Non-commutative Noetherian Unique Factorisation Domains

by

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Abstract

The commutative theory of Unique Factorisation Domains (UFDs) is well-developed (see, for example, Zariski-Samuel[75], Chapter 1, and Cohn[21], Chapter 11). This thesis is concerned with classes of non-commutative Noetherian rings which are generalisations of the commutative idea of UFD.

We may characterise commutative Unique Factorisation Domains amongst commutative domains as those whose height-1 prime ideals P are all principal (and completely prime ie R/P is a domain). In Chatters[13], A.W.Chatters proposed to extend this definition to non-commutative Noetherian domains by the simple expedient of deleting the word commutative from the above.

In Section 2.1 we describe the definition and some of the basic theory of Noetherian UFDs, and in Sections 2.2, 2.3, and 2.4 demonstrate that large classes of naturally occuring Noetherian rings are in fact Noetherian UFDs under this definition.

Chapter 3 develops some of the more surprising consequences of the theory by indicating that if a Noetherian UFD is not commutative then it has much better properties than if it were. All the work, unless otherwise indicated, of this Chapter is original and the main result of Section 3.1 appears in Gilchrist-Smith[30].

In the consideration of Unique Factorisation Domains the set C of elements of a UFD R which are regular modulo all the height-1 prime ideals of R plays a crucial role, akin to that

of the set of units in a commutative ring. The main motivation of Chapter 4 has been to generalise the commutative principal ideal theorem to non-commutative rings and so to enable us to draw conclusions about the set C. We develop this idea mainly in relation to two classes of prime Noetherian rings namely PI rings and bounded maximal orders.

Chapter 5 then returns to the theme of unique factorisation to consider firstly a more general notion to that of UFD, namely that of Unique Factorisation Ring (UFR) first proposed by Chatters-Jordan[17]. In Section 5.2 we prove some structural results for these rings and in particular an analogue of the decomposition $R = S \cap T$ for R a UFD. Finally Section 5.3 briefly sketches two other variations on the theme of unique factorisation due primarily to Cohn[20], and Beaurregard[4], and shows that in general these theories are distinct.

To my parents

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Contents

Chapter 0. Introduction.

The theory of non-commutative rings is now well established. Though much of the theory and many of the results arise from naturally occurring non-commutative Noetherian rings, it is also true that one of the most persistent themes in the development of the subject has been the extension and generalising of known results of commutative ring theory. This thesis is concerned with three such theories: the principal ideal theorem, Unique Factorisation Domains, and localisation.

One of the major impetuses for the development of commutative ring theory arose from the work of Kummer who showed that not all rings of algebraic integers are Unique Factorisation Domains. There developed a need to determine which domains are in fact UFDs and what could be said about those that were.

Several attempts have been made to extend this theory to non-commutative rings, notably by P.M.Cohn and R.A.Beauregard. In 1984 A.W.Chatters proposed a new definition. In Chatters[13], he defined a (not necessarily commutative) Noetherian Unique Factorisation Domain (UFD) to be a Noetherian domain R with at least one height-1 prime ideal such that (i) every height-1 prime ideal is principal, by the same element, on both sides, and (ii) every height-1 prime ideal P is completely prime, that is R/P is a domain. In Chatters[13] he proved some basic results concerning Noetherian UFDs and showed that certain classes of naturally occurring Noetherian rings are in fact Noetherian UFDs with this definition.

In Chapter 2 we describe some of these results and show, in particular, that both enveloping algebras of finite-dimensional Lie algebras and group rings of torsion-free polycyclic-by-finite groups are often Noetherian UFDs. The final section of Chapter 2 exhibits some quite natural results that seem to indicate that the definition of Noetherian UFD is a reasonable one. It includes a non-commutative version of Nagata's Theorem which characterises Noetherian UFDs in terms of certain localisations. It is the only original section of this chapter.

Suppose that R is a Noetherian UFD. Consider the set C of elements of R that are regular modulo every height-1 prime ideal of R. If R were commutative then C would simply consist of the set of units of R, U. In general however, the set C may be strictly larger than U. One of the fundamental results of Chatters[13] is that the set C is always an Ore set and hence that we may localise with respect to C. It turns out that this localisation T has particularly nice properties and in Chapter 3 we explore some of them.

Following the proposed definition of Noetherian UFD, A.W.Chatters, J.T.Stafford, and M.K.Smith showed that for certain classes of naturally occurring Noetherian UFD the localisation T is actually a principal ideal domain (PID). In Section 3.1 we show that this is the case in general, provided that R is not commutative. Clearly this indicates a quite serious and unexpected divergence in the theory since in the commutative case T is equal to R and hence need not be a PID. Perhaps one way to interpret this result is that it is an indication of how stringent a condition it is that a prime ideal of a non-commutative ring be completely prime. We should note that this result was proved independently by M.K.Smith and it appears in Gilchrist-Smith[30].

In Section 3.2 we extend this result by showing that in fact provided that R is not commutative then not only is T a PID, but also T has stable rank one. This means that we have a good grasp on the structure of T-modules since we have an array of "cancellation" results at our disposal.

On a closer examination of the proof of the main result of Section 3.2 it becomes clear that we have actually shown that the set of height-1 prime ideals of a Noetherian UFD satisfy the intersection condition. That is, if K is a right (or left) ideal with $K \cap C(P) \neq \emptyset$ for all height-1 prime ideals P then $K \cap C \neq \emptyset$.

This was quite unexpected and we explore the implications of this to localisation in Section 3.3. Broadly speaking, the question of localisation in non-commutative rings is still a vexed one and we have as yet only partial answers to the problem. In commutative ring theory it is always possible to localise at a prime ideal or set of prime ideals and it is often very convenient to do so.

However in non-commutative ring theory, it is often not possible to localise at a given prime ideal and the best we can hope for is to localise at a clique of prime ideals that are "linked" together. In the presence of a condition known as the second layer condition, a clique is localisable if and only if it satisfies the intersection condition. Consequently it becomes of great interest to determine when a set of prime ideals satisfies the intersection condition.

Generally, for Noetherian rings, results in this direction have needed to assume the existence of an uncountable central field in the ring in question. Then a counting argument may be employed because cliques consist of only (at most) countably many prime ideals. Using the method of Section 3.2, we are able to drop this uncountability hypothesis in certain situations. Unfortunately the results of Section 3.3 are very limited in scope: essentially they require that a clique X consists of completely prime ideals such that, denoting their intersection by Q, R/Q is not commutative and that for each prime ideal P of X, P/Q is a height-1 prime ideal of R/Q.

We have been unable to extend this result significantly, but in its present form we are able to show that cliques in enveloping algebras of certain solvable Lie algebras are localisable.

In Section 3.4 we consider the question of the centres of Noetherian UFDs and show that for R a Noetherian UFD Z(R) is always a Krull domain. Conversely, we also show, by an explicit construction, that every commutative Krull domain may be realised as a centre of a Noetherian UFD. Unless otherwise stated all the results of Chapter 3 are original.

It is readily apparent that the set C of elements regular modulo every height-1 prime ideal of a Noetherian UFD play an important part in Chapter 3. We can consider this set (renamed Γ in the general case) in the wider context of prime Noetherian rings. In a commutative Noetherian domain the set Γ consists solely of the set of units. In fact this statement is one of several equivalent formulations of the principal ideal theorem due to Krull. This observation motivates Chapter 4 which explores this theme by considering several different generalisations of the classical principal ideal theorem to non-commutative Noetherian rings. These different generalisations correspond to the different equivalent statements of the theorem due to Krull.

The results of Chapter 4 primarily apply to two main classes of prime Noetherian rings: PI rings and bounded maximal orders. Sections 4.2 and 4.3 deal primarily with results for Noetherian PI rings though we do prove them in a slightly wider context. In Section 4.4 we consider bounded maximal orders and in Section 4.5 we indicate how these results might be extended to more general classes of rings. This chapter is largely inspired by numerous discussions and correspondence with A.W.Chatters and many of the results of this chapter were proved independently by him. The main results are to appear in Chatters-Gilchrist[14].

The final chapter, Chapter 5, returns us to the theme of Unique Factorisation. In some respects, nice though the theory is, Noetherian UFDs are not an entirely satisfactory generalisation of the commutative case. If R is a commutative UFD then R[x] is also a UFD. However in Section 5.1, we exhibit an example of a (non-commutative) Noetherian UFD such that R[x] is not a Noetherian UFD. We note though that it is still true that all the height-1 prime ideals of R[x] are principal.

This motivates the definition of a Noetherian Unique Factorisation Ring (UFR) which was first proposed in Chatters-Jordan[17]. A prime Noetherian ring is a Noetherian

UFR if every non-zero prime ideal contains a height-1 prime ideal and every height-1 prime ideal is principal (by the same element) on both sides.

Section 5.1 devlops the basic theory of Noetherian UFRs and is mostly due to A.W.Chatters and D.A.Jordan, though we do, in passing, use some of the results of this section to construct examples of primitive Noetherian UFDs of any finite Krull or global dimension.

As might be anticipated, the theory of Noetherian UFRs is at one and the same time more natural and less tractable than for Noetherian UFDs. In the case of Noetherian UFDs the fact that the set C is Ore enables us to write any Noetherian UFD as the intersection of a simple Noetherian domain and a PID. The first part of Section 5.2 is devoted to proving that a Noetherian UFR R may be written as the intersection of a simple Noetherian ring and an ideal-principal ring. However since the latter is not an order in the full quotient ring of R this limits the utility of this result. For Noetherian UFRs we have, in general, been unable to show that Γ is an Ore set. For bounded Noetherian UFRs the results of Chapter 4 and the fact that Noetherian UFRs are maximal orders enables us to conclude that Γ is simply the set of units. We give some more sufficient conditions on Noetherian UFRs for Γ to be Ore.

We end Section 5.2 by presenting some preliminary results on the structure of one-sided reflexive ideals of a Noetherian UFR. These are inspired in part by similar results for hereditary Noetherian rings proved by Lenagan in his thesis [47]. All the work in this section, with the exception of Lemma 5.2.1, is original.

As we stated earlier, there have been several other attempts to generalise the definition of UFD to non-commutative rings. We end this thesis by answering a question of P.M.Cohn. We show, with examples, that in general the notion of Noetherian UFD is distinct from the definitions due to P.M.Cohn and R.A.Beauregard.

Chapter 1. Basic definitions and results.

Section 1.0. Summary.

In this chapter we shall recall the basic techniques and results of Noetherian ring theory which we will require subsequently. Very little here will be original and it is intended only to provide a ready source of reference and to serve as an introduction to certain classes of rings which we shall later discuss. Results will quite often not be stated in their full generality, since, for the most part, we shall generally have two-sided conditions present when often one-sided ones would do. For this somewhat whistle-stop tour of Noetherian ring theory we will use Chatters-Hajarnavis[16], Cohn[21], and (to appear) McConnell-Robson[51] as general references and as a source for the precise statements of the results.

Throughout all rings will have a 1 and all modules will be unitary. Sub-rings will share the same unit element. Fields will always be commutative. The notation of this chapter is standard and will be used throughout this thesis.

Recall that, for a ring R, R is said to be right (left) Noetherian if it satisfies the ascending chain condition on right (left) ideals. A ring is Noetherian if it is both left and right Noetherian.

A ring R is right (left) Artinian if it satisfies the descending chain condition on right (left) ideals. A ring is Artinian if it is both left and right Artinian.

A ring R is prime if, given two ideals A and B of R, then if

$A.B = 0$ then either $A = 0$ or $B = 0$. Equivalently, if $aRb = 0$ then either $a=0$ or $b=0$. A ring is semi-prime if, for an ideal A, $A^2 = 0$ implies that $A = 0$; equivalently, $aRa = 0$ implies that $a=0$. We say that an ideal I is a prime (semi-prime) ideal if the factor ring R/I is prime (semi-prime). A ring is a domain if it has no non-zero zero-divisors. A prime ideal P is completely prime if R/P is a domain.

A right R-module M_R is faithful if, for an element a of R, $Ma = 0$ implies that $a=0$. A module is simple if it contains no non-trivial sub-modules. We say that a ring R is right (left) primitive if it has a simple faithful right (left) R-module.

A right (left) ideal is principal if it is of the form aR (Ra) for some element a of R. An ideal is principal if it is both left and right principal. We say that a domain R is a principal right (left) ideal domain if every right (left) ideal is principal, and a principal ideal domain if it is both a left and right principal ideal domain. A ring is ideal-right (ideal-left) principal if every ideal is right (left) principal. We say that a ring is ideal-principal if every ideal is both left and right principal.

A ring R is right (left) hereditary if every right (left) ideal is projective. A ring R is hereditary if it is both left and right hereditary.

Section 1.1. Quotient rings and the Ore condition.

In commutative ring theory the method of localisation is a very powerful and ubiquitous tool. Let R be a ring. Suppose that S is a multiplicatively closed saturated sub-set S such that $0 \notin S$. We seek to form a ring R_S which is universal with respect to the property that it is S-inverting. That is, (i) R_S is a ring with a ring homomorphism $\lambda : R \longrightarrow R_S$ such that $\lambda(s)$ is a unit in R_S, for all $s \in S$, and (ii) given any ring homomorphism $\mu : R \longrightarrow T$ with $\mu(s)$ a unit in T, for all $s \in S$, then there exists a unique ring homomorphism $f : R_S \longrightarrow T$ such that $\mu = \lambda f$.

In the case of a commutative ring such an R_S is easy to construct. We define on $R \times S$ the equivalence relation $\equiv$ by $(r,s) \equiv (r',s')$ if and only if there exists an element $t \in S$ with $t(rs'-sr')=0$. Then we can define R_S to be the set of equivalence classes of $R \times S$ with the operations of addition and multiplication defined by $[r,s]+[r',s']=[rs'+sr',ss']$ and $[r,s].[r',s']=[rr',ss']$.

This defines a ring R_S and a unique ring homomorphism $\lambda : R \longrightarrow R_S$ given by $\lambda(r)=[r,1]$ which is universal S-inverting. See for example Cohn[21], Theorem 11.3.1. For P a prime ideal of R, the set $R \backslash P = S$ is a multiplicatively closed set. In this case, R_S is a local ring with unique maximal ideal PR_S; that is we have localised at P.

In the non-commutative case we have nowhere near the ease of this theory. In fact it seems reasonable to say that non-commutative ring theory has not yet quite surmounted this first step. It is true to say that, given any ring R and a multiplicatively closed subset S of R, that there does exist a universal S-inverting ring R_S. However, in general, it is almost impossible to do anything with this ring.

To have much hope of a useful theory of localisation we need a simplifying idea due to O.Ore (and independently E.Noether). This concept enables us to write R_S as a set of elements of the form $\lambda(r).\lambda(s)^{-1}$ where the forms of addition and multiplication are then easy to write down. However in order to do this we need to be able to write $\lambda(s)^{-1}.\lambda(r)$ in the form $\lambda(r').\lambda(s')^{-1}$, for some $r'\in R$ and $s'\in S$. It is this problem which the next theorem addresses.

1.1.1. Theorem: Let R be a ring. Let S be a multiplicatively closed saturated subset of R such that:

(i) for all $a\in R$ and $s\in S$, $aS\cap sR \neq \emptyset$;

(ii) for each $a\in R$ and $s\in S$ with $sa=0$, there exists $t\in S$ with $at=0$.

Then the elements of the univeral S-inverting ring R_S can be constructed as fractions a/s, where $a/s = a'/s'$ if and only if $au=a'u'$ and $su=s'u'$, for some $u,u'\in R$.

Moreover, the kernel of the canonical ring homomorphism $\lambda:R\longrightarrow R_S$ is $\mathrm{Ker}\lambda = \{a\in R: at=0 \text{ for some } t\in S\}$.

Proof: See Cohn[21], Theorem 12.1.2.

Suppose that S is a multiplicatively closed saturated subset of a ring R. We say that S is a <u>right denominator set</u> if it satisfies (i) and (ii) of Theorem 1.1.1. We may define a <u>left denominator set</u> in a symmetrical fashion. In general, the two notions are distinct. If S consists of non-zero-divisors then (ii) is superfluous and $\mathrm{Ker}\lambda = 0$.

Let S be a set of non-zero divisors in a ring R. We say that S satisfies the right Ore condition if, given $a \in R$ and $s \in S$, there exists $b \in R$ and $t \in S$ with $at = sb$. In this situation we say that S is a right Ore set. The left Ore condition is defined similarly. We say that S satisfies the Ore condition if it satisfies both the left and right Ore conditions. In this situation, we say that S is an Ore set.

1.1.2. Theorem: Let R be a ring. Suppose that S is a multiplicatively closed subset of non-zero-divisors satisfying the right Ore condition. Then S is a right denominator set and the canonical homomorphism $\lambda : R \longrightarrow R_S$ is injective.

In this situation, we call the ring R_S the right partial quotient ring of R with respect to S. For a left Ore set T, ${}_TR$ denotes the left partial ring of quotients with respect to T. If S is an Ore set then ${}_SR \cong R_S$.

For a commutative domain R one of the most useful constructions is the field of fractions of R in which we invert all the non-zero elements of the ring. For a general ring R, we say that it has a right full ring of quotients Q(R) which, if it exists satisfies:

(i) R is a sub-ring of Q(R);

(ii) each regular element of R is a unit of Q(R);

(iii) each element of Q(R) can be written in the form ac^{-1}, for elements $a, c \in R$ with c regular.

Thus $Q(R)$ is the right localisation of R with respect to the set of regular elements of R.

In this situation, we say that R is a right order in $Q(R)$. A left order is defined similarly and if both exist they are equal. A right or left full quotient ring of quotients need not always exist. However, in an important class of rings A.W.Goldie has determined necessary and sufficient conditions for a full quotient ring to exist.

Let R be a ring. Suppose that M is an R-module. A sub-module N is <u>essential</u> in M if any non-zero sub-module of M has non-zero intersection with N. A module M is uniform if every non-zero sub-module is essential. We say that a module M has <u>finite Goldie rank n</u> if there exists a direct sum of n uniform submodules of M which is essential. The right (left) Goldie rank of a ring R is the right (left) Goldie rank of R as a right (left) R-module.

In a ring R, a right ideal I is a <u>right annihilator ideal</u> if $I = r(S) = \{r : sr=0 \text{ for all } s\in S\}$ for a subset S of R. A ring R is said to be a <u>right Goldie ring</u> if (i) R has finite right Goldie rank, and (ii) it satisfies the ascending chain condition on right annihilator ideals.

Then we have the folowing very important result, due to A.W.Goldie.

1.1.3. Theorem: Let R be any ring. Then R has a right full quotient ring which is semi-simple Artinian if and only if R is semi-prime right Goldie.

Further, R has a right full quotient ring which is simple Artinian if and only if R is prime right Goldie.

In a ring R, we say that an element c is <u>right regular</u> if $cx=0$ implies that $x=0$; left regular is similarly defined and a regular element is both left and right regular.

For an ideal I of R, we shall use the notation $C'(I)$ (respectively $'C(I)$) to denote the elements of R whose images in the factor ring R/I are right regular (respectively left regular). $C(I) = C'(I) \cap 'C(I)$ is the set of elements of R whose images in R/I are regular. We shall say that a prime ideal P of a ring R is <u>right (left) Goldie</u> if R/P is right (left) Goldie. Clearly, in a Noetherian ring, every prime ideal is both left and right Goldie.

The following two results will be used implicitly in all that follows. They are well known, but we record them here explicitly. The first result is primarily due to A.T.Ludgate.

1.1.4. Theorem: Let R be a ring. Suppose that S is an Ore set of non-zero-divisors. Let I be an ideal of R. Then:

(i) IR_S is an ideal of R_S if and only if $S \subseteq C'(IR_S \cap R)$;

(ii) $IR_S \cap R = I$ if and only if $S \subseteq 'C(I)$.

Furthermore, if R is Noetherian then IR_S is an ideal of R_S.

Proof: (i) Suppose that $S \subseteq C'(IR_S \cap R)$. Suppose that $c \in S$ and $i \in R$. Since S is an Ore set, there exist $d \in S$ and $r \in R$ such that $cr = id$. Now $c \in C'(IR_S \cap R)$ and so $r \in IR_S$. Then $c^{-1}i = rd^{-1} \in IR_S$.

Conversely, suppose that IR_S is an ideal of R_S. Suppose that $c \in S$ and $x \in R$ are such that $cx \in IR_S \cap R$. Then $x \in IR_S$, and hence $x \in IR_S \cap R$. That is, $c \in C'(IR_S \cap R)$.

(ii) Suppose that $IR_S \cap R = I$. Suppose that $c \in S$ and $x \in R$

are such that $xc \in I$. Then $x \in IR_S \cap R = I$. Hence $c \in {}'C(I)$.

Conversely, suppose that $S \subseteq {}'C(I)$. Suppose that for an element $x \in R$, $x = ic^{-1} \in IR_S \cap R$. Then $xc = i \in I$, and so $x \in I$.

The final statement is proved as in Chatters-Hajarnavis[16], Theorem 1.31.

1.1.5. Theorem: Let R be a ring. Let S be a right Ore set of non-zero-divisors.

(i) For every right Goldie prime ideal P' of R_S, the intersection $P' \cap R$ is a right Goldie prime ideal of R with $P' = (P' \cap R)R_S$, and $R/P' \cap R$ and R_S/P' have the same full quotient ring.

(ii) For every right Goldie prime ideal P of R disjoint from S, the localisation PR_S is a right Goldie prime ideal of R_S with $P = PR_S \cap R$, and R/P and R_S/PR_S have the same full quotient ring.

(iii) For every prime ideal P of R such that the elements of S are all regular modulo P, the localisation PR_S is a prime ideal of R_S with $P = PR_S \cap R$. If PR_S is right Goldie then P is right Goldie. If R_S is right Noetherian then it is enough to assume that the elements of S are left regular modulo P: in this case P is always right Goldie.

Proof: Bell[6], Proposition 2.3.

A prime ideal P is <u>localisable</u> if C(P) is Ore.

1.1.6. Lemma: Let R be a prime Noetherian ring. Let I be an essential right ideal of R. Suppose that a is an element of R. Then a + I contains a regular element of R.

Proof: Since R is left Noetherian, R satisfies d.c.c. on right annihilators. Let $x \in I$ be with $r(a+x)$ minimal. Let $c = a+x$. Let B be a right ideal of R with $B \cap cR = 0$. Choose $0 \neq b \in B \cap I$. Then $c+b \in a+I$. Because $r(c+b) = r(c) \cap r(b)$, $r(c+b) \subseteq r(c)$. By our choice of c, $r(c+b) = r(c)$. But b was chosen arbitrarily such that $b \in B \cap I$. So $r(c) \subseteq r(b)$, for all $b \in B \cap I$. So $(B \cap I)r(c) = 0$. Since R is prime, either $r(c) = 0$ or $B \cap I = 0$. If $B \cap I = 0$ then $B = 0$, since I is essential. Therefore cR is essential. By Chatters-Hajarnavis[16], Theorem 1.10 and Corollary 1.13, cR contains a regular element of R. Hence c is regular.

1.1.7. Lemma: Let R be a right Artinian ring. Then a right regular element of R is a unit.

Proof: Let c be a right regular element of R. Consider the descending chain of right ideals $cR \supseteq c^2R \supseteq c^3R \supseteq \dots$. Since R is right Artinian, there exists an integer n such that $c^nR = c^{n+1}R = \dots$. So there exists $x \in R$ with $c^{n+1}x = c^n$. By the right regularity of c, $cx = 1$. Also, $c(xc-1) = 0$. So $xc = 1$.

We also record here a result which we will often have cause to use. We say that an element s of a ring R is <u>normal</u> if $sR = Rs$.

1.1.8. Lemma: Let R be a ring. Let S be a multiplicatively closed set of normal regular elements. Then S is an Ore set.

Proof: Suppose that $a \in R$ and $s \in S$. Since s is normal, there exist elements $a', a'' \in R$ such that $as = sa'$, and $sa = a''s$.

Section 1.2. Orders and maximal orders.

In this section we consider the situation of a ring R with a full (left and right) ring of quotients $Q = Q(R)$. We say that R is an order in Q. We refer to Maury-Raynaud [52] for further background material for this section.

Suppose that two rings R and S are both orders in Q. Then we say that R and S are order-equivalent, or often just equivalent, if there exist units $\alpha, \beta, \alpha', \beta'$ of Q such that $\alpha R\beta \subseteq S$ and $\alpha' S\beta' \subseteq R$. Clearly, α and β may be chosen to lie in R, and α' and β' to lie in S. Order-equivalence defines an equivalence relation on the set of orders of Q.

An order R in Q is said to be maximal if R is maximal in the equivalence class of R. That is, R is a maximal order if R is contained in no other order of Q to which it is equivalent. For an ideal I of R, let $O_r(I) = \{q\in Q: Iq \subseteq I\}$ and let $O_l(I) = \{q\in Q: qI \subseteq I\}$. We have the following characterisation of maximal orders.

1.2.1. Theorem: Let R be an order in Q. Then the following are equivalent:

(i) R is a maximal order;

(ii) For all non-zero ideals I of R, $O_r(I) = R = O_l(I)$.

Proof: Maury-Raynaud [52], Proposition 1.3.1.

1.2.2. Definition: Let R be an order in $Q(R) = Q$. Then a <u>right (left) R-ideal</u> of Q is a right (left) R-sub-module of Q, I, such that $I\cap U(Q) \neq \emptyset$ and there exists $\lambda \in U(Q)$ such that $I\lambda \subseteq R$

($I\lambda \subseteq R$). An R-ideal is a left and right R-ideal.

An order R is an <u>Asano order</u> if the R-ideals of Q form a group under multiplication. Equivalently, for every R-ideal I contained in R, there exists an R-ideal I^{-1} such that $I.I^{-1} = I^{-1}.I = R$.

Suppose that R is an order in its quotient ring Q. Let I be a one-sided R-ideal of Q. We can define $I^* = \{q \in Q: qI \subseteq R\}$ and $^*I = \{q \in Q: Iq \subseteq R\}$ and these are both one-sided R-ideals of Q. Let I be a right (left) R-ideal of R. Then I is <u>reflexive</u> if $I = {}^*(I^*)$ ($I = ({}^*I)^*$).

If R is a maximal order then for any R-ideal of Q, I, $I^* = {}^*I$. For an ideal I of a maximal order R, $(I^*)^* = I^{**}$ is an ideal of R containing I. We say that I is reflexive if $I = I^{**}$. We say that an ideal I of R is invertible if there exists an R-ideal of Q, denoted by I^{-1}, with $I.I^{-1} = I^{-1}.I = R$. Clearly any invertible ideal is reflexive.

A ring R is said to be <u>right (left) bounded</u> if every essential right (left) ideal contains an essential two-sided ideal. R is <u>right (left) fully bounded</u> if every prime factor ring of R is right (left) bounded. Bounded and fully bounded rings are defined in the obvious way.

1.2.3. Theorem: Suppose that R is an order in Q. Then the following are equivalent:

(i) R is bounded.

(ii) Suppose that S is a non-empty subset of Q such that there exist units λ and μ of Q with $\lambda S \mu \subseteq R$. Then there exist α and β in R, units of Q, such that $\alpha S \subseteq R$ and $S\beta \subseteq R$.

Further, let R be a bounded order in Q. If R is equivalent to an order S, then S is bounded.

Proof: Maury-Raynaud [52], Propositions 1.4.1 and 1.4.2.

1.2.4. Theorem: Let R be a maximal order in Q. Then $R[x]$ is a maximal order in $Q(R[x])$. Further, suppose that P is a reflexive prime ideal in R. Then $P[x]$ is a reflexive prime ideal in $R[x]$.

Proof: The first statement follows from Maury-Raynaud [52], Proposition V.2.5. Now suppose that $P = P^{**}$ is a reflexive prime ideal of R. Clearly $P[x]$ is a prime ideal of $R[x]$ and so it remains to show that $P[x]$ is reflexive.

Clearly $P^*[x] \subseteq P[x]^*$. Hence $P[x]^{**} \subseteq P^*[x]^*$. Also $R[x] \subseteq P^*[x]$ and so $P^*[x]^* \subseteq R[x]$. Clearly also, since $P^*[x]^*.P^*[x] \subseteq R[x]$, we obtain that

$$P^*[x]^*.P^*[x].P[x] \subseteq P[x].$$

Since R is a maximal order, if $P^*[x].P[x] \subseteq P[x]$ then $P^*[x] \subseteq R[x]$. But this contradicts the fact that P is reflexive. So we deduce that

$$P^*[x]^* \subseteq P[x] \subseteq P[x]^{**} \subseteq P^*[x]^*.$$

Therefore $P[x] = P^*[x]^* = P[x]^{**}$.

Section 1.3. PI rings.

Amongst non-commutative rings there is a large class of rings whose analysis has proved to be more tractable than most because they are in some sense "close" to being commutative rings. The theory of these rings is both wide and deep , but we shall draw out only a few of the salient and useful features of these rings in order to apply them later on. We shall follow McConnell-Robson[51] in our treatment.

Let R be a ring. Let be F the free algebra on countably many generators over the integers, Z. So $F = Z\langle x_1, x_2, \ldots\rangle$. Let $r^* = (r_1, r_2, \ldots)$ be any infinite sequence of elements of R. Then r^* defines a ring homomorphism $\theta: F \longrightarrow R$ given by $x_i \longmapsto r_i$. Conversely, any ring homomorphism from F to R is of this form. The image of $f \in F$ under θ we will write as $f(r^*)$. For an element $f \in F$, we define its degree in the normal way. We say that $f \in F$ is multi-linear if $f = f(x_1, \ldots, x_n) = \sum a_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}$, where $a_\sigma \in Z$, and where the sum is over all $\sigma \in S_n$, the symmetric group on n letters.

We say that f is an identity of R if $f(r^*)=0$, for all choices of r^*. Then R is a PI ring if it has a multi-linear identity which has at least one of its coefficients equal to ± 1.

1.3.1. Theorem: Let R be a primitive PI algebra of degree d. Then R is a simple algebra of finite dimension n^2 over its centre, where $n \leq d/2$.

Proof: See for example Cohn[21], Theorem 12.5.6.

For our purposes, the properties of PI rings we most require are their relationship to their centres.

1.3.2. Theorem: Let R be a prime PI ring with centre Z(R). Then every non-zero ideal of R intersects the centre non-trivially. Further,

(i) R has a left and right full quotient ring Q(R)=Q;

(ii) Q can be obtained by inverting the non-zero central elements of R;

(iii) Q is a f.d. central simple algebra;

(iv) R is Goldie;

(v) Any multi-linear identity of R is an identity of Q.

Proof: See Cohn[21], Theorems 12.6.7 and 12.6.8.

1.3.4. Corollary: Let R be a PI ring. Then R is a fully bounded ring.

With every prime PI ring R, we may associate with it a ring T(R) known as the trace ring of R. Its construction is as follows. Let Q be the quotient ring of R. Then, for some integer n, Q has dimension n^2 over its centre K. Then, if $A \in Q$, A may be associated with θ_A, an element of the endomorphism ring of Q, by left multiplication by A. So we may regard θ_A as an $n^2 x n^2$ matrix over K. The matrix θ_A satisfies its characteristic polynomial over K which has degree n^2. Hence A satisfies the same polynomial over k.

Let T be the subring of K generated by Z(R) and the coefficients of the characteristic polynomials of A, as A runs through all the elements of R. Then T is a commutative subring of K. Now let $T(R) = T.R$. Then $T(R)$ is the <u>trace ring</u> of R. We have the following result which indicates that $T(R)$ is sometimes an easier ring to deal with than R.

1.3.5. Theorem: Let R be a Λ-affine prime PI ring. Then:

(i) $T(R)$ is a f.g. T-module.

Further if Λ is Noetherian then:

(ii) both T and $T(R)$ are Λ-affine and Noetherian, and the centre of $T(R)$ is also Noetherian.

Proof: See Small[63], Definition 52.

There is then a close relationship between prime PI rings and their centres. One class of PI rings have a particularly nice relation to their centres.

A ring R is an <u>Azumaya algebra</u> (over $Z(R)$) of rank t if $[R:Z(R)]$ is finite, $R^{op}\otimes R \cong \mathrm{End}_{Z(R)}(R)$, and, for every prime ideal P of $Z(R)$, R_P is a free $Z(R)_P$-module of rank t.

A ring R is properly maximal central of rank t if R is an Azumaya algebra and R is a free $Z(R)$-module of rank t.

1.3.6. Theorem: Suppose that R is an Azumaya algebra. Then

(i) If I is an ideal of R then $I = (I\cap Z(R))R$;

(ii) if J is an ideal of $Z(R)$ then $J = JR\cap Z(R)$.

Proof: McConnell-Robson[51], Proposition 13.7.4 or Auslander-Goldman[3], Corollary 3.2.

To investigate PI rings further the notion of the generic matrix rings was introduced. Formally, the ring of d n×n generic matrices is the ring $R = F_{(n)} = k\langle x_1, \ldots, x_d\rangle$ such that the following holds: Let S be any n×n-matrix ring over a commutative k-algebra. R is universal with respect to the property that every mapping $x_i \longmapsto a_i$, $a_i \in S$, may be extended to a unique k-algebra homomorphism $R \longrightarrow S$.

An explicit construction of $F_{(n)}$ is obtained as follows. We adjoin to the field k the dn^2 commuting indeterminants x_{ij}^{μ}, $\mu = 1, \ldots, d$; $i,j = 1, \ldots, n$. In the n×n-matrix ring $M_n(k[x_{ij}^{\mu}])$, consider R the sub-algebra generated by the d matrices (x_{ij}^{μ}), $\mu = 1, \ldots, d$. Then R is the generic matrix ring of d n×n matrices.

Section 1.4. Dimension and rank techniques.

We review briefly two "measures" on Noetherian modules and rings which are used extensively in Noetherian ring theory. We record some results which will be useful later on. We will use Gordon-Robson[32] and Chatters-Hajarnavis[16] as our main sources.

The notion of Krull dimension for non-commutative rings was first proposed by Rentschler and Gabriel and extended to infinite ordinals by Krause. Let R be a ring. Let M_R be a right R-module. Then the Krull dimension of M may be defined by transfinite induction as follows. If $M = 0$, then $\mathrm{Kdim}(M)=-1$; if α is an ordinal and $\mathrm{Kdim}(M)\not<\alpha$, then $\mathrm{Kdim}(M)=\alpha$ if every descending chain $M = M_0 > M_1 > M_2 > \ldots$ of R-sub-modules of M with $\mathrm{Kdim}(M_i/M_{i+1})\not<\alpha$ terminates. For example, Artinian modules are precisely those modules with Krull dimension 0. For a module M_R, it is possible that there exists no such ordinal in which case we say that M has no Krull dimension. A ring R has Krull dimension α, where $\mathrm{Kdim}(R_R)=\alpha$.

1.4.1. Lemma: Let R be a ring.

(i) If N is a submodule of M then, if either side exists, $\mathrm{Kdim}(M) = \sup\{\mathrm{Kdim}(N),\mathrm{Kdim}(M/N)\}$.

(ii) $\mathrm{Kdim}(R) = \sup\{\mathrm{Kdim}(M)\colon M \text{ f.g. R-module}\}$ if either side exists.

(iii) Every factor ring of a ring R with Krull dimension has Krull dimension $\leq \mathrm{Kdim}(R)$.

Proof: Gordon-Robson[32], Lemmas 1.1 and 1.2(i).

1.4.2. Theorem: (i) Every Noetherian module has Krull dimension.

(ii) Every module with Krull dimension has finite uniform rank.

Proof: Gordon-Robson[32], Propositions 1.3 and 1.4.

We will apply the theory of Krull dimension to rings and regular elements in rings.

1.4.3. Theorem: Let R be a ring with Krull dimension. If $c \in R$ is regular then $\mathrm{Kdim}(R/cR) < \mathrm{Kdim}(R)$.

Proof: Suppose that $\mathrm{Kdim}(R/cR) = \mathrm{Kdim}(R)$. Consider the infinite descending chain of right ideals of R, $R > cR > c^2R >$ ... . Each factor in the chain is isomorphic to R/cR. This contradicts our definition of Kdim(R).

1.4.4. Theorem: Let R be a ring with Krull dimension.

(i) Suppose that $P_1 < P_2$ are prime ideals of R. Then $\mathrm{Kdim}(R/P_2) < \mathrm{Kdim}(R/P_1)$.

(ii) R satisfies the ascending chain condition on prime ideals.

Proof: Gordon-Robson[32], Theorem 7.1 and Corollary 7.2.

1.4.5. Theorem: Let R be a right Noetherian ring. Suppose that A and B are two ideals of R. Then

$\mathrm{Kdim}(R/AB) = \sup\{\mathrm{Kdim}(R/A), \mathrm{Kdim}(R/B)\}$.

Proof: McConnell-Robson[51], Lemma 6.3.3.

1.4.6. Corollary: Let R be a right Noetherian ring. Let N be the nilpotent radical of R. Then $\mathrm{Kdim}(R) = \mathrm{Kdim}(R/N)$.

1.4.7. Theorem: Let R be a right Noetherian fully bounded Noetherian ring. Let M_R be a f.g. faithful right R-module. Then $\mathrm{Kdim}(M) = \mathrm{Kdim}(R)$.

Proof: Jategaonkar[40],Lemma 2.1.

1.4.8. Theorem: Let R be a right fully bounded Noetherian ring. Let S be an arbitrary ring. Let ${}_SM_R$ be an S-R-bimodule which is f.g. as a right R-module. Then the Krull dimension of the partially ordered set of all S-R-bi-submodules of M is $\mathrm{Kdim}(M_R)$.

Let R and S be fully bounded Noetherian rings. Suppose that there exists an R-S-bimodule which is f.g. and faithful on both sides. Then $\mathrm{Kdim}(R) = \mathrm{Kdim}(S)$.

Proof: Jategaonkar[40], Lemma 2.2 and Theorem 2.3.

Let R be a ring with Krull dimension. We say that an ideal I is <u>weak ideal invariant</u>, w.i.i., (strictly speaking right weak ideal invariant) if, for every f.g. right R-module M with $\mathrm{Kdim}(M) < \mathrm{Kdim}(R/I)$, we have $\mathrm{Kdim}(M \otimes I) < \mathrm{Kdim}(R/I)$. Equivalently, if K is a right ideal of R with $\mathrm{Kdim}(R/K) <$

$\mathrm{Kdim}(R/I)$, then $\mathrm{Kdim}(I/KI) < \mathrm{Kdim}(R/I)$. We say that R is <u>ideal invariant</u> if given any f.g. R-module M_R and an ideal I of R, then $\mathrm{Kdim}(M \otimes I) \leq \mathrm{Kdim}(M)$.

Note that not all Noetherian rings are w.i.i. For example, J.T.Stafford has shown that the the enveloping algebra (see Section 1.8) $U(sl_2 \times sl_2)$ is not w.i.i., see Stafford[71]. The (slightly stronger) notion of ideal invariance was first introduced in Krause-Lenagan-Stafford[45], and that of weak ideal invariance in Stafford[67].

1.4.9. Theorem: A Noetherian fully bounded ring is weak ideal invariant.

Proof: Suppose that I is an ideal of R. Suppose that T is a right ideal of R such that $\mathrm{Kdim}(R/T) < \mathrm{Kdim}(R/I)$. If r-ann$(R/T) = L$ then, by Theorem 1.4.7, $\mathrm{Kdim}(R/T) = \mathrm{Kdim}(R/L)$. Denote the left-hand analogue of Krull dimension by l-Kdim. Then we have, using Theorem 1.4.8, $\mathrm{Kdim}(I/TI) \leq \mathrm{Kdim}(I/LI) = \text{l-}\mathrm{Kdim}(I/LI) \leq \text{l-}\mathrm{Kdim}(R/L) = \mathrm{Kdim}(R/T) < \mathrm{Kdim}(R/I)$.

We now turn to another measure, known as the <u>Goldie rank</u> or <u>reduced rank</u> of a module, which was first introduced by A.W.Goldie. Here we follow Chatters-Hajarnavis[16]. Throughout M_R will be a f.g. R-module over a right Noetherian ring.

First, suppose that R is semi-prime. Then, by Theorem 1.1.3, the full right quotient ring of R, Q, exists and is semi-simple Artinian. Then $M \otimes Q$ is a semi-simple Q-module of finite length over Q. We set $\rho(M) = \mathrm{length}_Q(M \otimes Q)$.

Alternatively, we could define $\rho(M)$ to be the uniform rank of $M/T(M)$, where $T(M) = \{m \in M : mc=0$ for some regular element c in $R\}$.

1.4.10. Theorem: Let R be a right Noetherian semi-prime ring and M a f.g. R-module. If K is a submodule of M then

$$\rho(M) = \rho(K) + \rho(M/K).$$

Proof: Chatters-Hajarnavis[16], Lemma 2.1.

For a general right Noetherian ring recall that the nilpotent radical N of R satisfies $N^k = 0$, for some integer k and that R/N is semi-prime.

Let R be a right Noetherian ring with nilpotent radical N. Let M be a f.g. right R-module. Suppose that $N^k = 0$. Define $\rho(M) = \sum_{i=1}^{i=k} \rho_{R/N}(MN^{i-1}/MN^i)$ which is well-defined since each MN^{i-1}/MN^i is a f.g. R/N-module.

1.4.11. Theorem: Let R be a right Noetherian ring with nilpotent radical N. Let M be a f.g. R-module. Then:

(i) if K is a submodule of M then $\rho(M) = \rho(K) + \rho(M/K)$;

(ii) $\rho(M) = 0$ if and only if, for all $m \in M$, there exists $c \in \mathcal{C}(N)$ such that $mc=0$.

Proof: Chatters-Hajarnavis [16], Theorem 2.2.

Section 1.5. The Artin-Rees property.

Let I be an ideal of a ring R. We say that I has the right Artin-Rees property (AR-property for short) if, for each right ideal K of R, there exists n, a positive integer, such that $K \cap I^n \subseteq KI$. A ring R has the right AR-property if every ideal of R has the right AR-property. Left AR-properties are defined analogously. A ring R has the AR-property if every ideal has both the left and right-AR property. The following result gives us a useful criterion for an ideal I to have the AR-property.

1.5.1. Definition: An ideal I has a centralising set of generators if $I = a_1R + a_2R + \ldots + a_nR$, where $a_1 \in Z(R)$ and for each $i > 1$, the image of a_i is central in $R/(a_1R + \ldots + a_{i-1}R)$.

1.5.2. Theorem: Let R be a right Noetherian ring. Let I be an ideal of R.

(i) If I has a centralising set of generators then I has the AR-property.

(ii) If I has a single normal generator, that is $I = aR = Ra$, for some $a \in R$, then I has the AR-property.

Proof: Chatters-Hajarnavis[16], Theorem 11.7.

As the next results illustrate, the AR-property is closely related to the problem of localising at prime ideals.

1.5.3. Theorem: Let R be a right Noetherian ring. Let I be an ideal which has the right AR-property. Suppose that, for each positive integer n, the ring R/I^n satisfies the right Ore condition with respect to $C(I/I^n)$. Then R satisfies the right Ore condition with respect to $C(I)$.

Proof: Smith[66], Proposition 2.1.

1.5.4. Theorem: Let R be a Noetherian AR-ring. Let P be a semi-prime ideal of R. Then R satisfies the Ore condition with respect to $C(P)$.

Proof: Smith[66], Propostion 3.4.

1.5.5. Theorem: Let R be a Noetherian ring. Let P be a prime ideal of R which has the AR-property. Suppose that $C(P) = C(P^n)$, for all positive integers n. Then P is localisable.

Proof: We use Smith[66], Proposition 2.1 which says that P is localisable if and only if, for all positive integers n, $C(P) \subseteq C(P^n)$, and the set $K_n=\{r\epsilon R: rc\epsilon P^n$ for some $c\epsilon C(P)\}$ is an ideal.

Here $K_n=\{r\epsilon R: rc\epsilon P^n$ for some $c\epsilon C(P^n)\} = P^n$. So, clearly, K_n is an ideal, for all n, and $C(P) = C(P^n)$.

To end this section, we note a result which we will use often.

1.5.6. Lemma: Let R be a prime Noetherian ring. Let P be a localisable prime ideal of R. Let Q be a prime ideal of R with

$Q \subset P$. Then $C(P) \subseteq C(Q)$. In particular, $C(P) \subseteq C(0)$.

Proof: Let $K = \{r : rc \in Q$, for some $c \in C(P)\}$. Since $C(P)$ is Ore, K is an ideal of R. So $K = \sum_{i=1}^{i=n} Ra_i$, for some elements a_i. So there exists $c \in C(P)$ with $Kc \subseteq Q$. Since Q is a prime ideal, either $c \in Q$ or $K \subseteq Q$. But $Q \subset P$ and so $c \notin Q$. Therefore $K \subseteq Q$. Hence $C(P) \subseteq C(Q)$.

Section 1.6. Stable Range.

In commutative Noetherian ring theory powerful results have been proved by Forster-Swan, Bass, and Serre concerning cancellation properties. These may be expressed as "large" projectives have free direct summands and free summands can be cancelled if the module is "large".

We wish to determine some bounds on the "largeness" required for these statements. The bounds rely on the notions of stable range and general stable range. We will follow McConnell-Robson[51] in our treatment.

Let R be any ring. Let M be a f.g. right R-module. We say that n is in the stable range of M if, for all s≥1, if $\sum_{i=1}^{i=n+s} m_i R = M$, then there exist elements $f_i \in R$ for $i=1,\ldots,n+s-1$ such that

$$M = \sum_{i=1}^{i=n+s-1} (m_i + m_{n+s} f_i)R.$$

The least n in the stable range of M is known as the stable rank of M and denoted by sr(M). For a ring R, $sr(R) = sr(R_R)$.

We will call a row $x^* = (x_1,\ldots,x_t) \in R^t$ right unimodular if $\sum x_i R = R$. Then n is in the stable range of R if and only if, for all s≥1, for every unimodular row $(x_1,\ldots,x_{n+s})$ in R^{n+s}, there exist elements $f_i \in R$ such that the row $(x_1 + x_{n+s} f_1, \ldots, x_{n+s-1} + x_{n+s} f_{n+s-1})$ is right unimodular in R^{n+s-1}. In this situation we say that $(x_1,\ldots,x_{n+s})$ is stable. So sr(R) = n if and only if every right unimodular row of length ≥ n+1 is stable.

We can define two other closely related ranks as follows. For n≥1, define $GL_n(R)$ to be the group of n×n invertible matrices over R. An element $A \in GL_n(R)$ is elementary if it is of the form $I_n + ae_{ij}$, where $a \in R$ and e_{ij} is the matrix with a 1 in the ijth position and zeroes everywhere else. Then $E_n(R)$ is the subgroup of $GL_n(R)$ generated by all the elementary matrices.

We say that n is in the general stable range of R if and only if, for all $s \geq 1$, $GL_{n+s}(R)$ acts transitively on the set of right unimodular rows of R^{n+s}. The least such n is the <u>general stable rank</u> of R, and is denoted by gsr(R). The elementary stable range and <u>elementary stable rank</u>, esr(R), are defined analogously, replacing $GL_{n+s}(R)$, by $E_{n+s}(R)$.

1.6.1. Theorem: For any ring R, $gsr(R) \leq esr(R) \leq sr(R)$.

Proof: McConnell-Robson[51], Theorem 11.3.1.

At first glance, it appears that these stable ranks rely on whether we consider the right or left unimodular rows. The next result shows that they are independent of side.

1.6.2. Theorem: Let R be a ring. Then the left and right stable ranks of R are equal. The corresponding results also hold for the general stable rank and the elementary stable rank.

Proof: McConnell-Robson[51], Theorem 11.3.4.

The significance of these ranks are demonstrated by the following two theorems whose proofs are similar to that of McConnell-Robson[51], Theorem 11.1.12.

1.6.3. Theorem: Let S be a ring. Let M be a left S-module. Suppose that ${}_SM$ has endomorphism ring $\mathrm{End}_S(M) \simeq R$.

(i) Suppose that $x^*=(x_1,\ldots,x_t)\epsilon R^t$ is a stable right unimodular row. Then the cokernel of the split monomorphism $\theta_x : M \longrightarrow M^t$ given by $m \longrightarrow \Sigma m x_i$ is isomorphic to M^{t-1}.

(ii) Suppose that $y^*=(y_1,\ldots,y_t)\epsilon R^t$ is a stable left unimodular row. Then the kernel of the split epimorphism $\theta_y : M^t \longrightarrow M$, given by $(m_1,\ldots,m_t) \longrightarrow \Sigma m_i y_i$, is isomorphic to M^{t-1}.

1.6.4. Theorem: For t a positive integer the following are equivalent:

(i) $t > \mathrm{gsr}(R)$;

(ii) If M_S satisfies $\mathrm{End}(M_S) \simeq R$ and $M \oplus N \simeq M^t$ then $N \simeq M^{t-1}$;

(iii) If X_R satisfies $R \oplus X \simeq R^t$ then $X \simeq R^{t-1}$;

(iv) If $x^* \epsilon R^t$ is left unimodular then x^* is a column of an invertible matrix in $M_t(R)$.

Suppose that M_R is an R-module such that $M \oplus R^m \simeq R^n$, for some integers m and n. Then we say that M is stably free of rank (n-m).

1.6.5. Corollary: R has general stable rank $\leq n$ if and only if all stably free R-modules of rank $\geq n$ are free.

For stable rank, we have a stronger result. First a definition. Let M_S be a module over a ring S. We will say that M has the <u>n-substitution</u> property if given any split endomorphism $\pi: M^n \oplus N \longrightarrow M$ there exist S-module homomorphisms $\psi: M \longrightarrow M^n \oplus N$ and $\theta: M^n \oplus N \longrightarrow M$ such that $\pi\psi = \theta\psi = 1_M$ and $N \subseteq \mathrm{Ker}\theta$.

1.6.6. Theorem: Let M_S be a module over a ring S. Suppose that $\mathrm{End}(M_S) \cong R$. Then the following are equivalent:

(i) n is in the stable range of R;

(ii) M_S has the n-substitution property.

1.6.7. Corollary: Let M_S be a module over a ring S. Suppose that $\mathrm{End}(M_S) \cong R$. Suppose that n is in the stable range of R and that $M^{n+1} \oplus X \cong M \oplus Y$. Then $M^n \oplus X \cong Y$.

To apply these results we need to be able to calculate suitable upper bounds for gsr(R), esr(R), and sr(R) for a ring R. One of the results obtained by J.T.Stafford is given here.

1.6.8. Theorem: Let R be a right Noetherian ring. Suppose that $\mathrm{Kdim}(R) = n$. Then $\mathrm{sr}(R) \leqslant n+1$.

Proof: Stafford [68], Theorem.

Clearly, the smaller the stable rank of a ring the better to describe the structure of R-modules. For example, taking an example from algebraic K-theory, if $\mathrm{sr}(R) = n$, then $K_1(R) \cong GL_{n+2}(R)/E_{n+2}(R)$. So a bound on the stable rank of a ring is very useful.

In the case of stable rank one we can even sharpen the results of this section a little further. Since in Chapter 3 we do show that certain rings have stable rank one we note briefly a couple of these results.

For a ring R, suppose that M is f.g. R-module. Let $g(M)$ denote the minimal number of generators of M and let $sr(M)$, as before, denote the stable rank of M. We say that M is uniquely presentable by a projective module P if there is an epimorphism $P \longrightarrow M$ and that any two such epimorphisms are right equivalent. That is, if f and g are any two such epimorphisms then there exists an isomorphism $\psi : P \longrightarrow P$ such that $f = g\psi$. Let $u(M)$ be the least integer m such that M is uniquely presentable by R^m.

1.6.9. Theorem: Let R be a ring. Suppose that M is a f.g. R-module with $sr(M)$ finite. Then $u(M) \leq g(M) + sr(M)$.

Proof: Warfield[73], Proposition 3.

1.6.10. Theorem: Let R be a ring with stable rank one. Then $g(M) = u(M) = sr(M)$.

Proof: Warfield[73], Theorem 7.

We can also sharpen Corollary 1.6.7 as follows.

1.6.11. Theorem: Let R be a ring with stable rank one. Let S be an arbitrary ring. Suppose that M is a S-module whose endomorphism ring is isomorphic to R. Suppose that there exist S-modules X and Y such that

$$M \oplus X \simeq M \oplus Y.$$

Then $X \simeq Y$.

Proof: McConnell-Robson[51], Theorem 11.4.9.

Section 1.7. Lattice conditions and factorisation.

In this section we recall some definitions and results that arise when we consider the "factorisation" of elements of a ring. We introduce the notion of modular lattice and Bezout domain which we will use in Sections 3.4 and 5.3.

Let $(X,\leq)$ be a partially ordered set (poset). We say that two elements x and y of X have a <u>least upper bound</u> or <u>sup</u>, denoted by $x\nabla y$, if there exists an element z with $x \leq z$ and $y \leq z$, and if w is any other upper bound on x and y, then $z \leq w$. Dually, we may define a <u>greatest lower bound</u> or <u>inf</u>, denoted by $x\Delta y$, in the obvious way.

A <u>lattice</u> is a poset such that any two elements have a sup and an inf. A lattice is <u>complete</u> if any set of elements has a sup and an inf. In particular, a complete lattice has a greatest and a least element, denoted by 1 and 0 respectively.

Let L be a complete lattice. Given any two elements $a,b\in L$, with $a \leq b$, we may define the <u>interval</u>, $[a,b]$, as $\{x\in L: a \leq x \leq b\}$. We say a lattice is <u>modular</u> if $a\nabla(b\Delta c) = (a\nabla b)\Delta(a\nabla c)$, for all elements a,b,c, with $a \leq c$. For example, the lattice of sub-modules for a module M is a modular lattice. In particular, the lattice of right ideals of a ring is a modular lattice.

Suppose that a and b are two elements of a lattice. Let $I = [a\Delta b,a]$ and let $J = [b,a\nabla b]$. We may define order-preserving maps $\alpha:I\longrightarrow J$ and $\beta:J\longrightarrow I$ by $\alpha(x) = x\nabla b$, and $\beta(y) = a\Delta y$. If L is modular, then $\alpha\beta = Id_I$, and $\beta\alpha = Id_J$. See, for example, Cohn[21], Section 2.1.

In an interval $[a,b]$, a <u>complement</u> of $c\in[a,b]$ is an element d with $c\Delta d = a$ and $c\nabla d = b$. We have the following criterion for a lattice to be modular.

1.7.1. Lemma: A lattice L is modular if and only if for each interval I of L any two comparable elements of I which have a common complement are equal.

Proof: Cohn[21], Proposition 2.1.3.

We now turn to the question of factorisation. In a ring R, an atom is an element which cannot be written as the product of two non-units. A domain is atomic if every non-zero element can be written as the product of a finite number of atoms. To investigate the factorisation of elements of a domain into atoms we want to investigate when the sub-lattice of principal right ideals of the lattice of right ideals is a modular lattice.

We define a right Bezout domain to be a domain R in which for any two elements $a,b \in R$, $aR+bR = cR$, for some element $c \in R$. A left Bezout domain and a Bezout domain are defined in the obvious way.

1.7.2. Lemma: A domain R is a principal right ideal domain if and only if R is an atomic right Bezout domain.

Proof: Suppose that R is an atomic right Bezout domain. Let I be a right ideal of R. Since R is atomic, we may choose an element $a \in I$ such that aR is a maximal principal right ideal contained in I. If $aR \neq I$, choose $b \in I \setminus aR$. Then $aR+bR = cR \leq I$, which contradicts our choice of a. Thus I is principal.

The converse is clear.

We shall, in Section 3.4, use this criterion to construct a commutative principal ideal domain.

Section 1.8. Some classes of Noetherian rings.

In this section we introduce some of the classes of Noetherian rings to be considered in the subsequent chapters. We will set out the basic definitions and properties of these rings and draw upon this section as required.

In commutative ring theory, the polynomial ring $R[x]$ in an indeterminate plays a basic role. The corresponding concept for non-commutative ring theory is the skew polynomial ring.

Let R be any ring. Let θ be an automorphism of R and let δ be a θ-derivation of R. That is; $\delta(ab) = \delta(a)\theta(b)+a\delta(b)$. We may define the ring $S = R[x:\theta;\delta]$ as follows. Let S be the free right R-module on the generators $\{1,x,x^2,\ldots\}$, where x is an indeterminate. Define on S the multiplication determined by $ax = x\theta(a) + \delta(a)$ and extended by the ring axioms. Then S is a skew polynomial extension of R. Note that every element of S can be expressed uniquely in the form $\sum_{i=0}^{i=n} x^i a_i$, for $a_i \in R$.

With R,θ,δ as above, we have the following useful result.

1.8.1. Let R be a right Noetherian domain. Then $R[x:\theta;\delta]$ is a right Noetherian domain.

Proof: McConnell-Robson[51], Theorem 1.2.9.

Let θ be an automorphism of R. Then $S = R[x,x^{-1}:\theta]$ denotes the ring of polynomials over R in x and x^{-1} subject to the relation that $ax = x\theta(a)$. This is a skew Laurent extension of R. Let $T = R[x:\theta]$ be the skew polynomial extension of R. Then, since $xR = Rx$, the set $C = \{1,x,x^2,\ldots\}$ is Ore, by Lemma 1.1.8.

It is easy to see that $S \cong T_C$.

1.8.2. Corollary: Let R be a right Noetherian domain. Then $R[x,x^{-1}:\theta]$ is a right Noetherian domain.

Many rings may be characterised either as skew polynomial extensions or skew Laurent extensions. Here, we note just a few examples that we shall use later on.

Let R be a ring. Let $S = R[x]$. Let δ be the derivation on S such that $\delta(r) = 0$, for all $r \in R$, and $\delta(x) = 1$. Let $A_1(R) = S[y:\delta]$. We say that $A_1(R)$ is the <u>first Weyl algebra over R</u>. Clearly, $A_1(R)$ may be thought of as the ring freely generated over R by x and y subject only to the relation $xy-yx=1$. Inductively, for any positive integer n, we may define $A_n(R) = A_1(A_{n-1}(R))$. By Lemma 1.8.1, if R is a right Noetherian domain, then $A_n(R)$ is a right Noetherian domain.

Suppose that λ is a unit of R. Let $T = R[x]$. Let θ be the R-automorphism on T which sends x to λx. Let $B_\lambda(R) = T[y:\theta]$. Then $B_\lambda(R)$ may be thought of as the ring freely generated over R by x and y subject only to the relation $xy=\lambda yx$.

Let R be a right Noetherian domain. Then $B_\lambda(R)$ is a right Noetherian domain, by Lemma 1.8.1.

We now turn to two major sub-classes of naturally occuring Noetherian rings.

First, suppose that L is a f.d. Lie algebra over a field of characteristic zero. A sub-space of L, K, is an ideal of L if $[L,K] \leq K$. A Lie algebra is simple if it has dimension greater than 1 and contains no non-trivial ideals. A Lie algebra is semi-simple if it may be written as a direct sum ΣK_i, where the K_i are ideals of L and simple Lie algebras themselves.

Let L be a Lie algebra. Define, for positive integers n, the ideals of L, $C^n(L)$ and $D^n(L)$, as follows. Let $C^1(L) = L$; and $C^{i+1}(L) = [L, C^1(L)]$, for $i \geq 1$. Let $D^0(L) = L$; and $D^{i+1}(L) = [D^i(L), D^i(L)]$, for $i \geq 0$. We say that L is nilpotent if, for some integer c, $C^c(L) = 0$, and the least such c is the nilpotency class of L. We say that L is solvable if $D^n(L) = 0$, for some integer n.

For a f.d. Lie algebra over a field k, we can construct a associative k-algebra U(L) and a unique k-linear map $\lambda: L \longrightarrow U(L)$, which is universal with respect to the property that $\lambda([x,y]) = \lambda(x)\lambda(y) - \lambda(y)\lambda(x)$, for all $x, y \in L$ (*). That is, given any other k-linear map from L to a k-algebra A, $\mu: L \longrightarrow A$ satisfying (*), then there exists a unique k-algebra homomorphism $f: U(L) \longrightarrow A$ with $\mu = \lambda f$.

For a more explicit construction, we follow Dixmier[25], Chapter 2. We define T to be the tensor algebra of L; that is $T = L^0 \oplus L^1 \oplus L^2 \oplus \ldots$, where L^n denotes the n-fold tensor product of L over k. Let J be the ideal of T generated by all the terms of the form $x \otimes y - y \otimes x - [x,y]$, as x,y run over L.

Then define $U(L) = T/J$. Note that, if L is Abelian, then U(L) is isomorphic to the polynomial ring $k[x_1, \ldots, x_n]$, where $n = \dim_K L$. The following result is a corollary of the Poincare-Birkhoff-Witt theorem.

1.8.3. Theorem: The map $\lambda: L \longrightarrow U(L)$ is injective.

Proof: Dixmier[25], Proposition 2.1.9.

1.8.4. Theorem: Let L be a finite-dimensional Lie algebra over a field of characteristic zero. Then U(L) is a Noetherian domain.

Proof: Dixmier[25], Corollaires 2.3.8 and 2.3.9.

With U(L) we can also define Z(L), the centre of U(L). Sometimes, however, this is a slightly too restrictive notion and we need to consider the semi-centre of U(L) which is defined as follows. Recall that for a Lie agebra L, we can define $L^* = \mathrm{Hom}_k(L,k)$.

Since U(L) is a Noetherian domain, we may construct its division ring of fractions D(L). For each $\lambda \in L^*$, let $D(L)_\lambda$ be defined by $D(L)_\lambda = \{u \in D(L): xu-ux=\lambda(x)u \text{ for all } x \in L\}$. Clearly, $D_\lambda(L)D_\mu(L) \subseteq D_{\lambda+\mu}(L)$, for all $\lambda,\mu \in L^*$; and it is easy to see that $D_\lambda(L) \cap D_\mu(L) = 0$, for all $\lambda \neq \mu$. Then the sum $\sum D_\lambda(L)$, over all $\lambda \in L^*$, is direct and is a sub-algebra SZ(D(L)) of D(L). Now put $U(L)_\lambda = D(L)_\lambda \cap U(L)$. Let $SZ(L) = \sum U(L)_\lambda$. This again is a direct sum and defines a sub-algebra of U(L), the <u>semi-centre</u> of U(L). If char(k)=0 and L is finite-dimensional, then SZ(L) is a commutative ring (Dixmier[25], Proposition 4.3.5). If k is algebraically closed, then SZ(L) is a unique factorisation domain. In general, SZ(L) is contained in a unique factorisation domain (Delvaux-Nauwelaerts-Ooms[24], Theorem 1.2).

Now we turn to group rings. For a group G and a ring K we can form the group ring, R = KG, which is the set of finite sums of the form $\Sigma k_g.g$, for $k_g \in K$ and $g \in G$. Addition is defined co-ordinate-wise and multiplication is defined by $rg.sh=rs.gh$ for $r,s\in K$ and $g,h\in G$, and extended by linearity.

Generally, we shall suppose that K is commutative.

Like universal enveloping algebras, the class of group rings is a broad and interesting class of rings. We will only make some basic definitions and record some useful results for later.

For a group G, we define the following sub-groups: $Z(G)$ is the centre of G; $\Delta(G)$ is the set of elements of G with only finitely many conjugates; and $\Delta^+(G)$ is the intersection of all the finite normal subgroups of G. For any group G, $\Delta^+(G)$ is a characteristic sub-group contained in $\Delta(G)$ and $\Delta(G)/\Delta^+(G)$ is torsion-free Abelian.

Let G be any group. We may define the sugroups of G, $Z^i(G)$, for all integers i, as follows. Let $Z^1(G) = Z(G)$. For $i > 1$, let $Z^{i+1}(G) = \{g\in G:\bar{g}\in Z(G/Z^i(G))\}$. If there exists an integer n with $Z^n(G) = G$, then we say that G is <u>nilpotent</u> and if the least such integer is c, we say that G has nilpotency class c.

Let Π denote a class of groups. Then a group G is said to be poly-Π if there exists a sub-normal chain of sub-groups in G such that $\langle 1\rangle = G_0 \leqslant G_1 \leqslant \ldots \leqslant G_n = G$, with, for each i, G_i normal in G_{i+1} and G_{i+1}/G_i belonging to Π. Let Π and Λ be two classes of groups. Then a group G is said to be a Π-by-Λ group, if there exists a normal sub-group H of G such that H belongs to Π and G/H belongs to Λ.

Let G be a poly-cyclic group. Then the <u>Hirsch number</u> of G, $h(G)$, is the minimal number of infinite-cyclic groups that occur in a sub-normal chain of G.

We record the following two results.

1.8.5. Theorem: Let K be a commutative Noetherian domain. Let G be a torsion-free poly-infinite-cyclic-by-finite group. Then KG is a Noetherian domain.

Proof: Passman[54], Theorem 10.2.7, and Farkas-Snider[28], Main Theorem (for the case char(K)=0) and Cliff[18], Theorem 2 (for char(K)=p≠0).

1.8.6. Theorem: Let G be a torsion-free nilpotent group. Then G/Z(G) is torsion-free nilpotent.

Proof: Passman[54], Lemma 11.1.3.

For a group ring KG we can define the augmentation ideal, I, by $I = \{\Sigma k_g g : \Sigma k_g = 0\}$.

We can define a more general construction than a group ring known as the skew group ring as follows. Let K be a ring and G a group. Suppose that there exists a group homomorphism $\theta: G \longrightarrow \mathrm{Aut}(K)$. Then denote $\theta(g)(r)$ by $g(r)$. We may then define K*G to be the free K-module with the elements of G as the free basis with addition co-ordinate-wise and multiplication defined by $rg.sh = rg(s).gh$, for all $r,s \in K$ and $g,h \in G$. The group ring KG is then simply the skew group ring K*G, where θ sends every element of G to the identity automorphism.

1.8.7. Theorem: Let K be a commutative Noetherian domain of characteristic zero. Let G be a poly-infinite-cyclic group. Then K*G is a Noetherian domain.

Proof: We proceed by induction on the Hirsch number of G. The result is clear for G an infinite-cyclic group.

Suppose that N is a normal poly-infinite-cyclic sub-group of G such that G/N is infinite-cyclic. By McConnell-Robson [51], Proposition 1.5.11, $R*G \cong R*N[x,x^{-1}:\theta]$, for some automorphism θ of R*N. By induction, R*N is a Noetherian domain. By Corollary 1.8.2, R*G is a Noetherian domain.

Chapter 2. Unique Factorisation Domains.

2.0. Summary.

In this chapter we consider a non-commutative analogue of the commutative unique factorisation domain, proposed by A. W.Chatters, and give examples of rings satisfying these conditions.

Section 2.1 will outline the definition and basic properties of UFDs. This work is entirely due to A.W.Chatters. Sections 2.2, 2.3, and 2.4 will show that though the definition of UFDs appear very restrictive in fact large classes of naturally occuring Noetherian rings actually satisfy the conditions. So the development of a theory of UFDs may well help in the study of these rings. Section 2.2 is effectively in the literature and Section 2.3 is a simplified account of results due to K.A.Brown. Section 2.4 gives a simple variation on a theme used by many people and a non-commutative analogue of Nagata's Theorem. It comprises the only original section of this chapter.

Section 2.1. Definition of Unique Factorisation Domains.

Let R be a prime Noetherian ring. A height-1 prime ideal is a prime minimal amongst the set of non-zero prime ideals of R. We will call a non-zero element p of R a prime element if pR = Rp is a prime ideal of R and R/pR is a domain. Following Chatters, let $C(R) = \cap C(P)$, where the intersection is over all the height-1 prime ideals of R, be the set of all elements of R which are regular modulo all the height-1 prime ideals of R. If there is no risk of confusion we shall write C for C(R). Note that for a prime element p, $C(pR) = R \backslash pR$.
Then we have (Chatters[13], Proposition 2.1)

2.1.1. Theorem: Let R be a prime Noetherian ring with at least one height-1 prime ideal. Then the following conditions on R are equivalent:

(1) Every height-1 prime ideal of R is of the form pR = Rp for some prime element of R.

(2) R is a domain and every non-zero element of R is of the form $cp_1p_2\cdots p_n$, for some $c \in C$ (as defined above) and for some finite sequence of prime elements p_i of R. Note that we will follow the convention that the product of an empty set of prime elements is 1.

Proof: Suppose that R satisfies (1). Let P be a height-1 prime of R. Then P = pR for some prime element of R. For every positive integer n, we have $P^n = p^nR = Rp^n$. Also $px=0$ implies that $Rpx = pRx = 0$. Since R is prime, $x = 0$, and hence p is regular. Suppose that $\cap P^n = I \neq 0$. Then there exists a regular element $x\in\bar{I}$. Thus for each n, $x\in\bar{P}^n$; that is for each n, $x = p^n x_n$ for some x_n regular in R.

But then $x_1 = px_2$, and $Rx_1 \subseteq Rx_2$. But $R \neq pR$ and x_2 is regular and so $Rx_1 = Rpx_2 \neq Rx_2$. Similarly $Rx_2 = Rpx_3 \neq Rx_3$, and so on. Thus the left ideals Rx_n form an infinite strictly ascending sequence which contradicts our assumption that R is Noetherian.

Now let $c\in C(P)$. We shall show by induction that $c\in C(P^n)$ for every postive integer n. Suppose that $c\in C(P^n)$ and that $r\in R$ is such that $cr\in P^{n+1}$. Then certainly $cr\in P^n$ and so $r=sp^n$, for some $s\in R$. Then $csp^n\in Rp^{n+1}$ and so $cs\in Rp$. Hence $s\in Rp$ and $r\in P^{n+1}$. By a symmetrical argument it follows that $c\in C(P^{n+1})$, and hence, by induction, that $c\in C(P^n)$ for all n. Also if $cr=0$ then $r\in P^n$ for all n and hence $r=0$. Therefore $c\in C(0)$.

Let a be non-zero element of R. Then there is a positive integer n such that $a\in P^n$ and $a\notin P^{n+1}$. Thus $a = p^n b$ for some $b\in C(P)$ and it follows that a is regular. Therefore R is a domain.

Let x be a non-zero element of R. Because R is Noetherian there are only finitely many prime ideals minimal over RxR. So x lies in only finitely many height-1 prime ideals of R. Therefore there exist prime elements $p_1, p_2, \ldots, p_n$ such that $x = cp_1p_2\ldots p_n$ for some $c\in C(R)$. Thus R satisfies condition (2).

Now suppose that R satisfies condition (2). Let P be a height-1 prime ideal of R. Let x be a non-zero element of P. Then $x = cp_1p_2\ldots p_n$ for some $c\in C$ and prime elements p_i. Because $c\in C(P)$ and $p_iR = Rp_i$, for each i, it follows that $p_i\in P$ for some i, and that $P = p_iR$. So R satisfies (1).

2.1.2. Definition: A prime Noetherian ring such that every non-zero prime ideal contains a height-1 prime ideal and which satisfies one of the two equivalent conditions of Theorem 2.1.1 is a <u>Noetherian Unique Factorisation Domain (UFD).</u>

It might be thought that the requirement that each height-1 prime ideal is generated by the same element on each side is unnecessarily restrictive and that we could make do with each each height-1 prime ideal being of the form P = pR = Rq. However it is easy to see that this gives us no more generality. We have p=uq and q=pv for some elements u and v in R. Then up=pw=uqw=upvw. But u and p are both regular and so 1=vw; that is v is a unit and similarly so is u.

We shall call a prime Noetherian ring R a Unique Factorisation Ring (UFR) if we simply require that every non-zero prime ideal of R contains a height-1 prime ideal which is principal on both sides. We shall discuss Noetherian UFRs more fully in Chapter 5.

Note that the condition that every non-zero prime ideal contains a height-1 prime ideal is not necessary for much of the following.It is unknown, in general, whether Noetherian rings satisfy the descending chain condition on prime ideals.

A.V.Jategaonkar has, though, constructed examples of right Noetherian (even principal right ideal domains) rings which do not have height-1 prime ideals. See Jategaonkar[37] for the details.

Second, we remark that, when R is a commutative ring this definition of a Noetherian UFD coincides with the classical definition of UFD. Thus this definition is a plausible generalisation of the commutative case. If R were commutative then the set $C = \cap C(P)$, defined as above, would simply be the set of units of R, by the classical commutative principal ideal theorem. But when R is not commutative, the set C can be strictly larger than the set of units of R. The question of when C is precisely the set of units will concern us more in Chapter 4 . For the moment we may consider two "generic" examples. Let $R_1 = k[x,y]$, be the ring of polynomials in two commuting variables over the field k of the complex numbers; and let ring R_2 be the enveloping algebra of the complex two-dimensional solvable Lie algebra, which we may write as $k[x,y:xy-yx=y]$.

2.1.3. Lemma: Let k be the field of the complex numbers. Let R $= k[x,y:xy-yx=y]$. Then R is a Noetherian domain. R has a unique height-1 prime ideal $P = yR = Ry$. Further, R/P is isomorphic to the polynomial ring in one indeterminate over k. Each height-2 prime ideal of R is of the form $(x-\alpha)R + yR$, for some $\alpha \in k$.

Proof: By Lemma 1.8.1, R is a Noetherian domain. Suppose that $f(x) \in R[x]$ and $g(y) \in k[y]$. Then $f(x)y = yf(x+1)$ and $xg(y)-g(y)x = yg'(y)$, where g' denotes the y-derivative of g.

Clearly, $P = yR = Ry$ is a height-1 prime ideal of R and $R/P \cong k[x]$. Now suppose that Q is any non-zero prime ideal of R. We shall show that $y \in Q$. First, we claim that there exists $f(x) \in Q$, for some $0 \neq f(x) \in k[x]$. Suppose not. Choose $f = f(x,y) =$

$\Sigma_{i=0}^{i=n} f_i(x)y^i$, for $f_i(x) \epsilon k[x]$, for least n.

Then $\quad xf(x,y)-f(x,y)x = \Sigma_{i=0}^{i=n}[xf_i(x)-f_i(x)(x+i)]y^i$

$$= \Sigma_{i=1}^{i=n} f_i(x)iy^i$$

$$= [\Sigma_{i=0}^{i=n-1} h_i(x)y^i]y$$

$= h(x,y)y$, where the y-degree of $h(x,y)$ is n-1. However, $xf-fx \epsilon Q$. Therefore, if $y \notin Q$, then $h(x,y) \epsilon Q$, which contradicts our choice of $f(x,y)$. So there exists $0 \neq f(x) \epsilon Q$. Now choose $0 \neq f(x) \epsilon Q$ of least degree in x. Then, by a similar argument to the above, $f(x)y-yf(x) = h(x)y$, where $h(x)$ is of lower degree in x. If $y \notin Q$, then $h(x) \epsilon Q$, which contradicts our choice of $f(x)$. So we are forced to conclude that $y \epsilon Q$. Hence, Q is the unique height-1 prime ideal of R. The rest follows easily.

So both R_1 and R_2 are Noetherian UFDs as defined above. Then $C(R_1)$ is the set of non-zero elements of k ie the units of R_1. However, since $yR_2 = R_2y$ is the only height-1 prime ideal of R_2, $C(R_2)$ is the set of all polynomials not contained in $yR_2 = R_2y$. Thus $C(R_2)$ is a very much larger set than the units of R_2.

We shall now show that for R a Noetherian UFD, C(R) is always Ore. The following result is a very slight generalisation of the result by Chatters (Chatters[13],Proposition 2.5), but the extra generality will be useful in Section 2.4.

2.1.4. Theorem: Let R be a Noetherian domain, X a set of height-1 prime ideals that are principal on both sides and completely prime. Let $C(X) = \cap C(P)$, where the intersection runs over all $P \in X$. Then $C(X)$ is Ore.

Proof: It is easy to see that every non-zero element d of R can be written as $d = ep_1p_2 \ldots p_n$, where $e \in C(X)$ and each p_iR is a member of X. Let $c \in C$ and $a \in R$. Since R is a Noetherian domain there exist y and $x \in R$ such that $cx = ay$. Now y may be written as $y = ep_1p_2 \ldots p_n$, where $e \in C(X)$ and $p_iR \in X$. But $c \in C(p_iR)$ for all i so $x = wp_1p_2 \ldots p_n$. Therefore $cw = ae$ as required. A symmetrical argument will then show that $C(X)$ is Ore.

2.1.5. Corollary: Let R be a Noetherian UFD. Let C be the set of elements of R which are regular modulo all the height-1 prime ideals of R. Then C is Ore.

2.1.6. Theorem: Let R be a Noetherian UFD and let T be the partial quotient ring of R with respect to C. Then:

(i) T is a Noetherian UFD.

(ii) The elements of $C(T)$ are units of T.

(iii) Every one-sided ideal of T is two-sided.

(iv) $AB = BA$ for all ideals A and B of T.

Proof: Let p be a prime element of R. Because $C \subseteq C(pR)$ it is clear that $pT = Tp$ and that T/pT is a domain. Also the height-1 primes of T are precisely the extensions to T of the height-1 prime ideals of R. This proves (i).

Let $t \in C(T)$. Then $t = ac^{-1}$ for some $a \in R$ and $c \in C$. Thus $a = tc$ where c is a unit of T. Then $a \in C(R)$ and so a, and hence t, is a unit of T.

Let x be a non-zero element of T. Then, by (i) and (ii), x =

$up_1p_2\ldots p_n$, for some unit u of T and prime elements p_i of T. Clearly then xT = Tx and so every one-sided ideal of T is two-sided.

Finally, if pT and qT are distinct prime ideals of T then $pTqT = pT \cap qT = qTpT$ and it follows easily that the multiplication of ideals is commutative.

To finish this section we prove some results concerning UFDs and maximal orders. Recall from Section 1.2, that R is a maximal order in its full ring of quotients Q if, whenever $q \in Q$ and I a non-zero ideal of R are such that either $qI \subseteq I$ or $Iq \subseteq I$, then $q \in R$.

2.1.7. Theorem: Let R be a Noetherian UFD and let pR be a height-1 prime ideal of R. Then the classical localisation $R_{C(pR)}$ exists and is a maximal order.

Proof: The first statement follows from Theorem 2.1.4. Let S = $R_{C(pR)}$. Now observe that pS is the unique maximal ideal of S and every ideal of S is of the form p^nS, for some integer n. Then it is easy to see that S is a maximal order and also that S is a local Noetherian ring with Jacobson radical pS.

2.1.8. Theorem: Let R be a Noetherian UFD. Let T be the partial quotient ring of R with respect to C. Then $T = \cap R_{C(P)}$, where the intersection ranges over all the height-1 prime ideals P of R. Further T is a maximal order.

Proof: Let $U = \cap R_{C(P)}$, as above. Then $T \subseteq U$ because $C \subseteq C(P)$ for all height-1 prime ideals of R. Observe then that $R_{C(P)} = T_{C(PT)}$ for any height-1 prime ideal of R and that $U = \cap T_{C(PT)}$.

Let $u \in U$. Then $xu \in T$ for some non-zero element x of T. By the above we may assume that $x = p_1 p_2 \dots p_n$, for prime elements p_i of T. Then $p_1 p_2 \dots p_n u \in T$ and $v = p_2 \dots p_n u \in U$. Hence $vuc \in T$ for some $c \in C(p_1 T)$. Then $p_1 vc \in p_1 T$ so $p_1 v \in p_1 T$ and hence $p_2 \dots p_n u \in T$. Proceeding, by induction on n, gives $u \in T$.

Now let I be a non-zero ideal of T and let y be an element of the quotient ring of T such that $yI \subseteq I$. Let Q be any height-1 prime ideal of T. Then $yIT_{C(Q)} \subseteq IT_{C(Q)}$. So, by Theorem 2.1.7, $y \in T_{C(Q)}$. So $y \in \cap T_{C(Q)} = T$. Therefore T is a maximal order.

2.1.9. Theorem: Let R be a Noetherian UFD such that every non-zero prime ideal of R contains a height-1 prime ideal. Then R is a maximal order.

Proof: Let D be the multiplicatively closed set generated by the prime elements of R. Then $dR = Rd$ for elements d in D. Hence R satisfies the Ore condition with respect to D, by Lemma 1.1.8. $S = R_D$ is simple, since if I is a non-zero ideal of R then $I \cap D \neq \emptyset$ and so $IS = S$.

Now let I be a non-zero ideal of R and let q be an element of the quotient ring of R such that $qI \subseteq I$. Then $qIS \subseteq IS$; that is $qS \subseteq S$ and so $q \in S$. If T is the partial quotient ring of R with respect to C(R), then $qIT \subseteq IT$. But IT is an ideal of

T and, since T is a maximal order, $q \in T$.

We shall complete the proof by showing that $R = T \cap S$, since then $q \in R$ is immediate. Let $u \in T \cap S$. As $u \in S$, there are prime elements $p_1, p_2, \ldots, p_n$ of R such that $p_1 p_2 \ldots p_n u \in R$. Also $v = p_2 \ldots p_n u \in T$ and so $vc \in R$, for some $c \in C(R)$. Then we have $p_1 vc \in p_1 R$ and hence $p_1 v \in p_1 R$. So $p_2 \ldots p_n u \in R$. Proceeding, by induction on n, gives $u \in R$.

2.1.10. Corollary: Let R be a Noetherian domain such that every non-zero prime ideal contains a height-1 prime ideal. Then R is a Noetherian UFD if and only if:

(i) R is a maximal order;

(ii) every height-1 prime ideal is principal on one side and is completely prime.

Proof: Immediate from Theorem 2.1.9 and Maury-Raynaud[52], Proposition I.3.5.

Section 2.2. Universal enveloping algebras are often UFDs.

In this section we will show that two major sub-classes of the class of enveloping algebras of Lie algebras are Noetherian UFDS as defined in Section 2.1. This section will use the material of Section 1.8 without further reference.

Following the definition of UFDs, A.W.Chatters observed that for complex solvable Lie Algebras their enveloping Algebras satisfied the conditions of Theorem 2.1.1. M.K.Smith subsequently pointed out that the same is true for semi-simple Lie algebras for any field of characteristic zero.

2.2.1. Theorem: Let L be a f.d. solvable Lie algebra over the field of the complex numbers. Then U(L) is a Noetherian UFD.

Proof: First we remark that every prime ideal of U(L) is completely prime by Dixmier[25], Theorem 3.7.2.

Now suppose that P is a minimal non-zero prime ideal of U(L). By Dixmier[25], Theorem 4.4.1, P has a non-zero intersection with the semi-centre SZ(L) of U(L). So by Moeglin[53], Theorem III.3, there exists a non-zero element $p \in P \cap SZ(L)$ which is irreducible as an element of U(L); that is $ab = p$ implies that either $a \in pu(L)$ or $b \in pU(L)$. Finally Moeglin [53], Proposition IV.4 tells us that $pU(L) = U(L)p$ is a two-sided prime ideal of U(L) and so $P = pU(L) = U(L)p$.

2.2.2. Theorem: Let L be a semi-simple Lie algebra over a field k of characteristic zero. Then U(L) is a Noetherian UFD.

Proof: Suppose that P is a minimal non-zero prime ideal of U(L). By Dixmier[25], Proposition 4.2.2, $P \cap Z(L) = Q$ is non-zero. Conversely, Conze[23], Theorem 11.2 says that if J is a prime ideal of Z(L) then JU(L) is a completely prime ideal of U(L).

Now Q is clearly a prime ideal of Z(L) and hence must be of height 1, since any non-zero prime ideal contained in Q would generate a prime ideal of U(L) strictly contained in P. Finally Dixmier[25], Theorem 7.3.8(ii), implies that Z(L) is a polynomial ring over k and hence is a commutative UFD. So Q is a principal prime ideal and P = QU(L) is also principal.

Section 2.3: Group rings are sometimes UFDs.

In this section we consider the question of when group rings are UFDs. We shall use the notation and definitions of Section 1.8 without comment. Following the definition of Noetherian UFD M.K.Smith observed the following.

2.3.1. Theorem: For k an abitrary field and G a f.g. torsion-free nilpotent group, the group ring R = kG is a Noetherian UFD.

Proof: Since G is nilpotent, Z(G) is a non-trivial f.g. (by Carter[10], Theorem 4.9) torsion-free Abelian group. Hence kZ(G) is isomorphic to a polynomial ring over k in finitely many variables localised at the powers of the variables and is then a commutative Noetherian UFD.

Recall, from Passman [54], Lemma 11.1.3 that G/Z(G) is torsion-free and nilpotent. Now suppose that P is a height-1 prime ideal of R. Then P' = P∩kZ(G) is a non-zero (by Roseblade-Smith[61], Theorems B and C) prime ideal of kZ(G).

Given any prime ideal of kZ(G), Q, we have that kG/QkG is isomorphic to (kZ(G)/Q)*G/Z(G) the skew group ring of kZ(G)/Q and G/Z(G), and this is a domain by Lemma 1.8.7. But then P'kG is a completely prime ideal of kG and is contained in P and hence is equal to P. Finally, P' must be a height-1 prime of kZ(G) and so is principal. Therefore P is principal.

More recently K.A.Brown has considered the problem in the more general setting of K a commutative Noetherian domain and

G a polycyclic-by-finite ring. What follows in this section is entirely due to him. Throughout K and G will be as here.

The problem really reduces to finding an appropiate control subgroup of G; that is a subgroup S(G) with the property that, if P is a height-1 prime ideal of KG, then P = $(P\cap KS(G))KG$. Recall from Section 1.8 the definitions of $\Delta(G)$ and $\Delta^+(G)$ for a group G. First a result which we will have cause to use several times

2.3.2. Theorem: For the group ring KG, K a domain, the following are equivalent:

(i) KG is prime;

(ii) Z(KG) is prime;

(iii) G has no non-trivial finite normal subgroup;

(iv) $\Delta(G)$ is torsion-free Abelian;

(v) $\Delta^+(G) = 1$.

Proof: Passman [54], Theorem 4.2.10.

2.3.3. Definition: A subgroup of H of a group G is <u>orbital</u> if it has only finitely many conjugates (equivalently $|G:N_G(H)|$ is finite). A <u>plinth</u> of G is a torsion-free Abelian orbital subgroup A of G such that $A\otimes Q$ is an irreducible QT-module for every subgroup T of $N_G(A)$ of finite index.

We say that a plinth is <u>centric</u> if $|G:C_G(A)|$ is finite (equivalently A has rank one). Otherwise A is <u>eccentric</u>.

Denote by P(G) the <u>plinth socle</u> as defined by I.Musson, the subgroup of G generated by the plinths of G. For H an orbital subgroup of G, we can define the <u>isolator</u> of H, Is(H), to be the subgroup generated by all the orbital subgroups of G containing H as a subgroup of finite index. A subgroup is <u>isolated</u> in G if Is(H) = H. Now define S(G) to be Is(P(G)); that is $S(G) = \{x \in G: \bar{x} \in \Delta^{+}(G/P(G))\}$.

Then S(G) is a characteristic Abelian-by-finite subgroup of G. The next theorem shows us that S(G) is the control group we are interested in.

2.3.3. Theorem: If P is a height-1 prime ideal of KG then $P = (P \cap KS(G))KG$.

Proof: Brown [9], Theorem A.

Before we consider the affirmative results we should observe the following. Define $D = \langle a,b : a^{-1}ba = b^{-1}, a^{2} = 1 \rangle$ the infinite dihedral group.

2.3.4. Lemma: Suppose that G has a subgroup isomorphic to D which is orbital and isolated. Let Q equal the augmentation ideal of KD, and let $P = \cap Q^{g}KG$, where the intersection is over all the elements of G (though note that this reduces to a finite intersection). Then P is a height-1 prime ideal of KG which is not principal.

Proof: Brown [9], Lemma 2.2.

Clearly then in order to have any hope that KG is a Noetherian UFD we must avoid this situation. We say that G is "dihedral-free" if G contains no orbital subgroup isomorphic to D.

The real crux to the argument comes in the next two theorems, Theorems B and C of Brown [9].

2.3.5. Theorem: Suppose that K is a commutative Noetherian UFD, that $\Delta^{+}(G) = 1$, and that G is dihedral-free. Let P be a height-1 prime ideal of KG. Set $J = P \cap KS(G)$ so that $J = \cap Q^{g}$, where Q is a G-orbital prime of KS(G) and the intersection runs over all the elements of G. Then the following are equivalent:

(i) P is right principal;

(ii) Q has height one;

(iii) P contains a non-zero central element;

(iv) P contains a non-zero normal element;

(v) P contains an invertible ideal of KG.

Proof. Brown [9], Theorem B.

The next result makes it clear why it is that for KG to be a UFD every plinth of G must be centric, and so we reproduce the proof in full.

2.3.6. Theorem: Suppose that $\Delta^{+}(G) = 1$. Then the following are equivalent:

(i) every non-zero ideal of KG contains a non-zero invertible ideal;

(ii) every non-zero ideal of KG contains a non-zero normal element;

(iii) every non-zero ideal of KG contains a non-zero central element;

(iv) every plinth of G is centric.

Proof: It is clear that (iii) holds implies that (ii) holds, and that (ii) implies that (i) holds.

Now suppose that (i) holds and suppose that A is an eccentric plinth. Let A^o be Is(A) and let I be the augmentation ideal of KA^o. Then I^G is a prime ideal of KG and $Q = I^{S(G)}$ is prime since A^o is isolated in G. It can be shown that height(Q) = h(A) which is greater than 1 by assumption. But the equivalence of (ii) and (v) of Theorem 2.3.5 implies that P contains no non-zero invertible ideal of KG, a contradiction.

Finally, suppose that (iv) holds. Then $S(G) = \Delta^o(G)$, the isolator of $\Delta(G)$. Suppose that P is a height-1 prime ideal of KG. Then by Theorem 2.3.3, $P = (P \cap K\Delta^o(G))KG$ and so $P \cap K\Delta^o(G)$ is non-zero. But using Theorem 2.3.2 and that $|\Delta^o(G):\Delta(G)|$ is finite we can deduce that there exists a non-zero element x of $P \cap K\Delta(G)$.

But x has only finitely many conjugates. So $\prod x^g$ is a non-zero central element of P. Since KG is Noetherian it is now easy to deduce that every non-zero ideal of KG has non-zero intersection with the centre of KG.

Now we have

2.3.7. Theorem: Let K be a commutative Noetherian UFD, and let G be a polycyclic-by-finite group. Then KG is a Noetherian UFR if and only if the following conditions hold:

(i) $\Delta^{+}(G) = 1$,

(ii) G is dihedral-free,

(iii) every plinth is centric.

Proof: Suppose that KG is a Noetherian UFR. Then KG is prime so by Theorem 2.3.2 (i) follows. G must be dihedral-free otherwise there would exist some non-principal height-1 prime ideal of KG and (iii) comes from Theorem 2.3.6.

Conversly if (i),(ii),and (iii) hold then KG is a prime Noetherian ring. That every height-1 prime ideal is principal comes from (iii) and Theorems 2.3.5 and 2.3.6.

2.3.8. Theorem: Let K be a commutative Noetherian UFD and let G be a polycyclic-by-finite group. Then (a) KG is a Noetherian UFD only if the following conditions hold:

(i) G is torsion-free,

(ii) all plinths are actually central (ie $C_G(A) = G$), and (iii) $G/\Delta(G)$ is torsion-free;

(b) if (i), (ii), and (iii)' $G/\Delta(G)$ is poly-(infinite cyclic) hold then R is a Noetherian UFD.

Proof: (a) Suppose that KG is a Noetherian UFD. Since KG is a domain, G is torsion-free (D.Passman [54], Lemma 13.1.1). If $G/\Delta(G)$ is not torsion-free then there exists $g\in G\backslash\Delta(G)$ with $x=g^{n}\in\Delta(G)$ for some n. Then there exists a height-1 prime ideal P of KG containing the central element $\prod(x^{y}-1)$, the product over the distinct (finitely many) conjugates of $(x-1)$. Since P is completely prime by assumption, we may assume that $(x-1)$

lies in P. But, if T is a transversal of $\langle x\rangle$ in $\langle g\rangle$, then set $s = \Sigma t$, sum over $t \in T$. Clearly then $s(g-1) \in (x-1)KG \subseteq P$. But neither s or $(g-1)$ lie in P since $P = (P \cap \Delta(G))KG$. This contradicts our assumption that P is completely prime.

If $\Delta(G)$ is not central then it is possible to construct a height-1 prime ideal of KG which is not completely prime so we can deduce that (ii) must hold.

(b) Suppose that (i), (ii), and (iii)' hold. First, KG is a domain, by Theorem 1.8.5. If P is a height-1 prime ideal of KG then P is principal by Theorem 2.3.7 and in fact $P = pKG$ for p some non-zero element of $K\Delta(G)$. Then $KG/P = KG/pKG$ is isomorphic to $K\Delta(G)/pK\Delta(G)*G/\Delta(G)$ and hence is a domain by Lemma 1.8.7.

Remark: It remains an open question as to whether (iii) would be sufficient in (b) of the above theorem. It would follow if one could prove a "twisted" version of the zero-divisor conjecture.

Section 2.4. Some constructions of Noetherian UFDs.

In this section we consider a "new for old" technique to enable us to construct UFDs from other UFDs. For the definition of $A_n(R)$ for a given ring R see Section 1.8.

Theorem 2.4.1: Let R be a Noetherian UFD such that char(R)=0. Then $A_1(R)$ is also a Noetherian UFD.

Proof: Recall that if P is a prime ideal of R, then $A_1(R/P)$ is isomorphic to $A_1(R)/PA_1(R)$ and so is prime. In particular, if P is completely prime then $PA_1(R)$ is a completely prime ideal of $A_1(R)$.

Suppose that P is a height-1 prime ideal of $A_1(R)$. Then we shall show that $P \cap R = Q$ is a non-zero height-1 prime ideal of R. Suppose that $Q = 0$ and that $r \in P$ is the element of least degree in y in P.

So $r = \sum_{j=0}^{j=n} f_j(x)y^j$, for some $f_j(x) \in R[x]$.

Then $rx-xr = \sum_{j=1}^{j=n} f_j(x) j y^{j-1}$.

But $0 \neq rx-xr \in P$ and is of lower degree in y. Now $f_j(x)j=0$ implies that $f_j(x)=0$ and so we deduce that $r=f_0(x)$. A similar argument in the x-degree using $ry-yr$ will then force $r \in R$.

So $P \cap R = Q$ is a non-zero prime ideal of R and it is easy to see that it must be height-1. Hence $Q = pR = Rp$ for some prime element of R and $P = pA_1(R) = A_1(R)p$ is a completely prime ideal of $A_1(R)$.

2.4.2. Corollary: Let R be a Noetherian UFD. Then $A_n(R)$ is a Noetherian UFD.

Proof: By induction.

2.4.3. Remarks: (a) Suppose that R is a Noetherian UFD. Then essentially the same proof as for Theorem 2.4.1 would work for the ring (as defined in Section 1.8) $B_\lambda(R)$, for λ a central unit of R such that $1-\lambda^n$ is a unit for all integers n. Note that here we do not need to assume that char(R)=0.

(b) Anticipating ourselves briefly, we exhibit in Chapter 5 an example of a Noetherian UFD R such that R[x] is a UFR and not a UFD. Since we may regard $A_1(R)$ as a skew polynomial extension of R[x], it is clear that R[x] is an example of a Noetherian UFR which has a skew polynomial extension which is a Noetherian UFD.

(c) If R is a Noetherian UFD with char(R) = p $\neq$ 0, then we have the following

2.4.4. Theorem: Let R be a Noetherian domain with non-zero characteristic p. Then in $A_1(R)$, x^p generates a height-1 prime ideal which is not completely prime.

Proof: Let $I = (x^p)A_1(R)$. Suppose that A and B are ideals of $A_1(R)$ containing I such that $A.B \subseteq I$. If $I \subset A$ then choose an element $f(x,y) = \Sigma_{i=0}^{i=n} f_i(x)y^i \in A\backslash I$ of least degree in y. Then $fx-xf = \Sigma_{i=1}^{i=n} f_i(x)iy^{i-1}$ also lies in A and is of lower degree in y. We may deduce then that $f(x,y) = g(x,y^p) + h(x,y)$, where $h(x,y)\in I$. Commuting f(x,y) with y we can similarly deduce that there exists $f(x,y)\in A$ such that $f(x,y) = u(y^p) + v(x,y)$, where $v(x,y)\in I$. But $A.B \subseteq I$ and hence $B \subseteq I$.

Therefore x^p generates a prime ideal, which must be of height 1 by Jategaonkar[39], Theorem 3.1. Clearly I is not completely prime.

We can also use Theorem 2.1.4 to construct UFDs from any prime Noetherian ring containing at least one prime element.

2.4.5 Theorem: Let R be a prime Noetherian ring. Suppose that X, the set of prime elements of R, is non-empty. Let $C = \cap C(pR)$, where the intersection runs over all the prime elements p in X. Then C is Ore and the partial quotient ring R_C is a UFD.

Proof: C is Ore just as in Theorem 2.1.4. Now suppose that P is a height-1 prime ideal of $T = R_C$. Then $P = QT$ for some height-1 prime ideal Q of R. If Q is not generated by a prime element then a simple argument considering a non-zero element of Q shows that $Q \cap C$ is non-empty. Thus all height-1 prime ideals of T are generated by prime elements of R.

This result means that results about Noetherian UFDs will give us information about prime elements in more general Noetherian domains.

In the commutative case there is a useful criterion to determine when R is a UFD. In a commutative Noetherian ring R suppose that S is a multiplicative set generated by prime elements. Then Nagata's Theorem says that R is a Noetherian UFD if and only if R_S is a Noetherian UFD. See, for example, Cohn[21], Theorem 11.3.5. To conclude this section we extend this result to non-commutative Noetherian UFDs.

2.4.6. Theorem: Let R be a prime Noetherian ring. Suppose that S is a set of prime elements of R. Let D be the multiplicatively closed set generated by S. Then R is a Noetherian UFD if and only if R_D is a Noetherian UFD.

Proof: If R is a Noetherian UFD then it is easy to see that then so is R_D.

Conversely, suppose that R_D is a Noetherian UFD and let P be a height-1 prime ideal of R. If $PR_D = R_D$ then $D \cap P \neq \emptyset$ and so some prime element of S lies in P. Hence PR_D is generated by a prime element. If $PR_D \neq R_D$, then PR_D is a height-1 prime ideal of R_D. Since R_D is a Noetherian UFD, $PR_D = pR_D = R_Dp$, where p is of the form qd^{-1}, for some $d \in D$ and $q \in R$. So without loss of generality P is generated by an element $q \in R$ such that $qR_D = R_Dq$. Choose $q \in P$ with $qR = Rq$, and such that qR is maximal. Then $q \notin p_iR$, for any prime element p_i of R. Let $Q = qR = Rq$. We claim that $P = Q$. Suppose not, then choose $r \in P \backslash Q$. Since $r \in PR_D$, there exists $s \in R$ with $r = d^{-1}sq$. Therefore, $rd = sq$. But $q \notin p_iR$, for any prime element of p_i of R. By a simple argument, based on an induction on the n such that $d = p_1 \ldots p_n$, $s \in dR$. Hence $r \in Rq$, which contradicts our choice of r. Therefore, $P = qR = Rq$.

2.4.7. Corollary: Let R be a Noetherian UFD. Let D be the multiplicative set generated by all the prime elements of R. Let $S = R_D$. Then R[x] is a Noetherian UFD if and only if S[x] is a Noetherian UFD.

Chapter 3. Some results on Noetherian UFDs.

3.0. Summary.

The preceding chapter makes it clear that Noetherian UFDs are objects of interest to study, both for their own sake and for the approach they offer to answer questions about some well-known classes of Noetherian rings. However, seen as a possible analogoue of the commutative case, the results of this chapter are surprising. We are able to prove actually stronger results about Noetherian UFDs when we know that they are not commutative. This enables us to draw conclusions about PI or bounded Noetherian UFDs which are true provided that the ring is not actually commutative. This appears to be a curious state of affairs, and is an indication of how strong a condition it is to require a prime ideal in a non-commutative ring to be completely prime.

Let R be a Noetherian UFD, and $C = C(R)$. In Section 3.1, we show that if R is not commutative, then the ring $T = R_C$ is always a principal ideal domain. This is a result first proved for the enveloping algebra of a solvable Lie algebra by A.W.Chatters and for group rings of torsion-free nilpotent groups by M.K.Smith. We should remark that the result was proved independently by M.K.Smith and it appears in Gilchrist-Smith [30].

Section 3.2 uses the result of the previous section to improve the bound on the stable rank of some UFDs using a similar technique to that of Section 3.1.

Section 3.3 applies the technique of Section 3.2 to the problem of localising at cliques of completely prime ideals. Whilst not directly concerned with the theory of Noetherian UFDs, this section may have applications to the theory of enveloping algebras of Lie algebras.

Finally, Section 3.4 considers the centres of Noetherian UFDs.

In this chapter we will use the following notation. Throughout, R will be a Noetherian UFD. C will denote the set $\cap C(P)$, where the intersection runs over all the height-1 prime ideals P of R. D will denote the multiplicatively closed set generated by the prime elements of R. The partial quotient rings R_C and R_D will be denoted by T and S respectively.

Section 3.1. Noetherian UFDs are often PIDs.

In Chatters[13], Theorem 3.3 shows that, if L is a non-Abelian solvable Lie algebra, $U(L) = R$, and $T = R_C$, then T is a principal ideal domain. M.K.Smith in a letter to J.T.Stafford showed that the same is true for the group ring over a field k, $R = kG$, where G is a torsion-free nilpotent group which is not Abelian.

We prove that this is the case in general.

3.1.1. Theorem: Suppose that R is a Noetherian UFD which is not commutative. Then every ideal which is contained in no height-1 prime ideal has non-empty intersection with C.

Proof: It clearly suffices to prove the theorem for a prime ideal, say P, whose height is greater than 1. So suppose that P is such a prime ideal.

First, suppose that P contains no height-1 prime ideal. Let a be any non-zero element of P. Then $a = cp_1p_2\cdots p_n$, for some $c \in C$ and prime elements $p_1, p_2, \ldots, p_n$. Here $p_n \notin P$ and $aR = cp_1p_2\cdots p_nR = cp_1p_2\cdots p_{n-1}Rp_n \subseteq P$. So $cp_1p_2\cdots p_{n-1} \in P$. By induction on n, we deduce that $c \in P$.

If P contains exactly one height-1 prime ideal, say $pR = Rp$, then choose $a \in P\backslash pR$ and proceed as in the first case.

So suppose that P contains two distinct height-1 prime ideals of R, $pR = Rp$ and $qR = Rq$. For each positive integer n and a fixed $r \in R$ (to be specified later) define the element

$$t_n = p+q(r+q^n) \in P.$$

Suppose that the theorem is false. Since each $t_n = cp_1\cdots p_m$, for $c \in C$ and prime elements p_i, then at least one of the $p_i \in P$. Hence $t_n \in I_n$, where I_n is a height-1 prime ideal of R contained in P. Note that $q \notin I_n$, since $q \in I_n$ would imply also that $p \in I_n$ which contradicts our assumption that p and q generate distinct height-1 prime ideals.

Suppose that $I_m = I_n$, for distinct integers m and n, $m < n$. So t_m, $t_n \in I_m$. Hence $t_m - t_n \in I_m$. So

$$q^{m+1} - q^{n+1} = q^{m+1}(1-q^{n-m}) \in I_m.$$

Since $q \notin I_m$, we conclude that $(1-q^{n-m}) \in I_m \subseteq P$. But, since $q \in P$, this would imply that $1 \in P$, a contradiction.

Thus the set of I_n's is infinite. Since R is Noetherian, only finitely many height-1 primes lie over any non-zero element of R and so $\cap I_n = 0$, where the intersection runs over all n.

We shall obtain a contradiction from this by exhibiting a non-zero element of $\cap I_n$. It is here that we have to use the fact that R is not commutative.

The proof splits into three cases:

(a) Suppose that both p and q are both central. Choose $r \in R$ to be any non-central element. Then there exists $s \in R$ such that $sr - rs \neq 0$. Then $[t_n, s] = t_n s - s t_n = qrs - sqr = q(rs - sr)$. Since $q \notin I_n$, $0 \neq (rs - sr) \in I_n$. This is true for all n.

(b) Suppose that $pq = qp$, but that q is not central. Then there exists $r \in R$ such that $qr - rq \neq 0$. Then $[t_n, q] = q(rq - qr) \neq 0$. So $(rq - qr) \in I_n$, for all n.

(c) Finally, suppose that $pq \neq qp$. Let $r = 0$. We have $[t_n, q] = (pq - qp) \in I_n$, for all n.

3.1.2. Corollary: Let R be a Noetherian UFD which is not commutative. Then T is a principal ideal domain.

Proof: Since T is Noetherian, it suffices to consider a right ideal $I = rT + sT$. By Theorem 2.1.5,(iii), I is an ideal and so I is contained in some maximal ideal M of T. Now $M = (M \cap R)T$ and $M \cap R$ is a proper prime ideal of R. By Theorem 3.1.1, $M \cap R$ is a height-1 prime ideal of R. Therefore $M = qT$, for some prime element q of R.

But now the ideal $J = q^{-1}I \subseteq T$. By a Noetherian induction, J is principal, say $J = dT$. Thus $I = qdT$ is principal, as required.

Remark: This seems to be a surprising result. If we consider a polynomial ring $R = k[x_1, \ldots, x_n]$ over a field K, then R is a Noetherian UFD and it is easy to see in this case that $T = R$. But clearly, if $n \geq 2$, then T is not a principal ideal domain. It seems to indicate that in the study of Noetherian UFDs there are going to be significant differences in results depending on whether or not R is commutative. To some extent this is borne out by Section 3.2 when we come to consider the stable range of some Noetherian UFDs.

3.1.3. Corollary: Let R be a bounded Noetherian UFD which is not commutative. Suppose that every prime ideal of R contains a height-1 prime ideal. Then R is a principal ideal domain and every one-sided ideal is two-sided.

Proof:

By the proof of Theorem 2.1.9, $R = T \cap S$ and $I \cap D \neq \emptyset$, for all non-zero ideals I. Suppose that $c \in R$ is a non-zero non-unit of R. If I is the bound of cR, then $I \cap D \neq \emptyset$, and so IS = S. Thus cS = S and c is a unit of S.

But then S is the full quotient ring of R. Hence $T \subseteq S$. Therefore R = T and the result follows from Theorem 2.1.6 and Corollary 3.1.2.

3.1.4. Corollary: Let R be a Noetherian UFD satisfying a polynomial identity. Suppose that R is not commutative. Then R is a principal ideal domain and every one-sided ideal is two-sided.

Proof: By Corollary 1.3.4, R is bounded. By Rowen[62], Theorem 5.2.19, R satisfies DCC on prime ideals. The result now follows from Corollary 3.1.3.

3.1.5. Corollary: Let R be a Noetherian Azumaya algebra. Suppose that Z(R) is a Unique Factorisation Domain. If Kdim(Z(R)) > 2 then at least one of the height-1 prime ideals of R is not completely prime.

Proof: By Theorem 1.3.6, every height-1 prime ideal of R is principal. If every height-1 prime ideal of R were completely prime then R would be a principal ideal domain, by Corollary 3.1.4. This is clearly not the case. Therefore, at least one of the height-1 prime ideals of R is not completely prime.

Remark: Let R be an arbitrary Noetherian PI domain. Let C = $\cap C(pR)$, where the intersection runs over all the prime elements p of R. By Theorem 2.1.4, the set C is Ore, and by, Corollary 3.1.4, R_C is a principal ideal domain. Thus by localising at C, we have automatically localised away all the prime ideals of height greater than 1.

Suppose that D is a division ring. Provided that D is not commutative, not all one-sided ideals of D[x] are two-sided. Suppose that $d \notin Z(D)$. Then, if $e \in D$ is such that $de \neq ed$, then $(ed-de) \in D[x](x-d)D[x]$. That is, $(x-d)D[x] \neq D[x](x-d)D[x]$. In particular, if D is a PI division ring, then D[x] is not a Noetherian UFD. This enables us to give an amusing proof of the following result.

3.1.6. Corollary: Let D be a PI division ring whose centre K is not the whole of D. Then there exists a polynomial in K[x], irreducible over K, but which is reducible over D.

Proof: R = D[x] is a PI principal ideal domain which by the foregoing remarks cannot be a Noetherian UFD. There exists therefore a height-1 prime ideal P of D[x] generated by p(x) in K[x] which is not completely prime. Clearly p(x) is irreducible as an element of K[x]. We shall show that p(X) is not irreducible as an element of D[x].

Since p(x)R is not completely prime, there exists a right ideal I of R such that $p(x)R \subset I \neq R$. Since R is a principal ideal domain, $I = a(x)R$, for some non-unit $a(x) \in R$. Hence $p(x) = a(x).b(x)$, for some non-unit $b(x) \in R$. Thus p(x) is reducible in D[x].

Remark: Of course, another proof of Corollary 3.1.6, may be obtained by noting that $K \leqslant F \leqslant D$, for some splitting field F of K. That is, some irreducible polynomial in $K[x]$ is reducible in $F[x]$.

3.1.7. Corollary: Let R be a hereditary Noetherian UFD. Then R is a principal ideal domain or primitive.

Proof: Immediate from Corollary 3.1.3 and Lenagan[47], Proposition 5.1.9.

Note that primitive Noetherian UFDs do exist. Let D be the quotient ring of the Weyl algebra $A_1(k)$, k the complex numbers. Then $D[x]$ is primitive (Amitsur-Small[1], Theorem 3) and it is not hard to see that the height-1 prime ideals of $D[x]$ are generated by central irreducible polynomials in $k[x]$. Since these are all of the form $(x-\alpha)$, for α a complex number, the result follows. We shall significantly improve this observation in Section 5.1.

Section 3.2. The stable rank of T.

In Section 1.6 we discussed possible bounds for the stable rank of a ring R and in particular recalled that, for a Noetherian ring R, $sr(R) \leq Kdim(R) + 1$. For a Noetherian UFD R, the partial localisation T, is a principal ideal domain by Theorem 3.1.2. So we have $sr(T) \leq 2$. It is thus a natural question to ask whether in fact $sr(T) = 1$. In this section we show that this is indeed the case. Hence, in particular, if R is a bounded Noetherian UFD which is not commutative, then $sr(R) = 1$.

First we have to prove a number-theoretic lemma

3.2.1. Lemma: Let k be a positive integer. Then there exist k positive integers $1 \leq a_1 < a_2 < \ldots < a_k$ satisfying the condition:

(*) $(a_i - a_j)$ divides a_i (and also a_j), for all pairs $i > j$.

Proof: By induction on k. Clearly $a_1 = 1$, $a_2 = 2$ satisfy (*) for $k = 2$.

Now suppose that, for the integer k, we have a set $a_1 < \ldots < a_k$ satisfying (*). Let $b_0 = \Pi a_i$. Let $b_i = b_0 + a_i$, for $i = 1, \ldots, k$. Then for $i > 0$, $(b_i - b_0) = a_i$ divides b_i; and for $0 < j < i$, $(b_i - b_j) = (a_i - a_j)$ divides a_i and hence divides b_i.

So $b_0 < b_1 < \ldots < b_k$ is a set of $(k+1)$ numbers satisfying (*).

Remark: A similar argument works with $b_0 = \Pi q^{a_j}$ and

$$b_i = b_0.(q^{a_i}-1)/q^{a_i},$$

where q is any positive integer > 1.

Let R be a Noetherian UFD which is not commutative. Then, from results in Section 2.1 and Section 3.1, $R_C = T$ is a Noetherian UFD whose non-zero prime ideals are of the form pT = Tp, for p a prime element of R, and C(T) is the set of units of T. Thus every non-unit of T lies in a height-1 prime ideal of T.

The following Theorem relies heavily on this simple result.

3.2.2. Lemma: Let m and n be positive integers with m dividing n. Suppose that r is an element of a ring R. Then

$$(r^n-1) \in (r^m-1)R.$$

Proof: $(r^n-1) = (r^m-1)(1+r^m+r^{2m}+\ldots+r^{(d-1)m})$, where n = m.d.

3.2.3. Theorem: Let R be a Noetherian domain. Suppose that every non-unit of R lies in a height-1 prime ideal and that every height-1 prime ideal is completely prime. Suppose also that R is not commutative. Then sr(R) = 1.

Proof: Suppose that aR + bR = R. If a is a unit of R then we have (a + b.0)R = R. If b is a unit of R, there exists c with bc = 1. Then (a + b.(c(1-a)))R = R. So, without loss of generality, we may assume that neither a nor b is a unit.

The idea of the proof is similar to that of Theorem 3.1.1, but a little more care is needed. We consider elements of the form $f_n = a + b.c_n$, for suitable choice of elements $c_n \epsilon R$ and n a positive integer. If none of the f_n are units of R then, by hypothesis, each f_n must lie in at least one height-1 prime ideal, say I_n. Then by commuting each f_n with a suitable element $r \epsilon R$, we obtain a non-zero element $[f_n,r]$ of R which is independent of n. Since R is Noetherian, $[f_n,r]$ can lie in only k height-1 prime ideals, for some integer k. So if we choose k+1 distinct elements f_n then at least two of them, say f_m and f_n, lie in the same height-1 prime ideal, I_m. Then (f_m-f_n) lies in I_m and from this we will be able to deduce a contradiction. We remark that a and b cannot both lie in the same height-1 prime ideal of R since they generate R. Similarly, the elements a and (a^r-1), for r a positive integer, cannot both lie in the same height-1 prime ideal.

The proof has to consider several cases:

(i) First, suppose that $ab \neq ba$. Then (ab-ba) lies in exactly k height-1 prime ideals of R for some integer k. By Lemma 3.2.1, we can choose a sequence of positive integers $a_1 < a_2 < \ldots < a_{k+1}$ satisfying the condition (*) of Lemma 3.2.1.

Define $f_n = a + b.(a^{a_n} - 1)$, for $n = 1,\ldots,k+1$. We may suppose that, for each n, $f_n \epsilon I_n$, for some height-1 prime ideal I_n.

Then $[f_n,a] = (ba-ab)(a^{a_n}-1) \epsilon I_n$. If $(a^{a_n}-1) \epsilon I_n$, then $a \epsilon I_n$, a contradiction. So $(ab-ba) \epsilon I_n$, for all n. We can deduce that there exist f_m and f_n, $m > n$, such that $f_m \epsilon I_m$ and $f_n \epsilon I_m$.

Now $f_m-f_n = b.(a^{a_m}-a^{a_n}) = b.a^{a_n}.(a^{a_m-a_n}-1) \epsilon I_m$. Suppose that $b \in I_m$, then $a \in I_m$; and if $a \in I_m$ then either $b \in I_m$ or $(a^{a_n}-1) \in I_m$. These all lead to contradictions. We are forced to deduce that $(a^{a_m-a_n}-1) \in I_m$. But (a_m-a_n) divides a_m and hence, by Lemma 3.2.2, $(a^{a_m}-1) \in I_m$. But $f_m = a + b.(a^{a_m}-1) \in I_m$, and so $a \in I_m$, a contradiction. This finishes case (i).

(ii) Now suppose that $ab = ba$. This case splits into three sub-cases.

(a) Suppose that b is not central. There exists $c \in R$ such that $bc \neq cb$ and, by replacing c by $(ac+1)$ if necessary, we may assume that if $a \in J$, for some proper ideal J of R then $c \notin J$ Suppose that $(bc-cb)$ lies in exactly k height-1 prime ideals. Choose, as before, positive integers, $a_1 < \ldots < a_{k+1}$ satisfying condition (*) of Lemma 3.2.1.

Let $f_n = a + b.(a^{a_n}-1).c$, for $n = 1,\ldots,k+1$. For each n, we may suppose that $f_n \in I_n$, for some height-1 prime ideal I_n of R.

Then $[f_n, b] = b.(a^{a_n}-1).(cb-bc) \in I_n$. As in case (i), if either $b \in I_n$ or $(a^{a_n}-1) \in I_n$, we can derive a contradiction. Hence $(cb-bc) \in I_n$, for $n = 1,\ldots,k+1$.

So there exist integers m and n, $m > n$, such that $f_m \in I_m$ and $f_n \in I_m$. Consequently,

$$f_m - f_n = b.(a^{a_m}-a^{a_n}).c = b.a^{a_n}.(a^{a_m-a_n}-1).c \in I_m.$$

Clearly, $b \notin I_m$, and $a^{a_n} \notin I_m$. Similarly $c \in I_m$ is ruled out by our choice of c. We are forced to deduce that $(a^{a_m-a_n}-1) \in I_m$. Since (a_m-a_n) divides a_m, we can proceed just as in case (a) to derive a contradiction.

(b) Suppose now that b is central and that a is not central. Then there exists $d \in R$ such that $ad \neq da$. Suppose that $(ad-da)$ lies in exactly k height-1 prime ideals. As before choose positive integers $a_1 < \dots < a_{k+1}$ satisfying condition (*) of Lemma 3.2.1.

Let $f_n = a - b^{a_n}.(a-1)$, for $n = 1,\dots, k+1$. We may suppose that, for each n, $f_n \in I_n$, for some height-1 prime ideal I_n.

Then $[f_n,d] = (ad-da).(1-b^{a_n}) \in I_n$. If $(1-b^{a_n}) \in I_n$, then $1 = a - (a-1) \in I_n$, a contradiction. So $(ad-da) \in I_n$ for $n = 1,\dots,k+1$.

Then there exist integers m and n, $m > n$, with $f_m \in I_m$ and $f_n \in I_m$. This implies that

$$f_m - f_n = (b^{a_m} - b^{a_n}).(a-1) = b^{a_n}.(b^{a_m - a_n} - 1).(a-1) \in I_m.$$

Clearly $b \notin I_m$, and $(a-1) \notin I_m$. Hence $(b^{a_m-a_n}-1) \in I_m$. Then $(b^{a_m}-1) \in I_m$ and this leads to a contradiction as before.

(c) Finally, suppose that both a and b are central. Since R is not commutative there exist elements of R, c and d, with $cd \neq dc$. As in (a) we may assume that if $a \in J$, for some ideal J of R then $c \notin J$. Suppose that $(cd-dc)$ lies in exactly k height-1 prime ideals. As before, choose positive integers $a_1 < \dots < a_{k+1}$ satisfying condition (*) of Lemma 3.2.1.

Let $f_n = a + b.(a^{a_n}-1).c$, for $n = 1,\dots,k+1$. We may suppose that, for each n, $f_n \in I_n$, for some height-1 prime ideal I_n of R. Then $[f_n,d] = b.(a^{a_n}-1).(cd-dc) \in I_n$. Now the proof of (ii)(a) goes through almost word for word.

Since the four sub-cases we have considered cover all possibilities the proof is complete.

3.2.4. Corollary: Let R be a Noetherian UFD which is not commutative. Then T has stable rank 1.

Proof: As we remarked earlier, T satisfies the conditions of

Theorem 3.2.3.

3.2.5. Corollary: Let R be a bounded Noetherian UFD which is not commutative. Suppose that every non-zero prime ideal of R contains a height-1 prime ideal. Then R has stable rank 1.

Proof: By Corollary 3.1.3, R = T.

3.2.6. Corollary: Let R be a Noetherian UFD satisfying a polynomial identity. Suppose that R is not commutative. Then R has stable rank 1.

Remark: Just as in Section 3.1, this result is stronger than one would expect from the commutative case. For example both the integers Z and the domain K[x] for some field K are principal ideal domains, but they both have stable range 2. We have that (5,7) is a unimodular row over Z which is not stable and the row $(x,1-x^2)$ is unimodular and not stable in K[x].

J.T.Stafford has conjectured that all affine PI rings have stable rank at least 2, which is known for commutative affine rings. Thus Corollary 3.2.6 is even more surprising than at first sight.

Suppose that D is a division ring. Then D[x] is a principal ideal domain and it is easy to see that $(x,1-x^2)$ is a unimodular row which is not stable. So D[x] has stable range 2. This would give another proof of Corollary 3.1.6.

To finish this section it might be interesting to note some of the properties of the sets of integers satisfying (*)

of Lemma 3.2.1. Let us call a set $\{a_1,...,a_k\}$ a k-*-set if it satisfies (*). Call a k-*-set coprime if the highest common factor of the a_i is 1. Then we can observe the following:

(a) Any subset of size m of a k-*-set is an m-*-set;

(b) There exists an infinite number of coprime k-*-sets for all integers k;

(c) There does not exist an infinite *-set.

Certainly (a) is true. To prove (b), note that it cetainly holds for k = 1. For k > 1, use the inductive construction of Lemma 3.2.1. To prove (c) suppose that a_1 is the first element in an infinite *-set. Then for all n, (a_n-a_1) divides a_n. But this means that $(a_n-a_1) < a_n/2$. Thus $a_n < 2a_1$ and so there exist at most a_1+1 terms in the *-set whose first term is a_1.

Finally, for k = 2 to 7 the smallest k-*-sets are as follows:

k=2 {1,2}

k=3 {2,3,4}

k=4 {6,8,9,12}

k=5 {36,40,42,45,48}

k=6 {210,216,220,224,225,240}

k=7 {14976,14980,14994,15000,15008,15015,15120}

I would like to thank R.Everson for his enthusiasm in determining these values.

Section 3.3. Cliques and localisation.

This section represents something of a meander in the flow of this thesis, and is not directly concerned with the theory of Noetherian UFDs. It arises from the observation that, in proving Theorem 3.2.3, we are in effect proving the "intersection condition" (which we shall define later) for a set of completely prime prime ideals. The method of Theorem 3.2.3 seems to be of independent interest and in this section we shall discuss one possible application: to that of cliques and localisation. We shall follow Warfield[74] in our treatment of cliques.

The problem of localising at a prime ideal reduces to that of showing that $C(P)$ is left or right Ore. However it is often the case that $C(P)$ is not Ore and it therefore becomes of interest to determine the largest subset of $C(P)$ which is Ore. We shall denote this set by $S(P)$.

Let R be a ring. The notion of a "link" between two prime ideals P and Q of R was first introduced by A.V.Jategaonkar in Jategaonkar[38]. Suppose that J is an ideal of R with $QP \subseteq J \subset Q \cap P$ such that $Q \cap P/J$ is torsion-free as a left R/Q-module and as a right R/P-module. In this situation we say that there is a <u>second layer link</u>, denoted by $Q \rightsquigarrow P$. The next Lemma shows that the existence of a link between prime ideals constitutes an "obstruction" to being able to localise at the prime ideals.

3.3.1.Lemma: Let R be a Noetherian ring. Suppose that $Q \rightsquigarrow P$ is a second layer link between the prime ideals P and Q of R.

(i) Suppose that C is a left Ore set with $C \subset C(Q)$. Then $C \subset C(P)$.

(ii) Suppose that D is a right Ore set with $D \subset C(P)$. Then $D \subset C(Q)$.

(iii) If E is an Ore set, then $E \subset C(Q)$ if and only if $E \subset C(P)$.

Proof: Suppose that $QP \subseteq J \subset Q \cap P$ is the second layer link. That is, ${}_{R/Q}(Q \cap P/J)_{R/P}$ is an R/Q-R/P-bimodule torsion-free on both sides. In particular, $l\text{-ann}_R(Q \cap P/J) = Q$ and $r\text{-ann}_R(Q \cap P/J) = P$. Suppose that $c \in C$ and $r \in R$ are such that $cr \in P$. We aim to show that $r \in P$.

Choose $b \in Q \cap P$. Since C is left Ore, there exist $b' \in R$ and $c' \in C$ with $b'c = c'b$. Clearly $b' \in Q$. We have $c'br = b'cr \in QP \subseteq J$. But $(Q \cap P/J)$ is torsion-free as a left R/Q-module. Therefore, since $c' \in C$, $br \in J$. But b was chosen arbitrarily in $Q \cap P$. Hence $(Q \cap P).r \subseteq J$. We conclude therefore that $r \in P$, as required.

The second statement of the Lemma follows by a symmetrical argument to the above. The last statement then follows immediately.

The <u>graph of links</u> of R is the directed graph whose vertices are the prime ideals of R and whose arrows are given by the second layer links. A <u>clique</u> of R is a connected component of the (undirected) graph of links. If P is a prime ideal of R then $Cl(P)$ is the unique clique containing P. By Lemma 3.3.1, it is immediate that $S(P) \subset C(Q)$, for all the prime ideals $Q \in Cl(P)$. It becomes, therefore, a natural question to ask when $S(P) = \cap C(Q)$, where the intersection runs over all the prime ideals $Q \in Cl(P)$.

For a set X of prime ideals of a ring R, let $C(X) = \cap C(P)$, where the intersection runs over all the prime ideals in X.

A.V.Jategaonkar has determined necessary and sufficient conditions for when C(X) is Ore and the ring $R_{C(X)} = R_X$ has particularly nice properties. We say that a clique X is <u>classical</u> or <u>classically localisable</u> if the following conditions hold.

(i) C(X) is a right and left Ore set;

(ii) for every prime ideal $Q \in X$, R_X/QR_X is naturally isomorphic to the Goldie quotient ring of R/Q;

(iii) for $Q \in X$, the prime ideals QR_X of R_X are precisely the primitive ideals of R_X;

(iv) the R_X-injective hull of every simple R_X-module is the union of its socle sequence.

Note that if X is finite and condition (i) is satisfied, then (ii) and (iii) are automatically satisfied. They are added here to ensure that, in the case when X is infinite, the ring R_X has "nice" properties. Condition (iv) is a useful condition that may hold in general and certainly holds in all well-known examples.

A prime ideal in a Noetherian ring satisfies the <u>second layer condition</u> if the injective hull $E(R/P)_R$ contains no f.g. sub-modules whose annihilator is a prime ideal other than P. We say that a set X of prime ideals satisfies the second layer condition if every member of X satisfies it. A ring R satisfies the second layer condition if Spec(R) satisfies the second layer condition.

The classes of Noetherian rings satisfying the second layer condition include the class of fully bounded Noetherian rings, the class of enveloping algebras of solvable Lie algebras (Brown[8], Theorem 3.2, and Heinicke[35], Theorem 1), and the class of group rings KG, where G is a poly-cyclic-by-finite group and K is a field of characteristic zero (Brown[7], Theorem 4.2).

Let R be a ring. A set of prime ideals X in R is said to satisfy the <u>right (left) intersection condition</u> if, given a right (left) ideal of R such that $I \cap C(P) \neq \emptyset$, for all $P \in X$, then $I \cap C(X) \neq \emptyset$. X satisfies the intersection condition if it satisfies both the left and right intersection conditions. Observe that, for example, if X is a localisable set of prime ideals, then C(X) is Ore.

3.3.2. Lemma: Let R be a ring. Let X be a set of localisable prime ideals of R. Suppose that X satisfies the intersection condition. Then C(X) is Ore.

Proof: Choose $c \in C(X)$ and $a \in R$. Let $K = \{r : ar \in cR\}$. Then K is a right ideal of R. Since, for each $P \in X$, C(P) satisfies the Ore condition, $K \cap C(P) \neq \emptyset$. Therefore, $K \cap C(X) \neq \emptyset$. The left Ore condition follows similarly.

The significance of the second layer and intersection conditions is indicated by the following result.

3.3.3. Theorem: Let R be a prime Noetherian ring. Let X be a clique of prime ideals which satisfies the second layer condition and the intersection condition. Then $C(X)$ is Ore and X is classical.

Proof: See Jategaonkar[43], Theorem 7.1.5 and Lemma 7.2.1.

It is clear, therefore, that if R satisfies the second layer condition, then it becomes of great interest to determine when a set of prime ideals satisfies the intersection condition.

For X a finite set of prime ideals we have the following.

3.3.4. Theorem: Let R be a prime right Noetherian ring. Let $X = \{P_1,\ldots,P_n\}$ be a finite set of prime ideals. If I is a right ideal of R with $I\cap C(P_i) \neq \emptyset$, for $i = 1,\ldots,n$, then $I\cap C(X) \neq \emptyset$.

Proof: Order the prime ideals so that $P_k \subset P_j$ implies that $k > j$. Then $P_1\cap\ldots\cap P_i \not\subset P_{i+1}$, for $i = 1,\ldots, n-1$. Suppose, by induction, that there exists $a\in I\cap C(P_1)\cap\ldots\cap C(P_i)$. Then the image of $aR + I.(P_1\cap\ldots\cap P_n)$ generates an essential right ideal of R/P_{i+1}. By Lemma 1.1.6, there exists $b\in I(P_1\cap\ldots\cap P_i)$ such that $a+b\in C(P_{i+1})$. Since $b\in P_1\cap\ldots\cap P_i$, it is clear that $a+b\in C(P_j)$, for $j = 1,\ldots,i+1$. Induction completes the proof.

However, for infinite sets of prime ideals, the problem of proving the intersection condition is more difficult and there is as yet no complete answer. There are partial solutions to this problem. However, these mostly rely on the existence of an uncountable set F of central units such that for any two distinct elements $a,b\in F$, $(a-b)\in F$. In particular, this condition is satisfied if the ring under consideration is an algebra over the complex numbers. We have, for example, the following result due to J.T.Stafford and R.B.Warfield,Jr.

3.3.5. Lemma: Let R be prime Noetherian ring. Let X be a set of prime ideals such that there is a uniform bound on the Goldie ranks of the rings R/Q, $Q \in X$. Suppose that R contains a central subfield K such that $|K| > |X|$. Then X satisfies the intersection condition.

Proof: See Warfield[74], Lemmas 1 and 6.

3.3.6. Theorem: Let R be prime Noetherian ring which contains an uncountable central subfield. Let X be a clique of prime ideals which satisfies the second layer condition. Suppose that there is a uniform bound on the Goldie ranks of the rings R/Q, for $Q \in X$. Then X is classical.

Proof: Since R is Noetherian, X is at most countable, by Stafford[70], Corollary 3.13. The result follows immediately from Lemma 3.3.5 and Theorem 3.3.3.

To extend this result we would like to be able to remove the condition that R contains an uncountable subfield. The natural test-case to consider would be to assume that the X is a clique in which all the prime ideals are completely prime. Our aim in this section is to give some (partial) results on localisations of cliques with this (very strong) condition.

3.3.7. Theorem: Let R be a right Noetherian domain which is not commutative. Let X be an infinite set of prime ideals, P_i, $i \in I$, which are all completely prime. Suppose that, for any infinite subset J of I, $\cap P_j = 0$, where the intersection runs over all $j \in J$. Suppose that K is a right ideal of R with $K \cap C(P_i) \neq \emptyset$ for all $P_i \in X$. Then $K \cap C(X) \neq \emptyset$.

Proof: The idea of the proof is similar to that of Theorem 3.2.3, but we have to do a little preparation first. Observe that the conditions in the statement of the Theorem imply that each non-zero element of R can lie in only finitely many members of X.

If K is a cyclic right ideal cR, then $c \in C(X)$, and we are done.

Since K is finitely generated, we may write $K = \Sigma_{i=1}^{i=n} a_i R$. We shall proceed by induction on n. Let us suppose that we have proved the Theorem for n = 2. Let $K_i = a_i R + a_n R$. Now define the sets $I_i = \{j : K_i \cap C(P_j) \neq \emptyset\}$. If there exists $P_j \in X$ with $j \notin I_i$ for i = 1,...,n-1, then $(a_i R + a_n R) \subseteq P_j$ for i = 1,...,n-1. But this contradicts our assumption that $K \cap C(P_j) \neq \emptyset$. Thus $I = \cup I_i$.

Let $X_i = \{P_j : P_j \in I_i\}$. For each K_i and set X_i, either X_i is finite or the conditions of the Theorem still hold. So, assuming that we have proved for the Theorem for right ideals K generated by two elements, we may deduce that $K_i \cap C(X_i) \neq \emptyset$. Choose $b_i \in K_i \cap C(X_i)$. Let $K' = \Sigma_{i=1}^{i=n-1} b_i R$. Then $K' \cap C(P_i) \neq \emptyset$, for all i. Therefore, by induction, $K' \cap C(X) \neq \emptyset$. We can therefore reduce to the case where K = aR + bR.

The proof now proceeds exactly as in Theoren 3.2.3. We consider one case only and leave the rest for the reader.

Suppose that $ab \neq ba$. Then $(ab-ba) \in P_i$, for exactly k members of X, for some integer k. By Lemma 3.2.1, we may choose a sequence of positive integers $1 < a_1 < \ldots < a_{k+1}$ satisfying $(a_i - a_j)$ divides a_i, for all choices of i and j with $j < i$.

Let $f_n = a + b.(a^{a_n}-1)$, for $n = 1, \ldots, k+1$. If the Theorem is false then each $f_n \in P_n$, for some prime ideal $P_n \in X$. Then $[f_n, a] = (ba-ab)(a^{a_n}-1) \in P_n$. If $(a^{a_n}-1) \in P_n$, then $a \in P_n$, which is clearly a contradiction. Thus $(ba-ab) \in P_n$, for $n = 1, \ldots, k+1$. By our choice of k, there exist integers m and n, $m > n$, with $f_m, f_n \in P_m$.

However, $f_m, f_n \in P_m$ implies that $f_m - f_n = b.a^{a_n}.(a^{a_m - a_n}-1) \in P_m$. If $b \in P_m$, then $a \in P_m$, which contradicts our hypothesis that $(aR+bR) \cap C(P_m) \neq \emptyset$. If $a \in P_m$, then either $b \in P_m$ or $(a^{a_m}-1) \in P_m$, and both of these lead to contradictions. Since P_m is completely prime, we are forced to conclude that $(a^{a_m - a_n}-1) \in P_m$. But then, by Lemma 3.2.2 and our choice of a_i, $(a^{a_m}-1) \in P_m$, and so $a \in P_m$, which again is a contradiction. Thus, for some n, $a + b.(a^{a_n}-1) \in C(P_i)$, for all $P_i \in X$.

The other cases we have to consider (when $ab = ba$) go through just as in Theorem 3.2.3 and we omit the details. This finishes the case $n = 2$ and we are done.

To apply this result we will use the following definition. We shall say that an infinite set of prime ideals $X = \{P_i : i \in I\}$ has the <u>infinite-intersection property</u> if, for all infinite subsets $J \subseteq I$, $\cap P_j = \cap P_i$, where the first intersection runs over all $j \in J$ and the second over all $i \in I$, that is, all the prime ideals in X.

3.3.8. Lemma: Let R be a ring. Let X be an infinite set of prime ideals which has the infinite-intersection property. Let $Q = \cap P$, where the intersection runs over all $P \in X$. Then Q is a prime ideal. Further, if every prime ideal $P \in X$ is completely prime, then Q is completely prime.

Proof: Suppose that $a, b \in R$ are such that $aRb \subseteq Q$. Suppose that $a \notin Q$. Then $a \in P_i$, for only finitely many $P_i \in X$. Hence $b \in P_i$, for infinitely many $P_i \in X$. So $b \in Q$.

The second statement proceeds almost identically.

3.3.9. Theorem: Let R be a Noetherian ring. Let X be an infinite clique of completely prime ideals. Suppose that X satisfies the second layer condition and has the infinite-intersection property. Let $Q = \cap P$, where the intersection runs over all $P \in X$. Suppose that R/Q is not commutative. Then X is classically localisable.

Proof: By Theorem 3.3.3, it suffices to show that X satisfies the intersection condition. Suppose that K is a right ideal of R such that $K \cap C(P) \neq \emptyset$, for all $P \in X$. Then $(K + Q/Q) \cap C(P/Q) \neq \emptyset$, for all $P \in X$. By Lemma 3.3.8, R/Q is a domain. We are in a position to apply Theorem 3.3.7, and hence there exists $x \in K+Q$ such that in R/Q, $\bar{x} \in \cap C(P/Q)$, where the intersection runs over all $P \in X$. So there exists $y \in K$ such that $y \in C(P)$, for all $P \in X$. Hence X satisfies the right intersection condition. By symmetry, X also satisfies the left intersection condition.

3.3.10. Corollary: Let R be a prime Noetherian ring. Let X be a clique of completely prime height-1 prime ideals. Suppose that X satisfies the second layer condition. Then X is classically localisable.

Proof: If R is commutative then X consists of a single height-1 prime and the result follows. If X consists of only finitely many prime ideals we may use Theorem 3.3.4. If X consists of infinitely many prime ideals then X certainly has the infinite-intersection property. The result follows immediately from Theorem 3.3.9.

We finish this section by applying these results to the enveloping algebras of solvable Lie algebras. We shall require a series of Lemmas whose proofs, for the sake of brevity, we shall omit.

3.3.11. Lemma: Let Q be a simple Noetherian domain. Let δ be a derivation on Q. Let $S = Q[x:\delta]$. Then any non-zero prime ideal of S has height 1.

3.3.12. Lemma: Let R be a Noetherian domain. Let δ be a derivation on R. Let $T = R[x:\delta]$. Suppose that P is a prime ideal of T such that $P \cap R = 0$. Then height(P) ≤ 1.

3.3.13: Lemma: Let R be a ring. Let δ be a derivation on R. Let $T = R[x:\delta]$. Suppose that C is an Ore set in R. Then C is an Ore set with respect to T.

3.3.14: Lemma: Let L be a Lie algebra. Let N be an ideal such that $\dim(L/N) = 1$. Then $U(L) \cong U(N)[x:\delta]$, for some derivation of $U(N)$.

3.3.15: Theorem: Let L be a solvable Lie algebra over a field of characteristic zero. Then every prime ideal of $U(L)$ is completely prime.

Proof: Dixmier[25], Theoreme 3.7.2.

3.3.16. Theorem: Let N be a nilpotent Lie algebra over a field of characteristic zero. Then every prime ideal of $U(N)$ is localisable.

Proof: McConnell[50], Theorem 3.2, and Theorem 1.5.4.

3.3.17. Lemma: Let L be a solvable Lie algebra over a field of characteristic zero. Let N be a nilpotent ideal of L with $\dim(L/N) = 1$. Let $X = \{P_\lambda : \lambda\in\Lambda\}$ be a clique of prime ideals in $U(L)$. Then $P_\lambda \cap U(N) = Q$ is independent of our choice of $\lambda\in\Lambda$.

Proof: Choose $\lambda\in\Lambda$. Let $Q = P_\lambda \cap U(N)$. Then, by Dixmier[25], Proposition 3.3.4, Q is a prime ideal of $U(N)$. By Theorem 3.3.16, Q is localisable. Hence $C = C_{U(N)}(Q) = U(N)\backslash Q$ is Ore in $U(N)$. By Lemmas 3.3.13 and 3.3.14, C is Ore in $U(L)$.

Since X is a clique, by Lemma 3.3.1,(iii), $C \subset C_{U(L)}(P_\mu) = U(L)\backslash P_\mu$, for all $\mu\in\Lambda$. Therefore,

$$U(N)\backslash U(N)\cap P_\lambda \subseteq U(N)\backslash U(N)\cap P_\mu, \text{ for all } \mu\in\Lambda.$$

But our choice of $\lambda\in\Lambda$ was arbitrary and the result follows.

3.3.18. Theorem: Let L be a solvable Lie algebra over a field of characteristic zero. Suppose that there exists a nilpotent ideal N of L with $\dim(L/N) = 1$. Let $X = \{P_\lambda : \lambda\epsilon\Lambda\}$ be a clique of prime ideals of $U(L)$. Let $Q = P_\lambda \cap U(N)$, for any (and hence all) $\lambda\epsilon\Lambda$. Suppose that $U(L)/QU(L)$ is not commutative. Then X is classically localisable.

Proof: By the remark before Lemma 3.3.2, $U(L)$ satisfies the second layer condition. So it suffices to show that X satisfies the intersection condition.

Let $Q = U(N) \cap P_\lambda$, which, by Lemma 3.3.17, is independent of our choice of $\lambda\epsilon\Lambda$. Then Q is a prime ideal of $U(N)$. By Lemma 3.3.14, $U(L) = U(N)[x:\delta]$, for some derivation δ of $U(N)$. Then Q is a δ-stable ideal of $U(N)$. Hence $P = QU(L) = QU(N)[x:\delta]$ is a prime ideal of $U(L)$. Further, $U(L)/P \approx (U(N)/Q)[x:\delta]$.

In $U(L)/P$, $\bar{P}_\lambda \cap U(N)/Q = 0$, by the definition of Q. Hence, by Lemma 3.3.12, $\bar{P}_\lambda$ is a height-1 prime ideal. Hence X has the infinite-intersection property. Finally, we may apply Theorem 3.3.9 to deduce that X satisfies the intersection condition. Therefore X is classically localisable.

Remark: Note that a clique in an enveloping algebra over a solvable Lie algebra need not have the infinite-intersection property. Let L be the Lie algebra $kx+ky+ka+kb$, over the field k of the complex numbers, where $[x,y] = x$ and $[a,b] = a$, and all other products zero. Then L is solvable. However, it can be shown that $\{\langle x,a,(y-n),(b-m)\rangle : m,n \text{ integers}\}$ form a clique in $U(L)$. It is easy to see that X does not have the infinite-intersection property.

Section 3.4: The centres of Noetherian UFDs.

Following the notation of Section 3.1, suppose that R is a Noetherian UFD and that T and S are the partial localisations R_C and R_D, respectively, as before. Throughout this section we shall assume that every prime ideal of R of height greater than 1 contains a height-1 prime ideal. Denote by Z(R) the centre of a ring R.

3.4.1. Lemma: Let R be a Noetherian UFD. Then $Z(R) = Z(T)$.

Proof: By Theorem 2.1.9, we know that $R = S\cap T$. So, if $z\in Z(R)$ then clearly $z\in Z(T)\cap Z(S)$. Conversely, if $z\in Z(T)\cap Z(S)$ then $z\in R$ and is central in R. Now suppose that $q\in Z(T)$. Let $0 \neq I = \{s\in S : qs\in S\}$. Then I is an ideal of S and, since S is simple, is equal to S. So $q\in S$ and hence $q\in R$.

Recall that a Krull domain is a commutative domain A with a field of fractions K with the following properties.

(i) For all height-1 prime ideals P of A, A_P is a principal ideal domain, and

(ii) for all non-zero $x\in A$, $x\in P$, for only finitely many height-1 prime ideals P.

With each height-1 prime ideal P of a Krull domain we may associate an integer-valued valuation, v_P, on A, defined by $v_P(x) = \max\{n : x\in P^n\}$ and $v_P(x) = 0$ if $x\notin P$. Thus a Krull domain may be characterised as a commutative domain A such that there exists a family of integer-valued valuations with the following properties.

(iii) For any non-zero element x of A, $v(x) \geq 0$, for all $v \in V$, with equality for all, but possibly finitely many v, and

(iv) $A = \cap K_v$, where the intersection runs over all the valuations $v \in V$, and $K_v = \{k \in K : v(k) \geq 0\}$ for $v \in V$.

3.4.2. Theorem: Let R be a principal ideal domain. Then Z(R) is a Krull domain.

Proof: Cohn[19], Theorem 6.2.4.

3.4.3. Corollary: Let R be a Noetherian UFD. Then Z(R) is a Krull domain.

Proof: If R is commutative then R is certainly a Krull domain (see, for example, Cohn[19], Section 6.2). Otherwise, $T = R_C$ is a principal ideal domain and the result follows from Lemma 3.4.1.

We can in fact show a converse to the above result by showing that any commutative Krull domain can be realised as the centre of a Noetherian UFD. That is, Corollary 3.4.3 is best possible. We will use a construction due to P.M.Cohn. For a given Krull domain C, we shall construct a commutative principal ideal domain A and an automorphism of infinite order, σ, of A, whose fixed ring $A^{\sigma} = C$.

3.4.4. Theorem: Let C be a commutative Krull domain. Then there exists a Noetherian UFD R with $Z(R) = C$.

Proof: We follow Cohn[19], Section 6.3. Let F be the field of fractions of C. Then $C = \cap F_v$, where the intersection runs over the set of Z-valued valuations V on F induced by the height-1 prime ideals of C.

We form the polynomial extension of F in infinitely many variables, $K = F[\ldots,t_{-1},t_0,t_1,\ldots]$. Observe that we can extend each valuation v on F to a valuation on K by

$$v(k)=\min\{v(a_i)\}, \text{ where } k = \sum a_i t_{i_1} t_{i_2} \ldots t_{i_n}.$$

Let L be the field of fractions of K. Then we can extend each valuation $v \in V$ to a valuation on L. For each valuation $v \in V$, we can define

$$L_v = \{f/g : v(f) \geq v(g), g \neq 0\}.$$

Then define $A = \cap L_v$, where the intersection runs over all $v \in V$. We claim that A is a principal ideal domain. If $f,g \in A$, by multiplying by a suitable common denominator, we may assume that $f,g \in C[\ldots,t_{-1},t_0,t_1,\ldots]$. Now take n to be any integer greater than the total degree of f. Let $h = f + t_0^n g$. We have $v(h) \leq v(f)$ and $v(h) \leq v(g)$, for all $v \in V$. Hence $f/h, g/h \in A$, and so $fA+gA = hA$. Therefore A is a Bezout domain as defined in Section 1.7. By a simple degree argument, it is easy to see that A is atomic. By Theorem 1.7.2, A is a principal ideal domain. Now define the F-automorphism on K, σ, by $\sigma(t_i) = t_{i+1}$. Clearly the fixed ring of σ is F, and $v(f) = v(\sigma(f))$, for all $v \in V$.

We may extend σ to an automorphism on A. The fixed ring of σ acting on A is $A^\sigma = K^\sigma \cap A = F \cap A = C$.

To recap so far; given a Krull domain C, we have constructed a principal ideal domain A, and an automorphism σ of A with infinite order, such that $A^\sigma = C$. We now proceed to construct a Noetherian UFD.

Let R be the skew Laurent extension $R = A[x,x^{-1}:\sigma]$, as defined in Section 1.8. Then, for all $a \in A$, $ax = x\sigma(a)$. In particular, $t_i x = x t_{i+1}$. We shall show that R is a Noetherian UFD, and that the centre of R is C.

First, suppose that P is a height-1 prime ideal of R. If $P \cap A = P' \neq 0$, then P' is principal. That is, $P' = pA$, and pR is a prime ideal which is completely prime contained in P. Therefore, $P = pR$.

Suppose that, instead, $P' = 0$. We shall derive a contradiction from this. Choose $f(x) = \Sigma_{i=0}^{i=n} x^i a_i \in P$, for least n.

$$\begin{aligned} \text{But then } t_0 f(x) - f(x) t_n &= \Sigma_{i=0}^{i=n} t_0 x^i a_i - x^i a_i t_n \\ &= \Sigma_{i=0}^{i=n} x^i (a_i t_i - a_i t_n) \\ &= \Sigma_{i=0}^{i=n-1} x^i (a_i t_i - a_i t_n), \end{aligned}$$

which is a polynomial of x-degree less than n, in P. This contradicts our choice of f(x).

Therefore, R is a Noetherian UFD. Now suppose that $z(x) = \Sigma_{i=m}^{i=n} x^i a_i$ is a central element of R. Then

$$0 = xz(x) - z(x)x = \Sigma_{i=m}^{i=n} x^{i+1} (a_i - \sigma(a_i)).$$

Thus, $a_i = \sigma(a_i)$, for all $i = m, \ldots, n$.

Also, $0 = t_0 z(x) - z(x) t_0 = \Sigma_{i=m}^{i=n} x^i (a_i t_i - a_i t_0)$. This implies that $a_i = 0$, for all i, except possibly $i = 0$. Therefore, $Z(R) = A^{\sigma} = C$.

This type of construction of UFD has enabled M.K.Smith to give a simple answer, answering in the negative, to a question of G.Bergman (among others). He asked whether, for any principal ideal domain R, the centre of the full ring of quotients of R was equal to the quotient ring of the centre of

R. This question has also been answered, using similar constructions by M.Chamarie and , independently, P.M.Cohn and A.Schofield. See Chamarie[12] and Cohn-Schofield[22].

Chapter 4. The principal ideal theorem.

Section 4.0. Summary: This chapter forms the other main theme of this work. It stems from the obvious interest that the set C of the previous two chapters has for us and a desire to see how the principal ideal theorem of commutative ring theory can best be generalised to non-commutative ring theory. Much of the material of this chapter was first suggested by A.W. Chatters and it consists of joint work with him. Most of the results are to appear in Chatters-Gilchrist [14].

In the study of a prime Noetherian ring it is natural to consider the set Γ of elements which are regular modulo all the height-1 prime ideals of the ring. The elements of Γ can be thought of informally as being those elements of the ring with no prime factor. As we saw in Chapters 2 and 3, they can play the role that in the case of a commutative ring is played by the units.

It is then a natural question to ask: Let R be a Noetherian ring. When is the set Γ the set of units? If R is a commutative ring, the statement that Γ is the set of units is proved using the principal ideal theorem due to Krull. In fact, it is one of a number of equivalent formulations of the classical principal ideal theorem. In Section 4.1, we shall consider some of these equivalent statements. We shall show, with examples that, even with quite strong conditions on these rings, that the statements are not, in general, equivalent.

We should recall that there have been several other formulations of possible generalisations of the principal ideal theorem notably due to Jategaonkar and to Chatters-

Goldie-Hajarnavis-Lenagan. We shall recall some of the variations on the principal ideal theorem in Section 4.1.

In Sections 4.2, 4.3, and 4.4, we shall prove some positive generalisations of the classical principal ideal theorem. These results will be shown notably for prime PI rings and for bounded maximal orders. In Section 4.5, we discuss some tentative extensions of these results to larger classes of rings.

For the purposes of the formulation of some of the results of this chapter, we shall assume throughout this chapter that all Noetherian rings considered satisfy d.c.c. on prime ideals. Thus all non-zero prime ideals considered will contain at least one height-1 prime ideal.

Section 4.1: Formulations of the principal ideal theorem.

We start by giving some equivalent forms of the classical principal ideal theorem of commutative algebra. Throughout, for a ring R, the set Γ (or $\Gamma(R)$ to distinguish between rings if necessary) will denote the set of elements regular modulo every height-1 prime ideal of R. That is, $\Gamma = \cap C(P)$, where the intersection ranges over the height-1 prime ideals P of R.

4.1.1. Theorem: Let R be a commutative domain. Then the following are equivalent:

(i) If a is a non-zero non-unit of R and P is a prime ideal minimal over a then height(P) = 1;

(ii) The elements of $\Gamma(R)$ are units;

(iii) Every non-zero prime ideal is the union of the height-1 prime ideals which it contains;

(iv) If P is a non-zero prime ideal of R then $C(P) = \cap C(Q)$, where the intersection runs over those height-1 prime ideals Q of R that P contains;

(iv)' If P is a non-zero prime ideal of R then $C(P) \supseteq \cap C(Q)$, with the same notation as (iv).

Proof: For any prime ideal P of R and any $a \in R$, $a \in C(P)$ if and only if $a \notin P$.

4.1.2. Theorem: Let R be a commutative Noetherian domain. Then one, and hence all, of the above statements are true.

Proof: See, for example, Kaplansky[44], Theorem 142.

The equivalent statements of Theorem 4.1.1 all suggest non-commutative generalisations which are likely to be inequivalent and not generally true. Here, we give two examples of non-commutative Noetherian rings which demonstrate the inequivalence of some of these statements.

Note that, in general, it is not true that, for an element $c \in R$ and prime ideal P, $c \notin C(P)$ implies that $c \in P$. This means that by formulating statements in terms of $a \notin C(P)$ rather than $a \notin P$ may help us in generalising the statements of results in commutative Noetherian ring theory. In particular it seems likely that statements (ii) and (iv) are far more likely to hold in non-commutative rings than (i) or (iii). The next example illustrates this point.

4.1.3. Example: Let $S = k[x,y]$ the commutative polynomial ring in two variables over k a field and set $R = M_2(S)$, the ring of 2×2 matrices over S. In S, let P be the prime ideal generated by x and y then P is a height-2 prime ideal of S and $M_2(P)$ is a height-2 prime ideal of R. Let $a = \mathrm{diag}(x,y)$ in R. Then $RaR = M_2(P)$ and hence $M_2(P)$ is a minimal prime ideal over a. Clearly then this provides a counter-example to statements (i) and (iii) of Theorem 4.1.1, but as we shall see later R does in fact satisfy (ii), (iv), and (iv)'. Further note that, for an integer n, had we taken $S = k[x_1,\ldots,x_n]$ and $R = M_n(S)$, then the element $\mathrm{diag}(x_1,..,x_n)$ generates a height-n prime ideal.

4.1.4. Example: Let R be the universal enveloping algebra of the complex two-dimensional non-Abelian solvable Lie algebra. Let k be the field of the complex numbers F. Then R is the k-algebra generated by x and y subject to the condition that $xy-yx=y$. By Lemma 2.1.3, R is a Noetherian domain with a unique height-1 prime ideal $P = Ry = yR$. Further, P is completely prime and the maximal ideals of R all have the form $Q = (x-c)R + yR$, for some $c \in k$.

So in this case, $\Gamma(R) = R \backslash yR$. In particular, $x \in \Gamma$. Since x is not a unit and $RxR = xR + yR$, it is clear that none of the statements of Theorem 4.1.1 are true for this Noetherian ring.

Clearly, we shall have to impose some extra conditions on the ring R in order to have any hope of obtaining a suitable principal ideal theorem.

We record here statements of principal ideal theorems for non-commutative Noetherian rings that are already in the literature.

4.1.5. Theorem: Let R be a right Noetherian ring. Let X be an invertible ideal of R with $X \neq R$. Let P be a prime ideal minimal over X. Then height$(P) \leq 1$.

Proof: See Chatters-Hajarnavis[16], Theorem 3.4.

4.1.6. Corollary: Let R be a right Noetherian ring. Let x be a normalising element of R which is not a unit. Let P be a prime ideal minimal over xR. Then height$(P) \leq 1$.

Proof: See Jategaonkar[39], Theorem 3.1, or Chatters-

Hajarnavis[16], Corollary 3.5.

4.1.7. Theorem: Let R be a prime Noetherian PI ring. Let c be a regular non-unit of R. Let B be the largest two-sided ideal contained in cR. Suppose that P is a prime ideal minimal over B and that $c \notin C(P)$. Then height(P)=1.

Proof: Chatters-Goldie-Hajarnavis-Lenagan[15], Theorem 4.8.

Remark: Note that such a prime ideal P always exists, by an argument due to A.W.Goldie. Effectively the same proof will be used to prove Theorem 4.2.4.

The next two sections will concern statement (ii) of Theorem 4.1.1 in the context of fully bounded Noetherian rings all of whose non-zero ideals intersect the centre non-trivially. We note that this class of rings is closed under forming partial localisations at Ore sets.

There are two large classes of Noetherian rings which satisfy these conditions.

4.1.8. Theorem: Let R be a prime Noetherian PI ring. Then R is fully bounded, and every non-zero ideal of R intersects the centre of R non-trivially.

Proof: Immediate from Theorem 1.3.2.

4.1.9. Let R be a prime Noetherian ring which is integral over its centre Z(R). Then R is fully bounded and every non-zero ideal of R intersects the centre non-trivially.

Proof: Suppose that P is a prime ideal of R. Choose an element $c\epsilon C_{R/P}(0)$. By Lemma 1.1.6, there exists $d\epsilon C_R(0)$ such that the image of d in R/P is c. By hypothesis, there exists elements $a_1,\ldots,a_n \epsilon Z(R)$, such that $d^n+a_1d^{n-1}+\ldots+a_n=0$. Then, for least n, suppose that $d^n+a_1d^{n-1}+\ldots+a_n\epsilon P$. If $a_n\epsilon P$, then $d.(d^{n-1}+\ldots+a_{n-1})=-a_n\epsilon P$.

But $d\epsilon C(P)$, and this forces $d^{n-1}+\ldots+a_{n-1}\epsilon P$, which contradicts our choice of n. Therefore, a_n generates a non-zero ideal in R/P. Hence R/P is bounded. By a similar argument, it is easy to see that every non-zero ideal of R intersects the centre non-trivially.

Section 4.2. The elements of Γ.

We recall our standard notation $\Gamma = \cap C(P)$, where the intersection ranges over the height-1 prime ideals P of R.

4.2.1. Lemma: Let R be a prime right Noetherian ring. Suppose that R contains an infinite number of height-1 prime ideals. Then Γ consists of regular elements.

Proof: Suppose that $c \in \Gamma$ and $0 \neq r \in R$ are such that $cr=0$. Then $r \in P$ for all the height-1 prime ideals P of R. Hence every height-1 prime ideal of R is minimal over $RrR \neq 0$. This contradicts the fact that R is Noetherian.

However, if R contains only finitely many height-1 primes, then it is possible for a zero-divisor to be in Γ.

4.2.2. Example: Let R be a Noetherian prime ring with a unique proper ideal P, see for example J.C.Robson [59], Example 7.3. Then $P = P^2$. Suppose that $P = x_1R + \ldots + x_nR$, for some $x_i \in P$. In $M_n(R)$, the only height-1 prime ideal is $M_n(P)$.

Then, for each $i = 1,\ldots,n$, $x_i = \Sigma_{i=1}^{i=n} x_j \alpha_{ji}$, for some elements $\alpha_{ji} \in P$. In $M_n(R)$, let A be the matrix (α_{ij}). Clearly, $(A-I_n) \in C(M_n(P))$. Let X be the matrix all of whose rows are $(x_1 \ldots x_n)$. Then $X.(A-I_n) = 0$. Hence $(A-I_n)$ is not regular, but clearly $(A-I_n) \in \Gamma$.

The following Lemma is well-known and is a consequence of Krause-Lenagan-Stafford [45], Lemma 3.

4.2.3. Lemma: Let R be a fully bounded Noetherian ring with nilpotent radical N. Let $c \in C(N)$. Then cR contains a non-zero ideal of R.

Proof: For a right R-module M, denote the Krull dimension of M by $|M|$. Since R is fully bounded, N has weak ideal invariance, by Theorem 1.4.9. That is, if K is a right ideal of R , then $|R/K| < |R/N|$ implies that $|N/KN| < |R/N|$.

Now suppose that cR contains no non-zero ideal of R. Then $|R/cR| = |R| = |R/N|$, by Corollary 1.4.6 and Theorem 1.4.7. But $c \in C(N)$ and so $|R/cR+N| < |R/N|$, by Theorem 1.4.3. We shall proceed, by induction on k, to show that $|R/cR+N^k| < |R/N|$, for all integers k. Suppose that we have shown that

$$|R/cR+N^{k-1}| < |R/N|.$$

Then, by weak ideal invariance, $|N/(cR+N^{k-1})N| < |R/N|$. Hence

$$|cR+N/cR+N^k| \leq |N/cN+N^k| \leq |N/(cR+N^{k-1})N| < |R/N|.$$

Thus, combining these inequalities, using Lemma 1.4.1, we have

$$|R/cR+N^k| = \sup\{|R/cR+N|, |cR+N/cR+N^k|\} < |R/N|.$$

But, for some integer m, $N^m = 0$. So $|R/cR| < |R/N|$, a contradiction.

Remark: Note that this is the only place in this thesis that we use Krull dimension. It seems likely that a more elementary proof exists, but we have been unable to find such a proof.

Recall that the bound of a right ideal I of R is the largest ideal of R contained in I.

4.2.4. Theorem: Let R be a prime fully bounded Noetherian ring such that every non-zero ideal of R intersects the centre non-trivially. Then the elements of Γ are units.

Proof: Choose $c\in\Gamma$. Suppose that R contains an infinite number of height-1 prime ideals. By Lemma 4.2.1, $c\in\Gamma$ implies that c is regular. Suppose instead that R contains only finitely many height-1 prime ideals, $Q_1,\ldots,Q_n$. Let $I = Q_1\cap\ldots\cap Q_n \neq 0$. Observe that $\Gamma = C(I)$. By Lemma 1.1.6, there exists $x\in I$ such that $d=c+x\in C(0)$. Suppose that we have shown that d is a unit. Then R/I is Artinian and I is the Jacobson radical of R. Thus R is 1-dimensional and semi-local. Hence c is a unit of R. Thus we may assume, without loss of generality, that c is regular.

Suppose that c is not a unit. Let B be the bound of cR. Then cR/B contains no non-zero ideal of R/B. Suppose that N/B is the nilpotent radical of R/B, where N is an ideal of R. From Lemma 4.2.3, we deduce that $c\notin C(N/B)$. So there exists a prime ideal P minimal over B with $c\notin C(P)$. Suppose that P is not a height-1 prime ideal. Then let Q be a non-zero prime ideal of R contained in P.

Suppose that $0 \neq d\in Q\cap Z(R)$. We now use a reduced rank argument to arrive at a contradiction.

For the positive integers n, let $I_n = \{r:c^n r\in dR\}$. Then I_n is a right ideal of R. For $n = 1,2,\ldots$, the I_n form an ascending sequence of right ideals. Hence, there exists n, such that $I_n = I_{n+1} = \ldots$. So replacing c by c^n, we may suppose that $I_1 = I_2$ and hence that $dR\cap(c^2R+cdR) = cdR$.

Let $\rho(.)$ denote the reduced rank of a f.g. R/B-module. Then we have the following equalities

$\rho(cR+dR/c^2R+dR) + \rho(c^2R+dR/c^2R+cdR) = \rho(cR+dR/c^2R+cdR)$ and

$$\begin{aligned}\rho(cR+dR/c^2R+cdR) &= \rho(cR+dR/cR) + \rho(cR/c^2R+cdR)\\ &= \rho(cR+dR/cR) + \rho(R/cR+dR)\\ &= \rho(R/cR)\end{aligned}$$

But $\rho(c^2R+dR/c^2R+cdR) = \rho(dR/(dR\cap(c^2R+cdR))) = \rho(dR/cdR)$

$= \rho(R/cR)$

So $\rho(cR+dR/c^2R+dR) = 0$

Therefore, there exists $e\in C(N) \subseteq C(P)$ such that $ce = c^2x+dy$. But $B \not\subseteq Q$ and so $c\in C(Q)$. Then $c(e-cx) = dy$ implies that $(e-cx)\in Q$. If $wc\in P$, then $we\in P$. By choice of e, this implies that $w\in P$. Hence $c\in C(P)$. But this contradicts our choice of P.

Thus height$(P) = 1$. Since $c\in\Gamma$, this is a contradiction.

4.2.5. Corollary: Suppose that R is a prime Noetherian ring which is either PI or is integral over its centre. Then $\Gamma(R)$ is the set of units.

Proof: By Theorems 4.1.8 and 4.1.9, both classes of rings satisfy the hypotheses of Theorem 4.2.4.

Remarks: (a) Essentially, this argument is no more than a recasting of Goldie [31], Theorem 2.13.

(b) S.A.Amitsur and L.W.Small have previously shown that if R is prime Noetherian PI and has only finitely many height-1 prime ideals, then R is 1-dimensional. See Amitsur-Small[2], Theorem 5.1.

(c) L.W.Small and J.T.Stafford (Small-Stafford [64], Example 3) have constructed an example of a prime Noetherian PI ring R with an element $c \in C(0)$ and a height-2 prime ideal P such that P is minimal over the bound of cR. In this case, $c \in C(P)$.

Finally, note that we cannot drop the assumption that R is fully bounded. For example, suppose that R is the enveloping algebra of a non-Abelian nilpotent Lie algebra, Then every ideal of R has a centralising set of generators, but yet Γ does not consist only of the units. If Γ were the set of units then, by Theorem 2.2.1 and Corollary 3.1.2, R would be a principal ideal domain, which contradicts our choice of R.

Section 4.3. The height-1 prime ideals related to a given prime ideal.

We now turn to the statements of (iv) and (iv)' of Theorem 4.1.1. For any ring R the statements (i) and (iii) are equivalent. It might be thought that statements (ii) and (iv) are also equivalent. It is however possible to give an example of a prime Noetherian ring which satisfies (iv) and not (ii).

4.3.1. Example: Let S be a simple Noetherian domain with an algebraically closed centre k. Let $R = S[x]$. Then R is a Noetherian domain and every non-zero prime ideal of R is of the form $P = (x-\alpha)R = R(x-\alpha)$, where $\alpha \in k$. Clearly each non-zero prime ideal of R has height-1 and hence condition (iv) is trivially satisfied. However, provided that S is not a division ring there exist non-zero non-units of S in R. Choose one such element c. Then $c \in \cap C(P)$, where the intersection runs over all the height-1 prime ideals of R, but is not a unit. Hence condition (ii) is not satisfied.

It is not hard to see that $S[x,y]$ would provide another (slightly less trivial) example of a ring satisfying (iv) and not (ii).

Secondly, it is important to note that, even in well-behaved rings, it is possible for condition (iv)' to hold when condition (iv) fails. The next example demonstrates this. We exhibit a prime Noetherian PI ring R with a height-1 prime ideal Q contained in a height-2 prime ideal P and a regular element c such that $c \in C(P)$ and $c \notin C(Q)$.

4.3.2. Example: Let k be a field of characteristic zero. Let R be the ring generated by two 2×2 generic matrices X and Y over k (see Section 1.3 for details). Let $T = T(R)$ be the trace ring of R. Then we know that T is a Noetherian PI domain and a maximal order (Small-Stafford [65]).

Let tr( ) denote the trace of a given matrix and det( ) the determinant. By Formanek-Halpin-Li[29], Theorem 6 and Lemma 2, $T.(XY-YX)$ is a height-1 prime ideal of T. Further, $T/T.(XY-YX)$ is isomorphic to a polynomial ring over k generated by the images of $X, Y, \mathrm{tr}(X)$, and $\mathrm{tr}(Y)$.

Now $\det(X) = X.(\mathrm{tr}(X)-X)$ by the Cayley-Hamilton Theorem. Let $P = T.X + T.(XY-YX)$. Then P is a height-2 prime ideal of T. Let $a = \det(X)$. Then $a\in P$ and a is central. Thus, by Jategaonkar's principal ideal theorem (Jategaonkar[41], Theorem 2) there exists a height-1 prime ideal $Q \subseteq P$ such that $a\in Q$. Observe that $X\notin Q$. Then $c = (\mathrm{tr}(X)-X)\in C(P)\cap C(O)$ and $c\notin C(Q)$.

Note that in this example condition (ii) holds, by Theorem 4.2.4.

Now we turn to the positive results of this section. They rely heavily on being able to reduce to the situation of Section 4.2 and then being able to apply Theorem 4.2.4.

4.3.3. Theorem: Let R be a prime fully bounded Noetherian ring such that every non-zero ideal contains a non-zero central element. Let P be a non-zero localisable prime ideal of R. Then $C(P) = \cap C(Q)$, where the intersection ranges over all the height-1 prime ideals contained in P.

Proof: We have that R satisfies the Ore condition with respect to C(P). By Lemma 1.5.6, $C(P) \subseteq C(Q)$, for every prime ideal $Q \subseteq P$. In particular, C(P) consists of regular elements. Let S be the partial quotient ring of R with respect to C(P).

Let $c \in \cap C(Q)$, as above. The height-1 prime ideals of S are of the form QS, where Q is a height-1 prime ideal of R contained in P. Thus c is regular modulo all the height-1 prime ideals of S. Hence c is a unit of S, by Theorem 4.2.4. So $1 = cad^{-1}$, for some $a \in R$ and $d \in C(P)$. That is, $ca = d \in C(P)$. Hence $c \in C(P)$.

For a general prime ideal of R we have a slightly weaker result.

4.3.4. Theorem: Let R be as in Theorem 4.3.3. Let P be a non-zero prime ideal of R. Then $C(P) \cap C(0) \supseteq \cap C(Q)$, where the intersection ranges over all the height-1 prime ideals Q with $Q \cap Z(R) \subseteq P \cap Z(R)$.

Proof: Let $P' = P \cap Z(R)$. Let S be the partial quotient ring of R formed by inverting the elements of $Z \backslash P'$.

Let $c \in \cap C(Q)$, as above. The height-1 prime ideals of S are of the form QS, where Q is a height-1 prime ideal of R with $Q \cap Z(R) \subseteq P'$. Thus c is regular modulo all the height-1 prime ideals of S. Hence c is a unit of S, by Theorem 4.2.4. So $1 = cad^{-1}$, for some $a \in R$ and $d \in Z \backslash P'$. So $ca = d \in C(P) \cap C(0)$. Therefore $c \in C(P) \cap C(0)$.

We have been unable to answer the following question, which is perhaps the most natural formulation of the principal ideal theorem for non-commutative rings. Let R be as in Theorem 4.3.3. Let P be a non-zero prime ideal of R. Is it true that $C(P) \cap C(0) \supseteq \cap C(Q)$, where the intersection runs over all height-1 prime ideals $Q \subseteq P$?

A positive answer to this question would imply the following: Let R be as in Theorem 4.3.3. If P is a prime ideal with height(P)$\geqslant$2, then P contains an infinite number of height-1 prime ideals of R. This is a result which is known in the PI case (Resco-Small-Stafford[57]), but is an open question, in general, for fully bounded Noetherian rings.

Section 4.4. The principal ideal theorem in bounded maximal orders.

We turn now to consider the situation in which R is a prime Noetherian bounded maximal order. In this section, we show that both conditions (ii) and (iv)$'$ hold for this class of rings. Primarily, this is because, for a regular element c, we are able to show that a prime ideal minimal over the bound of cR must have height 1. This is in marked distinction to the situation of Section 4.2, where the Small-Stafford example shows us that this need not be the case.

Let R be a prime Noetherian ring with full quotient ring $Q(R) = Q$. Recall, from Section 1.2, that R is a maximal order if, given $q \in Q$ such that either $qI \subseteq I$ or $Iq \subseteq I$, for some non-zero ideal I of R, then $q \in R$. For further details we refer to Section 1.2.

Suppose that R is an order in its full quotient ring Q. For I an ideal of R, we set $I^* = \{q \in Q : qI \subseteq R\}$. If R is a maximal order then $I^* = {}^*I = \{q \in Q : Iq \subseteq R\}$. Further, I^{**} is an ideal of R which contains I.

Recall that, for a regular element c of R, the bound of cR is the largest two-sided ideal of R contained in cR.

4.4.1. Lemma: Let R be a prime Noetherian maximal order. Let B be a reflexive ideal of R. Suppose that P is a prime ideal minimal over B. Then P is reflexive and height(P) = 1.

Proof: Suppose that P is minimal over B. By Goldie[31], Proposition 1.06, P/B is a middle annihilator prime ideal in

R/B. That is, there exist ideals $X,Y \supset B$ such that $XY \not\subset B$ and $XPY \subseteq B$.

Then $XPYB^* \subseteq R$. Since R is a maximal order this implies that $PYB^*X \subseteq R$ and so $YB^*X \subseteq P^*$. Suppose that P is not reflexive; that is, $P \neq P^{**}$. But then $P^*P^{**} \subseteq R$ implies that $P.P^*.P^{**} \subseteq P$. Hence $P.P^* \subseteq P$ and so $P^* \subseteq R$.

So $YB^*X \subseteq R$. Then $B^*XY \subseteq R$. But this implies that $XY \subseteq B^{**} = B$, which contradicts our choice of X and Y. Therefore $P = P^{**}$.

Suppose that Q is a non-zero prime ideal of R with $Q \subset P$. Then $P^*Q \subseteq P^*P \subseteq R$. Also $PP^* \subseteq R$ implies that $P.P^*.Q \subseteq Q$. Since $P \not\subset Q$, we conclude that $P^*Q \subseteq Q$. But this implies that $P^* \subseteq R$ which contradicts that fact that P is reflexive. Therefore height$(P)=1$.

4.4.2. Lemma: Let R be a prime Noetherian maximal order. Let c be a regular element of R such that the bound B of cR is non-zero. Let P be a prime ideal of R minimal over B. Then $c \notin C(P)$, P is reflexive, and height$(P)=1$.

Proof: We have $c^{-1}B \subseteq R$ and so $c^{-1} \in B^*$. Thus $c^{-1}B^{**} \subseteq R$; that is, $B^{**} \subseteq cR$. It follows that $B = B^{**}$. By Lemma 4.4.1, P is reflexive and height$(P)=1$. Finally, suppose that $c \in C(P)$. Then $cR \cap P = cP \supseteq B$. But then $B.P^* \subseteq cP.P^* = cR$. This implies that $B.P^* = B$. Since R is a maximal order, this implies that $P^* = R$. But this contradicts the fact that P is reflexive. Hence $c \notin C(P)$.

4.4.3. Theorem: Let R be a prime Noetherian bounded maximal order. Then the elements of Γ are units of R.

Proof: Choose $c\in\Gamma$. First, suppose that c is regular. Then cR has a non-zero bound B. If c is not a unit, then pick a non-zero prime ideal of R minimal over B. By Lemma 4.4.2, height(P)=1 and $c\notin C(P)$. But this contradicts our choice of c.

To obtain the result, we have to consider two cases. Suppose that R has infinitely many height-1 prime ideals. Then $c\in\Gamma$ is regular, by Lemma 4.2.1. Hence, by the first paragraph, c is a unit.

If R has only finitely many height-1 prime ideals $Q_1,\ldots,Q_n$, set $I = Q_1\cap\ldots\cap Q_n \neq 0$. We note that $\Gamma = C(I)$. Let $c\in C(I)$. By Lemma 1.1.6, c+x is regular, for some $x\in I$. Therefore, by the first paragraph, c+x is a unit. Hence R/I is Artinian and I is the Jacobson radical of R. Hence c is also a unit. Further, R is 1-dimensional and semi-local.

This shows that statement (ii) of Theorem 4.1.1 holds for bounded Noetherian maximal orders. Now we show that (iv)' also holds for these rings (note that (iv) need not hold, by Example 4.3.2).

4.4.4. Theorem: Let R be a prime Noetherian bounded maximal order. Let P be non-zero prime ideal of R. Then $C(P) \supseteq \cap C(Q)$, where the intersection ranges over those height-1 prime ideals $Q \subseteq P$.

Proof: Let $c\in \cap C(Q)$, as above. If P contains infinitely many height-1 prime ideals then, by Lemma 4.2.1, c is regular.

Suppose that P contains only finitely many height-1 prime ideals $Q_1,\dots,Q_n$. Let $I = Q_1\cap\dots\cap Q_n \neq 0$. By Lemma 1.1.6, there exists $x\in I$ such that $c+x$ regular. Further, $c+x\in C(P)$ if and only if $c\in C(P)$. Therefore, without loss of generality, we may assume that c is regular.

Let B be the bound of cR. Suppose that $c\notin C(P)$. Then we must have that $B \subseteq P$. Hence, by Lemma 4.4.1, there exists a height-1 prime ideal $Q \subseteq P$ such that $B \subseteq Q$ and $c\notin C(Q)$. This contradicts our choice of c.

4.4.5. Corollary: Let R be a prime Noetherian bounded maximal order. Let P be a prime ideal with height$(P)\geqslant 2$. Then P contains an infinite number of height-1 prime ideals.

Proof: Suppose that P contains only finitely many height-1 prime ideals $Q_1,\dots,Q_n$. Let $I = Q_1\cap\dots\cap Q_n$. Then P/I is a non-minimal prime ideal of the semi-prime Noetherian ring R/I. Therefore, by Goldie's Theorem (see, for example, Chatters-Hajarnavis[16], Theorem 1.10), there exists $c\in P$ such that $c\in C(I)$. By Theorem 4.4.4, $c\in C(P)$, a contradiction.

Note that, in view of Example 4.1.4, we cannot delete the word "bounded" from the statements of the results in this section.

Section 4.5: Partial results on the principal ideal theorem.

Let us recast the statement (iv)' of Theorem 4.1.1 as "suppose that c is a regular element of a prime ring R and P a prime ideal of R minimal with respect to the property that $c \notin C(P)$, then height(P)=1". Let R be a prime Noetherian ring. For the purposes of this section, we shall say that a prime ideal P of R satisfies PIT (principal ideal theorem) if C(P) $\supseteq \cap C(Q)$, where the intersection runs over the height-1 prime ideals Q contained in P. A set X of prime ideals satisfies PIT if all members of X satisfy PIT. We shall say that R satisfies PIT if Spec(R) satisfies PIT.

For an ideal I of R, define $\mathrm{Spec}_I(R)$ to be the set of prime ideals not containing I.

In this section, we wish to extend the results of the previous sections to rings which have a reasonably close relationship to the rings of those sections.

4.5.1. Lemma: Suppose that R is a prime fully bounded Noetherian ring. Let c be a regular element of R. Then the set of prime ideals P, minimal with respect to $c \notin C(P)$, is finite.

Proof:

By a Noetherian induction, we may suppose that the result is true for any proper prime factor ring of R. Let B be the bound of cR. If P is a prime ideal of R which does not contain B, then $c \in C(P)$. There exist only finitely many prime ideals minimal over B. Suppose that $c \in C(P)$ for one of these prime ideals. Then, by our inductive assumption, in R/P, there exist only finitely many prime ideals P' minimal with respect to

$c \notin C(P'/P)$. The result follows.

Note that we cannot drop the condition that R is fully bounded in the preceding result.

4.5.2. Example: Let $R = A_1(Z) = Z[x,y:xy-yx=1]$, the first Weyl algebra over the integers Z. By Theorem 2.4.1, R is a Noetherian domain and the height-1 prime ideals of R are of the form $pR = Rp$, for p prime in Z. $R/pR \cong A_1(F_p)$, where F_p is the field of p elements. So, clearly, $x \in C(pR)$, for all p primes of Z.

However, by Theorem 2.4.4, $\bar{x}^p$ generates a height-1 prime ideal of R/pR, for all p primes of Z. Let $Q_p = pR + x^pR$. Then $x \notin C(Q_p)$, for all primes p of Z. Clearly, each Q_p is minimal with respect to this property. So, (by Euclid[27], IX.2) there are infinitely many prime ideals P minimal with respect to the property that $x \notin C(P)$.

4.5.3. Lemma: Suppose that R is a sub-ring of a ring S. Suppose that R and S have a common non-zero ideal I. Then there exists an order-preserving bijection between $\mathrm{Spec}_I(R)$ and $\mathrm{Spec}_I(S)$.

Proof: Define maps $\theta:\mathrm{Spec}_I(R) \longrightarrow \mathrm{Spec}_I(S)$

$$\sigma:\mathrm{Spec}_I(S) \longrightarrow \mathrm{Spec}_I(R)$$

by $\theta(P) = \{s \in S: IsI \subseteq P\}$ and

$\sigma(Q) = \{r \in R: IrI \subseteq Q\} = R \cap Q$.

Then the proof that these maps define an order-preserving bijection is now straightforward.

We are now in a position to extend the results of Sections 4.3 and 4.4 somewhat.

4.5.4. Theorem: Suppose that R is a subring of a ring S. Suppose that R and S have a common non-zero ideal I. If P is a prime ideal of $\mathrm{Spec}_I(S)$ satisfying PIT, then $\sigma(P)$ is a prime ideal of R satisfying PIT.

Proof: Suppose that c is a regular element of R. Suppose that $cx \in \sigma(P)$, for some $x \notin \sigma(P)$. Then $x \notin P$ and so $c \notin C(P)$. Then $c \notin C(Q)$, for some Q a height-1 prime ideal of S contained in P. So there exists $y \notin Q$ with $cy \in Q$. But then $cyI \subseteq \sigma(Q)$ and $yI \not\subseteq \sigma(Q)$. So $c \notin C(\sigma(Q))$. Since $\sigma(Q)$ is a height-1 prime ideal of R contained in $\sigma(P)$, the result follows.

We may use this in several situations.

4.5.5. Theorem: Suppose that R is a prime bounded Noetherian ring which is a subring of a prime Noetherian maximal order S. Suppose that R is order-equivalent to S. Then R and S have a common non-zero ideal I and $\mathrm{Spec}_I(R)$ satisfies PIT.

Proof: From Maury-Raynaud[52], Proposition 1.4.1, we have that $\alpha S \subseteq R$ for some unit α of their common full quotient ring Q(R). Suppose that $\alpha = d^{-1}c$, where c and d are regular elements of S. Then $cS \subseteq R$. But, from Maury-Raynaud[52], Proposition 1.4.2, S is also a bounded ring. So cS contains a non-zero ideal I of S. Clearly, I is also an ideal of R.

By Theorem 4.4.4, S satisfies PIT. Hence from Theorem 4.5.4, $\mathrm{Spec}_I(R)$ also satisfies PIT.

One would of course like to show that Spec(R) satisfies PIT in the above situation. In some cases, we may use Theorem 4.5.5 together with some slightly more detailed consideration of the ring in question to show this.

4.5.6. Example: Suppose that S is a commutative Noetherian domain of Krull dimension at least two. Suppose that I is a non-zero ideal of S.

Let $R = \begin{bmatrix} S & I \\ I & S \end{bmatrix}$

Then R is a subring of T, the ring of 2×2 matrices over S. Further, R and T have the common non-zero ideal $M_2(I)$. By Theorem 4.3.3 or 4.3.4, T satisfies PIT and Spec(T) is in an order-preserving correspondence to Spec(S).

We can write $\mathrm{Spec}(R) = \mathrm{Spec}_I(R) \cup S_1 \cup S_2$, where S_1 is the set of prime ideals of R containing I of the form $\begin{bmatrix} P & I \\ I & S \end{bmatrix}$,

and S_2 is the set of prime ideals of the form $\begin{bmatrix} S & I \\ I & P \end{bmatrix}$,

where P is a prime ideal of S.

Now, by Theorem 4.5.4, $\mathrm{Spec}_I(R)$ satisfies PIT. Also, for any prime ideal Q of S_1, R/Q is isomorphic to S/Q', where Q' is the top left hand corner of Q. It is then easy to deduce that Q satisfies PIT. By symmetry, the same is true for S_2 and hence R satisfies PIT.

A similar method will work to determine that certain prime ideals of prime PI rings satisfy PIT by using the trace ring construction T(R).

To end this section, if we can show that a particular prime ideal of a ring satisfies PIT, then we have the following interesting conclusion.

4.5.7. Corollary: Let R be a right Noetherian ring. Suppose that P is a prime ideal of R with height(P)$\geq$ 2. If P satisfies PIT then P contains an infinite number of height-1 prime ideals.

Proof: Suppose that P contains only finitely many height-1 prime ideals $Q_1, \ldots Q_n$. Let $I = Q_1 \cap \ldots \cap Q_n$. Then P/I is a non-minimal prime ideal of the semi-prime right Noetherian ring R/I. So, by Goldie's Theorem (see Chatters-Hajarnavis[16], Theorem 1.10), P contains an element $c \in C(I)$. But if P satisfies PIT, then $c \in C(P)$, a contradiction.

Chapter 5. Unique Factorisation Rings and alternative Unique Factorisation Domains.

5.0. Summary.

Here we return to the theme of unique factorisation to look at some other related topics. In Sections 5.1 and 5.2 we look at a generalisation of the notion of Noetherian UFD. Most of Section 5.1 is due to A.W.Chatters and D.A.Jordan, but we do in passing give some more examples of primitive Noetherian UFDs. Section 5.2 we believe to be original and in it we provide some structural results for Noetherian UFRs including an analogue of part of the result of Theorem 2.1.9 that if R is a Noetherian UFD then R is the intersection of a simple Noetherian domain and a principal ideal ring. The main stumbling block to further progress is that we do not always know whether $C = \cap C(P)$, where the intersection runs over all the height-1 primes of R, is Ore. We do however provide some sufficient conditions for this to be so.

Finally in Section 5.3 we consider a question of P.M.Cohn who asked if the definition of Noetherian UFD is different from two other notions of Unique Factorisation Domain, namely similarity- and projectivity-UFDs. We show that the answer is "Yes".

Section 5.1. Unique Factorisation Rings in the sense of Chatters-Jordan.

In many respects the definition of Noetherian UFD of Chapter 2 has proved to be a satisfactory generalisation of the commutative definition and has provided a useful tool in considering certain classes of non-commutative Noetherian rings. However there are several aspects of the theory in which the definition of UFD seems to be too restrictive.

In the commutative case, if R is a Noetherian UFD then the polynomial extension R[x] is also a Noetherian UFD (see, for example, Cohn[21], Theorem 11.3.7). However it is possible for a non-commutative UFD to have a polynomial extension which is not a UFD.

5.1.1. Example: Let D be the division ring of the real quaternions and let R be the ring $D[x]_{(x)}$, the polynomial extension of D localised at the completely prime height-1 prime ideal (x). Then R is a Noetherian UFD. However, in R[y] the height-1 prime ideal $(y^2+1)R[y] = R[y](y^2+1)$ is principal, but is not completely prime since $(y-i)(y+i)=(y^2+1)$.

In fact, as we saw in Corollary 3.1.6, this is completely typical behaviour for a PI division ring.

However, as we shall see later, it is true that if R is a Noetherian UFD then the height-1 prime ideals of R[x] are principal on both sides. This inspires the following definition.

5.1.2. Definition: A prime Noetherian ring R is called a <u>unique factorisation ring</u> (UFR) if every non-zero prime ideal of R contains a non-zero prime ideal which is principal on both sides. In particular every height-1 prime ideal is of the form $pR = Rp$ for some regular element p of R.

Clearly, the class of Noetherian UFRs includes: (a) the class of commutative UFDs; (b) the class of Noetherian UFDs; (c) matrix ring extensions $M_n(R)$, for a Noetherian UFR R, and for any positive integer n.

We shall not develop the theory of Noetherian UFRs very far, referring the reader to Chatters-Jordan[17] for further details, but a few preliminary results are in order.

Let R be a Noetherian UFR. Then, just as for a UFD, let $D = \{\Pi p_i : p_iR = Rp_i$ a height-1 prime ideal of $R\}$, be the multiplicatively closed set generated by the generators of height-1 prime ideals. Then, by Lemma 1.1.8, D is an Ore set.

5.1.3. Lemma: Suppose that R is a Noetherian UFR, D as above. Let $S = R_D$. Then S is a simple Noetherian ring.

Proof: Chatters-Jordan[17], Lemma 2.1.

5.1.4. Lemma: Suppose that R is a Noetherian UFR and that P is a height-1 prime ideal of R. Then P is localisable.

Proof: It is easy to prove that $C(P) = C(P^n)$, for all positive integers n. Then we may apply Theorem 1.5.5 to conclude that C(P) is Ore.

5.1.5. Theorem: Let R be a Noetherian UFR and $S = R_D$ as above. Then $R = S\cap(\cap R_{pR})$, where the second intersection runs over the family of partial localisations at the height-1 prime ideals of R.

Proof: Chatters-Jordan[17], Theorem 2.3.

5.1.6. Theorem: Let R be a Noetherian UFR. Then R is a maximal order.

Proof: Essentially this is immediate after observing that S and each R_{pR} are maximal orders. See Chatters-Jordan[17], Theorem 2.4.

5.1.7. Corollary: Let R be a Noetherian UFR. Let P be a height-1 prime ideal of R. Then the partial quotient ring $R_{C(P)}$ is a principal ideal ring and is a bounded Asano order. Further, $PR_{C(P)}$ is the Jacobson radical of $R_{C(P)}$ and the ring $R_{C(P)}/PR_{C(P)}$ is simple Artinian.

Proof: Immediate from Maury-Raynaud[52], Theorem IV.2.15 and Theorem IV.1.5.

5.1.8. Theorem: Let R be a Noetherian UFR. Then R[x] is a Noetherian UFR.

Proof: Recall that S is a simple Noetherian ring. Then any ideal I of S[x] is of the form $I = f(x)S[x]$ for some central polynomial $f(x)$ of $S[x]$.

Now suppose that P is a height-1 prime ideal of $R[x]$. If $P' = P \cap R \neq 0$, then P' is a height-1 prime ideal of R and so is principal. Then $P' = pR = Rp$ for some $p \in R$. Thus $P = pR[x] = R[x]p$.

Suppose now that $P \cap R = 0$. Consider $PS[x]$ which is a prime ideal of $S[x]$. Then $PS[x] = f(x)S[x]$, for some central polynomial $f(x)$ of $S[x]$. Then $f(x) = g(x)d^{-1}$ for some $g(x) \in P$ and $d \in D$. Note that $dR = Rd$. Then $Rg(x) = Rf(x)d = f(x)Rd = f(x)dR = g(x)R$. Also $g(x)x = xg(x)$. Thus $g(x)R[x] = R[x]g(x)$, and $g(x)S[x] = S[x]g(x)$. Suppose that $g(x) \in P$ is chosen such that $g(x)R[x] = R[x]g(x)$ is maximal with respect to $PS[x] = g(x)S[x] = S[x]g(x)$. Suppose that $g(x)R[x] \neq P$. Choose $h(x) \in P \backslash g(x)R[x]$. Then $h(x) \in PS[x] = g(x)S[x]$. So, there exists $d' \in D$ such that $h(x)d' \in g(x)R[x]$. Then d' is a product of elements p_i which generate height-1 prime ideals. By an induction on the number of p_i such that $d' = \Pi p_i$, we may suppose that $d' = p$, where $pR = Rp$ is a height-1 prime ideal. Then $h(x)p = g(x)b(x)$, for some $b(x) \in R[x]$. Now $g(x)R[x]b(x) = R[x]g(x)b(x) = R[x]h(x)p$. If $b(x) \in R[x]p$, then $h(x) \in g(x)R[x]$, which contradicts our choice of $h(x)$. So $g(x) \in R[x]p$. But then $g(x) = g'(x)p$, for some $g'(x) \in R[x]$. It is not hard to see that $g'(x) \in P$ and that $g'(x)R[x] = R[x]g'(x)$. But this contradicts our choice of $g(x)$. Therefore $h(x) \in g(x)R[x]$ and $g(x)R[x] = P = R[x]g(x)$.

It is possible to generalise this last result to some skew polynomial extensions of R by either automorphisms or derivations, but we shall not go into this. For further details we refer the reader instead to Chatters-Jordan[17], Sections 4 and 5.

To finish this section we shall as promised return to the question of primtive UFDs. We use a result of A.D.Bell for which I would like to thank S.Walters for bringing to my attention.

5.1.9. Lemma: Let D be a division ring such that char(Z(D))=0. If D[x] is a Noetherian UFD then $D[x_1,...,x_n]$ is a Noetherian UFD for all integers n. Further, if Z(D) is algebraically closed then D[x] is a Noetherian UFD.

Proof: $D[x_1,...,x_n]$ is a Noetherian UFR by Theorem 5.1.8 and induction on n. Since D[x] is a Noetherian UFD, by hypothesis, all the prime factor rings of D[x] are domains. By Bell[6], Theorem A, for any integer n, every prime factor ring of $D[x_1,...,x_n]$ is a domain. In particular, the height-1 prime ideals of $D[x_1,..,x_n]$ are completely prime. So $D[x_1,...,x_n]$ is a Noetherian UFD.

If Z(D) is algebraically closed, then the height-1 prime ideals of D[x] are all generated by elements of the form (x-k), for $k \in Z(D)$. Hence all the height-1 prime ideals of D[x] are completely prime. The second statement of the theorem follows immediately.

5.1.10. Theorem: For any integer n, there exists a primitive Noetherian UFD whose Krull and global dimension are both n.

Proof: Let $A_n(k)$ be the nth Weyl algebra over the complex numbers, k. Let $D_n = D$ be its quotient division ring. Then the centre of D_n is k. By Lemma 5.1.9, $D[x_1,...,x_m]$ is a Noetherian UFD for all m. In particular $R = D[x_1,...,x_n]$ is a Noetherian UFD. By Amitsur-Small[1], Theorem 3, R is primitive. Finally, by Resco[55], Theorem 4.2, R has Krull and global dimension n.

Section 5.2. The set C revisited.

For R a Noetherian UFR define, as before, $C = \cap C(P)$, where the intersection runs over all the height-1 prime ideals of R. If R is a UFD then Theorem 2.1.4 tells us that C is an Ore set. However, in general it is unknown whether C is Ore or not. If R is bounded, then together Theorems 4.4.3 and 5.1.6 tell us that C is simply the set of units (and is trivially Ore). Taking our cue from the Noetherian UFD case, we might expect, for a general Noetherian UFR R, that C is Ore and that $R_C = \cap R_{pR}$, where the intersection runs over the partial quotient rings of R at the height-1 prime ideals of R. This would give us, in particular, that R_C is bounded from Maury-Raynaud[52], Theorem IV.2.17. We could then also conclude that the set C is equal to the units of R if and only if R is bounded. Unfortunately this is all conjecture.

We should also recall from Sections 2.1.and 3.1 that if R is a Noetherian UFD which is not commutative, then R can be written as the intersection of a simple Noetherian ring and a principal ideal domain. In this section, we shall prove a result which can be regarded as an analogue for Noetherian UFRs though it is significantly weaker.

Let R be a ring and let R[x] be the polynomial extension of R in one variable. For an element $c(x) = c_r x^r + \ldots + c_{n+r} x^{n+r}$, where $c_r \neq 0 \neq c_{n+r}$, define $T(c(x)) = c_r$, and $L(c_{n+r}) = c_{n+r}$. Let $T(0) = L(0) = 0$. For a right ideal I of R[x], define $T(I) = \{T(c(x)) : c(x) \in I\}$, and $L(I) = \{L(c(x)) : c(x) \in I\}$. Clearly, both T(I) and L(I) are right ideals of R.

5.2.1. Lemma: Let R be a semi-prime right Goldie ring. Let R[x] be the polynomial extension of R in one variable. Suppose that $c = c(x) \in R[x]$. Then $c \in C_{R[x]}(0)$

(i) if and only if $L(cR[x])$ is an essential right ideal of R,

(ii) if and only if $T(cR[x])$ is an essential right ideal of R.

Proof: Recall that a right ideal in a semi-prime right Goldie ring is essential if and only if it contains a regular element (Chatters-Hajarnavis[16], Theorem 1.10 and Lemma 1.11). Observe that R[x] is also a semi-prime right Goldie ring.

Suppose first that $c \in C_{R[x]}(0)$. Then $cR[x]$ is an essential right ideal of R[x]. Suppose that $0 \neq J$ is a right ideal of R. Then there exists $0 \neq p(x) \in cR[x] \cap JR[x]$. Suppose that $p(x) = a_r x^r + \ldots + a_{n+r} x^{n+r}$, where a_r and a_{n+r} are both non-zero. Then $0 \neq a_r \in T(cR[x]) \cap J$, and $0 \neq a_{n+r} \in L(cR[x]) \cap J$. Since we chose J arbitrarily, both $L(cR[x])$ and $T(cR[x])$ are essential.

Suppose now that there exists a polynomial $0 \neq a(x) = a_s x^s + \ldots + a_{m+s} x^{m+s} \in R[x]$ such that $a(x)c(x) = 0$. Then, for all $d(x) \in R[x]$, $a(x)c(x)d(x) = 0$. In particular, $a_r T(cR[x]) = 0$, and $a_{m+s} L(cR[x]) = 0$. Hence both $T(cR[x])$ and $L(cR[x])$ are not essential.

5.2.2. Lemma: Suppose that $b(x), c(x), d(x) \in R[x]$. Suppose that $c(x).b(x) \neq 0 \neq d(x)b(x)$. Let w be any integer such that $w > \deg(b)+\deg(c)$. Then

(i) $L(d(x)b(x)) = L((c(x)+x^w d(x))b(x))$, and

(ii) $T(c(x)b(x)) = T((c(x)+x^w d(x))b(x))$.

Proof: Immediate from the definitions of L and T.

5.2.3. Theorem: Let R be a Noetherian UFR. Let X be the set of height-1 prime ideals of R[x] induced from R. That is, X = {pR[x]: pR = Rp a height-1 prime ideal of R}. Then X satisfies the intersection condition and C(X) is Ore.

Proof: Suppose that K is a right ideal of R[x] such that $K \cap C(pR[x]) \neq \emptyset$, for all prime ideals pR[x] of X. By Theorem 5.1.8 and Lemma 5.1.4, each pR[x] is localisable. Hence, by Lemma 1.5.6, C(pR[x]) consists of regular elements. That is, $K \cap C_{R[x]}(0) \neq \emptyset$. Choose $c = c(x) \in K \cap C_{R[x]}(0)$. By Lemma 5.2.1, T(cR[x]) is an essential right ideal of R. So T(cR[x]) contains a regular element c' of R. By Chamarie[11], Proposition 1.8, $c' \in C(pR)$, for all, but finitely many height-1 prime ideals of R. By another application of Lemma 5.2.1, $c(x) \in C(pR[x])$, for all prime ideals in X, except possibly a finite set $\{p_1R[x],\ldots,p_nR[x]\}$.

Now, by Theorem 3.3.4, there exists $d(x) \in K \cap (\cap C(p_iR[x]))$, where the second intersection runs over i = 1,...,n.

Using the fact that R is Noetherian, T(cR[x]) = $\Sigma_{j=1}^{j=m} T(cb_j)R$, for some $b_j = b_j(x) \in R[x]$. Similarly, for each i = 1,...,n, there exist $e_{is} = e_{is}(x) \in R[x]$, such that $L(dR[x]+P_iR[x]/p_iR[x]) = \Sigma_{s=1}^{s=t} L(de_{is})+p_iR/p_iR)$. We may suppose that $de_{is} \notin p_iR[x]$, for all i and s.

Let $n = \max\{\deg(b_j),\deg(e_{is})\} + \deg(c)$. Let w = n+1. Let a = $a(x) = c(x) + x^w d(x)$. Then $T(aR[x]) \supseteq \Sigma T(ab_i)R = \Sigma T(cb_i)R$, by Lemma 5.2.2. Hence $T(aR[x]) \supseteq T(cR[x])$. So, by Lemma 5.2.1, $a(x) \in C(pR[x])$, for all pR[x], except possibly $p_iR[x]$, for i = 1,...,n.

However, we also have that, for $i = 1,\dots,n$,
$L(aR[x]+p_iR[x]/p_iR) \supseteq \Sigma L(ae_{is})R+p_iR/p_iR =$
$\Sigma L(de_{is})R+p_i R/p_iR$, by Lemma 5.2.2. Hence
$L(aR[x]+p_iR[x])/p_iR \supseteq L(dR[x]+p_iR[x])/p_iR$. By Lemma 5.2.1 again, $a(x)\in C(p_iR[x])$, for all $i = 1,\dots,n$. Thus $a(x)\in C(pR[x])$, for all $pR[x]\in X$. Then X satisfies the right intersection condition. By a symmetrical argument, X satisfies the left intersection condition. By Lemma 3.3.2, C(X) is Ore.

Remark: Note that the hypotheses of the theorem could be weakened slightly. Let R be a prime Noetherian maximal order. Let X be the collecton of reflexive prime ideals of R. By Theorem 1.2.3, R[x] is a maximal order. Let X[x] = {P[x]: P a reflexive prime ideal of R}. Then, again by Theorem 1.2.3, the set X[x] is a set of reflexive prime ideals. Finally, by Hajarnavis-Williams[34], Lemma 2.1 and Corollary 3.4, each $P[x]\in X[x]$ is a height-1 prime ideal and localisable. Then exactly the same argument as in Theorem 5.2.3 will work to show that X[x] satisfies the intersection condition in R[x].

5.2.4. Corollary: Let R be a Noetherian UFR. Then $R = S\cap T_0$, where S is a simple Noetherian ring and T_0 is a prime Noetherian ideal-principal bounded hereditary order.

Proof: Let $S = R_D$. Then, by Lemma 5.1.3, S is a simple Noetherian ring. With the notation of Theorem 5.2.3, let $T_0 = R[x]_{C(X)}$. Then we claim that $R = S\cap T_0$.

Clearly $R \subseteq S \cap T_0$. Now suppose that, in $Q(R[x])$, we have $f(x).g(x)^{-1} = (\Pi p_i)^{-1}.r \in S \cap T_0$, where $g(x) \in C(X)$ and each p_i is a generator of a height-1 prime ideal of R.

Then $p_1p_2\dots p_n f(x) = r.g(x)$. Since $g(x) \in C(X)$, $r = p_1\dots p_n r'$, where $r' \in R$. But then $f(x)g(x)^{-1} = r' \in R$.

It remains to show that T_0 has the properties claimed. Suppose that I is an ideal of T_0. Suppose that I is contained in no height-1 prime ideal of the form $pT_0 = T_0p$ of T, for $p \in R$ a generator of a height-1 prime ideal of R, $pR = Rp$. Therefore $I \cap C(pR[x]) \neq \emptyset$, for all height-1 prime ideals pR of R. Hence, by Theorem 5.2.3, $I \cap C(X) \neq \emptyset$. So $I = T_0$.

If I is contained in a height-1 prime ideal of T_0, say $pT_0 = T_0p$, then $I = pJ$ for some ideal J of T_0. By a Noetherian induction, we may assume that $J = dT_0 = T_0d$, for some $d = p_1\dots p_n$, where the p_i generate prime ideals of R. Then $I = pp_1\dots p_nT_0 = T_0pp_1\dots p_n$.

By Maury-Raynaud[52], Proposition 3.2.1, T_0 is an Asano order. Further, $T_0 = \cap T_{0C(pT_0)}$, where the intersection runs over all the height-1 prime ideals of T_0. Thus T_0 is a bounded order by Hajarnavis-Lenagan[33], Theorem 3.5. Finally, Lenagan[46], Theorem and Corollary tells us that T_0 is hereditary.

Remark: Suppose that the full ring of quotients of R, $Q(R) = Q$, is embedded in $Q(R[x])$ in the obvious way. Then $Q \cap T_0 = T$, where T is the intersection of the partial localisations of R at the height-1 prime ideals of R.

In many ways this result is unsatisfactory since it does not give us much information on the structure of R within its ring of fractions Q(R). The rest of this section is devoted to presenting some preliminary and tentative results in this direction.

First we consider what extra conditions we could impose on R to obtain that C is Ore. We need a definition.

5.2.5. Definition: A set X of prime ideals of a ring R is said to satisfy the <u>right reflexive-intersection condition</u> if, given a reflexive right ideal I such that $I \cap C(P) \neq \emptyset$, for all P in X, then $I \cap C(X) \neq \emptyset$. We define the <u>left reflexive-intersection condition</u> in a like manner.

Let R be a Noetherian UFR. Let $S = R_D$ as before. Let $T = \cap R_{C(P)}$, where the intersection runs over all the height-1 prime ideals of R.

5.2.6. Lemma: Let R be a Noetherian UFR. Let I be a reflexive right ideal of R. Then

(i) $I = IT \cap IS$.

(ii) $IT = \cap IR_{C(P)}$, where the intersection runs over all the height-1 prime ideals P of R.

Proof: We use $A.(B \cap C) \subseteq A.B \cap A.C$ for any subsets of R. Clearly $I \subseteq \cap IR_{C(P)} \cap IS$. By the same token, we have $I^*(\cap IR_{C(P)} \cap IS) \subseteq \cap I^*IR_{C(P)} \cap I^*IS \subseteq \cap R_{C(P)} \cap S = R$.

So $I \subseteq \cap IR_{C(P)} \cap IS \subseteq I^{**} = I$. This proves (i). We prove (ii) in a similar fashion.

5.2.7. Lemma: Let R be a prime Noetherian maximal order.

Suppose that a and c are elements of R with c regular. Then the right ideal $K = \{r : ar \in cR\}$ is reflexive.

Proof: $K^* = \{q : qK \subset R\} \supseteq Rc^{-1}a$. Therefore

$$K^{**} \subseteq R \cap (Rc^{-1}a)^* = \{r : ar \in cR\} = K.$$

Let $X = \{P : P = pR = Rp$ a height-1 prime ideal of $R\}$. Consider the following seven conditions:

(i) R is bounded;

(ii) X satisfies the intersection condition;

(iii) C is the set of units;

(iv) X satisfies the reflexive-intersection condition;

(v) C is Ore and $T = R_C$;

(vi) C is Ore and R_C is bounded;

(vii) C is Ore.

5.2.8. Theorem: Let R be a Noetherian UFR. Then we have the following diagram of implications between the conditions (i),...,(vii):

$$\begin{tikzcd}
 & (i) \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & & & & (ii) \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
(iii) \arrow[drrr, Rightarrow] & & (vi) \arrow[rr, Leftrightarrow] \arrow[dr, Rightarrow] & & (v) \arrow[rr, Leftrightarrow] \arrow[dl, Rightarrow] & & (iv) \\
 & & & (vii) & & &
\end{tikzcd}$$

Proof: Note that all the conditions are left-right symmetric so it suffices to consider one-sided conditions only. Suppose that condition (i) holds, then by Theorem 4.4.2, (iii) holds, and (vi) is clear.

If condition (ii) holds then it is immediate that (iv) holds. It is obvious that any of the conditions (iii), (v), or (vi) imply that (vii) holds.

So it remains to prove the equivalence of conditions (iv), (v), and (vi). Suppose that (iv) holds. Suppose that $a \in R$ and $c \in \mathcal{C}$. Let $K = \{r : ar \in cR\}$. Then, by Lemma 5.2.7, K is reflexive. Each height-1 prime ideal in X is localisable, and hence $K \cap \mathcal{C}(P) \neq \emptyset$, for all P in X. Condition (iv) then implies that $K \cap \mathcal{C}(X) \neq \emptyset$. Therefore $\mathcal{C}(X) = \mathcal{C}$ is Ore. Further, if $q \in \cap R_{\mathcal{C}(P)}$, let $J = \{r : qr \in R\}$. Then $J \cap \mathcal{C}(P) \neq \emptyset$, for all $P \in X$, and J is reflexive. Thus $J \cap \mathcal{C}(X)$ is non-empty and now both (v) and (vi) follow. Using Maury-Raynaud[52], Proposition IV.2.17, conditions (v) and (vi) are clearly equivalent. So it remains to prove that (vi) implies (iv).

Suppose that (vi) holds. Suppose that I is a reflexive right ideal such that $I \cap \mathcal{C}(P) \neq \emptyset$, for all $P \in X$. Then, by Lemma 5.2.6, $IT = \cap IR_{\mathcal{C}(P)} = \cap R_{\mathcal{C}(P)} = T = R_{\mathcal{C}}$. Therefore $I \cap \mathcal{C}(X) \neq \emptyset$.

We can in fact show that $\mathcal{C}(X)$ is Ore in some naturally occurring situations by using a counting argument very similar to that of J.T.Stafford and R.B.Warfield in showing that certain cliques in Noetherian rings are localisable.

5.2.9. Theorem: Let R be a Noetherian UFR. Let X be the set of height-1 prime ideals of R. Suppose that R contains a central sub-field such that $|X| < |F|$. Then $\mathcal{C}(X)$ is Ore.

Proof: By Lemma 3.3.2, it is enough to show that X satisfies the intersection condition. If X is finite then we are done by

Theorem 3.3.4. So suppose that X is infinite.

Suppose that K is a right ideal of R with $K \cap C(P) \neq \emptyset$ for all $P \in X$. By Lemma 1.5.6, we may choose $c \in K$ with c regular. By Chamarie[11], Proposition 1.8, $c \in C(P)$ for all, but finitely many $P \in X$. By Theorem 3.3.4, we may choose $d \in K$ such that for each $P \in X$ either $c \in C(P)$ or $d \in C(P)$.

By Jategaonkar[43], Lemma 7.2.10, there exists $\alpha \in F$ with $c+d\alpha \in C(X)$. That is $K \cap C(X) \neq \emptyset$. The left intersection condition is proved in a similar fashion.

Remark: It seems likely that $C(X)$ is Ore in most naturally occuring examples of Noetherian UFRs. In particular K. McKenzie has shown that $C(X)$ is Ore for a large class of group rings which are UFRs and it seems reasonable to make the following conjecture.

5.2.10. Conjecture: Let R be a Noetherian UFR. Let X be the set of height-1 prime ideals of R. Suppose that the set $\{$Goldie rank$(R/P) : P \in X\}$ is bounded above in the integers. Then $C(X)$ is Ore.

We conclude this section with two structural results inspired by corresponding results in Lenagan[47], Chapter 4 for prime hereditary Noetherian rings.

5.2.10. Definition: We shall call a right ideal I of R <u>completely reflexive-faithful</u> if, for all $I \leq J < K \leq R$, where J and K are reflexive right ideals, the module K/J is faithful.

5.2.12. Lemma: Let R be a prime Noetherian maximal order. Suppose that S is an Ore set of non-zero-divisors such that R_S is also a maximal order of Q(R). Denote for a right (respectively left) R_S-ideal of Q(R) J, $J' = \{q \in Q : qJ \subset R_S\}$ (respectively $J' = \{q \in Q : Jq \subset R_S\}$).

Let I be a right ideal of R. Then $R_S I^* = (IR_S)'$, and $I^{**}R_S = (IR_S)''$. In particular, if I is reflexive then IR_S is reflexive.

Proof: Clearly $R_S I^* \subseteq (IR_S)'$. Now suppose that $q \in Q(R)$ is such that $qIR_S \subset R_S$. Write $I = \Sigma i_j R$ for a finite set $\{j=1,\ldots,m\}$. Then for each j, $qi_j = c_j^{-1}s_j$, for some $c_j \in S$ and $s_j \in R$. Then there exists $c \in S$ such that $c_j^{-1}s_j = c^{-1}t_j$, for $t_j \in R$. Thus $cqI \subseteq R$. So $cq \in I^*$ and so $q \in R_S I^*$. The corresponding result for left R-ideals of Q(R) is clear.

Then $I^{**}R_S = (R_S I^*)' = IR_S''$.

5.2.13. Theorem: Let R be a Noetherian UFR. Then $S = \cup B^*$, where the union is over all non-zero ideals B, and $T = \cup I^*$, where the union runs over all the completely reflexive-faithful right ideals of R.

Proof: The first result is clear. Now suppose that I is a completely reflexive-faithful right ideal of R. Suppose that pR is a height-1 prime ideal of R. Let $J = (I + pR)$. If $J^{**} \neq R$, then R/J^{**} is not faithful, a contradiction. Thus $J^{**} = R$. Then, by Corollary 5.1.7 and Lemma 5.2.12, $JR_{C(pR)} = R_{C(pR)}$. So $J \cap C(pR) \neq \emptyset$ and hence $I \cap C(pR) \neq \emptyset$. Therefore $I^* \subseteq Rc^{-1}$, for some $c \in C(pR)$. So $I^* \subseteq R_{C(pR)}$. But pR was chosen arbitrarily and so $I^* \subseteq \cap R_{C(pR)} = T$.

Conversely, choose $q \in T$. Then, for each $pR \in X$, there exists $c_p \in C(pR)$ with $qc_p \in R$. Let $I = \Sigma c_p R$. Then $q \in I^*$, and we claim that I is completely reflexive-faithful. Suppose not, then there exist reflexive right ideals J and K with $I \leq J < K \leq R$ and a non-zero ideal A such that $(K/J)A = 0$. But then $JS = KS$ and also $JT = KT = T$. So $J = JT \cap JS = KT \cap KS = K$, which contradicts our choice of J and K.

We end this section with what we might consider a "decomposition" result.

5.2.14: Theorem: Let R be a Noetherian UFR. Let I be a reflexive right ideal of R. Then $I = J \cap K$ where:

(i) $J = J^{**} = JS \cap R$ and $JS = IS$;

(ii) $K = K^{**} = KT \cap R$ and $KT = IT$. Further $K^* \subset S$.

Proof: Choose J to be a right ideal maximal with respect to the conditions that $I < J$, that J is reflexive and that J/I is unfaithful. Choose K to be a right ideal maximal with respect to the conditions that $I < K$, that K is reflexive and that $KT = IT$.

By Maury-Raynaud[52], Proposition I.3,7, $J \cap K$ is reflexive. So, by Lemma 5.2.6, $J \cap K = (J \cap K)T \cap (J \cap K)S$. Since J/I is unfaithful, $JS = IS$. Hence $(J \cap K)S = IS$. By our choice of K, $(J \cap K)T = IT$. Therefore $(J \cap K) = IS \cap IT = I$.

Suppose that $J \neq JS \cap R$. Then $J \neq (JS \cap R)^{**}$. But, by Lemma 5.2.12, $(JS \cap R)^{**}S = ((JS \cap R)S)'' = (JS)'' = (IS)'' = I^{**}S = IS$ and this contradicts our choice of J. Therefore $J = JS \cap R$.

The proof that $K = KT \cap R$ is similar. Finally, since T is a bounded ring, K is bounded. By Theorem 5.2.13, $K^* \subset S$.

Section 5.3. Alternative Unique Factorisation domains.

In this section, we briefly consider two other variations on the theme of generalising the notion of Unique Factorisation domain from the commutative case. Whilst, in some cases, these definitions are equivalent to the definition of Chapter 2, they are in general distinct. The first was proposed by P.M.Cohn and the second is a natural generalisation proposed by R.A.Beauregard. Both are essentially lattice-theoretic notions. We shall give examples to indicate that these definitions are in general different from our notion of Noetherian UFD. This answers a question by P.M.Cohn.

There are two features common to all notions of unique factorisation, the first being that of a distinction between atomic and prime elements, and the second that of a factorisation of elements into primes in some form. Recall that, in a ring R, an atom is an element which cannot be written as a product of two non-units. A domain R is atomic if every element may be written as a product of atoms. In a commutative ring R, a prime p is an element such that if $a.b \in pR$ for some two elements a and b of R then either $a \in pR$ or $b \in pR$. Even in quite well-behaved commutative rings the two notions are distinct. For example in $Z[\sqrt{-5}]$, we have $6 = 2.3 = (1+\sqrt{-5})(1-\sqrt{-5})$, where all the factors are atoms, but none are primes. In a commutative ring, we say that two elements a and b are associates if $a = u.b$ for some unit u of R . A commutative UFD may be characterised by the property that all atoms are primes and every element has a (necessarily) unique factorisation as a product of atoms, up to order and associates.

We refer the reader to Section 1.7 for the definitions of lattice and modular lattice. P.M.Cohn considered the notion of similarity as a generalisation of associate as follows. Let R be a domain. Two elements a and b of R are said to be <u>similar</u> if $R/aR \cong R/bR$ as R-modules. The apparent asymmetry of this definition is resolved by the following.

5.3.1. Theorem: Let R be a domain. Suppose that a and b are two elements of R. Then $R/aR \cong R/bR$ as right R-modules if and only if $R/Ra \cong R/Rb$ as left R-modules.

Proof: See Cohn[19], Corollary 2 to Theorem 3.2.1.

Note that, if R is commutative, then two elements are similar if and only if they are associates because then they generate the same ideal of R. Let R be a domain. Given an element $c \in R$, we say that $c = a_1 \ldots a_n$ is an <u>atomic factorisation</u> of c if a_i is an atom, for all i.

Suppose that $c = a_1 a_2 \ldots a_n = b_1 b_2 \ldots b_m$ are two atomic factorisations of an element c. We shall say that these two factorisations are <u>similarity-isomorphic</u> if and only if $m = n$ and for some $\sigma \in S_n$, for each $i = 1, \ldots, n$, a_i is similar to $b_{\sigma(i)}$. We call a domain a <u>similarity-UFD</u> if R is an atomic domain in which any two atomic factorisations of an element are similarity-isomorphic. It is clear that if R is commutative then this reduces to the classical definition of UFD. We have, in terms of lattices, the following useful criterion to determine if a domain is a similarity-UFD.

5.3.2. Theorem: A domain R is a similarity-UFD if, for each $c \in R$, the set $L(cR,R)$ of principal right ideals between cR and R is a modular sub-lattice of finite length of the lattice of right ideals of R.

Proof: See Cohn[20], Theorem 5.6.

Let R be a domain. It is not hard to see that $a, a' \in R$ are similar if and only if there exists an element b of R with $aR + bR = R$ and $aR \cap bR = ba'R$. R.A.Beauregard uses this to generalise the notion of similarity as follows.

Suppose that a and b are two elements of a domain R. We define $(a,b)_r$ to be the element (if it exists) d such that $aR + bR = dR$. We define also $[a,b]_r$ to be the element (if it exists) c such that $aR \cap bR = cR$. Define the corresponding elements on the left, $(a,b)_l$ and $[a,b]_l$, in the obvious way. Then we say that two elements a and a' of R are <u>transitive</u> if there exists an element b such that $(a,b)_l = 1$ and $[a,b]_r = ba'$, and we write a tr a'. This relation is not necessarily symmetric, but we may use it to define an equivalence relation. We say that two elements a and a' of R are <u>projectively equivalent</u> if there exist elements $a_0, a_1, \ldots, a_n$ of R such that $a = a_0$, $a' = a_n$, and, for each $i = 1, \ldots n$, either a_{i-1} tr a_i or a_i tr a_{i-1}. We write a pr a'.

It is easy to see that if two elements are similar then they are projectively equivalent. In the case of a commutative domain, if two elements are projectively equivalent, then they are associates. If R is a Bezout domain then two projectively equivalent elements are similar.

Let R be a domain. Let c be an element of R. If $c = a_1 \ldots a_n = b_1 \ldots b_m$ are two atomic factorisations of c, then we say that they are <u>projective-isomorphic</u> if $m = n$ and for some element σ of S_n, for each $i = 1, \ldots, n$, a_i pr $b_{\sigma(i)}$. Just as in the case of a similarity-UFD, we say that an atomic domain R is a <u>projectivity-UFD</u> if all atomic factorisations of an element c of R are projective-isomorphic. Again, if R is commutative this reduces to the classical definition of a UFD.

It is clear that any PID is a similarity-UFD and a projectivity-UFD. But, as we saw in Section 3.2, if D is the division ring of real quaternions then D[x] is not a Noetherian UFD in the sense of Chapter 2. Let $R = A_1(Z)$, the first Weyl algebra over the integers. Then in R we have the atomic factorisations $c = (xy+1)x = x^2y$. Thus in R not even the number of factors in an atomic factorisation need be constant. But, by Theorem 2.1.4, $A_1(Z)$ is a Noetherian UFD in the sense of Chapter 2. These two examples make it clear that these generalisations of UFD are distinct.

In Beauregard [5], R.A.Beauregard proves an analogue of Nagata's Theorem for projectivity-UFDs. To do so he has to introduce a notion of prime element and it is perhaps ironic to observe that he uses a definition of prime element identical to that of Section 2.1.

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