum_{j=0}^{n}\sum_{k=0}^{k_2} a_{kj}^{-}\mathcal{M}_\infty \frac{\partial^k \psi_-(\mu)}{\partial\mu^k}\bigg|_{\mu=\lambda-j}$$

$$= e^{2\pi i\lambda}\sum_{j=0}^{n}\sum_{k=0}^{k_1} a_{kj}^{+}\sum_{p=0}^{k}\binom{k}{p}(2\pi i)^p \frac{\partial^{k-p}\psi_+(\mu)}{\partial\mu^{k-p}}\bigg|_{\mu=\lambda-j}$$

$$+e^{-2\pi i\lambda}\sum_{j=0}^{n}\sum_{k=0}^{k_2} a_{kj}^{-}\sum_{p=0}^{k}\binom{k}{p}(-2\pi i)^p \frac{\partial^{k-p}\psi_-(\mu)}{\partial\mu^{k-p}}\bigg|_{\mu=\lambda-j}$$

Apply $\mathcal{M}_\infty - e^{-2\pi i\lambda}$ to $\phi(x)$ to obtain

$$\sum_{j=0}^{n}\sum_{k=0}^{k_1} a_{kj}^{+}\left[e^{2\pi i\lambda}\sum_{p=0}^{k}\binom{k}{p}(2\pi i)^p \frac{\partial^{k-p}\psi_+(\mu)}{\partial\mu^{k-p}}\bigg|_{\mu=\lambda-j} - e^{-2\pi i\lambda}\frac{\partial^k\psi_+(\mu)}{\partial\mu^k}\bigg|_{\mu=\lambda-j}\right]$$

$$+\sum_{j=0}^{n}\sum_{k=0}^{k_2} a_{kj}^{-}\left[e^{-2\pi i\lambda}\sum_{p=0}^{k}\binom{k}{p}(-2\pi i)^p \frac{\partial^{k-p}\psi_-(\mu)}{\partial\mu^{k-p}}\bigg|_{\mu=\lambda-j} - e^{-2\pi i\lambda}\frac{\partial^k\psi_-(\mu)}{\partial\mu^k}\bigg|_{\mu=\lambda-j}\right]$$

Note that the highest derivative of ψ_+ at $\mu=\lambda$ has the coefficient

$$a_{k_1 0}^{+}\left(e^{2\pi i\lambda} - e^{-2\pi i\lambda}\right) \neq 0.$$

Regarding the terms with ψ_-, we have for each k,

$$e^{-2\pi i\lambda}\sum_{p=0}^{k}\binom{k}{p}(-2\pi i)^p \frac{\partial^{k-p}\psi_-(\mu)}{\partial\mu^{k-p}} - e^{-2\pi i\lambda}\frac{\partial^k\psi_-(\mu)}{\partial\mu^k}$$

$$= e^{-2\pi i\lambda}\left[\sum_{p=1}^{k}\binom{k}{p}(-2\pi i)^p \frac{\partial^{k-p}\psi_-(\mu)}{\partial\mu^{k-p}} + \binom{k}{0}(-2\pi i)^0\frac{\partial^{k-0}\psi_-(\mu)}{\partial\mu^{k-0}} - \frac{\partial^k\psi_-(\mu)}{\partial\mu^k}\right]$$

$$= e^{-2\pi i\lambda}\left[\sum_{p=1}^{k}\binom{k}{p}(-2\pi i)^p \frac{\partial^{k-p}\psi_-(\mu)}{\partial\mu^{k-p}} + \frac{\partial^k\psi_-(\mu)}{\partial\mu^k} - \frac{\partial^k\psi_-(\mu)}{\partial\mu^k}\right]$$

$$= e^{-2\pi i\lambda}\sum_{p=1}^{k}\binom{k}{p}(-2\pi i)^p \frac{\partial^{k-p}\psi_-(\mu)}{\partial\mu^{k-p}}.$$

The above shows that when we apply $\mathcal{M}_\infty - e^{-2\pi i\lambda}I$ to $\phi(x)$, the highest derivatives of ψ_- disappear. The resulting function still has derivatives of ψ_+ up to $k = k_1$ as long as

$2\lambda \neq \mathbb{Z}$, but now only has $k_2 - 1$ derivatives of ψ_- left.

We can repeat this procedure k_2-1 more times (that is, apply operator $(\mathcal{M}_\infty - e^{-2\pi i\lambda})^{k_2-1}$) to get rid of all the derivatives $\partial_\mu^k \psi_-(\mu)|_{\mu=\lambda-j}$. This will leave behind what was previously referred to as $\phi^{(+,1)}$, which is a linear combination of $\partial_\mu^k \psi_+, 0 \leq k \leq k_1$. This is in the kernel of P because for each $f \in \ker P$, $\mathcal{M}_\infty f \in \ker P$ and obviously $e^{-2\pi i\lambda} f \in \ker P$, so $\mathcal{M}_\infty f - e^{-2\pi i\lambda} f$ is a linear combination of elements of $\ker P$.

The above algorithm can be applied to *all* $\phi \in \ker P$. The span of the resulting functions would be denoted as W_+.

$$W_+ \subset \operatorname{span}\left\{ \frac{\partial^k}{\partial \mu^k} \psi_+(\mu) \Big|_{\mu=\lambda-j} : 0 \leq k \leq m_j - 1, 0 \leq j \leq n \right\}.$$

This process can also be done to remove all of the derivatives of ϕ_+ from any element of $\ker P$. So we also get

$$W_- \subset \operatorname{span}\left\{ \frac{\partial^k}{\partial \mu^k} \psi_-(\mu) \Big|_{\mu=\lambda-j} : 0 \leq k \leq m_j - 1, 0 \leq j \leq n \right\}$$

and

$$\ker P = W_+ \oplus W_-.$$

Ker P is symmetric:

Having established the breakdown of $\ker P$ into a direct sum $W_+ \oplus W_-$, we will now show that for every element in W_+, we can find an element in W_- which is otherwise the same, but has every instance of ψ_+ replaced by ψ_-. Let

$$f^+ = \sum_{j=0}^{n} \sum_{k=0}^{m_j-1} a_{kj} \frac{\partial^k \psi_+(\mu)}{\partial \mu^k} \Big|_{\mu=\lambda-j} \in W_+.$$

Apply $\mathcal{M}_0$ to this function.

$$\mathcal{M}_0 f^+ = \sum_{j=0}^{n} \sum_{k=0}^{m_j-1} a_{kj} \mathcal{M}_0 \frac{\partial^k \psi_+(\mu)}{\partial \mu^k} \Big|_{\mu=\lambda-j}$$

$$= \sum_{j=0}^{n} \sum_{k=0}^{m_j-1} a_{kj} \sum_{i=0}^{k} \binom{k}{i} \left[A^{(i)} \frac{\partial^{k-i}\psi_+(\mu)}{\partial\mu^{k-i}}\bigg|_{\mu=\lambda-j} + B^{(i)} \frac{\partial^{k-i}\psi_-(\mu)}{\partial\mu^{k-i}}\bigg|_{\mu=\lambda-j} \right]$$

For $i = 0$, the object multiplied by $B^{(0)} = B \neq 0$ is

$$\sum_{j=0}^{n} \sum_{k=0}^{m_j-1} a_{kj} \frac{\partial^{k}\psi_-(\mu)}{\partial\mu^{k}}\bigg|_{\mu=\lambda-j} \tag{4.29}$$

which is the required combination. To isolate it, we need to subtract off everything else whilst staying within $\ker P$. The ψ_+ terms are not a problem since they are in W_+ and can be eliminated using $\mathcal{M}_\infty - e^{2\pi i\lambda}$. To eliminate the unwanted ψ_- terms for $i \geq 1$, we apply $\mathcal{M}_\infty$ to the following combination.

$$f = \sum_{j=0}^{n} \sum_{k=0}^{m_j-1} a_{kj} \sum_{i=0}^{k} \frac{\partial^{k-i}\psi_-(\mu)}{\partial\mu^{k-i}}\bigg|_{\mu=\lambda-j}.$$

We would produce lower order terms in W_-. Those could then be subtracted off from the above combination to isolate (4.29). This completes this step of the proof.

The last thing to do is to check that a canonical basis of specific form (4.23) can be found for $\ker P$. Let f^+ be as above. Since $\mathcal{M}_\infty(\ker P) \subset \ker P$, $\mathcal{M}_\infty f^+ \in \ker P$.

$$\mathcal{M}_\infty f^+ = \sum_{j=0}^{n} \sum_{k=0}^{m_j-1} a_{kj} \mathcal{M}_\infty \frac{\partial^{k}\psi_+(\mu)}{\partial\mu^{k}}\bigg|_{\mu=\lambda-j}$$

$$= \sum_{j=0}^{n} \sum_{k=0}^{m_j-1} a_{kj} e^{2\pi i\lambda} \sum_{p=0}^{k} \binom{k}{p} (2\pi i)^p \frac{\partial^{k-p}\psi_+(\mu)}{\partial\mu^{k-p}}\bigg|_{\mu=\lambda-j}$$

$$= \sum_{j=0}^{n} \sum_{k=0}^{m_j-1} a_{kj} e^{2\pi i\lambda} \left[\frac{\partial^{k}\psi_+(\mu)}{\partial\mu^{k}}\bigg|_{\mu=\lambda-j} + \sum_{p=1}^{k} \binom{k}{p} (2\pi i)^p \frac{\partial^{k-p}\psi_+(\mu)}{\partial\mu^{k-p}}\bigg|_{\mu=\lambda-j} \right]$$

$$\implies e^{-2\pi i\lambda}\mathcal{M}_\infty f^+ - f^+ = \sum_{j=0}^{n} \sum_{k=0}^{m_j-1} a_{kj} \sum_{p=1}^{k} \binom{k}{p} (2\pi i)^p \frac{\partial^{k-p}\psi_+(\mu)}{\partial\mu^{k-p}}\bigg|_{\mu=\lambda-j}$$

The term corresponding to each index p is

$$(2\pi i)^p \sum_{j=0}^{n} \sum_{k=0}^{m_j-1} a_{kj} \binom{k}{p} \frac{\partial^{k-p}\psi_+(\mu)}{\partial\mu^{k-p}}\bigg|_{\mu=\lambda-j},$$

which is exactly the kind of basis function in (4.23), and can be isolated by performing a finite number of subtractions and applications of $e^{-2\pi i\lambda}\mathcal{M}_\infty - I$. ∎

Remark: Theorem 4.5 and its proof are similar to theorem 3.3 in [9].

4.4 Types of Invariant Subspaces for $\ker(P)$

The analysis in the previous section can be extended to the general case. We will not provide a comprehensive treatment because already for type I λ our answer was rather cumbersome, as shown in section 4.3. Instead, the material below only illustrates some new features that appear for other types of λ. The full analysis will be given in module theoretic terms in chapter 5.

4.4.1 Type I λ, No Integer Shifts

The most trivial situation is when factorising $h\big(L_x\big) = \big(L_x - \lambda^2\big)^m$. Here we are assuming that $2\lambda \notin \mathbb{Z}$, and none of $\lambda + \frac{g+h}{2}$, $\lambda - \frac{g+h}{2}$, $\frac{1}{2}\big(1+g-h-2\lambda\big)$ and $\frac{1}{2}\big(1+g-h+2\lambda\big)$ is an integer.

Using the method in section 4.3, it is fairly straightforward to show that the only subspaces of $S_m = \ker\big(L_x - \lambda^2\big)^m$ which are monodromy invariant are $S_i = \ker\big(L_x - \lambda^2\big)^i$ for $i = 0, 1, 2, ..., m$. This generalises the $h(L_x) = (L_x - \lambda^2)^2$ example above.

Therefore, the only factorisations of $\big(L_x - \lambda^2\big)^m$ are just $\big(L_x - \lambda^2\big)^{m-i} \circ \big(L_x - \lambda^2\big)^i$ which do not give us new operators when the factors are interchanged.

4.4.2 Type I λ, Integer Shifts

A more general non-trivial construction is applicable when we have integer shifts in λ:

$$h(L_x) = \prod_{j=0}^{n} \left(L_x - (\lambda - j)^2\right)^{m_j}. \tag{4.30}$$

The kernel of this operator is

$$\left\{ {\partial_\lambda}^k \psi_\pm(\lambda - j); 0 \leq k \leq m_j - 1, 0 \leq j \leq n \right\}.$$

Here, the benefit of making the monodromy matrices periodic in $\lambda \longrightarrow \lambda + 1$ becomes apparent. No matter the value of j, all monodromy matrices do the same thing to all the terms regardless of j; $M_\infty - e^{2\pi i(\lambda+\rho)}I$ and $M_\infty - e^{2\pi i(-\lambda+\rho)}I$ remove highest derivatives of ψ_+ and ψ_- respectively across all j. Therefore the periodicity of monodromy matrices shows that it is possible to construct $\mathbb{Z}_2$-invariant P with trigonometric coefficients by choosing basis functions which are linear combinations of terms which sit across different values of j.

We apply the procedure in section 4.3 to find a canonical basis of the subspace. The basis functions are of the form:

$$f_l^\pm = \sum_{j=0}^{n} \sum_{k=0}^{m_j-1} \binom{k}{l} a_{j,k} {\partial_\lambda}^{k-l} \psi_\pm(\lambda - j). \tag{4.31}$$

Here, l is an index which goes from 0, 1 etc. to some k for which all $a^\pm_{j,k}$ are zero.

4.4.3 Type I λ, Non-integer differences

Here we discuss a more general operator:

$$h(L_x) = \prod_{r=1}^{n} \prod_{j=0}^{n_r} \left(L_x - (\lambda_r - j)^2\right)^{m_{r,j}}. \tag{4.32}$$

We still have 2λ, $\lambda+\frac{g+h}{2}$, $-\lambda+\frac{g+h}{2}$, $\frac{1}{2}\big(1+g-h-2\lambda\big)$, $\frac{1}{2}\big(1+g-h+2\lambda\big)\notin\mathbb{Z}$. If $r\neq s$ then $\lambda_r-\lambda_s\notin\mathbb{Z}$. Other than that, each λ_r may carry a few integer shifts as encapsulated by $j=0,1,...n_r$.

Applying any monodromy transformation to anything in the kernel of $h\big(L_x\big)$ preserves the highest order derivatives from each term up to some multiple. This means that our usual method from section 4.3 still works.

This case is therefore simply a generalisation of the case 4.4.2, except we are now allowing for different values of λ which may not differ by an integer. The basis functions would be of the form:

$$f_l^{\pm}=\sum_{r=1}^{n}\sum_{j=0}^{n_r}\sum_{k=0}^{m_j-1}\binom{k}{l}a_{r,j,k}\partial_\lambda{}^{k-l}\psi_\pm\big(\lambda_r-j\big). \tag{4.33}$$

4.4.4 Type II λ

The functions (4.4) are obtained as Frobenius series solutions of the hypergeometric differential equation (2.12). This involves solving an indicial equation. The two roots of the indicial equation differ by 2λ.

If the two roots of an indicial equation differ by an integer, then logarithms need to be used. We propose that instead of using the basis $\{\psi_+,\psi_-\}$, we should change the basis and use the following basis of hypergeometric solutions near $z=0$:

$$\Psi_1=\frac{\Gamma(2\lambda)}{c(-\lambda,k)}\Big(\psi_++\psi_-\Big),\qquad \Psi_2=\frac{c(\lambda,k)}{\Gamma(-2\lambda)}\Big(\psi_+'+\psi_-'\Big). \tag{4.34}$$

To switch basis from $\{\psi_+,\psi+_-\}$ to $\big\{\Psi_1,\Psi_2\big\}$, all monodromy matrices would have to be conjugated by the matrix $C\big(\lambda\big)A$, where

$$C(\lambda)=\begin{bmatrix} c(\lambda,k) & c(\lambda,k') \\ c(-\lambda,k) & c(-\lambda,k') \end{bmatrix}\quad\text{and}\quad A=\begin{bmatrix} \frac{\Gamma(2\lambda)}{c(-\lambda,k)} & 0 \\ 0 & \frac{c(\lambda,k)}{\Gamma(-2\lambda)} \end{bmatrix}.$$

The matrices for loops around $z = 0$ and ∞ would become:

$$M_0 = A^{-1}C(\lambda)^{-1}\left[C(\lambda)\begin{bmatrix} 1 & 0 \\ 0 & -e^{2\pi i g}\end{bmatrix}C(\lambda)^{-1}\right]C(\lambda)A = \begin{bmatrix} 1 & 0 \\ 0 & -e^{2\pi i g}\end{bmatrix}, \tag{4.35}$$

$$M_\infty = e^{2\pi i\rho}A^{-1}C(\lambda)^{-1}\begin{bmatrix} e^{2\pi i\lambda} & 0 \\ 0 & e^{-2\pi i\lambda}\end{bmatrix}C(\lambda)A,$$

$$= \frac{1}{|C(\lambda)|}\begin{bmatrix} e^{2\pi i\lambda}c_+c'_- - e^{-2\pi i\lambda}c'_+c_- & 2ic_+c'_+c_-c'_-(\Gamma(2\lambda)\Gamma(-2\lambda))^{-1}\sin(2\pi\lambda) \\ -2i\Gamma(2\lambda)\Gamma(-2\lambda)\sin(2\pi\lambda) & e^{-2\pi i\lambda}c_+c'_- - e^{2\pi i\lambda}c'_+c_- \end{bmatrix}. \tag{4.36}$$

If $2\lambda \in \mathbb{Z}$, then one of $\pm 2\lambda$ is a positive integer and the other one is a negative integer. Looking back at the formulae (4.8), there are instances of $\Gamma\big(\pm 2\lambda\big)$ in the numerators. The way we made the matrices periodic with respect to λ in (4.16) and (4.17) meant that each $\Gamma\big(2\lambda\big)$ was accompanied by $\Gamma\big(-2\lambda\big)$.

The gamma function Γ has simple poles at non-positive integers. Since there are occurrences of $\Gamma\big(2\lambda\big)\Gamma\big(-2\lambda\big)$ in the formulas and one of the $\pm 2\lambda$ is a non-positive integer, it means that there are simple poles.

Our claim is that the new monodromy matrices (4.35) and (4.4.4) do not have any singularities and are periodic. Conjugating with $C(\lambda)$ removes any poles from our functions. Conjugating with A subsequently gives us periodicity in $\lambda \mapsto \lambda + 1$.

To understand why there would be no singularities, consider the above formulas for general λ. The determinant $|C(\lambda)|$ is

$$c_+c'_- - c'_+c_- = 4\Gamma\bigg(\frac{1}{2}+g\bigg)\Gamma\bigg(\frac{3}{2}-g\bigg)\Gamma(2\lambda)\Gamma(-2\lambda)$$

$$\times\frac{-\pi^2\sin\Big(\pi\Big(\frac{1}{2}+g\Big)\Big)\sin(2\pi\lambda)}{\sin\Big(\pi\Big(\lambda+\frac{g+h}{2}\Big)\Big)\sin\Big(\pi\Big(-\lambda+\frac{g+h}{2}\Big)\Big)\sin\Big(\pi\Big(-\lambda+\frac{1+g-h}{2}\Big)\Big)\sin\Big(\pi(\lambda+\frac{1+g-h}{2}\Big)\Big)}$$

$$\times\frac{\sin\Big(\pi\Big(\lambda+\frac{g+h}{2}\Big)\Big)\sin\Big(\pi\Big(-\lambda+\frac{g+h}{2}\Big)\Big)\sin\Big(\pi\Big(-\lambda+\frac{1+g-h}{2}\Big)\Big)\sin\Big(\pi(\lambda+\frac{1+g-h}{2}\Big)\Big)}{\pi^4}$$

$$= -\frac{4}{\pi^2}\Gamma\Big(\frac{1}{2}+g\Big)\Gamma\Big(\frac{3}{2}-g\Big)\sin\Big(\pi\Big(\frac{1}{2}+g\Big)\Big)\Gamma(2\lambda)\Gamma(-2\lambda)\sin(2\pi\lambda)$$

Then the bottom left entry becomes

$$[M_\infty]_{21}\frac{-2i\Gamma(2\lambda)\Gamma(-2\lambda)\sin(2\pi\lambda)}{|C(\lambda)|}$$

$$= \frac{2i\pi^2}{4}\frac{\Gamma(2\lambda)\Gamma(-2\lambda)\sin(2\pi\lambda)}{\Gamma\Big(\frac{1}{2}+g\Big)\Gamma\Big(\frac{3}{2}-g\Big)\sin\Big(\pi\Big(\frac{1}{2}+g\Big)\Big)\Gamma(2\lambda)\Gamma(-2\lambda)\sin(2\pi\lambda)}$$

$$= \frac{i\pi^2}{2}\frac{1}{\Gamma\Big(\frac{1}{2}+g\Big)\Gamma\Big(\frac{3}{2}-g\Big)\sin\Big(\pi\Big(\frac{1}{2}+g\Big)\Big)}$$

$$= \frac{i\pi}{2}\frac{\Gamma(\frac{1}{2}-g)}{\Gamma(\frac{3}{2}-g)} = \frac{i\pi}{2}\frac{2}{1-2g}$$

2λ is not to be found anywhere in this bottom left entry. So in the basis $\{\Psi_1, \Psi_2\}$, there is no dependence on λ in general. This guarantees that this matrix entry is periodic in λ. Specialising λ to half-integers would not result in any poles or zeroes.

We can calculate the explicit forms of the other entries as well. They are

$$[M_\infty]_{11} = \frac{i\Big[(-1)^{2\lambda+1}ie^{-i\pi g} - \cos\Big(\pi\Big(\frac{1}{2}-h\Big)\Big)\Big]}{\sin\Big(\pi\Big(\frac{1}{2}+g\Big)\Big)},$$

$$[M_\infty]_{12} = -\frac{8i\Gamma\Big(\frac{1}{2}+g\Big)\Gamma\Big(\frac{3}{2}-g\Big)}{\pi^2\sin\Big(\pi\Big(\frac{1}{2}+g\Big)\Big)}\sin\Big(\pi\Big(\lambda+\frac{g+h}{2}\Big)\Big)\sin\Big(\pi\Big(-\lambda+\frac{g+h}{2}\Big)\Big)$$

$$\times\sin\Big(\pi\Big(-\lambda+\frac{1+g-h}{2}\Big)\Big)\sin\Big(\pi\Big(\lambda+\frac{1+g-h}{2}\Big)\Big),$$

$$\text{and } [M_\infty]_{22} = \frac{i\Big[\cos\Big(\pi\Big(\frac{1}{2}-h\Big)\Big) - (-1)^{2\lambda}ie^{i\pi g}\Big]}{\sin\Big(\pi\Big(\frac{1}{2}+g\Big)\Big)}.$$

For generic λ, the above terms are periodic in λ. In the limit as λ tends to any half-integer, we would not get any zeroes or poles.

Other than that, the matrix M_0 (4.35) is now conveniently a diagonal matrix, whereas

M_∞ is no longer diagonal. So the basis functions for kernel of P can be calculated as usual using the process in section 4.3 with M_0 and M_∞ interchanging their roles.

4.4.5 Type III λ

Now we turn to the case when λ takes a value which makes one of $\lambda + \frac{g+h}{2}$, $-\lambda + \frac{g+h}{2}$, $\frac{1}{2}\big(1+g-h-2\lambda\big)$ or $\frac{1}{2}\big(1+g-h+2\lambda\big)$ an integer. In this case the monodromy group becomes reducible.

Again, looking back at the coefficients (4.8), the aforementioned terms are in the denominators as arguments of the gamma function. Not only that, but one minus the aforementioned terms are also arguments of the gamma function in the denominators. So if one of those terms is an integer, then necessarily there would be a pole somewhere. One of the off-diagonal entries in the matrix (4.16) would be 0, and so we would have, without loss of generality, upper triangular matrices.

In the previous cases the subspaces were all even dimensional. Here, we can now have odd dimensional subspaces. Suppose without loss of generality that $-\lambda + (g+h)/2$ was a negative integer. The matrices of the form (4.11) would be upper triangular. Then we can have a space in which the number of basis functions $\phi^{(+,i)}$ is one higher than the number of the basis functions $\phi^{(-,i)}$. Canonically, the basis functions would be:

$$f_l^+ = \sum_{j=0}^{n} \sum_{k=0}^{m_j - 1} \binom{k}{l} a_{j,k} {\partial_\lambda}^{k-l} \psi_+\big(\lambda - j\big). \tag{4.37}$$

$$f_l^- = \sum_{j=0}^{n} \sum_{k=0}^{m_j - 1} \binom{k}{l+1} a_{j,k} {\partial_\lambda}^{k-l} \psi_-\big(\lambda - j\big). \tag{4.38}$$

4.4.6 Type III λ, Several Integer Shifts

This is a continuation of the previous case. If there were too many integer shifts in λ, then it might well happen that one of those four quantities might be positive integers for some of the integer shifts j, zero at some specific j and then negative integers for the rest of the integer shifts. This can be made precise in the following way. Let ρ be one of the following:

$$\pm\frac{g+h}{2}, \quad \pm\frac{1+g-h}{2}.$$

Then we would have a set of type III λ's with integer shifts $\Lambda = \{\rho + \mathbb{Z}\} = \Lambda_+ \sqcup \Lambda_-$ where

$$\Lambda_+ = \{\rho + \mathbb{Z}_{\geq 0}\}, \quad \Lambda_- = \{\rho + \mathbb{Z}_{<0}\}$$

The idea is the same as in the previous case: the only extra thing to do here would be to identify where the sign change happens, and treat the arithmetic progression $\lambda - j$ as two separate progressions Λ_+ and Λ_-. We switch between the bases $\{\psi_+, \psi_-\}$ and $\{\psi'_+, \psi'_-\}$ when going from one progression to the other. This gives two odd dimensional subspaces: in one of them, there is one more $\phi^{(+,i)}$, and in the other space there would be one more $\phi^{(-,i)}$.

4.4.7 Type III λ, Mixing Two Progressions

For type III λ, another obscure type of monodromy invariant solution space arises. This happens when we factorise

$$h(L_x) = (L_x - \lambda^2)^n (L_x - \tilde{\lambda}^2)^m, \tag{4.39}$$

where λ and $\tilde{\lambda}$ are both of type III and $\lambda - \tilde{\lambda} \in \mathbb{Z}$, but they are from two different arithmetic progressions Λ_+ and Λ_-. So for example $\lambda + (g+h)/2 \leq 0$ and $\tilde{\lambda} + (g+h)/2 > 0$.

To illustrate this, we study the following example. Let $W \subset h(L_x)$ be a space generated by the basis $\{\psi'_+ + \alpha\tilde{\psi}_+, \alpha\tilde{\psi}_-\}$, where $\psi'_\pm = c'(\pm\lambda)\mathrm{g}^{-1}\phi_\pm(\lambda)$ and $\tilde{\psi}_\pm = c(\pm\tilde{\lambda})\mathrm{g}^{-1}\phi_\pm(\tilde{\lambda})$. W would be monodromy invariant if its basis functions are monodromy invariant. As usual we abbreviate M_0.

$$M_0(\lambda) = \begin{pmatrix} A(\lambda) & C(\lambda) \\ 0 & D(\lambda) \end{pmatrix}, \quad M_0(\tilde{\lambda}) = \begin{pmatrix} A(\tilde{\lambda}) & 0 \\ B(\tilde{\lambda}) & D(\tilde{\lambda}) \end{pmatrix}.$$

Due to monodromy invariance, we are able to set $A := A(\lambda) = A(\tilde{\lambda})$ and $D := D(\lambda) = D(\tilde{\lambda})$. We check for monodromy invariance.

$$\mathcal{M}_\infty(\psi'_+ + \alpha\tilde{\psi}_+) = e^{2\pi i\lambda}(\psi'_+ + \alpha\tilde{\psi}_+) \in W.$$

$$\mathcal{M}_\infty(\alpha\tilde{\psi}_-) = e^{-2\pi i\lambda}(\alpha\tilde{\psi}_-) \in W.$$

$$\mathcal{M}_0(\psi'_+ + \alpha\tilde{\psi}_+) = \mathcal{M}_0\psi'_+ + \alpha\mathcal{M}_0\tilde{\psi}_+$$

$$\sim \begin{pmatrix} A & C \\ 0 & D \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} (\lambda) + \alpha \begin{pmatrix} A & 0 \\ B & D \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} (\tilde{\lambda}) = \begin{pmatrix} A \\ 0 \end{pmatrix} (\lambda) + \alpha \begin{pmatrix} A \\ B \end{pmatrix} (\tilde{\lambda})$$

$$\sim A(\psi'_+ + \alpha\tilde{\psi}_+) + B(\alpha\tilde{\psi}_-) \in W.$$

$$\mathcal{M}_0(\alpha\tilde{\psi}) = D(\alpha\tilde{\psi}_-) \in W.$$

Therefore, we have discovered the monodromy invariant supspace $W = \text{span}\{\psi'_+ + \alpha\tilde{\psi}_+, \alpha\tilde{\psi}_-\}$ of $\ker(L_x - \lambda^2)(L_x - \tilde{\lambda}^2)$.

Without loss of generality, assume that $\lambda + (g+h)/2$. In general, given an operator of form (4.39), we would have a monodromy invariant solution subspace of the form

$$W = \text{span}\{f_0^+ + g_0^+, g_0^-\} \oplus \text{span}\{f_p^\pm : 1 \le p \le n-1\} \oplus \text{span}\{g_p^\pm : 1 \le p \le n-1\},$$

where

$$f_p^\pm = \sum_{k=0}^{n-1} a_k \binom{k}{p} \left(\frac{\partial}{\partial\lambda}\right)^{k-p} \psi'_\pm, \tag{4.40}$$

$$g_p^\pm = \sum_{k=0}^{m-1} b_k \binom{k}{p} \left(\frac{\partial}{\partial\lambda}\right)^{k-p} \tilde{\psi}_\pm. \tag{4.41}$$

In the above example, $n = m = 1$, $a_0 = 1$ and $b_0 = \alpha$. So

$$f_0^+ = \psi'_+, \quad g_0^+ = \alpha\tilde{\psi}_+, \quad g_0^- = \alpha\tilde{\psi}_- \quad \text{and} \quad W = \text{span}\{f_0^+ + g_0^+, g_0^-\}.$$

Remark: Note that the most general case would involve several factors in (4.39). We will not describe the general case. However, a complete description will follow from our analysis in chapter 5.

4.5 Example: Darboux Transformation of a Fourth Order Operator Factorisation

Here, we bring together everything that we have learnt so far in to a simple example of a non-trivial Darboux factorisation. Suppose we have the polynomial

$$h\big(L_x\big) = \big(L_x - \lambda^2\big)\big(L_x - (\lambda+1)^2\big). \tag{4.42}$$

Here we assume that λ is of type I, that is, we are dealing with the case 4.4.2. The kernel of $h\big(L_x\big)$ is:

$$\ker h\big(L_x\big) = \text{span}\big\{\psi_+ = \psi_+(\lambda), \psi_- = \psi_-(\lambda), \tilde{\psi}_+ = \psi_+(\lambda+1), \tilde{\psi}_- = \psi_-(\lambda+1)\big\}. \tag{4.43}$$

This is a 4 dimensional space. Following (4.31), we propose the following basis functions:

$$f_+ = \alpha\psi_+ + \beta\tilde{\psi}_+, \tag{4.44}$$

$$f_- = \alpha\psi_- + \beta\tilde{\psi}_-. \tag{4.45}$$

Monodromy transformations in this situation are represented by matrices (4.16) and (4.17). Because we only have an integer shift and the monodromy matrices are periodic in $\lambda \mapsto \lambda + 1$, the matrices (4.16) and (4.17) act in the same way on $\tilde{\psi}_\pm$ as they do on $\psi_\pm$. Therefore the basis $\big\{f_+, f_-\big\}$ is monodromy invariant and by theorem 4.4, P is expected to be $\mathbb{Z}_2$-invariant with trigonometric coefficients.

To calculate P, we need to evaluate:

$$P\varphi = \frac{Wr\{f_+, f_-, \varphi\}}{Wr\{f_+, f_-\}} = \frac{1}{\begin{vmatrix} f_+ & f_- \\ \partial_x f_+ & \partial_x f_- \end{vmatrix}} \begin{vmatrix} f_+ & f_- & 1 \\ \partial_x f_+ & \partial_x f_- & \partial_x \\ {\partial_x}^2 f_+ & {\partial_x}^2 f_- & {\partial_x}^2 \end{vmatrix} \varphi. \tag{4.46}$$

To calculate the Wronskians above, we use the following method. Note that the coefficients of ${\partial_x}^2$, ∂_x and the non-derivative term in P are determinants of 2×2 matrices, divided by the 2×2 determinant $Wr\{f_+, f_-\}$.

For different λ's, it would be useful to know how eigenfunctions are related to each other. This is because if we can write $\partial_x^n f_\pm = A\psi_\pm + B\partial_x \psi_\pm$, then all the 2×2 matrices in (4.46) can be factorised in the following way:

$$\begin{pmatrix} {\partial_x}^n f_+ & {\partial_x}^n f_- \\ {\partial_x}^m f_+ & {\partial_x}^m f_- \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \psi_+ & \psi_- \\ \partial_x \psi_+ & \partial_x \psi_- \end{pmatrix} \tag{4.47}$$

where $n, m \geq 0$, and A, B, C and D are coefficients from $\partial_x^n f_\pm = A\psi_\pm + B\partial_x\psi_\pm$ and $\partial_x^m f_\pm = C\psi_\pm + D\partial_x\psi_\pm$. If M_1 and M_2 are two square matrices then $\det\big(M_1 M_2\big) = \det\big(M_1\big)\det\big(M_2\big)$. So $Wr\{\psi_+, \psi_-\}$ would cancel off everywhere, thereby allowing us to compute the operator P. In the matrix factorisation (4.47), $\psi_\pm$ and $\partial_x\psi_\pm$ appear, but there are no occurrences of $\tilde{\psi}_\pm$. Therefore, in order to eliminate $\tilde{\psi}_\pm$ from (4.46), we need the so-called *creation operator* (see [28]).

4.5.1 Creation and Annihilation Operators

We are looking for a differential operator D_+ (depending also on λ) such that

$$D_+\psi_\pm(x, \lambda) = \xi\psi_\pm(x, \lambda + 1). \tag{4.48}$$

Our goal is to prove the following result:

Lemma 4.7. *Let* $D_+ = \sin x\,\partial_x + \lambda\cos x + d$ *where*

$$d = \frac{g\big(g-1\big) - h\big(h-1\big)}{2\big(2\lambda+1\big)}.$$

Then (4.48) *holds with*

$$\xi = \frac{(2\lambda+g+h)(2\lambda+g-h+1)}{2(2\lambda+1)}. \tag{4.49}$$

Proof. This proof will use ideas similar to the ones in theorem 2.1. We begin by showing that $F(x,\lambda)$ from (2.23) satisfies $DF(x,\lambda) = \xi_\pm F(x,\lambda)$. Equation 3.12 from [29] tells us that if $\lambda = n + (g+h)/2$, then

$$D_+ \mathrm{g}^{-1} P_n^{\alpha,\beta}(\cos x) = \frac{2(n+1)(2n+2\alpha+2\beta+2)}{2(2n+\alpha+\beta+2)} \mathrm{g}^{-1} P_{n+1}^{\alpha,\beta}(\cos x), \tag{4.50}$$

where P_n is a Jacobi polynomial and

$$\alpha = g - \frac{1}{2}, \qquad \beta = h - \frac{1}{2}. \tag{4.51}$$

This relation is true for general λ. We also have:

$${}_2F_1(-n, n+\alpha+\beta+1, \alpha+1; \sin^2(x/2)) = \frac{n!}{(\alpha+1)_n} P_n^{\alpha,\beta}(\cos x). \tag{4.52}$$

Substitute (4.51) and (4.52) in to (4.50).

$$D_+F(x,\lambda) = \frac{2(n+1)(2n+2\alpha+2\beta+2)}{2(2n+\alpha+\beta+2)} \mathrm{g}^{-1} \frac{n!}{(\alpha+1)_n} P_{n+1}^{\alpha,\beta}(\cos x)$$

$$= \frac{2(n+1)(2n+2\alpha+2\beta+2)}{2(2n+\alpha+\beta+2)} \mathrm{g}^{-1} \frac{n+\alpha+1}{n+1} \frac{(n+1)!}{(\alpha+1)_{n+1}} P_{n+1}^{\alpha,\beta}(\cos x)$$

$$= \frac{(2n+2\alpha+2)(2n+2\alpha+2\beta+2)}{2(2n+\alpha+\beta+2)} \left[\mathrm{g}^{-1} \frac{(n+1)!}{(\alpha+1)_{n+1}} P_{n+1}^{\alpha,\beta}(\cos x)\right]$$

$$= \frac{(2\lambda+g+h)(2\lambda+g-h+1)}{2(2\lambda+1)} F(x,\lambda+1) := \xi F(x,\lambda+1).$$

Now we translate x by 2π.

$$D_+ F(x+2\pi,\lambda) = \xi F(x+2\pi,\lambda).$$

We also have $D_+(e^{-2\pi i\lambda}F(x,\lambda)) = \xi(e^{-2\pi i\lambda}F(x,\lambda))$. Therefore from (2.28), we would obtain

$$D_+\psi_+ = D_+\left[\frac{F(x+2\pi,\lambda)-e^{-2\pi i\lambda}F(x,\lambda)}{2i\sin(2\pi\lambda)}\right] = \xi\tilde{\psi}_+.$$

Furthermore, we would get

$$D_+\psi_- = D_+\left[\frac{e^{2\pi i\lambda}F(x,\lambda)-F(x+2\pi,\lambda)}{2i\sin(2\pi\lambda)}\right] = \xi\tilde{\psi}_-.$$

■

D_+ is the creation operator for eigenfunctions of the DPT operator. We are going to use this to replace $\psi_\pm(\lambda+1)$ by $\psi_\pm(\lambda)$ and $\partial_x\psi_\pm(\lambda)$.

If $h(L_x)$ was a different operator such that we had $\psi_\pm(\lambda-1)\in\ker h(L_x)$, then we would use *annihilation operator* D_- to replace $\psi_\pm(\lambda-1)$ by $\psi_\pm(\lambda)$ and $\partial_x\psi_\pm(\lambda)$. The annihilation operator is:

$$D_- = -\sin x\partial_x + \lambda\cos x + \frac{g(g-1)-h(h-1)}{2(2\lambda-1)}.$$

The functions $\psi_\pm(\lambda-1)$ would satisfy the following equation:

$$D_-\psi_\pm(x,\lambda) = \frac{(2\lambda+g-h-1)(2\lambda-g+h-1)}{2(2\lambda-1)}\psi_\pm(x,\lambda-1).$$

4.5.2 Calculation of P and Q

In order to express $\partial_x^n f_\pm$ as $A\psi_\pm + B\partial_x\psi_\pm$, we use

- the creation operator D_+ to eliminate $\tilde{\psi}_\pm$ terms, followed by
- ${\partial_x}^2\psi(\lambda) = \big(u-\lambda^2\big)\psi(\lambda)$ to eliminate derivatives of order higher than 1.

Equations of the form $\partial_x^n f_\pm = A\psi_\pm + B\partial_x\psi_\pm$ are calculated below.

$$f_\pm = \alpha\psi_\pm + \beta\tilde{\psi_\pm} = \alpha\psi_\pm + \frac{\beta}{\xi}D_+\psi_\pm$$

$$= \left[\alpha + \frac{\beta}{\xi}(\lambda\cos x + d)\right]\psi_\pm + \left[\frac{\beta}{\xi}\sin x\right]\partial_x\psi_\pm.$$

$$\text{Set } U := \alpha + \frac{\beta}{\xi}(\lambda\cos x + d),\ V := \frac{\beta}{\xi}\sin x. \tag{4.53}$$

$$\partial_x f_\pm = -\frac{\beta}{\xi}\lambda\sin x\psi_\pm + \left[\alpha + \frac{\beta}{\xi}(\lambda\cos x + d)\right]\partial_x\psi_\pm + \frac{\beta}{\xi}\cos x\partial_x\psi_\pm + \frac{\beta}{\xi}\sin x\partial_x^2\psi_\pm$$

$$= -\frac{\beta}{\xi}\lambda\sin x\psi_\pm + \left[\alpha + \frac{\beta}{\xi}((\lambda+1)\cos x + d)\right]\partial_x\psi_\pm + \frac{\beta}{\xi}(u - \lambda^2)\sin x\psi_\pm$$

$$= \left[\frac{\beta}{\xi}(u - \lambda^2 - \lambda)\sin x\right]\psi_\pm + \left[\alpha + \frac{\beta}{\xi}((\lambda+1)\cos x + d)\right]\partial_x\psi_\pm.$$

$$\text{Set } W := \frac{\beta}{\xi}(u - \lambda^2 - \lambda)\sin x,\ X := \alpha + \frac{\beta}{\xi}((\lambda+1)\cos x + d). \tag{4.54}$$

$$\partial_x^2 f_\pm = \partial_x^2[\alpha\psi_\pm(\lambda) + \beta\psi_\pm(\lambda+1)] = \alpha\partial_x^2\psi_\pm(\lambda) + \beta\partial_x^2\psi_\pm(\lambda+1)]$$

$$= \alpha(u - \lambda^2)\psi_\pm(\lambda) + \beta(u - (\lambda+1)^2)\psi_\pm(\lambda+1).$$

$$= \left[\alpha(u - \lambda^2) + \frac{\beta}{\xi}(u - (\lambda+1)^2)(\lambda\cos x + d)\right]\psi_\pm + \left[\frac{\beta}{\xi}(u - (\lambda+1)^2)\sin x\right]\partial_x\psi_\pm.$$

$$\text{Set } Y := \alpha(u - \lambda^2) + \frac{\beta}{\xi}(u - (\lambda+1)^2)(\lambda\cos x + d),\ Z := \frac{\beta}{\xi}(u - (\lambda+1)^2)\sin x. \tag{4.55}$$

Expanding (4.46) gives

$$P = \partial_x^2 - \frac{\begin{vmatrix} f_+ & f_- \\ \partial_x^2 f_+ & \partial_x^2 f_- \end{vmatrix}}{\begin{vmatrix} f_+ & f_- \\ \partial_x f_+ & \partial_x f_- \end{vmatrix}}\partial_x + \frac{\begin{vmatrix} \partial_x f_+ & \partial_x f_- \\ \partial_x^2 f_+ & \partial_x^2 f_- \end{vmatrix}}{\begin{vmatrix} f_+ & f_- \\ \partial_x f_+ & \partial_x f_- \end{vmatrix}}. \tag{4.56}$$

Substitute (4.47) in to (4.56) to get

$$P = \partial_x^2 - \frac{\begin{vmatrix} U & V \\ Y & Z \end{vmatrix}}{\begin{vmatrix} U & V \\ W & X \end{vmatrix}}\partial_x + \frac{\begin{vmatrix} W & X \\ Y & Z \end{vmatrix}}{\begin{vmatrix} U & V \\ W & X \end{vmatrix}}. \tag{4.57}$$

Finally, substitute (4.53 - 4.55) in to (4.57) to obtain

$$P(x, \partial_x) = {\partial_x}^2 + \frac{f_1 \xi \alpha \beta}{\xi^2 \alpha^2 + f_2 \xi \alpha \beta + \sigma \beta^2}\partial_x - \frac{f_3 \xi^2 \alpha^2 + f_4 \xi \alpha \beta + f_5 \sigma \beta^2}{\xi \alpha^2 + f_2 \xi \alpha \beta + \sigma \beta^2}. \tag{4.58}$$

where the functions f_i are the following:

$$f_1 = \big(2\lambda + 1\big) \sin x, \quad f_2 = 2d + (2\lambda + 1) \cos x,$$

$$f_3 = u - \lambda^2, \quad f_4 = u - \big(\lambda + 1\big)^2,$$

$$f_5 = f_3 \cos x + \big(\lambda \cos x + d\big)\big(f_3 + f_4\big).$$

d and σ are the following constants:

$$d = \frac{g\big(g-1\big) - h\big(h-1\big)}{2\big(2\lambda + 1\big)},$$

$$\sigma = \lambda\big(\lambda + 1\big) - \frac{g\big(g-1\big) + h\big(h-1\big)}{2} + d^2.$$

Note here that P has trigonometric coefficients and is $\mathbb{Z}_2$-invariant as expected.

From $h\big(L_x\big) = Q(x, \partial_x) \circ P(x, \partial_x)$, $Q(x, \partial_x)$ can be calculated directly. We find that $Q(x, \partial_x)$ can be written in terms of $P(x, \partial_x)$ in a closed form as follows: define the anti-isomorphism $* : x \mapsto x$ and $\partial_x \mapsto -\partial_x$. Then if we write $P(x, \partial_x) = P(\alpha, \beta)$, then $Q(x, \partial_x) = P^*(\sqrt{\sigma}\beta, \alpha/\sqrt{\sigma})$. It is unclear if such a formula can be written or proved for higher order $h\big(L_x\big)$.

Set $\hat{L}_x = P(\alpha,\beta) \circ P^*(\sqrt{\sigma}\beta, \alpha/\sqrt{\sigma})$. To show that this operator is bispectral, we follow the method in [2]. We use the theorems 3.1 and 3.2.

We recognise that P in (4.58) is of the form $\Theta^{-1}V$ where $\Theta = 2\big(\alpha^2 + \alpha\beta f_2 + \sigma\beta^2\big)$ because $P = {\partial_x}^2 + p_1\partial_x + p_0 = \big({\partial_x}^2 - u\big) + p_1\partial_x + p_0 + u = -L_x + p_1\partial_x + p_0 + u = -\Theta^{-1}\big[\Theta L_x + \alpha\beta(2\lambda+1)\sin x\partial_x - \big(f_3\alpha^2 + f_4\alpha\beta + f_5\sigma\beta^2\big) + \Theta u\big]$. ΘL_x, $\big(f_3\alpha^2+f_4\alpha\beta+f_5\sigma\beta^2\big)$ and Θu are all obviously in $\mathcal{B}$. $\sin x \circ \partial_x$ is also in $\mathcal{B}$ because of lemma 3.8.

Therefore P is of the correct form $\Theta^{-1}V$. Then $Q = P^* = \big(\Theta^{-1}V\big)^* = V^*\big(\Theta^{-1}\big)^* = U\Gamma^{-1}$. Then we get:

$$\hat{L}_x = P \circ Q. \tag{4.59}$$

The eigenvalue for this new operator is $\big(\mu^2 - \lambda^2\big)\big(\mu^2 - \big(\lambda+1\big)^2\big)$. Eigenfunction is

$$\hat{\psi} = P\psi. \tag{4.60}$$

The eigenvalue for the new difference operator $\hat{A}_\mu$ would be

$$\Theta\Gamma = \Gamma^2 = 4\big(\alpha^2 + \alpha\beta f_2 + \sigma\beta^2\big)^2. \tag{4.61}$$

In chapter 6, we will see a generalisation of this example to higher order case.

4.5.3 Concluding Remarks

Comparing the above example with the construction in theorem 4.6 we see that this example constitutes the simplest choice of $\ker P$. However, it is already rather involved. The methods in this chapter do not offer a general algorithm for calculating the Darboux factorisations explicitly. Note that a natural generalisation of the above simplest example would be to take

$$h(L_x) = \prod_{i=1}^{n}(L_x - \lambda_i^2)(L_x - (\lambda_i + 1)^2)$$

for chosen generic $\lambda_1, \dots, \lambda_n$. A factorisation of $h(L_x)$ can be made by choosing $W = \ker P$ in the form

$$W = \mathrm{span}\{f_{i\pm} = \alpha_i \psi_\pm(\lambda_i) + \beta_i \psi_\pm(\lambda_i + 1) : 1 \leq i \leq n\} \tag{4.62}$$

for chosen arbitrary parameters $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_n$.

In chapter 6, we will explain how one can calculate the operator P in this situation. We will see that this problem can be reduced to solving an explicit system of linear equations.

In this chapter, we did not achieve a full classification of all possible Darboux factorisations. In principle, this could have been done using the methods described above. However such an analysis would be rather long and involved. A full classification will be obtained in the next chapter using methods of module theory.

Chapter 5

Module-Theoretic Classification of Darboux Factorisations

In this chapter, we provide a full classification of all possible Darboux factorisations. This is achieved by using the methods of module theory. Module theoretic arguments offer a more elegant reformulation of what we achieved in the previous chapters. They also allow us to formulate the final results in a uniform and compact way.

5.1 Solution Spaces as Modules $S_n(\lambda)$

The methods described in the previous sections provide a complete, algorithmic way to write proofs and perform computations. However, the proofs were quite long and difficult. They involved working with lengthy sums and repeated use of the generalised product rule.

We look for a more concise way to present the monodromy invariant subspaces. We reformulate ideas about types of monodromy subspaces entirely in these simpler terms. The proof of theorem (4.6) works for type I λ as it is. However we still need to provide formal proofs for other types of λ.

Consider the equation

$$h(L_z)\psi = 0, \quad \text{with} \quad h\big(L_z\big) = \prod_{r=1}^{n} \big(L_z - \lambda_r^2\big)^{m_r}, \tag{5.1}$$

where L_z is as in (2.13). Let V denote the space of solutions of (5.1) which are analytic in the neighbourhood of some chosen point $z_0 \neq 0, 1, \infty$. We will usually denote this space as $V = \ker h(L_z)$.

The monodromy representation of (5.1) is completely described by just two transformations $\mathcal{M}_0$ and $\mathcal{M}_\infty$ which correspond to the loops around 0 and ∞. Let us introduce the algebra $\mathcal{A} = \mathbb{C}\langle s, t\rangle$, the free algebra on two letters s and t. The letters s and t represent loops around $z = 0$ and $z = \infty$ repectively (see the figure on page 49). We may view $V = \ker h(L_x)$ as a module over $\mathcal{A}$ with s and t acting by $\mathcal{M}_0$ and $\mathcal{M}_\infty$ respectively. Recall that for a unital algebra $\mathcal{A}$, a vector space V is an $\mathcal{A}$-module if there is an operation (action)

$$\mathcal{A} \times V \to V, \quad \big(a, \phi\big) \mapsto a\phi, \quad \forall a \in \mathcal{A} \text{ and } \forall \phi \in V.$$

This operation needs to satisfy the following properties for all a, $a' \in \mathcal{A}$, ϕ, ϕ_1, $\phi_2 \in V$ and $\mu \in \mathbb{C}$:

1. Associativity: $a\big(a'\phi\big) = \big(aa'\big)\phi$.

2. Linearity: $a\big(\phi_1 + \mu\phi_2\big) = a\phi_1 + \mu a\phi_2$ and $\big(a + \mu a'\big)\phi = a\phi + \mu a'\phi$.

3. Unitality: $1\phi = \phi$.

In this case V is said to be a *left $\mathcal{A}$-module* (elements of V are multiplied with elements in $\mathcal{A}$ on the left to give a new element of V). *Right* $\mathcal{A}$-modules are defined in a similar way. Since we are only working with left $\mathcal{A}$-modules, from now on they would simply be referred to as $\mathcal{A}$-modules.

In this language, the subspaces $W \subset V$ which are monodromy invariant are nothing but *submodules* of V, i.e. subspaces $W \subseteq V$ such that $a\phi \in W$ for all $a \in \mathcal{A}$ and $\phi \in W$.

In this chapter, we reformulate the results of section 4.4 using the language of module theory. We make use of concepts such as module homomorphisms, quotient modules, simple modules, Jordan-Hölder (JH) theorem and the Splitting Lemma to express the module subspaces as composition series. This would give us that the only submodules of V are the ones that we found earlier. To do this, we should first describe all the aforementioned concepts.

Definition 5.1. Let S and T be $\mathcal{A}$-modules. Then a function $f : S \to T$ is called an *$\mathcal{A}$-module homomorphism* if for all $\phi_1, \phi_2 \in S$ and $a_1, a_2 \in \mathcal{A}$,

$$f(a_1\phi_1 + a_2\phi_2) = a_1 f(\phi_1) + a_2 f(\phi_2). \tag{5.2}$$

The property (5.2) of a function f is called *$\mathcal{A}$-linearity*; we say that f is $\mathcal{A}$-linear if it satisfies (5.2). f is an $\mathcal{A}$-module *isomorphism* if there exists another homomorphism $f^{-1} : T \to S$ such that $f \circ f^{-1} = \mathrm{id}_T$ and $f^{-1} \circ f = \mathrm{id}_S$. We write $S \cong T$ to indicate that S and T are isomorphic.

Definition 5.2. Let T be an $\mathcal{A}$-submodule of S. The *quotient module* S/T is the set of cosets of T

$$S/T = \big\{\phi + T : s \in S\big\}$$

with addition defined as $(\phi_1 + T) + (\phi_2 + T) = \phi_1 + \phi_2 + T$ and multiplication defined as $a(\phi_1 + T) = a\phi_1 + T$ for all $\phi_1, \phi_2 \in S$ and $a \in \mathcal{A}$.

Theorem 5.3 (First Isomorphism Theorem)**.** [30] *Let $f : S \longrightarrow R$ be an $\mathcal{A}$-module homomorphism between modules S and R. Then the following are true.*

- *The kernel of f is a submodule of S.*
- *The image of f is a submodule of R.*
- Image $f \cong S/\ker f$.

Definition 5.4. S' is a *proper* submodule of S if $S' \subsetneq S$. S is a *simple* $\mathcal{A}$-module if it has no non-zero proper submodules.

Definition 5.5. Let S and R be $\mathcal{A}$-modules. Then the sum of S and R is the module

$$S + R = \{\phi_s + \phi_r : \phi_s \in S \text{ and } \phi_r \in R\}.$$

$S + R$ is called a *direct sum* if $S \cap R = \{0\}$. If this is the case, then it is denoted by $S \oplus R$.

Definition 5.6. The *socle* of a module is the sum of its simple submodules.

Socle can be expressed as a direct sum of simples because simple submodules either coincide or have a zero intersection.

Definition 5.7. The chain of submodule inclusions

$$0 = S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_n = S$$

is called a *Jordan-Hölder series* if for every i, the *composition factor* S_{i+1}/S_i is a simple quotient module.

In general a Jordan-Hölder series for a given module may not exist. However we will only be dealing with finite dimensional modules, which always admit a Jordan-Hölder decomposition.

Jordan-Hölder Theorem: [30] *Any two Jordan-Hölder series for a module are equivalent; they are of the same length, and their composition factors are the same up to isomorphism and reordering.*

Definition 5.8. A sequence of $\mathcal{A}$-module homomorphisms $f_i : A_i \to A_{i+1}$ is said to be an *exact sequence* if for all i, image(f_i) = kernel(f_{i+1}). Exact sequences of the form

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{5.3}$$

which begin and end with the zero module $0 := \{0\}$ and have $\mathcal{A}$-module homomorphisms $f : A \to B$ and $g : B \to C$ are called *short exact sequences.* f would be injective and g would be surjective.

Definition 5.9. The short exact sequence

$$0 \to A \to B \to C \to 0$$

is said to *split* if $B \cong A \oplus C$.

Splitting Lemma: *Consider the short exact sequence*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0. \tag{5.4}$$

where the arrows represent $\mathcal{A}$-module homomorphisms $f : A \to B$ and $g : B \to C$. The following statements are equivalent [30].

- *There exists a map $\alpha : B \to A$ such that $\alpha \circ f = \mathrm{id}_A$.*
- *There exists a map $\beta : C \to B$ such that $g \circ \beta = \mathrm{id}_C$.*
- *The sequence* (5.4) *splits and $B \cong A \oplus C$. f is an inclusion map $A \hookrightarrow B$ and g is a natural projection of B on to C.*

$\mathcal{A}$-module homomorphisms are linear maps on modules. Let $f : S \to T$ be an $\mathcal{A}$-module homomorphism. Then kernel and image of f are submodules of S and T respectively.

Dimension of $\ker f$ (viewed as a vector space) is denoted by null f, whereas the dimension of image of f is called the *rank* of f. By rank-nullity theorem,

$$\dim_{\mathbb{C}} S = \text{null } f + \text{rank } f.$$

5.2 Uniseriality of $S_n(\lambda)$

In this section, we will prove an important property of our modules. We will show that they are *uniserial*; this means that if R_1 and R_2 are two submodules of a module, then either $R_1 \subseteq R_2$ or $R_2 \subseteq R_1$. To do this, we begin by presenting the proof for case 4.4.1 (the trivial case). Consider the following space:

$$S_n(\lambda) := \ker(L_x - \lambda^2)^n = \text{span}\left\{\frac{\partial^j}{\partial \lambda^j}\phi(\pm\lambda, z) : 0 \leq j \leq n-1\right\}. \tag{5.5}$$

We show that the only submodules of the $\mathcal{A}$-module S_n are S_r where $0 \leq r \leq n$. The method presented below can be modified to work on the rest of the cases as well.

First the following lemmas have to be proved:

Lemma 5.10. *The quotient module S_2/S_1 is isomorphic to S_1.*

Proof. Consider the following sequence.

$$0 \hookrightarrow S_1 \overset{i}{\hookrightarrow} S_2 \overset{l}{\longrightarrow} S_1 \to 0$$

where i is the canonical inclusion and $l = L - \lambda^2$. This is a short exact sequence of $\mathbb{C}$-modules because image$(i) = S_1 = \ker(L - \lambda^2) = \ker(l)$. It is also an $\mathcal{A}$-module because i and l are $\mathcal{A}$-module homomorphisms.

$\mathcal{A}$-linearity of i is obvious. To check $\mathcal{A}$-linearity of l, take any $a \in \mathcal{A}$ and $\phi \in S_1$. Let

$a\phi = A\phi_+ + B\phi_-$ where A and B are functions of λ. $\partial_\lambda \phi \in S_2$ and $l\partial_\lambda \phi = 2\lambda\phi \in S_1$.

$$l \circ a(\partial_\lambda \phi) = l\partial_\lambda(a\phi) = l\partial_\lambda(A\phi_+ + B\phi_-)$$

$$= l(A'\phi_+ + A\partial_\lambda\phi_+ + B'\phi_- + B\partial_\lambda\phi_-)$$

$$= Al\partial_\lambda\phi_+ + Bl\partial_\lambda\phi_- = 2\lambda(A\phi_+ + B\phi_-).$$

$$a \circ l(\partial_\lambda\phi) = a(2\lambda\phi) = 2\lambda(A\phi_+ + B\phi_-).$$

So $l \circ a = a \circ l \ \forall a \in \mathcal{A}$ and l is an $\mathcal{A}$-module homomorphism. By rank-nullity theorem, l is surjective because dimension of S_2 is 4 and $\ker l = S_1$ so nullity of l is 2. Hence the rank of l is also 2 and its image must be S_1 itself.

We can make the kernel of l trivial (and therefore make l injective) by taking the quotient module S_2/S_1. Therefore $l : S_2/S_1 \to S_1$ is an isomorphism and $S_2/S_1 \cong S_1$. ■

Lemma 5.10 generalises to S_k for any k.

Lemma 5.11. *The quotient module S_{k+1}/S_k is isomorphic to S_1 for all $k \in \mathbb{N}$.*

Proof. The proof follows the same pattern as above for the sequence

$$0 \hookrightarrow S_k \overset{i}{\hookrightarrow} S_{k+1} \xrightarrow{l^k} S_1 \to 0.$$

Lemma 3.1 tells us that $\ker l^k = S_k$. By rank-nullity theorem, rank $l^k = 2$ and so the image of l^k is S_1 and it is surjective.

We have to check $\mathcal{A}$-linearity condition $l^k \circ a = a \circ l^k$. Recall the equation (3.8) which tells us that $l^k \partial_\lambda^k \phi = c\phi$ where c is some constant.

$$l^k \circ a(\partial_\lambda^k \phi) = l^k \partial_\lambda^k(a\phi) = l^k \partial_\lambda^k(A\phi_+ + B\phi_-)$$

$$= l^k(A\partial_\lambda^k\phi_+ + B\partial_\lambda^k\phi_- + \text{lower order terms in } S_k) = c(A\phi_+ + B\phi_-).$$

$$a \circ l^k(\partial_\lambda^k\phi) = a(c\phi) = c(A\phi_+ + B\phi_-).$$

So $l^k \circ a = a \circ l^k$. The homomorphism $l^k : S_{k+1}/S_k \to S_1$ would be injective and therefore $S_{k+1}/S_k \cong S_1$. ■

We can further generalise lemma 5.11 in the following way.

Lemma 5.12. *The quotient module S_{k+r}/S_k is isomorphic to S_r for all $k, r \in \mathbb{N}$.*

Proof. The rest of the arguments are the same as above. The calculation for $\mathcal{A}$-linearity is provided here.

Checking $a \circ l^k = l^k \circ a$ is difficult in this case. Indeed, we only really need to confirm $a \circ l = l \circ a$ as this would imply $a \circ l^k = l^k \circ a$. Recall equation (3.6):

$$l\partial_\lambda^p\phi_\pm = 2\lambda p\partial_\lambda^{p-1}\phi_\pm + p(p-1)\partial_\lambda^{p-2}\phi_\pm.$$

$$\implies a \circ l(\partial_\lambda^p\phi) = 2\lambda p\sum_{i=0}^{p-1}\binom{p-1}{i}\Big(A^{(i)}\partial_\lambda^{p-i-1}\phi_+ + B^{(i)}\partial_\lambda^{p-i-1}\phi_-\Big)$$
$$+p(p-1)\sum_{i=0}^{p-2}\binom{p-2}{i}\Big(A^{(i)}\partial_\lambda^{p-i-2}\phi_+ + B^{(i)}\partial_\lambda^{p-i-2}\phi_-\Big).$$

$$l \circ a(\partial_\lambda^p\phi) = \sum_{i=0}^{p}\binom{p}{i}\Big[A^{(i)}\Big(2\lambda(p-i)\partial_\lambda^{p-i-1}\phi_+ + (p-i)(p-i-1)\partial_\lambda^{p-i-2}\phi_+\Big)$$
$$+B^{(i)}\Big(2\lambda(p-i)\partial_\lambda^{p-i-1}\phi_- + (p-i)(p-i-1)\partial_\lambda^{p-i-2}\phi_-\Big)\Big].$$

Here,

$$p\binom{p-1}{i} = p\frac{(p-1)!}{i!(p-i-1)!} = \frac{p!}{i!(p-i-1)!} = \binom{p}{i}(p-i),$$

and

$$p(p-1)\binom{p-2}{i} = p(p-1)\frac{(p-2)!}{i!(p-i-2)} = \frac{p!}{i!(p-i-2)} = \binom{p}{i}(p-i)(p-i-1).$$

Therefore $a \circ l(\partial_\lambda^p \phi) = l \circ a(\partial_\lambda^p \phi)$. ■

Lemma 5.13. *The short exact sequence* $0 \longrightarrow S_1 \longrightarrow S_2 \longrightarrow S_2/S_1 \longrightarrow 0$ *of* $\mathcal{A}$*-module homomorphisms is non-split. That is,* $S_2 \ncong S_1 \oplus \tilde{S}$ *for any submodule* $\tilde{S} \subset S_2$*.*

Proof. By splitting lemma, the short exact sequence

$$0 \longrightarrow S_1 \xrightarrow{f} S_2 \longrightarrow S_2/S_1 \longrightarrow 0 \tag{5.6}$$

is split if and only if there exists a homomorphism $\alpha : S_2 \to S_1$ such that $\alpha \circ f = \mathrm{id}_{S_1}$.

So in other words, to show that the above sequence is not split-exact, we need to show that for all homomorphisms $\alpha : S_2 \to S_1$ and for all homomorphisms $f : S_1 \to S_2$, $\alpha \circ f \neq \mathrm{id}_{S_1}$.

α and f are linear maps. So they can be represented by matrices. f is 4×2 whereas α is 2×4.

First we investigate all possible homomorphisms $f : S_1 \to S_2$. Since they are all linear maps, they automatically satisfy $f(m_1 + m_2) = f(m_1) + f(m_2)\ \forall m_1, m_2 \in S_1$. To be a homomorphism, f further needs to satisfy $\mathcal{A}$-linearity condition $f(a(m)) = a(f(m))\ \forall a \in \mathcal{A}$. This gives us a system of eight linear equations in eight unknowns f_i, where

$$f = \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \\ f_5 & f_6 \\ f_7 & f_8 \end{pmatrix}.$$

To solve this 8×8 linear system, we use the following shortcut: we realise that $\mathcal{A}$ is generated by merely two analytic continuations, which are represented by (4.5) and (4.6). So instead of checking $f \circ a = a \circ f$ for generic $a \in \mathcal{A}$, we instead impose $f \circ M_0 = M_0 \circ f$

and $f \circ M_\infty = M_\infty \circ f$. Since M_∞ is diagonal for λ not of type II, it dramatically simplifies the computation. We comment on the proof for type II λ case at the end.

First note that on S_2,

$$M_\infty = \begin{pmatrix} \mathbf{m}_\infty & \mathbf{m}'_\infty \\ \mathbf{0} & \mathbf{m}_\infty \end{pmatrix} \quad \text{where} \quad \mathbf{m}_\infty = \begin{pmatrix} e^{2\pi i\lambda} & 0 \\ 0 & e^{-2\pi i\lambda} \end{pmatrix}.$$

$$M_0 = \begin{pmatrix} \mathbf{m}_0 & \mathbf{m}'_0 \\ \mathbf{0} & \mathbf{m}_0 \end{pmatrix} \quad \text{where} \quad \mathbf{m}_0 = \begin{pmatrix} A & C \\ B & D \end{pmatrix}.$$

However on S_1, $\mathcal{M}_\infty$ and $\mathcal{M}_0$ are represented by just 2×2 matrices $\mathbf{m}_\infty$ and $\mathbf{m}_0$ respectively.

$$f \times M_\infty = \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \\ f_5 & f_6 \\ f_7 & f_8 \end{pmatrix} \times \begin{pmatrix} e^{2\pi i\lambda} & 0 \\ 0 & e^{-2\pi i\lambda} \end{pmatrix} = \begin{pmatrix} e^{2\pi i\lambda} f_1 & e^{-2\pi i\lambda} f_2 \\ e^{2\pi i\lambda} f_3 & e^{-2\pi i\lambda} f_4 \\ e^{2\pi i\lambda} f_5 & e^{-2\pi i\lambda} f_6 \\ e^{2\pi i\lambda} f_7 & e^{-2\pi i\lambda} f_8 \end{pmatrix}. \tag{5.7}$$

$$M_\infty \times f = \begin{pmatrix} e^{2\pi i\lambda} & 0 & 2\pi i e^{2\pi i\lambda} & 0 \\ 0 & e^{-2\pi i\lambda} & 0 & -2\pi i\lambda e^{-2\pi i\lambda} \\ 0 & 0 & e^{2\pi i\lambda} & 0 \\ 0 & 0 & 0 & e^{-2\pi i\lambda} \end{pmatrix} \times \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \\ f_5 & f_6 \\ f_7 & f_8 \end{pmatrix}$$

$$= \begin{pmatrix} e^{2\pi i\lambda} f_1 + 2\pi i e^{2\pi i\lambda} f_5 & e^{2\pi i\lambda} f_2 + 2\pi i e^{2\pi i\lambda} f_6 \\ e^{-2\pi i\lambda} f_3 - 2\pi i e^{-2\pi i\lambda} f_7 & e^{-2\pi i\lambda} f_4 - 2\pi i e^{-2\pi i\lambda} f_8 \\ e^{2\pi i\lambda} f_5 & e^{2\pi i\lambda} f_6 \\ e^{-2\pi i\lambda} f_7 & e^{-2\pi i\lambda} f_8 \end{pmatrix}. \tag{5.8}$$

Set (5.7) equal to (5.8) and compare entries. The bottom left entry gives

$$e^{2\pi i\lambda} f_7 = e^{-2\pi i\lambda} f_7 \implies 2i \sin(2\pi\lambda) f_7 = 0 \implies f_7 = 0,$$

since $\sin(2\pi\lambda) \neq 0$ because λ is not of type II. Likewise, f_6 is also 0.

The top left entry gives

$$e^{2\pi i\lambda} f_1 = e^{2\pi i\lambda} f_1 + 2\pi i e^{2\pi i\lambda} f_5 \implies 2\pi i e^{2\pi i\lambda} f_5 = 0 \implies f_5 = 0.$$

Similarly, the 22-entry gives us that $f_8 = 0$. Substituting $f_7 = 0$ in 21-entry shows that $f_3 = 0$. Similarly $f_6 = 0$ implies that $f_2 = 0$. So $f \times M_\infty = M_\infty \times f$ tells us that f has the following structure.

$$f = \begin{pmatrix} f_1 & 0 \\ 0 & f_4 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Next, we impose $f \times M_0 = M_0 \times f$.

$$f \times M_0 = \begin{pmatrix} f_1 & 0 \\ 0 & f_4 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} A & C \\ B & D \end{pmatrix} = \begin{pmatrix} Af_1 & Cf_1 \\ Bf_4 & Df_4 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.9}$$

$$M_0 \times f = \begin{pmatrix} A & C & A' & C' \\ B & D & B' & D' \\ 0 & 0 & A & C \\ 0 & 0 & B & D \end{pmatrix} \times \begin{pmatrix} f_1 & 0 \\ 0 & f_4 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} Af_1 & Cf_4 \\ Bf_1 & Df_4 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{5.10}$$

From the explicit formulas of B and C, we can see that $B, C \neq 0$ for type I λ and at least one of B or C is non-zero for type III λ. Then comparing the entries in (5.9) with those in (5.10) gives $f_1 = f_4$. Solving this linear system in the above way leads to the conclusion that all homomorphisms $f : S_1 \to S_2$ must necessarily be inclusion maps up multiplication by $n \in \mathbb{C}$.

$$f = \begin{pmatrix} n & 0 \\ 0 & n \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = n \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = n \begin{pmatrix} I \\ \mathbf{0} \end{pmatrix}. \tag{5.11}$$

The above calculation is for type I λ. For type II λ, we switch to the basis (4.34). With respect to that basis, M_∞ and M_0 reverse their roles: M_0 now gets diagonal blocks, but M_∞ does not. So we would first work with $f \times M_0 = M_0 \times f$, followed by $f \times M_\infty = M_\infty \times f$.

Same kind of method as above shows that α must be a canonical projection map of S_2 on to S_1 up to some multiplier.

$$\alpha = \begin{pmatrix} 0 & 0 & m & 0 \\ 0 & 0 & 0 & m \end{pmatrix} = m \begin{pmatrix} \mathbf{0} & I \end{pmatrix}. \tag{5.12}$$

Finally we compute $\alpha \circ f$ using (5.11) and (5.12).

$$\alpha \circ f = \begin{pmatrix} 0 & 0 & m & 0 \\ 0 & 0 & 0 & m \end{pmatrix} \begin{pmatrix} n & 0 \\ 0 & n \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.13}$$

So for all n and m, $\alpha \circ f = \mathbf{0}$ and never id_{S_1}.

Therefore there exists no homomorphism $\alpha : S_2 \to S_1$ for which $\alpha \circ f = \mathrm{id}_{S_1}$. By the contrapositive of the splitting lemma, the short exact sequence (5.6) does not split. ■

Lemma 5.13 can be generalised as follows.

Lemma 5.14. *More generally, the short exact sequence*

$$0 \longrightarrow S_{k+1}/S_k \longrightarrow S_{k+2}/S_k \longrightarrow (S_{k+2}/S_k) \,/\, (S_{k+1}/S_k) \longrightarrow 0, \tag{5.14}$$

is non-split.

Proof. Assume towards contradiction: suppose (5.14) splits.

$$S_{k+2}/S_k = S_{k+1}/S_k \oplus \tilde{S}/S_k.$$

By lemmas 5.11 and 5.12,

$$S_{k+1}/S_k \cong S_1 \quad \text{and} \quad S_{k+2}/S_k \cong S_2. \tag{5.15}$$

So

$$S_2 \cong S_{k+1}/S_k \oplus \tilde{S}/S_k \cong S_1 \oplus \hat{S}. \tag{5.16}$$

So our assumption would cause the sequence (5.6) to split, which contradicts lemma (5.13). Therefore (5.14) does not split. ∎

With the above lemmas, we now prove the result of section 4.4.1.

Theorem 5.15. *For type I λ, the only submodules of S_n are S_r where $r = 0, 1, 2, ..., n$.*

Proof. We prove this by induction on n.

<u>Induction Base</u>: When $n = 1$, $S_1 = \ker\left(L_x - \lambda^2\right)$. S_1 is a simple module (because the monodromy group is irreducible). So the only submodules are $S_0 = \{0\}$ (or just 0) and S_1 itself.

When $n = 2$, we have $0 \subsetneq S_1 \subsetneq S_2$. Let $N \subseteq S_2$ be a submodule. Consider $N \cap S_1 \subseteq S_1$. Since S_1 is a simple module and $N \cap S_1$ is a submodule of S_1, $N \cap S_1 = 0$ or $N \cap S_1 = S_1$

If $N \cap S_1 = 0$, then set $N' = N \oplus S_1 \subseteq S_2$. So we have a composition series $0 \subsetneq S_1 \subseteq N' = N \oplus S_1 \subseteq S_2$. By JH theorem, all composition series are the same up to isomorphisms and ordering. So either $N \oplus S_1 = S_1$, which implies $N = 0$, or $N \oplus S_1 = S_2$, which means that the short exact sequence $0 \longrightarrow S_1 \longrightarrow S_2 = N \oplus S_1 \longrightarrow S_2/S_1 \longrightarrow 0$ splits. This contradicts lemma 5.13, so $N = 0$ is the only possibility.

If on the other hand $N \cap S_1 = S_1$, then $S_1 \subseteq N$. Thus we get the composition series

$0 \subseteq S_1 \subseteq N \subseteq S_2$. By JH theorem, either $N = S_1$ or $N = S_2$.

Induction Hypothesis: Suppose all submodules of S_n (for a given natural number n) are of the form S_r where $0 \leq r \leq n$.

Induction Step: We are looking for submodules of S_{n+1}. Let $N \subseteq S_{n+1}$. Consider $N \cap S_n \subseteq S_n$. Due to the induction hypothesis, $N \cap S_n = S_r$ for some $0 \leq r \leq n$. We would have three cases.

Case 1: $r = 0$. So $N \cap S_n = S_0 = 0$. Let $N' = N \oplus S_n \subseteq S_{n+1}$. We get a composition series $0 \subset S_1 \subset S_2 \subset ... \subset S_n \subseteq N' = N \oplus S_n \subseteq S_{n+1}$. Then either $N \oplus S_n = S_n$ or $N \oplus S_n = S_{n+1}$.

If $N \oplus S_n = S_n$, then $N = 0$. So that is fine.

If $N \oplus S_n = S_{n+1}$, then the short exact sequence $0 \longrightarrow S_n \longrightarrow S_{n+1} = N \oplus S_n \longrightarrow S_{n+1}/S_n \longrightarrow 0$ splits, which contradicts lemma 5.13. So $n = 0$ is the only possibility.

Case 2: $r = n$. So $N \cap S_n = S_n$, which implies that $S_n \subset N$. We obtain the composition series: $0 \subset S_1 \subset S_2 \subset ... \subset S_n \subseteq N \subseteq S_{n+1}$. By JH theorem, either $N = S_n$ or $N = S_{n+1}$, and we are done.

Case 3: $0 < r < n$. So $N \cap S_n = S_r$. Take quotients by S_r : $\big(N \cap S_n\big)/S_r = \big(N/S_r\big) \cap \big(S_n/S_r\big) = 0$.

We have that $N/S_r \subseteq S_{n+1}/S_r \cong S_{n+1-r}$ and $S_n/S_r \cong S_{n-r}$. Consider $\big(N/S_r\big) \oplus \big(S_n/S_r\big)$.

$$0 \subset S_{r+1}/S_r \subset S_{r+2}/S_r \subset ... \subset S_n/S_r \subseteq \big(N/S_r\big) \oplus \big(S_n/S_r\big) \subseteq S_{n+1}/S_r.$$

By lemma 4.2.1, this composition series is equivalent to

$$0 \subset S_1 \subset S_2 \subset ... \subset S_{n-r} \subseteq \hat{N} \subseteq S_{n+1-r},$$

where $\hat{N}$ is some submodule of S_{n+1-r} that is isomorphic to $\big(N/S_r\big) \oplus \big(S_n/S_r\big)$.

By JH theorem, we get that either $\hat{N} = S_{n-r}$ or $\hat{N} = S_{n+1-r}$.

If $\hat{N} = S_{n-r}$, then $\big(N/S_r\big) \oplus \big(S_n/S_r\big) = S_n/S_r$. So for all $n \in \mathbb{N}$ and $s \in S_n$, $\big(n + S_r\big) + \big(s + S_r\big) = n + s + S_r \in S_n/S_r$. This implies that $n \in S_n$. But since $N \cap S_n = S_r$, $n \in S_r$.

So $N \subseteq S_r$. We already have $N \cap S_n = S_r$ which implies that $S_r \subseteq N$. Therefore $N = S_r$.

On the other hand, if $\hat{N} = S_{n+1-r}$, then $\big(N/S_r\big) \oplus \big(S_n/S_r\big) = S_{n+1}/S_r$. We obtain the short exact sequence:

$$0 \longrightarrow S_n/S_r \longrightarrow \big(N/S_r\big) \oplus \big(S_n/S_r\big) = S_{n+1}/S_r \longrightarrow \big(S_{n+1}/S_r\big)/S_n/S_r \longrightarrow 0.$$

By lemma 5.12 this is equivalent to:

$$0 \longrightarrow S_{n-r} \longrightarrow S_{n+1-r} \cong \big(N/S_r\big) \oplus \big(S_n/S_r\big) \longrightarrow S_{n+1-r}/S_n \longrightarrow 0.$$

This sequence splits, which violates lemma 5.13. Therefore, the only possibility is $N = S_r$. ■

We have shown that the only submodules of S_n are S_r where $0 \leq r \leq n$. In general, $S_q \subsetneq S_r$ for all $q < r$. This reveals the following important property of our modules.

Definition 5.16. A module is said to be *uniserial* if for any two of its submodules N_1 and N_2, either $N_1 \subseteq N_2$ or $N_2 \subseteq N_1$.

Definition 5.17. A module is called a *serial* module if it is a direct sum of uniserial modules.

For type II lambda (section 4.4.4), resonance happens. So we have to use the basis $\psi_+ + \psi_-, \psi'_+ + \psi'_-$ instead of ψ_+, ψ_-. S_1 is still a simple module. So the decomposition remains:

$$0 \subsetneq S_1 \subsetneq S_2 \subsetneq S_3 \subsetneq ... \subsetneq S_{n-1} \subsetneq S_n.$$

M_∞ is no longer diagonal; M_0 however is diagonal in this basis. We have established that the homomorphisms $S_1 \to S_2$, respectively $S_2 \to S_1$ are inclusions, respectively projections, as a result of one monodromy matrix being diagonal and the other one being non-diagonal.

So the analogue of lemma 5.13 is proved in the same way for $2\lambda \in \mathbb{Z}$ case as it is for non-special λ. The proof of uniseriality of S_n would then be the same.

Theorem 5.18. *For type II λ, the only submodules of S_n are S_r where $r = 0, 1, 2, ..., n$.*

For type III λ (4.4.5), where there is an odd number of basis functions for $\ker P$, we introduce the notation $S_{r+1/2}$ for submodules which have such a basis. Without loss of generality, suppose $-\lambda + \frac{g+h}{2}$ is a non-positive integer, resulting in upper triangular monodromy matrices. We use the following notation:

$$S_{n+\frac{1}{2}} = \left\{ \frac{\partial^k}{\partial \lambda^k} \psi_+(\lambda, x) : 1 \le k \le n \right\} \bigoplus \left\{ \frac{\partial^k}{\partial \lambda^k} \psi_+(\lambda, x) : 1 \le k \le n-1 \right\} \tag{5.17}$$

$$R_{\frac{1}{2}} = S_1 / S_{\frac{1}{2}} = \left\{ a\psi_- + S_{\frac{1}{2}} : a \in \mathbb{C} \right\} \tag{5.18}$$

$$R_1 = S_{\frac{3}{2}} / S_{\frac{1}{2}} = \left\{ a_-\psi_- + a_+\partial_\lambda \psi_+ + S_{\frac{1}{2}} : a \in \mathbb{C} \right\} \tag{5.19}$$

$$R_{\frac{3}{2}} = S_2 / S_{\frac{1}{2}} = \left\{ b\psi_- + a_+\partial_\lambda \psi_+ + a_-\partial_\lambda \psi_- + S_{\frac{1}{2}} : a \in \mathbb{C} \right\} \tag{5.20}$$

We need to prove a number of technical results.

Lemma 5.19. *The sequence* $0 \longrightarrow S_1 \longrightarrow S_2 \longrightarrow S_2/S_1 \longrightarrow 0$ *is non-split.*

Proof. The proof follows the same pattern as for lemma 4.2.3. The only difference is, the matrix for analytic continuation around $z = 0$ is upper triangular. ■

These next lemma lists all the short exact sequences which do not split. These are needed to prove the uniseriality of S_n. Their proofs are all similar; only the proof for sequence 1 is provided as an example.

Lemma 5.20. *None of the following short exact sequences split.*

Sequence 1. $0 \longrightarrow S_1 \longrightarrow S_{\frac{3}{2}} \longrightarrow S_{\frac{3}{2}}/S_1 \longrightarrow 0.$

Sequence 2. $0 \longrightarrow S_{\frac{1}{2}} \longrightarrow S_1 \longrightarrow R_{\frac{1}{2}} \longrightarrow 0.$

Sequence 3. $0 \longrightarrow R_{\frac{1}{2}} \longrightarrow R_1 \longrightarrow R_1/R_{\frac{1}{2}} \longrightarrow 0.$

Sequence 4. $0 \longrightarrow R_{\frac{1}{2}} \longrightarrow R_{\frac{3}{2}} \longrightarrow R_{\frac{3}{2}}/R_{\frac{1}{2}} \longrightarrow 0.$

Sequence 5. $0 \longrightarrow R_1 \longrightarrow R_{\frac{3}{2}} \longrightarrow R_{\frac{3}{2}}/R_1 \longrightarrow 0.$

Sequence 6. $0 \longrightarrow R_1/R_{\frac{1}{2}} \longrightarrow R_{\frac{3}{2}}/R_{\frac{1}{2}} \longrightarrow \Big(R_{\frac{3}{2}}/R_{\frac{1}{2}}\Big)/\Big(R_1/R_{\frac{1}{2}}\Big) \longrightarrow 0.$

Proof for Sequence 1. $S_{\frac{3}{2}} = \text{span}\Big\{\psi_+, \psi_-, \partial_\lambda \psi_+\Big\}.$

Using $\mathcal{A}$-linearity, we find that homomorphisms $f : S_1 \to S_{\frac{3}{2}}$ are inclusions of the form

$$f = n \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{5.21}$$

Moreover, the homomorphisms $\alpha : S_{\frac{3}{2}} \to S_1$ are projections of the form

$$\alpha = m \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.22}$$

$$\implies \alpha \circ f \propto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \neq I_{S_1}. \tag{5.23}$$

Therefore by the contrapositive of the splitting lemma, the sequence

$$0 \longrightarrow S_1 \longrightarrow S_{\frac{3}{2}} \longrightarrow S_{\frac{3}{2}}/S_1 \longrightarrow 0$$

does not split. ■

We also need to prove some lemmas about submodules of modules with small dimensions.

Lemma 5.21. *The only submodules of* S_1 *are* $S_0 = 0, S_{\frac{1}{2}}$ *and* S_1.

Proof. Let N be a submodule of S_1. Consider $N \cap S_{\frac{1}{2}} \subseteq S_{\frac{1}{2}}$. Since $S_{\frac{1}{2}}$ is simple, $N \cap S_{\frac{1}{2}}$ must be simple.

$$\implies N \cap S_{\frac{1}{2}} = 0 \text{ or } N \cap S_{\frac{1}{2}} = S_{\frac{1}{2}}.$$

If $N \cap S_{\frac{1}{2}} = 0$, then we obtain the composition series $0 \subsetneq S_{\frac{1}{2}} \subsetneq N \oplus S_{\frac{1}{2}} \subsetneq S_1$.

By JH theorem, all composition series are the same up to isomorphisms and ordering. So either $N \oplus S_{\frac{1}{2}} = S_{\frac{1}{2}}$ or $N \oplus S_{\frac{1}{2}} = S_1$.

- If $N \oplus S_{\frac{1}{2}} = S_{\frac{1}{2}}$, then $N = 0$, which is fine.
- If $N \oplus S_{\frac{1}{2}} = S_1$, then $0 \longrightarrow S_{\frac{1}{2}} \longrightarrow S_1 = N \oplus S_{\frac{1}{2}} \longrightarrow R_{\frac{1}{2}} \longrightarrow 0$ splits. This is a contradiction as it would split sequence 2 in lemma 5.20.

Therefore, if $N \cap S_{\frac{1}{2}} = 0$, then N necessarily has to be 0.

If on the other hand $N \cap S_{\frac{1}{2}} = S_{\frac{1}{2}}$ then $S_{\frac{1}{2}} \subseteq N$. Hence we get the composition series $0 \subsetneq S_{\frac{1}{2}} \subseteq N \subseteq S_1$. By JH theorem, either $N = S_{\frac{1}{2}}$ or $N = S_1$. ■

Lemma 5.22. *The only submodules of S_{i+1}/S_i for all i are 0, $S_{i+\frac{1}{2}}/S_i$ and S_{i+1}/S_i.*

Proof. Let $N \subseteq S_{i+1}$.

$$N/S_i \cap S_{i+\frac{1}{2}}/S_i \subseteq S_{i+\frac{1}{2}} \cong S_{\frac{1}{2}}, \text{ which is simple.}$$

So either $N/S_i \cap S_{i+\frac{1}{2}}/S_i = 0$ or $S_{i+\frac{1}{2}}/S_i$.

If $N/S_i \cap S_{i+\frac{1}{2}}/S_i = 0$, then

$$0 \subsetneq S_{i+\frac{1}{2}}/S_i \subseteq N/S_i \oplus S_{i+\frac{1}{2}}/S_i \subseteq S_{i+1}/S_i.$$

So either $N/S_i \oplus S_{i+\frac{1}{2}} = S_{i+\frac{1}{2}}/S_i \implies N/S_i = 0$, or $N/S_i \oplus S_{i+\frac{1}{2}} = S_{i+1}/S_i$, in which case the short exact sequence

$$0 \longrightarrow S_{i+\frac{1}{2}}/S_i \longrightarrow S_{i+1}/S_i = N/S_i \oplus S_{i+\frac{1}{2}}/S_i \longrightarrow R_{\frac{1}{2}} \longrightarrow 0$$

splits, which is not allowed to happen because of lemma 5.20, sequence 2. So $N/S_i = 0$.

Alternatively if $N/S_i \cap S_{i+\frac{1}{2}}/S_i = S_{i+\frac{1}{2}}/S_i$ then $S_{i+\frac{1}{2}}/S_i \subseteq N/S_i$

By JH theorem, $N/S_i = S_{i+\frac{1}{2}}/S_i$ or $N/S_i = S_{i+1}/S_i$. ■

Lemma 5.23. *If $S_i \subseteq S \subseteq S_{i+1}$, then $S = S_i, S_{i+\frac{1}{2}}$ or S_{i+1}.*

Proof. Consider S/S_i. By lemma 5.22, $S/S_i = 0, S_{i+\frac{1}{2}}/S_i$ or S_{i+1}/S_i.

If $S/S_i = 0$, then $S \subseteq S_i \subseteq S$. So $S = S_i$.

If $S/S_i = S_{i+\frac{1}{2}}/S_i$ (or S_{i+1}/S_i), then $\forall p \in S$, $\exists m \in S_{i+\frac{1}{2}}$ (or S_{i+1}) such that $p - m \in S_i \subseteq S_{i+\frac{1}{2}}$ (or S_{i+1}) which implies that $m + p - m = p \in S_{i+\frac{1}{2}}$ (or S_{i+1}). So $S \subseteq S_{i+\frac{1}{2}}$ (or S_{i+1}).

Also, $\forall m \in S_{i+\frac{1}{2}}$ (or S_{i+1}), $\exists p \in S$ such that $m - p \in S_i \subseteq S$. So $p + m - p = m \in S$. So $S_{i+\frac{1}{2}}$ (or S_{i+1}) $\subseteq S$. Therefore $S = S_{i+\frac{1}{2}}$ or S_{i+1}. ■

Lemma 5.24. *If $S_{i-\frac{1}{2}} \subseteq S \subseteq S_{i+\frac{1}{2}}$, then $S = S_{i-\frac{1}{2}}, S_i$ or $S_{i+\frac{1}{2}}$.*

Proof. Similar to lemma 5.22, we would show first that the only submodules of $S_{i+\frac{1}{2}}/S_{i-\frac{1}{2}}$ are $0, S_i/S_{i-\frac{1}{2}}$ and $S_{i+\frac{1}{2}}/S_{i-\frac{1}{2}}$.

Subsequently, the method for proving lemma 5.23 would be used to complete the rest of the proof. ■

With all of the above, we can prove the following result.

Theorem 5.25. *For type III λ, the only submodules of S_n are $S_{\frac{1}{2}r}$ where $r = 0, 1, 2, ..., 2n$.*

Proof. We use induction on n.

Induction Base: This is shown in lemma 5.21.

Induction Hypothesis: Suppose the claimed result is true for some specific $n \in \mathbb{N}$, that is, the only submodules of S_n are $S_{\frac{1}{2}r}$ where $r = 0, 1, 2, ..., 2n$.

<u>Induction Step</u>: Let $N \subseteq S_{n+1}$. $N \cap S_n \subseteq S_n$, so by induction hypothesis, $N \cap S_n = S_{\frac{1}{2}r}$ for some r.

Case 1: $r = 0$. $N \cap S_n = 0$. $S_n \subseteq N \oplus S_n \subseteq S_{n+1}$.

By lemma 5.23,

- either $N \oplus S_n = S_n \implies N = S_0$, which is fine,
- or $N \oplus S_n = S_{n+\frac{1}{2}} \implies 0 \to S_n \to S_{n+\frac{1}{2}} \to S_{n+\frac{1}{2}}/S_n \to 0$ splits,
- or $N \oplus S_n = S_{n+1} \implies 0 \to S_n \to S_{n+1} \to S_{n+1}/S_n \to 0$ splits.

So $N = S_0$.

Case 2: $r = 2n$. $N \cap S_n = S_n$. So $S_n \subseteq N \subseteq S_{n+1}$. By lemma 5.23, $N = S_n, S_{n+\frac{1}{2}}$ or S_n.

Case 3.1: $0 < r < n$, r is a whole number.

$$S_n \cap N = S_r \implies N/S_r \cap S_n/S_r = 0.$$

$$\begin{array}{ccccccc} S_n/S_r & \subseteq & S_n/S_r & \oplus & N/S_r & \subseteq & S_{n+1}/S_r. \\ \wr\!\| & & \wr\!\| & & \wr\!\| & & \wr\!\| \\ S_{n-r} & \subseteq & S_{n-r} & \oplus & \hat{N} & \subseteq & S_{n+1-r}. \end{array}$$

So by lemma 5.23, either

- $S_{n-r} \oplus \hat{N} = S_{n-r} \implies \hat{N} = 0 \implies N/S_r = 0 \implies N \subseteq S_r \subseteq N$ giving us $N = S_r$, which is fine,
- or $S_{n-r} \oplus \hat{N} = S_{n-r+\frac{1}{2}} \implies 0 \to S_{n-r} \to S_{n-r+\frac{1}{2}} \to S_{\frac{1}{2}} \to 0$ splits,
- or $S_{n-r} \oplus \hat{N} = S_{n+1-r} \implies 0 \to S_{n-r} \to S_{n-r+1} \to S_1 \to 0$ splits.

So $N = S_r$.

Case 3.2: $N \cap S_n = S_{r-\frac{1}{2}}$, r is a whole number.

$$\implies N/S_{r-\frac{1}{2}} \cap S_n/S_{r-\frac{1}{2}} = 0.$$

$$\begin{array}{ccccccc} S_n/S_{r-\frac{1}{2}} & \subseteq & S_n/S_{r-\frac{1}{2}} & \oplus & N/S_{r-\frac{1}{2}} & \subseteq & S_{n+1}/S_{r-\frac{1}{2}}. \\ \cong & & \cong & & \cong & & \cong \\ S_{n-r+\frac{1}{2}} & \subseteq & S_{n-r+\frac{1}{2}} & \oplus & \hat{N} & \subseteq & S_{n+\frac{3}{2}-r}. \end{array}$$

By lemma 5.24, either

- $\hat{N} \oplus S_{n-r+\frac{1}{2}} = S_{n-r+\frac{1}{2}} \implies \hat{N} = 0 \implies N \subseteq S_{r-\frac{1}{2}} \subseteq N$, which means $N = S_{r-\frac{1}{2}}$, which is fine,
- or $\hat{N} \oplus S_{n-r+\frac{1}{2}} = S_{n-r}$ which would result in the contradiction of lemma 3,
- or $\hat{N} \oplus S_{n-r+\frac{1}{2}} = S_{n-r+\frac{3}{2}}$ which would result in the contradiction of lemma 4.

Therefore the only possibility is $N = S_{r-\frac{1}{2}}$. ■

5.3 Homomorphism Spaces

Thus far, we have only dealt with composition series of modules with a single λ. For non-trivial Darboux factorisations, we need to mix different values of λ. The rest of this chapter will be devoted to figuring out what kind of submodules are allowed.

We will eventually show in the next subsection that given a general module of the form

$$M_k = \bigoplus_{\lambda \in \Lambda} S_k(\lambda),$$

where $\Lambda \in \mathbb{C}$ is a finite set, all of its submodules will be isomorphic to a canonical serial module.

$$N \subset \bigoplus_{\lambda \in \Lambda} S_k(\lambda) \implies N \cong \bigoplus_{\lambda \in \Lambda} S_{l(\lambda)}(\lambda), \quad 1 \leq l(\lambda) \leq k.$$

We will also study the isomorphism $N \cong \bigoplus_{\lambda\in\Lambda} S_{l(\lambda)}(\lambda)$. We will describe all possible submodules $N \subset M_k$ which are isomorphic to a given canonical submodule $\bigoplus_{\lambda\in\Lambda} S_{l(\lambda)}(\lambda)$. The two aforementioned results will give us a complete description of all possible submodules of $h(L_x)$ which will give us bispectral Darboux factorisations.

To prove the above structural results, we must understand the object $\mathrm{Hom}_{\mathcal{A}}(S, R)$, which is the set of all $\mathcal{A}$-module homomorphisms from S to R.

Let $S' = \bigoplus_{\lambda\in\Lambda} S_{l(\lambda)}(\lambda) \subset M_k$. Any N isomorphic to S' must be related to S' by an injective homomorphism, which is an injective element of $\mathrm{Hom}_{\mathcal{A}}(S', M_k)$. So the problem is: for any given $S' \subset M_k$, find the set of all injective elements of $\mathrm{Hom}_{\mathcal{A}}(S', M_k)$. This in turn requires us to determine what the set $\mathrm{Hom}_{\mathcal{A}}(S', M_k)$ itself looks like. To describe such sets explicitly, we need the *lowering map.*

Definition 5.26. In the short exact sequence $0 \to S_1 \to S_2 \overset{\varphi}{\to} S_1 \to 0$, φ satisfies $\ker\varphi = \text{image } \varphi = S_1$. We call this is the *lowering map* because it lowers multiplicity: $\ker(L-\lambda^2)^2 \to \ker(L-\lambda^2)^1$. We will eventually find that it reduces multiplicity at any level, so $\ker(L-\lambda^2)^n \to \ker(L-\lambda^2)^{n-1}$ for all n.

Lemma 5.27. *For all λ, all surjective module homomorphisms from S_2 to S_1 are lowering maps up to a multiple. That is, if $h : S_2 \to S_1$ with* $\ker h = S_1$ *and* image $h = S_1$, *then $h = u\varphi$ where $u \in \mathbb{C}$.*

Proof. φ is the composition $f \circ q$, where q is the quotient map $q : S_2 \to S_2/S_1$, $q(a) = a + S_1$ for all $a \in S_2$ and $f : S_2/S_1 \to S_1$ is an isomorphism.

To show that $h : S_2 \to S_1$ is a lowering map, we need to show that it can be expressed as $h = uf \circ q$, where $u \in \mathbb{C}$.

We know that image $h = S_1$. Take any $\phi \in$ image $h = S_1$ with $\phi \neq 0$. It's pre-image under h, namely $h^{-1}(\phi) \notin S_1$. This is because $S_1 = \ker h$. So if $h^{-1}(\phi) \in S_1$, then $h^{-1}(\phi) \in \ker h$ so $hh^{-1}(\phi) = \phi = 0$.

So for all the pre-images we can form non-trivial cosets $h^{-1}(\phi) + S_1 \in S_2/S_1$.

The space of pre-images is at least as big as image $h = S_1$. So the space of cosets is at least as big as $S_1 \cong S_2/S_1$. Therefore the space of cosets is S_2/S_1 and the formation of these cosets is exactly the map $q : S_2 \to S_2/S_1$. For all $\psi \in S_2$, there exists a $\phi \in S_1$ with $h(\psi) = \phi$. So $\psi = h^{-1}(\phi)$. $q(\psi) = q(h^{-1}(\phi)) = h^{-1}(\phi) + S_1$.

Then take an isomorphism $f : S_2/S_1 \to S_1$ such that for all $\phi \in \text{im } h$, $f(h^{-1}(\phi)+S_1) = \phi$. Such an isomorphism is guaranteed to exist thanks to lemma 5.10. Then $h \propto f \circ q = \varphi$. ■

To understand how to describe subsets of $\text{Hom}_{\mathcal{A}}$, let us begin with the simplest non-trivial example of S_2. We know that the only submodules of S_2 for type I and II λ are 0, S_1 and S_2. Lemma 5.13 tells us that the only homomorphisms from S_1 to S_2 are injective inclusions. The set of those inclusions can be written as

$$\text{Hom}_{\mathcal{A}}(S_1, S_2) = \{u \,\text{id} : u \in \mathbb{C}\}.$$

Proposition 5.28. *For type I or II λ,*

$$\text{Hom}_{\mathcal{A}}(S_2(\lambda), S_2(\lambda)) = \{u_0 \,\text{id} + u_1 \varphi : u_0, u_1 \in \mathbb{C}\},$$

where $\varphi : S_2 \to S_2$, $\varphi(S_2) = S_1$ and $\ker \varphi = S_1$.

Proof. $\text{Hom}_{\mathcal{A}}(N, M)$ is the space of all $\mathcal{A}$-module homomorphisms from N to M. Let $h \in \text{Hom}_{\mathcal{A}}(S_2(\lambda), S_2(\lambda))$. Since h is a linear map, it can be represented by a matrix H. Let $\{\psi_+, \psi_-\}$ be a basis of $S_2(\lambda)$. Then,

$$M_\infty = e^{2\pi i\rho} \begin{bmatrix} e^{2\pi i\lambda} & 0 & 2\pi i e^{2\pi i\lambda} & 0 \\ 0 & e^{-2\pi i\lambda} & 0 & -2\pi i e^{-2\pi i\lambda} \\ 0 & 0 & e^{2\pi i\lambda} & 0 \\ 0 & 0 & 0 & e^{-2\pi i\lambda} \end{bmatrix} := \begin{bmatrix} \mathbf{m}_\infty & \mathbf{m}'_\infty \\ 0 & \mathbf{m}_\infty \end{bmatrix}.$$

$$M_0 = \begin{bmatrix} A & C & A' & C' \\ B & D & B' & D' \\ 0 & 0 & A & C \\ 0 & 0 & B & D \end{bmatrix} := \begin{bmatrix} \mathbf{m}_0 & \mathbf{m}_0' \\ \mathbf{0} & \mathbf{m}_0 \end{bmatrix}.$$

Let

$$H := \begin{bmatrix} \mathbf{H}_1 & \mathbf{H}_2 \\ \mathbf{H}_3 & \mathbf{H}_4 \end{bmatrix},$$

where H_i are 2×2 blocks. We impose the $\mathcal{A}$-linearity conditions $HM_\infty = M_\infty H$ and $HM_0 = M_0 H$. This 16×16 system can be solved with a calculation similar to what was done in lemma 5.13. The result is

$$H = \begin{bmatrix} u_0 I & u_1 I \\ \mathbf{0} & u_0 I \end{bmatrix} = u_0 \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} + u_1 \begin{bmatrix} \mathbf{0} & I \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

The first matrix obviously corresponds to $\mathrm{id} \in \mathrm{Hom}_{\mathcal{A}}(S_2(\lambda), S_2(\lambda))$. The second matrix corresponds to the homomorphism

$$a_0^+\psi_+ + a_0^-\psi_- + a_1^+\partial_\lambda\psi_+ + a_1^-\partial_\lambda\psi_- \mapsto a_1^+\psi_+ + a_1^-\psi_-.$$

Therefore it represents a map $\varphi \in \mathrm{Hom}_{\mathcal{A}}(S_2(\lambda), S_2(\lambda))$ with $\varphi(S_2) = S_1$, and $\ker\varphi = S_1$.

Thus $\mathrm{Hom}_{\mathcal{A}}(S_2(\lambda), S_2(\lambda)) = \{u_0\mathrm{id} + u_1\varphi : u_0, u_1 \in \mathbb{C}\}$. ■

We know $\mathrm{Hom}_{\mathcal{A}}(S_1, (\lambda), S_1(\lambda))$ and $\mathrm{Hom}_{\mathcal{A}}(S_2, (\lambda), S_2(\lambda))$. We would like to know what $\mathrm{Hom}_{\mathcal{A}}(S_n(\lambda), S_n(\lambda))$ is for any n. To this end, we must extend the lowering map φ to larger spaces.

Theorem 5.29. *Let* $\varphi : S_n \to S_n$ *with* $\ker\varphi = S_1$. *Then for all* $1 \le k \le n$, $\varphi^k : S_n \to S_n$ *has* $\varphi^k(S_n) = S_{n-k}$ *and* $\ker\varphi^k = S_k$.

Proof. We prove this by induction on k.

Initial Case: When $k = 1$, $\ker\varphi$ is already S_1. So by rank-nullity theorem and uniseriality,

$\varphi(S_n) = S_{n-1}$.

Induction Hypothesis: Let $\varphi^k(S_n) = S_{n-k}$ and $\ker \varphi^k = S_k$ for some k.

Induction Step: $\varphi^{k+1}(S_n) = \varphi \circ \varphi^k(S_n) = \varphi(S_{n-k})$ by induction hypothesis. $S_1 = \ker \varphi \subset S_{n-k}$. So nullity of $\varphi|_{S_{n-k}}$ is 2. Dimension of S_{n-k} is $2n - 2k$.

By rank-nullity theorem,

$$\dim S_{n-k} = \text{null } \varphi|_{S_{n-k}} + \text{rank } \varphi|_{S_{n-k}},$$

$$\implies 2n - 2k = 2 + \text{rank } \varphi|_{S_{n-k}},$$

$$\implies \text{rank } \varphi|_{S_{n-k}} = 2(n - k - 1).$$

$\varphi(S_{n-k}) = S_i$ for $1 \leq i \leq n-k$ because S_{n-k} is uniserial. Also $\dim \varphi(S_{n-k}) = 2(n-k-1)$. So $\varphi^{k+1}(S_n) = \varphi(S_{n-k}) = S_{n-k-1}$.

Furthermore, $\dim S_n = 2n$. By rank-nullity,

$$2n = 2n - 2(k + 1) + \text{null } \varphi^{k+1},$$

$$\implies \text{null } \varphi^{k+1} = 2(k + 1).$$

Kernel of φ^{k+1} is a submodule of S_n, so by uniseriality of S_n, $\ker \varphi^{k+1} = S_i$, $1 \leq i \leq n$. Dim $\ker \varphi^{k+1} = 2(k + 1) \implies \ker \varphi^{k+1} = S_{k+1}$.

Therefore by mathematical induction, $\varphi : S_n \to S_n$, $\ker \varphi = S_1$ implies that $\varphi^k(S_n) = S_{n-k}$, $\ker \varphi^k = S_k$. ■

Returning to $\varphi : S_2 \to S_2$, recall that the image of φ is $S_1 \neq \emptyset$ and so is the kernel. So non-zero elements in the image of φ must have pre-images in S_2 and not in S_1, because $a \in S_1 = \ker \varphi \implies \varphi(a) = 0$. So for all $b \in S_1$, if $b \neq 0$, then there exists $a \in S_2$, but $a \notin S_1$ such that $\varphi(a) = b \in S_1$. Therefore φ reduces the multiplicity of $(L_x - \lambda^2)^2$. This is true for all multiplicities, as shown in the theorem 5.31.

Lemma 5.30. *Suppose $a \in S_n$ but $a \notin S_{n-1}$. The set $\{a\}$ generates the module S_n.*

Proof. The module generated by $\{a\}$ is the smallest module which contains $\{a\}$ as a subset. It is the intersection of all the modules which contain a.

$\{a\} \subset S_n$. But $\{a\} \not\subseteq S_k$ for $k < n$. From uniseriality, we also know that for all $k \geq n$, $S_n \subseteq S_k$. So $\{a\} \subset S_k$ for $k \geq n$.

We know that the only submodules of S_k are S_0, S_1, ..., S_{k-1} and S_k. So by taking arbitrarily large k, we conclude that the only modules which contain $\{a\}$ are of the form S_k, $k \geq n$. Take intersection over all those modules.

$$\langle a \rangle = \bigcap_{k=n}^{\infty} S_k = S_n.$$

Therefore, $\{a\}$ generates S_n. ■

Theorem 5.31. *Let $\varphi : S_n \to S_n$ with $\ker \varphi = S_1$ be a lowering map. Then φ reduces multiplicity for all n: if $a \in S_n$ and $a \notin S_{n-1}$, then $\varphi(a) \in S_{n-1}$, $\varphi(a) \notin S_{n-2}$.*

Proof. This can also be proved by induction.

Initial Case: Let $n = 2$. Let $a \in S_2$, $a \notin S_1$. Kernel of φ is S_1, so $a \notin \ker \varphi$. This implies that $\varphi(a) \neq 0$. $\varphi(S_2) = S_1$, so $\varphi(a) \in S_1$, but $\varphi(a) \notin S_0 = 0$.

Induction Hypothesis: For some $n = k$, let $a \in S_k$, $a \notin S_{k-1}$ imply that $\varphi(a) \in S_{k-1}$ but $\varphi(a) \notin S_{k-2}$.

Induction Step: For $n = k+1$, let $a \in S_{k+1}$, $a \notin S_k$. We want to show that $\varphi(a) \in S_k$, $\varphi(a) \notin S_{k-1}$. Assume towards contradiction: $\varphi(a) \in S_{k-1}$.

By lemma 5.30, the set $\{a\}$ generates the module S_{k+1}. That is, for all $b \in S_{k+1}$ there exists $m \in \mathcal{A}$ such that $b = ma$. In particular, for all $b \in S_{k+1}$, $b \notin S_k$, there exists an

$m \in \mathcal{A}$, such that $b = ma$.

$$\implies \varphi(b) = \varphi(ma) = m\varphi(a) \text{ by } \mathcal{A}\text{-linearity.}$$

$\varphi(a) \in S_{k-1}$. S_{k-1} is an $\mathcal{A}$-module, so for all $m \in \mathcal{A}$, $m(S_{k-1}) \subseteq S_{k-1}$. So $m\varphi(a) \in S_{k-1}$. Thus $\varphi(b) \in S_{k-1}$ for any $b \in S_{k+1}, b \notin S_k$. By induction hypothesis, for all $c \in S_k$, $\varphi(c) \in S_{k-1}$, since $\varphi(S_{k-1}) \subseteq S_{k-1}$.

So for all $b \in S_{k+1}$, $\varphi(b) \subseteq S_{k-1} \implies \varphi(S_{k+1}) \subseteq S_{k-1}$.

By uniseriality, $\varphi(S_{k+1}) = S_i$, $1 \leq i \leq k-1$.

$$\dim S_{k+1} = 2k+2, \quad \dim S_{k-1} = 2k-2, \quad \text{rank } \varphi(S_{k-1}) \leq 2k-2.$$

By rank-nullity theorem,

$$\dim S_{k+1} = \text{null } \varphi(S_{k+1}) + \text{rank } \varphi(S_{k+1}) \leq \text{null } \varphi(S_{k+1}) + 2k - 2,$$

$$\implies 2k+2 \leq \text{null } \varphi + 2k - 2 \implies \text{null } \varphi \geq 2k+2-2k+2 = 4,$$

$$\implies \ker \varphi(S_{k+1}) = S_2 \text{ or bigger.}$$

But $\ker \varphi = S_1$. This is a contradiction. Therefore $a \in S_{k+1}, a \notin S_k \implies \varphi(a) \in S_k,\ \varphi(a) \notin S_{k-1}$.

By mathematical induction, for all n, $a \in S_n, a \notin S_{n-1} \implies \varphi(a) \in S_{n-1}$, $\varphi(a) \notin S_{n-2}$. ■

Theorem 5.32. *For type I or II λ,*

$$\text{Hom}_{\mathcal{A}}(S_n(\lambda), S_n(\lambda)) = \left\{u_0 \,\text{id} + u_1\varphi + \ldots + u_{n-1}\varphi^{n-1} : u_i \in \mathbb{C}\right\} \cong \mathbb{C}^n.$$

Proof. We know that

$$\text{Hom}_{\mathcal{A}}(S_1(\lambda), S_1(\lambda)) = \{u_0 \,\text{id} : u_0 \in \mathbb{C}\} \cong \mathbb{C},$$

and

$$\mathrm{Hom}_{\mathcal{A}}(S_2(\lambda), S_2(\lambda)) = \{u_0 \,\mathrm{id} + u_1\varphi : u_i \in \mathbb{C}\} \cong \mathbb{C}^2.$$

This suggests we can use proof by induction.

Induction Hypothesis: Assume that the following is true.

$$\mathrm{Hom}_{\mathcal{A}}(S_k(\lambda), S_k(\lambda)) = \left\{ \sum_{i=0}^{k-1} u_i\varphi^i ; u_i \in \mathbb{C} \right\} \cong \mathbb{C}^k.$$

Induction Step: Let $h \in \mathrm{Hom}_{\mathcal{A}}(S_{k+1}(\lambda), S_{k+1}(\lambda))$. We study what h does to $S_k \subsetneq S_{k+1}$.

The restriction $h : S_k \to S_{k+1}$ is still an $\mathcal{A}$-module homomorphism. By rank-nullity theorem, $h(S_k)$ is a proper submodule of S_{k+1}. Because of uniseriality of S_{k+1}, $h(S_k) = S_i$ where $1 \leq i \leq k$.

Therefore there exists a $\hat{h} \in \mathrm{Hom}_{\mathcal{A}}(S_k(\lambda), S_k(\lambda))$, such that $\forall \phi \in S_k \subsetneq S_{k+1}$, $\hat{h}(\phi) = h(\phi)$. By induction assumption,

$$\hat{h} = u_0 \,\mathrm{id} + ... + u_{k-1}\varphi^{k-1}.$$

For all $\phi \in S_k$, $h(\phi) = \hat{h}(\phi) \implies (h - \hat{h})\phi = 0$. Set $\xi := h - \hat{h} : S_{k+1} \to S_{k+1}$. Since $\xi\phi = 0 \ \forall \phi \in S_k$, $S_k \subseteq \ker \xi$.

We would like to show that $S_k = \ker \xi$ and so $\xi \propto \varphi^k$.

With respect to basis $\{\partial_\lambda^i \psi_\pm : 0 \leq i \leq k\}$, ξ is represented by the matrix:

$$\xi = \begin{pmatrix} E_{1,1} & . \ . \ . & E_{1,k} & E_{1,k+1} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ E_{k+1,1} & . \ . \ . & E_{k+1,k} & E_{k+1,k+1} \end{pmatrix}.$$

Here, $E_{i,j}$ are 2×2 blocks. Make use of $S_k \subseteq \ker \xi$: for all $\phi \in S_k$, $\phi \mapsto (\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k, \mathbf{0})$,

$\xi\phi = 0$. $\mathbf{v}_i$ are 2 dimensional vectors.

$$\begin{pmatrix} E_{1,1} & . & . & . & E_{1,k} & E_{1,k+1} \\ . & . & & & . & . \\ . & & . & & . & . \\ . & & & . & . & . \\ E_{k,1} & . & . & . & E_{k,k} & E_{k,k+1} \\ E_{k+1,1} & . & . & . & E_{k+1,k} & E_{k+1,k+1} \end{pmatrix} \begin{pmatrix} \mathbf{v}_1 \\ . \\ . \\ . \\ \mathbf{v}_k \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ . \\ . \\ . \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}.$$

$$\Longrightarrow \begin{pmatrix} E_{1,1} & . & . & . & E_{1,k} \\ . & . & & & . \\ . & & . & & . \\ . & & & . & . \\ E_{k+1,1} & . & . & . & E_{k+1,k} \end{pmatrix} \begin{pmatrix} \mathbf{v}_1 \\ . \\ . \\ . \\ \mathbf{v}_k \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ . \\ . \\ . \\ \mathbf{0} \end{pmatrix}.$$

$\mathbf{v}_i$ are completely arbitrary. So the truncated matrix $(E_{i,j})$, $1 \le i \le k+1$, $1 \le j \le k$ annihilates an entire $2k$ dimensional space. Therefore

$$\xi = \begin{pmatrix} \mathbf{0} & . & . & . & \mathbf{0} & E_{1,k+1} \\ . & . & & & . & . \\ . & & . & & . & . \\ . & & & . & . & . \\ \mathbf{0} & . & . & . & \mathbf{0} & E_{k+1,k+1} \end{pmatrix}.$$

h and $\hat{h}$ are $\mathcal{A}$-linear. So is ξ. We can impose $\xi \circ M_\infty = M_\infty \circ \xi$ and $\xi \circ M_0 = M_0 \circ \xi$.

$$\xi \circ M_\infty = \begin{pmatrix} \mathbf{0} & . & . & . & \mathbf{0} & E_{1,k+1} \\ . & . & & & . & . \\ . & & . & & . & . \\ . & & & . & . & . \\ \mathbf{0} & . & . & . & \mathbf{0} & E_{k+1,k+1} \end{pmatrix} \begin{pmatrix} \mathbf{m}_\infty & \mathbf{m}'_\infty & . & . & . & \mathbf{m}_\infty^{(k)} \\ \mathbf{0} & \mathbf{m}_\infty & . & . & . & \binom{k}{1}\mathbf{m}_\infty^{(k-1)} \\ . & & . & & & . \\ . & & & . & & . \\ . & & & & . & . \\ \mathbf{0} & & . & . & . & \mathbf{m}_\infty \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{0} & . & . & . & \mathbf{0} & E_{1,k+1}\mathbf{m}_\infty \\ . & . & & & . & . \\ . & & . & & . & . \\ . & & & . & . & . \\ \mathbf{0} & . & . & . & \mathbf{0} & E_{k+1,k+1}\mathbf{m}_\infty \end{pmatrix}.$$

$$M_\infty \circ \xi = \begin{pmatrix} \mathbf{0} & . & . & . & \mathbf{0} & \sum_{i=1}^{k+1} \mathbf{m}_\infty^{(i-1)} E_{i,k+1} \\ . & . & & & . & . \\ . & & . & & . & . \\ . & & & . & . & . \\ \mathbf{0} & . & . & . & \mathbf{0} & \mathbf{m}_\infty E_{k+1,k+1} \end{pmatrix}.$$

Similarly $\xi \circ M_0 = M_0 \circ \xi$ gives

$$\begin{pmatrix} \mathbf{0} & . & . & . & \mathbf{0} & E_{1,k+1}\mathbf{m}_0 \\ . & . & & & . & . \\ . & & . & & . & . \\ . & & & . & . & . \\ \mathbf{0} & . & . & . & \mathbf{0} & E_{k+1,k+1}\mathbf{m}_0 \end{pmatrix} = \begin{pmatrix} \mathbf{0} & . & . & . & \mathbf{0} & \sum_{i=1}^{k+1} \mathbf{m}_0^{(i-1)} E_{i,k+1} \\ . & . & & & . & . \\ . & & . & & . & . \\ . & & & . & . & . \\ \mathbf{0} & . & . & . & \mathbf{0} & \mathbf{m}_0 E_{k+1,k+1} \end{pmatrix}.$$

The bottom-right entries of the above matrices are

$$E_{k+1,k+1}\mathbf{m}_\infty = \mathbf{m}_\infty E_{k+1,k+1}, \tag{5.24}$$

$$E_{k+1,k+1}\mathbf{m}_0 = \mathbf{m}_0 E_{k+1,k+1}. \tag{5.25}$$

Solving the above equations simultaneously gives $E_{k+1,k+1} = w_{k+1} I$ where $w_{k+1} \in \mathbb{C}$. The entries just above the bottom right entries of the matrices are

$$E_{k,k+1}\mathbf{m}_\infty = \mathbf{m}_\infty E_{k,k+1} + \binom{k}{k-1} \mathbf{m}'_\infty E_{k+1,k+1}, \tag{5.26}$$

$$E_{k,k+1}\mathbf{m}_0 = \mathbf{m}_0 E_{k,k+1} + \binom{k}{k-1} \mathbf{m}'_0 E_{k+1,k+1}. \tag{5.27}$$

Substituting $E_{k+1,k+1} = w_{k+1}I$ into (5.26) and (5.27) and solving those equations simultaneously gives

$$w_{k+1} = 0 \implies E_{k+1,k+1} = \mathbf{0} \text{ and } E_{k,k+1} = w_k I.$$

Iteratively we obtain

$$E_{2,k+1} = E_{3,k+1} = ... = E_{k,k+1} = E_{k+1,k+1} = \mathbf{0} \text{ and } E_{1,k+1} = u_k I.$$

$$\implies \xi \mapsto u_k \varphi^k.$$

$$\xi = h - \hat{h} = u_k\varphi^k \implies h = \hat{h} + u_k\varphi^k = u_0 \operatorname{id} + ... + u_{k-1}\varphi^{k-1} + u_k\varphi^k.$$

Therefore by the principle of mathematical induction,

$$\operatorname{Hom}_{\mathcal{A}}(S_n, S_n) = \left\{ \sum_{i=0}^{n-1} u_i\varphi^i : u_i \in \mathbb{C} \right\} \cong \mathbb{C}^n.$$

■

The next result is needed to justify our use of Darboux factorisations with proper submodules for serial modules.

Proposition 5.33. *Suppose $n > m$. Then* $\operatorname{Hom}_{\mathcal{A}}(S_m, S_n) = \operatorname{Hom}_{\mathcal{A}}(S_m, S_m) \cong \mathbb{C}^m$.

Proof. Let $h \in \operatorname{Hom}_{\mathcal{A}}(S_m, S_n)$. Then the image $h(S_m) \subseteq S_n$. S_n is uniserial. So $h(S_m) = S_i$, where $1 \le i \le n$.

Assume towards contradiction that $i > m$. Then rank $h = \dim h(S_m) > \dim S_m$. But since h is a linear map, we also have the rank-nullity theorem:

$$\text{rank } h \le \text{rank } h + \text{null } h = \dim S_m.$$

So rank $h \le \dim S_m$. But this contradicts rank $h > \dim S_m$, and hence $i \le m$.

So $h(S_m) \subseteq S_m$. Since h is a homomorphism, $h \in \operatorname{Hom}_{\mathcal{A}}(S_m, S_m)$. So $\operatorname{Hom}_{\mathcal{A}}(S_m, S_n) \subseteq$

$\mathrm{Hom}_{\mathcal{A}}(S_m, S_m)$. Also, since $S_m \subset S_n$, $\mathrm{Hom}_{\mathcal{A}}(S_m, S_m) \subseteq \mathrm{Hom}_{\mathcal{A}}(S_m, S_n)$. So we have the inclusions:

$$\mathrm{Hom}_{\mathcal{A}}(S_m, S_n) \subseteq \mathrm{Hom}_{\mathcal{A}}(S_m, S_m) \subseteq \mathrm{Hom}_{\mathcal{A}}(S_m, S_n),$$

$$\implies \mathrm{Hom}_{\mathcal{A}}(S_m, S_n) = \mathrm{Hom}_{\mathcal{A}}(S_m, S_m) \cong \mathbb{C}^m.$$

■

We have shown that $\mathrm{Hom}_{\mathcal{A}}(S_n(\lambda), S_n(\lambda)) \cong \mathbb{C}^n$ for type I and II λ. This is actually true for type III λ as well.

Let $S_n(\lambda) = \ker(L - \lambda^2)^n$ where λ is of type III. Without loss of generality, let $S_{n+\frac{1}{2}}(\lambda) = \mathrm{span}\{\partial_\lambda^n \psi_+(\lambda)\} \cup \ker(L - \lambda^2)^n$.

Theorem 5.34. *For all $n \in \mathbb{N}$,*

$$\mathrm{Hom}_{\mathcal{A}}(S_{n-\frac{1}{2}}(\lambda), S_n(\lambda)) = \mathrm{Hom}_{\mathcal{A}}(S_n(\lambda), S_n(\lambda)) \cong \mathbb{C}^n.$$

Proof. This is the same as the proof of theorem 5.32. ■

We have looked at Hom spaces for uniserial modules. For non-trivial Darboux transformations, we factorise serial modules, where integer shifts in λ are expected to give us non-trivial factorisations. We now extend the above ideas about Hom spaces to serial modules.

Theorem 5.35. *Let $\{S_i\}_i$ and $\{R_j\}$ be families of $\mathcal{A}$-modules. Then*

$$\mathrm{Hom}_{\mathcal{A}}\left(\bigoplus_i S_i, \bigoplus_j R_j\right) \cong \bigoplus_i \bigoplus_j \mathrm{Hom}_{\mathcal{A}}(S_i, R_j).$$

Proof. See [30, Proposition 3.15]. ■

Theorem 5.35 can be understood by looking back at the example in section 4.5. There,

$$\ker P = \mathrm{span}\{\alpha\psi_+ + \beta\tilde{\psi}_+, \alpha\psi_- + \beta\tilde{\psi}_-\} \cong \mathrm{span}\{\psi_+, \psi_-\} = S_1(\lambda).$$

So essentially, we looked for modules in $\ker(L_x - \lambda^2)(L_x - \tilde{\lambda}) = S_1(\lambda) \oplus S_1(\tilde{\lambda})$ which were isomorphic to $S_1(\lambda)$ (or even $S_1(\tilde{\lambda})$; it does not matter). Let $f : S_1 \longrightarrow \ker P$ be that isomorphism. By theorem 5.35,

$$f \in \mathrm{Hom}_{\mathcal{A}}\left(S_1(\lambda), S_1(\lambda) \oplus S_1(\tilde{\lambda})\right) \cong \mathrm{Hom}_{\mathcal{A}}\left(S_1(\lambda), S_1(\lambda)\right) \oplus \mathrm{Hom}_{\mathcal{A}}\left(S_1(\lambda), S_1(\tilde{\lambda})\right).$$

f can be seen component-wise in the direct sum of Hom spaces.

$$f = f_\lambda \oplus f_{\tilde{\lambda}}, \quad f_\lambda \in \mathrm{Hom}_{\mathcal{A}}\left(S_1(\lambda), S_1(\lambda)\right), \quad f_{\tilde{\lambda}} \in \mathrm{Hom}_{\mathcal{A}}\left(S_1(\lambda), S_1(\tilde{\lambda})\right).$$

$$\forall \psi(\lambda) \in S_1(\lambda), \quad f_\lambda(\psi(\lambda)) = \alpha\psi(\lambda), \quad f_{\tilde{\lambda}}(\psi(\lambda)) = \beta\psi(\tilde{\lambda}).$$

Theorem 5.35 tells us that Hom spaces over series modules breakdown in to component Hom spaces.

$$\mathrm{Hom}_{\mathcal{A}}\left(\bigoplus_i S_q(\lambda_i), \bigoplus_j S_r(\lambda_j)\right) \cong \bigoplus_i \bigoplus_j \mathrm{Hom}_{\mathcal{A}}(S_q(\lambda_i), S_r(\lambda_j))$$

Therefore, in our search for injective homomorphisms, we must find out all possibilities for the individual components $\mathrm{Hom}_{\mathcal{A}}(S_q(\lambda), S_r(\tilde{\lambda}))$ and see how they depend on the spectral parameter λ.

We will assume that $q \leq r$, because otherwise we get non-injective homomorphisms which we are not interested in because they give us no information on how to embed standard modules in to bigger ones.

<u>Case 1</u>: <u>$\lambda = \tilde{\lambda}$</u>. We have already dealt with this case in theorem 5.32 and proposition 5.33 for types I and II λ and theorem 5.34 for type III λ. For all λ, $\lambda = \tilde{\lambda}$ implies that

$$\mathrm{Hom}_{\mathcal{A}}(S_q(\lambda), S_r(\tilde{\lambda})) = \mathrm{Hom}_{\mathcal{A}}(S_q(\lambda), S_q(\lambda)) = \left\{\sum_{i=0}^{q-1} u_i\varphi^i : u_i \in \mathbb{C}\right\} \cong \mathbb{C}^q.$$

<u>Case 2</u>: <u>$\lambda \pm \tilde{\lambda} \notin \mathbb{Z}$</u>. For this case, we need the following lemma.

Lemma 5.36. *Suppose λ and/or $\tilde{\lambda}$ are of type I or II. Then $S_1(\lambda) \cong S_1(\tilde{\lambda})$ if and only if*

$\lambda \pm \tilde{\lambda} \in \mathbb{Z}$.

Proof of the if part.

$\underline{\lambda - \tilde{\lambda} \in \mathbb{Z}}$:

In $S_1(\lambda)$, the monodromy matrices are $M_0(\lambda)$ and $M_\infty(\lambda)$. In $S_1(\tilde{\lambda})$, the monodromy matrices are $M_0(\tilde{\lambda})$ and $M_\infty(\tilde{\lambda})$.

With respect to basis in (4.15) (or (4.34) for type II λ), the monodromy matrices are invariant under the transformation $\lambda \mapsto \lambda + n$, $n \in \mathbb{Z}$. So $M_0(\lambda) = M_0(\tilde{\lambda})$ and $M_\infty(\lambda) = M_\infty(\tilde{\lambda})$.

Put simply, there exists $F \in GL(2n, \mathbb{C})$ (with $F = I$), such that $M_0(\lambda) = FM_0(\lambda)F^{-1}$ and $M_\infty(\lambda) = FM_\infty(\lambda)F^{-1}$. Hence $S_1(\lambda) \cong S_1(\tilde{\lambda})$.

$\underline{\lambda + \tilde{\lambda} \in \mathbb{Z}}$:

$$S_1(-\lambda) = \ker(L_x - (-\lambda)^2) = \ker(L_x - \lambda^2) = S_1(\lambda).$$

So $S_1(-\lambda) = S_1(\lambda)$. By periodicity arguments given in the $\tilde{\lambda} - \lambda \in \mathbb{Z}$ case above, $S_1(-\lambda) \cong S_1(-\lambda + n)$ where $n \in \mathbb{Z}$. Set $\tilde{\lambda} := -\lambda + n$. This gives us $S_1(\lambda) \cong S_1(\tilde{\lambda})$, with $\tilde{\lambda} + \lambda = n \in \mathbb{Z}$.

Proof of the only if part.

$\underline{\lambda - \tilde{\lambda} \notin \mathbb{Z}}$:

Suppose $\tilde{\lambda}$ is of type I. Since $\lambda - \tilde{\lambda} \notin \mathbb{Z}$, $\tilde{\lambda} = \lambda + \mu$, where $\mu \notin \mathbb{Z}$. Assume towards contradiction that $S_1(\lambda) \cong S_1(\tilde{\lambda})$. In $S_1(\lambda)$ and $S_1(\tilde{\lambda})$,

$$M_\infty(\lambda) = e^{2\pi i \rho} \begin{bmatrix} e^{2\pi i \lambda} & 0 \\ 0 & e^{-2\pi i \lambda} \end{bmatrix} \quad \text{and} \quad M_\infty(\tilde{\lambda}) = e^{2\pi i \rho} \begin{bmatrix} e^{2\pi i \tilde{\lambda}} & 0 \\ 0 & e^{-2\pi i \tilde{\lambda}} \end{bmatrix}$$

Since we are assuming that $S_1(\lambda) \cong S_1(\tilde{\lambda})$, there must exist $F \in GL(2, \mathbb{C})$ such that

$M_\infty(\lambda) = FM_\infty(\tilde{\lambda})F^{-1}$. Let

$$F = \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix},$$

and solve for F.

$$\begin{bmatrix} e^{2\pi i\lambda} & 0 \\ 0 & e^{-2\pi i\lambda} \end{bmatrix} = \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix} \begin{bmatrix} e^{2\pi i\tilde{\lambda}} & 0 \\ 0 & e^{-2\pi i\tilde{\lambda}} \end{bmatrix} \begin{bmatrix} F_4 & -F_2 \\ -F_3 & F_1 \end{bmatrix} \frac{1}{\det F}$$

$$= \begin{bmatrix} F_1F_4e^{2\pi i\tilde{\lambda}} - F_2F_3e^{-2\pi i\tilde{\lambda}} & F_1F_2\left(e^{-2\pi i\tilde{\lambda}} - e^{2\pi i\tilde{\lambda}}\right) \\ F_3F_4\left(e^{2\pi i\tilde{\lambda}} - e^{-2\pi i\tilde{\lambda}}\right) & F_1F_4e^{-2\pi i\tilde{\lambda}} - F_2F_3e^{2\pi i\tilde{\lambda}} \end{bmatrix} \frac{1}{\det F}$$

12-entry: $F_1F_2\left(e^{-2\pi i\tilde{\lambda}} - e^{2\pi i\tilde{\lambda}}\right) = 0$. Since $\tilde{\lambda}$ is not a half integer (it is of type I), it can be anything, we must have $F_1F_2 = 0 \implies F_1 = 0$ or $F_2 = 0$.

21-entry: $F_3F_4\left(e^{2\pi i\tilde{\lambda}} - e^{-2\pi i\tilde{\lambda}}\right) = 0$. Since $\tilde{\lambda}$ can be anything, we must have $F_3 = 0$ or $F_4 = 0$.

Case 1: $F_1 = F_3 = 0$. Then $\det F = F_1F_4 - F_2F_3 = 0$. So $F \notin GL(2, \mathbb{C})$, which means that F does not represent an $\mathcal{A}$-module isomorphism. This contradicts $S_1(\lambda) \cong S_1(\tilde{\lambda})$.

Case 2: $F_2 = F_4 = 0$. Again, this would mean that $\det F = 0$, giving us a contradiction.

Case 3: $F_2 = F_3 = 0$. In this case, check the 11-entry:

$$e^{2\pi i\lambda} = \frac{F_1F_4e^{2\pi i\tilde{\lambda}} - 0}{F_1F_4 - 0} = e^{2\pi i\tilde{\lambda}}.$$

$$\implies e^{2\pi i\lambda} = e^{2\pi i(\lambda+\mu)} = e^{2\pi i\lambda}e^{2\pi i\mu} \implies e^{2\pi i\mu} = 1.$$

$\implies \mu \in \mathbb{Z}$. But $\mu = \tilde{\lambda} - \lambda \notin \mathbb{Z}$. This is a contradiction.

Case 4: $F_1 = F_4 = 0$. In this case, check the 11-entry:

$$e^{2\pi i\lambda} = \frac{0 - F_2F_3e^{-2\pi i\tilde{\lambda}}}{0 - F_2F_3} = e^{-2\pi i\tilde{\lambda}}.$$

$$\implies e^{2\pi i\lambda} = e^{-2\pi i(\lambda+\mu)} = e^{-2\pi i\lambda}e^{-2\pi i\mu} \implies e^{2\pi i\mu} = e^{-4\pi i\lambda}$$

$$\implies \mu = -2\lambda + n, \text{ where } n \in \mathbb{Z}.$$

Then $\tilde{\lambda} = \lambda + \mu = \lambda - 2\lambda + n = -\lambda + n$. This implies that $\tilde{\lambda} + \lambda \in \mathbb{Z}$, so we are just back to that case where we know that isomorphism exists.

We can check through explicit calculation that the relation $M_0(\lambda) = FM_0(\tilde{\lambda})F^{-1}$ also works for this choice μ.

All of the other cases lead to contradictions. Therefore we obtain $\lambda - \tilde{\lambda} \notin \mathbb{Z} \implies S_1(\lambda) \ncong S_1(\tilde{\lambda})$ for type I $\tilde{\lambda}$.

For type II $\tilde{\lambda}$, we would not use the above matrix for $M_\infty(\tilde{\lambda})$ because we have to use a different basis (see section 4.4.4 for the amended basis). However we still would not have isomorphism because the eigenvalues of M_∞ (which are $e^{2\pi i\lambda}$ and $e^{-2\pi i\lambda}$) would remain unchanged in the new basis. Looking back at the proof for type I case, it is this inequality of eigenvalues which makes isomorphism between $S_1(\lambda)$ and $S_1(\tilde{\lambda})$ impossible.

Through a similar calculation, we can show that $\lambda + \tilde{\lambda} \notin \mathbb{Z} \implies S_1(\lambda) \ncong S_1(\tilde{\lambda})$. ■

The if part of lemma 5.36 generalises to $S_n(\lambda)$ for any type I and type II λ using the same argument.

Corollary 5.37. *For all type I and type II λ, if $\lambda \pm \tilde{\lambda} \in \mathbb{Z}$, then $S_n(\lambda) \cong S_n(\tilde{\lambda})$.*

Theorem 5.38. *If λ is of type I or II and $\lambda \pm \tilde{\lambda} \notin \mathbb{Z}$, then*

$$\mathrm{Hom}_{\mathcal{A}}(S_q(\lambda), S_r(\tilde{\lambda})) = 0.$$

Proof. Let $f : S_q(\lambda) \to S_r(\tilde{\lambda})$, $f(S_q(\lambda)) \neq 0$. From uniseriality, $\ker f = S_k(\lambda)$ for some $0 \leq k \leq q$. Then by first isomorphism theorem, the image of f satisfies

$$\text{image } f \cong S_q(\lambda)/S_k(\lambda) \cong S_{q-k}(\lambda),$$

and is a submodule in $S_r(\tilde{\lambda})$. By uniseriality of $S_r(\tilde{\lambda})$, it must be $S_l(\tilde{\lambda})$ for some $l \leq r$.

$$\implies S_{q-k}(\lambda) \cong S_l(\tilde{\lambda}).$$

In particular, their simple modules must be isomorphic as well.

$$\implies S_1(\lambda) \cong S_1(\tilde{\lambda}).$$

By lemma 5.36, $\lambda \pm \tilde{\lambda} \in \mathbb{Z}$.

So if $\mathrm{Hom}_{\mathcal{A}}(S_q(\lambda), S_r(\tilde{\lambda}))$ contained non-zero f, then $\lambda \pm \tilde{\lambda} \in \mathbb{Z}$.

Therefore, if $\lambda \pm \tilde{\lambda} \notin \mathbb{Z}$, then $\mathrm{Hom}_{\mathcal{A}}(S_q(\lambda), S_r(\tilde{\lambda})) = 0$. ■

Within the case $\lambda \pm \tilde{\lambda} \notin \mathbb{Z}$, we can also cover the possibility that $\tilde{\lambda}$ is of type III and λ is not. If $\tilde{\lambda}$ is of type III, then one of the following must be an integer.

$$\tilde{\lambda} + \frac{g+h}{2}, \quad -\tilde{\lambda} + \frac{g+h}{2}, \quad \tilde{\lambda} + \frac{1+g-h}{2}, \quad -\tilde{\lambda} + \frac{1+g-h}{2}.$$

If λ is not of type III, then none of the following is an integer.

$$\lambda + \frac{g+h}{2}, \quad \lambda - \frac{g+h}{2}, \quad \lambda + \frac{1+g-h}{2}, \quad \lambda - \frac{1+g-h}{2}.$$

Assume without loss of generality that $\tilde{\lambda} + \frac{g+h}{2} \in \mathbb{Z}$. Then

$$\left(\lambda - \frac{g+h}{2}\right) + \left(\tilde{\lambda} + \frac{g+h}{2}\right) = \lambda + \tilde{\lambda} \notin \mathbb{Z}.$$

$$\left(\lambda + \frac{g+h}{2}\right) - \left(\tilde{\lambda} + \frac{g+h}{2}\right) = \lambda - \tilde{\lambda} \notin \mathbb{Z}.$$

$$\implies \lambda \pm \tilde{\lambda} \notin \mathbb{Z}.$$

Therefore if one of λ or $\tilde{\lambda}$ is of type III and the other one is not, then $\lambda \pm \tilde{\lambda} \notin \mathbb{Z}$.

Theorem 5.39. *If λ is of type III and $\tilde{\lambda}$ is not of type III then*

$$\mathrm{Hom}_{\mathcal{A}}(S_q(\lambda), S_r(\tilde{\lambda})) = 0.$$

Proof. Assume towards contradiction that $\mathrm{Hom}_{\mathcal{A}}(S_q(\lambda), S_r(\tilde{\lambda})) \neq 0$.

Let $f : S_q(\lambda) \to S_r(\tilde{\lambda})$ with $f(S_q(\lambda)) \neq 0$. Then image of f is $S_l(\tilde{\lambda})$ for some $l \leq r$.

On the other hand, by first isomorphism theorem,

$$\text{image } f \cong S_q(\lambda)/\ker f = S_l(\lambda) \text{ or } R_l(\lambda).$$

So the image of f has a two dimensional simple socle $S_1(\tilde{\lambda})$. However it is simultaneously isomorphic to $S_l(\lambda)$ or $R_l(\lambda)$ which has a one dimensional socle $S_{1/2}(\lambda)$ or $R_{1/2}(\lambda)$. This is a contradiction. Therefore $f \equiv 0$ and

$$\mathrm{Hom}_{\mathcal{A}}(S_q(\lambda), S_r(\tilde{\lambda})) = 0.$$

■

Theorem 5.40. *If λ is not of type III but $\tilde{\lambda}$ is, then*

$$\mathrm{Hom}_{\mathcal{A}}(S_q(\lambda), S_r(\tilde{\lambda})) = 0.$$

Proof. Assume towards contradiction that $\mathrm{Hom}_{\mathcal{A}}(S_q(\lambda), S_r(\tilde{\lambda})) \neq 0$.

Let $f : S_q(\lambda) \to S_r(\tilde{\lambda})$ with $f(S_q(\lambda)) \neq 0$. Then the image of f is $S_{l/2}(\tilde{\lambda})$ for some $0 \leq l \leq 2r$, whereas the kernel of f is $S_m(\lambda)$ for some integer $m \leq q$ because $\ker f \subset S_q(\lambda)$. By first isomorphism theorem,

$$\text{image } f \cong S_q(\lambda)/\ker f = S_q(\lambda)/S_m(\lambda) \cong S_{q-m}(\lambda).$$

The socle of image of f is the one dimensional module $S_{1/2}(\tilde{\lambda})$. However, it is also isomorphic to $S_{q-m}(\lambda)$ whose socle is the two dimensional module $S_2(\lambda)$.

This is a contradiction. Therefore $f \equiv 0$ and

$$\mathrm{Hom}_{\mathcal{A}}(S_q(\lambda), S_r(\tilde{\lambda})) = 0.$$

■

Theorem 5.41. *If λ and $\tilde{\lambda}$ are both type III and $\lambda \pm \tilde{\lambda} \notin \mathbb{Z}$ then*

$$\mathrm{Hom}_{\mathcal{A}}(S_q(\lambda), S_r(\tilde{\lambda})) = 0, \quad \textit{for any} \quad q, r \in \tfrac{1}{2}\mathbb{Z}.$$

Proof. Once again, we look for a non-zero homomorphism $f : S_q(\lambda) \to S_r(\tilde{\lambda})$.

$$\ker f = S_l(\lambda), \qquad l \in \tfrac{1}{2}\mathbb{Z}.$$

$$\text{image } f = S_m(\tilde{\lambda}), \qquad m \in \tfrac{1}{2}\mathbb{Z}$$

$$S_q(\lambda)/\ker f \cong S_q(\lambda)/S_l(\lambda) \cong S_m(\tilde{\lambda}).$$

But $S_q(\lambda)/S_l(\lambda) \cong S_{q-l}(\lambda)$ or $R_{q-l}(\lambda)$. So $S_m(\tilde{\lambda})$ is isomorphic to one of $S_{q-l}(\lambda)$ or $R_{q-l}(\lambda)$.

This would mean that their socles would be isomorphic. This is not possible because M_∞ has different eigenvalues ($e^{2\pi i\lambda}$ and $e^{2\pi i\tilde{\lambda}}$) on the one dimensional socle. This is a contradiction. Therefore,

$$\mathrm{Hom}_{\mathcal{A}}(S_q(\lambda), S_r(\tilde{\lambda})) = 0.$$

■

In summary, whatever the values of λ and $\tilde{\lambda}$ maybe,

$$\lambda \pm \tilde{\lambda} \notin \mathbb{Z} \implies \mathrm{Hom}_{\mathcal{A}}(S_q(\lambda), S_r(\tilde{\lambda})) = 0.$$

<u>**Case 3:**</u> <u>$\lambda \pm \tilde{\lambda} \in \mathbb{Z}$</u>. If λ is of type I or II, then by corollary 5.37, $S_q(\tilde{\lambda}) \cong S_q(\lambda)$. Therefore,

$$\mathrm{Hom}_{\mathcal{A}}(S_q(\lambda), S_r(\tilde{\lambda})) \cong \mathrm{Hom}_{\mathcal{A}}(S_q(\lambda), S_q(\lambda)) = \left\{ \sum_{i=0}^{q-1} u_i \varphi^i : u_i \in \mathbb{C} \right\} \cong \mathbb{C}^q.$$

As an aside, by lemma 5.12, we know that $S_n/S_1 \cong S_{n-1}$. So we can say something about the Hom spaces for quotient modules.

$$\mathrm{Hom}_{\mathcal{A}}(S_q(\lambda)/S_1(\lambda), S_r(\tilde{\lambda})/S_1(\tilde{\lambda})) \cong \mathrm{Hom}_{\mathcal{A}}(S_{q-1}(\lambda), S_{r-1}(\tilde{\lambda})) = \left\{ \sum_{i=0}^{q-2} u_i \varphi^i : u_i \in \mathbb{C} \right\}. \tag{5.28}$$

For type III λ, we have to keep in mind the fact that the arithmetic progression in type III λ breaks down in to two. If for example, $\lambda + (g+h)/2 \in \mathbb{Z}$ and $\lambda - \tilde{\lambda} \in \mathbb{Z}$, then $S(\lambda) \cong S(\tilde{\lambda})$ only if $\lambda + (g+h)/2$ and $\tilde{\lambda} + (g+h)/2$ are both strictly positive or are both non-positive. If they have different signs, then the one dimensional socles become non-isomorphic: for instance, one would be span$\{\psi_+\}$ and the other one would be span$\{\psi'_-\}$. They would have different eigenvalues for the analytic continuation around $z = 0$. (see section 4.4.5 for prior discussion on this). For simplification, let ρ be one of the following four:

$$\pm\frac{g+h}{2}, \quad \pm\frac{1-g+h}{2}.$$

We note that $S(\lambda) = S(-\lambda)$. Therefore for type III we will only concentrate on the $\lambda - \tilde{\lambda} \in \mathbb{Z}$ case.

Definition 5.42. Let us call $(\lambda, \tilde{\lambda})$ a *type III isomorphism pair* if λ and $\tilde{\lambda}$ are both of type III, $\lambda - \tilde{\lambda} \in \mathbb{Z}$ and one of the following is true.

- $\lambda > -\rho$ and $\tilde{\lambda} > -\rho$.
- $\lambda \leq -\rho$ and $\tilde{\lambda} \leq -\rho$.

We can similary define type III non-isomorphism pairs $(\lambda, \tilde{\lambda})$.

Definition 5.43. Let us call $(\lambda, \tilde{\lambda})$ a *type III non-isomorphism pair* if λ and $\tilde{\lambda}$ are both of type III, $\lambda - \tilde{\lambda} \in \mathbb{Z}$ and one of the following is true.

- $\lambda > -\rho$ and $\tilde{\lambda} \leq -\rho$.
- $\lambda \leq -\rho$ and $\tilde{\lambda} > -\rho$.

Lemma 5.44. *Suppose λ and $\tilde{\lambda}$ are of type III. Then*

$$S_{1/2}(\lambda) \cong S_{1/2}(\tilde{\lambda})$$

if and only if $(\lambda, \tilde{\lambda})$ is a type III isomorphism pair.

Proof. If $(\lambda, \tilde{\lambda})$ is a type III non-isomorophism pair, then suppose we are in the case $\lambda > -\rho$ and $\tilde{\lambda} \leq -\rho$. In this case,

$$S_{1/2}(\lambda) = \text{span}\{\psi_-(\lambda)\}, \quad \text{whereas} \quad S_{1/2}(\tilde{\lambda}) = \text{span}\{\psi'_+(\tilde{\lambda})\}.$$

$S_{1/2}(\lambda) \ncong S_{1/2}(\tilde{\lambda})$ because $M_\infty \psi_-(\lambda) = e^{-2\pi i \lambda}\psi_-(\lambda)$ but $M_\infty \psi'_+(\tilde{\lambda}) = e^{2\pi i \lambda}\psi'_+(\tilde{\lambda})$. The eigenvalues of M_∞ are different.

If $(\lambda, \tilde{\lambda})$ is a type III isomorphism pair then assume that $S_{1/2}(\lambda) = \text{span}\{\psi_+(\lambda)\}$ and $S_{1/2}(\tilde{\lambda}) = \text{span}\{\psi_+(\tilde{\lambda})\}$ (so $\rho = (g+h)/2$). Then,

$$M_\infty \psi_+(\lambda) = e^{2\pi i \lambda}\psi_+(\lambda).$$

$$M_\infty \psi_+(\tilde{\lambda}) = e^{2\pi i \tilde{\lambda}}\psi_+(\tilde{\lambda}).$$

$$e^{2\pi i \lambda} = e^{2\pi i \tilde{\lambda}} \quad \text{for} \quad \lambda - \tilde{\lambda} \in \mathbb{Z}.$$

$$M_0 \psi_+(\lambda) = A(\lambda)\psi_+(\lambda).$$

$$M_0 \psi_+(\tilde{\lambda}) = A(\tilde{\lambda})\psi_+(\tilde{\lambda})$$

$$\text{where } A(\lambda) = A(\tilde{\lambda}) = \frac{c_+c'_- + e^{2\pi i g}c'_+c_-}{c_+c'_- - c'_+c_-} \text{ because of periodicity of } A.$$

Therefore $S_{1/2}(\lambda) \cong S_{1/2}(\tilde{\lambda})$. ■

Corollary 5.45. *Suppose λ and $\tilde{\lambda}$ are of type III. Then $S_n(\lambda) \cong S_n(\tilde{\lambda})$ if and only if*

$(\lambda, \tilde{\lambda})$ is a type III isomorphism pair.

Theorem 5.46. *Suppose $(\lambda, \tilde{\lambda})$ is a type III isomorphism pair. Then*

$$\mathrm{Hom}_{\mathcal{A}}(S_q(\lambda), S_r(\tilde{\lambda})) \cong \mathrm{Hom}_{\mathcal{A}}(S_q(\lambda), S_q(\lambda)) = \left\{ \sum_{i=0}^{q-1} u_i \varphi^i : u_i \in \mathbb{C} \right\}.$$

Proof. By corollary 5.45, $S_r(\tilde{\lambda}) \cong S_r(\lambda)$, so $\mathrm{Hom}_{\mathcal{A}}(S_q(\lambda), S_r(\tilde{\lambda})) \cong \mathrm{Hom}_{\mathcal{A}}(S_q(\lambda), S_r(\lambda))$. By uniseriality, $f(S_q(\lambda)) = S_{l/2}(\lambda)$ for $l \leq 2q$. So $\mathrm{Hom}_{\mathcal{A}}(S_q(\lambda), S_r(\lambda)) = \mathrm{Hom}_{\mathcal{A}}(S_q(\lambda), S_q(\lambda))$. Finally we complete the proof by reaching the right hand side of the above equation using theorem 5.34. ∎

We do find non-trivial subspaces for type III non-isomorphic pairs (see section 4.4.7). For this we need to rely on the following isomorphism.

Lemma 5.47. *Suppose $(\lambda, \tilde{\lambda})$ is type III non-isomorphism pair, that is, $\lambda \pm \tilde{\lambda} \in \mathbb{Z}$, λ and $\tilde{\lambda}$ are of type III and $S_n(\lambda) \ncong S_n(\tilde{\lambda})$. Then the following relation holds.*

$$S_n(\tilde{\lambda})/S_{\frac{1}{2}}(\tilde{\lambda}) = R_{n-\frac{1}{2}}(\tilde{\lambda}) \cong S_{n-\frac{1}{2}}(\lambda). \tag{5.29}$$

Proof. Let $d_\pm(\lambda) = c'_\pm(\lambda)/c_\pm(\lambda)$. Then

$$\partial_\lambda^k \psi'_\pm = \partial_\lambda^k (d_\pm(\lambda)\psi_\pm) = \sum_{j=0}^{k} \binom{k}{j} \partial_\lambda^{k-j}(d_\pm(\lambda)) \partial_\lambda^j(\psi_\pm). \tag{5.30}$$

When λ and $\tilde{\lambda}$ are not of type III with $\lambda \pm \tilde{\lambda} \in \mathbb{Z}$, we have the isomorphism:

$$f : S_n(\tilde{\lambda}) \to S_n(\lambda), \qquad \partial_\lambda^k \psi'_\pm(\tilde{\lambda}) \mapsto \partial_\lambda^k \psi'_\pm(\lambda), \qquad k = 0, 1, \ldots n-1.$$

This is due to corollary 5.37. Using (5.30), this isomorphism can be written as

$$f : \partial_\lambda^k \psi'_\pm(\tilde{\lambda}) \mapsto \sum_{j=0}^{k} \binom{k}{j} \partial_\lambda^{k-j}(d_\pm(\lambda)) \partial_\lambda^j \psi_\pm(\lambda).$$

Being an isomorphism, this f satisfies the matrix similarity equation:

$$fM'(\tilde{\lambda}) = M(\lambda)f, \tag{5.31}$$

where $M'(\tilde{\lambda})$ is a matrix representing any monodromy transformation with respect to the basis $\left\{\partial_\lambda^k \psi'_\pm(\tilde{\lambda})\right\}$, and $M(\lambda)$ is the matrix representing the same transformation with respect to the basis $\left\{\partial_\lambda^k \psi_\pm(\lambda)\right\}$.

The equation (5.31) remains valid for type III λ and $\tilde{\lambda}$ as long as $d_\pm(\lambda)$ are well defined. In this case it means that f is still a homomorphism, but it may no longer be invertible.

Assume that $\tilde{\lambda} + (g+h)/2$ is a non-positive integer, so that $S_{1/2}(\tilde{\lambda}) = \text{span}\{\psi'_+(\tilde{\lambda})\}$ and $\lambda + (g+h)/2$ is a strictly positive integer so that $S_{1/2}(\lambda) = \text{span}\{\psi_-(\lambda)\}$. We claim that $\ker f = S_{1/2}(\tilde{\lambda})$. This can be checked as follows. We have the following formula:

$$f\left(\partial_\lambda^k \psi'_\pm(\tilde{\lambda})\right) = \sum_{j=0}^{k} \binom{k}{j} \partial_\lambda^{k-j}(d_\pm(\lambda))\partial_\lambda^j \psi_\pm(\lambda).$$

Set $k=0$. $\partial_\lambda^0 \psi'_+(\tilde{\lambda}) = \psi'_+(\tilde{\lambda}) \in S_{1/2}(\tilde{\lambda})$.

$$\text{RHS} = \sum_{j=0}^{0} \binom{k}{j} \partial_\lambda^{k-j}(d_+(\lambda))\partial_\lambda^j \psi_+(\lambda) = \binom{0}{0} \partial_\lambda^{0-0}(d_+(\lambda))\partial_\lambda^0 \psi_+(\lambda) = d_+(\lambda)\psi_+(\lambda).$$

So

$$f\left(\psi'_+(\tilde{\lambda})\right) = d_+(\lambda)\psi_+(\lambda).$$

This is true for any λ. If λ takes a type III value, $c'_+(\lambda) = 0$. So $d_+(\lambda) = c'_+(\lambda)/c_+(\lambda) = 0$. Therefore,

$$f\left(\psi'_+(\tilde{\lambda})\right) = 0 \times \psi_+(\lambda) = 0 \qquad \Longrightarrow \psi'_+(\tilde{\lambda}) \in \ker f \quad \text{and} \quad S_{1/2}(\tilde{\lambda}) \subseteq \ker f.$$

On the other hand, $\psi'_-(\tilde{\lambda}) \notin S_{1/2}(\tilde{\lambda})$ but it is in $S_1(\tilde{\lambda})$. $\partial_\lambda^0 \psi'_-(\tilde{\lambda}) = \psi'_-(\tilde{\lambda}) \in S_1$, whereas

$$\text{RHS} = \sum_{j=0}^{0} \binom{k}{j} \partial_\lambda^{k-j}(d_-(\lambda))\partial_\lambda^j \psi_-(\lambda) = \binom{0}{0} \partial_\lambda^{0-0}(d_-(\lambda))\partial_\lambda^0 \psi_-(\lambda) = d_-(\lambda)\psi_-(\lambda).$$

So,

$$f\left(\psi'_{-}(\tilde{\lambda})\right) = d_{-}(\lambda)\psi_{-}(\lambda) \neq 0.$$

In other words, $\psi'_{-}(\tilde{\lambda}) \notin \ker f$. By uniseriality, $\ker f = S_{i/2}(\tilde{\lambda})$, where $i = 0, 1, 2, \ldots, 2n$. However, if $i \geq 2$, then it would contain $\psi'_{-}(\tilde{\lambda})$ which is not the case. We conclude that $\ker f = S_{1/2}(\tilde{\lambda})$.

All of this implies that we have a short exact sequence:

$$0 \longrightarrow S_{1/2}(\tilde{\lambda}) \longrightarrow S_n(\tilde{\lambda}) \xrightarrow{f} \text{image}\, f \longrightarrow 0. \tag{5.32}$$

Image of f is a submodule of $S_n(\lambda)$. Its dimension is $2n-1$ because f has the one dimensional kernel $S_{1/2}(\tilde{\lambda})$. Therefore by rank-nullity theorem and uniseriality, image $f = S_{n-\frac{1}{2}}(\lambda)$. Then by first isomorphism theorem,

$$\text{Image}\, f \cong S_n(\tilde{\lambda})/\ker f \quad \Longrightarrow \; S_{n-\frac{1}{2}}(\lambda) \cong S_n(\tilde{\lambda})/S_{1/2}(\tilde{\lambda}) = R_{n-\frac{1}{2}}(\tilde{\lambda}).$$

■

Suppose $0 \neq f : S_q(\lambda) \to S_r(\tilde{\lambda})$. Then

$$\ker f = S_l(\lambda), \qquad \text{image } f = S_m(\tilde{\lambda}), \qquad l, m \in \tfrac{1}{2}\mathbb{Z}.$$

Again, by first isomorphism theorem,

$$S_q(\lambda)/\ker f = S_q(\lambda)/S_l(\lambda) \cong S_m(\tilde{\lambda}).$$

Here $S_q(\lambda)/S_l(\lambda) \cong S_{q-l}(\lambda)$ or $R_{q-l}(\lambda)$. $S_q(\lambda)/S_l(\lambda) \ncong S_{q-l}(\lambda)$ because that would mean that $S_{q-l}(\lambda) \cong S_m(\tilde{\lambda})$ which is not true for type III non-isomorphism pairs $(\lambda, \tilde{\lambda})$. Thus the only possibility is that $l \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$ and $m = q - l$.

Introduce the $\mathcal{A}$-module homomorphism

$$g : S_q(\lambda) \to S_{q-\frac{1}{2}}(\tilde{\lambda}),$$

such that image $g = S_{q-\frac{1}{2}}(\tilde{\lambda})$ and $\ker g = S_{\frac{1}{2}}(\lambda)$.

Theorem 5.48. *Assume that* $(\lambda, \tilde{\lambda})$ *is a type III non-isomorphism pair. Then*

$$\mathrm{Hom}_{\mathcal{A}}(S_q(\lambda), S_r(\tilde{\lambda})) = \left\{ \sum_{i=0}^{q-1} u_i g \circ \varphi^i : u_i \in \mathbb{C} \right\}.$$

Proof. Let $0 \neq f : S_q(\lambda) \to S_r(\tilde{\lambda})$. As we saw above, $\ker f = S_l(\lambda)$ where $l \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$. Obviously $S_{1/2}(\lambda) \subset \ker f$. Consider

$$\bar{f} : S_q(\lambda)/S_{1/2}(\lambda) \to S_r(\tilde{\lambda}).$$

The space of such $\bar{f}$ is isomorphic to

$$\mathrm{Hom}_{\mathcal{A}}(R_{q-\frac{1}{2}}(\lambda), S_r(\tilde{\lambda})) \cong \mathrm{Hom}_{\mathcal{A}}(S_{q-\frac{1}{2}}(\tilde{\lambda}), S_r(\tilde{\lambda})) \cong \mathbb{C}^q.$$

Thus we have a linear map

$$\mathrm{Hom}_{\mathcal{A}}(S_q(\lambda), S_r(\tilde{\lambda})) \to \mathrm{Hom}_{\mathcal{A}}(S_{q-\frac{1}{2}}(\tilde{\lambda}), S_r(\tilde{\lambda})) \cong \mathbb{C}^q, \qquad f \mapsto \bar{f}.$$

The Hom space on the left hand side must have dimension less than or equal to q. On the other hand, $g \circ \varphi^i$ for $i = 0, 1, 2, ...q-1$ are linearly independent elements of $\mathrm{Hom}_{\mathcal{A}}(S_q(\lambda), S_r(\tilde{\lambda}))$ and there are q of them. Therefore,

$$\mathrm{Hom}_{\mathcal{A}}(S_q(\lambda), S_r(\tilde{\lambda})) = \mathrm{span}\Big\{ g \circ \varphi^i : 0 \leq i \leq q-1 \Big\} = \left\{ \sum_{i=0}^{q-1} u_i g \circ \varphi^i : u_i \in \mathbb{C} \right\}.$$

■

Here is a summary of the results above.

<u>$\lambda \pm \tilde{\lambda} \notin \mathbb{Z}$</u>: For all λ and $\tilde{\lambda}$,

$$\mathrm{Hom}_{\mathcal{A}}(S_q(\lambda), S_r(\tilde{\lambda})) = 0.$$

<u>$\lambda \pm \tilde{\lambda} \in \mathbb{Z}$</u>:

- $\lambda = \tilde{\lambda} \implies \mathrm{Hom}_{\mathcal{A}}(S_q(\lambda), S_r(\tilde{\lambda})) = \Big\{ \sum_{i=0}^{q-1} u_i \varphi^i : u_i \in \mathbb{C} \Big\} \cong \mathbb{C}^q.$
- λ is of type I or II $\implies \mathrm{Hom}_{\mathcal{A}}(S_q(\lambda), S_r(\tilde{\lambda})) \cong \Big\{ \sum_{i=0}^{q-1} u_i \varphi^i : u_i \in \mathbb{C} \Big\} \cong \mathbb{C}^q.$
- $(\lambda, \tilde{\lambda})$ is a type III isomorphism pair

$$\implies \mathrm{Hom}_{\mathcal{A}}(S_q(\lambda), S_r(\tilde{\lambda})) \cong \Big\{ \sum_{i=0}^{q-1} u_i \varphi^i : u_i \in \mathbb{C} \Big\} \cong \mathbb{C}^q.$$

- $(\lambda, \tilde{\lambda})$ is a type III non-isomorphism pair,

$$\implies \mathrm{Hom}_{\mathcal{A}}(S_q(\lambda), S_r(\tilde{\lambda})) \cong \Big\{ \sum_{i=0}^{q-1} u_i g \circ \varphi^i : u_i \in \mathbb{C} \Big\} \cong \mathbb{C}^q$$

  where $g : S_q(\lambda) \to S_{q-\frac{1}{2}}(\tilde{\lambda})$, image $g = S_{q-\frac{1}{2}}(\tilde{\lambda})$ and $\ker g = S_{\frac{1}{2}}(\lambda)$.

5.4 Possible Submodules of Solution Spaces

Now that we know a complete description of Hom spaces, we will think about the structure of all possible submodules of $\bigoplus_{\lambda \in \Lambda} S_k(\lambda)$. In particular, we will show in the next theorem that all submodules of $\bigoplus_{\lambda \in \Lambda} S_k(\lambda)$ are isomorphic to a canonical submodule $\bigoplus_{\lambda \in \Lambda} S_{l(\lambda)}(\lambda)$ where $l(\lambda) \leq k$. But first, let us describe what the elements of Hom spaces look like.

We want to write down the general form of homomorphisms in $\mathrm{Hom}(\bigoplus_i S_{m_i}(\lambda_i), \bigoplus_j S_n(\lambda_j))$. Introduce $\tau_{ij} : S_k(\lambda_i) \to S_k(\lambda_j)$ by $\tau_{ij} = 0$ if $\lambda_i \pm \lambda_j \notin \mathbb{Z}$ and an isomorphism $\psi(\lambda_i) \mapsto \psi(\lambda_j)$ if $\lambda_i \pm \lambda_j \in \mathbb{Z}$. If λ_i is of type III then we include the additional requirement that (λ_i, λ_j) is a type III isomorphism pair.

If (λ_i, λ_j) is a type III non-isomorphism pair, then τ_{ij} would be modelled on the short exact sequence

$$0 \longrightarrow S_{1/2}(\lambda_i) \longrightarrow S_k(\lambda_i) \xrightarrow{\tau_{ij}} R_{k-\frac{1}{2}}(\lambda_i) \cong S_{k-\frac{1}{2}}(\lambda_j) \longrightarrow 0.$$

For each uniserial component $S_{m_i}(\lambda_i)$, use φ_i to denote its lowering map.

$$\varphi_i : S_{m_i}(\lambda_i) \to S_{m_i}(\lambda_i), \qquad 0 \longrightarrow S_1(\lambda_i) \longrightarrow S_{m_i}(\lambda_i) \xrightarrow{\varphi_i} S_{m_i}(\lambda_i)/S_1(\lambda_i) \longrightarrow 0.$$

Then any $f : \bigoplus_i S_{m_i}(\lambda_i) \longrightarrow \bigoplus_j S_{n_j}(\lambda_j)$ can be written as

$$f = \bigoplus_{i,j} f_{i,j}, \qquad f_{i,j} = \sum_{l=0}^{m_i-1} u_{ijl}\tau_{ij} \circ \varphi_i^l, \qquad u_{ijl} \in \mathbb{C}. \tag{5.33}$$

The terms $u_{ijl}\tau_{ij}$ can be thought of as entries of matrices U_l in the following way.

$$[U_l]_{ij} = u_{ijl}\tau_{ij}.$$

The number of summands S_{m_i} in the domain is the number of columns in U_l. The number of summands S_{n_i} in the codomain is the number of rows of U_l.

In general, the number of rows in U_l is greater than the number of columns because the domain is a submodule of the codomain. If the number of summands is the same on both sides then the number of rows is equal to the number of columns and U_l's become square matrices. Additionally if $m_i = n_i$ for all i and the determinant of U_0 is non-zero, then f is *invertible.*

Lemma 5.49. *Let M_k be the module*

$$M_k = \bigoplus_i S_k(\lambda_i).$$

For all $\bar{f} \in \mathrm{Hom}_{\mathcal{A}}\Big(M_k/(\mathrm{soc}\ M_k), M_k/(\mathrm{soc}\ M_k)\Big)$, there exists an $f \in \mathrm{Hom}_{\mathcal{A}}\Big(M_k, M_k\Big)$ such that f agrees with $\bar{f}$ on the quotient $M_k/(soc\ M_k)$. Furthermore, if $\bar{f}$ is invertible, *then so is f.*

Proof. For type I and II λ, soc $M_k = M_1$.

$$\mathrm{Hom}_{\mathcal{A}}(M_k, M_k) \cong \bigoplus_{i,j} \mathrm{Hom}_{\mathcal{A}}(S_k(\lambda_i), S_k(\lambda_j)).$$

Let $f \in \mathrm{Hom}_{\mathcal{A}}(M_k, M_k)$. Then f acts by

$$f = \bigoplus_{i,j} f_{i,j}, \qquad f_{i,j} = \sum_{l=0}^{k-1} u_{ijl}\tau_{ij} \circ \varphi_i^l, \qquad u_{ijl} \in \mathbb{C}. \tag{5.34}$$

By equation (5.28),

$$\mathrm{Hom}_{\mathcal{A}}(S_k(\lambda_i)/S_1(\lambda_i), S_k(\lambda_j)/S_1(\lambda_j)) \cong \mathrm{Hom}_{\mathcal{A}}(S_{k-1}(\lambda_i), S_{k-1}(\lambda_j))$$

$$= \left\{ \sum_{l=0}^{k-2} u_{ijl}\tau_{ij} \circ \varphi_i^l : u_{ijl} \in \mathbb{C} \right\}.$$

$$\implies \mathrm{Hom}_{\mathcal{A}}(M_k/M_1, M_k/M_1) = \bigoplus_{i,j} \mathrm{Hom}_{\mathcal{A}}(S_k(\lambda_i)/S_1(\lambda_i), S_k(\lambda_j)/S_1(\lambda_j))$$

$$\cong \bigoplus_{i,j} \mathrm{Hom}_{\mathcal{A}}(S_{k-1}(\lambda_i), S_{k-1}(\lambda_j)).$$

Let $\bar{f} \in \mathrm{Hom}_{\mathcal{A}}(M_k/M_1, M_k/M_1)$. $\bar{f}$ acts by

$$\bar{f}(\phi + M_1) = \bigoplus_{i,j} \bar{f}_{i,j}(\phi + M_1), \quad \bar{f}_{i,j}(\phi + M_1) = \sum_{l=0}^{k-2} u_{ijl}\tau_{ij} \circ \varphi_i^l(\phi) + M_1 \quad \forall\ \phi \in M_k.$$

For such an $\bar{f}$, there obviously exists $f \in \mathrm{Hom}_{\mathcal{A}}(M_k, M_k)$ such that f is of the form (5.34) and

$$\forall\ \phi \in M_k, \quad \bar{f}(\phi + M_1) = f(\phi) + M_1.$$

Invertibility of $\bar{f}$ depends on invertibility of the matrix $U_0 = (u_{ij0})$. This matrix is the same for both $\bar{f}$ and f. If U_0 is invertible for $\bar{f}$, then it is also invertible for f. Therefore if $\bar{f}$ is invertible, then so is f.

To include the possibility of λ_i of type III, we can introduce additionally $\tau_{ij} : S_k(\lambda_i) \to$

$S_k(\lambda_j)$ in the case when (λ_i, λ_j) are an isomorphism pair. Namely, in this case, $S_{k-\frac{1}{2}}(\lambda_i) \cong R_{k-\frac{1}{2}}(\lambda_j)$ and τ_{ij} modelled on the short exact sequence

$$0 \longrightarrow S_{1/2} \longrightarrow S_k \xrightarrow{\tau} R_{k-\frac{1}{2}}.$$

With this modification, any $f \in \mathrm{Hom}_{\mathcal{A}}(M_k, M_k)$ can still be written in the form (5.34). The rest of the proof carries over with slight adjustment. ■

Theorem 5.50. *Suppose we take the module*

$$M_k = \bigoplus_{\lambda \in \Lambda} S_k(\lambda),$$

where Λ is a finite subset of $\mathbb{C}$. Then for all submodules $N \subseteq M_k$,

$$N \cong \bigoplus_i N_i, \tag{5.35}$$

$$\text{where } N_i \cong S_{l_i}(\lambda_i),$$

for some $\lambda_i \in \Lambda_0 \subset \Lambda$ and with $1 \leq l_i \leq k$. Furthermore, there exists an automorphism f on M_k such that $f(N) = \bigoplus_i S_{l_i}(\lambda_i)$.

Proof. We prove this by induction on k. For simplicity let us first assume that all λ's are of type I and/or II.

<u>Initial Case</u>: When $k = 1$, $N \subset M_1 = \bigoplus_{\lambda \in \Lambda} S_1(\lambda)$. S_1 is simple for all λ. Therefore M_1 is semisimple. So N must be semisimple and a direct summand in M_1 [31, section 20].

$$\Longrightarrow N = \bigoplus_i N_i,$$

$$\Longrightarrow N_i \cong S_1(\lambda_i)$$

for some λ_i by Jordan-Hölder theorem.

Induction Step: Suppose the statement is true for $k-1$. Let $N \subset M_k$. Consider

$$\bar{N} \subset M_k/M_1 \cong \bigoplus_{\lambda \in \Lambda} S_{k-1}(\lambda), \;\; \bar{N} = N/(N \cap M_1).$$

By induction hypothesis, there exists automorphism $\bar{f}$ on M_k/M_1 such that

$$\bar{f}(\bar{N}) = \bigoplus_i S_{l_i}(\lambda_i)/S_1(\lambda_i).$$

From lemma 5.49, we know that there exists an automorphism $f : M_k \to M_k$ which agrees with $\bar{f}$.

Clearly, it is sufficient to check the statement of the theorem for $f(N) \subset M_k$. For simplicity, we will call $f(N)$ just N from now on and so

$$\bar{N} = \bigoplus_i S_{l_i}(\lambda_i)/S_1(\lambda_i), \;\; \bar{N} = N/(N \cap M_1). \tag{5.36}$$

We now split M_k in to a direct sum $M_k = L \oplus L'$, where

$$L = \bigoplus_i S_k(\lambda_i), \quad L' = \bigoplus_{\lambda \neq \lambda_i} S_k(\lambda).$$

With this direct sum decomposition, we can associate two projection maps: $p : M_k \to L$ and $p' : M_k \to L'$.

From (5.36) and uniseriality of $S_{l_i}(\lambda_i)$, we see that

$$p(N) = \bigoplus_i S_{l_i}(\lambda_i).$$

Let $N' = p'(N) \subset L'$, $K = N \cap L'$. Clearly $K \subset N'$. Note that K can be equivalently described as the kernel of p restricted to N. We have the short exact sequence

$$0 \longrightarrow K \longrightarrow N \xrightarrow{p} \bigoplus_i S_{l_i}(\lambda_i) \longrightarrow 0. \tag{5.37}$$

It follows from (5.36) that $K \subset N' \subset M_1$. Since M_1 is semisimple, K is a direct summand in N'; $N' = K \oplus K'$ for some submodule K'. Now we set $N_0 = N \cap (p')^{-1}(K')$; this is a submodule of N. Then $p'(N_0) = K'$ and so $N = N_0 \oplus K$. Combining this with (5.37) gives us an isomorphism $N_0 \cong \bigoplus_i S_{l_i}(\lambda_i)$. Also, since K is a submodule of M_1 (socle of M_k), it is isomorphic to sum of simples. Up to an automorphism of L', we may assume that $K = \bigoplus_j S_1(\lambda_j)$ for some collection of $\lambda_j \notin \Lambda_0$. As a result,

$$N = N_0 \oplus K \cong \Big(\bigoplus_i S_{l_i}(\lambda_i)\Big) \oplus \Big(\bigoplus_j S_1(\lambda_j)\Big).$$

This proves the first statement of the theorem.

It remains to explain why we may make N_0 equal to $\bigoplus_i S_{l_i}(\lambda_i)$ by a suitable automorphism. For an arbitrary element $a \in N_0$, we can present it as

$$a = p(a) + (a - p(a)),$$

where $p(a) \in \bigoplus_i S_{l_i}(\lambda_i)$ and $a - p(a) \in K'$. Note that the projection p acts bijectively between N_0 and $\bigoplus_i S_{l_i}(\lambda_i)$. Therefore we have a well defined map

$$\phi : \bigoplus_i S_{l_i}(\lambda_i) \to K', \qquad p(a) \mapsto a - p(a).$$

It can be presented in the following form:

$$\phi = \bigoplus_{i,j} c_{ij} \tau_{ij} \circ \varphi_i^{l_i - 1}.$$

Here, $c_{ij} \in \mathbb{C}$, $\varphi_i \in \mathrm{End} S_k(\lambda_i)$ are the lowering maps and τ_{ij} are a set of chosen isomorphisms $\tau_{ij} : S_k(\lambda_i) \to S_k(\lambda_j)$. Then the automorphism f of M_k given by $f = \mathrm{id} - \phi$ will map N_0 to $\bigoplus_i S_{l_i}(\lambda_i)$.

Finally, in the case when there are some type III λ's, the proof is essentially the same but in that case, one takes a quotient by the socle of M_k which will contain some summands of the form $S_{1/2}$.

$\blacksquare$

Now that we know that

$$N \subset \bigoplus_{\lambda \in \Lambda} S_k(\lambda) \implies N \cong \bigoplus_{\lambda \in \Lambda} S_{l(\lambda)}(\lambda), \quad 1 \leq l(\lambda) \leq k,$$

the question arises: what are all the possible N that can fit inside $\bigoplus_{\lambda \in \Lambda} S_k(\lambda)$? Since N must be isomorphic to some standard submodule $\bigoplus_{\lambda \in \Lambda} S_{l(\lambda)}(\lambda)$, our objective reduces to finding all possible N which are isomorphic to this standard module. To make a step towards solving this problem, we study the simple example of submodules of S_2, where λ is not of type III.

The injective elements of $\mathrm{Hom}_{\mathcal{A}}(S_2, S_2)$ are isomorphisms which provide submodules of S_2 which are isomorphic to S_2 (in this simplest case, we know that the only possibility for any such submodules would just be S_2 itself). For any submodule $N \subseteq S_2$, there must exist $f \in \mathrm{Hom}_{\mathcal{A}}(S_2, S_2)$ such that f is injective and $f(S_2) = N$. Proposition 5.28 tells us that in general, $f = u_0 \,\mathrm{id} + u_1\varphi$.

- If $u_0 \neq 0$ and $u_1 \neq 0$ then $f(S_2) = S_2$, so f is injective and this case is fine.
- If $u_0 \neq 0$ but $u_1 = 0$, then $f = u_0 \,\mathrm{id}$. So f is injective and this case is fine as well.
- If $u_0 = 0$ and u_1 is anything, then $f = u_1\varphi$. So $f(S_2) = S_1$ and $\ker f = S_1 \neq 0$. In this case, f is not injective.

Therefore all submodules of S_2 which are isomorphic (in this case, simply equal) to S_2 have isomorphism $f = u_0 \,\mathrm{id} + u_1\varphi$ where $u_0 \neq 0$. The term $u_0 \,\mathrm{id}$ acts on S_1, which is the socle of S_2. So $u_0 \neq 0$ is equivalent to the restriction of f on socle S_1 being injective.

This idea that the first coefficient must not be zero and the homomorphism's restriction onto the socle must be injective generalises as explained below. Our claim is: for a homomorphism to be injective on a module, it must be injective on the socle of that module. Indeed, if $\ker f$ is non-zero, then it contains a simple submodule and it intersects non-trivially with the socle. Recall that the socle of a module is the sum of its simple submodules.

The simple submodules of $\bigoplus_i S_{m_i}(\lambda_i)$ are $S_1(\lambda_i)$ (or $S_{1/2}(\lambda_i)$ if λ_i is of type III) for each i because of the uniserial structure of $S_{m_i}(\lambda_i)$. Hence

$$\operatorname{soc} \bigoplus_i S_{m_i}(\lambda_i) = \bigoplus_i \operatorname{soc} S_{m_i}(\lambda_i),$$

where $\operatorname{soc} S_{m_i}(\lambda_i) = S_{1/2}(\lambda_i)$ if λ is of type III and $\operatorname{soc} S_{m_i}(\lambda_i) = S_1(\lambda_i)$ if λ is not of type III. Let

$$f \in \operatorname{Hom}\Big(\bigoplus_i S_{m_i}(\lambda_i), \bigoplus_j S_{n_j}(\lambda_j)\Big), \quad m_i \leq n_i$$

$$\cong \bigoplus_i \bigoplus_j \operatorname{Hom}\Big(S_{m_i}(\lambda_i), S_{n_j}(\lambda_j)\Big) \quad \text{by theorem 5.35,}$$

$$f = \bigoplus_{i,j} f_{i,j}, \qquad f_{i,j} : S_{m_i}(\lambda_i) \to S_{n_j}(\lambda_j)).$$

Denote the restriction of f on to the socle by f_{soc}:

$$f_{\text{soc}} : \bigoplus_i \operatorname{soc} S_{m_i}(\lambda_i) \to \bigoplus_j S_{n_j}(\lambda_j).$$

Theorem 5.51. *The map $f : \bigoplus_i S_{m_i}(\lambda_i) \to \bigoplus_j S_{n_j}(\lambda_j)$ is injective if and only if $f_{\text{soc}} : \bigoplus_i \operatorname{soc} S_{m_i}(\lambda_i) \to \bigoplus_j S_{n_j}(\lambda_j)$ is injective.*

Proof. If f is injective, then it is injective over soc $\bigoplus_i S_{m_i}(\lambda_i)$. But $f = f_{\text{soc}}$ on $\operatorname{soc} \bigoplus_i S_{m_i}(\lambda_i)$, so f_{soc} is injective.

On the other hand, if f is not injective, then $\ker f \neq 0$. Any finite dimensional non-zero $\mathcal{A}$-module contains a simple submodule, by Jordan-Hölder decomposition. Therefore $\ker f$ intersects non-trivially with the socle and so f_{soc} would not be injective.

■

The above theorem shows that to check whether $f : \bigoplus_i S_{m_i}(\lambda_i) \to \bigoplus_i S_{n_i}(\lambda_i)$ is injective, we need to check that $f_{\text{soc}} : \bigoplus_i \operatorname{soc} S_{m_i}(\lambda_i) \to \bigoplus_j S_{n_j}(\lambda_j)$ is injective.

For all $\phi \in$ soc $\bigoplus_i S_{m_i}(\lambda_i)$, $f(\phi) = f_{\text{soc}}(\phi)$. Soc $\bigoplus_i S_{m_i}(\lambda_i) = \ker \underline{\varphi} \subseteq \ker \varphi^k$ for $k \geq 1$.

So

$$f(\phi) = \bigoplus_{i,j} \left(u_{ij0}\tau_{ij}\,\mathrm{id}(\phi) + \left(\sum_{l=1}^{m_i-1} u_{ijl}\tau_{ij}\varphi_i^{l-1} \right) \varphi_i(\phi) \right)$$

$$= \bigoplus_{i,j} \left(u_{ij0}(\tau_{ij}\phi) + \left(\sum_{l=1}^{m_i-1} u_{ijl}\tau_{ij}\varphi_i^{l-1} \right) \times 0 \right) = \bigoplus_{i,j} u_{ij0}(\tau_{ij}\phi) = f_{\mathrm{soc}}(\phi) \text{ as } \phi \in \mathrm{soc} \bigoplus_i S_{m_i}(\lambda_i).$$

Therefore we need the following to be injective:

$$f_{\mathrm{soc}} = \bigoplus_{i,j} u_{ij0}\tau_{ij}\,\mathrm{id}\,. \tag{5.38}$$

Note that because of the results at the end of section 5.3, $\tau_{ij} = 0$ if $\lambda_i \pm \lambda_j \notin \mathbb{Z}$, so such terms can be dropped. Moreover, if (λ_i, λ_j) is a type III non-isomorphism group, then $\tau_{ij} = 0$ on the socle, so this case can be discarded as well.

We claim that the homomorphism (5.38) would have a trivial kernel if the matrix $U_0 = (u_{ij0}\tau_{ij})$ has maximal rank; the maximal rank would be the number of summands $S_{m_i}(\lambda_i)$ in the domain. Let

$$f : \bigoplus_{i=p}^{r} S_{m_i}(\lambda_i) \to \bigoplus_{j=1}^{t} S_{n_j}(\lambda_j)$$

be an injective homomorphism with $1 \le p \le r \le t$ and $0 \le m_i \le n_i$. By theorem 5.51, f_{soc} would be injective. τ_{ij} is an $\mathcal{A}$-module homomorphism. It is either zero, or by $\mathcal{A}$-linearity, it must be identity (up to a multiple), at least on $S_1(\lambda)$ with respect to any of the standard bases $\{\psi_\pm\}$, $\{\psi'_\pm\}$ or $\{\Psi_1, \Psi_2\}$ (4.34).

f_{soc} is the following homomorphism:

$$f_{\mathrm{soc}} = \bigoplus_{i,j} u_{ij0}\tau_{ij}\,\mathrm{id}, \qquad p \le i \le r, \quad 1 \le j \le t.$$

Let

$$\phi = \sum_{i=p}^{r} \left(a_i^+ \psi_+(\lambda_i) + a_i^- \psi_-(\lambda_i) \right) \in \mathrm{soc} \bigoplus_{i=p}^{r} S_{m_i}(\lambda_i).$$

Then the action of f_{soc} on ϕ can be seen as the following matrix multiplication.

$$f_{\text{soc}}\phi = \begin{pmatrix} u_{p10}\tau_{p1}I & \dots & u_{r10}\tau_{r1}I \\ \vdots & \ddots & \vdots \\ u_{pt0}\tau_{pt}I & \dots & u_{rt0}\tau_{rt}I \\ \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} a_p^+ \\ a_p^- \\ a_{p+1}^+ \\ a_{p+1}^- \\ \vdots \\ a_r^+ \\ a_r^- \end{pmatrix},$$

where $u_{ij0}\tau_{ij}I$ are 2×2 blocks. We can write

$$S_1(\lambda_i) = S_{1/2}^+(\lambda_i) + S_{1/2}^-(\lambda_i),$$

where $S_{1/2}^+(\lambda_i) = \text{span}\{\psi_+(\lambda_i)\}$ and $S_{1/2}^-(\lambda_i) = \text{span}\{\psi_-(\lambda_i)\}$. So $S_{1/2}^+(\lambda_i) \subset S_1(\lambda_i)$ not necessarily as a submodule, but certainly as a subspace.

As a linear map, f_{soc} should be injective on $\bigoplus_{i=p}^r S_{1/2}^+(\lambda_i)$ if it is injective on $\bigoplus_{i=p}^r S_1(\lambda_i)$. The action of the restriction

$$f_{\text{soc}}^+ : \bigoplus_{i=p}^{r} S_{1/2}^+(\lambda_i) \to \bigoplus_{j=1}^{t} S_1(\lambda_j)$$

is represented by the matrix multiplication

$$f_{\text{soc}}^+ \begin{pmatrix} a_p^+ \\ a_{p+1}^+ \\ \vdots \\ a_r^+ \end{pmatrix} = \begin{pmatrix} u_{p10}\tau_{p1} & \dots & u_{r10}\tau_{r1} \\ \vdots & \ddots & \vdots \\ u_{pt0}\tau_{pt} & \dots & u_{rt0}\tau_{rt} \\ 0 & \dots & 0 \\ \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} a_p^+ \\ a_{p+1}^+ \\ \vdots \\ a_r^+ \end{pmatrix}.$$

Then $f_{\text{soc}}^+ : \bigoplus_{i=p}^r S_{1/2}^+(\lambda_i) \to \bigoplus_{j=1}^t S_1(\lambda_j)$ is injective if and only if the matrix representing f_{soc}^+ above has a maximal rank. This is a standard fact in linear algebra.

This matrix also represents the action of $f^-_{\text{soc}} : \bigoplus_{i=p}^r S^-_{1/2}(\lambda_i) \to \bigoplus_{j=1}^t S_1(\lambda_j)$. So $U_0 = (u_{ij0}\tau_{ij})$ has maximal rank if and only if f^+_{soc} and f^-_{soc} are both injective on their own.

Finally, we note that the images of f^+_{soc} and f^-_{soc} do not overlap non-trivially. This is because τ_{ij} is either zero or identity. So,

$$f^+_{\text{soc}}\Big(\bigoplus_{i=p}^r S^+_{1/2}(\lambda_i)\Big) \subset \bigoplus_{j=1}^t S^+_{1/2}(\lambda_j), \qquad f^-_{\text{soc}}\Big(\bigoplus_{i=p}^r S^-_{1/2}(\lambda_i)\Big) \subset \bigoplus_{j=1}^t S^-_{1/2}(\lambda_j),$$

$$\Longrightarrow f^+_{\text{soc}}\Big(\bigoplus_{i=p}^r S^+_{1/2}(\lambda_i)\Big) \cap f^-_{\text{soc}}\Big(\bigoplus_{i=p}^r S^-_{1/2}(\lambda_i)\Big) = 0. \tag{5.39}$$

Equation (5.39) is useful because it helps us see injectiveness of f_{soc}. Let $\phi \in \bigoplus_{i=p}^r \operatorname{soc} S_1(\lambda_i)$ with $\phi \neq 0$. Then

$$\phi = \phi_+ + \phi_-, \quad \text{where} \quad \phi_\pm \in \bigoplus_{i=p}^r S^\pm_{1/2}(\lambda_i).$$

$$f_{\text{soc}}(\phi) = f^+_{\text{soc}}(\phi_+) + f^-_{\text{soc}}(\phi_-).$$

By (5.39), there is no possibility that $f^-_{\text{soc}}(\phi_-)$ might be something which cancels out $f^+_{\text{soc}}(\phi_+)$. Therefore for all $\phi \neq 0$, $f_{\text{soc}}(\phi) \neq 0$ and $\phi \notin \ker f_{\text{soc}}$.

The above reasoning shows that f_{soc} is injective if and only if the matrix $U_0 = (u_{ij0}\tau_{ij})$ has maximal rank. This brings us to the following result.

Theorem 5.52. *Let*

$$\bigoplus_i S_{m_i}(\lambda_i) \subset \bigoplus_j S_{n_j}(\lambda_j), \quad 0 \leq m_i \leq n_i.$$

Then for all the submodules N of $\bigoplus_j S_{n_j}(\lambda_j)$, if

$$N \cong \bigoplus_i S_{m_i}(\lambda_i),$$

then there exists an isomorphism

$$f \in \operatorname{Hom}_{\mathcal{A}}\Big(\bigoplus_i S_{m_i}(\lambda_i), \bigoplus_j S_{n_j}(\lambda_j)\Big),$$

$$\textit{such that } f : \bigoplus_i S_{m_i}(\lambda_i) \to N, \quad f = \bigoplus_{i,j} f_{i,j}, \qquad f_{i,j} = \sum_{l=0}^{m_i-1} u_{ijl}\tau_{ij} \circ \varphi_i^l,$$

where $u_{ijl} \in \mathbb{C}$, $U_l = (u_{ijl}\tau_{ij})$ *are matrices and* $U_0 = (u_{ij0}\tau_{ij})$ *has a maximal rank.*

Let us now link the above results to our analysis in chapter 4.

Consider the case in section 4.4.2. Suppose $N \cong S_k(\lambda)$, where λ is of type I, and N is embedded into $M = \bigoplus_{j\in\mathbb{Z}} S_k(\lambda - j)$. We may identify $S_k(\lambda)$ to each $S_k(\lambda - j)$ by the map

$$\tau_j : S_k(\lambda) \to S_k(\lambda - j), \qquad \partial_\lambda^k \psi_\pm(\lambda) \mapsto \partial_\lambda^k \psi_\pm(\lambda - j).$$

We also require a lowering map φ on $S_k(\lambda)$. With respect to the basis $\{\psi_+, \psi_-\}$, φ can be represented as a matrix of the form

$$\varphi = \begin{pmatrix} \mathbf{0} & \varphi_{12} & \dots & \varphi_{1k} \\ \mathbf{0} & \mathbf{0} & \dots & \varphi_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \end{pmatrix},$$

where φ_{ij} and $\mathbf{0}$ are 2×2 blocks each. There are many different possibilities for φ. As long as $\varphi \times M_0 = M_0 \times \varphi$ and $\varphi \times M_\infty = M_\infty \times \varphi$, any φ of the above form would work.

One idea comes from section 4.3. We know that $(M_\infty - e^{2\pi i\lambda})$ reduces the order of derivatives of ψ_+ by one, and $(M_\infty - e^{-2\pi i\lambda})$ reduces the order of derivatives of ψ_- by one. Therefore one option is to use

$$\varphi := (M_\infty - e^{2\pi i\lambda}) \circ (M_\infty - e^{-2\pi i\lambda}).$$

Another option is to use the following:

$$\varphi = \begin{pmatrix} \mathbf{0} & \beta_1 I & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \beta_2 I & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \beta_{k-2} I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & I \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \end{pmatrix}, \qquad \beta_r = \frac{r}{k-1}.$$

We have the spaces:

$$S_k(\lambda) = \operatorname{span}\Big\{ \partial_\lambda^{k-1-l} \psi_\pm(\lambda) : \quad 0 \le l \le k-1 \Big\},$$

$$N = \operatorname{span}\left\{ \sum_{j=0}^{n} \sum_{p=0}^{k-1} \binom{p}{l} a_{jp} \partial_\lambda^{p-l} \psi_\pm(\lambda - j) : \quad 0 \le l \le k-1 \right\}.$$

Then $f : S_k(\lambda) \longrightarrow N$, $f = \bigoplus f_j$, where

$$f_j = \sum_{m=0}^{k-1} u_{jm} \tau_j \varphi^m,$$

where $u_{jm} \neq 0$ depend on a_{jp} and the choice of φ.

5.5 Link to Exceptional Jacobi Polynomials

Let us explain how the above work relates to the theory of exceptional Jacobi polynomials. In that theory, one constructs "rational extensions" of the DPT operator by a sequence of first order Darboux transformations (see [29], [32] and [33]). The resulting operators are known to be bispectral by the result of [13]. As we will explain, they fit into our scheme as a very special case.

Let us first discuss the notion of the spectral algebra related to polynomial Darboux transformations following [1] and [24]. Suppose we have a factorisation $h(L_x) = Q \circ P$ in our scheme. This ensures that the operator $\mathcal{L} = P \circ Q$ is bispectral, with eigenfunctions $\hat{\psi} = P\psi$ as in theorem 3.3. It may however be possible to have additional differential operators sharing the same eigenfunctions with $\mathcal{L}$. This feature is encapsulated by the following definition (cf. [1, proposition 1.5]):

Definition 5.53. *We have the following algebra of polynomials:*

$$A_W = \{u \in \mathbb{C}[\lambda^2] : u(L_x)W \subset W\}.$$

We also have the following algebra:

$$\mathcal{A}_W = \{P \circ u(L_x) \circ P^{-1} : u \in A_W\}.$$

Remark 1: Elements of $\mathcal{A}_W$ are differential operators. This is because the operator $P \circ u(L_x)$ is divisible on the right by P due to the condition that $u(L_x)W \subset W$.

Remark 2: Elements of $\mathcal{A}_W$ are commuting differential operators since the operators $u(L_x)$ commute. We also have $\left(Pu(L_x)P^{-1}\right)\hat{\psi} = u(\lambda^2)\hat{\psi}$ which shows that these operators share the same family of eigenfunctions and so they will all be bispectral.

Remark 3: It is clear that $h \in A_W$ but there might be smaller degree polynomials u in A_W.

Remark 4: Suppose $P = P' \circ f(L_x)$ for some polynomial f. Let $W' = \ker P'$. Then $Pu(L_x)P^{-1} = P'u(L_x)(P')^{-1}$. Therefore, the algebras $\mathcal{A}_W$ and $\mathcal{A}_{W'}$ coincide. Thus, one may restrict to those P that do not factorise as $P = P' \circ f(L_x)$. This is equivalent to requiring that $\ker(L_x - \lambda^2) \not\subseteq W$ for any λ.

The question arises: when do we have a Schrödinger operator as a member of $\mathcal{A}_W$? The answer is given by the following result.

Theorem 5.54. *The algebra $\mathcal{A}_W$ contains a second order differential operator if and only if $W = \bigoplus_i S_{n_i}(\lambda_i)$ for some λ_i and n_i. Here n_i can be a half-integer for type III λ_i. Furthermore, if we impose the conditions $\ker(L_x - \lambda_i^2) \not\subseteq W$ according to remark 4, then* *we must have* $W = \bigoplus_i S_{1/2}(\lambda_i)$.

Proof. It is clear that $\mathcal{A}_W$ contains a second order differential operator if and only if $\lambda^2 \in A_W$, i.e. if $L_x(W) \subset W$. Since W is finite-dimensional, it must decompose as a direct sum of generalised eigenfunctions:

$$W = \bigoplus W_i,$$

where $W_i \subset \ker(L_x - \lambda_i^2)^{m_i}$ for sufficiently large m_i. By uniseriality of $\ker(L_x - \lambda_i^2)^n$, we must then have

$$W_i = S_{n_i}(\lambda_i),$$

where n_i can be in $\frac{1}{2}\mathbb{Z}$ if λ_i is of type III. This proves the first claim of the theorem.

It remains to notice that imposing the requirement that $S_1(\lambda_i) \not\subseteq W$ means that all n_i must be equal to $\frac{1}{2}$. ■

In more concrete terms, the subspaces W allowed by the above theorem admit a basis of the form $\{\psi_\pm(\lambda_i)\}$, where λ_i are of type III and $\psi_\pm(\lambda_i)$ is the corresponding elementary eigenfunction.

For example, let us choose $\lambda_i = \frac{g+h}{2} + d_i$ for some $0 \leq d_1 < d_2 < ... < d_M$. Then the corresponding elementary eigenfunctions are of the form

$$f_i = \left(\sin\frac{x}{2}\right)^g \left(\cos\frac{x}{2}\right)^h P_{d_i}(\cos x),$$

where $P_n(\cos x)$ is a Jacobi polynomial with parameters $\alpha = g - \frac{1}{2}$ and $\beta = h - \frac{1}{2}$.

The parameters $\lambda_1, ..., \lambda_k \in \left\{\frac{g+h}{2} + \mathbb{Z}_{\geq 0}\right\}$ and so each of ψ_+ is a Jacobi polynomial. Let P be the monic differential operator with the kernel $W = \mathrm{span}\left\{f_1, ..., f_M\right\}$. By the classical result of [34], we then have

$$\hat{L} = PL_xP^{-1} = L_x - 2\frac{d^2}{dx^2}\log Wr,$$

where Wr is the Wronskian $Wr\Big(f_1, ..., f_M\Big)$. These are precisely the rational extensions of the DPT operator known in the theory of exceptional Jacobi polynomials ([29], [32] and [33]).

Further families can be constructed by mixing λ_i of type III from 4 different progressions.

$$\Big\{\lambda_1^{(1)}, ..., \lambda_{k_1}^{(1)}\Big\} \subset \Big\{\rho + \mathbb{Z}_{\geq 0}\Big\},$$

$$\Big\{\lambda_1^{(2)}, ..., \lambda_{k_2}^{(2)}\Big\} \subset \Big\{\rho + \mathbb{Z}_{< 0}\Big\},$$

$$\Big\{\lambda_1^{(3)}, ..., \lambda_{k_3}^{(3)}\Big\} \subset \Big\{\rho - \frac{1}{2} - g + \mathbb{Z}_{\geq 0}\Big\},$$

$$\Big\{\lambda_1^{(4)}, ..., \lambda_{k_4}^{(4)}\Big\} \subset \Big\{\rho - \frac{1}{2} - g + \mathbb{Z}_{< 0}\Big\}.$$

Here, ρ is one of the following:

$$\frac{g+h}{2}, \quad \frac{1-g+h}{2}.$$

These are discussed in e.g. [32].

Chapter 6

Miscellaneous Results

In addition to constructing bispectral extensions for Pöschl-Teller operator, we studied other topics as well. Our motivation for the results in this chapter was to try to find a link to integrable particle dynamics of Calogero–Moser type. Although we did not find definitive answers, our results clearly indicate the possibility of such a connection. To get a better understanding of the matter we also looked at similar questions for the Hermite differential operator.

6.1 Hermite Functions and Quantum Harmonic Oscillators

Let us study how our approach to bispectrality might be applied to Hermite functions. We begin by recalling some basic facts about the Hermite differential equation. It appears in various contexts, for instance, it describes the harmonic oscillator in quantum mechanics.

In standard form, the quantum harmonic oscillator is described by the equation:

$$\left(\frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2\right)\phi(x,n) = (2n+1)\frac{\hbar}{2}\omega\phi(x,n), \tag{6.1}$$

where

- m is mass,
- ω is angular frequency,
- x is position,
- $\hbar$ is the reduced Planck constant,
- p is momentum,
- and $n \in \mathbb{N}$.

$$p = -i\hbar\frac{\partial}{\partial x} \implies p^2 = -\hbar^2\frac{\partial^2}{\partial x^2}.$$

Rearranging (6.1) gives

$$\left[\frac{\partial^2}{\partial x^2} - \left(\frac{m\omega}{\hbar}\right)^2 x^2 + (2n+1)\left(\frac{m\omega}{\hbar}\right)\right]\phi(x,n) = 0. \tag{6.2}$$

The solution to this equation is

$$\phi(x,n) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{\hbar}\frac{x^2}{2}\right)H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) \tag{6.3}$$

where H_n is the *nth Hermite polynomial.* $H_n(z)$ is a solution to the second order ODE:

$$\frac{d^2y}{dz^2} - 2z\frac{dy}{dz} + 2ny = 0. \tag{6.4}$$

Substitute $\lambda = 2n$. Then $H_{\lambda/2}$ would satisfy

$$\frac{d^2y}{dz^2} - 2z\frac{dy}{dz} + \lambda y = 0. \tag{6.5}$$

The other linearly independent solution to (6.5) is the confluent hypergeometric series:

$${}_1F_1\Big(-\frac{1}{4}\lambda, \frac{1}{2}, z^2\Big) = \sum_{k=0}^{\infty} \frac{(-\frac{1}{4}\lambda)_k}{(\frac{1}{2})_k}\frac{z^k}{k!}.$$

We can renormalise (6.2) by replacing $m\omega/\hbar$ by ω.

$$\left[\frac{\partial^2}{\partial x^2} - \omega^2 x^2 + (2n+1)\omega\right]\phi(x,n) = 0 \implies \phi(x,n) = e^{-\frac{\omega x^2}{2}} H_n(\omega^{1/2}x).$$

For simplicity, we can set $\omega = 1$. Then the harmonic oscillator is described by the following eigenvalue problem.

$$L_x\phi(x,\lambda) = -\lambda\phi(x,\lambda), \text{ where } L_x = \frac{\partial^2}{\partial x^2} - x^2 + 1. \tag{6.6}$$

The solutions to (6.6) are

$$\phi_+(\lambda) := e^{-x^2/2}{}_1F_1\Big(-\frac{\lambda}{4}, \frac{1}{2}, x^2\Big) \text{ and} \tag{6.7}$$

$$\phi_-(\lambda) := xe^{-x^2/2}{}_1F_1\Big(-\frac{\lambda-2}{4}, \frac{3}{2}, x^2\Big). \tag{6.8}$$

ϕ_+ and ϕ_- are even and odd functions respectively. For special values of λ these turn into Hermite polynomials:

$$H_{2n}(x) = (-1)^{-n}\frac{(2n)!}{n!}{}_1F_1\Big(-n, \frac{1}{2}, x^2\Big), \tag{6.9}$$

$$H_{2n+1}(x) = 2x(-1)^{-n}\frac{(2n+1)!}{n!}{}_1F_1\Big(-n, \frac{3}{2}, x^2\Big). \tag{6.10}$$

The operator (6.6) admits creation and annihilation operators given by $A_\pm = \pm x - \partial_x$. This is because $L_x = A_+ \circ A_-$ and $L_x - 2 = A_- \circ A_+$. As a result,

$$L_x \circ A_+ = A_+ \circ (L_x - 2), \quad \text{and} \quad (L_x - 2) \circ A_- = A_- \circ L_x.$$

So A_+ maps $\text{span}\{\phi_+(\lambda), \phi_-(\lambda)\}$ to $\text{span}\{\phi_+(\lambda+2), \phi_-(\lambda+2)\}$. More precisely, we have

$$A_+\phi_+(\lambda) = (\lambda+2)\phi_-(\lambda+2), \qquad A_+\phi_-(\lambda) = -\phi_+(\lambda+2). \tag{6.11}$$

In the basis $\{\phi_+, \phi_-\}$, the operator A_+ does not produce the same coefficients for both basis functions. To get the same coefficients, we need to renormalise our basis functions as follows:

$$\psi_+(\lambda) = (-1)^{-\lambda/4} e^{-x^2/2} \frac{\Gamma\left(\frac{\lambda+2}{2}\right)}{\Gamma\left(\frac{\lambda+4}{4}\right)} {}_1F_1\Big(-\frac{\lambda}{4}, \frac{1}{2}, x^2\Big),$$

$$\psi_-(\lambda) = (-1)^{-(\lambda-2)/4} e^{-x^2/2} \frac{\Gamma\left(\frac{\lambda+2}{2}\right)}{\Gamma\left(\frac{\lambda+2}{4}\right)} {}_1F_1\Big(-\frac{\lambda-2}{4}, \frac{3}{2}, x^2\Big) 2x.$$

In the basis $\{\psi_+, \psi_-\}$, we obtain:

$$A\psi_\pm(\lambda) = \psi_\mp(\lambda+2). \tag{6.12}$$

There is a clear parallel between these and the Hermite polynomials (6.9 - 6.10).

In this new basis, we get the same coefficient for both ψ_+ and ψ_-. However, A_+ still changes sign, so that $\psi_+(\lambda) \mapsto \psi_-(\lambda+2)$ and vice versa. Define the new basis,

$$\varphi_+(\lambda) = e^{\frac{i\pi\lambda}{4}} \Big(\cos\Big(\frac{\pi\lambda}{4}\Big)\psi_+(\lambda) - i\sin\Big(\frac{\pi\lambda}{4}\Big)\psi_-(\lambda)\Big),$$

$$\varphi_-(\lambda) = e^{\frac{i\pi\lambda}{4}} \Big(- i\sin\Big(\frac{\pi\lambda}{4}\Big)\psi_+(\lambda) + \cos\Big(\frac{\pi\lambda}{4}\Big)\psi_-(\lambda)\Big). \tag{6.13}$$

In the basis (6.13),

$$A_+\varphi_\pm(\lambda) = \varphi_\pm(\lambda+2).$$

Furthermore, we have the three term recurrence relation (and therefore bispectrality):

$$A_\lambda \varphi_\pm(\lambda) = x\varphi_\pm(\lambda), \tag{6.14}$$

where $A_\lambda = (T + \lambda T^{-1})/2$ and T is the shift operator $\lambda \mapsto \lambda + 2$.

6.2 An Example of a Darboux Factorisation

As with Jacobi case, we look into possible factorisations of $h(L_x)$. Here is an example similar to the one in section 4.5.

Consider the operator

$$h(L_x) = (L_x + \lambda)(L_x + \lambda + 2).$$

This is similar to the operator (4.42). Factorise $h(L_x)$ as $Q \circ P$.

$$\ker P \subset \ker h(L_x) = \mathrm{span}\{\varphi_\pm(\lambda), \varphi_\pm(\lambda + 2)\}.$$

$$\ker P = \mathrm{span}\{\alpha\varphi_\pm(\lambda) + \beta\varphi_\pm(\lambda + 2)\}.$$

With the help of the creation operator $A_+ = x - \partial_x$, we can follow the exact same procedure as in section 4.5 to calculate P.

$$P = \partial_x^2 - \frac{2\alpha\beta}{\alpha^2 + 2\alpha\beta x + \beta^2(\lambda+2)}\partial_x$$

$$+\frac{\alpha^2(\lambda+1) + \beta^2(\lambda+2)(\lambda+3) + 2\alpha\beta(\lambda+2)x - (\alpha^2 + \beta^2(\lambda+2))x^2 - 2\alpha\beta x^3}{\alpha^2 + 2\alpha\beta x + \beta^2(\lambda+2)}. \tag{6.15}$$

What is needed now is an equivalent of theorem 3.3 as well conditions on P which make P satisfy the theorem. We did not pursue this; instead we look in to generalisation of this example. A natural idea would be to take

$$\ker P = \mathrm{span}\{\alpha_i\varphi_\pm(\lambda_i) + \beta_i\varphi_\pm(\lambda_i + 2) : 1 \le i \le n\}.$$

Such choice of kernel of P depends on $2n$ parameters: λ_i and $\gamma_i = \beta_i/\alpha_i$ for $1 \le i \le n$. We expect that there is a dynamical system in which λ_i's play the role of positions of n particles and γ_i are related to their momenta. In the next section we explore the structure of P, and the matrices which will appear in that analysis are expected to be related to Lax matrix of such a dynamical system. However, we will not pursue it any further.

6.3 Generic Darboux Factorisations

6.3.1 Factorisation of $h(L_x) = \prod_{i=1}^n (L_x + \lambda_i)(L_x + \lambda_i + 2)$

Let us factorise

$$h(L_x) = \prod_{i=1}^n (L_x + \lambda_i)(L_x + \lambda_i + 2) = Q \circ P \text{ where } L_x = \partial_x^2 - x^2 + 1. \tag{6.16}$$

We will assume that the values λ_i are in generic positions. Clearly,

$$\ker h(L_x) = \operatorname{span}\{\varphi_\pm(\lambda_i) : 1 \le i \le n\} \oplus \operatorname{span}\{\varphi_\pm(\lambda_i + 2) : 1 \le i \le n\}.$$

Set $W = \ker P$, where

$$W := \operatorname{span}\{f_{i\pm} = \varphi_\pm(\lambda_i) + \gamma_i \varphi_\pm(\lambda_i + 2) : 1 \le i \le n\}.$$

Let us assume that P has the following form:

$$P = \prod_{i=1}^n (L_x + \lambda_i) + \sum_{i=1}^n \Big[b_i \prod_{j \ne i} (L_x + \lambda_j) A_+ + c_i \prod_{j \ne i} (L_x + \lambda_j) \Big]. \tag{6.17}$$

Since P annihilates W,

$$P(f_{k\pm}) = P(\varphi_\pm(\lambda_k)) + \gamma_k P(\varphi_\pm(\lambda_k + 2)) = 0. \tag{6.18}$$

Here,

$$P(\varphi_\pm(\lambda_k)) = \prod_{i=1}^{n}(L_x+\lambda_i)\varphi_\pm(\lambda_k) + \sum_{i=1}^{n}\Big[b_i\prod_{j\neq i}(L_x+\lambda_j)A_+ + c_i\prod_{j\neq i}(L_x+\lambda_j)\Big]\varphi_\pm(\lambda_k)$$

$$= \sum_{i=1}^{n}\Big[b_i\prod_{j\neq i}(\lambda_j-\lambda_k-2)\varphi_\pm(\lambda_k+2)\Big] + c_k\prod_{j\neq k}(\lambda_j-\lambda_k)\varphi_\pm(\lambda_k),$$

and

$$P(\varphi_\pm(\lambda_k+2)) = \left[\prod_{i=1}^{n}(\lambda_i-\lambda_k-2)\right]\varphi(\lambda_k+2) + \sum_{i=1}^{n}2xb_i\left[\prod_{j\neq i}(\lambda_j-\lambda_k-4)\right]\varphi_\pm(\lambda_k+2)$$

$$-(\lambda_k+2)\sum_{i=1}^{n}b_i\left[\prod_{j\neq i}(\lambda_j-\lambda_k-4)\right]\varphi(\lambda_k) + \sum_{i=1}^{n}c_i\left[\prod_{j\neq i}(\lambda_j-\lambda_k-2)\right]\varphi(\lambda_k+2).$$

In the calculation of $P(\varphi_\pm(\lambda_k+2))$ we used the three term recurrence relation (6.14) to eliminate $\varphi_\pm(\lambda_k+4)$.

Put these in to the equation (6.18). The resulting expression is a combination of $\varphi_\pm(\lambda_k)$ and $\varphi_\pm(\lambda_k+2)$. Equating their coefficients to zero gives the following equations:

$$c_k\prod_{j\neq k}(\lambda_j-\lambda_k) - \sum_{i=1}^{n}(\lambda_k+2)b_i\gamma_k\left[\prod_{j\neq i}(\lambda_j-\lambda_k-4)\right] = 0, \tag{6.19}$$

$$\sum_{i=1}^{n}\left[\prod_{j\neq i}(\lambda_j-\lambda_k-2) + 2x\prod_{j\neq i}(\lambda_j-\lambda_k-4)\gamma_k\right]b_i$$
$$+\sum_{i=1}^{n}\left[\prod_{j\neq i}(\lambda_j-\lambda_k-2)\gamma_k\right]c_i + \left[\prod_{i=1}^{n}(\lambda_i-\lambda_k-2)\right]\gamma_k = 0. \tag{6.20}$$

The equations (6.19 and 6.20) can be arranged in to the following matrix equation:

$$\begin{pmatrix} B_1 & C_1 \\ B_2 & C_2 \end{pmatrix}\begin{pmatrix} \mathbf{b} \\ \mathbf{c} \end{pmatrix} = \begin{pmatrix} \mathbf{v} \\ \mathbf{0} \end{pmatrix}. \tag{6.21}$$

Here, $\mathbf{b} = (b_1, ..., b_n), \mathbf{c} = (c_1, ..., c_n)$ and $\mathbf{0} = (0, ..., 0)$. The vector $\mathbf{v}$ is represented by

$$[\mathbf{v}]_k = -\gamma_k \prod_{i=1}^{n} (\lambda_i - \lambda_k - 2), \qquad 1 \le k \le n.$$

We treat k as the row number and i as the column number in each of the four $n \times n$ blocks. These blocks are as follows:

$$[B_1]_{ki} = \prod_{j \ne i} (\lambda_j - \lambda_k - 2) + 2\gamma_k x \prod_{j \ne i} (\lambda_j - \lambda_k - 4),$$

$$[B_2]_{ki} = -(\lambda_k + 2)\gamma_k \prod_{j \ne i} (\lambda_j - \lambda_k - 4),$$

$$[C_1]_{ki} = \gamma_k \prod_{j \ne i} (\lambda_j - \lambda_k - 2), \qquad [C_2]_{ki} = \delta_{ki} \prod_{j \ne i} (\lambda_j - \lambda_k).$$

Here, δ_{ki} is the Kronecker delta.

We conclude that to calculate the operator P in (6.17), one needs to solve the explicit linear system (6.21). The matrix of this linear system has a linear dependence on x. Therefore the coefficients b_i and c_i will be rational functions of x. The singularities of P in the x variable are therefore found from the equation

$$\det \begin{pmatrix} B_1 & C_1 \\ B_2 & C_2 \end{pmatrix} = 0.$$

6.3.2 Factorisation of $h(A_\lambda) = \prod_{i=1}^{n} (A_\lambda - x_i)^2$

Using similar ideas to the ones in previous section, we can also perform factorisation on the dual side. In this case one takes h with repeated roots: $h(A_\lambda) = \prod_{i=1}^{n} (A_\lambda - x_i)^2$. Here, $A_\lambda = (T + \lambda T^{-1})/2$. We have

$$\ker h(A_\lambda) = \mathrm{span}\{\varphi_\pm(x_i, \lambda)\} \oplus \mathrm{span}\left\{ \frac{d}{dx} \varphi_\pm(x, \lambda) \Big|_{x = x_i} \right\}.$$

We want to find a factorisation $h(A_\lambda) = Q_b \circ P_b$. We take the following basis for kernel of P_b.

$$\left\{ f_{i\pm} = \xi_i \varphi_\pm(x_i, \lambda) + \frac{d}{dx}\varphi_\pm(x, \lambda)\Big|_{x=x_i} : 1 \le i \le n \right\}. \tag{6.22}$$

Altogether we have $2n$ parameters x_i and ξ_i $(1 \le i \le n)$. A reasonable guess for P_b looks as follows:

$$P_b = \prod_{i=1}^{n}(A_\lambda - x_i) + \sum_{i=1}^{n}\left[(b_i T + c_i)\prod_{j \ne i}(A_\lambda - x_j)\right]. \tag{6.23}$$

Here b_i and c_i are functions of λ to be determined.

$$P_b\left(\xi_k \varphi(x_k, \lambda) + \partial_x \varphi_\pm(x, \lambda)|_{x=x_k}\right) = 0 \tag{6.24}$$

The equation (6.24) leads to the matrix equation

$$\begin{pmatrix} B_1 & C_1 \\ B_2 & C_2 \end{pmatrix}\begin{pmatrix} \mathbf{b} \\ \mathbf{c} \end{pmatrix} = \begin{pmatrix} \mathbf{v} \\ \mathbf{0} \end{pmatrix}. \tag{6.25}$$

Here, $\mathbf{b} = (b_1, ..., b_n)$, $\mathbf{c} = (c_1, ..., c_n)$ and $\mathbf{0} = (0, 0, ..., 0)$ and

$$[\mathbf{v}]_k = -1, \qquad [B_1]_{ki} = \delta_{ki}(\lambda + 2), \qquad [C_2]_{ki} = -\delta_{ki}, \tag{6.26}$$

$$[B_2]_{ki} = \begin{cases} \xi_k - x_k + \sum_{j \ne k}(x_k - x_j)^{-1} & \text{if } i = k, \\ (x_k - x_i)^{-1} & \text{if } i \ne k. \end{cases}$$

$$[C_1]_{ki} = \begin{cases} \xi_k + x_k + \sum_{j \ne k}(x_k - x_j)^{-1} & \text{if } i = k, \\ (x_k - x_i)^{-1} & \text{if } i \ne k. \end{cases}$$

This shows the possibility of non-trivial higher order Darboux factorisations in the Hermite case. In the next section we will perform a calculation similar to the one in the section 6.3.1. This demonstrates similarities between the Jacobi and the Hermite cases. Therefore one can hope that the results which we obtained earlier for Jacobi case can be extended to the Hermite case. This is an attractive open problem for a future study. Note that in contrast with the Jacobi case, the Hermite differential equation has an essential singularity.

This means that extending our theory to Hermite case is not straightforward.

Our motivation for above calculations was to find a link between Darboux factorisations and particle systems of Calogero–Moser type. Some links of that sort are known from earlier works on bispectral problem (see [7] - [12] and [35]). However such a link in our case is not immediately clear. This is still an open problem.

6.4 Jacobi Case

6.4.1 Factorisation of $h(L_x) = \prod_{i=1}^n (L_x - \lambda_i^2)(L_x - (\lambda_i + 1)^2)$

We follow a similar approach to the one in section 6.3.1 in order to perform the factorisation

$$h(L_x) = \prod_{i=1}^{n}(L_x - \lambda_i^2)(L_x - (\lambda_i + 1)^2) = Q \circ P, \quad \text{where} \quad L_x = -\partial_x^2 + u. \tag{6.27}$$

$$\implies \ker h(L_x) = \operatorname{span}\{\psi_\pm(\lambda_i) : 1 \le i \le n\} \oplus \operatorname{span}\{\psi_\pm(\lambda_i + 1) : 1 \le i \le n\}.$$

Set $W = \ker P$, where

$$W = \operatorname{span}\{f_{i\pm} = \psi_\pm(\lambda_i) + \gamma_i \psi_\pm(\lambda_i + 1) : 1 \le i \le n\}.$$

Note that this is essentially the same choice as in (4.62) because one can always rescale α_i and β_i simultaneously and achieve $\alpha_i = 1$.

We assume that P is of the form:

$$P = \prod_{i=1}^{n}(L_x - \lambda_i^2) + \sum_{i=1}^{n}(a_i \sin x\partial_x + b_i)\prod_{j\neq i}(L_x - \lambda_j^2), \tag{6.28}$$

We place $\sin x\partial_x$ in (6.28) because it is found in the following equation:

$$D_+\psi_\pm(\lambda) = \Big[\sin x\partial_x + \lambda\cos x + d(\lambda)\Big]\psi_\pm(\lambda) = \xi(\lambda)\psi_\pm(\lambda + 1),$$

where

$$d(\lambda) = \frac{g(g-1) - h(h-1)}{2(2\lambda+1)} \quad \text{and} \quad \xi(\lambda) = \frac{(2\lambda+g+h)(2\lambda+g-h+1)}{2(2\lambda+1)}.$$

This implies that for all $\lambda \in \mathbb{C}$,

$$\sin x \partial_x \psi_\pm(\lambda) = \xi(\lambda)\psi_\pm(\lambda+1) - \Big[\lambda \cos x + d(\lambda)\Big]\psi_\pm(\lambda).$$

P must annihilate $f_{k\pm}$ for all $1 \le k \le n$.

$$P(f_{k\pm}) = P(\psi_\pm(\lambda_k)) + \gamma_k P(\psi_\pm(\lambda_k+1)) = 0. \tag{6.29}$$

Here,

$$\begin{aligned} P(\psi_\pm(\lambda_k)) = a_k \prod_{j\neq k}(\lambda_k^2 - \lambda_j^2)\xi(\lambda_k)\psi_\pm(\lambda_k+1) + b_k \prod_{j\neq k}(\lambda_k^2 - \lambda_j^2)\psi_\pm(\lambda_k) \\ - a_k \prod_{j\neq k}(\lambda_k^2 - \lambda_j^2)\Big[\lambda_k \cos x + d(\lambda_k)\Big]\psi_\pm(\lambda_k), \end{aligned}$$

and

$$\begin{aligned} P(\psi_\pm(\lambda_k+1)) = &\prod_{i=1}^{n}((\lambda_k+1)^2 - \lambda_i^2)\psi_\pm(\lambda_k+1) + \sum_{i=1}^{n} b_i \prod_{j\neq i}((\lambda_k+1)^2 - \lambda_j^2)\psi_\pm(\lambda_k+1) \\ &- \sum_{i=1}^{n} a_i \prod_{j\neq i}((\lambda_k+1)^2 - \lambda_j^2)\xi(\lambda_k+1)\left[\frac{4\sin^2(x/2) + A_0(\lambda_k+1)}{A_+(\lambda_k+1)}\right]\psi_\pm(\lambda_k+1) \\ &- \sum_{i=1}^{n} a_i \prod_{j\neq i}((\lambda_k+1)^2 - \lambda_j^2)\xi(\lambda_k+1)\left[\frac{A_-(\lambda_k+1)}{A_+(\lambda_k+1)}\right]\psi_\pm(\lambda_k), \\ &- \sum_{i=1}^{n} a_i \prod_{j\neq i}((\lambda_k+1)^2 - \lambda_j^2)\Big[(\lambda_k+1)\cos x + d(\lambda_k+1)\Big]\psi_\pm(\lambda_k+1), \end{aligned}$$

where $A_+(\lambda)$, $A_-(\lambda)$ and $A_0(\lambda)$ are the coefficients in the three term recurrence relation,

$$A_\lambda \psi_\pm(\lambda) = [A_+(\lambda)T + A_0(\lambda) + A_-(\lambda)T^{-1}]\psi_\pm(\lambda) = -4\sin^2\left(\frac{x}{2}\right)\psi_\pm(\lambda),$$

$$A_\pm(\lambda) = \left(1 \pm \frac{g+h}{2\lambda}\right)\left(1 \pm \frac{g-h}{2\lambda \pm 1}\right), \qquad A_0(\lambda) = -A_+(\lambda) - A_-(\lambda).$$

Substitute $P(\psi_\pm(\lambda_k))$ and $P(\psi_\pm(\lambda_k+1))$ in to (6.29). The resulting equation is a combination of $\psi_\pm(\lambda_k)$ and $\psi_\pm(\lambda_k+1)$. Equating their coefficients to zero results in the following equations:

$$b_k \prod_{j\neq k}(\lambda_k^2-\lambda_j^2) - \sum_{i=1}^{n}\gamma_k a_i \prod_{j\neq i}((\lambda_k+1)^2-\lambda_j^2)\xi(\lambda_k+1)\left[\frac{A_-(\lambda_k+1)}{A_+(\lambda_k+1)}\right]$$
$$-a_k \prod_{j\neq k}(\lambda_k^2-\lambda_j^2)\Big[\lambda_k \cos x + d(\lambda_k)\Big] = 0, \qquad (6.30)$$

$$a_k \prod_{j\neq k}(\lambda_k^2-\lambda_j^2)\xi(\lambda_k) + \gamma_k \prod_{i=1}^{n}((\lambda_k+1)^2-\lambda_i^2) + \sum_{i=1}^{n}\gamma_k b_i \prod_{j\neq i}((\lambda_k+1)^2-\lambda_j^2)$$
$$-\sum_{i=1}^{n}\gamma_k a_i \prod_{j\neq i}((\lambda_k+1)^2-\lambda_j^2)\xi(\lambda_k+1)\left[\frac{4\sin^2(x/2)+A_0(\lambda_k+1)}{A_+(\lambda_k+1)}\right]$$
$$-\sum_{i=1}^{n}\gamma_k a_i \prod_{j\neq i}((\lambda_k+1)^2-\lambda_j^2)\Big[(\lambda_k+1)\cos x + d(\lambda_k+1)\Big] = 0. \qquad (6.31)$$

The equations (6.30) and (6.31) together can be represented by the following matrix equation:

$$\begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix}\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} \mathbf{v} \\ \mathbf{0} \end{pmatrix}. \qquad (6.32)$$

Here, $\mathbf{a} = (a_1, ..., a_n), \mathbf{b} = (b_1, ..., b_n)$ and $\mathbf{0} = (0, ..., 0)$. The vector $\mathbf{v}$ is represented by

$$[\mathbf{v}]_k = -\gamma_k \prod_{i=1}^{n}((\lambda_k+1)^2-\lambda_i^2), \qquad 1 \le k \le n.$$

Once again, in the $n \times n$ blocks A_i and B_i, k is a row number whereas i is a column number.

$$
\begin{aligned}
[A_1]_{ki} = {} & \delta_{ki} \prod_{j \neq k} (\lambda_k^2 - \lambda_j^2) \xi(\lambda_k) - \gamma_k \prod_{j \neq i} ((\lambda_k + 1)^2 - \lambda_j^2) \xi(\lambda_k + 1) \left[\frac{4 \sin^2(x/2) + A_0(\lambda_k + 1)}{A_+(\lambda_k + 1)} \right] \\
& - \gamma_k \prod_{j \neq i} ((\lambda_k + 1)^2 - \lambda_j^2) \Big[(\lambda_k + 1) \cos x + d(\lambda_k + 1) \Big], \\
[A_2]_{ki} = {} & - \gamma_k \prod_{j \neq i} ((\lambda_k + 1)^2 - \lambda_j^2) \xi(\lambda_k + 1) \frac{A_-(\lambda_k + 1)}{A_+(\lambda_k + 1)} - \delta_{ki} \prod_{j \neq k} (\lambda_k^2 - \lambda_j^2) \Big[\lambda_k \cos x + d(\lambda_k) \Big], \\
[B_1]_{ki} = {} & \gamma_k \prod_{j \neq i} ((\lambda_k + 1)^2 - \lambda_j^2), \\
[B_2]_{ki} = {} & \delta_{ki} \prod_{j \neq k} (\lambda_k^2 - \lambda_j^2).
\end{aligned}
$$

The remarks made at the end of section 6.3.2 are fully applicable to the Jacobi case as well. Furthermore, based on an analogy with Lax matrices for the Ruijsenaars-Schneider model [20], we may expect the variables $\lambda_1, \ldots, \lambda_n$ to play the role of the particle positions while the variables $\gamma_1, \ldots, \gamma_n$ should depend exponentially on the momenta. The block structure of the matrix in (6.32) suggests that the relevant particle system should be of the BC_n type, cf. [36].

6.4.2 Matrix Jacobi Function and Bispectrality

The matrices that we obtained above look rather complicated. As indicated in the remarks at the end the previous section, we expected them to be related to the Lax matrix of the rational Koorwinder–van Diejen system [36], but this is not yet clear. One possibile further simplification may arise if we replace L_x, a second order differential operator, with a first order operator. To do that we have to use matrix operators. We hope that this result will be useful in the future study of this problem.

Let us introduce the following differential–reflection operator:

$$y := i(\partial_x + (f + a)s), \tag{6.33}$$

where

$$f=-\frac{1}{2}g\cot\frac{x}{2}+\frac{1}{2}h\tan\frac{x}{2},\qquad a=i\left(\frac{g+h}{2}\right). \tag{6.34}$$

The symbol s represents the transformation $x\mapsto -x$. Squaring the operator y gives us something familiar.

$$y^2=i^2(\partial_x+(f+a)s)^2=-\partial_x^2-f's+f^2-a^2$$

$$=-\partial_x^2+\frac{g(g-s)}{4\sin^2\frac{x}{2}}+\frac{h(h-s)}{4\cos^2\frac{x}{2}}-\left(\frac{g+h}{2}\right)^2-a^2$$

$$=-\partial_x^2+\frac{g(g-s)}{4\sin^2\frac{x}{2}}+\frac{h(h-s)}{4\cos^2\frac{x}{2}}.$$

For $s=1$, this is the Pöschl-Teller operator (2.17). For $s=-1$, this is the Pöschl-Teller operator with $g\mapsto g+1$ and $h\mapsto h+1$. We can interpret y as a matrix differential operator.

$$y=i(\partial_x+(f+a)s)\ \mapsto\ i\begin{pmatrix}0&\partial_x\\ \partial_x&0\end{pmatrix}+i\begin{pmatrix}a&f\\ f&a\end{pmatrix}\begin{pmatrix}1&0\\ 0&-1\end{pmatrix}. \tag{6.35}$$

It is easy to check that taking the square of this matrix operator gives

$$y^2\mapsto-\begin{pmatrix}(\partial_x-f)(\partial_x+f)+a^2&0\\ 0&(\partial_x+f)(\partial_x-f)+a^2\end{pmatrix}$$

$$=\begin{pmatrix}-\partial_x^2+\frac{g(g-1)}{4\sin^2\frac{x}{2}}+\frac{h(h-1)}{4\cos^2\frac{x}{2}}&0\\ 0&-\partial_x^2+\frac{g(g+1)}{4\sin^2\frac{x}{2}}+\frac{h(h+1)}{4\cos^2\frac{x}{2}}\end{pmatrix}:=\begin{pmatrix}L_x^{g,h}&0\\ 0&L_x^{g+1,h+1}\end{pmatrix}$$

We can solve the eigenvalue problem for the matrix representing y.

$$y\underline{\phi}=\lambda\underline{\phi},\qquad \underline{\phi}=\begin{pmatrix}\phi_0\\ \phi_1\end{pmatrix}. \tag{6.36}$$

Since $y^2\underline{\phi} = \lambda^2\underline{\phi}$, it follows that

$$\phi_0 = a_+\psi_+(g,h) + a_-\psi_-(g,h) \in \ker(L_x^{g,h} - \lambda^2),$$
$$\phi_1 = b_+\psi_+(g+1,h+1) + b_-\psi_-(g+1,h+1) \in \ker(L_x^{g+1,h+1} - \lambda^2).$$

If we allow a_+ and a_- to be arbitrary, then that would fix the other pair of constants, $b_\pm$, which need to be computed. The functions $\psi_\pm$ are given as the series (cf. [18]):

$$\psi_\pm(g,h) = C(\pm\lambda)\sin^g\frac{x}{2}\cos^h\frac{x}{2}\sum_{\nu\geq 0}\Gamma_\nu e^{ix(\pm\lambda+\frac{g+h}{2}+\nu)}.$$

We need only to work with the first term of this series.

$$\psi_+(g,h) = C(\lambda)\Big(-\frac{1}{2i}\Big)^g\Big(\frac{1}{2}\Big)^h e^{ix\lambda} + ... \tag{6.37}$$

$$\psi_-(g,h) = C(-\lambda)\Big(-\frac{1}{2i}\Big)^g\Big(\frac{1}{2}\Big)^h e^{-ix\lambda} + ... \tag{6.38}$$

We would also need to know the first term of the infinite series for f (6.34).

$$\cot\frac{x}{2} = \frac{\cos(x/2)}{\sin(x/2)} = i\frac{e^{ix/2}+e^{-ix/2}}{e^{ix/2}-e^{-ix/2}} = -i(1+2e^{ix}+2e^{2ix}+...), \tag{6.39}$$

$$\tan\frac{x}{2} = \frac{\sin(x/2)}{\cos(x/2)} = \frac{1}{i}\frac{e^{ix/2}-e^{-ix/2}}{e^{ix/2}+e^{-ix/2}} = i(1-2e^{ix}+2e^{2ix}+...). \tag{6.40}$$

Putting (6.37 - 6.40) into (6.36) tells us how the coefficients are related.

$$b_\pm = -i\Big(\frac{2\lambda+g+h}{2g+1}\Big)a_\pm.$$

Recall the coefficient (4.49).

$$\xi(g,h,\lambda) = \frac{(2\lambda+g+h)(2\lambda+1+g-h)}{2(2\lambda+1)}.$$

With all of the above, we calculated the following equations.

$$T^{-1}\begin{pmatrix}\phi_0(\lambda)\\ \phi_1(\lambda)\end{pmatrix}=\begin{pmatrix}\phi_0(\lambda-1)\\ \phi_1(\lambda-1)\end{pmatrix}=\frac{1}{\xi(-g,-h,\lambda-1)}\times$$

$$\begin{pmatrix}\left(\lambda-1-\frac{g+h}{2}\right)\left(\cos x-\frac{g-h}{2\lambda-1}\right) & \left(\lambda-1-\frac{g+h}{2}\right)i\sin x\\ \left(\lambda-1+\frac{g+h}{2}\right)i\sin x & \left(\lambda-1+\frac{g+h}{2}\right)\left(\cos x+\frac{g-h}{2\lambda-1}\right)\end{pmatrix}\begin{pmatrix}\phi_0(\lambda)\\ \phi_1(\lambda)\end{pmatrix}. \tag{6.41}$$

$$T\begin{pmatrix}\phi_0(\lambda)\\ \phi_1(\lambda)\end{pmatrix}=\begin{pmatrix}\phi_0(\lambda+1)\\ \phi_1(\lambda+1)\end{pmatrix}=\frac{1}{\xi(g,h,\lambda)}\times$$

$$\begin{pmatrix}\left(\lambda+\frac{g+h}{2}\right)\left(\cos x+\frac{g-h}{2\lambda+1}\right) & -\left(\lambda-\frac{g+h}{2}\right)i\sin x\\ -\left(\lambda+\frac{g+h}{2}\right)i\sin x & \left(\lambda-\frac{g+h}{2}\right)\left(\cos x-\frac{g-h}{2\lambda+1}\right)\end{pmatrix}\begin{pmatrix}\phi_0(\lambda)\\ \phi_1(\lambda)\end{pmatrix}. \tag{6.42}$$

These formulas provide a reformulation of the bispectral pair L_x (1.4) and A_λ (1.5) as a system of first order matrix equation. We can use them to re-derive the formula for A_λ. Namely, combining (6.41) and (6.42), gives (2.21) in matrix form.

$$\begin{pmatrix}A_\lambda^{g,h} & 0\\ 0 & \left(\lambda+\frac{g+h}{2}\right)\circ A_\lambda^{g+1,h+1}\circ\left(\lambda+\frac{g+h}{2}\right)^{-1}\end{pmatrix}\underline{\phi}(\lambda)=-4\sin^2\frac{x}{2}\underline{\phi}(\lambda). \tag{6.43}$$

Here,

$$\begin{pmatrix}A_\lambda^{g,h} & 0\\ 0 & \left(\lambda+\frac{g+h}{2}\right)\circ A_\lambda^{g+1,h+1}\circ\left(\lambda+\frac{g+h}{2}\right)^{-1}\end{pmatrix}$$

$$=\begin{pmatrix}A_+(g,h) & 0\\ 0 & \left(\frac{2\lambda+g+h}{2\lambda+2+g+h}\right)A_+(g+1,h+1)\end{pmatrix}T$$

$$-\begin{pmatrix}A_+(g,h)+A_-(g,h) & 0\\ 0 & A_+(g+1,h+1)+A_-(g+1,h+1)\end{pmatrix}$$

$$+\begin{pmatrix}A_-(g,h) & 0\\ 0 & \left(\frac{2\lambda+g+h}{2\lambda-2+g+h}\right)A_-(g+1,h+1)\end{pmatrix}T^{-1}.$$

$$A_+(g,h) = \left(1 + \frac{g+h}{2\lambda}\right)\left(1 + \frac{g-h}{2\lambda+1}\right),$$

$$A_-(g,h) = \left(1 - \frac{g+h}{2\lambda}\right)\left(1 - \frac{g-h}{2\lambda-1}\right).$$

This can be seen as an alternate proof for theorem 2.1.

The next step would have been to try to express the operator P in (6.28) solely in terms of variable x and operator y (6.33) with the hope of obtaining a simpler structure. For the time being, our understanding of this matter remains incomplete.

Bibliography

[1] Bakalov, B., Horozov, E. and Yakimov, M. Bispectral Algebras of Commuting Ordinary Differential Operators. *Communications in Mathematical Physics*. 1997, **190**(2), pp.331-373.

[2] Haine, L. and Iliev, P. Askey-Wilson Type Functions, With Bound States. *Ramanujan Journal*. 2006, **11**(3), pp.285-329.

[3] Grünbaum, F.A. and Yakimov, M. Discrete Bispectral Darboux Transformations From Jacobi Operators. *Pacific Journal Of Mathematics*. 2002, **204**(2), pp.395-431.

[4] Iliev, P. Bispectral extensions of the Askey–Wilson polynomials. *Journal Of Functional Analysis*. 2014, **266**(4), pp.2294-2318.

[5] Duistermaat, J.J. and Grünbaum, F.A. Differential Equations in the Spectral Parameter. *Communications In Mathematical Physics*. 1986, **103**(2), pp.177-240.

[6] Wilson, G. Bispectral Commutative Ordinary Differential-Operators. *Journal Fur Die Reine Und Angewandte Mathematik*. 1993, **442**, pp.177-204.

[7] Kasman, A. Bispectral KP Solutions and Linearization of Calogero-Moser Particle Systems. *Communications in Mathematical Physics*. 1995, **172**(2), pp.427-448.

[8] Haine, L. and Iliev, P. Commutative Rings of Difference Operators and an Adelic Flag Manifold. *International Mathematics Research Notices*. 2000, no. 6, pp.281-323.

[9] Haine, L., Horozov, E. and Iliev, P. Trigonometric Darboux Transformations and Calogero-Moser Matrices. *Glasgow Mathematical Journal*. 2009, **51A**, pp.95-106.

[10] Haine, L. KP Trigonometric Solitons and an Adelic Flag Manifold. *Symmetry, Integrability and Geometry: Methods and Applications*. 2007, **3**, article number: 015.

[11] Chalykh, O.A. and Nijhoff, F.V. Bispectral Rings of Difference Operators. *Russian Mathematical Surveys*. 1999, **54**(3), pp.644-645.

[12] Iliev, P. q-KP Hierarchy, Bispectrality and Calogero-Moser Systems. *Journal of Geometry and Physics*. 2000, **35**(2-3), pp.157-182.

[13] Odake, S. Recurrence Relations of the Multi-Indexed Orthogonal Polynomials. *Journal of Mathematical Physics*. 2013, **54**(8), article number: 083506.

[14] Szegö, G. Orthogonal Polynomials. Colloquium Publications, American Mathematical Society. 1939, 23.

[15] Andrews, G.E., Askey, R. and Roy, R. *Special Functions*. Cambridge: Cambridge University Press, 1999.

[16] Grünbaum, F.A. and Haine, L. Associated Polynomials, Spectral Matrices and the Bispectral Problem. *Methods and Applications of Analysis*. 1999, **6**(2), pp.209–224.

[17] Grünbaum, F.A. Haine, L. and Horozov, E. Some Functions That Generalize The Krall-Laguerre Polynomials. *Journal of Computational and Applied Mathematics*. 1999, **106**(2), pp.271-297.

[18] Heckman, J.G. and Opdam, E.M. Root systems and hypergeometric functions. *Compositio Mathematica*. 1987, **64**(3), pp.329-352.

[19] Pöschl, G. and Teller, E. Bemerkungen zur quantenmechanik des anharmonischen oszillators. *Zeitschrift für Physik*. 1933, **83**(3-4), 143-151.

[20] Ruijsenaars, S. Finite-dimensional soliton systems. *Integrable and superintegrable systems* (B. Kupershmidt, Ed.). World Scientific, Singapore, 1990. pp.165-206.

[21] Chalykh, O.A. Bispectrality for the quantum Ruijsenaars model and its integrable deformation. Journal Of Mathematical Physics. 2000, **41**(8), pp.5139-5167.

[22] Whittaker, E. T. and Watson, G. N. *A Course of Modern Analysis*. 3rd ed. Cambridge: Cambridge University Press, 1920.

[23] Ince, E.L. *Ordinary Differential Equations*. New York: Dover Publications Inc. 2003.

[24] Bakalov, B., Horozov, E. and Yakimov, M. General Methods for Constructing Bispectral Operators. *Physics Letters A*. 1996, **222**(1-2), pp.59-66.

[25] Brown, J.W. and Churchill, R.V. *Complex Variables and Applications*. 8th ed. New York: McGraw-Hill Higher Education, 2008.

[26] Greene, R.E. and Krantz, S.G. *Function Theory of One Complex Variable*. 3rd ed. Providence: American Mathematical Society, 2006.

[27] Beukers, F. *Notes on differential equations and hypergeometric functions*. [Online]. 2009. [Accessed 20 April 2018]. Available from: http://pages.uoregon.edu/njp/beukers.pdf

[28] Odake, S. and Sasaki, R. Unified theory of annihilation-creation operators for solvable (discrete) quantum mechanics. *Journal of Mathematical Physics*. 2006, **47**(10), [no pagination].

[29] Sasaki, R., Tsujimoto, S. and Zhedanov, A. Exceptional Laguerre and Jacobi polynomials and the corresponding potentials through Darboux-Crum transformations. *Journal of Physics A: Mathematical and Theoretical*. 2010, **43**(31), [no pagination].

[30] Adkins, W. and Weintraub, S. *Algebra: An Approach via Module Theory*. New York: Springer-Verlag, 1992.

[31] Wisbauer, R. *Foundations of Module and Ring Theory: A Handbook for Study and Research*. Reading: Gordon and Breach Science Publishers, 1991.

[32] Odake, S. and Sasaki, R. Exactly Solvable Quantum Mechanics and Infinite Families of Multi-indexed Orthogonal Polynomials. *Physics Letters B*. 2011, **702**(2-3), pp.164-170.

[33] Gómez-Ullate, D., Kamran, N. and Milson, R. An extension of Bochner's problem: Exceptional invariant subspaces. *Journal of Approximation Theory*. 2010, **162**(5), pp.987-1006.

[34] Crum, M. Associated Sturm-Liouville Systems. *The Quarterly Journal of Mathematics*. 1955, **6**(1), pp.121-127.

[35] Wilson, G. Collisions of Calogero-Moser Particles and an Adelic Grassmannian. *Inventiones mathematicae*. 1998, **133**(1), pp.1-41.

[36] Pusztai, B. G. The Hyperbolic BC_n Sutherland and the Rational BC_n Ruijsenaars-Schneider-van Diejen Models: Lax matrices and Duality. *Nuclear Physics B*. 2011, **856**(2), pp.528-551.