One-parameter Groups of Möbius Maps in Two-Dimensional Real Commutative Algebra

Khawlah Ali Mustafa

The University of Leeds

Department of Pure Mathematics

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The candidate confirms that the work submitted is her own, and that appropriate credit has been given where reference has been made to the work of others.

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Abstract

Möbius transformations have been thoughtfully studied over the field of complex numbers. In this thesis, we investigate Möbius transformations over two rings which are not fields: the ring of double numbers $\mathbb{O}$ and the ring of dual numbers $\mathbb{D}$. We will see certain similarity between the cases of fields and rings along with some significant distinctions.

After the introduction and necessary background material, given in the first two chapters, I introduce general linear groups, projective lines and Möbius transformations over several rings such us the ring of integer numbers, the Cartesian product ring and the two rings $\mathbb{O}$ and $\mathbb{D}$.

In the following chapters, we consider in details metrics, classification of Möbius maps based on the number of fixed points, connected continuous one-parameter subgroups and an application of Möbius maps.

Contents

Chapter 1

Introduction

Möbius transformations have been studied over the field of complex numbers; see [7, Ch.13;54;62;73, CH.8;75, Ch.9] for a comprehensive presentation. The purpose of this work is to expand these ideas to double and dual numbers. Some new and unexpected phenomena will appear in those cases. Relying on the four types of continuous one-parameter subgroups of $SL_2(\mathbb{R})$, I built all different types of continuous one-parameter subgroups of $GL_2(\mathbb{O})$, $SL_2(\mathbb{O})$, $GL_2(\mathbb{D})$ and $SL_2(\mathbb{D})$, up to similarity and rescaling. The rest of the introduction gives an overview of this work. In the first section of this introduction, we give a brief history of double numbers, dual numbers and Möbius transformations. Section 2 introduces the physical meaning of double and dual numbers. In Section 3, we present a review of Möbius transformations over the real and complex field. The fourth section introduces the projective lines and Möbius transformations over two rings, the ring of double and dual numbers. The last section presents the outline of the thesis.

1.1 A Brief History of Double Numbers, Dual Numbers and Möbius Transformations

Double (dual) numbers form a two-dimensional commutative associative algebra with identity. They are spanned by a basis consisting of 1 and a hypercomplex unit ι. The square of ι is 1 for double numbers and 0 for dual numbers. To place our work into a historic perspective we provide a brief account here.

1.1.1 A brief history of double numbers

The use of double numbers dates back to 1848 when James Cockle revealed his tessarines [20]. About thirty years later, William Clifford introduced the use of double numbers, now called split-biquaternions, in a quaternion algebra. He called its elements "motors", a term in parallel with the "rotor" action of a complex number taken from the circle group.

Since the early 20th century, the double multiplication has commonly been seen as a Lorentz boost of a spacetime plane [3]. A further generalisation to split-octonions was done by Adrian Albert, Richard Schafer, and others [12].

In 1935, J.C. Vignaux and A. Durañona y Vedia developed the double geometric algebra and function theory in four articles in Contribución a las Ciencias Físicas y Matemáticas (National University of La Plata,) República Argentina. In 1941, E.F. Allen used the double geometric arithmetic to establish the nine-point hyperbola of a triangle inscribed in $zz^* = 1$ [1]. In 1956 Mieczyslaw Warmus published "Calculus of Approximations" in Bulletin de l'Academie Polanaise des Sciences. He developed two algebraic systems, each of which he called "approximate numbers", the second of which forms a real algebra [77]. D. H. Lehmer reviewed the article in Mathematical Reviews and observed that this second system was isomorphic to the "double" numbers. At the beginning of the sixties of the last century, Warmus continued his exposition, referring to the components of an approximate number as midpoint and radius of the interval denoted.

1.1.2 A brief history of dual numbers

Dual numbers were introduced in 1873 by William Clifford and were used at the beginning of the 20th century by the German mathematician Eduard Study, who used them to represent the dual angle which measures the relative position of two skew lines in space. Study defined a dual angle as $a + d\epsilon$, where a is the angle between the directions of two lines in three-dimensional space and d is a distance between them.

The idea of a projective line over dual numbers was proposed by Grünwald [30]. Yaglom shows that the cycle $Z = \{z : y = \alpha x^2\}$, in the dual numbers plane is invariant under a cyclic rotation [78, Ch.2]. The concept of cyclic rotation has been further developed by V. V. Kisil [49].

There is a recent interest to double and dual numbers in different areas: differential geometry [4, 8, 9, 14, 15, 25], modal logic [60], quantum mechanics [39, 40, 55, 63], space-time geometry [13, 28, 33, 34, 64, 78], hypercomplex analysis [17, 23, 24, 41–43, 43–45, 47].

1.1.3 A brief history of Möbius transformations

Projective geometry appeared in the 17th century, and its earliest inventors were artists and architects who were interested in imaging, perspective, etc. Initially, projective geometry was the geometry "of projections", and this explains its name. Möbius (17 November 1790-26 September 1868) was the first who introduced homogeneous coordinates into projective geometry [65]. The approaches related to fractional linear transformations appeared in the nineteenth century [6].

The theory of linear groups arose in the middle of the 19th century and was developed in close connection with the theory of Lie groups and Galois theory. A systematic investigation of linear groups was started by C. Jordan [38].

An isomorphism of the complex Möbius group with the Lorentz group was noted by several authors. Felix Klein (1893, 1897) worked on automorphic functions related to hyperbolic geometry and Möbius geometry [59]. Gustav Herglotz (1909) showed that hyperbolic motions (i.e. isometric automorphisms of a hyperbolic space) transforming the unit sphere into itself correspond to Lorentz transformations [32]. Then, Herglotz was able to classify the one-parameter Lorentz transformations into loxodromic, elliptic, hyperbolic and parabolic groups. Roger Penrose and Wolfgang Rindler have described the relation between Lorentz transformations and Möbius transformations [67]. They develop the 2-spinor calculus in considerable detail and show how it may be viewed either a useful supplement or as a practical alternative to the more familiar world-tensor calculus.

Among numerous applications of Möbius transformations we mention Kleinian group. A Kleinian group is a discrete subgroup of $PGL_2(\mathbb{C})$, The group $PGL_2(\mathbb{C})$, of 2 by 2 complex matrices of determinant 1 modulo its center has several natural representations: as conformal transformations of the Riemann sphere, and as orientation-preserving isometries of 3-dimensional hyperbolic space H^3, and as orientation preserving conformal maps of the open unit ball B^3 in $\mathbb{R}^3$ to itself. Therefore, a Kleinian group can be regarded as a discrete subgroup acting on one of these spaces. By considering the ball's boundary, a Kleinian group is defined as a subgroup of $PGL_2(\mathbb{C})$, the complex projective linear group, which acts by Möbius transformations on the Riemann sphere. The theory of general Kleinian groups was founded by Felix Klein (1883) and Henri Poincaré (1883). The complex dynamics, which is defined by the iteration of complex Möbius transformation, is another interested application of Möbius maps. It is also worth to mention continued fraction as an application of Möbius maps.

1.2 The Physical Meaning of Double and Dual Numbers

Double and dual numbers have physical applications. The use of double numbers is motivated by special relativity. Since multiplication of double numbers respects the indefinite form $x^2 - y^2$ of two-dimensions Minkowski space-time.

Dual numbers find applications in physics too, where they frame one of the simplest non-trivial examples of a superspace. Equivalently, they are supernumbers with just one generator. Supernumbers are the elements of Grassmann algebra [22, Sec.1.1].

The motivation behind introducing dual numbers into physics follows from the Pauli exclusion principle for fermions. The direction along ϵ is termed the "fermionic" direction, and the real component is termed the "bosonic" direction. The fermionic direction models the Pauli exclusion principle for fermions: under the exchange of coordinates, the quantum mechanical wave function changes sign, and thus vanishes if two coordinates are brought together; this physical idea is captured by the algebraic relation $\epsilon^2 = 0$. Gromov and Kuratov employ dual numbers for quantum kinematics [29].

1.3 Real and Complex Projective Lines and Möbius Transformations

Our results for dual and double numbers will be compared with the known construction in $\mathbb{R}$ and $\mathbb{C}$. For the reader's convenience we briefly remind main points in a suitable form, further particularities can be found in [75]. Let $\mathbb{K}$ be a field of real or complex numbers. Let $\sim$ be an equivalence relation on $\mathbb{K}^2\backslash\{(0,0)\}$ defined as follows: $(z_1, z_2){\sim}(z_3, z_4)$ if and only if there exists a non-zero number $u \in \mathbb{K}$ such that $z_1 = uz_3$ and $z_2 = uz_4$. The set of all equivalence classes

$\mathbb{K}^2/\sim$ is called the projective line over $\mathbb{K}$, denoted by $\mathbb{K}^2/\sim$ and by $\mathbb{P}(\mathbb{K})$. The point of the projective line corresponding to a vector $\begin{pmatrix} x \\ y \end{pmatrix}$ is denoted by $[x : y]$. There is a natural embedding $x \mapsto [x : 1]$ of the field $\mathbb{K}$ to the projective line. The only point, $[1 : 0]$, not covered by this embedding is associated with infinity [6;54, Ch.8;57;76]. Any additional element added to a set in order to eliminate special cases is often called an ideal element, so infinity is an example of ideal element added to the real line.

A linear transformation of $\mathbb{K}^2$ can be represented by multiplication of 2×2-matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to two-dimensional vectors. The transformation is not degenerate (invertible) if $ad - bc \neq 0$. The collection of all 2×2-matrices, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $ad - bc \neq 0$, is a group denoted by $GL_2(\mathbb{K})$. The collection of all 2×2-matrices, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $ad - bc = 1$, is denoted by $SL_2(\mathbb{K})$, which is a subgroup of $GL_2(\mathbb{K})$. A linear map $\mathbb{K}^2 \to \mathbb{K}^2$ is a class invariant for $\sim$. Therefore, the linear transformation of $\mathbb{K}^2$ produces the map $\mathbb{P}(\mathbb{K}) \to \mathbb{P}(\mathbb{K})$ as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} [x : y] = [ax + by : cx + dy], \quad ad - bc \neq 0,$$

where $a, b, c, d \in \mathbb{K}$. For any $u \neq 0$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} ua & ub \\ uc & ud \end{pmatrix}$ define the same map of $\mathbb{P}(\mathbb{K})$. In other words, for all $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{K})$, such that $ad - bc > 0$ for $\mathbb{K} = \mathbb{R}$ and $ad - bc \neq 0$ for $\mathbb{K} = \mathbb{C}$, there is $A' = \frac{1}{\sqrt{\det(A)}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{K})$ such that both A, A' define the same map. It is clear that $\mathbb{P}(\mathbb{K})$ equals to the orbit of $[1 : 0]$ with respect to $SL_2(\mathbb{K})$. Which means $SL_2(\mathbb{K})$ acts transitively on

$\mathbb{P}(\mathbb{K})$.

For $[x:1] \in \mathbb{P}(\mathbb{K})$, its image is $[ax+b : cx+d]$. If $cx+d \neq 0$, then $[ax+b : cx+d] \sim [\frac{ax+b}{cx+d} : 1]$. Therefore, the map $[x:1] \mapsto [\frac{ax+b}{cx+d} : 1]$ can be abbreviated to $g(x) = \frac{ax+b}{cx+d}$ [54, Ch.2;56;57]. That means g is a map from $\mathbb{K} \backslash \{x : cx+d = 0\}$ to $\mathbb{K}$.

In the following, this formula will be used as a notation for more accurate discussion in terms of the projective line.

Given a Möbius map g, $\operatorname{tr}^2(g)$ is defined as $\operatorname{tr}^2(g) = (a+d)^2$, where the representative matrix of g is $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL_2(\mathbb{K})$. Suppose that g is not the identity map. Then, eigenvalues of A are solutions $\lambda_{1,2} = \dfrac{d+a \mp \sqrt{(d+a)^2-4}}{2}$ of the quadratic characteristic equation with $(a+d)^2 = \operatorname{tr}^2 A$ being the principal part of the discriminant. An eigenvector $\begin{pmatrix} x \\ y \end{pmatrix}$ of A corresponds to a fixed point $[x:y]$ of g. Then we can classify Möbius maps through eigenvalues of A.

1. A has two different complex-conjugated eigenvalues if and only if $0 \leq \operatorname{tr}^2(g) < 4$. That means, g fixes two distinct complex-conjugated points in $\mathbb{P}(\mathbb{C})$ and fixes no point in $\mathbb{P}(\mathbb{R})$. Such a map is called elliptic.

2. A has a double eigenvalue if and only if $\operatorname{tr}^2(g) = 4$. That means, g fixes a double point. Such a map is called parabolic.

3. A has two distinct real eigenvalues if and only if $\operatorname{tr}^2(g) > 4$. That means, g fixes two distinct points. Such a map is called hyperbolic.

4. For $\mathbb{K} = \mathbb{C}$, there is an extra class as follows. A has two distinct non-real eigenvalues if and only if $\operatorname{tr}^2(g) \notin [0, \infty)$. In other words, A has two distinct complex eigenvalues if $Im(\sqrt{\operatorname{tr}^2(g)}) \neq 0$. That means, g fixes two distinct complex points. Such a map is called strictly loxodromic.

The last type of transformation is not possible for $\mathbb{K} = \mathbb{R}$. The class, which contains the classes of hyperbolic and strictly loxodromic maps, is called the class of loxodromic maps.

Obviously, $SL_2(\mathbb{C})$ is the disjoint union of $\{I\}$ and the above four classes (parabolic, elliptic, hyperbolic and strictly loxodromic) of maps. $SL_2(\mathbb{R})$, which is a subgroup of $SL_2(\mathbb{C})$, splits into the disjoint union of $\{I\}$ and the three classes of parabolic, elliptic and hyperbolic maps. It is important that continuous one-parameter subgroups of $SL_2(\mathbb{K})$ consist only of maps of the same type [5, Ch.4;37;54, Ch.3].

1.4 Double and Dual Projective Lines and Möbius Transformations

The double numbers $\mathbb{O}$ are a two-dimensional commutative algebra over $\mathbb{R}$ spanned by the multiplicative identity element 1 and another element j that satisfies $j^2 = 1$. The dual numbers form a two-dimensional commutative algebra over $\mathbb{R}$ spanned by 1 and ϵ such that $\epsilon^2 = 0$. [54, App.A1;74]. The set of all double (dual) numbers is denoted by $\mathbb{O}$ ($\mathbb{D}$) respectively. It is known that any two-dimensional commutative algebra over $\mathbb{R}$ is isomorphic to either $\mathbb{C}$, $\mathbb{D}$ or $\mathbb{O}$ [11;54, App.A1]. $\mathbb{D}$ and $\mathbb{O}$ are interesting complements to the field $\mathbb{C}$ because they contain nilpotent and idempotent elements and are the simplest models for more complicated rings. The hypercomplex number systems are strongly connected to the theory of Clifford algebras and Lie groups [16, 18, 19, 26, 27, 29, 35, 36, 68, 69, 71, 76]. Algebraic properties of higher dimensional geometric spaces can be investigated in terms of hypercomplex matrix representations of Clifford algebras [76]. Our main aim is to investigate Möbius transformation over these two commutative algebras over $\mathbb{R}$. Furthermore, some of our results are true for a general ring R.

Let $\mathbb{A}$ be a ring of complex, double or dual numbers. Let $\sim$ be an equivalence relation on $\mathbb{A}^2\backslash\{(0,0)\}$ defined as follows: $(z_1, z_2)\sim(z_3, z_4)$ if and only if there exists a unit (an invertible element) $u \in \mathbb{A}$ such that $z_1 = uz_3$ and $z_2 = uz_4$. The set of all equivalence classes is denoted by $\mathbb{A}^2/\sim$. The point of $\mathbb{A}^2/\sim$ corresponding to a vector $\begin{pmatrix} x \\ y \end{pmatrix}$ is denoted by $[x : y]$. Therefore, $\mathbb{A}^2/\sim$ contains the following two types of equivalence classes.

1. $[x : y]$, such that $x\mathbb{A} + y\mathbb{A} = \mathbb{A}$.

2. $[x : y]$, such that $x\mathbb{A} + y\mathbb{A} \neq \mathbb{A}$.

Equivalence classes of the first type are points of the projective line over $\mathbb{A}$, which is denoted by $\mathbb{P}(\mathbb{A})$. There is a natural embedding $r : x \mapsto [x : 1]$ of the $\mathbb{A}$ to the projective line. For $\mathbb{O}$ and $\mathbb{D}$, $\mathbb{P}(\mathbb{A})\backslash r(\mathbb{A})$ has more than one ideal element [78, Suppl.C].

A linear transformation of $\mathbb{A}^2$ can be represented by multiplication of 2×2-matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to two-dimensional vectors. The transformation is not degenerate if $ad - bc$ is a unit in $\mathbb{A}$. The collection of all 2×2-matrices, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, such that $ad - bc$ is a unit, is a group denoted by $GL_2(\mathbb{A})$. The collection of all 2×2-matrices, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $ad - bc = 1$, is a group denoted by $SL_2(\mathbb{A})$, which is a subgroup of $GL_2(\mathbb{A})$. A linear map $\mathbb{A}^2 \to \mathbb{A}^2$ is a class invariant for $\sim$. Therefore, the linear transformation of $\mathbb{A}^2$ produces the map $\mathbb{A}^2/\sim \to \mathbb{A}^2/\sim$ as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} [x : y] = [ax + by : cx + dy], \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{A}). \tag{1.1}$$

For any unit $u \in \mathbb{A}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} ua & ub \\ uc & ud \end{pmatrix}$ define the same map (1.1). This map is called an $\mathbb{A}$-Möbius map. Recall that $\mathbb{K}^2/\sim$ is the only $GL_2(\mathbb{K})$-orbit. In contrast, $\mathbb{A}^2/\sim$ has more than one $GL_2(\mathbb{A})$-orbit. The two groups $GL_2(\mathbb{K})$ and

$SL_2(\mathbb{K})$ have the same number of types of non-equivalence connected continuous one-parameter subgroups. The two groups $GL_2(\mathbb{A})$ and $SL_2(\mathbb{A})$ also have the same number of types of non-equivalence connected continuous one-parameter subgroups.

A wider context for our work is provided by the Erlangen programme of F.Klein, cf. [53, 54]. Similarly to the case of $SL_2(\mathbb{R})$ [50, 52, 54], we want to characterise all non-equivalence homogeneous spaces G/H, where G is one of the groups $GL_2(\mathbb{O})$, $GL_2(\mathbb{D})$, $SL_2(\mathbb{O})$, $SL_2(\mathbb{D})$ and H is a closed continuous subgroup of G. The natural action of G on a homogeneous space G/H is geometrically represented by Möbius transformations. The respective conformal geometry is intimately connected with various physical models [51;54;58;78, Suppl.C]. Geometrical language provides an enlightening environment for many related questions, e.g. continued fractions [43], analytic functions [42, 43], spectral theory [46, 48, 53], etc.

1.5 Overview of The Thesis

We start in Chapter 2 by giving some background on commutative rings with identity. As well as fixing notation, this material is needed for the subsequent chapters. We provide a summary of useful definitions, facts and properties of commutative rings with identity in general and we pay more attention to the ring of double and dual numbers.

Chapter 3 is concerned with general linear groups, special linear groups, projective lines and Möbius maps for the following rings: the ring of integer numbers, Cartesian product ring, the ring of double numbers and the ring of dual numbers.

The $\mathbb{C}$-Möbius maps are continuous in a topology induced by some metric d. Any $\mathbb{C}$-Möbius map has maximum two different fixed points. In Chapter 4 we define metric over the projective line $\mathbb{P}(\mathbb{A})$, where $\mathbb{A}$ is the ring of double or dual numbers. We will also show that $\mathbb{A}$-Möbius maps are continuous. Moreover, we show that the set of fixed points of an $\mathbb{A}$-Möbius map contains maximum two or infinity different fixed points (four or infinity different fixed points) if $\mathbb{A}$ is the ring of dual (double) numbers. Finally, we provide a classification of fixed points based on Jacobian matrices.

The final chapter gives a full descriptions of the number of connected continuous one-parameter subgroups of real (double, dual) general linear groups. In the last section, we present an application of Möbius transformations the canonical triple of points.

Another overview of the thesis shows in the next table. The white part in the table refers to known objects. The gray part refer to new results in the thesis. That means our results are as follows:

1. There are three $SL_2(\mathbb{O})$-orbits and two $SL_2(\mathbb{D})$-orbits.

2. Double and dual Möbius maps are continuous maps.

3. The number of fixed points of double Möbius map is either $0, 1, 2, 4$ or infinity fixed points. The number of fixed points of double Möbius map is either $0, 1, 2$ or infinity fixed points.

4. There are six different types of fixed points of double and dual Möbius map.

5. There are six different types, up to similarity and rescaling, of continuous one-parameter subgroups of $SL_2(\mathbb{O})$, and three different types of continuous one-parameter subgroups of $SL_2(\mathbb{D})$.

6. There are five different canonical triples of points in $\mathbb{P}(\mathbb{O})$, and two different canonical triples of points in $\mathbb{P}(\mathbb{D})$.

The blue part gives some ideas for future works.

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{O}$	$\mathbb{D}$
Imaginary unit		$\iota^2 = -1$	$\iota^2 = 1$	$\iota^2 = 0$
The set of non-zero nilpotent element	$\emptyset$	$\emptyset$	$\emptyset$	$\{\epsilon a : a \in \mathbb{R}\}$
The set of non-trivial idempotent element	$\emptyset$	$\emptyset$	$\{P_+ = \frac{1}{2}(1+j), P_- = \frac{1}{2}(1-j)\}$	$\emptyset$
Proper ideal			$\{aP_+ : a \in \mathbb{R}\}$, $\{aP_- : a \in \mathbb{R}\}$	$\{\epsilon a : a \in \mathbb{R}\}$
$M\ddot{o}b(\mathbb{A})$	$SL_2(\mathbb{R})/\{\pm I\}$	$SL_2(\mathbb{C})/\{\pm I\}$	$SL_2(\mathbb{O})/\{\pm I, \pm jI\}$	$SL_2(\mathbb{D})/\{\pm I\}$
$\mathbb{P}(\mathbb{A})$	$= \mathbb{R}^2/\sim$	$= \mathbb{C}^2/\sim$	$\subset \mathbb{O}^2/\sim$	$\subset \mathbb{D}^2/\sim$
$SL_2(\mathbb{A})$-orbits	$\mathbb{P}(\mathbb{R})$	$\mathbb{P}(\mathbb{C})$	$\mathbb{P}(\mathbb{O}), \mathbb{P}(\mathbb{R}), \mathbb{P}(\mathbb{R})$	$\mathbb{P}(\mathbb{D}), \mathbb{P}(\mathbb{R})$
Continuity	Continuous	Continuous	Continuous	Continuous
Conformal	Conformal	Conformal	Need investigation	Need investigation
No. of distinct fixed points	$0, 1$ or 2	1 or 2	$0, 1, 2, 4$ or ∞	$0, 1, 2$ or ∞
Type of fixed points	Att., Rep., Ind.	Att., Rep., Ind., Spiral (Att., Rep., Ind.), Center	Att., Rep., Ind., Saddle(Att., Rep., Ind.)	Att., Rep., Ind., Improper(Att., Rep., Ind.)
Continuous one-parameter subgroups	$H_\sigma(t)$ elliptic, parabolic and hyperbolic	$H_\sigma(t)$ and $SL(t)$	$H_{\sigma_+}(t)P_+ + H_{\sigma_-}(at)P_-$	$\widetilde{H}_\sigma(t)$
Canonical triple of points	$\{0, 1, \infty\}$	$\{0, 1, \infty\}$	$\{0, 1, \infty\}$, $\{\infty, 0, \sigma_1\}$, $\{\infty, 0, \omega_1\}$, $\{\infty, 0, \omega_2\}$, $\{0, P_+, \sigma_1\}$, $\{0, P_-, \sigma_2\}$	$\{0, 1, \infty\}$, $\{\infty, 0, \omega\}$, $\{0, \epsilon, \epsilon 2\}$
Kleinian group		Studied	Need investigation	Need investigation
Continued fractions	Studied	Studied	Need investigation	Need investigation

Table1.1: Overview of The Thesis

Chapter 2

Background

In this chapter, we briefly present fundamentals on two topics -rings and modules- that are needed for the rest of our study. We present the essential definitions and properties, illustrated by a collection of examples. We follow the basic notation and terminology on algebraic structures such as rings and ideals [2, 31, 72], to which the reader is referred for further material.

2.1 Rings and Semirings

The semiring and the ring are ones of useful and fundamental concepts that generalise the idea of arithmetic of numbers.

Definition 2.1.1. Let R be a non-empty set and let $+, \cdot$ (often called addition and multiplication) be two binary operations defined on R such that the following conditions are satisfied:

1. $(R, +)$ is a commutative semigroup with identity.
2. $(R, \cdot)$ is a semigroup with identity.
3. $\forall a, b, c \in R$, we have $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$. (distributive law)

4. Multiplication by 0 annihilates R, which means that $\forall a \in R, a \cdot 0 = 0 \cdot a = 0$.

Then $(R, +, \cdot)$ is called a *semiring*.

Example 2.1.2. Let $\mathbb{N}$ be the set of all natural numbers with $\{0, \infty\}$. We defined two binary operations addition and multiplication $\oplus$ and $\otimes$ on $\mathbb{N}$ as follows:

for all $n, m \in \mathbb{N} \cup \{0, \infty\}$

1. $n \oplus m = \min\{n, m\}$,

2. $n \otimes m = n + m$.

It is easy to see that $(\mathbb{N} \cup \{\infty\}, \oplus, \otimes)$ is a commutative semiring, which is denoted by $\mathbb{N}_{\min}$. It is also called a *Tropical semiring.* Infinity is the identity element of $\oplus$ and zero is the identity element of $\otimes$ [70].

Definition 2.1.3. Let R be a non-empty set and $+, \cdot$ (often called addition and multiplication) be two binary operations defined on R. Then $(R, +, \cdot)$ is a *ring* if the following are satisfied:

1. $(R, +)$ is a commutative group.

2. $(R, \cdot)$ is a semi-group.

3. $\forall a, b, c \in R$ we have $a \cdot (b + c) = a \cdot b + a \cdot c$, and $(b + c) \cdot a = b \cdot a + c \cdot a$.

If $(R, \cdot)$ has an identity element which is denoted by 1, then $(R, +, \cdot)$ is called a *ring with identity*, and if $(R \backslash \{0\}, \cdot)$ is a group then $(R, +, \cdot)$ is called a *field.* Hereafter, instead of $a \cdot b$ and $a \oplus b$ we may write ab.

Remark 2.1.4. The fourth axiom in the Definition 2.1.1 is melted in the third one in the Definition 2.1.3.

Definition 2.1.5. An *algebra* over a field K is a set A, which is both a ring and vector space over K in such a manner that the additive group structures are same and the axiom $\lambda(ab) = a(\lambda b) = \lambda(ab)$ is satisfied for all $\lambda \in K$ and $a, b \in A$.

Example 2.1.6. The double numbers form the two dimensional commutative and associative real algebra with unit spanned by 1 and j, where j has the property $j^2 = 1$.

If we define $P_+ = \frac{1}{2}(1 + j)$ and $P_- = \frac{1}{2}(1 - j)$, then we can write the set of all double numbers by $\mathbb{O} = \{a_+P_+ + a_-P_- : a_+, a_- \in \mathbb{R}\}$, where for all $a = a_1 + ja_2$ there exists $a_\pm = a_1 \pm a_2$ such that $a = a_+P_+ + a_-P_-$. Therefore, the following properties hold:

1. $P_+^2 = P_+$ (projection).

2. $P_-^2 = P_-$ (projection).

3. $P_+ \cdot P_- = 0$ (orthogonal).

Example 2.1.7. The dual numbers form the two dimensional commutative and associative real algebra with unit spanned by 1 and ϵ, where ϵ has the property $\epsilon^2 = 0$.

Definition 2.1.8. Let R be a ring. An element $a \in R$ is a *unit (an invertible element)* if there exists $b \in R$ such that $a \cdot b = b \cdot a = 1$ (the identity element of multiplication).

Example 2.1.9. 1. Zero is the only invertible element in $\mathbb{N}_{\min}$.

2. Every non-zero real number a is an invertible element in $\mathbb{R}$, and the inverse of a is $\frac{1}{a}$.

3. Every non-zero complex number $z = a + ib$ is an invertible element, and the inverse of z is $z^{-1} = \dfrac{a}{a^2 + b^2} - i\dfrac{b}{a^2 + b^2}$.

4. The number $a = a_+P_+ + a_-P_-$ is an invertible double number if and only if $a_+ \cdot a_- \neq 0$. The inverse of a is $a^{-1} = a_+^{-1}P_+ + a_-^{-1}P_-$ [78, Suppl.C]. In

other words, if $a = a_+P_+ + a_-P_- \in \mathbb{O}$, then

$$a^{-1} = \begin{cases} a_+^{-1}P_+ + a_-^{-1}P_-, & \text{if } a_+ \cdot a_- \neq 0; \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

5. The number $a = a_1 + \epsilon a_2$ is an invertible dual number if and only if $a_1 \neq 0$. The inverse of a is $a^{-1} = a_1^{-1} - \epsilon(a_1^{-1})^2 a_2$ [78, Suppl.C]. In other words, if $a = a_1 + \epsilon a_2 \in \mathbb{D}$, then

$$a^{-1} = \begin{cases} a_1^{-1} - \epsilon(a_1^{-1})^2 a_2, & \text{if } a_1 \neq 0; \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Definition 2.1.10. Let R be a ring and a, b be two non-zero elements in R. If $a \cdot b = 0$, then both a and b are called *zero divisors*.

Example 2.1.11.

1. Let $z = a + jb$ be a non-zero double number. The number z is zero divisor if and only if $a = \pm b$ [78, Suppl.C].

2. Let $z = a + \epsilon b$ be a non-zero dual number. The number z is zero divisor if and only if $a = 0$ [78, Suppl.C].

Definition 2.1.12. Let R be a ring. The *centre* of R is $Cent(R) = \{x \in R : a \cdot x = x \cdot a, \forall a \in R\}$.

Definition 2.1.13. Let R be a ring. R is called a *commutative ring* if $Cent(R) = R$.

Each one of the previous rings is a commutative ring with identity. Hereafter, all rings are going to be commutative with identity.

Definition 2.1.14. Let R be a ring. Let S be a subset of R. S is called a *subring* if S is a ring in its own right, using the same operations as defined on R.

Because not every element in the rings is invertible, we need the following definition.

Definition 2.1.15. Let R be a ring. The set $I \subseteq R$ is said to be a *two-sided ideal* or shortly an *ideal* if $(I, +)$ is a subgroup of $(R, +)$ and for all $a \in R$ and $i \in I$ we have $ai \in I$ and $ia \in I$. I is a left *ideal* if $(I, +)$ is a subgroup of $(R, +)$ and for all $a \in R,\ i \in I,\ ai \in I$. I is a right *ideal* if $(I, +)$ is a subgroup of $(R, +)$ and for all $a \in R,\ i \in I,\ ia \in I$.

Remark 2.1.16.

1. Any ring contains at least two ideals $\{0\}$, and R. They are called *trivial ideals*. Any other ideal is called *proper ideal*.
2. The same definition of ideal is suitable for semiring and semisubgroup respectively.

Example 2.1.17. Let $I_+ = \{aP_+ : a \in \mathbb{R}\}$ and $I_- = \{aP_- : a \in \mathbb{R}\}$, which are two subsets of $\mathbb{O}$. I_+ and I_- are two proper ideals of the ring of double numbers.

Example 2.1.18. Let $I = \{\epsilon a : a \in \mathbb{R}\}$ which is a subset of $\mathbb{D}$. I is a proper ideal of the ring of dual numbers.

Proposition 2.1.19.

1. Let R be a commutative ring with identity. Let I be an ideal of R. If $1 \in I$, then $I = R$.
2. Let $(R, +, \cdot)$ be a commutative ring with identity. Let a be a non-zero element in R. If a is not an invertible element in R, then aR is a proper ideal.
3. Let $\mathbb{F}$ be a field. Let $I \subset \mathbb{F}$. If I is an ideal of $\mathbb{F}$, then I is a trivial ideal.

I omit the proofs as they are straightforward results.

Definition 2.1.20. Let R be a commutative ring with identity. R is called a *simple ring* if $\{0\}$ and R are the only ideals of R.

Definition 2.1.21. Let R be a ring. An element $e \in R$ is an *idempotent* element if $e^2 = e$.

Remark 2.1.22. Any ring with identity always has at least two idempotent elements, zero and one, which are called *trivial idempotent* elements.

Proposition 2.1.23. The ring $(\mathbb{O}, +, \cdot)$ contains only four idempotent elements $\{0, 1, P_+, P_-\}$.

Proof. Let $a = a_+P_+ + a_-P_-$ be an idempotent element in $\mathbb{O}$. This means that $a^2 = a$, which in turn implies that

$$\begin{aligned}
(a_+P_+ + a_-P_-)^2 &= a_+P_+ + a_-P_- \\
a_+^2P_+ + a_-^2P_- &= a_+P_+ + a_-P_- \\
a_+^2 = a_+ \quad \text{and} &\quad a_-^2 = a_- \\
\text{so, } a_+ = 0, 1 \quad \text{and} &\quad a_- = 0, 1.
\end{aligned}$$

Therefore, a is equal to $0, 1, P_+$ or P_-. $\square$

Proposition 2.1.24. Let a be a dual number. If a is an idempotent element, then a is a trivial idempotent element.

Proof. Let $a = a_1 + \epsilon a_2$ be a dual number. Let

$$\begin{aligned}
a^2 &= a; \\
a_1^2 + \epsilon 2a_1a_2 &= a_1 + \epsilon a_2; \\
\text{this means that } a_1^2 = a_1 \quad \text{and} &\quad 2a_1a_2 = a_2; \\
a_1(a_1 - 1) = 0 \quad \text{and} &\quad a_2(a_1 - 1) = 0
\end{aligned}$$

Therefore, $a_1 = 0$ or 1. If $a_1 = 0$, then $a_2 = 0$. Thus, $a = 0$. If $a_1 = 1$, then $a_2 = 0$. So, $a = 1$. That means a is a trivial idempotent element. $\square$

Proposition 2.1.25. Let $\mathbb{F}$ be a field. Then it does not contain proper idempotent elements.

Proof. Let f be a non-zero element in $\mathbb{F}$. Assume that f is an idempotent element. Then, $f^2 = f$. Since $f \neq 0$ and $\mathbb{F}$ is a field then $f - 1 = 0 \Rightarrow f = 1$. □

Definition 2.1.26. Let R be a ring. An element $x \in R$ is called a *nilpotent* element of R if there exists $n \in \mathbb{N}$ such that $x^n = 0$.

Any ring contains at least one nilpotent element which is 0.

Proposition 2.1.27. The ring $(\mathbb{O}, +, \cdot)$ does not contain any non-zero nilpotent elements.

Proof. Suppose that, $a = a_+P_+ + a_-P_-$ is a non-zero nilpotent element. Then, there exists $n \in \mathbb{N}$ such that $a^n = 0$. This means that

$$(a_+P_+ + a_-P_-)^n = a_+^nP_+ + a_-^nP_- = 0$$

which means that $a_+^n = 0$ and $a_-^n = 0$. So, $a = 0$ which means the only nilpotent element is zero. □

Proposition 2.1.28. A dual number k is a nilpotent element if and only if $k = \epsilon a$, for some $a \in \mathbb{R}$.

Proof. Zero is a nilpotent element and $0 = \epsilon 0$. Let $k = k_1 + \epsilon k_2$ be a non-zero dual number and $n \in \mathbb{N}$ such that $k^n = 0$. Therefore, $(k_1 + \epsilon k_2)^n = k_1^n + \epsilon k_1^{n-1}k_2 = 0$. That means, $k_1 = 0$ which means $k = \epsilon k_2$.

Conversely, let $k = \epsilon a$. Clearly, $k^2 = (\epsilon a)^2 = \epsilon^2 a^2 = 0$, i.e k is a nilpotent element. □

Proposition 2.1.29. Let R be a commutative ring with identity. Let $a \in R$. If a is an idempotent and nilpotent element in the same time, then $a = 0$.

I omit the proof as it is a standard result.

Proposition 2.1.30. Let R be a commutative ring with identity. Let $a \in R$. If a is a proper idempotent or non-zero nilpotent element in R, then a is a zero divisor.

Proof. If a is a proper idempotent element, then $a(a-1) = 0$, which means that a is a zero divisor. If a is a non-zero nilpotent element, then there exists $n \in \mathbb{N}$ such that $a(a^{n-1}) = 0$, which means that a is a zero divisor. □

Proposition 2.1.31. Let R be a commutative ring with identity. Let $e \in R$. Then

1. If e is an idempotent element in R, then $1 - e$ is an idempotent element too.

2. If e is a non-trivial idempotent element in R, then eR is a proper ideal of R.

3. If e is a non-zero nilpotent element in R, then eR is a proper ideal of R.

I omit proofs as they are straightforward results.

Lemma 2.1.32. Let $a = a_+P_+ + a_-P_- \in \mathbb{O}$. Then

1. The conjugate $\overline{a} = a_1 - ja_2$ of $a = a_1 + a_2j$ is $\overline{a} = a_-P_+ + a_+P_-$.

2. The square root of a, that is solutions of the equation $a = x^2$, has up to four values:

$$\pm(\sqrt{a_+}P_+ + \sqrt{a_-}P_-) \text{ or } \pm(\sqrt{a_+}P_+ - \sqrt{a_-}P_-)$$

   if $a_+, a_- \geq 0$ and is not defined otherwise.

I omit proofs as they are straightforward results. It is easy to show that the function $f : \mathbb{O} \to \mathbb{O}$, which is defined by $f(a) = \bar{a}$, is a ring isomorphism.

Lemma 2.1.33. Let $a = a_1 + \epsilon a_2 \in \mathbb{D}$. The square root of a is, that is solutions of the equation $a = x^2$,

$$\sqrt{a} = \begin{cases} \pm(\sqrt{a_1} + \epsilon\dfrac{a_2}{2\sqrt{a_1}}), & \text{if } a_1 > 0; \\ 0, & \text{if } a_1 = a_2 = 0; \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

I omit proofs as it is a straightforward result.

2.2 Examples

This section presents interesting examples of rings which are more complicated than those shown in the previous section.

2.2.1 Cartesian product rings

Let $R_1,\ R_2$ be two commutative ring with identity. Let $R = R_1 \times R_2$.

For all $r_1 = (a_1, b_1),\ r_2 = (a_2, b_2) \in R$, addition and multiplication on R are defined as follows:

1. $r_1 + r_2 = (a_1 + a_2, b_1 + b_2)$
2. $r_1 \cdot r_2 = (a_1 \cdot a_2, b_1 \cdot b_2)$

The triple $(R, +, \cdot)$ is a commutative ring with identity. It is called a Cartesian product ring. Let $(R, +_R, \cdot_R)$ be a Cartesian product ring. Then

1. $0 = (0, 0)$ is the zero element of R and $1 = (1, 1)$ is the identity element of R.
2. $u = (u_1, u_2)$ is a unit in R if and only if u_1 and u_2 is a unit in R_1 and R_2 respectively.

3. $z = (z_1, z_2) \neq (0, 0)$ is a zero divisor in R if and only if at least one of z_i is zero or zero divisor in R_i.

4. $e = (e_1, e_2)$ is an idempotent element in R if and only if e_1 and e_2 are idempotent elements in R_1 and R_2 respectively. For example, $(0, 1)$ is anon-trivial idempotent element in R.

5. $x = (x_1, x_2)$ is a nilpotent element in R if and only if $x_1$and x_2 is a nilpotent element in R_1 and R_2 respectively.

Proposition 2.2.1. Let R_1, R_2 be two commutative rings with identity and $R = R_1 \times R_2$. Then R contains non-zero nilpotent elements if and only if R_1 or R_2 does.

Proof. For necessity. Let (a, b) be a non-zero element in R. This means that either $a \neq 0$ or $b \neq 0$. Let $n \in \mathbb{N}$ such that $(a, b)^n = (0, 0)$, which means $a^n = 0$ and $b^n = 0$. That means, a and b are a nilpotent elements in R_1 and R_2, respectively.

For sufficiency. Let $a \in R_1$ be a non-zero nilpotent element such that $a^n = 0$, for some $n \in \mathbb{N}$. Obviously, $(a, 0) \in R$ and $(a, 0)^n = (a^n, 0) = (0, 0)$. This means that $(a, 0)$ is a non-zero nilpotent element in R. $\square$

Previous results are easily generalized on $R = R_1 \times \ldots \times R_k$, for some $k \in \mathbb{N}$. The ring of double numbers is an example of the product ring, $\mathbb{O} = \mathbb{R} \times \mathbb{R}$.

2.2.2 The ring of $n \times n$-matrices

The set of all $n \times n$-matrices, with entries from a commutative ring with identity R, with matrix addition and multiplication, is a ring with identity. It is denoted by $(M_n(R), +, \cdot)$. Here we interested in 2×2-matrices.

Definition 2.2.2. Let R be a commutative ring with identity. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(R)$. Then:

1. $\det(A) = ad - bc$ (determinant).

2. $\operatorname{tr} A = a + d$ (trace).

3. The element $\lambda \in R$ is called an eigenvalue of A if $\det(A - \lambda I) = 0$, where I is the identity matrix in $M_2(R)$.

4. $\widehat{A} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Proposition 2.2.3. Let R be a commutative ring with identity. Let $A_1,\ A_2$ be two matrices in $M_2(R)$. Then, $\det(A_1 \cdot A_2) = \det(A_1) \cdot \det(A_2)$.

I omit the proof as it is a standard result.

Corollary 2.2.4. Let $A \in M_2(R)$. Then, for some $\lambda \in R$, $\det(\lambda A) = \lambda^2 \det(A)$.

I omit the proof as it is a standard result. The following propositions prove that $A \in M_2(R)$ is invertible if and only if $\det(A)$ is invertible.

Proposition 2.2.5. Let R be a commutative ring with identity. Let $A \in M_2(R)$. A is an invertible matrix if and only if $\det(A)$ is an invertible element in R.

I omit the proof as it is a standard result.

There are ring isomorphisms from $\mathbb{C}, \mathbb{O}$and $\mathbb{D}$ to three subsets of $M_2(\mathbb{R})$, as the following example shows.

Example 2.2.6. The map

$$\varphi : \mathbb{C} \to \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R} \right\},$$

which is defined via

$$\varphi(x + iy) = \begin{pmatrix} x & y \\ -y & x \end{pmatrix},$$

is a ring isomorphism.

Example 2.2.7. The map

$$\varphi : \mathbb{O} \to \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a, b \in \mathbb{R} \right\},$$

which is defined via

$$\varphi(x + jy) = \begin{pmatrix} x & y \\ y & x \end{pmatrix},$$

is a ring isomorphism.

Example 2.2.8. The map

$$\varphi : \mathbb{D} \to \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in \mathbb{R} \right\},$$

which is defined via

$$\varphi(x + \epsilon y) = \begin{pmatrix} x & y \\ 0 & x \end{pmatrix},$$

is a ring isomorphism.

Proposition 2.2.9. Let R be a commutative ring with identity. Let $A, B \in M_2(R)$. Then, the followings are satisfied:

1. $\operatorname{tr}(A + B) = \operatorname{tr} A + \operatorname{tr} B$.
2. Let $r \in R$. Then $\operatorname{tr}(rA) = r(\operatorname{tr} A)$.
3. $\operatorname{tr} AB = \operatorname{tr} BA$.
4. If R is algebraically closed, then the trace of A is the sum of the eigenvalues of A.

I omit the proof as it is a standard result. The next definition is an important one in our study.

Definition 2.2.10. Let R be a commutative ring with identity. Let $A, B \in M_2(R)$. A is similar to B if there exists an invertible $C \in M_2(R)$ such that $C^{-1}AC = B$.

Proposition 2.2.11. Let R be a commutative ring with identity. Let $A, B \in M_2(R)$. If A is similar to B, then

1. $\det(A) = \det(B)$.

2. $\operatorname{tr} A = \operatorname{tr} B$.

3. both A and B have the same eigenvalue.

I omit the proof as it is a standard result. The converse of 1 and 2 in 2.2.11 do not hold, as shown by the next example.

Example 2.2.12. $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are two elements in $M_2(\mathbb{R})$. $\det(A) = \det(B)$ and $\operatorname{tr} A = \operatorname{tr} B$ but A is not similar to B.

Next, we investigate some properties of $A \in M_2(\mathbb{O})$.

Proposition 2.2.13.
1. If $A_+, A_- \in M_2(\mathbb{R})$, then $A = A_+P_+ + A_-P_- \in M_2(\mathbb{O})$.

2. The identity matrix in $M_2(\mathbb{O})$ is $I = I_+P_+ + I_-P_-$, where $I_+ = I_-$ is the identity matrix in $M_2(\mathbb{R})$.

3. The matrix $A = A_+P_+ + A_-P_- \in M_2(\mathbb{O})$ is an invertible matrix if and only if both A_+, A_- are invertible matrices in $M_2(\mathbb{R})$. The inverse of A is $A^{-1} = A_+^{-1}P_+ + A_-^{-1}P_-$.

4. Let $A = A_+P_+ + A_-P_- \in M_2(\mathbb{O})$, for some $A_+, A_- \in M_2(\mathbb{R})$. Then, $\operatorname{tr} A = \operatorname{tr} A_+P_+ + \operatorname{tr} A_-P_-$.

I omit the proof as it is a straightforward result.

Proposition 2.2.14. Let $A_{\pm} \in M_2(\mathbb{R})$. Then $A = A_+P_+ + A_-P_- \in M_2(\mathbb{O})$ and

$$\det(A) = \det(A_+)P_+ + \det(A_-)P_-.$$

Proof. Let $A_+ = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$, $A_- = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in M_2(\mathbb{R})$. So,

$$A = A_+P_+ + A_-P_- = \begin{pmatrix} a_1P_+ + a_2P_- & b_1P_+ + b_2P_- \\ c_1P_+ + c_2P_- & d_1P_+ + d_2P_- \end{pmatrix} \in M_2(\mathbb{O}).$$

$\det(A) = (a_1d_1 - c_1b_1)P_+ + (a_2d_2 - b_2c_2)P_- = \det(A_+)P_+ + \det(A_-)P_-.$ □

Next, we investigate some properties of $A \in M_2(\mathbb{D})$.

Proposition 2.2.15.

1. Let $A_1, A_2 \in M_2(\mathbb{R})$. Then $A = A_1 + \epsilon A_2$ in $M_2(\mathbb{D})$.
2. Let A_1, $A_2 \in M_2(\mathbb{R})$. $A = A_1 + \epsilon A_2 \in M_2(\mathbb{D})$ is an invertible element if and only if A_1 is an invertible element in $M_2(\mathbb{R})$. The inverse of A is $A^{-1} = A_1^{-1} + \epsilon A_1^{-1}(-A_2A_1^{-1})$.
3. Let $A = A_1 + \epsilon A_2 \in M_2(\mathbb{D})$, for some $A_1, A_2 \in M_2(\mathbb{R})$. Then $\operatorname{tr} A = \operatorname{tr} A_1 + \epsilon \operatorname{tr} A_2$.

I omit the proof as it is a straightforward result.

Proposition 2.2.16. If $A = A_1 + \epsilon A_2 \in M_2(\mathbb{D})$, where $A_1, A_2 \in M_2(\mathbb{R})$, then $\det(A) = \det(A_1) + \epsilon \operatorname{tr}(A_1\widehat{A}_2)$.

Proof. $\det(A) = a_1d_1 - b_1c_1 + \epsilon(a_2d_1 + d_2a_1 - c_2b_1 - b_2c_1) = \det(A_1) + \epsilon \operatorname{tr}(A_1\widehat{A}_2)$. □

Example 2.2.17. Let $R = R_1 \times \ldots \times R_n$ be a Cartesian product ring.

$$M_2(R) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in R \right\} = \{(A_1, \ldots, , A_n) : A_i \in M_2(R_i)\}.$$

Let $A \in M_2(R)$. Then $\det(A) = (\det(A_1), \ldots, \det(A_n))$. A is an invertible matrix in $M_2(R)$ if and only if, for all i, A_i is an invertible matrix in $M_2(R_i)$.

2.3 Modules

Rings are the generalisation of fields $\mathbb{C}$ and $\mathbb{R}$. There is a well known linear algebra in vector spaces over fields $\mathbb{C}$ and $\mathbb{R}$. Now we define an object which will generalise vector space for arbitrary semiring and ring.

Definition 2.3.1. Let R be a ring with identity. A *left R-module* (or a module over R) is a non-empty set M, together with two operations: addition, $+ : M \times M \to M$ via $(m_1, m_2) \to m_1 + m_2$, and scalar multiplication, $\cdot : R \times M \to M$ via $(r, m) \to rm$. Furthermore, the following properties must hold:

1. $(M, +)$ is an abelian group.
2. For all $r_1, r_2 \in R$ and $m_1, m_2 \in M$
   (a) $r_1(m_1 + m_2) = r_1m_1 + r_1m_2$,
   (b) $(r_1 + r_2)m_1 = r_1m_1 + r_2m_1$,
   (c) $(r_1r_2)m_1 = r_1(r_2m_1)$,
   (d) $1m_1 = m_1$.

We can define a *right R-module* in the same way with $\cdot : M \times R \to M$.

A left (right) semimodule is just a left (right) module over a semiring. The formal definition is exactly as above, but we relax the requirement that R be a ring, and instead, allow an arbitrary semiring.

Definition 2.3.2. If a module is defined over a field, then it is called a *vector space*.

Every ring with identity is a left and right module over itself if the scalar multiplication of the module coincides with the ring multiplication. This means that $\cdot : R \times M \to M$ is the same as ring multiplication for elements $M = R$.

Theorem 2.3.3. Let R be a commutative ring with identity. Every left (right) ideal of R is a left (right) R-module respectively.

I omit the proof as it is a standard result.

Example 2.3.4.

1. $\mathbb{N} \cup \{0\infty\}$ is an $\mathbb{N}_{\min}-$semimodule.
2. $\mathbb{Z}$ is a module over the ring $\mathbb{Z}$.
3. $\mathbb{C}$ is a vector space over the field $\mathbb{C}$.

Definition 2.3.5. Let R be a commutative ring with identity. Let M be a left R-module. Let m be a non-zero element in M. If there is a non-zero element r in R such that $r \cdot m = 0$, then m is called a *torsion* element.

Obviously, any zero divisor in R is a torsion element in a left R-module R. If every element in M is a torsion, then M is called a *torsion module*. If M does not contain any torsion element, then it is called a *torsion free* module.

Example 2.3.6.

1. $\mathbb{Z}$ is a torsion free $\mathbb{Z}$-module.
2. $\mathbb{O}$ and $\mathbb{D}$ is not a torsion free $\mathbb{O}$-module and $\mathbb{D}$-module respectively, but also $\mathbb{O}$ and $\mathbb{D}$ are not a torsion $\mathbb{O}$-module and $\mathbb{D}$-module.
3. Let $M = \{aP_+ : a \in \mathbb{R}\}$. M is a torsion module over $\mathbb{O}$.

The following definitions provide some properties of modules:

Definition 2.3.7. Let R be a commutative ring with identity. Let M be a module over R. $N \subseteq M$ is a *submodule* of M in case it is a module over R in its own right.

Remark 2.3.8. Every module contains at least two submodules, $\{0\}$ and M, which are called *trivial submodules* of M. Any other submodule is called *proper submodule.*

Definition 2.3.9. Let N be a proper submodule of M. Suppose that there is no proper submodule of M that contains N. Then N is called a *maximal submodule.*

Definition 2.3.10. A left R-module M is *simple* if it does not have any proper submodule.

Example 2.3.11. $\{aP_+ : a \in \mathbb{R}\}$ is a proper submodule of $\mathbb{O}$ over $\mathbb{O}$.

Definition 2.3.12. Let M be an R-module. Let $m \in M$, we say that m is a *linear combination* from $m_1, m_2, \dots, m_n \in M$ if there are $\lambda_1, \lambda_2, \dots, \lambda_n \in R$ such that $m = \lambda_1 \cdot m_1 + \lambda_2 \cdot m_2 + \cdots + \lambda_n \cdot m_n$.

Definition 2.3.13. The set of all linear combinations of the set

$$S = \{m_1, m_2, \dots, m_n\} \subseteq M$$

is called the *span* of S. It is denoted by $span(S)$. If $span(S) = M$, then we say that S is spanning M.

Definition 2.3.14. The linear combination is called *linearly independent* if and only if $\lambda_1 \cdot m_1 + \lambda_2 \cdot m_2 + \cdots + \lambda_n \cdot m_n = 0$. Then $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$.

Any set with torsion is linearly dependent.

Definition 2.3.15. Let M be an R-module. A set $S = \{m_1, m_2, \dots, m_n\} \subseteq M$ is a *basis* of M if S satisfies the following conditions:

1. S is linearly independent.

2. $Span(S) = M$.

Definition 2.3.16. If an R-module M has a basis, then it is called a *free module*. The number of elements in the basis of M is called the *rank* of M.

Proposition 2.3.17. Every ring with identity R considered as a module over itself is a free module.

Proof. Since R is a module over itself then it is clear that $\{1\} \subset R$ is a basis of R. □

Therefore, the previous modules $\mathbb{Z}$, $\mathbb{Z}_n$, $\mathbb{O}$ and $\mathbb{D}$ are free modules. Next examples give modules which do not have a basis.

Example 2.3.18. The set $\mathbb{M} = \{aP_+ : a \in \mathbb{R}\}$ is an ideal of $\mathbb{O}$. Therefore, it is a left $\mathbb{O}$-module by Theorem 2.3.3. The set $B = \{P_+\}$ is spanning $\mathbb{M}$ since for all $aP_+ \in \mathbb{M}$ there exists $aP_+ + bP_- \in \mathbb{O}$ such that $aP_+ = (aP_+ + bP_-)P_+$. Instead, $B = \{P_+\}$ is not linearly independent because there exists a non-zero double number P_- and $P_-P_+ = 0$. Thus B is not a basis for M. In the same way, every $B = \{aP_+\}$ is spanning $\mathbb{M}$ but is not a basis.

Example 2.3.19. The set $\mathbb{M} = \{\epsilon a : a \in \mathbb{R}\} \subseteq \mathbb{D}$ is an ideal of $\mathbb{D}$. This means that $\mathbb{M}$ is a left $\mathbb{D}$-module. $B = \{\epsilon\}$ is spanning $\mathbb{M}$, but $\epsilon \cdot z = 0$ where $z \in \mathbb{D}$ does not necessary imply $z = 0$, because we have $\epsilon \in \mathbb{D}$ and $\epsilon \cdot \epsilon = \epsilon^2 = 0$. Then B is not a basis of $\mathbb{M}$, and this is true for all subsets of $\mathbb{M}$, i.e. $\mathbb{M}$ is a non-free $\mathbb{D}$-module.

Definition 2.3.20. Let M and N be two modules over R. A function $T : M \to N$ is called *R-homomorphism* in case $T(rv + su) = rT(v) + sT(u)$, for all $r, s \in R$ and $v, u \in M$.

Definition 2.3.21. An *R-endomorphism* is an R-homomorphism from M to M.

The set of all $R-$endomorphisms of M is denoted by

$$End(M) = \{\lambda : M \to M \text{ such that } \lambda \text{ is an } R-\text{homomorphism}\}.$$

The set of all endomorphism with map addition and map composition makes a ring with identity.

Proposition 2.3.22. Let $\mathbb{M}$ be a free module over R of the rank n. Then there exists an isomorphism between $End(\mathbb{M})$ and $M_n(R)$ $(End(\mathbb{M}) \cong M_n(R))$.

I omit the proof as it is a standard result.

The next example is a simple application of Proposition 2.3.22.

Example 2.3.23. $\mathbb{O}^2$ is a free $\mathbb{O}$-module with the basis

$$B = \left\{ e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

$End(\mathbb{O}^2) \cong M_2(\mathbb{O})$ because if $\lambda \in End(\mathbb{O}^2)$, then

$\lambda(e_1) = c_{11}e_1 + c_{12}e_2$ and $\lambda(e_2) = c_{21}e_1 + c_{22}e_2$. That means $M = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$.

Assume $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{O})$. Thus,

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} e_1 &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \\ \text{and} \begin{pmatrix} a & b \\ c & d \end{pmatrix} e_2 &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix} \end{aligned}$$

but

$$\begin{aligned} \begin{pmatrix} a \\ c \end{pmatrix} &= a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \text{and} \begin{pmatrix} b \\ d \end{pmatrix} &= b \begin{pmatrix} 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

Therefore, for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ there exists $\lambda : End(\mathbb{M}) \rightarrow End(\mathbb{M})$ such that $\lambda(e_1) = ae_1 + ce_2$ and $\lambda(e_2) = be_1 + de_2$.

Remark 2.3.24. Generally, if R is a commutative ring with identity, then R^2 is a left free R-module.

The following notions will be relevant to our study:

Definition 2.3.25. Let R be a ring. Let M be a left R-module. Let $\{M_i : i = 1, \dots, n\}$ be simple submodules of M such that $M = M_1 + \cdots + M_n$ and for all $i \neq j$ we have $M_i \cap M_j = \{0\}$. Then the module M is said to be *semi-simple*.

Example 2.3.26. $\mathbb{Z}/p\mathbb{Z}$, for some positive prime $p \in \mathbb{Z}$ is a simple module, and $\oplus_{i \in I}\mathbb{Z}/p_i\mathbb{Z}$, for some set I, is a semi-simple module.

Chapter 3

General Linear Groups, Projective Lines and Möbius maps

This chapter presents the relationships between the general linear groups and, respectively, the projective lines (section one) and the Möbius maps (section two) of a commutative ring with identity.

3.1 General Linear Group And The Projective Line

Through this section, R is a commutative ring with identity. By Proposition 2.3.22 R-endomorphism can be presented by matrices. We can apply it to $End(R^2)$, and represent the endomorphism by 2×2-matrices. Then the following definitions are important ones.

Definition 3.1.1. Let R be a ring with identity. $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(R)$ is a *non-singular* matrix if and only if $\det(M) = ad - bc$ is not zero nor zero divisors.

Definition 3.1.2. Let R be a ring with identity.

$$GP_2(R) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in R, \text{ and } A \text{ is a non-singular matrix} \right\}.$$

The multiplication on $GP_2(R)$ is defined as a usual multiplication on matrices.

Since for all two matrices A and B, $\det(A \cdot B) = \det(A) \cdot \det(B)$, $GP_2(R)$ is closed under the multiplication. The identity matrix is the neutral element of $GP_2(R)$. That means, for any commutative ring with identity R, $(GL_2(R), \cdot)$ is a semigroup. The following notation is at the core of our study.

Definition 3.1.3. Let R be a commutative ring with identity.

$$GL_2(R) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in R, \text{ and } ad - bc \text{ is invertible} \right\} [54].$$

The multiplication on $GL_2(R)$ is defined as a usual multiplication on matrices.

In the same way before, $GL_2(R)$ is a semigroup with identity, and for all $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(R)$ there is $B = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ in $GL_2(R)$ and $A \cdot B$ is the identity matrix. Therefore, for any commutative ring with identity R, $(GL_2(R), \cdot)$ is a group. The group $(GL_2(R), \cdot)$ is called a *general linear group* of R.

Remark 3.1.4.

1. Let R be a commutative ring with identity. $GL_2(R) = GP_2(R)$ if and only if every non-zero element in R is either unit or zero divisor.

2. Let $\mathbb{F}$ be a field. Then,

$$GL_2(\mathbb{F}) = GP_2(\mathbb{F}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc \neq 0 \right\}.$$

Example 3.1.5.

$$GP_2(\mathbb{Z}) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc \neq 0\right\}.$$

$$GL_2(\mathbb{Z}) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = \pm 1\right\}.$$

Here we can see that $GP_2(\mathbb{Z}) \neq GL_2(\mathbb{Z})$ because every non-zero element in $\mathbb{Z}$ except ± 1 is neither an invertible nor a zero divisor element.

Proposition 3.1.6. Let R be a commutative ring with identity. Let $a,\ b \in R$ and $aR,\ bR$ be two ideals of R generated by a and b respectively. $aR + bR = R$ if and only if there exists $c, d \in R$ such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(R)$.

Proof. Let $aR + bR = R$. This means that there exists $r_1, r_2 \in R$ such that $1 = ar_1 + br_2$. Thus $\begin{pmatrix} a & b \\ -r_2 & r_1 \end{pmatrix} \in GL_2(R)$.

Conversely, let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(R)$. Let $ad - bc = u$ which means $u \in aR + bR$. So, $1 \in aR + bR$. Then, from 1 in Proposition 2.1.19 $aR + bR = R$. □

The following notion will be relevant to our study:

Definition 3.1.7. Let R be a commutative ring with identity. The pair $(a, b) \in R^2$ is called *admissible*, if there exists $c, d \in R$ such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an invertible matrix. Equivalently (a, b) is admissible if $aR + bR = R$, where aR, bR are ideals of R generated by a and b respectively [10].

Example 3.1.8. The pair $(1, 5) \in \mathbb{O}^2$ is admissible because $1, 0 \in \mathbb{O}$ and $\begin{pmatrix} 1 & 5 \\ 1 & 0 \end{pmatrix}$ is invertible, while $(P_+, 5P_+) \in \mathbb{O}^2$ is not admissible.

Proposition 3.1.9. Let R be a commutative ring with identity. Let $(a,b) \in R^2$ be an admissible pair and $c, d \in R$. If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(R)$, then (c,d) is an admissible pair too.

Proof. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(R)$. Let $ad - bc = u$, which means that $cb - ad = -u$. Therefore, $\begin{pmatrix} c & d \\ a & b \end{pmatrix} \in GL_2(R)$. □

Definition 3.1.10. Let R be a commutative ring with identity. We define an equivalence relation on $R^2 \backslash \{(0,0)\}$ as follows: $(z_1, z_2) \sim (z_3, z_4)$ if and only if there exists a unit $u \in R$ such that $z_1 = uz_3$ and $z_2 = uz_4$.

An endomorphism $R^2 \to R^2$ is a class invariant for $\sim$. By $R^2/\sim$, we mean the set of all equivalence classes. The point of $R^2/\sim$ corresponding to a vector $\begin{pmatrix} x \\ y \end{pmatrix}$ is denoted by $[x : y]$.

Definition 3.1.11. Let R be a commutative ring with identity. $\mathbb{P}(R) = \{[a : b] : (a,b) \in R^2$, is admissible $\}$ is the *projective line* over the ring R. Equivalently, the projective line over R is defined as follows: $\mathbb{P}(R) = \{[a : b] : aR + bR = R\}$, where aR, bR are ideals of R generated by a and b respectively [10].

The projective line over a ring is an extension of the concept of projective line over a field. Note that $\mathbb{P}(R) \subset R^2/\sim$.

Let $\mathbb{F}$ be a field. Let $\sim$ be a relation over $\mathbb{F}^2 \backslash \{(0,0)\}$ given in Definition 3.1.10. Therefore, we obtain only the following two types of equivalence classes $[f : 1]$ and $[1 : 0]$. Clearly, both $(f,1)$, $(1,0)$ are admissible pairs. Therefore, any point $[a : b]$ of the projective line, $\mathbb{P}(\mathbb{F})$, belongs to exactly one of those two classes. There is a natural embedding $x \mapsto [x : 1]$ of the field $\mathbb{F}$ to the projective line. The only point, $[1 : 0]$, not covered by this embedding is associated with infinity (ideal element) [6;54, Ch.8;57].

Thus, $\mathbb{F}$ parametrises the main part of the projective line of $\mathbb{F}$ as $[f : 1]$, for all $f \in \mathbb{F}$, except the class $[1 : 0]$.

In other words, $\mathbb{F}$ may be identified with the subset of the projective line given by $\{[f : 1] : f \in \mathbb{F}\}$. This subset covers all points in $\mathbb{P}(\mathbb{F})$ except one point, the one we call infinity. Therefore, ∞ is the only point added to $\mathbb{F}$ by $\mathbb{P}(\mathbb{F})$. From the preceding, we can see that the projective lines over $\mathbb{C}$, $\mathbb{Z}_p$ where p is prime, are $\mathbb{P}(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ and $\mathbb{P}(\mathbb{Z}_p) = \mathbb{Z}_p \cup \{\infty\}$ respectively.

The next propositions give us examples of the projective lines over several well-known rings:

Proposition 3.1.12. The projective line over the ring of integer numbers is:

$$\mathbb{P}(\mathbb{Z}) = \{[a : b] : \text{ for all } a, b \in \mathbb{Z} \text{ such that } \gcd(a, b) = 1\}.$$

Proof. $\mathbb{P}(\mathbb{Z})$ consists of $[a : b]$ for all admissible pairs (a, b). A pair (a, b) is admissible if and only if there exists c, d such that $ad + b(-c) = 1$. The last identity is equivalent to $\gcd(a, b) = 1$. Thus the projective line consists of all $[a, b]$ such that $\gcd(a, b) = 1$.

$\square$

Notation 3.1.13. A suggestive notation for $[1 : 0]$ is ∞ and for $[a : b]$ is $\frac{a}{b} \in \mathbb{Q}$ (the set of all rational numbers).

Proposition 3.1.14. For $i = 1, \cdots, k$, let R_i be a commutative ring with identity such that every element in R_i is either unit, zero or zero divisor. Let $R = R_1 \times \cdots \times R_k$ be the Cartesian product ring. Then, any point $[a : b]$ of the projective line, $\mathbb{P}(R)$, belongs exactly to one of the following three different classes:

1. $[z : 1]$, for all $z \in R$;
2. $[u : x]$, where u is a unit in R and x is zero or zero divisors in R;
3. $[x : y]$, where $x,\ y$ are two non-units in R such that, for all i, (x_i, y_i) is an admissible pair in R_i^2.

Proof. Let $\sim$ be an equivalence relation on $R^2\backslash\{(0,0)\}$ given in Definition 3.1.10. The set of all R elements is a disjoint union of the following three sets U (the set of all invertible elements, $\{0\}$ and Z_d (the set of all zero divisors). Then, we have, in $R^2/\!\sim$, the following three different types of equivalence classes:

1. If $a \in R$ and $b \in U$, then $(a, b) \in [z : 1]$, where $z = a \cdot b^{-1}$.

2. If $a, \lambda \in U$ and $b = \lambda x \in Z_d \cup \{0\}$, then $(a, b) \in [u : x]$, where $u = a \cdot \lambda^{-1}$.

3. $[a : b]$, where both a and b in $Z_d \cup \{0\}$.

The third type of equivalence classes splits into the following two categories:

1. $[x : x']$, such that, for all i, (x_i, x'_i) is an admissible pair in R_i^2.

2. $[x : y]$, such that, there is i, (x_i, y_i) is not an admissible pair in R_i^2.

We are going to show why the former belong to $\mathbb{P}(R)$. Clearly, any point in $R^2/\!\sim$ belongs to one of the following four different classes:

$$[z : 1],\ [u : x],\ [x : x'],\ [x : y],$$

where $z \in R$, $u \in U$ and $x, x', y \in Z_d \cup \{0\}$. Recall, from Example 2.2.17,

$$GL_2(R) = \{A = (A_1, \cdots, A_k) : A_i \in GL_2(R_i)\},$$

and $\det(A) = (\det(A_1), \ldots, \det(A_n))$. Obviously,

$$\begin{pmatrix} z & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} u & x \\ 0 & 1 \end{pmatrix} \in GL_2(R),$$

and there are $c, d \in R$ such that $\begin{pmatrix} x & x' \\ c & d \end{pmatrix} \in GL_2(R)$, while for all $a, b \in R$, $\begin{pmatrix} x & y \\ a & b \end{pmatrix} \notin GL_2(R)$. This means that the three pairs, $(z, 1)$, (u, x), (x, x') are admissible pairs while (x, y) is not. $\square$

Clearly, $\{[z : 1] : z \in R\}$ is isomorphic to R. Also, $\{[1 : z] : z \in R\}$ is isomorphic to R. Therefore, the only extra isolated points are $\{[x : x'] : x, x' \in Z_d\}$. For all composite numbers n, the projective line over $\mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_k}$, where $\mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_k}$ is isomorphic to $\mathbb{Z}_n$, is a special case of the ring R in the previous proposition.

The next proposition is regarded as a particular case of the previous one, where $\mathbb{O} = \mathbb{R} \times \mathbb{R}$. Yaglom, in [78, Suppl.C], gives the result of the proposition without proof. Here we give it in a simple way.

Proposition 3.1.15. Any point $[a : b]$ of the projective line, $\mathbb{P}(\mathbb{O})$, belongs to exactly one of the following six distinct classes:

$$[1 : 0],\ [z : 1],\ [1 : \lambda P_+],\ [1 : \lambda P_-],\ [P_+ : P_-] \text{ and } [P_- : P_+],$$

where $z \in \mathbb{O}$ and $\lambda \in \mathbb{R}\backslash\{0\}$.

Proof. Let $\sim$ be an equivalence relation on $\mathbb{O}^2\backslash\{(0,0)\}$ given in Definition 3.1.10. The set of all double numbers is a disjoint union of the following three sets $\overline{\mathbb{O}}$ (the set of all invertible elements), $\{0\}$ and $\widetilde{\mathbb{O}}$ (the set of all zero divisors). We can also split $\widetilde{\mathbb{O}}$ into two disjoint sets $\{aP_+\}$ and $\{aP_-\}$, where a is a non-zero real number. Then, we have, in $\mathbb{O}^2/\sim$, the following eight different types of equivalence classes:

1. If $a \in \overline{\mathbb{O}}$ and $b = 0$, then $(a, b) \in [1 : 0]$.

2. If $a \in \mathbb{O}$ and $b \in \overline{\mathbb{O}}$, then $(a, b) \in [z : 1]$, where $z = \frac{a}{b}$.

3. If $a = a_+P_+ + a_-P_- \in \overline{\mathbb{O}}$ and $b = b_-P_- \in \widetilde{\mathbb{O}}$, then $(a, b) \in [1 : \lambda P_-]$, where $\lambda = \frac{b_-}{a_-}$.

4. If $a = a_+P_+ + a_-P_- \in \overline{\mathbb{O}}$ and $b = b_+P_+ \in \widetilde{\mathbb{O}}$, then $(a, b) \in [1 : \lambda P_+]$, where $\lambda = \frac{b_+}{a_+}$.

5. If both a, $b \in \{\lambda P_+\} \cup \{0\}$, then $(a, b) \in [a_1P_+ : b_1P_+]$.

6. If $a = a_1P_+$, $b = b_1P_-$, then $(a, b) \in [P_+ : P_-]$.

7. If $a = a_1P_-$, $b = b_1P_+$, then $(a, b) \in [P_- : P_+]$.

8. If both a, $b \in \{\lambda P_-\} \cup \{0\}$, then $(a, b) \in [a_1P_- : b_1P_-]$.

Therefore, any point in $\mathbb{O}^2/\!\sim$ belongs to one of the following eight different classes:

$$[1 : 0],\ [z : 1],\ [1 : \lambda P_+],\ [1 : \lambda P_-],\ [a_1P_+ : b_1P_+],\ [P_+ : P_-],$$
$$[P_- : P_+] \text{ and } [a_1P_- : b_1P_-],$$

where $z \in \mathbb{O}$, $\lambda \in \mathbb{R}\backslash\{0\}$ and $a_1, b_1 \in \mathbb{R}$. Here, we are going to show why $[a_1P_\pm : b_1P_\pm]$, for all $a_1, b_1 \in \mathbb{R}$, does not belong to the projective line $\mathbb{P}(\mathbb{O})$. Recall,

$$GL_2(\mathbb{O}) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{O} \text{ and } \det(A) \text{ is a unit} \right\}.$$

Clearly,

$$I, \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & \lambda P_\pm \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} P_\pm & P_\mp \\ 1 & 1 \end{pmatrix} \in GL_2(\mathbb{O})$$

while for all $a, b \in \mathbb{O}$,

$$\begin{pmatrix} a_1P_\pm & b_1P_\pm \\ a & b \end{pmatrix} \notin GL_2(\mathbb{O}),$$

because its determinant is $P_\pm(a_1b - b_1a)$, which is a zero divisor. Thus, the pairs, $(1, 0)$, $(a, 1)$, $(1, \lambda P_\pm)$, $(P_\pm, P_\mp)$, are admissible pairs while the pairs $(a_1P_\pm : b_1P_\pm)$ are not. □

Thus, $\mathbb{O}$ parametrises the main part of the projective line over $\mathbb{O}$ as $[a : 1]$, for all $a \in \mathbb{O}$, except classes $[1 : 0], [P_+ : P_-], [P_- : P_+], [1 : \lambda P_-]$ and $[1 : \lambda P_+]$ for all non-zero $\lambda \in \mathbb{R}$.

Notation 3.1.16. A suggestive notation for $[a : 1]$ ($[1 : 0]$, $[1 : a_1P_-]$, $[1 : a_1P_+]$, $[P_+ : P_-]$, $[P_- : P_+]$) is a (∞, $\frac{1}{a_1}\omega_1$, $\frac{1}{a_1}\omega_2$, σ_1, σ_2) respectively, where $a \in \mathbb{O}$ and a_1 is a non-zero real number [54, Ch.8;78, Suppl.C]. In other words, $\mathbb{P}(\mathbb{O}) = \mathbb{O} \cup \{\infty, \sigma_1, \sigma_2\} \cup \{a\omega_1 : a \in \mathbb{R}\backslash\{0\}\} \cup \{a\omega_2 : a \in \mathbb{R} \setminus \{0\}\}$ [54, Ch.8;78, Suppl.C].

Yaglom, in [78, Suppl.C], gives the result of the next proposition without proof. Here we give it in a simple way.

Proposition 3.1.17. Any point $[a : b]$ of the projective line, $\mathbb{P}(\mathbb{D})$, belongs to exactly one of the following three classes:

$$[1 : 0], \ [z : 1] \text{ and } [1 : \epsilon a_1],$$

where $z \in \mathbb{D}$ and $a_1 \in \mathbb{R}\backslash\{0\}$.

Proof. Let $\sim$ be an equivalence relation on $\mathbb{D}^2\backslash\{(0,0)\}$ given in Definition 3.1.10. Let $\overline{\mathbb{D}}$ be the set of all invertible elements in $\mathbb{D}$, and $\widetilde{\mathbb{D}} \cup \{0\}$ be the rest.

Then, we have, in $\mathbb{D}^2/\sim$, the following four different types of equivalence classes:

1. If $a \in \mathbb{D}$ and $b \in \overline{\mathbb{D}}$, then $(a, b) \in [z : 1]$, where $z = \frac{a}{b}$.

2. If $a \in \overline{\mathbb{D}}$ and $b = 0$, then $(a, b) \in [1 : 0]$.

3. If $a = a_1 + \epsilon a_2 \in \overline{\mathbb{D}}$ and $b = \epsilon b_1 \in \widetilde{\mathbb{D}}$, then $(a, b) \in [1 : \epsilon\lambda]$, where $\lambda = \frac{b_1}{a_1}$.

4. If both a and b in $\widetilde{\mathbb{D}} \cup \{0\}$, then $(a, b) \in [\epsilon\lambda_1 : \epsilon\lambda_2]$, where $\lambda_{1,2} \in \mathbb{R}$.

So, any point in $\mathbb{D}^2/\sim$ belongs to one of the following four distinct classes:

$$[a : 1], \ [1 : 0], \ [1 : \epsilon\lambda] \text{ and } [\epsilon\lambda_1 : \epsilon\lambda_2],$$

where $a \in \mathbb{D}$, λ is a non-zero real number and $\lambda_{1,2} \in \mathbb{R}$. The points $[\epsilon\lambda_1 : \epsilon\lambda_2]$ are not points in the projective line, as we are going to see. Recall,

$$GL_2(\mathbb{D}) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{D} \text{ and } \det(A) \text{ is a unit} \right\}.$$

Obviously,

$$I, \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & \epsilon\lambda \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{D})$$

while for all $a, b \in \mathbb{D}$,

$$\begin{pmatrix} \epsilon\lambda_1 & \epsilon\lambda_2 \\ a & b \end{pmatrix} \notin GL_2(\mathbb{D}),$$

because its determinant is $\epsilon(\lambda_1 b - \lambda_2 a)$, which is a zero divisor. Therefore, the pairs, $(1, 0)$, $(a, 1)$, $(1, \epsilon\lambda)$, are admissible while the pairs $(\epsilon\lambda_1, \epsilon\lambda_2)$ are not. $\square$

Thus, $\mathbb{D}$ parametrises the main part of the projective line over $\mathbb{D}$ as $[a : 1]$, for all $a \in \mathbb{D}$, except classes $[1 : 0]$, and $[1 : \epsilon\lambda]$ for all non-zero $\lambda \in \mathbb{R}$.

Notation 3.1.18. A suggestive notation for $[a : 1]$ ($[1 : 0]$, $[1 : \epsilon\lambda]$) is a (∞, $\frac{1}{\lambda}\omega$) respectively, where $a \in \mathbb{D}$ and λ is a non-zero real number [54, Ch.8;78, Suppl.C]. In other words, $\mathbb{P}(\mathbb{D}) = \mathbb{D} \cup \{\infty\} \cup \{a\omega : a \in \mathbb{R}\backslash\{0\}\}$ [54, Ch.8;78, Suppl.C].

Generally, let R be an arbitrary commutative ring with identity. It is clear that R parametrises the main part of the projective line of R. In other words, R can be identified with the subset of the projective line given by $\{[r : 1] : r \in R\}$.

3.2 Möbius Transformations

Let R be a commutative ring with identity. In this section, we study $\mathbb{P}(R)$ as a $GL_2(R)$-homogeneous space.

Definition 3.2.1. Let $f : R \to R$ be a transformation. We say that $X \subseteq R$ is an *f-invariant* if $f(X) \subseteq X$ [54, Ch.2].

Definition 3.2.2. Let G be a group acting on R. A subset X of R is called *G-invariant* if $f(X) \subseteq X$ for all $f \in G$.

Definition 3.2.3. Let R be a commutative ring with identity. The action of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(R)$ on R^2 is a map $f : R^2 \to R^2$ defined via:

$$f((x, y)) = (ax + by, cx + dy).$$

The action of $A \in GL_2(R)$ on R^2 can be pulled to $R^2/\sim$ due to the following proposition.

Proposition 3.2.4. Let R be a commutative ring with identity. Let $M \in GL_2(R)$ and $(x, y), (v, w) \in R^2$. If $(x, y)\sim(v, w)$, then $M((x, y))\sim M((v, w))$.

Proof. Let $(x, y), (v, w) \in GL_2(R)$ and $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $(x, y)\sim(v, w)$, then there exists an invertible element $u \in R$ such that $x = uv,\ y = uw$.

$$M(x, y) = (ax + by, cx + dy) = (auv + buw, : cuv + duw) = uM(v, w).$$

Therefore, $M((x, y))\sim M((v, w))$.

□

The following lemma shows that $\mathbb{P}(R)$ is a $GL_2(R)$-invariant set.

Lemma 3.2.5. Let R be a commutative ring with identity. If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(R)$ and $\begin{pmatrix} x \\ y \end{pmatrix} \in R^2$ is an admissible pair, then $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$ is an admissible pair too.

Proof. Let $\begin{pmatrix} x \\ y \end{pmatrix} \in R^2$ be an admissible pair and $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(R)$ such that $M\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$. Since $\begin{pmatrix} x \\ y \end{pmatrix}$ is an admissible pair, there exists $\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \in R^2$ such that $M' = \begin{pmatrix} x & s_1 \\ y & s_2 \end{pmatrix} \in GL_2(R)$. The multiplication of M and M' is a matrix in $GL_2(R)$. Therefore, $M \cdot M' = \begin{pmatrix} w_1 & t_1 \\ w_2 & t_2 \end{pmatrix} \in GL_2(R)$, where $t_1 = as_1 + bs_2$ and $t_2 = cs_1 + ds_2$. □

Corollary 3.2.6. Let R be a commutative ring with identity. If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(R)$ and $\begin{pmatrix} x \\ y \end{pmatrix} \in R^2$ is not an admissible pair, then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$ is not an admissible pair either.

Proof. Let $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ be an admissible pair. We know that $\begin{pmatrix} x \\ y \end{pmatrix} = M^{-1}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$. Therefore, from the previous lemma $\begin{pmatrix} x \\ y \end{pmatrix}$ is an admissible pair. □

Definition 3.2.7. Let R be a commutative ring with identity. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(R)$. $T_M : \mathbb{P}(R) \to \mathbb{P}(R)$ is a function defined by $T_M([x : y]) = M[x : y] = [ax + by : cx + dy]$. The map T_M is called *Möbius transformation*.

For a commutative ring R, the set of all Möbius map is denoted by $M\ddot{o}b(R)$.

For $[x : 1] \in \mathbb{P}(R)$, its image is $[ax + b : cx + d]$. If $cx + d$ is unit, then $[ax + b : cx + d] \sim [\frac{ax+b}{cx+d} : 1]$. Therefore, the map $[x : 1] \mapsto [\frac{ax+b}{cx+d} : 1]$ can be abbreviated to $g(x) = \frac{ax+b}{cx+d}$ [54, Ch.2;56;57]. That means g is a map from $R \setminus \{x : cx + d \text{ is zero or zero divisor}\}$ to R. In the following, this formula will be used as a notation for a more accurate discussion in terms of the projective line.

Let $\mathbb{A}$ be one of $\mathbb{R}, \mathbb{O}, \mathbb{D}$. Let $A, A' \in GL_2(\mathbb{A})$ such that $A = uA'$, where u is a unit. If Möbius transformations are considered, then A and A' define the same map. The algebraic structure of $\mathbb{A}$ shows that for any invertible matrix $A \in M_2(\mathbb{A})$, such that $\det(A) = u^2$ and u is an invertible element in $\mathbb{A}$, there is $A' = \frac{1}{u}A \in M_2(\mathbb{A})$ such that $\det(A') = 1$.

Thus, we define an

$$SL_2(\mathbb{A}) = \{A \in GL_2(\mathbb{A}) : \det(A) = 1\},$$

which is a subgroup of $GL_2(\mathbb{A})$.

Proposition 3.2.8. Let $\mathbb{A}$ be one of $\mathbb{R}, \mathbb{O}, \mathbb{D}$. Let $\pi : SL_2(\mathbb{A}) \to M\ddot{o}b(\mathbb{A})$ be a map such that $\pi(A) = T_A$. π is a group homomorphism.

By a direct check, $\pi(AB) = T_{AB} = T_A \circ T_B$. We obtain the following fact:

1. If $\mathbb{A} = \mathbb{R}$ then the kernel of π is $\{\pm I\}$. Therefore $M\ddot{o}b(\mathbb{R}) \cong SL_2(\mathbb{R})/\{\pm I\}$ [6, 66].

2. If $\mathbb{A} = \mathbb{O}$ then the kernel of π is $\{\pm I, \pm jI\}$. Therefore

$$M\ddot{o}b(\mathbb{O}) \cong SL_2(\mathbb{O})/\{\pm I, \pm jI\}.$$

3. If $\mathbb{A} = \mathbb{D}$ then the kernel of π is $\{\pm I\}$. Therefore $M\ddot{o}b(\mathbb{D}) \cong SL_2(\mathbb{D})/\{\pm I\}$.

The proof of the next lemma follows immediately from Lemma 3.2.5.

Lemma 3.2.9. $GL_2(R)$ acts transitively on $\mathbb{P}(R)$.

Next, we investigate $GL_2(R)$ of several known rings, and the actions of $GL_2(R)$ over $\mathbb{P}(R)$.

Example 3.2.10. Recall from Example 3.1.5,

$$GL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = \pm 1 \right\}.$$

By Lemma 3.2.9, $\mathbb{P}(\mathbb{Z})$ is the orbit of $[1:0]$ concerning $GL_2(\mathbb{Z})$. By Lemma 3.2.5, the complement of $\mathbb{P}(\mathbb{Z})$, $\{[a:b]:(a,b)\neq 1\}$, is a $GL_2(\mathbb{Z})$-invariant subset of $\mathbb{Z}^2/\!\sim$.

Example 3.2.11. Let $R = R_1\times,\cdots,R_k$.

$$GL_2(R) = \{A = (A_1,\cdots,A_k) : A_i \in GL_2(R_i)\}.$$

From Lemma 3.2.5, $R^2/\!\sim$ is a disjoint union of the following two $GL_2(\mathbb{R})$-invariant subsets: the projective line $\mathbb{P}(R)$, and the complement $\{[x:y]:(x,y)$ is not an admissible pair $\}$.

The previous example does not give us the exact number of disjoint $GL_2(R)$-invariant subsets, which depends on the number k and also the rings R_k themselves.

The presence of the ideal $\{aP_\pm : a\in\mathbb{R}\}$ in $\mathbb{O}$ makes $GL_2(\mathbb{O})$-action on $\mathbb{O}^2/\!\sim$ not transitive.

Theorem 3.2.12. The set $\mathbb{O}^2/\!\sim$ is a disjoint union of the following three $GL_2(\mathbb{O})$-orbits:

1. The orbit of $[1:0]$ is the projective line over $\mathbb{O}$.
2. The orbit of $[P_+:0]$ is the set $P\mathbb{R}_+ = \{[\lambda_1P_+ : \lambda_2P_+] : \lambda_1$ and λ_2 are real numbers and not both $0\}$.
3. The orbit of $[P_-:0]$ is the set $P\mathbb{R}_- = \{[\lambda_1P_- : \lambda_2P_-] : \lambda_1$ and λ_2 are real numbers and not both $0\}$.

Proof. Recall, $GL_2(\mathbb{O}) = \left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix} : a,b,c,d\in\mathbb{O},\ \text{and } ad-bc \text{ is a unit}\right\}.$

1. Immediate from Lemma 3.2.9.

2. For all $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{O})$, $A[P_+ : 0] = [aP_+ : cP_+]$. Therefore, the orbit of $[P_+ : 0]$ is a subset of $P\mathbb{R}_+$.

   Conversely, let $[\lambda P_+ : \mu P_+]$ be any element in $P\mathbb{R}_+$. Clearly, $A[P_+ : 0] = [\lambda P_+ : \mu P_+]$, where $A = \begin{pmatrix} \lambda P_+ & P_- \\ \mu P_+ + P_- & P_+ \end{pmatrix}$ or $\begin{pmatrix} \lambda P_+ + P_- & P_+ \\ \mu P_+ & P_- \end{pmatrix}$. Therefore, $[\lambda P_+ : \mu P_+]$ is in the orbit of $[P_+ : 0]$, i.e. $P\mathbb{R}_+$ is a subset of the orbit of $[P_+ : 0]$. So, the orbit of $[P_+ : 0]$ equals to the set $P\mathbb{R}_+$.

3. For all $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{O})$, $A[P_- : 0] = [aP_- : cP_-]$. Therefore, the orbit of $[P_- : 0]$ is a subset of $P\mathbb{R}_-$.

   Conversely, let $[\lambda P_- : \mu P_-]$ be any element in $P\mathbb{R}_-$ Clearly, $A[P_- : 0] = [\lambda P_- : \mu P_-]$, where $A = \begin{pmatrix} \lambda P_- & P_+ \\ P_+ + \mu P_- & P_- \end{pmatrix}$ or $\begin{pmatrix} P_+ + \lambda P_- & P_- \\ \mu P_- & P_+ \end{pmatrix}$. Therefore, $[\lambda P_- : \mu P_-]$ is in the orbit of $[P_- : 0]$, i.e. $P\mathbb{R}_-$ is a subset of the orbit of $[P_- : 0]$. So, the orbit of $[P_- : 0]$ equals to the set $P\mathbb{R}_-$.

□

From the previous proposition, we can split $\mathbb{O}^2/\sim$ into three sets: The orbit of $[1 : 0]$, the orbit of $[P_+ : 0]$ and the orbit of $[P_- : 0]$. The next proposition explains an isomorphism between the orbit of $[P_\pm : 0]$ and $\mathbb{P}(\mathbb{R})$.

Proposition 3.2.13. Let X be the $GL_2(\mathbb{O})$-orbit of $[P_\pm : 0]$. There is a projection $p_\pm : SL_2(\mathbb{O}) \to SL_2(\mathbb{R})$ defined by $p_\pm(g) = g_\pm$ for $g = g_+P_+ + g_-P_- \in SL_2(\mathbb{O})$ and a bijection $f : \mathbb{P}(\mathbb{R}) \to X$ define by,

$$f[x : y] = [xP_\pm : yP_\pm], \quad \text{for } x, y \in \mathbb{R}.$$

I omit the proof as it is a straightforward result.

To make our work easier, we present the following lemma and proposition. Their proofs are straightforward results and the above stated properties of $GL_2(\mathbb{O})$ and $SL_2(\mathbb{O})$.

Proposition 3.2.14. Let $A_+, A_- \in M_2(\mathbb{R})$. $A = A_+P_+ + A_-P_- \in GL_2(\mathbb{O})$ if and only if both A_+ and A_- in $GL_2(\mathbb{R})$.

Clearly, $A = A_+P_+ + A_-P_- \in SL_2(\mathbb{O})$ if and only if $A_\pm \in SL_2(\mathbb{R})$.

Proposition 3.2.15. The function $f : SL_2(\mathbb{O}) \to SL_2(\mathbb{O})$ which is defined by $f(X_+P_+ + X_-P_-) = X_-P_+ + X_+P_-$ is a group homomorphism.

The presence of the ideal $\{\epsilon a : a \in \mathbb{R}\}$ in $\mathbb{D}$ makes $GL_2(\mathbb{D})$-action on $\mathbb{D}^2/\!\sim$ not transitive.

Theorem 3.2.16. The set $\mathbb{D}^2/\!\sim$ is a disjoint union of the following two $GL_2(\mathbb{D})$-orbits:

1. The orbit of $[1 : 0]$, is the projective line over $\mathbb{D}$.
2. The orbit of $[\epsilon a : 0]$, is the set $P\mathbb{R} = \{[\epsilon\lambda_1 : \epsilon\lambda_2] : \lambda_1$ and λ_2 are real numbers and not both $0\}$.

Proof. Recall, $GL_2(\mathbb{D}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{D}, \text{ and } ad - bc \text{ is a unit} \right\}$.

1. Immediate from Lemma 3.2.9.
2. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{D})$. $A[\epsilon : 0] = [\epsilon a : \epsilon c]$. Therefore, the orbit of $[\epsilon : 0]$ is a subset of $P\mathbb{R}$.

   Conversely, let $[\epsilon\lambda : \epsilon\mu]$ be any element in $P\mathbb{R}$. Clearly $A[\epsilon : 0] = [\epsilon\lambda : \epsilon\mu]$, where $A = \begin{pmatrix} \lambda + \epsilon & -\mu \\ \mu + \epsilon & \lambda \end{pmatrix}$, i.e. $P\mathbb{R}$ is a subset of the orbit of $[\epsilon : 0]$. So, the orbit of $[\epsilon : 0]$ equals to the set $P\mathbb{R}$.

□

As a consequence of the above proposition, we obtain that $\mathbb{D}^2/\sim$ splits into two sets: the orbit of $[1:0]$ and the orbit of $[\epsilon:0]$. The next proposition explains an isomorphism between the orbit of $[\epsilon:0]$ and $\mathbb{P}(\mathbb{R})$.

Proposition 3.2.17. Let X be the $GL_2(\mathbb{D})$-orbit of $[\epsilon:0]$. There is a projection $p: SL_2(\mathbb{D}) \to SL_2(\mathbb{R})$ defined by $p(g) = g_1$ for $g = g_1 + \epsilon g_2 \in SL_2(\mathbb{D})$, and a bijection $f: \mathbb{P}(\mathbb{R}) \to X$ define by, for all $[x:y] \in \mathbb{P}(\mathbb{R})$,

$$f[x:y] = [\epsilon x : \epsilon y], \quad \text{for } x, y \in \mathbb{R}.$$

I omit the proof as it is a straightforward result.

To make our work easier, we present the following lemma. Its proof is a straightforward result and the above stated properties of $GL_2(\mathbb{D})$.

Proposition 3.2.18. Let $A_1, A_2 \in M_2(\mathbb{R})$. $A = A_1 + \epsilon A_2 \in M_2(\mathbb{D})$ if and only if A_1 in $GL_2(\mathbb{R})$.

Clearly, $A = A_1 + \epsilon A_2 \in SL_2(\mathbb{D})$ if and only if $A_1 \in SL_2(\mathbb{R})$ and $\mathrm{tr}(A_1 \widehat{A_2}) = 0$. In the following chapters, we focus on the rings of double and dual numbers. We leave the other rings for further studies.

Chapter 4

Metric Properties of Möbius Maps

This chapter explores the metric properties of Möbius maps. The first section of this chapter defines a metric on $\mathbb{P}(\mathbb{A})$, where $\mathbb{A}$ is one of $\mathbb{R}$, $\mathbb{C}$, $\mathbb{O}$ or $\mathbb{D}$. We prove that a Möbius map T_A is a continuous map for all $A \in SL_2(\mathbb{A})$, in the respective topologies. The second section gives the general formula of the fixed points of T_A (if any exists). The last section provides a classification of fixed points based on considered metrics. It is useful for iterations of Möbius maps.

4.1 Metric Space

In this section, we use the standard definition of metric, topology and continuous maps.

Lemma 4.1.1. Let d and d' be two metrics defined on a space X. The following are equivalent:

1. The metrics d and d' are equivalent.

2. For each x there exists $0 < m,\ m' < \infty$ such that for every y, $m'd(x,y) \leq d'(x,y) \leq md(x,y)$.

3. Both identity maps $Id : (X, d) \to (X, d')$ and $Id' : (X, d') \to (X, d)$ are continuous.

I omit the prove as it is a standard result.

A real or complex Möbius map $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is continuous at every point except x such that $cx + d = 0$. The next definition provides an alteration of the Euclidean metric, such that $M : \mathbb{P}(\mathbb{A}) \to \mathbb{P}(\mathbb{A})$ is continuous at every point.

Definition 4.1.2. Let $\mathbb{A}$ be one of the field of real or complex numbers. A map $d_{\mathbb{A}} : \mathbb{P}(\mathbb{A}) \times \mathbb{P}(\mathbb{A}) \to \mathbb{R}$ is defined by

1.
$$d_{\mathbb{R}}([x : y], [x' : y']) = \frac{|xy' - yx'|}{\max\{|x|, |y|\} \max\{|x'|, |y'|\}}.$$

2.
$$d_{\mathbb{C}}([x : y], [x' : y']) = \frac{|xy' - yx'|}{\max\{|x|, |y|\} \max\{|x'|, |y'|\}}.$$

By using the properties of absolute value, we can prove that both $d_{\mathbb{R}}$ and $d_{\mathbb{C}}$ are metric over $\mathbb{P}(\mathbb{R})$ and $\mathbb{P}(\mathbb{C})$ respectively. Moreover, $\mathbb{P}(\mathbb{R})$ with $d_{\mathbb{R}}$ is isomorphic to the circle endowed with the Euclidean metric and $\mathbb{P}(\mathbb{C})$ with $d_{\mathbb{C}}$ is isomorphic to the Riemann sphere endowed with the Euclidean metric.

The technique of (P_+, P_-)-decomposition provides a bijective map $f : \mathbb{O} \to \mathbb{R}^2$, which is defined by $f(aP_+ + bP_-) = (a, b)$. So, we use two copies of $d_{\mathbb{R}}$ to define metric over $\mathbb{O}$.

Definition 4.1.3. A map $d_{\mathbb{O}} : \mathbb{P}(\mathbb{O}) \times \mathbb{P}(\mathbb{O}) \to \mathbb{R}$ is defined by

$$\begin{aligned} d_{\mathbb{O}}(z, z') &= \frac{\left|x_+y'_+ - y_+x'_+\right|}{\max\{|x_+|, |y_+|\} \max\{|x'_+|, |y'_+|\}} \\ &+ \frac{\left|x_-y'_- - y_-x'_-\right|}{\max\{|x_-|, |y_-|\} \max\{|x'_-|, |y'_-|\}}, \end{aligned}$$

where $z = [x_+P_+ + x_-P_- : y_+P_+ + y_-P_-]$, $z' = [x'_+P_+ + x'_-P_- : y'_+P_+ + y'_-P_-] \in \mathbb{P}(\mathbb{O})$.

Obviously, $d_{\mathbb{O}}$ is a metric over $\mathbb{O}$. Moreover, the projective line $\mathbb{P}(\mathbb{O})$ with the metric $d_{\mathbb{O}}$ is isomorphic to the one-sheet hyperboloid endowed with the Euclidean metric [54, Fig. 8.1 c].

For any $a \in \mathbb{R}$, the vertical line $\{a + \epsilon b : b \in \mathbb{R}\}$ is a Möbius invariant set. Thus, we define the distance between two points in two ways. If two points do not belong to the same vertical line, then the definition depends only on the real parts of the points. If two points belong to the same vertical line, then the definition depends on both the real parts and imaginary parts of the points. The formal definition is as follows:

Definition 4.1.4. A map $d_{\mathbb{D}} : \mathbb{P}(\mathbb{D}) \times \mathbb{P}(\mathbb{D}) \to \mathbb{R}$ is defined by

$$d_{\mathbb{D}}(z, z') = \begin{cases} \frac{\left|x_1 y_1' - y_1 x_1'\right|}{\max\{|x_1|,|y_1|\} \max\{\left|x_1'\right|,\left|y_1'\right|\}}, & \text{if } x_1 y_1' - y_1 x_1' \neq 0; \\ \left|\frac{x_2 y_1 - y_2 x_1}{\max\{x_1^2, y_1^2\}} - \frac{x_2' y_1' - y_2' x_1'}{\max\{(x_1')^2, (y_1')^2\}}\right|, & \text{otherwise.} \end{cases},$$

where $z = [x_1 + \epsilon x_2 : y_1 + \epsilon y_2],\ z' = [x_1' + \epsilon x_2' : y_1' + \epsilon y_2'] \in \mathbb{P}(\mathbb{D})$.

By using the properties of absolute value, we can prove that the map $d_{\mathbb{D}}$ is a metric. Further more, $\mathbb{P}(\mathbb{D})$ with $d_{\mathbb{D}}$ is isomorphic a cylinder endowed with the Euclidean metric [54, Fig. 8.1 b;78, Suppl.A].

Definition 4.1.5. Let $\mathbb{A}$ be one of the field of real, complex or the ring of dual numbers. A map $d'_{\mathbb{A}} : \mathbb{A} \times \mathbb{A} \to \mathbb{R}$ is defined by

1. For $\mathbb{A} = \mathbb{R}$, $d'_{\mathbb{A}}(x, y) = |x - y|$.
2. For $\mathbb{A} = \mathbb{C}$, $d'_{\mathbb{A}}(x_1 + ix_2, y_1 + iy_2) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$.
3. For $\mathbb{A} = \mathbb{D}$, $d'_{\mathbb{A}}(x_1 + \epsilon x_2, y_1 + \epsilon y_2) = |x_1 - y_1|$.

The map $d'_{\mathbb{A}}$ is a metric if $\mathbb{A} = \mathbb{R}$ or $\mathbb{C}$ and a pseudometric if $\mathbb{A} = \mathbb{D}$.

Definition 4.1.6. A map $dm : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$dm(x, y) = |x_1 - y_1| + |x_2 - y_2|, \quad x = (x_1, x_2),\ y = (y_1, y_2).$$

The map dm is a metric on $\mathbb{R}^2$, called the product (Manhattan) metric. Recall, $\mathbb{O}$ is isomorphic to $\mathbb{R}^2$. Thus, dm is a metric over $\mathbb{O}$.

Recall, for any commutative ring with identity $\mathbb{A}$, $\mathbb{A}' = \{[x : 1] : x \in \mathbb{A}\} \subset \mathbb{P}(\mathbb{A})$ is isomorphic to $\mathbb{A}$.

Lemma 4.1.7. Let $\mathbb{A}$ be one of $\mathbb{R}, \mathbb{C}, \mathbb{O}, \mathbb{D}$. The restriction of $d_{\mathbb{A}}$ over $\mathbb{A}' = \{[x : 1] : x \in \mathbb{A}\} \subset \mathbb{P}(\mathbb{A})$ is equivalent to the usual metric over $\mathbb{A}$ if $\mathbb{A}$ is a real or complex numbers, equivalent to the product metric over $\mathbb{A}$ if $\mathbb{A}$ double and equivalent to the usual pseudometric over $\mathbb{A}$ if $\mathbb{A}$ dual numbers.

Proof. Let $f : \mathbb{A}' \to \mathbb{A}$ be a map defined by $f([x : 1]) = x$. Let $d' : \mathbb{A} \times \mathbb{A} \to \mathbb{R}$ be a map defined by $d'(x, y) = d_{\mathbb{A}}(f^{-1}(x), f^{-1}(y))$. Then the calculation shows that:

1. For $\mathbb{A} = \mathbb{R}$ or $\mathbb{C}$ and $x, y \in \mathbb{A}$,

$$d'(x, y) = \frac{|x - y|}{k}, \qquad \text{where } k = \max\{|x|, 1\} \cdot \max\{|y|, 1\}.$$

2. For $\mathbb{A} = \mathbb{O}$, and $x = x_+P_+ + x_-P_-, y = y_+P_+ + y_-P_- \in \mathbb{A}$,

$$d'(x, y) = \frac{|x_+ - y_+|}{k_+} + \frac{|x_- - y_-|}{k_-}, \qquad \text{where } k_\pm = \max\{|x_\pm|, 1\} \cdot \max\{|y_\pm|, 1\}.$$

3. For $\mathbb{A} = \mathbb{D}$, and $x = x_1 + \epsilon x_2, y = y_1 + \epsilon y_2 \in \mathbb{A}$,

$$d'(x, y) = \frac{|x_1 - y_1|}{k}, \qquad \text{where } k = \max\{|x_1|, 1\} \cdot \max\{|y_1|, 1\}.$$

The map d' is the restriction of $d_{\mathbb{A}}$ on $\mathbb{A}'$. Long calculations prove that d' is a metric over $\mathbb{A}$ if $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{O}$ and a pseudometric if $\mathbb{A} = \mathbb{D}$. For $\mathbb{A} = \mathbb{R}, \mathbb{C}$, let d be the usual metric over $\mathbb{A}$. For $\mathbb{A} = \mathbb{O}$, let d be the product metric over $\mathbb{O}$. For $\mathbb{A} = \mathbb{D}$, let d be the usual pseudometric over $\mathbb{D}$. Long calculations show that the two identity maps $Id' : (\mathbb{A}, d') \to (\mathbb{A}, d)$ and $Id : (\mathbb{A}, d) \to (\mathbb{A}, d')$ are continuous maps, for each commutative ring with identity. Therefore, d' is equivalent to d by Lemma 4.1.1. □

Lemma 4.1.8. The inversion $i : [x : y] \to [y : x]$ is a continuous function in $d_{\mathbb{A}}$.

Proof. Clearly, regarding $d_{\mathbb{A}}$, the inversion i is an isometry. So, it is continuous. □

To prove the next proposition, I am going to use the standard definition of continuity of maps between metric spaces.

Proposition 4.1.9. Let $\mathbb{A}$ be one of $\mathbb{R}, \mathbb{C}, \mathbb{O}, \mathbb{D}$. An $\mathbb{A}$-Möbius transformation $T_A : \mathbb{P}(\mathbb{A}) \to \mathbb{P}(\mathbb{A})$ is a continuous map.

Proof. Given four elements $a,\ b,\ c,\ d$ in $\mathbb{A}$, let $f_1 : \mathbb{A} \to \mathbb{A}$ ($f_2 : \mathbb{A} \to \mathbb{A}$) be two functions defined as follows $f_1(x) = ax + b$ ($f_2(x) = cx + d$). Clearly, both $f_1,\ f_2$ are continuous functions in the usual metric (pseudometric) over $\mathbb{A}$, if $\mathbb{A}$ is a real, complex or (dual) numbers. They are also continuous functions in the product metric if $\mathbb{A}$ is the ring of double numbers. Let $\mathbb{A}'' = \{x : f_2(x) \text{ is invertible}\}$. Let $f : \mathbb{A}'' \to \mathbb{A}$ be a function defined by $f(x) = \frac{f_1(x)}{f_2(x)}$. The function f is also a continuous map regarding the usual (product) metric over $\mathbb{A}$ if $\mathbb{A}$ is a real, complex (double) numbers and it is also a continuous map regarding usual pseudometric over $\mathbb{A}$ if $\mathbb{A}$ is dual numbers. If $d,$ and d' are equivalent, then they determine the same topology. Hence continuity with respect to either metric is the same. By Lemma 4.1.7, the continuity of $T_A : \mathbb{P}(\mathbb{A}) \to \mathbb{P}(\mathbb{A})$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, on $\mathbb{A}''$ has been proved.

Because of inversion is continuous at $[1 : x]$, T_A is continuous at $i([1 : x])$, $T_A \circ i$ is continuous at $[1 : x]$. Therefore, T_A is continuous at the points $[1 : x]$, i.e. real, complex and dual Möbius maps are continuous. If we prove the continuity of $\mathbb{O}$-Möbius maps at $\sigma_1 = [P_+ : P_-] \in \mathbb{P}(\mathbb{O})$, then $\mathbb{O}$-Möbius maps are continuous too.

Let $z_0 = [P_+ : P_-]$, $z = [x_+P_+ + x_-P_- : y_+P_+ + y_-P_-]$ and

$$A = \begin{pmatrix} a_+P_+ + a_-P_- & b_+P_+ + b_-P_- \\ c_+P_+ + c_-P_- & d_+P_+ + d_-P_- \end{pmatrix}.$$

Obviously,

$$T_A(z_0) = [a_+P_+ + b_-P_- : c_+P_+ + d_-P_-].$$

Let

$$k_+ = \max\{|a_+x_+ + b_+y_+|, |c_+x_+ + d_+y_+|\} \cdot \max\{|a_+|, |c_+|\}$$

and

$$k_- = \max\{|a_-x_- + b_-y_-|, |c_-x_- + d_-y_-|\} \cdot \max\{|b_-|, |d_-|\}.$$

Note that both $k_\pm$ are greater than zero, otherwise T_A would not be an $\mathbb{O}$-Möbius map. Then

$$d_{\mathbb{O}}(T_A(z), T_A(z_0)) = \frac{|y_+|}{k_+} + \frac{|x_-|}{k_-}.$$

Let

$$\frac{\epsilon'_+}{2} < \frac{|y_+|}{k_+} < \frac{\epsilon_+}{2}, \tag{4.1}$$

$$\frac{\epsilon'_-}{2} < \frac{|x_-|}{k_-} < \frac{\epsilon_-}{2}. \tag{4.2}$$

Therefore, $\epsilon' < d_{\mathbb{O}}(T_A(z), T_A(z_0)) < \epsilon$, where $\epsilon = \max\{\epsilon_+, \epsilon_-\}$, $\epsilon' = \min\{\epsilon'_+, \epsilon'_-\}$.

Let $l_\pm = \max\{|x_\pm|, |y_\pm|\}$. Obviously, both $l_\pm$ are greater than zero, otherwise z would not be belong to $\mathbb{P}(\mathbb{O})$. From the equations (4.1) and (4.2) we obtain

$$\frac{\delta'_+}{2} = \frac{k_+}{l_+}\frac{\epsilon'_+}{2} < \frac{|y_+|}{l_+} < \frac{k_+}{l_+}\frac{\epsilon_+}{2} = \frac{\delta_+}{2}, \tag{4.3}$$

$$\frac{\delta'_-}{2} = \frac{k_-}{l_-}\frac{\epsilon'_-}{2} < \frac{|x_-|}{l_-} < \frac{k_-}{l_-}\frac{\epsilon_-}{2} = \frac{\delta_-}{2}, \tag{4.4}$$

respectively. By adding (4.3) and (4.4), we obtain

$$\delta' < \frac{\delta'_+}{2} + \frac{\delta'_-}{2} < d_{\mathbb{O}}(z, z_0) < \frac{\delta_+}{2} + \frac{\delta_-}{2} < \delta,$$

where $\delta = \max\{\delta_+, \delta_-\}$, $\delta' = \min\{\delta'_+, \delta'_-\}$. So, T_A is continuous at σ_1. □

Next, we are going to define a metric over $GL_2(\mathbb{A})$. This metric which is worth to define here, but we are going to use it in the next chapter.

Definition 4.1.10. Let $\mathbb{A}$ be one of $\mathbb{R}, \mathbb{C}, \mathbb{O}, \mathbb{D}$. A map $d_{GL_2(\mathbb{A})} : GL_2(\mathbb{A}) \times GL_2(\mathbb{A}) \to \mathbb{R}$ is defined by

$$\begin{aligned} d_{GL_2(\mathbb{A})}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right) &= d_{\mathbb{A}}([a:1],[a':1]) + d_{\mathbb{A}}([b:1],[b':1]) + \\ & \quad d_{\mathbb{A}}([c:1],[c':1]) + d_{\mathbb{A}}([d:1],[d':1]). \end{aligned}$$

This metric prove the continuity of one-parameter subgroups of $GL_2(\mathbb{A})$, where $\mathbb{A}$ is one of $\mathbb{R}, \mathbb{O}, \mathbb{D}$.

4.2 Fixed Points and Möbius Maps

Let $\mathbb{A}$ be one of $\mathbb{R}$, $\mathbb{C}$, $\mathbb{O}$, $\mathbb{D}$. Let T_A be a non-trivial map in Möb($\mathbb{A}$). Assume that $T_A[x:y] = [x:y]$, which means that $[x:y]$ is a fixed point under T_A. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the representative matrix of T_A. Thus $[ax+by : cx+dy] = [x:y]$. This means that there exists a unit element $\lambda \in \mathbb{A}$ such that

$$ax + by = \lambda x, \tag{4.5}$$

$$cx + dy = \lambda y. \tag{4.6}$$

The values of λ are the roots of the equation $(a-\lambda)(d-\lambda) - bc = 0$. Solving the quadratic equation we obtain that $\lambda = \dfrac{d+a+\sqrt{I}}{2}$, where $I = \operatorname{tr}^2 A - 4\det(A)$, and $\sqrt{I}$ is one of the square roots of I, if such square roots exist. The exact meaning of the square root is explained in the following sections case by case. If $\mathbb{A} = \mathbb{R}$ or $\mathbb{C}$, then from equation (4.7), we obtain the explicit fixed point

$$[x:y] = [-2b : (d-a) + \sqrt{I}],$$

and from equation (4.8), we obtain the explicit fixed point

$$[x:y] = [(a-d)+\sqrt{I} : -2c]$$

[61, Ch.5]. If T_A is not trivial Möbius map, then the two previous forms provide all fixed points of T_A.

Let $\mathbb{A} = \mathbb{O}$ or $\mathbb{D}$. Substituting λ in the above system we get the two following equations:

$$by = \frac{1}{2}(d-a+\sqrt{I})x \tag{4.7}$$

$$cx = \frac{1}{2}(a-d+\sqrt{I})y. \tag{4.8}$$

The number x^2+y^2 is a unit because of the pair $[x:y] \in \mathbb{P}(\mathbb{A})$. The number $(d-a+\sqrt{I})^2+4b^2$ is a unite because of $A \in GL_2(\mathbb{A})$ and $\sqrt{I} \in \mathbb{A}$. Thus, $u^2 = \frac{x^2+y^2}{(d-a+\sqrt{I})^2+4b^2}$ is a unit. The square root of u^2 is defined because of both the numerator and denominator are sum of squares. Lets come back to the equation From 4.7, we obtain

$$\begin{aligned}
4b^2y^2 &= (d-a+\sqrt{I})^2x^2, \\
4b^2x^2+4b^2y^2 &= (d-a+\sqrt{I})^2x^2+4b^2x^2, \\
4b^2(x^2+y^2) &= ((d-a+\sqrt{I})^2+4b^2)x^2, \\
x^2 &= 4b^2\frac{x^2+y^2}{(d-a+\sqrt{I})^2+4b^2}, \\
x^2 &= 4b^2u^2, \\
x &= 2bu.
\end{aligned}$$

where $2bu$ is one of the square roots $4b^2u^2$ and

$$\begin{aligned}
4b^2y^2 &= (d-a+\sqrt{I})^2x^2, \\
(d-a+\sqrt{I})^2y^2+4b^2y^2 &= (d-a+\sqrt{I})^2x^2+(d-a+\sqrt{I})^2y^2, \\
((d-a+\sqrt{I})^2+4b^2)y^2 &= (d-a+\sqrt{I})^2(x^2+y^2), \\
y^2 &= (d-a+\sqrt{I})^2\frac{x^2+y^2}{(d-a+\sqrt{I})^2+4b^2}, \\
y^2 &= (d-a+\sqrt{I})^2u^2, \\
y &= (d-a+\sqrt{I})u,
\end{aligned}$$

where $(d-a+\sqrt{I})u$ is one of the square roots of $(d-a+\sqrt{I})^2u^2$. Thus, from equation (4.7), we obtain the explicit fixed point

$$[x:y]=[2b:(d-a)+\sqrt{I}],$$

and in the same way, from equation (4.8), we obtain the explicit fixed point

$$[x:y]=[(a-d)+\sqrt{I}:2c].$$

The unit for the second form is defined by u such that $u^2=\frac{x^2+y^2}{(a-d+\sqrt{I})^2+4c^2}$. Let T_A be a non-trivial Möbius map. If $\sqrt{I}$ is defined and

$$[x:y]=[2b:(d-a)+\sqrt{I}]\notin\mathbb{P}(\mathbb{A}),$$

then T_A fixes infinite points. In the same way, if

$$[x:y]=[(a-d)+\sqrt{I}:2c]\notin\mathbb{P}(\mathbb{A}),$$

then T_A fixes infinite points.

- If $\mathbb{A}=\mathbb{O}$, T_A is a non-trivial Möbius map and $[2b:(d-a)+\sqrt{I}]\notin\mathbb{P}(\mathbb{O})$, then The fixed point are $\{[rP_-:1],[1:rP_-],[rP_+:1],[1:rP_+]:r\in\mathbb{R}\}$.

- If $\mathbb{A} = \mathbb{D}$, T_A is a non-trivial Möbius map and $[2b : (d-a) + \sqrt{I}] \notin \mathbb{P}(\mathbb{D})$, then The fixed point are $\{[r\epsilon : 1] : r \in \mathbb{R}\}$.

Thus the two previous forms provide all fixed points of T_A when the number of fixed points are finite.

4.2.1 Fixed points of an $\mathbb{R}$-Möbius transformation

Let T_A be a non-trivial map in Möb($\mathbb{R}$) with representative matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$. Let $I = \operatorname{tr}^2 A - 4$. That means the square root of I is in $\mathbb{R}$ if $\operatorname{tr}^2 A \geq 4$. Thus, we have the following three cases:

1. If $\operatorname{tr}^2 A - 4 < 0$, then T_A fixes no point.

2. If $\operatorname{tr}^2 A - 4 = 0$, then T_A fixes a double fixed point.

3. If $\operatorname{tr}^2 A - 4 > 0$, then T_A fixes two distinct points.

Example 4.2.1.

1. Let $A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. The trace square $\operatorname{tr}^2 A = 4$. T_A fixes one point, which is $[0 : 1]$.

2. Let $A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$. The trace square $\operatorname{tr}^2 A > 4$. T_A fixes two distinct points, which are $[\sqrt{3} : 1]$ and $[-\sqrt{3} : 1]$.

3. Let $A = \begin{pmatrix} \frac{1}{2} & -3 \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}$. The trace square $\operatorname{tr}^2 A < 4$. T_A fixes no point.

4.2.2 Fixed points of a $\mathbb{C}$-Möbius transformation

Let T_A be a non-trivial map in Möb($\mathbb{C}$) with a representative matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C})$. Let $I = \operatorname{tr}^2(A) - 4$. Thus we have two cases:

1. If $\operatorname{tr}^2(A) = 4$, then T_A fixes a double fixed point.

2. If $\operatorname{tr}^2(A) \neq 4$, then T_A fixes two distinct fixed points.

Example 4.2.2. 1. Let $A = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}$. T_A fixes $[0 : 1]$.

2. Let $A = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$. T_A fixes $[\frac{1}{2}(1 \pm \sqrt{3}) : 1]$.

3. Let $A = \begin{pmatrix} \frac{1}{2} & -\frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$. T_A fixes $[\pm i\sqrt{3} : 1]$.

4.2.3 Fixed points of an $\mathbb{O}$-Möbius transformation

Let T_A be a non-trivial map in $\text{Möb}(\mathbb{O})$ with a representative matrix $A = A_+P_+ + A_-P_- \in GL_2(\mathbb{O})$, where $A_\pm \in GL_2(\mathbb{R})$ and none of $A_\pm$ is the identity matrix. Let $I = \operatorname{tr}^2 A - 4\det(A)$. Then, from Lemma 2.1.32, $\sqrt{I}$ has the following values:

$$\pm(\sqrt{\operatorname{tr}^2 A_+ - 4\det(A_+)}P_+ + \sqrt{\operatorname{tr}^2 A_- - 4\det(A_-)}P_-),$$

$$\pm(\sqrt{\operatorname{tr}^2 A_+ - 4\det(A_+)}P_+ - \sqrt{\operatorname{tr}^2 A_- - 4\det(A_-)}P_-),$$

if $\operatorname{tr}^2 A_\pm \geq 4\det(A_\pm)$ and undefined otherwise.

Thus, we have the following four cases:

1. If $\operatorname{tr}^2(A_\pm) = 4\det(A_\pm)$, then T_A fixes a quadratic fixed point.

2. If $\operatorname{tr}^2(A_\pm) = 4\det(A_\pm)$ and $\operatorname{tr}^2(A_\mp) > 4\det(A_\mp)$, then T_A fixes two distinct points.

3. If $\operatorname{tr}^2(A_\pm) > 4\det(A_\pm)$, then T_A fixes four distinct points.

4. If one of the $\operatorname{tr}^2(A_\pm)$ is less than 4, then T_A has no fixed points.

If $\operatorname{tr}^2 A_+ \geq 4\det(A_+)$ and A_- is the identity matrix, then T_A fixes infinite number of points.

Example 4.2.3. 1. Let $A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} P_+ + \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} P_-$. The trace square $\operatorname{tr}^2 A_\pm = 4$. T_A fixes only one point, which is $[0:1]$.

2. Let $A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} P_+ + \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} P_-$. The trace square $\operatorname{tr}^2 A_+ = 4$ and $\operatorname{tr}^2 A_- > 4$. T_A fixes two points, which are $[\sqrt{3}P_- : 1]$ and $[-\sqrt{3}P_- : 1]$.

3. Let $A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} P_+ + \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} P_-$. The trace square $\operatorname{tr}^2 A_+ > 4$ and $\operatorname{tr}^2 A_- = 4$. T_A fixes two points, which are $[\sqrt{3}P_+ : 1]$ and $[-\sqrt{3}P_+ : 1]$.

4. Let $A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} P_+ + \begin{pmatrix} -5 & -3 \\ 2 & 1 \end{pmatrix} P_-$. The trace square $\operatorname{tr}^2 A_\pm > 4$. T_A fixes four points, which are $[\sqrt{3}P_+ - (\frac{3}{2} + \frac{\sqrt{3}}{2})P_- : 1]$, $[\sqrt{3}P_+ - (\frac{3}{2} - \frac{\sqrt{3}}{2})P_- : 1]$, $[-\sqrt{3}P_+ - (\frac{3}{2} + \frac{\sqrt{3}}{2})P_- : 1]$ and $[-\sqrt{3}P_+ - (\frac{3}{2} - \frac{\sqrt{3}}{2})P_- : 1]$.

5. Let $A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} P_+ + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The $\operatorname{tr}^2 A_+ = 4$ and A_- is the identity matrix. T_A fixes every point in the set $\{[\lambda P_- : 1] : \lambda \in \mathbb{R}\}$.

The following Möbius transformation fix no point.

6. Let $A = \begin{pmatrix} \frac{1}{2} & -3 \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix} P_+ + \begin{pmatrix} 1 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix} P_-$. T_A fixes no point because $\operatorname{tr}^2 A_\mp < 4$.

4.2.4 Fixed points of a $\mathbb{D}$-Möbius transformation

Let T_A be a non-identity map in $\text{Möb}(\mathbb{D})$ with a representative matrix $A = A_1 + \epsilon A_2 = \begin{pmatrix} a_1 + \epsilon a_2 & b_1 + \epsilon b_2 \\ c_1 + \epsilon c_2 & d_1 + \epsilon d_2 \end{pmatrix} \in SL_2(\mathbb{D})$, where $A_1 \in SL_2(\mathbb{R})$ and A_1 is not the identity matrix.

Let $I = \operatorname{tr}^2 A - 4\det(A)$. Let $x = 2\operatorname{tr} A_1 \operatorname{tr} A_2 - 4\operatorname{tr}(A_1\widehat{A_2})$. Then, from Lemma

2.1.33,

$$\sqrt{I} = \begin{cases} \pm(\sqrt{\operatorname{tr}^2 A_1 - 4\det(A_1)} + \epsilon\dfrac{x}{2\sqrt{\operatorname{tr}^2 A_1 - 4\det(A_1)}}), & \text{if } \operatorname{tr}^2 A_1 > 4\det(A_1); \\ 0, & \text{if } \operatorname{tr}^2 A_1 = 4\det(A_1), \\ & \text{and } x = 0; \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Therefore, we have four cases:

1. If $\operatorname{tr}^2(A_1) < 4\det(A_1)$, then T_A fixes no point.
2. If $\operatorname{tr}^2(A_1) = 4\det(A_1)$ and $x \neq 0$, then T_A fixes no point.
3. If $\operatorname{tr}^2(A_1) = 4\det(A_1)$ and $x = 0$, then
   (a) T_A fixes a double fixed point if $a_1 \neq d_1$.
   (b) T_A fixes points which make vertical line if $a_1 = d_1$.
4. If $\operatorname{tr}^2(A_1) > 4\det(A_1)$ then, T_A fixes two distinct points.

Example 4.2.4. 1. $A = \begin{pmatrix} \frac{1}{2}+\epsilon & -3+\epsilon 2 \\ \frac{1}{4}+\epsilon & \frac{1}{2}+\epsilon \end{pmatrix} \in GL_2(\mathbb{D})$, $\operatorname{tr}^2 A_1 < 4\det(A_1)$. T_A fixes no point.

2. $A = \begin{pmatrix} 1+\epsilon & \epsilon \\ 1 & 1+\epsilon \end{pmatrix} \in GL_2(\mathbb{D})$, $\operatorname{tr}^2 A_1 = 4\det(A_1)$ and $x \neq 0$. T_A fixes no point.

3. $A = \begin{pmatrix} 1+\epsilon & 0 \\ 1-\epsilon & 1+\epsilon \end{pmatrix} \in GL_2(\mathbb{D})$, $\operatorname{tr}^2 A_1 = 4\det(A_1)$, $x = 0$ and $a_1 = d_1$. For all $r \in \mathbb{R}$, $[\epsilon r : 1]$ is a fixed point of T_A.

4. $A = \begin{pmatrix} 1 & -1 \\ \frac{1}{4} & 2 \end{pmatrix} \in GL_2(\mathbb{D})$, $\operatorname{tr}^2 A_1 = 4\det(A_1)$, $x = 0$ and $a_1 \neq d_1$. The only fixed point of T_A is $[-2 : 1]$.

5. $A = \begin{pmatrix} 2+\epsilon & 3+\epsilon \\ 1+\epsilon & 2+\epsilon \end{pmatrix} \in GL_2(\mathbb{D})$, $\operatorname{tr}^2 A_1 > 4\det(A_1)$. T_A fixes both $[\sqrt{3} - \epsilon\frac{1}{\sqrt{3}} : 1]$ and $[-\sqrt{3} + \epsilon\frac{1}{\sqrt{3}} : 1]$.

4.3 Attracting, Repelling, Non-attracting Non-repelling Fixed Points

Let $\mathbb{A}$ be one of $\mathbb{R}$, $\mathbb{C}$, $\mathbb{O}$, $\mathbb{D}$. This section focuses on the types of fixed points under a non-trivial Möbius map. So, we start with the following basic definitions.

Definition 4.3.1. Let (X, d) be a metric space and $f : (X, d) \to (X, d)$ be a function. Let $z_0 \in X$ such that $f(z_0) = z_0$. Let D be a neighbourhood of z_0 and $z \in D$.

1. If, for all $z \in D$, $d(f(z), z_0) < d(z, z_0)$, then z_0 is called an *attracting* fixed point.
2. If, for all $z \in D$, $d(f(z), z_0) > d(z, z_0)$, then z_0 is called a *repelling* fixed point.
3. If, for all $z \in D$, $d(f(z), z_0) = d(z, z_0)$, then z_0 is called a *center*.
4. If z_0 is none of the previous, then it is called an *indifferent* fixed point.

Theorem 4.3.2. Let X be a metric space. Let $f : X \to X$ be a function with attracting fixed point z_0. Then there is an $\epsilon > 0$ such that if $d(z, z_0) < \epsilon$, then $d(f^n(z), z_0) < \epsilon$ for all positive n, and moreover $d(f^n(z), z_0) \to 0$ as $n \to \infty$.

Proof. [21, Ch.5]. □

Theorem 4.3.3. Let X be a metric space. Let $f : X \to X$ be a function with repelling fixed point z_0. Then there is an $\epsilon > 0$ such that if $d(z, z_0) < \epsilon$, then there is a positive integer N such that $d(f^N(z), z_0) > \epsilon$.

Proof. [21, Ch.5]. □

Informally, for a repelling fixed point, an orbit with an initial condition starting even very close to z_0 will eventually need to move away from z_0. Note that the orbit does not have to go to infinity or anywhere in particular. It just has to move away from z_0.

Example 4.3.4. Let f be an $\mathbb{R}$-Möbuis map. The following hold:

1. If f is a parabolic, then the fixed point is an indifferent fixed point.

2. If f is a hyperbolic, then one of the fixed points is an attracting fixed point and the other is a repelling fixed point.

Definition 4.3.5. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable function such that $f(x_1, \dots, x_n) = (f_1(x_1, \dots, x_n), \dots, f_n(x_1, \dots, x_n))$. $J_z = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(z) & \dots & \frac{\partial f_1}{\partial x_n}(z) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(z) & \dots & \frac{\partial f_n}{\partial x_n}(z) \end{pmatrix}$ is called a Jacobian matrix of f. For a fixed point z_0, the Jacobian J_{z_0} determines its type.

Definition 4.3.6. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable function, and J_z its Jacobian matrix. Let z be a fixed point of f, and let J_z have two eigenvalues say $\lambda_1,\ \lambda_2$. Then, the fixed point z is one of the following types:

1. If $\lambda_{1,2}$ are two real numbers and $|\lambda_{1,2}| < 1$, then the fixed point z is an attracting fixed point.

2. If $\lambda_{1,2}$ are two real numbers and $|\lambda_{1,2}| > 1$, then the fixed point z is a repelling fixed point.

3. If $\lambda_{1,2}$ are two real numbers such that $|\lambda_1| > 1$ and $|\lambda_2| < 1$, then the fixed point z is a saddle fixed point.

4. If $\lambda_{1,2}$ are two real numbers such that $|\lambda_1| = 1$ and $|\lambda_2| < 1$, then the fixed point z is an attracting saddle fixed point.

5. If $\lambda_{1,2}$ are two real numbers such that $|\lambda_1| = 1$ and $|\lambda_2| > 1$, then the fixed point z is a repelling saddle fixed point.

6. If $\lambda_{1,2}$ are two real numbers and $|\lambda_{1,2}| = 1$, then the fixed point z is an indifferent saddle fixed point.

7. If $\lambda_{1,2}$ are two conjugate complex numbers and $|\lambda_{1,2}| < 1$, then the fixed point z is an attracting spiral fixed point.

8. If $\lambda_{1,2}$ are two conjugate complex numbers and $|\lambda_{1,2}| > 1$, then the fixed point z is a repelling spiral fixed point.

9. If $\lambda_{1,2}$ are two conjugate complex numbers and $|\lambda_{1,2}| = 1$, then the fixed point z is a centre.

See Figures 4.1-4.9.

Suppose $\lambda_1 = \lambda_2 = \lambda$. There are two possibilities:

1. There are two independent eigenvectors corresponding to λ.

2. There is only one eigenvector corresponding to λ.

If there are two eigenvectors, then they span the plane and so every vector is an eigenvector with the same eigenvalue λ, i.e. $Ax = \lambda x$. If $|\lambda| < 1$ ($|\lambda| > 1$, $|\lambda| = 1$) then the fixed point is an attracting (repelling, indifferent) fixed point. The other possibility is that there is only one eigenvector, for example, any matrix of the form $A = \begin{pmatrix} \lambda & 0 \\ c & \lambda \end{pmatrix}$, with $c \neq 0$. When there is only one eigenvector, the fixed point is an improper fixed point [61, Ch.6].

Definition 4.3.7. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable function, and J_z its Jacobian matrix. Let z be a fixed point of f, and let J_z have only one eigenvalue say λ and there is only one eigenvector corresponding to λ. Then, the fixed point z is one of the following types:

1. If $|\lambda| < 1$, then the fixed point z an attracting improper fixed point.

2. If $|\lambda| > 1$, then the fixed point z a repelling improper fixed point.

3. If $|\lambda| = 1$, then the fixed point z an indifferent improper fixed point.

See Figures 4.10-4.12.

Let $\mathbb{A}$ be one of $\mathbb{C}, \mathbb{O}, \mathbb{D}$. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{A})$.

Let $A = az + b$ and $B = cz + d$ be a unit. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable map defined by $f(z) = \frac{A}{B}$. Clearly $\frac{\partial A}{\partial x}(z) = a$, $\frac{\partial A}{\partial y}(z) = \iota a$, $\frac{\partial B}{\partial x}(z) = c$, and $\frac{\partial B}{\partial y}(z) = \iota c$. Then,

$$\frac{\partial f}{\partial x}(z) = \frac{B\frac{\partial A}{\partial x} - A\frac{\partial B}{\partial x}}{B^2} = \frac{\det M}{B^2} \text{ and } \frac{\partial f}{\partial y}(z) = \frac{B\frac{\partial A}{\partial y} - A\frac{\partial B}{\partial y}}{B^2} = \iota\frac{\det M}{B^2}.$$

For double number, the formula will be as follow:

$$\frac{\partial f}{\partial x}(z) = \frac{\det M}{B^2}P_+ \text{ and } \frac{\partial f}{\partial y}(z) = \frac{\det M}{B^2}P_-$$

when our technique is (P_+, P_-) decomposition. Therefore,

$$\frac{\partial f}{\partial x}(z) = \frac{\det M}{B^2} = \frac{\partial f_1}{\partial x}(z) + \iota\frac{\partial f_2}{\partial x}(z),$$

$$\frac{\partial f}{\partial y}(z) = \iota\frac{\det M}{B^2} = \frac{\partial f_2}{\partial x}(z)\iota^2 + \iota\frac{\partial f_1}{\partial x}(z).$$

Thus, Cauchy-Riemann equations for complex number is

$$\frac{\partial f_1}{\partial y}(z) = -\frac{\partial f_2}{\partial x}(z), \quad \frac{\partial f_2}{\partial y}(z) = \frac{\partial f_1}{\partial x}(z)$$

for double number is

$$\frac{\partial f_1}{\partial y}(z) = \frac{\partial f_2}{\partial x}(z), \quad \frac{\partial f_2}{\partial y}(z) = \frac{\partial f_1}{\partial x}(z)$$

or

$$\frac{\partial f_1}{\partial y}(z) = \frac{\partial f_2}{\partial x}(z) = 0$$

for dual number is

$$\frac{\partial f_1}{\partial y}(z) = 0, \quad \frac{\partial f_2}{\partial y}(z) = \frac{\partial f_1}{\partial x}(z)$$

The numbers $\mathbb{A}$ are a two-dimensional commutative algebra over $\mathbb{R}$ spanned by the multiplicative identity element 1 and another element ι. Thus, we view a Möbius map $\mathbb{A} \to \mathbb{A}$ as a function $\mathbb{R}^2 \to \mathbb{R}^2$. In particular, for complex number the Jacobian matrix is:

$$J_z = \begin{pmatrix} \frac{\partial f_1}{\partial x}(z) & \frac{\partial f_2}{\partial x}(z) \\ -\frac{\partial f_2}{\partial x}(z) & \frac{\partial f_1}{\partial x}(z) \end{pmatrix},$$

where $z \in \mathbb{R}^2$. Then:

1. If $\frac{\partial f_2}{\partial x}(z) = 0$, then J_z is a scaling.
2. If $\frac{\partial f_1}{\partial x}(z) = 0$, then J_z is a composition of scaling and x-axis reflection.
3. A rotation otherwise.

Clearly, $\lambda_{1,2} = \frac{\partial f_1}{\partial x}(z) \pm i\frac{\partial f_2}{\partial x}(z)$. Therefore, there are the following cases:

1. Let $\frac{\partial f_2}{\partial x}(z) = 0$. That is, $\lambda_1 = \lambda_2 = \lambda$ with two independent eigenvectors. If z is a fixed point of f and $|\lambda| < 1$, ($|\lambda| > 1$, $|\lambda| = 1$), then z is an attracting (repelling, indifferent) fixed point.
2. Let $\frac{\partial f_2}{\partial x}(z) \neq 0$. If z is a fixed point of f and $|\lambda_{1,2}| < 1$ ($|\lambda_{1,2}| > 1$, $|\lambda_{1,2}| = 1$), then z is an attracting spiral (repelling spiral, center) fixed point.

Example 4.3.8. 1. From Example 4.2.2, $[0 : 1]$ is an indifferent fixed point of T_A, where $A = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}$.

2. From Example 4.2.2, $[\frac{1}{2}(1+\sqrt{3}) : 1]$ is an attracting fixed point of T_A, where $A = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$.

3. From Example 4.2.2, $[\frac{1}{2}(1-\sqrt{3}) : 1]$ is a repelling fixed point of T_A, where $A = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$.

4. From Example 4.2.2, $[i\sqrt{3} : 1]$ is a center fixed point T_A, where $A = \begin{pmatrix} \frac{1}{2} & -\frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

Example 4.3.9. Let $A = \begin{pmatrix} \frac{1}{2} & -\frac{7}{4} \\ \frac{1}{2} & \frac{1}{4} \end{pmatrix}$. T_A fixes $[\frac{1}{4}(1 \pm i\sqrt{7}) : 1]$. Both $[\frac{1}{4}(1 \pm i\sqrt{7}) : 1]$ are spiral fixed points.

Let f be an $\mathbb{O}$-Möbius map. The Jacobian matrix of $f : \mathbb{R}^2 \to \mathbb{R}^2$ is

$$J_z = \begin{pmatrix} \frac{\partial f_1}{\partial x}(z) & 0 \\ 0 & \frac{\partial f_2}{\partial y}(z) \end{pmatrix},$$

where $z \in \mathbb{R}^2$. Obviously, J_z is a composition of x-scaling and y-scaling.

Clearly, $\lambda_1 = \frac{\partial f_1}{\partial x}(z)$ and $\lambda_2 = \frac{\partial f_2}{\partial y}(z)$. Therefore, there are the following cases: if z is a fixed point of f, then z is an attracting fixed point if $|\lambda_{1,2}| < 1$, a repelling fixed point if $|\lambda_{1,2}| > 1$, a saddle fixed point if $|\lambda_1| < 1$ and $|\lambda_2| > 1$, an attracting saddle fixed point $|\lambda_1| = 1$ and $|\lambda_2| < 1$, a repelling saddle fixed point if $|\lambda_1| = 1$ and $|\lambda_2| > 1$, and an indifferent saddle fixed point if $|\lambda_1| = |\lambda_2| = 1$.

Example 4.3.10.

1. From (1) in Example 4.2.3, $[0 : 1]$ is an indifferent saddle fixed point of T_A, where $A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} P_+ + \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} P_-$.

2. From (2) in Example 4.2.3, $[\sqrt{3}P_- : 1]$ is an attracting saddle fixed point of T_A, where $A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} P_+ + \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} P_-$.

3. From (4) in Example 4.2.3, $[\sqrt{3}P_+ - \frac{3+\sqrt{3}}{2}P_- : 1]$ is an attracting fixed point of T_A, where $A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} P_+ + \begin{pmatrix} -5 & -3 \\ 2 & 1 \end{pmatrix} P_-$.

Recall, the ring of double numbers $\mathbb{O}$ can be represented as two copies of the field of real numbers. If $z \in \mathbb{O}$ is a fixed point of a $\mathbb{O}$-Möbius map f, then the sets $z + \lambda P_+$ and $z + \lambda P_-$, where $\lambda \in \mathbb{R}$ are f-invariant. The restriction of the map f, $f|_{z+\lambda P_+} : z + \lambda P_+ \to z + \lambda P_+$ coincides with $f_+ : \mathbb{P}(\mathbb{R}) \to \mathbb{P}(\mathbb{R})$ in Möb($\mathbb{R}$).

From example 4.3.4, if f_+ coincides with a parabolic, then f_+ fixes only one point (indifferent fixed point). If f_+ coincides with a hyperbolic, then f_+ fixes two fixed points (an attracting and a repelling fixed point). In the same way, $f|_{z+\lambda P_-} : z + \lambda P_- \to z + \lambda P_-$ coincides with $f_- : \mathbb{P}(\mathbb{R}) \to \mathbb{P}(\mathbb{R})$ in Möb($\mathbb{R}$).

Therefore, the classification of the fixed point of f in Möb($\mathbb{O}$) is based on the classification of a pair of fixed points in Möb($\mathbb{R}$), as the next table shows:

	Repelling (R)	Attracting (A)	Indifferent (IN)
Repelling (R)	(R,R)	(R,A)	(R,IN)
Attracting (A)	(A,R)	(A,A)	(A,IN)
Indifferent (IN)	-	-	(IN,IN)

Table4.1: Type of fixed points of $f \in$ Möb($\mathbb{O}$)

In other words, there are the following types of fixed points:

1. If a fixed point of f is from the type (A,A), then it is an attracting fixed point.

2. If a fixed point of f is from the type (R,R), then it is a repelling fixed point.

3. If a fixed point of f is from the type (A,R) or (R,A), then it is a saddle fixed point.

4. If a fixed point of f is from the type (A,IN), then it is an attracting saddle fixed point.

5. If a fixed point of f is from the type (R,IN), then it is a repelling saddle fixed point.

6. If a fixed point of f is from the type (IN,IN), then it is an indifferent saddle fixed point.

Clearly, this is the same classification that has been derived from the real Jacobian.

Using again the Jacobian matrix, next shows the types of fixed points of f, where f is a $\mathbb{D}$-Möbius map. Clearly, the Jacobian matrix is

$$J_z = \begin{pmatrix} \frac{\partial f_1}{\partial x}(z) & 0 \\ \frac{\partial f_2}{\partial x}(z) & \frac{\partial f_1}{\partial x}(z) \end{pmatrix},$$

where $z \in \mathbb{R}^2$.

Then:

1. If $\frac{\partial f_2}{\partial x}(z) = 0$, then J_z is a scaling .

2. Otherwise, J_z is a composition of scaling and y-shear.

Clearly, $\lambda_{1,2} = \frac{\partial f_1}{\partial x}(z)$. That is, $\lambda_1 = \lambda_2 = \lambda$. Therefore, there are the following cases:

1. Let $\frac{\partial f_2}{\partial x}(z) = 0$. There are two independent eigenvectors corresponding to λ. If z is a fixed point of f and $|\lambda| < 1$, $(|\lambda| > 1,\ |\lambda| = 1)$, then z is an attracting (repelling, indifferent) fixed point .

2. Let $\frac{\partial f_2}{\partial x}(z) \neq 0$. There is only one eigenvector corresponding to λ. If z is a fixed point of f and $|\lambda_1| < 1$, $(|\lambda_1| > 1,\ |\lambda_1| = 1)$, then z is an attracting (repelling, indifferent) improper fixed point.

Example 4.3.11. 1. From Example 4, $[0 : 1]$ is an attracting fixed point of T_A, where $A = \begin{pmatrix} 1+\epsilon & 0 \\ 1-\epsilon & 2+\epsilon \end{pmatrix}$ $(\frac{\partial f_2}{\partial x}([0 : 1]) = 0)$.

2. From Example 5, $[\sqrt{3} - \epsilon\frac{1}{\sqrt{3}} : 1]$ is an attracting improper fixed point and $[-\sqrt{3} + \epsilon\frac{1}{\sqrt{3}} : 1]$ is a repelling improper fixed point of T_A, where $A = \begin{pmatrix} 2+\epsilon & 3+\epsilon \\ 1+\epsilon & 2+\epsilon \end{pmatrix}$ $(\frac{\partial f_2}{\partial x}([\sqrt{3} - \epsilon\frac{1}{\sqrt{3}} : 1]) \neq 0$ and $\frac{\partial f_2}{\partial x}([\sqrt{3} - \epsilon\frac{1}{\sqrt{3}} : 1]) \neq 0)$.

To sum up, we obtain the following fact:

1. The spiral fixed points appear only for complex Möbius maps.

2. The saddle fixed points appear only for double Möbius maps.

3. The improper fixed points appear only for dual Möbius maps.

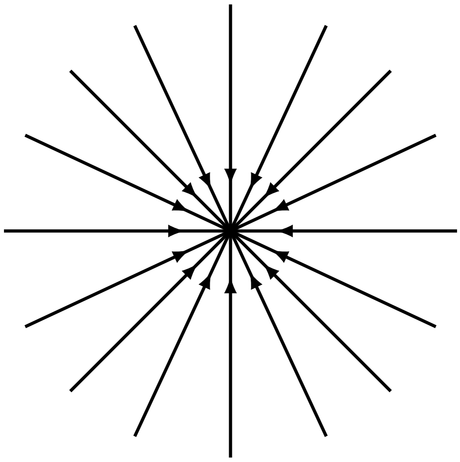

Figure4.1: An attracting fixed point.

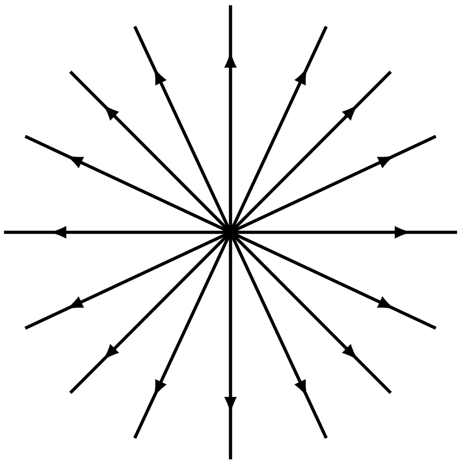

Figure4.2: A repelling fixed point.

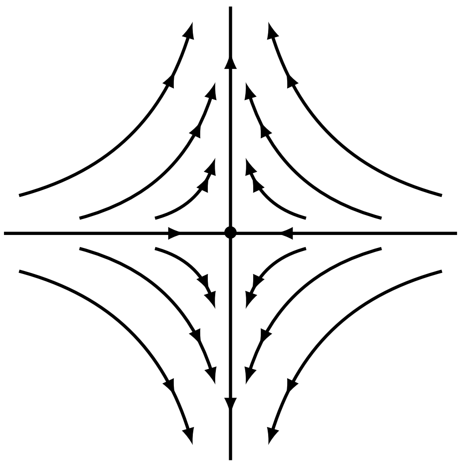

Figure4.3: A saddle fixed point.

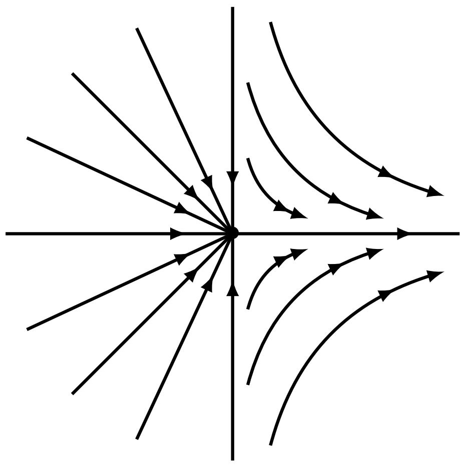

Figure4.4: An attracting saddle fixed point.

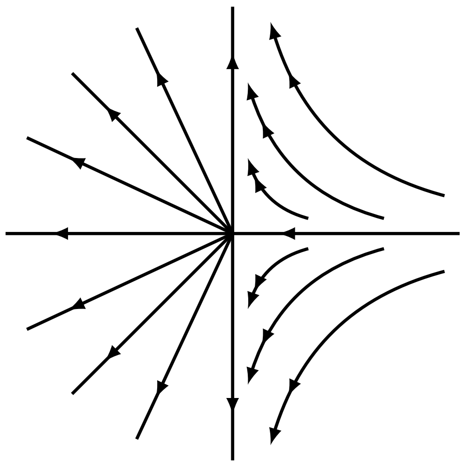

Figure4.5: A repelling saddle fixed point.

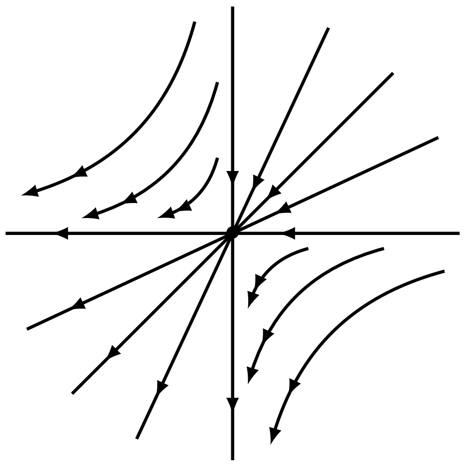

Figure4.6: An indifferent saddle fixed point.

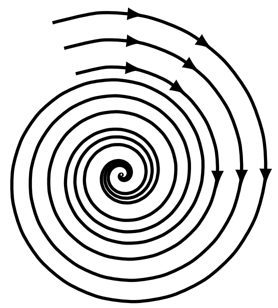

Figure4.7: An attracting spiral fixed point.

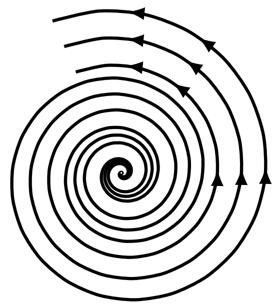

Figure4.8: A repelling spiral fixed point.

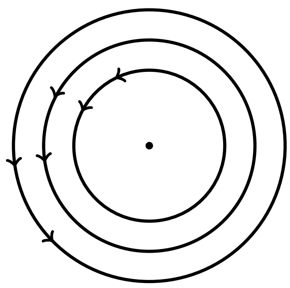

Figure4.9: A center fixed point.

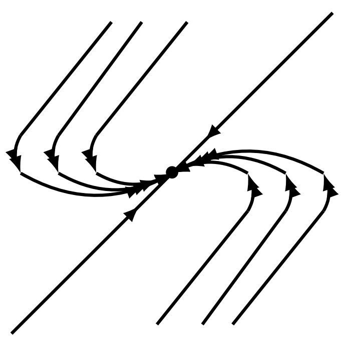

Figure4.10: An attracting improper fixed point.

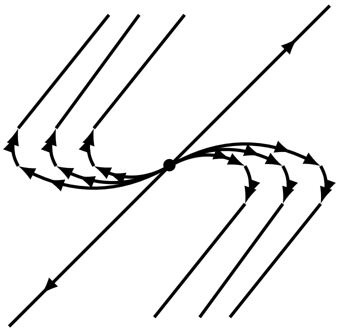

Figure4.11: A repelling improper fixed point.

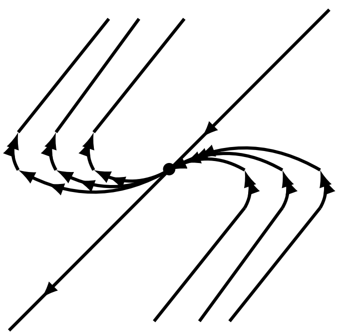

Figure4.12: An indifferent improper fixed point.

Chapter 5

Continuous One-parameter Subgroups

This chapter presents the different types of continuous one parameter subgroups of $GL_2(\mathbb{A})$ and $SL_2(\mathbb{A})$, up to similarity and rescaling. Section 1 presents several facts on fixed points and continuous one-parameter subgroups, which will be useful in the later sections. We consider continuous one-parameter subgroups for real Möbius maps in Section 2. The next two sections deal with double and dual numbers respectively. The well-known properties of Möbius maps on $\mathbb{C}$ or $\mathbb{R}$ is three transitivity. It is absent for $\mathbb{O}$ and $\mathbb{D}$. In the last section we introduce the concept of canonical triple of points, which can be viewed as an adjustment of three transitive for double and dual numbers.

5.1 Fixed Points and One-parameter Subgroups

Definition 5.1.1. A *continuous one-parameter group* is a continuous group homomorphism $\phi : \mathbb{R} \to G$, where G is a topological group and we have:

1. $\phi(t_1 + t_2) = \phi(t_1) \cdot \phi(t_2)$, for $t_1, t_2 \in \mathbb{R}$,

2. $\phi(0) = e$, where e is the identity element in G.

Here, we interested in $\phi(\mathbb{R})$ when $G = GL_2(\mathbb{A})$ or $SL_2(\mathbb{A})$, where $\mathbb{A} = \mathbb{O}$ or $\mathbb{D}$. Let g_t be a continuous one-parameter subgroup. Clearly, for any z, $z(t) = g_t z$ is a continuous function of t. Thus, the next result has a general nature.

Lemma 5.1.2. Let g_t be a continuous one-parameter group. If

1. for any $t \neq 0$, g_t is not the identity map,

2. there exists $t_0 \neq 0$, such that g_{t_0} fixes exactly k points $\{z_1, \cdots, z_k\}$,

then g_t fixes points z_i, $i = 1, \ldots, k$, for all t.

Proof. Let $\{z_1, \cdots, z_k\} \subset X$ be the set of all fixed points of $g_{t_0} : X \to X$, for $t_0 \neq 0$. Let us assume, for $t_1 \neq 0$, $g_{t_1} : X \to X$ does not fix z_0. From Assumption 1, let $z' = g_{t_1}(z_0)$. Since g_t is a subgroup, $g_{t_0}(z') = g_{t_0} g_{t_1}(z_0) = g_{t_1} g_{t_0}(z_0) = z'$. Therefore, from Assumption 2, $z' \in \{z_1, \cdots, z_k\}$. That is, for any t, g_t restricted to $\{z_1, \cdots, z_k\}$ is its permutation. So we have a homomorphism φ of a continuous group g_t to the discrete group of permutations S_k. A continuous map with a discrete range is constant. Since $\varphi(g_{t_0}) = I$, then $\varphi(g_t) = I$ for any t. □

Definition 5.1.3. Let $f,\ g$ be two maps from $\mathbb{A}$ to $\mathbb{A}$. We say f is similar to g if there exists an invertible map $h : \mathbb{A} \to \mathbb{A}$ such that $h \circ f = g \circ h$.

Proposition 5.1.4. Let $T_A,\ T_B$ be two real Möbius maps neither of which is the identity. T_A is similar to T_B if and only if $\operatorname{tr}^2 A = \operatorname{tr}^2 B$.

Proof. [6]. □

Definition 5.1.5. Let g_t be a continuous one-parameter subgroup, for $t \in \mathbb{R}$. If $a \neq 0$ in $\mathbb{R}$, then g_{at} is continuous one-parameter subgroup called the rescaling of g_t.

Recall that, from Section 2.2, there are more than one matrices in $GL_2(\mathbb{A})$ that define the same $\mathbb{A}$-Möbius map. The set of all matrices that define the identity (the trivial) map is $\{\lambda I : \lambda \text{ is units in } \mathbb{A}\}$. Any other map is called a non-trivial map. Here, we work on continuous one-parameter subgroups of $GL_2(\mathbb{A})$ and $SL_2(\mathbb{A})$, which are associated with a non-trivial $\mathbb{A}$-Möbius map, unless we say otherwise. Our classification will be done up to similarity and rescaling.

5.2 Continuous One-parameter Subgroups of $GL_2(\mathbb{R})$ and $SL_2(\mathbb{R})$

It is shown that there are only four different equivalent classes, up to similarity and rescaling, of continuous one-parameter subgroups of $SL_2(\mathbb{R})$ [54, Ch.3;57]. Therefore, there are only the following four types of continuous one-parameter subgroups of $GL_2(\mathbb{R})$, up to similarity and rescaling.

$$\hat{A}_\lambda = \left\{ e^{\lambda t} \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} : t \in \mathbb{R} \right\},$$

$$\hat{N}_\lambda = \left\{ e^{\lambda t} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} : t \in \mathbb{R} \right\},$$

$$\hat{K}_\lambda = \left\{ e^{\lambda t} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} : t \in \mathbb{R} \right\}$$

$$\text{and } \hat{I}_\lambda = \left\{ e^{\lambda} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

where λ is an arbitrary real constant. To give a unified form, which works for all continuous one-parameter subgroups of $GL_2(\mathbb{R})$, we use the extended set of values for assume that $\sigma \in \{-1, 0, 1, r\}$, where $\hat{H}_r$ means the trivial subgroups $\hat{I}_\lambda$, and introduce the following notation [54, Ch.9]:

1.
$$\cos_\sigma t = \begin{cases} \cos t, & \text{if } \sigma = -1; \\ 1, & \text{if } \sigma = 0; \\ \cosh t, & \text{if } \sigma = 1. \end{cases}$$

2.
$$\sin_\sigma t = \begin{cases} \sin t, & \text{if } \sigma = -1; \\ t, & \text{if } \sigma = 0; \\ \sinh t, & \text{if } \sigma = 1. \end{cases}$$

3. The two notation $\cos_\sigma t$ and $\sin_\sigma t$ give us a uniform presentation of different types of continuous one-parameter subgroups:
$$\hat{H}_{\lambda,\sigma} = \left\{ e^{\lambda t} \begin{pmatrix} \cos_\sigma t & \sigma \sin_\sigma t \\ \sin_\sigma t & \cos_\sigma t \end{pmatrix} : t \in \mathbb{R}, \sigma \in \{-1, 0, 1, r\} \right\}.$$
Therefore,
$$\hat{H}_{\lambda,\sigma} = \begin{cases} \hat{K}_\lambda, & \text{if } \sigma = -1; \\ \hat{N}_\lambda, & \text{if } \sigma = 0; \\ \hat{A}_\lambda, & \text{if } \sigma = 1; \\ \hat{I}_\lambda, & \text{if } \sigma = r. \end{cases} : t \in \mathbb{R}.$$

Clearly, there are the following three types of non-trivial continuous one-parameter subgroups of $SL_2(\mathbb{R})$, up to similarity and rescaling [54, Ch.3]:
$$H_\sigma = \left\{ \begin{pmatrix} \cos_\sigma t & \sigma \sin_\sigma t \\ \sin_\sigma t & \cos_\sigma t \end{pmatrix} : t \in \mathbb{R} \right\}$$

The two map $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(x) = \sin_\sigma t$ and $g(x) = \cos_\sigma t$ are continuous maps. So, $\phi : \mathbb{R} \to H_\sigma$ is a continuous map regarding to $d_{GL_2(\mathbb{R})}$ in Definition 4.1.10. Moreover, it is a continuous one-parameter group. Therefore,

$$H_\sigma = \begin{cases} K, & \text{if } \sigma = -1; \\ N, & \text{if } \sigma = 0; \\ A, & \text{if } \sigma = 1. \end{cases} \quad : t \in \mathbb{R}.$$

In other words, the continuous one-parameter subgroup K of $SL_2(\mathbb{R})$ is the set of $\mathbb{R}$-Möbius maps that fix no point, N is the set of $\mathbb{R}$-Möbius maps that fix only one point and A is the set of $\mathbb{R}$-Möbius maps that fix two distinct points. Hereafter, we use the notion $H_\sigma(t)$, for continuous one parameter subgroups of $GL_2(\mathbb{A})$ to highlight the parameter t and the scaling a.

5.3 Continuous One-parameter Subgroups of $GL_2(\mathbb{O})$ and $SL_2(\mathbb{O})$

We are going to classify all different types of continuous one-parameter subgroups in $GL_2(\mathbb{O})$ and $SL_2(\mathbb{O})$. Our main technique is (P_+, P_-) decomposition. As can be expected, a subgroup of $GL_2(\mathbb{O})$ is formed by a pair of subgroups of $GL_2(\mathbb{R})$, however, we need to take care of that.

Proposition 5.3.1. Let $B_+(t)$ and $B_-(t)$ be two one-parameter subsets of $GL_2(\mathbb{R})$ and $B(t) = B_+(t)P_+ + B_-(t)P_-$ be a corresponding one-parameter subset of $GL_2(\mathbb{O})$. $B(t)$ is a continuous one-parameter subgroup of $GL_2(\mathbb{O})$ if and only if both $B_+(t)$ and $B_-(t)$ are continuous one-parameter subgroups of $GL_2(\mathbb{R})$. Furthermore, $B(t)$ is a non-trivial if and only if at least one of $B_+(t)$ or $B_-(t)$ is non-trivial.

Proof. 1. For necessity. Let $B(t) = B_+(t)P_+ + B_-(t)P_-$ be a continuous one-parameter subgroup of $GL_2(\mathbb{O})$. Therefore, for any two real numbers t_1, t_2, $B(t_1 + t_2) = B(t_1)B(t_2)$. Therefore,

$$\begin{aligned} B_+(t_1)B_+(t_2)P_+ + B_-(t_1)B_-(t_2)P_- &= B_+(t_1+t_2)P_+ + B_-(t_1+t_2)P_-, \\ B(t_1)B(t_2) &= B_+(t_1+t_2)P_+ + B_-(t_1+t_2)P_-, \end{aligned}$$

which means $B_+(t_1)B_+(t_2) = B_+(t_1+t_2)$ and $B_-(t_1)B_-(t_2) = B_-(t_1+t_2)$. So, both $B_+(t)$ and $B_-(t)$ are continuous one-parameter subgroups.

2. For sufficiency. Let $B_+(t), B_-(t)$ be two continuous one-parameter subgroups of $GL_2(\mathbb{R})$. This means that, for any two real numbers t_1, t_2, $B_+(t_1)B_+(t_2) = B_+(t_1+t_2)$ and $B_-(t_1)B_-(t_2) = B_-(t_1+t_2)$. This leads to

$$\begin{aligned} B_+(t_1)B_+(t_2)P_+ + B_-(t_1)B_-(t_2)P_- &= B_+(t_1+t_2)P_+ + B_-(t_1+t_2)P_-, \\ B(t_1)B(t_2) &= B_+(t_1+t_2)P_+ + B_-(t_1+t_2)P_-, \end{aligned}$$

i.e. $B(t_1)B(t_2) = B(t_1+t_2)$. Thus, $B(t)$ is a continuous one-parameter subgroup of $GL_2(\mathbb{O})$.

Let $B(t) = B_+(t)P_+ + B_-(t)P_-$ be a non-trivial continuous one-parameter subgroup of $GL_2(\mathbb{O})$ and both $B_+(t), B_-(t)$ be two trivial continuous one-parameter subgroups of $GL_2(\mathbb{R})$. That means that, $B(t) = \hat{I}(t)P_+ + \hat{I}(t)P_-$. Thus, $B(t)$ is a trivial continuous one-parameter subgroup of $GL_2(\mathbb{O})$. This is a contradiction of our assumption. Thus at least one of $B_+(t)$ or $B_-(t)$ is non-trivial continuous one-parameter subgroup of $GL_2(\mathbb{R})$. The opposite statement is obvious as well: if at leat one of $B_+(t)$ or $B_-(t)$ is not trivial then $B(t) = B_+(t)P_+ + B_-(t)P_-$ is not trivial. $\square$

Similarity of subgroups is also formulated for the respective components.

Proposition 5.3.2. Let $B(t) = B_+(t)P_+ + B_-(t)P_-$ and $\tilde{B}(t) = \tilde{B}_+(t)P_+ + \tilde{B}_-(t)P_-$ be two continuous one-parameter subgroups of $GL_2(\mathbb{O})$. $B(t)$ is similar to $\tilde{B}(t)$ if and only if there exists $K_+, K_- \in GL_2(\mathbb{R})$ such that $K_+B_+(t)K_+^{-1} = \tilde{B}_+(t)$, and $K_-B_-(t)K_-^{-1} = \tilde{B}_-(t)$, in such a case

$$B(t) = (K_+(t)P_+ + K_-(t)P_-)\tilde{B}(t)(K_+^{-1}(t)P_+ + K_-^{-1}(t)P_-).$$

Proof. Let $B(t)$ be similar to $\tilde{B}(t)$. This means that there exists a $K = K_+P_+ + K_-P_- \in GL_2(\mathbb{O})$, where$K_+, K_- \in GL_2(\mathbb{R})$, such that $KB(t)K^{-1} = \tilde{B}(t)$. Thus,

$$K_+B_+(t)K_+^{-1}P_+ + K_-B_-(t)K_-^{-1}P_- = \tilde{B}_+(t)P_+ + \tilde{B}_-(t)P_-.$$

Therefore, there exists $K_+, K_- \in GL_2(\mathbb{R})$ such that $K_+B_+(t)K_+^{-1} = \tilde{B}_+(t)$ and $K_-B_-(t)K_-^{-1} = \tilde{B}_-(t)$. So, $B_+(t)$ is similar to $\tilde{B}_+(t)$ and $B_-(t)$ is similar to $\tilde{B}_-(t)$.

Conversely, let $B_+(t)$ be similar to $\tilde{B}_+(t)$ and $B_-(t)$ be similar to $\tilde{B}_-(t)$. That means, there exist two invertible $K_\pm \in GL_2(\mathbb{R})$ such that $K_+B_+(t)K_+^{-1} = \tilde{B}_+(t)$ and $K_-B_-(t)K_-^{-1} = \tilde{B}_-(t)$. Obviously, $K = K_+P_+ + K_-P_- \in GL_2(\mathbb{O})$ and $K^{-1} = K_+^{-1}P_+ + K_-^{-1}P_-$. Therefore,

$$\begin{aligned} KB(t)K^{-1} &= (C_+P_+ + C_-P_-)(B_+(t)P_+ + B_-(t)P_-)(C_+^{-1}P_+ + C_-^{-1}P_-) \\ &= \tilde{B}_+(t)P_+ + \tilde{B}_-(t)P_-. \end{aligned}$$

Thus, $B(t)$ is similar to $\tilde{B}(t)$. □

It is easy to show that the function $f : GL_2(\mathbb{O}) \to GL_2(\mathbb{O})$, which is defined by

$$f(X_+(t)P_+ + X_-(at)P_-) = X_-(at)P_+ + X_+(t)P_-,$$

is a group homomorphism. If we replace $GL_2(\mathbb{O})$ with $SL_2(\mathbb{O})$, the Propositions 5.3.1 and 5.3.2 remain valid.

Theorem 5.3.3. Any continuous one-parameter subgroups of $GL_2(\mathbb{O})$ has, up to similarity and rescaling, the following form

$$H_+(t)P_+ + H_-(at)P_-,$$

where $H_\pm$ are a subgroup similar to $\hat{H}_{\sigma_\pm}$, for $\sigma_\pm \in \{-1, 0, 1, r\}$.

Proof. 1. If $B(t)$ is a trivial continuous one-parameter subgroup then $B(t) = \hat{I}P_+ + \hat{I}P_-$.

2. Let $B(t)$ be a non-trivial continuous one-parameter subgroup. Then either $B_+(t)$ or $B_-(t)$ does not equal to $\hat{I}$.

   (a) If $B_+(t) \neq \hat{I}$, then up to scaling it is similar to $\hat{H}_{\sigma_+}(t)$, where $\sigma \in \{-1, 0, 1\}$. Then $B_-(t)$ is either $\hat{I}$ or, up to rescaling with $a \neq 0$ and similarity, $\hat{H}_{\sigma_-}(t)$, where $\sigma \in \{-1, 0, 1\}$.

   (b) The case of $B_-(t) \neq \hat{I}$ is treated in the same way.

□

Corollary 5.3.4. Any continuous one-parameter subgroups of $SL_2(\mathbb{O})$ has, up to similarity and rescaling, the following form

$$H_+(t)P_+ + H_-(at)P_-,$$

where $H_\pm$ are a subgroup similar to $H_{\sigma_\pm}$, for $\sigma_\pm \in \{-1, 0, 1, r\}$.

Proof. 1. If $B(t)$ is a trivial continuous one-parameter subgroup then $B(t) = IP_+ + IP_-$.

2. Let $B(t)$ be a non-trivial continuous one-parameter subgroup. Then either $B_+(t)$ or $B_-(t)$ does not equal to I.

   (a) If $B_+(t) \neq I$, then up to scaling it is similar to $H_{\sigma_+}(t)$, where $\sigma \in \{-1, 0, 1\}$. Then $B_-(t)$ is either I or, up to rescaling with $a \neq 0$ and similarity, $H_{\sigma_-}(t)$, where $\sigma \in \{-1, 0, 1\}$.

(b) The case of $B_-(t) \neq I$ is treated in the same way.

$\square$

Clearly, we do not lose any interesting types of continuous one-parameter subgroups when we move from $GL_2(\mathbb{O})$ to $SL_2(\mathbb{O})$.

Obviously, the continuous one-parameter subgroup $H_{\sigma_+}(t)P_+ + H_{\sigma_-}(t)P_-$ of $SL_2(\mathbb{O})$ is the set of $\mathbb{O}$-Möbius maps which fix no point if one of $\sigma_{+,-}$ equal -1, the set of $\mathbb{O}$-Möbius maps which fix only one point if both $\sigma_{+,-}$ equal 0, the set of $\mathbb{O}$-Möbius maps which fix two distinct points if one of $\sigma_{+,-}$ equals 0 and the other equals 1, the set of $\mathbb{O}$-Möbius maps which fix four distinct points if both $\sigma_{+,-}$ equals 1. Moreover, if $\sigma \in \{0,1\}$, then $H_\sigma(t)P_+ + IP_-$ is the set of $\mathbb{O}$-Möbius maps which fix infinite number of points.

Let $f = [y_+P_+ + y_-P_- : 1]$ be an arbitrary non-fixed point in $\mathbb{P}(\mathbb{O})$. Next proposition gives the $H_{\sigma_+}(t_+)P_+ + H_{\sigma_-}(t_-)P_-$-orbit of f, where $t_- = at_+$. The following notation is one of the objects that is used in the proof and agrees with the two notations 1 and 2 in Section 1.

$$\tan_\sigma t = \begin{cases} \tan t, & \text{if } \sigma = -1; \\ t, & \text{if } \sigma = 0; \\ \tanh t, & \text{if } \sigma = 1. \end{cases}$$

Proposition 5.3.5. Let $B(t) = H_{\sigma_+}(t)P_+ + H_{\sigma_-}(at)P_-$ be a continuous one-parameter subgroup of $SL_2(\mathbb{O})$, where a is a non-zero real number. If a point $[u + jv : 1] \in \mathbb{P}(\mathbb{O})$ belongs to the $B(t)$-orbit of a point $[y_+P_+ + y_-P_- : 1] \in \mathbb{P}(\mathbb{O})$, then

$$\tan^{-1}_{\sigma_+} \frac{y_+ - (u+v)}{y_+(u+v) - \sigma_+} = \frac{1}{a} \tan^{-1}_{\sigma_-} \frac{y_- - (u-v)}{y_-(u-v) - \sigma_-}.$$

Proof. Let a be a non-zero real number.

$$B(t)[y_+P_+ + y_-P_- : 1] = \left[\frac{y_+ + \sigma_+ \tan_{\sigma_+}(t)}{y_+ \tan_{\sigma_+}(t) + 1} P_+ + \frac{y_- + \sigma_- \tan_{\sigma_-}(at)}{y_- \tan_{\sigma_-}(at) + 1} P_- : 1 \right].$$

Let us define,

$$u' = \frac{y_+ + \sigma_+ \tan_{\sigma_+}(t)}{y_+ \tan_{\sigma_+}(t) + 1}, \quad v' = \frac{y_- + \sigma_- \tan_{\sigma_-}(at)}{y_- \tan_{\sigma_-}(at) + 1}.$$

A simple calculation leads us to

$$t = \tan_{\sigma_+}^{-1} \frac{y_+ - u'}{u'y_+ - \sigma_+}, \quad at = \tan_{\sigma_+}^{-1} \frac{y_- - v'}{v'y_- - \sigma_-}.$$

$u' = u + v$ and $v' = u - v$. Thus, we obtain

$$\tan_{\sigma_+}^{-1} \frac{y_+ - (u+v)}{y_+(u+v) - \sigma_+} = \frac{1}{a} \tan_{\sigma_-}^{-1} \frac{y_- - (u-v)}{y_-(u-v) - \sigma_-}.$$

□

There are several cases which admit a simpler description.

Corollary 5.3.6. Let $B(t) = N(t)P_+ + N(at)P_-$ be a continuous one-parameter subgroup of $SL_2(\mathbb{O})$, where a is a non-zero real number. If a point $[u + jv : 1] \in \mathbb{P}(\mathbb{O})$ belongs to the $B(t)$-orbit of a point $[y_+P_+ + y_-P_- : 1] \in \mathbb{P}(\mathbb{O})$, then

$$u^2 - v^2 + \frac{(a-1)y_+y_-}{y_+ - ay_-}u - \frac{(a+1)y_+y_-}{y_+ - ay_-}v = 0.$$

Proof. The proof follows immediately from Proposition 5.3.5. □

If one of the components is the identity matrix, then the orbit of $[y_+P_+ + y_-P_- : 1] \in \mathbb{P}(\mathbb{O})$ is a line as we are going to see in the next proposition.

Proposition 5.3.7. Let $B(t) = H_\sigma(t)P_+ + IP_-$ be a continuous one-parameter subgroup of $GL_2(\mathbb{O})$, where a is a non-zero real number. If a point $[u + jv : 1] \in \mathbb{P}(\mathbb{O})$ belongs to the $B(t)$-orbit of a point $[y_+P_+ + y_-P_- : 1] \in \mathbb{P}(\mathbb{O})$, then

$$2v = u^2 - v^2 - y_-(u - v).$$

Proof. Clearly,

$$[B(t)[y_+P_+ + y_-P_- : 1] = \left[\frac{y_+ \cos_\sigma t + \sigma \sin_\sigma t}{y_+ \sin_\sigma t + \cos_\sigma t} P_+ + y_-P_- : 1\right].$$

Let us define

$$u' = \frac{y_+ \cos_\sigma t + \sigma \sin_\sigma t}{y_+ \sin_\sigma t + \cos_\sigma t}, \text{ which means that } \tan_\sigma t = \frac{y_+ - u'}{y_+ u' - \sigma}, \text{ and } v' = y_-.$$

Since $u' = u + v, \quad v' = u - v$, therefore

$$\frac{1}{v'} - \frac{1}{u'} = \frac{u' - v'}{u'v'} = \frac{2v}{u^2 - v^2},$$

and this obtains

$$\frac{2v}{u^2 - v^2} = \frac{u + v - y_-}{y_-(y_-(u+v)}.$$

A simple calculation drives us to

$$2v = u^2 - v^2 - y_-(u - v).$$

□

5.4 Continuous One-Parameter Subgroups of $GL_2(\mathbb{D})$ and $SL_2(\mathbb{D})$

This subsection shows that $GL_2(\mathbb{D})$ ($SL_2(\mathbb{D})$) has three types of continuous one-parameter subgroups.

Lemma 5.4.1. Let $A(t)$ be a continuous non-trivial one-parameter subgroup of $GL_2(\mathbb{R})$, and let $\sigma \in \{-1, 0, 1\}$ such that $A(t)$ similar and re-scalable to $\hat{H}_\sigma$. Let B be any constant matrix in $GL_2(\mathbb{R})$. Then, $A(s)B = BA(s)$, for some s such that $\sin_\sigma(s) \neq 0$, if and only if, for some s_0 and a non-zero real number λ,

$B = \lambda A(s_0)$. Therefore, B belongs to the centralizer of $A(t)$.

Proof. 1. For necessity.

Let $B = C^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} C \in GL_2(\mathbb{R})$. Let $A(s) = C^{-1}e^{\lambda s}\begin{pmatrix} \cos_\sigma s & \sigma \sin_\sigma s \\ \sin_\sigma s & \cos_\sigma s \end{pmatrix} C$ be a continuous one-parameter subgroup of $GL_2(\mathbb{R})$. Assume that $BA(s) = A(s)B$, i.e.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \cos_\sigma s & \sigma \sin_\sigma s \\ \sin_\sigma s & \cos_\sigma s \end{pmatrix} = \begin{pmatrix} \cos_\sigma s & \sigma \sin_\sigma s \\ \sin_\sigma s & \cos_\sigma s \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Therefore,

$$a \cos_\sigma s + b \sin_\sigma s = a \cos_\sigma s + c\sigma \sin_\sigma s, \tag{5.1}$$

$$a\sigma \sin_\sigma s + b \cos_\sigma s = b \cos_\sigma s + d\sigma \sin_\sigma s, \tag{5.2}$$

$$c \cos_\sigma s + d \sin_\sigma s = a \sin_\sigma s + c \cos_\sigma s, \tag{5.3}$$

$$c\sigma \sin_\sigma s + d \cos_\sigma s = b \sin_\sigma s + d \cos_\sigma s. \tag{5.4}$$

From equations (5.1) or equivalently (5.4), $c\sigma \sin_\sigma s = b \sin_\sigma s$ i.e. $b = c\sigma$. And, from (5.2) or equivalently (5.3), $d \sin_\sigma s = a \sin_\sigma s$ i.e. $a = d$. Therefore, $B = \begin{pmatrix} a & \sigma c \\ c & a \end{pmatrix}$. Therefore, $B = \lambda A(s_0)$, where $\tan_\sigma^{-1}(s_0) = \frac{c}{a}$. Because of $A(t)$ is a continuous one parameter subgroup of $GL_2(\mathbb{R})$, B is commuting with every element of $A(t)$. Thus, B belongs to the centralizer of $A(t)$ of $GL_2(\mathbb{R})$.

2. The demonstration of the sufficiency is straightforward.

$\square$

Theorem 5.4.2. Any continuous one-parameter subgroup of $GL_2(\mathbb{D})$ has, up to similarity and rescaling, the form $\widetilde{\hat{H}_\sigma}(t) = H(t) + \epsilon\lambda H(t+t_0)t$, where $t_0 \in \mathbb{R}$, $\lambda \in \mathbb{R}\backslash\{0\}$ and H is a subgroup similar to $\hat{H}_\sigma$, for $\sigma \in \{-1, 0, 1, r\}$.

Proof. Let $B(t) = B_1(t) + \epsilon B_2(t)$ be a continuous one-parameter subgroup in $GL_2(\mathbb{D})$. That means that $B(t) = e^{Bt}$ and $B'(0) = B$, where $B \in M_2(\mathbb{D})$. Let $B = B_1 + \epsilon B_2$, for some $B_1, B_2 \in M(\mathbb{R})$. Therefore, the continuous one-parameter subgroup $B(t) = B_1(t) + \epsilon t B_1(t) B_2$, where $B_1(t)$ is a continuous one-parameter subgroup of $GL_2(\mathbb{R})$ and B_2 is a constant matrix in $M_2(\mathbb{R})$. If $B_1(t)$ is a trivial continuous one-parameter subgroup of $GL_2(\mathbb{R})$, then $B(t) = \hat{I}$. If $B_1(t)$ is a non-trivial continuous one-parameter subgroup of of $GL_2(\mathbb{R})$, then $B(t)$ is similar to $\hat{H}_\sigma(t) + \epsilon\hat{H}_\sigma(t)B_2 t$ where $\sigma \in \{-1, 0, 1\}$. For any $s_0, s_1 \in \mathbb{R}$, $B(s_0) = \hat{H}_\sigma(s_0) + \epsilon\hat{H}_\sigma(s_0)B_2 s_0$, and $B(s_1) = \hat{H}_\sigma(s_1) + \epsilon\hat{H}_\sigma(s_1)B_2 s_1$ are two continuous one parameter subgroups of $GL_2(\mathbb{D})$. Then

$$\begin{aligned}
B(s_0)B(s_1) &= \hat{H}_\sigma(s_0)\hat{H}_\sigma(s_1) + \epsilon(\hat{H}_\sigma(s_0)\hat{H}_\sigma(s_1)B_2 s_1 + \hat{H}_\sigma(s_0)B_2 s_0 \hat{H}_\sigma(s_0)) \\
B(t+s) &= \hat{H}_\sigma(s_0+s_1) + \epsilon(\hat{H}_\sigma(s_0+s_1)(B_2 \cdot (s_0+s_1)).
\end{aligned}$$

Because of $B(t)$ is a non-trivial continuous one-parameter subgroup of $GL_2(\mathbb{D})$, $B(t)B(s) = B(t+s)$. This means that

$$B_1(t)B_1(s) = B_1(t+s)$$

and

$$\begin{aligned}
\hat{H}_\sigma(s_0+s_1)(B_2 \cdot (s_0+s_1)) &= \hat{H}_\sigma(s_0)\hat{H}_\sigma(s_1)(B_2 s_1) + \hat{H}_\sigma(s_0)(B_2 s_0)\hat{H}_\sigma(s_1)) \\
\hat{H}_\sigma(s_1)(B_2 \cdot (s_0+s_1)) &= \hat{H}_\sigma(s_1)(B_2 s_1) + (B_2 s_0)\hat{H}_\sigma(s_1) \\
\text{then} \quad \hat{H}_\sigma(s_1)B_2 s_0 &= B_2 s_0 \hat{H}_\sigma(s_1).
\end{aligned}$$

That means $B_2\hat{H}_\sigma(s_1) = \hat{H}_\sigma(s_1)B_2$. Therefore,by Lemma 5.4.1,there is $t_0 \in \mathbb{R}$ and $\lambda \in \mathbb{R}\backslash\{0\}$ such that $B_2 = \lambda\hat{H}_\sigma(t_0)$. Thus,

$$B_2(s_1) = \lambda\hat{H}_\sigma(s_1)\hat{H}_\sigma(t_0)s_1 \text{ and } B(s) = \hat{H}_\sigma(s_1) + \epsilon\lambda\hat{H}_\sigma(s_1)\hat{H}_\sigma(t_0)s_1 \in GL_2(\mathbb{D}).$$

□

From preceding, we obtain that, for all $t_0 \in \mathbb{R}$, there is the following types of continuous one-parameter subgroups of $GL_2(\mathbb{D})$:

$$\widetilde{\hat{H}_\sigma}(t) = \hat{H}_\sigma(t) + \epsilon\lambda\hat{H}_\sigma(t+t_0)t;$$

where $\hat{H}_\sigma(t)$ is similar to one of $\hat{A}(t)$, $\hat{N}(t)$, $\hat{K}(t)$ or $\hat{I}$ and $\lambda \in \mathbb{R}\backslash\{0\}$.

Corollary 5.4.3. Any non-trivial continuous one-parameter subgroup of $SL_2(\mathbb{D})$ has the following form:

$$\widetilde{H_\sigma}(t) = H_\sigma(t) + \epsilon\lambda te^{\lambda_1 t_0}(H_\sigma(t+t_0) - \cos_\sigma(2t+t_0)H_\sigma(t));$$

where $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 0, 1\}$.

Proof. Let $\widetilde{\hat{H}_\sigma}(t) = \hat{H}_\sigma(t) + \epsilon\lambda\hat{H}_\sigma(t+t_0)t = e^{\lambda_1 t}H_\sigma(t) + \epsilon\lambda te^{\lambda_1(t+t_0)}H_\sigma(t+t_0)$ be a non-trivial continuous one-parameter subgroup of $GL_2(\mathbb{D})$. Clearly,

$$\begin{aligned}
\det(\widetilde{\hat{H}_\sigma}(t)) &= e^{2\lambda_1 t} + \epsilon\lambda 2te^{\lambda_1(2t+t_0)}\cos_\sigma(2t+t_0); \\
\sqrt{\det(\widetilde{\hat{H}_\sigma}(t))} &= \pm e^{\lambda_1 t} \pm \epsilon\lambda te^{\lambda_1(t+t_0)}\cos_\sigma(2t+t_0); \\
\frac{1}{\sqrt{\det(\widetilde{\hat{H}_\sigma}(t))}} &= \pm e^{-\lambda_1 t} \mp \epsilon\lambda te^{\lambda_1(t_0-t)}\cos_\sigma(2t+t_0); \\
\frac{1}{\sqrt{\det(\widetilde{\hat{H}_\sigma}(t))}}\widetilde{\hat{H}_\sigma}(t) &= H_\sigma(t) + \epsilon\lambda te^{\lambda_1 t_0}(H_\sigma(t+t_0) - \cos_\sigma(2t+t_0)H_\sigma(t)).
\end{aligned}$$

□

Obviously, the non-trivial continuous one-parameter subgroup

$$\widetilde{H_\sigma}(t) = H_\sigma(t) + \epsilon\lambda t e^{\lambda_1 t_0}(H_\sigma(t+t_0) - \cos_\sigma(2t+t_0)H_\sigma(t));$$

is the set of $\mathbb{D}$-Möbius maps which fix no point if $\sigma = -1$, the set of $\mathbb{D}$-Möbius maps which fix only one point if $\sigma = 0$ and the set of $\mathbb{D}$-Möbius maps which fix two distinct points if $\sigma = 1$. The non-trivial continuous one-parameter subgroup $\widetilde{H_\sigma}(t)$ is the set of $\mathbb{D}$-Möbius maps which fix infinite points if $\sigma = 0$ and they are not $\mathbb{R}$-Möbius maps.

Next proposition gives the sufficient conditions for the similarity between two continuous one-parameter subgroups in $GL_2(\mathbb{D})$.

Proposition 5.4.4. Let $B_1(t)$, $\hat{B}_1(t)$ be two continuous one-parameter subgroups of $GL_2(\mathbb{R})$. Then, $B(t) = B_1(t)+\epsilon\lambda B_1(t+t_0)t$, $\hat{B}(t) = \hat{B}_1(t)+\epsilon\lambda\hat{B}_1(t+t_0)t$ are two continuous one-parameter subgroups of $GL_2(\mathbb{D})$. $\hat{B}(t)$ is similar to $B(t)$ if and only if there exists an invertible $C \in GL_2(\mathbb{R})$ such that $\hat{B}_1(t) = CB_1(t)C^{-1}$.

Proof. Let $\hat{B}(t)$ be similar to $B(t)$. This means that there exists an invertible $M = C + \epsilon D \in GL_2(\mathbb{D})$ such that $\hat{B}(t) = MB(t)M^{-1}$. This leads to

$$\begin{aligned}
\hat{B}_1(t) + \epsilon\lambda\hat{B}_1(t+t_0)t &= (C+\epsilon D)(B_1(t) + \epsilon\lambda B_1(t+t_0)t)(C^{-1} - \epsilon C^{-1}DC^{-1}) \\
&= (CB_1(t) + \epsilon(DB_1(t) + C\lambda B_1(t+t_0)t))(C^{-1} - \epsilon C^{-1}DC^{-1}) \\
&= CB_1(t)C^{-1} + \epsilon(DB_1(t)C^{-1} + C\lambda B_1(t+t_0)tC^{-1} - \\
&\quad DB_1(t)C^{-1}DC^{-1})
\end{aligned}$$

A simple calculation shows that $\hat{B}_1(t) = CB_1(t)C^{-1}$.

Conversely, let $\hat{B}_1(t) = CB_1(t)C^{-1}$. Therefore, $\hat{B}_1(t+t_0) = CB_1(t+t_0)C^{-1}$.

$$\begin{aligned}
\hat{B}(t) = \hat{B}_1(t) + \epsilon\lambda\hat{B}_1(t+t_0)t &= CB_1(t)C^{-1} + \epsilon C\lambda B_1(t+t_0)C^{-1}, \\
&= C(B_1(t) + \epsilon\lambda B_1(t+t_0)t)C^{-1}, \\
&= CB(t)C^{-1}.
\end{aligned}$$

$\square$

Recall that Proposition 5.3.5 gives the formula of $H_{\sigma_+}(t_+)P_+ + H_{\sigma_-}(t_-)P_-$-orbits. Similarly, the next proposition shows the formula of $\widetilde{H_\sigma}(t)$-orbits.

Proposition 5.4.5. Let $\widetilde{H_\sigma}(t)$ be a continuous one-parameter subgroup of $SL_2(\mathbb{D})$. If $[u+\epsilon v : 1]$ belongs to the $\widetilde{H_\sigma}(t)$-orbits of $[a+\epsilon b : 1]$, then

$$\begin{aligned} v &- \lambda e^{\lambda_1 t_0} \tan_\sigma^{-1}\left(\frac{a-u}{au-\sigma}\right)\left(\frac{(b-a)(u^2-\sigma)}{(a^2-\sigma)} - a^2\left(\left(\frac{(u^2+\sigma)(a^2+\sigma)-4\sigma au}{(a^2-\sigma)^2}\cos_\sigma t_0\right.\right.\right. \\ &+ 2\sigma\frac{(a-u)(au-\sigma)}{(a^2-\sigma)^2}\sin_\sigma t_0) - \sigma(2\frac{(a-u)(au-\sigma)^3}{(u^2-\sigma)(a^2-\sigma)^3}\cos_\sigma t_0 \\ &+ \frac{((u^2+\sigma)(a^2+\sigma)-4\sigma au)(au-\sigma)^2}{(u^2-\sigma)(a^2-\sigma)^3}\sin_\sigma t_0) \\ &\quad (\frac{((u^2+\sigma)(a^2+\sigma)-4\sigma au)(au-\sigma)^2}{(u^2-\sigma)(a^2-\sigma)^3}\cos_\sigma t_0 + 2\sigma\frac{(a-u)(au-\sigma)^3}{(u^2-\sigma)(a^2-\sigma)^3}\sin_\sigma t_0) = 0. \end{aligned}$$

Proof. For an arbitrary non-zero point $t \in \mathbb{R}$ and $t_0 \in \mathbb{R}$,

$$\widetilde{H_\sigma}(t)f = \left[\frac{a\cos_\sigma t + \sigma \sin_\sigma t}{a\sin_\sigma t + \cos_\sigma t} + \epsilon\lambda t e^{\lambda_1 t_0}\frac{b-a-a^2\cos_\sigma(2t+t_0)\sin t_0 - \frac{\sigma}{2}\sin(4t+2t_0))}{(a\sin_\sigma t+\cos_\sigma t)^2} : 1\right].$$

Let us define,

$$u = \frac{a\cos_\sigma t + \sigma\sin_\sigma t}{a\sin_\sigma t + \cos_\sigma t} = \frac{a+\sigma\tan_\sigma t}{a\tan_\sigma t + 1}.$$

A simple calculation gives $\tan_\sigma t = \frac{a-u}{au-\sigma}$.

$$\begin{aligned}
v &= \lambda t e^{\lambda_1 t_0} \frac{b - a - a^2 \cos_\sigma(2t+t_0)\sin t_0 - \frac{\sigma}{2}\sin(2(2t+t_0))}{(a\sin_\sigma t + \cos_\sigma t)^2} \\
&= \frac{\lambda t e^{\lambda_1 t_0}}{(a\sin_\sigma t + \cos_\sigma t)^2}(b - a - a^2((\cos^2_\sigma t + \sigma\sin^2_\sigma t)\cos_\sigma t_0 + 2\sigma\sin_\sigma t\cos_\sigma t\sin_\sigma t_0) \\
&\quad -\sigma((2\sin_\sigma t\cos_\sigma t\cos_\sigma t_0 + (\cos^2_\sigma t + \sigma\sin^2_\sigma t)\sin_\sigma t_0)((\cos^2_\sigma t + \sigma\sin^2_\sigma t)\cos_\sigma t_0 \\
&\quad +2\sigma\sin_\sigma t\cos_\sigma t\sin_\sigma t_0) \\
&= \frac{\lambda t e^{\lambda_1 t_0}}{(1-\sigma\tan^2_\sigma t)(a\tan_\sigma t + 1)^2}((b-a)(1-\sigma\tan^2_\sigma t)^2 - a^2((1+\sigma\tan^2_\sigma t)(1-\sigma\tan^2_\sigma t) \\
&\quad \cos_\sigma t_0 + 2\sigma\tan_\sigma t(1-\sigma\tan^2_\sigma t)\sin_\sigma t_0) - \sigma((2\tan_\sigma t\cos_\sigma t_0 + (1+\sigma\tan^2_\sigma t)\sin_\sigma t_0) \\
&\quad ((1+\sigma\tan^2_\sigma t)\cos_\sigma t_0 + 2\sigma\tan_\sigma t\sin_\sigma t_0).
\end{aligned}$$

Substituting $\tan_\sigma t$ in v with simple calculations obtains the following result.

$$\begin{aligned}
v = \; & \frac{\lambda\tan^{-1}_\sigma \frac{a-u}{au-\sigma} e^{\lambda_1 t_0}(au-\sigma)^4}{(u^2-\sigma)(a^2-\sigma)^3}((b-a)\frac{(u^2-\sigma)^2(a^2-\sigma)^2}{(au-\sigma)^4} - a^2(\\
& (\frac{((u^2+\sigma)(a^2+\sigma)-4\sigma au)(u^2-\sigma)(a^2-\sigma)}{(au-\sigma)^4}\cos_\sigma t_0 + 2\sigma\frac{(a-u)(u^2-\sigma)(a^2-\sigma)}{(au-\sigma)^3} \\
& \sin_\sigma t_0) - \sigma(2\frac{a-u}{au-\sigma}\cos_\sigma t_0 + \frac{(u^2+\sigma)(a^2+\sigma)-4\sigma au}{(au-\sigma)^2}\sin_\sigma t_0) \\
& (\frac{(u^2+\sigma)(a^2+\sigma)-4\sigma au}{(au-\sigma)^2}\cos_\sigma t_0 + 2\sigma\frac{a-u}{au-\sigma}\sin_\sigma t_0) \\
= \; & \lambda e^{\lambda_1 t_0}\tan^{-1}_\sigma(\frac{a-u}{au-\sigma})(\frac{(b-a)(u^2-\sigma)}{(a^2-\sigma)} - a^2((\frac{(u^2+\sigma)(a^2+\sigma)-4\sigma au}{(a^2-\sigma)^2}\cos_\sigma t_0 \\
& +2\sigma\frac{(a-u)(au-\sigma)}{(a^2-\sigma)^2}\sin_\sigma t_0) - \sigma(2\frac{(a-u)(au-\sigma)^3}{(u^2-\sigma)(a^2-\sigma)^3}\cos_\sigma t_0 \\
& +\frac{((u^2+\sigma)(a^2+\sigma)-4\sigma au)(au-\sigma)^2}{(u^2-\sigma)(a^2-\sigma)^3}\sin_\sigma t_0) \\
& (\frac{((u^2+\sigma)(a^2+\sigma)-4\sigma au)(au-\sigma)^2}{(u^2-\sigma)(a^2-\sigma)^3}\cos_\sigma t_0 + 2\sigma\frac{(a-u)(au-\sigma)^3}{(u^2-\sigma)(a^2-\sigma)^3}\sin_\sigma t_0).
\end{aligned}$$

□

5.5 Canonical Triples of Points

The canonical triple of points in $\mathbb{P}(\mathbb{C})$ is $(0, 1, \infty)$. Namely, for each triple of points $Z = (z_1, z_2, z_3)$ there is a Möbius map M such that $MZ = (0, 1, \infty)$. In this section, we investigate several types of canonical triples of points in $\mathbb{P}(\mathbb{A})$, where $\mathbb{A}$ is $\mathbb{O}$ or $\mathbb{D}$.

Definition 5.5.1. A symplectic form $w : \mathbb{A}^2 \times \mathbb{A}^2 \to \mathbb{A}$ is defined by $w(z_1, z_2) = x_1 y_2 - y_1 x_2$, where $z_i = (x_i, y_i) \in \mathbb{A}^2$.

Proposition 5.5.2. Let $\mathbb{A}$ be one of $\mathbb{C}, \mathbb{O}, \mathbb{D}$ and $z_1 = [x_1 : y_1], z_2[x_2 : y_2]$ be two points in $\mathbb{P}(\mathbb{A})$. $z_1 = z_2$ if and only if $w((x_1, y_1), (x_2, y_2)) = 0$.

Proof. If $z_1 = z_2$, the there exists $\lambda \in \mathbb{A}$ such that $x_2 = \lambda x_1$ and $y_2 = \lambda y_1$. Therefore, $w((x_1, y_1), (x_2, y_2)) = x_1 y_2 - y_1 x_2 = x_1 \lambda y_1 - y_1 \lambda x_1 = 0$.

Conversely, If $w((x_1, y_1), (x_2, y_2)) = 0$, then $x_1 y_2 = y_1 x_2$. If $\mathbb{A} = \mathbb{R}, \mathbb{C}$ or $\mathbb{D}$, then either x_1 or y_1 is a unit. If y_1 is a unit, then $u = \frac{y_2}{y_1}$ is a unit too, otherwise $z_2 \notin \mathbb{P}(\mathbb{A})$. Obviously, $x_2 = u x_1$ and $y_2 = u y_1$. Thus, $z_1 = z_2$. In the same way if x_1 is a unit, then $u = \frac{x_2}{x_1}$. Let $\mathbb{A} = \mathbb{O}$, $z_1 = [P_+ : P_-]$ and $x_1 y_2 = y_1 x_2$. Then $y_2 P_+ = x_2 P_-$. This is valid if and only if $z_2 = [P_+ : P_-]$ i.e. $z_1 = z_2$. In the same way, if $z_1 = [P_- : P_+]$ and $w((x_1, y_1), (x_2, y_2)) = 0$, then $z_1 = z_2$. □

Definition 5.5.3. Let $z_1 = [x_1 : y_1], z_2 = [x_2 : y_2]$ be two different points in $\mathbb{P}(\mathbb{O})$. We say:

1. The two points z_1, z_2 make an arrangement of type 1, if $w((x_1, y_1), (x_2, y_2))$ is an invertible element in $\mathbb{O}$.

2. The two points z_1, z_2 make an arrangement of type P_+, if there exists a non-zero $\lambda \in \mathbb{R}$ such that $w((x_1, y_1), (x_2, y_2)) = \lambda P_+$.

3. The two points z_1, z_2 make an arrangement of type P_-, if there exists a non-zero $\lambda \in \mathbb{R}$ such that $w((x_1, y_1), (x_2, y_2)) = \lambda P_-$.

Definition 5.5.4. Let $z_1 = [x_1 : y_1], z_2 = [x_2 : y_2]$ be two different points in $\mathbb{P}(\mathbb{D})$, we say:

1. The two points z_1, z_2 make an arrangement of type 1, if $w((x_1, y_1), (x_2, y_2))$ is an invertible element in $\mathbb{D}$.

2. The two points z_1, z_2 make an arrangement of type ϵ, if there exists a non-zero $\lambda \in \mathbb{R}$ such that $w((x_1, y_1), (x_2, y_2)) = \lambda\epsilon$.

Next, we show that the type of arrangement is preserved under Möbius transformation.

Proposition 5.5.5. Let z_i be points in $\mathbb{P}(\mathbb{A})$. Let $A \in SL_2(\mathbb{A})$. Then, for all $i \neq k$ the types of arrangements of (z_i, z_k) and of (Az_i, Az_j) are the same.

Proof. Let $z_i = [x_i : y_i], z_j = [x_j : y_j]$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Clearly, $w((x_i, y_i), (x_j, y_j)) = x_i y_j - y_i x_j$.

$$\begin{aligned} & w((ax_i + by_i, cx_i + dy_i), (ax_j + by_j, cx_j + dy_j)) \\ = \; & (ax_i + by_i)(cx_j + by_j) - (ax_j + by_j)(cx_i + by_i) \\ = \; & (ad - bc)(x_i y_j - y_i x_j) = w((x_i, y_i), (x_j, y_j)). \end{aligned}$$

□

Proposition 5.5.6. Let $z_1 = [x_1 : y_1], z_2 = [x_2 : y_2]$ be two different points in $\mathbb{P}(\mathbb{O})$. Then

1. If z_1, z_2 make an arrangement of type 1, then there exists a an $A \in SL_2(\mathbb{O})$ such that $Az_1 = [1 : 0]$ and $Az_2 = [0 : 1]$.

2. If z_1, z_2 make an arrangement of type P_-, then there exists an $A \in SL_2(\mathbb{O})$ such that $Az_1 = [0 : 1]$ and $Az_2 = [P_- : P_+]$.

3. If z_1, z_2 make an arrangement of type P_+, then there exists an $A \in SL_2(\mathbb{O})$ such that $Az_1 = [0 : 1]$ and $Az_2 = [P_+ : P_-]$.

Proof. The required matrix for the first point is

$$A_1 = \begin{pmatrix} \lambda & -\vartheta \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{pmatrix}, \text{ where } \lambda = x_1y_2 - x_2y_1 \text{ and } \vartheta = x_1x_2 + y_1y_2.$$

The required matrix for the second point is $A_2 = \begin{pmatrix} 1 & 0 \\ \vartheta & P_+ + \lambda P_- \end{pmatrix} \begin{pmatrix} y_1 & -x_1 \\ x_1 & y_1 \end{pmatrix}$, where $\lambda P_- = x_1y_2 - x_2y_1$ and $\vartheta = x_1x_2 + y_1y_2$.

The required matrix for the third point is $A_3 = \begin{pmatrix} 1 & 0 \\ \vartheta & \lambda P_+ + P_- \end{pmatrix} \begin{pmatrix} y_1 & -x_1 \\ x_1 & y_1 \end{pmatrix}$, where $\lambda P_+ = x_1y_2 - x_2y_1$ and $\vartheta = x_1x_2 + y_1y_2$.

$\square$

Clearly, the above maps $A \in SL_2(\mathbb{O})$ is not unique. Any two such maps A_1 and A_2 define $A_1 \cdot A_2^{-1}$ which fixes the respective canonical pairs

$$(([1 : 0], [0, 1]), ([0, 1], [P_- : P_+]), ([0, 1], (P_+ : P_-])).$$

Proposition 5.5.7. Let $z_1 = [x_1 : y_1], z_2 = [x_2 : y_2]$ be two different points in $\mathbb{P}(\mathbb{D})$. Then

1. If z_1, z_2 make an arrangement of type 1, then there exists a an $A \in SL_2(\mathbb{D})$ such that $Az_1 = [1 : 0]$ and $Az_2 = [0 : 1]$.

2. If z_1, z_2 make an arrangement of type ϵ, then there exists an $A \in SL_2(\mathbb{D})$ such that $Az_1 = [0 : 1]$ and $Az_2 = [\epsilon : 1]$.

Proof. The required matrix for the first point is

$$A_1 = \begin{pmatrix} \lambda & -\vartheta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{pmatrix}, \text{ where } \lambda = x_1y_2 - x_2y_1 \text{ and } \vartheta = x_1x_2 + y_1y_2.$$

The required matrix for the second point is $A_2 = \begin{pmatrix} -\frac{\vartheta}{\lambda} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 & -x_1 \\ x_1 & y_1 \end{pmatrix}$, where $\lambda\epsilon = x_1y_2 - x_2y_1$ and $\vartheta = x_1x_2 + y_1y_2$.

□

Clearly, the above maps $A \in SL_2(\mathbb{D})$ is not unique. Any two such maps A_1 and A_2 define $A_1 \cdot A_2^{-1}$ which fixes the respective canonical pairs

$$(([1:0],[0,1]),([0,1],[\epsilon:1])).$$

The triple set $Z = (z_1, z_2, z_3)$ of the type 111 if $w(z_i, z_j)$ is a unit for all $i \neq j$. The next proposition proves that if $Z = (z_1, z_2, z_3)$ of the type 111, then there exists a unique $A \in SL_2(\mathbb{A})$ such that $W = AZ$ is of the type 111 too [11].

Proposition 5.5.8. Let z_1, z_2, z_3 be three pair-wise different points in $\mathbb{P}(\mathbb{A})$, such that any two of them make an arrangement of type 1. Let w_1, w_2, w_3 be another three different points in $\mathbb{P}(\mathbb{A})$. Then, there exists a unique $A \in SL_2(\mathbb{A})$ such that $Az_i = w_i$ if and only if any two points from w_1, w_2, w_3 make an arrangement of type 1.

5.5.1 Types of canonical triples of points in $\mathbb{P}(\mathbb{O})$

This subsection shows that there are six different canonical triples of points in $\mathbb{P}(\mathbb{O})$.

Definition 5.5.9. Let $\{z_1 = [x_1 : y_1], z_2 = [x_2 : y_2], z_3 = [x_3 : y_3]\}$ be a set of three pair-wise different points in $\mathbb{P}(\mathbb{O})$. We say that (z_1, z_2, z_3) is:

1. of the type 111, if $w((x_1, y_1), (x_2, y_2)) = \lambda_1$, $w((x_2, y_2), (x_3, y_3)) = \lambda_2$, $w((x_3, y_3), (x_1, y_1)) = \lambda_3$,
2. of the type $11P_+$, if $w((x_1, y_1), (x_2, y_2)) = \lambda_1$ $w((x_2, y_2), (x_3, y_3)) = \lambda_2$, $w((x_3, y_3), (x_1, y_1)) = \lambda_3 P_+$,

3. of the type $11P_-$, if $w((x_1,y_1),(x_2,y_2)) = \lambda_1$, $w((x_2,y_2),(x_3,y_3)) = \lambda_2$, $w((x_3,y_3),(x_1,y_1)) = \lambda_3 P_-$,

4. of the type $1P_+P_-$, if $w((x_1,y_1),(x_2,y_2)) = \lambda_1$, $w((x_2,y_2),(x_3,y_3)) = \lambda_2 P_+$, $w((x_3,y_3),(x_1,y_1)) = \lambda_3 P_-$,

5. of the type $P_+P_+P_+$, if $w((x_1,y_1),(x_2,y_2)) = \lambda_1 P_+$, $w((x_2,y_2),(x_3,y_3)) = \lambda_2 P_+$, $w((x_3,y_3),(x_1,y_1)) = \lambda_3 P_+$,

6. of the type $P_-P_-P_-$, if $w((x_1,y_1),(x_2,y_2)) = \lambda_1 P_-$, $w((x_2,y_2),(x_3,y_3)) = \lambda_2 P_-$, $w((x_3,y_3),(x_1,y_1)) = \lambda_3 P_-$,

where $\lambda_{1,2,3}$ are units in $\mathbb{O}$.

Permutations of previous types are defined in the same way. Clearly, the above six types of points are a disjoint sets. The following examples shows that every type is a non-empty set.

Example 5.5.10. We are calling the following triples as a canonical triples of the respective type:

1. $(0,1,\infty)$ is of the type 111.

2. $(\infty,0,\sigma_1)$ is of the type $1P_+P_-$, where $\sigma_1 = [P_+ : P_-]$.

3. $(\infty,0,\omega_1)$ is of the type $11P_-$, where $\omega_1 = [1 : P_-]$.

4. $(\infty,0,\omega_2)$ is of the type $11P_+$, where $\omega_2 = [1 : P_+]$.

5. $(0,P_+,\sigma_1)$ is of the type $P_+P_+P_+$.

6. $(0,P_-,\sigma_2)$ is of the type $P_-P_-P_-$, where $\sigma_2 = [P_- : P_+]$.

Theorem 5.5.11. Let $\{z_1,z_2,z_3\}$ be a set of three pair-wise different points in $\mathbb{P}(\mathbb{O})$. Then, the triple (z_1,z_2,z_3) is only of the one of six types from Definition 5.5.9.

Proof. To demonstrate the Theorem we only need to show that there is no triples of types $P_+P_+P_-, P_+P_-P_-, 1P_+P_+, 1P_-P_-$ which are together with the defined six types make all possible arrangements (combinations of letters $1, P_+, P_-$), up to permutation.

Let $z_1 = [x_i : y_i]$, where $x_i, y_i \in \mathbb{O}$ such that $x_i = r_iP_+ + u_iP_-$, $y_i = l_iP_+ + k_iP_-$, $i = 1, 2, 3$. Let z_1, z_2 make an arrangements of type P_+, i.e. there is $\lambda \in \mathbb{R}\backslash\{0\}$ such that $w((x_1, y_1), (x_2, y_2)) = \lambda P_+$. Therefore, $u_1k_2 - k_1u_2 = 0$, which means $[u_1 : k_1] = [u_2 : k_2]$. Let z_2, z_3 make an arrangements of type P_+. In the same way, we obtain $[u_3 : k_3] = [u_2 : k_2]$. So, $z_1 = [r_1 : l_1]P_+ + [u_1 : k_1]P_-, z_2 = [r_2 : l_2]P_+ + [u_1 : k_1]P_-, z_3 = [r_3 : l_3]P_+ + [u_1 : k_1]P_-$ Clearly, $w((x_3, y_3), (x_1, y_1)) = (r_3l_1 - l_3r_1)P_+ + (u_1k_1 - k_1u_1)P_- = \lambda_3P_+$, where $\lambda_3 = r_3l_1 - l_3r_1$. Therefore, there is no three points of type P_+P_+1 or $P_+P_+P_-$. In the same way, there is no three points of type P_-P_-1 or $P_-P_-P_+$. □

The next proposition says that $[1 : 0]$, $[0 : 1]$, $[P_+ : P_-]$ can be taken as a canonical triple of points of the type $1P_+P_-$.

Proposition 5.5.12. Let $\{z_1, z_2, z_3\}$ be three pair-wise different points in $\mathbb{P}(\mathbb{O})$. If (z_1, z_2, z_3) is of the type $1P_+P_-$, then there exists an $A \in SL_2(\mathbb{O})$ such that $Az_1 = \infty, Az_2 = 0, Az_3 = \sigma_1$.

Proof. Let $\{z_1, z_2, z_3\}$ be three pair-wise different points in $\mathbb{P}(\mathbb{O})$, i.e. there are two triple sets $\{r_1, r_2, r_3\}$ and $\{u_1, u_2, u_3\}$ in $\mathbb{P}(\mathbb{R})$ such that $z_i = r_iP_+ + u_iP_-$. If (z_1, z_2, z_3) is of the type $1P_+P_-$, then $r_1 \neq r_2$, $r_2 \neq r_3$, $u_1 \neq u_2$, $u_1 \neq u_3$ but $r_1 = r_3$ and $u_2 \neq u_3$. Clearly, there are two matrices $A_+ \in SL_2(\mathbb{R})$ such that $A_+r_1 = \infty, A_+r_2 = 0, A_+r_3 = A_+r_1 = \infty$ and $A_- \in SL_2(\mathbb{R})$ such that $A_-u_1 = \infty, A_-u_2 = 0, A_-u_3 = A_-u_2 = 0$. Therefore, $A = A_+P_+ + A_-P_- \in SL_2(\mathbb{O})$ such that $Az_1 = \infty, Az_2 = 0, Az_3 = \sigma_1$. □

Clearly A in the previous proposition is not unique. Actually, very matrix with the form $A = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} A_+P_+ + \begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix} A_-P_-$ maps z_1 to ∞, z_2 to 0,

z_3 to $[P_+ : P_-]$. The next proposition shows that for two triple of points Z, W of the type $1P_+P_-$, there exists a map M such that $MZ = W$.

Proposition 5.5.13. Let $Z = (z_1, z_2, z_3)$ and $W = (w_1, w_2, w_3)$ be two triples of pair-wise different points in $\mathbb{P}(\mathbb{O})$. Let Z be of the type $1P_+P_-$. W is of the type $1P_+P_-$ if and only if there exists an $M \in SL_2(\mathbb{O})$ such that $Mz_i = w_i$.

Proof. Let $Z = (z_1, z_2, z_3)$, and $W = (w_1, w_2, w_3)$ be two triples of pair-wise different points of the type $1P_+P_-$. Therefore, there are two maps A and B such that $AZ = (\infty, 0, \sigma_1)$ and $BW = (\infty, 0, \sigma_1)$ by Proposition 5.5.12. That is, A is sending (z_1, z_2, z_3) to $(\infty, 0, \sigma_1)$ and B^{-1} is sending $(\infty, 0, \sigma_1)$ to (w_1, w_2, w_3), respectively. That means, $B^{-1}A$ is the required map.

Conversely, let M be a map such that $MZ = W$. Therefore, from Proposition 5.5.5, any two different points in Z and any two different points in W make arrangements of the same type.

□

The set $(\infty, 0, \sigma_1)$ is called a canonical triple of points. The canonical triple of points depends on the type of the triple of points as the next theorem shows:

Corollary 5.5.14. Let $Z = \{z_1, z_2, z_3\}$ be a pair-wise different triple of points in $\mathbb{P}(\mathbb{O})$. There are the only the following five types of canonical triple of points.

1. If the triple (z_1, z_2, z_3) is of type 111, then the canonical triple of points is $(0, 1, \infty)$.

2. If the triple (z_1, z_2, z_3) is of type $1P_+P_-$, then the canonical triple of points is $(\infty, 0, \sigma_1)$.

3. If the triple (z_1, z_2, z_3) is of type $11P_-$, then the canonical triple of points is $(\infty, 0, \omega_1)$.

4. If the triple (z_1, z_2, z_3) is of type $11P_+$, then the canonical triple of points is $(\infty, 0, \omega_2)$.

5. If the triple (z_1, z_2, z_3) is of type $P_+P_+P_+$, then the canonical triple of points is $(0, P_+, \sigma_1)$.

6. If the triple (z_1, z_2, z_3) is of type $P_-P_-P_-$, then the canonical triple of points is $(0, P_-, \sigma_2)$.

Proof. 1. From Proposition 5.5.8, the first one is proved.

2. From Proposition 5.5.12, the second one is proved.

3. By using the same way in the proof of Proposition 5.5.12, the other cases are proved.

□

The other cases are obtained by permutation of the canonical triples of points.

5.5.2 Types of canonical triples of points in $\mathbb{P}(\mathbb{D})$

This subsection shows that there are three different canonical triples of points in $\mathbb{P}(\mathbb{D})$.

Definition 5.5.15. Let $Z = \{z_1 = [x_1 : y_1], z_2 = [x_2 : y_2], z_3 = [x_3 : y_3]\}$ be a set of three pair-wise different points in $\mathbb{P}(\mathbb{D})$. We say that (z_1, z_2, z_3) is:

1. of the type 111 if $w((x_1, y_1), (x_2, y_2)) = \lambda_1$, $w((x_2, y_2), (x_3, y_3)) = \lambda_2$, $w((x_3, y_3), (x_1, y_1)) = \lambda_3$,

2. of the type 11ϵ if $w((x_1, y_1), (x_2, y_2)) = \lambda_1$, $w((x_2, y_2), (x_3, y_3)) = \lambda_2$, $w((x_3, y_3), (x_1, y_1)) = \lambda_3\epsilon$,

3. of the type $\epsilon\epsilon\epsilon$ if $w((x_1, y_1), (x_2, y_2)) = \lambda_1\epsilon$, $w((x_2, y_2), (x_3, y_3)) = \lambda_2\epsilon$, $w((x_3, y_3), (x_1, y_1)) = \lambda_3\epsilon$,

where $\lambda_{1,2,3}$ are unit dual numbers.

Permutations of the second type are defined in the same way. Clearly, the above three types of points are a disjoint sets. The following examples shows that every type is a non-empty set.

Example 5.5.16. We are calling the following triples as a canonical triples of the respective type:

1. $(0, 1, \infty)$ is of the type 111.

2. $(\infty, 0, \omega)$ is of the type 11ϵ, where $\omega = [1 : \epsilon]$.

3. $(0, \epsilon, \epsilon 2)$ is of the type $\epsilon\epsilon\epsilon$.

Theorem 5.5.17. Let $\{z_1, z_2, z_3\}$ be a set of three pair-wise different points in $\mathbb{P}(\mathbb{D})$. Then, the triple (z_1, z_2, z_3) is only of the one of three types in the Definition 5.5.15.

Proof. To demonstrate the Theorem we only need to show that there is no triple of type $1\epsilon\epsilon$ which are together with the defined three types make all possible arrangements (combinations of letters $1, \epsilon$), up to permutation.

Let $z_1 = [x_i : y_i]$, where $x_i, y_i \in \mathbb{D}$ such that $x_i = a_i + \epsilon b_i$, $y_i = c_i + \epsilon d_i$, $i = 1, 2, 3$. Let z_1, z_2 make an arrangements of type ϵ, i.e. there is $\lambda \in \mathbb{R}\backslash\{0\}$ such that $w((x_1, y_1), (x_2, y_2)) = \epsilon\lambda$. Therefore, $a_1c_2 - a_2c_1 = 0$, which means $[a_1 : c_1] = [a_2 : c_2]$. Let z_2, z_3 make an arrangements of type ϵ. In the same way, we obtain $[a_3 : c_3] = [a_2 : c_2]$. So, $z_1 = [a_1 + \epsilon b_1 : c_1 + \epsilon d_1], z_2 = [\lambda_1 a_1 + \epsilon b_2 : \lambda_1 c_1 + \epsilon d_2], z_3 = [\lambda_2 a_1 + \epsilon b_3 : \lambda_2 c_1 + \epsilon d_3]$, for some non-zero real numbers λ_1, λ_2. Clearly, $w((x_3, y_3), (x_1, y_1)) = \lambda_2 c_1 a_1 - \lambda_2 a_1 c_1 + \epsilon\lambda_3 = \epsilon\lambda_3$, where $\lambda_3 = c_1(b_3 - \lambda_2 b_1) + a_1(\lambda_2 d_1 - d_3)$. Therefore, there is no three points of type $\epsilon\epsilon 1$. $\square$

Proposition 5.5.18. Let z_1, z_2, z_3 be three pair-wise different points in $\mathbb{P}(\mathbb{D})$. If (z_1, z_2, z_3) is of type 11ϵ, then there exists $A \in SL_2(\mathbb{D})$ such that $Az_1 = \infty, Az_2 = 0, Az_3 = \omega$.

Proof. Let $z_i = [x_i : y_i]$, where $i = 1, 2, 3$. If the triple of points (z_1, z_2, z_3) is of the type 11ϵ, then there exist three units $\lambda_1, \lambda_2, \lambda_3$ such that:

1. $L_1 = x_1y_2 - y_1x_2 = \lambda_1$,

2. $L_2 = x_2y_3 - y_2x_3 = \lambda_2$,

3. $L_3 = x_3y_1 - y_3x_1 = \lambda_3\epsilon$.

Let us define

1. $k_2 = x_1x_2 + y_1y_2$,

2. $k_3 = x_1x_3 + y_1y_3$.

Let $x_1^2 + y_1^2 = u$. The Dual number u is a unit because z_1 is in the projective line of dual numbers. It is has a square roots because it is a sum of squares. Thus $A_1 = \frac{1}{\sqrt{u}}\begin{pmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{pmatrix} \in SL_2(\mathbb{D})$. The image of z_1, z_2, z_3 under A_1 is $[1 : 0]$, $[k_2 : L_1]$, $[k_3 : -L_3]$, respectively. Let $A_2 = \begin{pmatrix} L_1 & -k_2 \\ 0 & L_1 \end{pmatrix}$. A_2 fixes ∞, and takes A_1z_2 and A_1z_3 to $[0 : 1]$ and $[-L_2 : -L_1L_3]$, respectively. Let $A_3 = \begin{pmatrix} \lambda_3L_1 & 0 \\ 0 & L_2 \end{pmatrix}$. A_3 takes $A_2A_1z_3$ to ω and fixes both ∞ and zero. Let $A = A_3A_2A_1$. It follows that $Az_1 = [1 : 0]$, $Az_2 = [0 : 1]$, $Az_3 = [1 : \epsilon]$. □

Clearly A in the previous proposition is not unique. Actually, very matrix with the form $A_1 = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} A$ maps z_1 to ∞, z_2 to 0, z_3 to $[1 : \epsilon]$. The next proposition shows that for two triple of points Z, W of the type 11ϵ, there exists a map M such that $MZ = W$.

Proposition 5.5.19. Let $Z = (z_1, z_2, z_3)$ and $W = (w_1, w_2, w_3)$ be two triples of pair-wise different points in $\mathbb{P}(\mathbb{D})$. Let Z be of the type 11ϵ. W is of the type 11ϵ if and only if there exists $M \in SL_2(\mathbb{D})$ such that $Mz_i = w_i$.

We obtain the result in the same way as in the proposition 5.5.13.

Corollary 5.5.20. Let $Z = \{z_1, z_2, z_3\}$ be a pair-wise different triple of points in $\mathbb{P}(\mathbb{D})$. There are the only the following two types of canonical triple of points.

1. If (z_1, z_2, z_3) of the type 111, then the canonical triple of points is $(0, 1, \infty)$.
2. If (z_1, z_2, z_3) of the type 11ϵ, then the canonical triple of points is $(\infty, 0, \omega)$.
3. If (z_1, z_2, z_3) of the type $\epsilon\epsilon\epsilon$, then the canonical triple of points is $(0, \epsilon, \epsilon 2)$.

Proof.

1. From Proposition 5.5.8, the first one is proved.
2. From Proposition 5.5.18, the second one is proved.
3. By using the same way in the proof of 5.5.18.

□

Conclusion and Outlook

For several rings $\mathbb{A}$, Chapter 3 defines the general linear groups $GL_2(\mathbb{A})$, the special linear groups $SL_2(\mathbb{A})$, the projective lines $\mathbb{P}(\mathbb{A})$ and $\mathbb{A}$-Möbius maps.

From Chapter 4 we restrict our work to a ring $\mathbb{A}$, where $\mathbb{A}$ is the field of real or complex numbers, the ring of double or dual numbers. Chapter 4 defines metrics over $\mathbb{P}(\mathbb{A})$, such that $\mathbb{A}$-Möbius maps are continuous maps in these metrics. It also describes the number and types of fixed points of $\mathbb{A}$-Möbius maps.

For $\mathbb{A}$ being one of the rings of double or dual numbers, Chapter 5 presents the number of non-isomorphic types of non-trivial connected continuous one-parameter subgroups of $GL_2(\mathbb{A})$. All possible types of canonical triples of points of $\mathbb{A}$-Möbius maps are also given.

For further study, it is worth to work on the tropical semiring $\mathbb{N}_{\min}$. $GL_2(\mathbb{N}_{\min})$, $\mathbb{P}(\mathbb{N}_{\min})$ and $\mathbb{N}_{\min}$-Möbius maps are interesting objects of study. An equally promising subject is Kleinian group of $GL_2(\mathbb{O})$ and $GL_2(\mathbb{D})$. The other subject is the definition of continued fractions in a ring $\mathbb{A}$ and of their relations with $\mathbb{A}$-Möbius maps, where $\mathbb{A}$ is the ring of double or dual numbers.

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