Triple vector bundles in Differential Geometry

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Ἀεὶ ὁ Θεὸς ὁ Μέγας γεωμετρεῖ

Τὸ κύκλου μῆκος ἵνα ὁρίσῃ διαμέτρῳ

Παρήγαγεν ἀριθμὸν ἀπέραντον

καὶ ὅν φεῦ! οὐδέποτε ὅλον θνητοὶ θὰ εὕρωσι.

ΝΙΚΟΛΑΟΣ ΧΑΤΖΗΔΑΚΙΣ

Abstract

The triple tangent bundle T^3M of a manifold M is a prime example of a triple vector bundle. The definition of a general triple vector bundle is a cube of vector bundles that commute in the strict categorical sense. We investigate the intrinsic features of such cubical structures, introducing systematic notation, and further studying linear double sections; a generalization of sections of vector bundles.

A set of three linear double sections on a triple vector bundle E yields a total of six different routes from the base manifold M of E to the total space E. The underlying commutativity of the vector bundle structures of E leads to the concepts of warp and ultrawarp, concepts that measure the noncommutativity of the six routes. The main theorem shows that despite this noncommutativity, there is a strong relation between the ultrawarps. The methods developed to prove the theorem rely heavily on the analysis of the core double vector bundles and of the ultracore vector bundle of E.

This theorem provides a conceptual proof of the Jacobi identity, and a new interpretation of the curvature of a connection ∇ on a vector bundle A. We expect these methods to be capable of further development, and to apply in a wider variety of situations.

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Contents

List of Tables

Glossary

- **grid on a double vector bundle** D: a pair of linear sections, see Definition 0.2.1.
- **warp of a grid on** D: a section of the core vector bundle of D, see Definition 0.2.2. It measures the lack of commutativity of the grid; see diagram (10).
- **struts over** A: non-linear sections of $D \to A$, defined by sections of the core; see Definition 1.1.3. These project to the zero section 0^B, and were called *core sections over* A in [25, p. 347].
- **bolt of** φ: a linear section $\varphi^{\natural}$ of D that projects to the zero section 0^B, defined by a vector bundle map $\varphi : A \to C$ over M, see Definition 4.2.6.

Introduction

0.1 Some history

Double vector bundles have been implicitly present in the literature of Mathematics since at least Dieudonné's treatment of connection theory in [5]. Pradines was the first to give a systematic and general treatment of the subject in [33]. Since the early 1990s, double vector bundles have been used in several areas. A few (but by no means a complete list of) such areas are the following:

- Poisson geometry has used double structures extensively, since at least the early 1990s, for example, see [28], [29], [21], [37].
- Double vector bundles and their relation to Lie algebroid theory have been studied in [20], [23], [26], [15], [12], and [3].
- Classical mechanics has also used double vector bundles in formulations and applications, for example, see [10], [35], [11].

Besides applications, double vector bundles have their own rich theory. Their *duality* was introduced and developed by Mackenzie [21], and has surprising properties. A recent account with references can be found in [25, Chap. 9] and [13].

Another important feature is the *warp*, introduced in [27]. Once linear sections on double vector bundles are defined, warps emerge naturally. Various identities of differential geometry can be then described as applications of this concept.

The concepts of warp and linear sections can be extended to triple vector bundles. The first serious treatments of triple vector bundles were given in [24], [13], and [38]. Our primary objective in this thesis is the systematic treatment of these concepts.

A general double vector bundle is quite distinct from a (strict) 2-vector bundle; double vector bundles are double structures in the sense of Ehresmann, [7]. We are also not considering relations between double vector bundles and graded vector bundles.

So what is a double vector bundle? The definition of a double vector bundle consists of three parts: (i) the algebraic compatibility conditions, (ii) the *double source condition*, and (iii) the existence of *sigma maps*.

For part (i), the algebraic conditions are efficiently covered by the following definition from [25, Definition 9.1.1].

Definition 0.1.1. (Part (i)) A *double vector bundle* $(D; A, B; M)$ is a system of four vector bundle structures

$$\begin{array}{ccc} D & \xrightarrow{q^D_B} & B \\ {\scriptstyle q^D_A}\downarrow & & \downarrow{\scriptstyle q_B} \\ A & \xrightarrow{q_A} & M, \end{array} \tag{1}$$

in which D has two vector bundle structures, on bases A and B, and both A and B are vector bundles on M. In addition, each of the four structure maps of each vector bundle structure on D, that is, the bundle projection, addition, scalar multiplication and the zero section, is a morphism of vector bundles with respect to the other structure.

This is a well-established and widely used definition of a general double vector bundle D, and in practice, the algebraic compatibility conditions are the ones we check when establishing that a square of vector bundles is a double vector bundle. For the purpose of this thesis though, Definition 0.1.1 is not strong enough. We present parts (ii) and (iii) in Section 1.1.1, and further discuss their significance.

An equivalent and often more practical way of describing the algebraic compatibility conditions on D are the *interchange laws*, presented in Section 1.1.

Definition 0.1.2. Given an element $d \in D$, let $q^D_A(d) = a$, $q^D_B(d) = b$, and $q_A(a) = q_B(b) = m$. The first diagram in (2) comprised by these projections is called the *outline* of d.

Given another element d' as shown, the sum over A has the outline shown in the third figure.

$$\begin{array}{ccc} d & \longmapsto & b \\ \downarrow & & \downarrow \\ a & \longmapsto & m, \end{array} \qquad \begin{array}{ccc} d' & \longmapsto & b' \\ \downarrow & & \downarrow \\ a & \longmapsto & m, \end{array} \qquad \begin{array}{ccc} d \underset{A}{+} d' & \longmapsto & b + b' \\ \downarrow & & \downarrow \\ a & \longmapsto & m. \end{array} \tag{2}$$

For elements d which project to zeros under both bundle projections, that is, elements that are in the intersection $\operatorname{Ker}(q^D_A) \cap \operatorname{Ker}(q^D_B)$, the two additions and the two scalar multiplications coincide. Under these operations the set of such elements forms a vector bundle over M, called the *core of* D [33], usually denoted by C. More details on C in Section 1.1.2.

Now suppose that $(d; a, b; m)$ and $(d'; a', b'; m)$ have $a = a'$ and $b = b'$. Then there is a unique $c \in C$ such that

$$d = d' \underset{A}{+} (c \underset{B}{+} 0^D_a) = d' \underset{B}{+} (c \underset{A}{+} 0^D_b). \tag{3}$$

In equations of this type, what is important is that the difference $d - d'$, calculated in either structure, results to the same core element c plus an appropriate zero. We will indicate this by

$$d - d' \triangleright c, \tag{4}$$

and we use this notation from subsection 3.1.2 onwards.

The following are two fundamental examples that arise from an arbitrary vector bundle A; the *tangent and the cotangent double vector bundle.*

$$\begin{array}{ccc} TA & \xrightarrow{T(q)} & TM \\ {\scriptstyle p_A}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ A & \xrightarrow{q} & M, \end{array} \qquad \begin{array}{ccc} T^*A & \xrightarrow{r} & A^* \\ {\scriptstyle c_A}\big\downarrow & & \big\downarrow{\scriptstyle q_*} \\ A & \xrightarrow{q} & M. \end{array} \tag{5}$$

The tangent bundle TA of an arbitrary vector bundle $A \to M$ has two vector bundle structures: the usual tangent bundle structure $TA \xrightarrow{p_A} A$, and the *tangent prolongation structure* $TA \xrightarrow{T(q)} TM$. The latter structure is obtained once we apply the tangent functor to all the vector bundle operations of $A \to M$. The tangent double vector bundle TA is described in detail in [25, Section 3.4] and in [34, Ch.9]. Specifically about the vector bundle structure of TA over TM, see [5, (16.15.7)].

As the following will be used again and again throughout calculations in most chapters, we briefly state how we add two elements $\xi_1, \xi_2 \in T(q)^{-1}(v)$ in the same fibre of $TA \to TM$. Since $T(q)(\xi_1) = T(q)(\xi_2)$, we can write

$$\xi_1 = \frac{d}{dt}a_1(t)\Big|_{t=0}, \quad \xi_2 = \frac{d}{dt}a_2(t)\Big|_{t=0},$$

for $a_1(t)$ and $a_2(t)$ two curves in A, with $q(a_1(t)) = q(a_2(t))$, for t near zero, see Proposition 1.2.2. Define:

$$\xi_1 \underset{TM}{+} \xi_2 = \frac{d}{dt}(a_1(t) + a_2(t))\Big|_{t=0}. \tag{6}$$

More specifically, for $F \in C^\infty(TA)$:

$$(\xi_1 \underset{TM}{+} \xi_2)(F) = \frac{d}{dt}F(a_1(t) + a_2(t))\Big|_{t=0}. \tag{7}$$

Dualizing TA over A yields the *cotangent double vector bundle* T^*A. This is described in detail in [28] and [25, Section 9.4]. We briefly present the necessary formulas for our work in Section 2.4.5.

Of course, as with all Mathematics, there is more than one way of working with double and with triple vector bundles.

- Focusing on the intrinsic structure of these geometric objects, as in [25, Chapter 9], and in [24]. In this way of working, core vector bundles and core double vector bundles are crucial. The main bulk of our work is done in this fashion, see Chapters 2, 3, 4.

- Using decompositions, as in [27]. We explain this in detail in Sections 1.1.1 and 2.4.1.

- Using local coordinates. A few papers that apply this method of work are [36], [37], [35], and [3] for double vector bundles, and [38] for triple vector bundles.

  We present T^2M, TA, a general double vector bundle D, a general triple vector bundle E, and then T^2A in local coordinates. Some of the key concepts are described using local coordinates, see Sections 1.2.1, 1.2.3, 1.1.3, 2.1.3, and 2.4.4.

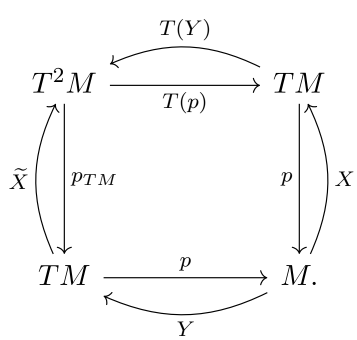
- Dual frames, as in [15, p.5], and in [30]. This is another way of working locally; once a decomposition and local coordinates on D are chosen, one can describe sections of the vector bundle structures of D using dual frames.

0.2 Warps and grids in double vector bundles

The original motivating example for the concepts of grid and warp lies in [1, p.297], where the authors give the following formula for the Lie bracket of vector fields X and Y on a manifold M,

$$T(Y)(X(m)) - \widetilde{X}(Y(m)) = ([X,Y])^{\uparrow}(Y(m)). \tag{8}$$

Here $\widetilde{X}$ denotes the complete lift of X to a vector field on TM and the uparrow denotes the vertical lift to TM of the vector $[X,Y](m)$ to $Y(m)$. The complete lift, or tangent lift, $\widetilde{X}$ is $J_M \circ T(X)$ where $J_M : T^2M \to T^2M$ is the canonical involution which interchanges the two bundle structures on T^2M. The double vector bundle T^2M is the tangent double vector bundle of $TM \xrightarrow{p} M$, called the *double tangent bundle*. Its core vector bundle is yet a third copy of TM. We elaborate on the double vector bundle T^2M, on the J_M map, and on the vector fields $\widetilde{X}$, $X^{\uparrow}$ in Section 1.2. The left hand side of (8) is encapsulated in (9).



(9)

If we look at the elements $T(Y)(X(m))$ and $\widetilde{X}(Y(m))$, we see that they have the same outlines

$$\begin{array}{ccc} T(Y)(X(m)) & \longmapsto & X(m) \\ \downarrow & & \downarrow \\ Y(m) & \longmapsto & m, \end{array} \qquad \begin{array}{ccc} \widetilde{X}(Y(m)) & \longmapsto & X(m) \\ \downarrow & & \downarrow \\ Y(m) & \longmapsto & m. \end{array}$$

The two elements therefore determine a core element $\bar{c} \in TM$. Taking $d = T(Y)(X(m))$ and $d' = \widetilde{X}(Y(m))$ in (3), we have

$$T(Y)(X(m)) - \widetilde{X}(Y(m)) = \bar{c} \underset{T(p)}{+} 0^{T^2M}_{Y(m)},$$

where the subtraction on the left is the usual subtraction of vectors which are tangent to TM at $Y(m)$, and the addition on the right is addition in $T(p) : T^2M \to TM$. That is, $\bar{c} \underset{T(p)}{+} 0^{T^2M}_{Y(m)}$ is the vertical lift of c to $Y(m)$ and so, by (8), $c = [X, Y](m)$.

A comment on the notation of the last equation. In the case of a general double vector bundle D, the two additions $\underset{A}{+}$ and $\underset{B}{+}$ are clearly distinct. In the case of T^2M however, both side bundles are copies of TM. To distinguish between the two additions, we use the projection maps, for example, addition in $T^2M \xrightarrow{T(p)} TM$ will be denoted by $\underset{T(p)}{+}$. We adopt this notation whenever necessary, especially in Sections 4.5 and 4.6.

Equation (8) can be proved either in local coordinates, as in [31, Section 8.14], or in terms of the action of vector fields on linear and pullback functions, by applying directly [25, Theorem 3.4.5] for $D = L_X$, the Lie derivative of the vector field X.

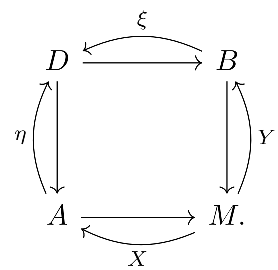
The use of (9) expresses the result in a compact conceptual way. To the best of our knowledge, the first time equation (8) appeared in the literature of Mathematics is the 1988 edition of the book [1] by Abraham, Marsden and Raţiu.

We now generalize the picture (9) to any double vector bundle D. The following is [25, Definition 10.3.1].

Definition 0.2.1. A pair of sections $X \in \Gamma A$ and $\xi \in \Gamma_B D$ form a *linear section* of D if ξ is a morphism of vector bundles over X.

A *grid on* D is a pair of linear sections (ξ, X) and (η, Y) as shown in (10).

A section $\xi \in \Gamma_A D$ of $D \to A$ is *q-projectable* if there exists a section $X \in \Gamma B$ such that $q^D_B \circ \xi = X \circ q_A$. A linear section (ξ, X) projects to its *base section* $X \in \Gamma B$. See [15, p.6] for more details.



(10)

For each $m \in M$, $\xi(Y(m))$ and $\eta(X(m))$ have the same outline. They therefore determine an element of the core C and, as m varies, a section of C which we denote $\mathrm{w}(\xi, \eta)$. More precisely,

$$\begin{aligned} \xi(Y(m)) \underset{A}{-} \eta(X(m)) &= \mathrm{w}(\xi,\eta)(m) \underset{B}{+} 0^D_{X(m)}, \\ \xi(Y(m)) \underset{B}{-} \eta(X(m)) &= \mathrm{w}(\xi,\eta)(m) \underset{A}{+} 0^D_{Y(m)}. \end{aligned} \tag{11}$$

Definition 0.2.2. The *warp* of the grid consisting of (ξ, X) and (η, Y) is $\mathrm{w}(\xi, \eta) \in \Gamma C$.

Equation (8) can now be expressed as saying that *the warp of (9) is* $[X, Y]$.

Adopting the notation introduced in (4), we may write (11) succinctly as

$$\xi(Y(m)) - \eta(X(m)) \triangleright \mathrm{w}(\xi,\eta)(m). \tag{12}$$

The sign of the warp $\mathrm{w}(\xi, \eta)$ changes if ξ and η are interchanged. Our convention gives the positive sign to the counterclockwise composition $\xi \circ Y$.

The question of signs — or orientations — is omnipresent throughout the thesis. In the double vector bundle setting, certain rules follow from established conventions of Differential geometry, as in (9). Later on, we will see that in many cases, arbitrary but consistent choices must be made to determine which difference to take as the positive warp (see Remark 3.1.5).

0.3 Main results

The main theorem of the thesis is Theorem 3.1.4. This result first appeared in [27] with a proof that relied on the use of decompositions for triple vector bundles. In Section 3.2 we give a different and genuinely geometric proof, based on a new technique using exclusively the intrinsic structure of triple vector bundles, developed in Chapters 2 and 3.

As an application of Theorem 3.1.4, we derive Definition 4.5.1 in Chapter 4. There are a variety of formulations of the definition of curvature of a connection depending on whether one is working with vector bundles, principal bundles or general fibre bundles.

A guiding principle which holds in all these cases is that the curvature measures the difference between the bracket of horizontal lifts and the horizontal lift of the bracket of two vector fields. This is implicit in many treatments of connection theory; see, for example [5]. In Chapter 4, using the warp, bolts, and grids language, we give a new explicit proof that this principle, which we formulate as Definition 4.5.1, leads to one of the standard formulas of curvature.

In a double vector bundle D, the $C^\infty(B)$-module of sections $\Gamma_B D$ is generated by the *linear sections* and the sections which arise from sections of the core; these latter were called *core sections* of D in [26, Proposition 3.2]. In Definition 1.1.3 we introduce the less confusing term 'strut' for them. These two kinds of sections of D have been largely used in the literature, [26], [15], etc. The linear sections that project to the zero section in particular, called *core-linear sections*, were introduced in [12] and [17]. These sections are central to Definition 4.5.1, and in Chapter 4 we call these the *bolt sections.* Bolt sections arise naturally when taking the difference of the horizontal lift and the complete lift of a vector field, see Section 4.2.4. We also give the corresponding analog of these sections in the triple vector bundle setting, which we call *double bolt sections.*

Some of the results of this thesis, notably, the proof of Theorem 3.1.4 and Definition 4.5.1 appear in [8].

0.4 Outline of thesis

In Chapter 1, the Background chapter, we present all the necessary theory concerning double vector bundles on which we build in the following chapters. Most of the material can be found in references given therein. Section 1.1.1, where we prove the existence of nontrivial grids in a double vector bundle, is presented there for the first time.

In Chapter 2 we give a systematic treatment of the intrinsic structure of triple vector bundles, which does not rely on decompositions or local coordinates. We set up the notation and the operations on triple vector bundles. This is a nontrivial extension of double vector bundle theory.

In Chapter 3, we formulate and prove Theorem 3.1.4, which we call the *warp-grid theorem.* This is the heart of the thesis. The proof is a lengthy and intricate application of the techniques developed in Chapter 2. Despite the technical nature of the proof, we believe Theorem 3.1.4 is a natural result, and we explain the grounds for our belief fully in Remark 3.2.1. A detailed outline of the proof is given in Section 3.2.

In Chapter 4, we present the *bolt* and the *double bolt* sections of a double vector bundle and of a triple vector bundle respectively, and in Section 4.4 we present a class of examples of grids on a triple vector bundle E invovling two double bolt sections. Section 4.5 examines a grid on T^2A obtained from a connection ∇ on the vector bundle $A \xrightarrow{q} M$, and the curvature of ∇. Section 4.6 describes the first instance of the warp-grid theorem, the Jacobi identity, which was introduced in [27, Section 3].

0.5 Future developments

We have not considered here the question of grids and warps in 4-fold vector bundles or the general case of n-fold vector bundles. We expect however that the cases of n odd and n even will exhibit different behaviour.

The question of bracket structures on triple vector bundles will be treated in a separate publication.

0.6 Notation and Conventions

All manifolds are smooth, real, finite dimensional, Hausdorff and second-countable.

All vector bundles are smooth, real, and of finite rank. We denote a vector bundle by $A \xrightarrow{q} M$. The dual vector bundle to $A \xrightarrow{q} M$ is denoted by $A^* \xrightarrow{q_*} M$.

In conclusion

In conclusion, I would like to express my best thanks to Yvette Kosmann-Schwarzbach, Theodore Voronov, and Ping Xu for lengthy conversations at different stages of the thesis.

Chapter 1

Background

1.1 Preliminaries in double vector bundles

As mentioned in the Introduction, the definition of a double vector bundle has three parts. And the first part, which is that the operations of $D \to A$ be vector bundle morphisms with respect to $D \to B$ (or equivalently, that the operations of $D \to B$ be vector bundle morphisms with respect to $D \to A$), is equivalent to interchange laws.

Indeed, let us draw our attention to the addition in $D \to A$. That the addition in $D \to A$ is a morphism of vector bundles with respect to the structure $D \to B$:

$$\begin{array}{ccc} D \underset{A}{\times} D & \xrightarrow{\underset{A}{+}} & D \\ \Big\downarrow{\scriptstyle q^D_B * q^D_B} & & \Big\downarrow{\scriptstyle q^D_B} \\ B \underset{M}{\times} B & \xrightarrow[+]{} & B, \end{array}$$

where $D \times_A D$ is a vector bundle over $B \times_M B$, means that fibrewise

$$\begin{aligned} \underset{A}{+} : D \underset{A}{\times} D\Big|_{(b_1,b_2)} &\to D\Big|_{b_1+b_2}, \\ (d_1, d_2) &\mapsto d_1 \underset{A}{+} d_2, \end{aligned}$$

is a linear map. Hence for $(d_1, d_2), (d_3, d_4) \in D \times_A D\Big|_{(b_1,b_2)}$, two elements in the same fibre over $(b_1, b_2) \in B \times_M B$, we have

$$\underset{A}{+}\left((d_1, d_2) \underset{B \underset{M}{\times} B}{+} (d_3, d_4)\right) = \left(\underset{A}{+}(d_1, d_2)\right) \underset{B}{+} \left(\underset{A}{+}(d_3, d_4)\right),$$

which we rewrite as the following *interchange law*:

$$(d_1 \underset{A}{+} d_2) \underset{B}{+} (d_3 \underset{A}{+} d_4) = (d_1 \underset{B}{+} d_3) \underset{A}{+} (d_2 \underset{B}{+} d_4). \tag{1.1}$$

Of course since $(d_1, d_2) \in D \times_A D$ we have that $q_A^D(d_1) = q_A^D(d_2) = a_1$, and similarly for $(d_3, d_4) \in D \times_A D$, $q_A^D(d_3) = q_A^D(d_4) = a_3$. In total, $(d_i; a_i, b_i; m)$, $i = 1, \ldots, 4$, have $a_1 = a_2$, $a_3 = a_4$, $b_1 = b_3$ and $b_2 = b_4$. The outlines of the four elements:

$$\begin{array}{ccccccccccc} d_1 & \longmapsto & b_1 & & d_2 & \longmapsto & b_2 & & d_3 & \longmapsto & b_1 & & d_4 & \longmapsto & b_2 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ a_1 & \longmapsto & m, & & a_1 & \longmapsto & m, & & a_2 & \longmapsto & m, & & a_2 & \longmapsto & m. \end{array}$$

Similar conditions involving scalar multiplications:

- $t \underset{A}{\cdot} (d_1 \underset{B}{+} d_2) = t \underset{A}{\cdot} d_1 \underset{B}{+} t \underset{A}{\cdot} d_2$, for $t \in \mathbb{R}$, and with $q_B^D(d_1) = q_B^D(d_2)$,
- $t \underset{B}{\cdot} (d_1 \underset{A}{+} d_2) = t \underset{B}{\cdot} d_1 \underset{A}{+} t \underset{B}{\cdot} d_2$, for $t \in \mathbb{R}$, and with $q_A^D(d_1) = q_A^D(d_2)$,
- $t \underset{A}{\cdot} (u \underset{B}{\cdot} d) = u \underset{B}{\cdot} (t \underset{A}{\cdot} d)$, for $t, u \in \mathbb{R}$, $d \in D$.

The zero section of $A \to M$ is denoted by 0^A, and the zero section of $B \to M$ is denoted by 0^B. We denote the zero of D over $a \in A$ by 0_a^D, and the zero of D over $b \in B$ by 0_b^D. Consequently, the following equations hold:

- $0_{a+a'}^D = 0_a^D \underset{B}{+} 0_{a'}^D$, for $a, a' \in A_m$,
- $0_{ta}^D = t \underset{B}{\cdot} 0_a^D$,
- $0_{b+b'}^D = 0_b^D \underset{A}{+} 0_{b'}^D$, for $b, b' \in B_m$,
- $0_{tb}^D = t \underset{A}{\cdot} 0_b^D$.

We write $\odot_m^D$ for the *double zero of* D, $\odot_m^D$, that is

$$\odot_m^D := 0_{0_m^A}^D = 0_{0_m^B}^D.$$

The notation $d \underset{A}{-} d'$ is short hand notation for $d \underset{A}{+} (-1) \underset{A}{\cdot} d'$. A useful variation of the interchange law (1.1), starting with four elements d_i, $i = 1, 2, 3, 4$ as in (1.1), is the following:

$$(d_1 \underset{A}{-} d_2) \underset{B}{-} (d_3 \underset{A}{-} d_4) = (d_1 \underset{B}{-} d_3) \underset{A}{-} (d_2 \underset{B}{-} d_4). \tag{1.2}$$

To see this, rewrite the left hand side of (1.2) as

$$\begin{aligned}(d_1 \underset{A}{-} d_2) \underset{B}{-} (d_3 \underset{A}{-} d_4) &= \left(d_1 \underset{A}{+} \left((-1) \underset{A}{\cdot} d_2\right)\right) \underset{B}{+} (-1) \underset{B}{\cdot} \left(d_3 \underset{A}{+} \left((-1) \underset{A}{\cdot} d_4\right)\right) \\ &= \left(d_1 \underset{A}{+} \left((-1) \underset{A}{\cdot} d_2\right)\right) \underset{B}{+} \left((-1) \underset{B}{\cdot} d_3 \underset{A}{+} (-1) \underset{B}{\cdot} \left((-1) \underset{A}{\cdot} d_4\right)\right) \qquad (1.3)\end{aligned}$$

where the outlines of the four elements are:

$$\begin{array}{ccc} d_1 & \longmapsto & b_1 \\ \downarrow & & \downarrow \\ a_1 & \longmapsto & m, \end{array} \quad \begin{array}{ccc} (-1) \underset{A}{\cdot} d_2 & \longmapsto & -b_2 \\ \downarrow & & \downarrow \\ a_1 & \longmapsto & m, \end{array} \quad \begin{array}{ccc} (-1) \underset{B}{\cdot} d_3 & \longmapsto & b_1 \\ \downarrow & & \downarrow \\ -a_2 & \longmapsto & m, \end{array} \quad \begin{array}{ccc} (-1) \underset{B}{\cdot} ((-1) \underset{A}{\cdot} d_4) & \longmapsto & -b_2 \\ \downarrow & & \downarrow \\ -a_2 & \longmapsto & m. \end{array}$$

We see that $q^D_A(d_1) = q^D_A((-1) \underset{A}{\cdot} d_2)$, $q^D_A((-1) \underset{B}{\cdot} d_3) = q^D_A((-1) \underset{B}{\cdot} ((-1) \underset{A}{\cdot} d_4))$, and that $q^D_B(d_1) = q^D_B((-1) \underset{B}{\cdot} d_3)$, $q^D_B((-1) \underset{A}{\cdot} d_2) = q^D_B((-1) \underset{B}{\cdot} ((-1) \underset{A}{\cdot} d_4))$. Apply the interchange law (1.1) to (1.3):

$$\begin{aligned}&\left(d_1 \underset{A}{+} \left((-1) \underset{A}{\cdot} d_2\right)\right) \underset{B}{+} \left((-1) \underset{B}{\cdot} d_3 \underset{A}{+} (-1) \underset{B}{\cdot} \left((-1) \underset{A}{\cdot} d_4\right)\right) \\ =\ &\left(d_1 \underset{B}{+} \left((-1) \underset{B}{\cdot} d_3\right)\right) \underset{A}{+} \left((-1) \underset{A}{\cdot} d_2 \underset{B}{+} (-1) \underset{B}{\cdot} \left((-1) \underset{A}{\cdot} d_4\right)\right) \\ =\ &\left(d_1 \underset{B}{-} d_3\right) \underset{A}{+} \left((-1) \underset{A}{\cdot} d_2 \underset{B}{+} (-1) \underset{A}{\cdot} \left((-1) \underset{B}{\cdot} d_4\right)\right) \\ =\ &\left(d_1 \underset{B}{-} d_3\right) \underset{A}{+} (-1) \underset{A}{\cdot} \left(d_2 \underset{B}{+} ((-1) \underset{B}{\cdot} d_4)\right) \\ =\ &\left(d_1 \underset{B}{-} d_3\right) \underset{A}{-} \left(d_2 \underset{B}{-} d_4\right),\end{aligned}$$

and this proves (1.2).

We won't include as much detail in future calculations.

1.1.1 Double source map and sigma maps

Let us return to Definition 0.1.1, and discuss why we include parts (ii) and (iii) in the definition of a double vector bundle.

The *double source condition*, part (ii) of the definition of a double vector bundle (part (i) was Definition 0.1.1), is that the *double source map*, the double vector bundle morphism which we denote by $\natural : D \to A \times_M B$, $d \mapsto (q^D_A(d), q^D_B(d))$, be a surjective submersion[1].

[1]In [19], the authors prove in Lemma 2, Appendix A, that part (ii) of the definition of a double vector bundle follows from part (i) of the definition.

As we have already mentioned in the Introduction, the core C of a double vector bundle D is the intersection of the kernels of the two projections of D. To ensure that C is a well-defined closed embedded submanifold of D, we require that $\natural$ be a surjective submersion.

The last part of the definition of a double vector bundle, part (iii), is the existence of the *sigma maps* $\Sigma : A \times_M B \to D$, previously called the *splitting maps*, see [13, Definition 1.1, p.178]. A sigma map is a double vector bundle morphism which is a right-inverse to $\natural : D \to A \times_M B$: $\natural(\Sigma(a,b)) = (a,b)$. It follows that it preserves the side bundles A and B. We use sigma maps to prove the existence of nontrivial grids on D.

Some authors prove the existence of sigma maps from the first part of the definition of double vector bundle, e.g. [11]. Our policy is to take the existence of this sigma map as part of the definition of a double vector bundle. And as all examples of double vector bundles known so far satisfy this requirement, and the operations of tangent and cotangent prolongation, and the dualization processes preserve the sigma maps, in practice, it is enough to check part (i) of the definition of a double vector bundle D. When using local coordinates on D, one implicitly assumes the existence of the sigma map. Pradines [33] did so, building explicit charts for D, which he called *double charts*.

So far, we have described a double vector bundle as (i) interchange laws, (ii) the double source condition, and (iii) the existence of sigma maps. Alternatively, one can equally describe a double vector bundle by (i), (ii), and the *decomposition map* $\Omega : D \to A \times_M B \times_M C$ instead of the sigma map. We will show that there is a bijective correspondence between the Σ and Ω maps. And Ω is a double vector bundle morphism. We denote the inverse of the decomposition map by $\mho : A \times_M B \times_M C \to D$.

Decompositions are very helpful, as they provide insight to the following. In the setting of vector bundles, we only have one "level" of local triviality, the level of local charts. In the setting of double vector bundles, we have two "levels" of local triviality: the first level which is charts on the constituent bundles A, B, and C, and the second level which is decompositions $\Omega : D \to A \times_M B \times_M C$ which play a role for double vector bundles comparable to local charts for ordinary vector bundles. The second level is separate from the first level.

Bijective correspondence between Σ and Ω

Start with a *sigma map* $\Sigma : A \times_M B \to D$. For $(a,b) \in A \times_M B$, the outline of $\Sigma(a,b)$ is $(\Sigma(a,b); a, b; m)$, and by Σ's definition: $\natural(\Sigma(a,b)) = (a,b)$. Take any $d \in D$ with outline $(d; a, b; m)$. Then:

$$d \underset{A}{-} \Sigma(a,b) = c \underset{B}{+} 0^D_a,$$

for a unique $c \in C$. Define

$$\begin{aligned} \Omega : D &\to A \times_M B \times_M C \\ d &\to (a, b, (d \underset{A}{-} \Sigma(a,b)) \underset{B}{-} 0_a^D). \end{aligned} \tag{1.4}$$

The inverse of this map is

$$\begin{aligned} \mho : A \times_M B \times_M C &\to D \\ (a, b, c) &\mapsto \Sigma(a,b) \underset{A}{+} (c \underset{B}{+} 0_a^D). \end{aligned} \tag{1.5}$$

Both Ω and $\mho$ are smooth, as a combination of Σ and operations, all of which are smooth. And Ω and $\mho$ are mutually inverse.

Ω is a diffeomorphism, and if we regard $A \times_M B \times_M C$ as a double vector bundle, then Ω is an isomorphism of double vector bundles, and is the identity map on the side bundles A, B, and on the core C. Using terminology of [13, Definition 2.2, p.181], Ω is a *statomorphism*.

Therefore, given a sigma double vector bundle map Σ, there exists a unique double vector bundle map Ω, defined by (1.4), which is a decomposition of D.

Equivalently, starting with an Ω, we can define a Σ using the inverse of Ω:

$$\Sigma : A \times_M B \to D, \quad (a,b) \mapsto \mho(a, b, 0_m^C).$$

Therefore, there exists a bijective correspondence between the sigma double vector bundle Σ and the decomposition Ω maps of D.

Parenthesis

The following is a result from [13, p.181], and we will use it to prove the existence of nontrivial grids on a triple vector bundle E, in Section 3.1.1. First, we need the following [25, Definition 9.1.2].

Definition 1.1.1. A *double vector bundle morphism*

$$(\varphi; \varphi_A, \varphi_B; f) : (D; A, B; M) \to (D'; A', B'; M')$$

consists of maps $\varphi : D \to D'$, $\varphi_A : A \to A'$, $\varphi_B : B \to B'$, $f : M \to M'$, such that each of (φ, φ_A), (φ, φ_B), (φ_A, f) and (φ_B, f) is a morphism of the relevant vector bundles.

Proposition 1.1.2. *Take $\varphi : D \to D'$ a double vector bundle morphism $(\varphi; \varphi_A, \varphi_B; f)$, and denote its core morphism, the restriction of φ to the core vector bundles, by $\varphi_C : C \to C'$. If $D = A \times_M B \times_M C$ and $D' = A' \times_{M'} B' \times_{M'} C'$, then we can write*

$$\varphi(a, b, c) = (\varphi_A(a), \varphi_B(b), \lambda(a,b) + \varphi_C(c)),$$

where $\lambda : A \times_M B \to C'$ is a bilinear map.

Proof. Write $\varphi(a,b,c) = (\varphi_A(a), \varphi_B(b), f(a,b,c))$, with $f : A\times_M B\times_M C \to C'$. Then $\varphi(0^A_m, 0^B_m, c) = (0^{A'}_{f(m)}, 0^{B'}_{f(m)}, f(0^A_m, 0^B_m, c))$, so

$$\varphi_C(c) = f(0^A_m, 0^B_m, c).$$

For $d_1 = (a, b_1, c_1)$, $d_2 = (a, b_2, c_2) \in D$, since

$$\varphi(d_1 \underset{A}{+} d_2) = \varphi(d_1) \underset{A'}{+} \varphi(d_2),$$

it follows that

$$f(a, b_1 + b_2, c_1 + c_2) = f(a, b_1, c_1) + f(a, b_2, c_2). \tag{1.6}$$

Similarly, for $d'_1 = (a_1, b, c_1)$, $d'_2 = (a_2, b, c_2) \in D$, by

$$\varphi(d'_1 \underset{B}{+} d'_2) = \varphi(d'_1) \underset{B'}{+} \varphi(d'_2),$$

we obtain,

$$f(a_1 + a_2, b, c_1 + c_2) = f(a_1, b, c_1) + f(a_2, b, c_2). \tag{1.7}$$

Note that subscripts for the additions in equations (1.6) and (1.7) are not necessary as both additions coincide in the core vector bundle $C' \to M$.

In (1.6), taking $b_1 = b_2 = 0^B_m$ and $c_1 = c_2 = 0^C_m$:

$$f(a, 0^B_m, 0^C_m) = f(a, 0^B_m, 0^C_m) + f(a, 0^B_m, 0^C_m),$$

hence

$$f(a, 0^B_m, 0^C_m) = 0^{C'}_{f(m)},$$

the zero of the fibre $C'_{f(m)}$. Similarly, we obtain

$$f(0^A_m, b, 0^C_m) = 0^{C'}_{f(m)}.$$

Hence, from (1.6), if we set $b_1 = b$ and $b_2 = 0^B_m$, and $c_1 = 0^C_m$ and $c_2 = c$,

$$\begin{aligned} f(a,b,c) = f(a,b,0^C_m) + f(a, 0^B_m, c) &\overset{(1.7)}{=} f(a,b,0^C_m) + f(a, 0^B_m, 0^C_m) + f(0^A_m, 0^B_m, c) \\ &= f(a,b,0^C_m) + f(0^A_m, 0^B_m, c) = f(a,b,0^C_m) + \varphi_C(c), \end{aligned}$$

since $f(a, 0^B_m, 0^C_m) = 0^{C'}_{f(m)}$. The bilinear map in question is then $\lambda : A\times_M B \to C'$, $(a,b) \mapsto \lambda(a,b) = f(a,b,0^C_m)$. And this completes the proof. □

Take a sigma map $\Sigma : A\times_M B \to D$, and a decomposition $\Omega' : D \to A\times_M B\times_M C$, not necessarily the one corresponding to the given sigma map. The core morphism of Σ is the zero map, because the core of $A\times_M B$ is the zero vector bundle. The core morphism of Ω' is the identity map, therefore, the core morphism of the composition

$\Omega' \circ \Sigma$ is the zero map. According to Proposition 1.1.2, there exists a bilinear map $\lambda : A \times_M B \to C$, such that:

$$\begin{aligned} \Omega' \circ \Sigma : A \times_M B &\to A \times_M B \times_M C \\ (a,b) &\mapsto (a, b, \lambda(a,b)). \end{aligned}$$

As the decomposition Ω' corresponds to some sigma map Σ', we have,

$$\begin{aligned} \Omega' \circ \Sigma : A \times_M B &\to A \times_M B \times_M C \\ (a,b) &\mapsto (a, b, (\Sigma(a,b) \underset{A}{-} \Sigma'(a,b)) \underset{B}{-} 0^D_a), \end{aligned}$$

so

$$\lambda(a,b) = (\Sigma(a,b) \underset{A}{-} \Sigma'(a,b)) \underset{B}{-} 0^D_a = (c \underset{B}{+} 0^D_a) \underset{B}{-} 0^D_a = c.$$

We call this $\lambda(a,b)$ the *core component* of the element $\Sigma(a,b) \in D$ with respect to Ω'; $\lambda(a,b)$ is not intrinsically defined.

In the case where Ω is the decomposition corresponding to the given Σ, then

$$\begin{aligned} \Omega \circ \Sigma : A \times_M B &\to A \times_M B \times_M C \\ (a,b) &\mapsto (a, b, (\Sigma(a,b) \underset{A}{-} \Sigma(a,b)) \underset{B}{-} 0^D_a), \end{aligned}$$

and since $\Sigma(a,b) \underset{A}{-} \Sigma(a,b) = 0^D_a$, the core component of the element $\Sigma(a,b) \in D$ with respect to Ω is $\lambda \equiv 0$.

Therefore, we see that even though the core morphism of the double vector bundle Σ is zero, the core component of the element $\Sigma(a,b) \in D$ depends on the decomposition chosen, and is not necessarily zero.

Nontrivial grids on D

The sigma map $\Sigma : A \times_M B \to D$ guarantees the existence of nontrivial grids on D. To see this, take a $X \in \Gamma A$ and $\psi : B \to C$ a vector bundle map over M. Then using (1.5),

$$\xi(b) = \mho(X(m), b, \psi(b)), \tag{1.8}$$

is a linear section of $D \to B$ over X.

Some calculations:

$$
\begin{aligned}
&\xi(b_1) \underset{A}{+} \xi(b_2) \\
&\quad = \mho(X(m), b_1, \psi(b_1)) \underset{A}{+} \mho(X(m), b_2, \psi(b_2)) \\
&\quad = \left(\Sigma(X(m), b_1) \underset{A}{+} \left(\psi(b_1) \underset{B}{+} 0^D_{X(m)}\right)\right) \underset{A}{+} \left(\Sigma(X(m), b_2) \underset{A}{+} \left(\psi(b_2) \underset{B}{+} 0^D_{X(m)}\right)\right) \\
&\quad = \left(\Sigma(X(m), b_1) \underset{A}{+} \Sigma(X(m), b_2)\right) \underset{A}{+} \left(\left(\psi(b_1) \underset{B}{+} 0^D_{X(m)}\right) \underset{A}{+} \left(\psi(b_2) \underset{B}{+} 0^D_{X(m)}\right)\right) \\
&\quad = \Sigma(X(m), b_1 + b_2) \underset{A}{+} \left(\left(\psi(b_1) \underset{A}{+} \psi(b_2)\right) \underset{B}{+} \left(0^D_{X(m)} \underset{A}{+} 0^D_{X(m)}\right)\right) \\
&\quad = \Sigma(X(m), b_1 + b_2) \underset{A}{+} \left(\psi(b_1 + b_2) \underset{B}{+} 0^D_{X(m)}\right) \\
&\quad = \mho(X(m), b_1 + b_2, \psi(b_1 + b_2)) \\
&\quad = \xi(b_1 + b_2).
\end{aligned}
$$

That $\Sigma(X(m), b_1) \underset{A}{+} \Sigma(X(m), b_2) = \Sigma(X(m), b_1 + b_2)$ follows since Σ is a double vector bundle map. And $q^D_A(\xi(b)) = X(m)$, hence ξ projects to $X \in \Gamma A$.

Now take a $Y \in \Gamma B$, and $\varphi : A \to C$ a vector bundle map over M. Then

$$\eta(a) = \mho(a, Y(m), \varphi(a)),$$

is a linear section of $D \to A$ over Y. Therefore, (ξ, X) and (η, Y) is a nontrivial grid on D. The warp of this grid:

$$
\begin{aligned}
&\xi(Y(m)) \underset{A}{-} \eta(X(m)) = \\
&\quad = \mho(X(m), Y(m), \psi(Y(m))) \underset{A}{-} \mho(X(m), Y(m), \varphi(X(m))) \\
&\quad = \left(\Sigma(X(m), Y(m)) \underset{A}{+} \left(\psi(Y(m)) \underset{B}{+} 0^D_{X(m)}\right)\right) \underset{A}{-} \left(\Sigma(X(m), Y(m)) \underset{A}{+} \left(\varphi(X(m)) \underset{B}{+} 0^D_{X(m)}\right)\right) \\
&\quad = \left(\Sigma(X(m), Y(m)) \underset{A}{-} \Sigma(X(m), Y(m))\right) \underset{A}{+} \left(\left(\psi(Y(m)) \underset{B}{+} 0^D_{X(m)}\right) \underset{A}{-} \left(\varphi(X(m)) \underset{B}{+} 0^D_{X(m)}\right)\right) \\
&\quad = \Sigma(X(m), 0^B_m) \underset{A}{+} \left((\psi(Y(m)) \underset{A}{-} \varphi(X(m)) \underset{B}{+} (0^D_{X(m)} \underset{A}{-} 0^D_{X(m)})\right) \\
&\quad = 0^D_{X(m)} \underset{A}{+} \left((\psi(Y(m)) \underset{A}{-} \varphi(X(m)) \underset{B}{+} 0^D_{X(m)}\right) \\
&\quad = \left(0^D_{X(m)} \underset{B}{+} \odot^D_m\right) \underset{A}{+} \left((\psi(Y(m)) \underset{A}{-} \varphi(X(m)) \underset{B}{+} 0^D_{X(m)}\right) \\
&\quad = \left(0^D_{X(m)} \underset{A}{+} 0^D_{X(m)}\right) \underset{B}{+} \left((\psi(Y(m)) \underset{A}{-} \varphi(X(m)) \underset{A}{+} \odot^D_m\right) \\
&\quad = 0^D_{X(m)} \underset{B}{+} \left(\psi(Y(m)) \underset{A}{-} \varphi(X(m)\right),
\end{aligned}
$$

hence $\mathrm{w}(\xi, \eta) = \psi(Y(m)) \underset{A/B}{-} \varphi(X(m))$, as the two subtractions coincide in the core vector bundle.

Since $d \underset{A}{-} d = 0^D_a$ and $d \underset{B}{-} d = 0^D_b$, by linearity of Σ it follows that,

$$\Sigma(0^A_m, b) = \Sigma(a, b) \underset{B}{-} \Sigma(a, b) = 0^D_b,$$

for $a, b \in A \times_M B$. Similarly $\Sigma(a, 0^B_m) = 0^D_a$, and finally, $\Sigma(0^A_m, 0^B_m) = \odot^D_m$.

1.1.2 Core vector bundle

In this section we describe in detail everything concerning the core vector bundle of a double vector bundle.

By definition, the *core* C of the double vector bundle D is the intersection of the kernels of the two projections q^D_A and q^D_B of D, i.e.,

$$C := \mathrm{Ker}(q^D_A) \cap \mathrm{Ker}(q^D_B).$$

The core C is a closed embedded submanifold of D, as the preimage of the closed submanifold $Z = \{(0^A_m, 0^B_m) \mid m \in M\} \subseteq A \times_M B$ via the double source map $\natural : D \to A \times_M B$. The core C is a submanifold of D, but not a subvector bundle of D. It is however a vector bundle over M. My best thanks to Madeleine Jotz Lean for explaining the use of pullbacks to define the vector bundle structure of C over M.

The kernel of the vector bundle morphism (q^D_B, q_A) is a subvector bundle of D over A:

$$\mathrm{Ker}(q^D_B) = \bigcup_{a \in A} \mathrm{Ker}\left(q^D_B\Big|_a\right) = \bigcup_{a \in A} \{d \in D\Big|_a \mid q^D_B(d) = 0^B_m\}.$$

Now take the pullback of $\mathrm{Ker}(q^D_B) \to A$ across the zero section $0^A \in \Gamma A$. This is now a vector bundle over M,

$$\begin{array}{ccc} 0^{A^!}\mathrm{Ker}(q^D_B) & \longrightarrow & \mathrm{Ker}(q^D_B) \\ {\scriptstyle q_C}\Big\downarrow & & \Big\downarrow{\scriptstyle q^D_A} \\ M & \underset{0^A}{\longrightarrow} & A. \end{array}$$

This pullback bundle $0^{A^!}\mathrm{Ker}(q^D_B) \to M$ is the core vector bundle $C \to M$,

$$\begin{aligned} 0^{A^!}\mathrm{Ker}(q^D_B) &= \{(d, m) \in \mathrm{Ker}(q^D_B) \times M \mid q^D_A(d) = 0^A_m\} \\ &= \{d \in D \mid q^D_A(d) = 0^A_m,\ q^D_B(d) = 0^B_m, m \in M\} = C, \end{aligned}$$

and this is the vector bundle structure the core inherits from D. Indeed, if $d \in 0^{A^!}\mathrm{Ker}(q_B^D)$, then $q_A^D(d) = 0_m^A$, hence,

$$q_C(d) = m = q_A(0_m^A) = q_A(q_A^D(d)).$$

Applying the same method the other way around, that is, starting with the kernel $\mathrm{Ker}(q_A^D)$ of (q_A^D, q_B), and taking its pullback across the zero section $0^B \in \Gamma B$, we obtain the vector bundle

$$\begin{array}{ccc} 0^{B^!}\mathrm{Ker}(q_A^D) & \longrightarrow & \mathrm{Ker}(q_A^D) \\ {\scriptstyle q'_C}\downarrow & & \downarrow{\scriptstyle q_B^D} \\ M & \underset{0^B}{\longrightarrow} & B. \end{array}$$

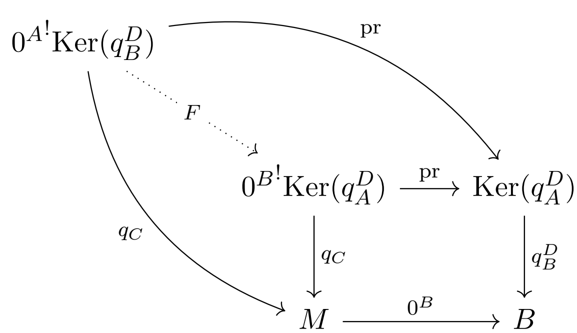
For $d' \in 0^{B^!}\mathrm{Ker}(q_A^D)$, then $q_B^D(d') = 0_m^B$, and

$$q'_C(d') = m = q_B(0_m^B) = q_B(q_B^D(d')).$$

As sets, the two manifolds $0^{A^!}\mathrm{Ker}(q_B^D)$ and $0^{B^!}\mathrm{Ker}(q_A^D)$ are both equal to the core C of D. Do the two pullback bundles define a unique vector bundle structure on C? For a d in $C = 0^{A^!}\mathrm{Ker}(q_B^D) = 0^{B^!}\mathrm{Ker}(q_A^D)$, since $q_A \circ q_A^D = q_B \circ q_B^D$,

$$q_C(d) = q_A(q_A^D(d)) = q_B(q_B^D(d)) = q'_C(d),$$

that is, $q_C = q'_C$. Also, in the following diagram, both the inner and the outer square diagrams commute, hence the unique map $F : 0^{A^!}\mathrm{Ker}(q_B^D) \to 0^{B^!}\mathrm{Ker}(q_A^D)$ is a smooth map, and in fact, a vector bundle map over M.



We can similarly define a map $G : 0^{B^!}\mathrm{Ker}(q_A^D) \to 0^{A^!}\mathrm{Ker}(q_B^D)$, and we see that F and G are mutual inverses, hence F is a diffeomorphism. Finally, about the two additions and scalar multiplications that both pullback vector bundles inherit from D coincide. To see this, take $d_1, d_2 \in C = 0^{A^!}\mathrm{Ker}(q_B^D) = 0^{B^!}\mathrm{Ker}(q_A^D)$, with $q_A(q_A^D(d_1)) = q_A(q_A^D(d_2)) = m$. Then, by the interchange law (1.1):

$$d_1 \underset{B}{+} d_2 = (d_1 \underset{A}{+} \odot_m^D) \underset{B}{+} (\odot_m^D \underset{A}{+} d_2) = (d_1 \underset{B}{+} \odot_m^D) \underset{A}{+} (\odot_m^D \underset{B}{+} d_2) = d_1 \underset{A}{+} d_2.$$

And similarly for the scalar multiplication. Therefore, both pullback vector bundles define the same vector bundle structure on the core.

From now on we denote a core element by $c \in C$. When working with examples, we can usually identify the core vector bundle with a familiar vector bundle. For example, the core vector bundle of TA can be canonically identified with $A \to M$, as described in the following subsection, see also [25, 9.1.7]. However, it can be important to distinguish between these two pictures: the elements in C (i) as elements in D, and (ii) as elements of the familiar bundle with which we identified $C \to M$. To indicate that an element $c \in C$ is viewed as in (i), we write $\bar{c}$, a bar over c. For instance, in the case of TA, an element of the core vector bundle can be viewed either as an element $\bar{a} \in TA$, or as an element $a \in A$. This distinction is usually not necessary for general double vector bundles and triple vector bundles, so in Section 2.3 of Chapter 2 and in Chapter 3, we do not write bars over core elements. The bar notation is used repeatedly in Section 2.4 and in Chapter 4.

As mentioned in the Introduction, two elements $d, d' \in D$ with the same outlines differ by a unique core element $c \in C$, as in (3). Indeed, take $(d; a, b; m)$ and $(d'; a, b, m)$. Then $q^D_A(d \underset{A}{-} d') = a$ and $q^D_B(d \underset{A}{-} d') = 0^B_m$. Subtracting 0^D_a over B yields the element $(d \underset{A}{-} d') \underset{B}{-} 0^D_a$, and since $q^D_A((d \underset{A}{-} d') \underset{B}{-} 0^D_a) = 0^A_m$ and $q^D_B((d \underset{A}{-} d') \underset{B}{-} 0^D_a) = 0^B_m$, this is precisely an element $c \in C$. In other words,

$$d \underset{A}{-} d' = 0^D_a \underset{B}{+} c.$$

Of course if we take their difference over B, we obtain the same core element $c \in C$. To see this, start with

$$d = d' \underset{A}{+} (0^D_a \underset{B}{+} c),$$

and apply the interchange law (1.1):

$$d' \underset{A}{+} (0^D_a \underset{B}{+} c) = (d' \underset{B}{+} 0^D_b) \underset{A}{+} (0^D_a \underset{B}{+} c) = (d' \underset{A}{+} 0^D_a) \underset{B}{+} (0^D_b \underset{A}{+} c) = d' \underset{B}{+} (0^D_b \underset{A}{+} c).$$

Therefore, we see that core elements naturally arise when combinations of operations over different structures occur.

Struts

Given a section $c \in \Gamma C$ of the core vector bundle we can define $c^A \in \Gamma_A D$ and $c^B \in \Gamma_B D$, called the *core sections over A and over B respectively, corresponding to* c, see [25, Section 9.1]. From now on, we call these sections *struts.*

Definition 1.1.3. For a section $c \in \Gamma C$, define

$$c^A : A \to D, \quad a \mapsto c(q_A(a)) \underset{B}{+} 0^D_a,$$

and call c^A the strut of c over A, and similarly,

$$c^B : B \to D, \quad b \mapsto c(q_B(b)) \underset{A}{+} 0^D_b,$$

the strut c^B of c over B.

Struts are q-projectable sections, and they project to the zero section.

The core of TA

We briefly recall the identification of the core of TA with the vector bundle $A \to M$ itself, see [25, Sections 3.4 and 9.7.1].

The kernel of the vector bundle map $T(q) : TA \to TM$ over $q : A \to M$, consists of vectors $\xi \in T_aA$ with $T(q)(\xi) = 0^{TM}_{q(a)}$, that is, vectors with base point a, and fibre component $0^{TM}_{q(a)}$ in TM. This means that ξ is tangent along the fibre $A_{q(a)}$, i.e., $\xi \in T_aA_m$. Therefore, the kernel of $T(q)$ consists of the *vertical tangent vectors of* TA, and is the usual vertical bundle $T^qA \to A$.

The kernel of the vector bundle morphism $p_A : TA \to A$ over $p : TM \to M$ is a vector bundle over TM, and it consists of the vectors $\xi \in TA$ with $p_A(\xi) = 0^A_m$. And these are the vectors that are based on the zeros 0^A_m of A.

The core of TA is the intersection of the two kernels, that is, it consists of the *vertical tangent vectors of* TA *based at the zeros* 0^A_m *of* A. Fibrewise, we canonically identify the tangent space $T_{0^A_m}A_m$ with A_m, hence the core vector bundle of TA can be identified with $A \to M$.

Conversely, when $a \in A_m$ is in the core of TA, we view it as an element in TA,

$$\bar{a} = \frac{d}{dt}(ta)\Big|_{t=0} \in T_{0^A_m}A, \tag{1.9}$$

where the curve ta is entirely in the fibre A_m. Therefore

$$T(q)(\bar{a}) = \frac{d}{dt}q(ta)\Big|_{t=0} = \frac{d}{dt}m\Big|_{t=0} = 0^{TM}_m.$$

In short, we view $\bar{a}$ as the velocity vector at the point 0^A_m, the zero of the fibre A_m, of the curve ta.

For a section $\mu \in \Gamma A$ of the core vector bundle of TA, the strut $\mu^\uparrow$ of μ over A is,

$$\mu^\uparrow(F)(a) = \frac{d}{dt}F(a + t\mu(q(a)))\Big|_{t=0}, \tag{1.10}$$

for $F \in C^\infty(A)$, $a \in A$. It is a vector field on A.

Finally, as we will use the following repeatedly, we write the two different zeros of TA with respect to its two different vector bundle structures. For $v \in T_mM$, with $v = \frac{d}{dt}\gamma(t)\Big|_{t=0}$, where $\gamma : I \to M$, $t \mapsto \gamma(t)$ is a curve in M, then,

$$T(0^A)(v) = \frac{d}{dt}0^A(\gamma(t))\Big|_{t=0} \in T_{0^A_m}A. \tag{1.11}$$

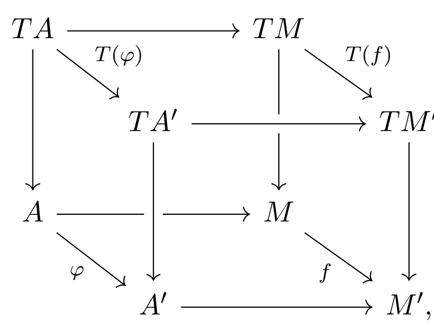
If $a \in A_m$, then

$$0^{TA}_a = \frac{d}{dt}a\Big|_{t=0} \in T_aA. \tag{1.12}$$

Something that will be needed later on is the following. Take a vector bundle map (φ, f),

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & A' \\ \downarrow & & \downarrow \\ M & \xrightarrow[f]{} & M'. \end{array}$$

Then the morphism $T(\varphi)$ of the tangent bundles is in fact a double vector bundle morphism $(T(\varphi); \varphi, T(f); f)$:



and its core morphism is the vector bundle map (φ, f). Indeed, for $a \in A$ a core element of TA, write it as $\bar{a} = \frac{d}{dt}ta\Big|_{t=0}$. Then,

$$T(\varphi)(\bar{a}) = \frac{d}{dt}(\varphi(ta))\Big|_{t=0} = \frac{d}{dt}t(\varphi(a))\Big|_{t=0} = \overline{\varphi(a)},$$

since $\varphi\Big|_a : A_m \to A'_{f(m)}$ is linear.

1.1.3 Some local coordinates on D

It is customary in Differential Geometry to work in local coordinates. Double vector bundles are no exception to this custom, and indeed Pradines in [33] introduced appropriate charts for double vector bundles, see also [38] and [36]. We now present corresponding notation for this technique.

Take a double vector bundle D. Denote by r_A the rank of $A \to M$, by r_B the rank of $B \to M$, by r_{D_A} the rank of $D \to A$, and finally by r_{D_B} the rank of $D \to B$. The dimension of the base manifold M is $\dim M = n$. The dimension of the total space D is then $n + r_A + r_{D_A} = n + r_B + r_{D_B}$, that is, $r_A + r_{D_A} = r_B + r_{D_B}$. Denote by $r_C := r_{D_B} - r_A = r_{D_A} - r_B$.

Now denote a local coordinate system for D by,

$$(x^1, \ldots, x^n, a^1, \ldots, a^{r_A}, b^1, \ldots, b^{r_B}, z^1, \ldots, z^{r_C}),$$

or in shorthand notation, (x, a, b, z). In particular, (a) are fibre coordinates for the vector bundle $A \to M$, and (b) are fibre coordinates for the vector bundle $B \to M$.

From [33], in the intersection of overlapping charts on D, the coordinates change as follows. For $m \in M$ in the intersection of two charts on D, for the fibre coordinates of A and of B we have, respectively,

$$\tilde{a}^i = P_A(m)^i_j a^j, \quad \text{and} \quad \tilde{b}^\ell = P_B(m)^\ell_k b^k,$$

where j and k are summation indices, $i, j = 1, \ldots, r_A$, $\ell, k = 1, \ldots, r_B$, $P_A(m) \in \mathrm{GL}(r_A, \mathbb{R})$ and $P_B(m) \in \mathrm{GL}(r_B, \mathbb{R})$. We denote by $P_A(m)^i_j$ the element in the i-th row and j-th column of the matrix $P_A(m)$, and similarly for $P_B(m)^\ell_k$, the element in the ℓ-th row and k-th column of the matrix $P_B(m)$. The transition laws for the fibre coordinates (z),

$$\tilde{z}^\mu = P_C(m)^\mu_\nu z^\nu + P(m)^\mu_{jk} a^j b^k, \tag{1.13}$$

where $P_C(m) \in \mathrm{GL}(r_C, \mathbb{R})$, and $P(m)^\mu_{jk}$ are the components of a bilinear map $P(m) : A_m \times B_m \to C_m$, $(a^j, b^k) \mapsto P(m)^\mu_{jk} a^j b^k$, and $\mu, \nu = 1, \ldots, r_C$, $j = 1, \ldots, r_A$, and $k = 1, \ldots, r_B$. To see this in matrix form, for the fibre coordinates of $D \to A$,

$$\begin{bmatrix} \tilde{a}^1 \\ \vdots \\ \tilde{a}^{r_A} \\ \tilde{z}^1 \\ \vdots \\ \tilde{z}^{r_C} \end{bmatrix} = \left[\begin{array}{ccc|c} & P_A(m)_{(r_A \times r_A)} & & \mathbf{0}_{(r_A \times r_C)} \\ \hline P(m)^1_{1k} b^k & \ldots & P(m)^1_{r_A k} b^k & \\ \vdots & \ddots & \vdots & P_C(m)_{(r_C \times r_C)} \\ P(m)^{r_C}_{1k} b^k & \ldots & P(m)^{r_C}_{r_A k} b^k & \end{array}\right] \begin{bmatrix} a^1 \\ \vdots \\ a^{r_A} \\ z^1 \\ \vdots \\ z^{r_C} \end{bmatrix}$$

For the fibre coordinates of $D \to B$,

$$\begin{bmatrix} \tilde{b}^1 \\ \vdots \\ \tilde{b}^{r_B} \\ \tilde{z}^1 \\ \vdots \\ \tilde{z}^{r_C} \end{bmatrix} = \left[\begin{array}{ccc|c} & P_B(m)_{(r_B \times r_B)} & & \mathbf{0}_{(r_B \times r_C)} \\ \hline P(m)^1_{j1} a^j & \ldots & P(m)^1_{j r_B} a^j & \\ \vdots & \ddots & \vdots & P_C(m)_{(r_C \times r_C)} \\ P(m)^{r_C}_{j1} a^j & \ldots & P(m)^{r_C}_{j r_B} a^j & \end{array}\right] \begin{bmatrix} b^1 \\ \vdots \\ b^{r_B} \\ z^1 \\ \vdots \\ z^{r_C} \end{bmatrix}.$$

And of course we have

$$\tilde{z}^1 = P(m)^1_{1k}a^1b^k + \ldots + P(m)^1_{r_Ak}a^{r_A}b^k = P(m)^1_{j1}a^jb^1 + \ldots + P(m)^1_{jr_B}a^jb^{r_B}.$$

Local coordinates for the core vector bundle are (x, z), where the fibre coordinates (z) change as $(\tilde{z})$ do when additionally we set a^j and b^k to zero for $j = 1, \ldots, r_A$, $k = 1, \ldots, r_B$ in (1.13).

Let us describe struts in local coordinates. Any $c \in \Gamma C$,

$$(x^1, \ldots, x^n) \mapsto (x^1, \ldots, x^n, z^1(x), \ldots, z^{r_C}(x)), \tag{1.14}$$

hence the strut $c^A \in \Gamma_A D$ in local coordinates,

$$(x^1, \ldots, x^n, a^1, \ldots, a^{r_A}) \mapsto (x^1, \ldots, x^n, a^1, \ldots, a^{r_A}, 0^1, \ldots, 0^{r_B}, z^1(x), \ldots, z^{r_C}(x)). \tag{1.15}$$

The zero of D over $a \in A$ in coordinates,

$$0^D_a = (x^1, \ldots, x^n, a^1, \ldots, a^{r_A}, 0^1, \ldots, 0^{r_B}, 0^1, \ldots, 0^{r_C}),$$

hence $c^A(a) = c(m) \underset{B}{+} 0^D_a$. From this we see directly that $c^A(a_1 + a_2) \neq c^A(a_1) \underset{B}{+} c^A(a_2)$. This will come up again later on.

We present the warp of a grid on a general double vector bundle D using local coordinates.

Example 1.1.4. Any section $\xi \in \Gamma_B D$ is described in local coordinates as follows,

$$(x^1, \ldots, x^n, b^1, \ldots, b^{r_B}) \mapsto (x^1, \ldots, x^n, a^1(x), \ldots, a^{r_A}(x), b^1, \ldots, b^{r_B}, z^1(x, b), \ldots, z^{r_C}(x, b)).$$

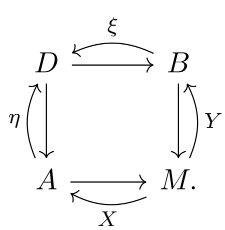
By Definition (0.2.1), a linear section $\xi \in \Gamma_B D$ is a vector bundle morphism over $X \in \Gamma A$. In local coordinates $X \in \Gamma A$ is written,

$$(x^1, \ldots, x^n) \mapsto (x^1, \ldots, x^n, a^1(x), \ldots, a^{r_A}(x)),$$

and $\xi \in \Gamma_B D$ has the following expression,

$$(x^1, \ldots, x^n, b^1, \ldots, b^{r_B}) \mapsto (x^1, \ldots, x^n, a^1(x), \ldots, a^{r_A}(x), b^1, \ldots, b^{r_B}, z^1_k(x)b^k, \ldots, z^{r_C}_k(x)b^k). \tag{1.16}$$

Now take a grid on D as in (10),



Then for the linear section $\eta \in \Gamma_A D$,

$$(x^1, \dots, x^n, a^1, \dots, a^{r_A}) \mapsto (x^1, \dots, x^n, a^1, \dots, a^{r_A}, b^1(x), \dots, b^{r_B}(x), z^1_j(x)a^j, \dots, z^{r_C}_j(x)a^j),$$

over $Y \in \Gamma B$,

$$(x^1, \dots, x^n) \mapsto (x^1, \dots, x^n, b^1(x), \dots, b^{r_B}(x)).$$

Therefore,

$$\xi(Y(m)) = (x^1, \dots, x^n, a^1(x), \dots, a^{r_A}(x), b^1(x), \dots, b^{r_B}(x), z^1_k(x)b^k(x), \dots, z^{r_C}_k(x)b^k(x))$$

$$\eta(X(m)) = (x^1, \dots, x^n, a^1(x), \dots, a^{r_A}(x), b^1(x), \dots, b^{r_B}(x), z^1_j(x)a^j(x), \dots, z^{r_C}_j(x)a^j(x)).$$

Their difference $\xi(Y(m)) \underset{A}{-} \eta(X(m))$ over A,

$$(x^1, \dots, x^n, a^1(x), \dots, a^{r_A}(x), 0^1, \dots, 0^{r_B}, z^1_k(x)b^k(x) - z^1_j(x)a^j(x), \dots, z^{r_C}_k(x)b^k(x) - z^{r_C}_j(x)a^j(x)),$$

and this defines a section of the core C,

$$\mathrm{w}(\xi, \eta) : M \to C,\ (x^1, \dots, x^n) \mapsto (x^1, \dots, x^n, z^1(x), \dots, z^{r_C}(x)),$$

where $z^\mu(x) = z^\mu_k(x)b^k(x) - z^\mu_j(x)a^j(x)$, $\mu = 1, \dots, r_C$. This is exactly the warp of (ξ, X) and of (η, Y).

1.2 Double tangent bundle et al

In this section we focus on the double tangent bundle T^2M for a manifold M, and we describe everything concerning T^2M needed for the work that follows in later Chapters. Main references for this subsection are [18] and [6], and for a treatment of T^2M in synthetic terms, see [32, Section 4.1]. First, we set up the notation for local coordinates on T^2M. We then proceed with the canonical involution J_M, and we describe some of its most important properties. We then set up notation for local coordinates on TA (relevant sources are [36, Section 3], [34, Section 9], and [3]), and we describe in detail a technical result from [34, Chapter 9]. Finally, we include the connection theory needed for the examples of grids on T^2M and on TA in Section 4.2.4.

1.2.1 Local coordinates for T^2M

We introduce the notation we need in local coordinates for a smooth manifold M with dimension $\dim M = n$, for the tangent bundle $TM \xrightarrow{p} M$, and for the *double tangent bundle* $T^2M = T(TM)$. To describe how local coordinates change from one chart to another, when the charts overlap in the first place, we iterate the well-known construction of building local coordinates for the tangent bundle $TM \to M$ from a chart (U, φ) on M.

Step 1

Start with a chart (U, φ) on M, with associated local coordinates $(x^1, \ldots, x^n)$. Denote a point $m \in U$ by $(x^1, \ldots, x^n)$, or by (x) in shorthand notation.

From the coordinates $(x^1, \ldots, x^n)$, we build the basis $\left(\frac{\partial}{\partial x^1}\Big|_m, \ldots, \frac{\partial}{\partial x^n}\Big|_m\right)$, shorthand notation $\left(\frac{\partial}{\partial x}\Big|_m\right)$, for the tangent space T_mM.

Take a chart (U, φ) with local coordinates $(x^1, \ldots, x^n)$ and another one (V, ψ) with local coordinates $(\tilde{x}^1, \ldots, \tilde{x}^n)$, such that $U \cap V \neq \emptyset$. The transition map on the intersection of these overlapping charts,

$$\begin{aligned} \psi \circ \varphi^{-1} : \varphi(U \cap V) &\rightarrow \psi(U \cap V), \\ (x^1, \ldots, x^n) &\mapsto (\tilde{x}^1, \ldots, \tilde{x}^n). \end{aligned}$$

For points $m \in U \cap V$, we have two bases $\left(\frac{\partial}{\partial x}\Big|_m\right)$ and $\left(\frac{\partial}{\partial \tilde{x}}\Big|_m\right)$ of the tangent space T_mM. The relation between these two bases, for $i = 1, \ldots, n$:

$$\frac{\partial}{\partial x^i}\Big|_m = \frac{\partial \tilde{x}^j}{\partial x^i}(m) \frac{\partial}{\partial \tilde{x}^j}\Big|_m, \tag{1.17}$$

where j is the summation index. Equivalently, in matrix form:

$$\begin{bmatrix} \frac{\partial}{\partial x^1}\Big|_m & \cdots & \frac{\partial}{\partial x^n}\Big|_m \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial \tilde{x}^1}\Big|_m & \cdots & \frac{\partial}{\partial \tilde{x}^n}\Big|_m \end{bmatrix} \begin{bmatrix} \frac{\partial \tilde{x}^1}{\partial x^1}(m) & \cdots & \frac{\partial \tilde{x}^1}{\partial x^n}(m) \\ \vdots & \ddots & \vdots \\ \frac{\partial \tilde{x}^n}{\partial x^1}(m) & \cdots & \frac{\partial \tilde{x}^n}{\partial x^n}(m) \end{bmatrix}.$$

Step 2

From a chart (U, φ) on M with associated local coordinates $(x^1, \ldots, x^n)$, we can build a chart on TM as usual, $(p^{-1}(U), \widetilde{\varphi})$, see [18] for example. We denote by $v \in T_mM$ a single tangent vector to M at point $m \in M$, and the corresponding chart on TM,

$$\begin{aligned} \tilde{\varphi} : p^{-1}(U) &\rightarrow \varphi(U) \times \mathbb{R}^n \subseteq \mathbb{R}^{2n}, \\ v^1 \frac{\partial}{\partial x^1}\Big|_m + \ldots + v^n \frac{\partial}{\partial x^n}\Big|_m &\mapsto (x^1, \ldots, x^n, v^1, \ldots, v^n). \end{aligned} \tag{1.18}$$

Hence, in local coordinates we write $v \in T_mM$ as $(x^1, \ldots, x^n, v^1, \ldots, v^n)$, or as (x, v).

Now take (U, φ) and (V, ψ) two charts on M, with $U \cap V \neq \emptyset$, and with corresponding local coordinates $(x^1, \ldots, x^n)$, and $(\tilde{x}^1, \ldots, \tilde{x}^n)$, respectively. The transition map on the region of intersection $p^{-1}(U) \cap p^{-1}(V) = p^{-1}(U \cap V)$ of the two charts $(p^{-1}(U), \widetilde{\varphi})$ and $(p^{-1}(V), \widetilde{\psi})$ of TM,

$$\begin{aligned} \tilde{\psi} \circ \tilde{\varphi}^{-1} : \varphi(U \cap V) \times \mathbb{R}^n &\rightarrow \psi(U \cap V) \times \mathbb{R}^n, \\ (x^1, \ldots, x^n, v^1, \ldots, v^n) &\mapsto (\tilde{x}^1, \ldots, \tilde{x}^n, \tilde{v}^1, \ldots, \tilde{v}^n). \end{aligned} \tag{1.19}$$

The transformation laws for the coordinates $(\tilde{v}^1, \ldots, \tilde{v}^n)$ on the intersection of the two charts

$$\tilde{v}^i = \frac{\partial \tilde{x}^i}{\partial x^j}(m) v^j, \quad i = 1, \ldots, n, \tag{1.20}$$

where (1.20) follows from (1.17), the relation between the two bases $\left(\frac{\partial}{\partial x}\Big|_m\right)$ and $\left(\frac{\partial}{\partial \tilde{x}}\Big|_m\right)$ of $T_m M$. Therefore, (1.20) in matrix form:

$$\begin{bmatrix} \tilde{v}^1 \\ \vdots \\ \tilde{v}^n \end{bmatrix} = \begin{bmatrix} \frac{\partial \tilde{x}^1}{\partial x^1}(m) & \ldots & \frac{\partial \tilde{x}^1}{\partial x^n}(m) \\ \vdots & \ddots & \vdots \\ \frac{\partial \tilde{x}^n}{\partial x^1}(m) & \ldots & \frac{\partial \tilde{x}^n}{\partial x^n}(m) \end{bmatrix} \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}.$$

Now, the two local coordinate systems, for $v \in p^{-1}(U \cap V)$, define two bases of $T_v TM$,

$$\left(\frac{\partial}{\partial x^1}\Big|_m, \ldots, \frac{\partial}{\partial x^n}\Big|_m, \frac{\partial}{\partial v^1}\Big|_v, \ldots, \frac{\partial}{\partial v^n}\Big|_v\right), \qquad \text{and}$$
$$\left(\frac{\partial}{\partial \tilde{x}^1}\Big|_m, \ldots, \frac{\partial}{\partial \tilde{x}^n}\Big|_m, \frac{\partial}{\partial \tilde{v}^1}\Big|_v, \ldots, \frac{\partial}{\partial \tilde{v}^n}\Big|_v\right).$$

The Jacobian matrix of the transition map (1.19),

$$T\left(\tilde{\psi} \circ \tilde{\varphi}^{-1}\right) = \left[\begin{array}{c|c} \left(\frac{\partial \tilde{x}^i}{\partial x^j}\right) & \mathbf{0}_{(n \times n)} \\ \hline \left(\frac{\partial \tilde{v}^i}{\partial x^j}\right) & \left(\frac{\partial \tilde{v}^i}{\partial v^j}\right) \end{array}\right] \overset{(1.20)}{=} \left[\begin{array}{c|c} \left(\frac{\partial \tilde{x}^i}{\partial x^j}\right) & \mathbf{0}_{(n \times n)} \\ \hline \left(\frac{\partial^2 \tilde{x}^i}{\partial x^j \partial x^k} v^k\right) & \left(\frac{\partial \tilde{x}^i}{\partial x^j}\right) \end{array}\right], \tag{1.21}$$

describes the relation between the two bases of $T_v TM$, i.e.,

$$\begin{aligned} \frac{\partial}{\partial x^i}\Big|_m &= \frac{\partial \tilde{x}^j}{\partial x^i}(m) \frac{\partial}{\partial \tilde{x}^j}\Big|_m + \frac{\partial^2 \tilde{x}^j}{\partial x^i \partial x^k}(m) v^k(v) \frac{\partial}{\partial \tilde{v}^j}\Big|_v, \\ \frac{\partial}{\partial v^i}\Big|_v &= \frac{\partial \tilde{x}^j}{\partial x^i}(m) \frac{\partial}{\partial \tilde{v}^j}\Big|_v. \end{aligned}$$

Step 3

We now build charts for the tangent bundle structure $p_{TM} : T^2 M \to TM$. Given a chart $(\widetilde{U}, \widetilde{\varphi})$[2] on TM, we can define a chart for the vector bundle structure $T(TM) \xrightarrow{p_{TM}} TM$:

$$\Phi : p_{TM}^{-1}(\widetilde{U}) \to \tilde{\varphi}(\widetilde{U}) \times \mathbb{R}^{2n} \subseteq \mathbb{R}^{4n},$$

and for an element $v = (x, v) \in \widetilde{U}$ write an element of $p_{TM}^{-1}(\widetilde{U})$ as a linear combination of the basis vectors of $T_v TM$:

$$\dot{x}^1 \frac{\partial}{\partial x^1}\Big|_m + \ldots + \dot{x}^n \frac{\partial}{\partial x^n}\Big|_m + \dot{v}^1 \frac{\partial}{\partial v^1}\Big|_v + \ldots + \dot{v}^n \frac{\partial}{\partial v^n}\Big|_v.$$

[2] We have written $\widetilde{U}$ for $p^{-1}(U)$

Then Φ maps it to

$$(x^1, \ldots, x^n, v^1, \ldots, v^n, \dot{x}^1, \ldots, \dot{x}^n, \dot{v}^1, \ldots, \dot{v}^n).$$

And this is how we denote a local coordinate system on T^2M, and in shorthand notation: $(x, v, \dot{x}, \dot{v})$.

For two charts $(\widetilde{U}, \widetilde{\varphi})$ and $(\widetilde{V}, \widetilde{\psi})$ on TM, with $\widetilde{U} \cap \widetilde{V} \neq \emptyset$, denote by $(x, v, \dot{x}, \dot{v})$ and by $(\tilde{x}, \tilde{v}, \dot{\tilde{x}}, \dot{\tilde{v}})$ the corresponding coordinates on T^2M, respectively. The transition map on the region of intersection $p_{TM}^{-1}(\widetilde{U}) \cap p_{TM}^{-1}(\widetilde{V})$ of the two corresponding charts on T^2M,

$$(x, v, \dot{x}, \dot{v}) \mapsto (\tilde{x}, \tilde{v}, \dot{\tilde{x}}, \dot{\tilde{v}}).$$

As in the case of (1.20), the transformation laws for the coordinates $(\dot{\tilde{x}})$ and $(\dot{\tilde{v}})$ on the intersection of the two charts of T^2M follow by (1.21):

$$\dot{\tilde{x}}^i = \frac{\partial \tilde{x}^i}{\partial x^j}(m)\dot{x}^j, \quad \dot{\tilde{v}}^i = \frac{\partial^2 \tilde{x}^i}{\partial x^j \partial x^k}(m)\dot{x}^j v^k + \frac{\partial \tilde{x}^i}{\partial x^j}(m)\dot{v}^j. \tag{1.22}$$

And this completes Step 3.

As mentioned in Subsection 1.1.3, the coordinates of the core vector bundle change as the coordinates $(\dot{\tilde{v}})$ change when we set $\dot{x}^j$ and v^k, $j, k = 1, \ldots, n$, to zero in the second equation of (1.22). This is another way of describing the canonical identification of the core vector bundle of T^2M with $TM \to M$.

The two bundle projections, p_{TM} and $T(p)$ in local coordinates $(x, v, \dot{x}, \dot{v})$,

$$p_{TM} : T^2M \to TM, \quad (x, v, \dot{x}, \dot{v}) \mapsto (x, v),$$

and

$$T(p) : T^2M \to TM, \quad (x, v, \dot{x}, \dot{v}) \mapsto (x, \dot{x}).$$

For the second projection, take the tangent of the bundle projection of the tangent bundle $p : TM \to M$, $(x, v) \mapsto (x)$, for $v \in T_mM$,

$$T_v(p) : T_vTM \to T_mM,$$

which is described by the matrix:

$$\left[\begin{array}{ccc|ccc} \frac{\partial x^1}{\partial x^1} & \cdots & \frac{\partial x^1}{\partial x^n} & \frac{\partial x^1}{\partial v^1} & \cdots & \frac{\partial x^1}{\partial v^n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x^n}{\partial x^1} & \cdots & \frac{\partial x^n}{\partial x^n} & \frac{\partial x^n}{\partial v^1} & \cdots & \frac{\partial x^n}{\partial v^n} \end{array}\right] = \left[\begin{array}{c|c} I_{(n\times n)} & \mathbf{0}_{(n\times n)} \end{array}\right].$$

For $\xi \in T_v(TM)$, written as $\xi = \dot{x}^1 \frac{\partial}{\partial x^1}\Big|_m + \ldots + \dot{v}^n \frac{\partial}{\partial v^n}\Big|_v$, we then have,

$$T_v(p)(\xi) = \left[\begin{array}{c|c} I_{(n\times n)} & \mathbf{0}_{(n\times n)} \end{array}\right] \begin{bmatrix} \dot{x}^1 \\ \vdots \\ \dot{x}^n \\ \dot{v}^1 \\ \vdots \\ \dot{v}^n \end{bmatrix} = \begin{bmatrix} \dot{x}^1 \\ \vdots \\ \dot{x}^n \end{bmatrix} = \dot{x}^1 \frac{\partial}{\partial x^1} + \ldots + \dot{x}^n \frac{\partial}{\partial x^n},$$

hence

$$T(p): T^2M \to TM, \quad (x, v, \dot{x}, \dot{v}) \mapsto (x, \dot{x}).$$

The two additions in T^2M, for $(x, v, \dot{x}, \dot{v}), (x, v, \dot{x}', \dot{v}') \in T_{(x,v)}TM$:

$$(x, v, \dot{x}, \dot{v}) \underset{p_{TM}}{+} (x, v, \dot{x}', \dot{v}') = (x, v, \dot{x} + \dot{x}', \dot{v} + \dot{v}'),$$

and for $(x, v, \dot{x}, \dot{v})$, $(x, v', \dot{x}, \dot{v}') \in T(p)^{-1}(x, \dot{x})$:

$$(x, v, \dot{x}, \dot{v}) \underset{T(p)}{+} (x, v', \dot{x}, \dot{v}') = (x, v + v', \dot{x}, \dot{v} + \dot{v}').$$

In the rest of the thesis, we denote a single tangent vector on M at point $m \in M$ either by $v = (x, v) = (x^1, \ldots, x^n, v^1, \ldots, v^n) \in T_mM$, or by $X_m \in T_mM$. For $X \in \mathfrak{X}(M)$ a vector field, we denote its value at point $m \in M$ by $X(m)$.

1.2.2 The canonical involution $J_M : T^2M \to T^2M$

For any vector bundle (A, q, M), viewing A as a manifold, the corresponding canonical involution $J_A : T^2A \to T^2A$ is of paramount importance to Section 4.5. Hence, a preliminary exposition on the properties of J_M is necessary.

A detailed exposition in local coordinates can be found in [34, Section 10]. Other main references are [1, Exercises 3.3B and 6.4G], [31, Section 8.13 and Section 8.14], [25, Section 9.6], and [2, Chapter 1].

Definition of J_M

To begin with, $J_M : T^2M \to T^2M$ is a map from T^2M to itself. There are two definitions of J_M. The first one is given via local coordinates, and the second one via second derivatives. We briefly describe both.

First, the definition via local coordinates. This is the definition given in both [1, Exercise 3.3B] and [31, Section 8.13]. Locally, the *canonical involution* J_M is described by

$$J_M(x, v, \dot{x}, \dot{v}) = (x, \dot{x}, v, \dot{v}).$$

This definition is invariant under changes of charts, [31, p.107].

The second definition via second derivatives, as in [25, Section 9.6]. When $\xi \in T^2M$, we can write it as

$$\xi = \frac{\partial}{\partial t}\left(\frac{\partial}{\partial s}\mu(t, s)\Big|_{s=0}\right)\Big|_{t=0}, \tag{1.23}$$

where $\mu : D \to M$ is a smooth square of elements of M, $D \subseteq \mathbb{R} \times \mathbb{R}$ an open subset of $\mathbb{R} \times \mathbb{R}$, with $(0, 0) \in D$.

Then J_M is the map that interchanges the order of differentiation, i.e.,

$$J_M(\xi) := \frac{\partial}{\partial s}\left(\frac{\partial}{\partial t}\mu(t,s)\Big|_{t=0}\right)\Big|_{s=0}. \tag{1.24}$$

The use of partial derivatives in this setting can be misleading; we explain in detail this definition in the following subsection.

Of course, the most immediate consequence of this definition is that $J_M^2 = J_M \circ J_M = \mathrm{id}_{T^2M}$, i.e., J_M is an involution.

J_M interchanges the two bundle structures on T^2M

Indeed, we will show that

$$T(p) \circ J_M = p_{TM}, \quad p_{TM} \circ J_M = T(p),$$

using the definition of J_M via second derivatives.

Take a $\mu : D \to M$, $(t,s) \mapsto \mu(t,s)$, a smooth square of elements of M. First fix $t \in I$, $I \subseteq \mathbb{R}$ an open interval of $\mathbb{R}$ such that for $t \in I$ then $(t,s) \in D$, and of course $0 \in I$. Then, for every t, we obtain a curve in M:

$$\mu_t : I' \to M, \quad s \mapsto \mu_t(s) = \mu(t,s),$$

where $I' \subseteq \mathbb{R}$ an open interval of $\mathbb{R}$, such that, if $s \in I'$ and $t \in I$, then $(t,s) \in D$, and additionally, $0 \in I'$. Take the velocity vector Y_t of each of these curves at the point $\mu_t(0) = \mu(t,0)$,

$$Y_t := \frac{d}{ds}\mu_t(s)\Big|_{s=0} \in T_{\mu(t,0)}M.$$

These Y_t, $t \in I$, form a smooth curve $Y : I \to TM$, $t \mapsto Y_t$ in TM , with $p(Y_t) = \mu_t(0) = \mu(t,0)$, i.e., its projection on M is the curve $\mu(\cdot,0) : I \to M$, $t \mapsto \mu(t,0)$.

Take the velocity vector of the curve Y in TM. This is exactly ξ as in (1.23):

$$\xi = \frac{d}{dt}Y_t\Big|_{t=0} \in T_{Y_0}(TM).$$

The relevant projections:

$$\begin{aligned} p_{TM}(\xi) &= Y_0 = \frac{d}{ds}\mu_0(s)\Big|_{s=0} \in T_{\mu(0,0)}M, \\ T(p)(\xi) &= T(p)\left(\frac{d}{dt}Y_t\Big|_{t=0}\right) = \frac{d}{dt}p(Y_t)\Big|_{t=0} = \frac{d}{dt}\mu(t,0)\Big|_{t=0} \in T_{\mu(0,0)}M. \end{aligned} \tag{1.25}$$

Now we describe $J_M(\xi)$ in detail. All we have to do is switch the roles of s and t. Start again with $\mu : D \to M$, $(t,s) \mapsto \mu(t,s)$. Fix $s \in I'$, I' as before. Then again, for every $s \in I'$ we get a curve in M:

$$\mu_s : I \to M, \quad t \mapsto \mu_s(t) = \mu(t,s),$$

and again, the velocity vector of μ_s at point $\mu_s(0) = \mu(0,s)$, denoted by X_s:

$$X_s := \frac{d}{dt}\mu_s(t)\Big|_{t=0} \in T_{\mu(0,s)}M.$$

The X_s, $s \in I'$, form a smooth curve $X : I' \to TM$, $s \to X_s$, in TM, with $p(X_s) = \mu_s(0) = \mu(0,s)$. The projection of X on M is the curve $\mu(0,\cdot) : I' \to M$, $s \mapsto \mu(0,s)$.

The velocity vector of the curve X at $s = 0$ is exactly $J_M(\xi)$ as in (1.24),

$$\frac{d}{ds}X_s\Big|_{s=0} = \frac{d}{ds}\left(\frac{d}{dt}\mu_s(t)\Big|_{t=0}\right)\Big|_{s=0} = J_M(\xi).$$

Observe that

$$\begin{aligned} p_{TM}(J_M(\xi)) &= X_0 = \frac{d}{dt}\mu(t,0)\Big|_{t=0} \overset{(1.25)}{=} T(p)(\xi), \\ T(p)(J_M(\xi)) &= T(p)\left(\frac{d}{ds}X_s\Big|_{s=0}\right) = \frac{d}{ds}p(X_s)\Big|_{s=0} = \frac{d}{ds}\mu(0,s)\Big|_{s=0} = Y_0 = p_{TM}(\xi). \end{aligned}$$

The outlines of the two elements ξ and $J_M(\xi)$:

$$\begin{array}{ccc} \xi & \overset{T(p)}{\longmapsto} & X_0 \\ {\scriptstyle p_{TM}}\downarrow & & \downarrow \\ Y_0 & \longmapsto & \mu(0,0), \end{array} \qquad \begin{array}{ccc} J_M(\xi) & \overset{T(p)}{\longmapsto} & Y_0 \\ {\scriptstyle p_{TM}}\downarrow & & \downarrow \\ X_0 & \longmapsto & \mu(0,0). \end{array}$$

Therefore, we see that J_M interchanges the two bundle structures on T^2M:

$$p_{TM} \circ J_M = T(p), \quad T(p) \circ J_M = p_{TM}. \tag{1.26}$$

In particular, this shows that as a vector bundle map $J_M : T^2M \to T^2M$ induces the identity map on the bases TM:

$$\begin{array}{ccc} T^2M & \xrightarrow{J_M} & T^2M \\ {\scriptstyle T(p)}\downarrow & & \downarrow{\scriptstyle p_{TM}} \\ TM & = & TM, \end{array} \qquad \begin{array}{ccc} T^2M & \xrightarrow{J_M} & T^2M \\ {\scriptstyle p_{TM}}\downarrow & & \downarrow{\scriptstyle T(p)} \\ TM & = & TM, \end{array} \tag{1.27}$$

hence J_M *preserves the side bundles* TM of T^2M.

Focus now on the first diagram of (1.27). Fibrewise linearity for $\xi, \xi' \in T(p)^{-1}(v)$, with $v \in T_mM$:

$$J_M(\xi \underset{T(p)}{+} \xi') = J_M(\xi) \underset{p_{TM}}{+} J_M(\xi'). \tag{1.28}$$

This follows immediately by J_M's definition in local coordinates. Since $\xi, \xi' \in T(p)^{-1}(v)$:

$$\xi = (x, v, \dot{x}, \dot{v}), \quad \xi' = (x, v', \dot{x}, \dot{v}'),$$

therefore,

$$J_M(\xi \underset{T(p)}{+} \xi') = J_M(x, v+v', \dot{x}, \dot{v}+\dot{v}') = (x, \dot{x}, v+v', \dot{v}+\dot{v}')$$
$$= (x, \dot{x}, v, \dot{v}) \underset{p_{TM}}{+} (x, \dot{x}, v', \dot{v}') = J_M(\xi) \underset{p_{TM}}{+} J_M(\xi').$$

Of course the same is true the other way around. For $\xi, \xi' \in T^2M$ with $p_{TM}(\xi) = p_{TM}(\xi')$,

$$J_M(\xi \underset{p_{TM}}{+} \xi') = J_M(\xi) \underset{T(p)}{+} J_M(\xi'). \tag{1.29}$$

Core morphism of J_M

What is the core morphism of J_M? Take a tangent vector $v \in T_mM$ in the core of T^2M. How do we express $\overline{v} \in T^2M$ in terms of $\mu : D \to M$ (recall that we write $\overline{v}$ when we view a core element in the double vector bundle T^2M, and we simply write v when we view it in TM, the familiar vector bundle with which we have identified the core of T^2M)?

Since $v \in T_mM$, we can write $v = \frac{d}{dt}\gamma(t)\Big|_{t=0}$, for $\gamma : I \to M$, $t \mapsto \gamma(t)$ a curve in M with $\gamma(0) = m$, and γ at least twice differentiable. Consider $u : D \to I$, $(t, s) \mapsto t + s$, and set $\mu(t, s) = \gamma(u(t, s))$. Then it follows that

$$\overline{v} = \frac{\partial}{\partial t}\left(\frac{\partial}{\partial s}\mu(t,s)\Big|_{s=0}\right)\Big|_{t=0} = \frac{\partial}{\partial t}\left(\frac{\partial}{\partial s}\gamma(u(t,s))\Big|_{s=0}\right)\Big|_{t=0} = \frac{\partial}{\partial t}\left(\frac{d\gamma}{du}\Big|_{u(t,0)}\frac{du}{ds}\Big|_{s=0}\right)\Big|_{t=0}$$
$$= \frac{\partial}{\partial t}\left(\frac{d\gamma}{du}\Big|_{u(t,0)}\right)\Big|_{t=0} = \frac{\partial}{\partial t}\left(\frac{d\gamma}{dt}(t)\right)\Big|_{t=0} = \frac{d^2\gamma}{dt^2}\Big|_{t=0}.$$

If we now switch the order of the variables s and t, then $\mu(s, t) = \gamma(s + t)$ is the same curve, therefore $J_M(\overline{v}) = \overline{v}$, see [2, Section 1.20].

Another way of understanding why J_M is the identity map on the core is the following. The core of the double tangent bundle T^2M we start with is the intersection of the two kernels $\mathrm{Ker}(p_{TM}) \cap \mathrm{Ker}(T(p))$. The map J_M interchanges the two bundle structures $T(p) : T^2M \to TM$ and $p_{TM} : T^2M \to TM$. Therefore, J_M maps one kernel to the other, i.e., $J_M(\mathrm{Ker}(p_{TM})) = \mathrm{Ker}(T(p))$, and $J_M(\mathrm{Ker}(T(p))) = \mathrm{Ker}(p_{TM})$. Consequently, J_M leaves the intersection of the kernels unchanged. Hence, the core of $J_M(T^2M) = T^2M$ will be again the intersection $\mathrm{Ker}(T(p)) \cap \mathrm{Ker}(p_{TM})$, i.e. the same intersection that defines the core vector bundle of the initial T^2M.

Combining the last two subsections, it follows that J_M is a double vector bundle isomorphism $J_M : T^2M \to T^2M$ that induces the identity map on the core vector bundles, see Theorem [25, 9.6.1].

Local coordinates for $\widetilde{X}$ and $T(X)$

The canonical involution J_M when applied to sections of the vector bundle structure $T^2M \xrightarrow{T(p)} TM$, which are not vector fields on TM, yields sections of the vector bundle structure $T^2M \xrightarrow{p_{TM}} TM$, that is, it yields vector fields on TM. Specifically, the complete lift $\widetilde{X}$ (sometimes also called the "tangent lift") of a vector field $X \in \mathfrak{X}(M)$, is a vector field on TM such that

$$\widetilde{X} = J_M \circ T(X), \quad T(X) = J_M \circ \widetilde{X}. \tag{1.30}$$

Take two vector fields $X, Y \in \mathfrak{X}(M)$:

$$X = X^i \frac{\partial}{\partial x^i} = (x^1, \ldots, x^n, X^1, \ldots, X^n), \quad Y = Y^i \frac{\partial}{\partial x^i} = (x^1, \ldots, x^n, Y^1, \ldots, Y^n),$$

where for each $i = 1, \ldots, n$, $X^i, Y^i \in C^\infty(M)$.

Take a chart $(p^{-1}(U), \widetilde{\varphi})$ for TM as in (1.18). We consider X as a map from M to TM, $m \to (m, X(m))$. Then the tangent map $T_m X : T_m M \to T_{X(m)} TM$ maps the vector $Y(m) \in T_m M$ to,

$$T_m X(Y(m)) = \begin{bmatrix} I_{n\times n} \\ \hline \begin{matrix} \frac{\partial X^1}{\partial x^1}(m) & \ldots & \frac{\partial X^1}{\partial x^n}(m) \\ \vdots & \ddots & \vdots \\ \frac{\partial X^n}{\partial x^1}(m) & \ldots & \frac{\partial X^n}{\partial x^n}(m) \end{matrix} \end{bmatrix} \begin{bmatrix} Y^1(m) \\ \vdots \\ Y^n(m) \end{bmatrix} = \begin{bmatrix} Y^1(m) \\ \vdots \\ Y^n(m) \\ Y^i(m) \frac{\partial X^1}{\partial x^i}(m) \\ \vdots \\ Y^i(m) \frac{\partial X^n}{\partial x^i}(m) \end{bmatrix},$$

hence

$$T_m X(Y(m)) = \left(x^1(m), \ldots, x^n(m), X^1(m), \ldots, X^n(m), Y^1(m), \ldots, Y^n(m), Y^i(m) \frac{\partial X^1}{\partial x^i}(m), \ldots, Y^i(m) \frac{\partial X^n}{\partial x^i}(m) \right), \tag{1.31}$$

and of course $T_m X(Y(m)) \in T_{X(m)} TM$. Similarly about $T_m Y$:

$$T_m Y(X(m)) = \left(x^1(m), \ldots, x^n(m), Y^1(m), \ldots, Y^n(m), X^1(m), \ldots, X^n(m), X^i(m) \frac{\partial Y^1}{\partial x^i}(m), \ldots, X^i(m) \frac{\partial Y^n}{\partial x^i}(m) \right), \tag{1.32}$$

and $T_mY(X(m)) \in T_{Y(m)}TM$. Applying J_M to $T_mX(Y(m))$ interchanges the two vector bundle structures, therefore,

$$J_M(T_mX(Y(m))) = \left(x^1(m),\dots,x^n(m),Y^1(m),\dots,Y^n(m),X^1(m),\dots,X^n(m),Y^i(m)\frac{\partial X^1}{\partial x^i}(m),\dots,Y^i(m)\frac{\partial X^n}{\partial x^i}(m)\right), \tag{1.33}$$

and now $J_M(T_mX(Y(m))) \in T_{Y(m)}TM$. The complete lift $\widetilde{X}$ of $X \in \mathfrak{X}(M)$ is a vector field on TM:

$$\begin{aligned} \widetilde{X} : TM &\to T^2M, \\ Y_m &\mapsto \widetilde{X}(Y_m) = X^i(m)\frac{\partial}{\partial x^i}\Big|_m + Y^k(m)\frac{\partial X^i}{\partial x^k}(m)\frac{\partial}{\partial v^i}\Big|_{Y(m)} \in T_{Y(m)}TM. \end{aligned} \tag{1.34}$$

In local coordinates this is precisely (1.33), and of course

$$p_{TM}(\widetilde{X}(Y(m))) = Y(m), \quad T(p)(\widetilde{X}(Y(m))) = X(m),$$

and

$$T(p)(T_mX(Y(m))) = Y(m), \quad p_{TM}(T_mX(Y(m))) = X(m).$$

Flows of complete lift

In this subsection we answer the question "$\widetilde{X}$ is velocity vector of which curve?", see [1, Exercise 6.4G(ii)] and [25, Proposition 9.6.6] for further reading.

Denote by $\varphi : \Omega \to M$ the (local) flow of the vector field X, Ω being an appropriate open subset of $\mathbb{R}\times M$, $(t,m) \to \varphi(t,m)$, and by $\{\varphi_t\}$ the one-parameter group of (local) diffeomorphisms of M defined by φ. Then,

- for $I \subseteq \mathbb{R}$ an open subset of $\mathbb{R}$, s.t. $I \times M \subset \Omega$, $\varphi^m : I \to M$, $t \to \varphi^m(t)$, denotes the unique integral curve of X, starting at $m \in M$;
- for each $t \in I$, $\varphi_t : M \to M$, $m \mapsto \varphi_t(m) = \varphi(t,m) = \varphi^m(t)$, sends each $m \in M$ to the point obtained by following for time t the integral curve starting at m;
- and $X(m) = \frac{d}{dt}\varphi^m(t)\Big|_{t=0} = \frac{d}{dt}\varphi(t,m)\Big|_{t=0}$, denotes the tangent vector of the curve φ^m at the point m.

The flow properties:

- For any $t,s \in \mathbb{R}$, s.t. (s,m), $(t,\varphi_s(m))$, and $(t+s,m) \in \Omega$, we have $\varphi_{t+s}(m) = (\varphi_t \circ \varphi_s)(m)$, and

- at $t = 0$, $\varphi_0(m) = m$, $\forall m \in M$, i.e., $\varphi_0 = \mathrm{id}_M$.

We will show that $\{T(\varphi_t)\}$ is the one-parameter group of (local) diffeomorphisms of TM, corresponding to the (local) flow of the complete lift $\widetilde{X}$. To begin with, the tangent functor preserves the properties of the flow:

- Since $\varphi_{t+s} = \varphi_t \circ \varphi_s$, it follows that
$$T(\varphi_{t+s}) = T(\varphi_t) \circ T(\varphi_s).$$

- Since $\varphi_0 = \mathrm{id}_M$, it follows immediately that $T(\varphi_0) = \mathrm{id}_{TM}$.

Now take any $v \in T_m M$, and some curve γ in M, $\gamma : (-\epsilon, \epsilon) \to M$, $u \mapsto \gamma(u)$, with $\gamma(0) = m$, and with $v = \frac{d}{du}\gamma(u)\Big|_{u=0}$. Let $\mathscr{O}$ be an open subset of $\mathbb{R}^2$ such that, for any $(t, u) \in \mathscr{O}$, $(t, \gamma(u)) \in \Omega$. Consider the map $(t, u) \mapsto \varphi(t, \gamma(u)) = \varphi^{\gamma(u)}(t) = \varphi_t(\gamma(u))$ from $\mathscr{O}$ to M. Then,

$$T_m(\varphi_t)(v) = T_m(\varphi_t)\left(\frac{d}{du}\gamma(u)\Big|_{u=0}\right) = \frac{d}{du}\varphi_t(\gamma(u))\Big|_{u=0}.$$

Denote by ξ the velocity vector of the curve $t \mapsto T_m(\varphi_t)(v)$,

$$\xi = \frac{d}{dt}T_m(\varphi_t)(v)\Big|_{t=0} = \frac{d}{dt}\left(\frac{d}{du}\varphi_t(\gamma(u))\Big|_{u=0}\right)\Big|_{t=0} = \frac{d}{dt}\left(\frac{d}{du}\varphi(t, \gamma(u))\Big|_{u=0}\right)\Big|_{t=0}. \tag{1.35}$$

By (1.24) it follows that

$$J_M(\xi) = \frac{d}{du}\left(\frac{d}{dt}\varphi(t, \gamma(u))\Big|_{t=0}\right)\Big|_{u=0} = \frac{d}{du}X(\gamma(u))\Big|_{u=0} = T(X)(v),$$

and since $J_M^2 = \mathrm{id}_{TM}$, we have that

$$(J_M \circ T(X))(v) = \xi = \frac{d}{dt}T_m(\varphi_t)(v)\Big|_{t=0}.$$

Therefore, the vector field for which $\{T(\varphi_t)\}$ is the one-parameter group of (local) diffeomorphisms of TM, is $J_M \circ T(X)$, and from (1.30), this is exactly $\widetilde{X}$.

Vertical lift $X^\uparrow$

For completeness, we present here $X^\uparrow$. To begin with, one can define the vertical lift of a single tangent vector. It isn't necessary to start with a vector field, contrary to complete lifts.

As a picture, you start with a vector $X_m \in T_m M$. Choose a vector $Y_m \in T_m M$, and we want to "lift" X_m to a vector on the fibre of $T^2 M \xrightarrow{p_{TM}} TM$ over Y_m. This we do by simply taking X_m, placing its tail at Y_m (which we view as $0^{T^2M}_{Y_m}$, the zero vector of the fibre over Y_m), and asking it to be tangent along the fibre T_mM, as follows. Take the velocity vector of the path $Y_m + tX_m \in T_mM$:

$$X^{\uparrow}_m(Y_m) = \frac{d}{dt}(Y_m + tX_m)\Big|_{t=0}.$$

Given $X \in \mathfrak{X}(M)$, the vertical lift of $X^{\uparrow}$ is a vector field on TM,

$$\begin{aligned} X^{\uparrow} : TM &\rightarrow TTM \\ Y_m &\mapsto X^{\uparrow}(Y_m) \in T_{Y_m}TM, \end{aligned}$$

defined as follows. If $X = X^i\frac{\partial}{\partial x^i} = (x^1, \ldots, x^n, X^1, \ldots, X^n)$, and $Y = Y^i\frac{\partial}{\partial x^i} = (x^1, \ldots, x^n, Y^1, \ldots, Y^n)$, then

$$X^{\uparrow}(Y_m) = X^i\frac{\partial}{\partial v^i} = (x^1, \ldots, x^n, Y^1, \ldots, Y^n, 0, \ldots, 0, X^1, \ldots, X^n) \in T_{Y_m}TM,$$

or, in double vector bundle language:

$$\begin{aligned} X^{\uparrow}(Y_m) = \overline{X} \underset{T(p)}{+} 0^{T^2M}_{Y_m} &= (x^1, \ldots, x^n, 0, \ldots, 0, 0, \ldots, 0, X^1, \ldots, X^n) \\ &\underset{T(p)}{+} (x^1, \ldots, x^n, Y^1, \ldots, Y^n, 0, \ldots, 0, 0, \ldots, 0). \end{aligned} \quad (1.36)$$

Also $T(p)(X^{\uparrow}(Y_m)) = 0^{TM}_m$ and, of course, $p_{TM}(X^{\uparrow}(Y_m)) = Y_m$.

The core vector bundle of T^2M is a copy of $TM \rightarrow M$. A section of this vector bundle is a vector field X on M. From Definition 1.1.3, the strut of X with respect to p_{TM} is:

$$X^{p_{TM}} : TM \rightarrow T(TM), \quad Y_m \mapsto X^{p_{TM}}(Y_m) = 0^{T^2M}_{Y_m} \underset{T(p)}{+} \overline{X(m)},$$

and by (1.36), this is precisely the vertical lift of $X(m)$ at point Y_m: $X^{p_{TM}}(Y_m) = X^{\uparrow}(Y_m)$. In order to distinguish between the struts X^A and X^B in this case, since $A = B = TM$, we again use the projections. So instead of X^{TM} we write $X^{p_{TM}}$ and $X^{T(p)}$. And $X^{p_{TM}} = X^{\uparrow}$.

Naturality of J_M

The following Lemma regarding the naturality property of the canonical involution will be needed later on, see [1, Exercise 3.3B(ii)], and [31, Section 8.13(1)].

Lemma 1.2.1. *Let M and N be smooth manifolds, and $F : M \rightarrow N$ a smooth map. Then $T^2(F) \circ J_M = J_N \circ T^2(F)$, where $T^2(F) = T(T(F))$ is the tangent of the tangent map $T(F)$.*

Proof. Take a $\xi = \frac{d}{dt}\left(\frac{d}{ds}\mu(t,s)\Big|_{s=0}\right)\Big|_{t=0}$, where $\mu : D \to M$ a smooth square of elements on M, where $D \subseteq \mathbb{R}\times\mathbb{R}$ an open subset of $\mathbb{R}\times\mathbb{R}$ with $(0,0)\in D$. Write,

$$\begin{aligned} T^2(F)\circ J_M(\xi) &= T^2(F)\left(\frac{d}{ds}\left(\frac{d}{dt}\mu(t,s)\Big|_{t=0}\right)\Big|_{s=0}\right) \\ &= \frac{d}{ds}\left(T(F)\left(\frac{d}{dt}\mu(t,s)\Big|_{t=0}\right)\right)\Big|_{s=0} \\ &= \frac{d}{ds}\left(\frac{d}{dt}F(\mu(t,s))\Big|_{t=0}\right)\Big|_{s=0}, \end{aligned}$$

and $F\circ\mu : D \to N$ a smooth square of elements on N, so

$$\begin{aligned} \frac{d}{ds}\left(\frac{d}{dt}F(\mu(t,s))\Big|_{t=0}\right)\Big|_{s=0} &= J_N\left(\frac{d}{dt}\left(\frac{d}{ds}F(\mu(t,s))\Big|_{s=0}\right)\Big|_{t=0}\right) \\ &= J_N\left(\frac{d}{dt}T(F)\left(\frac{d}{ds}\mu(t,s)\Big|_{s=0}\right)\Big|_{t=0}\right) \\ &= J_N\circ T^2(F)\left(\frac{d}{dt}\left(\frac{d}{ds}\mu(t,s)\Big|_{s=0}\right)\Big|_{t=0}\right) \\ &= J_N\circ T^2(F)(\xi), \end{aligned}$$

and this completes the proof. □

1.2.3 TA: Curves and Tulczyjew

Let $A \xrightarrow{q} M$ be a vector bundle of rank r, and denote its fibre over $m\in M$ by A_m. Take a local coordinate chart (U,φ) on M with associated local coordinates $(x^1,\dots,x^n)$, and a smooth local frame $(s_1,\dots,s_r)$ for A over U. Write an element $a\in A_m$, $m\in U$, as $a^1s_1(m)+\ldots+a^rs_r(m)$. Then $(q^{-1}(U),\Phi)$, the associated local chart on A,

$$\begin{aligned} \Phi : q^{-1}(U) &\to \varphi(U)\times\mathbb{R}^r, \\ a^1s_1(m)+\ldots+a^rs_r(m) &\mapsto (x^1,\dots,x^n,a^1,\dots,a^r). \end{aligned}$$

Therefore, we write an element $a\in A_m$ in local coordinates as $(x^1,\dots,x^n,a^1,\dots,a^r)$, or as (x,a).

Now take two local coordinate charts (U,φ) and (V,ψ) on M, with $U\cap V\neq\emptyset$. In addition, take two smooth local frames for A, $(s_1,\dots,s_r)$ over U, and $(\tilde{s}_1,\dots,\tilde{s}_r)$ over V, and the associated local coordinate charts $(q^{-1}(U),\Phi)$ and $(q^{-1}(V),\Psi)$ on A, with corresponding local coordinates, $(x^1,\dots,x^n,a^1,\dots,a^r)$, and $(\tilde{x}^1,\dots,\tilde{x}^n,\tilde{a}^1,\dots,\tilde{a}^r)$. The transition map on the region of intersection:

$$\begin{aligned} \Psi\circ\Phi^{-1} : \varphi(U\cap V)\times\mathbb{R}^r &\to \psi(U\cap V)\times\mathbb{R}^r, \\ (x^1,\dots,x^n,a^1,\dots,a^r) &\mapsto (\tilde{x}^1,\dots,\tilde{x}^n,\tilde{a}^1,\dots,\tilde{a}^r). \end{aligned} \tag{1.37}$$

Denote by $P : U \cap V \to \mathrm{GL}(r, \mathbb{R})$, the transition function defined by (1.37). Then for $a \in q^{-1}(U) \cap q^{-1}(V)$:

$$a = (\tilde{x}^1, \ldots, \tilde{x}^n, \tilde{a}^1, \ldots, \tilde{a}^r) = (\tilde{x}^1, \ldots, \tilde{x}^n, P^1_k(m)a^k, \ldots, P^r_k(m)a^k).$$

As in Section 1.2.1, the last equation in matrix form,

$$\begin{bmatrix} \tilde{a}^1 \\ \vdots \\ \tilde{a}^r \end{bmatrix} = \begin{bmatrix} P^1_1(m) & \ldots & P^1_r(m) \\ \vdots & \ddots & \vdots \\ P^r_1(m) & \ldots & P^r_r(m) \end{bmatrix} \begin{bmatrix} a^1 \\ \vdots \\ a^r \end{bmatrix},$$

of course in P^ℓ_k, ℓ denotes the row and k denotes the column of the matrix P. Therefore, the transformation laws for the fibre coordinates $(\tilde{a})$:

$$\tilde{a}^\ell = P^\ell_k a^k, \quad \ell = 1, \ldots, r. \tag{1.38}$$

As in the case of T^2M, denote a local coordinate system for TA by:

$$(x^1, \ldots, x^n, a^1, \ldots, a^r, \dot{x}^1, \ldots, \dot{x}^n, \dot{a}^1, \ldots, \dot{a}^r),$$

and shorthand notation $(x, a, \dot{x}, \dot{a})$. If we take two overlapping charts on TA, with associated local coordinates $(x, a, \dot{x}, \dot{a})$ and $(\tilde{x}, \tilde{a}, \dot{\tilde{x}}, \dot{\tilde{a}})$, we use the Jacobian of the transition map (1.37) to show how the coordinates change. In particular, the Jacobian of (1.37),

$$\left[\begin{array}{ccc|ccc} \frac{\partial \tilde{x}^1}{\partial x^1} & \cdots & \frac{\partial \tilde{x}^1}{\partial x^n} & \frac{\partial \tilde{x}^1}{\partial a^1} & \cdots & \frac{\partial \tilde{x}^1}{\partial a^r} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \tilde{x}^n}{\partial x^1} & \cdots & \frac{\partial \tilde{x}^n}{\partial x^n} & \frac{\partial \tilde{x}^n}{\partial a^1} & \cdots & \frac{\partial \tilde{x}^n}{\partial a^r} \\ \hline \frac{\partial \tilde{a}^1}{\partial x^1} & \cdots & \frac{\partial \tilde{a}^1}{\partial x^n} & \frac{\partial \tilde{a}^1}{\partial a^1} & \cdots & \frac{\partial \tilde{a}^1}{\partial a^r} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \tilde{a}^r}{\partial x^1} & \cdots & \frac{\partial \tilde{a}^r}{\partial x^n} & \frac{\partial \tilde{a}^r}{\partial a^1} & \cdots & \frac{\partial \tilde{a}^r}{\partial a^r} \end{array}\right] = \left[\begin{array}{c|c} \left(\frac{\partial \tilde{x}^i}{\partial x^j}\right) & \mathbf{0}_{(n\times r)} \\ \hline \left(\frac{\partial \tilde{a}^\ell}{\partial x^j}\right) & \left(\frac{\partial \tilde{a}^\ell}{\partial a^k}\right) \end{array}\right],$$

where $i, j = 1, \ldots, n$, and $\ell, k = 1, \ldots, r$. From (1.38), it follows that

$$\frac{\partial \tilde{a}^\ell}{\partial x^j} = \frac{\partial P^\ell_s}{\partial x^j} a^s, \quad \frac{\partial \tilde{a}^\ell}{\partial a^k} = P^\ell_k,$$

where s is a summation index, $s = 1, \ldots, r$. Therefore,

$$\left[\begin{array}{c|c} \left(\frac{\partial \tilde{x}^i}{\partial x^j}\right) & \mathbf{0}_{(n\times r)} \\ \hline \left(\frac{\partial \tilde{a}^\ell}{\partial x^j}\right) & \left(\frac{\partial \tilde{a}^\ell}{\partial a^k}\right) \end{array}\right] = \left[\begin{array}{c|c} \left(\frac{\partial \tilde{x}^i}{\partial x^j}\right)_{(n\times n)} & \mathbf{0}_{(n\times r)} \\ \hline \left(\frac{\partial P^\ell_s}{\partial x^j} a^s\right)_{(r\times n)} & (P^\ell_k)_{(r\times r)} \end{array}\right],$$

hence, we have the following change of coordinates for $(x, a, \dot{x}, \dot{a})$ and $(\tilde{x}, \tilde{a}, \dot{\tilde{x}}, \dot{\tilde{a}})$,

$$\tilde{a}^\ell = P^\ell_k(m) a^k, \tag{1.39}$$

$$\dot{\tilde{x}}^i = \frac{\partial \tilde{x}^i}{\partial x^j}(m) \dot{x}^j, \tag{1.40}$$

$$\dot{\tilde{a}}^\ell = P^\ell_k(m) \dot{a}^k + \frac{\partial P^\ell_s}{\partial x^j}(m) \dot{x}^j a^s. \tag{1.41}$$

The coordinates for the core vector bundle here change as $(\dot{\tilde{a}})$ change, when we set $\dot{x}^j$ and a^s, $j = 1, \ldots, n$, $s = 1, \ldots, r$ to zero in the last equation, hence again we see that the core vector bundle can be canonically identified with $A \to M$.

The two bundle projections p_A and $T(q)$ in local coordinates $(x, a, \dot{x}, \dot{a})$,

$$p_A : TA \to A, \quad (x, a, \dot{x}, \dot{a}) \mapsto (x, a),$$

and

$$T(q) : TA \to TM, \quad (x, a, \dot{x}, \dot{a}) \mapsto (x, \dot{x}),$$

where the tangent prolongation of $q : A \to M$ follows similarly as $T(p)$ in Section 1.2.1.

Now, for any curve $\gamma : I \to M$, $I \subseteq \mathbb{R}$ an open interval of $\mathbb{R}$ with $0 \in I$, what is $\frac{d}{dt}\gamma(t)\Big|_{t=0}$ in local coordinates of TM?

First off, take a chart (U, φ) in M, with $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n$, $m \mapsto (x^1(m), \ldots, x^n(m))$. Write γ in local coordinates:

$$\gamma(t) = (x^1(t), \ldots, x^n(t)).$$

Then the velocity vector of this curve at $t = 0$:

$$\gamma'(0) = \frac{dx^i}{dt}(0) \frac{\partial}{\partial x^i}\Big|_{\gamma(0)} \in T_{\gamma(0)}M,$$

and in the corresponding local coordinates on TM defined by (U, φ):

$$\gamma'(0) = (x^1(0), \ldots, x^n(0), \frac{dx^1}{dt}(0), \ldots, \frac{dx^n}{dt}(0)).$$

Now take a curve $a : I \to A$ in A, $I \subseteq \mathbb{R}$ an open interval of $\mathbb{R}$, with $0 \in I$. Then $a'(0) = \frac{d}{dt}a(t)\Big|_{t=0} \in T_{a(0)}A$. Take a chart (U, φ) on M, a local frame (s_i) for A over U, and the associated local chart $(q^{-1}(U), \Phi)$ on A. The curve $a(t)$ in local coordinates,

$$a(t) = (x^1(t), \ldots, x^n(t), a^1(t), \ldots, a^r(t)) = (x(t), a(t)),$$

and its velocity vector at $t = 0$,

$$\begin{aligned} a'(0) = \frac{d}{dt}a(t)\Big|_{t=0} &= \frac{dx^i}{dt}(0) \frac{\partial}{\partial x^i}\Big|_{x(0)} + \frac{da^i}{dt}(0) \frac{\partial}{\partial a^i}\Big|_{a(0)} \\ &= \left(x^1(0), \ldots, x^n(0), a^1(0), \ldots, a^r(0), \frac{dx^1}{dt}(0), \ldots, \frac{dx^n}{dt}(0), \frac{da^1}{dt}(0), \ldots, \frac{da^r}{dt}(0) \right). \end{aligned}$$

The following is [34, Proposition 1, p.81]. It is a result we use repeatedly throughout the following Chapters and we include its proof for completeness.

Proposition 1.2.2. *Take two vectors $\xi_1, \xi_2 \in TA$ with $T(q)(\xi_1) = T(q)(\xi_2)$. Then there exist curves a_1, $a_2 : I \to A$ such that $\xi_1 = \frac{d}{dt}a_1(t)\Big|_{t=0}$ and $\xi_2 = \frac{d}{dt}a_2(t)\Big|_{t=0}$, with $q(a_1(t)) = q(a_2(t))$ for t near zero.*

Proof. Let $m = p(T(q)(\xi_1)) = p(T(q)(\xi_2)) = (x^1(m), \ldots, x^n(m))$ be in the domain of U of the chart (U, φ), with coordinates $(x^1, \ldots, x^n)$. Write

$$\begin{aligned}
\xi_1 &= (x^1(m), \ldots, x^n(m), a_1^1(m), \ldots, a_1^r(m), \dot{x}_1^1(m), \ldots, \dot{x}_1^n(m), \dot{a}_1^1(m), \ldots, \dot{a}_1^r(m)) \\
\xi_2 &= (x^1(m), \ldots, x^n(m), a_2^1(m), \ldots, a_2^r(m), \dot{x}_2^1(m), \ldots, \dot{x}_2^n(m), \dot{a}_2^i(m), \ldots, \dot{a}_2^r(m)),
\end{aligned}$$

for some local coordinates on TA as described above. The following curves $a_1, a_2 : \mathbb{R} \to A$,

$$\begin{aligned}
a_1(t) &= \left(x^1(m) + t\dot{x}_1^1(m), \ldots, x^n(m) + t\dot{x}_1^n(m), a_1^1(m) + t\dot{a}_1^1(m), \ldots, a_1^r(m) + t\dot{a}_1^r(m)\right), \\
a_2(t) &= \left(x^1(m) + t\dot{x}_2^1(m), \ldots, x^n(m) + t\dot{x}_2^n(m), a_2^1(m) + t\dot{a}_2^1(m), \ldots, a_2^r(m) + t\dot{a}_2^r(m)\right),
\end{aligned}$$

for t sufficiently close to 0, satisfy the requirements. Indeed, we immediately see that

$$\frac{d}{dt}a_1(t)\Big|_{t=0} = \xi_1, \quad \frac{d}{dt}a_2(t)\Big|_{t=0} = \xi_2.$$

The condition $T(q)(\xi_1) = T(q)(\xi_2)$ implies additionally that

$$\dot{x}_1^i(m) = \dot{x}_2^i(m), \quad i = 1, \ldots, n,$$

hence from the formulas of $a_1(t)$ and $a_2(t)$ it follows that $q(a_1(t)) = q(a_2(t))$, for t near zero. □

Two types of functions on A

A small parenthesis on recalling a useful technique. Two types of functions defined on A, linear and pullback functions, are of particular importance. A section $\varphi \in \Gamma A^*$ defines a *linear function* ℓ_φ on A:

$$\begin{aligned}
\ell_\varphi : A &\to \mathbb{R}, \\
a &\mapsto \ell_\varphi(a) = \langle \varphi(q(a)), a \rangle.
\end{aligned}$$

For $f \in C^\infty(M)$, its *pullback function* $f \circ q \in C^\infty(A)$ on A is constant on the fibres of A.

To define either a vector field or a tangent vector on a vector bundle $A \xrightarrow{q} M$, it is enough to check how it "behaves" when applied to linear and pullback functions of A. A proof of this is given in Appendix A.2.1.

For example, in the case of $p_{TM} : T^2M \to TM$, for $\omega \in \Gamma(T^*M)$ the corresponding linear function on TM is,

$$\ell_\omega : TM \to \mathbb{R}, \quad X_m \mapsto \langle \omega(m), X_m \rangle.$$

From the definition (1.34) of the complete lift $\widetilde{X}$ of a vector field $X \in M$, the action of $\widetilde{X}$ on linear and pullback functions, for $\omega \in \Gamma(T^*M)$ and $f \in C^\infty(M)$ is,

$$\widetilde{X}(f \circ p) = X(f) \circ p, \quad \widetilde{X}(\ell_\omega) = \ell_{L_X(\omega)}.$$

Also, for $f \in C^\infty(M)$, $\omega \in \Gamma(T^*M)$, it follows directly from the definition of $X^\uparrow$, see Section 1.2.2, that

$$X^\uparrow(f \circ p) = 0, \quad X^\uparrow(\ell_\omega) = \langle \omega, X \rangle \circ p.$$

For a section $\mu \in \Gamma A$, a section of the core of TA, the strut $\mu^\uparrow$ of μ over A is given by (1.10). Applied to the two types of functions on A, the linear functions ℓ_φ, for $\varphi \in \Gamma A^*$, and the pullbacks $f \circ q$ for $f \in C^\infty(M)$, it follows directly from (1.10) that

$$\mu^\uparrow(\ell_\varphi) = \langle \varphi, \mu \rangle \circ q, \quad \mu^\uparrow(f \circ q) = 0. \tag{1.42}$$

1.2.4 Connections in A and in TM

In this section we present all the basic concepts and formulas from Connection theory needed in later sections.

Given a connection ∇ in a vector bundle $A \xrightarrow{q} M$, the *dual* connection $\nabla^{(*)}$ in the dual vector bundle $A^* \xrightarrow{q_*} M$ is defined by,

$$\langle \nabla^{(*)}_X(\varphi), \mu \rangle = X(\langle \varphi, \mu \rangle) - \langle \varphi, \nabla_X(\mu) \rangle, \tag{1.43}$$

where $\mu \in \Gamma A$, $\varphi \in \Gamma A^*$, and $X \in \mathfrak{X}(M)$. Further reading in [25, Section 3.4], [16, p.320].

Given a vector field $X \in \mathfrak{X}(M)$, denote by X^H its horizontal lift on A with respect to ∇. Then (X^H, X) is a linear vector field on A. The action of the horizontal lift X^H on the linear and the pullback functions,

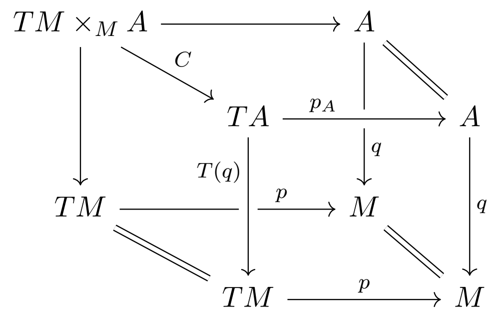
$$X^H(f \circ q) = X(f) \circ q, \quad X^H(\ell_\varphi) = \ell_{\nabla^{(*)}_X(\varphi)}, \tag{1.44}$$

for $f \in C^\infty(M)$ and $\varphi \in \Gamma A^*$.

That a connection in a vector bundle A is equivalent to a double vector bundle morphism

$$C : TM \times_M A \to TA, \quad (X_m, a) \mapsto (X_m)^H(a),$$

is described in detail in [5, p.324-8 and Problem 4, p.337]. Other references include [25, Section 5.2], and for the particular case $A = TM$ in [31, Section 22.8], and in implicit form in [4, p.334].



The double vector bundle morphism C is a smooth right-inverse to $(T(q), p_A) : TA \to TM \times_M A$, and is linear in both arguments, i.e., for $a_1, a_2 \in A_m$ and $X \in T_mM$,

$$C(X, a_1 + a_2) = X^H(a_1 + a_2) = X^H(a_1) \underset{T(q)}{+} X^H(a_2) = C(X, a_1) \underset{T(q)}{+} C(X, a_2),$$

and for $X_1, X_2 \in T_mM$ two tangent vectors on M at point $m \in M$, and $a \in A_m$:

$$C(X_1+X_2, a) = (X_1+X_2)^H(a) = ({X_1}^H \underset{p_A}{+} {X_2}^H)(a) = {X_1}^H(a) \underset{p_A}{+} {X_2}^H(a) = C(X_1, a) \underset{p_A}{+} C(X_2, a).$$

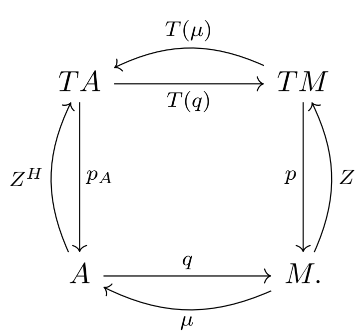
This formulation of a connection in A will show up once again towards the end of this section.

Example 1.2.3. The following example is central to Section 4.5. It is a subcase of [25, Theorem 3.4.5].

Consider the tangent double vector bundle TA, and let ∇ be a connection in A. Take a vector field $Z \in \mathfrak{X}(M)$, and take its horizontal lift Z^H with respect to ∇. Take also any $\mu \in \Gamma A$ and form the grid shown in (1.46). Then the warp of the grid is $\nabla_Z \mu$; that is, for $m \in M$,

$$T(\mu)(Z(m)) \underset{A}{-} Z^H(\mu(m)) = (\nabla_Z \mu)^\uparrow(\mu(m)), \tag{1.45}$$

where the right hand side is the vertical lift of $(\nabla_Z \mu)(m) \in A_m$ to $T_{\mu(m)}A$.



(1.46)

It is enough to check that the right hand side and the left hand side of (1.45) are equal when applied to the linear and the pullback functions of A. Starting from the right hand side of (1.45), when applied to a linear function ℓ_φ, $\varphi \in \Gamma A^*$:

$$\begin{aligned}
&\left(T(\mu)(Z(m)) \underset{A}{-} Z^H(\mu(m))\right)(\ell_\varphi) \\
&= \quad T(\mu)(Z(m))(\ell_\varphi) - Z^H(\mu(m))(\ell_\varphi) = Z(m)(\ell_\varphi \circ \mu) - (Z^H(\ell_\varphi))(\mu(m)) \\
&\overset{(1.44)}{=} \quad (Z(\langle\varphi,\mu\rangle))(m) - \ell_{\nabla^{(*)}_Z\varphi}(\mu(m)) = (Z(\langle\varphi,\mu\rangle))(m) - \langle\mu(m), (\nabla^{(*)}_Z\varphi)(m)\rangle \\
&\overset{(1.43)}{=} \quad \langle\varphi(m), (\nabla_Z\mu)(m)\rangle = \langle\varphi, \nabla_Z\mu\rangle(m) = \langle\varphi, \nabla_Z\mu\rangle(q(\mu(m))) \overset{(1.42)}{=} (\nabla_Z\mu)^\uparrow(\ell_\varphi)(\mu(m)) \\
&= \quad \left((\nabla_Z\mu)^\uparrow(\mu(m))\right)(\ell_\varphi),
\end{aligned}$$

and this is exactly the left hand side of (1.45) acting on ℓ_φ. About pullback functions $f \circ q$, $f \in C^\infty(M)$, it follows that

$$T(\mu)(Z(m))(f \circ q) = (Z(m))(f \circ q \circ \mu) = (Z(m))(f \circ \mathrm{id}_M) = (Z(f))(m),$$

and

$$Z^H(\mu(m))(f \circ q) = (Z^H(f \circ q))(\mu(m)) \overset{(1.44)}{=} (Z(f) \circ q)(\mu(m)) = Z(f)(q(\mu(m))) = (Z(f))(m),$$

and finally, by (1.42), it follows that (1.45) is true for pullback functions too. And this completes the proof.

Connections in TM

Now let us consider the special case of TM as a vector bundle over M. Take a connection ∇ in $TM \xrightarrow{p} M$. Denote by $\Gamma^k_{ij} \in C^\infty(M)$ the Christoffel symbols of ∇, $i, j, k = 1, \ldots, n$,

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k},$$

where we're summing over k. For $X, Y \in \mathfrak{X}(M)$ two vector fields on M with local coordinate expressions $X = X^i \frac{\partial}{\partial x^i}$ and $Y = Y^j \frac{\partial}{\partial x^j}$, $X^i, Y^j \in C^\infty(M)$. The covariant derivative of Y with respect to X:

$$\nabla_X Y = \left(X^i \Gamma^k_{ij} Y^j + X^i \frac{\partial Y^k}{\partial x^i}\right) \frac{\partial}{\partial x^k},$$

from where we deduce that $\nabla_X Y \in TM$ has coordinates

$$\nabla_X Y = \left(x^1, \ldots, x^n, X^i \Gamma^1_{ij} Y^j + X^i \frac{\partial Y^1}{\partial x^i}, \ldots, X^i \Gamma^n_{ij} Y^j + X^i \frac{\partial Y^n}{\partial x^i}\right),$$

where we write $X^i \Gamma^k_{ij} Y^j$ to distinguish between the order of indices. Given a vector field $X \in \mathfrak{X}(M)$ we can take its horizontal lift X^H with respect to ∇, a vector field on TM:

$$\begin{aligned}
X^H : TM &\to T^2 M, \\
v &\mapsto X^H(v),
\end{aligned}$$

and in local coordinates, writing $v = (x^1, \ldots, x^n, v^1, \ldots, v^n)$, we have,

$$X^H(v) = X^i \frac{\partial}{\partial x^i} - X^i \Gamma^k_{ij} v^j \frac{\partial}{\partial v^k} = (x^1, \ldots, x^n, v^1, \ldots, v^n, X^1, \ldots, X^n, -X^i\Gamma^1_{ij}v^j, \ldots, -X^i\Gamma^n_{ij}v^j). \tag{1.47}$$

Conjugate connection on TM

We consider TM as a special case for the following reason as well. Given a connection ∇ on TM, we can define the *conjugate connection* $\widehat{\nabla}$ on TM:

$$\widehat{\nabla}_X Y = \nabla_Y X + [X, Y], \tag{1.48}$$

see [16, p.319] for further details. The conjugate connection is only defined for the tangent bundle, and not for any arbitrary vector bundle $A \to M$. From (1.48) it follows immediately that, if Γ^k_{ij} are the Christoffel symbols of ∇, then for the Christoffel symbols $\widehat{\Gamma}^k_{ij}$ of $\widehat{\nabla}$ we have

$$\widehat{\Gamma}^k_{ij} \frac{\partial}{\partial x^k} = \widehat{\nabla}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} + \left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = \Gamma^k_{ji} \frac{\partial}{\partial x^k}.$$

Therefore,

$$\widehat{\nabla}_X Y = \left(X^i \widehat{\Gamma}^k_{ij} Y^j + X^i \frac{\partial Y^k}{\partial x^i}\right) \frac{\partial}{\partial x^k} = \left(Y^j \Gamma^k_{ji} X^i + X^i \frac{\partial Y^k}{\partial x^i}\right) \frac{\partial}{\partial x^k}. \tag{1.49}$$

Denote by $X^{\hat{H}} \in \mathfrak{X}(TM)$ the horizontal lift on TM of a vector field X on M with respect to the conjugate connection $\widehat{\nabla}$. For $v \in TM$,

$$X^{\hat{H}}(v) = X^i \frac{\partial}{\partial x^i} - X^i \widehat{\Gamma}^k_{ij} v^j \frac{\partial}{\partial v^k} = X^i \frac{\partial}{\partial x^i} - v^j \Gamma^k_{ji} X^i \frac{\partial}{\partial v^k}.$$

Finally, using the double vector bundle formulation, if we denote by $C : TM \times_M TM \to T^2M$ the double vector bundle morphism that corresponds to the connection ∇ in TM, then the conjugate connection $\widehat{\nabla}$ is described by $\widehat{C} = J_M \circ C \circ J_0$, where J_M is the canonical involution in T^2M, and $J_0 : TM \oplus TM \to TM \oplus TM$ interchanges the arguments, see [22, p.7].

Chapter 2

Triple vector bundles

Triple vector bundles were introduced in [24],[13], and [27]. In this chapter we start with the definition of a triple vector bundle, describe the basic operations and examples, and set up the notation.

2.1 Definition of triple vector bundle

As the definition of a double vector bundle has three parts, so does the definition of a triple vector bundle consist of (i) the algebraic compatibility conditions, (ii) the triple source condition[1], and (iii) the existence of sigma maps.

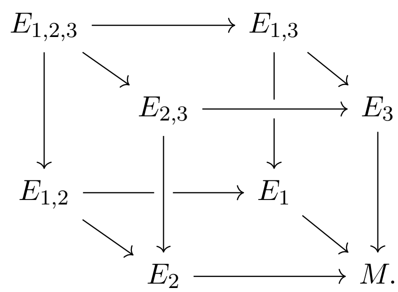
We start with part (i). We consider a cube of vector bundles as in (2.1). We refer to the faces of $E_{1,2,3}$ by the names

Back, Front, Left, Right, Up, Down.

The Back, Left, and Up faces are called *upper faces*, and the Front, Right, and Down faces are called *lower faces*. The total space of (2.1) should be denoted, for consistency with the labelling scheme, by $E_{1,2,3}$ but we will usually denote it by E.

Definition 2.1.1. (Part (i)). A *triple vector bundle* is a cube of vector bundle structures, as in (2.1), such that each face is a double vector bundle, and such that the vector bundle operations in $E \to E_{1,2}$ are morphisms of double vector bundles from the Up face of E to the Down face of E. Similarly for the other vector bundle structures in E.

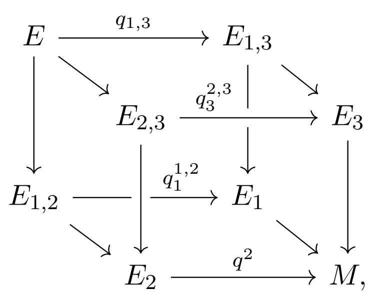
[1] With an argument analogous to the one in [19], it is proved in [9] that part (ii) of the definition of a triple vector bundle follows from part (i) of the definition.



(2.1)

2.1.1 Projection maps

The general rule of indices of a projection map q is as follows: superscripts denote the domain and subscripts denote the target. We omit superscripts when the domain is the total space E, and omit the subscript when the target is M. For example, the projection from the Left to the Right face of E,



and altogether, we denote this double vector bundle morphism by $(q_{1,3}; q_1^{1,2}, q_3^{2,3}; q^2)$.

2.1.2 Triple source condition

Before proceeding with part (ii) of the definition of a triple vector bundle, we need to establish the following.

Proposition 2.1.2. *Given a triple vector bundle E, write W for the set of all*

$$(e_{1,2}, e_{2,3}, e_{1,3}) \in E_{1,2} \times E_{2,3} \times E_{1,3}$$

such that

$$q_2^{1,2}(e_{1,2}) = q_2^{2,3}(e_{2,3}), \quad q_3^{2,3}(e_{2,3}) = q_3^{1,3}(e_{1,3}), \quad q_1^{1,3}(e_{1,3}) = q_1^{1,2}(e_{1,2}). \tag{2.2}$$

This is a submanifold of $E_{1,2} \times E_{2,3} \times E_{1,3}$.

Proof. To see this, write W as the preimage of a submanifold under a surjective submersion.

Define $F : E_{1,2} \times E_{2,3} \times E_{1,3} \to E_1 \times E_2 \times E_2 \times E_3 \times E_1 \times E_3$ by

$$F(e_{1,2}, e_{2,3}, e_{1,3}) = (q_1^{1,2}(e_{1,2}), q_2^{1,2}(e_{1,2}), q_2^{2,3}(e_{2,3}), q_3^{2,3}(e_{2,3}), q_1^{1,3}(e_{1,3}), q_3^{1,3}(e_{1,3})).$$

The domain of F is simply the cartesian product of the three submanifolds $E_{1,2}$, $E_{1,3}$, $E_{2,3}$. The target of F may be restricted to the following cartesian product:

$$(E_1 \times_M E_2) \times (E_2 \times_M E_3) \times (E_1 \times_M E_3)\,.$$

If the target of F were simply the cartesian product of the six manifolds (twice each copy of E_i, $i = 1, 2, 3$), for example, for $(e_1, e_2) \in E_1 \times E_2$, there is no guarantee that there exists an $e_{1,2} \in E_{1,2}$ such that

$$q_1^{1,2}(e_{1,2}) = e_1, \quad q_2^{1,2}(e_{1,2}) = e_2.$$

We can now view F as the product of the three double source maps:

$$F := (\natural_{1,2}, \natural_{2,3}, \natural_{1,3}) : E_{1,2} \times E_{2,3} \times E_{1,3} \to (E_1 \times_M E_2) \times (E_2 \times_M E_3) \times (E_1 \times_M E_3)\,.$$

By Definition 2.1.1, the lower faces of E satisfy the double source condition, hence each double source map $\natural_{1,2}$, $\natural_{2,3}$, and $\natural_{1,3}$ is a surjective submersion. It follows that F is a surjective submersion.

Hence, for any $(e_1, e_2, f_2, e_3, f_1, f_3) \in (E_1 \times_M E_2) \times (E_2 \times_M E_3) \times (E_1 \times_M E_3)$, we see that there exist $e_{1,2} \in E_{1,2}$, $e_{2,3} \in E_{2,3}$, $e_{1,3} \in E_{1,3}$, such that

$$F(e_{1,2}, e_{2,3}, e_{1,3}) = (e_1, e_2, f_2, e_3, f_1, f_3).$$

Now choose $e_1 = f_1$, $e_2 = f_2$, and $e_3 = f_3$. Then,

$$\Delta = \{(e_1, e_2, e_2, e_3, e_1, e_3) \,|\, e_2 \in E_2, e_3 \in E_3, e_1 \in E_1\}.$$

This Δ is a submanifold of the target of F and F is a surjective submersion, so $F^{-1}(\Delta)$ is a submanifold of $E_{1,2} \times E_{2,3} \times E_{1,3}$.

To show that $F^{-1}(\Delta) = W$ it is necessary to be sure that if $(e_{1,2}, e_{2,3}, e_{1,3}) \in F^{-1}(\Delta)$ then $q^1(q_1^{1,2}(e_{1,2})) = q^2(q_2^{2,3}(e_{2,3})) = q^3(q_3^{1,3}(e_{1,3})) = m$, all three elements project to the same element of M.

Given $(e_{1,2}, e_{2,3}, e_{1,3}) \in F^{-1}(\Delta)$, write e_1, e_2, e_3 as above and write $m = q^1(e_1)$. Then

$$q^3(e_3) = q^3(q_3^{1,3}(e_{1,3})) = q^1(q_1^{1,3}(e_{1,3})) = q^1(e_1) = m.$$

Likewise $q^2(e_2) = m$. So $(e_{1,2}, e_{2,3}, e_{1,3}) \in W$.

This completes the proof. □

The manifold W is a triple vector bundle, with zero ultracore. We point out that this proof also gives, for any three double vector bundles with matching side bundles, a triple vector bundle with zero ultracore. To see the structure on $W \to E_{1,2}$, take $(e_{1,2}, e_{2,3}, e_{1,3})$ and $(e_{1,2}, e'_{2,3}, e'_{1,3})$ in W, and define

$$(e_{1,2}, e_{2,3}, e_{1,3}) \underset{1,2}{+} (e_{1,2}, e'_{2,3}, e'_{1,3}) = (e_{1,2}, e_{2,3} \underset{E_2}{+} e'_{2,3}, e_{1,3} \underset{E_1}{+} e'_{1,3}).$$

The additions on the right hand side are defined, thanks to the definition of W. Scalar multiplication is defined likewise.

We can now state the following natural condition we impose on a triple vector bundle E.

Definition 2.1.3. (Part (ii)) A general triple vector bundle E satisfies the *triple source condition* if the *triple source map*

$$\widetilde{\natural} : E \to W, \quad e \mapsto (q_{1,2}(e), q_{2,3}(e), q_{1,3}(e)), \tag{2.3}$$

is a surjective submersion.

2.1.3 Local coordinates on E

As with double vector bundles, one can also introduce and work with local coordinates on a triple vector bundle. For completeness we present some notation, following [38, Example 6.3]. We will resort to this technique in Section 2.4.4, to prove Lemma 2.4.6.

We denote a local coordinate system on E by

$$(x, v_{(1)}, v_{(2)}, v_{(3)}, v_{(12)}, v_{(13)}, v_{(23)}, v_{(123)}),$$

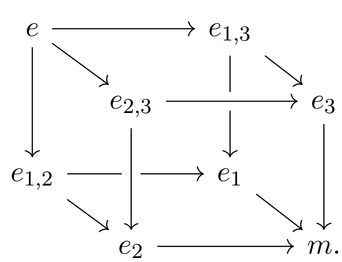
where (x) is shorthand notation for $(x^1, \dots, x^n)$, local coordinates on the base manifold M of E, and the subsequent $(v_{(1)}), \dots, (v_{(123)})$ are fibre coordinates for the constituent vector bundles of E. Denoting by r_1, for example, the rank of $E_1 \to M$, then $(v_{(1)}) = (v^1_{(1)}, \dots, v^{r_1}_{(1)})$, and so forth.

On the intersection of two overlapping charts, the transformation laws for the seven type fibre coordinates, are the following,

$$
\begin{aligned}
\tilde{v}_{(1)}^{i_1} &= P_{(1)}{}^{i_1}_{j_1} v_{(1)}^{j_1}, \\
\tilde{v}_{(2)}^{i_2} &= P_{(2)}{}^{i_2}_{j_2} v_{(2)}^{j_2}, \\
\tilde{v}_{(3)}^{i_3} &= P_{(3)}{}^{i_3}_{j_3} v_{(3)}^{j_3}, \\
\tilde{v}_{(12)}^{i_{12}} &= P_{(12)}{}^{i_{12}}_{j_{12}} v_{(12)}^{j_{12}} + P_{(1,2)}{}^{i_{12}}_{j_1\, j_2} v_{(1)}^{j_1} v_{(2)}^{j_2}, \\
\tilde{v}_{(13)}^{i_{13}} &= P_{(13)}{}^{i_{13}}_{j_{13}} v_{(13)}^{j_{13}} + P_{(1,3)}{}^{i_{13}}_{j_1\, j_3} v_{(1)}^{j_1} v_{(3)}^{j_3}, \\
\tilde{v}_{(23)}^{i_{23}} &= P_{(23)}{}^{i_{23}}_{j_{23}} v_{(23)}^{j_{23}} + P_{(2,3)}{}^{i_{23}}_{j_2\, j_3} v_{(2)}^{j_2} v_{(3)}^{j_3}, \\
\tilde{v}_{(123)}^{i_{123}} &= P_{(123)}{}^{i_{123}}_{j_{123}} v_{(123)}^{j_{123}} + P_{(23,1)}{}^{i_{123}}_{j_{23}\, j_1} v_{(23)}^{j_{23}} v_{(1)}^{j_1} \\
&\quad + P_{(13,2)}{}^{i_{123}}_{j_{13}\, j_2} v_{(13)}^{j_{13}} v_{(2)}^{j_2} + P_{(12,3)}{}^{i_{123}}_{j_{12}\, j_3} v_{(12)}^{j_{12}} v_{(3)}^{j_3} \\
&\quad + P_{(1,2,3)}{}^{i_{123}}_{j_1\, j_2\, j_3} v_{(1)}^{j_1} v_{(2)}^{j_2} v_{(3)}^{j_3}.
\end{aligned}
\tag{2.4}
$$

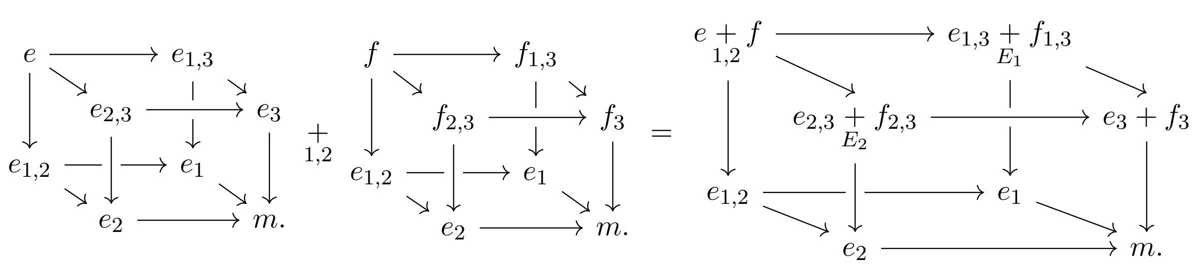
2.2 Basic apparatus on triple vector bundles

We now establish the notation and the basic operations in the triple vector bundle setting. The outline of a single element $e \in E$:

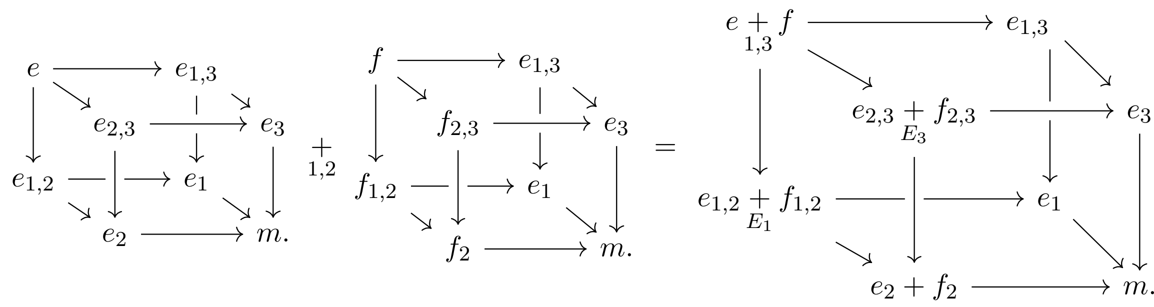


2.2.1 Addition and scalar multiplication

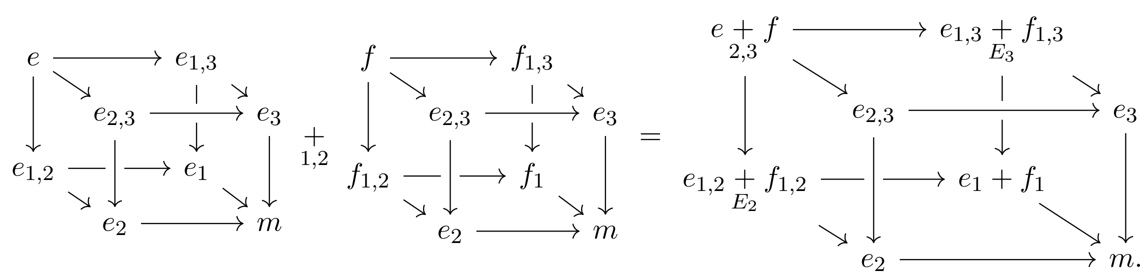
How do we add elements in triple vector bundles? Addition in each upper vector bundle structure is a double vector bundle morphism, therefore, if $e, f \in E$ lie over the same point of $E_{1,2}$, their sum has the outline:



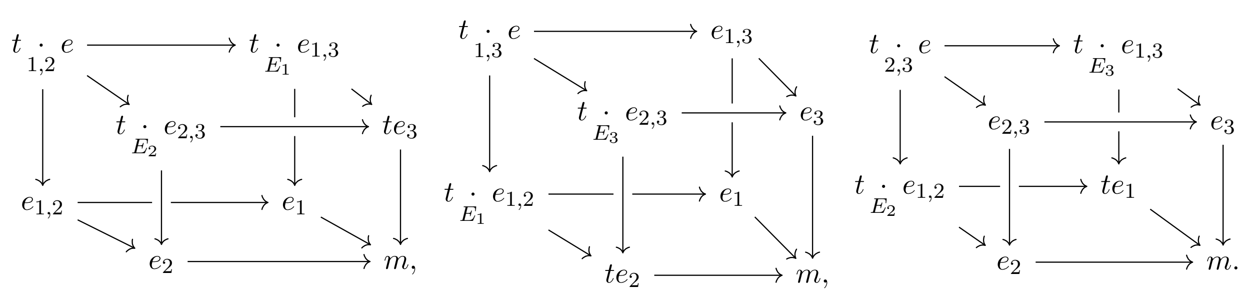
Similarly, if e and f lie over the same element in $E_{1,3}$, the outline of their sum:



Finally, addition of $e, f \in E$ over the same point in $E_{2,3}$,



Scalar multiplication follows in a similar way. If $t \in \mathbb{R}$ is a scalar, then scalar multiplication over the three vector bundle structures of E is



2.2.2 Interchange laws

In the double case, the interchange laws encompass the structure of the double vector bundle (see [25, Section 9.1]). We write similar laws for a triple vector bundle E.

There are two types of interchange laws in a triple vector bundle, the *"small" interchange laws*, and the *"big" interchange law*. The small interchange laws are direct generalizations of the double case; they involve only two out of the three vector bundle structures of the total space E.

Small interchange laws

We descibe in detail the interchange law in the Left face.

As in the double vector bundle case, we have the following vector bundle morphism

$$\begin{array}{ccc} E \underset{E_{1,2}}{\times} E & \xrightarrow{\underset{1,2}{+}} & E \\ \Big\downarrow & & \Big\downarrow \\ E_{2,3} \underset{E_2}{\times} E_{2,3} & \xrightarrow[\underset{E_2}{+}]{} & E_{2,3}. \end{array}$$

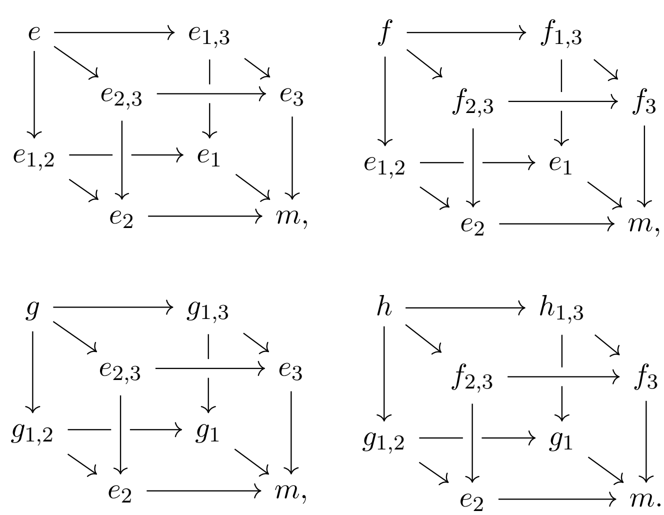
For $(e,f),(g,h) \in E \underset{E_{1,2}}{\times} E\Big|_{(e_{2,3},f_{2,3})}$ we have:

$$\underset{1,2}{+}\left((e,f) \underset{E_{2,3} \underset{E_2}{\times} E_{2,3}}{+} (g,h)\right) = \left(\underset{1,2}{+}(e,f)\right) \underset{2,3}{+} \left(\underset{1,2}{+}(g,h)\right)$$

and expanding both the right hand side and the left hand side we have the following interchange law:

$$(e \underset{2,3}{+} g) \underset{1,2}{+} (f \underset{2,3}{+} h) = (e \underset{1,2}{+} f) \underset{2,3}{+} (g \underset{1,2}{+} h), \tag{2.5}$$

where $q_{1,2}(e) = q_{1,2}(f)$, $q_{1,2}(g) = q_{1,2}(h)$, and $q_{2,3}(e) = q_{2,3}(g)$ and $q_{2,3}(f) = q_{2,3}(h)$. The outlines of e, f, g, h are,



How do e, f, g, and h project to the Right face? Use the double vector bundle morphism $(q_{1,3}; q_1^{1,2}, q_3^{2,3}; q^2)$, the projection from the Left to the Right face of E. In particular, the vector bundle morphism $(q_{1,3}, q_1^{1,2})$,

$$\begin{array}{ccc} E & \xrightarrow{q_{1,3}} & E_{1,3} \\ \big\downarrow & & \big\downarrow \\ E_{1,2} & \xrightarrow[q_1^{1,2}]{} & E_1. \end{array}$$

For $e, f \in E\Big|_{e_{1,2}}$, by fibrewise linearity we have:

$$q_{1,3}(e \underset{1,2}{+} f) = q_{1,3}(e) \underset{E_1}{+} q_{1,3}(f) = e_{1,3} \underset{E_1}{+} f_{1,3}. \tag{2.6}$$

Similarly, the vector bundle morphism $(q_{1,3}, q_3^{2,3})$,

$$\begin{array}{ccc} E & \xrightarrow{q_{1,3}} & E_{1,3} \\ \big\downarrow & & \big\downarrow \\ E_{2,3} & \xrightarrow[q_3^{2,3}]{} & E_3, \end{array}$$

for $e, g \in E\Big|_{e_{2,3}}$, by fibrewise linearity again we have:

$$q_{1,3}(e \underset{2,3}{+} g) = q_{1,3}(e) \underset{E_3}{+} q_{1,3}(g) = e_{1,3} \underset{E_3}{+} g_{1,3}. \tag{2.7}$$

Therefore, since both $e \underset{2,3}{+} g$ and $f \underset{2,3}{+} h$ project to the same $e_{1,2} \underset{E_2}{+} g_{1,2} \in E_{1,2}$, from (2.6),

$$\begin{aligned} q_{1,3}\left((e \underset{2,3}{+} g) \underset{1,2}{+} (f \underset{2,3}{+} h)\right) &= q_{1,3}(e \underset{2,3}{+} g) \underset{E_1}{+} q_{1,3}(f \underset{2,3}{+} h) \\ &\overset{(2.7)}{=} \left(q_{1,3}(e) \underset{E_3}{+} q_{1,3}(g)\right) \underset{E_1}{+} \left(q_{1,3}(f) \underset{E_3}{+} q_{1,3}(h)\right) = (e_{1,3} \underset{E_3}{+} g_{1,3}) \underset{E_1}{+} (f_{1,3} \underset{E_3}{+} h_{1,3}). \end{aligned}$$

Now $e \underset{1,2}{+} f$ and $g \underset{1,2}{+} h$ project to $e_{2,3} \underset{E_2}{+} f_{2,3}$, hence from (2.7),

$$\begin{aligned} q_{1,3}\left((e \underset{1,2}{+} f) \underset{2,3}{+} (g \underset{1,2}{+} h)\right) &= q_{1,3}(e \underset{1,2}{+} f) \underset{E_3}{+} q_{1,3}(g \underset{1,2}{+} h) \\ &\overset{(2.6)}{=} \left(q_{1,3}(e) \underset{E_1}{+} q_{1,3}(f)\right) \underset{E_3}{+} \left(q_{1,3}(g) \underset{E_1}{+} q_{1,3}(h)\right) = (e_{1,3} \underset{E_1}{+} f_{1,3}) \underset{E_3}{+} (g_{1,3} \underset{E_1}{+} h_{1,3}). \end{aligned}$$

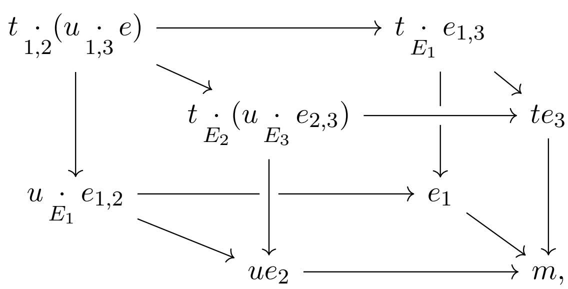
Applying $q_{1,3}$ to (2.5), we see that the interchange law in the Right face holds for $e_{1,3}$, $f_{1,3}$, $g_{1,3}$, and $h_{1,3} \in E_{1,3}$.

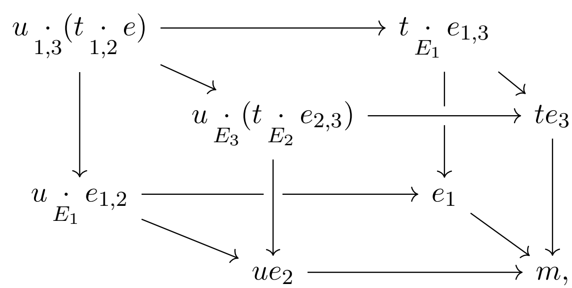
A small remark. When we look at elements in the Left face, we focus entirely on that face, and then *project* to the Right face. We do not look at the four elements e, f, g, and h as being over the same $e_{1,3} \in E_{1,3}$; we're looking at e, f, g, and h in E over $e_2 \in E_2$, and *how* they project in $E_{1,3}$.

Let us state an interchange law for scalar multiplication. For $e \in E$, and $t, u \in \mathbb{R}$, we have

$$t \underset{1,2}{\cdot} (u \underset{1,3}{\cdot} e) = u \underset{1,3}{\cdot} (t \underset{1,2}{\cdot} e).$$

Some outlines,





in other words, as in the case of addition, the interchange law in the Back face for scalar multiplication projects to the interchange law in the Front face.

Variations of interchange laws

In Chapter 1 we described in detail (1.2), a variation of the interchange law for the two additions in D. A similar identity holds in the triple vector bundle setting. Take e, f, g, h with outlines as in (2.5). Then

$$(e \underset{2,3}{-} g) \underset{1,2}{-} (f \underset{2,3}{-} h) = (e \underset{1,2}{-} f) \underset{2,3}{-} (g \underset{1,2}{-} h) \tag{2.8}$$

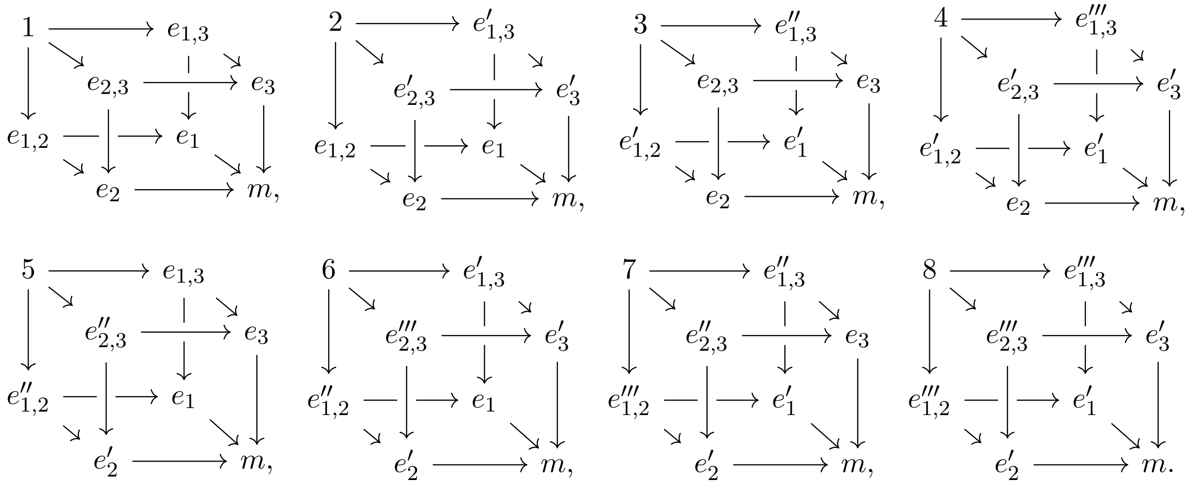
is a variation of (2.5), of the interchange law for the two additions in the Left face of E. Of course, taking the projection $q_{1,3}$ of the previous identity yields

$$(e_{1,3} \underset{E_3}{-} g_{1,3}) \underset{E_1}{-} (f_{1,3} \underset{E_3}{-} h_{1,3}) = (e_{1,3} \underset{E_1}{-} f_{1,3}) \underset{E_3}{-} (g_{1,3} \underset{E_1}{-} h_{1,3}),$$

a variation of the interchange law for the additions in the Right face of E, an example of (1.2).

Big interchange law

The big interchange law involves all three vector bundle structures of E. We need eight elements, with the following outlines,



Start with

$$\left((1 \underset{1,2}{+} 2) \underset{2,3}{+} (3 \underset{1,2}{+} 4)\right) \underset{1,3}{+} \left((5 \underset{1,2}{+} 6) \underset{2,3}{+} (7 \underset{1,2}{+} 8)\right),$$

and in each parenthesis, apply the interchange law in the Left face of E:

$$\left((1 \underset{2,3}{+} 3) \underset{1,2}{+} (2 \underset{2,3}{+} 4)\right) \underset{1,3}{+} \left((5 \underset{2,3}{+} 7) \underset{1,2}{+} (6 \underset{2,3}{+} 8)\right).$$

Now apply the interchange law in the Back face of E in the outer parentheses:

$$\left((1 \underset{2,3}{+} 3) \underset{1,3}{+} (5 \underset{2,3}{+} 7)\right) \underset{1,2}{+} \left((2 \underset{2,3}{+} 4) \underset{1,3}{+} (6 \underset{2,3}{+} 8)\right),$$

and in each parenthesis apply the interchange law in the Up face,

$$\left((1 \underset{1,3}{+} 5) \underset{2,3}{+} (3 \underset{1,3}{+} 7)\right) \underset{1,2}{+} \left((2 \underset{1,3}{+} 6) \underset{2,3}{+} (4 \underset{1,3}{+} 8)\right).$$

Applying the interchange law in the Left face of E in the outer parentheses,

$$\left((1 \underset{1,3}{+} 5) \underset{1,2}{+} (2 \underset{1,3}{+} 6)\right) \underset{2,3}{+} \left((3 \underset{1,3}{+} 7) \underset{1,2}{+} (4 \underset{1,3}{+} 8)\right).$$

Finally, apply the interchange law in the Back face in each parenthesis,

$$\left((1 \underset{1,2}{+} 2) \underset{1,3}{+} (5 \underset{1,2}{+} 6)\right) \underset{2,3}{+} \left((3 \underset{1,2}{+} 4) \underset{1,3}{+} (7 \underset{1,2}{+} 8)\right).$$

Applying the interchange law in the Up face in the outer parentheses, takes us back to original expression. Hence we have these six expressions that are equal.

In practice, we will be using small interchange laws and variations thereof in what follows.

2.2.3 Zero sections of E

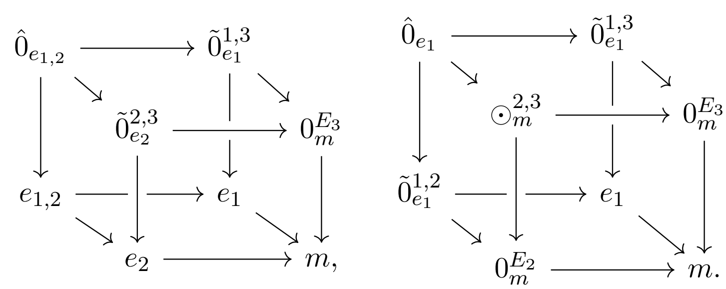
We have a lot of zero sections; the zero section of each invidual vector bundle structure, but also, of each double vector bundle structure.

We denote the zero section of E_1 by $0^{E_1} : M \to E_1$, $m \mapsto 0^{E_1}_m$, and similarly for E_2 and E_3.

The zero section of $E_{1,2} \to E_1$ is denoted by $\tilde{0}^{1,2} : E_1 \to E_{1,2}$, $e_1 \mapsto \tilde{0}^{1,2}_{e_1}$. The double zero of $E_{1,2}$ is denoted by $\odot^{1,2}_m$, with similar notations for the other vector bundle structures.

We denote the zero section of $E \to E_{1,2}$ by $\hat{0} : E_{1,2} \to E$, $e_{1,2} \mapsto \hat{0}_{e_{1,2}}$. Note that the subscripts of the element $e_{1,2}$ are enough to indicate that this is the zero section of E over $E_{1,2}$, therefore, there is no need for superscripts on $\hat{0}$, see the first diagram of (2.9).

We denote the special case $\hat{0}_{\tilde{0}^{1,2}_{e_1}} = \hat{0}_{\tilde{0}^{1,3}_{e_1}}$ simply by $\hat{0}_{e_1}$, as in the second diagram of (2.9).



(2.9)

The triple zero of E is denoted by $\odot^3_m$.

2.2.4 Useful operations with zeros – Part 1

The following operations appear all the time, and we describe them in detail in this subsection. We write out explicit formulas for $E_{1,3}$. Similar formulas for the other structures follow, and we include them for completion.

Take $e_{1,3}, e'_{1,3} \in E_{1,3}$, with outlines $(e_{1,3}; e_1, e_3; m)$ and $(e'_{1,3}; e'_1, e'_3; m)$. We have the following cases.

1. If $e_{1,3}, e'_{1,3} \in E_{1,3}$ are over the same $e_1 \in E_1$, then:

$$\hat{0}_{e_{1,3} \underset{E_1}{+} e'_{1,3}} = \hat{0}_{e_{1,3}} \underset{1,2}{+} \hat{0}_{e'_{1,3}}. \tag{2.10}$$

   To see this, note that we have two additions in the Right face, $\underset{E_1}{+}$ and $\underset{E_3}{+}$. The zero section $\hat{0} \in \Gamma_{E_{1,3}} E$ is a double vector bundle morphism from the Right to the

Left face. The two additions in the Left face are $\underset{1,2}{+}$ and $\underset{2,3}{+}$. And $e_{1,3}$ and $e'_{1,3}$ are over the same $e_1 \in E_1$. Therefore, $\hat{0}_{e_{1,3}}$ and $\hat{0}_{e'_{1,3}}$ are over the same $\tilde{0}^{1,2}_{e_1} \in E_{1,2}$, hence, we take their sum over $E_{1,2}$.

If $e'_{1,3} = \tilde{0}^{1,3}_{e_1}$, then since $e_{1,3} \underset{E_1}{+} \tilde{0}^{1,3}_{e_1} = e_{1,3}$, and $\hat{0}_{\tilde{0}^{1,3}_{e_1}} = \hat{0}_{e_1}$, we have:

$$\hat{0}_{e_{1,3}} \underset{1,2}{+} \hat{0}_{e_1} = \hat{0}_{e_{1,3} \underset{E_1}{+} \tilde{0}^{1,3}_{e_1}} = \hat{0}_{e_{1,3}}. \tag{2.11}$$

Another way of describing (2.11) is the following: $\hat{0}_{e_1}$ is the double zero of the Back face over $e_1 \in E_1$.

2. Similarly, if $e_{1,3}, e'_{1,3} \in E_{1,3}$ are over the same $e_3 \in E_3$, then:

$$\hat{0}_{e_{1,3} \underset{E_3}{+} e'_{1,3}} = \hat{0}_{e_{1,3}} \underset{2,3}{+} \hat{0}_{e'_{1,3}}.$$

In case $e'_{1,3} = \tilde{0}^{1,3}_{e_3}$, then since $e_{1,3} \underset{E_3}{+} \tilde{0}^{1,3}_{e_3} = e_{1,3}$, and $\hat{0}_{\tilde{0}^{1,3}_{e_3}} = \hat{0}_{e_3}$, we have:

$$\hat{0}_{e_{1,3}} \underset{2,3}{+} \hat{0}_{e_3} = \hat{0}_{e_{1,3} \underset{E_3}{+} \tilde{0}^{1,3}_{e_3}} = \hat{0}_{e_{1,3}}. \tag{2.12}$$

Again, $\hat{0}_{e_3}$ is the double zero of the Up face over $e_3 \in E_3$.

3. In case $e_{1,3} = e'_{1,3}$:

$$\hat{0}_{e_{1,3}} \underset{1,3}{+} \hat{0}_{e_{1,3}} = \hat{0}_{e_{1,3}}, \tag{2.13a}$$

$$\hat{0}_{e_{1,3}} \underset{1,2}{+} \hat{0}_{e_{1,3}} = \hat{0}_{e_{1,3} \underset{E_1}{+} e_{1,3}} = \hat{0}_{2 \underset{E_1}{\cdot} e_{1,3}}, \tag{2.13b}$$

$$\hat{0}_{e_{1,3}} \underset{2,3}{+} \hat{0}_{e_{1,3}} = \hat{0}_{e_{1,3} \underset{E_3}{+} e_{1,3}} = \hat{0}_{2 \underset{E_3}{\cdot} e_{1,3}}. \tag{2.13c}$$

4. In case $e_{1,3} = e'_{1,3} = \tilde{0}^{1,3}_{e_1}$:

$$\hat{0}_{e_1} \underset{1,3}{+} \hat{0}_{e_1} = \hat{0}_{e_1}, \tag{2.14a}$$

$$\hat{0}_{e_1} \underset{1,2}{+} \hat{0}_{e_1} = \hat{0}_{2 \underset{E_1}{\cdot} \tilde{0}^{1,3}_{e_1}} = \hat{0}_{\tilde{0}^{1,3}_{e_1}} = \hat{0}_{e_1}, \tag{2.14b}$$

$$\hat{0}_{e_1} \underset{2,3}{+} \hat{0}_{e_1} = \hat{0}_{2 \underset{E_3}{\cdot} \tilde{0}^{1,3}_{e_1}} = \hat{0}_{\tilde{0}^{1,3}_{2e_1}} = \hat{0}_{2e_1}. \tag{2.14c}$$

5. In case $e_{1,3} = e'_{1,3} = \tilde{0}^{1,3}_{e_3}$:

$$\hat{0}_{e_3} \underset{1,3}{+} \hat{0}_{e_3} = \hat{0}_{e_3}, \tag{2.15a}$$

$$\hat{0}_{e_3} \underset{1,2}{+} \hat{0}_{e_3} = \hat{0}_{2 \underset{E_1}{\cdot} \tilde{0}^{1,3}_{e_3}} = \hat{0}_{\tilde{0}^{1,3}_{2e_3}} = \hat{0}_{2e_3}, \tag{2.15b}$$

$$\hat{0}_{e_3} \underset{2,3}{+} \hat{0}_{e_3} = \hat{0}_{2 \underset{E_3}{\cdot} \tilde{0}^{1,3}_{e_3}} = \hat{0}_{\tilde{0}^{1,3}_{e_3}} = \hat{0}_{e_3}. \tag{2.15c}$$

About (2.15a) and (2.15c): $\hat{0}_{e_3}$ is the double zero of the Up face over $e_3 \in E_3$. But since $\hat{0}_{e_3}$ is not a double zero in the Left or in the Back face, (2.15b) follows.

Similar calculations for $e_{1,2}, e'_{1,2} \in E_{1,2}$, with outlines $(e_{1,2}; e_1, e_2; m)$, and $(e'_{1,2}; e'_1, e'_2; m)$.

1. If $e_1 = e'_1 \in E_1$, then:
$$\hat{0}_{e_{1,2} \underset{E_1}{+} e'_{1,2}} = \hat{0}_{e_{1,2}} \underset{1,3}{+} \hat{0}_{e'_{1,2}}. \tag{2.16}$$
If $e'_{1,2} = \tilde{0}^{1,2}_{e_1}$, then as $e_{1,2} \underset{E_1}{+} \tilde{0}^{1,2}_{e_1} = e_{1,2}$, we have:
$$\hat{0}_{e_{1,2}} \underset{1,3}{+} \hat{0}_{e_1} = \hat{0}_{e_{1,2} \underset{E_1}{+} \tilde{0}^{1,2}_{e_1}} = \hat{0}_{e_{1,2}}. \tag{2.17}$$

2. If $e_2 = e'_2 \in E_2$, then:
$$\hat{0}_{e_{1,2} \underset{E_2}{+} e'_{1,2}} = \hat{0}_{e_{1,2}} \underset{2,3}{+} \hat{0}_{e'_{1,2}}. \tag{2.18}$$
If $e'_{1,2} = \tilde{0}^{1,2}_{e_2}$, then as $e_{1,2} \underset{E_2}{+} \tilde{0}^{1,2}_{e_2} = e_{1,2}$, we have:
$$\hat{0}_{e_{1,2}} \underset{2,3}{+} \hat{0}_{e_2} = \hat{0}_{e_{1,2} \underset{E_2}{+} \tilde{0}^{1,2}_{e_2}} = \hat{0}_{e_{1,2}}. \tag{2.19}$$

3. In case $e_{1,2} = e'_{1,2}$:
$$\hat{0}_{e_{1,2}} \underset{1,3}{+} \hat{0}_{e_{1,2}} = \hat{0}_{e_{1,2} \underset{E_1}{+} e_{1,2}} = \hat{0}_{2 \underset{E_1}{\cdot} e_{1,2}}, \tag{2.20a}$$
$$\hat{0}_{e_{1,2}} \underset{1,2}{+} \hat{0}_{e_{1,2}} = \hat{0}_{e_{1,2}}, \tag{2.20b}$$
$$\hat{0}_{e_{1,2}} \underset{2,3}{+} \hat{0}_{e_{1,2}} = \hat{0}_{e_{1,2} \underset{E_2}{+} e_{1,2}} = \hat{0}_{2 \underset{E_2}{\cdot} e_{1,2}}. \tag{2.20c}$$

4. In case $e_{1,2} = e'_{1,2} = \tilde{0}^{1,2}_{e_1}$, we obtain (2.14).

5. In case $e_{1,2} = e'_{1,2} = \tilde{0}^{1,2}_{e_2}$:
$$\hat{0}_{e_2} \underset{1,3}{+} \hat{0}_{e_2} = \hat{0}_{2e_2}, \tag{2.21a}$$
$$\hat{0}_{e_2} \underset{1,2}{+} \hat{0}_{e_2} = \hat{0}_{e_2}, \tag{2.21b}$$
$$\hat{0}_{e_2} \underset{2,3}{+} \hat{0}_{e_2} = \hat{0}_{e_2}. \tag{2.21c}$$

Finally, for $e_{2,3}, e'_{2,3} \in E_{2,3}$ with outlines $(e_{2,3}; e_2, e_3; m)$ and $(e'_{2,3}; e'_2, e'_3; m)$:

1. If $e_2 = e_2' \in E_2$, then:
$$\hat{0}_{e_{2,3} \underset{E_2}{+} e'_{2,3}} = \hat{0}_{e_{2,3}} \underset{1,2}{+} \hat{0}_{e'_{2,3}}. \tag{2.22}$$
If $e'_{2,3} = \tilde{0}^{2,3}_{e_2}$, then as $e_{2,3} \underset{E_2}{+} \tilde{0}^{2,3}_{e_2} = e_{2,3}$, we have:
$$\hat{0}_{e_{2,3}} \underset{1,2}{+} \hat{0}_{e_2} = \hat{0}_{e_{2,3} \underset{E_2}{+} \tilde{0}^{2,3}_{e_2}} = \hat{0}_{e_{2,3}}. \tag{2.23}$$

2. If $e_3 = e_3' \in E_3$, then:
$$\hat{0}_{e_{2,3} \underset{E_3}{+} e'_{2,3}} = \hat{0}_{e_{2,3}} \underset{1,3}{+} \hat{0}_{e'_{2,3}}. \tag{2.24}$$
If $e'_{2,3} = \tilde{0}^{2,3}_{e_3}$, then since $e_{2,3} \underset{E_3}{+} \tilde{0}^{2,3}_{e_3} = e_{2,3}$, we have:
$$\hat{0}_{e_{2,3}} \underset{1,3}{+} \hat{0}_{e_3} = \hat{0}_{e_{2,3} \underset{E_3}{+} \tilde{0}^{2,3}_{e_3}} = \hat{0}_{e_{2,3}}. \tag{2.25}$$

3. In case $e_{2,3} = e'_{2,3}$:
$$\hat{0}_{e_{2,3}} \underset{1,3}{+} \hat{0}_{e_{2,3}} = \hat{0}_{e_{2,3} \underset{E_3}{+} e_{2,3}} = \hat{0}_{2 \underset{E_3}{\cdot} e_{2,3}}, \tag{2.26a}$$
$$\hat{0}_{e_{2,3}} \underset{1,2}{+} \hat{0}_{e_{2,3}} = \hat{0}_{e_{2,3} \underset{E_2}{+} e_{2,3}} = \hat{0}_{2 \underset{E_2}{\cdot} e_{2,3}}, \tag{2.26b}$$
$$\hat{0}_{e_{2,3}} \underset{2,3}{+} \hat{0}_{e_{2,3}} = \hat{0}_{e_{2,3}}. \tag{2.26c}$$

4. In case $e_{2,3} = e'_{2,3} = \tilde{0}^{2,3}_{e_2}$, we obtain (2.21).

5. In case $e_{2,3} = e'_{2,3} = \tilde{0}^{2,3}_{e_3}$, we obtain (2.15).

2.3 How to subtract elements in E

So far we have seen how to add elements in E. In this section we will investigate the operation of subtraction. We will obtain formulas expressing the difference of two elements of a triple vector bundle in terms of core, ultracore, and zero elements. These formulas are a significant part of the technical work needed for the proof of the warp-grid theorem.

First, we need to describe the cores and the ultracore of a triple vector bundle E.

2.3.1 Core double vector bundles and the ultracore

Since each face of E is a double vector bundle, each face has a core vector bundle. The cores of the lower faces $E_{i,j}$ are denoted E_{ij} with the comma removed. The core of the upper face with base manifold E_k is denoted $E_{ij,k}$. (This convention comes from [14] and from [13]).

Focus on the core vector bundles of the Left and of the Right faces. The Left face projects to the Right face via the double vector bundle morphism which consists of the bundle projections $E \to E_{1,3}$, $E_{1,2} \to E_1$, $E_{2,3} \to E_3$ and $E_2 \to M$. The restriction of $E \to E_{1,3}$ to $E_{13,2}$ goes into E_{13} and inherits the vector bundle structure of $E \to E_{1,3}$. The total space $E_{13,2}$ with the usual vector bundle structure over E_2 (as the core of the Left face of E), and with the vector bundle structure over E_{13}, yields another double vector bundle, which we call the *(L-R) core double vector bundle*.

We denote the core morphism of the projection double vector bundle morphsim $(q_{1,3}; q_1^{1,2}, q_3^{2,3}; q^2)$ from the Left to the Right face by (q_{13}, q^2).

The (L-R) core double vector bundle is the following,

$$\begin{array}{ccc} E_{13,2} & \xrightarrow{q_{13}} & E_{13} \\ {\scriptstyle q_2^{13,2}}\downarrow & & \downarrow{\scriptstyle q^{13}} \\ E_2 & \xrightarrow[q^2]{} & M. \end{array} \tag{2.27}$$

The addition in $E_{13,2} \to E_{13}$ is the usual addition in $E \to E_{1,3}$. If $k_1, k_2 \in E_{13,2}$ are over the same $w_{13} \in E_{13}$, then

$$k_1 \underset{1,3}{+} k_2, \tag{2.28}$$

is their usual sum in $E \to E_{1,3}$. For $k_1, k_2 \in E_{13,2}$ over the same $e_2 \in E_2$, then

$$k_1 \underset{E_2}{+} k_2 = k_1 \underset{1,2/2,3}{+} k_2. \tag{2.29}$$

Here we write $k_1 \underset{1,2/2,3}{+} k_2$ to denote that $k_1 \underset{1,2}{+} k_2 = k_1 \underset{2,3}{+} k_2$.

The algebraic compatibility conditions for the (L-R) core double vector bundle follow easily using the apparatus set up earlier in the chapter. The core double vector bundle satisfies part (ii) of the definition of a double vector bundle as well. For example, take any $(e_2, w_{13}) \in E_2 \times_M E_{13}$. Then this is an element $(\tilde{0}_{e_2}^{1,2}, \tilde{0}_{e_2}^{2,3}, w_{13}) \in W$, and since the triple source map $\widetilde{\natural} : E \to W$ is a surjection, there exists an $e \in E$ such that $\widetilde{\natural}(e) = (\tilde{0}_{e_2}^{1,2}, \tilde{0}_{e_2}^{2,3}, w_{13})$. That $e \in E_{13,2}$ follows immediately from its outline. Hence the double source map $\natural : E_{13,2} \to E_2 \times_M E_{13}$ of the (L-R) core double vector bundle is surjective. That it is a submersion, follows again from $\widetilde{\natural} : E \to W$ being a submersion. Part (iii) of the definition of a double vector bundle for the (L-R) core double vector bundle is explained towards the end of Section 2.4.1.

Of course this can also be done for the other two pairs of parallel faces. So there are three core double vector bundles, shown in (2.30).

$$\begin{array}{ccccccccccc} E_{23,1} & \longrightarrow & E_{23} & & E_{13,2} & \longrightarrow & E_{13} & & E_{12,3} & \longrightarrow & E_{12} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ E_1 & \longrightarrow & M, & & E_2 & \longrightarrow & M, & & E_3 & \longrightarrow & M. \end{array} \tag{2.30}$$

Elements of the core of $E_{12,3}$ project to zeros in the Down face. In the Up face they project to zeros over the zero in E_3. It follows that an element of the core of $E_{12,3}$ projects to zero in every bundle structure. Equally the cores of the (B-F) and (L-R) double vector bundles consist of the elements of $E_{1,2,3}$ which project to zeros in every bundle structure. Thus each double vector bundle in (2.30) has the same core. This is denoted E_{123} (without commas) and called the *ultracore of* E.

From the interchange laws it follows that the three additions on E, namely $\underset{1,2}{+}$, $\underset{1,3}{+}$, and $\underset{2,3}{+}$, coincide on the ultracore and give it the structure of a vector bundle over M.

The triple zero $\odot^3_m$ of E is the zero of the ultracore vector bundle $E_{123} \to M$.

To see the core double vector bundles and ultracore vector bundle in local coordinates, take a local coordinate system on E as described in Section 2.1.3. By setting $(v_{(1)})$, $(v_{(2)})$, $(v_{(13)})$, $(v_{(23)})$ to zero in the equations (2.4), we obtain,

$$\begin{aligned} \tilde{v}^{i_3}_{(3)} &= P_{(3)}{}^{i_3}_{j_3} v^{j_3}_{(3)}, \\ \tilde{v}^{i_{12}}_{(12)} &= P_{(12)}{}^{i_{12}}_{j_{12}} v^{j_{12}}_{(12)}, \\ \tilde{v}^{i_{123}}_{(123)} &= P_{(123)}{}^{i_{123}}_{j_{123}} v^{j_{123}}_{(123)} + P_{(12,3)}{}^{i_{123}}_{j_{12}\ j_3} v^{j_{12}}_{(12)} v^{j_3}_{(3)}, \end{aligned}$$

that is, local coordinates for the (U-D) core double vector bundle, $(E_{12,3}; E_3, E_{12}; M)$. Setting additionally $(v_{(3)})$ and $(v_{(12)})$ to zero, we obtain a single vector bundle,

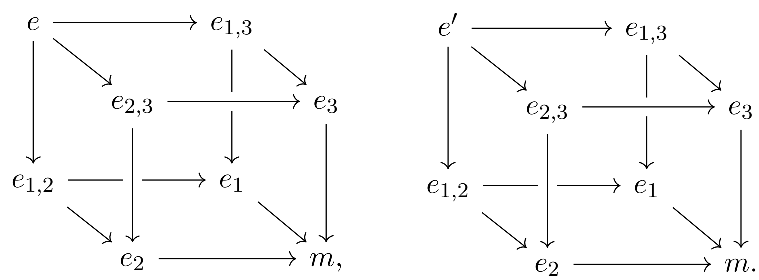
$$\tilde{v}^{i_{123}}_{(123)} = P_{(123)}{}^{i_{123}}_{j_{123}} v^{j_{123}}_{(123)},$$

and this is precisely the ultracore vector bundle $E_{123} \to M$ of E.

2.3.2 First case: two elements that have the same outline

Now we are ready to investigate subtraction. There are three cases to consider; two elements of a triple vector bundle that can be subtracted, may admit exactly one, or two, or all three, of the subtractions $\underset{1,2}{-}$, $\underset{1,3}{-}$, and $\underset{2,3}{-}$.

We begin with the case where e and e' have exactly the same outline



Then all three differences $e \underset{1,2}{-} e'$, $e \underset{1,3}{-} e'$, $e \underset{2,3}{-} e'$ are defined.

Step 1. Focus on the Back faces of e and e'

$$\begin{array}{ccc} e, e' & \longmapsto & e_{1,3} \\ \big\downarrow & & \big\downarrow \\ e_{1,2} & \longmapsto & e_1. \end{array}$$

Then, from double vector bundle theory, we can write

$$e \underset{1,2}{-} e' = k_1 \underset{1,3}{+} \hat{0}_{e_{1,2}}, \qquad e \underset{1,3}{-} e' = k_1 \underset{1,2}{+} \hat{0}_{e_{1,3}},$$

where $k_1 \in E_{23,1}$, the core of the Back face, with outline

$$\begin{array}{ccc} E_{23,1} \ni k_1 & \longmapsto & w_{23} \in E_{23} \\ \big\downarrow & & \big\downarrow \\ e_1 & \longmapsto & m. \end{array}$$

Step 2. Show that $w_{23} = \odot_m^{2,3}$.

Use the morphism $q_{2,3} : E \to E_{2,3}$. We know that $q_{2,3}(e \underset{1,3}{-} e') = \tilde{0}_{e_3}^{2,3}$ and

$$q_{2,3}(k_1 \underset{1,2}{+} \hat{0}_{e_{1,3}}) = q_{2,3}(k_1) \underset{E_2}{+} q_{2,3}(\hat{0}_{e_{1,3}}) = w_{23} \underset{E_2}{+} \tilde{0}_{e_3}^{2,3}.$$

Therefore

$$w_{23} \underset{E_2}{+} \tilde{0}_{e_3}^{2,3} = \tilde{0}_{e_3}^{2,3},$$

and, since $\tilde{0}^{2,3}_{e_3} \underset{E_2}{-} \tilde{0}^{2,3}_{e_3} = \odot^{2,3}_m$, we have that $w_{23} = \tilde{0}^{2,3}_{0^{E_2}_m} = \odot^{2,3}_m$. So k_1 has the outline

$$\begin{array}{ccc} k_1 & \longmapsto & \odot^{2,3}_m \\ \big\downarrow & & \big\downarrow \\ e_1 & \longmapsto & m. \end{array}$$

Step 3. Applying double vector bundle theory again, we get

$$k_1 = u_1 \underset{2,3}{+} \hat{0}_{e_1},$$

where u_1 is an ultracore element.

Step 4. Apply the same procedure to Left and Up faces of e and e'.

Focus on the Left faces of e and e'

$$e \underset{2,3}{-} e' = k_2 \underset{1,2}{+} \hat{0}_{e_{2,3}}, \qquad e \underset{1,2}{-} e' = k_2 \underset{2,3}{+} \hat{0}_{e_{1,2}},$$

where $k_2 \in E_{13,2}$, core of the Left face, with outline

$$\begin{array}{ccc} k_2 & \longmapsto & \odot^{1,3}_m \\ \big\downarrow & & \big\downarrow \\ e_2 & \longmapsto & m. \end{array}$$

So, we can write

$$k_2 = u_2 \underset{1,3}{+} \hat{0}_{e_2},$$

where u_2 is an ultracore element. Similarly for the Up faces, we have

$$e \underset{1,3}{-} e' = k_3 \underset{2,3}{+} \hat{0}_{e_{1,3}}, \qquad e \underset{2,3}{-} e' = k_3 \underset{1,3}{+} \hat{0}_{e_{2,3}},$$

where $k_3 \in E_{12,3}$, core of the Up face, with outline

$$\begin{array}{ccc} k_3 & \longmapsto & \odot^{1,2}_m \\ \big\downarrow & & \big\downarrow \\ e_3 & \longmapsto & m, \end{array}$$

so $k_3 = u_3 \underset{1,2}{+} \hat{0}_{e_3}$ with u_3 an ultracore element.

Step 5. Show that $u_1 = u_2 = u_3$.

We show that $u_1 = u_3$. So far, we have two expressions for $e \underset{1,3}{-} e'$, namely:

$$k_1 \underset{1,2}{+} \hat{0}_{e_{1,3}} = k_3 \underset{2,3}{+} \hat{0}_{e_{1,3}}. \tag{2.31}$$

Expand the left hand side of (2.31), mimicking the double vector bundle case:

$$\begin{aligned}\hat{0}_{e_{1,3}} \underset{1,2}{+} (\hat{0}_{e_1} \underset{2,3}{+} u_1) &= (\hat{0}_{e_{1,3}} \underset{2,3}{+} \hat{0}_{e_3}) \underset{1,2}{+} (\hat{0}_{e_1} \underset{2,3}{+} u_1) \\ &= (\hat{0}_{e_{1,3}} \underset{1,2}{+} \hat{0}_{e_1}) \underset{2,3}{+} (\hat{0}_{e_3} \underset{1,2}{+} u_1) = \hat{0}_{e_{1,3}} \underset{2,3}{+} (\hat{0}_{e_3} \underset{1,2}{+} u_1).\end{aligned}$$

Therefore, we see that (2.31) can be rewritten as:

$$\hat{0}_{e_{1,3}} \underset{2,3}{+} (\hat{0}_{e_3} \underset{1,2}{+} u_1) = \hat{0}_{e_{1,3}} \underset{2,3}{+} (\hat{0}_{e_3} \underset{1,2}{+} u_3),$$

from where it follows that $u_1 = u_3$. Similarly, we can show that $u_2 = u_3$.

At this point write $u_1 = u_2 = u_3$ to be u.

Step 6. We obtain six formulas for the differences between e and e'.

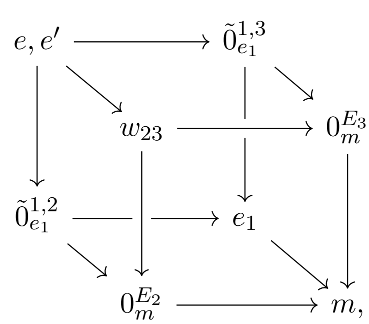
Proposition 2.3.1. *With the above notation, two elements e and e' which have the same outline are related by*

$$\begin{aligned}e \underset{1,3}{-} e' &= \hat{0}_{e_{1,3}} \underset{1,2}{+} (\hat{0}_{e_1} \underset{2,3}{+} u) = \hat{0}_{e_{1,3}} \underset{2,3}{+} (\hat{0}_{e_3} \underset{1,2}{+} u), \\ e \underset{1,2}{-} e' &= \hat{0}_{e_{1,2}} \underset{1,3}{+} (\hat{0}_{e_1} \underset{2,3}{+} u) = \hat{0}_{e_{1,2}} \underset{2,3}{+} (\hat{0}_{e_2} \underset{1,3}{+} u), \\ e \underset{2,3}{-} e' &= \hat{0}_{e_{2,3}} \underset{1,3}{+} (\hat{0}_{e_3} \underset{1,2}{+} u) = \hat{0}_{e_{2,3}} \underset{1,2}{+} (\hat{0}_{e_2} \underset{1,3}{+} u).\end{aligned} \tag{2.32}$$

What is important here is that the subtraction with respect to each structure results in the same ultracore element u.

Special case: when e, e' are in a core double vector bundle

If e, e' are in one of the core double vector bundles the preceding equations simplify. For example if $e, e' \in E_{23,1}$, with outline



then from (2.32) we have

$$e \underset{2,3}{-} e' = \hat{0}_{w_{23}} \underset{1,3}{+} (\hat{0}_{0^{E_3}_m} \underset{1,2}{+} u) = \hat{0}_{w_{23}} \underset{1,3}{+} (\odot^3_m \underset{1,2}{+} u) = \hat{0}_{w_{23}} \underset{1,3}{+} u$$

and

$$e \underset{2,3}{-} e' = \hat{0}_{w_{23}} \underset{1,2}{+} (\hat{0}_{0^{E_2}_m} \underset{1,3}{+} u) = \hat{0}_{w_{23}} \underset{1,2}{+} (\odot^3_m \underset{1,3}{+} u) = \hat{0}_{w_{23}} \underset{1,2}{+} u,$$

and therefore

$$\hat{0}_{w_{23}} \underset{1,3}{+} u = \hat{0}_{w_{23}} \underset{1,2}{+} u. \tag{2.33}$$

Also, the following will be needed in Subsection 3.2.2. Again, using (2.32) we see that

$$\begin{aligned} e \underset{1,3}{-} e' &= \hat{0}_{\tilde{0}^{1,3}_{e_1}} \underset{1,2}{+} (\hat{0}_{e_1} \underset{2,3}{+} u) = \hat{0}_{e_1} \underset{1,2}{+} (\hat{0}_{e_1} \underset{2,3}{+} u) \\ &= (\hat{0}_{e_1} \underset{2,3}{+} \odot^3_m) \underset{1,2}{+} (\hat{0}_{e_1} \underset{2,3}{+} u) = (\hat{0}_{e_1} \underset{1,2}{+} \hat{0}_{e_1}) \underset{2,3}{+} (\odot^3_m \underset{1,2}{+} u) \overset{(2.14b)}{=} \hat{0}_{e_1} \underset{2,3}{+} u, \end{aligned}$$

or, equivalently,

$$e \underset{1,3}{-} e' = \hat{0}_{\tilde{0}^{1,3}_{e_1}} \underset{2,3}{+} (\hat{0}_{0^{E_3}_m} \underset{1,2}{+} u) = \hat{0}_{e_1} \underset{2,3}{+} (\odot^3_m \underset{1,2}{+} u) = \hat{0}_{e_1} \underset{2,3}{+} u.$$

For the last difference, by (2.32):

$$\begin{aligned} e \underset{1,2}{-} e' &= \hat{0}_{\tilde{0}^{1,2}_{e_1}} \underset{1,3}{+} (\hat{0}_{e_1} \underset{2,3}{+} u) = \hat{0}_{e_1} \underset{1,3}{+} (\hat{0}_{e_1} \underset{2,3}{+} u) \\ &= (\hat{0}_{e_1} \underset{2,3}{+} \odot^3_m) \underset{1,3}{+} (\hat{0}_{e_1} \underset{2,3}{+} u) = (\hat{0}_{e_1} \underset{1,3}{+} \hat{0}_{e_1}) \underset{2,3}{+} (\odot^3_m \underset{1,3}{+} u) \overset{(2.14a)}{=} \hat{0}_{e_1} \underset{2,3}{+} u, \end{aligned}$$

and finally,

$$e \underset{1,2}{-} e' = \hat{0}_{\tilde{0}^{1,2}_{e_1}} \underset{2,3}{+} (\hat{0}_{0^{E_2}_m} \underset{1,3}{+} u) = \hat{0}_{e_1} \underset{2,3}{+} (\odot^3_m \underset{1,3}{+} u) = \hat{0}_{e_1} \underset{2,3}{+} u.$$

2.3.3 Second case: two elements that have two lower faces in common

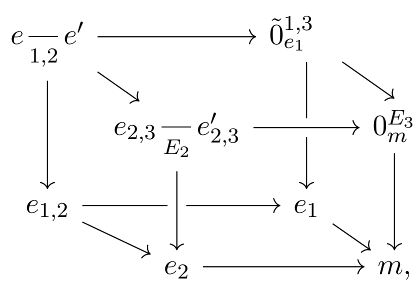
What happens if e and e' have only two of the lower faces in common? Then only two of the three subtractions are defined. There are three cases to consider, each of which arises later.

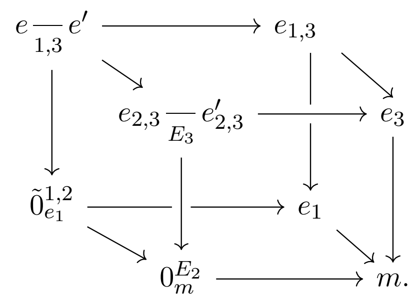
If e and e' have the same Right and Down face

Since e and e' project to the same $e_{1,2}$ and $e_{1,3}$, it follows that they project to the same e_1, e_2 and e_3. However e and e' will differ at $e_{2,3}$ and $e'_{2,3}$, and these will differ by a core element $w_{23} \in E_{23}$ of the core of the Front face, that is

$$e_{2,3} \underset{E_2}{-} e'_{2,3} = w_{23} \underset{E_3}{+} \tilde{0}^{2,3}_{e_2}, \qquad e_{2,3} \underset{E_3}{-} e'_{2,3} = w_{23} \underset{E_2}{+} \tilde{0}^{2,3}_{e_3}. \tag{2.34}$$

It is useful to write out the outlines of these differences





Since e and e' have the same Back face, again by applying double vector bundle theory, we can write

$$e \underset{1,2}{-} e' = k \underset{1,3}{+} \hat{0}_{e_{1,2}}, \qquad e \underset{1,3}{-} e' = k \underset{1,2}{+} \hat{0}_{e_{1,3}}, \tag{2.35}$$

where $k \in E_{23,1}$, the core of the Back face.

Also, using the morphism $q_{2,3} : E \to E_{2,3}$, we show that $q_{2,3}(k) = w_{23}$. First,

$$q_{2,3}(e \underset{1,2}{-} e') = e_{2,3} \underset{E_2}{-} e'_{2,3} = w_{23} \underset{E_3}{+} \tilde{0}^{2,3}_{e_2},$$

and

$$q_{2,3}(k \underset{1,3}{+} \hat{0}_{e_{1,2}}) = q_{2,3}(k) \underset{E_3}{+} \tilde{0}^{2,3}_{e_2},$$

hence $q_{2,3}(k) = w_{23}$. Therefore, k has outline

$$\begin{array}{ccc} E_{23,1} \ni k & \longmapsto & w_{23} \in E_{23} \\ \downarrow & & \downarrow \\ e_1 & \longmapsto & m. \end{array}$$

Example 2.3.2. Special case: $\hat{0}_{e_{2,3}}$ and $\hat{0}_{e'_{2,3}}$.

Recall (2.24) and (2.22),

$$\hat{0}_{e_{2,3}} \underset{1,3}{+} \hat{0}_{e'_{2,3}} = \hat{0}_{e_{2,3} \underset{E_3}{+} e'_{2,3}}, \qquad \hat{0}_{e_{2,3}} \underset{1,2}{+} \hat{0}_{e'_{2,3}} = \hat{0}_{e_{2,3} \underset{E_2}{+} e'_{2,3}},$$

and $(-1) \underset{1,3}{\cdot} \hat{0}_{e_{2,3}} = \hat{0}_{f_{2,3}}$ where $f_{2,3} = \underset{E_3}{-} e_{2,3}$.

Suppose we have two elements $e_{2,3}$ and $e'_{2,3}$ of $E_{2,3}$ that differ by a core element $w_{23} \in E_{23}$, as in (2.34). The differences we are interested in are

$$\hat{0}_{e_{2,3}} \underset{1,2}{-} \hat{0}_{e'_{2,3}} = \hat{0}_{e_{2,3}} \underset{1,2}{+} \hat{0}_{\underset{E_2}{-} e'_{2,3}} = \hat{0}_{e_{2,3} \underset{E_2}{-} e'_{2,3}} = \hat{0}_{w_{23} \underset{E_3}{+} \tilde{0}^{2,3}_{e_2}} = \hat{0}_{w_{23}} \underset{1,3}{+} \hat{0}_{e_2}, \tag{2.36}$$

and

$$\hat{0}_{e_{2,3}} \underset{1,3}{-} \hat{0}_{e'_{2,3}} = \hat{0}_{e_{2,3}} \underset{1,3}{+} \hat{0}_{\underset{E_3}{-} e'_{2,3}} = \hat{0}_{e_{2,3} \underset{E_3}{-} e'_{2,3}} = \hat{0}_{w_{23} \underset{E_2}{+} \tilde{0}^{2,3}_{e_3}} = \hat{0}_{w_{23}} \underset{1,2}{+} \hat{0}_{e_3}. \tag{2.37}$$

If e and e' have the same Front and Down face

In this case, the elements $e_{1,3}$ and $e'_{1,3}$ differ by a core element w_{13} of E_{13}

$$e_{1,3} \underset{E_1}{-} e'_{1,3} = w_{13} \underset{E_3}{+} \tilde{0}^{1,3}_{e_1}, \qquad e_{1,3} \underset{E_3}{-} e'_{1,3} = w_{13} \underset{E_1}{+} \tilde{0}^{1,3}_{e_3}. \tag{2.38}$$

As before, we can write

$$e \underset{1,2}{-} e' = k \underset{2,3}{+} \hat{0}_{e_{1,2}}, \qquad e \underset{2,3}{-} e' = k \underset{1,2}{+} \hat{0}_{e_{2,3}}, \tag{2.39}$$

with k an element of the core of the Left face with outline

$$\begin{array}{ccc} E_{13,2} \ni k & \longmapsto & w_{13} \in E_{13} \\ \downarrow & & \downarrow \\ e_2 & \longmapsto & m. \end{array}$$

Example 2.3.3. Special case: $\hat{0}_{e_{1,3}}$ and $\hat{0}_{e'_{1,3}}$.

Suppose two elements $e_{1,3}$ and $e'_{1,3}$ of $E_{1,3}$ differ by a core element $w_{13} \in E_{13}$, as in (2.38). The differences we will need are the following:

$$\hat{0}_{e_{1,3}} \underset{1,2}{-} \hat{0}_{e'_{1,3}} = \hat{0}_{e_{1,3}} \underset{1,2}{+} \hat{0}_{\underset{E_1}{-} e'_{1,3}} = \hat{0}_{e_{1,3} \underset{E_1}{-} e'_{1,3}} = \hat{0}_{w_{13} \underset{E_3}{+} \tilde{0}^{1,3}_{e_1}} = \hat{0}_{w_{13}} \underset{2,3}{+} \hat{0}_{e_1}, \tag{2.40}$$

and

$$\hat{0}_{e_{1,3}} \underset{2,3}{-} \hat{0}_{e'_{1,3}} = \hat{0}_{e_{1,3}} \underset{2,3}{+} \hat{0}_{\underset{E_3}{-} e'_{1,3}} = \hat{0}_{e_{1,3} \underset{E_3}{-} e'_{1,3}} = \hat{0}_{w_{13} \underset{E_1}{+} \tilde{0}^{1,3}_{e_3}} = \hat{0}_{w_{13}} \underset{1,2}{+} \hat{0}_{e_3}. \tag{2.41}$$

If e and e' have the same Front and Right face

In this case, $e_{1,2}$ and $e'_{1,2}$ will differ by an element $w_{12} \in E_{12}$ of the core of the Down face

$$e_{1,2} \underset{E_1}{-} e'_{1,2} = w_{12} \underset{E_2}{+} \tilde{0}^{1,2}_{e_1}, \qquad e_{1,2} \underset{E_2}{-} e'_{1,2} = w_{12} \underset{E_1}{+} \tilde{0}^{1,2}_{e_2}, \tag{2.42}$$

and as before

$$e \underset{1,3}{-} e' = k \underset{2,3}{+} \hat{0}_{e_{1,3}}, \qquad e \underset{2,3}{-} e' = k \underset{1,3}{+} \hat{0}_{e_{2,3}}, \tag{2.43}$$

where k is an element of the core of the Up face with outline

$$\begin{array}{ccc} E_{12,3} \ni k & \longmapsto & w_{12} \in E_{12} \\ \downarrow & & \downarrow \\ e_3 & \longmapsto & m. \end{array}$$

Example 2.3.4. Special case: $\hat{0}_{e_{1,2}}$ and $\hat{0}_{e'_{1,2}}$.

Suppose two elements $e_{1,2}$ and $e'_{1,2}$ of $E_{1,2}$ differ by a core element $w_{12} \in E_{12}$, as in (2.42). The differences we need to work out are

$$\hat{0}_{e_{1,2}} \underset{1,3}{-} \hat{0}_{e'_{1,2}} = \hat{0}_{e_{1,2}} \underset{1,3}{+} \hat{0}_{\underset{E_1}{-} e'_{1,2}} = \hat{0}_{e_{1,2} \underset{E_1}{-} e'_{1,2}} = \hat{0}_{w_{12} \underset{E_2}{+} \tilde{0}^{1,2}_{e_1}} = \hat{0}_{w_{12}} \underset{2,3}{+} \hat{0}_{e_1}, \tag{2.44}$$

and

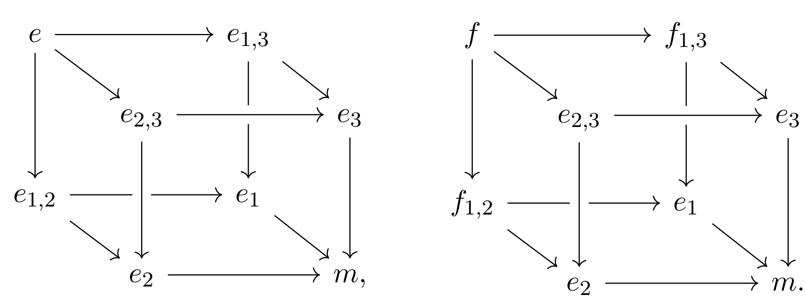
$$\hat{0}_{e_{1,2}} \underset{2,3}{-} \hat{0}_{e'_{1,2}} = \hat{0}_{e_{1,2}} \underset{2,3}{+} \hat{0}_{\underset{E_2}{-} e'_{1,2}} = \hat{0}_{e_{1,2} \underset{E_2}{-} e'_{1,2}} = \hat{0}_{w_{12} \underset{E_1}{+} \tilde{0}^{1,2}_{e_2}} = \hat{0}_{w_{12}} \underset{1,3}{+} \hat{0}_{e_2}. \tag{2.45}$$

2.3.4 Third case: two elements that have one lower face in common

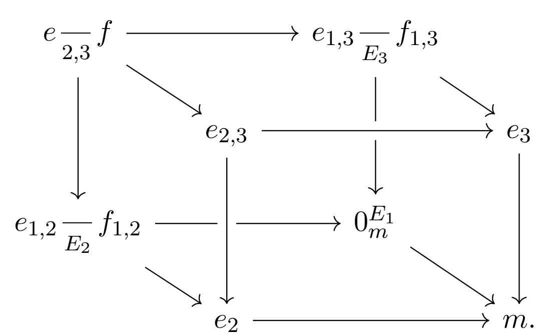
This case is directly relevant to Step 2 of Section 3.2.3.

So far we have seen that two elements of E with the same outlines differ by a unique ultracore element, and that two elements with two of the lower faces in common differ by a unique element λ which lies in the relevant core double vector bundle.

What happens in the case where e, f have only one lower face in common, for example, if they have only the Front face in common? Then only the difference $e \underset{2,3}{-} f$ is defined. The elements e and f project to the same $e_2 \in E_2$, and $e_3 \in E_3$ as they have the same Front face. In the case we are interested in Step 2 of Section 3.2.3, both e and f project to the same $e_1 \in E_1$. Then $q_{1,2}(e) = e_{1,2}$ and $q_{1,2}(f) = f_{1,2}$ will have the same outlines and hence will differ by a unique core element $w_{12} \in E_{12}$. Likewise for $q_{1,3}(e)$ and $q_{1,3}(f)$. The outlines of e and f:



Their difference:



(2.46)

Since $e_{1,3}$ and $f_{1,3}$ differ by a core element $w_{13} \in E_{13}$:

$$e_{1,3} \underset{E_1}{-} f_{1,3} = w_{13} \underset{E_3}{+} \tilde{0}^{1,3}_{e_1}, \quad e_{1,3} \underset{E_3}{-} f_{1,3} = w_{13} \underset{E_1}{+} \tilde{0}^{1,3}_{e_3},$$

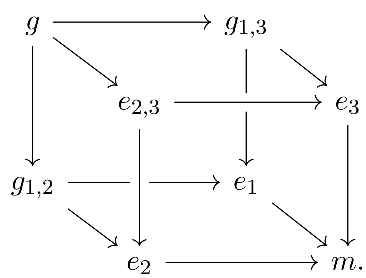
and $e_{1,2}$ and $f_{1,2}$ differ by a core element $w_{12} \in E_{12}$:

$$e_{1,2} \underset{E_1}{-} f_{1,2} = w_{12} \underset{E_2}{+} \tilde{0}^{1,2}_{e_1}, \quad e_{1,2} \underset{E_2}{-} f_{1,2} = w_{12} \underset{E_1}{+} \tilde{0}^{1,2}_{e_2}.$$

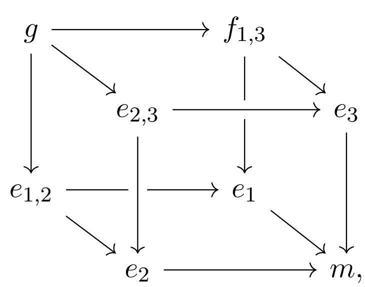
Working in the ordinary vector bundle $E \to E_{2,3}$, we have:

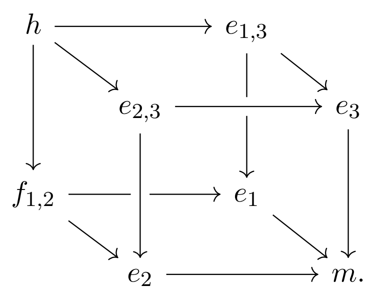
$$e \underset{2,3}{-} f = (e \underset{2,3}{-} g) \underset{2,3}{+} (g \underset{2,3}{-} f), \tag{2.47}$$

for any $g \in E$ with outline:



However, as we want to make the calculation (2.46) easier, it makes sense to either take $g \in E$ with $q_{1,3}(g) = f_{1,3}$ and $q_{1,2}(g) = e_{1,2}$, or to choose an $h \in E$ with $q_{1,3}(h) = e_{1,3}$ and $q_{1,2}(h) = f_{1,2}$. In total, the outlines of g and h will be:





We see that g has two lower faces in common with e and two lower faces in common with f. The same is true of h.

Let's start with g. Then from (2.39) and from (2.43), we can rewrite (2.47) as:

$$e \underset{2,3}{-} f = (e \underset{2,3}{-} g) \underset{2,3}{+} (g \underset{2,3}{-} f) = (k_1 \underset{1,2}{+} \hat{0}_{e_{2,3}}) \underset{2,3}{+} (k_2 \underset{1,3}{+} \hat{0}_{e_{2,3}}), \tag{2.48}$$

where $k_1 \in E_{13,2}$, $k_2 \in E_{12,3}$ with outlines:

$$\begin{array}{ccc} E_{13,2} \ni k_1 & \longmapsto & w_{13} \in E_{13} \\ \downarrow & & \downarrow \\ e_2 & \longmapsto & m, \end{array} \qquad \begin{array}{ccc} E_{12,3} \ni k_2 & \longmapsto & w_{12} \in E_{12} \\ \downarrow & & \downarrow \\ e_3 & \longmapsto & m. \end{array}$$

Using h in (2.47), and again from (2.39) and (2.43):

$$e \underset{2,3}{-} f = (e \underset{2,3}{-} h) \underset{2,3}{+} (h \underset{2,3}{-} f) = (\lambda_1 \underset{1,3}{+} \hat{0}_{e_{2,3}}) \underset{2,3}{+} (\lambda_2 \underset{1,2}{+} \hat{0}_{e_{2,3}}), \tag{2.49}$$

where now $\lambda_1 \in E_{12,3}$ and $\lambda_2 \in E_{13,2}$ with outlines:

$$\begin{array}{ccc} E_{12,3} \ni \lambda_1 & \longmapsto & w_{12} \in E_{12} \\ \Big\downarrow & & \Big\downarrow \\ e_3 & \longmapsto & m, \end{array} \qquad \begin{array}{ccc} E_{13,2} \ni \lambda_2 & \longmapsto & w_{13} \in E_{13} \\ \Big\downarrow & & \Big\downarrow \\ e_2 & \longmapsto & m. \end{array}$$

Both k_1 and λ_2 project to the same $w_{13} \in E_{13}$. This follows from (2.39), since $q_{1,3}(e \underset{2,3}{-} g) = q_{1,3}(h \underset{2,3}{-} f)$. Similarly for k_2 and λ_1.

Of course since (2.47) and (2.49) are equal, there will be a relation between the k_i's and the λ_i's, $i = 1, 2$.

We investigate this relation further towards the end of Chapter 3.

2.3.5 Useful operations with zeros – Part 2

We include equations for the various zero elements, as they show up again and again. Note that these equations follow directly from the algebraic compatibility conditions of the triple vector bundle; at this point, we do not use the methods developed in the previous sections of this chapter.

1. Since $(-1) \underset{1,3}{\cdot} \hat{0}_{e_1} = \hat{0}_{e_1}$, we have

$$\hat{0}_{e_1} \underset{1,3}{-} \hat{0}_{e_1} = \hat{0}_{e_1} \underset{1,3}{+} (-1) \underset{1,3}{\cdot} \hat{0}_{e_1} = \hat{0}_{e_1} \underset{1,3}{+} \hat{0}_{e_1} \overset{(2.14a)}{=} \hat{0}_{e_1}.$$

In total,

$$\hat{0}_{e_1} \underset{1,3}{-} \hat{0}_{e_1} = \hat{0}_{e_1}, \tag{2.50a}$$

$$\hat{0}_{e_1} \underset{1,2}{-} \hat{0}_{e_1} = \hat{0}_{e_1}, \tag{2.50b}$$

$$\hat{0}_{e_1} \underset{2,3}{-} \hat{0}_{e_1} = \odot^3_m. \tag{2.50c}$$

Similarly,

$$\hat{0}_{e_2} \underset{1,3}{-} \hat{0}_{e_2} = \odot^3_m, \tag{2.51a}$$

$$\hat{0}_{e_2} \underset{1,2}{-} \hat{0}_{e_2} = \hat{0}_{e_2}, \tag{2.51b}$$

$$\hat{0}_{e_2} \underset{2,3}{-} \hat{0}_{e_2} = \hat{0}_{e_2}. \tag{2.51c}$$

And finally,

$$\hat{0}_{e_3} \underset{1,3}{-} \hat{0}_{e_3} = \hat{0}_{e_3}, \tag{2.52a}$$

$$\hat{0}_{e_3} \underset{1,2}{-} \hat{0}_{e_3} = \odot^3_m, \tag{2.52b}$$

$$\hat{0}_{e_3} \underset{2,3}{-} \hat{0}_{e_3} = \hat{0}_{e_3}. \tag{2.52c}$$

2. Since $(-1) \underset{1,3}{\cdot} \hat{0}_{e_{1,2}} = \hat{0}_{(-1) \underset{E_1}{\cdot} e_{1,2}}$, we have

$$\hat{0}_{e_{1,2}} \underset{1,3}{-} \hat{0}_{e_{1,2}} = \hat{0}_{e_{1,2}} \underset{1,3}{+} \hat{0}_{(-1) \underset{E_1}{\cdot} e_{1,2}} \overset{(2.16)}{=} \hat{0}_{e_{1,2} \underset{E_1}{+} (-1) \underset{E_1}{\cdot} e_{1,2}} = \hat{0}_{e_1}.$$

Altogether,

$$\hat{0}_{e_{1,2}} \underset{1,3}{-} \hat{0}_{e_{1,2}} = \hat{0}_{e_1}, \tag{2.53a}$$

$$\hat{0}_{e_{1,2}} \underset{1,2}{-} \hat{0}_{e_{1,2}} = \hat{0}_{e_{1,2}}, \tag{2.53b}$$

$$\hat{0}_{e_{1,2}} \underset{2,3}{-} \hat{0}_{e_{1,2}} = \hat{0}_{e_2}. \tag{2.53c}$$

About $\hat{0}_{e_{1,3}}$,

$$\hat{0}_{e_{1,3}} \underset{1,3}{-} \hat{0}_{e_{1,3}} = \hat{0}_{e_{1,3}}, \tag{2.54a}$$

$$\hat{0}_{e_{1,3}} \underset{1,2}{-} \hat{0}_{e_{1,3}} = \hat{0}_{e_1}, \tag{2.54b}$$

$$\hat{0}_{e_{1,3}} \underset{2,3}{-} \hat{0}_{e_{1,3}} = \hat{0}_{e_3}. \tag{2.54c}$$

Finally, about $\hat{0}_{e_{2,3}}$,

$$\hat{0}_{e_{2,3}} \underset{1,3}{-} \hat{0}_{e_{2,3}} = \hat{0}_{e_3}, \tag{2.55a}$$

$$\hat{0}_{e_{2,3}} \underset{1,2}{-} \hat{0}_{e_{2,3}} = \hat{0}_{e_2}, \tag{2.55b}$$

$$\hat{0}_{e_{2,3}} \underset{2,3}{-} \hat{0}_{e_{2,3}} = \hat{0}_{e_{2,3}}. \tag{2.55c}$$

3. The following is also used extensively throughout calculations:

$$\hat{0}_{e_{1,3}} \underset{2,3}{-} \hat{0}_{e_3} = \hat{0}_{e_{1,3}}.$$

This follows because $(-1) \underset{2,3}{\cdot} \hat{0}_{e_3} = \hat{0}_{e_3}$:

$$\hat{0}_{e_{1,3}} \underset{2,3}{-} \hat{0}_{e_3} = \hat{0}_{e_{1,3}} \underset{2,3}{+} (-1) \underset{2,3}{\cdot} \hat{0}_{e_3} = \hat{0}_{e_{1,3}} \underset{2,3}{+} \hat{0}_{e_3} \overset{(2.12)}{=} \hat{0}_{e_{1,3}}.$$

Similarly,

$$\hat{0}_{e_{1,3}} \underset{1,2}{-} \hat{0}_{e_1} = \hat{0}_{e_{1,3}},$$

and,

$$\hat{0}_{e_{1,2}} \underset{1,3}{-} \hat{0}_{e_1} = \hat{0}_{e_{1,2}}, \quad \hat{0}_{e_{1,2}} \underset{2,3}{-} \hat{0}_{e_2} = \hat{0}_{e_{1,2}}.$$

And finally,

$$\hat{0}_{e_{2,3}} \underset{1,2}{-} \hat{0}_{e_2} = \hat{0}_{e_{2,3}}, \quad \hat{0}_{e_{2,3}} \underset{1,3}{-} \hat{0}_{e_3} = \hat{0}_{e_{2,3}}.$$

2.4 Examples of triple vector bundles

We now present the fundamental examples of triple vector bundles.

2.4.1 Decomposed triple vector bundles

First let us work on part (iii) of the definition of a triple vector bundle[2], the existence of the corresponding sigma maps. We will use these maps to establish the existence of nontrivial grids on E.

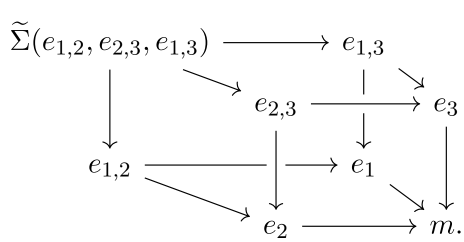
In the triple vector bundle setting there are three steps to defining sigma and omega maps.

Step 1. In the first step, we are decomposing E into $W \times_M E_{123}$.

Recall part (ii) of the definition of a triple vector bundle, Definition 2.1.3.

Definition 2.4.1. (Part (iii)) Given a triple vector bundle E, a *sigma triple vector bundle map* is a triple vector bundle map $\widetilde{\Sigma} : W \to E$ that is right inverse to $\widetilde{\natural} : E \to W$.

If $(e_{1,2}, e_{2,3}, e_{1,3}) \in W$, the outline of $\widetilde{\Sigma}(e_{1,2}, e_{2,3}, e_{1,3})$,



For any $e \in E$ with $q_{1,2}(e) = e_{1,2}$, $q_{2,3}(e) = e_{2,3}$ and $q_{1,3}(e) = e_{1,3}$, from Proposition 2.3.1 we can write

$$e \underset{1,2}{-} \widetilde{\Sigma}(e_{1,2}, e_{2,3}, e_{1,3}) = \hat{0}_{e_{1,2}} \underset{1,3}{+} (\hat{0}_{e_1} \underset{2,3}{+} u),$$

[2] As with double vector bundles, this part of the definition of a triple vector bundle may follow from part (i) of the definition.

for a unique $u \in E_{123}$. Using $\widetilde{\Sigma} : W \to E$, define

$$\begin{aligned} \widetilde{\Omega} : E &\to W \underset{M}{\times} E_{123}, \\ e &\to \left(e_{1,2}, e_{2,3}, e_{1,3}, \left((e \underset{1,2}{-} \widetilde{\Sigma}(e_{1,2}, e_{1,3}, e_{2,3})) \underset{1,3}{-} \hat{0}_{e_{1,2}} \right) \underset{2,3}{-} \hat{0}_{e_1} \right). \end{aligned} \tag{2.56}$$

The inverse of $\widetilde{\Omega}$:

$$\begin{aligned} \widetilde{\mho} : W \underset{M}{\times} E_{123} &\to E \\ (e_{1,2}, e_{2,3}, e_{1,3}, u) &\to \widetilde{\Sigma}(e_{1,2}, e_{2,3}, e_{1,3}) \underset{1,2}{+} \left(\hat{0}_{e_{1,2}} \underset{1,3}{+} (\hat{0}_{e_1} \underset{2,3}{+} u) \right). \end{aligned} \tag{2.57}$$

Conversely, given $\widetilde{\Omega}$, define a unique (that is, so that the $\widetilde{\Omega}$ corresponding to $\widetilde{\Sigma}$ is the given one),

$$\begin{aligned} \widetilde{\Sigma} : W &\to E \\ (e_{1,2}, e_{2,3}, e_{1,3}) &\to \widetilde{\mho}(e_{1,2}, e_{2,3}, e_{1,3}, \odot^3_m). \end{aligned}$$

So we see that there is a bijective correspondence between $\widetilde{\Sigma}$ and $\widetilde{\Omega}$, as triple vector bundle maps.

Step 2. In the second step, we are decomposing W. Denote by

$$\overline{E} := E_1 \times_M E_2 \times_M E_3 \times_M E_{12} \times_M E_{23} \times_M E_{13} \times_M E_{123},$$

the pullback manifold, the *decomposed triple vector bundle*. The various vector bundle structures are pullbacks, as with decomposed double vector bundles. Denote by

$$\overline{E}' := E_1 \times_M E_2 \times_M E_3 \times_M E_{12} \times_M E_{23} \times_M E_{13}.$$

This is a triple vector bundle with zero ultracore.

As starting with a triple vector bundle E we do not assume decompositions of $E_{1,2}$, $E_{2,3}$, and $E_{1,3}$, we need to choose decompositions of the three lower faces:

$$\begin{aligned} \Omega_{1,2} : E_{1,2} &\to E_1 \underset{M}{\times} E_2 \underset{M}{\times} E_{12}, \\ \Omega_{2,3} : E_{2,3} &\to E_2 \underset{M}{\times} E_3 \underset{M}{\times} E_{23}, \\ \Omega_{1,3} : E_{1,3} &\to E_1 \underset{M}{\times} E_3 \underset{M}{\times} E_{13}. \end{aligned}$$

Using these maps we can define the following Ω_W triple vector bundle map from W to the $\overline{E}'$,

$$\begin{aligned} \Omega_W : W &\to E_1 \underset{M}{\times} E_2 \underset{M}{\times} E_3 \underset{M}{\times} E_{12} \underset{M}{\times} E_{23} \underset{M}{\times} E_{13}, \\ (e_{1,2}, e_{2,3}, e_{1,3}) &\mapsto (e_1, e_2, e_3, w_{12}, w_{23}, w_{13}), \end{aligned} \tag{2.58}$$

where

$$\begin{aligned} w_{12} &= (e_{1,2} \underset{E_1}{-} \Sigma_{1,2}(e_1,e_2)) \underset{E_2}{-} \tilde{0}^{1,2}_{e_1}, \\ w_{23} &= (e_{2,3} \underset{E_2}{-} \Sigma_{2,3}(e_2,e_3)) \underset{E_3}{-} \tilde{0}^{2,3}_{e_2}, \\ w_{13} &= (e_{1,3} \underset{E_3}{-} \Sigma_{1,3}(e_1,e_3)) \underset{E_1}{-} \tilde{0}^{1,3}_{e_3}, \end{aligned}$$

$\Sigma_{1,2}$ being the sigma map corresponding to $\Omega_{1,2}$, and same for $\Sigma_{2,3}, \Sigma_{1,3}$.

The inverse of this Ω_W is $\mho_W$:

$$\mho_W : E_1 \underset{M}{\times} E_2 \underset{M}{\times} E_3 \underset{M}{\times} E_{12} \underset{M}{\times} E_{23} \underset{M}{\times} E_{13} \to W.$$

Step 3. In the final step, we decompose E. To define a map from E to the decomposed triple vector bundle $\overline{E}$, take the composition of the following,

$$E \xrightarrow{\widetilde{\Omega}} W \underset{M}{\times} E_{123} \xrightarrow{\Omega_W \underset{M}{\times} \mathrm{id}} \overline{E}. \tag{2.59}$$

Denote the composition $\Omega := (\Omega_W \underset{M}{\times} \mathrm{id}) \circ \widetilde{\Omega}$. This is a *decomposition map* of E.

Denote the inverse of Ω by $\mho : \overline{E} \to E$:

$$\overline{E} \xrightarrow{\mho_W \underset{M}{\times} \mathrm{id}} W \underset{M}{\times} E_{123} \xrightarrow{\widetilde{\mho}} E.$$

The Σ that corresponds to this Ω, from $\overline{E}'$ to E,

$$(e_1, e_2, e_3, w_{12}, w_{23}, w_{13}) \to \mho(e_1, e_2, e_3, w_{12}, w_{23}, w_{13}, \odot^3_m). \tag{2.60}$$

Remark 2.4.2. The following shows why choosing decompositions of the lower faces is necessary.

Taking an $e \in E$ with the same outline as $\Sigma(e_{1,2}, e_{2,3}, e_{1,3})$ only defines a $u \in E_{123}$, and no $w'_{ij}s \in E_{ij}$, elements in the cores of the lower faces.

Question: What if we compare $\Sigma(e_{1,2}, e_{2,3}, e_{1,3})$ with an $f \in E$ that has the same $e_{1,2}$ and $e_{1,3}$, but different $e_{2,3}$?

Then $e_{2,3} \underset{E_2}{-} e'_{2,3} = w_{23} \underset{E_3}{+} \tilde{0}^{2,3}_{e_2}$, where $w_{23} \in E_{23}$, and

$$f \underset{1,2}{-} \Sigma(e_{1,2}, e_{2,3}, e_{1,3}) = k \underset{1,3}{+} \hat{0}_{e_{1,2}},$$

where $k \in E_{23,1}$ has outline $(k; e_1, w_{23}; m)$. The problem is that both $k \in E_{23,1}$ and $w_{23} \in E_{23}$ depend on f. How do we choose f? We would need a map to choose it in a "canonical" way.

Therefore, we cannot define a w_{23} from $\widetilde{\Sigma}$. So we see that we need decompositions of the lower faces. △

To sum up, we have the following maps associated with a triple vector bundle E:

Step 1.

- $\widetilde{\natural} : E \to W$, the triple source map,
- $\widetilde{\Sigma} : W \to E$, a right-inverse to $\widetilde{\natural}$,
- $\widetilde{\Omega} : E \to W \times_M E_{123}$, defined by (2.56),
- $\widetilde{\mho} : W \times_M E_{123} \to E$, the inverse of $\widetilde{\Omega}$, defined by (2.57),

Step 2.

- $\Omega_W : W \to \overline{E}'$, defined by (2.58),
- $\mho_W : \overline{E}' \to W$, inverse of Ω_W,

Step 3.

- $\Omega : E \to \overline{E}$, defined by (2.59), $\Omega := (\Omega_W \underset{M}{\times} \mathrm{id}) \circ \widetilde{\Omega}$, a *decomposition* of E.
- $\mho : \overline{E} \to E$, inverse to Ω, and finally,
- $\Sigma : \overline{E}' \to E$, defined by (2.60).

A choice of decomposition map of E, namely $\Omega : E \to \overline{E}$, determines decompositions of upper faces. To see this, starting with an $\Omega : E \to \overline{E}$, rearrange $\overline{E}$ to

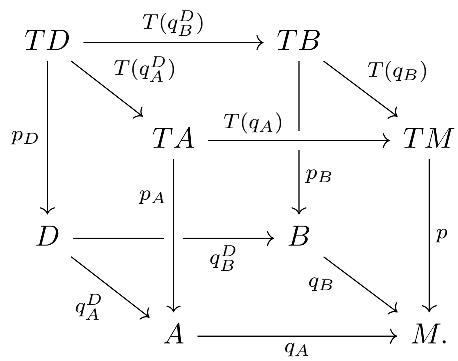
$$(E_2 \times_M E_1 \times_M E_{12}) \times_{E_2} (E_2 \times_M E_3 \times_M E_{23}) \times_{E_2} (E_2 \times_M E_{13} \times_M E_{123}),$$

and this is precisely a decomposition of the Left face.

The choice of Ω also determines decompositions of the core double vector bundles. For example, for $E_{13,2}$, start with a decomposition $\Omega : E \to \overline{E}$, and then restrict to E_2, E_{13}, and E_{123}, and set the other building vector bundles to zero. So we see that the core double vector bundles satisfy all three parts of the definition of a double vector bundle.

2.4.2 The tangent TD of a double vector bundle D

Appplying the tangent functor to a vector bundle $A \to M$, we obtain the tangent double vector bundle TA. Starting with a double vector bundle D, applying the tangent functor to each structure in D yields the triple vector bundle TD as shown in (2.61).



(2.61)

The Down face of (2.61) is D itself, the Front face is the tangent double vector bundle TA of $A \to M$, and the Left, the Back and the Right faces are the tangent double vector bundles of $D \to A$, of $D \to B$, and of $B \to M$ respectively. These faces are known double vector bundles. We need to check that the Up face of (2.61) is also a double vector bundle.

Proposition 2.4.3. *The Up face of (2.61) is a double vector bundle, with core vector bundle $TC \to C$.*

Proof. The algebraic compatibility conditions for the Up face of (2.61) are straightforward. And as the tangent functor preserves the double source map and the sigma map, parts (ii) and (iii) of the definition of a double vector bundle follow immediately. What we need to describe in detail is the core of this double vector bundle.

The core of the Up face of (2.61) is the tangent of the core of the Down face, that is, $TC \to TM$. To see this, first take any $W \in T_cC$. Denote by $\nu : I \to C$, a path in C whose velocity vector at $t = 0$ is W, with $\nu(0) = c$,

$$W = \frac{d}{dt}\nu(t)\Big|_{t=0}.$$

Since $\nu(t)$ is a path in C, it is also a path in D. What is $T(q_A^D)(W)$? For any $f \in C^\infty(A)$, $f \circ q_A^D \in C^\infty(D)$ and we have:

$$T(q_A^D)(W)(f) = W(f \circ q_A^D) = \frac{d}{dt}(f \circ q_A^D)(\nu(t))\Big|_{t=0}.$$

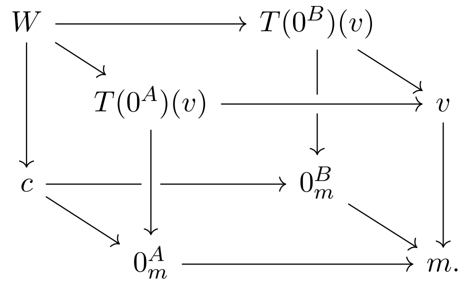
Since $\nu(t)$ is a core element of D for $t \in I$, $q_A^D(\nu(t)) = 0^A(m(t))$, where $m(t) = (q_A \circ q_A^D)(\nu(t))$ is a path in M. Denote by $v = \frac{d}{d}m(t)\Big|_{t=0}$. Then continuing from where we left off:

$$\frac{d}{dt}(f \circ q_A^D)(\nu(t))\Big|_{t=0} = \frac{d}{dt}(f \circ 0^A)(m(t))\Big|_{t=0} = T(0^A)(v)(f),$$

and this is true for every $f \in C^\infty(A)$ so $T(q_A^D)(W) = T(0^A)(v)$. And since

$$(q_B \circ q_B^D)(\nu(t)) = (q_A \circ q_A^D)(\nu(t)) = m(t),$$

similarly, $T(q_B^D)(W) = T(0^B)(v)$. Therefore, W is in the core of the Up face of (2.61). The outline of W,



(2.62)

Conversely, take a ξ in the core of the Up face of (2.61). We will show that $\xi \in T_cC$.

Take a $\xi \in T_cD$. Then $\xi = \frac{d}{dt}\varphi(t)\Big|_{t=0}$, for some path $\varphi : I \to D$ in D, with $\varphi(0) = c \in D$. By hypothesis $T(q_A^D)(\xi) = T(0^A)(v)$, and $T(q_B^D)(\xi) = T(0^B)(v)$, where $v = \frac{d}{d}m(t)\Big|_{t=0} \in T_{m(0)}M$, with $m(t) = q_A(q_A^D(\varphi(t))) = q_B(q_B^D(\varphi(t)))$, a curve in M.

Since $T(q_A^D)(\xi) = T(0^A)(v)$,

$$\frac{d}{dt}(q_A^D(\varphi(t)))\Big|_{t=0} = \frac{d}{dt}0^A(m(t))\Big|_{t=0}.$$

With a similar argument as in the proof of Proposition 1.2.2, we can arrange for $q_A^D(\varphi(t)) = 0^A(m(t))$ for t near zero. Likewise, we can additionally arrange for $q_B^D(\varphi(t)) = 0^B(m(t))$ for t near zero. Hence, we can arrange for $\varphi(t)$ to be a path in C, and therefore $\xi \in T_cC$. □

So far we have shown that each face of TD is a double vector bundle. To ensure that this is a triple vector bundle, we need to check parts (i), (ii), (iii) of the definition of a triple vector bundle.

The algebraic compatibility conditions follow easily. What is interesting in this case of TD is part (iii). The following is [13, Proposition 3.4].

Proposition 2.4.4. *If a double vector bundle D satisfies part (iii) of the definition of a double vector bundle, then its tangent prolongation TD satisfies part (iii) of the definition of a triple vector bundle.*

Proof. Since D satisfies part (iii) of the definition of a double vector bundle, there exist decomposition $\Omega : D \to A \times_M B \times_M C$ of D. The tangent of Ω,

$$T(\Omega) : TD \to TA \times_{TM} TB \times_{TM} TC,$$

and if we choose decompositions of TA, TB, and TC, then we obtain a map from TD to

$$(A \times_M A \times_M TM) \times_{TM} (B \times_M B \times_M TM) \times_{TM} (C \times_M C \times_M TM)$$

we can rearrange this to

$$A \times_M A \times_M B \times_M B \times_M C \times_M C \times_M TM = \overline{TD}.$$

□

Having established that decompositions of TD exist, we can show that the map $\widetilde{\natural} : TD \to W$ is a surjective submersion.

Take any $(d, \xi_1, \xi_2) \in W$, where $d \in D$, $\xi_1 \in TA$, $\xi_2 \in TB$, with matching projections as in (2.2). Then $(d, \xi_1, \xi_2, 0^C_m) \in W \times_M C$, where $m = q_A(q^D_A(d))$. Since there exist decompositions $\Omega : TD \to \overline{TD}$, we have

$$\Phi := (\Omega_W \times_M \mathrm{id})(d, \xi_1, \xi_2, 0^C_m) \in \overline{TD}.$$

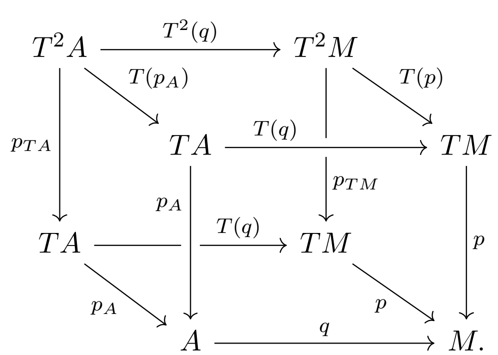
Then $\mho(\Phi) \in TD$ and $\widetilde{\natural}(\mho(\Phi)) = (d, \xi_1, \xi_2)$. Hence $\widetilde{\natural} : TD \to W$ is surjective. Submersion follows with a similar argument.

The three core double vector bundle of TD in the usual order, and the ultracore:

(Back-Front)	(Left-Right)	(Up-Down)	Ultracore
$\begin{array}{ccc} D & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & M \end{array}$	$\begin{array}{ccc} D & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & M \end{array}$	$\begin{array}{ccc} TC & \longrightarrow & C \\ \downarrow & & \downarrow \\ TM & \longrightarrow & M \end{array}$	$C \to M$

2.4.3 Special case: T^2A

In the case where $D = TA$ for a vector bundle (A, q, M), the triple vector bundle T^2A is as shown in (2.63).



(2.63)

The core double vector bundles of T^2A

The three core double vector bundles of (2.63) are shown in (2.64), in the usual order (B-F), (L-R), and (U-D), and arranged as in (2.30).

$$\begin{array}{ccc} TA & \xrightarrow{p_A} & A \\ {\scriptstyle T(q)}\downarrow & & \downarrow{\scriptstyle q} \\ TM & \xrightarrow[p]{} & M, \end{array} \qquad \begin{array}{ccc} TA & \xrightarrow{T(q)} & TM \\ {\scriptstyle p_A}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow[q]{} & M, \end{array} \qquad \begin{array}{ccc} TA & \xrightarrow{p_A} & A \\ {\scriptstyle T(q)}\downarrow & & \downarrow{\scriptstyle q} \\ TM & \xrightarrow[p]{} & M. \end{array} \tag{2.64}$$

These core double vector bundles are the same as abstract double vector bundles but are embedded differently in T^2A, as we show in what follows.

Take a $\xi \in T(q)^{-1}(v)$, $v \in T_mM$, in the core of the Back face. Denote by $a = p_A(\xi)$. So its outline in the (B-F) core double vector bundle is

$$\begin{array}{ccc} \xi & \longmapsto & a \\ \downarrow & & \downarrow \\ v & \longmapsto & m. \end{array}$$

Denote by $\bar{\xi}^B$ the corresponding element in T^2A defined by this $\xi \in T(q)^{-1}(v)$.

The Back face is the tangent double vector bundle for the tangent prolongation bundle $T(q): TA \to TM$. Follow the construction in Subsection 1.1.2. The corresponding curve to which $\bar{\xi}^B$ is a tangent vector at point $T(0^A)(v)$, is $t \underset{TM}{\cdot} \xi$:

$$\begin{array}{ccc} t \underset{TM}{\cdot} \xi & \longmapsto & ta \\ \downarrow & & \downarrow \\ v & \longmapsto & m, \end{array}$$

and this is entirely in the fibre $T(q)^{-1}(v)$. Therefore

$$\bar{\xi}^B = \frac{d}{dt}(t \underset{TM}{\cdot} \xi)\Big|_{t=0} \in T_{T(0^A)(v)}(TA) \tag{2.65}$$

Furthermore,

$$T^2(q)(\bar{\xi}^B) = \frac{d}{dt}T(q)(t \underset{TM}{\cdot} \xi)\Big|_{t=0} = \frac{d}{dt}v\Big|_{t=0} \overset{(1.12)}{=} 0_v^{T^2M},$$

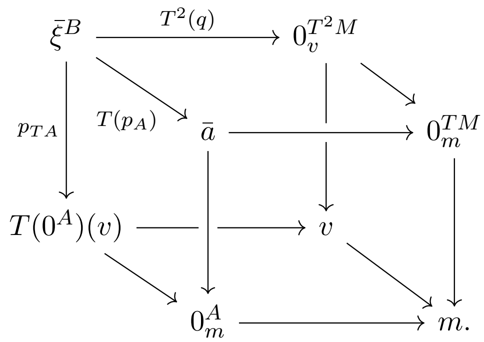
and

$$T(p_A)(\bar{\xi}^B) = \frac{d}{dt}p_A(t \underset{TM}{\cdot} \xi)\Big|_{t=0} = \frac{d}{dt}ta\Big|_{t=0} \overset{(1.9)}{=} \bar{a},$$

and of course

$$p_{TA}(\bar{\xi}^B) = T(0^A)(v).$$

The outline of $\bar{\xi}^B$ in T^2A:

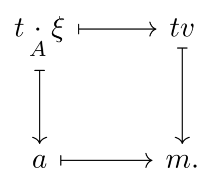


(2.66)

Now take a $\xi \in T_aA$ in the core of the Left face. The Left face is the double tangent bundle of the manifold A. If we denote by $\bar{\xi}^L$ the corresponding element in T^2A that $\xi \in T_aA$ determines, we have

$$\bar{\xi}^L = \frac{d}{dt}(t \underset{A}{\cdot} \xi)\Big|_{t=0}, \tag{2.67}$$

where the scalar multiplication is in the usual tangent bundle $TA \to A$. The curve $t \underset{A}{\cdot} \xi$ is in the fibre T_aA entirely:



It follows that

$$T^2(q)(\bar{\xi}^L) = \frac{d}{dt}T(q)(t \underset{A}{\cdot} \xi)\Big|_{t=0} = \frac{d}{dt}tv\Big|_{t=0} \overset{(1.9)}{=} \bar{v},$$

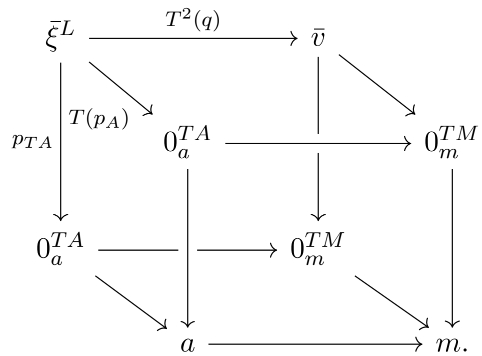
and

$$T(p_A)(\bar{\xi}^L) = \frac{d}{dt}p_A(t \underset{A}{\cdot} \xi)\Big|_{t=0} = \frac{d}{dt}a\Big|_{t=0} \overset{(1.12)}{=} 0_a^{TA},$$

and finally, the zero of the fibre T_aA is 0_a^{TA}, therefore,

$$p_{TA}(\bar{\xi}^L) = 0_a^{TA}.$$

Therefore, the outline of $\bar{\xi}^L$ in T^2A:



(2.68)

Finally, take a ξ in the (U-D) core double vector bundle with outline

$$\begin{array}{ccc} \xi & \longmapsto & a \\ \downarrow & & \downarrow \\ v & \longmapsto & m. \end{array}$$

Following the construction in Subsection 2.4.2, take a curve $a(t)$ in the core A of the Down face, with $a(0) = a$, $q(a(t)) = m(t)$ a curve in M with $v = \frac{d}{dt}m(t)\Big|_{t=0}$. Then

$$\xi = \frac{d}{dt}a(t)\Big|_{t=0} \in T_aA,$$

is in the core TA of the Up face. We view it as $\bar{\xi}^U$ in T^2A as follows. Take the curve $\overline{a(t)}$ in TA, where $\overline{a(t)}$ is the core element in TA corresponding to $a(t)$ for every t:

$$\overline{a(t)} = \frac{d}{ds}sa(t)\Big|_{s=0} \in T_{0^A(m(t))}A,$$

with outlines

$$\begin{array}{ccc} \overline{a(t)} & \longmapsto & 0^A(m(t)) \\ \downarrow & & \downarrow \\ 0^{TM}(m(t)) & \longmapsto & m(t). \end{array}$$

Therefore,

$$\bar{\xi}^U = \frac{d}{dt}\overline{a(t)}\Big|_{t=0} \in T_{\bar{a}}TA. \tag{2.69}$$

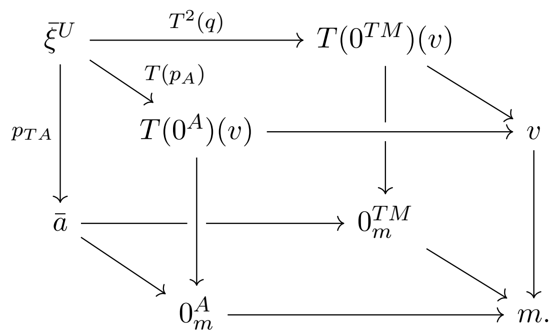
It follows that,

$$T^2(q)(\bar{\xi}^U) = \frac{d}{dt}T(q)(\overline{a(t)})\Big|_{t=0} = \frac{d}{dt}0^{TM}(m(t))\Big|_{t=0} = T(0^{TM})(v),$$

and

$$T(p_A)(\bar{\xi}^U) = \frac{d}{dt}p_A(\overline{a(t)})\Big|_{t=0} = \frac{d}{dt}0^A(m(t))\Big|_{t=0} = T(0^A)(\frac{d}{dt}m(t)\Big|_{t=0}) = T(0^A)(v),$$

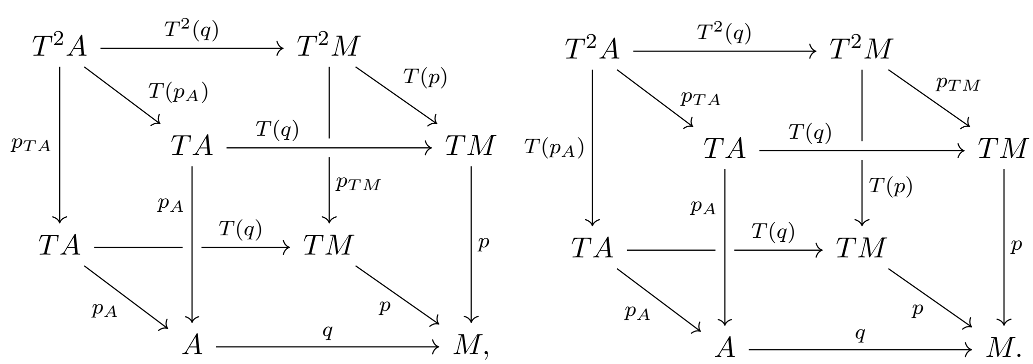
and of course $p_{TA}(\bar{\xi}^U) = \bar{a}$, see Subsection 2.4.2, (2.62). And the triple outline of $\bar{\xi}^U$ in T^2A:



2.4.4 The canonical involution on T^2A

The canonical involution $J_A : T^2A \to T^2A$ for the manifold A is an isomorphism from the double vector bundle T^2A to its flip. In what follows we will need to use it as a map of triple vector bundles.

Proposition 2.4.5. *The map J_A is an isomorphism of the triple vector bundles shown in* (2.70).



(2.70)

In (2.70) the Left faces are the double tangent bundles of the manifold A and J_A maps the Left face of the domain to its flip. It interchanges the Up and Back faces. The Right faces are the double tangent bundles of M, and as J_A induces $J_M : T^2M \to T^2M$ to the Right faces, it maps the Right face of the domain to its flip. The Front and Down faces are interchanged.

The proof of Proposition 2.4.5 relies on two lemmas. First, the naturality property of the canonical involution, Lemma 1.2.1. Secondly, we need to show that (J_A, J_M) is a vector bundle map. That the diagram

$$\begin{array}{ccc} T^2A & \xrightarrow{J_A} & T^2A \\ {\scriptstyle T^2(q)}\Big\downarrow & & \Big\downarrow{\scriptstyle T^2(q)} \\ T^2M & \xrightarrow[J_M]{} & T^2M, \end{array}$$

commutes follows from Lemma 1.2.1 for $q : A \to M$. It remains to show that J_A is linear fibrewise. At this point we need to work in local coordinates on T^2A.

In Section 1.2.3 we described a local coordinate system on TA for a vector bundle $A \to M$ of rank r. We denoted it by $(x, a, \dot{x}, \dot{a})$, with transformation laws (1.39). We now present a local coordinate system on T^2A.

As it is important to distinguish between different copies of the same thing, instead of using $(x, a, \dot{x}, \dot{a})$ on TA, we now write (x, a, v, w), where $\tilde{x} = \tilde{x}(x)$, $\tilde{a} = \tilde{a}(x, a)$, $\tilde{v} = \tilde{v}(x, v)$, $\tilde{w} = \tilde{w}(x, a, v, w)$ on the intersection of two overlapping charts on TA

$$(x, a, v, w) \to (\tilde{x}, \tilde{a}, \tilde{v}, \tilde{w}), \tag{2.71}$$

and as in (1.39), we have

$$\begin{aligned} \tilde{a}^\ell &= P^\ell_h(m)a^h, \\ \tilde{v}^i &= \frac{\partial \tilde{x}^i}{\partial x^p}(m)v^p, \\ \tilde{w}^\ell &= P^\ell_h(m)w^h + \frac{\partial P^\ell_s}{\partial x^p}(m)v^p a^s. \end{aligned}$$

A local coordinate system on T^2A is now, in shorthand notation,

$$(x, a, v, w, \dot{x}, \dot{a}, \dot{v}, \dot{w}),$$

where $(x) = (x^1, \dots, x^n)$, $(a) = (a^1, \dots, a^r)$, $(v) = (v^1, \dots, v^n)$, $(w) = (w^1, \dots, w^r)$, and corresponding indices for the respective dots.

Following the usual rule of calculating the Jacobian matrix of 2.71, the following 4×4 block matrix,

$$\begin{bmatrix} \left(\frac{\partial \tilde{x}}{\partial x}\right)_{(n\times n)} & \left(\frac{\partial \tilde{x}}{\partial a}\right)_{(n\times r)} & \left(\frac{\partial \tilde{x}}{\partial v}\right)_{(n\times n)} & \left(\frac{\partial \tilde{x}}{\partial w}\right)_{(n\times r)} \\ \left(\frac{\partial \tilde{a}}{\partial x}\right)_{(r\times n)} & \left(\frac{\partial \tilde{a}}{\partial a}\right)_{(r\times r)} & \left(\frac{\partial \tilde{a}}{\partial v}\right)_{(r\times n)} & \left(\frac{\partial \tilde{a}}{\partial w}\right)_{(r\times r)} \\ \left(\frac{\partial \tilde{v}}{\partial x}\right)_{(n\times n)} & \left(\frac{\partial \tilde{v}}{\partial a}\right)_{(n\times r)} & \left(\frac{\partial \tilde{v}}{\partial v}\right)_{(n\times n)} & \left(\frac{\partial \tilde{v}}{\partial w}\right)_{(n\times r)} \\ \left(\frac{\partial \tilde{w}}{\partial x}\right)_{(r\times n)} & \left(\frac{\partial \tilde{w}}{\partial a}\right)_{(r\times r)} & \left(\frac{\partial \tilde{w}}{\partial v}\right)_{(r\times n)} & \left(\frac{\partial \tilde{w}}{\partial w}\right)_{(r\times r)} \end{bmatrix}, \tag{2.72}$$

this matrix (2.72) describes how coordinates $(x, a, v, w, \dot{x}, \dot{a}, \dot{v}, \dot{w})$ and $(\tilde{x}, \tilde{a}, \tilde{v}, \tilde{w}, \dot{\tilde{x}}, \dot{\tilde{a}}, \dot{\tilde{v}}, \dot{\tilde{w}})$ on T^2A on overlapping charts change:

- First row of (2.72),

$$\frac{\partial \tilde{x}^i}{\partial x^j}, \quad \frac{\partial \tilde{x}^i}{\partial a^k} = 0, \quad \frac{\partial \tilde{x}^i}{\partial v^j} = 0, \quad \frac{\partial \tilde{x}^i}{\partial w^k} = 0.$$

- Second row of (2.72),

$$\frac{\partial \tilde{a}^\ell}{\partial x^j} = \frac{\partial P^\ell_s}{\partial x^j} a^s, \quad \frac{\partial \tilde{a}^\ell}{\partial a^k} = P^\ell_k, \quad \frac{\partial \tilde{a}^\ell}{\partial v^j} = 0, \quad \frac{\partial \tilde{a}^\ell}{\partial w^k} = 0.$$

- Third row of (2.72),

$$\frac{\partial \tilde{v}^i}{\partial x^j} = \frac{\partial^2 \tilde{x}^i}{\partial x^j \partial x^p} v^p, \quad \frac{\partial \tilde{v}^i}{\partial a^k} = 0, \quad \frac{\partial \tilde{v}^i}{\partial v^j} = \frac{\partial \tilde{x}^i}{\partial x^j}, \quad \frac{\partial \tilde{v}^i}{\partial w^k} = 0.$$

- Fourth row of (2.72),

$$\frac{\partial \tilde{w}^\ell}{\partial x^j} = \frac{\partial P^\ell_h}{\partial x^j} w^h + \frac{\partial^2 P^\ell_s}{\partial x^j \partial x^p} v^p a^s, \ \frac{\partial \tilde{w}^\ell}{\partial a^k} = \frac{\partial P^\ell_k}{\partial x^p} v^p, \ \frac{\partial \tilde{w}^\ell}{\partial v^j} = \frac{\partial P^\ell_s}{\partial x^j} a^s, \ \frac{\partial \tilde{w}^\ell}{\partial w^k} = P^\ell_k.$$

In total, about the seven fibre coordinates of $T^2 A$, transformation laws are the following:

$$\begin{aligned}
\tilde{a}^\ell &= P^\ell_k(m) a^k, \\
\tilde{v}^i &= \frac{\partial \tilde{x}^i}{\partial x^j}(m) v^j, \\
\tilde{w}^\ell &= P^\ell_k(m) w^k + \frac{\partial P^\ell_s}{\partial x^j}(m) a^s v^j, \\
\dot{\tilde{x}}^i &= \frac{\partial \tilde{x}^i}{\partial x^j}(m) \dot{x}^j, \\
\dot{\tilde{a}}^\ell &= \frac{\partial P^\ell_s}{\partial x^j}(m) a^s \dot{x}^j + P^\ell_k(m) \dot{a}^k, \\
\dot{\tilde{v}}^i &= \frac{\partial^2 \tilde{x}^i}{\partial x^j \partial x^p}(m) v^p \dot{x}^j + \frac{\partial \tilde{x}^i}{\partial x^j}(m) \dot{v}^j, \\
\dot{\tilde{w}}^\ell &= \frac{\partial P^\ell_h}{\partial x^j}(m) w^h \dot{x}^j + \frac{\partial^2 P^\ell_s}{\partial x^j \partial x^p}(m) v^p a^s \dot{x}^j + \frac{\partial P^\ell_k}{\partial x^p}(m) v^p \dot{a}^k + \frac{\partial P^\ell_s}{\partial x^j}(m) a^s \dot{v}^j + P^\ell_k(m) \dot{w}^k.
\end{aligned}$$

The three projections of $T^2 A$,

$$\begin{aligned}
p_{TA} : (x, a, v, w, \dot{x}, \dot{a}, \dot{v}, \dot{w}) &\mapsto (x, a, v, w), \quad \text{Down Face}, \\
T(p_A) : (x, a, v, w, \dot{x}, \dot{a}, \dot{v}, \dot{w}) &\mapsto (x, a, \dot{x}, \dot{a}), \quad \text{Front Face}, \\
T^2(q) : (x, a, v, w, \dot{x}, \dot{a}, \dot{v}, \dot{w}) &\mapsto (x, v, \dot{x}, \dot{v}), \quad \text{Right Face}.
\end{aligned}$$

The canonical involution $J_A : T^2 A \to T^2 A$ in local coordinates:

$$(x, a, v, w, \dot{x}, \dot{a}, \dot{v}, \dot{w}) \mapsto (x, a, \dot{x}, \dot{a}, v, w, \dot{v}, \dot{w}).$$

We need the following lemma in order to prove that J_A is a fibrewise linear map over J_M.

Lemma 2.4.6. *If $\Phi_1, \Phi_2 \in T^2A$, with $T^2(q)(\Phi_1) = T^2(q)(\Phi_2)$, then*

$$J_A\left(\Phi_1 \underset{T^2(q)}{+} \Phi_2\right) = J_A(\Phi_1) \underset{T^2(q)}{+} J_A(\Phi_2). \tag{2.73}$$

First we need the following "double version" of Proposition 1.2.2.

Proposition 2.4.7. *Take two vectors $\Phi_1,\ \Phi_2 \in T^2A$, with $T^2(q)(\Phi_1) = T^2(q)(\Phi_2)$. Then there exist smooth squares $\nu_1, \nu_2 : (-\epsilon,\epsilon)\times(-\epsilon,\epsilon) \to A$ such that $\Phi_1 = \frac{d}{dt}\left(\frac{d}{ds}\nu_1(t,s)\Big|_{s=0}\right)\Big|_{t=0}$ and $\Phi_2 = \frac{d}{dt}\left(\frac{d}{ds}\nu_2(t,s)\Big|_{s=0}\right)\Big|_{t=0}$, with $q(\nu_1(t,s)) = q(\nu_2(t,s))$, for t and s near zero.*

The proof is similar to the single case one, we present it here for reference.

Proof. Let $m = p(T(p)(T^2(q)(\Phi_1))) = p(T(p)(T^2(q)(\Phi_2)))$ be in the domain U of the chart (U,φ) on M with coordinates $(x^1,\ldots,x^n)$, shorthand notation (x). Write:

$$\begin{aligned}
\Phi_1 &= (x(m), a_1(m), v_1(m), w_1(m), \dot{x}_1(m), \dot{a}_1(m), \dot{v}_1(m), \dot{w}_1(m)),\\
\Phi_2 &= (x(m), a_2(m), v_2(m), w_2(m), \dot{x}_2(m), \dot{a}_2(m), \dot{v}_2(m), \dot{w}_2(m)),
\end{aligned}$$

for some local coordinates on T^2A. The following squares $\nu_1, \nu_2 : (-\epsilon,\epsilon)\times(-\epsilon,\epsilon) \to A$,

$$\begin{aligned}
\nu_1(t,s) &= \left(x(m) + t\dot{x}_1(m) + sv_1(m) + ts\dot{v}_1(m), a_1(m) + t\dot{a}_1(m) + sw_1(m) + ts\dot{w}_1(m)\right),\\
\nu_2(t,s) &= \left(x(m) + t\dot{x}_2(m) + sv_2(m) + ts\dot{v}_2(m), a_2(m) + t\dot{a}_2(m) + sw_2(m) + ts\dot{w}_2(m)\right).
\end{aligned}$$

Denote $\nu_1(t,s)$ succinctly by $(x(t,s), a(t,s))$. Then,

$$\begin{aligned}
\frac{d}{ds}\nu_1(t,s)\Big|_{s=0} &= \left(x(t,0), a(t,0), \frac{dx(t,s)}{ds}\Big|_{s=0}, \frac{da(t,s)}{ds}\Big|_{s=0}\right)\\
&= (x(m) + t\dot{x}_1(m), a_1(m) + t\dot{a}_1(m), v_1(m) + t\dot{v}_1(m), w_1(m) + t\dot{w}_1(m)),
\end{aligned}$$

and this is a curve $\frac{d}{ds}\nu_1(t,s)\Big|_{s=0} : (-\epsilon,\epsilon) \to TA$. Its velocity vector at $t = 0$,

$$\frac{d}{dt}\left(\frac{d}{ds}\nu_1(t,s)\Big|_{s=0}\right)\Big|_{t=0} = (x(m), a_1(m), v_1(m), w_1(m), \dot{x}_1(m), \dot{a}_1(m), \dot{v}_1(m), \dot{w}_1(m)) = \Phi_1.$$

And a similar calculation shows that

$$\frac{d}{dt}\left(\frac{d}{ds}\nu_2(t,s)\Big|_{s=0}\right)\Big|_{t=0} = \Phi_2.$$

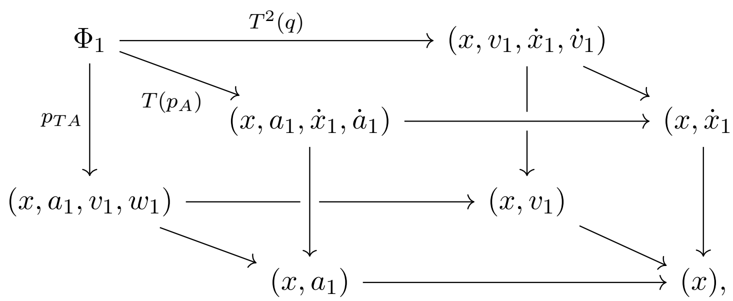
By hypothesis, $T^2(q)(\Phi_1) = T^2(q)(\Phi_2)$, which implies that

$$v_1(m) = v_2(m), \quad \dot{x}_1(m) = \dot{x}_2(m), \quad \dot{v}_1(m) = \dot{v}_2(m),$$

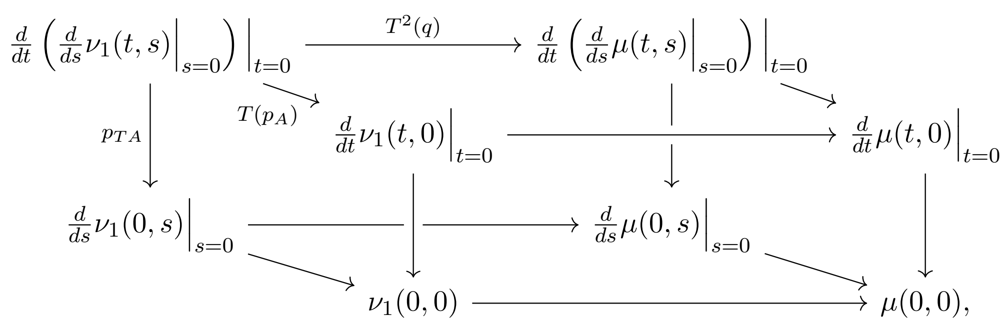
therefore,

$$q(\nu_1(t,s)) = x(m) + t\dot{x}_1(m) + sv_1(m) + ts\dot{v}_1(m)$$
$$= x(m) + t\dot{x}_2(m) + sv_2(m) + ts\dot{v}_2(m) = q(\nu_2(t,s)).$$

We include some outlines for reference. The triple outline of Φ_1 in local coordinates,



and the triple outline of $\frac{d}{dt}\left(\frac{d}{ds}\nu_1(t,s)\Big|_{s=0}\right)\Big|_{t=0}$, where we denote by $q \circ \nu_1 = \mu$, smooth square of elements of M:



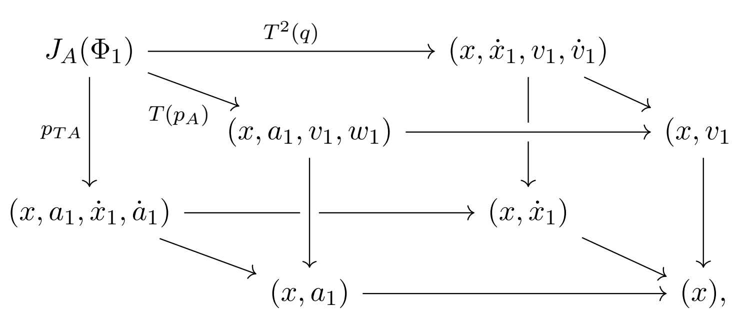
And of course

$$\frac{d}{ds}\nu_1(0,s)\Big|_{s=0} = (x, a_1, v_1, w_1).$$

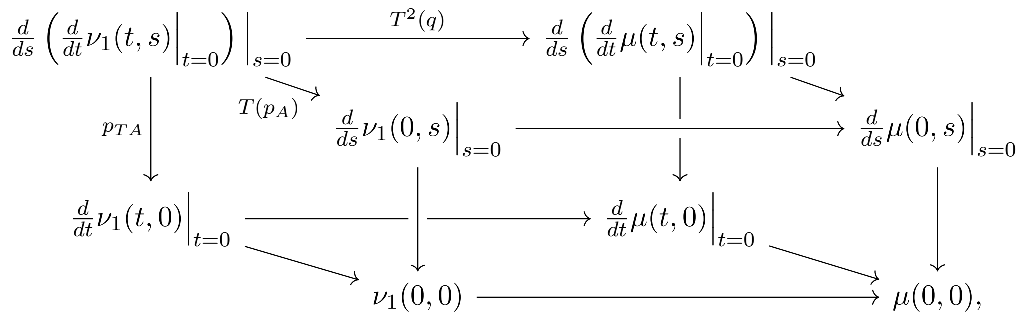
Additionally,

$$\nu_1(t,0) = (x + t\dot{x}_1, a_1 + t\dot{a}_1) \Rightarrow \frac{d}{dt}\nu_1(t,0)\Big|_{t=0} = (x, a_1, \dot{x}_1, \dot{a}_1).$$

The outline of $J_A(\Phi_1)$,



and the outline of $J_A\left(\frac{d}{dt}\left(\frac{d}{ds}\nu_1(t,s)\Big|_{s=0}\right)\Big|_{t=0}\right) = \frac{d}{ds}\left(\frac{d}{dt}\nu_1(t,s)\Big|_{t=0}\right)\Big|_{s=0}$,



□

We now proceed with the proof of Lemma 2.4.6.

Proof. By Proposition 2.4.7, there exist $\nu_1, \nu_2 : (-\epsilon,\epsilon)\times(-\epsilon,\epsilon) \to A$ smooth squares of elements of A, such that:

$$\Phi_1 = \frac{d}{dt}\left(\frac{d}{ds}\nu_1(t,s)\Big|_{s=0}\right)\Big|_{t=0}, \quad \Phi_2 = \frac{d}{dt}\left(\frac{d}{ds}\nu_2(t,s)\Big|_{s=0}\right)\Big|_{t=0},$$

with $q\circ\nu_1 = q\circ\nu_2 = \mu : (-\epsilon,\epsilon)\times(-\epsilon,\epsilon)\to M$, a smooth square of elements of M. Since $q\circ\nu_1 = q\circ\nu_2$, it follows that

$$T(q)\left(\frac{d}{ds}\nu_1(t,s)\Big|_{s=0}\right) = \frac{d}{ds}q(\nu_1(t,s))\Big|_{s=0} = \frac{d}{ds}q(\nu_2(t,s))\Big|_{s=0} = T(q)\left(\frac{d}{ds}\nu_2(t,s)\Big|_{s=0}\right).$$

In other words, for the two curves $Y_1(t) = \frac{d}{ds}\nu_1(t,s)\Big|_{s=0}$ and $Y_2(t) = \frac{d}{ds}\nu_2(t,s)\Big|_{s=0}$ in TA, we have that $T(q)(Y_1(t)) = T(q)(Y_2(t))$, for t near zero. Therefore,

$$\begin{aligned}\Phi_1 \underset{T^2(q)}{+} \Phi_2 = \frac{d}{dt}Y_1(t)\Big|_{t=0} \underset{T^2(q)}{+} \frac{d}{dt}Y_2(t)\Big|_{t=0} &= \frac{d}{dt}\left(Y_1(t)\underset{T(q)}{+}Y_2(t)\right)\Big|_{t=0} \\ &= \frac{d}{dt}\left(\frac{d}{ds}\nu_1(t,s)\Big|_{s=0}\underset{T(q)}{+}\frac{d}{ds}\nu_2(t,s)\Big|_{s=0}\right)\Big|_{t=0}.\end{aligned}$$

Again, due to our hypothesis, that $q\circ\nu_1 = q\circ\nu_2$, we have

$$\frac{d}{ds}\nu_1(t,s)\Big|_{s=0}\underset{T(q)}{+}\frac{d}{ds}\nu_2(t,s)\Big|_{s=0} = \frac{d}{ds}(\nu_1(t,s)+\nu_2(t,s))\Big|_{s=0}.$$

Altogether,

$$\Phi_1 \underset{T^2(q)}{+} \Phi_2 = \frac{d}{dt}\left(\frac{d}{ds}(\nu_1(t,s)+\nu_2(t,s))\Big|_{s=0}\right)\Big|_{t=0}.$$

Applying J_A to the previous equation,

$$J_A\left(\Phi_1 \underset{T^2(q)}{+} \Phi_2\right) = \frac{d}{ds}\left(\frac{d}{dt}\left(\nu_1(t,s)+\nu_2(t,s)\right)\Big|_{t=0}\right)\Big|_{s=0}. \tag{2.74}$$

About the right hand side of (2.73), again from the fact that $q(\nu_1(t,s)) = q(\nu_2(t,s))$, it follows that,

$$J_A(\Phi_1) \underset{T^2(q)}{+} J_A(\Phi_2) = \frac{d}{ds}\left(\frac{d}{dt}\nu_1(t,s)\Big|_{t=0}\right)\Big|_{s=0} \underset{T^2(q)}{+} \frac{d}{ds}\left(\frac{d}{dt}\nu_2(t,s)\Big|_{t=0}\right)\Big|_{s=0}$$
$$= \frac{d}{ds}\left(\frac{d}{dt}\nu_1(t,s)\Big|_{t=0} \underset{T(q)}{+} \frac{d}{dt}\nu_2(t,s)\Big|_{t=0}\right)\Big|_{s=0} = \frac{d}{ds}\left(\frac{d}{dt}(\nu_1(t,s)+\nu_2(t,s))\Big|_{t=0}\right)\Big|_{s=0},$$

and we see that this is equal to (2.74). □

Also, as we will need it later on, note that J_A as the canonical involution of the double tangent bundle T^2A, for the manifold A, interchanges the two additions, that is, recall (1.28) and (1.29). For $\Phi_1, \Phi_2 \in T^2A$ with $T(p_A)(\Phi_1) = T(p_A)(\Phi_2)$:

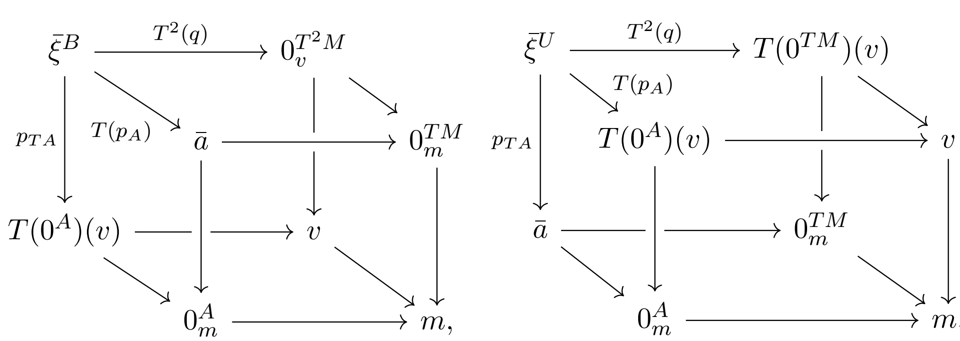
$$J_A(\Phi_1 \underset{T(p_A)}{+} \Phi_2) = J_A(\Phi_1) \underset{p_{TA}}{+} J_A(\Phi_2),$$

and for $\Phi_1, \Phi_2 \in T^2A$ with $p_{TA}(\Phi_1) = p_{TA}(\Phi_2)$:

$$J_A(\Phi_1 \underset{p_{TA}}{+} \Phi_2) = J_A(\Phi_1) \underset{T(p_A)}{+} J_A(\Phi_2).$$

Now consider the maps which J_A induces on the cores.

Take an element $\xi \in TA$ in the core of the Back face. Regarded as an element of T^2A this is $\bar{\xi}^B$, with outline shown on the left of (2.75).



(2.75)

It follows from (2.65) and (2.69) that

$$J_A(\bar{\xi}^B) = \bar{\xi}^U \quad \text{and} \quad J_A(\bar{\xi}^U) = \bar{\xi}^B, \tag{2.76}$$

since J_A^2 is the identity.

Since the Left faces in (2.70) are the double tangent bundle T^2A, the map on the cores of the Left faces is the identity and so

$$J_A(\bar{\xi}^L) = \bar{\xi}^L. \tag{2.77}$$

2.4.5 The cotangent T^*D

In this section we present in detail the cotangent T^*D of the tangent triple vector bundle TD. The cotangent T^*D is a triple vector bundle obtained through the process of *dualization*. First, some background on the duality of double vector bundles.

Double vector bundles and Duality

When we dualize the double vector bundle D with respect to its vector bundle structure over A, the resulting structure is denoted by $D \ast A$. That $D \ast A$ is a double vector bundle, this is described in detail in [25, Section 9.2].

$$\begin{array}{ccc} D \ast A & \xrightarrow{\gamma^A_{C^*}} & C^* \\ {\scriptstyle \gamma^A_A}\downarrow & & \downarrow \\ A & \longrightarrow & M. \end{array}$$

The core of this double vector bundle is $B^* \to M$.

As we will use it extensively in what follows, we write the formula for the unfamiliar projection $\gamma^A_{C^*} : D \ast A \to C^*$. From equation (16) of [25, p.348], this is

$$\langle \gamma^A_{C^*}(\Phi), c \rangle_{C^*} = \langle \Phi, 0^D_a \underset{B}{+} c \rangle_A, \tag{2.78}$$

where $c \in C_m$, $\Phi : (q^D_A)^{-1}(a) \to \mathbb{R}$, and $a \in A_m$. The zero above $\kappa \in C^*_m$ is denoted by $0^{D \ast A}_\kappa$ and is defined by

$$\langle 0^{D \ast A}_\kappa, 0^D_b \underset{A}{+} c \rangle_A = \langle \kappa, c \rangle_{C^*}, \tag{2.79}$$

where $b \in B_m$ and $c \in C_m$. The core element ψ corresponding to $\psi \in B^*_m$ is

$$\langle \psi, 0^D_b \underset{A}{+} c \rangle_A = \langle \psi, b \rangle_B. \tag{2.80}$$

The addition $\underset{C^*}{+}$ in $D \ast A \to C^*$ is defined by

$$\langle \Phi \underset{C^*}{+} \Phi', d \underset{B}{+} d' \rangle_A = \langle \Phi, d \rangle_A + \langle \Phi', d' \rangle_A, \tag{2.81}$$

where Φ and Φ' have outlines $(\Phi; a, \kappa; m)$ and $(\Phi'; a', \kappa; m)$, and d and d' have outlines $(d; a, b; m)$ and $(d'; a', b; m)$, see [25, p.348] for more details. Similarly for $D \ast B$.

The two duals $D \ast A$ and $D \ast B$ of D have a remarkable relation, namely, $D \ast A \to C^*$ and $D \ast B \to C^*$ are dual vector bundles. There exists a nondegenerate pairing between $D \ast A$ and $D \ast B$ over C^* [25, 9.2.2], denoted $\| \ , \ \|$, which is natural up to sign. Again, for details see [25, Section 9.2].

$$\begin{array}{ccc} D & \xrightarrow{q_B^D} & B \\ {\scriptstyle q_A^D}\downarrow & & \downarrow{\scriptstyle q_B} \\ A & \xrightarrow{q_A} & M, \end{array} \qquad \begin{array}{ccc} D \ast A & \xrightarrow{\gamma_{C^*}^A} & C^* \\ {\scriptstyle \gamma_A^A}\downarrow & & \downarrow \\ A & \longrightarrow & M, \end{array} \qquad \begin{array}{ccc} D \ast B & \xrightarrow{\gamma_B^B} & B \\ {\scriptstyle \gamma_{C^*}^B}\downarrow & & \downarrow \\ C^* & \longrightarrow & M. \end{array} \tag{2.82}$$

For $\Phi \in D \ast A$, and $\Psi \in D \ast B$, with outlines $(\Phi; a, \kappa; m)$ and $(\Psi; \kappa, b; m)$ respectively, and for any $d \in D$ with outline $(d; a, b; m)$, define

$$\| \Phi, \ \Psi \| = \langle \Phi, d \rangle_A - \langle \Psi, d \rangle_B. \tag{2.83}$$

Note that the pairing (2.83) is independent of the choice of $d \in D$. This pairing induces two double vector bundle isomorphisms, namely,

$$Z_A : D \ast A \to D \ast B \ast C^*, \qquad \langle Z_A(\Phi), \Psi \rangle_{C^*} = \| \Phi, \ \Psi \|, \tag{2.84}$$

and

$$Z_B : D \ast B \to D \ast A \ast C^*, \qquad \langle Z_B(\Psi), \Phi \rangle_{C^*} = \| \Phi, \ \Psi \|. \tag{2.85}$$

The core of T^*A

The cotangent double vector bundle T^*A:

$$\begin{array}{ccc} T^*A & \xrightarrow{r} & A^* \\ {\scriptstyle c_A}\downarrow & & \downarrow{\scriptstyle q_*} \\ A & \xrightarrow{q} & M, \end{array}$$

is a prime example of a dual double vector bundle. Since T^*A is the resulting double vector bundle after dualizing TA over A, its core vector bundle is $T^*M \to M$.

The unfamiliar projection $r : T^*A \to A^*$ in this case, using (2.78) we can write:

$$\langle r(\Phi), a' \rangle = \langle \Phi, 0_a^{TA} \underset{TM}{+} \bar{a}' \rangle_A, \tag{2.86}$$

for $\Phi \in T_a^*A$, $a \in A_m$, and for $a' \in A_m$ (where this copy of A_m is the core of TA).

By (2.81), the addition in $T^*A \to A^*$ is described by

$$\langle \Phi_1 \underset{A^*}{+} \Phi_2, \xi_1 \underset{TM}{+} \xi_2 \rangle_A = \langle \Phi_1, \xi_1 \rangle_A + \langle \Phi_2, \xi_2 \rangle_A, \tag{2.87}$$

for $\Phi_1 \in T^*_{a_1}A, \Phi_2 \in T^*_{a_2}A$, with $r(\Phi_1) = r(\Phi_2)$, and for $\xi_1 \in T_{a_1}A, \xi_2 \in T_{a_2}A$, with $T(q)(\xi_1) = T(q)(\xi_2)$.

For a single covector $\omega_m \in T^*_m M$, its image in T^*A is described by (2.80):

$$\langle \overline{\omega}, T(0^A)(v) \underset{A}{+} \bar{a} \rangle = \langle \omega, v \rangle,$$

for $v \in T_m M$. A section $\omega \in \Gamma(T^*M)$ of the core of T^*A, defines two sections of T^*A:

- $q^*(\omega) \in \Omega^1(A)$, the strut of ω over A, a section of the vector bundle $T^*A \to A$, and
- $\breve{\omega}$, the strut of ω over A^* which is a section of $T^*A \to A^*$.

More precisely,

$$\begin{array}{rcl} q^*(\omega) : A & \to & T^*A, \\ A_m \ni a & \mapsto & 0_a^{T^*A} \underset{A^*}{+} \overline{\omega(m)}, \end{array}$$

the pullback of $\omega(m) \in T^*_m M$ to A at the point $a \in A_m$.

About $\breve{\omega}$:

$$\begin{array}{rcl} \breve{\omega} : A^* & \to & T^*A, \\ A^*_m \ni \alpha & \mapsto & 0_\alpha^{T^*A} \underset{A}{+} \overline{\omega(m)}, \end{array}$$

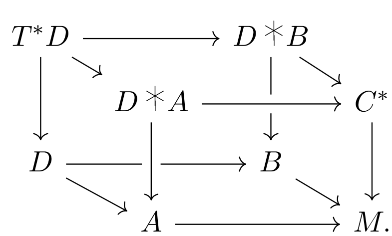
and from (2.79) and (2.80) it follows that

$$\begin{aligned} &\langle 0_\alpha^{T^*A} \underset{A}{+} \overline{\omega(m)}, T(0^A)(v) \underset{A}{+} \bar{a} \rangle_A \\ &\qquad = \langle 0_\alpha^{T^*A}, T(0^A)(v) \underset{A}{+} \bar{a} \rangle_A + \langle \overline{\omega(m)}, T(0^A)(v) \underset{A}{+} \bar{a} \rangle_A = \langle \alpha, a \rangle_A + \langle \omega, v \rangle_{TM}, \end{aligned}$$

for $v \in T_m M$, and $a \in A_m$.

The triple vector bundle T^*D

We are now ready to further investigate T^*D. This triple vector bundle was first introduced in [24]. Dualizing TD with respect to D, we obtain the following triple vector bundle,

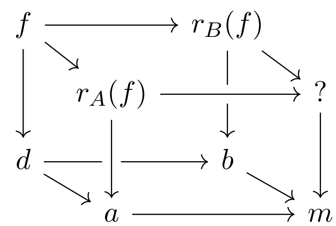


(2.88)

All faces of T^*D except for the Up face are known double vector bundles. So focus on the Up face,

$$\begin{array}{ccc} T^*D & \xrightarrow{r_B} & D \ast B \\ {\scriptstyle r_A}\downarrow & & \downarrow{\scriptstyle \gamma^B_{C^*}} \\ D \ast A & \xrightarrow[\gamma^A_{C^*}]{} & C^*. \end{array}$$

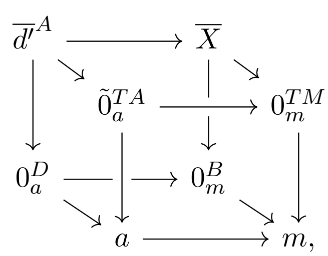
First we check that $\gamma^B_{C^*} \circ r_B = \gamma^A_{C^*} \circ r_A$. To begin with, let $f \in T^*_d D$, with triple outline:



Let us describe $r_A(f)$. The Left face of TD is the tangent double vector bundle of $D \to A$. Its dual with respect to D is the Left face of T^*D, see (2.88). And by (2.78), for $f \in T^*_d D$:

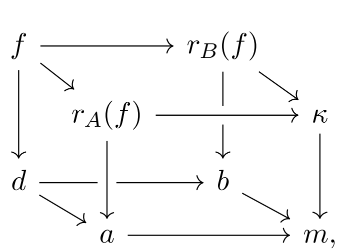
$$\langle r_A(f), d' \rangle = \langle f, \hat{0}_d \underset{TA}{+} \overline{d'}^A \rangle, \tag{2.89}$$

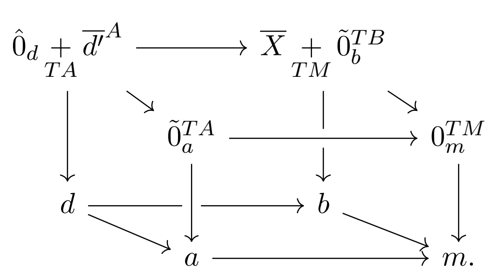
where d' is in the core of the Left face of TD and we denote by $\overline{d'}^A$ its image in TD. An element of the (L-R) core double vector bundle, its triple outline is



where $\overline{X} \in TB$ is in the core of the Right face.

Therefore, equation (2.89) shows how to pair an element $f \in T^*_d D$ with $\hat{0}_d \underset{TA}{+} \overline{d'}^A$:





Denote by $\kappa := \gamma^A_{C^*} \circ r_A(f)$. The outline of $r_A(f) \in D \ast A$ (the Front face of T^*D),

$$\begin{array}{ccc} r_A(f) & \longmapsto & \kappa \\ \downarrow & & \downarrow \\ a & \longmapsto & m, \end{array}$$

and again from (2.78), for any $c \in C_m$:

$$\langle \kappa, c \rangle = \langle r_A(f), 0^D_a \underset{B}{+} c \rangle. \tag{2.90}$$

Note the following. The element $c \in C_m$ lies in the ultracore C of TD: since $\kappa = \gamma^A_{C^*} \circ r_A(f) \in C^*$ is in the dual of the ultracore C of TD, we pair it with an element of the ultracore C of TD. As there are more than one copy of the vector bundle $C \to M$ in TD, it is important to state explicitly in which copy the element c belongs to.

Focus on the right hand side of (2.90): $r_A(f) \in D \ast A$, and $0^D_a \underset{B}{+} c$ is in the core of the Left face of TD (it plays the role of $d' \in D\Big|_a$ in the left hand side of (2.89)). That $0^D_a \underset{B}{+} c$ is in the core of the Left face of TD means that c is in the core of the (L-R) core double vector bundle, ergo, in the ultracore.

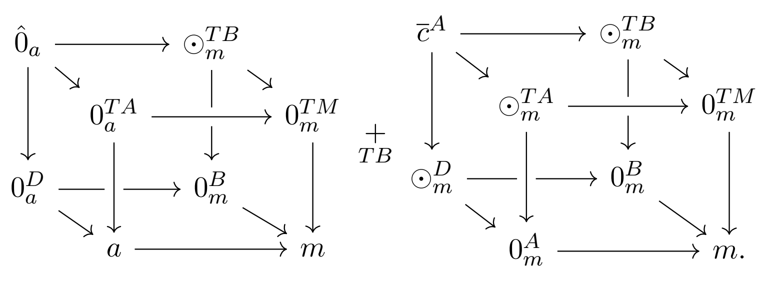
The image of $0^D_a \underset{B}{+} c$ in TD is from (1.9),

$$\overline{0^D_a \underset{B}{+} c}^A = \frac{d}{dt} t \underset{A}{\cdot} (0^D_a \underset{B}{+} c)\Big|_{t=0} = \frac{d}{dt}(0^D_a \underset{B}{+} t \underset{A}{\cdot} c)\Big|_{t=0} = \frac{d}{dt} 0^D_a\Big|_{t=0} \underset{TB}{+} \frac{d}{dt} t \underset{A}{\cdot} c\Big|_{t=0} = \hat{0}_a \underset{TB}{+} \overline{c}^A.$$

where we have denoted by $\overline{c}^A$ the image of the ultracore element in TD. Note that

$$\overline{c}^A = \frac{d}{dt} t \underset{A}{\cdot} c\Big|_{t=0} = \frac{d}{dt} t \underset{B}{\cdot} c\Big|_{t=0} = \overline{c}^B,$$

and their triple diagrams:



By (2.89), we obtain

$$\langle \kappa, c \rangle = \langle r_A(f), 0^D_a \underset{B}{+} c \rangle = \langle f, \hat{0}_d \underset{TA}{+} (\hat{0}_a \underset{TB}{+} \overline{c}^A) \rangle. \tag{2.91}$$

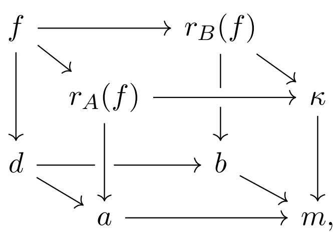
We do exactly the same for $r_B(f)$. In this case, we can write:

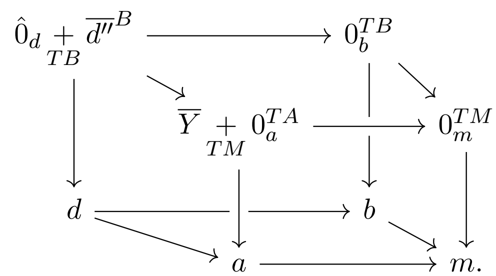
$$\langle r_B(f), d''\rangle = \langle f, \hat{0}_d \underset{TB}{+} \overline{d''}^B\rangle, \tag{2.92}$$

where d'' is now a core element of the Back face of TD. It belongs to the (B-F) core double vector bundle of TD, with outline:

$$\begin{array}{ccc} d'' & \longmapsto & Y \\ \downarrow & & \downarrow \\ b & \longmapsto & m, \end{array}$$

where $Y \in A$ is a core element of the Front face of TD. The triple outlines of the elements $f \in T^*_d D$ and $\hat{0}_d \underset{TB}{+} \overline{d''}^B$:

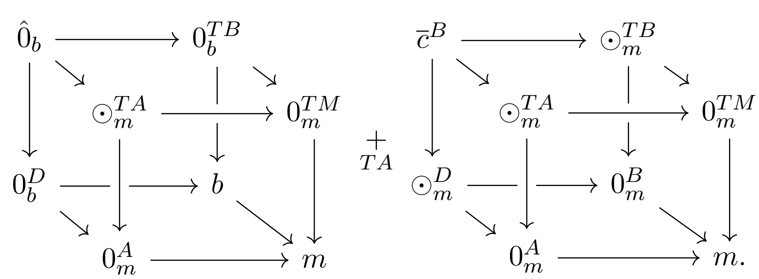




Denote now by $\kappa' := \gamma^B_{C^*} \circ r_B(f)$. Again by (2.78), for any $c' \in C_m$:

$$\langle \kappa', c'\rangle = \langle r_B(f), 0^D_b \underset{A}{+} c'\rangle,$$

and now as $0^D_b \underset{A}{+} c'$ plays the role of d'' in (2.92), it is an element of the (B-F) core double vector bundle of TD, and now c' in the core of the (B-F) core double vector bundle, an ultracore element. Choose the same $c \in C_m$ as we had chosen in (2.90), in the case of r_A. The image of $0^D_b \underset{A}{+} c'$ in TD is $\overline{0^D_b \underset{A}{+} c}^B$, and their triple diagrams:



Applying (2.92) for $d'' = 0^D_b \underset{A}{+} c$:

$$\langle \kappa', c\rangle = \langle r_B(f), 0^D_b \underset{A}{+} c\rangle = \langle f, \hat{0}_d \underset{TB}{+} (\hat{0}_b \underset{TA}{+} \overline{c}^B)\rangle. \tag{2.93}$$

In order to show that $\kappa = \kappa'$, by (2.91) and (2.93), it is enough to show that

$$\hat{0}_d \underset{TA}{+} (\hat{0}_a \underset{TB}{+} \bar{c}^A) = \hat{0}_d \underset{TB}{+} (\hat{0}_b \underset{TA}{+} \bar{c}^B),$$

and this follows directly using interchange laws:

$$\hat{0}_d \underset{TA}{+} (\hat{0}_a \underset{TB}{+} \bar{c}^A) = (\hat{0}_d \underset{TB}{+} \hat{0}_b) \underset{TA}{+} (\hat{0}_a \underset{TB}{+} \bar{c}^A) = (\hat{0}_d \underset{TA}{+} \hat{0}_a) \underset{TB}{+} (\hat{0}_b \underset{TA}{+} \bar{c}^A) = \hat{0}_d \underset{TB}{+} (\hat{0}_b \underset{TA}{+} \bar{c}^A),$$

and as we mentioned earlier, $\bar{c}^A = \bar{c}^B$.

Linearity of r_A

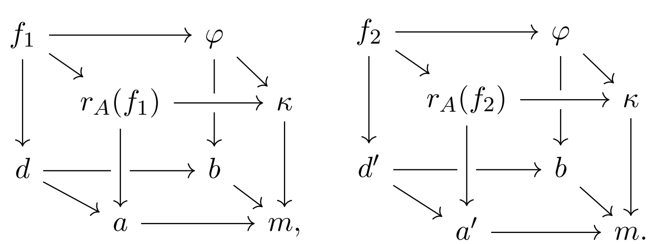
The Left face of T^*D is the dual double vector bundle of the Left face of TD, therefore, r_A is a morphism of vector bundles from $T^*D \to D$ to $D \ast A \to A$, i.e., for $f_1, f_2 \in T^*_d D$:

$$r_A(f_1 \underset{D}{+} f_2) = r_A(f_1) \underset{A}{+} r_A(f_2).$$

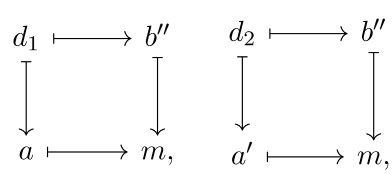
To check that r_A is a morphism of double vector bundles from the Back to the Front face of T^*D, we also need to check linearity over $D \ast B$, that is, assuming that $r_B(f_1) = r_B(f_2) = \varphi \in D \ast B$, does the following hold?

$$r_A(f_1 \underset{D\ast B}{+} f_2) = r_A(f_1) \underset{C^*}{+} r_A(f_2). \tag{2.94}$$

The outlines of f_1 and f_2:



The right hand side of (2.94): by (2.81), for $d_1, d_2 \in D$, with outlines:



we have

$$\langle r_A(f_1) \underset{C^*}{+} r_A(f_2), d_1 \underset{B}{+} d_2 \rangle = \langle r_A(f_1), d_1 \rangle + \langle r_A(f_2), d_2 \rangle. \tag{2.95}$$

Note that $b'' = q^D_B(d_1) = q^D_B(d_2)$ is not related to the $b = q^D_B(p_D(f_1)) = q^D_B(p_D(f_2))$.

About the left hand side of (2.94), we will use (2.89), with $d_1 \underset{B}{+} d_2$ in place of d'. So looking at $d_1 \underset{B}{+} d_2$ in the core of the Left face of TD, its image $\overline{d_1 \underset{B}{+} d_2}^A$ is in the (L-R) core double vector bundle with outline

$$\begin{array}{ccc}
\overline{d_1}^A \underset{TB}{+} \overline{d_2}^A & \longmapsto & b'' \\
\downarrow & & \downarrow \\
a + a' & \longmapsto & m,
\end{array}$$

where

$$\overline{d_1 \underset{B}{+} d_2}^A = \frac{d}{dt}\left(t \underset{A}{\cdot} (d_1 \underset{B}{+} d_2)\right)\Big|_{t=0} = \frac{d}{dt} t \underset{A}{\cdot} d_1 \Big|_{t=0} \underset{TB}{+} \frac{d}{dt} t \underset{A}{\cdot} d_2 \Big|_{t=0} = \overline{d_1}^A \underset{TB}{+} \overline{d_2}^A.$$

Applying now (2.89) for $f_1 \underset{D \ast B}{+} f_2 \in T^*_{d \underset{B}{+} d'} D$ and $r_A(f_1 \underset{D \ast B}{+} f_2) \in D \ast A \Big|_{a+a'}$, the left hand side of (2.94) can now be written,

$$\begin{aligned}
&\langle r_A(f_1 \underset{D \ast B}{+} f_2), d_1 \underset{B}{+} d_2 \rangle = \langle f_1 \underset{D \ast B}{+} f_2, \hat{0}_{d \underset{B}{+} d'} \underset{TA}{+} (\overline{d_1 \underset{B}{+} d_2})^A \rangle \\
&= \langle f_1 \underset{D \ast B}{+} f_2, (\hat{0}_d \underset{TB}{+} \hat{0}_{d'}) \underset{TA}{+} (\overline{d_1}^A \underset{TB}{+} \overline{d_2}^A) \rangle = \langle f_1 \underset{D \ast B}{+} f_2, (\hat{0}_d \underset{TA}{+} \overline{d_1}^A) \underset{TB}{+} (\hat{0}_{d'} \underset{TA}{+} \overline{d_2}^A) \rangle.
\end{aligned}$$

We need to describe the addition $\underset{D \ast B}{+}$ in $T^*D \to D \ast B$, using (2.81). This comes by the Back face of T^*D, the dual of the tangent double vector bundle of $D \to B$:

$$\begin{array}{ccc}
TD & \longrightarrow & TB \\
\downarrow & & \downarrow \\
D & \longrightarrow & B,
\end{array}
\quad \xrightarrow{\text{dualize over } D} \quad
\begin{array}{ccc}
T^*D & \longrightarrow & D \ast B \\
\downarrow & & \downarrow \\
D & \longrightarrow & B.
\end{array}$$

Therefore, the addition $f_1 \underset{D \ast B}{+} f_2$ is defined by pairing with elements $\xi_1, \xi_2 \in (TD; D, TB; TM)$ with outlines:

$$\begin{array}{ccc}
\xi_1 & \longmapsto & x \\
\downarrow & & \downarrow \\
d & \longmapsto & b,
\end{array}
\qquad
\begin{array}{ccc}
\xi_2 & \longmapsto & x \\
\downarrow & & \downarrow \\
d' & \longmapsto & b,
\end{array}$$

therefore, we can write:

$$\langle f_1 \underset{D \ast B}{+} f_2, \xi_1 \underset{TB}{+} \xi_2 \rangle = \langle f_1, \xi_1 \rangle + \langle f_2, \xi_2 \rangle.$$

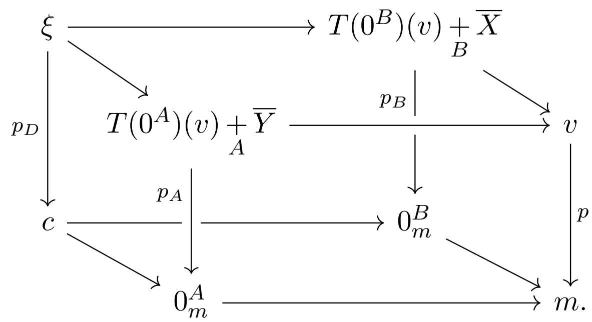
For $\xi_1 = \hat{0}_d \underset{TA}{+} \overline{d_1}^A$, and $\xi_2 = \hat{0}_{d'} \underset{TA}{+} \overline{d_2}^A$, we can rewrite the left hand side of (2.94),

$$\langle f_1, \hat{0}_d \underset{TA}{+} \overline{d_1}^A \rangle + \langle f_2, \hat{0}_{d'} \underset{TA}{+} \overline{d_2}^A \rangle \overset{(2.89)}{=} \langle r_A(f_1), d_1 \rangle + \langle r_A(f_2), d_2 \rangle,$$

and this is precisely (2.95), the right hand side of (2.94). The proof of the following can be found in [24, Proposition 5.4]. We present it here in detail.

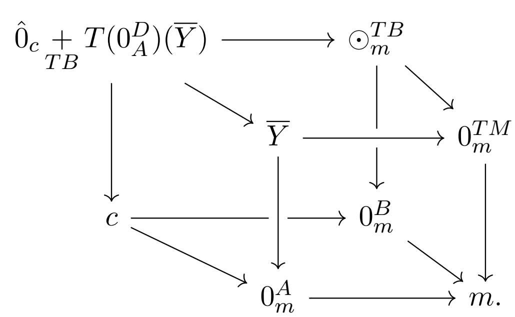
Proposition 2.4.8. *The core of the Up face of T^*D is $T^*C \to C^*$.*

Proof. Consider an $\omega \in T_c^*C$. To define its image $\widetilde{\omega}$ in T_c^*D, we first need to describe the elements $\xi \in T_cD$. The outline of such an element $\xi \in T_cD$ is

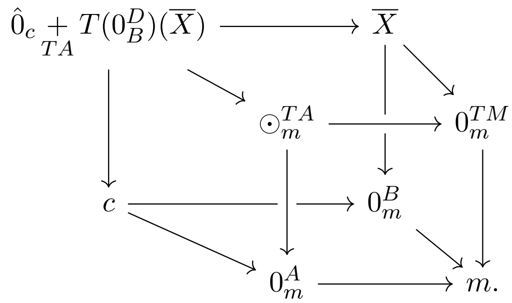


To see this, since $p_D(\xi) = c$, a core element of the Down face of TD, it follows that $q_A^D(c) = 0_m^A$ and $q_B^D(c) = 0_m^B$. Therefore, $p_A(T(q_A^D)(\xi)) = 0_m^A$, and $T(q_A)(T(q_A^D)(\xi)) = v \in T_mM$. From usual double vector bundle theory $T(q_A^D)(\xi) = T(0^A)(v) \underset{A}{+} \overline{Y}$, where $Y \in A$ lies in the core of the Front face of TD, and $\overline{Y}$ is its image in TA. Similarly for $T(q_B^D)(\xi) = T(0^B)(v) \underset{B}{+} \overline{X}$, where $X \in B$ is an element of the core of the Right face of TD.

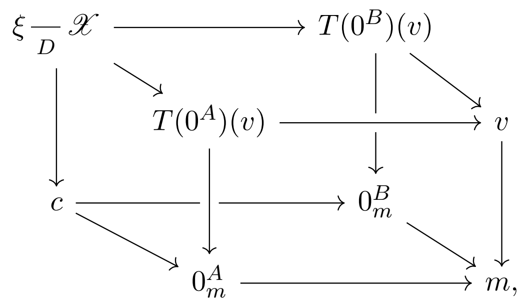
Consider the following cases, where $\xi = \hat{0}_c \underset{TB}{+} T(0_A^D)(\overline{Y})$. Then $v = 0_m^{TM}$ and $X = 0_m^B$, and ξ has outline:



Similarly, when $\xi = \hat{0}_c \underset{TA}{+} T(0^D_B)(\overline{X})$ then $v = 0^{TM}_m$ and $Y = 0^A_m$:



Denote by $\mathscr{X} = (\hat{0}_c \underset{TB}{+} T(0^D_A)(\overline{Y})) \underset{D}{+} (\hat{0}_c \underset{TA}{+} T(0^D_B)(\overline{X}))$. Then for any $\xi \in T_cD$, we see that $\xi \underset{D}{-} \mathscr{X}$ will have the following outline:



that is, $\xi \underset{D}{-} \mathscr{X}$ is in the (U-D) core double vector bundle of TD. Therefore, $\xi \underset{D}{-} \mathscr{X} = W$, for some $W \in T_cC$.

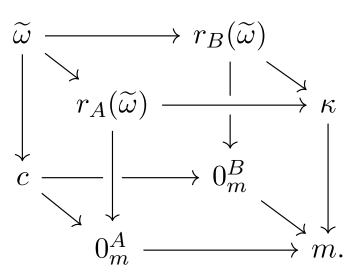
Define $\widetilde{\omega}$ in T^*_cD as follows. For any $\xi \in T_cD$:

$$\langle \widetilde{\omega}, \xi \rangle := \langle \omega, W \rangle, \tag{2.96}$$

where $W = \xi \underset{D}{-} \mathscr{X}$.

To show that $\widetilde{\omega}$ is in the core of the Up face of T^*D we need to show that $r_A(\widetilde{\omega}) = 0^{D \ast A}_\kappa$, and $r_B(\widetilde{\omega}) = 0^{D \ast B}_\kappa$.

So far, the outline of $\widetilde{\omega}$:



Using (2.89) we have:

$$\langle r_A(\widetilde{\omega}), d'\rangle = \langle \widetilde{\omega}, \hat{0}_c \underset{TA}{+} \overline{d'}^A\rangle,$$

for d' in the core of the Left face of TD, with outline in the (L-R) core double vector bundle:

$$\begin{array}{ccc} d' & \longmapsto & X \\ \Big\downarrow & & \Big\downarrow \\ 0_m^A & \longmapsto & m. \end{array}$$

We can write $d' = 0_X^D \underset{A}{+} c'$, for c' in the core of the (L-R) core double vector bundle of TD, i.e., an ultracore element. And we have

$$\overline{0_X^D \underset{A}{+} c'}^A = \hat{0}_{\overline{X}} \underset{TA}{+} \overline{c'}^A$$

Therefore, by (2.89) for $d' = 0_X^D \underset{A}{+} c'$:

$$\langle r_A(\widetilde{\omega}), 0_X^D \underset{A}{+} c'\rangle = \langle \widetilde{\omega}, \hat{0}_c \underset{TA}{+} (\hat{0}_{\overline{X}} \underset{TA}{+} \overline{c'}^A)\rangle = \langle \widetilde{\omega}, (\hat{0}_c \underset{TA}{+} \hat{0}_{\overline{X}}) \underset{D}{+} (\hat{0}_c \underset{TA}{+} \overline{c'}^A)\rangle.$$

The corresponding W defined here, noting that $\hat{0}_{\overline{X}} = T(0^B)(\overline{X})$, is

$$W = \left((\hat{0}_c \underset{TA}{+} T(0_B^D)(\overline{X})) \underset{D}{+} (\hat{0}_c \underset{TA}{+} \overline{c'}^A) \right) \underset{D}{-} (\hat{0}_c \underset{TA}{+} T(0_B^D)(\overline{X})) = (\hat{0}_c \underset{TA}{+} \overline{c'}^A),$$

hence $W = 0_c^{TC} \underset{TM}{+} c'$. Therefore, by (2.96) for $\xi, \mathscr{X}$, and W as just described:

$$\langle \widetilde{\omega}, (\hat{0}_c \underset{TA}{+} \hat{0}_{\overline{X}}) \underset{D}{+} (\hat{0}_c \underset{TA}{+} \overline{c'}^A)\rangle = \langle \omega, 0_c^{TC} \underset{TM}{+} c'\rangle.$$

Now using (2.78), the unfamiliar projection $r : T^*C \to C^*$, we have

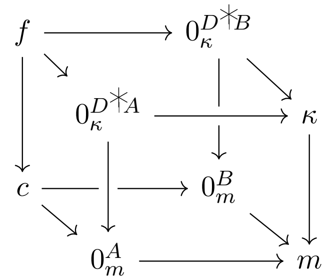
$$\langle \omega, 0_c^{TC} \underset{TM}{+} c'\rangle = \langle r(\omega), c'\rangle = \langle \kappa, c'\rangle.$$

In total,

$$\langle r_A(\widetilde{\omega}), 0_X^D \underset{A}{+} c'\rangle = \langle \kappa, c'\rangle \overset{(2.79)}{=} \langle 0_\kappa^{D \ast A}, 0_X^D \underset{A}{+} c'\rangle,$$

and this is true for every $c' \in C$ in the ultracore of TD. Therefore, $r_A(\widetilde{\omega}) = 0_\kappa^{D \ast A}$. And similarly for $r_B(\widetilde{\omega})$.

Conversely, take an element $f \in T_c^* D$ with outline:



We will show that $f = \widetilde{\omega}$, for some $\omega \in T_c^*C$.

By (2.89) we have:

$$\langle 0_\kappa^{D \ast A}, d' \rangle = \langle f, \hat{0}_c \underset{TA}{+} \overline{d'}^A \rangle,$$

for d' in the core of the Left face of TD with outline in the (L-R) core double vector bundle,

$$\begin{array}{ccc} d' & \longmapsto & X \\ \downarrow & & \downarrow \\ 0_m^A & \longmapsto & m \end{array}$$

therefore, as before we can write $d' = 0_X^D \underset{A}{+} c'$, for c' in the core of the (L-R) double vector bundle, i.e., in the ultracore. Hence:

$$\langle \kappa, c' \rangle \overset{(2.79)}{=} \langle 0_\kappa^{D \ast A}, 0_X^D \underset{A}{+} c' \rangle = \langle f, \hat{0}_c \underset{TA}{+} \overline{(0_X^D \underset{A}{+} c')}^A \rangle = \langle f, \hat{0}_c \underset{TA}{+} (T(0_B^D)(\overline{X}) \underset{TA}{+} \overline{c'}^A) \rangle. \tag{2.97}$$

Similarly for $r_B(f) = 0_\kappa^{D \ast B}$, use (2.92). The outline of d'' in (B-F) core double vector bundle of TD:

$$\begin{array}{ccc} d'' & \longmapsto & Y \\ \downarrow & & \downarrow \\ 0_m^B & \longmapsto & m \end{array}$$

and we can write $d'' = 0_Y^D \underset{B}{+} c''$, where again, c'' is an ultracore element (in the core of (B-F) core double vector bundle). Therefore,

$$\langle \kappa, c'' \rangle \overset{(2.79)}{=} \langle 0_\kappa^{D \ast B}, 0_Y^D \underset{B}{+} c'' \rangle = \langle f, \hat{0}_c \underset{TB}{+} \overline{(0_Y^D \underset{B}{+} c'')}^B \rangle = \langle f, \hat{0}_c \underset{TB}{+} (T(0_A^D)(\overline{Y}) \underset{TB}{+} \overline{c''}^B) \rangle. \tag{2.98}$$

By (2.97) and (2.98), we see that f vanishes on elements of type $\mathscr{X}$, since no ultracore elements appear in $\mathscr{X}$. Therefore, for any $\xi \in T_cD$:

$$\langle f, \xi \rangle = \langle f, \mathscr{X} \underset{D}{+} W \rangle = \langle f, \mathscr{X} \rangle + \langle f, W \rangle = \langle f, W \rangle.$$

Now define $\omega \in T_c^*C$ by

$$\omega(W) := \langle f, \xi \rangle,$$

where $\xi \in T_cD$, with $\xi = \mathscr{X} \underset{D}{+} W$. Now we need to check that $\widetilde{\omega} = f$. For any ξ, with $\xi = \mathscr{X} \underset{D}{+} W$, from the previous subsection, we extend ω as

$$\langle \widetilde{\omega}, \xi \rangle := \langle \omega, W \rangle,$$

so it follows directly that $\widetilde{\omega} = f$. □

Again one needs to check that T^*D satisfies parts (i), (ii), and (iii) of the definition of a triple vector bundle. Part (i) follows routinely. To see part (iii), start with a decomposition $\Omega : TD \to \overline{TD}$. Then the inverse of the following map

$$\Omega \ast D : \overline{TD} \ast D \to T^*D,$$

is a decomposition of T^*D. Finally, part (ii) follows as it did in the case of TD.

The three core double vector bundle of T^*D in the usual order, and the ultracore vector bundle:

(Back-Front)	(Left-Right)	(Up-Down)	Ultracore
$\begin{array}{ccc} T^*B & \longrightarrow & B^* \\ \downarrow & & \downarrow \\ B & \longrightarrow & M \end{array}$	$\begin{array}{ccc} T^*A & \longrightarrow & A^* \\ \downarrow & & \downarrow \\ A & \longrightarrow & M \end{array}$	$\begin{array}{ccc} T^*C & \longrightarrow & C \\ \downarrow & & \downarrow \\ C^* & \longrightarrow & M \end{array}$	$T^*M \to M$

Chapter 3

The warp-grid theorem

In this chapter, we formulate the main theorem of the thesis in the first section, and prove it in the second section using the techniques developed in Chapter 2.

3.1 The warp-grid theorem

3.1.1 Grids in triple vector bundles

A grid in a double vector bundle constitutes two linear sections. In a triple vector bundle the concept of grid requires what we call linear double sections. The following definition was first stated in [27, p.360].

Definition 3.1.1. A *down-up linear double section of E* is a collection of sections

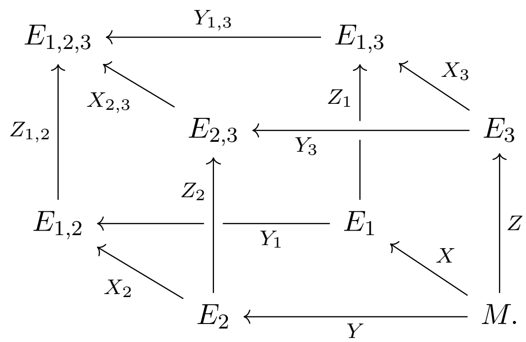
$$Z_{1,2} : E_{1,2} \to E_{1,2,3}, \quad Z_1 : E_1 \to E_{1,3}, \quad Z_2 : E_2 \to E_{2,3}, \quad Z : M \to E_3,$$

which form a morphism of double vector bundles from the Down face to the Up face.

The core morphism of $Z_{1,2}$ defines a vector bundle morphism from the core of the Down face to the core of the Up face. We denote this by $Z_{12} : E_{12} \to E_{12,3}$. It is a linear section over $Z : M \to E_3$.

In a similar fashion we define *right-left* and *front-back linear double sections* of E. We thus arrive to the following Definition, stated in [27].

Definition 3.1.2. A *grid on E* is a set of three linear double sections, one in each direction, as shown in (3.1).



(3.1)

Notation-wise, we write the linear double sections as:

$$(X_{2,3}; X_2, X_3; X),\ (Y_{1,3}; Y_1, Y_3; Y),\ (Z_{1,2}; Z_1, Z_2; Z).$$

By definition the linear double sections are double vector bundle morphisms, hence we have the following equations.

For $e_{2,3}$, $e'_{2,3}$ projecting to the same $e_2 \in E_2$,

$$X_{2,3}(e_{2,3} \underset{E_2}{-} e'_{2,3}) = X_{2,3}(e_{2,3}) \underset{1,2}{-} X_{2,3}(e'_{2,3}). \tag{3.2}$$

For $e_{2,3}$, $e'_{2,3}$ over $e_3 \in E_3$,

$$X_{2,3}(e_{2,3} \underset{E_3}{-} e'_{2,3}) = X_{2,3}(e_{2,3}) \underset{1,3}{-} X_{2,3}(e'_{2,3}). \tag{3.3}$$

For $e_{1,3}$, $e'_{1,3}$ over $e_1 \in E_1$,

$$Y_{1,3}(e_{1,3} \underset{E_1}{-} e'_{1,3}) = Y_{1,3}(e_{1,3}) \underset{1,2}{-} Y_{1,3}(e'_{1,3}). \tag{3.4}$$

For $e_{1,3}$, $e'_{1,3}$ over $e_3 \in E_3$,

$$Y_{1,3}(e_{1,3} \underset{E_3}{-} e'_{1,3}) = Y_{1,3}(e_{1,3}) \underset{2,3}{-} Y_{1,3}(e'_{1,3}). \tag{3.5}$$

For $e_{1,2}$, $e'_{1,2}$ over $e_1 \in E_1$,

$$Z_{1,2}(e_{1,2} \underset{E_1}{-} e'_{1,2}) = Z_{1,2}(e_{1,2}) \underset{1,3}{-} Z_{1,2}(e'_{1,2}). \tag{3.6}$$

For $e_{1,2}$, $e'_{1,2}$ over $e_2 \in E_2$,

$$Z_{1,2}(e_{1,2} \underset{E_2}{-} e'_{1,2}) = Z_{1,2}(e_{1,2}) \underset{2,3}{-} Z_{1,2}(e'_{1,2}). \tag{3.7}$$

Nontrivial grids on E

We now establish the existence of nontrivial grids on E.

A right-left linear double section $(Y_{1,3}; Y_1, Y_3; Y)$ is a double vector bundle morphism from the Right to the Left face of E. To define this map, we will use Proposition 1.1.2, decompositions of E, and the corresponding result from the double case.

Start with Y a section of $E_2 \to M$. Then from the double case, using (1.8), define the linear section (Y_1, Y)

$$\begin{array}{rcl} Y_1 : E_1 & \to & E_{1,2} \\ e_1 & \mapsto & \mho_{1,2}(Y(m), e_1, \varphi_1(e_1)), \end{array}$$

where $\varphi_1 : E_1 \to E_{12}$ is a vector bundle map over M, and $\mho_{1,2}$ is a decomposition of the Down face of E. Define the linear section (Y_3, Y)

$$\begin{array}{rcl} Y_3 : E_3 & \to & E_{2,3} \\ e_3 & \mapsto & \mho_{2,3}(Y(m), e_3, \varphi_3(e_3)), \end{array}$$

where $\varphi_3 : E_3 \to E_{23}$ is a vector bundle map over M, and $\mho_{2,3}$ is a decomposition of the Front face of E.

To define a linear double section $(Y_{1,3}; Y_1, Y_3; Y)$ on E, from Proposition 1.1.2, we can write:

$$\begin{array}{rcl} Y_{1,3} : E_{1,3} & \to & E \\ e_{1,3} & \mapsto & \widetilde{\mho}(Y_1(e_1), Y_3(e_3), e_{1,3}, \varphi(w_{13}) \underset{E_2}{+} \lambda(e_1, e_3)), \end{array}$$

where $\widetilde{\mho} : W \times E_{123} \to E$ is a decomposition of E. Initially, $\varphi : E_{13} \to E_{13,2}$ is a vector bundle map over $Y : M \to E_2$, and $\lambda : E_1 \times_M E_3 \to E_{13,2}$, a bilinear map. However, since $Y_{1,3}$ is a section of $E \xrightarrow{q_{1,3}} E_{1,3}$ it follows that $\varphi : E_{13} \to E_{123}$ is a vector bundle map over M, and the bilinear map $\lambda : E_1 \times_M E_3 \to E_{123}$. Hence,

$$\begin{array}{rcl} Y_{1,3} : E_{1,3} & \to & E \\ e_{1,3} & \mapsto & \widetilde{\mho}(Y_1(e_1), Y_3(e_3), e_{1,3}, \varphi(w_{13}) + \lambda(e_1, e_3)), \end{array}$$

We have also chosen a $\Sigma_{1,3} : E_1 \times_M E_3 \to E_{1,3}$, to write

$$w_{13} = (e_{1,3} \underset{E_1}{-} \Sigma_{1,3}(e_1, e_3)) \underset{E_3}{-} \tilde{0}^{1,3}_{e_1}.$$

For $e_{1,3}, f_{1,3}$ over $e_1 \in E_1$, denote by $w'_{13} = (f_{1,3} \underset{E_1}{-} \Sigma_{1,3}(e_1, f_3)) \underset{E_3}{-} \tilde{0}^{1,3}_{e_1}$.

$$
\begin{aligned}
& Y_{1,3}(e_{1,3}) \underset{1,2}{+} Y_{1,3}(f_{1,3}) \\
= \ & \widetilde{\mho}(Y_1(e_1), Y_3(e_3), e_{1,3}, \varphi(w_{13}) + \lambda(e_1, e_3)) \underset{1,2}{+} \widetilde{\mho}(Y_1(e_1), Y_3(f_3), f_{1,3}, \varphi(w'_{13}) + \lambda(e_1, f_3)) \\
= \ & \widetilde{\mho}(Y_1(e_1), Y_3(e_3) \underset{E_2}{+} Y_3(f_3), e_{1,3} \underset{E_1}{+} f_{1,3}, \varphi(w_{13}) + \varphi(w'_{13}) + \lambda(e_1, e_3) + \lambda(e_1, f_3)) \\
= \ & \widetilde{\mho}(Y_1(e_1), Y_3(e_3 + f_3), e_{1,3} \underset{E_1}{+} f_{1,3}, \varphi(w_{13} + w'_{13}) + \lambda(e_1, e_3 + f_3)) \\
= \ & Y_{1,3}(e_{1,3} \underset{E_1}{+} f_{1,3}),
\end{aligned}
$$

That $Y_3(e_3 + f_3) = Y_3(e_3) \underset{E_2}{+} Y_3(f_3)$, follows directly from the linearity of Y_3.

To see the core element in E_{13} of $e_{1,3} \underset{E_1}{+} f_{1,3}$ with respect to $\Sigma_{1,3}$,

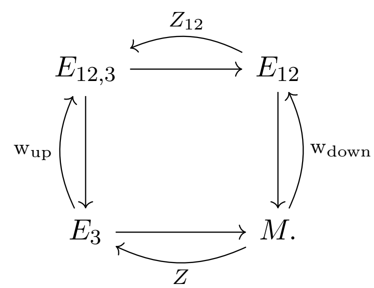
$$
\begin{aligned}
& \left((e_{1,3} \underset{E_1}{+} f_{1,3}) \underset{E_1}{-} \Sigma_{1,3}(e_1, e_3 + f_3) \right) \underset{E_3}{-} \tilde{0}^{1,3}_{e_1} \\
& \quad = \left((e_{1,3} \underset{E_1}{+} f_{1,3}) \underset{E_1}{-} (\Sigma_{1,3}(e_1, e_3) \underset{E_1}{+} \Sigma(e_1, f_3)) \right) \underset{E_3}{-} (\tilde{0}^{1,3}_{e_1} \underset{E_1}{+} \tilde{0}^{1,3}_{e_1}) \\
& = \left((e_{1,3} \underset{E_1}{-} \Sigma_{1,3}(e_1, e_3)) \underset{E_3}{-} \tilde{0}^{1,3}_{e_1} \right) \underset{E_1}{+} \left((f_{1,3} \underset{E_1}{-} \Sigma_{1,3}(e_1, f_3)) \underset{E_3}{-} \tilde{0}^{1,3}_{e_1} \right) = w_{13} \underset{E_1}{+} w'_{13}.
\end{aligned}
$$

For more details on grids on E using this technique, see [27].

3.1.2 Reformulation of the warp-grid theorem

In this subsection we first describe the original formulation of the warp-grid theorem, as stated in [27]. Introducing then a more succint notation, we work towards equation (3.23), a prototype of the kind of equations we will use in the second subsection to prove the warp-grid theorem.

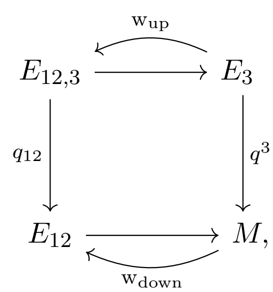
Start with a grid on E as in (3.1), and focus on the Up face of the triple vector bundle. Then $(Y_{1,3}, Y_3)$ and $(X_{2,3}, X_3)$ define a grid on the Up face. Denote its warp by w_{up}; this is a section of the core vector bundle of the Up face, $\mathrm{w}_{\mathrm{up}} : E_3 \to E_{12,3}$. Likewise, (Y_1, Y) and (X_2, X) define a grid on the Down face, and we denote its warp by $\mathrm{w}_{\mathrm{down}}$, a section of the core vector bundle of the Down face, $\mathrm{w}_{\mathrm{down}} : M \to E_{12}$. It follows that $(\mathrm{w}_{\mathrm{up}}, \mathrm{w}_{\mathrm{down}})$ is a linear section of the (U-D) core double vector bundle, see Proposition 3.1.3. In addition, the core morphism Z_{12} of the linear double section $Z_{1,2}$ defines another linear section of the (U-D) core double vector bundle. Therefore, we have the following induced grid on $E_{12,3}$:



We call the warp of this grid the *Up-Down ultrawarp* and denote it by u_{UD}. It is a section of the ultracore E_{123}.

Proposition 3.1.3. *The sections* w_{up} *and* w_{down} *as described earlier, form a linear section of the (U-D) core double vector bundle.*

Proof. That w_{up} and $\mathrm{w}_{\mathrm{down}}$ are sections of the corresponding vector bundle structures follows immediately from the definition of the warp. To show that $(\mathrm{w}_{\mathrm{up}}, \mathrm{w}_{\mathrm{down}})$ is a vector bundle morphism, we first check commutativity of the diagram:



where (q_{12}, q^3) is the core morphism of the projection map $(q_{1,2}; q_1^{1,3}, q_2^{2,3}; q^3)$ from the Up to the Down face of E, as in (2.27).

Using (11), for $e_3 \in E_3$,

$$Y_{1,3}(X_3(e_3)) \underset{1,3}{-} X_{2,3}(Y_3(e_3)) = \mathrm{w}_{\mathrm{up}}(e_3) \underset{2,3}{+} \hat{0}_{X_3(e_3)}. \tag{3.8}$$

Applying $q_{1,2}$ to both hand sides of the previous equation, writing $q^3(e_3) = m$, we obtain,

$$Y_1(X(m)) \underset{E_1}{-} X_2(Y(m)) = q_{1,2}(\mathrm{w}_{\mathrm{up}}(e_3)) \underset{E_2}{+} \tilde{0}^{1,2}_{X(m)}, \tag{3.9}$$

due to the following relations of the linear double sections,

$$q_{1,2} \circ Y_{1,3} = Y_1 \circ q_1^{1,3},\ q_{1,2} \circ X_{2,3} = X_2 \circ q_2^{2,3},\ q_1^{1,3} \circ X_3 = X \circ q^3,\ q_2^{2,3} \circ Y_3 = Y \circ q^3.$$

And (3.9) is precisely the equation that describes $\mathrm{w}_{\mathrm{down}}(m)$. From uniqueness of core elements, it follows that $q_{1,2}(\mathrm{w}_{\mathrm{up}}(e_3)) = \mathrm{w}_{\mathrm{down}}(m)$.

To check fibrewise linearity, we need to show that for $e_3, f_3 \in E_3$ over the same $m \in M$:

$$\mathrm{w}_{\mathrm{up}}(e_3 + f_3) = \mathrm{w}_{\mathrm{up}}(e_3) \underset{1,2}{+} \mathrm{w}_{\mathrm{up}}(f_3).$$

Rewrite (3.8) as,

$$\left(Y_{1,3}(X_3(e_3)) \underset{1,3}{-} X_{2,3}(Y_3(e_3))\right) \underset{2,3}{-} \hat{0}_{X_3(e_3)} = \mathrm{w}_{\mathrm{up}}(e_3) \underset{2,3}{+} \left(\hat{0}_{X_3(e_3)} \underset{2,3}{-} \hat{0}_{X_3(e_3)}\right) \overset{(2.54c)}{=} \mathrm{w}_{\mathrm{up}}(e_3) \underset{2,3}{+} \hat{0}_{e_3}.$$

Since $\hat{0}_{e_3}$ plays the role of the double zero of the Up face over e_3, it follows that $\mathrm{w}_{\mathrm{up}}(e_3) \underset{2,3}{+} \hat{0}_{e_3} = \mathrm{w}_{\mathrm{up}}(e_3)$.

Hence, (3.8) can now be stated as:

$$\mathrm{w}_{\mathrm{up}}(e_3) = \left(Y_{1,3}(X_3(e_3)) \underset{1,3}{-} X_{2,3}(Y_3(e_3))\right) \underset{2,3}{-} \hat{0}_{X_3(e_3)}. \tag{3.10}$$

Equation (3.10) for $e_3 + f_3 \in E_3$, with e_3, f_3 over the same $m \in M$:

$$\mathrm{w}_{\mathrm{up}}(e_3 + f_3) = \left(Y_{1,3}(X_3(e_3 + f_3)) \underset{1,3}{-} X_{2,3}(Y_3(e_3 + f_3))\right) \underset{2,3}{-} \hat{0}_{X_3(e_3+f_3)}. \tag{3.11}$$

Using the linearity of the various linear sections involved:

1. First, what is $\hat{0}_{X_3(e_3+f_3)}$? Since (X_3, X) is the following linear section of the Right face

$$\begin{array}{ccc} E_{1,3} & \underset{\overset{X_3}{\curvearrowleft}}{\longrightarrow} & E_3 \\ \downarrow & & \downarrow \\ E_1 & \overset{\longrightarrow}{\underset{X}{\curvearrowleft}} & M, \end{array}$$

   and we have $X_3(e_3 + f_3) = X_3(e_3) \underset{E_1}{+} X_3(f_3)$, by (2.10), we have that

$$\hat{0}_{X_3(e_3+f_3)} = \hat{0}_{X_3(e_3) \underset{E_1}{+} X_3(f_3)} = \hat{0}_{X_3(e_3)} \underset{1,2}{+} \hat{0}_{X_3(f_3)}.$$

2. Secondly, since both $X_3(e_3)$, and $X_3(f_3) \in E_{1,3}$ are over the same $X(m) \in E_1$, and $(Y_{1,3}, Y_1)$ is a linear section of the Back face:

$$\begin{array}{ccc} E & \underset{\overset{Y_{1,3}}{\curvearrowleft}}{\longrightarrow} & E_{1,3} \\ \downarrow & & \downarrow \\ E_{1,2} & \overset{\longrightarrow}{\underset{Y_1}{\curvearrowleft}} & E_1, \end{array}$$

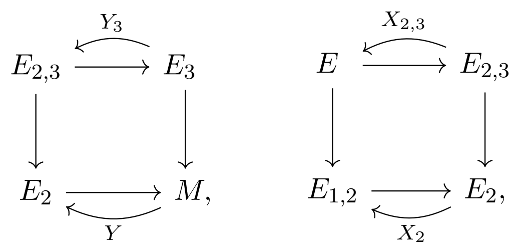
it follows that

$$Y_{1,3}(X_3(e_3+f_3)) = Y_{1,3}(X_3(e_3) \underset{E_1}{+} X_3(f_3)) = Y_{1,3}(X_3(e_3)) \underset{1,2}{+} Y_{1,3}(X_3(f_3)).$$

3. Similarly, since $Y_3(e_3)$, and $Y_3(f_3) \in E_{2,3}$ are over the same $Y(m) \in E_2$, it follows that

$$X_{2,3}(Y_3(e_3+f_3)) = X_{2,3}(Y_3(e_3) \underset{E_2}{+} Y_3(f_3)) = X_{2,3}(Y_3(e_3)) \underset{1,2}{+} X_{2,3}(Y_3(f_3)),$$

where we've used that $(X_{2,3}, X_2)$ is a linear section of the Left face:

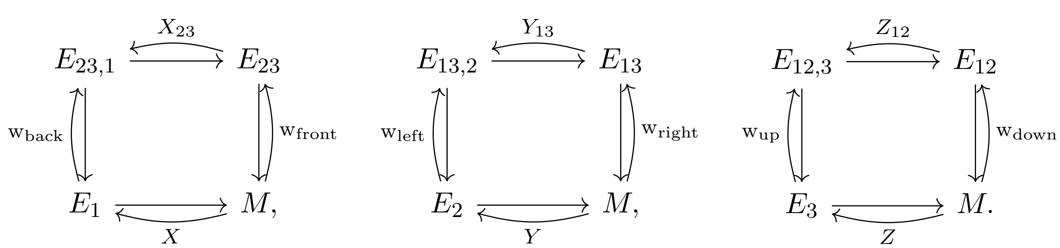


So now we can rewrite (3.11) as follows:

$$\begin{aligned}
&\mathrm{w}_{\mathrm{up}}(e_3+f_3)\\
&= \left(Y_{1,3}(X_3(e_3+f_3)) \underset{1,3}{-} X_{2,3}(Y_3(e_3+f_3))\right) \underset{2,3}{-} \hat{0}_{X_3(e_3+f_3)}\\
&= \left(\left[Y_{1,3}(X_3(e_3)) \underset{1,2}{+} Y_{1,3}(X_3(f_3))\right] \underset{1,3}{-} \left[X_{2,3}(Y_3(e_3)) \underset{1,2}{+} X_{2,3}(Y_3(f_3))\right]\right) \underset{2,3}{-} \left[\hat{0}_{X_3(e_3)} \underset{1,2}{+} \hat{0}_{X_3(f_3)}\right]\\
&= \left(\left[Y_{1,3}(X_3(e_3)) \underset{1,3}{-} X_{2,3}(Y_3(e_3))\right] \underset{1,2}{+} \left[Y_{1,3}(X_3(f_3)) \underset{1,3}{-} X_{2,3}(Y_3(f_3))\right]\right) \underset{2,3}{-} \left[\hat{0}_{X_3(e_3)} \underset{1,2}{+} \hat{0}_{X_3(f_3)}\right]\\
&= \left(\left[Y_{1,3}(X_3(e_3)) \underset{1,3}{-} X_{2,3}(Y_3(e_3))\right] \underset{2,3}{-} \hat{0}_{X_3(e_3)}\right) \underset{1,2}{+} \left(\left[Y_{1,3}(X_3(f_3)) \underset{1,3}{-} X_{2,3}(Y_3(f_3))\right] \underset{2,3}{-} \hat{0}_{X_3(f_3)}\right)\\
&= \mathrm{w}_{\mathrm{up}}(e_3) \underset{1,2}{+} \mathrm{w}_{\mathrm{up}}(f_3),
\end{aligned}$$

and this completes the proof. □

Of course we can build corresponding grids on the other two core double vector bundles. Therefore, a grid on E induces the following three ultrawarps,



(3.12)

Using the notation as in (12),

$$\begin{aligned} \mathrm{w}_{\mathrm{back}} \circ X - X_{23} \circ \mathrm{w}_{\mathrm{front}} &\triangleright \mathrm{u}_{\mathsf{BF}}, \\ \mathrm{w}_{\mathrm{left}} \circ Y - Y_{13} \circ \mathrm{w}_{\mathrm{right}} &\triangleright \mathrm{u}_{\mathsf{LR}}, \\ \mathrm{w}_{\mathrm{up}} \circ Z - Z_{12} \circ \mathrm{w}_{\mathrm{down}} &\triangleright \mathrm{u}_{\mathsf{UD}}. \end{aligned} \tag{3.13}$$

Note that the orientation we take in (3.13) is opposite to the one we take in (12). We explain this in Remark 3.1.5.

We can now state the main theorem about grids in triple vector bundles.

Theorem 3.1.4 (Warp-Grid Theorem)**.** *Given a triple vector bundle E and a grid in E as in* (3.1)*,*

$$\mathrm{u}_{\mathsf{BF}} + \mathrm{u}_{\mathsf{LR}} + \mathrm{u}_{\mathsf{UD}} = 0. \tag{3.14}$$

To give an intrinsic proof, we need to describe the ultrawarps in an alternative way. Focus on the ultrawarp u_{UD}. From the grid on the (U-D) core double vector bundle, for $m \in M$, by (11) we have that

$$(\mathrm{w}_{\mathrm{up}} \circ Z)(m) \underset{2,3}{-} (Z_{12} \circ \mathrm{w}_{\mathrm{down}})(m) = \hat{0}_{Z(m)} \underset{1,2}{+} \mathrm{u}_{\mathsf{UD}}(m). \tag{3.15}$$

How can we express $(\mathrm{w}_{\mathrm{up}} \circ Z)(m)$ and $(Z_{12} \circ \mathrm{w}_{\mathrm{down}})(m)$ in a more useful way? We have already written (3.8) for w_{up}, for any $e_3 \in E_3$,

$$Y_{1,3}(X_3(e_3)) \underset{1,3}{-} X_{2,3}(Y_3(e_3)) = \hat{0}_{X_3(e_3)} \underset{2,3}{+} \mathrm{w}_{\mathrm{up}}(e_3). \tag{3.16}$$

Putting $e_3 = Z(m)$, we have

$$Y_{1,3}(X_3(Z(m))) \underset{1,3}{-} X_{2,3}(Y_3(Z(m))) = \hat{0}_{X_3(Z(m))} \underset{2,3}{+} \mathrm{w}_{\mathrm{up}}(Z(m)). \tag{3.17}$$

We introduce a more succint notation, for use in calculations.

$$\begin{aligned} &\mathsf{ZYX} = Z_{1,2}(Y_1(X(m))),\ \mathsf{YZX} = Y_{1,3}(Z_1(X(m))),\ \mathsf{XZY} = X_{2,3}(Z_2(Y(m))), \\ &\mathsf{ZXY} = Z_{1,2}(X_2(Y(m))),\ \mathsf{YXZ} = Y_{1,3}(X_3(Z(m))),\ \mathsf{XYZ} = X_{2,3}(Y_3(Z(m))). \end{aligned} \tag{3.18}$$

Now (3.17) becomes

$$\mathsf{YXZ} \underset{1,3}{-} \mathsf{XYZ} = \hat{0}_{e'_{1,3}} \underset{2,3}{+} \lambda_3, \tag{3.19}$$

where $e'_{1,3} = X_3(Z(m))$ and $\lambda_3 = \mathrm{w}_{\mathrm{up}}(Z(m))$.

In the proof of Proposition 3.1.3, we rewrote (3.16) as equation (3.10). In a similar fashion, we rewrite (3.19) as

$$\lambda_3 = (\mathsf{YXZ} \underset{1,3}{-} \mathsf{XYZ}) \underset{2,3}{-} \hat{0}_{e'_{1,3}}. \tag{3.20}$$

About $(Z_{1,2} \circ \mathrm{w}_{\mathrm{down}})(m)$, first write $\mathrm{w}_{\mathrm{down}}(m)$ out using (11) as

$$Y_1(X(m)) \underset{E_1}{-} X_2(Y(m)) = \widetilde{0}^{1,2}_{X(m)} \underset{E_2}{+} \mathrm{w}_{\mathrm{down}}(m).$$

Apply $Z_{1,2}$ to this, and using (3.6) and (3.7), it follows that

$$Z_{1,2}(Y_1(X(m))) \underset{1,3}{-} Z_{1,2}(X_2(Y(m))) = \hat{0}_{Z_1(X(m))} \underset{2,3}{+} Z_{12}(\mathrm{w}_{\mathrm{down}}(m))$$

Again, for reasons of economy of space, rewrite this as

$$\mathsf{ZYX} \underset{1,3}{-} \mathsf{ZXY} = \hat{0}_{e_{1,3}} \underset{2,3}{+} k_3,$$

where $e_{1,3} = Z_1(X(m))$ and $k_3 = Z_{12}(\mathrm{w}_{\mathrm{down}}(m))$. Alternatively, as we did for λ_3,

$$k_3 = (\mathsf{ZYX} \underset{1,3}{-} \mathsf{ZXY}) \underset{2,3}{-} \hat{0}_{e_{1,3}}. \tag{3.21}$$

Let us go back to (3.15). We can rewrite this as

$$\lambda_3 \underset{2,3}{-} k_3 = \hat{0}_{e_3} \underset{1,2}{+} \mathrm{u}_{\mathsf{UD}}(m),$$

and using (3.20) and (3.21), we have that

$$\left((\mathsf{YXZ} \underset{1,3}{-} \mathsf{XYZ}) \underset{2,3}{-} \hat{0}_{e'_{1,3}}\right) \underset{2,3}{-} \left((\mathsf{ZYX} \underset{1,3}{-} \mathsf{ZXY}) \underset{2,3}{-} \hat{0}_{e_{1,3}}\right) = \hat{0}_{e_3} \underset{1,2}{+} \mathrm{u}_{\mathsf{UD}}(m) \tag{3.22}$$

or, more elegantly, using interchange laws,

$$\begin{aligned}(\mathsf{YXZ} \underset{1,3}{-} \mathsf{XYZ}) \underset{2,3}{-} (\mathsf{ZYX} \underset{1,3}{-} \mathsf{ZXY}) &= (\hat{0}_{e'_{1,3}} \underset{2,3}{+} \lambda_3) \underset{2,3}{-} (\hat{0}_{e_{1,3}} \underset{2,3}{+} k_3) \\ &= (\hat{0}_{e'_{1,3}} \underset{2,3}{-} \hat{0}_{e_{1,3}}) \underset{2,3}{+} (\lambda_3 \underset{2,3}{-} k_3) = (\hat{0}_{e'_{1,3}} \underset{2,3}{-} \hat{0}_{e_{1,3}}) \underset{2,3}{+} (\hat{0}_{e_3} \underset{1,2}{+} \mathrm{u}_{\mathsf{UD}}(m)).\end{aligned} \tag{3.23}$$

In calculations it is generally preferable to use equations of the form (3.19), and to avoid equations of the form (3.20).

Therefore, in order to describe ultrawarps such as $\mathrm{u}_{\mathsf{UD}}(m)$, we will use equations of the form (3.23) and we will often use the abbreviated notation

$$(\mathsf{YXZ} - \mathsf{XYZ}) - (\mathsf{ZYX} - \mathsf{ZXY}) \triangleright \mathrm{u}_{\mathsf{UD}}(m),$$

as introduced in (12).

It is worth emphasizing that the above arguments rely on the fact that core and ultra-core elements are uniquely determined by equations such as (11).

There are similar abbreviated equations for the other two ultrawarps. Altogether we have

$$(\mathsf{ZYX} - \mathsf{YZX}) - (\mathsf{XZY} - \mathsf{XYZ}) \triangleright \mathrm{u}_{\mathsf{BF}}(m), \tag{3.24a}$$
$$(\mathsf{XZY} - \mathsf{ZXY}) - (\mathsf{YXZ} - \mathsf{YZX}) \triangleright \mathrm{u}_{\mathsf{LR}}(m), \tag{3.24b}$$
$$(\mathsf{YXZ} - \mathsf{XYZ}) - (\mathsf{ZYX} - \mathsf{ZXY}) \triangleright \mathrm{u}_{\mathsf{UD}}(m), \tag{3.24c}$$

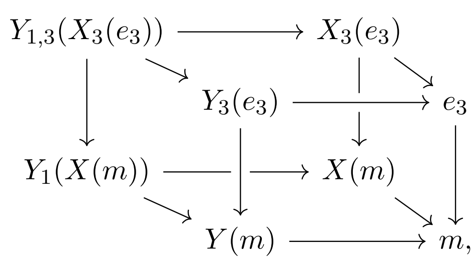
and from now on we will use a further shortening of the notation

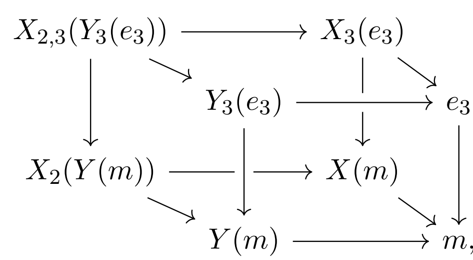
$$\mathrm{u}_{\mathsf{BF}}(m) = u_1, \qquad \mathrm{u}_{\mathsf{LR}}(m) = u_2, \qquad \mathrm{u}_{\mathsf{UD}}(m) = u_3.$$

The main difficulty in proving (3.14) is that we cannot simply add and subtract the expressions in (3.24), since the operations are over different vector bundle structures. The apparatus of the next section overcomes this difficulty.

Remark 3.1.5. A further problem arises from the fact that the warp of a grid in a double vector bundle is only defined up to sign. We now need to consider how to choose these signs consistently for a grid in a triple vector bundle. This is a question of fixing the *orientations of the grids.*

First, observe that the orientations of the grids on the upper faces determine the orientations of the grids on the corresponding lower faces. For example, asssume that the orientation of a grid on the Up face is as in (3.16). The triple outlines of the elements $Y_{1,3}(X_3(e_3))$ and $X_{2,3}(Y_3(e_3))$ are the following





(at this point we forget that we have three linear double sections on E, we are only interested on the grids on the Up and on the Down faces). Then we see that

$$Y_{1,3}(X_3(e_3)) \underset{1,3}{-} X_{2,3}(Y_3(e_3))$$

projects to

$$Y_1(X(m)) \underset{E_3}{-} X_2(Y(m)) \;\in\; E_{1,2}.$$

For this reason, we orient the corresponding lower faces so that the positive term in the warp defines the inward normal. We choose to orient each upper face so that the positive term in the formula for the warp defines the outward normal by the right-hand rule. In total, given a grid on E as in (3.1), the orientation of the grid on each face is the following:

- Back face: $Z_{1,2} \circ Y_1 - Y_{1,3} \circ Z_1$, Front face: $Z_2 \circ Y - Y_3 \circ Z$,
- Left face: $X_{2,3} \circ Z_2 - Z_{1,2} \circ X_2$, Right face: $X_3 \circ Z - Z_1 \circ X$,
- Up face: $Y_{1,3} \circ X_3 - X_{2,3} \circ Y_3$, Down face: $Y_1 \circ X - X_2 \circ Y$.

Thus we see that the orientation of the grid on the Up face determines the signs in the first subtraction in (3.24c), and the orientation of the Down face determines the signs in the second subtraction.

The "middle subtractions" in (3.24), that is, the orientations of the core double vector bundles, is an independent choice, equivalent to the choice of signs in (3.13). What matters here is consistency: if we took all three ultrawarps with the opposite signs, that would be fine.

We further explain the orientation of a grid and the meaning of the theorem at the end of this Chapter, see Remark 3.2.1. △

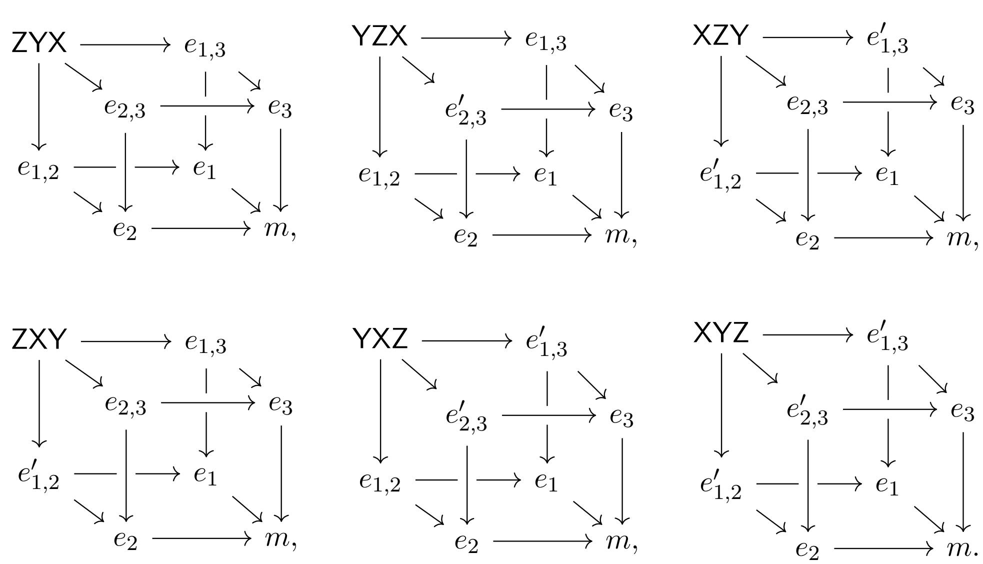
3.2 Proof of the theorem

3.2.1 Notation

In this section we prove Theorem 3.1.4. We will use the notation of (3.18). We further simplify the notation for elements of the lower faces and edges, as follows

$$\begin{aligned} X(m) &:= e_1, & Y(m) &:= e_2, & Z(m) &:= e_3, \\ Z_1(X(m)) &:= e_{1,3}, & X_3(Z(m)) &:= e'_{1,3}, & Z_2(Y(m)) &:= e_{2,3}, \\ Y_3(Z(m)) &:= e'_{2,3}, & Y_1(X(m)) &:= e_{1,2}, & X_2(Y(m)) &:= e'_{1,2}. \end{aligned}$$

The outlines of the elements in (3.18) are now written as follows



We will need the following relations for the core elements of the lower faces in detailed form.

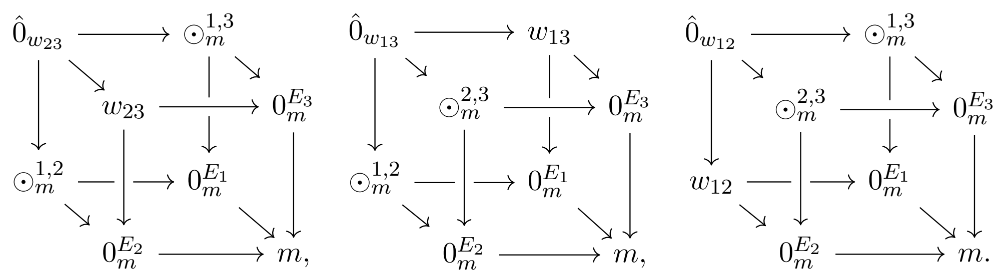
$$e_{2,3} \underset{E_2}{-} e'_{2,3} = \tilde{0}^{2,3}_{e_2} \underset{E_3}{+} w_{23}, \quad e_{2,3} \underset{E_3}{-} e'_{2,3} = \tilde{0}^{2,3}_{e_3} \underset{E_2}{+} w_{23}, \tag{3.25}$$

$$e'_{1,3} \underset{E_1}{-} e_{1,3} = \tilde{0}^{1,3}_{e_1} \underset{E_3}{+} w_{13}, \quad e'_{1,3} \underset{E_3}{-} e_{1,3} = \tilde{0}^{1,3}_{e_3} \underset{E_1}{+} w_{13}, \tag{3.26}$$

$$e_{1,2} \underset{E_1}{-} e'_{1,2} = \tilde{0}^{1,2}_{e_1} \underset{E_2}{+} w_{12}, \quad e_{1,2} \underset{E_2}{-} e'_{1,2} = \tilde{0}^{1,2}_{e_2} \underset{E_1}{+} w_{12}, \tag{3.27}$$

where $w_{23} \in E_{23}$, $w_{13} \in E_{13}$ and $w_{12} \in E_{12}$.

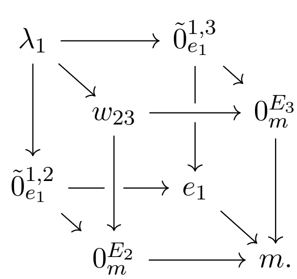
For the zeros of these w elements, the diagrams are



3.2.2 Core and ultracore elements arising from the grid

We collect here for reference the definitions and outlines of the core and ultracore elements arising from the grid.

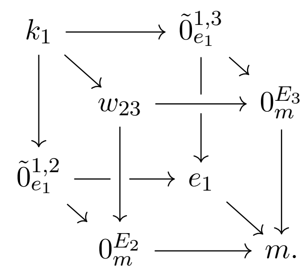
• **λ_1, k_1 and u_1.** The elements ZYX and YZX have the same Right and Back faces, and so their differences define an element $\lambda_1 \in E_{23,1}$ with outline



Using (2.35) the defining equations are

$$\mathsf{ZYX} \underset{1,2}{-} \mathsf{YZX} = \hat{0}_{e_{1,2}} \underset{1,3}{+} \lambda_1, \qquad \mathsf{ZYX} \underset{1,3}{-} \mathsf{YZX} = \hat{0}_{e_{1,3}} \underset{1,2}{+} \lambda_1. \tag{3.28}$$

If we look at XZY and XYZ, we see that they also have two faces in common, and their differences define a $k_1 \in E_{23,1}$, with outline



The two differences are, again using (2.35),

$$\mathsf{XZY} \underset{1,2}{-} \mathsf{XYZ} = \hat{0}_{e'_{1,2}} \underset{1,3}{+} k_1, \qquad \mathsf{XZY} \underset{1,3}{-} \mathsf{XYZ} = \hat{0}_{e'_{1,3}} \underset{1,2}{+} k_1. \tag{3.29}$$

We see that λ_1 and k_1 have the same outlines so they differ by an ultracore element $u_1 \in E_{123}$. By Subsection (2.3.2), "Special case: when e and e' are in a core double vector bundle", we have that:

$$\lambda_1 \underset{1,3}{-} k_1 = \hat{0}_{e_1} \underset{2,3}{+} u_1, \tag{3.30a}$$

$$\lambda_1 \underset{1,2}{-} k_1 = \hat{0}_{e_1} \underset{2,3}{+} u_1, \tag{3.30b}$$

$$\lambda_1 \underset{2,3}{-} k_1 = \hat{0}_{w_{23}} \underset{1,3}{+} u_1 = \hat{0}_{w_{23}} \underset{1,2}{+} u_1. \tag{3.30c}$$

There are four ways of describing the ultrawarp u_1. The full calculations are presented in detail in Appendix A.1.3.

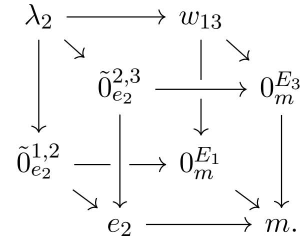
$$(\mathsf{ZYX} \underset{1,2}{-} \mathsf{YZX}) \underset{1,3}{-} (\mathsf{XZY} \underset{1,2}{-} \mathsf{XYZ}) = \hat{0}_{e_1} \underset{2,3}{+} (\hat{0}_{w_{12}} \underset{1,3/2,3}{+} u_1), \tag{3.31a}$$

$$(\mathsf{ZYX} \underset{1,2}{-} \mathsf{YZX}) \underset{2,3}{-} (\mathsf{XZY} \underset{1,2}{-} \mathsf{XYZ}) = (\hat{0}_{w_{12}} \underset{1,3}{+} \hat{0}_{w_{23}}) \underset{1,3}{+} (\hat{0}_{e_2} \underset{1,3}{+} u_1), \tag{3.31b}$$

$$(\mathsf{ZYX} \underset{1,3}{-} \mathsf{YZX}) \underset{1,2}{-} (\mathsf{XZY} \underset{1,3}{-} \mathsf{XYZ}) = \hat{0}_{e_1} \underset{2,3}{+} (\hat{0}_{-w_{13}} \underset{1,2/2,3}{+} u_1), \tag{3.31c}$$

$$(\mathsf{ZYX} \underset{1,3}{-} \mathsf{YZX}) \underset{2,3}{-} (\mathsf{XZY} \underset{1,3}{-} \mathsf{XYZ}) = (\hat{0}_{-w_{13}} \underset{1,2}{+} \hat{0}_{w_{23}}) \underset{1,2}{+} (\hat{0}_{e_3} \underset{1,2}{+} u_1). \tag{3.31d}$$

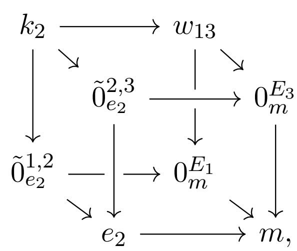
• **λ_2, k_2 and u_2.** The same procedure can be applied to XZY and ZXY; they have the same Front and Down faces, so their differences will define an element $\lambda_2 \in E_{13,2}$



The corresponding equations, using (2.39), are

$$\mathsf{XZY} \underset{1,2}{-} \mathsf{ZXY} = \hat{0}_{e'_{1,2}} \underset{2,3}{+} \lambda_2, \qquad \mathsf{XZY} \underset{2,3}{-} \mathsf{ZXY} = \hat{0}_{e_{2,3}} \underset{1,2}{+} \lambda_2. \tag{3.32}$$

If we look at YXZ and YZX, their differences define a $k_2 \in E_{13,2}$, with outline



and the differences defined are, due to (2.39),

$$\mathsf{YXZ} \underset{1,2}{-} \mathsf{YZX} = \hat{0}_{e_{1,2}} \underset{2,3}{+} k_2, \qquad \mathsf{YXZ} \underset{2,3}{-} \mathsf{YZX} = \hat{0}_{e'_{2,3}} \underset{1,2}{+} k_2. \tag{3.33}$$

Since λ_2 and k_2 have the same outlines, they differ by an ultracore element $u_2 \in E_{123}$,

$$\lambda_2 \underset{1,3}{-} k_2 = \hat{0}_{w_{13}} \underset{1,2/2,3}{+} u_2, \tag{3.34a}$$

$$\lambda_2 \underset{1,2}{-} k_2 = \hat{0}_{e_2} \underset{1,3}{+} u_2, \tag{3.34b}$$

$$\lambda_2 \underset{2,3}{-} k_2 = \hat{0}_{e_2} \underset{1,3}{+} u_2. \tag{3.34c}$$

Again there are four ways of describing the ultrawarp u_2, and relevant calculations are in Appendix A.1.4.

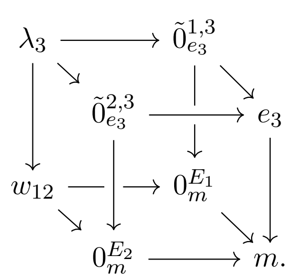
$$(\mathsf{XZY} \underset{1,2}{-} \mathsf{ZXY}) \underset{2,3}{-} (\mathsf{YXZ} \underset{1,2}{-} \mathsf{YZX}) = \hat{0}_{e_2} \underset{1,3}{+} (\hat{0}_{-w_{12}} \underset{1,3/2,3}{+} u_2), \tag{3.35a}$$

$$(\mathsf{XZY} \underset{1,2}{-} \mathsf{ZXY}) \underset{1,3}{-} (\mathsf{YXZ} \underset{1,2}{-} \mathsf{YZX}) = (\hat{0}_{w_{13}} \underset{2,3}{+} \hat{0}_{-w_{12}}) \underset{2,3}{+} (\hat{0}_{e_1} \underset{2,3}{+} u_2), \tag{3.35b}$$

$$(\mathsf{XZY} \underset{2,3}{-} \mathsf{ZXY}) \underset{1,2}{-} (\mathsf{YXZ} \underset{2,3}{-} \mathsf{YZX}) = \hat{0}_{e_2} \underset{1,3}{+} (\hat{0}_{w_{23}} \underset{1,2/1,3}{+} u_2), \tag{3.35c}$$

$$(\mathsf{XZY} \underset{2,3}{-} \mathsf{ZXY}) \underset{1,3}{-} (\mathsf{YXZ} \underset{2,3}{-} \mathsf{YZX}) = (\hat{0}_{w_{23}} \underset{1,2}{+} \hat{0}_{w_{13}}) \underset{1,2}{+} (\hat{0}_{e_3} \underset{1,2}{+} u_2). \tag{3.35d}$$

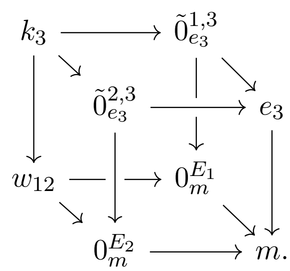
- **λ_3, k_3 and u_3.** Likewise YXZ and XYZ define $\lambda_3 \in E_{12,3}$ with outline



The corresponding relations are, due to (2.43),

$$\mathsf{YXZ} \underset{1,3}{-} \mathsf{XYZ} = \hat{0}_{e'_{1,3}} \underset{2,3}{+} \lambda_3, \qquad \mathsf{YXZ} \underset{2,3}{-} \mathsf{XYZ} = \hat{0}_{e'_{2,3}} \underset{1,3}{+} \lambda_3. \tag{3.36}$$

Likewise ZYX and ZXY define a $k_3 \in E_{12,3}$ with outline



The differences defined are, due to (2.43),

$$\mathsf{ZYX} \underset{1,3}{-} \mathsf{ZXY} = \hat{0}_{e_{1,3}} \underset{2,3}{+} k_3, \qquad \mathsf{ZYX} \underset{2,3}{-} \mathsf{ZXY} = \hat{0}_{e_{2,3}} \underset{1,3}{+} k_3. \tag{3.37}$$

The ultracore element $u_3 \in E_{123}$ defined by λ_3 and k_3 satisfies

$$\lambda_3 \underset{1,3}{-} k_3 = \hat{0}_{e_3} \underset{1,2}{+} u_3, \tag{3.38a}$$

$$\lambda_3 \underset{1,2}{-} k_3 = \hat{0}_{w_{12}} \underset{1,3/2,3}{+} u_3, \tag{3.38b}$$

$$\lambda_3 \underset{2,3}{-} k_3 = \hat{0}_{e_3} \underset{1,2}{+} u_3. \tag{3.38c}$$

The four relations in this case are the following, where the details can be found in Appendix A.1.5.

$$(\mathsf{YXZ} \underset{1,3}{-} \mathsf{XYZ}) \underset{2,3}{-} (\mathsf{ZYX} \underset{1,3}{-} \mathsf{ZXY}) = \hat{0}_{e_3} \underset{1,2}{+} (\hat{0}_{w_{13}} \underset{1,2/2,3}{+} u_3), \tag{3.39a}$$

$$(\mathsf{YXZ} \underset{1,3}{-} \mathsf{XYZ}) \underset{1,2}{-} (\mathsf{ZYX} \underset{1,3}{-} \mathsf{ZXY}) = (\hat{0}_{w_{13}} \underset{2,3}{+} \hat{0}_{w_{12}}) \underset{2,3}{+} (\hat{0}_{e_1} \underset{2,3}{+} u_3), \tag{3.39b}$$

$$(\mathsf{YXZ} \underset{2,3}{-} \mathsf{XYZ}) \underset{1,3}{-} (\mathsf{ZYX} \underset{2,3}{-} \mathsf{ZXY}) = \hat{0}_{e_3} \underset{1,2}{+} (\hat{0}_{-w_{23}} \underset{1,2/1,3}{+} u_3), \tag{3.39c}$$

$$(\mathsf{YXZ} \underset{2,3}{-} \mathsf{XYZ}) \underset{1,2}{-} (\mathsf{ZYX} \underset{2,3}{-} \mathsf{ZXY}) = (\hat{0}_{-w_{23}} \underset{1,3}{+} \hat{0}_{w_{12}}) \underset{1,3}{+} (\hat{0}_{e_2} \underset{1,3}{+} u_3). \tag{3.39d}$$

3.2.3 Proof of the warp-grid theorem

We will show that $u_1 + u_2 + u_3 = \odot^3_m$ by showing that $u_1 = -u_2 - u_3$. There are five steps.

Step 1. Rewrite (3.31b)

$$(\mathsf{ZYX} \underset{1,2}{-} \mathsf{YZX}) \underset{2,3}{-} (\mathsf{XZY} \underset{1,2}{-} \mathsf{XYZ})$$

as

$$(\mathsf{ZYX} \underset{2,3}{—} \mathsf{XZY}) \underset{1,2}{—} (\mathsf{YZX} \underset{2,3}{—} \mathsf{XYZ}),$$

using a variation of the double vector bundle interchange law in the Left face, as in (2.8). We know from (3.31b) that the ultracore element defined by the first expression is u_1, therefore, the ultracore element of the latter expression will also be u_1. We will show that the second expression has $-u_2 - u_3$ as its ultracore element, and this will show that $u_1 = -u_2 - u_3$.

Step 2. First, using (2.49), write $\mathsf{ZYX} \underset{2,3}{—} \mathsf{XZY}$ as

$$\mathsf{ZYX} \underset{2,3}{—} \mathsf{XZY} = (\mathsf{ZYX} \underset{2,3}{—} \mathsf{ZXY}) \underset{2,3}{—} (\mathsf{XZY} \underset{2,3}{—} \mathsf{ZXY}),$$

where we have from (3.37) and from (3.32),

$$\mathsf{ZYX} \underset{2,3}{—} \mathsf{ZXY} = \hat{0}_{e_{2,3}} \underset{1,3}{+} k_3, \qquad \mathsf{XZY} \underset{2,3}{—} \mathsf{ZXY} = \hat{0}_{e_{2,3}} \underset{1,2}{+} \lambda_2. \tag{3.40}$$

Step 3. Similarly, write $\mathsf{YZX} \underset{2,3}{—} \mathsf{XYZ}$ as

$$\mathsf{YZX} \underset{2,3}{—} \mathsf{XYZ} = (\mathsf{YXZ} \underset{2,3}{—} \mathsf{XYZ}) \underset{2,3}{—} (\mathsf{YXZ} \underset{2,3}{—} \mathsf{YZX}),$$

and we have

$$\mathsf{YXZ} \underset{2,3}{—} \mathsf{XYZ} = \hat{0}_{e'_{2,3}} \underset{1,3}{+} \lambda_3, \qquad \mathsf{YXZ} \underset{2,3}{—} \mathsf{YZX} = \hat{0}_{e'_{2,3}} \underset{1,2}{+} k_2. \tag{3.41}$$

Step 4. Since our convention is that $\lambda_3 - k_3$ defines u_3, it follows that $k_3 - \lambda_3$ defines $-u_3$. These conventions need to be revered. Indeed,

$$k_3 \underset{1,2}{—} \lambda_3 = (-1) \underset{1,2}{\cdot} (\lambda_3 \underset{1,2}{—} k_3) \overset{(3.38b)}{=} (-1) \underset{1,2}{\cdot} (\hat{0}_{w_{12}} \underset{1,3}{+} u_3) = (-1) \underset{1,2}{\cdot} \hat{0}_{w_{12}} \underset{1,3}{+} (-1) \underset{1,2}{\cdot} u_3.$$

Note that $u_3 \in E_{123}$, and the three multiplications $\underset{1,2}{\cdot}$, $\underset{1,3}{\cdot}$, and $\underset{2,3}{\cdot}$ coincide in the ultracore of E. And note that $(-1) \underset{1,2}{\cdot} \hat{0}_{w_{12}} = \hat{0}_{w_{12}}$. Therefore, rewrite the previous equation as:

$$k_3 \underset{1,2}{—} \lambda_3 = \hat{0}_{w_{12}} \underset{1,3}{—} u_3. \tag{3.42}$$

Step 5. We are finally able to complete the proof of Theorem 3.1.4. First, using operations in $E \to E_{2,3}$, we have

$$(\mathsf{ZYX} \underset{2,3}{—} \mathsf{XZY}) \underset{1,2}{—} (\mathsf{YZX} \underset{2,3}{—} \mathsf{XYZ}) =$$
$$[(\mathsf{ZYX} \underset{2,3}{—} \mathsf{ZXY}) \underset{2,3}{—} (\mathsf{XZY} \underset{2,3}{—} \mathsf{ZXY})] \underset{1,2}{—} [(\mathsf{YXZ} \underset{2,3}{—} \mathsf{XYZ}) \underset{2,3}{—} (\mathsf{YXZ} \underset{2,3}{—} \mathsf{YZX})].$$

Now, using (3.40) and (3.41), this is equal to

$$[(\hat{0}_{e_{2,3}} \underset{1,3}{+} k_3) \underset{2,3}{-} (\hat{0}_{e_{2,3}} \underset{1,2}{+} \lambda_2)] \underset{1,2}{-} [(\hat{0}_{e'_{2,3}} \underset{1,3}{+} \lambda_3) \underset{2,3}{-} (\hat{0}_{e'_{2,3}} \underset{1,2}{+} k_2)].$$

Applying the interchange law in the Left face to the outer operations, this becomes

$$[(\hat{0}_{e_{2,3}} \underset{1,3}{+} k_3) \underset{1,2}{-} (\hat{0}_{e'_{2,3}} \underset{1,3}{+} \lambda_3)] \underset{2,3}{-} [(\hat{0}_{e_{2,3}} \underset{1,2}{+} \lambda_2) \underset{1,2}{-} (\hat{0}_{e'_{2,3}} \underset{1,2}{+} k_2)].$$

Applying the interchange law in the Back face, in each [ ], we have

$$[(\hat{0}_{e_{2,3}} \underset{1,2}{-} \hat{0}_{e'_{2,3}}) \underset{1,3}{+} (k_3 \underset{1,2}{-} \lambda_3)] \underset{2,3}{-} [(\hat{0}_{e_{2,3}} \underset{1,2}{-} \hat{0}_{e'_{2,3}}) \underset{1,2}{+} (\lambda_2 \underset{1,2}{-} k_2)].$$

Now apply (2.36) to the first term in each [ ]. Then use (3.42) and (3.34b) This gives

$$[(\hat{0}_{w_{23}} \underset{1,3}{+} \hat{0}_{e_2}) \underset{1,3}{+} (\hat{0}_{w_{12}} \underset{1,3}{-} u_3)] \underset{2,3}{-} [(\hat{0}_{w_{23}} \underset{1,3}{+} \hat{0}_{e_2}) \underset{1,2}{+} (\hat{0}_{e_2} \underset{1,3}{+} u_2)].$$

Now apply the interchange law in the Back face to the second [ ] :

$$[\hat{0}_{w_{23}} \underset{1,3}{+} (\hat{0}_{e_2} \underset{1,3}{+} \hat{0}_{w_{12}} \underset{1,3}{-} u_3)] \underset{2,3}{-} [(\hat{0}_{w_{23}} \underset{1,2}{+} u_2) \underset{1,3}{+} (\hat{0}_{e_2} \underset{1,2}{+} \hat{0}_{e_2})].$$

Focus on the second [ ]. Using (2.33) in its first ( ), and (2.21b) in its second ( ), this in turn is equal to

$$[\hat{0}_{w_{23}} \underset{1,3}{+} (\hat{0}_{e_2} \underset{1,3}{+} \hat{0}_{w_{12}} \underset{1,3}{-} u_3)] \underset{2,3}{-} [(\hat{0}_{w_{23}} \underset{1,3}{+} u_2) \underset{1,3}{+} \hat{0}_{e_2}].$$

Rewrite this as :

$$[\hat{0}_{w_{23}} \underset{1,3}{+} (\hat{0}_{e_2} \underset{1,3}{+} \hat{0}_{w_{12}} \underset{1,3}{-} u_3)] \underset{2,3}{-} [\hat{0}_{w_{23}} \underset{1,3}{+} (\hat{0}_{e_2} \underset{1,3}{+} u_2)];$$

note that the second [ ] is in an ordinary vector bundle. Now use the interchange law in the Up face :

$$[\hat{0}_{w_{23}} \underset{2,3}{-} \hat{0}_{w_{23}}] \underset{1,3}{+} [(\hat{0}_{e_2} \underset{1,3}{+} \hat{0}_{w_{12}} \underset{1,3}{-} u_3) \underset{2,3}{-} (\hat{0}_{e_2} \underset{1,3}{+} u_2)].$$

and this is equal to

$$\hat{0}_{w_{23}} \underset{1,3}{+} [\hat{0}_{e_2} \underset{2,3}{-} \hat{0}_{e_2}] \underset{1,3}{+} [(\hat{0}_{w_{12}} \underset{1,3}{-} u_3) \underset{2,3}{-} u_2],$$

using the facts that the zeros $\hat{0}_{w_{23}}$ in the first [ ] are zeros over $E_{2,3}$, and then the interchange law in the Up face. Likewise, using the fact that the zeros $\hat{0}_{e_2}$ are zeros over $E_{2,3}$, this is equal to

$$\hat{0}_{w_{23}} \underset{1,3}{+} \hat{0}_{e_2} \underset{1,3}{+} [(\hat{0}_{w_{12}} \underset{2,3}{-} u_3) \underset{2,3}{-} u_2].$$

Finally, using an equation of the form (2.33), this becomes

$$\hat{0}_{w_{23}} \underset{1,3}{+} \hat{0}_{e_2} \underset{1,3}{+} [\hat{0}_{w_{12}} \underset{2,3}{-} (u_3 \underset{2,3}{+} u_2)] = \hat{0}_{w_{23}} \underset{1,3}{+} \hat{0}_{e_2} \underset{1,3}{+} \hat{0}_{w_{12}} \underset{1,3}{-} (u_3 \underset{1,3}{+} u_2),$$

from which we obtain $-(u_3 + u_2)$ as the ultracore element.

Comparing this with (3.31b),

$$(\mathsf{ZYX} \underset{1,2}{-} \mathsf{YZX}) \underset{2,3}{-} (\mathsf{XZY} \underset{1,2}{-} \mathsf{XYZ}) = (\hat{0}_{w_{12}} \underset{1,3}{+} \hat{0}_{w_{23}}) \underset{1,3}{+} (\hat{0}_{e_2} \underset{1,3}{+} u_1),$$

we have $u_1 = -(u_3 + u_2)$ as desired.

This completes the proof of the warp-grid theorem.

Remark 3.2.1. The strategy of this proof deserves some commentary.

What should the warp of a grid on a triple vector bundle be? Or, in other words, why are we interested in the ultrawarps of a grid of a triple vector bundle?

The warp of a grid in the double case is a section of the core vector bundle, and measures the non-commutativity of the two routes defined by the grid.

So far, we have seen that all operations on a triple vector bundle are iterations of operations defined in double vector bundles. The ultracore, for example, is the core of the core double vector bundles.

For these reasons, we would want the warp of a grid in the triple case to be a section of the ultracore vector bundle, and to measure the non-commutativity of routes defined by the grid.

Pick an upper face of E, for example the Up face. If we compare the two routes defined by the grid in this face, then we obtain an element of the (U-D) core double vector bundle, which we denoted by λ_3. Similarly for the other upper faces, the non-commutativity of the corresponding routes defines λ_1 and λ_2. The three λ's are elements of different spaces; therefore, if we tried to compare them, or indeed perform any sort of operation with them (such as adding them or subtracting them), we would see that such an operation could be algebraically possible but would not be geometrically meaningful.

The same applies for the three k_i defined by the comparison of the routes for the lower faces.

The λ_i's and the corresponding k_i's however, are elements of the same spaces, therefore, comparing them is a possibility, and indeed the only sensible operation. And by comparing them, we measure the non-commutativity of four routes, instead of two.

This can be done for the three pairs of λ_i and k_i, and so we obtain the three ultrawarps.

So what does the warp-grid theorem tell us?

Each ultrawarp measures the non-commutativity of four routes. In total, a grid on a triple vector bundle provides six different routes from M to E. The sum of the three ultrawarps takes into account each route twice, once with a positive and once with a negative sign, and this is the reason we orient the core double vector bundles the way we do. The warp-grid theorem tells us that these add up to zero, a result that seems reasonable. The different vector bundle structures over which the operations take place however, are the main obstacle here — as soon as one realizes that simple operations like addition and subtraction in the triple vector bundle setting are no longer simple. △

3.2.4 Promised calculation

In the end of Section 2.3.4 we mentioned that the λ_i's and the k_i's described in that section are in fact related. Applying the method described in Step 5 of the previous Section, we proceed with investigating the aforementioned relation.

First, recall the outlines of the λ_i's and the k_i's.

$$\begin{array}{ccc} E_{12,3} \ni \lambda_1 & \longmapsto & w_{12} \in E_{12} \\ \downarrow & & \downarrow \\ e_3 & \longmapsto & m, \end{array} \qquad \begin{array}{ccc} E_{13,2} \ni \lambda_2 & \longmapsto & w_{13} \in E_{13} \\ \downarrow & & \downarrow \\ e_2 & \longmapsto & m, \end{array}$$

$$\begin{array}{ccc} E_{13,2} \ni k_1 & \longmapsto & w_{13} \in E_{13} \\ \downarrow & & \downarrow \\ e_2 & \longmapsto & m, \end{array} \qquad \begin{array}{ccc} E_{12,3} \ni k_2 & \longmapsto & w_{12} \in E_{12} \\ \downarrow & & \downarrow \\ e_3 & \longmapsto & m. \end{array}$$

Since k_1 and λ_2 have the same outlines, they will differ by a unique ultracore element, call it $\varpi_1 \in E_{123}$. Similarly, k_2 and λ_1 will differ by a unique ultracore element, denote it by $\varpi_2 \in E_{123}$. Is there a relation between ϖ_1 and ϖ_2? The equations for $k_1 - \lambda_2 \triangleright \varpi_1$,

$$k_1 \underset{1,3}{-} \lambda_2 = \hat{0}_{w_{13}} \underset{1,2/2,3}{+} \varpi_1, \tag{3.43a}$$

$$k_1 \underset{1,2}{-} \lambda_2 = \hat{0}_{e_2} \underset{1,3}{+} \varpi_1, \tag{3.43b}$$

$$k_1 \underset{2,3}{-} \lambda_2 = \hat{0}_{e_2} \underset{1,3}{+} \varpi_1. \tag{3.43c}$$

and for $k_2 - \lambda_1 \triangleright \varpi_2$,

$$k_2 \underset{1,3}{-} \lambda_1 = \hat{0}_{e_3} \underset{1,2}{+} \varpi_2, \tag{3.44a}$$

$$k_2 \underset{1,2}{-} \lambda_1 = \hat{0}_{w_{12}} \underset{1,3/2,3}{+} \varpi_2, \tag{3.44b}$$

$$k_2 \underset{2,3}{-} \lambda_1 = \hat{0}_{e_3} \underset{1,2}{+} \varpi_2. \tag{3.44c}$$

As in the double case $d \underset{A}{-} d = 0^D_a$, in the triple case, $e \underset{2,3}{-} e = \hat{0}_{e_{2,3}}$, or in this case,

$$(e \underset{2,3}{-} f) \underset{2,3}{-} (e \underset{2,3}{-} f) = (2.48) \underset{2,3}{-} (2.49) = \hat{0}_{e_{2,3}}. \tag{3.45}$$

Start with the left hand side:

$$\begin{aligned}
(2.48) \underset{2,3}{-} (2.49) &= \left[(k_1 \underset{1,2}{+} \hat{0}_{e_{2,3}}) \underset{2,3}{+} (k_2 \underset{1,3}{+} \hat{0}_{e_{2,3}})\right] \underset{2,3}{-} \left[(\lambda_1 \underset{1,3}{+} \hat{0}_{e_{2,3}}) \underset{2,3}{+} (\lambda_2 \underset{1,2}{+} \hat{0}_{e_{2,3}})\right] \\
&= \left[(k_1 \underset{1,2}{+} \hat{0}_{e_{2,3}}) \underset{2,3}{-} (\lambda_2 \underset{1,2}{+} \hat{0}_{e_{2,3}})\right] \underset{2,3}{+} \left[(k_2 \underset{1,3}{+} \hat{0}_{e_{2,3}}) \underset{2,3}{-} (\lambda_1 \underset{1,3}{+} \hat{0}_{e_{2,3}})\right] \\
&= \left[(k_1 \underset{2,3}{-} \lambda_2) \underset{1,2}{+} (\hat{0}_{e_{2,3}} \underset{2,3}{-} \hat{0}_{e_{2,3}})\right] \underset{2,3}{+} \left[(k_2 \underset{2,3}{-} \lambda_1) \underset{1,3}{+} (\hat{0}_{e_{2,3}} \underset{2,3}{-} \hat{0}_{e_{2,3}})\right] \\
&\overset{(3.43c),(3.44c)}{=} \left[(\hat{0}_{e_2} \underset{1,3}{+} \varpi_1) \underset{1,2}{+} \hat{0}_{e_{2,3}}\right] \underset{2,3}{+} \left[(\hat{0}_{e_3} \underset{1,2}{+} \varpi_2) \underset{1,3}{+} \hat{0}_{e_{2,3}}\right].
\end{aligned} \tag{3.46}$$

Now rewrite the first bracket of (3.46),

$$(\hat{0}_{e_2} \underset{1,3}{+} \varpi_1) \underset{1,2}{+} \hat{0}_{e_{2,3}} = (\hat{0}_{e_2} \underset{1,3}{+} \varpi_1) \underset{1,2}{+} (\hat{0}_{e_{2,3}} \underset{1,3}{+} \hat{0}_{e_3}) = (\hat{0}_{e_2} \underset{1,2}{+} \hat{0}_{e_{2,3}}) \underset{1,3}{+} (\varpi_1 \underset{1,2}{+} \hat{0}_{e_3}) = \hat{0}_{e_{2,3}} \underset{1,3}{+} (\varpi_1 \underset{1,2}{+} \hat{0}_{e_3}).$$

Returning to (3.46):

$$\begin{aligned}
(2.48) \underset{2,3}{-} (2.49) &= \left[(\varpi_1 \underset{1,2}{+} \hat{0}_{e_3}) \underset{1,3}{+} \hat{0}_{e_{2,3}}\right] \underset{2,3}{+} \left[(\hat{0}_{e_3} \underset{1,2}{+} \varpi_2) \underset{1,3}{+} \hat{0}_{e_{2,3}}\right] \\
&= \left[(\varpi_1 \underset{1,2}{+} \hat{0}_{e_3}) \underset{2,3}{+} (\hat{0}_{e_3} \underset{1,2}{+} \varpi_2)\right] \underset{1,3}{+} \left[\hat{0}_{e_{2,3}} \underset{2,3}{+} \hat{0}_{e_{2,3}}\right] \\
&= \left[(\varpi_1 \underset{2,3}{+} \varpi_2) \underset{1,2}{+} (\hat{0}_{e_3} \underset{2,3}{+} \hat{0}_{e_3})\right] \underset{1,3}{+} \hat{0}_{e_{2,3}} \\
&= \left[(\varpi_1 \underset{2,3}{+} \varpi_2) \underset{1,2}{+} \hat{0}_{e_3}\right] \underset{1,3}{+} \hat{0}_{e_{2,3}}.
\end{aligned} \tag{3.47}$$

By (3.45),

$$(2.48) \underset{2,3}{-} (2.49) = \hat{0}_{e_{2,3}} = \hat{0}_{e_{2,3}} \underset{1,3}{+} \hat{0}_{e_3},$$

therefore, comparing (3.45) and (3.47), it follows that

$$\varpi_1 \underset{2,3}{+} \varpi_2 = \odot^3_m,$$

and this applies over any structure, hence, $\varpi_1 = -\varpi_2$.

Chapter 4

Warps, bolts and grids; Examples

We begin this chapter with an example of a grid and its warp on the cotangent double vector bundle T^*A. We proceed with further investigating properties of the warp and of the ultrawarp. We then develop bolt sections, and introduce double bolt sections, and give a class of examples of grids on E using them. We continue with examples of grids on T^2A and on T^3M. Finally, we give an alternative formula for the warp of a grid on D using the duality of D in Section 4.7.

4.1 The reversal isomorphism $R : T^*A^* \to T^*A$

Recall the cotangent double vector bundle T^*A, described in Section 2.4.5:

$$\begin{array}{ccc} T^*A & \xrightarrow{\ r\ } & A^* \\ {\scriptstyle c_A}\big\downarrow & & \big\downarrow{\scriptstyle q_*} \\ A & \xrightarrow{\ q\ } & M. \end{array}$$

To build a grid on T^*A we need to use the *reversal isomorphism* $R : T^*(A^*) \to T^*A$, a double vector bundle isomorphism introduced by Mackenzie and Xu in [28]. This map is a canonical diffeomorphism, which reverses the standard symplectic structures; see [28] and references given there. In (2.86) we have defined the unfamiliar projection $r : T^*A \to A^*$ using duality theory. Alternatively, one can use R^{-1} to transport the vector bundle structure of $T^*(A^*) \to A^*$ to $T^*A \to A^*$.

We need the following result concerning R from [28], or see [25, 9.5.1].

Proposition 4.1.1. *For all $\xi \in TA$, $\mathscr{X} \in T(A^*)$, $\mathfrak{F} \in T^*(A^*)$ such that ξ and $\mathscr{X}$ have the same projection into TM, $\mathscr{X}$ and $\mathfrak{F}$ have the same projection into A^*, and $\mathfrak{F}$ and ξ have the same projection into A,*

$$\langle\!\langle \mathscr{X}, \xi \rangle\!\rangle = \langle R(\mathfrak{F}), \xi \rangle_A + \langle \mathfrak{F}, \mathscr{X} \rangle_{A^*}. \tag{4.1}$$

To keep track of the various calculations, we present the outlines of the four elements involved in (4.1):

$$
\begin{array}{ccc}
TA \ni \xi & \longmapsto & v_0 \in TM \\
\big\downarrow & & \big\downarrow \\
A \ni a_0 & \longmapsto & m,
\end{array}
\qquad
\begin{array}{ccc}
TA^* \ni \mathscr{X} & \longmapsto & v_0 \in TM \\
\big\downarrow & & \big\downarrow \\
A^* \ni \varphi_0 & \longmapsto & m,
\end{array}
$$

$$
\begin{array}{ccc}
T^*A \ni R(\mathfrak{F}) & \longmapsto & \varphi_0 \in A^* \\
\big\downarrow & & \big\downarrow \\
A \ni a_0 & \longmapsto & m,
\end{array}
\qquad
\begin{array}{ccc}
T^*A^* \ni \mathfrak{F} & \longmapsto & a_0 \in A \\
\big\downarrow & & \big\downarrow \\
A^* \ni \varphi_0 & \longmapsto & m.
\end{array}
$$

In fact, R as a double vector bundle isomorphism preserves the side bundles A and A^*, and induces the $-\operatorname{id} : T^*M \to T^*M$ on the cores (see Appendix A.2.2 for proof). For a description of R in local coordinates, see [3] and [37, Theorem 7.1].

The two pairings on the right hand side of (4.1) are usual pairings between a vector bundle and its dual. Specifically, for $\langle R(\mathfrak{F}), \xi\rangle_A$ we have the usual pairing between $TA \to A$ and $T^*A \to A$, and for $\langle \mathfrak{F}, \mathscr{X}\rangle_{A^*}$ we have the usual pairing between $TA^* \to A^*$ and $T^*A^* \to A^*$.

The pairing $\langle\!\langle \mathscr{X}, \xi\rangle\!\rangle$ on the left hand side of (4.1) is described in detail in [25, p.117-18]. Briefly, given a vector bundle $A \to M$, the canonical pairing between $A \to M$ and $A^* \to M$:

$$\langle \cdot, \cdot, \rangle : A^* \underset{M}{\times} A \to \mathbb{R}, \quad (\alpha_m, a_m) \mapsto \langle \alpha_m, a_m\rangle = \alpha_m(a_m).$$

induces a pairing between TA and TA^* as vector bundles over TM, called the *tangent (prolongation) pairing* as follows. Take $\mathscr{X} \in TA^*$ and $\xi \in TA$ with $T(q)(\xi) = T(q_*)(\mathscr{X})$, and write

$$\mathscr{X} = \frac{d}{dt}\varphi(t)\Big|_{t=0} \in TA^*, \quad \xi = \frac{d}{dt}a(t)\Big|_{t=0} \in TA,$$

where $\varphi(t)$ is a curve in A^*, $a(t)$ is a curve in A, with $q_*(\varphi(t)) = q(a(t)) = m(t) \in M$, a curve in M for t near zero. Define the tangent pairing $\langle\!\langle\ ,\ \rangle\!\rangle$ by:

$$\langle\!\langle \mathscr{X}, \xi\rangle\!\rangle_{TM} = \frac{d}{dt}\langle \varphi(t), a(t)\rangle\Big|_{t=0}. \tag{4.2}$$

Equation (4.2) defines a non-degenerate pairing; one needs to check non-degeneracy, it does not follow automatically.

Example 4.1.2. Now we can build a grid on T^*A. Take $\mu \in \Gamma A$ and $\varphi \in \Gamma A^*$. These define two linear sections as follows.

First, take the 1-form $d\ell_\varphi \in \Omega^1(A)$ defined by the linear map $\ell_\varphi : A \to \mathbb{R}$,

$$
\begin{array}{rcl}
\ell_\varphi : A & \to & \mathbb{R}, \\
A_m \ni a_m & \mapsto & \langle \varphi(m), a_m\rangle.
\end{array}
$$

Then $d\ell_\varphi$ is a section of $T^*A \to A$. To see that $(d\ell_\varphi, \varphi)$ is a linear section,

$$\begin{array}{ccc} A & \xrightarrow{d\ell_\varphi} & T^*A \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle r} \\ M & \xrightarrow[\varphi]{} & A^* \end{array}$$

first, we check that for $a \in A_m$, $r(d\ell_\varphi(a)) = \varphi(m)$.

By (2.86), the definition of the r map, for any $a' \in A_m$:

$$\langle r(d\ell_\varphi(a)), a'\rangle = \langle d\ell_\varphi(a), 0_a^{TA} \underset{TM}{+} \bar{a}'\rangle = (0_a^{TA} \underset{TM}{+} \bar{a}')(\ell_\varphi).$$

By (1.12) we have that $0_a^{TA} = \frac{d}{dt}a\Big|_{t=0}$, and by (1.9) $\bar{a}' = \frac{d}{dt}ta'\Big|_{t=0}$, therefore we can write

$$\begin{aligned}\langle r(d\ell_\varphi(a)), a'\rangle = \frac{d}{dt}\ell_\varphi(a + ta')\Big|_{t=0} &= \frac{d}{dt}(\ell_\varphi(a) + \ell_\varphi(ta'))\Big|_{t=0} \\ &= \frac{d}{dt}t\ell_\varphi(a')\Big|_{t=0} = \ell_\varphi(a') = \langle\varphi(m), a'\rangle.\end{aligned}$$

This is true for any $a' \in A_m$, therefore, $r(d\ell_\varphi(a)) = \varphi(m)$.

Note that $\ell_\varphi(a') \in \mathbb{R}$, so $\frac{d}{dt}t\ell_\varphi(a')\Big|_{t=0} = \ell_\varphi(a')$ and not $\overline{\ell_\varphi(a')}$. Similarly, $\ell_\varphi(a) \in \mathbb{R}$ so $\frac{d}{dt}\ell_\varphi(a)\Big|_{t=0} = 0$, and not $0^{TA}_{\ell_\varphi(a)}$.

Secondly, we check linearity. Take $a_1, a_2 \in A_m$ and $\xi_1 \in T_{a_1}A$, $\xi_2 \in T_{a_2}A$ with $T(q)(\xi_1) = T(q)(\xi_2)$. As usual, see (6), we can arrange for $a_1(t), a_2(t)$ two curves in A, with $q(a_1(t)) = q(a_2(t))$ for t near zero, where $\xi_1 = \frac{d}{dt}a_1(t)\Big|_{t=0}$, and $\xi_2 = \frac{d}{dt}a_2(t)\Big|_{t=0}$. Of course $a_1(0) = a_1$ and $a_2(0) = a_2$. Hence,

$$\begin{aligned}(d\ell_\varphi(a_1 + a_2))(\xi_1 \underset{TM}{+} \xi_2) = (\xi_1 \underset{TM}{+} \xi_2)(\ell_\varphi) &= \frac{d}{dt}\ell_\varphi(a_1(t) + a_2(t))\Big|_{t=0} \\ &= \frac{d}{dt}\ell_\varphi(a_1(t))\Big|_{t=0} + \frac{d}{dt}\ell_\varphi(a_2(t))\Big|_{t=0} = \xi_1(\ell_\varphi) + \xi_2(\ell_\varphi) \\ &= (d\ell_\varphi(a_1))(\xi_1) + (d\ell_\varphi(a_2))(\xi_2) \overset{(2.87)}{=} \langle d\ell_\varphi(a_1) \underset{A^*}{+} d\ell_\varphi(a_2), \xi_1 \underset{TM}{+} \xi_2\rangle.\end{aligned}$$

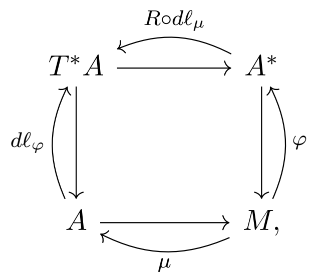
Hence $(d\ell_\varphi, \varphi)$ is a linear section of T^*A.

For $\mu \in \Gamma A$, the 1-form $d\ell_\mu \in \Omega^1(A^*)$ defined by the corresponding linear map $\ell_\mu : A^* \to \mathbb{R}$, is a section of $T^*A^* \to A^*$. Composing with the reversal isomorphism $R : T^*A^* \to T^*A$ it follows that $R(d\ell_\mu)$ is a section of $T^*A \to A^*$. To check that $(R(d\ell_\mu), \mu)$ is a linear section, since R is an isomorphism of double vector bundles, one only needs to check that $(d\ell_\mu, \mu)$ is a linear section, and this follows in a similar way as $(d\ell_\varphi, \varphi)$.

It was proved in [28] that

$$R(d\ell_\mu(\varphi(m))) \underset{A}{-} d\ell_\varphi(\mu(m)) = \underset{A}{-} q^*(d\langle\varphi,\mu\rangle)(\mu(m)). \tag{4.3}$$

(or see [25, 9.5.3]). Therefore, the warp of the grid described on T^*A,



is $\mathrm{w}(R\circ d\ell_\mu, d\ell_\varphi) = -d\langle\varphi,\mu\rangle$.

Remark 4.1.3. This is a good place to make the following remark. So far, we have seen struts, e.g in T^*A in (2.4.5), and examples of warps of grids, e.g. in TA, Example 1.2.3. It is important to distinguish between the two terms "struts" and "warps". The warp is a section of the core vector bundle, defined for any $m \in M$ by (11):

$$\xi(Y(m)) \underset{A}{-} \eta(X(m)) = \mathrm{w}(\xi,\eta)(m) \underset{B}{+} 0^D_{X(m)}.$$

Since the warp is a section of the core vector bundle, it will define two struts. Without loss of generality, take $\mathrm{w}(\xi,\eta)^A \in \Gamma_A D$, and from Definition 1.1.3, for $a \in A$:

$$\mathrm{w}(\xi,\eta)^A(a) = \mathrm{w}(\xi,\eta)(m) \underset{B}{+} 0^D_a. \tag{4.4}$$

Setting $a = X(m)$ at (4.4),

$$\mathrm{w}(\xi,\eta)^A(X(m)) = \mathrm{w}(\xi,\eta)(m) \underset{B}{+} 0^D_{X(m)}.$$

Comparing the previous equation with (11), one might write:

$$\mathrm{w}(\xi,\eta)^A(X(m)) = \xi(Y(m)) \underset{A}{-} \eta(X(m)).$$

However, this is an equality about specific elements, not about sections; we cannot say that the right hand side of (11) is equal to the strut $\mathrm{w}(\xi,\eta)^A$ of the warp $\mathrm{w}(\xi,\eta)$.

By specifying $a = X(m)$ in (4.4), $\mathrm{w}(\xi,\eta)^A(X(m))$ is no longer a map from A to D (as the strut $\mathrm{w}(\xi,\eta)^A$ is), but a map from M to D, just as $\xi(Y(m)) \underset{A}{-} \eta(X(m))$ is not a map from A to D and is not defined for any $a \in A$.

To illustrate this point clearly, let us use the Example 4.1.2 of the grid on T^*A. We have established that:

- the warp of $(R \circ d\ell_\mu, \mu)$ and of $(d\ell_\varphi, \varphi)$ is $-d\langle\varphi,\mu\rangle \in \Gamma(T^*M)$, a section of the core vector bundle of T^*A, defined by (4.3).
- By Section 2.4.5, one of the two struts defined by $-d\langle\varphi,\mu\rangle \in \Omega^1(M)$ is $q^*(-d\langle\varphi,\mu\rangle) \in \Omega^1(A)$, a section of the vector bundle $T^*A \to A$, where for $a \in A$:

$$q^*(-d\langle\varphi,\mu\rangle)(a) = 0^{T^*A}_a \underset{A^*}{-} d\langle\varphi,\mu\rangle(m) \tag{4.5}$$

By (4.3), we have that

$$R(d\ell_\mu(\varphi(m))) \underset{A}{-} d\ell_\varphi(\mu(m)) = \underset{A}{-} q^*(d\langle\varphi,\mu\rangle)(\mu(m)) = \underset{A}{-}\left(0^{T^*A}_{\mu(m)} \underset{A^*}{+} d\langle\varphi,\mu\rangle(m)\right).$$

The latter is a statement about elements of $T^*_{\mu(m)}A$, and it is misleading to state it as an equation for sections. After all, $-q^*(d\langle\varphi,\mu\rangle) \circ \mu$ is a map from M to T^*A, and we cannot compare it to (4.5), which describes the strut $q^*(d\langle\varphi,\mu\rangle)$ of $d\langle\varphi,\mu\rangle$.

In a similar note, take the very first example of warp, the Lie bracket of two vector fields $X, Y \in \mathfrak{X}(M)$ being the warp of $(T(Y), Y)$ and of $(\widetilde{X}, X)$, as described by (8). The vertical lift $[X,Y]^\uparrow \in \mathfrak{X}(TM)$ is the strut defined by the warp $[X,Y]$ (see Section 1.2.2), but the right hand side of (8), namely, $[X,Y]^\uparrow(Y(m))$ is the value at m of a map from M to $T(TM)$. △

4.2 Warps and Bolts

4.2.1 Properties of warps and ultrawarps

So far, we have seen examples of grids on double vector bundles and their warps. Are there any further operations one can perform with warps? We proceed with showing that the warp has various linearity properties.

Proposition 4.2.1. *Take (ξ, X), and (ξ_i, X_i), $i = 1, 2$ linear sections of the horizontal structure of D, where $\xi, \xi_i \in \Gamma_B D$, and $X, X_i \in \Gamma A$. And take (η, Y), and (η_i, Y_i), $i = 1, 2$, linear sections of the vertical structure of D, where $\eta, \eta_i \in \Gamma_A D$, and $Y, Y_i \in \Gamma B$. Then,*

1. $(\kappa \underset{A}{\cdot} \eta_1 \underset{A}{+} \lambda \underset{A}{\cdot} \eta_2, \kappa Y_1 + \lambda Y_2)$ *is a linear section of the vertical structure of D, where $\kappa, \lambda \in \mathbb{R}$,*
2. $\mathrm{w}(\xi, \eta_1 \underset{A}{+} \eta_2) = \mathrm{w}(\xi, \eta_1) + \mathrm{w}(\xi, \eta_2)$.
3. *For any $\kappa \in \mathbb{R}$:* $\mathrm{w}(\xi, \kappa \underset{A}{\cdot} \eta) = \kappa\, \mathrm{w}(\xi, \eta)$.
4. *For any $\lambda, \kappa \in \mathbb{R}$:* $\mathrm{w}(\lambda \underset{B}{\cdot} \xi, \kappa \underset{A}{\cdot} \eta) = \lambda\kappa\, \mathrm{w}(\xi, \eta)$.

5. *For any* $\lambda, \kappa \in \mathbb{R}$*:* $\mathrm{w}(\xi, \kappa \underset{A}{\cdot} \eta_1 \underset{A}{+} \lambda \underset{A}{\cdot} \eta_2) = \kappa\, \mathrm{w}(\xi, \eta_1) + \lambda\, \mathrm{w}(\xi, \eta_2)$.

6. *For any* $\mu, \nu, \lambda, \kappa \in \mathbb{R}$*:*

$$\begin{aligned} &\mathrm{w}(\mu \underset{B}{\cdot} \xi_1 \underset{B}{+} \nu \underset{B}{\cdot} \xi_2, \kappa \underset{A}{\cdot} \eta_1 \underset{A}{+} \lambda \underset{A}{\cdot} \eta_2) \\ &\qquad = \mu\kappa\, \mathrm{w}(\xi_1, \eta_1) + \nu\kappa\, \mathrm{w}(\xi_2, \eta_1) + \mu\lambda\, \mathrm{w}(\xi_1, \eta_2) + \nu\lambda\, \mathrm{w}(\xi_2, \eta_2). \end{aligned}$$

Proof. 1. For $a_1, a_2 \in A$,

$$\begin{aligned} &(\kappa \underset{A}{\cdot} \eta_1 \underset{A}{+} \lambda \underset{A}{\cdot} \eta_2)(a_1 + a_2) \\ &= (\kappa \underset{A}{\cdot} \eta_1(a_1 + a_2)) \underset{A}{+} (\lambda \underset{A}{\cdot} \eta_2(a_1 + a_2)) \\ &= (\kappa \underset{A}{\cdot} (\eta_1(a_1) \underset{B}{+} \eta_1(a_2))) \underset{A}{+} (\lambda \underset{A}{\cdot} (\eta_2(a_1) \underset{B}{+} \eta_2(a_2))) \\ &= (\kappa \underset{A}{\cdot} \eta_1(a_1) \underset{B}{+} \kappa \underset{A}{\cdot} \eta_1(a_2)) \underset{A}{+} (\lambda \underset{A}{\cdot} \eta_2(a_1) \underset{B}{+} \lambda \underset{A}{\cdot} \eta_2(a_2)) \\ &= (\kappa \underset{A}{\cdot} \eta_1(a_1) \underset{A}{+} \lambda \underset{A}{\cdot} \eta_2(a_1)) \underset{B}{+} (\kappa \underset{A}{\cdot} \eta_1(a_2) \underset{A}{+} \lambda \underset{A}{\cdot} \eta_2(a_2)) \\ &= (\kappa \underset{A}{\cdot} \eta_1 \underset{A}{+} \lambda \underset{A}{\cdot} \eta_2)(a_1) \underset{B}{+} (\kappa \underset{A}{\cdot} \eta_1 \underset{A}{+} \lambda \underset{A}{\cdot} \eta_2)(a_2). \end{aligned}$$

2. The warps of the two grids $\mathrm{w}(\xi, \eta_1)$ and $\mathrm{w}(\xi, \eta_2)$ for $m \in M$:

$$\begin{aligned} \xi(Y_1(m)) \underset{A}{-} \eta_1(X(m)) &= \mathrm{w}(\xi, \eta_1)(m) \underset{B}{+} 0^D_{X(m)}, \\ \xi(Y_2(m)) \underset{A}{-} \eta_2(X(m)) &= \mathrm{w}(\xi, \eta_2)(m) \underset{B}{+} 0^D_{X(m)}. \end{aligned}$$

What is $\mathrm{w}(\xi, \eta_1 \underset{A}{+} \eta_2)$?

$$\begin{aligned} &\xi((Y_1 + Y_2)(m)) \underset{A}{-} (\eta_1 \underset{A}{+} \eta_2)(X(m)) \\ &= (\xi(Y_1(m)) \underset{A}{+} \xi(Y_2(m))) \underset{A}{-} (\eta_1(X(m)) \underset{A}{+} \eta_2(X(m))) \\ &= (\xi(Y_1(m)) \underset{A}{-} \eta_1(X(m))) \underset{A}{+} (\xi(Y_2(m)) \underset{A}{-} \eta_2(X(m))) \\ &= (\mathrm{w}(\xi, \eta_1)(m) \underset{B}{+} 0^D_{X(m)}) \underset{A}{+} (\mathrm{w}(\xi, \eta_2)(m) \underset{B}{+} 0^D_{X(m)}) \\ &= (\mathrm{w}(\xi, \eta_1)(m) \underset{A}{+} \mathrm{w}(\xi, \eta_2)(m)) \underset{B}{+} (0^D_{X(m)} \underset{A}{+} 0^D_{X(m)}) \\ &= (\mathrm{w}(\xi, \eta_1) \underset{A}{+} \mathrm{w}(\xi, \eta_2))(m) \underset{B}{+} 0^D_{X(m)}, \end{aligned}$$

ergo, $\mathrm{w}(\xi, \eta_1 \underset{A}{+} \eta_2) = \mathrm{w}(\xi, \eta_1) \underset{A}{+} \mathrm{w}(\xi, \eta_2)$.

3. Since $\xi(\kappa Y(m)) = \kappa \underset{A}{\cdot} \xi(Y(m))$, for the warp $\mathrm{w}(\xi, \kappa \underset{A}{\cdot} \eta)$ we have:

$$\begin{aligned}
\xi(\kappa Y(m)) \underset{A}{-} \kappa \underset{A}{\cdot} \eta(X(m)) &= \kappa \underset{A}{\cdot} \xi(Y(m)) \underset{A}{-} \kappa \underset{A}{\cdot} \eta(X(m)) \\
&= \kappa \underset{A}{\cdot} (\xi(Y(m)) \underset{A}{-} \eta(X(m))) \\
&= \kappa \underset{A}{\cdot} (\mathrm{w}(\xi, \eta)(m) \underset{B}{+} 0^D_{X(m)}) \\
&= \kappa \underset{A}{\cdot} \mathrm{w}(\xi, \eta)(m) \underset{B}{+} \kappa \underset{A}{\cdot} 0^D_{X(m)} \\
&= \kappa \underset{A}{\cdot} \mathrm{w}(\xi, \eta)(m) \underset{B}{+} 0^D_{X(m)},
\end{aligned}$$

hence $\mathrm{w}(\xi, \kappa \underset{A}{\cdot} \eta) = \kappa \underset{A}{\cdot} \mathrm{w}(\xi, \eta)$.

4. The linear sections involved $(\lambda \underset{B}{\cdot} \xi, \lambda X)$ and $(\kappa \underset{A}{\cdot} \eta, \kappa Y)$.

$$\begin{aligned}
(\lambda \underset{B}{\cdot} \xi)(\kappa Y(m)) \underset{A}{-} (\kappa \underset{A}{\cdot} \eta)(\lambda X(m)) &= \lambda \underset{B}{\cdot} (\xi(\kappa Y(m))) \underset{A}{-} \kappa \underset{A}{\cdot} (\eta(\lambda X(m))) \\
&= \lambda \underset{B}{\cdot} (\kappa \underset{A}{\cdot} (\xi(Y(m))) \underset{A}{-} \kappa \underset{A}{\cdot} (\lambda \underset{B}{\cdot} (\eta(X(m))) \\
&= \lambda \underset{B}{\cdot} (\kappa \underset{A}{\cdot} (\xi(Y(m))) \underset{A}{-} \lambda \underset{B}{\cdot} (\kappa \underset{A}{\cdot} (\eta(X(m))) \\
&= \lambda \underset{B}{\cdot} \left(\kappa \underset{A}{\cdot} (\xi(Y(m)) \underset{A}{-} \eta(X(m))) \right) \\
&= \lambda \underset{B}{\cdot} \left(\kappa \underset{A}{\cdot} (\mathrm{w}(\xi, \eta)(m) \underset{B}{+} 0^D_{X(m)}) \right) \\
&= \lambda \underset{B}{\cdot} \left(\kappa \underset{A}{\cdot} \mathrm{w}(\xi, \eta)(m) \underset{B}{+} \kappa \underset{A}{\cdot} 0^D_{X(m)} \right) \\
&= \lambda \underset{B}{\cdot} \left(\kappa \underset{A}{\cdot} \mathrm{w}(\xi, \eta)(m) 0^D_{X(m)} \underset{B}{+} 0^D_{X(m)} \right) \\
&= \lambda \underset{B}{\cdot} (\kappa \underset{A}{\cdot} \mathrm{w}(\xi, \eta)(m)) \underset{B}{+} \lambda \underset{B}{\cdot} 0^D_{X(m)} \\
&= \lambda \kappa \, \mathrm{w}(\xi, \eta)(m) \underset{B}{+} 0^D_{\lambda X(m)},
\end{aligned}$$

and this ends the proof.

□

Now start with a grid (ξ, X), (η, Y) on D as in (10), that has warp $\mathrm{w}(\xi, \eta) \in \Gamma C$ as in (11). Applying the tangent functor to a double vector bundle D yields TD, the triple vector bundle we described in Section 2.4.2. In the following proposition we show that applying the tangent functor to a grid (ξ, X), (η, Y) on D yields another grid $(T(\xi), T(X))$, $(T(\eta), T(Y))$ on TD. In fact,

Proposition 4.2.2. *Let* (ξ, X) *and* (η, Y) *be a grid on a double vector bundle* D *with* *warp* $\mathrm{w}(\xi,\eta)\in\Gamma C$. *Then the warp of the following grid*

$$\begin{array}{ccc} TD & \xrightarrow{\quad} \atop \xleftarrow[\quad]{T(\xi)} & TB \\ {\scriptstyle T(\eta)}\uparrow\downarrow & & \downarrow\uparrow{\scriptstyle T(Y)} \\ TA & \xrightarrow{\quad} \atop \xleftarrow[T(X)]{\quad} & TM, \end{array} \tag{4.6}$$

is $T(\mathrm{w}(\xi,\eta))\in\Gamma_{TM}(TC)$.

Proof. From Proposition 2.4.3 we know that the Up face of TD is a double vector bundle with core $TC\to TM$. That $(T(\xi),T(X))$ and $(T(\eta),T(Y))$ are linear sections of TD follows immediately.

The warp $\mathrm{w}(\xi,\eta)\in\Gamma C$ of (ξ,X), (η,Y) is given as usual by (11). We calculate the warp of the tangent grid. From the definition of a warp, for $v\in T_mM$,

$$(T(\xi)\circ T(Y))(v)\underset{TA}{-}(T(\eta)\circ T(X))(v)=T(0^D\circ X)(x)\underset{TB}{+}\mathrm{w}(T(\xi),T(\eta))(v).$$

Write $v=\frac{d}{dt}m(t)\Big|_{t=0}$, for $m(t)$ a curve in M with tangent vector v at $t=0$. Then, for $F\in C^\infty(D)$,

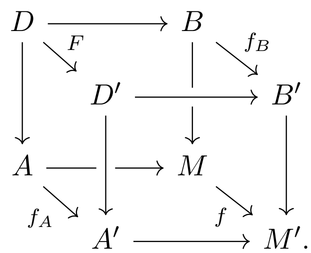
$$\begin{aligned}
&\left((T(\xi)\circ T(Y))\underset{TA}{-}(T(\eta)\circ T(X))\right)(v)(F)\\
&\overset{(7)}{=}\frac{d}{dt}F\left((\xi\circ Y)(m(t))\underset{A}{-}(\eta\circ X)(m(t))\right)\Big|_{t=0}\\
&=\frac{d}{dt}F\left(0^D_{X(m(t))}\underset{B}{+}\mathrm{w}(\xi,\eta)(m(t))\right)\Big|_{t=0}\\
&=\frac{d}{dt}F\left((0^D\circ X)(m(t))\underset{B}{+}\mathrm{w}(\xi,\eta)(m(t))\right)\Big|_{t=0}\\
&\overset{(7)}{=}\left(T(0^D\circ X)(v)\underset{TB}{+}T(\mathrm{w}(\xi,\eta))(v)\right)(F).
\end{aligned}\tag{4.7}$$

By uniqueness of the core element, it follows from (4.7) that

$$\mathrm{w}(T(\xi),T(\eta))(v)=T(\mathrm{w}(\xi,\eta))(v).$$

□

Proposition 4.2.3. *Let $F : D \to D'$ be a double vector bundle morphism,*



If (ξ, X) and (η, Y) is a grid on D, and (ξ', X') and (η', Y') is a grid on D', related by

$$F \circ \xi = \xi' \circ f_B, \quad F \circ \eta = \eta' \circ f_A, \quad f_B \circ Y = Y' \circ f, \quad f_A \circ X = X' \circ f. \tag{4.8}$$

Then

$$\mathrm{w}(\xi', \eta')(f(m)) = f_C(\mathrm{w}(\xi, \eta)(m)), \quad m \in M,$$

where $f_C = F\Big|_C$ is the core morphism of F.

Proof. For $m \in M$:

$$\xi'(Y'(m')) \underset{A'}{-} \eta'(X'(m')) = \mathrm{w}(\xi', \eta')(m') \underset{B'}{+} 0^{D'}_{X'(m')}.$$

For $m' = f(m)$:

$$\begin{aligned}
\xi'(Y'(f(m))) \underset{A'}{-} \eta'(X'(f(m))) &= \xi'(f_B(Y(m))) \underset{A'}{-} \eta'(f_A(X(m))) \\
&= F(\xi(Y(m))) \underset{A'}{-} F(\eta(X(m))) \\
&= F\left(\xi(Y(m)) \underset{A}{-} \eta(X(m))\right) \\
&= F\left(\mathrm{w}(\xi, \eta)(m) \underset{B}{+} 0^{D}_{X(m)}\right) \\
&= f_C(\mathrm{w}(\xi, \eta)(m)) \underset{B'}{+} 0^{D'}_{X'(f(m))}.
\end{aligned}$$

And the result follows. □

An extension of the previous result to triple vector bundles is immediate. If F is a triple vector bundle map from E to E', then we have six double vector bundle maps from each face of E to the corresponding face of E' — more details on the definition of a triple vector bundle morphism can be found in [13]. Assume that F maps the grid of E to a grid of E'. What happens to the induced grids on the three core double vector bundles and the ultrawarps? Exactly what we expect.

Proposition 4.2.4. *Let $F : E \to E'$ be a morphism of triple vector bundles, that is, a system of six double vector bundle morphisms between the corresponding upper and lower faces of E and of E'. If a grid on E as in (3.1) projects via F to a grid on E', and u_{BF}, u_{LR}, u_{UD} are the ultrawarps of the grid on E and $\mathrm{u}'_{\mathsf{BF}}$, $\mathrm{u}'_{\mathsf{LR}}$, $\mathrm{u}'_{\mathsf{UD}}$ are the ultrawarps of the grid on E', then*

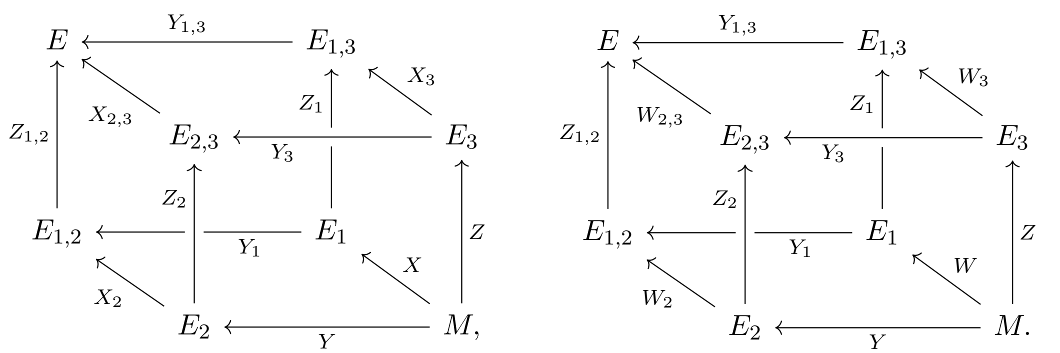
$$\mathrm{u}'_{\mathsf{BF}}(f(m)) = F\Big|_{E_{123}}((\mathrm{u}_{\mathsf{BF}})(m)), \quad m \in M,$$

where $F\Big|_{E_{123}} : E_{123} \to E'_{123}$ is the ultracore morphism of F over $f : M \to M'$, that is, the restriction of F to the ultacore vector bundles.

Proof. We sketch the proof for u_{UD}. The triple vector bundle map F induces a double vector bundle morphism from the (U-D) core double vector bundle of E to the (U-D) core double vector bundle of E'. Then the induced grid on the (U-D) core double vector bundle of E is related to the one induced on the (U-D) core double vector bundle of E' as in (4.8) in Proposition 4.2.3, via the induced core double vector bundle morphism of F. The result then follows immediately. □

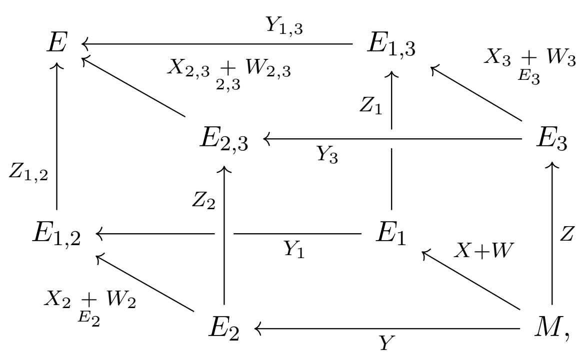
Now a proposition about ultrawarps, an extension of Proposition 4.2.1, (5).

Proposition 4.2.5. *Suppose the following two grids in (4.9) have corresponding ultrawarps $\mathrm{u}^X_{\mathsf{BF}}, \mathrm{u}^X_{\mathsf{LR}}, \mathrm{u}^X_{\mathsf{UD}}$ and $\mathrm{u}^W_{\mathsf{BF}}, \mathrm{u}^W_{\mathsf{LR}}, \mathrm{u}^W_{\mathsf{UD}}$,*



(4.9)

Then the following grid on E



(4.10)

where we have added the two front-back linear double sections $X_{2,3}$ and $W_{2,3}$, has ultrawarps

$$\mathrm{u}_{\mathsf{BF}}^{X+W} = \mathrm{u}_{\mathsf{BF}}^{X} + \mathrm{u}_{\mathsf{BF}}^{W}, \quad \mathrm{u}_{\mathsf{LR}}^{X+W} = \mathrm{u}_{\mathsf{LR}}^{X} + \mathrm{u}_{\mathsf{LR}}^{W}, \quad \mathrm{u}_{\mathsf{UD}}^{X+W} = \mathrm{u}_{\mathsf{UD}}^{X} + \mathrm{u}_{\mathsf{UD}}^{W}\,.$$

Proof. For the proof of this result, we abandon the succinct notation introduced in (3.18) and in Section 3.2.1. We use the original notation of the grids involved.

We will prove that $\mathrm{u}_{\mathsf{BF}}^{X+W} = \mathrm{u}_{\mathsf{BF}}^{X} + \mathrm{u}_{\mathsf{BF}}^{W}$, and the other two ultrawarps will follow likewise.

First, about $\mathrm{u}_{\mathsf{BF}}^{X}$, we will use a variation of the form (3.22) of equation (3.31a) to describe it:

$$\begin{aligned}
&\mathrm{w}_{\mathrm{back}}(X(m)) \underset{1,3}{-} X_{23}(\mathrm{w}_{\mathrm{front}}(m)) \\
&= \left[\left(Z_{1,2}(Y_1(X(m))) \underset{1,2}{-} Y_{1,3}(Z_1(X(m)))\right) \underset{1,3}{-} \hat{0}_{Y_1(X(m))}\right] \\
&\underset{1,3}{-} \left[\left(X_{2,3}(Z_2(Y(m))) \underset{1,2}{-} X_{2,3}(Y_3(Z(m)))\right) \underset{1,3}{-} \hat{0}_{X_2(Y(m))}\right] \\
&= \hat{0}_{X(m)} \underset{2,3}{+} \mathrm{u}_{\mathsf{BF}}^{X}(m).
\end{aligned}$$

A similar equation describes $\mathrm{u}_{\mathsf{BF}}^{W}(m)$.

Let us write the induced grid on the (B-F) core double vector bundle, for diagram (4.10),

$$\begin{array}{ccc}
E_{23,1} & \overset{X_{23}+W_{23}}{\rightleftarrows} & E_{23} \\
\mathrm{w}_{\mathrm{back}} \updownarrows & & \updownarrows \mathrm{w}_{\mathrm{front}} \\
E_1 & \underset{X+W}{\rightleftarrows} & M.
\end{array}$$

Denote by (X_{23}, X) the core morphism of the linear double section $X_{2,3}$, and by (W_{23}, W) the core morphism of $W_{2,3}$. It follows immediately that the core morphism of $(X_{2,3} \underset{2,3}{+} W_{2,3}; X_2 \underset{E_2}{+} W_2, X_3 \underset{E_3}{+} W_3; X + W)$ is $(X_{23} \underset{2,3}{+} W_{2,3}, X + W)$. For $w_{23} \in E_{23}$, the core of the Down face, then

$$(X_{2,3} \underset{2,3}{+} W_{2,3})(w_{23}) = X_{2,3}(w_{23}) \underset{2,3}{+} W_{2,3}(w_{23}) = X_{23}(w_{23}) \underset{2,3}{+} W_{23}(w_{23}).$$

Therefore, the equation that describes $\mathrm{u}_{\mathsf{BF}}^{X+W}(m)$, is

$$\begin{aligned}
&\mathrm{w}_{\text{back}}((X+W)(m)) \underset{1,3}{-} (X_{23} \underset{2,3}{+} W_{23})(\mathrm{w}_{\text{front}}(m)) \\
&= \Big[\Big(Z_{1,2}(Y_1((X+W)(m))) \underset{1,2}{-} Y_{1,3}(Z_1((X+W)(m)))\Big) \underset{1,3}{-} \hat{0}_{Y_1((X+W)(m))}\Big] \\
&\underset{1,3}{-} \Big[\Big((X_{2,3} \underset{2,3}{+} W_{2,3})(Z_2(Y(m))) \underset{1,2}{-} (X_{2,3} \underset{2,3}{+} W_{2,3})(Y_3(Z(m)))\Big) \underset{1,3}{-} \hat{0}_{(X_2 \underset{E_2}{+} W_2)(Y(m))}\Big] \\
&= \hat{0}_{(X+W)(m)} \underset{2,3}{+} \mathrm{u}_{\mathsf{BF}}^{X+W}(m).
\end{aligned} \tag{4.11}$$

Start from

$$\begin{aligned}
&\Big[\Big(Z_{1,2}(Y_1((X+W)(m))) \underset{1,2}{-} Y_{1,3}(Z_1((X+W)(m)))\Big) \underset{1,3}{-} \hat{0}_{Y_1((X+W)(m))}\Big] \\
\underset{1,3}{-} &\Big[\Big((X_{2,3} \underset{2,3}{+} W_{2,3})(Z_2(Y(m))) \underset{1,2}{-} (X_{2,3} \underset{2,3}{+} W_{2,3})(Y_3(Z(m)))\Big) \underset{1,3}{-} \hat{0}_{(X_2 \underset{E_2}{+} W_2)(Y(m))}\Big]
\end{aligned}$$

and using (3.7) and (3.5) in the first bracket, rewrite the previous equation as

$$\begin{aligned}
&\Big[\Big(\Big(Z_{1,2}(Y_1(X(m))) \underset{2,3}{+} Z_{1,2}(Y_1(W(m)))\Big) \underset{1,2}{-} \Big(Y_{1,3}(Z_1(X(m))) \underset{2,3}{+} Y_{1,3}(Z_1(W(m)))\Big)\Big) \\
&\qquad \underset{1,3}{-} \hat{0}_{Y_1(X(m)) \underset{E_2}{+} Y_1(W(m))}\Big] \\
\underset{1,3}{-} &\Big[\Big(\Big(X_{2,3}(Z_2(Y(m))) \underset{2,3}{+} W_{2,3}(Z_2(Y(m)))\Big) \underset{1,2}{-} \Big(X_{2,3}(Y_3(Z(m))) \underset{2,3}{+} W_{2,3}(Y_3(Z(m)))\Big)\Big) \\
&\qquad \underset{1,3}{-} \hat{0}_{X_2(Y(m)) \underset{E_2}{+} W_2(Y(m))}\Big].
\end{aligned}$$

Now i) use the interchange law in the Left face of E, in the first parentheses of each bracket, and ii) use (2.18) for the zeros,

$$\begin{aligned}
&\Big[\Big(\Big(Z_{1,2}(Y_1(X(m))) \underset{1,2}{-} Y_{1,3}(Z_1(X(m)))\Big) \underset{2,3}{+} \Big(Z_{1,2}(Y_1(W(m))) \underset{1,2}{-} Y_{1,3}(Z_1(W(m)))\Big)\Big) \\
&\qquad \underset{1,3}{-} \Big(\hat{0}_{Y_1(X(m))} \underset{2,3}{+} \hat{0}_{Y_1(W(m))}\Big)\Big] \\
\underset{1,3}{-} &\Big[\Big(\Big(X_{2,3}(Z_2(Y(m))) \underset{1,2}{-} X_{2,3}(Y_3(Z(m)))\Big) \underset{2,3}{+} \Big(W_{2,3}(Z_2(Y(m))) \underset{1,2}{-} W_{2,3}(Y_3(Z(m)))\Big)\Big) \\
&\qquad \underset{1,3}{-} \Big(\hat{0}_{X_2(Y(m))} \underset{2,3}{+} \hat{0}_{W_2(Y(m))}\Big)\Big].
\end{aligned}$$

Now, in each bracket, use the interchange law in the Up face of E

$$\begin{aligned}
&\Bigg[\bigg(\Big(Z_{1,2}(Y_1(X(m))) \underset{1,2}{-} Y_{1,3}(Z_1(X(m)))\Big) \underset{1,3}{-} \hat{0}_{Y_1(X(m))}\bigg)\\
\underset{2,3}{+}\ &\bigg(\Big(Z_{1,2}(Y_1(W(m))) \underset{1,2}{-} Y_{1,3}(Z_1(W(m)))\Big) \underset{1,3}{-} \hat{0}_{Y_1(W(m))}\bigg)\Bigg]\\
\underset{1,3}{-}\ &\Bigg[\bigg(\Big(X_{2,3}(Z_2(Y(m))) \underset{1,2}{-} X_{2,3}(Y_3(Z(m)))\Big) \underset{1,3}{-} \hat{0}_{X_2(Y(m))}\bigg)\\
\underset{2,3}{+}\ &\bigg(\Big(W_{2,3}(Z_2(Y(m))) \underset{1,2}{-} W_{2,3}(Y_3(Z(m)))\Big) \underset{1,3}{-} \hat{0}_{W_2(Y(m))}\bigg)\Bigg].
\end{aligned}$$

Finally, use once more the interchange law in the Up face of E,

$$\begin{aligned}
&\Bigg[\bigg(\Big(Z_{1,2}(Y_1(X(m))) \underset{1,2}{-} Y_{1,3}(Z_1(X(m)))\Big) \underset{1,3}{-} \hat{0}_{Y_1(X(m))}\bigg)\\
\underset{1,3}{-}\ &\bigg(\Big(X_{2,3}(Z_2(Y(m))) \underset{1,2}{-} X_{2,3}(Y_3(Z(m)))\Big) \underset{1,3}{-} \hat{0}_{X_2(Y(m))}\bigg)\Bigg]\\
\underset{2,3}{+}\ &\Bigg[\bigg(\Big(Z_{1,2}(Y_1(W(m))) \underset{1,2}{-} Y_{1,3}(Z_1(W(m)))\Big) \underset{1,3}{-} \hat{0}_{Y_1(W(m))}\bigg)\\
\underset{1,3}{-}\ &\bigg(\Big(W_{2,3}(Z_2(Y(m))) \underset{1,2}{-} W_{2,3}(Y_3(Z(m)))\Big) \underset{1,3}{-} \hat{0}_{W_2(Y(m))}\bigg)\Bigg].
\end{aligned}$$

The first bracket now is exactly the equation that describes the $\mathrm{u}_{\mathsf{BF}}^X(m)$ and the second bracket describes the $\mathrm{u}_{\mathsf{BF}}^W(m)$, hence the last equation is equal to,

$$\Big[\hat{0}_{X(m)} \underset{2,3}{+} \mathrm{u}_{\mathsf{BF}}^X(m)\Big] \underset{2,3}{+} \Big[\hat{0}_{W(m)} \underset{2,3}{+} \mathrm{u}_{\mathsf{BF}}^W(m)\Big] = \hat{0}_{X(m)+W(m)} \underset{2,3}{+} (\mathrm{u}_{\mathsf{BF}}^X(m) \underset{2,3}{+} \mathrm{u}_{\mathsf{BF}}^W(m)).$$

Comparing the last equation with (4.11), from the uniqueness of ultracore elements, it follows that $\mathrm{u}_{\mathsf{BF}}^{X+W}(m) = \mathrm{u}_{\mathsf{BF}}^X(m) + \mathrm{u}_{\mathsf{BF}}^W(m)$.

□

4.2.2 Bolts

So far we have seen in Chapter 1 that a section of the core C of a double vector bundle D defines the strut $c^A \in \Gamma_A D$ over A, see Definition 1.1.3,

$$c^A : A \to D, \quad a \mapsto c(q_a^D(a)) \underset{B}{+} 0_a^D.$$

For $a_1, a_2 \in A_m$,

$$\begin{aligned}
c^A(a_1 + a_2) = c(m) \underset{B}{+} 0^D_{a_1+a_2} &= c(m) \underset{B}{+} (0^D_{a_1} \underset{B}{+} 0^D_{a_2})\\
&\neq (c(m) \underset{B}{+} c(m)) \underset{B}{+} (0^D_{a_1} \underset{B}{+} 0^D_{a_2}) = c^A(a_1) \underset{B}{+} c^A(a_2),
\end{aligned}$$

in other words, c^A is not a linear section of $D \to A$, as we have already seen in local coordinates towards the end of Section 1.1.3. Similarly for the strut $c^B \in \Gamma_B D$. Except for the zero section, struts are not linear sections. This is how we arrive at the definition of the *bolt section.* Bolt sections are defined via the core vector bundle, and in addition, are linear.

Definition 4.2.6. Let $\varphi : A \to C$ be a vector bundle map over M. Define a section $\varphi^{\lightning} \in \Gamma_A D$ by

$$\varphi^{\lightning}(a) = \varphi(a) \underset{B}{+} 0_a^D.$$

We call $\varphi^{\lightning}$ the *bolt* of φ. First, note that $\varphi^{\lightning}$ projects to the zero section:

$$q_B^D \circ \varphi^{\lightning}(a) = q_B^D(\varphi(a) \underset{B}{+} 0_a^D) = 0_m^B.$$

Secondly, $(\varphi^{\lightning}, 0^B)$ is a linear section. Take $a_1, a_2 \in A_m$,

$$\begin{aligned}\varphi^{\lightning}(a_1 + a_2) = \varphi(a_1 + a_2) \underset{B}{+} 0^D_{a_1+a_2} &= (\varphi(a_1) \underset{B}{+} \varphi(a_2)) \underset{B}{+} (0^D_{a_1} \underset{B}{+} 0^D_{a_2}) \\ &= (\varphi(a_1) \underset{B}{+} 0^D_{a_1}) \underset{B}{+} (\varphi(a_2) \underset{B}{+} 0^D_{a_2}) = \varphi^{\lightning}(a_1) \underset{B}{+} \varphi^{\lightning}(a_2),\end{aligned}$$

and similarly for the scalar multiplication.

In a local coordinate system (x, a, b, z) on D the vector bundle map $\varphi : A \to C$ over M,

$$(x^1, \ldots, x^n, a^1, \ldots, a^{r_A}) \mapsto (x^1, \ldots, x^n, z^1_j(x)a^j, \ldots, z^{r_C}_j(x)a^j),$$

hence $\varphi^{\lightning} \in \Gamma_A D$,

$$(x^1, \ldots, x^n, a^1, \ldots, a^{r_A}) \mapsto (x^1, \ldots, x^n, a^1, \ldots, a^{r_A}, 0^1, \ldots, 0^{r_B}, z^1_j(x)a^j, \ldots, z^{r_C}_j(x)a^j). \tag{4.12}$$

Comparing the last equation with (1.15) we see directly how struts and bolts differ. Now rewrite the last equation as

$$\begin{aligned}&(x^1, \ldots, x^n, a^1, \ldots, a^{r_A}, 0^1, \ldots, 0^{r_B}, 0^1, \ldots, 0^{r_C}) \\ &\quad \underset{B}{+} (x^1, \ldots, x^n, 0^1, \ldots, 0^{r_A}, 0^1, \ldots, 0^{r_B}, z^1_j(x)a^j, \ldots, z^{r_C}_j(x)a^j) = 0^D_a \underset{B}{+} \varphi(a).\end{aligned}$$

Finally, we note that using decompositions, in particular using (1.5), a bolt section $(\varphi^{\lightning}, 0^B)$ can be described by

$$\varphi^{\lightning}(a) = \mho(a, 0, \varphi(a)).$$

If we start with a vector bundle map $\psi : B \to C$ over M, we can define in a similar fashion a bolt section $(\psi^{\lightning}, 0^A)$ of the vertical structure of D.

So far we have seen that given a vector bundle map $\varphi : A \to C$ over M, we can define a linear section $(\varphi^{\lightning}, 0^B)$ of $D \to A$. The converse is also true.

Lemma 4.2.7. *Every linear section $(\eta, 0^B)$ is a bolt section for a unique $\varphi : A \to C$.*

Proof. Suppose that $(\eta, 0^B)$ is a linear section. For $a \in A_m$ the outline of $\eta(a)$,

$$\begin{array}{ccc} \eta(a) & \longmapsto & 0^B_m \\ \downarrow & & \downarrow \\ a & \longmapsto & m. \end{array}$$

Therefore, we can write $\eta(a) = c \underset{B}{+} 0^D_a$, for a unique $c \in C_m$, see Section 1.1.2 and (3). Define $\varphi(a) = c$. Then

$$\eta(a) = \varphi(a) \underset{B}{+} 0^D_a.$$

To show that φ is a vector bundle morphism, take $a_1, a_2 \in A_m$. We will show that

$$\varphi(a_1 + a_2) = \varphi(a_1) + \varphi(a_2). \tag{4.13}$$

Write

$$\eta(a_1) = \varphi(a_1) \underset{B}{+} 0^D_{a_1}, \quad \eta(a_2) = \varphi(a_2) \underset{B}{+} 0^D_{a_2},$$

and since η is linear over 0^B,

$$\begin{aligned} \eta(a_1) \underset{B}{+} \eta(a_2) &= (\varphi(a_1) \underset{B}{+} 0^D_{a_1}) \underset{B}{+} (\varphi(a_2) \underset{B}{+} 0^D_{a_2}) \\ &= (\varphi(a_1) \underset{B}{+} \varphi(a_2)) \underset{B}{+} (0^D_{a_1} \underset{B}{+} 0^D_{a_2}) = (\varphi(a_1) \underset{B}{+} \varphi(a_2)) \underset{B}{+} 0^D_{a_1+a_2}. \end{aligned} \tag{4.14}$$

Again, since η is linear,

$$\eta(a_1 + a_2) = \varphi(a_1 + a_2) \underset{B}{+} 0^D_{a_1+a_2}, \tag{4.15}$$

therefore, comparing (4.14) and (4.15), we obtain (4.13). Following a similar calculation for the scalar multiplication, we establish that φ is a vector bundle map. And finally, we see directly that $\varphi^{\lightning} = \eta$. □

The following property is an immediate consequence of the definition of bolt sections.

Corollary 4.2.8. *If $(\varphi^{\lightning}, 0^B)$ is a bolt section, then*

$$\varphi^{\lightning}(0^A_m) = \odot^D_m,$$

where $\odot^D_m$ is the double zero of D over $m \in M$.

Proof. We have that

$$\varphi^{\lightning}(0^A_m) = \varphi(0^A_m) \underset{B}{+} 0^D_{0^A_m} = \varphi(0^A_m),$$

and since $\varphi : A \to C$ is a vector bundle map over M, it will send the zero of A_m to the zero of C_m, that is, $\varphi(0^A_m) = 0^C_m$. Therefore,

$$\varphi^{\lightning}(0^A_m) = 0^C_m = \odot^D_m.$$

□

The next proposition will prove very useful.

Proposition 4.2.9. *Take (η_1, Y) and (η_2, Y) two linear sections of D, where $Y \in \Gamma B$ and $\eta_1, \eta_2 \in \Gamma_A D$. Their difference $(\eta_1 \underset{A}{-} \eta_2, 0^B)$ defines a bolt section corresponding to a unique vector bundle map $\varphi : A \to C$.*

Proof. We follow the same procedure as in the proof of Lemma 4.2.7.

Looking at $\eta_1(a) \underset{A}{-} \eta_2(a)$, we can write it as

$$\eta_1(a) \underset{A}{-} \eta_2(a) = c \underset{B}{+} 0^D_a,$$

where $c \in C$ is unique. Define a map $\varphi : A \to C$ by $\varphi(a) = c$. With a similar calculation as in the proof of Lemma 4.2.7, it follows that φ is a vector bundle map. Finally, $\varphi^{\lightning}(a) = \varphi(a) \underset{B}{+} 0^D_a = \eta_1(a) \underset{A}{-} \eta_2(a)$, hence $\eta_1 \underset{A}{-} \eta_2 = \varphi^{\lightning}$. □

Remark 4.2.10. In Remark 4.1.3 we described how struts and warps differ. At this point we describe how bolts differ from warps. For example, if (ξ_1, X) and (ξ_2, X) are two linear sections of the horizontal structure of D, $\xi_1, \xi_2 \in \Gamma_B D$, $X \in \Gamma A$, then from Proposition 4.2.9 they define a bolt section $\psi^{\lightning} \in \Gamma_B D$:

$$\xi_1(b) \underset{B}{-} \xi_2(b) = \psi^{\lightning}(b) = 0^D_b \underset{A}{+} \psi(b).$$

If (ξ, X) and (η, Y) is the usual grid on D as in (10), then the warp $\mathrm{w}(\xi, \eta)$ is described by (11),

$$\xi(Y(m)) \underset{B}{-} \eta(X(m)) = 0^D_{Y(m)} \underset{A}{+} \mathrm{w}(\xi, \eta)(m).$$

Clearly, the blue arguments in the left hand side are different; the first term has argument $Y(m)$ and the second term has argument $X(m)$. Recall from Remark 4.1.3 that the right hand side of the last equation is not the strut defined by the warp $\mathrm{w}(\xi, \eta) \in \Gamma C$. △

4.2.3 Bolts and warps

We now proceed with what we are really interested in, warps of pairs of linear sections.

Proposition 4.2.11. *We have the following:*

1. *The warp of a bolt section and of the zero section is the zero section.*

2. *The warp of a linear section and of the zero section is the zero section.*

3. *The warp of two bolt sections is the zero section.*

4. *The warp of a linear section* (ξ, X)*, with* $\xi \in \Gamma_B D$*,* $X \in \Gamma A$*, and of a bolt section* $(\varphi^{\lightning}, 0^B)$*, is the section*

$$\mathrm{w}(\xi, \varphi^{\lightning}) = -\varphi \circ X. \tag{4.16}$$

Proof. For (1), take $(\psi^{\lightning}, 0^A)$ a bolt section with $\psi^{\lightning} \in \Gamma_B D$ and the linear zero section $(0^D_A, 0^B)$:

$$\begin{array}{ccc} D & \overset{\psi^{\lightning}}{\rightleftarrows} & B \\ {\scriptstyle 0^D_A}\upharpoonleft\downarrow & & \downarrow\upharpoonright{\scriptstyle 0^B} \\ A & \underset{0^A}{\rightleftarrows} & M \end{array}$$

What is the difference between $\psi^{\lightning}(0^B_m)$ and $0^D_A \circ 0^A_m = \odot^D_m$? From Corollary 4.2.8, $\psi^{\lightning}(0^B_m) = \odot^D_m$ as well, therefore, their warp is the zero section of the core vector bundle.

For (2), take (ξ, X) a linear section, with $\xi \in \Gamma_B D$, $X \in \Gamma A$, and the linear zero section $(0^D_A, 0^B)$:

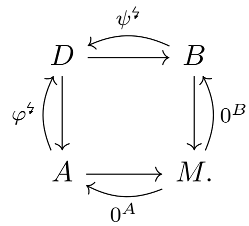
$$\begin{array}{ccc} D & \overset{\xi}{\rightleftarrows} & B \\ {\scriptstyle 0^D_A}\upharpoonleft\downarrow & & \downarrow\upharpoonright{\scriptstyle 0^B} \\ A & \underset{X}{\rightleftarrows} & M. \end{array}$$

In this case, the outlines of the two elements $\xi(0^B_m)$ and $0^D_A(X(m))$:

$$\begin{array}{ccc} \xi(0^B_m) = 0^D_{X(m)} & \longmapsto & 0^B_m \\ \downarrow & & \downarrow \\ X(m) & \longmapsto & m, \end{array} \qquad \begin{array}{ccc} 0^D_A(X(m)) = 0^D_{X(m)} & \longmapsto & 0^B_m \\ \downarrow & & \downarrow \\ X(m) & \longmapsto & m, \end{array}$$

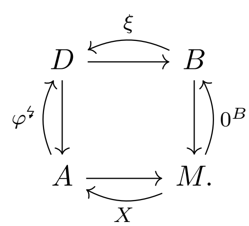
hence the two elements $\xi(0^B_m)$ and $0^D_A(X(m))$ are the same, therefore, their warp is the zero section of the core vector bundle.

For (3), take $(\psi^{\lightning}, 0^A)$ and $(\varphi^{\lightning}, 0^B)$ two bolt sections, with $\psi^{\lightning} \in \Gamma_B D$ and $\varphi^{\lightning} \in \Gamma_A D$. The corresponding grid on D,



Again, by Corollary 4.2.8, $\psi^{\lightning}(0^B_m) = \varphi^{\lightning}(0^A_m) = \odot^D_m$, therefore, their warp is the zero section of the core vector bundle.

Finally, for (4.16), the grid on D defined by the two linear sections:



The unique core element $c \in C$ their two differences define:

$$\begin{aligned} \xi(0^B_m) \underset{A}{-} \varphi^{\lightning}(X(m)) &= c \underset{B}{+} 0^D_{X(m)}, \\ \xi(0^B_m) \underset{B}{-} \varphi^{\lightning}(X(m)) &= c \underset{A}{+} 0^D_{0^B_m} = c. \end{aligned}$$

Write $\varphi^{\lightning}(X(m)) = \varphi(X(m)) \underset{B}{+} 0^D_{X(m)}$, where $\varphi : A \to C$ is the corresponding vector bundle map over M. Then,

$$\begin{aligned} \xi(0^B_m) \underset{A}{-} \varphi^{\lightning}(X(m)) &= 0^D_{X(m)} \underset{A}{-} \left(\varphi(X(m)) \underset{B}{+} 0^D_{X(m)}\right) \\ &= \left(0^D_{X(m)} \underset{B}{+} \odot^D_m\right) \underset{A}{-} \left(\varphi(X(m)) \underset{B}{+} 0^D_{X(m)}\right) \\ &= \left(0^D_{X(m)} \underset{A}{-} 0^D_{X(m)}\right) \underset{B}{+} \left(\odot^D_m \underset{A}{-} \varphi(X(m))\right) \\ &= 0^D_{X(m)} \underset{B}{-} \varphi(X(m)), \end{aligned}$$

and of course the same is true for the other difference:

$$\begin{aligned} \xi(0^B_m) \underset{B}{-} \varphi^{\lightning}(X(m)) = 0^D_{X(m)} \underset{B}{-} \left(\varphi(X(m)) \underset{B}{+} 0^D_{X(m)}\right) \\ = \left(0^D_{X(m)} \underset{B}{-} 0^D_{X(m)}\right) \underset{B}{-} \varphi(X(m)) = \odot^D_m \underset{B}{-} \varphi(X(m)) = \underset{B}{-} \varphi(X(m)). \end{aligned}$$

Hence $\mathrm{w}(\xi, \varphi^{\lightning}) = -\varphi \circ X \in \Gamma C$.

To see the last statement in local coordinates, in a local coordinate system (x, a, b, z) on D, from (1.16) we have,

$$\begin{aligned}\xi(0^B(m)) &= (x^1, \dots, x^n, a^1(x), \dots, a^{r_A}(x), 0^1, \dots, 0^{r_B}, z_k^1(x)0^k, \dots, z_k^{r_C}(x)0^k) \\ &= (x^1, \dots, x^n, a^1(x), \dots, a^{r_A}(x), 0^1, \dots, 0^{r_B}, 0^1, \dots, 0^{r_C}),\end{aligned}$$

and from (4.12),

$$\varphi^{\lightning}(X(m)) = (x^1, \dots, x^n, a^1(x), \dots, a^{r_A}(x), 0^1, \dots, 0^{r_B}, z_j^1(x)a^j(x), \dots, z^{r_C}(x)a^j(x)).$$

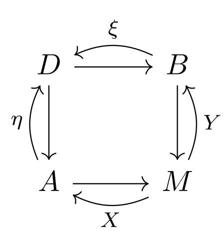
Hence

$$\begin{aligned}&\xi(0^B(m)) \underset{A}{-} \varphi^{\lightning}(X(m)) \\ &\quad = (x^1, \dots, x^n, a^1(x), \dots, a^{r_A}(x), 0^1, \dots, 0^{r_B}, -z_j^1(x)a^j(x), \dots, -z^{r_C}(x)a^j(x)) \\ &\qquad = 0^D_{X(m)} \underset{B}{-} \varphi(X(m))\end{aligned}$$

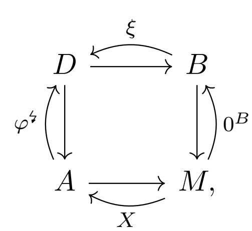
□

The following special case of Proposition 4.2.1, (2), that “the warp of the sum is the sum of the warp” is important to Section 4.5.3.

Proposition 4.2.12. *Given two grids* (ξ, X), (η, Y), *and* (ξ, X), $(\varphi^{\lightning}, 0^B)$ *on a double vector bundle* D,



and



with warps $\mathrm{w}(\xi, \eta)$ *and* $\mathrm{w}(\xi, \varphi^{\lightning})$, *then*

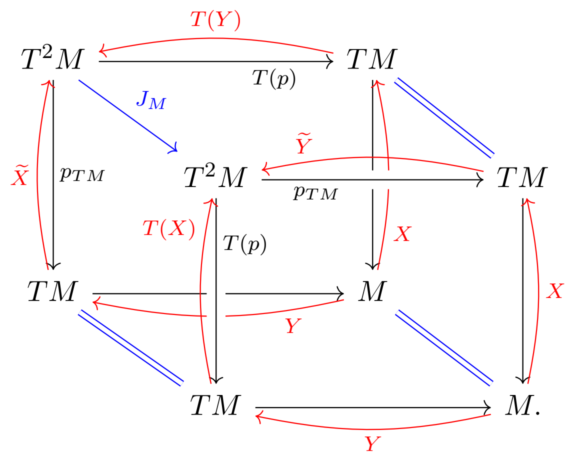
$$\mathrm{w}(\xi, \eta \underset{A}{+} \varphi^{\lightning}) = \mathrm{w}(\xi, \eta) + \mathrm{w}(\xi, \varphi^{\lightning}).$$

4.2.4 Grids on T^2M and TA

Using the canonical involution $J_M : T^2M \to T^2M$ and bolt sections, we further investigate grids on T^2M.

J_M and warps

Apply J_M to the double vector bundle T^2M and to the grid (9). We have:



(4.17)

Apply J_M to (8):

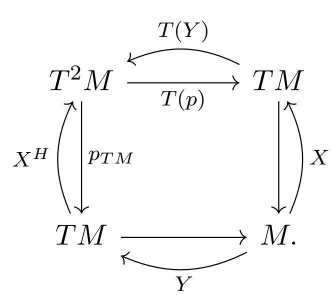
$$J_M\left(T_m(Y)(X(m)) \underset{p_{TM}}{-} \widetilde{X}(Y(m))\right) = J_M\left(0^{T^2M}_{Y(m)} \underset{T(p)}{+} \overline{[X,Y](m)}\right)$$
$$\Rightarrow J_M(T(Y)(X(m))) \underset{T(p)}{-} J_M(\widetilde{X}(Y(m))) = J_M(0^{T^2M}_{Y(m)}) \underset{p_{TM}}{+} J_M(\overline{[X,Y](m)})$$
$$\Rightarrow \widetilde{Y}(X(m)) \underset{T(p)}{-} T(X)(Y(m)) = T(0^{TM})(Y(m)) \underset{p_{TM}}{+} \overline{[X,Y](m)}, \quad (4.18)$$

so we see that the new grid has the same warp as grid (9) because J_M is the identity on the cores: $J_M(\overline{[X,Y](m)}) = \overline{[X,Y](m)}$.

Equivalently, by applying directly Proposition 4.2.3, it follows that the warp of the resulting grid of (4.17) is $[X,Y] \in \Gamma(TM)$. Both (8) and (4.18) are mentioned in [1, p.297].

Warps and conjugate connections

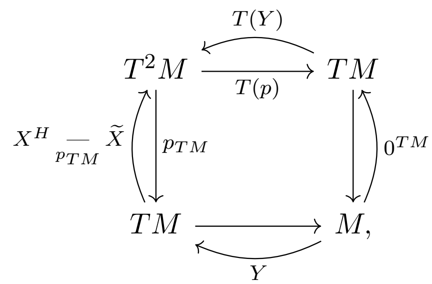
Take a connection ∇ in TM, and for $X, Y \in \mathfrak{X}(M)$ build the following grid on T^2M as in (1.46),



(4.19)

By (1.45), the warp of (4.19) has warp $\mathrm{w}(T(Y), X^H) = \nabla_X Y$.

Now take both grids: (4.19), and (9) on page xiv. Apply Proposition 4.2.1, (2). Then the following grid,



(4.20)

has warp

$$\mathrm{w}(T(Y), X^H) - \mathrm{w}(T(Y), \widetilde{X}) = \nabla_X Y - [X, Y],$$

which is exactly $\widehat{\nabla}_Y X$, see (1.48).

What is $X^H(Y(m)) \underset{p_{TM}}{-} \widetilde{X}(Y(m))$? Both the horizontal lift (X^H, X) and the complete lift $(\widetilde{X}, X)$ of a vector field $X \in \mathfrak{X}(M)$ are linear vector fields on TM that project to $X \in \mathfrak{X}(M)$. By Proposition 4.2.9, $X^H(Y(m)) \underset{p_{TM}}{-} \widetilde{X}(Y(m))$ is a bolt section,

$$X^H(Y(m)) \underset{p_{TM}}{-} \widetilde{X}(Y(m)) = 0^{T^2M}_{Y(m)} \underset{T(p)}{+} \overline{\varphi(Y(m))} = \varphi^{\lightning}(Y(m)),$$

a vertical vector field on TM. The vector bundle map $\varphi : TM \to TM$ over M is precisely the vector bundle map $-\widehat{\nabla}X : TM \to TM$, $Y(m) \mapsto -(\widehat{\nabla}X)(Y(m)) = -\widehat{\nabla}_{Y(m)}X$, whose corresponding map on the sections of TM is none other than the total covariant derivative $-\widehat{\nabla}X : \mathfrak{X}(M) \to \mathfrak{X}(M)$, $Y \to -\widehat{\nabla}_Y X$.

We work this out in local coordinates. Write $Y = Y^j \frac{\partial}{\partial x^j}$ and $X = X^i \frac{\partial}{\partial x^i}$, where $X^i, Y^j \in C^\infty(M)$. Then from (1.49),

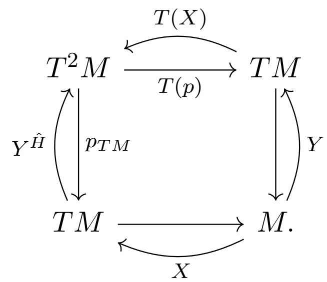
$$\widehat{\nabla}_Y X = \left(Y^j \frac{\partial X^i}{\partial x^j} + Y^j \widehat{\Gamma}^i_{jk} X^k \right) \frac{\partial}{\partial x^i} = \left(Y^j \frac{\partial X^i}{\partial x^j} + X^k \Gamma^i_{kj} Y^j \right) \frac{\partial}{\partial x^i}.$$

Additionally, from (1.47) and (1.33), and since $(\widehat{\nabla}_Y X)(m) = \widehat{\nabla}_{Y(m)} X = (\widehat{\nabla}X)(Y(m))$, it follows that,

$$\begin{aligned}
&X^H(Y(m)) \underset{p_{TM}}{-} \widetilde{X}(Y(m)) \\
&= \quad (x^1, .., x^n, Y^1, \ldots, Y^n, X^1, \ldots, X^n, -X^k \Gamma^1_{kj} Y^j, \ldots, -X^k \Gamma^n_{kj} Y^j) \\
&\underset{p_{TM}}{-} \quad (x^1, \ldots, x^n, Y^1, \ldots, Y^n, X^1, \ldots, X^n, Y^j \frac{\partial X^1}{\partial x^j}, \ldots, Y^j \frac{\partial X^n}{\partial x^j}) \\
&= \quad (x^1, \ldots, x^n, Y^1, \ldots, Y^n, 0^1, \ldots, 0^n, -X^k \Gamma^1_{kj} Y^j - Y^j \frac{\partial X^1}{\partial x^j}, \ldots, -X^k \Gamma^n_{kj} Y^j - Y^j \frac{\partial X^n}{\partial x^j}) \\
&= \quad 0^{T^2M}_{Y(m)} \underset{T(p)}{-} \overline{\widehat{\nabla}X(Y(m))},
\end{aligned}$$

that is, $\varphi = -\widehat{\nabla} X$.

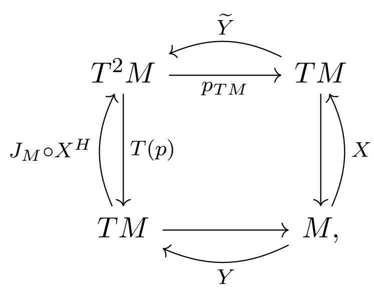
On the other hand, start directly with the conjugate connection $\widehat{\nabla}$ of ∇ in TM, and build the following grid on T^2M for $X, Y \in T^2M$ as in (1.46)



(4.21)

From (1.45), its warp is $\mathrm{w}(T(X), Y^{\hat{H}}) = \widehat{\nabla}_Y X$. Hence the two grids (4.20) and (4.21) have the same warp.

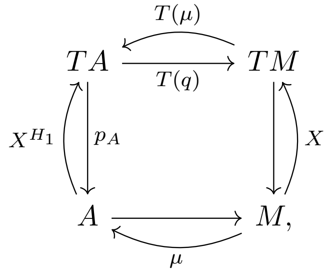
Applying J_M to the grid (4.19), we obtain the grid

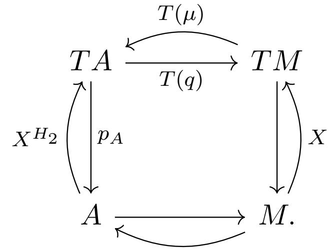


and again, since J_M is the identity on the core vector bundle, the warp of $(\widetilde{Y}, Y)$ and $(J_M \circ X^H, X)$ is also $\nabla_X Y \in \mathfrak{X}(M)$.

Affine space of connections on A

Take now two connections ∇^1 and ∇^2 in a vector bundle $A \to M$. For $\mu \in \Gamma A$ and $X \in \mathfrak{X}(M)$, we have the following two different grids on TA,





The warp of the first and of the second grid, using (1.45), are $\mathrm{w}(T(\mu), X^{H_1}) = \nabla^1_X\mu$, and $\mathrm{w}(T(\mu), X^{H_2}) = \nabla^2_X\mu$ repsectively. In particular,

$$\begin{aligned} T(\mu)(X(m)) \underset{A}{-} X^{H_1}(\mu(m)) &= 0^{TA}_{\mu(m)} \underset{TM}{+} (\nabla^1_X\mu)(m), \\ T(\mu)(X(m)) \underset{A}{-} X^{H_2}(\mu(m)) &= 0^{TA}_{\mu(m)} \underset{TM}{+} (\nabla^2_X\mu)(m). \end{aligned}$$

Both horizontal lifts X^{H_1} and X^{H_2} are linear vector fields over X, and therefore, by Proposition 4.2.9, their difference is a bolt section. To describe this bolt section, we need to describe the difference $(X^{H_1} \underset{A}{-} X^{H_2})(a)$, for any $a \in A_m$. For any $a \in A_m$, write $\mu(m) = a$ where $\mu \in \Gamma A$. Therefore, we can equivalently describe $(X^{H_1} \underset{A}{-} X^{H_2})(\mu(m))$:

$$X^{H_1}(\mu(m)) \underset{A}{-} X^{H_2}(\mu(m)) = 0^{TA}_{\mu(m)} \underset{TM}{+} \varphi(\mu(m)),$$

where $\varphi : A \to A$, a vector bundle map over M. Rewriting the previous two equations as

$$\begin{aligned} X^{H_1}(\mu(m)) &= T(\mu)(X(m)) \underset{A}{-} \left(0^{TA}_{\mu(m)} \underset{TM}{+} (\nabla^1_X\mu)(m)\right), \\ X^{H_2}(\mu(m)) &= T(\mu)(X(m)) \underset{A}{-} \left(0^{TA}_{\mu(m)} \underset{TM}{+} (\nabla^2_X\mu)(m)\right), \end{aligned}$$

it now follows,

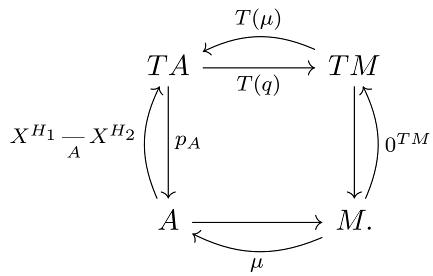
$$\begin{aligned} & X^{H_1}(\mu(m)) \underset{A}{-} X^{H_2}(\mu(m)) \\ = \;& \left[T(\mu)(X(m)) \underset{A}{-} \left(0^{TA}_{\mu(m)} \underset{TM}{+} (\nabla^1_X\mu)(m)\right)\right] \underset{A}{-} \left[T(\mu)(X(m)) \underset{A}{-} \left(0^{TA}_{\mu(m)} \underset{TM}{+} (\nabla^2_X\mu)(m)\right)\right] \\ = \;& \left[T(\mu)(X(m)) \underset{A}{-} T(\mu)(X(m))\right] \underset{A}{-} \left[\left(0^{TA}_{\mu(m)} \underset{TM}{+} (\nabla^1_X\mu)(m)\right) \underset{A}{-} \left(0^{TA}_{\mu(m)} \underset{TM}{+} (\nabla^2_X\mu)(m)\right)\right] \\ = \;& 0^{TA}_{\mu(m)} \underset{A}{-} \left[\left(0^{TA}_{\mu(m)} \underset{A}{-} 0^{TA}_{\mu(m)}\right) \underset{TM}{+} \left((\nabla^1_X\mu)(m) \underset{A}{-} (\nabla^2_X\mu)(m)\right)\right] \\ = \;& 0^{TA}_{\mu(m)} \underset{A}{-} \left[0^{TA}_{\mu(m)} \underset{TM}{+} \left((\nabla^1_X\mu)(m) \underset{A}{-} (\nabla^2_X\mu)(m)\right)\right] \\ = \;& 0^{TA}_{\mu(m)} \underset{TM}{-} \left((\nabla^1_X\mu)(m) \underset{A}{-} (\nabla^2_X\mu)(m)\right), \end{aligned}$$

where in the last step we applied the following variation of interchange laws (in general double vector bundle language),

$$0^D_a \underset{A}{-} (0^D_a \underset{B}{+} c) = (0^D_A \underset{B}{+} \odot^D_m) \underset{A}{-} (0^D_a \underset{B}{+} c) = (0^D_a \underset{A}{-} 0^D_a) \underset{B}{+} (\odot^D_m \underset{A}{-} c) = 0^D_a \underset{B}{-} c.$$

The difference of two connections is $C^\infty(M)$-linear in both X and μ. It defines a vector bundle map $TM \oplus A \to A$. Therefore, the corresponding vector bundle map is

$\varphi = -(\nabla^1_X - \nabla^2_X) : A \to A$ over M. Take the following grid on TA



By Proposition 4.2.1, (2), its warp is

$$\mathrm{w}(T(\mu), X^{H_1} \underset{A}{-} X^{H_2}) = \mathrm{w}(T(\mu), X^{H_1}) - \mathrm{w}(T(\mu), X^{H_2}) = \nabla^1_X\mu - \nabla^2_X\mu,$$

or, equivalently, from (4.16),

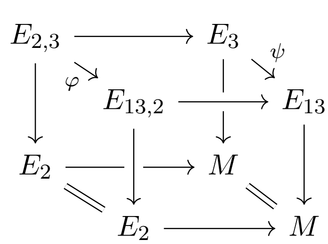
$$\mathrm{w}(T(\mu), X^{H_1} \underset{A}{-} X^{H_2}) = \mathrm{w}(T(\mu), \varphi^{\lightning}) = -(-(\nabla^1_X - \nabla^2_X)) \circ \mu = \nabla^1_X\mu - \nabla^2_X\mu.$$

4.3 Double Bolts

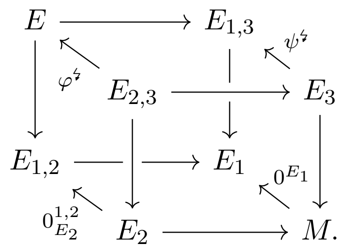
To define a single bolt section $(\varphi^{\lightning}, 0^B)$ in the double vector bundle setting, we used a vector bundle map $\varphi : A \to C$ over M. To define a *double bolt section*, we use a double vector bundle map from one of the lower faces of the triple vector bundle (Front, Right, Down), to a "corresponding" core double vector bundle. The "corresponding" cases are the following:

- Front $\to$ (L-R) core double vector bundle,
- Front $\to$ (U-D) core double vector bundle,
- Right $\to$ (B-F) core double vector bundle,
- Right $\to$ (U-D) core double vector bundle,
- Down $\to$ (B-F) core double vector bundle,
- Down $\to$ (L-R) core double vector bundle.

Definition 4.3.1. A double vector bundle map $(\varphi; \mathrm{id}_{E_2}, \psi; \mathrm{id}_M)$ from the Front face to the (L-R) core double vector bundle,



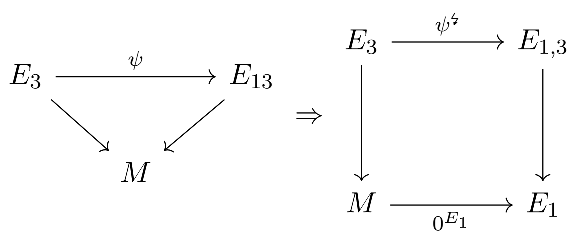
defines the following front-back linear double section



We call $(\varphi^{\lightning}; 0^{1,2}_{E_2}, \psi^{\lightning}; 0^{E_1})$ the double bolt section of the double vector bundle map $(\varphi; \mathrm{id}_{E_2}, \psi; \mathrm{id}_M)$.

We proceed with describing the double vector bundle morphism $(\varphi^{\lightning}; 0^{1,2}_{E_2}, \psi^{\lightning}; 0^{E_1})$ in detail, that is, the vector bundle maps $(\psi^{\lightning}, 0^{E_1})$, $(\varphi^{\lightning}, 0^{1,2}_{E_2})$ and $(\varphi^{\lightning}, \psi^{\lightning})$.

Start with $(\psi^{\lightning}, 0^{E_1})$. The vector bundle map $\psi : E_3 \to E_{13}$ over M defines the bolt section $(\psi^{\lightning}, 0^{E_1})$ of the Right face of E:



As we will need them later on, we write out explicit formulas. For every $e_3 \in E_3$:

$$\psi^{\lightning}(e_3) = \psi(e_3) \underset{E_1}{+} \tilde{0}^{1,3}_{e_3}, \tag{4.22}$$

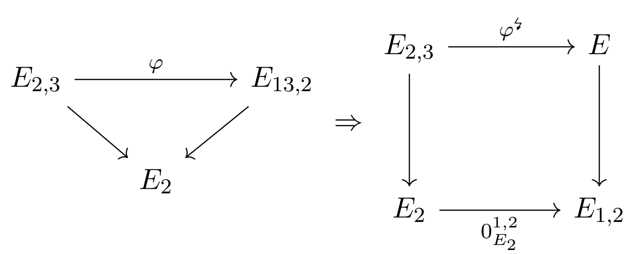
and from (2.10),

$$\hat{0}_{\psi^{\lightning}(e_3)} = \hat{0}_{\psi(e_3) \underset{E_1}{+} \tilde{0}^{1,3}_{e_3}} = \hat{0}_{\psi(e_3)} \underset{1,2}{+} \hat{0}_{e_3}. \tag{4.23}$$

The linearity condition for $e_3, e_3' \in E_3\Big|_m$:

$$\psi^{\lightning}(e_3 + e_3') = \psi^{\lightning}(e_3) \underset{E_1}{+} \psi^{\lightning}(e_3').$$

Continuing with $(\varphi^{\lightning}, 0^{1,2}_{E_2})$. The vector bundle map $\varphi : E_{2,3} \to E_{13,2}$ defines the bolt section $(\varphi^{\lightning}, 0^{1,2}_{E_2})$ of the Left face of E:



and for every $e_{2,3} \in E_{2,3}$:

$$\varphi^{\lightning}(e_{2,3}) = \varphi(e_{2,3}) \underset{1,2}{+} \hat{0}_{e_{2,3}}. \tag{4.24}$$

The outline of $\varphi(e_{2,3})$ in the (L-R) core double vector bundle,

$$\begin{array}{ccc} E_{13,2} \ni \varphi(e_{2,3}) & \longmapsto & \psi(e_3) \in E_{13} \\ \downarrow & & \downarrow \\ e_2 & \longmapsto & m. \end{array}$$

Equation (4.24) in outlines,

$$\begin{array}{ccccc} \varphi(e_{2,3}) & \longrightarrow & \psi(e_3) & & \\ & \searrow & & \searrow & \\ \downarrow & & \tilde{0}^{2,3}_{e_2} & \longrightarrow & 0^{E_3}_m \\ \tilde{0}^{1,2}_{e_2} & \longrightarrow & 0^{E_1}_m & & \downarrow \\ & \searrow & \downarrow & \searrow & \\ & & e_2 & \longrightarrow & m \end{array} \underset{1,2}{+} \begin{array}{ccccc} \hat{0}_{e_{2,3}} & \longrightarrow & \tilde{0}^{1,3}_{e_3} & & \\ & \searrow & & \searrow & \\ \downarrow & & e_{2,3} & \longrightarrow & e_3 \\ \tilde{0}^{1,2}_{e_2} & \longrightarrow & 0^{E_1}_m & & \downarrow \\ & \searrow & \downarrow & \searrow & \\ & & e_2 & \longrightarrow & m \end{array}$$

$$= \begin{array}{ccccc} \varphi(e_{2,3}) \underset{1,2}{+} \hat{0}_{e_{2,3}} & \longrightarrow & \psi(e_3) \underset{E_1}{+} \tilde{0}^{1,3}_{e_3} & & \\ & \searrow & & \searrow & \\ \downarrow & & e_{2,3} & \longrightarrow & e_3 \\ \tilde{0}^{1,2}_{e_2} & \longrightarrow & 0^{E_1}_m & & \downarrow \\ & \searrow & \downarrow & \searrow & \\ & & e_2 & \longrightarrow & m \end{array} = \begin{array}{ccccc} \varphi^{\lightning}(e_{2,3}) & \longrightarrow & \psi^{\lightning}(e_3) & & \\ & \searrow & & \searrow & \\ \downarrow & & e_{2,3} & \longrightarrow & e_3 \\ \tilde{0}^{1,2}_{e_2} & \longrightarrow & 0^{E_1}_m & & \downarrow \\ & \searrow & \downarrow & \searrow & \\ & & e_2 & \longrightarrow & m. \end{array}$$

The linearity condition for $\varphi^{\lightning}$, for $e_{2,3}, e'_{2,3} \in E_{2,3}$ that project to the same $e_2 \in E_2$:

$$\varphi^{\lightning}(e_{2,3} \underset{E_2}{+} e'_{2,3}) = \varphi^{\lightning}(e_{2,3}) \underset{1,2}{+} \varphi^{\lightning}(e'_{2,3}). \tag{4.25}$$

The third vector bundle map we need to describe is $(\varphi^{\lightning}, \psi^{\lightning})$. This is defined by the vector bundle map (φ, ψ),

$$\begin{array}{ccc} E_{2,3} & \xrightarrow{\varphi} & E_{13,2} \\ {\scriptstyle q^{2,3}_3}\downarrow & & \downarrow{\scriptstyle q_{13}} \\ E_3 & \xrightarrow[\psi]{} & E_{13} \end{array} \Rightarrow \begin{array}{ccc} E_{2,3} & \xrightarrow{\varphi^{\lightning}} & E \\ {\scriptstyle q^{2,3}_3}\downarrow & & \downarrow{\scriptstyle q_{1,3}} \\ E_3 & \xrightarrow[\psi^{\lightning}]{} & E_{1,3}. \end{array} \tag{4.26}$$

First, we need to show that $q_{1,3}(\varphi^{\lightning}(e_{2,3})) = \psi^{\lightning}(q^{2,3}_3(e_{2,3}))$. Writing $q^{2,3}_3(e_{2,3}) = e_3$, we need to show that,

$$q_{1,3}(\varphi^{\lightning}(e_{2,3})) = \psi^{\lightning}(e_3). \tag{4.27}$$

The left hand side of (4.27),

$$q_{1,3}(\varphi^{\lightning}(e_{2,3})) = q_{1,3}(\varphi(e_{2,3}) \underset{1,2}{+} \hat{0}_{e_{2,3}}) \overset{(2.6)}{=} q_{1,3}(\varphi(e_{2,3})) \underset{E_1}{+} q_{1,3}(\hat{0}_{e_{2,3}}) = q_{1,3}(\varphi(e_{2,3})) \underset{E_1}{+} \tilde{0}^{1,3}_{e_3}.$$

What is $q_{1,3}(\varphi(e_{2,3}))$? Since $\varphi(e_{2,3}) \in E_{13,2}$, by definition of the core morphism, Section 2.3.1,

$$q_{1,3}(\varphi(e_{2,3})) = q_{13}(\varphi(e_{2,3})).$$

And by the vector bundle map (φ, ψ), we have that $q_{13} \circ \varphi = \psi \circ q^{2,3}_3$, hence the left hand side of (4.27) can be written as,

$$q_{1,3}(\varphi^{\lightning}(e_{2,3})) = q_{13}(\varphi(e_{2,3})) \underset{E_1}{+} \tilde{0}^{1,3}_{e_3} = \psi(e_3) \underset{E_1}{+} \tilde{0}^{1,3}_{e_3} = \psi^{\lightning}(e_3) = \psi^{\lightning}(q^{2,3}_3(e_{2,3})),$$

and this establishes the commutativity of diagram (4.26). About fibrewise linearity of $\varphi^{\lightning}$ over $\psi^{\lightning}$, take $e_{2,3}$ and $f_{2,3} \in E_{2,3}$ projecting to the same $e_3 \in E_3$. Since (φ, ψ) is a vector bundle map, $\varphi(e_{2,3} \underset{E_3}{+} f_{2,3}) \overset{(2.28)}{=} \varphi(e_{2,3}) \underset{1,3}{+} \varphi(f_{2,3})$. Therefore,

$$\begin{aligned}
\varphi^{\lightning}(e_{2,3} \underset{E_3}{+} f_{2,3}) &\overset{(4.24)}{=} \varphi(e_{2,3} \underset{E_3}{+} f_{2,3}) \underset{1,2}{+} \hat{0}_{e_{2,3} \underset{E_3}{+} f_{2,3}} \\
&\overset{(2.24)}{=} \Big(\varphi(e_{2,3}) \underset{1,3}{+} \varphi(f_{2,3})\Big) \underset{1,2}{+} \Big(\hat{0}_{e_{2,3}} \underset{1,3}{+} \hat{0}_{f_{2,3}}\Big) \\
&= \Big(\varphi(e_{2,3}) \underset{1,2}{+} \hat{0}_{e_{2,3}}\Big) \underset{1,3}{+} \Big(\varphi(f_{2,3}) \underset{1,2}{+} \hat{0}_{f_{2,3}}\Big) \\
&= \varphi^{\lightning}(e_{2,3}) \underset{1,3}{+} \varphi^{\lightning}(f_{2,3}),
\end{aligned}$$

and scalar multiplication follows similarly.

4.3.1 Zeros and core morphism of double bolts

By Corollary 4.2.8, for $(\psi^{\lightning}, 0^{E_1})$ and $(\varphi^{\lightning}, 0^{1,2}_{E_2})$ we have respectively,

$$\psi^{\lightning}(0^{E_3}_m) = \odot^{1,3}_m, \quad \varphi^{\lightning}(\tilde{0}^{2,3}_{e_2}) = \hat{0}_{e_2}.$$

About the vector bundle map $(\varphi^{\lightning}, \psi^{\lightning})$, for $\tilde{0}^{2,3}_{e_3} \in E_{2,3}$, the zero in $E_{2,3}$ over e_3,

$$\begin{array}{ccc}
E_{2,3} \ni \tilde{0}^{2,3}_{e_3} & \longmapsto & \varphi^{\lightning}(\tilde{0}^{2,3}_{e_3}) \in E \\
\downarrow & & \downarrow \\
E_3 \ni e_3 & \longmapsto & \psi^{\lightning}(e_3) \in E_{1,3},
\end{array}$$

hence $\varphi^{\lightning}(\tilde{0}^{2,3}_{e_3})$ is the zero in E over $\psi^{\lightning}(e_3)$,

$$\varphi^{\lightning}(\tilde{0}^{2,3}_{e_3}) = \hat{0}_{\psi^{\lightning}(e_3)}. \tag{4.28}$$

Or, using (4.24) directly,

$$\varphi^{\lightning}(\tilde{0}^{2,3}_{e_3}) = \varphi(\tilde{0}^{2,3}_{e_3}) \underset{1,2}{+} \hat{0}_{\tilde{0}^{2,3}_{e_3}} = \varphi(\tilde{0}^{2,3}_{e_3}) \underset{1,2}{+} \hat{0}_{e_3}$$

What is $\varphi(\tilde{0}^{2,3}_{e_3})$? Going back to the vector bundle map (φ, ψ):

$$\begin{array}{ccc} E_{2,3} \ni \tilde{0}^{2,3}_{e_3} & \longmapsto & \varphi(\tilde{0}^{2,3}_{e_3}) \in E_{13,2} \\ \downarrow & & \downarrow \\ E_3 \ni e_3 & \longmapsto & \psi(e_3) \in E_{13}, \end{array}$$

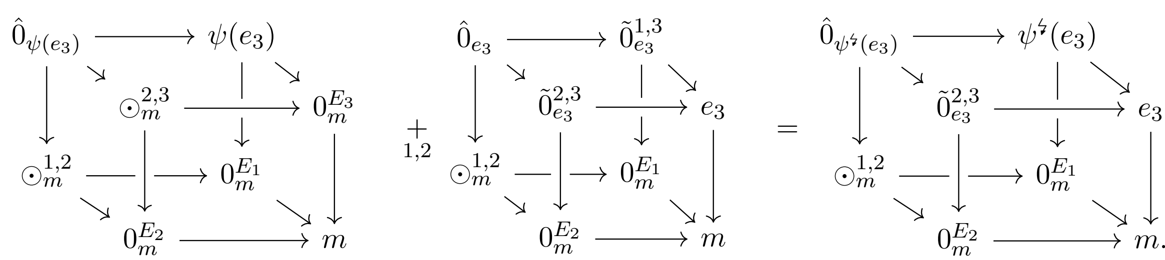
we see that $\varphi(\tilde{0}^{2,3}_{e_3})$ is the zero of $E_{13,2}$ over $\psi(e_3)$. Its image in E,

$$\varphi(\tilde{0}^{2,3}_{e_3}) = \hat{0}_{\psi(e_3)},$$

hence,

$$\varphi^{\lightning}(\tilde{0}^{2,3}_{e_3}) = \hat{0}_{\psi(e_3)} \underset{1,2}{+} \hat{0}_{e_3} \overset{(2.10)}{=} \hat{0}_{\psi(e_3) \underset{E_1}{+} \tilde{0}^{1,3}_{e_3}} = \hat{0}_{\psi^{\lightning}(e_3)},$$

which is exactly (4.28). Some outlines,



Last but not least of our calculations, the core morphism of the linear double section $(\varphi^{\lightning}; 0^{1,2}_{E_2}, \psi^{\lightning}; 0^{E_1})$.

To begin with, the core morphism of the double vector bundle morphism $(\varphi; \mathrm{id}_{E_2}, \psi; \mathrm{id}_M)$ is the restriction of φ to the core vector bundles,

$$\begin{array}{ccc} E_{23} & \xrightarrow{\ \varphi\ } & E_{123} \\ \downarrow & & \downarrow \\ M & = & M. \end{array}$$

Therefore, for $w_{23} \in E_{23}$, $\varphi(w_{23}) \in E_{123}$ is an ultracore element. Now we denote the restriction of φ on the cores by $\overline{\varphi}$. Notation-wise, we will write $\overline{\varphi}(w_{23}) \in E_{123}$. This is an ad-hoc notation; typically, we would write $\varphi\Big|_{E_{23}}(w_{23})$ for the restriction of φ to the cores, but this isn't practical to use in the calculations that follow. Therefore, we use $\overline{\varphi}$,

$$\begin{array}{ccc} E_{23} & \xrightarrow{\overline{\varphi}} & E_{123} \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

This core morphism $\overline{\varphi}$ defines a bolt section of the (B-F) core double vector bundle:

$$\begin{array}{ccc} E_{23} & \xrightarrow{\overline{\varphi}^{\lightning}} & E_{23,1} \\ \downarrow & & \downarrow \\ M & \xrightarrow[0^{E_1}]{} & E_1, \end{array}$$

and from (4.24)

$$\overline{\varphi}^{\lightning}(w_{23}) = \overline{\varphi}(w_{23}) \underset{E_1}{+} \hat{0}_{w_{23}} = \overline{\varphi}(w_{23}) \underset{1,2/1,3}{+} \hat{0}_{w_{23}}.$$

We arrive at the final statement of this section, that *the core morphism of a double bolt is the bolt of the core morphism.* This translates to the following proposition.

Proposition 4.3.2. *The core morphism of* $(\varphi^{\lightning}; 0^{1,2}_{E_2}, \psi^{\lightning}; 0^{E_1})$ *is the bolt section* $\overline{\varphi}^{\lightning}$ *just described.*

Proof. To calculate the core morphism of $(\varphi^{\lightning}; 0^{1,2}_{E_2}, \psi^{\lightning}; 0^{E_1})$, take a $w_{23} \in E_{23}$, in the core of $E_{2,3}$. By (4.24),

$$\varphi^{\lightning}(w_{23}) = \varphi(w_{23}) \underset{1,2}{+} \hat{0}_{w_{23}}.$$

Since $w_{23} \in E_{23}$, it follows that $\varphi(w_{23}) = \overline{\varphi}(w_{23}) \in E_{123}$, an ultracore element. And $\hat{0}_{w_{23}} \in E_{23,1}$ is the zero over $0^{E_1}_m$, so we can write:

$$\varphi^{\lightning}(w_{23}) = \overline{\varphi}(w_{23}) \underset{E_1}{+} \hat{0}_{w_{23}} = \overline{\varphi}^{\lightning}(w_{23}),$$

and this completes the proof. ∎

A few outlines:

$$\begin{array}{ccccc} \varphi^{\lightning}(w_{23}) & \longrightarrow & \odot^{1,3}_m & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ & & w_{23} & \longrightarrow & 0^{E_3}_m \\ \odot^{1,2}_m & \longrightarrow & 0^{E_1}_m & & \downarrow \\ & \searrow & \downarrow & \searrow & \\ & & 0^{E_2}_m & \longrightarrow & m, \end{array} = \begin{array}{ccccc} \overline{\varphi}(w_{23}) & \longrightarrow & \odot^{1,3}_m & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ & & \odot^{2,3}_m & \longrightarrow & 0^{E_3}_m \\ \odot^{1,2}_m & \longrightarrow & 0^{E_1}_m & & \downarrow \\ & \searrow & \downarrow & \searrow & \\ & & 0^{E_2}_m & \longrightarrow & m, \end{array} \underset{1,2}{+} \begin{array}{ccccc} \hat{0}_{w_{23}} & \longrightarrow & \odot^{1,3}_m & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ & & w_{23} & \longrightarrow & 0^{E_3}_m \\ \odot^{1,2}_m & \longrightarrow & 0^{E_1}_m & & \downarrow \\ & \searrow & \downarrow & \searrow & \\ & & 0^{E_2}_m & \longrightarrow & m. \end{array}$$

We saw in Proposition 4.2.9 that two linear sections $\eta_1, \eta_2 \in \Gamma_A D$ that project to the same section $Y \in \Gamma B$ differ by a bolt section. Analogously,

Proposition 4.3.3. *Given two front-back linear double sections,* $(X_{2,3}; X_2, X_3; X)$ *and* $(W_{2,3}; X_2, W_3; X)$*, then* $(X_{2,3} \underset{2,3}{-} W_{2,3}; 0^{1,2}_{E_2}, X_3 \underset{E_3}{-} W_3; 0^{E_1})$ *is a double bolt section.*

Proof. Focusing on the Right face of E, it follows immediately from Proposition 4.2.9 that $(X_3 \underset{E_3}{-} W_3; 0^{E_1})$ is a bolt section, and the vector bundle map $\psi : E_3 \to E_{13}$ over M that defines this bolt section is

$$\psi(e_3) = (X_3(e_3) \underset{E_3}{-} W_3(e_3)) \underset{E_1}{-} \tilde{0}^{1,3}_{e_3}$$

as follows from the proof of Proposition 4.2.9.

Likewise for the Left face, it follows immediately that $(X_{2,3} \underset{2,3}{-} W_{2,3}, 0^{1,2}_{E_2})$ is a bolt section defined by a vector bundle map $\varphi : E_{2,3} \to E_{13,2}$ over E_2,

$$\varphi(e_{2,3}) = (X_{2,3}(e_{2,3}) \underset{2,3}{-} W_{2,3}(e_{2,3})) \underset{1,2}{-} \hat{0}_{e_{2,3}}.$$

It remains to show that the φ just described is a vector bundle map over ψ,

$$\begin{array}{ccc} E_{2,3} & \xrightarrow{\ \varphi\ } & E_{13,2} \\ {\scriptstyle q^{2,3}_3}\downarrow & & \downarrow{\scriptstyle q_{13}} \\ E_3 & \xrightarrow[\ \psi\ ]{} & E_{13}, \end{array} \tag{4.29}$$

and this follows from properties of the linear double sections. About the commutativity of the diagram (4.29),

$$\begin{aligned} q_{13}(\varphi(e_{2,3})) &= q_{13}\left((X_{2,3}(e_{2,3}) \underset{2,3}{-} W_{2,3}(e_{2,3})) \underset{1,2}{-} \hat{0}_{e_{2,3}} \right) \\ &= q_{1,3}((X_{2,3}(e_{2,3}) \underset{2,3}{-} W_{2,3}(e_{2,3}))) \underset{E_1}{-} q_{1,3}(\hat{0}_{e_{2,3}}) \\ &= \left(q_{1,3}(X_{2,3}(e_{2,3})) \underset{E_3}{-} q_{1,3}(W_{2,3}(e_{2,3})) \right) \underset{E_1}{-} \tilde{0}^{1,3}_{e_3} \\ &= \psi(q^{2,3}_3(e_{2,3})). \end{aligned}$$

About fibrewise linearity, for $e_{2,3}, f_{2,3}$ over the same $e_3 \in E_3$, we have

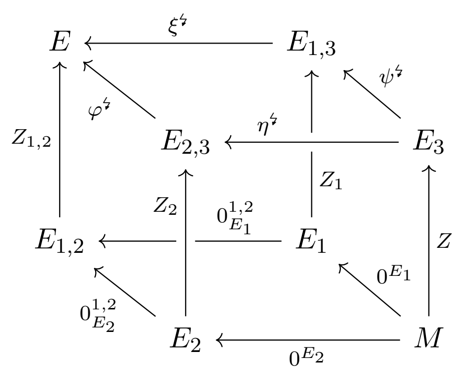
$$\begin{aligned}
&\varphi(e_{2,3} \underset{E_3}{+} f_{2,3}) \\
&\quad = \quad (X_{2,3}(e_{2,3} \underset{E_3}{+} f_{2,3}) \underset{2,3}{-} W_{2,3}(e_{2,3} \underset{E_3}{+} f_{2,3})) \underset{1,2}{-} \hat{0}_{e_{2,3} \underset{E_3}{+} f_{2,3}} \\
&\overset{(3.3),(2.24)}{=} \quad \Big((X_{2,3}(e_{2,3}) \underset{1,3}{+} X_{2,3}(f_{2,3})) \underset{2,3}{-} (W_{2,3}(e_{2,3}) \underset{1,3}{+} W_{2,3}(f_{2,3})) \Big) \underset{1,2}{-} (\hat{0}_{e_{2,3}} \underset{1,3}{+} \hat{0}_{f_{2,3}}) \\
&\quad = \quad \Big((X_{2,3}(e_{2,3}) \underset{2,3}{-} W_{2,3}(e_{2,3})) \underset{1,3}{+} (X_{2,3}(f_{2,3}) \underset{2,3}{-} W_{2,3}(f_{2,3})) \Big) \underset{1,2}{-} (\hat{0}_{e_{2,3}} \underset{1,3}{+} \hat{0}_{f_{2,3}}) \\
&\quad = \quad \Big((X_{2,3}(e_{2,3}) \underset{2,3}{-} W_{2,3}(e_{2,3})) \underset{1,2}{-} \hat{0}_{e_{2,3}} \Big) \underset{1,3}{+} \Big((X_{2,3}(f_{2,3}) \underset{2,3}{-} W_{2,3}(f_{2,3})) \underset{1,2}{-} \hat{0}_{f_{2,3}} \Big) \\
&\quad = \quad \varphi(e_{2,3}) \underset{1,3}{+} \varphi(f_{2,3}),
\end{aligned}$$

and this completes the proof. □

4.4 Class of examples with bolt sections; two double bolts

This is a confirmation of the general result of the warp-grid theorem in a special case, where special features of the following grid enable clear calculations.

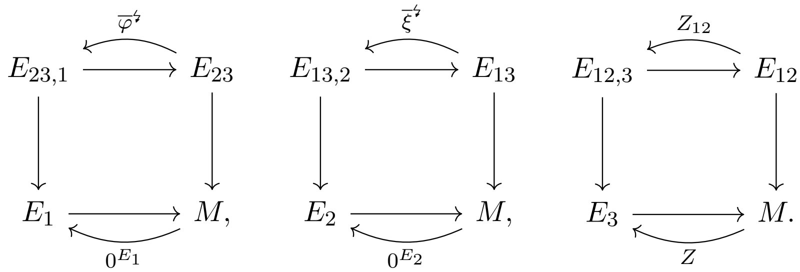
Take the following grid on E,



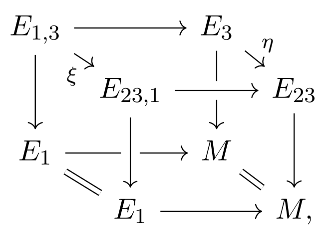
where $(Z_{1,2}; Z_1, Z_2; Z)$ is a down-up linear double section, $(\varphi^{\lightning}; 0^{1,2}_{E_2}, \psi^{\lightning}; 0^{E_1})$ is a front-back double bolt section, and $(\xi^{\lightning}; 0^{1,2}_{E_1}, \eta^{\lightning}; 0^{E_2})$ is a right-left double bolt section.

To calculate the three ultrawarps defined by this grid, we first calculate the core morphism of each linear double section. The core morphism of the double bolt $(\varphi^{\lightning}; 0^{1,2}_{E_2}, \psi^{\lightning}; 0^{E_1})$, as we calculated in Section 4.3.1, is the bolt section defined by $\overline{\varphi}$, a section of the (B-F) core double vector bundle. Similarly for $(\xi^{\lightning}; 0^{1,2}_{E_1}, \eta^{\lightning}; 0^{E_2})$, the core morphism will be the bolt section defined by $\overline{\xi}$, a section of the (L-R) core double vector bundle. The core morphism of the double vector bundle morphism $(Z_{1,2}; Z_1, Z_2; Z)$ is the vector bundle morphism (Z_{12}, Z), see Section 3.1.2.

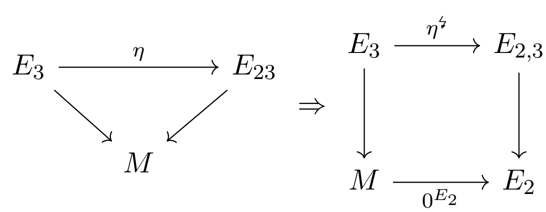
The three core double vector bundles in the usual order, (B-F), (L-R), and (U-D):



We also need some calculations about the right-left double bolt section. The double vector bundle map $(\xi; \mathrm{id}_{E_1}, \eta; \mathrm{id}_M)$ from the Right face to the (B-F) core double vector bundle,



defines the double bolt section $(\xi^{\lightning}; \eta^{\lightning}, 0^{1,2}_{E_2}; 0^{E_2})$. The vector bundle map η, and the bolt section it defines, along with the corresponding relations:



For $e_3 \in E_3$:

$$\eta^{\lightning}(e_3) = \eta(e_3) \underset{E_2}{+} \tilde{0}^{2,3}_{e_3}, \tag{4.30}$$

and as we will need it later, from (2.22),

$$\hat{0}_{\eta^{\lightning}(e_3)} = \hat{0}_{\eta(e_3) \underset{E_2}{+} \tilde{0}^{2,3}_{e_3}} = \hat{0}_{\eta(e_3)} \underset{1,2}{+} \hat{0}_{e_3}. \tag{4.31}$$

For $e_3, e_3' \in E_3\Big|_m$:

$$\eta^{\lightning}(e_3 + e_3') = \eta^{\lightning}(e_3) \underset{E_2}{+} \eta^{\lightning}(e_3').$$

The vector bundle map ξ defines the bolt section $\xi^{\lightning}$ of the Back face of E,

$$\begin{array}{ccc} E_{1,3} & \xrightarrow{\xi} & E_{23,1} \\ & \searrow \quad \swarrow & \\ & E_1 & \end{array} \quad \Rightarrow \quad \begin{array}{ccc} E_{1,3} & \xrightarrow{\xi^{\lightning}} & E \\ \downarrow & & \downarrow \\ E_1 & \xrightarrow[0^{1,2}_{E_1}]{} & E_{1,2}. \end{array}$$

For $e_{1,3} \in E_{1,3}$:

$$\xi^{\lightning}(e_{1,3}) = \xi(e_{1,3}) \underset{1,2}{+} \hat{0}_{e_{1,3}}. \tag{4.32}$$

For $e_{1,3}, e'_{1,3} \in E_{1,3}$ that are over the same $e_1 \in E_1$:

$$\xi^{\lightning}(e_{1,3} \underset{E_1}{+} e'_{1,3}) = \xi^{\lightning}(e_{1,3}) \underset{1,2}{+} \xi^{\lightning}(e'_{1,3}). \tag{4.33}$$

For $e_{1,3}, e'_{1,3} \in E_{1,3}$ that are over the same $e_3 \in E_3$, using the fact that (ξ, η) is a vector bundle map:

$$\xi^{\lightning}(e_{1,3} \underset{E_3}{+} e'_{1,3}) = \xi^{\lightning}(e_{1,3}) \underset{2,3}{+} \xi^{\lightning}(e'_{1,3}).$$

And the zeros defined:

$$\xi^{\lightning}(\tilde{0}^{1,3}_{e_1}) = \hat{0}_{e_1}, \qquad \xi^{\lightning}(\tilde{0}^{1,3}_{e_3}) = \hat{0}_{\eta^{\lightning}(e_3)}. \tag{4.34}$$

Finally, the core morphism of $(\xi^{\lightning}; 0^{1,2}_{E_2}, \eta^{\lightning}; 0^{E_2})$ will be the bolt section defined by the core morphism of $(\xi; \mathrm{id}_{E_1}, \eta; \mathrm{id}_M)$, which is $\overline{\xi}$:

$$\begin{array}{ccc} E_{13} & \xrightarrow{\overline{\xi}} & E_{123} \\ & \searrow \quad \swarrow & \\ & M & \end{array} \quad \Rightarrow \quad \begin{array}{ccc} E_{13} & \xrightarrow{\overline{\xi}^{\lightning}} & E_{13,2} \\ \downarrow & & \downarrow \\ M & \xrightarrow[0^{E_2}]{} & E_2, \end{array}$$

such that, for every $w_{13} \in E_{13}$:

$$\overline{\xi}^{\lightning}(w_{13}) = \overline{\xi}(w_{13}) \underset{E_2}{+} \hat{0}_{w_{13}} = \overline{\xi}(w_{13}) \underset{1,2/2,3}{+} \hat{0}_{w_{13}}.$$

4.4.1 Warp of grid of each face of E and ultrawarps

The only warp we need to calculate thoroughly is the warp of the Up face. The Down warp follows immediately that it is zero, and the remaining warps follow directly from Proposition 4.2.11, (4). In total, the six warps,

i. Back face: $\mathrm{w}_{\text{back}} = -\xi \circ Z_1 : E_1 \to E_{23,1}$,

ii. Front face: $\mathrm{w}_{\text{front}} = -\eta \circ Z : M \to E_{23}$,

iii. Left face: $\mathrm{w}_{\text{left}} = \varphi \circ Z_2 : E_2 \to E_{13,2}$,

iv. Right face: $\mathrm{w}_{\text{right}} = \psi \circ Z : M \to E_{13}$,

v. Up face: $\mathrm{w}_{\text{up}} = \overline{\xi} \circ \psi - \overline{\varphi} \circ \eta : E_3 \to E_{12,3}$,

vi. Down face: zero section $\mathrm{w}_{\text{down}} = 0^{12} : M \to E_{12}$.

Warp of Up face

We now have everything we need to calculate the warp of the Up face. For the two elements $\xi^{\lightning}(\psi^{\lightning}(e_3))$ and $\varphi^{\lightning}(\eta^{\lightning}(e_3))$, we need to calculate,

$$\begin{aligned} \xi^{\lightning}(\psi^{\lightning}(e_3)) \underset{1,3}{-} \varphi^{\lightning}(\eta^{\lightning}(e_3)) &= \mathrm{w}_{\text{up}}(e_3) \underset{2,3}{+} \hat{0}_{\psi^{\lightning}(e_3)}, \\ \xi^{\lightning}(\psi^{\lightning}(e_3)) \underset{2,3}{-} \varphi^{\lightning}(\eta^{\lightning}(e_3)) &= \mathrm{w}_{\text{up}}(e_3) \underset{1,3}{+} \hat{0}_{\eta^{\lightning}(e_3)}. \end{aligned} \tag{4.35}$$

First, focus on $\xi^{\lightning}(\psi^{\lightning}(e_3))$:

$$\begin{aligned} \xi^{\lightning}(\psi^{\lightning}(e_3)) &\overset{(4.22)}{=} \xi^{\lightning}(\psi(e_3) \underset{E_1}{+} \tilde{0}^{1,3}_{e_3}) \overset{(4.33)}{=} \xi^{\lightning}(\psi(e_3)) \underset{1,2}{+} \xi^{\lightning}(\tilde{0}^{1,3}_{e_3}) \\ &\overset{(4.32)}{=} \left(\xi(\psi(e_3)) \underset{1,2}{+} \hat{0}_{\psi(e_3)}\right) \underset{1,2}{+} \hat{0}_{\eta^{\lightning}(e_3)} \overset{(4.31)}{=} \left(\xi(\psi(e_3)) \underset{1,2}{+} \hat{0}_{\psi(e_3)}\right) \underset{1,2}{+} \left(\hat{0}_{\eta(e_3)} \underset{1,2}{+} \hat{0}_{e_3}\right). \end{aligned}$$

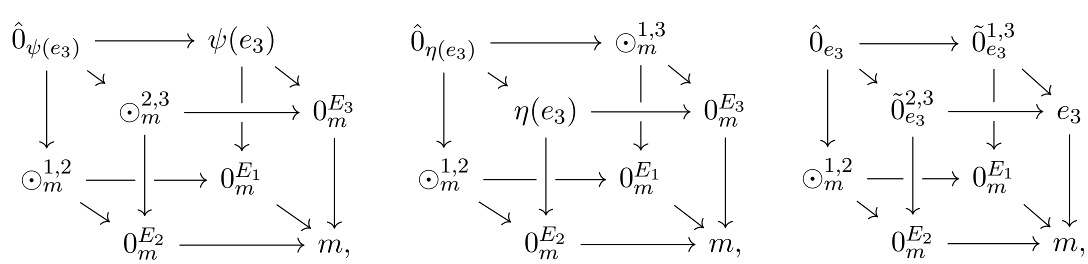
And since $\psi(e_3) \in E_{13}$, it follows that $\xi(\psi(e_3)) = \overline{\xi}(\psi(e_3))$, hence,

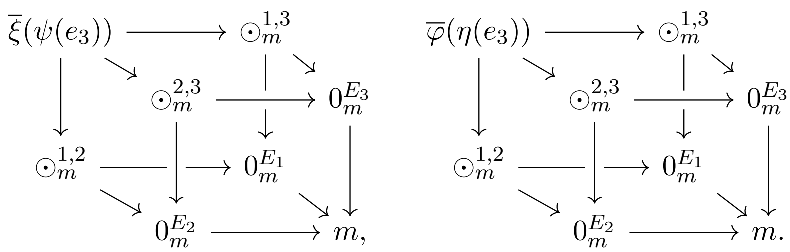
$$\xi^{\lightning}(\psi^{\lightning}(e_3)) = \left(\overline{\xi}(\psi(e_3)) \underset{1,2}{+} \hat{0}_{\psi(e_3)}\right) \underset{1,2}{+} \left(\hat{0}_{\eta(e_3)} \underset{1,2}{+} \hat{0}_{e_3}\right).$$

Similarly for $\varphi^{\lightning}(\eta^{\lightning}(e_3))$,

$$\begin{aligned} \varphi^{\lightning}(\eta^{\lightning}(e_3)) &\overset{(4.30)}{=} \varphi^{\lightning}(\eta(e_3) \underset{E_2}{+} \tilde{0}^{2,3}_{e_3}) = \varphi^{\lightning}(\eta(e_3)) \underset{1,2}{+} \varphi^{\lightning}(\tilde{0}^{2,3}_{e_3}) \\ &\overset{(4.24),(4.28)}{=} \left(\varphi(\eta(e_3)) \underset{1,2}{+} \hat{0}_{\eta(e_3)}\right) \underset{1,2}{+} \hat{0}_{\psi^{\lightning}(e_3)} \overset{(4.23)}{=} \left(\overline{\varphi}(\eta(e_3)) \underset{1,2}{+} \hat{0}_{\eta(e_3)}\right) \underset{1,2}{+} \left(\hat{0}_{\psi(e_3)} \underset{1,2}{+} \hat{0}_{e_3}\right), \end{aligned}$$

where we have used that $\eta(e_3) \in E_{23}$, hence $\varphi(\eta(e_3)) = \overline{\varphi}(\eta(e_3))$. In order to keep track of calculations, we add the outlines of the elements involved:





The left hand side of (4.35) can now be written as,

$$\begin{aligned}&\xi^{\lightning}(\psi^{\lightning}(e_3)) \underset{2,3}{—} \varphi^{\lightning}(\eta^{\lightning}(e_3))\\ &= \left[\left(\overline{\xi}(\psi(e_3)) \underset{1,2}{+} \hat{0}_{\psi(e_3)}\right) \underset{1,2}{+} \left(\hat{0}_{\eta(e_3)} \underset{1,2}{+} \hat{0}_{e_3}\right)\right] \underset{2,3}{—} \left[\left(\overline{\varphi}(\eta(e_3)) \underset{1,2}{+} \hat{0}_{\eta(e_3)}\right) \underset{1,2}{+} \left(\hat{0}_{\psi(e_3)} \underset{1,2}{+} \hat{0}_{e_3}\right)\right].\end{aligned}$$

Rearrange the second term of the previous equation,

$$\left[\left(\overline{\xi}(\psi(e_3)) \underset{1,2}{+} \hat{0}_{\psi(e_3)}\right) \underset{1,2}{+} \left(\hat{0}_{\eta(e_3)} \underset{1,2}{+} \hat{0}_{e_3}\right)\right] \underset{2,3}{—} \left[\left(\overline{\varphi}(\eta(e_3)) \underset{1,2}{+} \hat{0}_{\psi(e_3)}\right) \underset{1,2}{+} \left(\hat{0}_{\eta(e_3)} \underset{1,2}{+} \hat{0}_{e_3}\right)\right].$$

Now apply the interchange law in the Left face,

$$\left[\left(\overline{\xi}(\psi(e_3)) \underset{1,2}{+} \hat{0}_{\psi(e_3)}\right) \underset{2,3}{—} \left(\overline{\varphi}(\eta(e_3)) \underset{1,2}{+} \hat{0}_{\psi(e_3)}\right)\right] \underset{1,2}{+} \left[\left(\hat{0}_{\eta(e_3)} \underset{1,2}{+} \hat{0}_{e_3}\right) \underset{2,3}{—} \left(\hat{0}_{\eta(e_3)} \underset{1,2}{+} \hat{0}_{e_3}\right)\right]. \tag{4.36}$$

About the first bracket of (4.36), apply the interchange law in the Left face,

$$\begin{aligned}\left(\overline{\xi}(\psi(e_3)) \underset{1,2}{+} \hat{0}_{\psi(e_3)}\right) \underset{2,3}{—} \left(\overline{\varphi}(\eta(e_3)) \underset{1,2}{+} \hat{0}_{\psi(e_3)}\right) &= \left(\overline{\xi}(\psi(e_3)) \underset{2,3}{—} \overline{\varphi}(\eta(e_3))\right) \underset{1,2}{+} \left(\hat{0}_{\psi(e_3)} \underset{2,3}{—} \hat{0}_{\psi(e_3)}\right)\\ &\overset{(2.54c)}{=} \left(\overline{\xi}(\psi(e_3)) \underset{2,3}{—} \overline{\varphi}(\eta(e_3))\right) \underset{1,2}{+} \hat{0}_{0_m^{E_3}} = \overline{\xi}(\psi(e_3)) \underset{2,3}{—} \overline{\varphi}(\eta(e_3)).\end{aligned}$$

About the second bracket of (4.36),

$$\left(\hat{0}_{\eta(e_3)} \underset{1,2}{+} \hat{0}_{e_3}\right) \underset{2,3}{—} \left(\hat{0}_{\eta(e_3)} \underset{1,2}{+} \hat{0}_{e_3}\right) = \left(\hat{0}_{\eta(e_3)} \underset{2,3}{—} \hat{0}_{\eta(e_3)}\right) \underset{1,2}{+} \left(\hat{0}_{e_3} \underset{2,3}{—} \hat{0}_{e_3}\right) \overset{(2.55c),(2.52c)}{=} \hat{0}_{\eta(e_3)} \underset{1,2}{+} \hat{0}_{e_3}.$$

Therefore, we can rewrite (4.36) as

$$\left[\overline{\xi}(\psi(e_3)) \underset{2,3}{—} \overline{\varphi}(\eta(e_3))\right] \underset{1,2}{+} \left(\hat{0}_{\eta(e_3)} \underset{1,2}{+} \hat{0}_{e_3}\right) \overset{(4.31)}{=} \left[\overline{\xi}(\psi(e_3)) \underset{2,3}{—} \overline{\varphi}(\eta(e_3))\right] \underset{1,2}{+} \hat{0}_{\eta^{\lightning}(e_3)}.$$

Denote temporarily $\overline{\xi}(\psi(e_3)) \underset{2,3}{—} \overline{\varphi}(\eta(e_3))$ by $u \in E_{123}$. Therefore, the last expression is now,

$$u \underset{1,2}{+} \hat{0}_{\eta^{\lightning}(e_3)}.$$

Recall from (2.25) that $\hat{0}_{\eta^{\lightning}(e_3)} = \hat{0}_{\eta^{\lightning}(e_3)} \underset{1,3}{+} \hat{0}_{e_3}$. In total, the left hand side of (4.35) can now be rewritten, using the interchange law in the Back face,

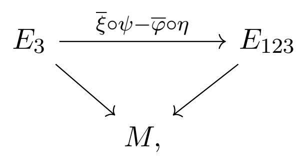
$$\xi^{\lightning}(\psi^{\lightning}(e_3)) \underset{2,3}{-} \varphi^{\lightning}(\eta^{\lightning}(e_3)) =$$

$$\left(u \underset{1,3}{+} \odot^3_m\right) \underset{1,2}{+} \left(\hat{0}_{\eta^{\lightning}(e_3)} \underset{1,3}{+} \hat{0}_{e_3}\right) = \left(u \underset{1,2}{+} \hat{0}_{e_3}\right) \underset{1,3}{+} \left(\hat{0}_{\eta^{\lightning}(e_3)} \underset{1,2}{+} \odot^3_m\right) = \left(u \underset{1,2}{+} \hat{0}_{e_3}\right) \underset{1,3}{+} \hat{0}_{\eta^{\lightning}(e_3)}.$$

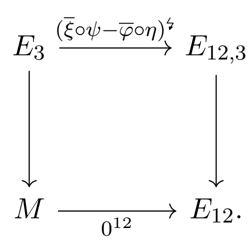
Comparing the previous equation with the right hand side of (4.35), we see that

$$\mathrm{w}_{\mathrm{up}} = \left(\overline{\xi}(\psi(e_3)) \underset{2,3}{-} \overline{\varphi}(\eta(e_3))\right) \underset{1,2}{+} \hat{0}_{e_3}.$$

Note that $\overline{\xi} \circ \psi - \overline{\varphi} \circ \eta$ is a vector bundle map. Indeed, both $\overline{\xi} \circ \psi$ and $\overline{\varphi} \circ \eta$ are compositions of vector bundle maps, and are both vector bundle maps $E_3 \to E_{123}$ over M. Hence $\overline{\xi} \circ \psi - \overline{\varphi} \circ \eta$ is a vector bundle map,



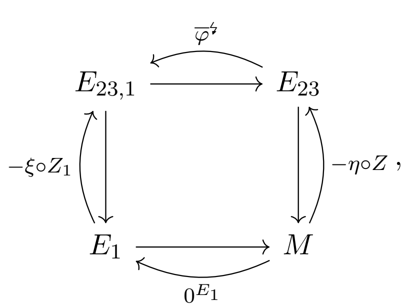
and this defines a bolt section in the (U-D) core double vector bundle:

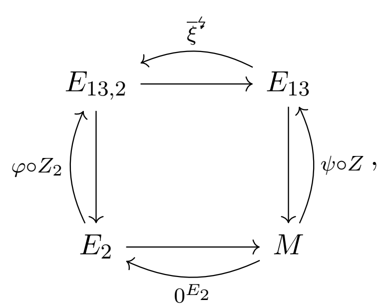


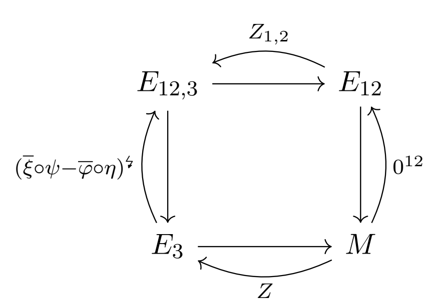
So finally, the warp of the Up face is the bolt section defined by $\overline{\xi} \circ \psi - \overline{\varphi} \circ \eta$.

Final calculation

The three core double vector bundles in the usual order, (B-F), (L-R), and (U-D):







In every core double vector bundle we have a bolt section and a linear section. Therefore, applying Proposition 4.2.1, (4) we obtain:

$$\begin{aligned}
(-\xi \circ Z_1) \circ (0^{E_1}) — \overline{\varphi}^{\lightning} \circ (-\eta \circ Z) \quad &\triangleright \quad \overline{\varphi} \circ \eta \circ Z, \\
(\varphi \circ Z_2) \circ (0^{E_2}) — \overline{\xi}^{\lightning} \circ (\psi \circ Z) \quad &\triangleright \quad -\overline{\xi} \circ \psi \circ Z, \\
(\overline{\xi} \circ \psi — \overline{\varphi} \circ \eta)^{\lightning} \circ Z - Z_{1,2} \circ (0^{12}) \quad &\triangleright \quad (\overline{\xi} \circ \psi - \overline{\varphi} \circ \eta) \circ Z.
\end{aligned}$$

Hence the three ultrawarps,

$$\mathrm{u}_{\mathsf{BF}} = \overline{\varphi} \circ \eta \circ Z, \quad \mathrm{u}_{\mathsf{LR}} = -\overline{\xi} \circ \psi \circ Z, \quad \mathrm{u}_{\mathsf{UD}} = (\overline{\xi} \circ \psi - \overline{\varphi} \circ \eta) \circ Z,$$

and since $\xi \circ \psi - \varphi \circ \eta$ is a vector bundle map, we can rewrite the last ultrawarp as:

$$\mathrm{u}_{\mathsf{UD}} = \overline{\xi} \circ \psi \circ Z - \overline{\varphi} \circ \eta \circ Z.$$

The sum of the three ultrawarps is zero.

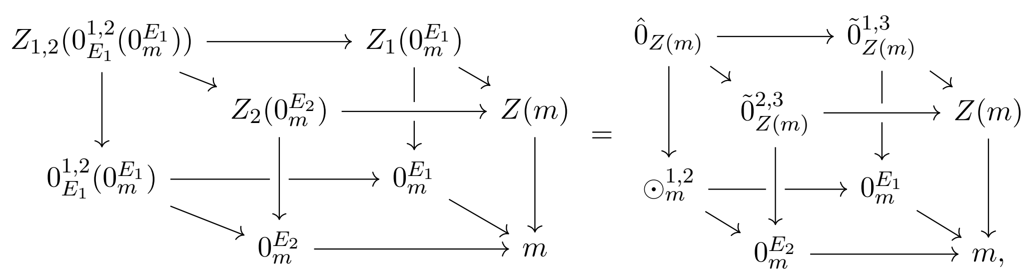
4.4.2 Six elements method

We will calculate the three ultrawarps via the six elements method, calculating the λ_i and k_i, for $i = 1, 2, 3$.

The six elements defined

The first element:

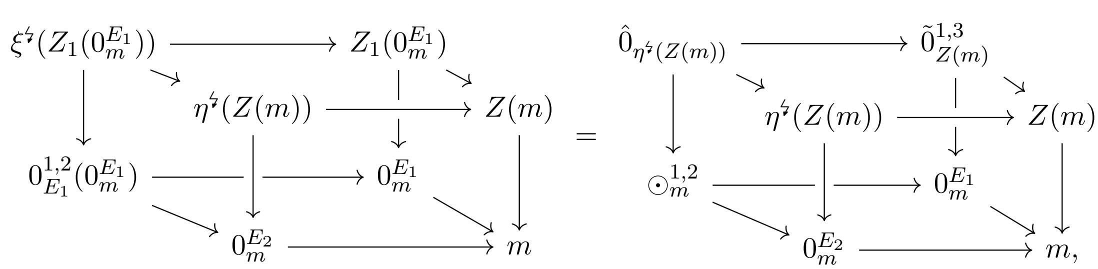
$\mathsf{ZYX} :=$



where we have used the fact that $Z_{1,2}$ is a double vector bundle map, hence $Z_2(0_m^{E_2}) = \tilde{0}^{2,3}_{Z(m)}$.

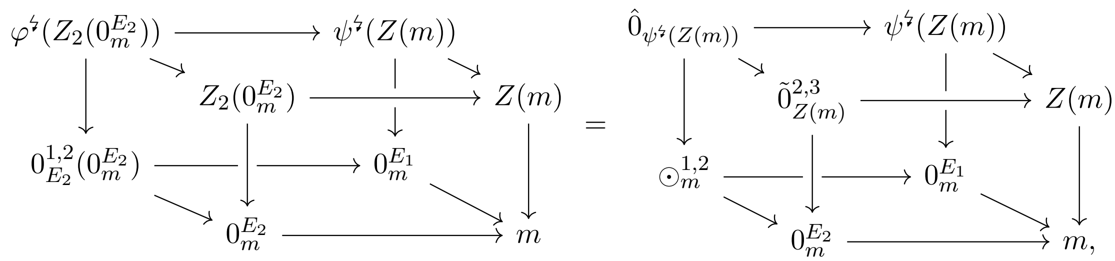
The second element:

$\mathsf{YZX} :=$



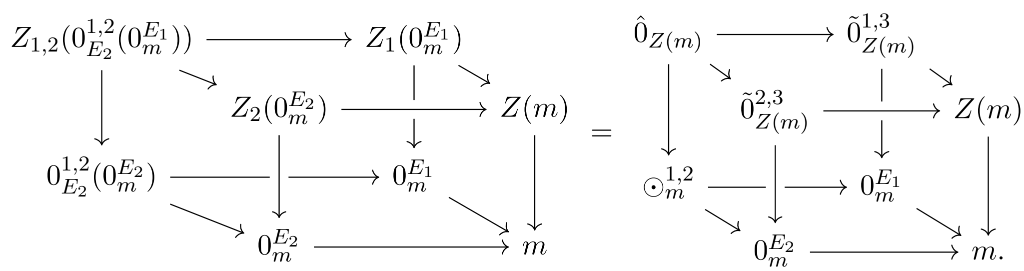
where we have used (4.34), $\xi^{\natural}(Z_1(0_m^{E_1})) = \xi^{\natural}(\tilde{0}^{1,3}_{Z(m)}) = \hat{0}_{\eta^{\natural}(Z(m))}$.

The third element:

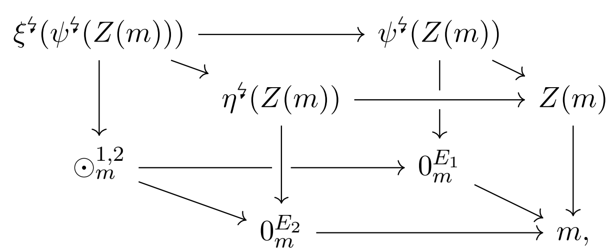
XZY :=


where we have used (4.28), $\varphi^{\natural}(Z_2(0_m^{E_2})) = \varphi^{\natural}(\tilde{0}^{2,3}_Z(m)) = \hat{0}_{\psi^{\natural}(Z(m))}$.

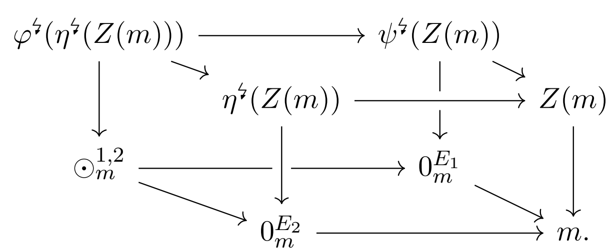
The fourth element:

ZXY :=


The fifth element:

YXZ :=


and finally the sixth element:

XYZ :=


u_1: the ultrawarp of induced grid on the (B-F) core dvb

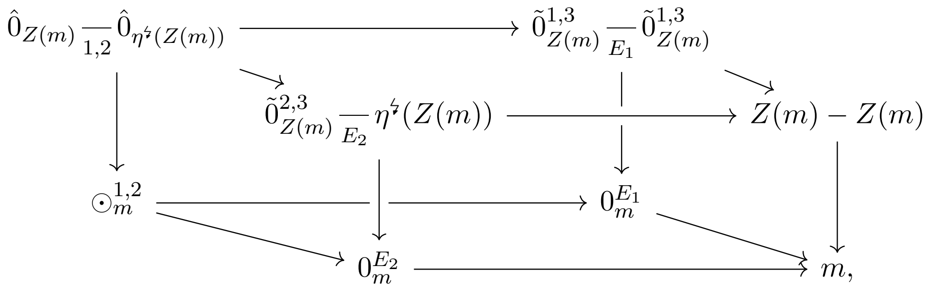
We need to calculate λ_1 and k_1. From (3.28) we have,

$$\mathsf{ZYX} \underset{1,2}{-} \mathsf{YZX} = \hat{0}_{e_{1,2}} \underset{1,3}{+} \lambda_1,$$

and since in this case $e_{1,2} = 0^{1,2}_{E_1}(0^{E_1}_m) = \odot^{1,2}_m$, we have that $\hat{0}_{e_{1,2}} = \odot^3_m$, the triple zero of E. Therefore,

$$\mathsf{ZYX} \underset{1,2}{-} \mathsf{YZX} = \lambda_1.$$

The outline of the difference $\mathsf{ZYX} \underset{1,2}{-} \mathsf{YZX}$,



and since

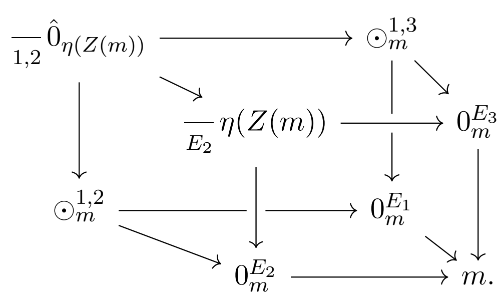
$$\tilde{0}^{2,3}_{Z(m)} \underset{E_2}{-} \eta^\natural(Z(m)) \overset{(4.30)}{=} \tilde{0}^{2,3}_{Z(m)} \underset{E_2}{-} (\eta(Z(m)) \underset{E_2}{+} \tilde{0}^{2,3}_{Z(m)})$$
$$= (\tilde{0}^{2,3}_{Z(m)} \underset{E_2}{-} \tilde{0}^{2,3}_{Z(m)}) \underset{E_2}{-} \eta(Z(m)) = \underset{E_2}{-} \eta(Z(m)),$$

and additionally,

$$\hat{0}_{Z(m)} \underset{1,2}{-} \hat{0}_{\eta^\natural(Z(m))} \overset{(4.31)}{=} \hat{0}_{Z(m)} \underset{1,2}{-} (\hat{0}_{\eta(Z(m))} \underset{1,2}{+} \hat{0}_{Z(m)})$$
$$= (\hat{0}_{Z(m)} \underset{1,2}{-} \hat{0}_{Z(m)}) \underset{1,2}{-} \hat{0}_{\eta(Z(m))} \overset{(2.52b)}{=} \underset{1,2}{-} \hat{0}_{\eta(Z(m))},$$

it follows that

$\mathsf{ZYX} \underset{1,2}{-} \mathsf{YZX} =$



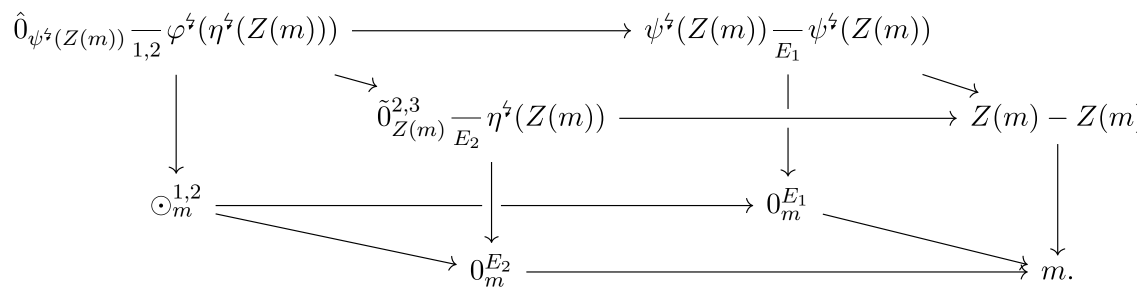
To calculate k_1, use (3.29),

$$\mathsf{XZY} \underset{1,2}{-} \mathsf{XYZ} = \hat{0}_{e'_{1,2}} \underset{1,3}{+} k_1.$$

In this case, $e'_{1,2} = 0^{1,2}_{E_2}(0^{E_2}_m) = \odot^{1,2}_m$ so $\hat{0}_{e'_{1,2}} = \odot^3_m$ the triple zero of E. So,

$$\mathsf{XZY} \underset{1,2}{-} \mathsf{XYZ} = k_1.$$

The outline of the difference between the third and the sixth element,



As for λ_1, $\tilde{0}^{2,3}_{Z(m)} \underset{E_2}{-} \eta^{\lightning}(Z(m)) = \underset{E_2}{-} \eta(Z(m))$, and of course, $\psi^{\lightning}(Z(m)) \underset{E_1}{-} \psi^{\lightning}(Z(m)) = \odot^{1,3}_m$. Also,

$$\hat{0}_{\psi^{\lightning}(Z(m))} \underset{1,2}{-} \varphi^{\lightning}(\eta^{\lightning}(Z(m))) \overset{(4.30)}{=} \hat{0}_{\psi^{\lightning}(Z(m))} \underset{1,2}{-} \varphi^{\lightning}(\eta(Z(m)) \underset{E_2}{+} \tilde{0}^{2,3}_{Z(m)})$$

$$= \hat{0}_{\psi^{\lightning}(Z(m))} \underset{1,2}{-} \left(\varphi^{\lightning}(\eta(Z(m))) \underset{1,2}{+} \varphi^{\lightning}\left(\tilde{0}^{2,3}_{Z(m)}\right)\right) \overset{(4.28)}{=} \hat{0}_{\psi^{\lightning}(Z(m))} \underset{1,2}{-} \left(\varphi^{\lightning}(\eta(Z(m))) \underset{1,2}{+} \hat{0}_{\psi^{\lightning}(Z(m))}\right)$$

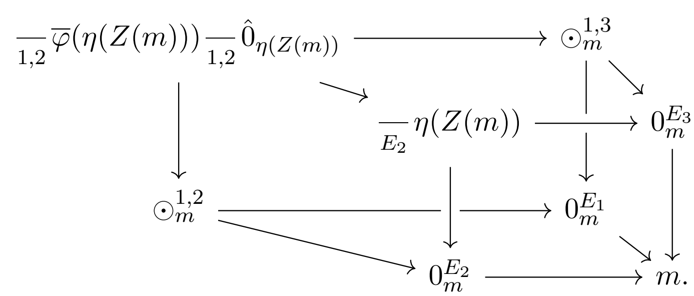
$$= \left(\hat{0}_{\psi^{\lightning}(Z(m))} \underset{1,2}{-} \hat{0}_{\psi^{\lightning}(Z(m))}\right) \underset{1,2}{-} \varphi^{\lightning}(\eta(Z(m))) = \underset{1,2}{-} \varphi^{\lightning}(\eta(Z(m)))$$

and we have that

$$\underset{1,2}{-} \varphi^{\lightning}(\eta(Z(m))) \overset{(4.24)}{=} \underset{1,2}{-} \left(\overline{\varphi}(\eta(Z(m))) \underset{1,2}{+} \hat{0}_{\eta(Z(m))}\right) = \underset{1,2}{-} \overline{\varphi}(\eta(Z(m))) \underset{1,2}{-} \hat{0}_{\eta(Z(m))},$$

so in total,

$$\mathsf{XZY} \underset{1,2}{-} \mathsf{XYZ} =$$



From (3.30b), since $e_1 = 0^{E_1}_m$, $\hat{0}_{e_1} = \odot^3_m$,

$$\lambda_1 \underset{1,2}{-} k_1 = \hat{0}_{e_1} \underset{2,3}{+} u_1 = u_1,$$

therefore,

$$\lambda_1 \underset{1,2}{-} k_1 = \underset{1,2}{-} \hat{0}_{\eta(Z(m))} \underset{1,2}{-} \left(\underset{1,2}{-} \overline{\varphi}(\eta(Z(m))) \underset{1,2}{-} \hat{0}_{\eta(Z(m))}\right) = \overline{\varphi}(\eta(Z(m)))$$

therefore,

$$u_1 = \overline{\varphi}(\eta(Z(m))).$$

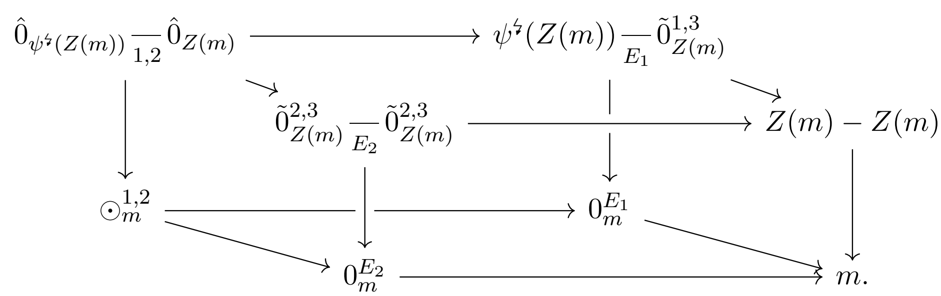
u_2: the ultrawarp of induced grid on the (L-R) core dvb

We now calculate λ_2 and k_2. For λ_2, from (3.32),

$$\mathsf{XZY} \underset{1,2}{-} \mathsf{ZXY} = \hat{0}_{e'_{1,2}} \underset{2,3}{+} \lambda_2 = \lambda_2,$$

since $e'_{1,2} = 0^{1,2}_{E_2}(0^{E_2}_m) = \odot^{1,2}_m$. The outline of the corresponding difference now is,

$\mathsf{XZY} \underset{1,2}{-} \mathsf{ZXY} =$



Of course we have that $\tilde{0}^{2,3}_{Z(m)} \underset{E_2}{-} \tilde{0}^{2,3}_{Z(m)} = \odot^{2,3}_m$, and

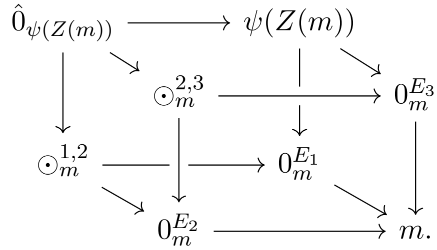
$$\psi^\natural(Z(m)) \underset{E_1}{-} \tilde{0}^{1,3}_{Z(m)} \overset{(4.22)}{=} (\psi(Z(m)) \underset{E_1}{+} \tilde{0}^{1,3}_{Z(m)}) \underset{E_1}{-} \tilde{0}^{1,3}_{Z(m)} = \psi(Z(m)), \tag{4.37}$$

and from (4.23),

$$\hat{0}_{\psi^\natural(Z(m))} \underset{1,2}{-} \hat{0}_{Z(m)} = (\hat{0}_{\psi(Z(m))} \underset{1,2}{+} \hat{0}_{Z(m)}) \underset{1,2}{-} \hat{0}_{Z(m)} = \hat{0}_{\psi(Z(m))} \underset{1,2}{+} (\hat{0}_{Z(m)} \underset{1,2}{-} \hat{0}_{Z(m)})$$
$$\overset{(2.52b)}{=} \hat{0}_{\psi(Z(m))} \underset{1,2}{+} \odot^3_m = \hat{0}_{\psi(Z(m))}.$$

The outline of λ_2,

$\mathsf{XZY} \underset{1,2}{-} \mathsf{ZXY} =$

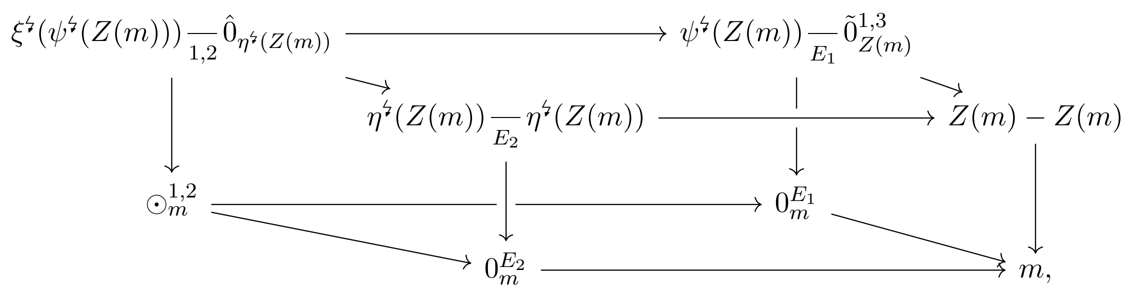


For k_2, from (3.33),

$$\mathsf{YXZ} \underset{1,2}{-} \mathsf{YZX} = \hat{0}_{e_{1,2}} \underset{2,3}{+} k_2 = k_2,$$

since $e_{1,2} = 0^{1,2}_{E_1}(0^{E_1}_m) = \odot^{1,2}_m$. The outline of $\mathsf{YXZ} \underset{1,2}{-} \mathsf{YZX}$:

$\mathsf{YXZ} \underset{1,2}{-} \mathsf{YZX} =$



and as usual, $\eta^{\lightning}(Z(m)) \underset{E_2}{-} \eta^{\lightning}(Z(m)) = \odot^{2,3}_m$, and from (4.37),

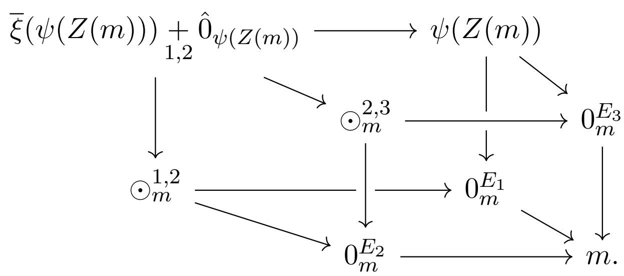
$$\psi^{\lightning}(Z(m)) \underset{E_1}{-} \tilde{0}^{1,3}_{Z(m)} = \psi(Z(m)).$$

Also,

$$\begin{aligned}
\xi^{\lightning}(\psi^{\lightning}(Z(m))) \underset{1,2}{-} \hat{0}_{\eta^{\lightning}(Z(m))} &\overset{(4.22)}{=} \xi^{\lightning}(\psi(Z(m)) \underset{E_1}{+} \tilde{0}^{1,3}_{Z(m)}) \underset{1,2}{-} \hat{0}_{\eta^{\lightning}(Z(m))} \\
\overset{(4.33)}{=} (\xi^{\lightning}(\psi(Z(m))) \underset{1,2}{+} \xi^{\lightning}(\tilde{0}^{1,3}_{Z(m)})) \underset{1,2}{-} \hat{0}_{\eta^{\lightning}(Z(m))} &\overset{(4.34)}{=} \xi^{\lightning}(\psi(Z(m))) \underset{1,2}{+} (\hat{0}_{\eta^{\lightning}(Z(m))} \underset{1,2}{-} \hat{0}_{\eta^{\lightning}(Z(m))}) \\
&\overset{(2.55b)}{=} \xi^{\lightning}(\psi(Z(m))) \overset{(4.32)}{=} \overline{\xi}(\psi(Z(m))) \underset{1,2}{+} \hat{0}_{\psi(Z(m))}.
\end{aligned}$$

hence the outline of k_2,

$\mathsf{YXZ} \underset{1,2}{-} \mathsf{YZX} =$



From (3.34b), we have

$$\lambda_2 \underset{1,2}{-} k_2 = \hat{0}_{e_2} \underset{1,3}{+} u_2 = u_2,$$

since in this case $e_2 = 0^{E_2}_m$. Therefore, as from (2.52b) we have $\hat{0}_{\psi(Z(m))} \underset{1,2}{-} \hat{0}_{\psi(Z(m))} = \odot^3_m$, it follows that,

$$\lambda_2 \underset{1,2}{-} k_2 = \hat{0}_{\psi(Z(m))} \underset{1,2}{-} \left(\overline{\xi}(\psi(Z(m))) \underset{1,2}{+} \hat{0}_{\psi(Z(m))} \right) = \underset{1,2}{-} \overline{\xi}(\psi(Z(m))),$$

that is,

$$u_2 = \underset{1,2}{-} \overline{\xi}(\psi(Z(m))).$$

u_3: the ultrawarp of induced grid on the (U-D) core dvb

To calculate λ_3 use (3.36),

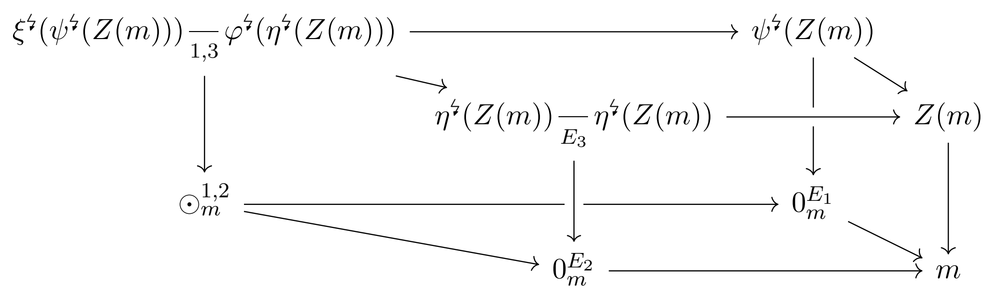
$$\mathsf{YXZ} \underset{1,3}{-} \mathsf{XYZ} = \hat{0}_{e'_{1,3}} \underset{2,3}{+} \lambda_3,$$

and $e'_{1,3} = \psi^{\lightning}(Z(m))$, so

$$\mathsf{YXZ} \underset{1,3}{-} \mathsf{XYZ} = \hat{0}_{\psi^{\lightning}(Z(m))} \underset{2,3}{+} \lambda_3.$$

The outline of the difference of the elements,

$\mathsf{YXZ} \underset{1,3}{-} \mathsf{XYZ} =$

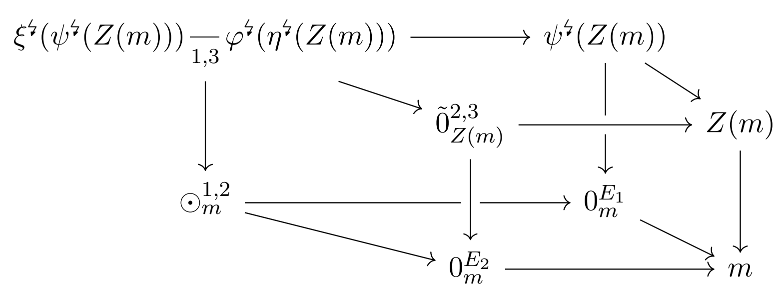


and since

$$\eta^{\lightning}(Z(m)) \underset{E_3}{-} \eta^{\lightning}(Z(m)) = \tilde{0}^{2,3}_{Z(m)},$$

it follows that:

$\mathsf{YXZ} \underset{1,3}{-} \mathsf{XYZ} =$



The following calculation is a variation on the calculation (4.35):

$$
\begin{aligned}
&\xi^{\lightning}(\psi^{\lightning}(Z(m))) \underset{1,3}{-} \varphi^{\lightning}(\eta^{\lightning}(Z(m))) \\
\overset{(4.22),(4.30)}{=}\quad & \xi^{\lightning}(\psi(Z(m)) \underset{E_1}{+} \tilde{0}^{1,3}_{Z(m)}) \underset{1,3}{-} \varphi^{\lightning}(\eta(Z(m)) \underset{E_2}{+} \tilde{0}^{2,3}_{Z(m)}) \\
\overset{(4.33),(4.25)}{=}\quad & \Big(\xi^{\lightning}(\psi(Z(m))) \underset{1,2}{+} \xi^{\lightning}(\tilde{0}^{1,3}_{Z(m)})\Big) \underset{1,3}{-} \Big(\varphi^{\lightning}(\eta(Z(m))) \underset{1,2}{+} \varphi^{\lightning}(\tilde{0}^{2,3}_{Z(m)})\Big) \\
\overset{(4.34),(4.28)}{=}\quad & \Big(\xi^{\lightning}(\psi(Z(m))) \underset{1,2}{+} \hat{0}_{\eta^{\lightning}(Z(m))}\Big) \underset{1,3}{-} \Big(\varphi^{\lightning}(\eta(Z(m))) \underset{1,2}{+} \hat{0}_{\psi^{\lightning}(Z(m))}\Big) \\
\overset{(4.32),(4.24)}{=}\quad & \Big(\overline{\xi}(\psi(Z(m))) \underset{1,2}{+} \hat{0}_{\psi(Z(m))} \underset{1,2}{+} \hat{0}_{\eta^{\lightning}(Z(m))}\Big) \\
& \underset{1,3}{-} \Big(\overline{\varphi}(\eta(Z(m))) \underset{1,2}{+} \hat{0}_{\eta(Z(m))} \underset{1,2}{+} \hat{0}_{\psi^{\lightning}(Z(m))}\Big) \\
\overset{(4.31),(4.23)}{=}\quad & \Big(\overline{\xi}(\psi(Z(m))) \underset{1,2}{+} \hat{0}_{\psi(Z(m))} \underset{1,2}{+} \hat{0}_{\eta(Z(m))} \underset{1,2}{+} \hat{0}_{Z(m)}\Big) \\
& \underset{1,3}{-} \Big(\overline{\varphi}(\eta(Z(m))) \underset{1,2}{+} \hat{0}_{\eta(Z(m))} \underset{1,2}{+} \hat{0}_{\psi(Z(m))} \underset{1,2}{+} \hat{0}_{Z(m)}\Big) \\
=\quad & \Big[\overline{\xi}(\psi(Z(m))) \underset{1,3}{-} \overline{\varphi}(\eta(Z(m)))\Big] \underset{1,2}{+} \Big[\hat{0}_{\psi(Z(m))} \underset{1,3}{-} \hat{0}_{\psi(Z(m))}\Big] \\
& \underset{1,2}{+} \Big[\hat{0}_{\eta(Z(m))} \underset{1,3}{-} \hat{0}_{\eta(Z(m))}\Big] \underset{1,2}{+} \Big[\hat{0}_{Z(m)} \underset{1,3}{-} \hat{0}_{Z(m)}\Big] \\
\overset{(2.54a),(2.55a),(2.52a)}{=}\quad & \Big[\overline{\xi}(\psi(Z(m))) \underset{1,3}{-} \overline{\varphi}(\eta(Z(m)))\Big] \underset{1,2}{+} \hat{0}_{\psi(Z(m))} \underset{1,2}{+} \odot^3_m \underset{1,2}{+} \hat{0}_{Z(m)} \\
\overset{(4.23)}{=}\quad & \Big[\overline{\xi}(\psi(Z(m))) \underset{1,3}{-} \overline{\varphi}(\eta(Z(m)))\Big] \underset{1,2}{+} \hat{0}_{\psi^{\lightning}(Z(m))}
\end{aligned}
$$

So far, we have written:

$$\mathsf{YXZ} \underset{1,3}{-} \mathsf{XYZ} = \Big(\overline{\xi}(\psi(Z(m))) \underset{1,3}{-} \overline{\varphi}(\eta(Z(m)))\Big) \underset{1,2}{+} \hat{0}_{\psi^{\lightning}(Z(m))}$$

Denote by $u := \overline{\xi}(\psi(Z(m))) \underset{1,3}{-} \overline{\varphi}(\eta(Z(m)))$ the ultracore element in the previous equation. Using interchange laws in the Left face of E, we can rewrite the right hand side of the previous equation as follows,

$$
\begin{aligned}
u \underset{1,2}{+} \hat{0}_{\psi^{\lightning}(Z(m))} &= (u \underset{2,3}{+} \odot^3_m) \underset{1,2}{+} (\hat{0}_{\psi^{\lightning}(Z(m))} \underset{2,3}{+} \hat{0}_{Z(m)}) \\
&= (u \underset{1,2}{+} \hat{0}_{Z(m)}) \underset{2,3}{+} (\odot^3_m \underset{1,2}{+} \hat{0}_{\psi^{\lightning}(Z(m))}) = (u \underset{1,2}{+} \hat{0}_{Z(m)}) \underset{2,3}{+} \hat{0}_{\psi^{\lightning}(Z(m))},
\end{aligned}
$$

in other words:

$$\mathsf{YXZ} \underset{1,3}{-} \mathsf{XYZ} = (u \underset{1,2}{+} \hat{0}_{Z(m)}) \underset{2,3}{+} \hat{0}_{\psi^{\lightning}(Z(m))},$$

from where it follows that

$$\lambda_3 = u \underset{1,2}{+} \hat{0}_{Z(m)}.$$

To calculate k_3, use (3.37),

$$\mathsf{ZYX} \underset{1,3}{-} \mathsf{ZXY} = \hat{0}_{e_{1,3}} \underset{2,3}{+} k_3 = \hat{0}_{Z(m)} \underset{2,3}{+} k_3,$$

since $e_{1,3} = Z_1(0_m^{E_1}) = \tilde{0}_{Z(m)}^{1,3}$. We have that

$$\mathsf{ZYX} \underset{1,3}{-} \mathsf{ZXY} = \hat{0}_{Z(m)} \underset{1,3}{-} \hat{0}_{Z(m)} \overset{(2.52a)}{=} \hat{0}_{Z(m)}.$$

Comparing the last two equations, we have

$$\hat{0}_{Z(m)} \underset{2,3}{+} k_3 = \hat{0}_{Z(m)}.$$

Since $\hat{0}_{Z(m)}$ is the double zero of the Up face over $Z(m)$, we have that $k_3 \underset{2,3}{+} \hat{0}_{Z(m)} = k_3$. Therefore, from the final equation we obtain $k_3 = \hat{0}_{Z(m)}$. In total, we have that $\lambda_3 = u \underset{1,2}{+} \hat{0}_{Z(m)}$ and $k_3 = \hat{0}_{Z(m)}$.

For u_3, from (3.38a),

$$\lambda_3 \underset{1,3}{-} k_3 = u_3 \underset{1,2}{+} \hat{0}_{e_3} = u_3 \underset{1,2}{+} \hat{0}_{Z(m)},$$

since $e_3 = Z(m)$. Finally,

$$\begin{aligned}\lambda_3 \underset{1,3}{-} k_3 = (u \underset{1,2}{+} \hat{0}_{Z(m)}) \underset{1,3}{-} \hat{0}_{Z(m)} &= (u \underset{1,2}{+} \hat{0}_{Z(m)}) \underset{1,3}{-} (\hat{0}_{Z(m)} \underset{1,2}{+} \odot_m^3) \\ &= (u \underset{1,3}{-} \odot_m^3) \underset{1,2}{+} (\hat{0}_{Z(m)} \underset{1,3}{-} \hat{0}_{Z(m)}) = u \underset{1,2}{+} \hat{0}_{Z(m)},\end{aligned}$$

therefore, from uniqueness of core elements, we have

$$u_3 = u = \overline{\xi}(\psi(Z(m))) \underset{1,3}{-} \overline{\varphi}(\eta(Z(m))).$$

We see that the ultrawarps we obtain with this method are the same as the ultrawarps obtained with the method in the previous section.

4.5 Connections in A and grids on T^2A

In this section and the next we examine two typical instances of the warp-grid theorem. In this section we consider grids which arise in T^2A from connections in A. In the following section we consider T^3M, the triple tangent bundle of a manifold M.

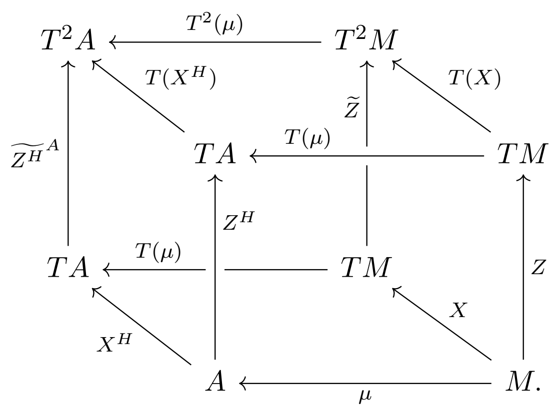
4.5.1 Grids on T^2A

Consider a connection ∇ in A. Recall that Example 1.2.3 gave a construction of a grid in TA for which the warp is $\nabla_X\mu$. We now extend this idea to define a grid in T^2A.

Let $X, Z \in \mathfrak{X}(M)$, and $\mu \in \Gamma A$. Define the following three linear double sections:

- From Front to Back face: $(T(X^H); X^H, T(X); X)$.
- From Right to Left face: $(T^2(\mu); T(\mu), T(\mu); \mu)$.
- From Down to Up face: $(\widetilde{Z^H}^A; \widetilde{Z}, Z^H; Z)$.

Here $\widetilde{Z} = J_M \circ T(Z)$ is the complete (or tangent) lift of Z to a vector field on TM. Likewise $\widetilde{Z^H}^A$ is the complete lift of $Z^H \in \mathfrak{X}(A)$ to a vector field on TA. The grid is shown in (4.38).



(4.38)

The front-back and right-left linear double sections are straightforward. For the down-up linear double section we need to show that $(\widetilde{Z^H}^A, \widetilde{Z})$ is a vector bundle map:

$$\begin{array}{ccc} TA & \xrightarrow{\widetilde{Z^H}^A} & T^2A \\ {\scriptstyle T(q)}\downarrow & & \downarrow{\scriptstyle T^2(q)} \\ TM & \xrightarrow[\widetilde{Z}]{} & T^2M. \end{array} \tag{4.39}$$

First, commutativity of the diagram. Using that $\widetilde{Z^H}^A = J_A \circ T(Z^H)$ we have,

$$T^2(q) \circ \widetilde{Z^H}^A = T^2(q) \circ J_A \circ T(Z^H),$$

and from Lemma 1.2.1:

$$T^2(q) \circ J_A = J_M \circ T^2(q),$$

so

$$T^2(q) \circ J_A \circ T(Z^H) = J_M \circ T^2(q) \circ T(Z^H) = J_M \circ T\left(T(q) \circ Z^H\right).$$

Since (Z^H, Z) is a vector bundle map we have that $T(q) \circ Z^H = Z \circ q$. Hence

$$J_M \circ T\left(T(q) \circ Z^H\right) = J_M \circ T(Z \circ q) = J_M \circ T(Z) \circ T(q) = \widetilde{Z} \circ T(q),$$

and this establishes the commutativity of the diagram (4.39).

Secondly, we need to check fibrewise linearity. Take $\xi_1, \xi_2 \in TA$ with $T(q)(\xi_1) = T(q)(\xi_2) = v$, $v \in TM$. Then as usual:

$$\xi_1 = \frac{d}{dt} a_1(t)\Big|_{t=0}, \quad \xi_2 = \frac{d}{dt} a_2(t)\Big|_{t=0},$$

where $a_1(t), a_2(t)$ are curves in A, with $q(a_1(t)) = q(a_2(t)) = m(t)$, a curve in M, for t near zero, and with $v = \frac{d}{dt} m(t)\Big|_{t=0}$. Now expand $\widetilde{Z^H}^A \left(\xi_1 \underset{T(q)}{+} \xi_2\right)$:

$$\widetilde{Z^H}^A \left(\xi_1 \underset{T(q)}{+} \xi_2\right) = (J_A \circ T(Z^H)) \left(\xi_1 \underset{T(q)}{+} \xi_2\right) \overset{(6)}{=} J_A \left(\frac{d}{dt} Z^H(a_1(t) + a_2(t))\Big|_{t=0}\right). \tag{4.40}$$

Using that Z^H is a linear vector field over Z:

$$\begin{array}{ccc} A & \xrightarrow{Z^H} & TA \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle T(q)} \\ M & \xrightarrow[Z]{} & TM, \end{array}$$

we have that $Z^H(a_1(t) + a_2(t)) = Z^H(a_1(t)) \underset{T(q)}{+} Z^H(a_2(t))$. Therefore, (4.40) can be rewritten as:

$$J_A \left(\frac{d}{dt}(Z^H(a_1(t)) \underset{T(q)}{+} Z^H(a_2(t)))\Big|_{t=0}\right). \tag{4.41}$$

We have that $Z^H(a_1(t))$ and $Z^H(a_2(t))$ are curves in TA with

$$T(q)(Z^H(a_1(t))) = Z(m(t)) = T(q)(Z^H(a_2(t))),$$

for t near zero. And since (Z^H, Z) is a linear vector field,

$$\begin{array}{ccc} Z^H(a_1(t)) & \longmapsto & Z(m(t)) \\ \downarrow & & \downarrow \\ a_1(t) & \longmapsto & m(t), \end{array} \qquad \begin{array}{ccc} Z^H(a_2(t)) & \longmapsto & Z(m(t)) \\ \downarrow & & \downarrow \\ a_2(t) & \longmapsto & m(t), \end{array}$$

and $Z^H(a_1(t)) \underset{T(q)}{+} Z^H(a_2(t))$ is the addition in $TA \xrightarrow{T(q)} TM$ of the two curves, therefore:

$$\frac{d}{dt}\left(Z^H(a_1(t)) \underset{T(q)}{+} Z^H(a_2(t))\right)\Big|_{t=0} = \frac{d}{dt}Z^H(a_1(t))\Big|_{t=0} \underset{T^2(q)}{+} \frac{d}{dt}Z^H(a_2(t))\Big|_{t=0}, \quad (4.42)$$

and note that

$$\begin{aligned} T^2(q)(\frac{d}{dt}Z^H(a^i(t))\Big|_{t=0}) &= \frac{d}{dt}T(q)(Z^H(a^i(t)))\Big|_{t=0} \\ &= \frac{d}{dt}Z(m(t))\Big|_{t=0} = T(Z)(\frac{d}{dt}m(t)\Big|_{t=0}) = T(Z)(v), \end{aligned}$$

where $\frac{d}{dt}Z^H(a^i(t)) \in T(TA)$, $i = 1, 2$, as the tangent double vector bundle of $TA \xrightarrow{T(q)} TM$, the Back face of 2.63:

$$\begin{array}{ccc} T(TA) & \xrightarrow{T^2(q)} & T(TM) \\ {\scriptstyle p_{TA}}\downarrow & & \downarrow{\scriptstyle p_{TM}} \\ TA & \xrightarrow[T(q)]{} & TM. \end{array}$$

Therefore, going back to (4.41), we can rewrite it as,

$$J_A\left(\frac{d}{dt}Z^H(a_1(t))\Big|_{t=0} \underset{T^2(q)}{+} \frac{d}{dt}Z^H(a_2(t))\Big|_{t=0}\right), \quad (4.43)$$

and since J_A preserves addition over $T^2(q)$, see Lemma 2.4.6, (4.43) is equal to,

$$\begin{aligned} &J_A\left(\frac{d}{dt}Z^H(a_1(t))\Big|_{t=0}\right) \underset{T^2(q)}{+} J_A\left(\frac{d}{dt}Z^H(a_2(t))\Big|_{t=0}\right) \\ &\qquad = J_A(T(Z^H)(\xi_1)) \underset{T^2(q)}{+} J_A(T(Z^H)(\xi_2)) = \widetilde{Z^H}^A(\xi_1) \underset{T^2(q)}{+} \widetilde{Z^H}^A(\xi_2), \end{aligned}$$

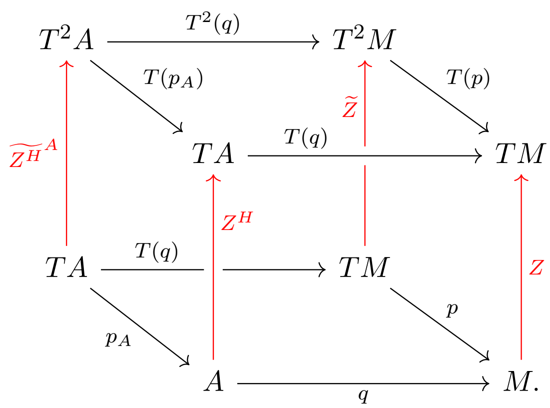
and this completes the proof that (4.39) is a vector bundle map.

The core morphisms of the linear double sections will be needed later:

- For $(T(X^H); X^H, T(X); X)$ the core morphism is (X^H, X).
- For $(T^2(\mu); T(\mu), T(\mu); \mu)$ the core morphism is $(T(\mu), \mu)$.
- For $(\widetilde{Z^H}^A; \widetilde{Z}, Z^H; Z)$ the core morphism is (Z^H, Z).

The first two cases are instances of the general fact that given a morphism (φ, f) of vector bundles, the core morphism of the double vector bundle map $(T(\varphi); \varphi, T(f); f)$ is (φ, f), see Section 1.1.2.

To calculate the core morphism of $(\widetilde{Z^H}^A; \widetilde{Z}, Z^H; Z)$, focus on (4.44). At this point we investigate this linear double section further; it is a double vector bundle morphism from the Down face to the Up face of T^2A. Note that (4.44) is not a triple vector bundle.



(4.44)

Take an element $a \in A$. As an element of the core of the Down face of (4.38), its image in the Down face is $\bar{a} = \frac{d}{dt}ta\Big|_{t=0} \in TA$.

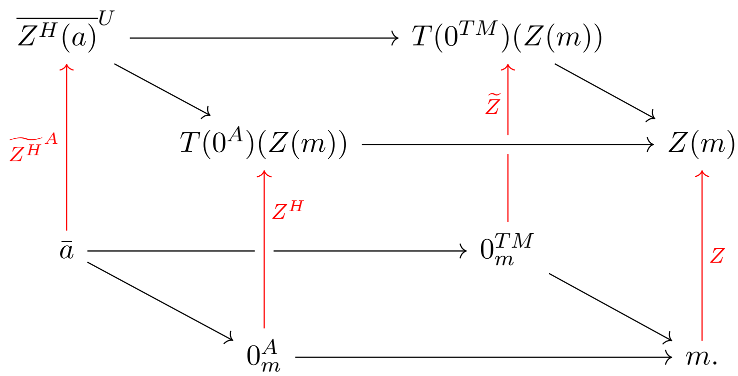
Using the fact that $(\widetilde{Z}, Z)$ is a vector bundle map, we have that $\widetilde{Z}(0_m^{TM}) = T(0^{TM})(Z(m))$. Similarly, using the fact that (Z^H, Z) is a vector bundle map, we have that $Z^H(0_m^A) = T(0^A)(Z(m))$. Finally,

$$
\begin{aligned}
&\widetilde{Z^H}^A(\bar{a}) \\
&= J_A\left(T(Z^H)(\bar{a})\right) = J_A\left(\frac{d}{dt}Z^H(ta)\Big|_{t=0}\right) = J_A\left(\frac{d}{dt}tZ^H(a)\Big|_{t=0}\right) = J_A\left(\overline{Z^H(a)}^B\right).
\end{aligned}
$$

Note the following. The element $\widetilde{Z^H}^A(\bar{a})$ is an element of the Up face of T^2A. And we can write it as $J_A\left(T(Z^H)(\bar{a})\right)$. As J_A maps the Back face of T^2A to the Up face of T^2A, it follows that $T(Z^H)(\bar{a})$ is an element of the Back face of T^2A. Hence, in $T(Z^H)(\bar{a})$, $\bar{a}$ is an element of the Front face of T^2A.

The maps $(T(Z^H); Z^H, T(Z); Z)$ form a double vector bundle morphism from the Front to the Back face of (2.63), with core morphism (Z^H, Z) as usual. Therefore, $T(Z^H)(\bar{a}) = \overline{Z^H(a)}^B$ is now in the core of the Back face. And by (2.76), it follows that $J_A\left(\overline{Z^H(a)}^B\right) = \overline{Z^H(a)}^U$.

In total, the triple outline of $\bar{a}$:



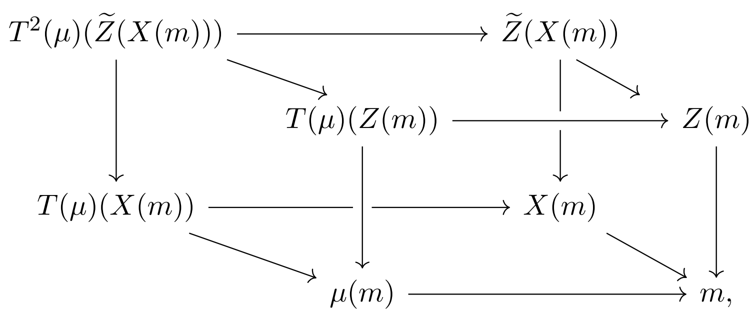
This completes the proof that the core morphism of $(\widetilde{Z^H}^A; \widetilde{Z}, Z^H; Z)$ is (Z^H, Z).

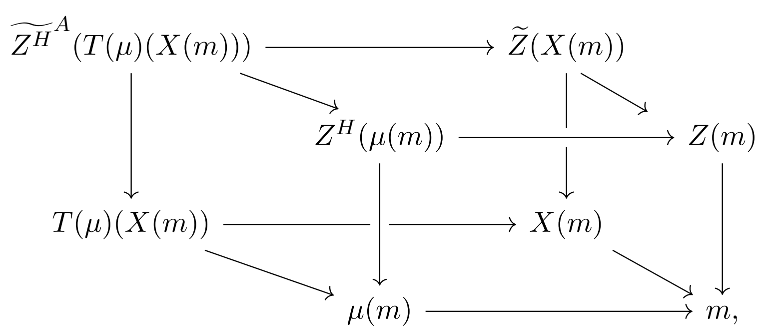
4.5.2 The warp of the Back face

The warp of the Back face is given by

$$T^2(\mu)(\widetilde{Z}(X(m))) \underset{p_{TA}}{-} \widetilde{Z^H}^A(T(\mu)(X(m))).$$

The outlines of the two elements are





(compare with the general triple outlines of the elements YZX and ZYX, of subsection 3.2.1).

Writing the complete lifts as $\widetilde{Z^H}^A = J_A \circ T(Z^H)$ and $\widetilde{Z} = J_M \circ T(Z)$, and using the naturality of J-maps (Lemma 1.2.1), we have that

$$
\begin{aligned}
& T^2(\mu)(\widetilde{Z}(X(m))) \underset{p_{TA}}{-} \widetilde{Z^H}^A(T(\mu)(X(m))) \\
= \;& T^2(\mu)\left(J_M(T(Z)(X(m)))\right) \underset{p_{TA}}{-} J_A\left(T(Z^H)(T(\mu)(X(m)))\right) \\
= \;& J_A(T^2(\mu)(T(Z)(X(m)))) \underset{p_{TA}}{-} J_A(T(Z^H)(T(\mu)(X(m)))). \qquad (4.45)
\end{aligned}
$$

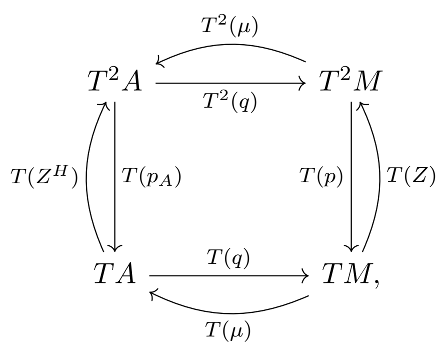
Since J_A interchanges the structures p_{TA} and $T(p_A)$, we can rewrite the last expression in (4.45) as

$$J_A\left(T^2(\mu)(T(Z)(X(m))) \underset{T(p_A)}{-} T(Z^H)(T(\mu)(X(m)))\right).$$

Focus on $T^2(\mu)(T(Z)(X(m))) \underset{T(p_A)}{-} T(Z^H)(T(\mu)(X(m)))$. This now describes the warp of the grid $(T^2(\mu), T(\mu))$ and $(T(Z^H), T(Z))$, a grid on the Up face of T^2A. We can rewrite this as

$$T(T(\mu) \circ Z)(X(m)) \underset{T(p_A)}{-} T(Z^H \circ \mu)(X(m)). \qquad (4.46)$$

At this point, we use Proposition 4.2.2, that the warp of the tangent of a grid is the tangent of the warp of the grid. We apply this to the grid $(T(\mu), \mu)$, (Z^H, Z) on the Down face of T^2A. From Example 1.2.3 the warp of this grid is $\nabla_Z(\mu)$ (see(1.45)). The tangent of this grid is a grid on the Up face of T^2A,



and so its warp is given, for any $v \in T_mM$, by Proposition 4.2.2,

$$
\begin{aligned}
& (T^2(\mu) \circ T(Z))(v) \underset{T(p_A)}{-} (T(Z^H) \circ T(\mu))(v) = \\
& \qquad\qquad \overline{T(\nabla_Z\mu)(v)}^U \underset{T^2(q)}{+} T(\tilde{0}^{TA})(T(\mu)(v)). \qquad (4.47)
\end{aligned}
$$

Having denoted by $\tilde{0}^{TA}$ the zero section of $TA \xrightarrow{p_A} A$, the zero section of $T^2A \xrightarrow{T(p_A)} TA$ is then $T(\tilde{0}^{TA})$. For $v = X(m)$, the left hand side of (4.47) is equal to (4.46). Therefore, (4.46) is equal to

$$\overline{T(\nabla_Z\mu)(X(m))}^U \underset{T^2(q)}{+} T(\tilde{0}^{TA})(T(\mu)(X(m))). \qquad (4.48)$$

We return now to our calculation of (4.45). Applying J_A to (4.48), we have that (4.45) is

$$J_A\left(\overline{T(\nabla_Z\mu)(X(m))}^U\right) \underset{T^2(q)}{+} J_A\left(T(\tilde{0}^{TA})(T(\mu)(X(m)))\right).$$

The addition over $T^2(q)$ does not change under J_A, by Lemma 2.4.6. From (2.76) we have

$$J_A\left(\overline{T(\nabla_Z\mu)(X(m))}^U\right) \underset{T^2(q)}{+} J_A\left(T(\tilde{0}^{TA})(T(\mu)(X(m)))\right) =$$
$$\overline{T(\nabla_Z\mu)(X(m))}^B \underset{T^2(q)}{+} 0^{T^2A}_{T(\mu)(X(m))}.$$

This completes the calculation of the warp of the Back face; taking into consideration the orientation of the Back face, the warp is $-T(\nabla_Z\mu) \in \Gamma_{TM}TA$.

4.5.3 The three ultrawarps

We now focus on the grids defined on the core double vector bundles of T^2A. We present a table with the results here, and outline the calculations in the following subsections.

Back-Front

The Back face is the tangent double vector bundle of the prolonged bundle $TA \to TM$ and by the results of subsection 4.5.2 we obtain

$$T^2(\mu) \circ \widetilde{Z} - \widetilde{Z^H}^A \circ T(\mu) \triangleright T(\nabla_Z\mu). \tag{4.49}$$

Taking into account the orientation of the Back face, the warp is $-T(\nabla_Z\mu) \in \Gamma_{TM}(TA)$. For the Front face, with the appropriate orientation, the warp is $-\nabla_Z\mu \in \Gamma A$, by Example 1.2.3. Therefore the ultrawarp for the Back-Front core double vector bundle (first row of Table 4.1) is, again using Example 1.2.3,

$$-T(\nabla_Z\mu) \circ X + X^H(\nabla_Z\mu) \triangleright -\nabla_X\nabla_Z\mu.$$

Left-Right

The Left face is the double tangent vector bundle T^2A for the manifold A. We therefore apply (8). Taking into account the orientation of the Left face, we have

$$T(X^H) \circ Z^H - \widetilde{Z^H}^A \circ X^H \triangleright [Z^H, X^H].$$

The Right face is T^2M so the warp is $[Z, X] \in \mathfrak{X}(M)$. So the warp of the core double vector bundle in the second row of Table 4.1 is defined by

$$T(\mu) \circ [Z, X] - [Z^H, X^H] \circ \mu. \tag{4.50}$$

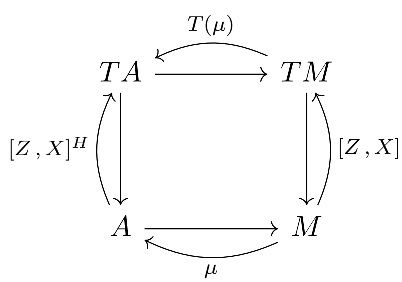
First, what is $[Z^H, X^H]$? In general, it is not equal to $[Z, X]^H$. Using the warp-grid theorem, we now show that $[Z^H, X^H] - [Z, X]^H$, for a connection in a vector bundle A, corresponds to the usual definition of R_∇ in terms of the covariant derivatives ∇_X; see (4.52) below.

Both $[Z^H, X^H]$ and $[Z, X]^H$ project to $[Z, X]$ and therefore their difference is a linear and vertical vector field on A, see Proposition 4.2.9. We now state the following definition.

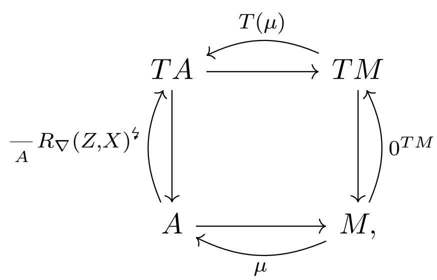
Definition 4.5.1. With the above notation, $R_\nabla(Z, X) : A \to A$ is the map such that

$$[Z, X]^H - [Z^H, X^H] = R_\nabla(Z, X)^{\lightning}. \tag{4.51}$$

In the rest of the section we show that this definition leads to the usual concept of curvature. We can rewrite the grid on the (L-R) core double vector bundle as the sum of the following two grids,



and



so (4.50) is now, from Proposition 4.2.12,

$$\big(T(\mu) \circ [Z, X] - [Z, X]^H \circ \mu\big) + \Big(T(\mu) \circ 0^{TM} - (\underset{A}{-} R_\nabla(Z, X)^{\lightning}) \circ \mu\Big).$$

Here we could cancel the minus signs in the second parenthesis, but we retain them both in order to make the application of (4.16) clear. From Example 1.2.3,

$$T(\mu) \circ [Z, X] - [Z, X]^H \circ \mu \triangleright \nabla_{[Z,X]}\mu,$$

and from (4.16)

$$T(\mu) \circ 0^{TM} - (\underset{A}{-} R_\nabla(Z, X)^{\lightning}) \circ \mu \triangleright +R_\nabla(Z, X)(\mu).$$

So in total, the warp of this core double vector bundle will be

$$\nabla_{[Z,X]}\mu + R_\nabla(Z, X)(\mu).$$

Taking into consideration the orientation of the core double vector bundle, take the opposite sign

$$-\nabla_{[Z,X]}\mu - R_\nabla(Z, X)(\mu).$$

Up-Down

The warp of the Down face is, again by Example 1.2.3, $\nabla_X\mu$. For the warp of the Up face, we use Proposition 4.2.2, and obtain $T(\nabla_X\mu) \in \Gamma_{TM}(TA)$. Therefore the warp of the grid in the third row of Table 4.1 is

$$\nabla_Z\nabla_X\mu.$$

This completes the exposition of Table 4.1.

The warp-grid theorem now gives us that

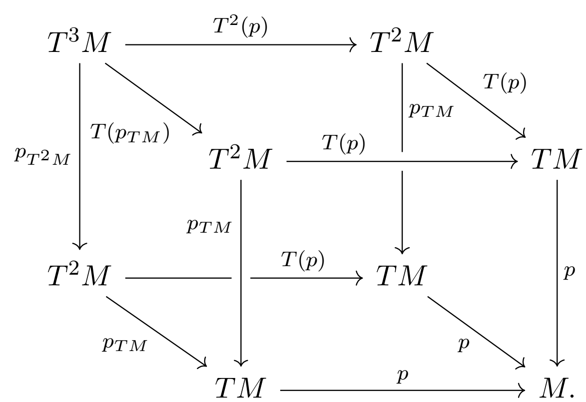
$$-\nabla_X\nabla_Z\mu - \nabla_{[Z,X]}\mu - R_\nabla(Z,X)(\mu) + \nabla_Z\nabla_X\mu = 0. \tag{4.52}$$

This is the usual definition of the curvature of ∇ via differential operators. Therefore, we have shown that if we start with the concept of a connection ∇, and apply the warp-grid theorem to the grid (4.38) in T^2A, we obtain the usual formula for $R_\nabla(Z,X)(\mu)$.

4.6 The triple tangent bundle T^3M and the Jacobi identity

In this section we consider the triple tangent bundle T^3M of a manifold M and construct a grid on it, for which the Jacobi identity emerges as a consequence of the warp-grid theorem. A version of this approach was given by Mackenzie [27]. We present here a clearer and more detailed calculation.

Take E to be T^3M, the triple tangent bundle. This is a special case of T^2A, for $A = TM$:



The three lower faces are copies of T^2M. The Left face is the double tangent bundle of the manifold TM. The Back face is not a double tangent bundle; it is the tangent double vector bundle of $T^2M \xrightarrow{T(p)} TM$. The Up face is obtained by applying the tangent functor to T^2M.

Starting with three vector fields X, Y, and Z, each a section of one of the three copies of TM, one can build a grid on T^3M as follows; see (4.53) below.

- The front-back linear double section $(T(\widetilde{X}); \widetilde{X}, T(X); X)$. Take the complete lift of X across the Down face, and apply the tangent functor to the linear section $(\widetilde{X}, X)$.

- The right-left linear double section $(T^2(Y); T(Y), T(Y); Y)$. Apply the tangent functor to Y and then to $T(Y)$.

- The down-up linear double section $(\widetilde{\widetilde{Z}}; \widetilde{Z}, \widetilde{Z}; Z)$. Take the complete lift of Z across the Front face, and the complete lift of this across the Left face. Likewise take the complete lift of Z across the Right face.

As in T^2A, we need to check that $(\widetilde{\widetilde{Z}}, \widetilde{Z})$ is indeed a linear section of the Back face. First, we need to check commutativity of

$$\begin{array}{ccc} T^2M & \xrightarrow{\widetilde{\widetilde{Z}}} & T^3M \\ {\scriptstyle T(p)}\downarrow & & \downarrow{\scriptstyle T^2(p)} \\ TM & \xrightarrow[\widetilde{Z}]{} & T^2M. \end{array}$$

Using Lemma 1.2.1 for $T^2(p) \circ J_{TM} = J_M \circ T^2(p)$, and that $(\widetilde{Z}, Z)$ is a linear vector field of T^2M, that is, $T(p) \circ \widetilde{Z} = Z \circ p$, it follows:

$$\begin{aligned} T^2(p) \circ \widetilde{\widetilde{Z}} = T^2(p) \circ J_{TM} \circ T(\widetilde{Z}) = J_M \circ T^2(p) \circ T(\widetilde{Z}) = J_M \circ T(T(p) \circ \widetilde{Z}) \\ = J_M \circ T(Z \circ p) = J \circ T(Z) \circ T(p) = \widetilde{Z} \circ T(p). \end{aligned}$$

To check fibrewise linearity, take $\xi_1, \xi_2 \in T^2M$, with $T(p)(\xi_1) = T(p)(\xi_2) = v$, for $v \in TM$. Write $\xi_1 = \frac{d}{dt}a_1(t)\Big|_{t=0}$, $\xi_2 = \frac{d}{dt}a_2(t)\Big|_{t=0}$, for $a_1(t), a_2(t)$ curves in TM, with $p(a_1(t)) = p(a_2(t)) = m(t)$, a curve in M, for t near zero, with $v = \frac{d}{dt}m(t)\Big|_{t=0}$. A similar calculation as in Section 4.5.1,

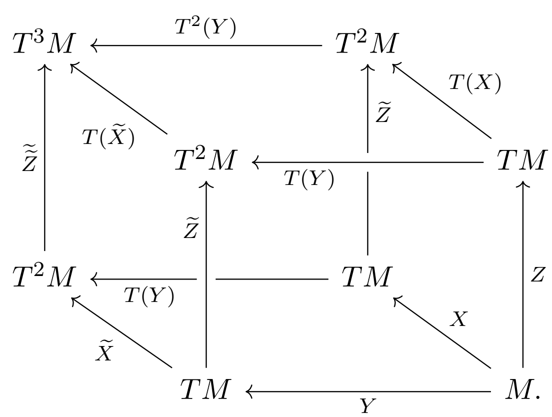
$$
\begin{aligned}
\widetilde{\widetilde{Z}}\left(\xi_1 \underset{T(p)}{+} \xi_2\right) &= (J_{TM}\circ T(\widetilde{Z}))\left(\xi_1 \underset{T(p)}{+} \xi_2\right)\\
&= J_{TM}\circ T(\widetilde{Z})\left(\frac{d}{dt}(a_1(t)+a_2(t))\Big|_{t=0}\right)\\
&= J_{TM}\left(\frac{d}{dt}\widetilde{Z}(a_1(t)+a_2(t))\Big|_{t=0}\right)\\
&= J_{TM}\left(\frac{d}{dt}\left(\widetilde{Z}(a_1(t)) \underset{T(p)}{+} \widetilde{Z}(a_2(t))\right)\Big|_{t=0}\right)\\
&= J_{TM}\left(\frac{d}{dt}\widetilde{Z}(a_1(t))\Big|_{t=0} \underset{T^2(p)}{+} \frac{d}{dt}\widetilde{Z}(a_2(t))\Big|_{t=0}\right)\\
&= J_{TM}\left(T(\widetilde{Z})(\xi_1) \underset{T^2(p)}{+} T(\widetilde{Z})(\xi_2)\right)\\
&= J_{TM}(T(\widetilde{Z})(\xi_1)) \underset{T^2(p)}{+} J_{TM}(T(\widetilde{Z})(\xi_2))\\
&= \widetilde{\widetilde{Z}}(\xi_1) \underset{T^2(p)}{+} \widetilde{\widetilde{Z}}(\xi_2),
\end{aligned}
$$

and this completes the proof.

That $\widetilde{Z}(a_1(t)+a_2(t)) = \widetilde{Z}(a_1(t)) \underset{T(p)}{+} \widetilde{Z}(a_2(t))$ follows from $(\widetilde{Z}, Z)$ being a linear vector field.

That $\frac{d}{dt}\left(\widetilde{Z}(a_1(t)) \underset{T(p)}{+} \widetilde{Z}(a_2(t))\right)\Big|_{t=0} = \frac{d}{dt}\widetilde{Z}(a_1(t))\Big|_{t=0} \underset{T^2(p)}{+} \frac{d}{dt}\widetilde{Z}(a_2(t))\Big|_{t=0}$, follows as in 4.42.

The diagram in (4.53) shows the entire grid.



(4.53)

We now calculate the three ultrawarps defined by this grid. To do this, we calculate the core morphisms of the three linear double sections, and the warps of the six faces.

First, the core morphisms. These follow in an analogous way as in the example of T^2A,

- The core morphism of $(T(\widetilde{X}); \widetilde{X}, T(X); X)$ is $(\widetilde{X}, X)$.
- The core morphism of $(T^2(Y); T(Y), T(Y); Y)$ is $(T(Y), Y)$.
- The core morphism of $(\widetilde{\widetilde{Z}}; \widetilde{Z}, \widetilde{Z}; Z)$ is $(\widetilde{Z}, Z)$.

To calculate the warps of the six faces, we take into consideration the orientation of the faces of a triple vector bundle. For the lower faces, by (8):

- For the Front face: $\widetilde{Z}(Y) - T(Y)(Z) \triangleright [Y, Z]$.
- For the Right face: $T(X)(Z) - \widetilde{Z}(X) \triangleright [Z, X]$.
- For the Down face: $T(Y)(X) - \widetilde{X}(Y) \triangleright [X, Y]$.

We now calculate the warps of the upper faces.

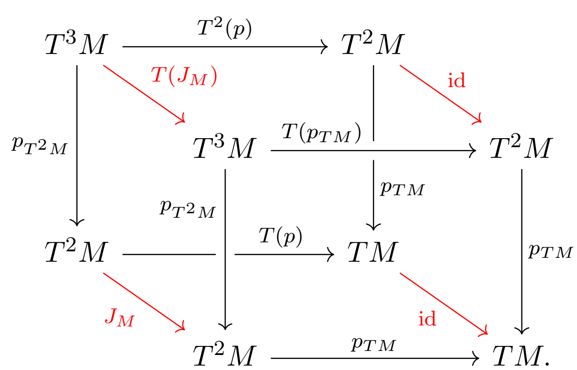
4.6.1 Upper faces

Back face

The warp of the Back face, for $v \in TM$, is given by

$$\widetilde{\widetilde{Z}} \circ T(Y)(v) \underset{T^2(p)}{-} T^2(Y) \circ \widetilde{Z}(v) = \mathrm{w}_{\mathrm{back}}(v) \underset{p_{T^2M}}{+} \hat{0}_{\widetilde{Z}(v)}. \tag{4.54}$$

As we mentioned, the Back face is the tangent double vector bundle of $T^2M \xrightarrow{T(p)} TM$. Apply $T(J_M)$ to it, the tangent of the canonical involution $J_M : T^2M \to T^2M$. The resulting double vector bundle is now the double tangent bundle of TM. In fact, $T(J_M)$ is a triple vector bundle morphism, and maps the Back face of T^3M to the double tangent bundle of TM as shown in (4.55).

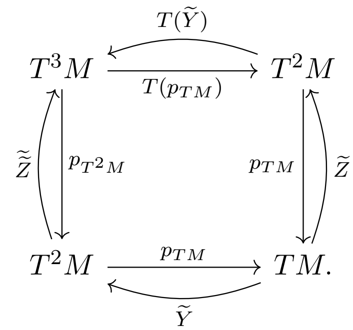


(4.55)

As usual, from Section 1.1.2, the core morphism of (4.55) is (J_M, id). Hence, applying $T(J_M)$ to (4.54),

$$T(J_M)\left(\widetilde{\widetilde{Z}} \circ T(Y)(v) \underset{T^2(p)}{-} T^2(Y) \circ \widetilde{Z}(v)\right) = J_M(\mathrm{w}_{\mathrm{back}}(v)) \underset{p_{T^2M}}{+} \hat{0}_{\widetilde{Z}(v)}. \tag{4.56}$$

Note that $T(J_M)$ changes the vector bundle structure over which the subtraction of the left hand side takes place, and $\underset{T^2(p)}{-}$ will become $\underset{T(p_{TM})}{-}$. Applying $T(J_M)$ to the grid of the Back face yields the following grid on the double tangent bundle of TM.



Therefore, expanding the left hand side of (4.56),

$$\begin{aligned} T(J_M)&\left(\widetilde{\widetilde{Z}} \circ T(Y)(v) \underset{T^2(p)}{-} T^2(Y) \circ \widetilde{Z}(v)\right) \\ &= T(J_M)((\widetilde{\widetilde{Z}} \circ T(Y))(v)) \underset{T(p_{TM})}{-} T(J_M)((T^2(Y) \circ \widetilde{Z})(v)). \end{aligned} \tag{4.57}$$

At this point we need to show that

$$T(J_M) \circ \widetilde{\widetilde{Z}} = \widetilde{\widetilde{Z}} \circ J_M. \tag{4.58}$$

This we do as follows. First, rewrite the left hand side of (4.58) as,

$$\begin{aligned} T(J_M) \circ \widetilde{\widetilde{Z}} = T(J_M) \circ (J_{TM} \circ T(\widetilde{Z})) &= T(J_M) \circ (J_{TM} \circ T(J_M \circ T(Z))) \\ &= T(J_M) \circ J_{TM} \circ T(J_M) \circ T^2(Z). \end{aligned}$$

Rewrite the right hand side of (4.58) as,

$$\widetilde{\widetilde{Z}} \circ J_M = J_{TM} \circ T(J_M) \circ T^2(Z) \circ J_M = J_{TM} \circ T(J_M) \circ J_{TM} \circ T^2(Z),$$

where in the last equality we have used that $T^2(Z) \circ J_M = J_{TM} \circ T^2(Z)$ applying Lemma 1.2.1. Therefore, it suffices to show that

$$T(J_M) \circ J_{TM} \circ T(J_M) = J_{TM} \circ T(J_M) \circ J_{TM}.$$

Take a $\Phi \in T^3M$, and write it as:

$$\frac{d}{dt}\frac{d}{ds}\frac{d}{du}m(t,s,u)\Big|_{u,s,t=0},$$

where $m : (-\epsilon,\epsilon)\times(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)\to M$, a smooth cube of elements of M. Then,

$$T(J_M)(\Phi) = \frac{d}{dt}\left(J_M(\frac{d}{ds}\frac{d}{du}m(t,s,u)\Big|_{u,s=0})\right)\Big|_{t=0} = \frac{d}{dt}\frac{d}{du}\frac{d}{ds}m(t,s,u)\Big|_{s,u,t=0},$$

and

$$J_{TM}(T(J_M)(\Phi)) = J_{TM}\left(\frac{d}{dt}\frac{d}{du}\frac{d}{ds}m(t,s,u)\Big|_{s,u,t=0}\right) = \frac{d}{du}\frac{d}{dt}\frac{d}{ds}m(t,s,u)\Big|_{s,t,u=0}$$

so finally

$$\begin{aligned} T(J_M)(J_{TM}(T(J_M)(\Phi))) &= T(J_M)\left(\frac{d}{du}\frac{d}{dt}\frac{d}{ds}m(t,s,u)\Big|_{s,t,u=0}\right) \\ &= \frac{d}{du}\left(J_M(\frac{d}{dt}\frac{d}{ds}m(t,s,u)\Big|_{s,t=0})\right)\Big|_{u=0} = \frac{d}{du}\frac{d}{ds}\frac{d}{dt}m(t,s,u)\Big|_{t,s,u=0} \qquad (4.59) \end{aligned}$$

Similarly,

$$\begin{aligned} (J_{TM}\circ T(J_M)\circ J_{TM})(\Phi) &= (J_{TM}\circ T(J_M))(J_{TM}(\Phi)) \\ &= (J_{TM}\circ T(J_M))\left(\frac{d}{ds}\frac{d}{dt}\frac{d}{du}m(t,s,u)\Big|_{u,t,s=0}\right) \\ &= J_{TM}\left(T(J_M)\left(\frac{d}{ds}\frac{d}{dt}\frac{d}{du}m(t,s,u)\Big|_{u,t,s=0}\right)\right) \\ &= J_{TM}\left(\frac{d}{ds}\frac{d}{du}\frac{d}{dt}m(t,s,u)\Big|_{t,u,s=0}\right) \\ &= \frac{d}{du}\frac{d}{ds}\frac{d}{dt}m(t,s,u)\Big|_{t,s,u=0} \end{aligned}$$

and we see that this is equal to (4.59). Therefore, using (4.58), we can rewrite (4.57) as,

$$(\widetilde{\widetilde{Z}}\circ\widetilde{Y})(v) \underset{T(p_{TM})}{-} (T(\widetilde{Y})\circ\widetilde{Z})(v) \overset{(8)}{=} -[\widetilde{Z},\widetilde{Y}](v) \underset{p_{T^2M}}{+} \hat{0}_{\widetilde{Z}(v)} = \widetilde{[Y,Z]}(v) \underset{p_{T^2M}}{+} \hat{0}_{\widetilde{Z}(v)}.$$

Substituting this into (4.56),

$$\widetilde{[Y,Z]}(v) \underset{p_{T^2M}}{+} \hat{0}_{\widetilde{Z}(v)} = J_M(\mathrm{w}_{\mathrm{back}}(v)) \underset{p_{T^2M}}{+} \hat{0}_{\widetilde{Z}(v)},$$

and using that $J_M^2 = \mathrm{id}$, we obtain

$$\mathrm{w}_{\mathrm{back}} = T([Y,Z]).$$

Left face

The Left face is the double tangent bundle of TM, so we simply apply (8) for the grid $(T(\widetilde{X}), \widetilde{X})$, $(\widetilde{\widetilde{Z}}, \widetilde{Z})$,

$$T(\widetilde{X}) \circ \widetilde{Z} - \widetilde{\widetilde{Z}} \circ \widetilde{X} \triangleright [\widetilde{Z}, \widetilde{X}] = \widetilde{[Z, X]},$$

so $\mathrm{w}_{\text{left}} = \widetilde{[Z, X]}$.

Up face

For the Up face, using Proposition 4.2.2, it follows directly that $\mathrm{w}_{\text{up}} = T([X, Y])$.

The three ultrawarps

The three core double vector bundles are all copies of T^2M, and their ultracore is $TM \to M$.

The three core double vector bundles in the usual order (B-F), (L-R), and (U-D), with the induced grids from the original grid on T^3M,

$$\begin{array}{ccc} T^2M & \overset{\widetilde{X}}{\rightleftarrows} & TM \\ {\scriptstyle \mathrm{w}_{\text{back}}}\updownarrows & & \updownarrows{\scriptstyle \mathrm{w}_{\text{front}}} \\ TM & \underset{X}{\rightleftarrows} & M, \end{array} \qquad \begin{array}{ccc} T^2M & \overset{T(Y)}{\rightleftarrows} & TM \\ {\scriptstyle \mathrm{w}_{\text{left}}}\updownarrows & & \updownarrows{\scriptstyle \mathrm{w}_{\text{right}}} \\ TM & \underset{Y}{\rightleftarrows} & M, \end{array} \qquad \begin{array}{ccc} T^2M & \overset{\widetilde{Z}}{\rightleftarrows} & TM \\ {\scriptstyle \mathrm{w}_{\text{up}}}\updownarrows & & \updownarrows{\scriptstyle \mathrm{w}_{\text{down}}} \\ TM & \underset{Z}{\rightleftarrows} & M. \end{array}$$

Finally, by (8), the ultracore elements are

$$\begin{aligned} \mathrm{w}_{\text{back}} \circ X - \widetilde{X} \circ \mathrm{w}_{\text{front}} &= \ T([Y, Z]) \circ X - \widetilde{X} \circ [Y, Z] \quad \triangleright [X, [Y, Z]], \\ \mathrm{w}_{\text{left}} \circ Y - T(Y) \circ \mathrm{w}_{\text{right}} &= \ \widetilde{[Z, X]} \circ Y - T(Y) \circ [Z, X] \quad \triangleright [Y, [Z, X]], \\ \mathrm{w}_{\text{up}} \circ Z - \widetilde{Z} \circ \mathrm{w}_{\text{down}} &= \ T([X, Y]) \circ Z - \widetilde{Z} \circ [X, Y] \quad \triangleright [Z, [X, Y]]. \end{aligned}$$

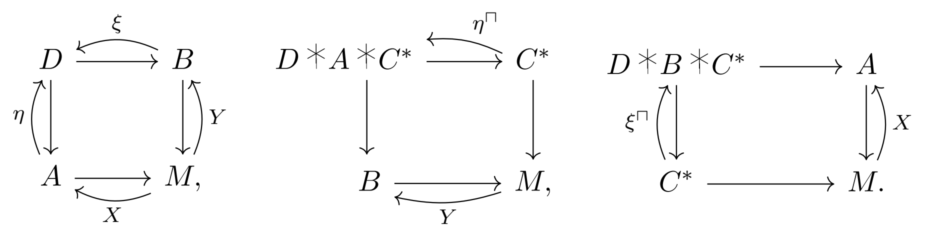
We see that in this way we have formulated the three terms of the Jacobi identity. And applying the warp-grid theorem, we obtain a diagrammatic proof of the Jacobi identity.

4.7 Warps and duality

In this subsection we present Theorem 4.7.2, an alternative formula for the warp which relies on the duality of double vector bundles. And in subsections 4.7.2 and 4.7.3 we verify Theorem 4.7.2 directly for the grids in (9) and in (1.46).

4.7.1 Squarecap sections and their pairings

For a double vector bundle D, in Subsection 2.4.5 we encountered its two duals $D \ast A$ and $D \ast B$. We now examine linear sections of these structures. In particular, starting with a grid (ξ, X) and (η, Y) on D, we will describe the following two linear sections on the iterated duals $D \ast A \ast C^*$ and $D \ast B \ast C^*$ of D:



(4.60)

A linear section (η, Y) of D as in the first diagram of (4.60) induces a linear map,

$$\begin{array}{rcl} \ell_\eta : D \ast A & \to & \mathbb{R} \\ \Phi & \mapsto & \langle \Phi, \eta(\gamma^A_A(\Phi))\rangle_A, \end{array}$$

where $\gamma^A_A : D \ast A \to A$, see middle diagram of (2.82). This map is automatically linear with respect to $D \ast A \to A$, that is, for $\Phi_1, \Phi_2 \in D \ast A$ with $\gamma^A_A(\Phi_1) = \gamma^A_A(\Phi_2) = a$:

$$\begin{aligned} \ell_\eta(\Phi_1 \underset{A}{+} \Phi_2) = \langle \Phi_1 \underset{A}{+} \Phi_2, \eta(\gamma^A_A(\Phi_1 \underset{A}{+} \Phi_2))\rangle_A &= \langle \Phi_1 \underset{A}{+} \Phi_2, \eta(a)\rangle_A \\ &= \langle \Phi_1, \eta(a)\rangle_A + \langle \Phi_2, \eta(a)\rangle_A = \ell_\eta(\Phi_1) + \ell_\eta(\Phi_2). \end{aligned}$$

In ([26, Proposition 3.1]), it is proved that ℓ_η is also linear with respect to the other vector bundle structure, $D \ast A \to C^*$. We include the proposition and its proof.

Proposition 4.7.1. *If (η, Y) is a linear section, then $\ell_\eta : D \ast A \to \mathbb{R}$ defined by*

$$\Phi \mapsto \langle \Phi, \eta(\gamma^A_A(\Phi))\rangle$$

is linear with respect to C^* as well as A, and the restriction of ℓ_η to the core of $D \ast A$ is $\ell_Y : B^* \to \mathbb{R}$.

Proof. If $\Phi_1, \Phi_2 \in D \ast A$ with $(\Phi_1; a_1, \kappa; m)$ and $(\Phi_2; a_2, \kappa; m)$, their sum over C^* has outline $(\Phi_1 \underset{C^*}{+} \Phi_2; a_1 + a_2, \kappa; m)$, therefore,

$$\ell_\eta(\Phi_1 \underset{C^*}{+} \Phi_2) = \langle \Phi_1 \underset{C^*}{+} \Phi_2, \eta(a_1 + a_2)\rangle_A.$$

Now since (η, Y) is a vector bundle morphism, for $a_1, a_2 \in A_m$: $\eta(a_1 + a_2) = \eta(a_1) \underset{B}{+} \eta(a_2)$, therefore

$$\langle \Phi \underset{C^*}{+} \Phi_2, \eta(a_1) \underset{B}{+} \eta(a_2)\rangle_A.$$

Using (2.81) the definition of $\underset{C^*}{+}$, we can write:

$$\ell_\eta(\Phi_1 \underset{C^*}{+} \Phi_2) = \langle \Phi_1, \eta(a_1)\rangle + \langle \Phi_2, \eta(a_2)\rangle = \ell_\eta(\Phi_1) + \ell_\eta(\Phi_2).$$

Similarly for scalar multiplication. Now given $\psi \in B_m^*$, recall by (2.80) the corresponding core element $\psi \in D \ast A$ is given by

$$\langle \psi, 0_b^D \underset{A}{+} c\rangle = \langle \psi, b\rangle,$$

for any $b \in B_m$, and any $c \in C_m$. Hence

$$\ell_\eta(\psi) = \langle \psi, \eta(0_m^A)\rangle = \langle \psi, Y(m)\rangle = \ell_Y(\psi),$$

where $\ell_Y : B^* \to \mathbb{R}$ denotes the linear map $\varphi \mapsto \langle \varphi, Y(q_{B^*}(\varphi))\rangle$, for $Y \in \Gamma B$, of (η, Y). $\square$

Therefore, we can define a linear section of $D \ast A \ast C^* \to C^*$, which we denote by $\eta^\sqcap$:

$$\begin{aligned} \eta^\sqcap : C^* &\to D \ast A \ast C^* \\ \kappa &\mapsto \eta^\sqcap(\kappa) \in D \ast A \ast C^*\Big|_\kappa. \end{aligned}$$

Define $\eta^\sqcap(\kappa) \in D \ast A \ast C^*\Big|_\kappa$ by defining its pairing with any $\Phi \in D \ast A\Big|_\kappa$ to be

$$\langle \eta^\sqcap(\kappa), \Phi\rangle_{C^*} := \ell_\eta(\Phi) = \langle \Phi, \eta(\gamma_A^A(\Phi))\rangle_A. \tag{4.61}$$

This $\eta^\sqcap$ is again a linear section over $Y \in \Gamma B$. We use notations such as $\Big|_\kappa$ on double vector bundles when the symbol for the base point makes clear which structure is meant. The corresponding linear function defined by $(\eta^\sqcap, Y)$ then is,

$$\begin{aligned} \ell_{\eta^\sqcap} : D \ast A &\to \mathbb{R} \\ \Phi &\mapsto \langle \eta^\sqcap(\gamma_{C^*}^A(\Phi)), \Phi\rangle_{C^*}, \end{aligned}$$

and due to (4.61), of course $\ell_{\eta^\sqcap} = \ell_\eta$.

Therefore, we see that there exists a one-to-one correspondence between linear sections (η, Y) of $D \to A$ and linear sections $(\eta^\sqcap, Y)$ of $D \ast A \ast C^* \to C^*$.

Similarly, there exists a one-to-one correspondence between linear sections (ξ, X) of $D \to B$ and linear sections $(\xi^\sqcap, X)$ of $D \ast B \ast C^* \to C^*$, given by

$$\langle \xi^\sqcap(\kappa), \Psi\rangle_{C^*} := \ell_\xi(\Psi) = \langle \Psi, \xi(\gamma_B^B(\Psi))\rangle_B, \tag{4.62}$$

where $\kappa \in C^*$ and $\Psi \in D \ast B\Big|_\kappa$.

We refer to $\xi^\sqcap$ and $\eta^\sqcap$ collectively as ‘squarecap sections.’

We now transform the pairing $\boldsymbol{|}\ ,\ \boldsymbol{|}$ into a pairing $[\![\ ,\ ]\!]$ between the bundles $D \ast B \ast C^* \to C^*$ and $D \ast A \ast C^* \to C^*$. Given a grid in D, applying $[\![\ ,\ ]\!]$ to the sections defined in (4.61) and (4.62) will give an alternative formula for the warp.

Since $D \ast A \ast C^*$ and $D \ast A$ are dual vector bundles over C^*, we have the usual nondegenerate pairing between them. We will use this pairing and the map $Z_A^{-1} : D \ast B \ast C^* \to D \ast A$ (see (2.84)) to define a pairing between $D \ast A \ast C^*$ and $D \ast B \ast C^*$ over C^*[1].

Take elements $\Lambda \in D \ast B \ast C^*$ and $\Sigma \in D \ast A \ast C^*$ with outlines $(\Lambda; \kappa, X; m)$ and $(\Sigma; Y, \kappa; m)$, and define

$$[\![\Lambda, \Sigma]\!] = \langle \Sigma, Z_A^{-1}(\Lambda)\rangle_{C^*}. \tag{4.63}$$

Equivalently, we can define the pairing (4.63) via the map $Z_B^{-1} : D \ast A \ast C^* \to D \ast B$, see (2.85), as follows

$$[\![\Lambda, \Sigma]\!] = \langle \Lambda, Z_B^{-1}(\Sigma)\rangle_{C^*}. \tag{4.64}$$

Both (4.63) and (4.64) define the same pairing. Indeed, rewrite the right hand side of (4.63) using (2.85):

$$\langle \Sigma, Z_A^{-1}(\Lambda)\rangle_{C^*} = \boldsymbol{|}\, Z_A^{-1}(\Lambda),\ Z_B^{-1}(\Sigma)\boldsymbol{|}.$$

And rewriting the right hand side of (4.64) using (2.84),

$$\langle \Lambda, Z_B^{-1}(\Sigma)\rangle_{C^*} = \boldsymbol{|}\, Z_A^{-1}(\Lambda),\ Z_B^{-1}(\Sigma)\boldsymbol{|},$$

so we see that they are equal.

The proof of the next result will take us to the end of the subsection.

Theorem 4.7.2. *Let (ξ, X) and (η, Y) be linear sections forming a grid on a double vector bundle D. Then*

$$[\![\xi^{\sqcap}, \eta^{\sqcap}]\!] = \ell_{\mathrm{w}(\xi,\eta)}.$$

Proof. The following outlines may help us to keep track of the various calculations,

$$\begin{array}{ccc} D \ast A \ni Z_A^{-1}(\xi^{\sqcap}(\kappa)) & \longmapsto & \kappa \\ \downarrow & & \downarrow \\ -X(m) & \longmapsto & m, \end{array} \qquad \begin{array}{ccc} D \ast B \ni Z_B^{-1}(\eta^{\sqcap}(\kappa)) & \longmapsto & Y(m) \\ \downarrow & & \downarrow \\ \kappa & \longmapsto & m. \end{array}$$

Note that the minus sign on $-X(m)$ of $Z_A^{-1}(\xi^{\sqcap}(\kappa))$ comes from the fact that Z_A induces $-\mathrm{id}_A : A \to A$ over M.

[1] *Parenthesis:* Suppose $\varphi \in V^*$ and $v \in V$. Take the usual pairing between V and V^*: $\langle \varphi, v\rangle \in \mathbb{R}$. Let $F : W \to V^*$ be an isomorphism of vector spaces. Then we can define a pairing between W and V as follows: $\langle w, v\rangle := \langle F(w), v\rangle$, for $w \in W$ and $v \in V$. This is what we do in this case.

We can now begin calculations. Start with (4.63),

$$\begin{aligned}
[\![\xi^{\sqcap}(\kappa),\eta^{\sqcap}(\kappa)]\!] &= \langle \eta^{\sqcap}(\kappa), Z_A^{-1}(\xi^{\sqcap}(\kappa))\rangle_{C^*} \\
&\overset{(4.61)}{=} \ell_\eta(Z_A^{-1}(\xi^{\sqcap}(\kappa))) \\
&= -\langle Z_A^{-1}(\xi^{\sqcap}(\kappa)), \eta(X(m))\rangle_A .
\end{aligned}$$

Now using (2.83), with $\Phi = Z_A^{-1}(\xi^{\sqcap}(\kappa))$, $\Psi = Z_B^{-1}(\eta^{\sqcap}(\kappa))$, and $d = \underset{B}{-}\eta(X(m))$, we have

$$\begin{aligned}
&\boldsymbol{|} Z_A^{-1}(\xi^{\sqcap}(\kappa)),\ Z_B^{-1}(\eta^{\sqcap}(\kappa))\boldsymbol{|} \\
&\qquad = \langle Z_A^{-1}(\xi^{\sqcap}(\kappa)), \underset{B}{-}\eta(X(m))\rangle_A - \langle Z_B^{-1}(\eta^{\sqcap}(\kappa)), \underset{B}{-}\eta(X(m))\rangle_B ,
\end{aligned}$$

and this implies that

$$\begin{aligned}
&-\langle Z_A^{-1}(\xi^{\sqcap}(\kappa)), \eta(X(m))\rangle_A \\
&\qquad = \boldsymbol{|} Z_A^{-1}(\xi^{\sqcap}(\kappa)),\ Z_B^{-1}(\eta^{\sqcap}(\kappa))\boldsymbol{|} - \langle Z_B^{-1}(\eta^{\sqcap}(\kappa)), \eta(X(m))\rangle_B .
\end{aligned}$$

Returning to the previous calculations

$$\begin{aligned}
[\![\xi^{\sqcap}(\kappa),\eta^{\sqcap}(\kappa)]\!] &= -\langle Z_A^{-1}(\xi^{\sqcap}(\kappa)), \eta(X(m))\rangle_A \\
&= \boldsymbol{|} Z_A^{-1}(\xi^{\sqcap}(\kappa)),\ Z_B^{-1}(\eta^{\sqcap}(\kappa))\boldsymbol{|} - \langle Z_B^{-1}(\eta^{\sqcap}(\kappa)), \eta(X(m))\rangle_B \\
&\overset{(2.84)}{=} \langle \xi^{\sqcap}(\kappa), Z_B^{-1}(\eta^{\sqcap}(\kappa))\rangle_{C^*} - \langle Z_B^{-1}(\eta^{\sqcap}(\kappa)), \eta(X(m))\rangle_B \qquad (4.65) \\
&\overset{(4.62)}{=} \langle Z_B^{-1}(\eta^{\sqcap}(\kappa)), \xi(Y(m))\rangle_B - \langle Z_B^{-1}(\eta^{\sqcap}(\kappa)), \eta(X(m))\rangle_B \\
&= \langle Z_B^{-1}(\eta^{\sqcap}(\kappa)), \xi(Y(m)) \underset{B}{-} \eta(X(m))\rangle_B \\
&\overset{(11)}{=} \langle Z_B^{-1}(\eta^{\sqcap}(\kappa)), \mathrm{w}(\xi,\eta)(m) \underset{A}{+} 0^D_{Y(m)}\rangle_B .
\end{aligned}$$

Using (2.78), we can rewrite the last expression of (4.65) as

$$\langle Z_B^{-1}(\eta^{\sqcap}(\kappa)), \mathrm{w}(\xi,\eta)(m) \underset{A}{+} 0^D_{Y(m)}\rangle_B = \langle \kappa, \mathrm{w}(\xi,\eta)(m)\rangle = \ell_{\mathrm{w}(\xi,\eta)}(\kappa),$$

since $\gamma^B_{C^*}(Z_B^{-1}(\eta^{\sqcap}(\kappa))) = \kappa$.

In total, we have shown that, for $\kappa \in C^*$, $[\![\xi^{\sqcap},\eta^{\sqcap}]\!](\kappa) = \ell_{\mathrm{w}(\xi,\eta)}(\kappa)$. And this completes the proof of Theorem 4.7.2. □

Note that by comparing (4.64) with (4.65), we see that $\langle Z_B^{-1}(\eta^{\sqcap}(\kappa)), \eta(X(m))\rangle_B = 0$. This can be proved directly. To see this, let $\Phi \in D \ast A$, with outline $(\Phi; X(m), \kappa; m)$. Then, via (2.83), rewrite $\langle Z_B^{-1}(\eta^{\sqcap}(\kappa)), \eta(X(m))\rangle_B$ as follows:

$$\begin{aligned}
\langle Z_B^{-1}(\eta^{\sqcap}(\kappa)), \eta(X(m))\rangle_B &= \langle \Phi, \eta(X(m))\rangle_A - \|\Phi, Z_B^{-1}(\eta^{\sqcap}(\kappa))\|_{C^*} \\
&\overset{(2.85)}{=} \langle \Phi, \eta(X(m))\rangle_A - \langle \eta^{\sqcap}(\kappa), \Phi\rangle_{C^*} \\
&\overset{(4.61)}{=} \langle \Phi, \eta(X(m))\rangle_A - \langle \Phi, \eta(X(m))\rangle_A \\
&= 0.
\end{aligned}$$

4.7.2 Example with T^2M

Consider the double vector bundle T^2M and the grid consisting of $(\widetilde{X}, X)$ and $(T(Y), Y)$, as in (9). What are the corresponding $\widetilde{X}^{\sqcap}$ and $T(Y)^{\sqcap}$? The two duals of T^2M are

$$\begin{array}{ccc} T^2M & \xrightarrow{T(p)} & TM \\ \downarrow{\scriptstyle p_{TM}} & & \downarrow{\scriptstyle p} \\ TM & \xrightarrow{p} & M, \end{array} \qquad \begin{array}{ccc} T^\bullet TM & \xrightarrow{T(p)_\bullet} & TM \\ \downarrow{\scriptstyle r_\bullet} & & \downarrow{\scriptstyle p} \\ T^*M & \xrightarrow{c_M} & M, \end{array} \qquad \begin{array}{ccc} T^*(TM) & \xrightarrow{r} & T^*M \\ \downarrow{\scriptstyle c_{TM}} & & \downarrow{\scriptstyle c_M} \\ TM & \xrightarrow{p} & M. \end{array} \tag{4.66}$$

The double vector bundle $T^\bullet TM$ is the prolongation dual of T^2M, in the notation of [25, Section 9.3]. It is canonically isomorphic as a double vector bundle to $T(T^*M)$ under the map I:

$$\begin{aligned} I : T(T^*M) &\to T^\bullet(TM), \\ \mathscr{X} &\mapsto I(\mathscr{X}), \end{aligned}$$

such that

$$\langle I(\mathscr{X}), \eta\rangle_{TM} = \langle\!\langle \mathscr{X}, \eta\rangle\!\rangle, \tag{4.67}$$

where $\mathscr{X} \in T(T^*M)$, $\eta \in T^2M$, and $\langle\!\langle\ ,\ \rangle\!\rangle$ is the tangent prolongation of the pairing of TM with T^*M, as we saw in (4.2). The map I induces the identity map on both side bundles and on the core vector bundle.

Now, given $(T(Y), Y)$, we will calculate $T(Y)^{\sqcap}$ using $\ell_{T(Y)}$,

$$\ell_{T(Y)} : T^\bullet(TM) \to \mathbb{R}, \qquad T^\bullet_v TM \ni \xi \mapsto \langle \xi, T(Y)(v)\rangle_{TM},$$

where $v \in TM$ and $T^\bullet_v TM$ is the fibre of $T^\bullet(TM)$ over $v \in TM$.

The function $\ell_{T(Y)}$ is linear with respect to both TM and T^*M as noted in general in subsection 4.7.1. Since it is linear with respect to T^*M, it defines a linear section of the dual of the vector bundle $T^\bullet(TM) \to T^*M$, that is, of $T^\bullet(TM) \overset{*}{\;} T^*M \to T^*M$. We use I to simplify this.

Take the function

$$\ell_{T(Y)} \circ I : T(T^*M) \to \mathbb{R}.$$

It follows directly that this is also linear with respect to T^*M.

Therefore, it will define a linear section $\mathfrak{Y}$ of the dual of the vector bundle $T(T^*M) \to T^*M$, that is, of the iterated cotangent $T^*(T^*M) \to T^*M$.

Consider $\mathfrak{Y}(\varphi) \in T^*(T^*M)$ for $\varphi \in T^*M$. Pair this with a $\xi \in T(T^*M)$ with outline $(\xi; \varphi, v; m)$, where $v \in TM$. Using (4.61),

$$\langle \mathfrak{Y}(\varphi), \xi\rangle_{T^*M} = (\ell_{T(Y)} \circ I)(\xi) = \langle I(\xi), T(Y)(v)\rangle_{TM}.$$

We now need the following result from [28]; see also [25, 3.4.6]. It is valid for an arbitrary vector bundle A.

Proposition 4.7.3. *Given* $(\xi;\mu(m),v;m)\in TA$ *and* $(\mathfrak{X};\varphi_m,v;m)\in T(A^*)$, *let* $\mu\in\Gamma(A)$ *and* $\varphi\in\Gamma(A^*)$ *be any sections taking the values* $\mu(m)$ *and* φ_m *at* m. *Then*

$$\langle\!\langle \mathfrak{X},\xi\rangle\!\rangle = \mathfrak{X}(\ell_\mu)+\xi(\ell_\varphi)-v(\langle\varphi,\mu\rangle). \tag{4.68}$$

Using Proposition 4.7.3 and (4.67), it follows that

$$\langle I(\xi), T(Y)(v)\rangle_{TM} = \langle d\ell_Y(\varphi),\xi\rangle. \tag{4.69}$$

This is true for any such $\xi\in T(T^*M)$ so it follows that

$$\mathfrak{Y}(\varphi) = (d\ell_Y)(\varphi),$$

and the linear section in question, $(T(Y)^\sqcap, Y)$, can be identified with $(d\ell_Y, Y)$.

Next consider the linear section $(\widetilde{X}, X)$ and the third double vector bundle in (4.66). As before, the function

$$\ell_{\widetilde{X}}\colon T^*(TM)\to\mathbb{R}.$$

is linear with respect to both TM and T^*M. Using the linearity over T^*M, we obtain a section of the dual of the vector bundle $T^*(TM)\to T^*M$; that is, of $T^*(TM)\ast T^*M\to T^*M$.

Again, this is not easy to work with, and in this case we need to use the reversal isomorphism R which we saw in Example 4.1.2. It follows that

$$\ell_{\widetilde{X}}\circ R\colon T^*(T^*M)\to\mathbb{R}$$

is also linear with respect to T^*M, and defines a section $\mathfrak{X}$ of the dual of the vector bundle $T^*(T^*M)\to T^*M$; that is, of the tangent bundle $T(T^*M)\to T^*M$.

Then for $\varphi\in T^*M$, and any $\mathfrak{F}\in T^*(T^*M)$ with outline $(\mathfrak{F};v,\varphi;m)$, with $v\in TM$, using (4.61),

$$\langle\mathfrak{X}(\varphi),\mathfrak{F}\rangle_{T^*M} = (\ell_{\widetilde{X}}\circ R)(\mathfrak{F}) = \ell_{\widetilde{X}}(R(\mathfrak{F})) = \langle R(\mathfrak{F}),\widetilde{X}(v)\rangle_{TM}.$$

At this point, we use the commutative diagram (4.70), in which each map is an isomorphism of double vector bundles and $(d\nu)^\sharp$ is the map associated to the canonical symplectic structure $d\nu$ on T^*M; see [28] or [25, p. 442].

$$\begin{array}{ccc} T^*(T^*M) & \xrightarrow{\;R\;} & T^*(TM) \\ {\scriptstyle (d\nu)^\sharp}\downarrow & & \uparrow{\scriptstyle J^*} \\ T(T^*M) & \xrightarrow[\;I\;]{} & T^\bullet(TM). \end{array} \tag{4.70}$$

Using $R = J^*\circ I\circ(d\nu)^\sharp$, we have

$$\begin{aligned}\langle R(\mathfrak{F}),\widetilde{X}(v)\rangle_{TM} &= \langle J^*(I((d\nu)^\sharp(\mathfrak{F}))),\widetilde{X}(v)\rangle_{TM}\\ &= \langle I((d\nu)^\sharp(\mathfrak{F})), J(\widetilde{X}(v))\rangle_{TM} = \langle I((d\nu)^\sharp(\mathfrak{F})), T(X)(v)\rangle_{TM}.\end{aligned}$$

As before, using (4.69),

$$\langle I((d\nu)^\sharp(\mathfrak{F})), T(X)(v)\rangle_{TM} = (d\nu)^\sharp(\mathfrak{F})(\ell_X) = \langle d\ell_X, (d\nu)^\sharp(\mathfrak{F})\rangle = -\langle (d\nu)^\sharp(d\ell_X), \mathfrak{F}\rangle,$$

so we see that $\mathfrak{X} = -(d\nu)^\sharp(d\ell_X)$; that is, it is the Hamiltonian vector field for the function ℓ_X. Denote it by H_{ℓ_X}. Finally,

$$[\![T(Y)^\sqcap, \widetilde{X}^\sqcap]\!]_{T^*M} = \langle d\ell_Y, H_{\ell_X}\rangle = \ell_{[X,Y]}.$$

4.7.3 Example with TA

In the case of Example 1.2.3, what are the corresponding sections $T(\mu)^\sqcap$ and $(X^H)^\sqcap$? Just as in the case of T^2M, $(T(\mu)^\sqcap, \mu)$ can be identified with $(d\ell_\mu, \mu)$.

For (X^H, X) a more elaborate calculation is needed. Again, we use $\ell_{X^H} \circ R : T^*A^* \to \mathbb{R}$, and its linearity with respect to A^*. This will define a section of the dual of $T^*A^* \to A^*$, that is, of $TA^* \to A^*$. Denote this vector field by Φ.

Given $\kappa \in A^*$, pair $\Phi(\kappa) \in TA^*$ with any $\Psi \in T^*A^*$ which has outline $(\Psi; \kappa, a; m)$. By Proposition 4.1.1, for suitable $\mathscr{X} \in TA^*$,

$$\langle \Phi(\kappa), \Psi\rangle_{A^*} = (\ell_{X^H} \circ R)(\Psi) = \langle R(\Psi), X^H(a)\rangle_A = \langle\!\langle \mathscr{X}, X^H(a)\rangle\!\rangle - \langle \Psi, \mathscr{X}\rangle_{A^*}. \quad (4.71)$$

The outlines for the elements involved are:

$$\begin{array}{ccc} T^*A \ni R(\Psi) & \longmapsto & a \in A \\ \downarrow & & \downarrow \\ A^* \ni \kappa & \longmapsto & m, \end{array} \qquad \begin{array}{ccc} TA \ni X^H(a) & \longmapsto & X(m) \in TM \\ \downarrow & & \downarrow \\ A \ni a & \longmapsto & m, \end{array}$$

$$\begin{array}{ccc} T^*A^* \ni \Psi & \longmapsto & a \in A \\ \downarrow & & \downarrow \\ A^* \ni \kappa & \longmapsto & m, \end{array} \qquad \begin{array}{ccc} TA^* \ni \mathscr{X} & \longmapsto & X(m) \in TM \\ \downarrow & & \downarrow \\ A^* \ni \kappa & \longmapsto & m. \end{array}$$

Now use Proposition 4.7.3 for the first term of (4.71). Choose a $\varphi \in \Gamma A^*$ with $\varphi(m) = \kappa$, and a $\mu \in \Gamma A$ with $\mu(m) = a$. We can also make the following choice; linear vector fields of a vector bundle A are in bijective correspondence with linear vector fields on its dual bundle A^* (see [25, 3.4.5]). Therefore, take $\mathscr{X}$ to be $X^{H_*}(\varphi(m))$, where X^{H_*} is the corresponding linear vector field to X^H. Then we can write

$$\begin{aligned} \langle\!\langle X^{H_*}(\varphi(m)), X^H(a)\rangle\!\rangle &- \langle \Psi, X^{H_*}(\varphi(m))\rangle_{A^*} \\ &= X^{H_*}(\varphi(m))(\ell_\mu) + X^H(\mu(m))(\ell_\varphi) \\ &\qquad - X(m)(\langle \varphi, \mu\rangle) - \langle \Psi, X^{H_*}(\varphi(m))\rangle_{A^*}. \quad (4.72) \end{aligned}$$

At this point, recall from (1.44) that for $\varphi \in \Gamma A^*$, and for $\mu \in \Gamma A$,

$$X^H(\ell_\varphi) = \ell_{\nabla^{(*)}_X(\varphi)} \in C^\infty(A), \qquad X^{H_*}(\ell_\mu) = \ell_{\nabla_X(\mu)} \in C^\infty(A^*),$$

and of course equation (1.43), the relation between ∇ and $\nabla^{(*)}$,

$$\langle \nabla^{(*)}_X(\varphi), \mu\rangle = X(\langle \varphi, \mu\rangle) - \langle \varphi, \nabla_X(\mu)\rangle,$$

and the latter equation can be rewritten as

$$\ell_{\nabla^{(*)}_X(\varphi)} \circ \mu = X(\langle \varphi, \mu\rangle) - \ell_{\nabla_X(\mu)} \circ \varphi.$$

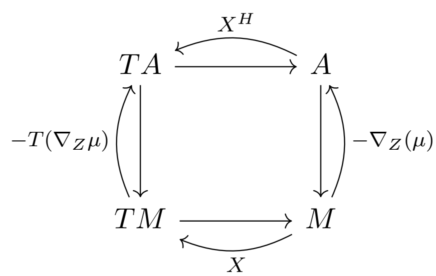
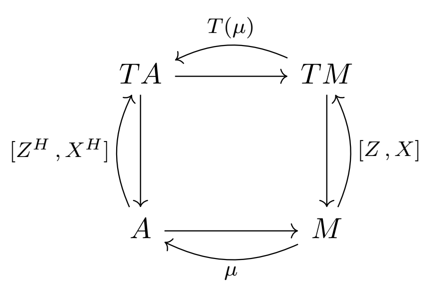
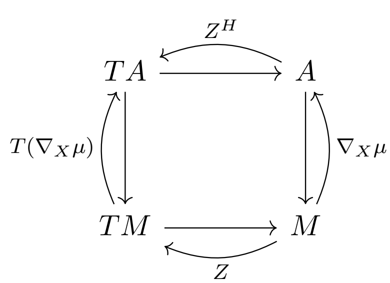
Returning to (4.72),

$$\begin{aligned} &\langle\langle X^{H_*}(\varphi(m)), X^H(a)\rangle\rangle - \langle \Psi, X^{H_*}(\varphi(m))\rangle_{A^*} \\ &\qquad = \ell_{\nabla_X(\mu)}(\varphi(m)) + \ell_{\nabla^{(*)}_X(\varphi)}(\mu(m)) - X(m)(\langle \varphi, \mu\rangle) - \langle \Psi, X^{H_*}(\varphi(m))\rangle_{A^*} \\ &\qquad\qquad = -\langle \Psi, X^{H_*}(\varphi(m))\rangle_{A^*}. \end{aligned}$$

Finally, we have shown that the pairing between $T(\mu)^\sqcap$, which we have shown can be identified with $(d\ell_\mu, \mu)$, and $(X^H)^\sqcap$, which can be identified with (X^{H_*}, X) is,

$$\langle X^{H_*}, d\ell_\mu\rangle = X^{H_*}(\ell_\mu) = \ell_{\nabla_X(\mu)}.$$

Table 4.1: Warps and ultrawarps in T^2A.

<table>
<tr><td>Back</td><td>Front</td><td>(B-F) core double vector bundle</td><td>u_{BF}</td></tr>
<tr><td>$-T(\nabla_Z\mu)$</td><td>$-\nabla_Z\mu$</td><td>
</td><td>$-\nabla_X\nabla_Z\mu$</td></tr>
<tr><td>Left</td><td>Right</td><td>(L-R) core double vector bundle</td><td>u_{LR}</td></tr>
<tr><td>$[Z^H, X^H]$</td><td>$[Z, X]$</td><td>
</td><td>$-\nabla_{[Z,X]}\mu - R_\nabla(Z,X)(\mu)$</td></tr>
<tr><td>Up</td><td>Down</td><td>(U-D) core double vector bundle</td><td>u_{UD}</td></tr>
<tr><td>$T(\nabla_X\mu)$</td><td>$\nabla_X\mu$</td><td>
</td><td>$\nabla_Z\nabla_X\mu$</td></tr>
</table>

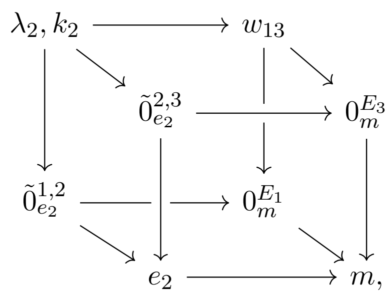
Appendix A

Additional calculations

A.1 Calculations for Section 3.2.2

A.1.1 Proofs of the equations (3.34a), (3.34b), and (3.34c)

From Paragraph 3.2.2, the outlines of λ_2 and k_2 are:



The calculations,

- $\lambda_2 \underset{1,3}{-} k_2 = \hat{0}_{w_{13}} \underset{1,2}{+} (\hat{0}_{0_m^{E_1}} \underset{2,3}{+} u_2) = \hat{0}_{w_{13}} \underset{1,2}{+} (\odot_m^3 \underset{2,3}{+} u_2) = \hat{0}_{w_{13}} \underset{1,2}{+} u_2,$

- $\lambda_2 \underset{1,3}{-} k_2 = \hat{0}_{w_{13}} \underset{2,3}{+} (\hat{0}_{0_m^{E_3}} \underset{1,2}{+} u_2) = \hat{0}_{w_{13}} \underset{2,3}{+} (\odot_m^3 \underset{1,2}{+} u_2) = \hat{0}_{w_{13}} \underset{2,3}{+} u_2,$

- $\lambda_2 \underset{1,2}{-} k_2 = \hat{0}_{\tilde{0}_{e_2}^{1,2}} \underset{1,3}{+} (\hat{0}_{0_m^{E_1}} \underset{2,3}{+} u_2) = \hat{0}_{e_2} \underset{1,3}{+} (\odot_m^3 \underset{2,3}{+} u_2) = \hat{0}_{e_2} \underset{1,3}{+} u_2,$

- And

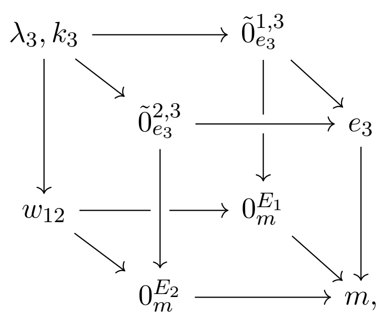
$$\begin{aligned}\lambda_2 \underset{1,2}{-} k_2 &= \hat{0}_{\tilde{0}_{e_2}^{1,2}} \underset{2,3}{+} (\hat{0}_{e_2} \underset{1,3}{+} u_2) = \hat{0}_{e_2} \underset{2,3}{+} (\hat{0}_{e_2} \underset{1,3}{+} u_2) \\ &= (\hat{0}_{e_2} \underset{1,3}{+} \odot_m^3) \underset{2,3}{+} (\hat{0}_{e_2} \underset{1,3}{+} u_2) = (\hat{0}_{e_2} \underset{2,3}{+} \hat{0}_{e_2}) \underset{1,3}{+} (\odot_m^3 \underset{2,3}{+} u_2) \overset{(2.21c)}{=} \hat{0}_{e_2} \underset{1,3}{+} u_2,\end{aligned}$$

- $\lambda_2 \underset{2,3}{\rule{1em}{0.4pt}} k_2 = \hat{0}_{\tilde{0}^{2,3}_{e_2}} \underset{1,3}{+} (\hat{0}_{0^{E_3}_m} \underset{1,2}{+} u_2) = \hat{0}_{e_2} \underset{1,3}{+} (\odot^3_m \underset{1,2}{+} u_2) = \hat{0}_{e_2} \underset{1,3}{+} u_2,$
- Finally,

$$\begin{aligned}\lambda_2 \underset{2,3}{\rule{1em}{0.4pt}} k_2 &= \hat{0}_{\tilde{0}^{2,3}_{e_2}} \underset{1,2}{+} (\hat{0}_{e_2} \underset{1,3}{+} u_2) = \hat{0}_{e_2} \underset{1,2}{+} (\hat{0}_{e_2} \underset{1,3}{+} u_2) \\ &= (\hat{0}_{e_2} \underset{1,3}{+} \odot^3_m) \underset{1,2}{+} (\hat{0}_{e_2} \underset{1,3}{+} u_2) = (\hat{0}_{e_2} \underset{1,2}{+} \hat{0}_{e_2}) \underset{1,3}{+} (\odot^3_m \underset{1,2}{+} u_2) \overset{(2.21b)}{=} \hat{0}_{e_2} \underset{1,3}{+} u_2.\end{aligned}$$

A.1.2 Proofs of the equations (3.38a), (3.38b), and (3.38c)

From Paragraph 3.2.2, the outlines of λ_3 and k_3 are:



And the relevant calculations,

- $\lambda_3 \underset{1,3}{\rule{1em}{0.4pt}} k_3 = \hat{0}_{\tilde{0}^{1,3}_{e_3}} \underset{1,2}{+} (\hat{0}_{0^{E_1}_m} \underset{2,3}{+} u_3) = \hat{0}_{e_3} \underset{1,2}{+} u_3,$
- And

$$\begin{aligned}\lambda_3 \underset{1,3}{\rule{1em}{0.4pt}} k_3 &= \hat{0}_{\tilde{0}^{1,3}_{e_3}} \underset{2,3}{+} (\hat{0}_{e_3} \underset{1,2}{+} u_3) = (\hat{0}_{e_3} \underset{1,2}{+} \odot^3_m) \underset{2,3}{+} (\hat{0}_{e_3} \underset{1,2}{+} u_3) \\ &= (\hat{0}_{e_3} \underset{2,3}{+} \hat{0}_{e_3}) \underset{1,2}{+} (\odot^3_m \underset{2,3}{+} u_3) \overset{(2.15c)}{=} \hat{0}_{e_3} \underset{1,2}{+} u_3,\end{aligned}$$

- $\lambda_3 \underset{1,2}{\rule{1em}{0.4pt}} k_3 = \hat{0}_{w_{12}} \underset{1,3}{+} (\hat{0}_{0^{E_1}_m} \underset{2,3}{+} u_3) = \hat{0}_{w_{12}} \underset{1,3}{+} u_3,$
- $\lambda_3 \underset{1,2}{\rule{1em}{0.4pt}} k_3 = \hat{0}_{w_{12}} \underset{2,3}{+} (\hat{0}_{0^{E_2}_m} \underset{1,3}{+} u_3) = \hat{0}_{w_{12}} \underset{2,3}{+} u_3,$
- And

$$\begin{aligned}\lambda_3 \underset{2,3}{\rule{1em}{0.4pt}} k_3 &= \hat{0}_{\tilde{0}^{2,3}_{e_3}} \underset{1,3}{+} (\hat{0}_{e_3} \underset{1,2}{+} u_3) = (\hat{0}_{e_3} \underset{1,2}{+} \odot^3_m) \underset{1,3}{+} (\hat{0}_{e_3} \underset{1,2}{+} u_3) \\ &= (\hat{0}_{e_3} \underset{1,3}{+} \hat{0}_{e_3}) \underset{1,2}{+} (\odot^3_m \underset{1,3}{+} u_3) \overset{(2.15a)}{=} \hat{0}_{e_3} \underset{1,2}{+} u_3,\end{aligned}$$

- $\lambda_3 \underset{2,3}{\rule{1em}{0.4pt}} k_3 = \hat{0}_{\tilde{0}^{2,3}_{e_3}} \underset{1,2}{+} (\hat{0}_{0^{E_2}_m} \underset{1,3}{+} u_3) = \hat{0}_{e_3} \underset{1,2}{+} u_3.$

A.1.3 Calculations for u_1

$\mathsf{ZYX}\underset{1,2}{-}\mathsf{YZX}$ and $\mathsf{XZY}\underset{1,2}{-}\mathsf{XYZ}$

Recall from Paragraph 3.2.2, that

$$\mathsf{ZYX}\underset{1,2}{-}\mathsf{YZX} = \hat{0}_{e_{1,2}} \underset{1,3}{+} \lambda_1, \qquad \mathsf{XZY}\underset{1,2}{-}\mathsf{XYZ} = \hat{0}_{e'_{1,2}} \underset{1,3}{+} k_1.$$

Using these, we can write:

$$\begin{aligned}
(\mathsf{ZYX}\underset{1,2}{-}\mathsf{YZX})\underset{1,3}{-}(\mathsf{XZY}\underset{1,2}{-}\mathsf{XYZ}) &= (\hat{0}_{e_{1,2}} \underset{1,3}{+} \lambda_1)\underset{1,3}{-}(\hat{0}_{e'_{1,2}} \underset{1,3}{+} k_1) \\
&= (\hat{0}_{e_{1,2}} \underset{1,3}{-} \hat{0}_{e'_{1,2}}) \underset{1,3}{+} (\lambda_1 \underset{1,3}{-} k_1) \\
&\overset{(2.44),(3.30a)}{=} (\hat{0}_{e_1} \underset{2,3}{+} \hat{0}_{w_{12}}) \underset{1,3}{+} (\hat{0}_{e_1} \underset{2,3}{+} u_1) \\
&= (\hat{0}_{e_1} \underset{1,3}{+} \hat{0}_{e_1}) \underset{2,3}{+} (\hat{0}_{w_{12}} \underset{1,3}{+} u_1) \\
&\overset{(2.14a)}{=} \hat{0}_{e_1} \underset{2,3}{+} (\hat{0}_{w_{12}} \underset{1,3/2,3}{+} u_1).
\end{aligned}$$

That $\hat{0}_{w_{12}} \underset{1,3}{+} u_1 = \hat{0}_{w_{12}} \underset{2,3}{+} u_1$ follows similarly as (2.33). And this proves (3.31a).

About (3.31b):

$$\begin{aligned}
(\mathsf{ZYX}\underset{1,2}{-}\mathsf{YZX})\underset{2,3}{-}(\mathsf{XZY}\underset{1,2}{-}\mathsf{XYZ}) &= (\hat{0}_{e_{1,2}} \underset{1,3}{+} \lambda_1)\underset{2,3}{-}(\hat{0}_{e'_{1,2}} \underset{1,3}{+} k_1) \\
&= (\hat{0}_{e_{1,2}} \underset{2,3}{-} \hat{0}_{e'_{1,2}}) \underset{1,3}{+} (\lambda_1 \underset{2,3}{-} k_1) \\
&\overset{(2.45),(3.30c)}{=} (\hat{0}_{e_2} \underset{1,3}{+} \hat{0}_{w_{12}}) \underset{1,3}{+} (\hat{0}_{w_{23}} \underset{1,3}{+} u_1) \\
&= (\hat{0}_{w_{12}} \underset{1,3}{+} \hat{0}_{w_{23}}) \underset{1,3}{+} (\hat{0}_{e_2} \underset{1,3}{+} u_1).
\end{aligned}$$

$\mathsf{ZYX}\underset{1,3}{-}\mathsf{YZX}$ and $\mathsf{XZY}\underset{1,3}{-}\mathsf{XYZ}$

Again, from Paragraph 3.2.2, we have:

$$\mathsf{ZYX}\underset{1,3}{-}\mathsf{YZX} = \hat{0}_{e_{1,3}} \underset{1,2}{+} \lambda_1, \qquad \mathsf{XZY}\underset{1,3}{-}\mathsf{XYZ} = \hat{0}_{e'_{1,3}} \underset{1,2}{+} k_1.$$

Therefore, we see that

$$\begin{aligned}
(\mathsf{ZYX} \underset{1,3}{-} \mathsf{YZX}) \underset{1,2}{-} (\mathsf{XZY} \underset{1,3}{-} \mathsf{XYZ}) &= (\hat{0}_{e_{1,3}} \underset{1,2}{+} \lambda_1) \underset{1,2}{-} (\hat{0}_{e'_{1,3}} \underset{1,2}{+} k_1) \\
&= (\hat{0}_{e_{1,3}} \underset{1,2}{-} \hat{0}_{e'_{1,3}}) \underset{1,2}{+} (\lambda_1 \underset{1,2}{-} k_1) \\
&\overset{(2.40),(3.30b)}{=} (\hat{0}_{e_1} \underset{2,3}{+} \hat{0}_{-w_{13}}) \underset{1,2}{+} (\hat{0}_{e_1} \underset{2,3}{+} u_1) \\
&= (\hat{0}_{e_1} \underset{1,2}{+} \hat{0}_{e_1}) \underset{2,3}{+} (\hat{0}_{-w_{13}} \underset{1,2}{+} u_1) \\
&\overset{(2.14b)}{=} \hat{0}_{e_1} \underset{2,3}{+} (\hat{0}_{-w_{13}} \underset{1,2/2,3}{+} u_1),
\end{aligned}$$

and this proves (3.31c).

Note that we used: $\hat{0}_{e_{1,3}} \underset{1,2}{-} \hat{0}_{e'_{1,3}} = \hat{0}_{e_1} \underset{2,3}{+} \hat{0}_{-w_{13}}$. This requires some explanation. Recall that by hypothesis (3.26):

$$e'_{1,3} \underset{E_1}{-} e_{1,3} = \tilde{0}^{1,3}_{e_1} \underset{E_3}{+} w_{13}, \quad e'_{1,3} \underset{E_3}{-} e_{1,3} = \tilde{0}^{1,3}_{e_3} \underset{E_1}{+} w_{13}.$$

As in (3.42), taking the difference of $e'_{1,3}$ and $e_{1,3}$ the other way around, the corresponding core element is:

$$e_{1,3} \underset{E_1}{-} e'_{1,3} = \tilde{0}^{1,3}_{e_1} \underset{E_3}{+} (-1) \underset{E_3}{\cdot} w_{13}, \quad e_{1,3} \underset{E_3}{-} e'_{1,3} = \tilde{0}^{1,3}_{e_3} \underset{E_1}{+} (-1) \underset{E_1}{\cdot} w_{13},$$

and since $w_{13} \in E_{13}$ is in the core of the Right face, the two scalar multiplications over E_3 and over E_1 coincide, hence we may denote by $-w_{13} := (-1) \underset{E_1}{\cdot} w_{13} = (-1) \underset{E_3}{\cdot} w_{13}$. Therefore, applying (2.40):

$$\hat{0}_{e_{1,3}} \underset{1,2}{-} \hat{0}_{e'_{1,3}} = \hat{0}_{e_{1,3} \underset{E_1}{-} e'_{1,3}} = \hat{0}_{\tilde{0}^{1,3}_{e_1} \underset{E_3}{+} (-1) \underset{E_3}{\cdot} w_{13}} = \hat{0}_{e_1} \underset{2,3}{+} \hat{0}_{(-1) \underset{E_3}{\cdot} w_{13}} = \hat{0}_{e_1} \underset{2,3}{+} \hat{0}_{-w_{13}}.$$

Similarly, applying (2.41):

$$\hat{0}_{e_{1,3}} \underset{2,3}{-} \hat{0}_{e'_{1,3}} = \hat{0}_{e_3} \underset{1,2}{+} \hat{0}_{-w_{13}}. \tag{A.1}$$

Finally, note that

$$(-1) \underset{2,3}{\cdot} \hat{0}_{w_{13}} = \hat{0}_{(-1) \underset{E_3}{\cdot} w_{13}} = \hat{0}_{(-1) \underset{E_1}{\cdot} w_{13}} = (-1) \underset{1,2}{\cdot} \hat{0}_{w_{13}} := \hat{0}_{-w_{13}}.$$

Writing:

$$\hat{0}_{e_{1,3}} \underset{1,2}{-} \hat{0}_{e'_{1,3}} = \hat{0}_{e_1} \underset{2,3}{+} (-1) \underset{2,3}{\cdot} \hat{0}_{w_{13}} = \hat{0}_{e_1} \underset{2,3}{-} \hat{0}_{w_{13}},$$

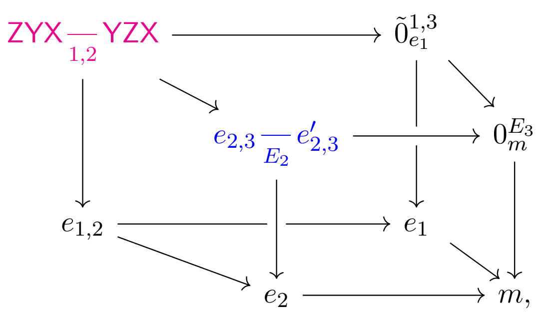
is not particularly useful in this case.

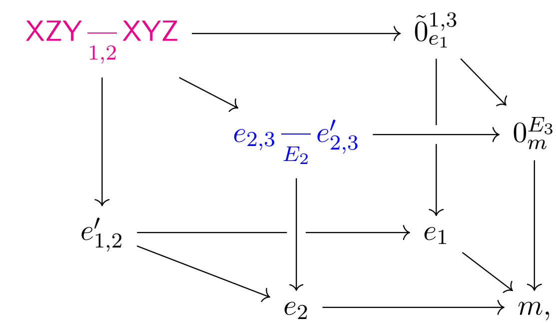
About the last equation (3.31d) of this kind:

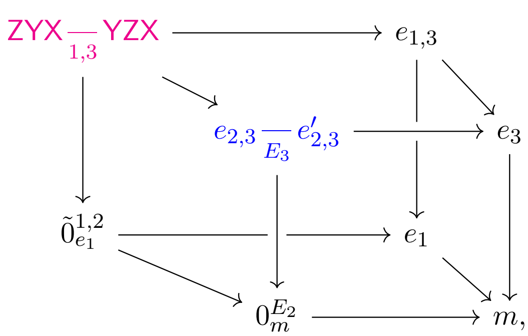
$$\begin{aligned}
(\mathsf{ZYX}\underset{1,3}{-}\mathsf{YZX})\underset{2,3}{-}(\mathsf{XZY}\underset{1,3}{-}\mathsf{XYZ}) &= (\hat{0}_{e_{1,3}}\underset{1,2}{+}\lambda_1)\underset{2,3}{-}(\hat{0}_{e'_{1,3}}\underset{1,2}{+}k_1) \\
&= (\hat{0}_{e_{1,3}}\underset{2,3}{-}\hat{0}_{e'_{1,3}})\underset{1,2}{+}(\lambda_1\underset{2,3}{-}k_1) \\
&\overset{(A.1),(3.30c)}{=} (\hat{0}_{e_3}\underset{1,2}{+}\hat{0}_{-w_{13}})\underset{1,2}{+}(\hat{0}_{w_{23}}\underset{1,2}{+}u_1) \\
&= (\hat{0}_{-w_{13}}\underset{1,2}{+}\hat{0}_{w_{23}})\underset{1,2}{+}(\hat{0}_{e_3}\underset{1,2}{+}u_1).
\end{aligned}$$

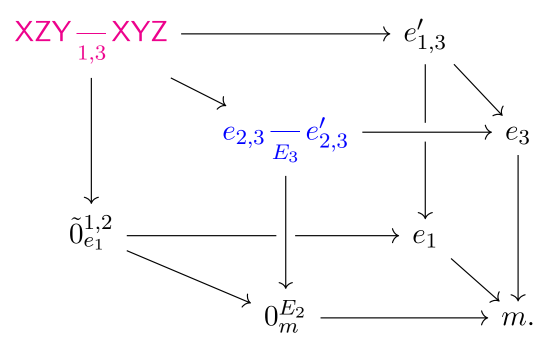
Relevant diagrams

For completeness, we include the following diagrams.









A.1.4 Calculations for u_2

$\mathsf{XZY}\underset{1,2}{-}\mathsf{ZXY}$ and $\mathsf{YXZ}\underset{1,2}{-}\mathsf{YZX}$

From Paragraph 3.2.2 we have:

$$\mathsf{XZY}\underset{1,2}{-}\mathsf{ZXY} = \hat{0}_{e'_{1,2}}\underset{2,3}{+}\lambda_2, \qquad \mathsf{YXZ}\underset{1,2}{-}\mathsf{YZX} = \hat{0}_{e_{1,2}}\underset{2,3}{+}k_2.$$

Therefore,

$$
\begin{aligned}
(\mathsf{XZY} \underset{1,2}{—} \mathsf{ZXY}) \underset{2,3}{—} (\mathsf{YXZ} \underset{1,2}{—} \mathsf{YZX}) &= (\hat{0}_{e'_{1,2}} \underset{2,3}{+} \lambda_2) \underset{2,3}{—} (\hat{0}_{e_{1,2}} \underset{2,3}{+} k_2) \\
&= (\hat{0}_{e'_{1,2}} \underset{2,3}{—} \hat{0}_{e_{1,2}}) \underset{2,3}{+} (\lambda_2 \underset{2,3}{—} k_2) \\
&\overset{(2.45),(3.34c)}{=} (\hat{0}_{-w_{12}} \underset{1,3}{+} \hat{0}_{e_2}) \underset{2,3}{+} (\hat{0}_{e_2} \underset{1,3}{+} u_2) \\
&= (\hat{0}_{e_2} \underset{2,3}{+} \hat{0}_{e_2}) \underset{1,3}{+} (\hat{0}_{-w_{12}} \underset{2,3}{+} u_2) \\
&\overset{(2.21c)}{=} \hat{0}_{e_2} \underset{1,3}{+} (\hat{0}_{-w_{12}} \underset{1,3/2,3}{+} u_2),
\end{aligned}
$$

and this proves (3.35a).

About (3.35b):

$$
\begin{aligned}
(\mathsf{XZY} \underset{1,2}{—} \mathsf{ZXY}) \underset{1,3}{—} (\mathsf{YXZ} \underset{1,2}{—} \mathsf{YZX}) &= (\hat{0}_{e'_{1,2}} \underset{2,3}{+} \lambda_2) \underset{1,3}{—} (\hat{0}_{e_{1,2}} \underset{2,3}{+} k_2) \\
&= (\hat{0}_{e'_{1,2}} \underset{1,3}{—} \hat{0}_{e_{1,2}}) \underset{2,3}{+} (\lambda_2 \underset{1,3}{—} k_2) \\
&\overset{(2.44),(3.34a)}{=} (\hat{0}_{-w_{12}} \underset{2,3}{+} \hat{0}_{e_1}) \underset{2,3}{+} (\hat{0}_{w_{13}} \underset{2,3}{+} u_2) \\
&= (\hat{0}_{w_{13}} \underset{2,3}{+} \hat{0}_{-w_{12}}) \underset{2,3}{+} (\hat{0}_{e_1} \underset{2,3}{+} u_2).
\end{aligned}
$$

Equations $\hat{0}_{e'_{1,2}} \underset{2,3}{—} \hat{0}_{e_{1,2}} = \hat{0}_{-w_{12}} \underset{1,3}{+} \hat{0}_{e_2}$, and $\hat{0}_{e'_{1,2}} \underset{1,3}{—} \hat{0}_{e_{1,2}} = \hat{0}_{-w_{12}} \underset{2,3}{+} \hat{0}_{e_1}$ follow as in page 184, in the previous subsection.

$\mathsf{XZY} \underset{2,3}{—} \mathsf{ZXY}$ and $\mathsf{YXZ} \underset{2,3}{—} \mathsf{YZX}$

From Paragraph 3.2.2 we have:

$$
\mathsf{XZY} \underset{2,3}{—} \mathsf{ZXY} = \hat{0}_{e_{2,3}} \underset{1,2}{+} \lambda_2, \qquad \mathsf{YXZ} \underset{2,3}{—} \mathsf{YZX} = \hat{0}_{e'_{2,3}} \underset{1,2}{+} k_2.
$$

About (3.35c):

$$
\begin{aligned}
(\mathsf{XZY} \underset{2,3}{—} \mathsf{ZXY}) \underset{1,2}{—} (\mathsf{YXZ} \underset{2,3}{—} \mathsf{YZX}) &= (\hat{0}_{e_{2,3}} \underset{1,2}{+} \lambda_2) \underset{1,2}{—} (\hat{0}_{e'_{2,3}} \underset{1,2}{+} k_2) \\
&= (\hat{0}_{e_{2,3}} \underset{1,2}{—} \hat{0}_{e'_{2,3}}) \underset{1,2}{+} (\lambda_2 \underset{1,2}{—} k_2) \\
&\overset{(2.36),(3.34b)}{=} (\hat{0}_{w_{23}} \underset{1,3}{+} \hat{0}_{e_2}) \underset{1,2}{+} (\hat{0}_{e_2} \underset{1,3}{+} u_2) \\
&= (\hat{0}_{e_2} \underset{1,2}{+} \hat{0}_{e_2}) \underset{1,3}{+} (\hat{0}_{w_{23}} \underset{1,2}{+} u_2) \\
&\overset{(2.21b)}{=} \hat{0}_{e_2} \underset{1,3}{+} (\hat{0}_{w_{23}} \underset{1,2/1,3}{+} u_2).
\end{aligned}
$$

Note: At this point, we make the following comment. Since $\mathsf{XZY}\underset{2,3}{—}\mathsf{ZXY}$ and $\mathsf{YXZ}\underset{2,3}{—}\mathsf{YZX}$ have the same Right and Down faces, we can use equations (2.35), with $e = \mathsf{XZY}\underset{2,3}{—}\mathsf{ZXY}$ and $e' = \mathsf{YXZ}\underset{2,3}{—}\mathsf{YZX}$. From (2.35):

$$e\underset{1,2}{—}e' = k \underset{1,3}{+} \hat{0}_{e_{1,2}},$$

and in this case, $q_{1,2}(e) = q_{1,2}(e') = \tilde{0}^{1,2}_{e_2}$, so

$$e\underset{1,2}{—}e' = k \underset{1,3}{+} \hat{0}_{e_2},$$

and comparing this with (3.35c), it follows that

$$k = \hat{0}_{w_{23}} \underset{1,2/1,3}{+} u_2 \in E_{23,1}$$

The second difference of e and e', from (2.35):

$$e\underset{1,3}{—}e' = k \underset{1,2}{+} \hat{0}_{e_{1,3}},$$

we know that $k = \hat{0}_{w_{23}} \underset{1,2}{+} u_2$, and in our case $q_{1,3}(e) = q_{1,3}(e') = e'_{1,3}\underset{E_3}{—}e_{1,3}$, therefore, by (2.41) we have that $\hat{0}_{e'_{1,3}\underset{E_3}{—}e_{1,3}} = \hat{0}_{w_{13}} \underset{1,2}{+} \hat{0}_{e_3}$, so

$$e\underset{1,3}{—}e' = (\hat{0}_{w_{23}} \underset{1,2}{+} u_2) \underset{1,2}{+} (\hat{0}_{w_{13}} \underset{1,2}{+} \hat{0}_{e_3}),$$

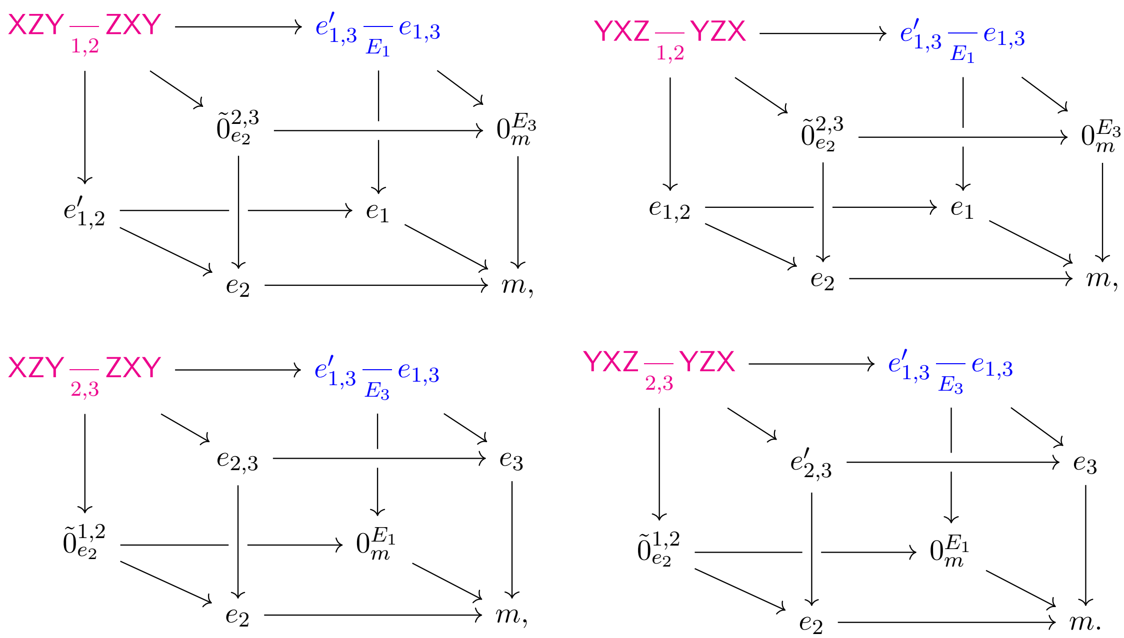
and this just a rearrangement of (3.35d).

Alternatively, proving (3.35d) the usual way:

$$\begin{aligned}
(\mathsf{XZY}\underset{2,3}{—}\mathsf{ZXY})\underset{1,3}{—}(\mathsf{YXZ}\underset{2,3}{—}\mathsf{YZX}) &= (\hat{0}_{e_{2,3}} \underset{1,2}{+} \lambda_2)\underset{1,3}{—}(\hat{0}_{e'_{2,3}} \underset{1,2}{+} k_2) \\
&= (\hat{0}_{e_{2,3}}\underset{1,3}{—}\hat{0}_{e'_{2,3}}) \underset{1,2}{+} (\lambda_2\underset{1,3}{—}k_2) \\
&\overset{(2.37),(3.34a)}{=} (\hat{0}_{w_{23}} \underset{1,2}{+} \hat{0}_{e_3}) \underset{1,2}{+} (\hat{0}_{w_{13}} \underset{1,2}{+} u_2) \\
&= (\hat{0}_{w_{23}} \underset{1,2}{+} \hat{0}_{w_{13}}) \underset{1,2}{+} (\hat{0}_{e_3} \underset{1,2}{+} u_2).
\end{aligned}$$

Relevant diagrams

Again, we include the following diagrams.



A.1.5 Calculations for u_3

$\mathsf{YXZ}\underset{1,3}{—}\mathsf{XYZ}$ and $\mathsf{ZYX}\underset{1,3}{—}\mathsf{ZXY}$

In Paragraph 3.2.2, we established that

$$\mathsf{YXZ}\underset{1,3}{—}\mathsf{XYZ} = \hat{0}_{e'_{1,3}} \underset{2,3}{+} \lambda_3, \qquad \mathsf{ZYX}\underset{1,3}{—}\mathsf{ZXY} = \hat{0}_{e_{1,3}} \underset{2,3}{+} k_3.$$

Write:

$$\begin{aligned}
(\mathsf{YXZ}\underset{1,3}{—}\mathsf{XYZ})\underset{2,3}{—}(\mathsf{ZYX}\underset{1,3}{—}\mathsf{ZXY}) &= (\hat{0}_{e'_{1,3}} \underset{2,3}{+} \lambda_3)\underset{2,3}{—}(\hat{0}_{e_{1,3}} \underset{2,3}{+} k_3) \\
&= (\hat{0}_{e'_{1,3}} \underset{2,3}{—} \hat{0}_{e_{1,3}}) \underset{2,3}{+} (\lambda_3 \underset{2,3}{—} k_3) \\
&\overset{(2.41),(3.38c)}{=} (\hat{0}_{w_{13}} \underset{1,2}{+} \hat{0}_{e_3}) \underset{2,3}{+} (\hat{0}_{e_3} \underset{1,2}{+} u_3) \\
&= (\hat{0}_{e_3} \underset{2,3}{+} \hat{0}_{e_3}) \underset{1,2}{+} (\hat{0}_{w_{13}} \underset{2,3}{+} u_3) \\
&\overset{(2.15c)}{=} \hat{0}_{e_3} \underset{1,2}{+} (\hat{0}_{w_{13}} \underset{1,2/2,3}{+} u_3),
\end{aligned}$$

and this proves (3.39a).

About (3.39b):

$$\begin{aligned}
(\mathsf{YXZ}\underset{1,3}{-}\mathsf{XYZ})\underset{1,2}{-}(\mathsf{ZYX}\underset{1,3}{-}\mathsf{ZXY}) &= (\hat{0}_{e'_{1,3}}\underset{2,3}{+}\lambda_3)\underset{1,2}{-}(\hat{0}_{e_{1,3}}\underset{2,3}{+}k_3)\\
&= (\hat{0}_{e'_{1,3}}\underset{1,2}{-}\hat{0}_{e_{1,3}})\underset{2,3}{+}(\lambda_3\underset{1,2}{-}k_3)\\
&\overset{(2.40),(3.38b)}{=} (\hat{0}_{w_{13}}\underset{2,3}{+}\hat{0}_{e_1})\underset{2,3}{+}(\hat{0}_{w_{12}}\underset{2,3}{+}u_3)\\
&= (\hat{0}_{w_{13}}\underset{2,3}{+}\hat{0}_{w_{12}})\underset{2,3}{+}(\hat{0}_{e_1}\underset{2,3}{+}u_3).
\end{aligned}$$

$\mathsf{YXZ}\underset{2,3}{-}\mathsf{XYZ}$ and $\mathsf{ZYX}\underset{2,3}{-}\mathsf{ZXY}$

Again, from Paragraph 3.2.2 we have:

$$\mathsf{YXZ}\underset{2,3}{-}\mathsf{XYZ}=\hat{0}_{e'_{2,3}}\underset{1,3}{+}\lambda_3, \qquad \mathsf{ZYX}\underset{2,3}{-}\mathsf{ZXY}=\hat{0}_{e_{2,3}}\underset{1,3}{+}k_3.$$

Write:

$$\begin{aligned}
(\mathsf{YXZ}\underset{2,3}{-}\mathsf{XYZ})\underset{1,3}{-}(\mathsf{ZYX}\underset{2,3}{-}\mathsf{ZXY}) &= (\hat{0}_{e'_{2,3}}\underset{1,3}{+}\lambda_3)\underset{1,3}{-}(\hat{0}_{e_{2,3}}\underset{1,3}{+}k_3)\\
&= (\hat{0}_{e'_{2,3}}\underset{1,3}{-}\hat{0}_{e_{2,3}})\underset{1,3}{+}(\lambda_3\underset{1,3}{-}k_3)\\
&\overset{(2.37),(3.38a)}{=} (\hat{0}_{-w_{23}}\underset{1,2}{+}\hat{0}_{e_3})\underset{1,3}{+}(\hat{0}_{e_3}\underset{1,2}{+}u_3)\\
&= (\hat{0}_{e_3}\underset{1,3}{+}\hat{0}_{e_3})\underset{1,2}{+}(\hat{0}_{-w_{23}}\underset{1,3}{+}u_3)\\
&\overset{(2.15a)}{=} \hat{0}_{e_3}\underset{1,2}{+}(\hat{0}_{-w_{23}}\underset{1,2/1,3}{+}u_3),
\end{aligned}$$

and this proves (3.39c).

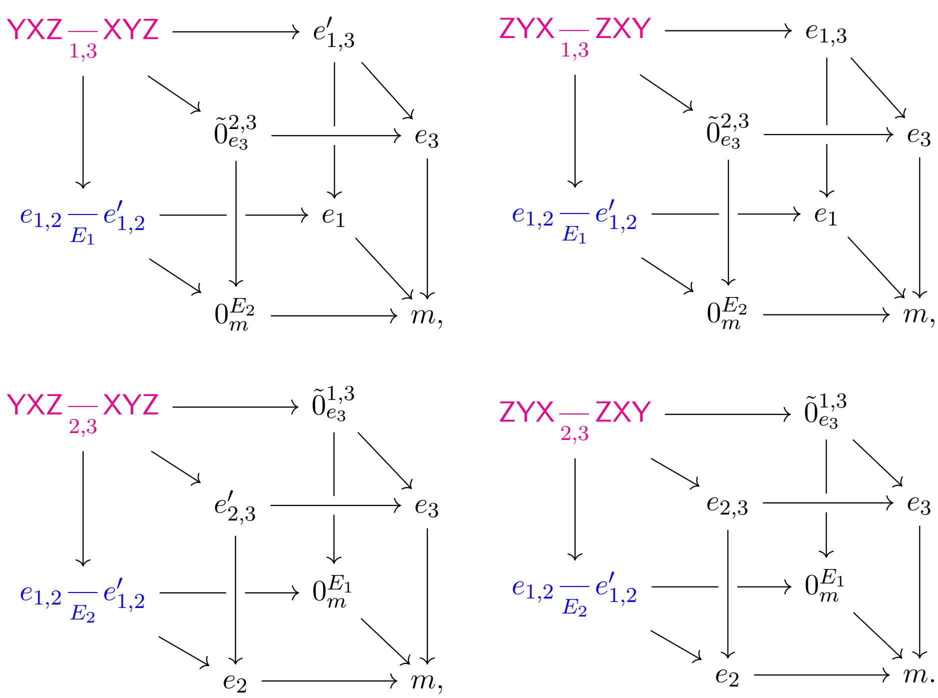
Finally, for (3.39d),

$$\begin{aligned}
(\mathsf{YXZ}\underset{2,3}{-}\mathsf{XYZ})\underset{1,2}{-}(\mathsf{ZYX}\underset{2,3}{-}\mathsf{ZXY}) &= (\hat{0}_{e'_{2,3}}\underset{1,3}{+}\lambda_3)\underset{1,2}{-}(\hat{0}_{e_{2,3}}\underset{1,3}{+}k_3)\\
&= (\hat{0}_{e'_{2,3}}\underset{1,2}{-}\hat{0}_{e_{2,3}})\underset{1,3}{+}(\lambda_3\underset{1,2}{-}k_3)\\
&\overset{(2.36),(3.38b)}{=} (\hat{0}_{-w_{23}}\underset{1,3}{+}\hat{0}_{e_2})\underset{1,3}{+}(\hat{0}_{w_{12}}\underset{1,3}{+}u_3)\\
&= (\hat{0}_{-w_{23}}\underset{1,3}{+}\hat{0}_{w_{12}})\underset{1,3}{+}(\hat{0}_{e_2}\underset{1,3}{+}u_3).
\end{aligned}$$

And again $\hat{0}_{e'_{2,3}}\underset{1,3}{-}\hat{0}_{e_{2,3}} = \hat{0}_{-w_{23}}\underset{1,2}{+}\hat{0}_{e_3}$, and $\hat{0}_{e'_{2,3}}\underset{1,2}{-}\hat{0}_{e_{2,3}} = \hat{0}_{-w_{23}}\underset{1,3}{+}\hat{0}_{e_2}$ follow as in page 184.

Relevant diagrams

Finally, the diagrams in this case are:



A.2 Functions on A and more on R

A.2.1 Classes of functions on $A \to M$

As mentioned in Section 1.2.3, to define either a vector field or a tangent vector on a vector bundle $A \xrightarrow{q} M$, it is enough to check how it "behaves" when applied to linear and pullback functions of A. It's not quite true to say that these classes of functions generate $C^\infty(A)$. What is true is that one can write any 1-form $\Phi \in \Omega^1(A)$ as a (not unique) sum of $d\ell_\varphi$ and of q^*df, where $\varphi \in \Gamma A^*$ and $f \in C^\infty(M)$.

The following is Proposition 9.4.1, [25].

Proposition A.2.1. *For $(\Phi; X, \varphi(m); m) \in T^*A$, a covector at $X \in A_m$, and any $\varphi \in \Gamma A^*$ which takes the value $\varphi(m)$, there exists $\omega \in \Omega^1(M)$ such that*

$$\Phi = d\ell_\varphi(X) \underset{A}{+} (q^*\omega)(X).$$

Proving that two tangent vectors (or two vector fields) $\xi_1, \xi_2 \in T_aA$ are equal, is equivalent to checking that for every covector $\Phi \in T^*_aA$: $\langle \Phi, \xi_1 \rangle = \langle \Phi, \xi_2 \rangle$. And by

Proposition A.2.1, it is enough to check that

$$\langle d\ell_\varphi, \xi_1 \rangle = \langle d\ell_\varphi, \xi_2 \rangle, \quad \forall \varphi \in \Gamma A^*, \tag{A.2}$$

and

$$\langle q^*\omega, \xi_1 \rangle = \langle q^*\omega, \xi_2 \rangle, \quad \forall \omega \in \Gamma(T^*M). \tag{A.3}$$

We can directly reformulate (A.2) to:

$$\xi_1(\ell_\varphi) = \xi_2(\ell_\varphi), \quad \forall \varphi \in \Gamma A^*.$$

Locally any 1-form on M can be written as a linear combination of differentials of functions $f \in C^\infty(M)$, so we can reformulate (A.3) as

$$\langle q^*(df), \xi_1 \rangle = \langle q^*(df), \xi_2 \rangle,$$

and since $q^*(df) = d(q^*f)$, rewrite the last equation as

$$\xi_1(f \circ q) = \xi_2(f \circ q).$$

And this is why linear and pullback functions are of special importance.

A.2.2 Core morphism of R

Recall by Proposition (4.1.1),

$$\langle\langle \mathscr{X}, \xi \rangle\rangle_{TM} = \langle R(\mathfrak{F}), \xi \rangle_A + \langle \mathfrak{F}, \mathscr{X} \rangle_{A^*},$$

for elements

$$\begin{array}{ccc} TA \ni \xi & \longmapsto & v_0 \in TM \\ \downarrow & & \downarrow \\ A \ni a_0 & \longmapsto & m, \end{array} \qquad \begin{array}{ccc} TA^* \ni \mathscr{X} & \longmapsto & v_0 \in TM \\ \downarrow & & \downarrow \\ A^* \ni \varphi_0 & \longmapsto & m, \end{array}$$

$$\begin{array}{ccc} T^*A \ni R(\mathfrak{F}) & \longmapsto & \varphi_0 \in A^* \\ \downarrow & & \downarrow \\ A \ni a_0 & \longmapsto & m, \end{array} \qquad \begin{array}{ccc} T^*A^* \ni \mathfrak{F} & \longmapsto & a_0 \in A \\ \downarrow & & \downarrow \\ A^* \ni \varphi_0 & \longmapsto & m. \end{array}$$

If $\mathfrak{F}$ is a core element of T^*A^*, i.e.,

$$\begin{array}{ccc} \mathfrak{F} = \overline{\omega} & \longmapsto & 0^A_m \\ \downarrow & & \downarrow \\ 0^{A^*}_m & \longmapsto & m \end{array}$$

which means that $\mathscr{X}$ will have outline:

$$\begin{array}{ccc} \mathscr{X} & \longmapsto & v \\ \downarrow & & \downarrow \\ 0_m^{A^*} & \longmapsto & m, \end{array}$$

therefore, we can write $\mathscr{X} = T(0^{A^*})(v) \underset{A^*}{+} \overline{\eta}$, for $\overline{\eta}$ a core element of TA^*. By (2.80), it follows that,

$$\langle \mathfrak{F}, \mathscr{X} \rangle_{A^*} = \langle \overline{\varpi}, T(0^{A^*})(v) \underset{A^*}{+} \overline{\eta} \rangle_{A^*} = \langle \omega, v \rangle.$$

The corresponding $\xi \in TA$ will have outline

$$\begin{array}{ccc} \xi & \longmapsto & v \\ \downarrow & & \downarrow \\ 0_m^{A} & \longmapsto & m, \end{array}$$

hence we can write $\xi = T(0^A)(v) \underset{A}{+} \bar{a}$, for $\bar{a}$ a core element of TA. Then, since R is a double vector bundle morphism, it will map core elements to core elements, hence $R(\overline{\varpi}) = \overline{R(\omega)}$, and again from (2.80),

$$\langle R(\mathfrak{F}), \xi \rangle_A = \langle \overline{R(\omega)}, T(0^A)(v) \underset{A}{+} \bar{a} \rangle_A = \langle R(\omega), v \rangle.$$

For $\mathscr{X} = T(0^{A^*})(v) \underset{A^*}{+} \overline{\eta}$ and $\xi = T(0^A)(v) \underset{A}{+} \bar{a}$, we can write:

$$\mathscr{X} = \frac{d}{dt}(m(t), t \cdot \eta)\Big|_{t=0}, \quad \xi = \frac{d}{dt}(m(t), t \cdot a)\Big|_{t=0}$$

where $\frac{d}{dt} m(t)\Big|_{t=0} = v$. Hence,

$$\langle\!\langle \mathscr{X}, \xi \rangle\!\rangle_{TM} = \frac{d}{dt}\langle t \cdot \eta, t \cdot a \rangle\Big|_{t=0} = \frac{d}{dt} t^2 \langle \eta, a \rangle\Big|_{t=0} = 0.$$

Substituting everything into (4.1):

$$0 = \langle \omega, v \rangle + \langle R(\omega), v \rangle = \langle \omega + R(\omega), v \rangle$$

and this is true for all $v \in T_m M$, therefore, $R(\omega) = -\omega$.

Bibliography

[1] R. Abraham, J. E. Marsden, and T. Raţiu. *Manifolds, tensor analysis, and applications*, volume 75 of *Applied Mathematical Sciences*. Springer-Verlag, New York, second edition, 1988.

[2] A. L. Besse. *Manifolds all of whose geodesics are closed*, volume 93 of *Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas]*. Springer-Verlag, Berlin-New York, 1978. With appendices by D. B. A. Epstein, J.-P. Bourguignon, L. Bérard-Bergery, M. Berger and J. L. Kazdan.

[3] H. Bursztyn, A. Cabrera, and C. Ortiz. Linear and multiplicative 2-forms. *Lett. Math. Phys.*, 90(1-3):59–83, 2009.

[4] M. Crampin and F. A. E. Pirani. *Applicable differential geometry*, volume 59 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1986.

[5] J. Dieudonné. *Treatise on analysis. Vol. III.* Academic Press, New York-London, 1972. Translated from the French by I. G. MacDonald, Pure and Applied Mathematics, Vol. 10-III.

[6] B. A. Dubrovin, A. T. Fomenko, and S. P. Novikov. *Modern geometry—methods and applications. Part II*, volume 104 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1985. The geometry and topology of manifolds, Translated from the Russian by Robert G. Burns.

[7] Ch. Ehresmann. Catégories doubles et catégories structurées. *C. R. Acad. Sci. Paris*, 256:1198–1201, 1963.

[8] M. K. Flari and K. C. H. Mackenzie. Warps, grids and curvature in triple vector bundles. arXiv:1705.01017v2.

[9] M. K. Flari and K. C. H. Mackenzie. Warps, grids and curvature in triple vector bundles. *Lett. Math. Phys.*, doi:10.1007/s11005-018-1103-y.

[10] K. Grabowska and M. Zając. The Tulczyjew triple in mechanics on a Lie group. *J. Geom. Mech.*, 8(4):413–435, 2016.

[11] J. Grabowski and M. Rotkiewicz. Higher vector bundles and multi-graded symplectic manifolds. *J. Geom. Phys.*, 59(9):1285–1305, 2009.

[12] A. Gracia-Saz, M. Jotz Lean, K. C. H. Mackenzie, and R. A. Mehta. Double Lie algebroids and representations up to homotopy. *J. Homotopy Relat. Struct.*, 2017. doi:10.1007/s40062-017-0183-1.

[13] A. Gracia-Saz and K. C. H. Mackenzie. Duality functors for triple vector bundles. *Lett. Math. Phys.*, 90(1-3):175–200, 2009.

[14] A. Gracia-Saz and K. C. H. Mackenzie. Duality functors for n-fold vector bundles. arXiv:1209.0027.

[15] A. Gracia-Saz and R. A. Mehta. Lie algebroid structures on double vector bundles and representation theory of Lie algebroids. *Adv. Math.*, 223(4):1236–1275, 2010.

[16] W. Greub, S. Halperin, and R. Vanstone. *Connections, curvature, and cohomology. Vol. II: Lie groups, principal bundles, and characteristic classes.* Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1973. Pure and Applied Mathematics, Vol. 47-II.

[17] M. Jotz Lean and K. Kirchhoff-Lukat. Natural lifts of Dorfman brackets. arXiv:1610.05986v2.

[18] J. M. Lee. *Introduction to smooth manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2013.

[19] D. Li-Bland and P. Ševera. Quasi-Hamiltonian groupoids and multiplicative Manin pairs. *Int. Math. Res. Not. IMRN*, (10):2295–2350, 2011.

[20] K. C. H. Mackenzie. Double Lie algebroids and second-order geometry. I. *Adv. Math.*, 94(2):180–239, 1992.

[21] K. C. H. Mackenzie. On symplectic double groupoids and the duality of Poisson groupoids. *Internat. J. Math.*, 10(4):435–456, 1999.

[22] K. C. H. Mackenzie. Affinoid structures and connections. In *Poisson geometry (Warsaw, 1998)*, volume 51 of *Banach Center Publ.*, pages 175–186. Polish Acad. Sci. Inst. Math., Warsaw, 2000.

[23] K. C. H. Mackenzie. Double Lie algebroids and second-order geometry. II. *Adv. Math.*, 154(1):46–75, 2000.

[24] K. C. H. Mackenzie. Duality and triple structures. In *The breadth of symplectic and Poisson geometry*, volume 232 of *Progr. Math.*, pages 455–481. Birkhäuser Boston, Boston, MA, 2005.

[25] K. C. H. Mackenzie. *General theory of Lie groupoids and Lie algebroids*, volume 213 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2005.

[26] K. C. H. Mackenzie. Ehresmann doubles and Drinfel'd doubles for Lie algebroids and Lie bialgebroids. *J. Reine Angew. Math.*, 658:193–245, 2011.

[27] K. C. H. Mackenzie. Proving the Jacobi identity the hard way. In *Geometric Methods in Physics. XXX Workshop 2011*, Trends in Mathematics, pages 357–366. Springer Basel AG, 2012.

[28] K. C. H. Mackenzie and P. Xu. Lie bialgebroids and Poisson groupoids. *Duke Math. J.*, 73(2):415–452, 1994.

[29] K. C. H. Mackenzie and P. Xu. Classical lifting processes and multiplicative vector fields. *Quart. J. Math. Oxford Ser. (2)*, 49(193):59–85, 1998.

[30] R. A. Mehta. *Supergroupoids, double structures, and equivariant cohomology.* ProQuest LLC, Ann Arbor, MI, 2006. Thesis (Ph.D.)–University of California, Berkeley.

[31] P. W. Michor. *Topics in differential geometry*, volume 93 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 2008.

[32] I. Moerdijk and G. E. Reyes. *Models for smooth infinitesimal analysis*, Springer-Verlag, New York, 1991.

[33] J. Pradines. *Fibrés vectoriels doubles et calcul des jets non holonomes*, volume 29 of *Esquisses Mathématiques [Mathematical Sketches].* Université d'Amiens, U.E.R. de Mathématiques, Amiens, 1977.

[34] W. M. Tulczyjew. *Geometric formulations of physical theories*, volume 11 of *Monographs and Textbooks in Physical Science. Lecture Notes.* Bibliopolis, Naples, 1989. Statics and dynamics of mechanical systems.

[35] P. Urbański. Double vector bundles in classical mechanics. *Rend. Sem. Mat. Univ. Politec. Torino*, 54(4):405–421, 1996. Geometrical structures for physical theories, II (Vietri, 1996).

[36] Th. Th. Voronov. Q-manifolds and Mackenzie theory: An overview. arXiv:0709.4232v1.

[37] Th. Th. Voronov. Graded manifolds and Drinfeld doubles for Lie bialgebroids. In *Quantization, Poisson brackets and beyond (Manchester, 2001)*, volume 315 of *Contemp. Math.*, pages 131–168. Amer. Math. Soc., Providence, RI, 2002.

[38] Th. Th. Voronov. Q-manifolds and Mackenzie theory. *Comm. Math. Phys.*, 315(2):279–310, 2012.

Index