A Study of SPDEs w.r.t. Compensated Poisson Random Measures and Related Topics

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Abstract

This thesis consists of two parts. In the first part, we define stochastic integrals w.r.t. the compensated Poisson random measures in a martingale type p, $1 \leq p \leq 2$ Banach space and establish a certain continuity, in substitution of the Itô isometry property, for the stochastic integrals. A version of Itô formula, as a generalization of the case studies in Ikeda and Watanabe [40], is derived. This Itô formula enables us to treat certain Lévy processes without Gaussion components. Moreover, using ideas in [63] a version of stochastic Fubini theorem for stochastic integrals w.r.t. compensated Poisson random measures in martingale type spaces is established. In addition, if we assume that E is a martingale type p Banach space with the q-th, $q \geq p$, power of the norm in C^2-class, then we prove a maximal inequality for a càdlàg modification $\tilde{u}$ of the stochastic convolution w.r.t. the compensated Poisson random measures of a contraction C_0-semigroups.

The second part of this thesis is concerned with the existence and uniqueness of global mild solutions for stochastic beam equations w.r.t. the compensated Poisson random measures. In view of Khas'minskii's test for nonexplosions, the Lyapunov function technique is used via the Yosida approximation approach. Moreover, the asymptotic stability of the zero solution is proved and the Markov property of the solution is verified.

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To my family,

Contents

Chapter 1

Introduction

The thesis is devoted to a systematic study of the construction of stochastic integrals with respect to compensated Poisson random measures in martingale type p, $1 \leq p \leq 2$, Banach spaces and to their's applications by proving the existence and uniqueness of stochastic beam equations with respect to compensated Poisson random measures. The notions of point processes, Poisson random measures and stochastic integrals w.r.t. compensated Poisson random measures can be traced back to P. Lévy [55] and K. Itô [41] as a fundamental of the constructing a process with independent, stationary increments and stochastic continuous paths, a Lévy process. K. Itô in [41] first formulated and proved the Lévy-Itô decomposition theorem, namely that every Lévy process can be expressed as a sum of two independent parts, a Brownian motion and a jump process which is an integral w.r.t. a compensated Poisson random measure, a random measure counting the jumps of the Lévy process. The Lévy-Itô decomposition theorem tells us that Brownian motions and Poisson random measures are the fundamental prerequisites for construction of any Banach valued Lévy processes. At this stage, it is worth pointing out here that the integral of a deterministic function x w.r.t. a compensated Poisson random measure in the Lévy-Itô decomposition coincides with the stochastic integral of the function x w.r.t. the compensated Poisson random measure associated to the Lévy process introduced in this thesis, see Remark 3.1.27 and Theorem 3.4.9. The Lévy-Itô decomposition formula has been intensively studied by many authors, see [73], [52], [7] [43] and also [44], [77], [78] and the references therein. In [28] and [1] the Lévy-Itô decomposition theorem was investigated for the case where the state space is a Banach space of type 2. Especially, for a detailed proof of the Lévy-Itô decomposition theorem in a Banach space, [1] may be consulted.

The extension of stochastic integration to the infinite dimensional spaces was exploited first by

Kunita in [51], where he investigated the stochastic integration w.r.t. the Hilbert-valued martingales and established the corresponding Itô formula. Later, Metivier and Pellaumail in [58] studied the stochastic integrals of operator-valued predictable processes w.r.t. a certain process, π-process, including the Hilbert-valued càdlàg square integrable martingales and Banach-valued processes with finite variation and by introducing the tensor quadratic variation, they derived a version of Itô formula for these stochastic integrals. See, in particular, [57] and [29]. The obstruction to extend the stochastic integrations to Banach spaces is that the Banach space-valued measurable functions may fail to be stochastically integrable. A classical counterexample for this defect was given by Yor in [82]. Neidhardt in his thesis [60] considered a certain class of Banach spaces, 2-uniformly-smooth Banach spaces, in which a stochastic integral with some certain continuity property, in substitution of the Itô isometry property, can be defined. Brzezniak in [12] investigated the stochastic integration theory in a martingale type 2 Banach space, which is in fact equal to the class of 2-smooth Banach spaces, see [65]. With the help of martingale type p Banach spaces setting, we may define the stochastic integral for a certain class of measurable Banach space-valued functions w.r.t. the compensated Poisson random measures.

In [40] the stochastic integrals of real-valued $\mathfrak{F}$-predictable functions w.r.t. the compensated Poisson random measure associated to a Poisson point process, in a terminology of simple p-integral in [71], is defined as a limit in $L^p(\Omega, \mathcal{F}, \mathbb{P})$ of Lebesgue integrals w.r.t. the compensated Poisson random measure over the approximating sets of a σ-finite Poisson point process. Here we call it the Ikeda-Watanabe stochastic integrable w.r.t. the compensated Poisson random measure. Similar definitions of stochastic integrals w.r.t. the compensated Poisson random measures using the Ikeda-Watanabe scheme can also be found in [66], [47] and [70]. Another common technical tool used to define the stochastic integral is approximation of general random functions by some random functions of simple structures, see [58], [56], [71], [4]. and [63]. Compared to the Ikeda-Watanabe stochastic integral, we will follow the approximation approach to define the stochastic integrals for a more general class of random functions, the $\mathfrak{F}$-progressively measurable functions or even the $\mathfrak{F}$-measurable and adapted functions, w.r.t. the compensated Poisson random measure in a martingale type p, $1 \leq p \leq 2$, Banach spaces.

Our study of stochastic integrals of martingale type p, $1 \leq p \leq 2$, Banach space-valued random functions w.r.t. the compensated Poisson random measure is related to the systematic study of stochastic integrals of real-valued random functions w.r.t. Poisson point process started by Ikeda and Watanabe in [40] in the sense that they are in fact equal when the integrand functions are $\mathfrak{F}$-predictable and take values in $\mathbb{R}$. Later in Theorem 3.3.4, we will show that all stochastic integrals of functions in the space $\mathcal{M}^p_{\mathcal{K}}(E)$, the space of all measurable and 'adapted' functions satisfying a certain integrability condition, are actually indistinguishable from the stochastic integrals of functions in the space $\mathcal{M}^p(\hat{\mathcal{P}}; E)$, the space of all $\mathfrak{F}$-predictable functions satisfying a certain integrability. We do not create new stochastic integrals for measurable and 'adapted' functions. In other words, the class $M^p_{\mathcal{K}}(E)$ of all equivalence classes of functions from $\mathcal{M}^p_{\mathcal{K}}(E)$ is isometric to the space $M^p(\hat{\mathcal{P}}; E)$ of all equivalence classes of functions from $\mathcal{M}^p(\hat{\mathcal{P}}; E)$. This fact demonstrates that there is no significant loss of generality in focusing on the class of $\mathfrak{F}$-predictable functions rather than on the class of the $\mathfrak{F}$-measurable and adapted functions. In [71], Rüdiger studied the stochastic integral w.r.t. the compensated Poisson random measure associated with a Lévy process, by means of the terminology of strong 2-integral, in a martingale type 2 Banach space. Moreover, it was shown in [71] that the strong 2- integrals coincide with the simple 2-integrals, or in our terminology the Ikeda-Watanabe stochastic integral, when the integrand function is left continuous. Analogically, we will show in this thesis that the Ikeda-Watanabe stochastic integrals are equal to our stochastic integrals w.r.t. the compensated Poisson random measure, when the integrand functions are $\mathfrak{F}$-predictable. Especially, the Bochner integrals w.r.t. the compensated Poisson random measure agree with the stochastic integrals w.r.t. the compensated Poisson random measure on sets of finite intensity measure when the integrand function is $\mathfrak{F}$-predictable. Furthermore, we give an example to illustrate that the two stochastic integrals may not be equal when the integrand function is only $\mathfrak{F}$-progressively measurable even on a set with bounded intensity measure.

The Itô formula was first formulated and proved by K. Itô [41] for real-valued stochastic integrals w.r.t. the Brownian motion. Subsequently, many other versions of Itô formulas for different types of stochastic integrals have been studied, see [51], [52], [40], [58], [57], [4], [29], etc. Especially, Rüdiger in [72] established a versions of Itô formula in a martingale type 2 Banach spaces for stochastic integrals w.r.t. a compensated Poisson random measure associated with a Lévy process. An Itô formula in a martingale type p, $1 < p \leq 2$, Banach spaces for stochastic integrals w.r.t. a compensated Poisson random measure associated with a Lévy process was derived by Hausenbals [36] by generalization a version of Itô formula introduced by Applebaum in [4]. In this thesis, we will prove that for the martingale type p, $1 < p \leq 2$, Banach space-valued processes of the form

$$X_t = X_0 + \int_0^t a(s)ds + \int_0^t \int_Z f(s,z)\tilde{N}(ds,dz) + \int_0^t \int_Z g(s,z)N(ds,dz), \tag{1.0.1}$$

the following Itô formula holds $\mathbb{P}$-a.s.

$$\begin{aligned}\phi(X_t) = \phi(X_0) &+ \int_0^t \phi'(X_s)(a(s))ds + \int_0^t \int_Z \Big[\phi(X_{s-} + g(s,z)) - \phi(X_{s-})\Big]N(ds,dz) \\ &+ \int_0^t \int_Z \Big[\phi(X_{s-} + f(s,z)) - \phi(X_{s-})\Big]\tilde{N}(ds,dz) \\ &+ \int_0^t \int_Z \Big[\phi(X_s + f(s,z)) - \phi(X_s) - \phi'(X_s)(f(s,z))\Big]\nu(dz)ds.\end{aligned} \tag{1.0.2}$$

Our contributions in this Itô formula include the following, firstly, in contrast to the Itô formula developed in [72] where the space was in a martingale type 2 space and the function ϕ was assumed to be twice Fréchet differentiable with uniformly bounded second Fréchet derivative, we consider the Itô formula in a martingale type p Banach space, $1 < p \leq 2$ and the function ϕ is assumed to be of class C^1 such that the first Fréchet derivative $\phi' : E \to L(E;G)$ is $(p-1)$-Hölder continuous. Secondly, in comparison to [36], we expand the stochastic process X to include the "big jumps" term $\int_0^t \int_Z g(s,z)N(ds,dz)$, which hence can be applied to all Lévy processes without Gaussion components, and the function f is assumed to be $\mathfrak{F}$-predictable which is weaker than the càglàg assumption in [36]. Thirdly, compared to both [72] and [36] the stochastic integral $\int_0^t \int_Z f(s,z)\tilde{N}(ds,dz)$ is defined w.r.t. the compensated Poisson random measure associated with a Poisson point process which in general includes the case when the stochastic integrals is defined w.r.t. the compensated Poisson random measure associated with a Lévy process, because the jump process of a Lévy process in $E \setminus \{0\}$ is actually a Poisson point process.

Some different versions of stochastic Fubini theorems have already been studied by many authors, see Bichteler [8], Curtain and Pritchard [25], Jacob [46], Da Prato and Zabczyk [26] and Protter [66], van Neerven and Veraar [79], ect. In this thesis a stochastic Fubini theorem for stochastic integrations of an extremely broad class of functions w.r.t. a compensated Poisson random measure will be established. Our approach was motivated by the proof of a version of stochastic Fubini theorem for Hilbert valued square integrable martingales as integrators in [63].

The maximal inequality for stochastic convolutions of a contraction C_0-semigroup and right continuous martingales in Hilbert spaces was studied by Ichikawa [39], see also [76] and [63], for more details. A submartingale type inequality for the stochastic convolutions of a contraction C_0-semigroup and square integrable martingales in Hilbert spaces were obtained by Kotelenez [50]. Kotelenez also proved the existence of a càdlàg version of the stochastic convolution processes for square integrable càdlàg martingales. In the paper by Brzeźniak and Peszat [17], the authors established a maximal inequality in a certain class of Banach spaces for stochastic convolution processes driven by a Wiener process. It is of interest to know whether the maximal inequality holds also for pure jump processes. Here we extend the results from [17] to the case where the stochastic convolution is driven by a compensated Poisson random measure. We work in the framework of stochastic integrals and convolutions driven by a compensated Poisson random measures recently introduced by the first two named authors in [16]. In this thesis, roughly speaking, we will show that the stochastic convolution process u has an E-valued càdlàg modification $\tilde{u}$ which satisfies the following maximal inequality, see Theorems 3.7.9 and 3.7.11,

$$\mathbb{E} \sup_{0\leq s\leq t} |\tilde{u}(s)|_E^{q'} \leq C\, \mathbb{E} \left(\int_0^t \int_Z |\xi(s,z)|_E^p N(ds,dz) \right)^{\frac{q'}{p}}, \; t \in [0,T]. \tag{1.0.3}$$

It is worth pointing out here that it is possible to derive inequality (1.0.3) by the same method as it has been applied to get inequality (4) in [38] whose authors used Szeköfalvi-Nagy's Theorem on unitary dilations. The latter result has recently been generalized to Banach space of finite cotype by Fröhlich and Weis [32]. However, this method works only for analytic semigroups of contraction type. The results from the current paper are valid for all C_0 semigroups of contraction type. To be more precise, assuming the setting before and the additional assumption that A generates an analytic semigroup, by nearly the same lines as in [38] it would follow

$$\mathbb{E} \sup_{0\leq s\leq t} |\tilde{u}(s)|_E^{q'} \leq C\, \mathbb{E} \left(\int_0^t \int_Z |\xi(s,z)|_E^p N(ds,dz) \right)^{\frac{q'}{p}}. \tag{1.0.4}$$

Another main focus of this thesis is to establish the existence and uniqueness of solutions to stochastic beam equation w.r.t. the compensated Poisson random measure. The Euler-Bernoulli beam equation

$$EI\frac{d^4u}{dx^4} = w$$

as a simplification of linear beam theory was first introduced in 1750 to describe the relationship between the deflection and applied load. The transversal deflection u of a hinged extensible beam of length l under an axial force H which satisfies the following form

$$\frac{\partial^2 u}{\partial t^2} + \frac{EI}{\rho}\frac{\partial^4 u}{\partial x^4} = \left(\frac{H}{\rho} + \frac{EA}{2\rho l}\int_0^l \left(\frac{\partial u}{\partial x}\right)^2 dx\right)\frac{\partial^2 u}{\partial x^2}. \tag{1.0.5}$$

was studied by S. Woinowsky-Krieger [81]. See also Eisley [31] and Burgreen [19] for more details. Chueshov [23] considered a problem of the following form

$$u_{tt} + \gamma u_t + A^2 u + m(\|A^{\frac{1}{2}}u\|^2)Au + Lu = p(t)$$

which arises in the nonlinear theory of oscillations of a plate in a supersonic gas flow moving along an x_1-axis described by

$$\frac{\partial^2 u}{\partial t^2} + \gamma\frac{\partial u}{\partial t} + \Delta^2 u + \left(\alpha - \int_D |\nabla u|^2 dx\right)\Delta u + \rho\frac{\partial u}{\partial x_1} = p(x,t), \quad x \in (x_1, x_2) \subset D,$$

where $u(x,t)$ measures the plate deflection at the point x and the moment t, $\gamma > 0$, $\rho \geq 0$ and function $p(x,t)$ describes the transverse load on the plate. In [61] Patcheu considered a model of (1.0.5) with a nonlinear friction force. The existence and uniqueness of global solutions of a nonlinear version of the Euler-Bernoulli with white noise arising from vibration of an aeroelastic panels

$$\frac{\partial^2 u}{\partial t^2} - \left(a + b\int_0^l \left(\frac{\partial u}{\partial x}\right)^2 dx\right)\frac{\partial^2 u}{\partial x^2} + \gamma\frac{\partial^4 u}{\partial x^4} + f\left(t, x, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}\right) + \sigma\left(t, x, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}\right)\dot{W}(t) = 0 \tag{1.0.6}$$

has been investigated by Chow and Menaldi in [21]. Z. Brzeźniak, B, Maslowski, J. Seidler (2004) considered and show the existence of global mild solutions of the following stochastic beam equations including a white noise type and a nonlinear random damping term in a Hilbert space H

$$u_{tt} + A^2 u + g(u, u_t) + m(\|B^{\frac{1}{2}}u\|^2)Bu = \sigma(u, u_t)\dot{W}, \tag{1.0.7}$$

where the operators A and B are self-adjoint and $\mathcal{D}(A) \subset \mathcal{D}(B)$.

It is of interest to know whether the theory can be extended to the problems with jump noise which is in some sense more realistic. In our paper, we consider a stochastic beam equation in some Hilbert space H with stochastic jump noise perturbations of the form

$$u_{tt} = -A^2 u - f(t, u, u_t) - m(\|B^{\frac{1}{2}}u\|^2)Bu + \int_Z g(t, u, u_t, z)\tilde{N}(t, dz), \tag{1.0.8}$$

where m is a nonnegative function in $C^1([0,\infty))$, A, B are self-adjoint operators and $\tilde{N}$ is a compensated Poisson random measure. We will show that under some suitable locally Lipschitz continuity and linear growth assumptions of the coefficients f and m, the stochastic beam equation (1.0.8) has a unique maximal local mild solution u which satisfies

$$\mathrm{u}(t \wedge \tau_n) = e^{t\mathcal{A}}\mathrm{u}_0 + \int_0^{t\wedge\tau_n} e^{(t\wedge\tau_n - s)\mathcal{A}} F(s, \mathrm{u}(s))ds + I_{\tau_n}(G(\mathrm{u}))(t \wedge \tau_n) \ \ \mathbb{P}\text{-a.s., for every } t \geq 0, \tag{1.0.9}$$

where $\{\tau_n\}_{n\in\mathbb{N}}$ is a sequence of stopping times and $I_{\tau_n}(G(\mathrm{u}))$ is a process defined by

$$I_{\tau_n}(G(\mathrm{u}))(t) = \int_0^t \int_Z 1_{[0,\tau_n]} e^{(t-s)\mathcal{A}} G(s, \mathrm{u}(s-), z)\tilde{N}(ds, dz).$$

We also show the nonexplosion of the local maximal solution. The basic method that we shall use in showing the nonexplosion is the Khas'minskii's test. For this aim, the essence is to be able to construct an appropriate Lyapunov function. One can first derive some estimates when u is in $\mathcal{D}(\mathcal{A})$, where $\mathcal{D}(\mathcal{A})$ is the domain of the generator $\mathcal{A}$. In fact, one can always approximating u by such functions in $\mathcal{D}(\mathcal{A})$ and pass the limit as in [76] to get the desired estimate of Lyapunov function. Moreover, the asymptotic stability and uniform boundedness of the solution has also been established in the same manner by a suitable choice of another Lyapunov function. We also show that under some natural conditions all the results in this paper we've achieved for (1.0.8) can be applied to a wide class of models including the following problem

$$\frac{\partial^2 u}{\partial t^2} - m\left(\int_D |\nabla u|^2 dx\right)\Delta u + \gamma\Delta^2 u + G\left(t, x, u, \frac{\partial u}{\partial t}, \nabla u\right) = \int_Z \Pi(t, x, u, \frac{\partial u}{\partial t}, \nabla u, z)\tilde{N}(t, du) \tag{1.0.10}$$

with either the clamped boundary conditions

$$u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial D, \tag{1.0.11}$$

or the hinged boundary conditions

$$u = \Delta u = 0 \text{ on } \partial D. \tag{1.0.12}$$

In the above $\frac{\partial}{\partial n}$ denotes the outer normal derivative.

The rest of this thesis is arranged as follows. The second chapter is devoted to studying systematically various types of measurabilities of processes and examining the relationships among these different types of measurabilities. The third chapter proceeds with the construction of the stochastic integral w.r.t. the compensated Poisson random measure. From the application point of view, both the Itô formula and the stochastic Fubini theorem for the stochastic integral w.r.t. the compensated Poisson random measure are established. Moreover, some maximal inequalities for stochastic convolutions w.r.t. a compensated Poisson random measure of a contraction C_0-semigroups are investigated. In the last chapter, we study a type of stochastic nonlinear beam equation w.r.t. the compensated Poisson random measure. By constructing a suitable Lyapunov function we can apply Khas'inskii's test to show the nonexplosion of the mild solutions. In addition, if we strengthen the linear growth hypothesis, the exponential stability of the solution can also be achieved.

Chapter 2

Preliminaries

2.1 Stochastic Processes

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A family of σ-fields $(\mathcal{F}_t)_{t\geq 0}$ is called a **filtration** if

$$\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}, \quad \text{for all } 0 \leq s \leq t < \infty.$$

We shall denote the filtration $(\mathcal{F}_t)_{t\geq 0}$ by an abbreviated symbol $\mathfrak{F}$. We say a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is filtered if it comes equipped with a filtration $\mathfrak{F}$. For the future convenience, it is necessary to add the σ-field $\mathcal{F}_\infty := \bigvee_{0\leq t<\infty} \mathcal{F}_t$ to the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. To a given filtration $\mathfrak{F}$, we always associate, for every $t \geq 0$, a σ-field $\mathcal{F}_{t-} := \bigvee_{s<t} \mathcal{F}_s$ which is the σ-field of events strictly prior to time t and a σ-field $\mathcal{F}_{t+} := \bigcap_{s>t} \mathcal{F}_s$ which is the σ-field of events after time t. For $t = 0$, we set $\mathcal{F}_{0-} = \mathcal{F}_0$ as usual. Note that the intersection of a family of σ-fields on the space Ω is also a σ-field. Hence the above definition of the σ-filed $\mathcal{F}_{t+}$ makes sense. However, a union of a family of σ-fields is not necessarily a σ-field. By the notation $\bigvee_{s<t} \mathcal{F}_s$ we mean the σ-field generated by $\cup_{s<t}\mathcal{F}_s$. The intersection $\bigcap_{s>t} \mathcal{F}_s$ can also be characterized by sequences. Let t_n be a sequence such that $t_n > t$ for all $n \in \mathbb{N}$ and $t_n \downarrow t$. Then we have $\mathcal{F}_{t+} = \bigcap_{n\in\mathbb{N}} \mathcal{F}_{t_n}$. A filtration $\mathfrak{F}$ is called to be **right-continuous** if $\mathcal{F}_t = \mathcal{F}_{t+}$ for each $t \geq 0$. A filtration $\mathfrak{F}$ is called to be **left-continuous** if $\mathcal{F}_t = \mathcal{F}_{t-}$ for each $t \geq 0$.

A probability space $(\Omega, \mathcal{F}, \mathfrak{F}, \mathbb{P})$ is said to be **complete** if the σ-field $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets in $\mathcal{F}$. A filtered probability space $(\Omega, \mathcal{F}, \mathfrak{F}, \mathbb{P})$ is said to satisfy the **usual hypotheses** if the right-continuity and the completeness conditions are fulfilled.

2.1.1 Stochastic Processes

Let $(\Omega, \mathcal{F}, \mathfrak{F}, \mathbb{P})$ be a filtered probability space satisfying the usual hypotheses. Let $(E, \|\cdot\|)$ be a separable Banach space with its σ-field $\mathcal{B}(E)$ of all Borel subsets. Let us fix $t \geq 0$. Let $\mathcal{L}^0(\Omega, \mathcal{F}_t, \mathbb{P}; E)$ be the set of all E-valued $\mathcal{F}_t$-measurable random variables. We say two random variables in $\mathcal{L}^0(\Omega, \mathcal{F}_t, \mathbb{P}; E)$ are **equivalent** if they are equal a.s. Let $L^0(\Omega, \mathcal{F}_t, \mathbb{P}; E)$ be the set of all equivalence classes of elements of $\mathcal{L}^0(\Omega, \mathfrak{F}, \mathbb{P}; E)$. Let us take $\xi, \eta \in \mathcal{L}^0(\Omega, \mathcal{F}_t, \mathbb{P}; E)$. Define

$$d(\xi, \eta) = \inf\{\varepsilon \geq 0 : \mathbb{P}(\|\xi - \eta\| > \varepsilon) \leq \varepsilon\}.$$

Then one can show, see Dudley [30] Theorem 9.2.2, that the function d is nonnegative, symmetric and satisfies the triangle inequality. Furthermore, $d(\xi, \eta) = 0$ if and only if $\xi = \eta$ a.s. This implies that d can be lifted to the space $L^0(\Omega, \mathcal{F}_t, \mathbb{P}; E)$ and this lifting, still denoted by the letter d, is a proper metric on the space $L^0(\Omega, \mathcal{F}_t, \mathbb{P}; E)$. The metric d is called the **Ky Fan metric**. It is known that it metrizes the convergence in probability. That is a sequence ξ_n converges in probability to ξ in $\mathcal{L}^0(\Omega, \mathcal{F}_t, \mathbb{P}; E)$ if and only if $d(\xi_n, \xi) \to 0$, as $n \to \infty$. Moreover, the space $L^0(\Omega, \mathcal{F}_t, \mathbb{P}; E)$ is complete with respect to the Ky Fan metric d, see Dudley [30] Theorem 9.2.3. It is worth pointing out that the almost sure convergence is not metrizable, in other words there is no topology on $\mathcal{L}^0(\Omega, \mathcal{F}_t, \mathbb{P}; E)$ such that the almost sure convergence is equivalent to convergent with respect to this topology.

We say that $X := (X_t)_{t \geq 0}$ is an **E-valued stochastic process** if for each $t \geq 0$, X_t is an E-valued random variable on $(\Omega, \mathcal{F})$. One can also regard X as an E-valued mapping defined on $\Omega \times \mathbb{R}_+$ through the formula $X(t, \omega) = X_t(\omega)$. We say that X is a **process defined up to modification** if X is a mapping from $\mathbb{R}_+$ to $L^0(\Omega, \mathcal{F}_t, \mathbb{P}; E)$.

For a fixed point $\omega \in \Omega$, the function $\mathbb{R}_+ \ni t \mapsto X_t(\omega) \in E$ is called a **path (trajectory)** of the process X associated with ω. We shall introduce some regularity properties of the paths of processes. An E-valued process X is said to be **continuous** (resp. right-continuous) if for every sample point $\omega \in \Omega$, the function $\mathbb{R}_+ \ni t \mapsto X_t(\omega) \in E$ is continuous (resp. right-continuous). Analogously, an E-valued process X is said to be **càdlàg** (right continuous with left limits) if for every $\omega \in \Omega$, the path $\mathbb{R}_+ \ni t \mapsto X_t(\omega) \in E$ is càdàg, namely for every $t \in \mathbb{R}_+$, $X_t(\omega) = \lim_{s \searrow t} X_s(\omega)$ and the left limit $X_{t-}(\omega) = \lim_{s \nearrow t} X_s(\omega)$ exists. Here the limits are with respect to the norm on E.

Consider two stochastic processes $X = (X_t)_{t\geq 0}$ and $Y = (Y_t)_{t\geq 0}$ defined on the same filtered probability space $(\Omega, \mathcal{F}, \mathfrak{F}, \mathbb{P})$ and taking values in the same space E. Since stochastic processes X and Y are functions on $\mathbb{R}_+ \times \Omega$, X and Y are equal if and only if $X_t(\omega) = Y_t(\omega)$, for every $t \geq 0$ and every $\omega \in \Omega$. One can see that the above requirements are quite strict, so we introduce the following two common definitions that weaken the above notion. Two E-valued processes X and Y are called **indistinguishable (or $\mathbb{P}$-equivalent)** if and only if

$$\mathbb{P}\{\omega \in \Omega : X_t(\omega) = Y_t(\omega);\ \forall\, t \in [0, \infty)\} = 1.$$

Since we will regard indistinguishable processes as equal, we say that an E-valued process X is continuous (resp. càdlàg) if it is indistinguishable with a continuous (resp. càdlàg) E-valued process. An E-valued process Y is said to be a **modification** of X if, for each $t \geq 0$,

$$\mathbb{P}\{\omega : X_t(\omega) = Y_t(\omega)\} = 1.$$

In the latter case, we say that the processes X and Y are **stochastically equivalent.**

It's easy to prove that if two processes X and Y are indistinguishable, then Y is a modification of X. Indeed, we have $1 \geq \mathbb{P}\{X_t = Y_t\} \geq \mathbb{P}\{X_t = Y_t, \forall\, t \in \mathbb{R}_+\} = 1$, for each $t \geq 0$. Conversely, we can easily find a stochastic process Y, see Example 2.1.1, that is a modification of X, but X and Y are not indistinguishable.

Example 2.1.1. Let τ be a positive random variable with a continuous distribution. Let $X_t \equiv 0$ and let $Y_t = 1$ if $t = \tau$, and $Y_t = 0$ if $t \neq \tau$. Since $\mathbb{P}\{X_t \neq Y_t\} = \mathbb{P}\{\tau = t\} = \lim_{h\to 0^+}(\mathbb{P}\{\tau \leq t\} - \mathbb{P}\{\tau \leq t-h\}) = 0$, then $\mathbb{P}\{X_t = Y_t\} = 1$, for each $t \geq 0$ which implies that Y is a modification of X. However, $\mathbb{P}\{X_t = Y_t, \forall t \geq 0\} = \mathbb{P}\{\tau \neq t, \forall t \geq o\} = \mathbb{P}\{\tau < 0\} = 0$. Each path of X_t is identically zero, but every path of Y_t has a jump at the point τ. This shows that these two stochastic processes have completely different sample paths. See [80].

Lemma 2.1.2. *Suppose that $X = (X_t)_{t\geq 0}$ and $Y = (Y_t)_{t\geq 0}$ are two E-valued stochastic processes both with right-continuous paths and Y is a modification of X. Then X and Y are indistinguishable.*

Proof. First we claim that

$$\bigcup_{t\geq 0}\{X_t \neq Y_t\} = \bigcup_{r\in\mathbb{Q}, r\geq 0}\{X_t \neq Y_t\}. \tag{2.1.1}$$

Indeed, it is clear that $LHS \supset RHS$. On the other hand, for any ω belonging to the left side of (2.1.1), i.e. $\omega \in \bigcup_{t\geq 0}\{X_t \neq Y_t\}$, there exists some $t_0 \in \mathbb{R}_+$ such that $X_{t_0}(\omega) \neq Y_{t_0}(\omega)$. Then we can find a sequence of rational numbers $\{r_n\}_{n\in\mathbb{N}}$ such that $r_n > t_0$, for any $n \in \mathbb{N}$. Since both X and Y have right-continuous sample paths, we infer that

$$\lim_{n\to\infty} X_{r_n}(\omega) = X_{t_0}(\omega), \tag{2.1.2}$$

$$\lim_{n\to\infty} Y_{r_n}(\omega) = Y_{t_0}(\omega). \tag{2.1.3}$$

However, since $X_{t_0}(\omega) \neq Y_{t_0}(\omega)$, it follows from (2.1.2) and (2.1.3) that there exists a natural number $n \in \mathbb{N}$ such that $X_{r_n}(\omega) \neq Y_{r_n}(\omega)$. This implies that $\omega \in \bigcup_{n\in\mathbb{N}}\{X_{r_n} \neq Y_{r_n}\} \subset \bigcup_{r\in\mathbb{Q},r\geq 0}\{X_t \neq Y_t\}$. Therefore, the equality (2.1.1) holds.

It follows that

$$0 \leq \mathbb{P}(\bigcup_{t\geq 0}\{X_t \neq Y_t\}) = \mathbb{P}(\bigcup_{r\in\mathbb{Q},r\geq 0}\{X_t \neq Y_t\}) \leq \sum_{r\in\mathbb{Q},r\geq 0}\{X_r \neq Y_r\} = 0,$$

which implies that

$$\mathbb{P}(\bigcup_{t\geq 0}\{X_t \neq Y_t\}) = 0.$$

Therefore, we have

$$\mathbb{P}\{X_t = Y_t, \forall t \geq 0\} = \mathbb{P}(\bigcap_{t\geq 0}\{X_t = Y_t\}) = 1.$$

□

In view of the above proof, one can derive the same conclusion for an E-valued process with left-continuous paths.

2.1.2 Measurability, Progressive Measurability and Predictability

Let $(\Omega, \mathcal{F}, \mathfrak{F}, \mathbb{P})$ be a filtered probability space satisfying the usual hypotheses.

Definition 2.1.3. An E-valued process X is said to be **adapted** with respect to $\mathfrak{F}$ if and only if for every $t \geq 0$, the random variable $X_t(\cdot) : \Omega \ni \omega \mapsto X_t(\omega) \in E$ is $\mathcal{F}_t/\mathcal{B}(E)$-measurable, for each $t \geq 0$.
An E-valued process $X = (X_t)_{t\geq 0}$ is said to be **measurable** if and only if the mapping

$$\mathbb{R}_+ \times \Omega \ni (t, \omega) \mapsto X_t(\omega) \in E$$

is $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}/\mathcal{B}(E)$-measurable.

Remark 2.1.4.

1. Since E is a separable Banach space, we know, see [26], that the Borel σ-fields $\mathcal{B}(E)$ is generated by all subsets of E of the form

$$\{x \in E : \phi(x) \leq \alpha\}, \ \phi \in E^*, \ \alpha \in \mathbb{R}.$$

2. Since the filtered probability space $(\Omega, \mathcal{F}, \mathfrak{F}, \mathbb{P})$ is complete, if two E-valued processes X and Y are modifications with each other, then X and Y cannot differ in adaptedness. In such a case, we say that a process X defined up to a modification is adapted if and only if for every $t \geq 0$, the random variable $X_t(\cdot)$ belongs to $L^0(\Omega, \mathcal{F}_t, \mathbb{P}; E)$. In particular, two indistinguishable processes have the same adaptedness and measurability properties.

Lemma 2.1.5. *If an E-valued process X is adapted to a complete filtration $\mathfrak{F}$ and an E-valued process Y is a modification of X, then Y is adapted to $\mathfrak{F}$.*

Proof. For any $B \in \mathcal{B}(E)$, observe that

$$\begin{aligned}\{Y_t \in B\} &= \{Y_t \in B\} \cap (\{X_t = Y_t\} \cup \{X_t \neq Y_t\}) \\ &= (\{X_t \in B\} \cap \{X_t = Y_t\}) \cup (\{Y_t \in B\} \cap \{X_t \neq Y_t\}).\end{aligned}$$

Since Y is a modification of X, we infer that $\{X_t \neq Y_t\}$ is a $\mathbb{P}$-null set. So the set $\{X_t \neq Y_t\}$ belongs to $\mathcal{F}_0$ which is a subset of $\mathcal{F}_t$. Hence the complement set $\{X_t = Y_t\}$ of $\{X_t \neq Y_t\}$ belongs to $\mathcal{F}_t$ as well. By the adaptedness of X, we have $\{X_t \in B\} \in \mathcal{F}_t$. Hence, $\{Y_t \in B\} \in \mathcal{F}_t$. □

Definition 2.1.6. An E-valued process X is said to be **progressively measurable** with respect to the filtration $\mathfrak{F}$ if and only if for every t, the restriction of X to $[0,t] \times \Omega$ is $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$-measurable, more precisely, if and only if for every $t \geq 0$, the mapping

$$[0,t] \times \Omega \ni (s,\omega) \mapsto X_s(\omega) \in E$$

is $\mathcal{B}([0,t]) \otimes \mathcal{F}_t/\mathcal{B}(E)$-measurable.

Lemma 2.1.7. *Let $\mathcal{BF}$ be a family of sets which is defined by*

$$\mathcal{BF} := \{A \subset \mathbb{R}_+ \times \Omega : \forall\, t \geq 0, A \cap ([0,t] \times \Omega) \in \mathcal{B}([0,t]) \otimes \mathcal{F}_t\}.$$

Then we have

(1) $\mathcal{BF}$ is a σ-field;

(2) If $X = (X_t)_{t\geq 0}$ is an E-valued process, then X is progressively measurable if and only if $X : \mathbb{R}_+ \times \Omega \to E$ is $\mathcal{BF}$-measurable.

Proof. To show that $\mathcal{BF}$ is a σ-field, we need to verify the three conditions of a σ-field.

(i) Since for every $t \geq 0$, the set $(\mathbb{R}_+ \times \Omega) \cap ([0,t] \times \Omega) = [0,t] \times \Omega \in \mathcal{B}([0,t]) \otimes \mathcal{F}_t$, then $\mathbb{R}_+ \times \Omega \in \mathcal{BF}$.

(ii) Take $A \in \mathcal{BF}$. Then we see that

$$\begin{aligned}A^c \cap ([0,t] \times \Omega) &= ((\mathbb{R}_+ \times \Omega) \setminus A) \cap ([0,t] \times \Omega) \\ &= ([0,t] \times \Omega) \setminus A = ([0,t] \times \Omega) \setminus (A \cap ([0,t] \times \Omega)).\end{aligned}$$

Since $A \in \mathcal{BF}$, we have that $A \cap ([0,t] \times \Omega) \in \mathcal{B}([0,t]) \otimes \mathcal{F}_t$ and $[0,t] \times \Omega \in \mathcal{B}([0,t]) \otimes \mathcal{F}_t$. Hence we infer that $A^c \cap ([0,t] \times \Omega) \in \mathcal{B}([0,t]) \times \mathcal{F}_t$.

(iii) If $A_1, A_2, \cdots \in \mathcal{BF}$, then we find out that

$$\Big(\bigcup_{i\in\mathbb{N}} A_i\Big) \bigcap ([0,t] \times \Omega) = \bigcup_{i\in\mathbb{N}} \{A_i \bigcap ([0,t] \times \Omega)\} \in \mathcal{B}([0,t]) \otimes \mathcal{F}_t.$$

Since A_i is $\mathcal{BF}$-measurable providing $A_i \cap ([0,t] \times \Omega) \in \mathcal{B}([0,t]) \otimes \mathcal{F}_t$, we have

$$\Big(\bigcup_{i\in\mathbb{N}} A_i\Big) \bigcap ([0,t] \times \Omega) \in \mathcal{B}([0,t]) \otimes \mathcal{F}_t$$

implying $\bigcup_{i\in\mathbb{N}} A_i \in \mathcal{BF}$.

Therefore, $\mathcal{BF}$ is a σ-field.

Note that the process X is progressively measurable if and only if for each $t \geq 0$ and every $B \in \mathcal{B}(E)$, the set

$$\{(s,\omega) : 0 \leq s \leq t, \omega \in \Omega, X_s(\omega) \in B\} \in \mathcal{B}([0,t]) \otimes \mathcal{F}_t. \tag{2.1.4}$$

Since we know that $\{(s,\omega) : 0 \leq s \leq t, \omega \in \Omega, X_s(\omega) \in B\} = \{(s,\omega) : s \geq 0, \omega \in \Omega, X_s(\omega) \in B\} \cap ([0,t] \times \Omega)$, the condition (2.1.4) is equivalent to

$$\{(s,\omega) : X_s(\omega) \in B\} \cap ([0,t] \times \Omega) \in \mathcal{B}([0,t]) \otimes \mathcal{F}_t.$$

This is nothing else but $\{(t,\omega) : X_t(\omega) \in B\} \in \mathcal{BF}$. □

We usually call this σ-field $\mathcal{BF}$ the progressive σ-field on $\mathbb{R}_+ \times \Omega$.

Theorem 2.1.8. *Let X be an E-valued process adapted to the filtration $\mathfrak{F}$. If the process X is right-continuous (or left-continuous), then X is progressively measurable.*

Proof. Suppose that the process X is right-continuous. We define a sequence of functions from $[0,s] \times \Omega$ to E by, for every $n \in \mathbb{N}$,

$$X^n(t,\omega) = 1_{\{0\}}(t)X_0(\omega) + \sum_{k=0}^{2^n-1} 1_{(\frac{ks}{2^n}, \frac{(k+1)s}{2^n}]}(t) X_{\frac{(k+1)s}{2^n}}(\omega), \quad (t,\omega) \in [0,s] \times \Omega.$$

By the adaptedness of the process X, we find out that X^n is $\mathcal{B}([0,s]) \otimes \mathcal{F}_s$-measurable.
Now we proceed to show that by the right-continuity of the process X, the sequence $(X^n)_{n\in\mathbb{N}}(t,\omega)$ converges to $X(t,\omega)$ as $n \to \infty$, for all $(t,\omega) \in [0,s] \times \Omega$. Let us fix $\omega \in \Omega$.

- If $t = 0$, then $X^n(0,\omega) = X_0(\omega)$, for each $n \in \mathbb{N}$.
- If $0 < t \leq s$, we can find a natural number $n_1 \in \mathbb{N}$ such that $\frac{s}{2^{n_1}} < t$. By the right-continuity of the process X, there exists a positive number δ such that for every $t' \in [t, t+\delta)$, we have

  $$\|X_{t'} - X_t\| < \epsilon.$$

  Choose next a natural number $n_0 \geq n_1$. Hence $\frac{s}{2^{n_0}} \leq \delta$ and for each $n \geq n_0$, we have $\frac{s}{2^n} \leq \frac{s}{2^{n_0}} \leq \delta$. Since the intervals $\{(\frac{ks}{2^n}, \frac{(k+1)s}{2^n}]\}_{k=1}^{2^n-1}$ form a pairwise disjoint sequence of subsets of $(0,s]$, the sequence of intervals covers $(0,s]$ and for every $n \geq n_0$, one can find k such that $t \in (\frac{ks}{2^n}, \frac{(k+1)s}{2^n}]$ implying $t < \frac{(k+1)\delta}{2^n} = \frac{ks}{2^n} + \frac{s}{2^n} < t+\delta$. Hence, by the right-continuity of the process X, we have

  $$\|X_{\frac{(k+1)s}{2^n}} - X_t\| < \epsilon.$$

  Since $X_t^n = X_{\frac{(k+1)s}{2^n}}$, for $t \in (\frac{ks}{2^n}, \frac{(k+1)s}{2^n})$, we infer that

  $$\|X_t^n - X_t\| < \epsilon.$$

  Therefore, we can conclude that $X^n(t,\omega) \to X(t,\omega)$ as $n \to \infty$ for all $(t,\omega) \in [0,s] \times \Omega$. It follows that X is $\mathcal{B}([0,s]) \otimes \mathcal{F}_s$-measurable when it is restricted to $[0,s] \times \Omega$. Thus, the process X is progressively measurable.

If X is left-continuous, for fixed $0 \leq s < \infty$, one can define a sequence of functions in the following way

$$X^n(t,\omega) = \sum_{k=0}^{2^n-1} 1_{[\frac{ks}{2^n}, \frac{(k+1)s}{2^n})}(t) X_{\frac{ks}{2^n}}(\omega).$$

In this case, it can be shown, by using a similar argument as before, that the left-continuous process X is progressively measurable.

□

Corollary 2.1.9. *If X is an E-valued right-continuous stochastic process, then X is measurable. The same conclusion holds when X is a left-continuous process.*

Proof. We only need to show that every right-continuous process can be approximated by a sequence of measurable processes of the following form

$$X^n(t,\omega) = 1_{\{0\}}(t)X_0(\omega) + \sum_{k\in\mathbb{N}} 1_{(\frac{ks}{2^n}, \frac{(k+1)s}{2^n}]}(t) X_{\frac{(k+1)s}{2^n}}(\omega), \ \ (t,\omega) \in \mathbb{R}_+ \times \Omega.$$

Similarly, one can show that every left-continuous process can be approximated by a sequence of measurable processes of the form

$$X^n(t,\omega) = \sum_{k\in\mathbb{N}} 1_{[\frac{ks}{2^n}, \frac{(k+1)s}{2^n})}(t) X_{\frac{ks}{2^n}}(\omega).$$

□

Lemma 2.1.10. *Every E-valued progressively measurable process is measurable and adapted.*

Proof. Every progressively measurable process is clearly measurable. Indeed, for any $A \in \mathcal{B}(E)$, we know that

$$\{(t,\omega) \in \mathbb{R}_+ \times \Omega : X_t(\omega) \in A\} = \bigcup_{r\in\mathbb{Q}, r\geq 0} \{(s,\omega) \subset [0,r] \times \Omega : X_s(\omega) \in A\}.$$

Furthermore, by progressive measurability, we have for each $r \in \mathbb{Q}$ and $r \geq 0$

$$\{(s,\omega) \in [0,r] \times \Omega : X_s(\omega) \in A\} \in \mathcal{B}([0,r]) \otimes \mathcal{F}_r \subset \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}.$$

Thus, we infer

$$\{(t,\omega) \in \mathbb{R}_+ \times \Omega : X_t(\omega) \in A\} \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F},$$

which implies X is measurable.
Since by the definition of progressive measurability, for every $t \geq 0$, $X : [0,t] \times \Omega \to E$ is $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$-measurable, it follows from the Fubini Theorem , see also Lemma 3.6.1, that for every $0 \leq s \leq t$, the function $\omega \mapsto X_s(\omega)$ is $\mathcal{F}_t$-measurable. Hence we infer that for every $t \geq 0$, X_t is $\mathcal{F}_t$-measurable which shows that X is adapted. The proof is thus complete. □

Typically, all the theorems above hold for $\mathbb{R}$-valued processes. Let $\mathcal{V}$ denote the σ-filed generated by all measurable and adapted E-valued process on $\mathbb{R}_+ \times \Omega$. One may ask whether every measurable and adapted process is progressively measurable. The answer is in a negative way. In fact, we can always find a measurable and adapted process which is not progressively measurable, see Example 2.1.22 in this section. But, the following theorem due to Meyer [59] says that the question raised above would be possible if we take a modification of this process.

Theorem 2.1.11. *If an $\mathbb{R}$-valued process X is measurable and adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$, then there exists a modification of X which is progressively measurable.*

Cohn in [24] showed that the above theorem actually holds for all measurable and adapted processes taking values in a compact metric space. If E is a separable Banach space, we have the following theorem which is an immediate consequence of Proposition 2.1 in [79].

Theorem 2.1.12. *Every measurable and adapted E-valued process has a progressively measurable modification.*

Definition 2.1.13. The **predictable σ-field** $\mathcal{P}$ is the σ-field generated by all adapted and left-continuous $\mathbb{R}$-valued processes. An E-valued process is called **predictable** if and only if it is measurable with respect to $\mathcal{P}$.

Definition 2.1.14. The **optional σ-field** $\mathcal{O}$ is the σ-field generated by all adapted and right-continuous $\mathbb{R}$-valued processes. An E-valued process is called **optional** if and only if it is measurable with respect to $\mathcal{O}$.

Now we are going to present a number of characterizations of the predictable σ-fields.

Theorem 2.1.15. *The predictable σ-field $\mathcal{P}$ is equal to each of the following σ-fields.*

(i) the σ-field $\mathcal{P}_1$ generated by all adapted and continuous $\mathbb{R}$-valued processes;

(ii) the σ-field $\mathcal{P}_2$ generated by all adapted and càglàd (left continuous with right limits) $\mathbb{R}$-valued processes;

(iii) the σ-field generated by the following families of sets.

$$\mathcal{R} := \{(s,t] \times F:\ 0 \leq s \leq t < \infty, F \in \mathcal{F}_s\} \cup \{\{0\} \times F, F \in \mathcal{F}_0\},$$
$$\mathcal{R}_1 := \{(s,t] \times F:\ 0 \leq s \leq t < \infty, F \in \mathcal{F}_{s-}\} \cup \{\{0\} \times F, F \in \mathcal{F}_0\},$$
$$\mathcal{R}_2 := \{[s,t) \times F:\ 0 \leq s \leq t < \infty, F \in \mathcal{F}_{s-}\}.$$

Proof. Since both continuous and càglàd processes are left-continuous, it is clear that $\mathcal{P}_1 \subset \mathcal{P}$ and $\mathcal{P}_2 \subset \mathcal{P}$.

Now we will show that $\mathcal{P} \subset \sigma(\mathcal{R})$. In fact, we only need to show that every adapted left-continuous process is $\sigma(\mathcal{R})$-measurable. For this, we define a sequence $\{X^n\}_{n=1}^{\infty}$ of simple functions by, for every $n \in \mathbb{N}$,

$$X^n(t,\omega) = 1_{\{0\}}(0)X(0,\omega) + \sum_{k=0}^{\infty} 1_{(\frac{k}{2^n},\frac{k+1}{2^n}]}(t)X(\frac{k}{2^n},\omega),\ (t,\omega) \in \mathbb{R}_+ \times \Omega.$$

Let B be a Borel set in $\mathcal{B}(\mathbb{R})$. We observe that

$$\{(t,\omega) : X^n(t,\omega) \in B\}$$
$$= \Big(\{0\} \times \{\omega : X(0,\omega) \in B\}\Big) \bigcup \Big(\bigcup_{k=0}^{\infty} \Big((\frac{k}{2^n}, \frac{k+1}{2^n}] \times \{\omega : X(\frac{k}{2^n},\omega) \in B\} \Big) \Big).$$

Since X is adapted, we infer that, for every $k \in \mathbb{N}$, $\{X(\frac{k}{2^n},\omega) \in B\} \in \mathcal{F}_{\frac{k}{2^n}}$. Thus the above set is a countable union of sets from the family $\mathcal{R}$, which implies that X^n is $\sigma(\mathcal{R})$-measurable.
By using a similar argument as in the proof of Theorem 2.1.8, one can show that the sequence $X^n(t,\omega)$ converges to $X(t,\omega)$ as $n \to \infty$, for all $(t,\omega) \in \mathbb{R}_+ \times \Omega$. Therefore, the process X is $\sigma(\mathcal{R})$-measurable. This proves $\mathcal{P} \subset \sigma(\mathcal{R})$.

Now we are in a position to show that $\sigma(\mathcal{R}) \subset \mathcal{P}_2$. Consider a set, from the family $\mathcal{R}$, of the form $(s,t] \times F$, where $F \in \mathcal{F}_s$. Clearly, the indicator function $1_{(s,t]\times F}$ of this set is an adapted càglàd process. Thus $1_{(s,t]\times F}$ is $\mathcal{P}_2$-measurable which proves the claim $\sigma(\mathcal{R}) \subset \mathcal{P}_2$.

Next we will show that $\sigma(\mathcal{R}) \subset \mathcal{P}_1$. To prove this inclusion, it is enough to show that the indicator functions $1_{\{0\}\times F}$, $F \in \mathcal{F}_0$ and $1_{(s,t]\times F}$, $F \in \mathcal{F}_s$ are pointwise limits of sequences of continuous adapted processes.

First we deal with the function $1_{\{0\}\times F}(t,\omega)$, $F \in \mathcal{F}_0$. To do this, we find a sequence of continuous functions $\{\varphi_n\}_{n=1}^{\infty}$, $\varphi : [0,\infty) \to \mathbb{R}$, of the form

$$\varphi_n(u) = \begin{cases} 1 - nu, & if\ 0 \le u < \frac{1}{n} \\ 0, & if\ u \ge \frac{1}{n}. \end{cases}$$

It is easy to see that $\lim_{n\to\infty} \varphi_n(t) = 1_{\{0\}}$ pointwise on $[0,\infty)$ and the processes

$$\mathbb{R}_+ \times \Omega \ni (t,\omega) \mapsto 1_F(\omega) \cdot \varphi_n(t) \in \mathbb{R},\ n \in \mathbb{N}$$

are adapted and continuous. Since $F \in \mathcal{F}_0$ and $1_F(\omega) \cdot \varphi_n(t) \to 1_{\{0\}\times F}(t,\omega)$, as $n \to \infty$, for all $(t,\omega) \in \mathbb{R}^+ \times \Omega$, we infer that $1_{\{0\}\times F}$ is $\mathcal{P}_1$-measurable, i.e. $\{0\} \times F \in \mathcal{P}_1$.

Next we consider the function $1_{(s,t]\times F}$, $F \in \mathcal{F}_s$. For this, we take another sequence $\{\varphi_n\}_{n=1}^{\infty}$, $\varphi : [0,\infty) \to \mathbb{R}$, of continuous functions

$$\varphi_n(u) = \begin{cases} 0, & if\ 0 \le u \le s \\ n(u-s), & if\ s < u \le s + \frac{1}{n} \\ 1, & if\ s - \frac{1}{n} < u < t \\ 1 - n(u-t), & if\ t < u \le t + \frac{1}{n} \\ 0, & if\ u > t + \frac{1}{n}. \end{cases}$$

Note that for every $n \in \mathbb{N}$, the function $\varphi_n(u)$ is continuous on $\mathbb{R}_+$ and $\lim_{n\to\infty} \varphi_n = 1_{(s,t]}$ pointwise on $[0,\infty)$. Hence $1_F(\omega) \cdot \varphi_n(u) \to 1_{(s,t]\times F}(u,\omega)$, as $n \to \infty$, for all $(t,\omega) \in \mathbb{R}^+ \times \Omega$. Since $F \in \mathcal{F}_s$, the processes $1_F(\omega) \cdot \varphi_n(u)$ are adapted. Thus the adaptedness together with the continuity shows that

$$\mathbb{R}_+ \times \Omega \ni (u,\omega) \mapsto 1_F(\omega) \cdot \varphi_n(u) \in \mathbb{R}$$

is $\mathcal{P}_1$-measurable. Therefore the limit $1_{(s,t]\times F}$ of that sequence is also $\mathcal{P}_1$-measurable. This implies that $(s,t] \times F \in \mathcal{P}_1$, which proves that $\sigma(\mathcal{R}) \subset \mathcal{P}_1$.

So far, we have shown that $\mathcal{P}_1 = \sigma(\mathcal{R}) = \mathcal{P}_2 = \mathcal{P}$. Now it remains to show that $\sigma(\mathcal{R}) = \sigma(\mathcal{R}_1) = \sigma(\mathcal{R}_2)$.

Since for each $t \ge 0$, $\mathcal{F}_{t-} \subset \mathcal{F}_t$, we have $\sigma(\mathcal{R}_1) \subset \sigma(\mathcal{R})$. For the inverse inclusion, let us take $(s,t] \times F \in \mathcal{R}$, where F is $\mathcal{F}_s$-measurable. Note that F is also $\mathcal{F}_{(s+\frac{1}{n})-}$-measurable. We consider a sequence of sets of the form $(s+\frac{1}{n}, t+\frac{1}{n}] \times F$. Clearly, $(s+\frac{1}{n}, t+\frac{1}{n}] \times F \in \sigma(\mathcal{R}_1)$. Also, observe that pointwise on $\mathbb{R}_+ \times \Omega$,

$$1_{(s,t]\times F} = \lim_{n\to\infty} 1_{(s+\frac{1}{n}, t+\frac{1}{n}]\times F}.$$

Hence we infer $(s,t] \times F \in \sigma(\mathcal{R}_1)$. This shows that $\sigma(\mathcal{R}) \subset \sigma(\mathcal{R}_1)$.

To prove $\sigma(\mathcal{R}_1) = \sigma(\mathcal{R}_2)$, let us take $(s,t] \times F \in \sigma(\mathcal{R}_1)$, where $F \in \mathcal{F}_{s-}$. We consider a sequence of sets $[s+\frac{1}{n}, t+\frac{1}{n}) \times F$. It is clear that $F \in \mathcal{F}_{(s+\frac{1}{n})-}$. Then $[s+\frac{1}{n}, t+\frac{1}{n}) \times F \in \sigma(\mathcal{R}_2)$, for all $n \in \mathbb{N}$. Since we have, pointwise on $\mathbb{R}_+ \times \Omega$,

$$1_{(s,t]\times F} = \lim_{n\to\infty} 1_{[s+\frac{1}{n}, t+\frac{1}{n})\times F},$$

we infer $(s,t] \times F \in \sigma(\mathcal{R}_2)$. This proves $\sigma(\mathcal{R}_1) \subset \sigma(\mathcal{R}_2)$. Similarly, the inverse inclusion follows immediately if we consider another sequence of sets of the form $(s-\frac{1}{n}, t-\frac{1}{n}] \times F$ and observe that $1_{[s,t)\times F} = \lim_{n\to\infty} 1_{(s-\frac{1}{n}, t-\frac{1}{n}]\times F}$.

□

Remark 2.1.16. 1. The sets in $\mathcal{R}$ are usually called the **predictable rectangles**.

2. Remark 2.1.4 tells us that an E-valued process X is predictable if and only if for every $\phi \in E^*$, the $\mathbb{R}$-valued process $\phi(X)$ is predictable. Indeed, clearly, if X is predictable, then for every $\phi \in \mathbb{E}^*$, which is a continuous mapping from E to $\mathbb{R}$, the composite mapping

   $$(t,\omega) \mapsto X(t,\omega) \mapsto \phi(X(t,\omega))$$

   is predictable. On the other hand, let us set $\mathcal{H} := \{B \in \mathcal{B}(E) : X^{-1}(B) \in \mathcal{P}\}$. It is easy to show that the family $\mathcal{H} \subset \mathcal{B}(E)$ is a σ-field. Moreover, since $\mathcal{B}(E)$ is generated by all subsets of E of the form

   $$A := \{x \in E : \phi(x) \le \alpha\}, \ \phi \in E^*, \ \alpha \in \mathbb{R}, \tag{2.1.5}$$

   we can see that for every set $A \in \mathcal{B}(E)$ given by (2.1.5)

   $$X^{-1}(A) = \{(t,\omega) : \phi(X(t,\omega) \le \alpha)\} \in \mathcal{P}.$$

   Hence we conclude that $\mathcal{H} = \mathcal{B}(E)$ which implies that X is $\mathcal{P}/\mathcal{B}(E)$-measurable.

3. If X is left-continuous and adapted, then for every $\phi \in E^*$, $\phi(X)$ is an $\mathbb{R}$-valued left-continuous and adapted process. It follows from the definition of predictability that the $\mathbb{R}$-valued process $\phi(X)$ is predictable. Therefore, every left-continuous and adapted E-valued process is predictable. However, predictable process need not to be left-continuous. For instance, suppose that X is a deterministic process given by $X(t,\omega) = f(t)$, $(t,\omega) \in \mathbb{R}_+ \times \Omega$. Clearly, it is predictable, but it may not be left-continuous.

Let us recall the definitions of semiring and ring.

Definition 2.1.17. A **semi-ring** $\mathcal{S}$ on Ω is a collection of subsets of Ω such that

(i) $\emptyset \in \mathcal{S}$;

(ii) if A, $B \in \mathcal{S}$, then $A \cap B \in \mathcal{S}$;

(iii) if $A,\ B \in \mathcal{S}$, then there exists $n \in \mathbb{N}$, and $A_i \in \mathcal{S}$, $i = 1, \cdots, n$, such that

$$A - B = \bigcup_{i=1}^{n} A_i.$$

A **ring** $\mathcal{A}$ is a collection of subsets of Ω satisfying

(i) $\emptyset \in \mathcal{A}$;

(ii) if $A,\ B \in \mathcal{A}$, then $A \cup B \in \mathcal{A}$;

(iii) if $A,\ B \in \mathcal{A}$, then $A - B \in \mathcal{A}$.

From the above definitions, one can easily conclude that a ring $\mathcal{A}$ is stable under operations of finite unions and finite differences. Note also that a ring is stable under operations of intersections of sets, because $A \cap B = A - (A - B)$.

Proposition 2.1.18. *Let $\mathcal{A}$ be the smallest ring generated by all predictable rectangles in $\mathcal{R}$. Then $\mathcal{R}$ is a semiring and $\mathcal{A}$ consists of all finite unions of disjoint rectangles in $\mathcal{R}$, i.e.*

$$\mathcal{A} = \{A = \cup_{i=1}^{n} A_i : \ \{A_i\}_{i=1}^{n} \subset \mathcal{R} \text{ are disjoint}, n \in \mathbb{N}\}.$$

Proof. Denote

$$\hat{\mathcal{A}} := \{A = \cup_{i=1}^{n} A_i : \ \{A_i\}_{i=1}^{n} \subset \mathcal{R} \text{ are disjoint}, \ n \in \mathbb{N}\}.$$

If $A \in \hat{\mathcal{A}}$, then $A = \uplus_{i=1}^{n} A_i$, $A_i \in \mathcal{R}$. Here we use the notation $\uplus$ to denote the union of pairwise disjoint sets. Since the ring $\mathcal{A}$ is stable under unions of sets, then $A \in \mathcal{A}$. Thus $\hat{\mathcal{A}} \subset \mathcal{A}$.

On the other hand, we will show that $\hat{\mathcal{A}}$ is a ring. To do this, we need to check the three conditions of a ring. Clearly, $\emptyset \in \mathcal{R}$, so $\emptyset \in \hat{\mathcal{R}}$. Before proving the condition (ii), we will verify the condition (iii) first.

(iii): Take $A,\ B \in \hat{\mathcal{A}}$. Then there exists two finite disjoint unions of sets $\{A_i\}_{i=1}^{n} \in \mathcal{R}$, $\{B_j\}_{j=1}^{m} \in \mathcal{R}$ such that $A = \uplus_{i=1}^{n} A_i$ and $B = \uplus_{j=1}^{m} B_j$. We see that $A \cap B = (\uplus_{i=1}^{n} A_i) \cap (\uplus_{j=1}^{m} B_j) = \uplus_{i=1}^{n} \uplus_{j=1}^{m} (A_i \cap B_j)$, since $A_i \cap B_j$ are pairwise disjoint for all $i = 1, \cdots, n$, $j = 1, \cdots, m$. Clearly, if $A_i \cap B_j \in \mathcal{R}$, then $A \cap B \in \mathcal{R}$. Thus $A \cap B \in \hat{\mathcal{A}}$.

Next we will show that if $M_1,\ M_2 \in \mathcal{R}$, then $M_1 - M_2 = \uplus_{k=1}^{n} D_k$, for some $B_k \in \mathcal{R}$. Let $M_1 = (s_1, t_1] \times F_1$ and $M_2 = (s_2, t_2] \times F_2$, where $F_1 \in \mathcal{F}_{s_1}$, $F_2 \in \mathcal{F}_{s_2}$. If $t_1 \leq s_2$, then $M_1 - M_2 = M_1$. If $s_1 < s_2 < t_1$, then we have

$$\begin{aligned} M_1 - M_2 &= (s_1, t_1] \times F_1 - (s_2, t_2] \times F_2 \\ &= ((s_1, s_2] \times F_1) \uplus ((s_2, t_1] \times F_2) \uplus ((s_2, t_1] \times (F_1 - F_2)) \\ &\quad - ((s_2, t_1] \times F_2) \uplus ((t_1, t_2] \times F_2) \\ &= ((s_1, s_2] \times F_1) \uplus ((s_2, t_1] \times (F_1 - F_2)), \end{aligned}$$

where $F_1 - F_2 \in \mathcal{F}_{s_2}$. Thus $M_1 - M_2 = D_1 \uplus D_2$, for $D_1,\ D_2 \in \mathcal{R}$. For the other cases, we can consider in a similar way to conclude that $M_1 - M_2 = \uplus_{k=1}^{n} D_k$, for some $D_k \in \mathcal{R}$. From this one can also deduce that $\mathcal{R}$ is a semiring.

Let $A,\ B \in \hat{\mathcal{A}}$, $A = \uplus_{i=1}^{n} A_i$ and $B = \uplus_{j=1}^{m} B_j$. Then

$$A - B = \cup_{i=1}^{n} A_i - \cup_{j=1}^{m} B_j = \cup_{i=1}^{n} (A_i - \cup_{j=1}^{m} B_j) = \cup_{i=1}^{n} \cap_{j=1}^{m} (A_i - A_j)$$

Observe that finite union of disjoint sets of $\hat{\mathcal{A}}$ is still in $\hat{\mathcal{A}}$. Since we have shown that $A_i - B_j = \uplus_{k=1}^{p} D_k$, for some $D_k \in \mathcal{R}$, we can conclude that $A - B \in \hat{\mathcal{A}}$.

To prove condition (ii), let $A,\ B \in \hat{\mathcal{A}}$. Then $A \cup B = (A - B) \uplus (A \cap B) \uplus (B - A) \in \hat{\mathcal{A}}$.

In conclusion, we showed that $\hat{\mathcal{A}}$ is a ring. Since we define $\mathcal{A}$ to be the smallest ring containing $\mathcal{R}$, we infer $\mathcal{A} \subset \hat{\mathcal{A}}$. □

Let us briefly summarize the relationships among the different types of measurability.

Theorem 2.1.19. $\mathcal{P} \subsetneq \mathcal{O} \subsetneq \mathcal{BF} \subsetneq \mathcal{V} \subsetneq \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$.

Proof. Recall from Theorem 2.1.15 that the σ-field $\mathcal{P}$ is generated by adapted and continuous $\mathbb{R}$-valued processes. Note that every adapted and continuous $\mathbb{R}$-valued process is also right-continuous and adapted. Since, by definition, the optional σ-field $\mathcal{O}$ is generated by all adapted right-continuous $\mathbb{R}$-valued processes, we infer that $\mathcal{P} \subset \mathcal{O}$. By Theorem 2.1.8, every right-continuous adapted $\mathbb{R}$-valued process is progressively measurable. Thus, we have $\mathcal{O} \subset \mathcal{BF}$. The inclusion $\mathcal{BF} \subset \mathcal{V}$ follows from Lemma 2.1.10, that is every E-valued progressively measurable process is measurable and adapted. We will give several examples to illustrate that all the inclusions are strict. □

Example 2.1.20 (Optional but not Predictiable). Let $N = (N_t)_{t\geq 0}$ be a Poisson process and $\mathfrak{F}$ be its associated filtration. Then N is optional but it is not predictable. See [58] for a detailed proof or Example 3.4.11 for an alternative proof. In fact, a theorem due to [58] tells us that every predictable right-continuous martingale is continuous.

Example 2.1.21 (Progressively Measurable but not Optional). This example is due to Dellacherie and Meyer. Let $(W_t)_{t\geq 0}$ be the standard Brownian Motion with continuous paths and let $\mathcal{F}_t := \sigma(W_t : 0 \leq s \leq t)$, for $0 \leq t < \infty$. Set $D := \{(t,\omega) : W_t(\omega) = 0\}$. Then the set D is closed and predictable. In fact, D is a.s. a set without interior points. In other words $D' := \{s : (s,\omega) \in D^c\}$ is the disjoint union of open intervals. Take $L := \{(s,\omega) \in D : \text{s is not isolated from the right}\}$. The set L may also be characterized as all the points (s,ω) of D such that s is the left-end point of an excursion intervals of D'. It was show in Dellacherie and Meyer that the indicator $X := 1_L$ is progressively measurable but not optional.

The following example from Chung and Willian [22] illustrates that there are processes that are measurable and adapted but not progressively measurable.

Example 2.1.22 (Measurable but not Progressively Measurable). Let $W = (W_t)_{t\geq 0}$ be a one-dimensional Brownian Motion and $\mathfrak{F} = (\mathcal{F}_t)_{t\geq 0}$ be a filtration generated by the Brownian Motion, i.e., $\mathcal{F}_t = \sigma(W_s;\ 0 \leq s \leq t)$ and augmented by $\mathbb{P}$-null sets from $\mathcal{F}$. Assume $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets in $\mathcal{F}$. We define

$$T(\omega) := \sup\{t \in [0,1] : W_t(\omega) = 0\},\ \omega \in \Omega.$$

Take $[T] = \{(T(\omega),\omega) :\ \omega \in \Omega\} \subset [0,T] \times \Omega$, i.e. $[T]$ is the graph of the random variable T, see Subsection 2.2.1. Let $X_t(\omega) = 1_{[T]}(t,\omega) = 1_{\{(T(\omega),\omega):\ \omega\in\Omega\}}(t,\omega)$. Then X is measurable and adapted but it is not progressively measurable.

Remark 2.1.23. By Lemma 2.1.10, every progressively measurable process is measurable and adapted. From the above Example 2.1.22 we can see the converse may not true. But Theorem 2.1.11 states that any measurable and adapted process has a progressively measurable modification. In this example, it is easily seen that process $Y \equiv 0$ is a modification of X which is progressively measurable. To see this, we note that for every $t \geq 0$,

$$\begin{aligned}\{\omega : X_t(\omega) = 1\} &= \{\omega : T(\omega) = t\} \\ &= \{W_t = 0\} \cap \{W_t \neq 0, \forall s \in (t, 1]\} \subset \{W_t = 0\}.\end{aligned}$$

Thus, we obtain that

$$\mathbb{P}(\{\omega : X_t(\omega) = 1\}) \leq \mathbb{P}(\{W_t = 0\}) = 0. \tag{2.1.6}$$

Hence, $\mathbb{P}(\{X_t = Y_t\}) = 1$, for all $t \geq 0$. This implies that $Y \equiv 0$ is a modification of X. Clearly, the process $Y \equiv 0$ is progressively measurable.

Example 2.1.24 (An adapted process which is not measurable). Let $\Omega = [0, 1]$ and let the $\mathcal{F}$ be the σ-field generated by all finite subsets of $[0, 1]$. Take the filtration $\mathfrak{F}$ to be that $\mathcal{F}_t = \mathcal{F}$ for every $0 \leq t < \infty$. Define a measure $\mathbb{P}$ by $\mathbb{P}(N) = 0$ if N is a countable set on $[0, 1]$ and $\mathbb{P}(N) = 1$, otherwise. Then $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$. Set $A = \{(t, \omega) \in [0, 1] \times \Omega : t = \omega\}$. Define a process $X_t(\omega) = 1_A(t, \omega)$, for $(t, \omega) \in [0, 1] \times \Omega$. Note that for a fixed time $t \in [0, 1]$, $\{\omega : X_t(\omega) = 1\} = \{\omega : \omega = t\} \in \mathcal{F}_t$ by the definition of the filtration $\mathfrak{F}$. Assume that $A \in \mathcal{B}([0, 1]) \otimes \mathcal{F}$. In other words, assume that X is measurable. Since the product set $B = [0, \frac{1}{2}] \times \Omega \in \mathcal{B}([0, 1]) \otimes \mathcal{F}$, we have $A \cap B \in \mathcal{B}([0, 1]) \otimes \mathcal{F}$. So the function $1_{A \cap B}$ is $\mathcal{B}([0, 1]) \otimes \mathcal{F}$-measurable. Since the measurability with respect to a product σ-field implies measurability with respect to all sections, the function $\omega \mapsto 1_{A \cap B}(t, \omega)$ is $\mathcal{F}$-measurable. Hence we have $[0, \frac{1}{2}] \in \mathcal{F}$ which is impossible by the definition of the σ-field $\mathcal{F}$. Therefore, the process X is not measurable.

2.2 Stopping Times

A random variable $\tau : \Omega \to [0, \infty]$ is called a **stopping time** w.r.t. the filtration $\mathfrak{F}$ if and only if for each $0 \leq t < \infty$, $\{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$. Clearly, τ is a stopping time if and only if the process $1_{[0,\tau)}$ is adapted. A random variable $\tau : \Omega \to [0, \infty]$ is called an **optional time** of the filtration $\mathfrak{F}$ if and only if $\{\tau < t\} \in \mathcal{F}_t$ for every $t \geq 0$.

Proposition 2.2.1 ([48]). *(i) Every stopping time is optional.*

(ii) A random variable τ is optional w.r.t. the filtration $(\mathcal{F}_t)_{t\geq 0}$ if and only if τ is a stopping time with respect to the filtration $(\mathcal{F}_{t+})_{t\geq 0}$.

(iii) In particular, if the filtration $(\mathcal{F}_t)_{t\geq 0}$ is right continuous, then τ is an optional time if and only if it is a stopping time.

Proposition 2.2.2 ([48]). *Let τ, σ be two stopping times on a filtered probability space $(\Omega, \mathcal{F}, (\mathfrak{F}, \mathbb{P})$.*

(i) If τ is a nonnegative constant, then τ is a stopping time.

(ii) Then random variables $\tau + \sigma$, $\tau \wedge \sigma$ and $\tau \vee \sigma$, $a\tau$, where $a \geq 1$, are stopping times.

(iii) If $\{\tau_n\}_{n=1}^{\infty}$ is a sequence of stopping times, then the random variable $\sup_{n\in\mathbb{N}} \tau_n$ is a stopping time.

There are many ways to produce new stopping times. Given a set $A \in \mathcal{B}(E)$ and an E-valued process $X = (X_t)_{t\geq 0}$ on $(\Omega, \mathcal{F})$. We define the first hitting time H_A by

$$H_A = \inf\{t \geq 0 : X_t \in A\},$$

where we adopt the convention that $\inf\{\emptyset\} = \infty$.

Proposition 2.2.3. *Let X be an E-valued stochastic process with right-continuous paths which is adapted to the filtration $\{\mathcal{F}_t\}$. If A is an open set, then H_A is an optional time (or H_A is an stopping time with respect to $\{\mathcal{F}_{t+}\}$).*

Proof. First we claim that for every $t \geq 0$,

$$\{\omega \in \Omega : H_A(\omega) < t\} = \bigcup_{0\leq s<t} \{\omega \in \Omega : X_s(\omega) \in A\}.$$

To proof this, let us suppose that $\omega \in \bigcup_{0\leq s<t}\{X_s \in A\}$. Then there exists a number $s \in [0,t)$ such that $X_s(\omega) \in A$ which implies that the first hitting time $H_A(\omega)$ of A should not be bigger than s. That is $H_A(\omega) \leq s < t$. Conversely, assuming that $\omega \in \{H_A < t\}$, we get $H_A(\omega) < t$. Let $u = H_A(\omega)$. Since $u = \inf\{t \geq 0, X_t(\omega) \in A\}$, we can find a number $s_0 \in (u,t)$ such that $X_{s_0}(\omega) \in A$, which shows that $\omega \in \{H_A < t\}$.

Next we claim that

$$\bigcup_{0\leq s<t} \{X_s \in A\} = \bigcup_{q\in\mathbb{Q}^+\cap[0,t)} \{X_q \in A\}.$$

Indeed, it is easy to see that $\bigcup_{0\leq s<t}\{X_s \in A\} \supset \bigcup_{q\in\mathbb{Q}^+\cap[0,t)}\{X_q \in A\}$. For the other conclusion, let us suppose that $\omega \in LHS$. Then there exists at least one $s' \in [0,t)$ such that $X_{s'}(\omega) \in A$. Obviously, we can find a non-increasing sequence $\{q_n\}_{n\in\mathbb{N}}$ of rational numbers such that $q_n \in [0,t)$ and $q_n \downarrow s'$. Since the process X, by assumption, has right-continuous paths, we infer that $\lim_{n\to\infty} X_{q_n}(\omega) = X_{s'}(\omega)$. Moreover, since the set A is open, we deduce that there exists at least one $n \in \mathbb{N}$ satisfying $X_{q_n}(\omega) \in A$, where $q_n \in [0,t)$. This shows that

$$\omega \in \bigcup_{n\in\mathbb{N}}\{X_{q_n}(\omega) \in A\} \subset \bigcup_{q\in\mathbb{Q}^+\cap[0,t)} \{X_q \in A\},$$

what proves the equality. In conclusion, we proved that

$$\{H_A < t\} = \bigcup_{0\leq s<t} \{X_s \in A\} = \bigcup_{q\in\mathbb{Q}^+\cap[0,t)} \{X_q \in A\} \in \mathcal{F}_t.$$

□

Remark 2.2.4. Note that even if we assume that all the paths of the stochastic process X are continuous, one still can not deduce that when the set A is open, H_A is a stopping time with respect to $\mathfrak{F} := (\mathcal{F}_t)_{t\geq 0}$ without the right-continuity assumption on the filtration $\mathfrak{F}$. For instance, take an open interval $A = (t, \infty)$, for some $t > 0$ and take a stochastic process X with $\Omega = \{\omega_1, \omega_2\}$,

$$X_s(\omega_1) = s$$

$$X_s(\omega_2) = \begin{cases} s, & if\ 0 \leq s \leq t \\ 2t - s, & if\ s \geq t. \end{cases}$$

Let $(\mathcal{F}_t^X)_{t\geq 0}$ be the filtration generated by the process X. Then we have $H_A(\omega_1) = \inf\{s : X_s(\omega_1) \in (t,\infty)\} = \inf\{s : s \in (t,\infty)\} = t$ and $H_A(\omega_2) = \inf\{s : X_s(\omega_2) \in (t,\infty)\} = \inf \emptyset = \infty$. However, for our $t \geq 0$ the set $\{H_A \leq t\} = \{\omega_1\} \notin \mathcal{F}_t^X$. Hence the usual hypotheses on the filtration $\mathfrak{F}$ are somehow reasonable.

Lemma 2.2.5. *Let $\rho(x,A) = \inf\{\|x-y\| : y \in A\}$, where $x \in E$ and A is a closed set of E. Denote by $A_n = \{x : \rho(x,A) < \frac{1}{n}\}$, $n \in \mathbb{N}$ the $\frac{1}{n}$-neighborhood of A. Then $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of open sets and we have $\bigcap_{n=1}^{\infty} A_n = A$.*

Proposition 2.2.6 ([80]). *Let X be a stochastic process with continuous paths which is adapted to the filtration $\{\mathcal{F}_t\}$. If A is a closed set, then H_A is a stopping time.*

Proof. By Lemma 2.2.5, for every $n \in \mathbb{N}$, the $\frac{1}{n}$-neighborhood $A_n = \{x : \rho(x,A) < \frac{1}{n}\}$ of the closed set A are open and $\bigcap_{n=1}^{\infty} A_n = A$. Define a sequence of the random variables by $H_{A_n} = \inf\{t \geq 0; X_t \in A_n\}$. From Proposition 2.2.3, it follows that H_{A_n} are optional. Since $\{A_n\}$ is a decreasing sequence of open sets, the optional times $\{H_{A_n}\}_{n=1}^{\infty}$ is a nondecreasing sequence. Hence the limit of $\{H_{A_n}\}$ exists, and we denote it by $H_{A_\infty} \in [0,\infty]$. Let $H_A = \inf\{t \geq 0 : X_t \in A\}$. We will show that $H_A = H_{A_\infty}$.

If $H_{A_\infty} = \infty$, then $H_A = \infty$. This is because $H_{A_n} \leq H_A$ implies that $H_{A_\infty} \leq H_A$.

If $H_{A_\infty} < \infty$, we will justify the result as follows. Suppose first that $H_A = 0$, which means the stochastic process X is contained in A at the beginning. Then $H_{A_n} = 0$ for every $n \in \mathbb{N}$. If $H_A > 0$, since the stochastic process X has continuous paths, we find out that $X_{H_{A_n}} \in \partial A_n$, where ∂A_n is the boundary of A_n. Clearly, $\partial A_n \subsetneq A_n$ but $\partial A_n \subset A_k$, for $k < n$. This gives that $X_{H_{A_n}} \notin A_n$ but $X_{H_{A_n}} \in A_k$ for $k < n$. It follows that $H_{A_n} < H_{A_{n+1}} < H_A$, for any $n \in \mathbb{N}$. Moreover, since $X_{H_{A_n}} \in \partial A_n \subset \{x : \rho(x,A) \leq \frac{1}{m}\}$ for any $m \leq n$ and the sets $\{x : \rho(x,A) \leq \frac{1}{m}\}$ are closed, the limit $X_{H_{A_\infty}}$ of the sequence $\{X_{H_{A_n}}\}_{n\geq m}$ is also contained in $\{x : \rho(x,A) \leq \frac{1}{m}\}$, for every $m \in \mathbb{N}$, i.e. $X_{H_{A_\infty}} \in \bigcap_{m=1}^{\infty}\{x : \rho(x,A) \leq \frac{1}{m}\}$. Observe that

$$\bigcap_{m=1}^{\infty} \{x : \rho(x,A) \leq \frac{1}{m}\} = A$$

It follows that $X_{H_{A_\infty}} \in A$. However $H_A = \inf\{t \geq 0; X_t \in A\}$. Hence we infer that $H_{A_\infty} \geq H_A$. This together with the observation $H_{A_\infty} \leq H_A$ yields that $\lim_{n\to\infty} H_{A_n} = H_{A_\infty} = H_A$.

Since we have shown above that $H_{A_n} < H_A$, for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} H_{A_n} = H_A$, we have for every $t > 0$, $\{H_A \leq t\} = \bigcap_{n=1}^{\infty}\{H_{A_n} < t\}$. Since the sets A_n are open, by Proposition 2.2.3, we know that $\{H_{A_n} < t\} \in \mathcal{F}_t$. Thus $\{H_A \leq t\} \in \mathcal{F}_t$, for every $t > 0$. When $t = 0$, since X is adapted to $\{\mathcal{F}_t\}$, $\{H_A \leq 0\} = \{X_0 \in A\} \in \mathcal{F}_0 = \mathcal{F}_t$.

□

Let $X = (X_t)_{t\geq 0}$ be a measurable E-valued process defined on $(\Omega, \mathcal{F}, \mathfrak{F}, \mathbb{P})$, and let τ be a random variable on Ω with values in $[0,\infty]$. We define a function $X_\tau : \Omega \to E$ by

$$X_\tau(\omega) := X_{\tau(\omega)}(\omega), \ \omega \in \Omega.$$

The function X_τ we defined above is a random variable. In fact, the mapping $\Omega \ni \omega \to (\tau(\omega), \omega) \in \mathbb{R} \times \Omega$ is $\mathcal{F}/\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$-measurable. Moreover, the mapping $(t,\omega) \to X_t(\omega)$ of the space $(\mathbb{R}_+ \times \Omega, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F})$ into $(E, \mathcal{B}(E))$ is measurable. As the composition of the two measurable mappings, the function X_τ is thus $\mathcal{F}/\mathcal{B}(E)$-measurable, i.e. X_τ is a random variable. The random time τ is also allowed to take the value $+\infty$ when $X_\infty(\omega)$ is well defined for all $\omega \in \Omega$. In such a case, we set $X_\tau(\omega) := X_\infty(\omega)$ on $\{\tau = \infty\}$.

Definition 2.2.7. Let T be a stopping time with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$. The family $\mathcal{F}_T$ containing all the events $A \in \mathcal{F}$ such that $A \cap \{T \leq t\} \in \mathcal{F}_t$ for each $t \geq 0$, is called to be the σ-field of events prior to T.

Proposition 2.2.8. *Let T be a stopping time of the filtration $(\mathcal{F}_t)_{t\geq 0}$. Then we have*

(i) $\mathcal{F}_T$ is a σ-field.

(ii) T is $\mathcal{F}_T$-measurable.

(iii) If the stopping time T is equal to a constant t, then $\mathcal{F}_T = \mathcal{F}_t$.

Proof. (i) If $A \in \mathcal{F}_T$, i.e. $A \cap \{T \leq t\} \in \mathcal{F}_t$ for every $t \geq 0$, then we have

$$A^c \cap \{T \geq t\} = \{T \geq t\} \backslash (A \cap \{T \geq t\}) \in \mathcal{F}_t, \ t \geq 0,$$

which implies that $A^c \in \mathcal{F}_T$. If $A_1, A_2, \cdots \in \mathcal{F}_T$, hence

$$(A_1 \cup A_2 \cup \cdots) \cap \{T \leq t\} = \cup_{n\in\mathbb{N}}(A_1 \cap \{T \leq t\}) \in \mathcal{F}_t,$$

for each $t \geq 0$.
Therefore, $\mathcal{F}_T$ is a σ-field.

(ii) Note that for each $t \geq 0$, the event $\{T \in (-\infty, t]\} \cap \{T \leq s\} = \{T \leq t\} \cap \{T \leq s\} \in \mathcal{F}_s$, $\forall s \geq 0$. This shows that T is $\mathcal{F}_T$-measurable.

(iii) We have to show that $\mathcal{F}_t \supset \mathcal{F}_T$ and $\mathcal{F}_t \subset \mathcal{F}_T$. For the first claim $\mathcal{F}_t \supset \mathcal{F}_T$, we observe that if $B \in \mathcal{F}_T$, then

$$\begin{aligned} B = B \cap \{T = t\} &= B \cap (\{T \leq t\} \cap \{T \geq t\}^c) \\ &= (B \cap \{T \leq t\}) \cap \{T < t\} \\ &= B \cap \{T \leq t\} \cap (\cup_{n\in\mathbb{N}}\{T \leq t - \frac{1}{n}\}) \in \mathcal{F}_t. \end{aligned}$$

On the other hand, if $B \in \mathcal{F}_t$, then $B \cap \{T \leq t\} = B \in \mathcal{F}_t$. Hence $B \in \mathcal{F}_T$.

□

Theorem 2.2.9. *Let S and T be two stopping times. For any element $A \in \mathcal{F}_S$, we have $A \cap \{S \leq T\} \in \mathcal{F}_T$.*

Proof. One only need to show that for any $A \in \mathcal{F}_S$, we have $A \cap \{S \leq T\} \cap \{T \leq t\} \in \mathcal{F}_t$, for each $t \geq 0$. Observe that

$$\begin{aligned} A \cap \{S \leq T\} \cap \{T \leq t\} &= A \cap \{S \leq t\} \cap \{T \leq t\} \cap \{S \leq T\} \\ &= (A \cap \{S \leq t\}) \cap \{T \leq t\} \cap \{S \wedge t \leq T \wedge t\}. \end{aligned}$$

Now we claim that for every stopping time T, $T \wedge t$ is $\mathcal{F}_t$-measurable. Indeed, this claim follows immediately from the fact that for every $B \in \mathcal{B}(\mathbb{R})$,

$$\{T \wedge t \leq s\} \ = (\{T \leq s\} \cup \{t \leq s\} = \begin{cases} \{T \leq s\} \in \mathcal{F}_s \subset \mathcal{F}_t, & if\ t > s \\ \Omega \in \mathcal{F}_t, & if\ t \leq s \end{cases}$$

Using the above claim, we obtain

$$A \cap \{S \le T\} \cap \{T \le t\} \in \mathcal{F}_t.$$

□

Proposition 2.2.10 ([58]). *Let X be an E-valued right-continuous and adapted process. Let τ be a stopping time and a be a positive number. Define for every $\omega \in \Omega$,*

$$\sigma(\tau)(\omega) := \inf\{t \ge 0 : t > \tau(\omega),\ \|X_t(\omega) - X_{\tau(\omega)}(\omega)\| > a\},$$

where $\inf \emptyset = \infty$ *as usual. Then σ is a stopping time with respect to the family $(\mathcal{F}_{t+})_{t\ge 0}$.*

2.2.1 Stochastic Intervals

Definition 2.2.11. Let S and T be two stopping times. We define the stochastic intervals $((S,T]]$, $[[S,T]]$, $[[S,T))$, $((S,T))$ and $[[T]]$ by

$$\begin{aligned}
((S,T]] &:= \{(s,\omega) \in \mathbb{R}_+ \times \Omega : S(\omega) < s \le T(\omega)\},\\
[[S,T]] &:= \{(s,\omega) \in \mathbb{R}_+ \times \Omega : S(\omega) \le s \le T(\omega)\},\\
[[S,T)) &:= \{(s,\omega) \in \mathbb{R}_+ \times \Omega : S(\omega) \le s < T(\omega)\},\\
((S,T)) &:= \{(s,\omega) \in \mathbb{R}_+ \times \Omega : S(\omega) < s < T(\omega)\},\\
[[T]] &:= \{(t,\omega) \in \mathbb{R}_+ \times \Omega : t = T(\omega) < \infty\}.
\end{aligned}$$

In particular, $[[T]]$ is called the graph of the stopping time T.

Note that the stochastic intervals are subsets of $\mathbb{R}_+ \times \Omega$ and it is easy to see, by definition, that they belong to $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$. Let $s, t \in \mathbb{R}_+$ be two constant stopping times. Then, according to the Definition 2.2.11, the stochastic interval $((s,t]]$ is equal to the set $(s,t] \times \Omega$.

Proposition 2.2.12. *The predictable σ-field $\mathcal{P}$ is generated by the family of stochastic intervals of the form*

$$\mathcal{S} := \{((S,T]] :\ S \text{ and } T \text{ stopping times}\} \cup \{\{0\} \times F,\ F \in \mathcal{F}_0\}.$$

Proof. Clearly, the processes $1_{((S,T]]}$ and $1_{\{0\}\times F}$ are left-continuous. Also, it is easy to see that the process $1_{((S,T]]}$ is adapted, since S and T are stopping times. Therefore, by the definition of the predictable σ-field $\mathcal{P}$, we infer that the processes $1_{((S,T]]}$ and $1_{\{0\}\times F}$ are predictable. Hence $((S,T]] \in \mathcal{P}$ and $\{0\} \times F \in \mathcal{P}$. This proves that $\mathcal{S} \subset \mathcal{P}$.

On the other hand, by Theorem 2.1.15, the predictable σ-field $\mathcal{P}$ is also generated by the set $\mathcal{R}$ of all predictable rectangles

$$\mathcal{R} := \{(s,t] \times F,\ 0 \le s \le t < \infty, F \in \mathcal{F}_s\} \cup \{\{0\} \times F, F \in \mathcal{F}_0\}.$$

So it is enough to show that every set of the form $\{(s,t] \times F :\ 0 \le s \le t < \infty, F \in \mathcal{F}_s\}$ is a stochastic interval. To see this, we set $S = s \cdot 1_F + t \cdot 1_{F^c}$ and $T = t$. Since $F \in \mathcal{F}_s$, the random variable S is a stopping time. Hence $((S,t]]$ is a stochastic interval. This proves the Proposition. □

Corollary 2.2.13. *The predictable σ-field $\mathcal{P}$ defined in Definition 2.1.13 is generated by the family of stochastic intervals of the form*

$$\mathcal{T} := \{[[0,T]] :\ T \text{ stopping time}\}.$$

Proof. Let T be a stopping time. Since the process $1_{[[0,T]]}$ is left-continuous and adapted, it is predictable, hence $[[0,T]] \in \mathcal{P}$. On the other hand, note that $((S,T]] = [[0,T]] - [[0,S]]$ and $\{0\} \times F = [[1_{F^c}, 0]]$. Thus $((S,T]] \in \mathcal{T}$ and $\{0\} \times F \in \mathcal{T}$. In other words, $\mathcal{S} \subset \mathcal{T}$. Therefore the proof follows immediately from the above proposition. □

2.2.2 Stopped Processes and Localization

Let X be an E-valued process. Let τ be a stopping time defined on $(\Omega, \mathcal{F})$. Define a process X^τ by

$$X_t^\tau(\omega) := X_{t\wedge\tau} = \begin{cases} X_t(\omega), & if\ t < \tau(\omega) \\ X_{\tau(\omega)}(\omega), & if\ t \geq \tau(\omega). \end{cases} \tag{2.2.1}$$

We call X^τ the process stopped at the random time τ. It can be easily seen that the stopped process preserves all continuity and adaptedness properties of the process X. In other words, if X is a right-continuous and adapted process, X^τ is also right-continuous and adapted.

However, the stopped process X^τ may have a jump exactly at time τ. In such cases, it is often technically convenient to have the following definition by replacing τ by $\tau-$ when the process X is càdlàg. Let X be an E-valued adapted and càdlàg process. Define

$$X_t^{\tau-}(\omega) := X_{t\wedge\tau-} = \begin{cases} X_t(\omega), & if\ t < \tau(\omega) \\ X_{\tau(\omega)-}(\omega), & if\ t \geq \tau(\omega), \end{cases} \tag{2.2.2}$$

which is called the process X stopped strictly before the stopping time τ. Here $(X_{t-})_{t\geq 0}$ is a càglàd process defined by for every $\omega \in \Omega$, $X_{t-}(\omega) := \lim_{s\nearrow t} X_s(\omega)$ for every $0 < t < \infty$ and $X_{0-}(\omega) = X_0(\omega)$. Note that the stopped process $X^{\tau-}$ inherits the adaptedness and càdlàg property from the process X. The reason for introducing above definition of stopped process $X^{\tau-}$ is that there are always processes without boundedness assumption. In such cases, we can define a localizing sequence of stopping times $\{\tau_n\}_{n\in\mathbb{N}}$ such that the associated processes X^{τ_n-} is bounded.

Remark 2.2.14. If the process X is left-continuous, then both definitions agree.

Chapter 3

Stochastic Integrals w.r.t. compensated Poisson random measures

3.1 Poisson Random Measures

Let $(\Omega, \mathbb{P}, \mathfrak{F}, \mathcal{F})$ be a filtered probability space. Let $(U, \mathcal{U})$ be a measurable space. Let $\mathbb{N} = \{0, 1, 2, \cdots\}$ and $\bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. Let $\mathbb{M}_{\bar{\mathbb{N}}}(U)$ denote the space of all $\bar{\mathbb{N}}$-valued measures on $(U, \mathcal{U})$. In other words, $\mathbb{M}_{\bar{\mathbb{N}}}(U)$ is the collection of all counting measures. Let $\mathcal{B}(\mathbb{M}_{\bar{\mathbb{N}}}(U))$ be the smallest σ-field on $\mathbb{M}_{\bar{\mathbb{N}}}(U)$ with respect to which the mappings $i_B : \mathbb{M}_{\bar{\mathbb{N}}}(U) \ni \mu \mapsto \mu(B) \in \bar{\mathbb{N}}$, $B \in \mathcal{U}$ are measurable.

Remark 3.1.1. If $\mu \in \mathbb{M}_{\bar{\mathbb{N}}}(U)$, then μ is a measure and so it satisfies

1. $\mu(\emptyset) = 0$;
2. (σ-additivity) for any sequence $\{A_n\}_{n \in \mathbb{N}}$ of disjoint sets in $\mathcal{U}$, $\mu\big(\bigcup_{n\in\mathbb{N}} A_n\big) = \sum_{n\in\mathbb{N}} \mu(A_n)$.

Note that the σ-field $\mathcal{B}(\mathbb{M}_{\bar{\mathbb{N}}}(U))$ is generated by sets of the form

$$i_B^{-1}(A), \quad A \in \mathcal{P}(\bar{\mathbb{N}}),\ B \in \mathcal{U},$$

where $i_B^{-1}(A) = \{\mu \in \mathbb{M}_{\bar{\mathbb{N}}}(U); \mu(B) \in A\}$.

Definition 3.1.2. A map $N : \Omega \times \mathcal{U} \to \bar{\mathbb{N}}$ is called an $\bar{\mathbb{N}}$-valued random measure if and only if for each $\omega \in \Omega$, $N(\omega, \cdot) \in \mathbb{M}_{\bar{\mathbb{N}}}(U)$ and for each $A \in \mathcal{U}$, $N(\cdot, A)$ is an $\bar{\mathbb{N}}$-valued random variable on the probability space $(\Omega, \mathbb{P}, \mathcal{F})$. We will oftern write $N(A)$ instead of $N(\cdot, A)$.

Remark 3.1.3. An $\bar{\mathbb{N}}$-valued random measure N can also be viewed as an $(\mathbb{M}_{\bar{\mathbb{N}}}(E), \mathcal{B}(\mathbb{M}_{\bar{\mathbb{N}}}(U)))$-valued random variable on the probability space $(\Omega, \mathbb{P}, \mathcal{F})$. In such a case, for each $A \in \mathcal{U}$, $N(\cdot, A) := i_A \circ N(\cdot) : \Omega \to \bar{\mathbb{N}}$ is an $\bar{\mathbb{N}}$-valued random variable.

Definition 3.1.4. An $\bar{\mathbb{N}}$-valued random measure N, is called a **Poisson random measure** if and only if

(1) for any $B \in \mathcal{U}$ provided $\mathbb{E}[N(B)] < \infty$, $N(B)$ is a random variable with Poisson distribution, i.e.

$$\mathbb{P}(N(B) = n) = e^{-\eta(B)} \frac{\eta(B)^n}{n!}, \quad n = 0, 1, 2, \cdots,$$

with $\eta(B) = \mathbb{E}(N(B))$.

(2) (independently scattered property) for any pairwise disjoint sets $B_1, \cdots, B_n \in \mathcal{U}$, the random variables

$$N(B_1), \ \cdots, \ N(B_n)$$

are independent.

Remark 3.1.5. Note that for every $\omega \in \Omega$, $N(\omega, \cdot)$ is an $\bar{\mathbb{N}}$-valued measure on $(U, \mathcal{U})$ and for every $B \in \mathcal{U}$, $N(\cdot, B)$ is a Poisson random variable. One can treat the Poisson random measure N as a collection of Poisson random variables, $\{N(\cdot, B); B \in \mathcal{U}\}$. It can also be viewed as a collection of counting measures in $\mathbb{M}_{\bar{\mathbb{N}}}(U)$, $\{N(\omega) : \omega \in \Omega\}$.

The following theorem, due to [41], shows that there exists a Poisson random measure, as defined above. See also Sato [73] p.122, Ikeda and Watanabe [40] and Kyprianow [53] for a detailed proof.

Theorem 3.1.6. *Given a σ-finite measure η on $(U, \mathcal{U})$, there exists a Poisson random measure N on $(U, \mathcal{U})$ over $(\Omega, \mathcal{F}, \mathbb{P})$ such that*

$$\mathbb{E}(N(B)) = \eta(B), \quad \textit{for all } B \in \mathscr{E}.$$

Outline of the Proof. If $\eta(U) = 0$, we can choose $N(B) = 0$, for all $B \in \mathcal{U}$. Now suppose that η is a finite measure, i.e. $0 < \eta(U) < \infty$. We can always construct a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which we will construct some random variables as follows:

1. a Poisson-distributed random variable M with parameter $\eta(U)$,
2. a sequence $\{X_i\}_{i \in \mathbb{N}}$ of independent random variables which is independent of N and each of the random variable X_i have the distribution

$$\mathbb{P}(X_i \in A) = \frac{\eta(A)}{\eta(U)}, \quad A \in \mathcal{U}.$$

For every $B \in \mathcal{U}$, define

$$N(B) := \sum_{i=1}^{M} 1_{\{X_i \in B\}}.$$

Clearly, $N(B)$ is a random variable with respect to $\mathcal{F}$, since M, X_1, $X_2, \cdots$ are all $\mathcal{F}$-random variable. Let $A_1, \cdots, A_2$ be disjoint sets in $\mathcal{U}$ and $n_1, \cdots, n_k \in \mathbb{N}$. Then we find that

$$\begin{aligned}
&\mathbb{P}(N(A_1) = n_1, \cdots, N(A_k) = n_k) \\
&= \sum_{n \in \mathbb{N}} \mathbb{P}(N(A_1) = n_1, \cdots, N(A_k) = n_k \big| M = n) \mathbb{P}(M = n) \\
&= \sum_{n \in \mathbb{N}} \mathbb{P}(\sum_{i=1}^{n} 1_{\{X_i \in A_1\}} = n_1, \cdots, \sum_{i=1}^{n} 1_{\{A_k \in B\}} = n_k) \mathbb{P}(M = n) \\
&= \sum_{n \in \mathbb{N}} \frac{n!}{n_1! \cdots n_k!} \prod_{i=1}^{k} \Big(\frac{\eta(A_i)}{\eta(U)}\Big)^{n_k} \frac{(\eta(U))^n}{n!} \\
&= \prod_{i=1}^{k} e^{-\eta(A_i)} \frac{\eta(A_i)^{n_k}}{n_k!},
\end{aligned}$$

which shows that the random variables $N(A_1), \cdots, N(A_k)$ are independent and Poisson distributed. Hence N is a Poisson random measure on U.

Suppose now that η is σ-finite. Let $\{U_n\}$ be a partition of U such that $0 < \eta(U_n) < \infty$, $\cup_n U_n = U$ and $U_n \cap U_m = \emptyset$, $n \neq m$. Set $\eta_n(\cdot) = \eta(\cdot \cap U_n)$, for every $n \in \mathbb{N}$. Clearly, η_n is a finite measure on $\mathcal{U}$. It follows from the first part of the argument that for every $n \in \mathbb{N}$, there exists a probability space $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$ and a Poisson random measure N_n defined on $(U_n, \mathcal{U} \cap U_n)$ over $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$. Define

$$N(A) = \sum_{n=1}^{\infty} N_n(A \cap U_n).$$

over the probability space $(\Omega, \mathcal{F}, \mathbb{P}) := \prod_{i \in \mathbb{N}} (\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$. Then one can show, see Kyprianow [53] for a detailed proof, that N is a Poisson random measure with $\mathbb{E}N(\cdot) = \eta(\cdot)$. □

Remark 3.1.7. Let us briefly review how we have constructed the Poisson random measure associated to a σ-finite measure. Given a σ-finite measure η on $(U, \mathcal{U})$, let $\{U_n\}$ be a partition of U such that $0 < \eta(U_n) < \infty$, $\cup_n U_n = U$ and $U_n \cap U_m = \emptyset$, $n \neq m$. We constructed in the proof of the Theorem 3.1.6 a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the following random variables on the probability space,

1. $\{M_n\}_{n \in \mathbb{N}}$ is a sequence of $\mathbb{N}$-valued random variables and each of the random variables M_n have a Poisson distribution with parameter $\eta(U_n)$;

2. for each n, $\{X_i^n\}_{i \in \mathbb{N}}$ is a sequence of U_n-valued random variables and each of the random variables X_i^n have distribution

$$\mathbb{P}(X_i^n \in A) = \frac{\eta(A)}{\eta(U_n)}, \quad A \in (\mathcal{U} \cap U_n);$$

3. the random variables M_n, X_i^n, $i = 1, \cdots, n = 1, \cdots$ are mutually independent.

Then for each $A \in \mathcal{U}$, the random variable $N(A)$ defined by

$$N(A) := \sum_{n=1}^{\infty} \sum_{i=1}^{M_n} 1_{A \cap U_n}(X_i^n) 1_{M_n \geq 1}.$$

is a Poisson random measure with $\eta(\cdot) = \mathbb{E}N(\cdot)$ on $(U, \mathcal{U}_n)$ over the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Take $A \in \mathcal{U}$. Let $\mathcal{U}_A := \{F \cap A : F \in \mathcal{U}\}$. In fact, $\mathcal{U}_A$ is a σ-field and we will call it the **trace σ-field** of $\mathcal{U}$ on A.

Proposition 3.1.8. *Suppose that N is a Poisson random measure on $(U, \mathcal{U})$ over the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then for every $A \in \mathcal{U}$, the mapping*

$$\Omega \times \mathcal{U}_A \ni (\omega, B) \mapsto N(\omega, B \cap A) \in \bar{\mathbb{N}}$$

is a Poisson random measure on $(A, \mathcal{U}_A)$.

Proof. To show that $N(\cdot \cap A)$ is a Poisson random measure, we have to verify that $N(\cdot \cap A)$ satisfies conditions (1) and (2) of Definition 3.1.4. Since $A \in \mathcal{U}$ and N is a Poisson random measure, for any $B \in \mathcal{U}$, $B \cap A \in \mathcal{U}$ and $N(\cdot, B \cap A)$ is a Poisson random variable. Let $B_1, B_2, \cdots, B_n$ be pairwise disjoint sets from $\mathcal{U}$. Then the sets $B_1 \cap A, B_2 \cap A, \cdots, B_n \cap A$ are pairwise disjoint and they are all in $\mathcal{U}$. Therefore, by the independently scattered property of Poisson random measure N, $N(\cdot, B_1 \cap A), \cdots, N(\cdot, B_n \cap A)$ are pairwise independent. Hence $N(\cdot \cap A)$ is a Poisson random measure. □

Let $(Z, \mathcal{Z}, \nu)$ be a measurable space, where ν is a nonnegative σ-finite measure. Let λ be the Lebesgue measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+)$. Then the product measure $\lambda \otimes \nu$ is σ-finite on $(\mathbb{R}_+ \times Z, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{Z})$.

Definition 3.1.9. 1. An $\bar{\mathbb{N}}$-valued random measure M defined on $(\mathbb{R}_+ \times Z, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{Z})$ is called **adapted** to the filtration $\mathfrak{F}$ if and only if for every $t \in \mathbb{R}_+$, the random variable $M(\cdot, A)$ is $\mathcal{F}_t$-measurable, for every $A \in \mathcal{B}([0,t]) \otimes \mathcal{Z}$.

2. An $\bar{\mathbb{N}}$-valued random measure M defined on $(\mathbb{R}_+ \times Z, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{Z})$ is said to be **σ-finite** if and only if there exists an increasing sequence $\{D_n\}_{n \in \mathbb{N}} \subset Z$ such that $\cup_n D_n = Z$ and $\mathbb{E}M((0,t] \times A) < \infty$, for all $t > 0$ and $n \in \mathbb{N}$.

3. An $\bar{\mathbb{N}}$-valued random measure M defined on $(\mathbb{R}_+ \times Z, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{Z})$ is called a **martingale random measure** if and only if for any $A \in \mathcal{Z}$ satisfying $\mathbb{E}(M((0,t] \times A)) < \infty$, $t > 0$, the process $M((0,t] \times A)$, $t \geq 0$ is a martingale.

Theorem 3.1.10. *Let M be an $\bar{\mathbb{N}}$-valued adapted random measure defined on $(\mathbb{R}_+ \times Z, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{Z})$. Then there exists an increasing sequence of stopping times $\{T_n\}_{n \in \mathbb{N}}$ and a Z-valued optional process p such that*

$$M(\omega, A) = \sum_{s \geq 0} 1_D(s, \omega) 1_A(s, p(s, \omega)), \quad \textit{for all } A \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{Z}, \tag{3.1.1}$$

where $D = \cup_n [[T_n]]$ and $[[T_n]]$ is the graph of the stopping time T_n, $n \in \mathbb{N}$.

See Proposition 1.14 in [45] or Theorem 3.4.3 in [47] for more details.

Remark 3.1.11. If M is a Poisson random measure associated to a Poisson point process π, see Section 3.1.1, then equality (3.1.1) holds with $p = \pi$ and

$$D = \cup_{n \in \mathbb{N}} \cup_{k \in \mathbb{N}} [[\tau_n^k]]$$

Here

$$\tau_n^k = \inf\{t \geq 0 : M((0,t] \times U_n) \geq k\}, \quad k \in \mathbb{N}, \ n \in \mathbb{N}.$$

If M is a Poisson random measure associated to a Lévy process L, see Section 3.1.2, then equality holds with $\pi = \triangle L$ and

$$D = \{(s, \omega) : \|\triangle L_s(\omega)\| > 0\}.$$

In particular, $D = \cup_{n \in \mathbb{N}} \cup_{k \in \mathbb{N}} [[\tau_n^k]]$, where $\tau_n^0 = 0, \cdots, \tau_n^k = \inf\{s > \tau_n^{k-1} : \|\triangle L_s\| \geq \frac{1}{2^n}\}$.

Remark 3.1.12. Notice that since the product measure $\nu \otimes \lambda$ is σ-finite on $(Z \times \mathbb{R}_+, \mathcal{Z} \otimes \mathcal{B}(\mathbb{R}_+))$, by Theorem 3.1.6, there exists a Poisson random measure N with $\mathbb{E}N(B) = \nu \otimes \lambda(B)$, $B \in \mathcal{Z} \otimes \mathcal{B}(\mathbb{R}_+)$. In particular, there exists a Poisson random measure N associated with a stationary Poisson point process on $(Z, \mathcal{Z})$ with an intensity measure ν, see Theorem 3.1.21. For the future convenience, we also impose the condition $N(\{0\} \times B) = 0$, for every $B \in \mathcal{Z}$, $n \in \mathbb{N}$.

For simplicity, we shall use the notation

$$N(t, B) := N((0,t] \times B), \quad t \in \mathbb{R}_+, \ B \in \mathcal{Z}.$$

Also, we employ the notation

$$\tilde{N}(\cdot) = N(\cdot) - \mathbb{E}(N(\cdot)) = N(\cdot) - \eta(\cdot)$$

to denote the compensated Poisson random measure of N. Similarly, we write $\tilde{N}(t,B)$ instead of $\tilde{N}((0,t]\times B)$ for simplicity of the notations. Since for every $\omega\in\Omega$, $N(\omega,\cdot)$ is a measure on $(U,\mathcal{U})$, and the sets $(0,s]\times B$ and $(s,t]\times B$ are disjoint, we infer that

$$N((0,t]\times B)=N\Big(((0,s]\times B)\cup((s,t]\times B)\Big)=N((0,s]\times B)+N((s,t]\times B).$$

Thus we have

$$N(t,B)-N(s,B))=N((s,t]\times B).$$

Let $\mathfrak{F}=(\mathcal{F}_t)_{t\geq 0}$ be a filtration on $(\Omega,\mathcal{F},\mathbb{P})$ satisfying the usual hypotheses such that $N(t,B)$, $B\in\mathcal{Z}$ is $\mathcal{F}_t$-measurable, for each $t\geq 0$ and $N((s,t]\times B)$, $B\in\mathcal{Z}$ is independent of $\mathcal{F}_s$, for all $s\leq t$.

Lemma 3.1.13. *For each $A\in\mathcal{Z}$, the process $(\tilde{N}(t,A))_{t\geq 0}$ is a mean 0 martingale . Furthermore, it has càdlàg trajectories. In particular, for each $A\in\mathcal{Z}$ with $\nu(A)<\infty$, the process $N(t,A),t\geq 0$ is a Poisson process with intensity $\nu(A)$.*

Proof. Note that

$$\mathbb{E}\tilde{N}(t,A)=\mathbb{E}N((0,t]\times A)-\mathbb{E}N((0,t]\times A)=0.$$

Let $0\leq s\leq t<T$. Let us fix $A\in\mathcal{Z}$. Observe that

$$\begin{aligned}\mathbb{E}(\tilde{N}(t,A)|\mathcal{F}_s)&=\mathbb{E}\big(N((0,t]\times A)-\mathbb{E}(N(0,t]\times A)|\mathcal{F}_s\big)\\&=\mathbb{E}\big(N((0,s]\times A)+N((s,t]\times A)|\mathcal{F}_s\big)-\mathbb{E}(N(0,t]\times A)\\&=N((0,s]\times A)+\mathbb{E}(N((s,t]\times A))-\mathbb{E}(N(0,t]\times A)\\&=N((0,s]\times A)-\mathbb{E}(N(0,s]\times A)=\tilde{N}(s,A),\end{aligned}$$

where we used the σ-additivity of the measure N in the second equality and measurability of $N((0,s]\times A)$ and independence of $N((s,t]\times A)$ with respect to $\mathcal{F}_s$. Therefore, we showed that the process $(\tilde{N}(t,A))_{t\geq 0}$ is a martingale with mean 0.

For the right-continuity, let us fix $t\in[0,T]$ and take a sequence $(t_n)_{n\in\mathbb{N}}$ of times such that $t_n\searrow t$. Then the decreasing sequence $(0,t_n]\times A$, $n\in\mathbb{N}$ of sets converges to the set $(0,t]\times A$. That is $\cap_{j\in\mathbb{N}}((0,t_n]\times A)=(0,t]\times A$. For each $\omega\in\Omega$, note that $\tilde{N}(\omega)$ is a measure. So by the continuity of the measure $\tilde{N}(\cdot)$, we have

$$\tilde{N}(t,A)=\tilde{N}((0,t]\times A)=\lim_{n\to\infty}\tilde{N}((0,t_n]\times A),\text{ for all }\omega\in\Omega.$$

Note that for any sequence $\{t_n\}$ such that $t_n<t$, $n\in\mathbb{N}$ and $t_n\nearrow t$, $\cap_n((0,t_n]\times A)=(0,t)\times A$, hence

$$\lim_{n\to\infty}\tilde{N}((0,t_n]\times A)=N((0,t)\times A)\neq N((0,t]\times A).$$

This shows that $N(t,A)$ has left limits but it may not be left-continuous, since $N((0,t)\times A)$ may not equal to $N((0,t]\times A)$ in some cases.

To show that $N(t,A)$, $t\geq 0$, is a Poisson process, we first observe that $N(0,A)=N(\emptyset)=0$. Since N is a Poisson random measure, $N(t,A)=N((0,t]\times A)$ is a Poisson random variable with parameter $\mathbb{E}(N((0,t]\times A))=\lambda((0,t])\nu(A)=t\nu(A)$. Now it remains to show that $N(t,A),t\geq 0$ has independent increments. Take $t_0\leq t_1\leq\cdots\leq t_{n-1}\leq t_n$. Then the sets $(t_0,t_1]\times A,\cdots,(t_{n-1},t_n]\times A$ are pairwise disjoint. Hence by the independent scattered property of Poisson random measure, we deduce that $N((t_0,t_1]\times A),\cdots,N((t_{n-1},t_n]\times A)$ are independent. This gives that the increments $N(t_1,A)-N(t_0,A),\cdots,N(t_n,A)-N(t_{n-1},A)$ are independent. Therefore, $N(t,A),t\geq 0$ is a Poisson process. □

Remark 3.1.14. From the above Lemma, we infer that for each $A \in \mathcal{Z}$, the process $N(t, A)$, $t \geq 0$ is a nondecreasing submartingale with càdlàg paths. Hence, by the Meyer-Doob decomposition, see [59], there exists a unique predictable increasing process, denoted by γ, such that

$$N(t, A) - \gamma(t, A), \quad t \geq 0$$

is a martingale. Moreover, the decomposition is unique if γ is predictable or natural. We call the process γ the Meyer process. By the uniqueness part of the Meyer-Doob decomposition, see Theorem VII.21 in [59], we conclude that

$$\gamma = \eta.$$

Proposition 3.1.15. *For each $A \in \mathcal{Z}$, $\tilde{N}(t, A)^2 - \eta(t, A)$, $t \geq 0$ is a martingale.*

Proof. Since $N((s, t] \times A)$ is a Poisson process, we have

$$\begin{aligned}
\mathbb{E}\big(\tilde{N}((s,t] \times A)\big)^2 &= \mathbb{E}\big(N((s,t] \times A) - \eta((s,t] \times A)\big)^2 \\
&= \mathbb{E}\big(N((s,t] \times A)\big)^2 - 2\eta((s,t] \times A)\mathbb{E}(N((s,t] \times A)) + \eta((s,t] \times A)^2 \\
&= \mathbb{E}\big(N((s,t] \times A)\big)^2 - \eta((s,t] \times A)^2 \\
&= \sum_{n=0}^{\infty} n^2 \mathbb{P}(\{N((s,t] \times A) = n\}) - \eta((s,t] \times A)^2 \\
&= \sum_{n=0}^{\infty} n^2 \frac{\eta((s,t] \times A)^n}{n!} e^{-\eta((s,t]\times A)} - \eta((s,t] \times A)^2 \\
&= \sum_{n=0}^{\infty} \frac{\eta((s,t] \times A)^n}{(n-2)!} e^{-\eta((s,t]\times A)} + \sum_{n=0}^{\infty} \frac{\eta((s,t] \times A)^n}{(n-1)!} e^{-\eta((s,t]\times A)} - \eta((s,t] \times A)^2
\end{aligned}$$

$$\begin{aligned}
&= \eta((s,t] \times A)^2 \sum_{n=0}^{\infty} \frac{\eta((s,t] \times A)^{n-2}}{(n-2)!} e^{-\eta((s,t]\times A)} \\
&\quad + \eta((s,t] \times A) \sum_{n=0}^{\infty} \frac{\eta((s,t] \times A)^{n-1}}{(n-1)!} e^{-\eta((s,t]\times A)} - \eta((s,t] \times A)^2 \\
&= \eta((s,t] \times A)^2 + \eta((s,t] \times A) - \eta((s,t] \times A)^2 \\
&= \eta((s,t] \times A),
\end{aligned}$$

It follows that for $0 \leq s < t < \infty$,

$$\begin{aligned}
\mathbb{E}(\tilde{N}(t,A)^2 - \eta(t,A) | \mathcal{F}_s) &= \mathbb{E}\left(\left(\tilde{N}(s,A) + \tilde{N}(t,A) - \tilde{N}(s,A)\right)^2 - \eta(t,A) \Big| \mathcal{F}_s\right) \\
&= \mathbb{E}\left(\tilde{N}(s,A)^2 + 2\tilde{N}(s,A)\left(\tilde{N}(t,A) - \tilde{N}(s,A)\right) + \left(\tilde{N}(t,A) - \tilde{N}(s,A)\right)^2 \Big| \mathcal{F}_s\right) \\
&\quad - \eta(t,A) \\
&= \tilde{N}(s,A)^2 - 2\tilde{N}(s,A)\mathbb{E}\left(\tilde{N}(t,A) - \tilde{N}(s,A)\right) + \mathbb{E}\left(\tilde{N}(t,A) - \tilde{N}(s,A)\right)^2 - \eta(t,A) \\
&= \tilde{N}(s,A)^2 + \mathbb{E}\left(\tilde{N}(t,A) - \tilde{N}(s,A)\right)^2 - \eta(t,A) \\
&= \tilde{N}(s,A)^2 + \eta((s,t] \times A) - \eta(t,A) \\
&= \tilde{N}(s,A)^2 - \eta(s,A).
\end{aligned}$$

which shows that $\tilde{N}(t, A)^2 - \eta(t, A)$, $t \geq 0$ is a martingale. □

Proposition 3.1.16. *For each $A \in \mathcal{Z}$ and each $0 \leq s \leq t$,*

$$\mathbb{E}\left(\left(\tilde{N}(t,A) - \tilde{N}(s,A)\right)^2 | \mathcal{F}_s\right) = \eta(t,A) - \eta(s,A).$$

Proof. Observe that

$$\begin{aligned}
\mathbb{E}\left(\left(\tilde{N}(t,A) - \tilde{N}(s,A)\right)^2 | \mathcal{F}_s\right) &= \mathbb{E}\left(\tilde{N}(t,A)^2 | \mathcal{F}_s\right) - 2\tilde{N}(s,A)\mathbb{E}\left(\tilde{N}(t,A)|\mathcal{F}_s\right) + \tilde{N}(s,A)^2 \\
&= \mathbb{E}\left(\tilde{N}(t,A)^2 - \eta(t,A)|\mathcal{F}_s\right) + \eta(t,A) - 2\tilde{N}(s,A)^2 + \tilde{N}(s,A)^2 \\
&= \tilde{N}(s,A)^2 - \eta(s,A) + \eta(t,A) - \tilde{N}(s,A)^2 \\
&= \eta(t,A) - \eta(s,A).
\end{aligned}$$

□

Proposition 3.1.17. *For every $A \in \mathcal{Z}$ with $\nu(A) < \infty$ and every $t \geq 0$,*

$$[\tilde{N}(\cdot,A), \tilde{N}(\cdot,A)]_t = N(t,A).$$

Remark 3.1.18. Here $[\tilde{N}(\cdot,A), \tilde{N}(\cdot,A)]_t$, $t \geq 0$, is the quadratic variation of the process $\tilde{N}(t,A)$, $t \geq 0$.

Proof of Proposition 3.1.17. We will use the definition of quadratic variation to show the assertion. Let $\pi^m = \{0 = t_0^m \leq t_1^m \leq \cdots \leq t_{n(m)}^m = t\}$, $m \in \mathbb{N}$, be a sequence of partitions of $[0,t]$. Let $\|\pi^m\| = \max_{0\leq i\leq n(m)-1} |t_{i+1}^m - t_i^m|$ be the mesh of π^m. Suppose that $\{\pi_m\}$ is a sequence of partitions satisfying $\lim_{m\to\infty} \|\pi_m\| = 0$. We observe that

$$\begin{aligned}
V_{\pi_m}^{(2)}(\tilde{N})(t) &= \sum_{i=0}^{m-1} \left(\tilde{N}(t_{i+1}^m, A) - \tilde{N}(t_i^m, A)\right)^2 \\
&= \sum_{i=0}^{m-1} \left(N((t_i^m, t_{i+1}^m] \times A) - (t_{i+1}^m - t_i^m)\nu(A)\right)^2 \\
&= \sum_{i=0}^{m-1} N((t_i^m, t_{i+1}^m] \times A)^2 - 2\sum_{i=0}^{m-1} N((t_i^m, t_{i+1}^m] \times A)(t_{i+1}^m - t_i^m)\nu(A) \\
&\qquad + \sum_{i=0}^{m-1} (t_{i+1}^m - t_i^m)^2 \nu(A)^2 \\
&\leq \sum_{i=0}^{m-1} N((t_i^m, t_{i+1}^m] \times A)^2 + 2\|\pi^m\| \sum_{i=0}^{m-1} N((t_i^m, t_{i+1}^m] \times A)\nu(A) + \|\pi^m\| t\nu(A)^2.
\end{aligned}$$

Clearly, the last term in the above inequality converges to 0 as $m \to \infty$. Notice also that since the sample path $t \mapsto N(t,A)(\omega)$ is piecewise constant and increases by jumps of size 1, when $\|\pi^m\| \to 0$, as $m \to \infty$, $N((t_i^m, t_{i+1}^m] \times A) \leq 1$ as $m \to \infty$. We note further that by the càdlàg property of the process $N(s,A)$, $s \geq 0$, $N(s,A)$ may only have finite jumps in the time interval $[0,t]$. Thus we infer that the second sum $\|\pi^m\| \sum_{i=0}^{m-1} N((t_i^m, t_{i+1}^m] \times A)\nu(A)$ converges to 0, $\mathbb{P}$-a.s., as $m \to \infty$. Since $[0,t] \ni s \mapsto N(s,A) \in \mathbb{N}$ is a Poisson process what means every path of N is a piecewise constant function of time and the jump size is of 1, we find out that $N(t_{i+1}^m, A) - N(t_i^n, A)$ takes only two values 0 and 1 for sufficient small $t_{i+1}^m - t_i^m$. Hence for m big enouth, $N(t_{i+1}^m, A) - N(t_i^n, A) = (N(t_{i+1}^m, A) - N(t_i^n, A))^2$. Therefore, by above considerations, we infer that

$$\begin{aligned}\lim_{m\to\infty} V_{\pi_m}^{(2)}(\tilde{N})(t) &= \lim_{m\to\infty} \sum_{i=0}^{m-1} N((t_i^m, t_{i+1}^m] \times A)^2 \\ &= \lim_{m\to\infty} \sum_{i=0}^{m-1} N((t_i^m, t_{i+1}^m] \times A) = N(t, A), \quad \mathbb{P}\text{-a.s.}\end{aligned}$$

which competes the proof. □

3.1.1 Poisson Point Processes

Let $(Z, \mathcal{Z})$ be a measurable space. A **point function** α on $(Z, \mathcal{Z})$ is a mapping $\alpha : \mathcal{D}(\alpha) \to Z$, where the domain $\mathcal{D}(\alpha) \subset (0, \infty)$ of α is a countable subset. Let Π_Z be the set of all point functions on Z. For each point function, we define a counting measure N by

$$N_\alpha(U) := \sharp\{s \in (0,\infty) \cap \mathcal{D}(\alpha) : \ (s, \alpha(s)) \in U\}, \quad U \in \mathcal{B}((0,\infty)) \otimes \mathcal{Z}, \quad 0 < t < \infty.$$

Let $\mathcal{Q}$ be the σ-field on Π_Z generated by all the subsets $\{\alpha \in \Pi_Z : \ N_\alpha(U) = k\}$, $U \in \mathcal{Z}$, $k = 0, 1, 2, \cdots$. A function $\pi : \Omega \to \Pi_Z$ is called a **point process** on Z if and only if it is $\mathcal{F}/\mathcal{Q}$-measurable. Let π be a point process in $(\Pi_Z, \mathcal{Q})$. Analogously, We define for every $\omega \in \Omega$,

$$N_\pi(U, \omega) = \sharp\{s \in \mathcal{D}(\pi(\omega)) : (s, \pi(s, \omega)) \in U\}, \quad U \in \mathcal{B}((0,\infty)) \otimes \mathcal{Z}. \tag{3.1.2}$$

In particular, we have

$$N_\pi((0,t] \times A, \omega) = \sharp\{s \in (0,t] \cap \mathcal{D}(\pi(\omega)) : \pi(s,\omega) \in A\}, \quad A \in \mathcal{Z}, \quad 0 < t < \infty. \tag{3.1.3}$$

Note that a difficulty related to this approach is that for each $\omega \in \Omega$, the domain with respect to the time t of the function $\pi(t, \omega)$ will be different. A point process π is called **finite** if $\mathbb{E}N_\pi((0,t] \times D) < \infty$, for every $0 < t < \infty$. The point process π is called **σ-finite** if there exists an increasing sequence $\{D_n\}_{n\in\mathbb{N}} \subset \mathcal{Z}$ such that $\cup_n D_n = Z$ and $\mathbb{E}N_\pi((0,t] \times D_n) < \infty$ for all $0 < t < \infty$ and $n \in \mathbb{N}$. A point process π is said to be **stationary** if and only if for every $t > 0$, π and $\theta_t\pi$ have the same probability laws. Here $\theta_t\pi$ is the shifted point process defined by

$$\begin{aligned}(\theta_t\pi)(s) &= \pi(s+t), \ s > 0; \\ \mathcal{D}(\theta_t\pi) &= \{s \in (0,\infty) : s + t \in \mathcal{D}(\pi)\}.\end{aligned}$$

Also, let us define the stopped point process $\alpha_t\pi$ by

$$(\alpha_t\pi)(s) = \pi(s), \text{ for } s \in \mathcal{D}(\alpha_t\pi);$$

$$\mathcal{D}(\alpha_t\pi) = (0,t] \cap \mathcal{D}(\pi).$$

It is easy to see that $\theta_t\pi$ and $\alpha_t\pi$ are still in Π_Z, for every $t > 0$. A point process π is said to be **renewal** if and only if it is stationary and for every $0 < t < \infty$, the point processes $\alpha_t\pi$ and $\theta_t\pi$ are independent. A point process π is said to be **adapted** to the filtration $\mathfrak{F}$ if for every $t > 0$ and $A \in \mathcal{Z}$, its counting measure $N_\pi((0,t] \times A)$ is $\mathcal{F}_t$-measurable.

A point process π is called a **Poisson point process** if and only if $N_\pi(\cdot)$ defined by (3.1.2) is a Poisson random measure on $((0,\infty) \times Z, \mathcal{B}((0,\infty)) \otimes \mathcal{Z})$, see Definition 3.1.4.

Theorem 3.1.19 (Theorem 3.1, [42]). *If a point process π is σ-finite and renewal, then for every $U \in \mathcal{B}(0,\infty) \otimes \mathcal{Z}$ with $\mathbb{E}N_\pi(U) < \infty$, the random variable $N_\pi(U)$ is Poisson distributed and for any pairwise disjoint sets $U_1, \cdots, U_n \in \mathcal{B}(0,\infty) \otimes \mathcal{Z}$, the random variables $N_\pi(U_1), \cdots N_\pi(U_n)$ are independent. In other words, N_π is a Poisson random measure.*

It can be shown that a Poisson point process is **stationary** if and only if there exists a nonnegative measure ν on $(Z, \mathcal{Z})$ such that

$$\mathbb{E}N_\pi((0,t] \times A) = t\nu(A), \quad t \geq 0, \quad A \in \mathcal{Z}. \tag{3.1.4}$$

In such a case, we say that the Poisson random measure N_π is time homogenous.

Remark 3.1.20. At this point, it should be mentioned that, in literature, some authors may use the above property (3.1.4) as the definition of stationary property of a Poisson point process. Actually, this is consistent with our earlier definition of a stationary point process. To see this, let us assume first that the Poisson point process is stationary, that is for every $r > 0$, π and $\theta_r\pi$ have the same probability laws. It follows that, for each $t \geq 0$, the random variable

$$N((0.t] \times A) = \sharp\{s \in (0,t] \cap \mathcal{D}(\pi) : \pi(s) \in A\},$$

has the same distribution as

$$\begin{aligned}\sharp\{s \in (0,t] \cap \mathcal{D}(\theta_r\pi) : \pi(s+r) \in A\} &= \sharp\{s \in (r,t+r] \cap \mathcal{D}(\pi) : \pi(s) \in A\} \\ &= N((0,t+r] \times A) - N((0,r] \times A).\end{aligned}$$

Hence we infer that $\mathbb{E}N((0,t]\times A) = \mathbb{E}N((0,t+r]\times A) - \mathbb{E}N((0,r]\times A)$, $t \geq 0$. Set $\phi(t) = \mathbb{E}N((0,t]\times A)$, $t \geq 0$. It is easy to see that ϕ is an additive function, i.e. it satisfies $\phi(t+s) = \phi(t) + \phi(s)$ for $t, s > 0$. Note that, by the local boundedness of the point process π, ϕ is bounded from above on a subset I of $(0,\infty)$ with the positive Lebesgue measure. Hence the function ϕ is of the form $\phi(t) = t\phi(1)$, $t \geq 0$, for some constiant C, see Bingham [10] Theorem 1.1.7b. This gives that

$$\mathbb{E}N((0,t] \times A) = t\mathbb{E}N((0,1] \times A).$$

Let us put $\nu(A) := \mathbb{E}N((0,1]\times A)$, $A \in \mathcal{Z}$. Since N_π is a Poisson random measure, ν is a nonnegative measure on $(Z, \mathcal{Z})$. To prove the other direction, suppose that $\mathbb{E}N_\pi((0,t]\times A) = t\nu(A)$ for $t \geq 0$ and $A \in \mathcal{Z}$. Then we have $\mathbb{E}N_\pi((t_1,t_2]\times A) = (t_2-t_1)\nu(A)$ and $\mathbb{E}N_\pi((t_1,t_2]\times A) = \mathbb{E}N_\pi((t_1+r,t_2+r]\times A)$ for $0 < t_1 < t_2 < \infty$ and $0 < r < \infty$. It follows that

$$\mathbb{E}\sum_{t_1<s\leq t_2} 1_A(\pi(s)) = \mathbb{E}\sum_{t_1+r<s\leq t_2+r} 1_A(\pi(s)) = \mathbb{E}\sum_{t_1<s\leq t_2} 1_A(\pi(s+r))$$

which shows that $\pi(\cdot)$ and $\pi(r+\cdot)$ have the same law for every $r > 0$. This shows that the process p is stationary.

Theorem 3.1.21. *Let ν be a σ-finite measure on $(Z, \mathcal{Z})$. Then there exists a Poisson point process π on $(Z, \mathcal{Z})$ with the intensity measure ν.*

Proof. Since ν is a σ-finite measure on $(Z, \mathcal{Z})$, there exists a disjoint partition $\{D_n\}_{n\in\mathbb{N}} \subset \mathcal{Z}$ of Z such that $\nu(D_n) < \infty$ for every $n \in \mathbb{N}$. Let X_i^n, $i = 1,2,\cdots$, $n = 1,2,\cdots$ be pairwise independent D_n-valued random variables with distribution $\mathbb{P}(X_i^n \in A) = \frac{\eta(A)}{\eta(D_n)}$, $A \in \mathcal{Z} \cap D_n$ defined on a probability space $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$. Note that $\lambda \otimes \nu$ is a σ-finite measure on the product space $(\mathbb{R}_+ \times Z, \mathcal{B}(\mathbb{R}_+)\otimes\mathcal{Z})$. By the Theorem 3.1.6, there exists a Poisson random measure M with $\mathbb{E}M(\cdot) = \lambda\otimes\nu(\cdot)$ defined on a probability space $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$. Then Lemma 3.1.13 tells us that for each $n \in \mathbb{N}$, $M_t^n := M(t, D_n)$, $t \geq 0$ is a Poisson process with intensity $\nu(U_n)$. Moreover, since D_n, $n \in \mathbb{N}$ are pairwise disjoint, the processes $M(t, D_n)$, $t \geq 0$, $n \in \mathbb{N}$ are mutually independent. Set

$$(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2, \mathbb{P}_1 \times \mathbb{P}_2).$$

For every $\omega = (\omega_1, \omega_2) \in \Omega$, set

$$X_i^n(\omega) = X_i^n(\omega_1)\ i = 1,\cdots,\ n = 1,\cdots \text{ and } M_t^n(\omega) = M_t^n(\omega_2),\ n = 1,\cdots.$$

It follows that X_i^n, M_n, $i = 1,\cdots$, $n = 1,\cdots$ are mutually independent. Define for every $n \in \mathbb{N}$,

$$T_i^n := \inf\{t > 0 : M(t, D_n) \geq i\},\ i = 1,2,\cdots.$$

Then for each $n \in \mathbb{N}$, $\{T_i^n\}_{i\in\mathbb{N}}$ is a sequence of stopping times and they are jump times of the Poisson processes $M(t, D_n)$, $n \in \mathbb{N}$. Let

$$\tau_1^n = T_1^n, \cdots,\ \tau_{i+1}^n = T_{i+1}^n - T_i^n, \cdots.$$

Then the random times τ_i^n, $i = 1,2,\cdots$, are independent random variables with exponential distribution, i.e. $\mathbb{P}(\tau_i^n > t) = e^{-tM(t,D_n)}$, for all $t > 0$. Now define

$$\mathcal{D}_\pi := \cup_{n=1}^\infty \cup_{i=1}^\infty \{T_i^n\}$$

and

$$\pi(T_i^n) = X_i^n,\ i,n = 1,2,\cdots.$$

Then the counting measure N_π associated to π defined by for $U \in \mathcal{B}(\mathbb{R}_+)\times\mathcal{Z}$,

$$\begin{aligned} N_\pi(U) &:= \sum_{n=1}^\infty\sum_{i=1}^\infty \delta_{(T_i^n, X_i^n)}(U) = \sum_{n=1}^\infty\sum_{i=1}^\infty 1_U(T_i^n, \pi(T_i^n)) \\ &= \sum_{s\geq 0}\sum_{n=1}^\infty\sum_{i=1}^\infty 1_{[T_i^n]} 1_U(s, \pi(s)) \end{aligned}$$

is a Poisson random measure over the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Recall that for $\omega \in \Omega$, $\mathcal{D}_{\pi(\omega)} = \cup_{n=1}^{\infty} \cup_{i=1}^{\infty} \{T_i^n(\omega)\}$, hence we infer

$$\begin{aligned} N_\pi(U)(\omega) &= \sum_{s\geq 0} \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} 1_{(T_i^n(\omega),\omega)}(s,\omega) 1_U(s, \pi(s,\omega)) \\ &= \sum_{s\in(0,\infty)\cap \mathcal{D}_{\pi(\omega)}} 1_U(s, \pi(s,\omega)). \end{aligned}$$

For the detialed proof, we refer readers to Theorem 54 [70]. In particular, for $B \in \mathcal{Z}$, we have

$$\begin{aligned} N_\pi((s,t] \times B) &= \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} 1_{[T_i^n]} 1_B(\pi(r)) 1_{(s,t]}(r) \\ &= \sharp\{r \in (s,t] \cap \mathcal{D}_\pi : \pi(r) \in B\}. \end{aligned}$$

□

We shall use the notation $N_\pi(t, A)$ as an abbreviation for the counting measure $N_\pi((0,t] \times A)$. In such a case we call $\nu(\cdot) = \mathbb{E}N_\pi(1,\cdot)$ the **intensity measure** (or characteristic measure) of the stationary Poisson point process π and the Poisson random measure N_π is called **time homogenous**. If the Poisson point process π is σ-finite, the intensity measure associated to π is a σ-finite measure. From now on, we suppose that π is a σ-finite stationary and adapted Poisson point process. For simplicity of notation, assuming that there will not be any confusion, a Poisson random measure associated with a Poisson point process will be often denoted by N instead of N_π. We use the notation $\tilde{N}(t, A) = N(t, A) - t\nu(A)$, $t \geq 0$, $A \in \mathcal{Z}$ to denote the compensated Poisson random measure associated with the Poisson point process π.

3.1.2 Lévy Processes

Let $(E; \mathcal{B}(E))$ be a separable Banach space with norm $\|\cdot\|$.

Definition 3.1.22 (**E-valued Lévy process**). A càdlàg process $L = (L_t)_{t\geq 0}$ with values in E is called a Lévy process if and only if

(1) $L_0 = 0$ a.s.

(2) L has independent increments, that is for every increasing sequence of times $t_0 < t_1 < \cdots < t_n$, the random vairables

$$L_{t_0}, L_{t_1} - L_{t_0}, \cdots, L_{t_n} - L_{t_{n-1}}$$

are independent.

(3) L has stationary increments, that is $L_{t+h} - L_t$ has the same distribution as L_t, for every $h \geq 0$.

(4) L is stochastically continuous, that is for all $\varepsilon > 0$ and all $s \geq 0$,

$$\lim_{t \to s} \mathbb{P}(\|L_t - L_s\| > \varepsilon) = 0. \tag{3.1.5}$$

Remark 3.1.23. (1) Even if we don't impose the càdlàg property in the definition of a Lévy process, one can always show that every E-valued process L satisfying the above four conditions (1)-(4) has a càdlàg modification, see Theorem 16.1 in [29]. Moreover, it can be seen that the càdlàg modification of L is also a Lévy process, i.e. satisfies the conditions (1)-(4). So without loss of generality, we can always assume the càdlàg property in the definition of Lévy process.

(2) On the basis of conditions (1), (2) and (3), the condition (3.1.5) is equivalent to the following two conditions

$$\lim_{h \to 0} \mathbb{P}(\|L_{t+h} - L_t\| > \varepsilon) = 0,$$
$$\lim_{t \searrow 0} \mathbb{P}(\|L_t\| > \varepsilon) = 0.$$

(3) Suppose that L is an E-valued càdlàg Levy process. The stochastic continuity implies that for every given time t, $\mathbb{P}(L_t \neq L_{t-}) = 0$.

Lemma 3.1.24. *Let $f : [0, \infty) \to E$ be a càdlàg function. Then for every $t > 0$ and $\varepsilon > 0$, the set*

$$B_\varepsilon = \{s \in [0,t] : \|f(s) - f(s-)\| > \varepsilon\}$$

is finite. Consequently, f has at most countable jumps on $[0,t]$. Moreover, f is bounded on $[0,t]$.

Proof. Let $\varepsilon > 0$. We will show it by contradiction. Suppose that the set B_ε has infinite number of points. Since the interval $[0,t]$ is compact, the set B_ε has a limit point in B_ε. Assume that p is this limit point in B_ε. By the *càdlàg* property of f, $f(p-)$ and $f(p+)$ both exist. Thus for $\frac{\varepsilon}{3} > 0$, there exists a number δ so that $s \in (p-\delta, p)$ implies $\|f(s) - f(p)\| < \frac{\varepsilon}{3}$, and $s \in (p, p+\delta)$ implies $\|f(s) - f(p)\| < \frac{\varepsilon}{3}$. Then for every $r \in (p-\delta, p)$, we can find two sequences $\{u_n\}_{n\in\mathbb{N}}, \{v_n\}_{n\in\mathbb{N}} \subset (p-\delta, p)$ such that $u_n < r \leq v_n$, $u_n \to r$ and $v_n \to r$ as $n \to \infty$. We have for each n,

$$\|f(u_n) - f(v_n)\| \leq \|f(u_n) - f(p)\| + \|f(v_n) - f(p)\| < \frac{2\varepsilon}{3}.$$

Let $n \to \infty$. It follows that for every $r \in (p-\delta, p)$,

$$\|f(r-) - f(r)\| \leq \frac{2\varepsilon}{3}.$$

Similarly, we can show that for every $r \in (p, p+\delta)$,

$$\|f(r-) - f(r)\| \leq \frac{2\varepsilon}{3}.$$

Thus for every $r \in (p-\delta, p) \cup (p, p+\delta)$, we find that $r \notin B_\varepsilon$. This contradicts the fact that p is a limit point in B_ε.
Now we will show that f has at most countable jumps on $[0,t]$. As the sets $B_{\frac{1}{n}}$ are increasing as $n \to \infty$, we find the set

$$\{s \in [0,t] : f(s-) \neq f(s)\} = \bigcup_{n=1}^{\infty} \{s \in [0,t] : \|f(s) - f(s-)\| > \frac{1}{n}\} = \bigcup_{n=1}^{\infty} B_{\frac{1}{n}}.$$

Since each set $B_{\frac{1}{n}}$ is finite for every $n \in \mathbb{N}$, the set $\{s \in [0,t] : f(s-) \neq f(s)\}$ is countable.
For the boundedness of f on $[0,t]$, we first suppose that f is not bounded on $[0,t]$. Then we can find a sequence of numbers $\{s_n\}_{n\in\mathbb{N}} \subset [0,t]$ such that $f(s_n)$ converges to infinite as $n \to \infty$. Since the interval $[0,t]$ is compact, the infinite set $\{s_n\}$ has a limit point s in $[0,t]$. This gives that $\lim_{n\to\infty} s_n = s$. Thus we can find a subsequence $\{s_{n_k}\}_{k=1}^{\infty}$ of $\{s_n\}$ such that $s_{n_k} \geq s$ for each $k \in \mathbb{N}$ or $s_{n_k} < s$ for each $k \in \mathbb{N}$. Thus the sequence $\{f(s_{n_k})\}_{k\in\mathbb{N}}$ converges to $f(s)$ or $f(s-)$ as $k \to \infty$, where $f(s)$ and $f(s-)$ both exist by the *càdlàg* property, so that the subsequence $\{f(s_{n_k})\}$ does not converge to infinite. Hence f is bounded on $[0,t]$. □

Let L be an E-valued Lévy process. Since every path of L is càdlàg, for every $\omega \in \Omega$, by Lemma 3.1.24, $L_s(\omega)$ has at most a countable number of jumps on $[0,t]$. Futhermore, since for every $\omega \in \Omega$,

$$\{s \in (0,\infty) : L_s(\omega) \neq L_{s-}(\omega)\} = \cup_{m\geq 1}\{s \in (0,m] : L_s(\omega) \neq L_{s-}(\omega)\},$$

we find that $L_s(\omega)$ has at most countable jumps over $(0,\infty)$. Thus in view of Section 3.1.1, it is easy to see that for every $\omega \in \Omega$, $\triangle L_\cdot(\omega)$ is a point function in $(E \setminus \{0\}, \mathcal{B}(E \setminus \{0\}))$. Here $\mathcal{B}(E \setminus \{0\})$ is the trace σ-field on $E \setminus \{0\}$ of the Borel σ-field of $\mathcal{B}(E)$, namely,

$$\mathcal{B}(E \setminus \{0\}) := \{(E \setminus \{0\}) \cap A : A \in \mathcal{B}(E)\}$$

which is a σ-field on $E \setminus \{0\}$. Let us define

$$N(U,\omega) = \sharp\{s \in (0,\infty) : (s, \triangle L_s(\omega)) \in U\}, \quad U \in \mathcal{B}((0,\infty)) \otimes \mathcal{B}(E \setminus \{0\}), \ \omega \in \Omega. \tag{3.1.6}$$

Note that since processes $(L_t)_{t>0}$ and $(L_{t-})_{t>0}$ are both progressively measurable, the process $(\triangle L_t)_{t>0}$ is also progressively measurable, and hence it is measurable. From Lemma 3.6.1, it follows that for every $s > 0$, the mapping $\omega \mapsto \triangle L_s(\omega)$ is $\mathcal{F}$-measurable. For every $U \in E \setminus \{0\}$, take a set $\{\alpha \in \Pi_Z : N_\alpha(U) = k\}$ from $\mathcal{Q}$. Then we have

$$\begin{aligned}\{\omega : \triangle L_\cdot(\omega) \in \{\alpha \in \Pi_Z : N_\alpha(U) = k\}\} &= \{\omega : \triangle L_\cdot(\omega) \in \Pi_Z : N_{\triangle L_\cdot(\omega)}(U) = k\} \\ &= \bigcup_{s\in\{r:\sharp\{(r,\triangle L_r(\omega))\in U\}=k\}} \{\omega : (s, \triangle L_s(\omega) \in U\}\}\end{aligned}$$

which is a finite union of sets in $\mathcal{F}$. Hence we infer that $\{\omega : \triangle L_\cdot(\omega) \in \{\alpha \in \Pi_Z : N_\alpha(U) = k\}\} \in \mathcal{F}$ which shows that $\triangle L : \Omega \to \Pi_Z$ is $\mathcal{F}/\mathcal{Q}$-measurable. Therefore, $\triangle L$ is a point process. Let us take $t > 0$. Let $\{h_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of positive numbers such that $0 < h_n < t$, $n \in \mathbb{N}$ and $\lim_{n\to\infty} h_n = 0$. Since the Lévy process L has independent and stationary increments, for every $s > 0$ and h_n, $n \in \mathbb{N}$, the random variables $L_s - L_{s-h_n}$ and $L_{t+s} - L_{t+s-h_n}$ are stationary and independent. Then the limits $L_s - L_{s-}$ and $L_{t+s} - L_{(t+s)-}$ inherit the stationary and independence properties from the random variables $L_s - L_{s-h_n}$ and $L_{t+s} - L_{t+s-h_n}$, $n \in \mathbb{N}$. Hence we infer that the point process $\triangle L$ is stationary and renewal. Furthermore, by taking $D_n = \{x \in E : \|x\| > \frac{1}{n}\}$, we see from Lemma 3.1.24 that the point process $\triangle L$ is σ-finite. On the basis of Theorem 3.1.19, we know that N defined by (3.1.6) is a stationary Poisson random measure with a nonnegative measure $\nu(\cdot)$ such that

$$\mathbb{E}N((0,t] \times A) = t\nu(A), \; t > 0, \; A \in \mathcal{B}(E \setminus \{0\}).$$

We say that a set $A \in \mathcal{B}(E \setminus \{0\})$ is **bounded away** from 0 if and only if $0 \in (\bar{A})^c$, where as usual $\bar{A}$ is the closure of the set A. Set

$$\mathcal{A} := \{A \in \mathcal{B}(E \setminus \{0\}) : 0 \notin \bar{A}\}. \tag{3.1.7}$$

Take $A \in \mathcal{A}$. Then there exists $n \in \mathbb{N}$ such that $A \subset \{x \in E : \|x\| > \frac{1}{n}\}$. By the càdlàg regularity, see Lemma 3.1.24, for every $\omega \in \Omega$, the function $L_\cdot(\omega)$ has finite numbers of jumps in the set A on $(0,t]$, $t > 0$. Thus we have

$$N((0,t] \times A, \omega) = \#\{r \in (0,t] : \triangle L_r \in A\} = \sum_{s<r\le t} 1_A(\triangle L_r) < \infty, \; \omega \in \Omega. \tag{3.1.8}$$

This random measure counts the number of jump times of the process L between times s and t with their jump sizes belonging to A. For simplicity of notation, we use notation $N(t, U)$ instead of $N((0,t] \times U)$.

Remark 3.1.25. In fact the family $\mathcal{A}$ is a ring, see Definition 2.1.17 and $\mathcal{B}(E \setminus \{0\})$ is a σ-field generated by $\mathcal{A}$.

To show that $\mathcal{A}$ is a ring, take $A, B \in \mathcal{A}$. Then $A = (E \setminus \{0\}) \cap A^*$, for some $A^* \in \mathcal{B}(E)$ and $B = (E \setminus \{0\}) \cap B^*$, for some $B^* \in \mathcal{B}(E)$ with $0 \notin \bar{A}$ and $0 \notin \bar{B}$. Observe that

$$\begin{aligned} A \cup B &= \big((E \setminus \{0\}) \cap A^*\big) \cup \big((E \setminus \{0\}) \cap B^*\big) \\ &= (E \setminus \{0\}) \cap (A^* \cup B^*) \in \mathcal{B}(E \setminus \{0\}), \end{aligned}$$

since $A^* \cap B^* \in \mathcal{B}(E)$. Meanwhile, since $0 \notin \bar{A}$ and $0 \notin \bar{B}$, $0 \notin \bar{A} \cup \bar{B} = \overline{A \cup B}$. Hence $A \cup B \in \mathcal{A}$. Further we find

$$\begin{aligned} A \setminus B &= \big((E \setminus \{0\}) \cap A^*\big) \setminus \big((E \setminus \{0\}) \cap B^*\big) \\ &= \big(((E \setminus \{0\}) \cap A^*) \setminus (E \setminus \{0\})\big) \cup \big(((E \setminus \{0\}) \cap A^*) \setminus B^*\big) \\ &= \big((E \setminus \{0\}) \cap A^*\big) \setminus \big((E \setminus \{0\}) \cap B^*\big) \in \mathcal{B}(E \setminus \{0\}). \end{aligned}$$

since $(E \setminus \{0\}) \cap A_1, (E \setminus \{0\}) \cap B_1 \in \mathcal{B}(E \setminus \{0\})$. Since $0 \notin \bar{A}$, $0 \notin \overline{A \setminus B} \subset \bar{A}$. This shows that $A \setminus B \in \mathcal{A}$. In conclusion, the family $\mathcal{A}$ is a ring. To prove the other claim that $\mathcal{B}(E \setminus \{0\})$ is the

σ-field generated by $\mathcal{A}$, first we note that $\mathcal{A} \subset \mathcal{B}(E \setminus \{0\})$. So take $F \in \mathcal{B}(E \setminus \{0\})$ and $F \notin \mathcal{A}$. Then the set F is given by $F = (E \setminus \{0\}) \cap G$ for some $G \in \mathcal{B}(E)$ and $0 \in \bar{F}$. Construct a sequence of sets $\{F_n\}_{n \in \mathbb{N}}$ in $\mathcal{A}$ by

$$F_0 = G \cap \{x \in E : \|x\| > 1\};$$
$$F_n = G \cap \{x \in E : 1/(n+1) < \|x\| \leq 1/n\}.$$

It's easy to see that $F = \cup_{n=0}^{\infty} F_n$. This implies that $\mathcal{A}$ is the generator of the σ-field $\mathcal{B}(E \setminus \{0\})$.

Let us now define the compensated Poisson random measure of the Lévy process L by

$$\tilde{N}(B) = N(B) - \int_0^{\infty} \int_Z 1_B(s,z)\nu(dz)ds, \quad B \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(E \setminus \{0\}).$$

Let F be a separable Banach space with the corresponding Borel σ-field $\mathcal{B}(F)$. Let $f : E \to F$ be a $\mathcal{B}(E)/\mathcal{B}(F)$-measurable function. Take $A \in \mathcal{A}$, where $\mathcal{A}$ is a ring defined by (3.1.7). Recall that $N(t, A) < \infty$ a.s. Hence we may define the Poisson integral on A of this deterministic function f by

$$\Big(\int_A f(x)N(t,dx)\Big)(\omega) = \sum_{x \in A} f(x)N(t,\{x\})(\omega), \quad \omega \in \Omega.$$

Since by the definition of N, $N(t, \{x\}) \neq 0$ if and only if there exists $u \in (0, t]$ such that $\triangle L_u = x$, we infer that for every $\omega \in \Omega$,

$$\begin{aligned}\int_A f(x)N(t,dx)(\omega) &= \sum_{x \in A} f(x) \sum_{0<s\leq t} 1_{\{x\}}(\triangle L_s)(\omega) \\ &= \sum_{0<s\leq t} \sum_{x \in A} f(x)1_{\{x\}}[(\triangle L_s)(\omega)] \\ &= \sum_{0<s\leq t} f(\triangle L_s(\omega))1_A(\triangle L_s(\omega)).\end{aligned} \tag{3.1.9}$$

Let $\mathcal{L}^1(E, \nu; F)$ be the space of all $\mathcal{B}(E)/\mathcal{B}(F)$-measurable functions $f : E \to F$ such that

$$\int \|f(x)\| \nu(dx) < \infty.$$

It is natural to define the compensated Poisson integral for function $f \in \mathcal{L}^1(E, \nu; F)$ by

$$\int_A f(x)\tilde{N}(t,dx) := \int_A f(x)N(t,dx) - t\int_A f(x)\nu(dx). \quad = \sum_{0<s\leq t} f(\triangle L_s)1_A(\triangle L_s) - t\int_A f(x)\nu(dx),$$

where the latter term is understood as a Bochner integral, see Section 3.2.3. For more detials, we refer the reader to [73] where $E = \mathbb{R}^d$ and see also [28], [1] where E is a separable Banach space.

Theorem 3.1.26 (Lévy-Itô decomposition [28]). *Let $L := (L_t)_{t\geq 0}$ be a Lévy process on a separable martingale type 2 Banach space E and ν be its Lévy measure of the Poisson random measure N defined by* (3.1.8) *satisfying*

$$\int_{E\setminus\{0\}} \frac{\|x\|^2}{1+\|x\|^2}\nu(dx) < \infty. \tag{3.1.10}$$

Then there exist an E-valued Brownian Motion which is independent of N and $\gamma \in E$ such that for all $t \geq 0$,

$$L_t = \gamma t + B_t + \int_0^t \int_{\|x\|\geq 1} xN(ds, dx) + \lim_{\varepsilon\downarrow 0} \int_0^t \int_{\varepsilon\leq\|x\|<1} x\tilde{N}(ds, dx).$$

Remark 3.1.27. (1) The integral $\int_0^t \int_{\varepsilon\leq\|x\|<1} x\tilde{N}(ds, dx)$ is given by

$$\int_0^t \int_{\varepsilon\leq\|x\|<1} x\tilde{N}(ds, dx) = \int_0^t \int_{\varepsilon\leq\|x\|<1} xN(ds, dx) - \int_0^t \int_{\varepsilon\leq\|x\|<1} x\nu(dx)ds.$$

The limit $\lim_{\varepsilon\downarrow 0} \int_0^t \int_{\varepsilon\leq\|x\|<1} x\tilde{N}(ds, dx)$ is usually denoted by $\int_0^t \int_{0<\|x\|<1} x\tilde{N}(ds, dx)$.

(2) The term $\int_0^t \int_{0<\|x\|<1} x\tilde{N}(ds, dx)$ in the above decomposition is usually called the compensated sum of small jumps and the last term $\int_0^t \int_{\|x\|>1} xN(ds, dz)$ is understood as the "big jumps" part. There are some other ways to get an equivalent version of the Lévy-Itô decomposition by rewriting the "big jumps" term with jumps bigger than K. Let $K > 0$. Then we can get a version of the Lévy-Itô decomposition

$$L_t = \gamma_K t + B_t + \int_0^t \int_{0<\|x\|<K} x\tilde{N}(ds, dx) + \int_0^t \int_{\|x\|\geq K} xN(ds, dx).$$

Here $\gamma_K = \gamma + \int_{1\leq\|x\|<K} x\nu(dz)ds$, if $K > 1$ and $\gamma_K = \gamma - \int_{K\leq\|x\|<1} x\nu(dz)ds$, if $0 < K < 1$.

(3) If ν satisfies the additional condition that $\int_{\|x\|>0} \|x\|\nu(dx) < \infty$, then the compensated sum of small jumps can be written as

$$\int_0^t \int_{0<\|x\|<1} x\tilde{N}(ds, dx) = \int_0^t \int_{0<\|x\|<1} xN(ds, dx) - \int_0^t \int_{0<\|x\|<1} x\nu(dz)ds, \quad t \geq 0. \tag{3.1.11}$$

This is because both two terms on the right side of above identity are finite as respectively the Lebesgue-Bochner integral and the Bochner integral.
Hence one can write

$$L_t = \gamma' t + B_t + \int_0^t \int_{0<\|x\|<\infty} xN(ds, dx).$$

where

$$\gamma' = \gamma - \int_{0<\|x\|<1} x\nu(dz)ds.$$

It has to be borne in mind that, if the condition $\int_{0<\|x\|<1} \|x\|\nu(dx) < \infty$ is not satisfied, the two terms

$$\int_0^t \int_{0<\|x\|<1} xN(ds, dx) \text{ and } \int_0^t \int_{0<\|x\|<1} x\nu(dz)ds$$

may diverge, but the compensated sum remains convergent. In such a case the identity (3.1.11) doesn't hold any more.

(4) Later in the Section 3.4, see Theorem 3.4.9, we will show that the integral $\int_0^t \int_{0<\|x\|<1} x\tilde{N}(ds,dx)$ in the Lévy-Itô decomposition, which is defined to be the limit $\lim_{\varepsilon\downarrow 0} \int_0^t \int_{\varepsilon\le\|x\|<1} x\tilde{N}(ds,dx)$, coincides with the stochastic integral $\int_0^t \int_{0<\|x\|<1} x\tilde{N}(ds,dx)$ defined in this thesis.

(5) Recall from (3.1.8) that for every $t \geq 0$,

$$\mathbb{E}N((0,t] \times \{\|x\| > 1\}) < \infty.$$

In other words, we have $\nu(\{\|x\| > 1\}) < \infty$. Hence in view of Proposition 3.4.5, if we impose the additional assumption that $\int_{\|x\|>1} \|x\|^2 \nu(dx) < \infty$, then the Lévy process L can be written in the following form, $t > 0$,

$$L_t = \gamma' t + B_t + \int_0^t \int_{E\setminus\{0\}} x\tilde{N}(ds,dx),$$

where $\gamma' = \gamma + \int_{\|x\|>1} x\nu(dx)$ and $\int_0^t \int_{E\setminus\{0\}} x\tilde{N}(ds,dx)$ is the stochastic integral defined in the thesis.

3.2 Stochastic Integrals w.r.t. Compensated Poisson Random Measures

Let E be an martingale type p Banach space with its corresponding σ-field $\mathcal{B}(E)$. Let $(\Omega, \mathcal{F}, \mathfrak{F}, \mathbb{P})$ be a complete filtered probability space.

3.2.1 Predictability and Progressive Measurability

Definition 3.2.1 (Predictability). Let $\hat{\mathcal{P}}$ denote the σ-field on $\mathbb{R}_+ \times \Omega \times Z$ generated all $\mathbb{R}$-valued functions $g : \mathbb{R}_+ \times \Omega \times Z \to \mathbb{R}$ satisfying the following properties

(1) for every $t \geq 0$, the mapping $\Omega \times Z \ni (\omega, z) \mapsto g(t,\omega,z) \in \mathbb{R}$ is $\mathcal{Z} \otimes \mathcal{F}_t/\mathcal{B}(\mathbb{R})$-measurable;

(2) for every $(\omega, z) \in \Omega \times Z$, the path $\mathbb{R}_+ \ni t \mapsto g(t,\omega,z) \in \mathbb{R}$ is left-continuous.

We say that an E-valued function $g : \mathbb{R}_+ \times \Omega \times Z \to E$ is **$\mathfrak{F}$-predictable** if it is $\hat{\mathcal{P}}/\mathcal{B}(E)$-measurable.

Proposition 3.2.2. *$\hat{\mathcal{P}} = \mathcal{P} \otimes \mathcal{Z}$. Furthermore, they are both equal to the σ-field generated by a family $\hat{\mathcal{R}}$ of the following form*

$$\hat{\mathcal{R}} = \{\{0\} \times F \times B : F \in \mathcal{F}_0, B \in \mathcal{Z}\} \cup \{(s,t] \times F \times B : F \in \mathcal{F}_s, B \in \mathcal{Z}, 0 \leq s < t < \infty\}.$$

In particular, $\hat{R}$ is a semi-ring.

Proof. Recall that the predictable σ-field $\mathcal{P}$, see Theorem 2.1.15, is generated by the following set

$$\mathcal{R} = \{\{0\} \times F, F \in \mathcal{F}_0\} \cup \{(s,t] \times F, F \in \mathcal{F}_s, 0 \leq s < t < \infty\}.$$

That is $\mathcal{P} = \sigma(\mathcal{R})$. Therefore, we infer that

$$\mathcal{P} \otimes \mathcal{Z} = (\sigma(\mathcal{R})) \otimes \mathcal{Z}.$$

Also, we know that $\mathcal{R} \times \mathcal{Z} = \hat{\mathcal{R}}$. Thus $\sigma(\mathcal{R} \times \mathcal{Z}) = \sigma(\hat{\mathcal{R}})$. In order to prove that $\mathcal{P} \otimes \mathcal{Z} = \sigma(\hat{\mathcal{R}})$, it suffices to show that $(\sigma(\mathcal{R})) \otimes \mathcal{Z} = \sigma(\mathcal{R} \times \mathcal{Z})$. The inclusion $\sigma(\mathcal{R} \times \mathcal{Z}) \subset (\sigma(\mathcal{R})) \otimes \mathcal{Z}$ is clear since $\mathcal{R} \subset \sigma(\mathcal{R})$. For the inclusion $(\sigma(\mathcal{R})) \otimes \mathcal{Z} \subset \sigma(\mathcal{R} \times \mathcal{Z})$, we consider the following family

$$\mathcal{A} = \{A \in \sigma(\mathcal{R}) : A \times B \in \sigma(\mathcal{R} \times \mathcal{Z}), B \in \mathcal{Z}\}.$$

We claim that $\mathcal{A}$ is a σ-field. Indeed, we find out that

(i) Since $[0, n) \times \Omega \in \mathcal{R}$, for all $B \in \mathcal{Z}$, $[0, n) \times \Omega \times B \in \mathcal{R} \times \mathcal{Z}$, so $\mathbb{R}_+ \times \Omega \times B = \cup_{n \in \mathbb{N}}\big([0, n) \times \Omega \times B\big) \in \sigma(\mathcal{R} \times \mathcal{Z})$. Thus $\mathbb{R}_+ \times \Omega \in \mathcal{A}$.

(ii) Let $A \in \mathcal{A}$. Then $A \times B \in \sigma(\mathcal{R} \times \mathcal{Z})$, for every $B \in \mathcal{Z}$. Observe that $A^c \times B = (\mathbb{R}_+ \times \Omega \times B) \backslash (A \times B)$. Hence $A^c \times B \in \sigma(\mathcal{R} \times \mathcal{Z})$. It follows that $A^c \in \mathcal{A}$.

(iii) Take a sequence $A_1, A_2, \cdots$ of sets in $\mathcal{A}$. Then for every $B \in \mathcal{Z}$, $A_j \times B \in \sigma(\mathcal{R} \times \mathcal{Z})$. It follows that

$$(\cup_{j \in \mathbb{N}} A_j) \times B = \cup_{j \in \mathbb{N}} (A_j \times B) \in \sigma(\mathcal{R} \times \mathcal{Z}).$$

Therefore $\cup_{j \in \mathbb{N}} A_j \in \mathcal{A}$.

This shows that $\mathcal{A}$ is a σ-field. Take $D \in \mathcal{R}$. Then $D \in \sigma(\mathcal{R})$ and for all $B \in \mathcal{Z}$, $D \times B \in \mathcal{R} \times \mathcal{Z} \subset \sigma(\mathcal{R} \times \mathcal{Z})$. So by the definition of family $\mathcal{A}$, we infer that $D \in \mathcal{A}$. Hence $\mathcal{R} \subset \mathcal{A} \subset \sigma(\mathcal{R})$. Since $\mathcal{A}$ is a σ-field, we conclude that $\mathcal{A} = \sigma(\mathcal{R})$. Therefore, by the definition of the σ-field $\mathcal{A}$, $A \times B \in \sigma(\mathcal{R} \times \mathcal{Z})$, for all $A \in \sigma(\mathcal{R})$ and $B \in \mathcal{Z}$. Hence $\sigma(\mathcal{R}) \times \mathcal{Z} \subset \sigma(\mathcal{R} \times \mathcal{Z})$. This concludes the proof of equality $(\sigma(\mathcal{R})) \otimes \mathcal{Z} = \sigma(\mathcal{R} \times \mathcal{Z})$.

Note that $\sigma(\hat{\mathcal{R}}) = \sigma(\tilde{\mathcal{R}})$, where

$$\tilde{\mathcal{R}} = \{\{0\} \times F', F' \in \mathcal{F}_s \otimes \mathcal{Z}\} \cup \{(s, t] \times F', F' \in \mathcal{F}_s \otimes \mathcal{Z}, 0 \leq s < t < \infty\}.$$

The inclusion $\sigma(\hat{\mathcal{R}}) \subset \sigma(\tilde{\mathcal{R}})$ is clear as $\hat{\mathcal{R}} \subset \tilde{\mathcal{R}}$. The other inclusion $\sigma(\tilde{\mathcal{R}}) \subset \hat{\mathcal{R}}$ follows from a similar argument as above by constructing a set $\mathcal{F}'_s = \{F' \in \mathcal{F}_s \otimes \mathcal{Z} : (s, t] \times F' \in \sigma(\hat{\mathcal{R}}), s \leq t\}$. We can show that $\mathcal{F}'_s$ is a σ-field for each $s \geq 0$. Moreover, we find that $\mathcal{F}_s \times \mathcal{Z} \subset \mathcal{F}'_s \subset \mathcal{F}_s \otimes \mathcal{Z}$. Since $\mathcal{F}'_s$ is a σ-filed, $\mathcal{F}_s \times \mathcal{Z} \subset \mathcal{F}'_s$. This gives that for every $F' \in \mathcal{F}_s \otimes \mathcal{Z}$ and $s \leq t$, $(s, t] \times F' \in \sigma(\hat{\mathcal{R}})$. So $\tilde{R} \subset \sigma(\hat{\mathcal{R}})$. Hence we infer that $\sigma(\hat{\mathcal{R}}) = \sigma(\tilde{\mathcal{R}})$.
Clearly, the indicator function $1_{(s,t]}(u) 1_{F'}(\omega, x)$, $(u, \omega, x) \in \mathbb{R}_+ \times \Omega \times Z$ is left-continuous for all $(\omega, x) \in \Omega \times Z$ and it is $\mathcal{F}_s \otimes \mathcal{Z}$-measurable. Thus it follows from the definition of $\mathfrak{F}$-predictability that the process $1_{(s,t]}(u) 1_{F'}(\omega, x)$ is $\mathfrak{F}$-predictable. This implies that $\sigma(\tilde{\mathcal{R}}) \subset \hat{\mathcal{P}}$. It remains to show that $\hat{\mathcal{P}} \subset \sigma(\tilde{\mathcal{R}})$. To prove this, it is enough to show that every $\mathbb{R}$-valued function X satisfying conditions (1) and (2) in the definition is $\sigma(\tilde{\mathcal{R}})$-measurable. Let us construct the following sequence of functions, $n \in \mathbb{N}$,

$$X^n(t, \omega, z) = 1_{\{0\}}(t) X(0, \omega, z) + \sum_{k=0}^{\infty} 1_{(\frac{k}{2^n}, \frac{k+1}{2^n}]}(t) X(\frac{k}{2^n}, \omega, z), \quad (t, \omega, x) \in \mathbb{R}_+ \times \Omega \times Z.$$

By the left-continuity of X, we infer that $X^n(t, \omega, z)$ converges to $X(t, \omega, z)$ for every $(t, \omega, z) \in \mathbb{R}_+ \times \Omega \times Z$. Take $B \in \mathcal{B}(\mathbb{R})$. We find out that

$$\begin{aligned}\{(t, \omega, z) :& X^n(t, \omega, z) \in B\} \\ &= (\{0\} \times \{(\omega, z) : X(0, \omega, z) \in B\}) \bigcup \Big(\bigcup_{k=0}^{\infty} (\frac{k}{2^n}, \frac{k+1}{2^n}] \times \{(\omega, z) : X(\frac{k}{2^n}, \omega, z) \in B\} \Big).\end{aligned}$$

Since for each t, $(\omega, z) \mapsto X(t, \omega, z)$ is $\mathcal{F}_t \otimes \mathcal{Z}$-measurable, so $X(\frac{k}{2^n}, \omega, z)$ is $\mathcal{F}_{\frac{k}{2^n}} \otimes \mathcal{Z}$-measurable for $k = 0, 1, 2, \cdots$. Hence the sets $\{(\omega, z) : X(\frac{k}{2^n}, \omega, z) \in B\} \in \mathcal{F}_{\frac{k}{2^n}} \otimes \mathcal{Z}$ for $k = 0, 1, 2, \cdots$. Therefore, the set $\{(t, \omega, z) : X^n(t, \omega, z) \in B\}$ as a countable union of sets from $\tilde{\mathcal{R}}$ is also in the σ-field $\sigma(\tilde{\mathcal{R}})$ of $\tilde{\mathcal{R}}$. This implies that X^n is $\sigma(\tilde{\mathcal{R}})$-measurable for each $n \in \mathbb{N}$. Therefore, the limit X is also $\sigma(\tilde{\mathcal{R}})$-measurable. This shows that $\hat{\mathcal{P}} \subset \sigma(\tilde{\mathcal{R}})$. Recall that $\sigma(\tilde{\mathcal{R}}) = \sigma(\hat{\mathcal{R}})$. In conclusion, we have $\hat{\mathcal{P}} = \sigma(\hat{\mathcal{R}})$ which completes our proof. The proof that $\hat{\mathcal{R}}$ is a semi-ring goes the same as in the proof of Proposition 2.1.18. □

Definition 3.2.3 ($\mathfrak{F}$-progressively measurability). An E-valued function $g : \mathbb{R}_+ \times \Omega \times Z \to E$ is called $\mathfrak{F}$-progressively measurable if the mapping

$$(s, \omega, x) \mapsto g(s, \omega, z) : [0, t] \times \Omega \times Z \to E$$

is $\mathcal{B}([0, t]) \otimes \mathcal{F}_t \otimes \mathcal{Z}/\mathcal{B}(E)$-measurable for each $t \geq 0$.

Remark 3.2.4. Set $Z = E$ and take S a singleton, e.g. $Z = \{z_0\}$. Define two functions

$$\begin{aligned} g &: \mathbb{R}_+ \times \Omega \ni (t, \omega) \to g(t, \omega) \in E \\ \tilde{g} &: \mathbb{R}_+ \times \Omega \times Z \ni (t, \omega, z_0) \to \tilde{g}(t, \omega, z_0) = g(t, \omega) \in E. \end{aligned}$$

Then one can see that g is progressively measurable if and only if $\tilde{g}$ is $\mathfrak{F}$-progressively measurable.

Now we state results analogous to Lemma 2.1.7.

Proposition 3.2.5. *Define two families of sets*

$$\begin{aligned} \mathcal{BFZ} &= \{A \subset \mathbb{R}_+ \times \Omega \times Z : A \cap ([0, t] \times \Omega \times Z) \in \mathcal{B}([0, t]) \otimes \mathcal{F}_t \otimes \mathcal{Z}\} \\ \mathcal{BF} &= \{A \subset \mathbb{R}_+ \times \Omega : A \cap ([0, t] \times \Omega) \in \mathcal{B}([0, t]) \otimes \mathcal{F}_t\}. \end{aligned}$$

Then $\mathcal{BFZ}$ and $\mathcal{BF}$ are σ-fields. A process $X : \mathbb{R}_+ \times \Omega \to E$ is progressively measurable if and only if X is $\mathcal{BF}$-measurable. A function $X : \mathbb{R}_+ \times \Omega \times Z \to E$ is $\mathfrak{F}$-progressively measurable if and only if X is $\mathcal{BFZ}$-measurable. Furthermore, $\mathcal{BF} \otimes \mathcal{Z} \subset \mathcal{BFZ}$.

Proof. The proof follows the lines of the proof of Lemma 2.1.7. So we omit the first part of the proof here. To show the inclusion $\mathcal{BF} \otimes \mathcal{Z} \subset \mathcal{BFZ}$, let us take $A \in \mathcal{BF}$ and $B \in \mathcal{Z}$. We will show that $A \times B \in \mathcal{BFZ}$. Indeed, since $A \in \mathcal{BF}$, $A \subset \mathbb{R}_+ \times \Omega$ and $A \cap ([0, t] \times \Omega) \in \mathcal{B}([0, t]) \otimes \mathcal{F}_t$, for every $t \geq 0$. Thus $A \times B \subset \mathbb{R}_+ \times \Omega \times Z$ and

$$\begin{aligned} (A \times B) \cap ([0, t] \times \Omega \times Z) &= (A \cap ([0, t] \times \Omega)) \times (B \cap Z) \\ &= (A \cap ([0, t] \times \Omega)) \times B \in \mathcal{B}([0, t]) \otimes \mathcal{F}_t \otimes \mathcal{Z}, \quad t \geq 0. \end{aligned}$$

Therefore, $A \times B \in \mathcal{BFZ}$. □

Remark 3.2.6. In view of Theorem 2.1.19, Proposition 3.2.2 and Propostion 3.2.5, we have $\hat{\mathcal{P}} = \mathcal{P} \otimes \mathcal{Z} \subset \mathcal{BF} \otimes \mathcal{Z} \subset \mathcal{BFZ}$.

Definition 3.2.7. Let $\mathcal{K}$ denote the class of all the functions $g : \mathbb{R}_+ \times \Omega \times Z \to E$ satisfying the following properties

(1) (measurability) the mapping $\mathbb{R}_+ \times \Omega \times Z \ni (t, \omega, z) \mapsto g(t, \omega, z) \in E$ is $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F} \times \mathcal{Z}/\mathcal{B}(E)$-measurable;

(2) (adaptedness) for every $t \geq 0$, the mapping $\Omega \times Z \ni (\omega, z) \mapsto g(t, \omega, z) \in E$ is $\mathcal{F}_t \otimes \mathcal{Z}/\mathcal{B}(E)$-measurable.

Proposition 3.2.8. *Every $\mathfrak{F}$-progressively measurable function $f : \mathbb{R}_+ \times \Omega \times Z \to E$ belongs to $\mathcal{K}$*

Proof. The proof follows immediately from the Toneli theorem (or Lemma 3.6.1). □

3.2.2 Martingale Type p Banach Spaces

An E-valued process $M := (M_t)_{t \geq 0}$ is an E-valued $\mathfrak{F}$-martingale if and only if M is an adapted process such that $\mathbb{E}(\|M_t\|) < \infty$ or all $t \geq 0$ and for every $0 \leq s < t < \infty$ and every $F \in \mathcal{F}_s$,

$$\mathbb{E}(1_F \cdot M_t) = \mathbb{E}(1_F \cdot M_s). \tag{3.2.1}$$

Equivalently, (3.2.1) can also be expressed in the following

$$\mathbb{E}(M_t|\mathcal{F}_s) = M_s, \quad \text{for all } 0 \leq s < t < \infty.$$

For more details of conditional expectation of a Banach valued process, please see [58].

Definition 3.2.9 (Martingale type p Banach space). A Banach space E with norm $\|\cdot\|$ is of martingale type p, for $p \in (0, \infty)$ if and only if there exists a constant $C_p(E) > 0$ such that for any E-valued discrete martingale $\{M_k\}_{k=1}^n$ the following inequality holds

$$\mathbb{E}\|M_n\|^p \leq C_p(E) \sum_{k=0}^{n} \mathbb{E}\|M_k - M_{k-1}\|^p, \tag{3.2.2}$$

with $M_{-1} = 0$ as usual.

Remark 3.2.10. Every Banach space has of martingale type 1. By using parallelogram law and properties of conditional expectation, it's easy to show that any separable Hilbert space is of martingale type 2 with

$$\mathbb{E}\|M_n\|^2 = \sum_{k=0}^{n} \mathbb{E}\|M_k - M_{k-1}\|^2.$$

If E and F are isomorphic Banach spaces, then E is of martingale type p if and only if F is of martingale type p.

Neidhardt in [60] studied a theory of stochastic integration on a certain class of Banach spaces which satisfies for all $x, y \in E$,

$$\|x + y\|^2 + \|x - y\|^2 \leq 2\|x\|^2 + K_2\|y\|^2, \tag{3.2.3}$$

with some constant $K_2 \geq 2$. We call Banach spaces satisfying (3.2.3) 2-uniformly-smooth (2-smooth) Banach spaces. Set $K(E) = \sup_{x,y \in E, y \neq 0} \left\{ \frac{\frac{1}{2}\|x+y\|^2 + \frac{1}{2}\|x-y\|^2 - \|x\|^2}{\|y\|^2} \right\}$ which is a constant measuring the smoothness of the norm of E. Then we see that if E is 2-uniformly-smooth, $K(E) < \infty$. The following equivalent definition of 2-smooth Banach spaces in term of asymptoticity of the modulus of smoothness of the norm can be found in [64], [65].

Definition 3.2.11 (p-smooth Banach space). A Banach space E is p-smooth if there exists an equivalent norm defined by the modulus of smoothness of $(E, \|\cdot\|)$

$$\rho_E(t) = \sup\{\frac{1}{2}(\|x+ty\| + \|x-ty\|) - 1 : \|x\| = \|y\| = 1\}$$

satisfying $\rho_E(t) \leq Kt^p$ for all $t > 0$ and some $K > 0$.

Remark 3.2.12. A Banach space is of martingale type p if and only if it is p-smooth, see [65]. Hence all spaces $L^q(\mu)$, for $q \in [p, \infty)$ and $q > 1$ with an arbitrary positive measure μ are of martingale type p. Note that any closed subspaces of martingale type p spaces are of martingale type p. The Sobolev spaces $W^{k,q}$, for $q \in [p, \infty)$ and $k > 0$ are of martingale type p.

3.2.3 Bochner Integrals

Throughout this section, let $(F, \mathcal{F}, \mu)$ be a measure space, where μ a nonnegative σ-finite measure. Let E be a separable Banach space. Let $\mathcal{B}(E)$ be the Borel σ-field, i.e. the smallest σ-field containing all open subsets of E. Note that the Borel σ-field $\mathcal{B}(E)$ is also generated by all sets of the form $\{x \in E : \phi(x) \leq a\}$, $\phi \in E^*$ and $a \in \mathbb{R}$.

Definition 3.2.13. We say a function $f : F \to E$ a **simple function** if f is $\mathcal{F}/\mathcal{B}(E)$-measurable and f is finite valued. In other words, there exist a finite number of disjoint sets $F_1, \cdots, F_n$ in $\mathcal{F}$ with $\mu(F_i) < \infty$, $i = 1, \cdots, n$ and a finite number of elements $x_1, \cdots, x_n$ in F such that

$$f(x) = \sum_{i=1}^{n} x_i 1_{F_i}(x), \ x \in F. \tag{3.2.4}$$

Then we can define the Bochner integral of a function f of the form 3.2.4 with respect to μ over a measurable subset A of F by

$$\int_A f(x)\mu(dx) = \sum_{i=1}^{n} x_i \mu(F_i \cap A).$$

Let f be a simple function of the form (3.2.4) and A be a measurable set in $\mathcal{F}$. Then we have

$$\|\int_A f(x)\mu(dx)\| \leq \int_A \|f(x)\|\mu(dx)$$

Indeed, we know that

$$\|\int_A f(x)\mu(dx)\| = \|\sum_{i=1}^{n} x_i \mu(F_i \cap A)\| \leq \sum_{i=1}^{n} \|x_i\|\mu(F_i \cap A) = \int_A \|f(x)\|\mu(dx).$$

Definition 3.2.14. An $\mathcal{F}/\mathcal{B}(E)$-measurable function f is said to be Bochner integrable if there exists a sequence of simple functions $\{f_n\}$ such that

$$\lim_{n\to\infty}\int_F \|f_n(x)-f(x)\|\mu(dx)=0.$$

Let f be a Bochner integrable function. Suppose that $\{f_n\}$ be a sequence of simple functions such that

$$\lim_{n\to\infty}\int_F \|f_n(x)-f(x)\|\mu(dx)=0.$$

It follows that

$$\begin{aligned}\|\int_F f_m(x)\mu(dx)-\int_F f_n(x)\mu(dx)\| &\le \int_F \|f_m(x)-f_n(x)\|\mu(dx)\\ &\le \int_F \|f(x)-f_m(x)\|\mu(dx)+\int_F \|f(x)-f_n(x)\|\mu(dx)\\ &\to 0, \text{ as } n,m\to\infty.\end{aligned}$$

This shows that $\{\int_F f_n(x)\mu(dx)\}$ is a Cauchy sequence in E. So it is convergent in E. Hence we may define the Bochner integral of f by

$$\int_F f(x)\mu(dx):=\lim_{n\to\infty}\int_F f_n(x)\mu(dx).$$

In this case, we have $\lim_{n\to\infty}\|\int_F f(x)\mu(dx)-\int_F f_n(x)\mu(dx)\|=0$.

If $A\in\mathcal{F}$ and f is Bochner integrable, it is easy to find that the function $1_A f$ is again Bochner integrable, and hence we define the Bochner integral of f on the set A by

$$\int_A f(x)\mu(dx):=\int_F 1_A(x)f(x)\mu(dx).$$

Proposition 3.2.15. *Let $f:F\to E$ be an $\mathcal{F}/\mathcal{B}(E)$-measurable function. Then there exists a sequence of simple functions $\{f_n\}$ of the form* (3.2.4) *such that*

$$\|f_n(x)-f(x)\|$$

monotonically decreases to 0.

Proof. Since E is a separable Banach space, suppose that $E_0=\{a_1,a_2,\cdots\}$ be a countable dense subset of E. We will construct a sequence of simple functions. Define, for each n, a function $K_n(x)$ by

$$K_n(x)=\min\left\{1\le k\le n:\|f(x)-a_k\|=\min\{\|f(x)-a_i\|,i=1,\cdots,n\}\right\}.$$

which is the least integer in $\{1,\cdots,n\}$ such that $a_{K_n(x)}$ is the closest one to f among $a_1,\cdots,a_n$. Set

$$A_{ni}=\{x\in F:K_n(x)=i\},\quad i=1,\cdots,n.$$

Then A_{in} are pairewise disjoint, $\cup_{i=1}^{n} A_{ni} = F$ and

$$A_{ni} = \Big(\bigcap_{j=1}^{i-1}\{\|f - a_i\| < \|f - a_j\|\}\Big)\bigcap\Big(\bigcap_{j=i+1}^{m}\{\|f - a_i\| \leq \|f - a_j\|\}\Big).$$

Observe that the set $\{a \in E : \|a - a_i\| > \|a - a_j\|\}$ belongs to $\mathcal{B}(E)$, since the pre-image of any open $\mathcal{B}(\mathbb{R})$-set under a continuous mapping is an $\mathcal{B}(E)$-open set and the mapping $x \mapsto \|a - a_i\| - \|a - a_j\|$ is continuous. Using this fact and the $\mathcal{F}/\mathcal{B}(E)$-measurability of f, we infer the set $\{x \in F : \|f(x) - a_i\| \leq \|f(x) - a_j\|\}$ belongs to $\mathcal{F}$, for each j. Thus $A_{ni} \in \mathcal{F}$, for $i = 1, \cdots, n$. For each x, we want to find an element from $\{a_1, \cdots, a_n\}$ which is closest to $f(x)$ and the subscript of which is the smallest. For this, let us define a sequence of functions from F to E by the following

$$g_n(x) = \sum_{i=1}^{n} a_i 1_{A_{ni}}(x), \quad x \in F$$

We find by the definition of the sets A_{ni} that

$$\begin{aligned}\|g_n(x) - f(x)\| &= \left\|\sum_{i=1}^{n} x_i 1_{A_{ni}}(x) - f(x)\right\| \\ &= \min\{\|f(x) - a_k\| : k = 1, \cdots, n\}.\end{aligned}$$

Note that since E_0 is the dense in E, for every $x \in F$, $f(x)$ can be approximated by an E_0-valued sequence. Hence we infer that $\min\{\|f(x) - a_k\| : k = 1, \cdots, n\}$ is pointwise monotonically decreasing to 0, as $n \to \infty$.

Since μ is σ-finite, there exists an increasing sequence of set $U_1 \subset U_2 \subset \cdots$ such that $\cup_n U_n = F$ and $\mu(U_n) < \infty$ for all $n \in \mathbb{N}$. Now define

$$f_n(x) = g_n(x) 1_{U_n}(x).$$

Clearly, for every $n \in \mathbb{N}$, f_n is a simple function. Moreover, $\|f_n(x) - f(x)\|$ is also pointwise monotonically decreasing to 0, as n goes to ∞. □

Proposition 3.2.16. *An $\mathcal{F}/\mathcal{B}(E)$-measurable function $f : F \to E$ is Bochner integrable if and only if $\int_F \|f(x)\|\mu(dx) < \infty$, in which case, we have*

$$\left\|\int_F f(x)\mu(dx)\right\| \leq \int_F \|f(x)\|\mu(dx). \tag{3.2.5}$$

Proof. Suppose that f is Bochner integrable. Let $\{f_n\}$ be a sequence of simple functions such that $\lim_{n\to\infty} \int_F \|f_n(x) - f(x)\|\mu(dx) = 0$. Hence we may choose $n \in \mathbb{N}$ big enough such that

$$\int_F \|f_n(x) - f(x)\|\mu(dx) < \infty.$$

It follows that

$$\int_F \|f(x)\|\mu(dx) \leq \int_F \|f(x) - f_n(x)\|\mu(dx) + \int_F \|f_n(x)\|\mu(dx) < \infty.$$

For the converse part, Let f be an $\mathcal{F}/\mathcal{B}(E)$-measurable function such that $\int_F \|f(x)\|\mu(dx) < \infty$. By Proposition 3.2.15, there exists a sequence $\{f_n\}$ of simple functions such that for every $x \in F$,

$$\|f_n(x) - f(x)\| \searrow 0, \text{ as } n \to \infty.$$

Then the monotone convergence theorem tells us that

$$\lim_{n\to\infty}\int_F \|f_n(x)-f(x)\|\mu(dx)=0.$$

To show the inequality (3.2.5), we know that the inequality (3.2.5) holds for simple functions $\{f_n\}$. Therefore,

$$\begin{aligned}\|\int_F f(x)\mu(dx)\| &\le \|\int_F f(x)\mu(dx)-\int_F f_n(x)\mu(dx)\|+\|\int_F f_n(x)\mu(dx)\|\\ &\le \|\int_F f(x)\mu(dx)-\int_F f_n(x)\mu(dx)\|+\int_F \|f_n(x)\|\mu(dx)\\ &\le \|\int_F f(x)\mu(dx)-\int_F f_n(x)\mu(dx)\|+\int_F \|f_n(x)-f(x)\|\mu(dx)\\ &+\int_F \|f(x)\|\mu(dx)\end{aligned}$$

Letting $n\to\infty$ in above inequality yields that the inequality (3.2.5). □

Let us state without proof some properties of the Bochner integrals.

Theorem 3.2.17. *1. Let T be a bounded linear operator from E to another separable Banach space G. Let $f : F \to E$ be a Bochner integrable function. Then $Tf : F \to G$ is Bochner integrable and*

$$\int_F Tf(x)\mu(dx)=T\int_F f(x)\mu(dx).$$

*2. **Lebesgue dominated convergence theorem** Let $(F,\mathcal{F},\mu)$ be a σ-finite measure space and let $f_n : F\to E$, $n\in\mathbb{N}$ be a sequence of Bochner integrable functions which converges μ-a.e. to a function f. Suppose that there exists an $\mathcal{F}$-measurable function $g : F\to\mathbb{R}$ with $\int |g(x)|\mu(dx)<\infty$ such that $\|f_n\|\le|g|$ for all $n\in\mathbb{N}$ μ-a.e. Then f is Bochner integrable and we have*

(a) $\lim_{n\to\infty}\int_F\|f_n(x)-f(x)\|\mu(dx)=0$;

(b) $\lim_{n\to\infty}\int_F\|f_n(x)\|\mu(dx)=\int_F\|f(x)\|\mu(dx)$.

3.2.4 Stochastic Integrals

Assume now that E is a martingale type p, $1\le p\le 2$ Banach space with the norm $\|\cdot\|$.

- Let $\mathcal{M}^p_{\mathcal{K}}(\mathbb{R}_+\times\Omega\times Z,\lambda\otimes\nu\otimes\mathbb{P};E)$ denote the linear space of all functions $f:\mathbb{R}_+\times Z\times\Omega\to E$ from $\mathcal{K}$, such that

$$\int_0^\infty\int_Z \mathbb{E}\|f(t,\cdot,z)\|^p\nu(dz)dt<\infty.$$

  Let us recall here that $\mathcal{K}$ is the class of all $\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{F}\times\mathcal{Z}$-measurable and $(\mathcal{F}_t\otimes\mathcal{Z})_{t\ge0}$-adapted functions. Sometimes the notation is simplified by dropping the the set $\mathbb{R}_+\times\Omega$ and the measure

$\lambda\otimes\mathbb{P}$, if they have been specified, to $\mathcal{M}^p_{\mathcal{K}}(E)$ in the sequel. We shall alternative freely between these two different notations and use the one which seems more clear or convenient.

- Let $\mathcal{M}^p(\mathbb{R}_+\times\Omega\times Z,\mathcal{BFZ},\lambda\otimes\nu\otimes\mathbb{P};E)$ denote the linear space of all $\mathfrak{F}$-progressively measurable functions $f:\mathbb{R}_+\times Z\times\Omega\to E$ such that

$$\int_0^\infty\int_Z \mathbb{E}\|f(t,\cdot,z)\|^p\nu(dz)dt<\infty.$$

Likewise, for simplicity, we adopt the notation $\mathcal{M}^p(\mathcal{BFZ};E)$ instead of $\mathcal{M}^p(\mathbb{R}_+\times\Omega\times Z,\mathcal{BF}\otimes\mathcal{Z},\lambda\otimes\nu\otimes\mathbb{P};E)$.

- Let $\mathcal{M}^p(\mathbb{R}_+\times\Omega\times Z,\hat{\mathcal{P}},\lambda\otimes\nu\otimes\mathbb{P};E)$ (or $\mathcal{M}^p(\hat{\mathcal{P}};E)$) denote the linear space of all $\mathfrak{F}$-predictable functions $f:\mathbb{R}_+\times Z\times\Omega\to E$ such that

$$\int_0^\infty\int_Z \mathbb{E}\|f(t,\cdot,z)\|^p\nu(dz)dt<\infty.$$

Remark 3.2.18. So far, we have considered three classes $\mathcal{M}^p_{\mathcal{K}}(E)$, $\mathcal{M}^p(\mathcal{BFZ};E)$ and $\mathcal{M}^p(\hat{\mathcal{P}};E)$ of functions on $\mathbb{R}_+\times\Omega\times\mathbb{P}$. A quick observation about the relationships between these three classes is that $\mathcal{M}^p(\hat{\mathcal{P}};E)\subset\mathcal{M}^p(\mathcal{BFZ};E)\subset\mathcal{M}^p_{\mathcal{K}}(E)$

Definition 3.2.19. We call f a **step function** if there is a finite sequence of numbers $0=t_0<t_1<\cdots<t_n<\infty$ and a sequence of disjoint sets A^k_{j-1}, $j=1,\cdots,n$, $k=1,\cdots,m$ in $\mathcal{Z}$ with $\nu(A^k_{j-1})<\infty$ such that

$$f(t,\omega,z)=\sum_{j=1}^n\sum_{k=1}^m\xi^k_{j-1}(\omega)1_{(t_{j-1},t_j]}(t)1_{A^k_{j-1}}(z),\tag{3.2.6}$$

where ξ^k_{j-1}, $j=1,\cdots,n$ and $k=1,\cdots,m$ are E-valued p-integrable and $\mathcal{F}_{t_{j-1}}$-measurable random vairables. The set of all such step functions will be denoted by $\mathcal{M}^p_{step}(\mathcal{Z};E)$.

Definition 3.2.20. The stochastic integral of a step function f in $\mathcal{M}^p_{step}(\mathcal{Z};E)$ of the form (3.2.6) is defined by

$$I(f):=\sum_{j=1}^n\sum_{k=1}^m\xi^k_{j-1}(\omega)\tilde{N}((t_{j-1},t_j]\times A^k_{j-1}).$$

Proposition 3.2.21. *Let $f\in\mathcal{M}^p_{step}(\mathcal{Z};E)$. Then the stochastic integral $I(f)$ of f is well defined. That is the value of $I_T(f)$ does not depend on the representation chosen for f. Furthermore, $I(f)$ is in $L^p(\Omega)$ and*

(1) $\mathbb{E}(I(f))=0$

(2) $\mathbb{E}\|I(f)\|^p\le C\,\mathbb{E}\int_0^\infty\int_Z\|f(t,z)\|^p\nu(dz)dt.$

Before starting the proof of the proposition 3.2.21, we establish an auxiliary lemma.

Lemma 3.2.22 ([16] Lemma C.3). *Let ξ be a Poisson random variable with parameter λ. Then for all $1 \leq p \leq 2$, we have*

$$\mathbb{E}|\xi - \lambda|^p \leq 2^{2-p}\lambda.$$

Proof. We state with the simple case $p = 1, 2$. For $p = 1$, by the triangle property, we have

$$\mathbb{E}|\xi - \lambda| \leq \mathbb{E}\xi + \lambda = 2\lambda.$$

If $p = 2$,

$$\mathbb{E}|\xi - \lambda^2| = var(\xi) = \lambda.$$

For the general case $1 < p < 2$, it follows from the Höder's inequality that

$$\begin{aligned}
\mathbb{E}|\xi - \lambda|^p &= \mathbb{E}\left(|\xi - \lambda|^{2(p-1)}|\xi - \lambda|^{2-p}\right) \\
&\leq \left[\mathbb{E}\Big(|\xi - \lambda|^{2(p-1)}\Big)^{1/(p-1)}\right]^{p-1} \left[\mathbb{E}\Big(|\xi - \lambda|^{2-p)}\Big)^{1/(2-p)}\right]^{2-p} \\
&= \left(\mathbb{E}|\xi - \lambda|^2\right)^{p-1} \left(\mathbb{E}|\xi - \lambda|\right)^{2-p} \\
&\leq \lambda^{p-1}(2\lambda)^{2-p} = 2^{2-p}\lambda,
\end{aligned}$$

which completes the proof. □

Proof of Proposition 3.2.21. Let f be of the form (3.2.6). Then the stochastic integral $I(f)$ of f is given by

$$I(f) := \sum_{j=1}^{n}\sum_{k=1}^{m} \xi_{j-1}^k(\omega)\tilde{N}((t_{j-1}, t_j] \times A_{j-1}^k).$$

Taking expectation of $I(f)$, we have

$$\begin{aligned}
\mathbb{E}(I(f)) &= \sum_{j=1}^{n}\sum_{k=1}^{m} \mathbb{E}\left(\xi_{j-1}^k \tilde{N}((t_{j-1}, t_j] \times A_{j-1}^k)\right) \\
&= \sum_{j=1}^{n}\sum_{k=1}^{m} \mathbb{E}\left(\xi_{j-1}^k\right) \mathbb{E}\left(\tilde{N}((t_{j-1}, t_j] \times A_{j-1}^k)\right) \\
&= 0,
\end{aligned}$$

where we used the independence of ξ_{j-1}^k and $\tilde{N}((t_{j-1}, t_j] \times A_{j-1}^k)$ in the second equality and the fact that $\mathbb{E}\left(\tilde{N}((t_{j-1}, t_j] \times A_{j-1}^k)\right) = 0$ for each j and k in the last equality.
Note that the sequence $\sum_{j=1}^{i}\sum_{k=1}^{n} \xi_{j-1}^k \tilde{N}((t_{j-1}, t_j] \times A_{j-1}^k)$, $i = 1, \cdots, m$ is a martingale with respect to the filtration $\{\mathcal{F}_{t_i}\}_{i=1}^m$ on $(\Omega, \mathcal{F}, \mathbb{P})$. Therefore, by using the martingale type p property of the space E, we find out that

$$\begin{aligned}
\mathbb{E}\|I(f)\|^p &= \mathbb{E}\left\|\sum_{j=1}^{n}\sum_{k=1}^{m} \xi_{j-1}^k \tilde{N}((t_{j-1}, t_j] \times A_{j-1}^k)\right\|^p \\
&\leq C_p(E)\sum_{j=1}^{n} \mathbb{E}\left\|\sum_{k=1}^{m} \xi_{j-1}^k \tilde{N}((t_{j-1}, t_j] \times A_{j-1}^k)\right\|^p . \qquad (3.2.7)
\end{aligned}$$

For the case $p = 1$, we have

$$
\begin{aligned}
\mathbb{E}\|I(f)\| &\le C_1(E)\sum_{j=1}^{n}\mathbb{E}\left\|\sum_{k=1}^{m}\xi_{j-1}^{k}\tilde{N}((t_{j-1},t_j]\times A_{j-1}^{k})\right\| \\
&\le C_1(E)\sum_{j=1}^{n}\sum_{k=1}^{m}\mathbb{E}\|\xi_{j-1}^{k}\|\left|\tilde{N}((t_{j-1},t_j]\times A_{j-1}^{k})\right| \\
&\le 2C_1(E)\sum_{j=1}^{n}\sum_{k=1}^{m}\mathbb{E}\|\xi_{j-1}^{k}\|\nu(A_{j-1}^{k})(t_j-t_{j-1}) \\
&= \mathbb{E}\int_0^{\infty}\int_S\|f(t,z)\|\nu(dz)dt,
\end{aligned}
$$

since

$$
\mathbb{E}\left|\tilde{N}((t_{j-1},t_j]\times A_{j-1}^{k})\right| \le \mathbb{E}N((t_{j-1},t_j]\times A_{j-1}^{k}) + \nu(A_{j-1}^{k})(t_j-t_{j-1}) = 2\nu(A_{j-1}^{k})(t_j-t_{j-1}).
$$

For the general case p, $1 \le p \le 2$, one way of doing this, due to [16], is that since for fixed j, the random variables ξ_{j-1}^{k}, $k = 1,\cdots,m$ are $\mathcal{F}_{j-1}$-measurable and the compensated poisson random variables $\tilde{N}((t_{j-1},t_j]\times A_{j-1}^{k}$, $k = 1,\cdots,m$ are independent of $\mathcal{F}_{j-1}$, hence we may suppose that the random variables ξ_{j-1}^{k}, $k = 1,\cdots,m$ are defined on a probability space $(\Omega_1,\mathcal{F}_1,\mathbb{P}_1)$ and $\tilde{N}((t_{j-1},t_j]\times A_{j-1}^{k})$, $k = 1,\cdots,m$ are defined on another probability space $(\Omega_2,\mathcal{F}_2,\mathbb{P}_2)$ such that $\Omega = \Omega_1\times\Omega_2$, $\mathcal{F} = \mathcal{F}_1\otimes\mathcal{F}_2$ and $\mathbb{P} = \mathcal{P}_1\otimes\mathbb{P}_2$. Let E_1 (resp. E_2) be the expectation on $(\Omega_1,\mathcal{F}_1)$ (resp. $(\Omega_1,\mathcal{F}_2)$ with respect to $\mathbb{P}_1$ (resp. $\mathbb{P}_2$). On the space $(\Omega_2,\mathcal{F}_2,\mathbb{P}_2)$ construct a filtration $\mathcal{G}_i$ of σ-fields by

$$
\mathcal{G}_i := \sigma\{N((t_{j-1},t_j]\times A_{j-1}^{k}) : 1\le k\le i\},\quad i = 1,\cdots,m.
$$

Therefore, by the measurability and independence of the random variables, it's easy to verify that the class $\sum_{k=1}^{i}\xi_{j-1}^{k}\tilde{N}((t_{j-1},t_j]\times A_{j-1}^{k})$, $i = 1,\cdots,m$ of random variables is a martingale with respect to the filtration $\{\mathcal{G}_i\}_{i=1}^{m}$ on $(\Omega_2,\mathcal{F}_2)$. Indeed,

$$
\begin{aligned}
\mathbb{E}_2\left(\sum_{k=1}^{i+1}\xi_{j-1}^{k}\tilde{N}((t_{j-1},t_j]\times A_{j-1}^{k})\middle|\mathcal{G}_i\right) &= \sum_{k=1}^{i+1}\left(\xi_{j-1}^{k}\mathbb{E}_2\left(\tilde{N}((t_{j-1},t_j]\times A_{j-1}^{k})\middle|\mathcal{G}_i\right)\right) \\
&= \sum_{k=1}^{i}\xi_{j-1}^{k}\tilde{N}((t_{j-1},t_j]\times A_{j-1}^{k}) \\
&\qquad + \xi_{j-1}^{i+1}\mathbb{E}_2\left(\tilde{N}((t_{j-1},t_j]\times A_{j-1}^{i+1})\right) \\
&= \sum_{k=1}^{i}\xi_{j-1}^{k}\tilde{N}((t_{j-1},t_j]\times A_{j-1}^{k}).
\end{aligned}
$$

By applying the martingale type p property of the space E to this martingale to (3.2.7), we have

$$
\begin{aligned}
\mathbb{E}\|I(f)\|^p &\le C_p(E)\sum_{j=1}^{n}\mathbb{E}\left\|\sum_{k=1}^{m}\xi_{j-1}^k\tilde{N}((t_{j-1},t_j]\times A_{j-1}^k)\right\|^p \\
&= C_p(E)\sum_{j=1}^{n}\mathbb{E}_1\mathbb{E}_2\left\|\sum_{k=1}^{m}\xi_{j-1}^k\tilde{N}((t_{j-1},t_j]\times A_{j-1}^k)\right\|^p \\
&\le C_p(E)^2\sum_{j=1}^{n}\mathbb{E}_1\left[\sum_{k=1}^{m}E_2\left\|\xi_{j-1}^k\tilde{N}((t_{j-1},t_j]\times A_{j-1}^k)\right\|^p\right] \\
&= C_p(E)^2\sum_{j=1}^{n}\mathbb{E}_1\sum_{k=1}^{m}\|\xi_{j-1}^k\|^pE_2\left|\tilde{N}((t_{j-1},t_j]\times A_{j-1}^k)\right|^p \\
&\le 2^{2-p}C_p(E)^2\sum_{j=1}^{n}\sum_{k=1}^{m}\mathbb{E}\|\xi_{j-1}^k\|^p\nu(A_{j-1}^k)(t_j-t_{j-1}),
\end{aligned}
$$

where we used the property $\mathbb{E}\Big(\tilde{N}((t_{j-1},t_j]\times A_{j-1}^k)^p \le 2^{2-p}\nu(A_{j-1}^k)(t_j-t_{j-1})$ by Lemma 3.2.22. □

Theorem 3.2.23. *$\mathcal{M}^p_{step}(\mathcal{Z};E)$ is dense in $\mathcal{M}^p(\hat{\mathcal{P}};E)$.*

Proof. We split the proof into two steps. In fact, the proof shows that E can be taken to be any separable Banach space.

We define a new class $\mathcal{M}$ of functions $f\in\mathcal{M}^p(\hat{\mathcal{P}};E)$ such that

$$f(t,\omega,z)=\sum_{i=1}^{m}e_i1_{M_i}(t,\omega,z),\quad (t,\omega,z)\in\mathbb{R}_+\times\Omega\times Z, \tag{3.2.8}$$

where $e_i\in E$, $i=1,\cdots,m$ and $M_i\subset\mathbb{R}_+\times\Omega\times Z$, $i=1,\cdots m$ are pairwise disjoint $\mathfrak{F}$-predictable sets. Our proof will be proceeded as follows. We first show that $\mathcal{M}^p_{step}(Z;E)$ is dense in $\mathcal{M}$ and then we show that $\mathcal{M}$ is dense in $\mathcal{M}^p(\hat{\mathcal{P}};E)$.

Step 1: $\mathcal{M}^p_{step}(Z;E)$ is dense in $\mathcal{M}$.

Note that the family $\hat{\mathcal{R}}$ of $\mathfrak{F}$-predictable rectangles

$$\hat{\mathcal{R}}=\{\{0\}\times F\times B: F\in\mathcal{F}_0,\ B\in\mathcal{Z}\}\cup\{(s,t]\times F\times B: 0\le s\le t<\infty,\ F\in\mathcal{F}_s,\ B\in\mathcal{Z}\}$$

is a semi-ring, see Proposition 3.2.2 and Definition 2.1.17.

Let $\mathcal{A}$ be the smallest ring generated by $\hat{\mathcal{R}}$. Then the elements of $\mathcal{A}$ are finite unions of disjoint rectangles from $\hat{\mathcal{R}}$, see [11]. Define $\mu:=\lambda\otimes\mathbb{P}\otimes\nu$. Take $M\in\hat{\mathcal{P}}$. According to Section 13 Theorem D in [34], for every $\varepsilon>0$, there exists $A\in\mathcal{A}$ such that $\mu(M\triangle A)<\varepsilon$, where $\triangle$ denotes the symmetric difference i.e. $M\triangle A=(M\backslash A)\cup(A\backslash M)$. Since $A\in\mathcal{A}$, there exists a finite sequence of pairwise disjoint $\hat{\mathcal{R}}$-sets $\hat{R}_1,\cdots,\hat{R}_n$ such that $A=\cup_{i=1}^n\hat{R}_i$ and

$$\mu\Big(M\triangle(\cup_{i=1}^n\hat{R}_n)\Big)<\varepsilon.$$

Since $|1_M-\sum_{i=1}^n1_{\hat{R}_i}|=1_{M\triangle(\cup_{i=1}^n\hat{R}_n)}$, we infer that

$$
\begin{aligned}
\mathbb{E}\int_0^\infty\int_Z|1_M(t,\omega,z)-\sum_{i=1}^n1_{\hat{R}_i}(t,\omega,z)|^p\nu(dz)dt &= \mathbb{E}\int_0^\infty\int_Z1_{M\triangle(\cup_{i=1}^n)\hat{R}_i}(t,\omega,z)\nu(dz)dt \\
&= \mu(M\triangle(\cup_{i=1}^n)\hat{R}_i)<\varepsilon.
\end{aligned}
$$

Take $f \in \mathcal{M}$ which is given by the form (3.2.8). Let us fix $\varepsilon > 0$. Put $\varepsilon' = \frac{\varepsilon}{\sum_{i=1}^{m} \|e_i\|_{\mathcal{H}}^p} > 0$. Then for each $M_i \in \hat{\mathcal{P}}$, $i = 1, \cdots, m$, we can find finitely many disjoint sets $\hat{R}_1^i, \cdots, \hat{R}_{n_i}^i$ in $\mathcal{R}$ which depend on ε such that

$$\mathbb{E}\left(\int_0^\infty \int_Z |1_{M_i}(t,\cdot,z) - \sum_{k=1}^{n_i} 1_{\hat{R}_k^i}(t,\cdot,z)|^2 \nu(dz)dt\right) < \varepsilon.$$

Define a function f^ε by

$$f^\varepsilon(t,\omega,z) = \sum_{i=1}^{m} e_i \sum_{k=1}^{n_i} 1_{\hat{R}_k^i}(t,\omega,z), \quad (t,\omega,z) \in \mathbb{R}_+ \times \Omega.$$

Since $\hat{R}_k^i \in \hat{\mathcal{R}}$, for all k, i, it is of the form $(s,u] \times F \times B$, F is $\mathcal{F}_s$-measurable and $B \in \mathcal{Z}$. Note that $1_{(s,u] \times F \times B}(t,\omega,z) = 1_{(s,u]}(t) 1_F(\omega) 1_B(z)$. Moreover,

$$\begin{aligned}
\mathbb{E}\int_0^\infty \int_Z \|f^\varepsilon(t,\cdot,z)\|^p \nu(dz)dt &= \mathbb{E}\int_0^T \int_Z \left\|\sum_{i=1}^{m} e_i \sum_{k=1}^{n_i} 1_{\hat{R}_k^i}(t,\cdot,z)\right\|^p \nu(dz)dt \\
&\le C\mathbb{E}\int_0^\infty \int_Z \sum_{i=1}^{m} (\sum_{k=1}^{n_i} 1_{\hat{R}_k^i}(t,\cdot,z))^p \|e_i\|^p \nu(dz)dt \\
&\le C\sum_{i=1}^{m} \|e_i\|^p \mathbb{E}\int_0^\infty \int_Z \sum_{k=1}^{n_i} 1_{\hat{R}_k^i}(t,\cdot,z)\nu(dz)dt < \infty.
\end{aligned}$$

Therefore, we infer that $f^\varepsilon \in \mathcal{M}_{step}^p(Z;E)$. Also, observe that

$$\begin{aligned}
&\mathbb{E}\int_0^\infty \int_Z \|f(t,\cdot,z) - f^\varepsilon(t,\cdot,z)\|^p \nu(dz)dt \\
&\le C\sum_{i=1}^{m} \|e_i\|^p \mathbb{E}\left(\int_0^\infty \int_Z |1_{M_i}(t,\omega) - \sum_{k=1}^{n_i} 1_{\hat{R}_k^i}(t,\cdot,z)|^p \nu(dz)dt\right) \\
&\le \left(\sum_{i=1}^{m} \|e_i\|^p\right)\varepsilon' = \varepsilon,
\end{aligned}$$

So we have

$$\mathbb{E}\int_0^\infty \int_Z \|f(t,\cdot,z) - f^\varepsilon(t,\cdot,z)\|^p \nu(dz)dt < \varepsilon.$$

Step 2: $\mathcal{M}$ is dense in $\mathcal{M}^p(\hat{\mathcal{P}};E)$.

Since the Banach space E is separable, there exists a countable dense subset $E_0 = \{x_1, x_2, \cdots\}$ in E. Take $f \in \mathcal{M}^p(\hat{\mathcal{P}};E)$. This means that f is an $\mathfrak{F}$-predictable function. By using the Propositon 3.2.15, we can construct a sequence of approximating functions in $\mathcal{M}$ of the following form

$$g^n(t,\omega,z) = \sum_{i=1}^{n} x_i 1_{M_{ni}}(t,\omega,z), \quad (t,\omega,z) \in \mathbb{R}_+ \times \Omega \times Z,$$

where M_{ni} are pairwise disjoint sets in $\hat{\mathcal{P}}$, $i = 1, \cdots, n$, $n \in \mathbb{N}$, such that

$$\|g^n(t,\omega,z) - f(t,\omega,z)\|$$

is pointwise monotonically decreasing to 0, as n goes to infinity for every $(t, \omega, z) \in \mathbb{R}_+ \times \Omega \times Z$. Therefore, we can apply the monotone convergence theorem to get

$$\lim_{n\to\infty} \mathbb{E} \int_0^\infty \int_Z \|g^n(t,\cdot,z) - f(t,\cdot,z)\|^p \nu(dz)dt = 0.$$

Step 2: $\mathcal{M}^p_{step}(Z;E)$ is dense in $\mathcal{M}^p(\hat{\mathcal{P}};E)$.
Take $\varepsilon > 0$ and $f \in \mathcal{M}^p(\hat{\mathcal{P}};E)$. Hence for n large enough, by the step 2 we can find $g^n \in \mathcal{M}$ such that

$$\mathbb{E} \int_0^\infty \int_Z |f(t,\cdot,z) - g^n(t,\cdot,z)|^p \nu(dz)dt < \frac{\varepsilon}{4}.$$

It then follows from the assertion 1 that for each $g^n \in \mathcal{M}$ and $\varepsilon > 0$ there is a corresponding function $f^n \in \mathcal{M}^p_{step}(Z;E)$ such that

$$\mathbb{E} \int_0^\infty \int_Z \|g^n(t,\cdot,z) - f^n(t,\cdot,z)\|^p \nu(dz)dt < \frac{\varepsilon}{4}.$$

Consequently,

$$\begin{aligned}
&\mathbb{E} \int_0^\infty \int_Z \|f(t,\cdot,z) - f^n(t,\cdot,z)\|^p \nu(dz)dt \\
&\le 2^{p-1}\mathbb{E} \int_0^\infty \int_Z \|f(t,\cdot,z) - g^n(t,\cdot,z)\|^p \nu(dz)dt + 2^{p-1}\mathbb{E} \int_0^\infty \int_Z \|g^n(t,\cdot,z) - f^n(t,\cdot,z)\|^p \nu(dz)dt \\
&\le 2^{p-1}\frac{\varepsilon}{4} + 2^{p-1}\frac{\varepsilon}{4} < \varepsilon,
\end{aligned}$$

completing the proof of the theorem.

□

Theorem 3.2.24. *$\mathcal{M}^p_{step}(Z;E)$ is dense in $\mathcal{M}^p_{\mathcal{K}}(E)$.*

Remark 3.2.25. In the proof of this theorem, we need the completion requirement of the family $(\mathcal{F}_t \otimes \mathcal{Z})_{t\ge 0}$ of σ-fields. That is for every $t \ge 0$, $\mathcal{F}_t \otimes \mathcal{Z}$ contains all the $\mathbb{P} \otimes \nu$-null sets in $\mathcal{F} \otimes \mathcal{Z}$.

Proof. Let $f \in \mathcal{M}^p_{\mathcal{K}}(E)$. Without loss of generality, we may assume that f is almost everywhere bounded on E. Indeed, for every $f \in \mathcal{M}^p_{\mathcal{K}}(E)$, we can define a sequence $\{g_j\}_{j\in\mathbb{N}}$ of functions

$$g_j(t,\omega,z) = f(t,\omega,z)1_{\{\|f\|\le j\}}(t,\omega,z), \quad (t,\omega,z) \in \mathbb{R}_+ \times \Omega \times Z.$$

Set $\mu = \lambda \otimes \mathbb{P} \otimes \nu$ and $N = \{\|f\| = \infty\}$. Then

$$\mu(N) = \lim_{j\to\infty} \mu(\{\|f\| > j\}) \le \lim_{j\to\infty} \left(\frac{1}{j^p} \int \|f\|^p d\mu\right) = 0.$$

Observe that $\lim_{j\to\infty} g_j(t,\omega,z) = \lim_{j\to\infty} f(t,\omega,z)1_{\{\|f\|\le j\}}(t,\omega,z) = f(t,\omega,z)1_{N^c}(t,\omega,z)$ for $(t,\omega,z) \in \mathbb{R}_+ \times \Omega \times Z$. It follows that $\lim_{j\to\infty} g_j(t,\omega,z) = f(t,\omega,z)$ μ-a.s. Moreover we find that $\|g_j(t,\omega,z) - f(t,\omega,z)\| \le 2\|f(t,\omega,z)\|$ for all $j \in \mathbb{N}$ and $(t,\omega,z) \in \mathbb{R}_+ \times \Omega \times Z$. Hence by the Lebesgue dominated convergence theorem, we have

$$\lim_{j\to\infty} \mathbb{E} \int_0^\infty \int_Z \|g^j(t,\omega,z) - f(t,\omega,z)\|^p \nu(dz)dt = 0,$$

where $\|g_j\| \leq j$ for every $j \in \mathbb{N}$. Hence we can assume that $\|f(t,\omega,z)\| \leq C$, for every $(t,\omega,z) \in \hat{\Omega}$, where $\hat{\Omega} \subset \mathbb{R}_+ \times \Omega \times Z$ and $\lambda \otimes \mathbb{P} \otimes \nu(\hat{\Omega}^c) = 0$.

We can also assume that f vanishes outside some finite interval $[0,T]$ and some set U of finite ν-measure. For this, define another sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions by

$$f_n(t,\omega,z) = f(t,\omega,z)1_{[0,n]}(t)1_{U_n}(z), \ (t,\omega,z) \in \mathbb{R}_+ \times \Omega \times Z,$$

where $\{U_n\}$ is an increasing sequence in $\mathcal{Z}$ such that $\cup_n U_n = Z$ and $\nu(U_n) < \infty$, since the measure ν is σ-finite. Note that $\|f_n(t,\omega,z) - f(t,\omega,z)\|$ monotonically decreasing to 0 as $n \to \infty$, μ-a.s.. Hence the monotone convergence theorem tells us that

$$\lim_{n\to\infty} \mathbb{E} \int_0^\infty \int_Z \|f_n(s,\cdot,z) - f(s,\cdot,z)\|^p \nu(dz)ds = 0.$$

Case 1: Assume that f is left-continuous and for every $t > 0$, $f(t,\cdot)$ is $\mathcal{F}_t \otimes \mathcal{Z}$-measurable.
Step 1 Let us fix $t_0 \in (0,T]$. Then the function $f(t_0,\cdot)$ is $\mathcal{F}_{t_0} \otimes \mathcal{Z}$-measurable. Since the Banach space E is separable, by Proposition 3.2.15, there exists a sequence g^n of approximating functions of the form

$$g^n(t_0,\omega,z) = \sum_{i=1}^n x_i 1_{A^i_{t_0}}(\omega,z), \ (\omega,z) \in \mathbb{R}_+ \times Z,$$

where $A^i_{t_0} \in \mathcal{F}_{t_0} \otimes \mathcal{Z}$ and $\mathbb{P} \otimes \nu(A^i_{t_0}) < \infty$, $i = 1, \cdot, n$, such that

$$\|g^n(t_0,\omega,z) - f(t_0,\omega,z)\|$$

is (ω,z)-pointwise monotonically decreasing to 0, as $n \to \infty$. Applying the monotone convergence theorem yields that

$$\lim_{n\to\infty} \mathbb{E} \int_Z \|g^n(t_0,\cdot,z) - f(t_0,\omega,z)\|^p \nu(dz) = 0.$$

Take $\varepsilon > 0$. Then for some sufficient big n_ε, we have

$$\mathbb{E} \int_Z \|g^{n_\varepsilon}(t_0,\cdot,z) - f(t_0,\omega,z)\|^p \nu(dz) < \varepsilon.$$

Note that $\mathcal{F}_{t_0} \times \mathcal{Z}$ is a semi-ring. By the Theorem D in Section 13 [34], for every set $A^i_{t_0}$ there exists a finite sequence $\{F_{k_i,t_0} \times B_{k_i}\}_{k_i=1}^{m_i}$ of pairwise disjoint sets in $\mathcal{F}_{t_0} \times \mathcal{Z}$ such that

$$\mathbb{P} \otimes \nu\big(A^i_{t_0} \Delta (\cup_{k=1}^m (F^i_{k,t_0} \times B^i_{k,t_0}))\big) < \varepsilon.$$

That is

$$\mathbb{E} \int_Z |1_{A^i_{t_0}}(\cdot,z) - 1_{\cup_{k=1}^m (F^i_{k,t_0} \times B^i_{k,t_0})}(\cdot,z)| \nu(dz)ds < \varepsilon.$$

Define

$$f^{n_\varepsilon}(t_0,\omega,z) = \sum_{i=1}^{n_\varepsilon} \sum_{k=1}^{m_i} x_i 1_{F^i_{k,t_0}}(\omega) 1_{B^i_{k,t_0}}(z), \ (\omega,z) \in \Omega \times Z. \tag{3.2.9}$$

Then it is straightforward to see that

$$
\begin{aligned}
&\mathbb{E}\int_Z \|f^{n_\varepsilon}(t_0,\cdot,z) - f(t_0,\cdot,z)\|^p \nu(dz) \\
&\le 2^{p-1}\mathbb{E}\int_Z \|f^{n_\varepsilon}(t_0,\cdot,z) - g^{n_\varepsilon}(t_0,\cdot,z)\|^p \nu(dz) + 2^{p-1}\mathbb{E}\int_Z \|g^{n_\varepsilon}(t_0,\cdot,z) - f(t_0,\cdot,z)\|^p \nu(dz) \\
&\le C_p \sum_{i=1}^{n_\varepsilon} \|x_i\|^p \mathbb{E}\int_Z \Big|\sum_{k=1}^{m_i} 1_{F^i_{k,t_0}}(\cdot) 1_{B^i_{k,t_0}}(z) - 1_{A^i_{t_0}}(\cdot,z)\Big| \nu(dz) + 2^{p-1}\varepsilon \\
&\le C_p\varepsilon
\end{aligned}
$$

Step 2. Since f is left-continuous in the time variable t, we construct the following sequence of functions

$$
f^n(t,\omega,z) = \sum_{j=0}^{2^n-1} f(\frac{jT}{2^n},\omega,z) 1_{(\frac{jT}{2^n},\frac{(j+1)T}{2^n}]}(t), \quad (t,\omega,z) \in \mathbb{R}_+ \times \Omega \times Z. \tag{3.2.10}
$$

We can show that $\lim_{n\to\infty} f^n(t,\omega,z) = f(t,\omega,z)$ for all $(t,\omega,z) \in \mathbb{R}_+ \times \Omega \times Z$. Indeed, let $\varepsilon > 0$. For every $t \in \mathbb{R}_+$, By the left-continuity of f, there exists some δ such that for every $t' \in (t-\delta, t]$, we have

$$
\|f(t',\omega,z) - f(t,\omega,z)\| < \varepsilon.
$$

Choose $N \in \mathbb{N}$ so that $\frac{T}{2^N} \le \delta$, then for each $n \ge N$, we have $\frac{1}{2^n} \le \frac{1}{2^N} \le \delta$. Since the sequence of intervals $\{(\frac{jT}{2^n}, \frac{(j+1)T}{2^n}]\}_{j=0}^{2^n-1}$ cover $(0,T]$ and they are pairwise disjoint, for every $n \ge N$, one can find k such that $t \in (\frac{kT}{2^n}, \frac{(k+1)T}{2^n}]$ implying $0 < t - \frac{kT}{2^n} < \frac{T}{2^n} < \delta$. By the left-continuity, we have

$$
\|f(t,\omega,z) - f(\frac{kT}{2^n},\omega,z)\| < \epsilon.
$$

This gives that

$$
\|f^n(t,\omega,z) - f(t,\omega,z)\| < \epsilon,
$$

since $f^n(t,\omega,z) = f(\frac{kT}{2^n},\omega,z)$, for $t \in (\frac{kT}{2^n}, \frac{(k+1)T}{2^n}]$. From this we obtain

$$
\lim_{n\to\infty} \|f^n(t,\omega,z) - f(t,\omega,z)\|^p = 0,
$$

for every $(t,\omega,z) \in \mathbb{R}_+ \times \Omega \times Z$. By the boundedness assumption of f, we have for almost all (t,ω),

$$
\|f^n(t,\omega,z) - f(t,\omega,z)\|^p \le 4C^p.
$$

Since by assumption f vanishes outside the set U of finite ν-measure, we may apply the Lebesgue dominated convergence theorem to get

$$
\lim_{n\to\infty} \mathbb{E}\int_Z \|f(t,\cdot,z) - f^n(t,\cdot,z)\|^p \nu(dz) = 0,
$$

where the set U were assumed above. Note that for every j, by the Step 1, the function $f(\frac{jT}{2^n},\omega,z)$ can be approximated by a sequence of functions of the form (3.2.9). This means that for every n, the function f^n can be approximated by a sequence $\{f^{n,m}\}_{m\in\mathbb{N}}$ of functions of the following form

$$
f^{n,m}(t,\omega,z) = \sum_{j=1}^{2^n-1}\sum_{i=1}^{m}\sum_{k=1}^{m_i} x_i 1_{F^i_{k,j}}(\omega) 1_{B^i_{k,j}}(z) 1_{(\frac{jT}{2^n},\frac{(j+1)T}{2^n}]}(t).
$$

Clearly, for every $n, m \in \mathbb{N}$, $f^{n,m}$ is a step function and we have

$$\lim_{n,m\to\infty} \mathbb{E} \int_Z \|f^{n,m}(t,\omega,z) - f(t,\omega,z)\|^p \nu(dz) = 0.$$

Since by the assumption f is bounded and vanishes outside some bounded interval $[0,T]$, again the Lebesgue dominated convergence theorem tells us that

$$\lim_{n,m\to\infty} \mathbb{E} \int_0^T \int_Z \|f(t,\cdot,z) - f^{n,m}(t,\cdot,z)\|^p \nu(dz)dt = 0.$$

Case 2: Assume now that f is $\mathfrak{F}$-progressively measurable. Then we define a function F by

$$F(t,\omega,z) = \int_0^t f(s,\omega,z)ds, \quad (t,\omega,z) \in \mathbb{R}_+ \times \Omega \times Z.$$

Note that the process F is well defined for $\mathbb{P}\otimes\nu$-a.s., since by the assumption the process f is bounded outside a $\lambda\otimes\mathbb{P}\otimes\nu$-null set. Also, we can see that F is continuous with respect to t, $\mathbb{P}\otimes\nu$-a.s. Since f is $\mathfrak{F}$-progressively measurable, for every $t > 0$, the function $f : [0,t] \times \Omega \times Z \to E$ is $\mathcal{B}([0,t])\otimes\mathcal{F}_t\otimes\mathcal{Z}$-measurable. Hence by the Tonelli theorem, we find that for every $t > 0$, $F(t,\cdot,z) = \int_0^t f(s,\cdot,z)ds$ is $\mathcal{F}_t \otimes \mathcal{Z}$-measurable. Note that the $\mathfrak{F}$-progressively measurability assumption here is the main point of achieving the $(\mathcal{F}_t \otimes \mathcal{Z})$-adaptedness of the process F. Define next a sequence of functions by

$$\tilde{f}_m(t,\omega,z) = m\left[F(t,\omega,z) - F\left((t-\frac{1}{m}) \vee 0,\omega,z\right)\right], \ (t,\omega) \in \mathbb{R}_+ \times \Omega. \tag{3.2.11}$$

Then one can see that $\tilde{f}_m$ is continuous in the variable t for almost all $(\omega,z) \in \Omega \times Z$. Moreover, since for every t, F is $\mathcal{F}_t \otimes \mathcal{Z}$-measurable, the same conclusion holds for the function $\tilde{f}_m$, $m \in \mathbb{N}$.

Observe that for almost all $t \geq 0$, the following holds for almost all $(\omega,z) \in \Omega \times Z$

$$\begin{aligned} f(t,\omega,z) &= \lim_{m\to\infty} m\left[F(t,\omega,z) - F\left(\left(t-\frac{1}{m}\right) \vee 0,\omega,z\right)\right] \\ &= \lim_{m\to\infty} \tilde{f}_m(t,\omega,z). \end{aligned}$$

By the Lebesgue dominated convergence theorem, we have

$$\lim_{m\to\infty} \mathbb{E} \int_0^\infty \int_Z \|\tilde{f}_m(t,\cdot,z) - f(t,\cdot,z)\|^p \nu(dz)dt = 0. \tag{3.2.12}$$

Let $\varepsilon > 0$. For every $f \in \mathcal{M}^p(\mathcal{BFZ};E)$, the above convergence (3.2.12) allows us to find $m \in \mathbb{N}$ big enough such that

$$\mathbb{E} \int_0^\infty \int_Z \|\tilde{f}_m(t,\cdot,z) - f(t,\cdot,z)\|^p \nu(dz)dt < \frac{\varepsilon}{8}.$$

Since $\tilde{f}_m$ is continuous in the time variable t and for every t, $\tilde{f}_m$ is $\mathcal{F}_t \otimes \mathcal{Z}$-measurable, by case 1 we can associate $\tilde{f}_m$ with a function $\tilde{f} \in \mathcal{M}^p_{step}(Z;E)$ such that

$$\mathbb{E} \int_0^\infty \|\tilde{f}_m(t,\cdot,z) - \tilde{f}(t,\cdot,z)\|^p \nu(dz)dt < \frac{\varepsilon}{8}.$$

It follows that

$$\mathbb{E} \int_0^\infty \int_Z \|f(t,\cdot,z) - \tilde{f}(t,\cdot,z)\|^p \nu(dz)dt < \varepsilon,$$

which completes the proof of this part.

Case 3: Let $f \in \mathcal{K}$. That is the function f is $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F} \otimes \mathcal{Z}$-measurable and for every $t > 0$, $f(t,\cdot)$ is $\mathcal{F}_t \otimes \mathcal{Z}$-measurable. Hence by Proposition 2.1 in [79], for every $t \geq 0$, there exists an $\mathcal{B}([0,t]) \otimes \mathcal{F}_t \otimes \mathcal{Z}$-measurable modification g of f. That is for every $t \geq 0$, the function $g : [0,t] \times \Omega \times Z \to E$ is $\mathcal{B}([0,t]) \otimes \mathcal{F}_t \otimes \mathcal{Z}$-measurable and for all $s \in [0,t]$, we have $f(s,\omega,z) = g(s,\omega,z)$ for almost all $(\omega,z) \in \Omega \times Z$. Define two continuous functions $G(t,\omega,z) = \int_0^t g(s,\omega,z)ds$ and $F(t,\omega,z) = \int_0^t f(s,\omega,z)ds$. By the Toneli Theorem, for every $t > 0$, the function G is $\mathcal{F}_t \otimes \mathcal{Z}$-measurable. We will show that for every $t \geq 0$, the process $F(t,\cdot)$ is $\mathcal{F}_t \otimes \mathcal{Z}$-measurable as well. Indeed, consider a function $\eta_t(\omega,z) = 1_{\{(\omega,z):f(t,\omega,z)\neq g(t,\omega,z)\}}(\omega)$. Note that $\mathbb{P}\otimes\nu\{(\omega,z) : f(t,\omega,z) \neq g(t,\omega,z)\} = 0$. Then we infer that for every $t \geq 0$, $\eta_t(\cdot,\cdot)$ is $\mathcal{F}_t\otimes\mathcal{Z}$-measurable. By using the Fubini theorem, we have

$$\begin{aligned}\mathbb{E}\int_0^\infty \int_Z \eta_t(\cdot,z)\nu(dz)dt &= \int_0^\infty \mathbb{E}\Big(\int_Z (1_{\{(\omega,z):f(t,\omega,z)\neq g(t,\omega,z)\}}(\omega,z)\nu(dz)\Big)dt \\ &= \int_0^\infty \mathbb{P}\otimes\nu((\omega,z) : f(t,\omega,z) \neq g(t,\omega,z))dt = 0.\end{aligned}$$

Hence we infer that $\int_0^\infty \eta_t(\omega,z)dt = 0$ $\mathbb{P}\otimes\nu$-a.e. We see that for every $t \geq 0$, the set $\{(\omega,z) : \int_0^\infty \eta_t(\omega,z)dt > 0\}$ is a $\mathbb{P}\otimes\nu$-null set. We shall show that for every $t \geq 0$,

$$\{(\omega,z) : F(t,\omega,z) \neq G(t,\omega,z)\} \subset \{(\omega,z) : \int_0^\infty \eta_t(\omega),zdt > 0\}.$$

To see this, we take $(\tilde{\omega},\tilde{z}) \in \{(\omega,z) : F(t,\omega,z) \neq G(t,\omega,z)\}$. Then the Lebesgue measure of the set $A_{(\tilde{\omega},\tilde{z})} = \{s : 0 \leq s \leq t, f(s,\tilde{\omega},\tilde{z}) \neq g(s,\tilde{\omega},\tilde{z})\}$ is positive. This gives that $\int_0^\infty \eta_t(\tilde{\omega},z)dt \geq \int_0^t 1_{A_{(\tilde{\omega},\tilde{z})}}(s)ds > 0$. Thus $(\tilde{\omega},\tilde{z}) \in \{(\omega,z) : \int_0^\infty \eta_t(\omega,z)dt > 0\}$.

It follows that for every $t > 0$ the set $\{(\omega,z) : F(t,\omega,z) \neq G(t,\omega,z)\}$ is a $\mathbb{P}\otimes\nu$-null set. Since we know that $G(t,\cdot,\cdot)$ is $\mathcal{F}_t\otimes\mathcal{Z}$-measurable, by the completion assumptions of the σ-fields $\mathcal{F}_t\otimes\mathcal{Z}$ we can conclude that $F(t,\cdot,\cdot)$ is also $\mathcal{F}_t\otimes\mathcal{Z}$-measurable. This together with continuity of F allows us to define an approximating sequence $\tilde{f}_m$ of $(\mathcal{F}_t\otimes\mathcal{Z})$-adapted and left-continuous functions as in 3.2.11, and hence the results achieved in the case 2 can be applied. □

The following corollary is an immediate consequences of the above Theorem.

Corollary 3.2.26. *$\mathcal{M}^p_{step}(\mathcal{Z};E)$ is dense in $\mathcal{M}^p(\mathcal{BFZ};E)$.*

Theorem 3.2.27. *Let f be in $\mathcal{M}^p_{\mathcal{K}}(E)$ (or $\mathcal{M}^p(\mathcal{BFZ};E)$, or $\mathcal{M}^p(\hat{\mathcal{P}};E)$). Let $\{f^n\} \subset \mathcal{M}^p_{step}(\mathcal{Z};E)$ be any sequence of step functions satisfying*

$$\mathbb{E}\int_0^\infty \int_Z \|f(t,\cdot,z) - f^n(t,\cdot,z)\|^p\nu(dz)dt \to 0, \quad as\ n \to \infty.$$

Then there exists a random variable, say $I(f)$, in $\mathcal{L}^p(\Omega;E)$ such that

$$\mathbb{E}\|I(f) - I(f^n)\|^p \to 0, \quad as\ n \to \infty.$$

Moreover, such random variable is uniquely defined $\mathbb{P}$-a.s., that is, it does not depend on the choice of approximating step function. We usually call $I(f)$ the stochastic integral of f with respect to the compensated Poisson random measure $\tilde{N}$.

Proof. Let us first introduce the following notations

$$\|f\|_{\mathcal{M}^p} = \left(\mathbb{E}\left(\int_0^\infty \int_Z \|f(t,\cdot,z)\|^p \nu(dz)dt\right)\right)^{1/p} \quad \text{and} \quad \|Y\|_{L^p(E)} = (\mathbb{E}\|Y\|^p)^{1/p}.$$

Since the class $\mathcal{M}^p_{step}(\mathcal{Z};E)$ is dense in $\mathcal{M}^p_{\mathcal{K}}(E)$, we can find a sequence of step functions $\{f^n\}_{n\in\mathbb{N}} \subset \mathcal{M}^p_{step}(\mathcal{Z};E)$ such that

$$\lim_{n\to\infty} \|f^n - f\|_{\mathcal{M}^p} = 0.$$

Then for every $\varepsilon > 0$, there exists some $N > 0, N \in \mathbb{N}$ such that for every $n, m > N$, $n, m \in \mathbb{N}$ we have

$$\|f^n - f\|^p_{\mathcal{M}^p} < \frac{\varepsilon}{2}, \quad and \quad \|f^m - f\|^p_{\mathcal{M}^p} < \frac{\varepsilon}{2}.$$

By Proposition 3.2.21, we observe that $\{I(f^n)\}_{n\in\mathbb{N}}$ is a sequence of random variables in $\mathcal{L}^p(\Omega)$. We shall show that this sequence $\{I(f^n)\}$ is a Cauchy sequence in $\mathcal{L}^p(\Omega)$. Using linearity and boundedness of mapping I we get

$$\begin{aligned}
\mathbb{E}\|I(f_n) - I(f_m)\|^p =& \mathbb{E}\|I_t(f_n - f_m)\|^p \\
\leq& C\|f^n - f^m\|^p_{\mathcal{M}^p} \\
=& \|(f - f^n) + (f - f^m)\|^p_{\mathcal{M}^p} \\
\leq& \|f - f^n\|^p_{\mathcal{M}^p} + \|f - f^m\|^p_{\mathcal{M}^p} \\
<& \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

which shows that $\{I(f^n)\}$ is Cauchy sequence in $\mathcal{L}^p(\Omega)$. Since the space $\mathcal{L}^p(\Omega)$ is complete, the Cauchy sequence converges to some limit which we denote by $I(f)$ in $\mathcal{L}^p(\Omega)$, i.e. $I(f) := \mathcal{L}^p\text{-}\lim I(f^n)$.

Now we proceed to show the uniqueness of the random variable $I(f)$. Assume that there are two sequence of random step functions $\{f^n\}_{n\in\mathbb{N}} \in \mathcal{M}^p_{step}(\mathcal{Z};E)$ and $\{g^n\}_{n\in\mathbb{N}} \in \mathcal{M}^p_{step}(\mathcal{Z};E)$ that both converge to f in $\mathcal{M}^p_{\mathcal{K}}(E)$. Consider the interlaced sequence $f^1, g^1, f^2, g^2, \cdots$, then we can see the sequence is also a Cauchy sequence in $\mathcal{M}^p_{\mathcal{K}}(E)$ and it converges to f.

Form above we have shown that $I(f^1), I(g^1), I(f^2), I(g^2), \cdots$ is a Cauchy sequence in $\mathcal{L}^2(\Omega)$ and by completeness of $\mathcal{L}^2(\Omega)$, it has a limit in $\mathcal{L}^2(\Omega)$. Since $\{I(f^n)\}_{n\in\mathbb{N}}$ and $\{I(g^n)\}_{n\in\mathbb{N}}$ are two subsequences of this interlaced sequence $I(f^1), I(g^1), I(f^2), I(g^2), \cdots$, then their limits must be the same.

□

3.3 Properties of the Stochastic integrals

For $0 \leq a \leq b \leq T$, $B \in \mathcal{Z}$ and $f \in \mathcal{M}^p_{\mathcal{K}}(E)$, since $1_{(a,b]}1_B f$ is also in $\mathcal{M}^p_{\mathcal{K}}(E)$, so we can define the stochastic integral from a to b of the function $f \in \mathcal{M}^p_{\mathcal{K}}(E)$ by

$$I^B_{a,b}(f) = \int_a^b \int_Z f(t,z)\tilde{N}(dx,dt) = I(1_{(a,b]}1_B f). \tag{3.3.1}$$

For simplicity, we denote

$$I_t(f) = \int_0^t \int_Z f(t,z)\tilde{N}(dz,dt) = I(1_{(0,t]}f).$$

Remark 3.3.1. Notice that a function of the form $1_{\{a\}}(t)\xi(t,\omega)$ with $f \in \mathcal{M}^p_{\mathcal{K}}(E)$ is equivalent to the identically zero process with respect to the measure $\lambda \otimes \mathbb{P} \times \nu$, so it has zero stochastic integral. Therefore, the inclusion or exclusion of the the point a in above definition will not influence the integral. In other words, the integral $\int_{(a,b]} \int_Z f(t,z)\tilde{N}(ds,dz)$ is indistinguishable to the integral $\int_{[a,b]} \int_Z f(t,z)\tilde{N}(ds,dz)$.

Theorem 3.3.2. *Let f, g be in $\mathcal{M}^p_{\mathcal{K}}(E)$. Then*

(1) For every $t \geq 0$, $I_t(\alpha f + \beta g) = \alpha I_t(f) + \beta I_t(g)$, where $\alpha, \beta \in \mathbb{R}$.

(2) For every $t \geq 0$, $\mathbb{E}(I_t(f)) = 0$.

(3) For every $t \geq 0$,

$$\mathbb{E}\|I_t(f)\|^p \leq C\mathbb{E}\int_0^t \int_Z \|f(t,\cdot,z)\|^p \nu(dz)dt. \tag{3.3.2}$$

(4) $I_t(f)$, $t \geq 0$ is a càdlàg p-integrable martingale. More precisely, $I_t(f)$ has a modification which has càdlàg trajectories.

Remark 3.3.3. When E is a Hilbert space, the inequality in (3.3.2) becomes an equality with $C = 1$, namely

$$\mathbb{E}\|I_t(f)\|^2 = \mathbb{E}\int_0^t \int_Z \|f(t,z)\|^2 \nu(dz)dt. \tag{3.3.3}$$

From now on, while considering the stochastic process $I_t(f)$, $t \geq 0$, it will be assumed that the process $I_t(f)$, $t \geq 0$ has càdlàg trajectories.

Proof. (1) If $f,\ g \in \mathcal{M}^p_{\mathcal{K}}(E)$, then $1_{(0,t]}f,\ 1_{(0,t]}g \in \mathcal{M}^p_{\mathcal{K}}(E)$. So we can find two sequences of simple function $\{f^n\}$ and $\{g^n\}$ in $\mathcal{M}^p_{step}(\mathcal{Z};E)$ such that

$$\|f^n - 1_{(0,t]}f\|_{\mathcal{M}^p} \to 0 \text{ and } \|g^n - g1_{(0,t]}\|_{\mathcal{M}^p} \to 0.$$

Hence

$$\|\alpha f^n + \beta g^n - (\alpha f + \beta g)1_{(0,t]}\|_{\mathcal{M}^p} \to 0.$$

By Proposition 3.2.21, we know that

$$I(\alpha f^n + \beta g^n) = \alpha I(f^n) + \beta I(g^n),$$

for each n. Taking the $\mathcal{L}^p$-limit on both side as $n \to \infty$, it follows that

$$I_t(\alpha f + \beta g) = \alpha I_t(f) + \beta I_t(g).$$

(2) See (4)

(3) By Theorem 3.2.24 there exists a sequence of step functions f^n in $\mathcal{M}^p_{step}(\mathcal{Z};E)$ such that

$$\lim_{n\to\infty} \|f^n - f\|_{\mathcal{M}^p} = 0.$$

It follows that for each $t \in \mathbb{R}_+$,

$$1_{(0,t]}f \in \mathcal{M}^p_{\mathcal{K}}(E),\ 1_{[0,t]}f^n \in \mathcal{M}^p_{step}(\mathcal{Z};E)$$

and

$$\lim_{n\to\infty} \|1_{[0,t]}f - 1_{(0,t]}f^n\|_{\mathcal{M}^p} = 0.$$

By Propositon 3.2.21 we have

$$\|I_t(f^n)\|_{\mathcal{L}^p} \le C\|1_{(0,t]}f^n\|_{\mathcal{M}^p}. \tag{3.3.4}$$

By taking the limit of (3.3.4) as $n \to \infty$ we get

$$\|I_t(f)\|_{\mathcal{L}^p} \le C\|1_{(0,t]}f\|_{\mathcal{M}^p} = \mathbb{E}\left(\int_0^t \int_Z \|f(s,\cdot,z)\|^p ds\right).$$

Since $f \in \mathcal{M}^p_{step}(\mathcal{Z};E)$, it gives that

$$\mathbb{E}\|I_t(f)\|^p \le C\mathbb{E}\left(\int_0^t \int_Z |f(s,\cdot,z)\|^p \nu(dz)ds\right) \le C\mathbb{E}\left(\int_0^\infty \int_Z \|f(s,\cdot,z)\|^p \nu(dz)ds\right) < \infty,$$

which implies $I_t(f) \in \mathcal{L}^p(\Omega)$. Moreover, we observe that

$$\sup_{t\ge0} \mathbb{E}\|I_t(f)\|^p \le C\int_0^\infty \int_Z \|f(s,\cdot,z)\|^p \nu(dz)ds < \infty.$$

This shows that $I_t(f)$ is p-integrable.

(4) Now we are going to show that the stochastic integral process $I_t(f)$, $t \ge 0$ is a martingale. Since by Theorem 3.2.27 there is a sequence of step function $\{f^n\}$ such that $I_t(f^n)$ converges to $I_t(f)$ in $\mathcal{L}^p(\Omega)$, we can find a subsequence of $\{I_t(f^n)\}$ such that it converges to $I_t(f)$ $\mathbb{P}$-a.s. for every $t \ge 0$. If $\mathcal{F}_0$ contains all $\mathbb{P}$-negligible sets in $\mathcal{F}$, then by the adaptedness of $I_t(f^n)$, $t \ge 0$, the process $I_t(f)$, $t \ge 0$ is also adapted to $\mathfrak{F}$.
Let $0 \le s < t < \infty$. First, we verify martingale property for step functions in $\mathcal{M}^p_{step}(\mathcal{Z};E)$. Let g be a step function of the form (3.2.6). Then the stochastic integral $I_t(g)$ of f is given by

$$I_t(g) = I(1_{[0,t]}g) = I(1_{(0,s]}g + 1_{(s,t]}g) = I(1_{(0,s]}g) + I(1_{(s,t]}g) = I_s(g) + I_{s,t}(g).$$

It is easy to see that $I_{s,t}(g)$ is independent from $\mathcal{F}_s$ by the independence of $\tilde{N}((s,t)\times A)$ with respect to $\mathcal{F}_s$. But $I_s(g)$ is $\mathcal{F}_s$-measurable. It follows that

$$\begin{aligned}\mathbb{E}(I_t(g)|\mathcal{F}_s) = \mathbb{E}(I_s(g) + I_{s,t}(g)|\mathcal{F}_s) &= \mathbb{E}(I_s(g)|\mathcal{F}_s) + \mathbb{E}(I_{s,t}(g)|\mathcal{F}_s)\\ &= I_s(g) + \mathbb{E}(I_{s,t}(g)) = I_s(g),\end{aligned}$$

which shows martingale property for step function. For each n, from above discussion we know that

$$\mathbb{E}(I_t(f^n)|\mathcal{F}_s) = I_s(f^n).$$

Therefore, it from the Jensen's inequality for conditional expectations that

$$\begin{aligned}\mathbb{E}\big\|I_s(f) - \mathbb{E}(I_t(f)|\mathcal{F}_s)\big\|^p &= \mathbb{E}\big\|I_s(f) - I_s(f^n) + \mathbb{E}(I_t(f^n)|\mathcal{F}_s) - \mathbb{E}(I_t(f)|\mathcal{F}_s)\big\|^p\\ &\le 2^p\mathbb{E}\|I_s(f) - I_s(f^n)\|^p + 2^p\mathbb{E}\big\|\mathbb{E}(I_t(f^n)|\mathcal{F}_s) - \mathbb{E}(I_t(f)|\mathcal{F}_s)\big\|^p\\ &= 2^p\mathbb{E}\|I_s(f) - I_s(f^n)\|^p + 2^p\mathbb{E}\big\|\mathbb{E}(I_t(f^n) - I_t(f)|\mathcal{F}_s)\big\|^p\\ &\le 2^p\mathbb{E}\|I_s(f) - I_s(f^n)\|^p + 2^p\mathbb{E}\left(\mathbb{E}(\|I_t(f^n) - I_t(f)\|^p|\mathcal{F}_s)\right)\\ &= 2^p\mathbb{E}\|I_s(f) - I_s(f^n)\|^p + 2^p\mathbb{E}\|I_t(f_n) - I_t(f)\|^p.\end{aligned}$$

Since for every $t \geq 0$, $\lim_{n\to\infty} \mathbb{E}\|I_t(f^n) - I_t(f)\|^p = 0$, we infer

$$\lim_{n\to\infty} \mathbb{E}\big\|I_s(f) - \mathbb{E}(I_t(f)|\mathcal{F}_s)\big\|^p = 0.$$

Therefore,

$$\mathbb{E}(I_t(f)|\mathcal{F}_s) = I_s(f) \ a.s.$$

It remains to show the càdlàg continuity of the paths of process $I_t(f)$, $t \geq 0$. For this, we have to show that there is a sequence of step functions $\{f^n\}_{n\in\mathbb{N}}$ in $\mathcal{M}^p_{step}(\mathcal{Z}; E)$ such that the stochastic integrals $I_t(f^n)$ converges uniformly to a modification of $I_t(f)$ on $\mathbb{R}_+$.

If $f \in \mathcal{M}^p_{\mathcal{K}}(E)$, then by Theorem 3.2.26 we can find a sequence of step functions $\{f^n\}_{n\in\mathbb{N}}$ in $\mathcal{M}^p_{step}(\mathcal{Z}; E)$ such that

$$\lim_{n\to\infty} \mathbb{E} \int_0^\infty \int_Z \|f^n(s,\cdot,z) - f(s,\cdot,z)\|^p \nu(dz)ds = 0.$$

Thus there exists a sequence of natural number $\{n_k\}_{k=1}^\infty$ such that

$$\mathbb{E} \int_0^\infty \int_Z \|f^{n_k}(s,\cdot,z) - f(s,\cdot,z)\|^p \nu(dz)ds < \frac{1}{10^k}, \ k \in \mathbb{N}.$$

We write k instead of n_k for brevity,

$$\mathbb{E} \int_0^\infty \int_Z \|f^{k}(s,\cdot,z) - f(s,\cdot,z)\|^p \nu(dz)ds < \frac{1}{10^k}, \ k \in \mathbb{N}.$$

It follows that

$$\begin{aligned}
\mathbb{E} \int_0^\infty & \int_Z \|f^{k+1}(s,\cdot,z) - f^k(s,\cdot,z)\|^p \nu(dz)ds \\
&\leq 2^p \mathbb{E} \int_0^\infty \int_S \|f^{k+1}(s,\cdot,z) - f(s,\cdot,z)\|^p \nu(dz)ds \\
&\quad + 2^p \mathbb{E} \int_0^\infty \int_Z \|f^{k}(s,\cdot,z) - f(s,\cdot,z)\|^p \nu(dz)ds \\
&< \frac{8}{10^k}, \ k \in \mathbb{N}.
\end{aligned}$$

By the Chebyshev inequality, we have that

$$\begin{aligned}
\mathbb{P}\Big\{\omega \in \Omega : \sup_{t\geq 0} \Big\| \int_0^t \int_Z & \left[f^{k+1}(s,z) - f^k(s,z)\right] \tilde{N}(ds,dz)\|(\omega) \geq \frac{1}{2^k}\Big\} \\
&\leq \frac{1}{(1/2^k)^p} \sup_{t\geq 0} \mathbb{E} \left\| \int_0^t \int_Z [f^{k+1}(s,z) - f^n(s,z)] \tilde{N}(ds,dz) \right\|^p \\
&\leq \frac{C}{(1/2^k)^p} \sup_{t\geq 0} \mathbb{E} \int_0^t \int_Z |f^{k+1}(s,z) - f^k(s,z)\|^p \nu(dz)ds
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{C}{(1/2^k)^p} \mathbb{E} \int_0^\infty \int_Z \|f^{k+1}(s,z) - f^k(s,z)\|^p \nu(dz)ds \\
&\leq \frac{8C/10^k}{(1/2^k)^p} < \frac{C}{2^{k-3}}, \ k \in \mathbb{N}.
\end{aligned}$$

Since the series $\sum_{k=1}^{\infty} \frac{C}{2^{k-3}}$ is convergent, we infer that

$$\sum_{k=1}^{\infty} \mathbb{P}\{\sup_{t\geq 0} \left\| \int_0^t \int_Z [f^{k+1}(s,z) - f^k(s,z)]\tilde{N}(ds,dz) \right\| \geq \frac{1}{2^k}\} < \infty.$$

Hence, in the view of the first Borel-Cantelli lemma (see Theorem 4.3 in [9]), it follows that

$$\mathbb{P}\left(\limsup_k \left\{ \omega \in \Omega : \sup_{t\geq 0} \left\| \left(\int_0^t \int_Z [f^{k+1}(s,z) - f^k(s,z)]\tilde{N}(ds,dz) \right)(\omega) \right\| \geq \frac{1}{2^k} \right\} \right) = 0.$$

This implies that there exists a set $\hat{\Omega} \subset \Omega$ such that $\mathbb{P}(\hat{\Omega}) = 1$ and for every $t \geq 0$, there exists some $j \in \mathbb{N}$ such that for all $k \geq j$ we have $\left\| \left(\int_0^t \int_Z [f^{k+1}(s,z) - f^k(s,z)]\tilde{N}(ds,dz) \right)(\omega) \right\| < \frac{1}{2^k}$ $\omega \in \hat{\Omega}$. Hence

$$\sum_{n=j}^{\infty} \left\| \left(\int_0^t \int_Z [f^{k+1}(s,z) - f^k(s,z)]\tilde{N}(ds,dz) \right)(\omega) \right\| < \infty, \quad \text{for all } 0 \leq t < \infty, \ \omega \in \hat{\Omega}.$$

Consider the series

$$\xi_t := \sum_{k=0}^{\infty} \int_0^t \int_Z [f^{k+1}(s,z) - f^k(s,z)]\tilde{N}(ds,dz),$$

where $f^0(s) \equiv 0$. Since for $k = 0, \cdots, j-1$,

$$\begin{aligned} &\mathbb{E} \left\| \int_0^t \int_Z [f^{k+1}(s,z) - f^k(s,z)]\tilde{N}(ds,dz) \right\|^p \\ &\leq C\, \mathbb{E} \int_0^t \int_Z \|f^{k+1}(s,z) - f^k(s,z)\|^p \nu(dz)ds \\ &< \infty, \end{aligned}$$

we have $\| \int_0^t \int_Z [f^{(k+1)}(s,z) - f^k(s,z)]\tilde{N}(ds,dz))\| < \infty$ on a set of probability measure 1, which we shall also denote by $\hat{\Omega}$ for simplicity of notation, for every $n = 0, \cdots, j-1$. Then we can conclude that $\sum_{k=0}^{\infty} \left\| \int_0^t \int_Z [f^{k+1}(s,z) - f^k(s,z)]\tilde{N}(ds,dz) \right\| < \infty$ on $\hat{\Omega}$. This shows that the series $\sum_{k=0}^{\infty} \int_0^t \int_Z (f^{k+1}(s,z) - f^k(s,z))\tilde{N}(ds,dz)$ is uniformly convergent on $\mathbb{R}_+$ for each $\omega \in \hat{\Omega}$. Now we define $\xi_t = \sum_{k=0}^{\infty} \int_0^t \int_Z (f^{k+1}(s,z) - f^k(s,z))\tilde{N}(ds,dz)$ when the sum converges, i.e.

$$\xi_t = \lim_{k\to\infty} \int_0^t \int_Z f^k(s,z)\tilde{N}(ds,dz),$$

and if this limit diverges, we replaced it by zero. Thus $I_t(f^k)$ converges uniformly on $(0,\infty)$ to ξ_t. On the other hand, since by Theorem 3.2.26 we know that for each $0 \leq t < \infty$,

$$\mathbb{E} \left\| \int_0^t \int_Z [f^k(s,z)\tilde{N}(ds,dz) - \int_0^t \int_Z f(s,z)\tilde{N}(ds,dz) \right\|^p \to 0, \quad as\ n \to \infty,$$

by taking a subsequence $\{f^{k_j}\}_{j\in\mathbb{N}}$ we have for every $0 \leq t < \infty$,

$$\lim_{j\to\infty} \int_0^t \int_Z f^{k_j}(s,z)\tilde{N}(ds,dz) = \int_0^t \int_Z f(s,z)\tilde{N}(ds,dz) \quad \text{on } \Omega_t \text{ with } \mathbb{P}(\Omega_t) = 1.$$

By the uniqueness of the limits, we infer that $I_t(f)$ is a modification of ξ_t.

Note that for each n, the process $I_t(f^n)$, $t \geq 0$ is càdlàg. Since the limit of a uniformly convergent sequence of càdlàg-continuous functions preserve the càdlàg-continuity, we can conclude that ξ_t, $t \geq 0$ is a modification of $I_t(f)$ with all paths càdlàg paths.

□

So far we have three classes of functions, $\mathcal{M}^p_{\mathcal{K}}(E)$, $\mathcal{M}^p(\mathcal{BFZ}; E)$ and $\mathcal{M}^p(\hat{\mathcal{P}}; E)$, for which we can define stochastic integrals. One may ask what is the relationship between these three classes. Clearly, we have $\mathcal{M}^p(\hat{\mathcal{P}}; E) \subset \mathcal{M}^p(\mathcal{BFZ}; E) \subset \mathcal{M}^p_{\mathcal{K}}(E)$. The precise relationship is the following which shows that both classes are in fact the same.

Theorem 3.3.4. *Assume that $f \in \mathcal{M}^p_{\mathcal{K}}(E)$. Then there exists a process $\hat{f} \in \mathcal{M}^p(\hat{\mathcal{P}}; E)$ such that*

$$\mathbb{E}\int_0^\infty \int_Z \|f(t,z) - \hat{f}(t,z)\|^p \nu(dz)dt = 0.$$

Furthermore, the stochastic integrals $\int_0^\infty \int_Z f(s,z)\tilde{N}(ds,dz)$ and $\int_0^\infty \int_Z \hat{f}(s,z)\tilde{N}(ds,dz)$ are $\mathbb{P}$-a.s. equal and the processes $\int_0^t \int_Z f(s,z)\tilde{N}(ds,dz)$, $t \geq 0$ and $\int_0^t \int_Z \hat{f}(s,z)\tilde{N}(ds,dz)$, $t \geq 0$ are modification of each other. Especially, if we take the càdlàg modifications of the processes $\int_0^t \int_Z f(s,z)\tilde{N}(ds,dz)$, $t \geq 0$ and $\int_0^t \int_Z \hat{f}(s,z)\tilde{N}(ds,dz)$, $t \geq 0$, then they are indistinguishable.

Proof. Let $f \in \mathcal{M}^p_{\mathcal{K}}(E)$. As we have shown in the proof of Theorem 3.2.24, one can find a sequence of step functions $\{f^n\}_{n=1}^\infty$, which are $\mathfrak{F}$-predictable by the special forms (3.2.6), such that

$$\lim_{n\to\infty} \mathbb{E}\int_0^\infty \int_Z |f(t,z) - f^n(t,z)|^p \nu(dz)dt = 0.$$

Hence by taking a subsequence $\{f^{n_k}\}_{k\in\mathbb{N}}$, we infer $f^{n_k} \to f$, $\lambda \otimes \mathbb{P} \times \nu$-a.e as $k \to \infty$.
Define

$$\hat{f}(t,\omega,z) = \limsup_{n\to\infty} f^{n_k}(t,\omega,z), \quad (t,\omega,z) = \mathbb{R}_+ \times \Omega \times Z.$$

Note that $\limsup_{n\to\infty} f^{n_k}$ inherits the measurability of the functions $\{f^n\}$. Therefore, $\hat{f}$ is also $\mathfrak{F}$-predictable and $f = \hat{f}$, $\lambda \otimes \mathbb{P} \times \nu$-a.e.. In other words $f - \hat{f} = 0$, $\lambda \otimes \mathbb{P} \times \nu$-a.e.. On the basis of Theorem (3.3.2), we have

$$\begin{aligned}
\mathbb{E}\left\|\int_0^\infty \int_Z f(t,\cdot,z)\tilde{N}(dt,dz) - \int_0^\infty \int_Z \hat{f}(t,\cdot,z)\tilde{N}(dt,dz)\right\|^p \\
\leq C\mathbb{E}\int_0^\infty \int_Z \|f(t,\cdot,z) - \hat{f}(t,\cdot,z)\|^p \nu(dz)dt \\
= C\int_{\mathbb{R}_+\times\Omega\times Z} |f - \hat{f}|^p d(\lambda \otimes \mathbb{P} \times \nu) = 0.
\end{aligned}$$

It follows that $\int_0^\infty \int_Z f(t,\cdot,z)\tilde{N}(dt,dz) = \int_0^\infty \int_Z \hat{f}(t,\cdot,z)\tilde{N}(dt,dz)$, $\mathbb{P}$-a.e. In particular, we have that for every $t \geq 0$, $1_{(0,t]}f = 1_{(0,t]}\hat{f}$, $\lambda \otimes \mathbb{P} \otimes \nu$-a.e.. Hence in view of inequality 3.3.2, we infer that

$$\begin{aligned}
\mathbb{E}\left\|\int_0^t \int_Z f(s,\cdot,z)\tilde{N}(ds,dz) - \int_0^t \int_Z \hat{f}(s,\cdot,z)\tilde{N}(ds,dz)\right\| \\
\leq C\mathbb{E}\int_0^\infty \int_Z 1_{(0,t]]}(s)\|f(s,\cdot,z) - \hat{f}(s,\cdot,z)\|^p \nu(dz)ds = 0.
\end{aligned}$$

This implies that the processes $\int_0^t \int_Z f(s,\cdot,z)\tilde{N}(ds,dz)$, $t \geq 0$ and $\int_0^t \int_Z \hat{f}(s,\cdot,z)\tilde{N}(ds,dz)$, $t \geq 0$ are modifications with each other. Therefore, by Lemma 2.1.2, the càdlàg modifications of the processes $\int_0^t \int_Z f(s,\cdot,z)\tilde{N}(ds,dz)$, $t \geq 0$ and $\int_0^t \int_Z \hat{f}(s,\cdot,z)\tilde{N}(ds,dz)$, $t \geq 0$ are indistinguishable.

□

Remark 3.3.5. Let $M^p_{\mathcal{K}}(E)$ be the set of equivalence classes of functions from $\mathcal{M}^p_{\mathcal{K}}(E)$, let $M^p(\mathcal{BFZ};E)$ be the set of equivalence classes of processes from $\mathcal{M}^p(\mathcal{BFZ};E)$ and let $M^p(\hat{\mathcal{P}};E)$ be the set of equivalence classes of processes from $\mathcal{M}^p(\hat{\mathcal{P}};E)$.

The above Theorem 3.3.4 indicates that all stochastic integrals of processes in $M^p_{\mathcal{K}}(E)$ are indistinguishable from the stochastic integrals of processes in $M^p(\hat{\mathcal{P}};E)$. We do not create new stochastic integrals for adapted measurable processes. The class of all stochastic integrals of adapted measurable processes coincides with the set of all stochastic integrals of predictable processes.

Corollary 3.3.6. *The spaces $M^p_{\mathcal{K}}(E)$ and $M^p(\hat{\mathcal{P}};E)$ are isometric.*

Proof. The proof follows immediately once we observe that $[f] = [\hat{f}]$, where $[f]$ (resp. $[\hat{f}]$) is the equivalence class induced by the function $f \in \mathcal{M}^p_{\mathcal{K}}(E)$ (resp. $\mathcal{M}^p(\hat{\mathcal{P}};E)$). □

3.4 Relationship between different types of stochastic integrals

Throughout this section let us fix $1 \leq p \leq 2$ and E be a martingale type p Banach space. Let N be a Poisson random measure associated to a Poisson point process π with intensity ν. Now we will introduce three classes of functions for which we can define stochastic integrals and Bochner integrals.

⋆ Let $\mathcal{M}^p_{loc}(\hat{\mathcal{P}},\nu;E)$, $1 < p \leq 2$, be the space of all $\mathfrak{F}$-predictable E-valued functions such that for each $T > 0$,

$$\mathbb{E}\int_0^T \int_Z \|f(s,\cdot,z)\|^p \nu(dz)ds < \infty. \tag{3.4.1}$$

Remark 3.4.1. This assumption (3.4.1) is somehow weaker than the assumption that

$$\mathbb{E}\int_0^\infty \int_Z \|f(s,\cdot,z)\|^p \nu(dz)ds < \infty.$$

But this assumption (3.4.1) provides that for every $T > 0$, the functions $f1_{(0,T]}$, $T > 0$ are all in $\mathcal{M}(\hat{\mathcal{P}};E)$, hence for every T, the stochastic integrals $\int_0^T \int_Z f(s,\cdot,z)\tilde{N}(ds,dz)$, $T > 0$ are well defined, on the basis of Theorems 3.2.27 and 3.3.2, and possess the properties in Theorem 3.3.2.

⋆ Let $\mathcal{M}^1_{loc}(\hat{\mathcal{P}},\nu;E)$ be the space of all $\mathfrak{F}$-predictable E-valued functions such that for each $T > 0$,

$$\mathbb{E}\int_0^T \int_Z \|f(s,\cdot,z)\|\nu(dz)ds < \infty.$$

Remark 3.4.2. In view of Section 3.2.3, the function f is Bochner integrable w.r.t. the measure $\nu \otimes \lambda$. Furthermore, in Proposition 3.4.7 we will show that the function f is Bochner integrable w.r.t. the compensated Poisson random measure N, $\mathbb{P}$-a.s.

⋆ Let $\mathcal{M}_{loc}(\hat{\mathcal{P}}, N; E)$ be the space of all $\mathfrak{F}$-predictable E-valued functions such that for each $T > 0$,

$$\mathbb{E} \int_0^T \int_Z \|f(s,\cdot,z)\| N(ds,dz) < \infty. \tag{3.4.2}$$

Here $\int_0^T \int_Z \|f(s,\cdot,z)\| N(ds,dz)(\omega)$ is defined as Lebesgue integral with respect to the measure $N(\omega,\cdot)$ for every $\omega \in \Omega$ and is equal to the convergent sum, see [40],

$$\int_0^T \int_Z \|f(s,\cdot,z)\| N(ds,dz) = \sum_{s \in (0,T] \cap \mathcal{D}(\pi)} \|f(s,\cdot,\pi(s))\|.$$

Remark 3.4.3. Theorem 4.6 in [42] tells us that the condition

$$\sum_{s \in (0,T] \cap \mathcal{D}(\pi)} \|f(s,\cdot,\pi(s))\| < \infty, \text{ a.s.}$$

is equivalent to the following two conditions

$$\int_0^T \int_Z \|f(s,\cdot,z)\| \wedge 1 \; \nu(dz) ds < \infty, \quad \text{a.s.}$$

and

$$\int_0^T \int_Z \Big(1 - exp\{-\|f(s,\cdot,z)\|\}\Big) \nu(dz) ds < \infty, \quad \text{a.s.}$$

In such a case, we may define the integral $\int_0^t \int_Z f(s,\omega,z) N(ds,dz)$ by the formula

$$\begin{aligned} \Big[\int_0^t \int_Z f(s,z) N(ds,dz)\Big](\omega) &:= \sum_{s \in (0,t] \cap \mathcal{D}(\pi)} f(s,\omega,\pi(s)) \\ &= \sum_{s \in (0,t] \cap \mathcal{D}(\pi(\omega))} f(s,\omega,\pi(s,\omega)), \end{aligned} \tag{3.4.3}$$

since the series is absolutely convergent.

Remark 3.4.4. Since for every $\omega \in \Omega$, $N(\cdot)(\omega)$ is a measure on $(\mathbb{R}_+ \times Z, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{Z})$, one can also define, for every $\omega \in \Omega$, the integral $\int_0^T \int_Z f(s,\omega,z) N(ds,dz)(\omega)$ in terms of the Bochner integral introduced in the Section 3.2.3. More precisely, if f is a simple function of the form

$$f(t,\omega,z) = \sum_{j=1}^{n} \sum_{i=1}^{m_j} a_j 1_{F_{i-1}^j}(\omega) 1_{(t_{i-1}^j, t_i^j]}(t) 1_{B_{i-1}^j}(z) \tag{3.4.4}$$

where $F^j_{i-1} \in \mathcal{F}_{t^j_{i-1}}$, $i = 1, \cdots, m_j$, $j = 1, \cdots, n$, and $\{B^j_{i-1}\}_{i,j} \subset \mathcal{Z}$ are pairwise disjoint and $\nu(B^j_{i-1}) < \infty$, $i = 1, \cdots, m_j$, $j = 1, \cdots, n$. We may define

$$\begin{aligned}\int_0^T \int_Z f(t,\omega,z)N(ds,dz) &:= \sum_{j=1}^n \sum_{i=1}^{m_j} a_j 1_{F^j_{i-1}}(\omega) N((t^j_{i-1} \wedge T, t^j_i \wedge T] \times B^j_{i-1}) \\ &= \sum_{j=1}^n \sum_{i=1}^{m_j} a_j 1_{F^j_{i-1}}(\omega) \sum_{s\in(t^j_{i-1}\wedge T, t^j_i \wedge T]\cap \mathcal{D}(\pi)} 1_{B^j_{i-1}}(\pi(s)) \\ &= \sum_{s\in(0,T]\cap\mathcal{D}(\pi)} f(s,\omega,\pi(s))\end{aligned}$$

It is easy to see that for simple function of the form (3.4.4), we have

$$\|\int_0^T \int_Z f(t,\omega,z)N(ds,dz)\| \leq \int_0^T \int_Z \|f(t,\omega,z)\| N(ds,dz).$$

From Theorem 3.2.23, recall that for every E-valued and $\hat{\mathcal{P}}$ function f, there exists a sequence $\{f_n\}$ of simple functions of the form (3.4.4) such that the sequence

$$\|f_n(t,\omega,z) - f(t,\omega,z)\|$$

converges to 0, for all $(t,\omega,z) \in M$, where $M \subset \mathbb{R}_+ \times \Omega \times Z$ and $\lambda \otimes \mathbb{P} \otimes \nu(M^c) = 0$. A similar argument as in the proof of the Proposition 3.4.5 we have

$$\|f_n(t,\omega,\pi(t,\omega)) - f(t,\omega,\pi(t,\omega))\|$$

converges to 0 $\mathbb{P}$-a.s. as $n \to \infty$, for all $t \geq 0$. Then the series

$$\sum_{s\in(0,T]\cap\mathcal{D}(\pi)} \|f^n(s,\cdot,\pi(s)) - f(s,\cdot,\pi(s))\|$$

converges to 0 $\mathbb{P}$-a.s. provided the series $\sum_{s\in(0,T]\cap\mathcal{D}(\pi)} \|f(s,\cdot,\pi(s))$ is convergent $\mathbb{P}$-a.s. We know by assumption that $\int_0^T \int_Z \|f(s,z)\| N(ds,dz) < \infty$, $\mathbb{P}$-a.s. Put it in other words, we have

$$\sum_{s\in(0,T]\cap\mathcal{D}(\pi)} \|f(s,\cdot,\pi(s))\| < \infty \ \mathbb{P}\text{-a.s.}$$

Hence we infer that the series $\sum_{s\in(0,T]\cap\mathcal{D}(\pi)} f^n(s,\cdot,\pi(s))$ converges to $\sum_{s\in(0,T]\cap\mathcal{D}(\pi)} f(s,\cdot,\pi(s))$ $\mathbb{P}$-a.s. and

$$\begin{aligned}&\lim_{n\to\infty} \int_0^T \int_Z \|f^n(t,\omega,z) - f(t,\omega,z)\| N(ds,dz) \\ &\quad = \lim_{n\to\infty} \sum_{s\in(0,T]\cap\mathcal{D}(\pi)} \|f^n(s,\cdot,\pi(s)) - f(s,\cdot,\pi(s))\| \to \infty, \ \mathbb{P}\text{-a.s. as}\end{aligned}$$

Now we may define the Bochner integral of f by

$$\int_0^T \int_Z f(s,\cdot,z)N(ds,dz) := \lim_{n\to} \int_0^T \int_Z f^n(s,\cdot,z)N(ds,dz) \ \mathbb{P}\text{-a.s.}$$

In this case,

$$\int_0^T \int_Z f(s,\cdot,z)N(ds,dz) = \sum_{s\in(0,T]\cap\mathcal{D}(\pi)} f(s,\cdot,\pi(s)), \ \mathbb{P}\text{-a.s.}$$

If $f \in \mathcal{M}^1_{loc}(\hat{\mathcal{P}}, \nu; E) \cap \mathcal{M}_{loc}(\hat{\mathcal{P}}, N; E)$, we may define the integral w.r.t. the compensated Poisson random measure N by

$$(B) \int_0^t \int_Z f(s,\cdot,z)\tilde{N}(ds,dz) := \int_0^t \int_Z f(s,\cdot,z)N(ds,dz) - \int_0^t \int_D f(s,\cdot,z)\nu(dz)ds, \ \mathbb{P}\text{-a.s.} \quad (3.4.5)$$

Here the second integral is understood as Bochner integral w.r.t. the measure $\lambda \otimes \nu$. We call $\int_0^t \int_Z \xi(s,\cdot,z)\tilde{N}(ds,dz)$ the Bochner integral w.r.t. the Compensated Poisson random measure $\tilde{N}$.

Let us now investigate some relationships between the above three classes of $\mathfrak{F}$-predictable functions. We first consider an important result connecting the stochastic integrals and the Bochner integrals on finite ν-measure sets.

Proposition 3.4.5. *Assume that* $1 \le p \le 2$. *Let* $D \in \mathcal{Z}$ *with* $\mathbb{E}(N(t,D)) < \infty$. *Suppose that* $\xi \in \mathcal{M}^p_{loc}(\hat{\mathcal{P}}, \nu; E)$. *Then for every* $t \ge 0$,

$$\begin{aligned}\int_0^t \int_D \xi(s,z)\tilde{N}(ds,dz) &= (B)\int_0^t \int_D \xi(s,\cdot,z)\tilde{N}(ds,dz) \\ &= \sum_{s\in(0,t]\cap\mathcal{D}(\pi)} \xi(s,\pi(s))1_D(\pi(s)) - \int_0^t \int_D \xi(s,z)\nu(dz)ds, \quad \mathbb{P}\text{-}a.s. \quad (3.4.6)\end{aligned}$$

Before proceeding the proof of Proposition 3.4.5, let us give an auxiliary Lemma.

Lemma 3.4.6. *Let* $M \subset (0,\infty)\times\Omega\times Z$ *with* $M \in \hat{\mathcal{P}}$ *and* $\lambda\otimes\mathbb{P}\otimes\nu(M^c) = 0$. *Then*

$$\mathbb{P}\{\omega \in \Omega : (s,\omega,\pi(s,\omega)) \in M^c\} = 0.$$

Proof of Lemma 3.4.6. In particular, if M^c is of the form $(t_1,t_2]\times F\times B$, $F \in \mathcal{F}_{t_1}$, $B \in \mathcal{Z}$, then we infer

$$\begin{aligned}\mathbb{P}\{\omega : (s,\omega,\pi(s,\omega)) \in (t_1,t_2]\times F\times B\} &= \mathbb{E}(1_{(t_1,t_2]}(s)1_B(\pi(s))1_F(\omega)) \\ &= \mathbb{P}\{\omega : s\in(t_1,t_2], \pi(s)\in B\}\mathbb{P}(F) \\ &\le \mathbb{P}(N((t_1,t_2]\times B) \ge 1)\mathbb{P}(F) \\ &= \mathbb{E}(1_{\{N((t_1,t_2]\times B)\ge 1\}})\mathbb{P}(F) \\ &\le \mathbb{E}(N((t_1,t_2)\times B)\mathbb{P}(F) \\ &= \lambda\otimes\mathbb{P}\otimes\nu((t_1,t_2]\times F\times B) = 0,\end{aligned}$$

which shows our assertion for $M^c = (t_1,t_2]\times F\times B$. Since $M^c \in \hat{\mathcal{P}}$, by Theorem D in [34] p.56, there exists a decreasing sequence $\{M_n\}_{n\in\mathbb{N}} \subset \hat{\mathcal{R}}$ such that $\cap_{n=1}^\infty M_n = M^c$ and

$$\lim_{n\to\infty} \lambda\otimes\mathbb{P}\otimes\nu(M_n) = \lambda\otimes\mathbb{P}\otimes\nu(M^c).$$

Moreover, each set M_n is a finite union of disjoint sets D_i^n, $i = 1,\cdots,k_n$ of the form

$$D_i^n = (t_i^n, t_{i+1}^n]\times F_i^n \times B_i^n.$$

This gives that

$$M^c = \cap_{n=1}^\infty \cup_{i=1}^{k_n} D_i^n.$$

It follows that

$$
\begin{aligned}
\mathbb{P}\{\omega : (s,\omega,\pi(s,\omega)) \in M^c\} &= \lim_{n\to\infty} \mathbb{P}\{\omega : (s,\omega,\pi(s,\omega)) \in M_n\} \\
&= \lim_{n\to\infty} \mathbb{P}\{\omega : (s,\omega,\pi(s,\omega)) \in \cup_{i=1}^{k_n} D_i^n\} \\
&= \lim_{n\to\infty} \sum_{i=1}^{k_n} \mathbb{P}\{\omega : (s,\omega,\pi(s,\omega)) \in (t_i^n, t_{i+1}^n] \times F_i^n \times B_i^n\} \\
&= \lim_{n\to\infty} \sum_{i=1}^{k_n} \mathbb{P}\{\omega : s \in (t_i^n, t_{i+1}^n], \pi(s) \in B_i^n\}\mathbb{P}(F_i^n) \\
&\leq \lim_{n\to\infty} \sum_{i=1}^{k_n} \mathbb{P}\{N((t_i^n, t_{i+1}^n] \times B_i^n) \geq 1\}\mathbb{P}(F_i^n) \\
&\leq \lim_{n\to\infty} \sum_{i=1}^{k_n} \mathbb{E}(N((t_i^n, t_{i+1}^n] \times B_i^n)\mathbb{P}(F_i^n) \\
&= \lim_{n\to\infty} \sum_{i=1}^{k_n} \lambda \otimes \mathbb{P} \otimes \nu((t_i^n, t_{i+1}^n] \times B_i^n \times F_i^n) \\
&= \lambda \otimes \mathbb{P} \otimes \nu(M^c) = 0.
\end{aligned}
$$

This completes the proof. □

Proof of Proposition 3.4.5. We first show that Equality (3.4.6) holds for step function of the form (3.2.6)

$$\xi(t,\omega,z) = \sum_{j=1}^{n}\sum_{k=1}^{m} \xi_{j-1}^k(\omega) 1_{(t_{j-1},t_j]}(t) 1_{A_{j-1}^k}(z).$$

Then the stochastic integral of ξ is given by

$$\int_0^t \int_D \xi(s,z)\tilde{N}(ds,dz) = \sum_{j=1}^{n}\sum_{k=1}^{m} \xi_{j-1}^k \tilde{N}((t_{j-1},t_j] \times (D \cap A_{j-1}^k)).$$

By the definition of compensated Poisson random measure, we obtain

$$
\begin{aligned}
\int_0^t \int_D \xi(s,z)\tilde{N}(ds,dz) &= \sum_{j=1}^{n}\sum_{k=1}^{m} \xi_{j-1}^k \tilde{N}((t_{j-1},t_j] \times (D \cap A_{j-1}^k)) \\
&= \sum_{j=1}^{n}\sum_{k=1}^{m} \xi_{j-1}^k N((t_{j-1},t_j] \times (D \cap A_{j-1}^k)) - \sum_{j=1}^{n}\sum_{k=1}^{m} \xi_{j-1}^k \nu(D \cap A_{j-1}^k)(t_j - t_{j-1}) \\
&= \sum_{j=1}^{n}\sum_{k=1}^{m} \sum_{s\in(t_{j-1},t_j]\cap\mathcal{D}(\pi)} \xi_{j-1}^k 1_{D\cap A_{j-1}^k}(\pi(s)) - \sum_{j=1}^{n}\sum_{k=1}^{m} \xi_{j-1}^k \nu(D \cap A_{j-1}^k)(t_j - t_{j-1}).
\end{aligned}
$$

On the other side, we have

$$
\begin{aligned}
&\sum_{s\in(0,t]\cap\mathcal{D}(\pi)} \xi(s,\pi(s))1_D(\pi(s)) - \int_0^t \int_D \xi(s,z)\nu(dz)ds \\
&= \sum_{s\in(0,t]\cap\mathcal{D}(\pi)} 1_D(\pi(s)) \sum_{j=1}^{n}\sum_{k=1}^{m} \xi_{j-1}^k 1_{(t_{j-1},t_j]}(s) 1_{A_{j-1}^k}(\pi(s)) - \sum_{j=1}^{n}\sum_{k=1}^{m} \xi_{j-1}^k \nu(D \cap A_{j-1}^k)(t_j - t_{j-1})
\end{aligned}
$$

$$= \sum_{j=1}^{n}\sum_{k=1}^{m} \sum_{s\in(0,t]\cap\mathcal{D}(\pi)} 1_D(\pi(s))\xi_{j-1}^k 1_{(t_{j-1},t_j]}(s)1_{A_{j-1}^k}(\pi(s)) - \sum_{j=1}^{n}\sum_{k=1}^{m}\xi_{j-1}^k\nu(D\cap A_{j-1}^k)(t_j-t_{j-1})$$

$$= \sum_{j=1}^{n}\sum_{k=1}^{m} \sum_{s\in(t_{j-1},t_j]\cap\mathcal{D}(\pi)} \xi_{j-1}^k 1_{D\cap A_{j-1}^k}(\pi(s)) - \sum_{j=1}^{n}\sum_{k=1}^{m}\xi_{j-1}^k\nu(D\cap A_{j-1}^k)(t_j-t_{j-1}).$$

Therefore, we infer

$$\int_0^t\int_D \xi(s,z)\tilde{N}(ds,dz) = \sum_{s\in(0,t]\cap\mathcal{D}(\pi)} \xi(s,\pi(s))1_D(\pi(s)) - \int_0^t\int_D \xi(s,z)\nu(dz)ds$$

which verifies (3.4.6) for step functions in $\mathcal{M}^p_{step}(\mathcal{Z};E)$.

Note that for every $\xi \in \mathcal{M}^p_{loc}(\hat{\mathcal{P}},\nu;E)$, since by assumption $t\nu(D) = \mathbb{E}N(t,D) < \infty$, we infer that

$$\int_0^t\int_D \|\xi(s,z)\|\nu(dz)ds \le \Big(\int_0^t\int_Z \|\xi(s,z)\|^p\nu(dz)ds\Big)^{\frac{1}{p}}(t\nu(D))^{1-\frac{1}{p}} < \infty,\ \mathbb{P}-a.s.$$

This implies that the integral $\int_0^t\int_D \xi(s,z)\nu(dz)ds$ is well defined as Bochner integral. Moreover, since $N(t,D) < \infty$, $\mathbb{P}$-a.s., we infer that the series $\sum_{s\in(0,t]\cap\mathcal{D}(\pi)}\xi(s,\pi(s))1_D(\pi(s))$ is convergent $\mathbb{P}$-a.s. Hence all the terms in the equality (3.4.6) are well defined.

Now we consider any function $\xi \in \mathcal{M}^p(\hat{\mathcal{P}};E)$ for which there exists a sequence $\{\xi_n\}_{n\in\mathbb{N}}$ of step functions in $\mathcal{M}^p_{step}(\mathcal{Z};E)$ such that

$$\mathbb{E}\int_0^t\int_Z 1_D(z)|\xi_n(s,z)-\xi(s,z)|_E^p\nu(dz)ds \to 0, \quad \text{as } n\to\infty. \tag{3.4.7}$$

As we have shown before, for each $n\in\mathbb{N}$, the Equality (3.4.6) holds, i.e.

$$\int_0^t\int_D \xi_n(s,z)\tilde{N}(ds,dz) = \sum_{s\in(0,t]\cap\mathcal{D}(\pi)} \xi_n(s,\pi(s))1_D(\pi(s)) - \int_0^t\int_D \xi_n(s,z)\nu(dz)ds.$$

So in order to establish Equality (3.4.6), we first observe the following

$$\begin{aligned}
&\mathbb{E}\left|\int_0^t\int_D \xi(s,z)\tilde{N}(ds,dz) - \sum_{s\in(0,t]\cap\mathcal{D}(\pi)} \xi(s,\pi(s))1_D(\pi(s)) + \int_0^t\int_D \xi(s,z)\nu(dz)ds.\right|_E^p \\
&\le 4^p\mathbb{E}\left|\int_0^t\int_D \xi(s,z)\tilde{N}(ds,dz) - \int_0^t\int_D \xi_n(s,z)\tilde{N}(ds,dz)\right|_E^p \\
&\quad + 4^p\mathbb{E}\left|\int_0^t\int_D \xi_n(s,z)\tilde{N}(ds,dz) - \sum_{s\in(0,t]\cap\mathcal{D}(\pi)} \xi_n(s,\pi(s))1_D(\pi(s)) + \int_0^t\int_D \xi_n(s,z)\nu(dz)ds\right|_E^p \\
&\quad + 4^p\mathbb{E}\left|\int_0^t\int_D \xi_n(s,z)\nu(dz)ds - \int_0^t\int_D \xi(s,z)\nu(dz)ds\right|_E^p \\
&\quad + 4^p\mathbb{E}\left|\sum_{s\in(0,t]\cap\mathcal{D}(\pi)} \xi_n(s,\pi(s))1_D(\pi(s)) - \sum_{s\in(0,t]\cap\mathcal{D}(\pi)} \xi(s,\pi(s))1_D(\pi(s))\right|_E^p
\end{aligned}$$

$$\leq 4^pC_p\mathbb{E}\int_0^t\int_D|\xi(s,z)-\xi_n(s,z)|_E^p\nu(dz)ds$$

$$+4^p\mathbb{E}\left|\int_0^t\int_D\xi_n(s,z)\tilde{N}(ds,dz)-\sum_{s\in(0,t]\cap\mathcal{D}(\pi)}\xi_n(s,\pi(s))1_D(\pi(s))+\int_0^t\int_D\xi_n(s,z)\nu(dz)ds\right|_E^p$$

$$+4^p(\nu(D)T)^{\frac{p}{p-1}}\mathbb{E}\int_0^t\int_D|\xi_n(s,z)-\xi(s,z)|_E^p\nu(dz)ds$$

$$+4^p\mathbb{E}\left|\sum_{s\in(0,t]\cap\mathcal{D}(\pi)}\xi_n(s,\pi(s))1_D(\pi(s))-\sum_{s\in(0,t]\cap\mathcal{D}(\pi)}\xi(s,\pi(s))1_D(\pi(s))\right|_E^p.$$

From above discussion, we see that the first three terms on the right side of above inequality are all converges to 0, so we only need to estimate the last term. For this, by (3.4.7) we can always find a subsequence, for simplicity of notation also denoted by $\{\xi_n\}_{n\in\mathbb{N}}$, such that

$$1_D(z)\xi_n(s,\omega,z)\to 1_D(z)\xi(s,\omega,z),\ \text{as}\ n\to\infty,\quad \text{for every}\ (s,\omega,z)\in M,$$

where $M\subset(0,t]\times\Omega\times Z$ with $M\in\hat{\mathcal{P}}$ and $\lambda\otimes\mathbb{P}\otimes\nu(M^c)=0$. In view of Lemma 3.4.6, we find that for every $s\in(0,t]$, the sequence $1_D(\pi(s,\omega))\xi_n(s,\omega,\pi(s,\omega))$ converges to $1_D(\pi(s,\omega))\xi(s,\omega,\pi(s,\omega))$ as $n\to\infty$ $\mathbb{P}$-a.s.

Moreover, we may always assume that $\|\xi_n\|\leq 2\|\xi\|$. Indeed, define $\phi_n:=1_{\{\|\xi_n\|\leq 2\|\xi\|\}}\xi_n$. Then ϕ_n is also a step function in $\mathcal{M}^p_{step}(\mathcal{Z};E)$ and we have $\lim_{n\to\infty}\phi_n=\xi$ in M and $\|\phi_n\|\leq 2\|\xi\|$.Since, by the assumption, $\mathbb{E}N(t,D)<\infty$, we know that the two serieses $\sum_{s\in(0,t]\cap\mathcal{D}(\pi)}\xi(s,\pi(s))1_D(\pi(s))$ and $\sum_{s\in(0,t]\cap\mathcal{D}(\pi)}\xi_n(s,\pi(s))1_D(\pi(s))$ are convergent, hence in the view of the above discussions and the Lebesgue Dominated Convergence theorem we infer that

$$\mathbb{E}\left|\sum_{s\in(0,t]\cap\mathcal{D}(\pi)}\xi_n(s,\pi(s))1_D(\pi(s))-\sum_{s\in(0,t]\cap\mathcal{D}(\pi)}\xi(s,\pi(s))1_D(\pi(s))\right|_E^p\to 0.$$

In conclusion, we obtain

$$\mathbb{E}\left|\int_0^t\int_D\xi(s,z)\tilde{N}(ds,dz)-\sum_{s\in(0,t]\cap\mathcal{D}(\pi)}\xi(s,\pi(s))1_D(\pi(s))+\int_0^t\int_D\xi(s,z)\nu(dz)ds\right|_E^p=0,$$

which completes the proof. □

Proposition 3.4.7. *If $f\in\mathcal{M}^1_{loc}(\hat{\mathcal{P}},\nu;E)$, then we have $f\in\mathcal{M}_{loc}(\hat{\mathcal{P}},N;E)$ and for each $t\geq 0$,*

$$\mathbb{E}\int_0^t\int_Z f(s,\cdot,z)N(ds,dz)=\mathbb{E}\int_0^t\int_Z f(s,\cdot,z)\nu(dz)ds. \tag{3.4.8}$$

In particular, if $f\in\mathcal{M}^1_{loc}(\hat{\mathcal{P}},\nu;E)\cap\mathcal{M}^p_{loc}(\hat{\mathcal{P}},\nu;E)$, then we have for each $t\geq 0$, $\mathbb{P}$-a.s.

$$\begin{aligned}\int_0^t\int_Z f(s,\cdot,z)\tilde{N}(ds,dz)&=(B)\int_0^t\int_Z f(s,\cdot,z)\tilde{N}(ds,dz)\\&=\sum_{s\in(0,t]\cap\mathcal{D}(\pi)}f(s,\cdot,\pi(s))-\int_0^t\int_Z f(s,\cdot,z)\nu(dz)ds.\end{aligned} \tag{3.4.9}$$

Here the integral $\int_0^t\int_Z f(s,\cdot,z)\tilde{N}(ds,dz)$ on the left side is understood as stochastic integral.

Proof. The proof could be done exactly the same manner as earlier in the proof of Proposition 3.4.5. First the Equality (3.4.8) can be verified for a class of step functions with particularly simple structure. Next, an approximating step allows us to extend the equality to a general $\mathfrak{F}$-predictable process in $\mathcal{M}^1_{loc}(\hat{\mathcal{P}}, \nu; E)$. To do this, suppose that $f \in \mathcal{M}_{step}(Z; E)$ of the form (3.2.6)

$$f(t, \omega, z) = \sum_{j=1}^{n} \sum_{k=1}^{m} f_{j-1}^k(\omega) 1_{(t_{j-1}, t_j]}(t) 1_{A_{j-1}^k}(z) 1_D(z),$$

where $D \in \mathcal{Z}$ with $\mathbb{E}N(t, D) < \infty$, for every $t \in \mathbb{R}_+$. It follows that

$$\begin{aligned}
\mathbb{E} \int_0^t \int_Z f(t, z) N(ds, dz) &= \mathbb{E} \sum_{s \in (0,t] \cap \mathcal{D}(\pi)} f(s, \pi(s)) \\
&= \mathbb{E} \sum_{s \in (0,t] \cap \mathcal{D}(\pi)} \sum_{j=1}^{n} \sum_{k=1}^{m} f_{j-1}^k 1_{(t_{j-1}, t_j]}(s) 1_{A_{j-1}^k \cap D_n}(\pi(s)) \\
&= \sum_{j=1}^{n} \sum_{k=1}^{m} \mathbb{E} \sum_{s \in (0,t] \cap \mathcal{D}(\pi)} f_{j-1}^k 1_{(t_{j-1}, t_j]}(s) 1_{A_{j-1}^k}(\pi(s)) \\
&= \sum_{j=1}^{n} \sum_{k=1}^{m} \mathbb{E} f_{j-1}^k N((t_{j-1} \wedge t, t_j \wedge t] \times (A_{j-1}^k \cap D_n)) \\
&= \sum_{j=1}^{n} \sum_{k=1}^{m} \mathbb{E} f_{j-1}^k \mathbb{E} N((t_{j-1} \wedge t, t_j \wedge t] \times (A_{j-1}^k \cap D_n)) \\
&= \sum_{j=1}^{n} \sum_{k=1}^{m} \mathbb{E} f_{j-1}^k \nu(A_{j-1}^k \cap D_n)(t_j \wedge t - t_{j-1} \wedge t) \\
&= \mathbb{E} \int_0^t \int_Z f(s, z) \nu(dz) ds.
\end{aligned}$$

A similar argument shows that

$$\mathbb{E} \int_0^t \int_Z \|f(t, z)\| N(ds, dz) = \mathbb{E} \int_0^t \int_Z \|f(s, z)\| \nu(dz) ds.$$

By Theorem 3.2.23 there exists a sequence of functions $\{g^n\} \subset \mathcal{M}^p_{step}(\mathcal{Z}; E)$ such that

$$\mathbb{E} \int_0^t \int_Z \|f(s, \cdot, z) - g^n(s, \cdot, z)\| \nu(dz) ds \to 0, \quad \text{as } n \to \infty.$$

Since π is σ-finite, there exists a sequence $\{D_n\}_{n \in \mathbb{N}} \subset Z$ of sets such that $\mathbb{E}N(t, D_n) < \infty$, for all $t \in \mathbb{R}_+$ and $n \in \mathbb{N}$ and $D_n \nearrow Z$. Define

$$f^n(s, \omega, z) = g^n(s, \omega, z) 1_{D_n}(z), \quad (t, \omega, z) \in \mathbb{R}_+ \times \Omega \times Z. \tag{3.4.10}$$

Clearly, we have

$$E \int_0^t \int_Z \|f(s, \cdot, z) - f^n(s, \cdot, z)\| \nu(dz) ds \to 0, \quad \text{as } n \to \infty.$$

Since $\mathbb{E}N(t, D_n) < \infty$, for all $t \in \mathbb{R}_+$ and $n \in \mathbb{N}$, it follows from Proposition 3.4.5 that

$$\begin{aligned} &\mathbb{E}\Big|\int_0^t \int_Z \|f^n(s,z)\| N(ds,dz) - \int_0^t \int_Z \|f^m(s,z)\| N(ds,dz)\Big| \\ &\le \mathbb{E}\int_0^t \int_Z \|f^n(s,z) - f^m(s,z)\| N(ds,dz) \\ &= \mathbb{E}\int_0^t \int_Z \|f^n(s,z) - f^m(s,z)\| \nu(dz)ds, \end{aligned} \tag{3.4.11}$$

which shows that $\int_0^t \int_Z \|f^n(s,z)\| N(ds,dz)$ is a Cauchy sequence in $L^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$. Hence the Cauchy sequence $\int_0^t \int_Z \|f^n(s,z)\| N(ds,dz)$ has a limit in $L^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$. From (3.4.10) and (3.4.11), it allows us to find a subsequence, still denoted by $\{f^n\}$, such that $\int_0^t \int_Z \|f^n(s,z)\| N(ds,dz)$ is convergent $\mathbb{P}$-a.s. and $f^n(t,\omega,z)$ converges to $f(t,\omega,z)$, for all $(t,\omega,z) \in M$, as $n \to \infty$, where $M \subset \mathbb{R}_+ \times \Omega \times Z$ with $M \in \hat{\mathcal{P}}$ and $\lambda \otimes \mathbb{P} \otimes \nu(M^c) = 0$. Therefore, by Lemma 3.4.6, we have

$$f_n(s,\omega,\pi(s,\omega)) \to f(s,\omega,\pi(s,\omega)), \text{ for all } s \ge 0, \text{ as } n \to \infty, \ \mathbb{P}\text{-a.s.}$$

As we have noted before, the sequence $\sum_{s\in(0,t]\cap\mathcal{D}} \|f^n(s,\omega,\pi(s,\omega))\|$ is convergent $\mathbb{P}$-a.s. Hence we conclude that

$$\sum_{s\in(0,t]\cap\mathcal{D}} \|f(s,\omega,\pi(s,\omega))\| < \infty, \ \mathbb{P}\text{-a.s.}$$

and

$$\sum_{s\in(0,t]\cap\mathcal{D}} f_n(s,\omega,\pi(s,\omega)) \to \sum_{s\in(0,t]\cap\mathcal{D}} f(s,\omega,\pi(s,\omega)), \quad \text{as } n \to \infty, \ \mathbb{P}\text{-a.s.}$$

which shows $\sum_{s\in(0,t]\cap\mathcal{D}(\pi)} \|f(s,\pi(s))\| < \infty$, $\mathbb{P}$-a.s. Hence applying the Lebesgue Dominated Convergence Theorem yields that

$$\lim_{n\to\infty} \mathbb{E}\Big\|\int_0^t \int_Z f^n(s,z)N(ds,dz) - \int_0^t \int_Z f(s,z)N(ds,dz)\Big\| = 0.$$

It follows that

$$\begin{aligned} &\mathbb{E}\Big[\sum_{s\in(0,t]\cap\mathcal{D}(\pi)} f(s,\pi(s)) - \int_0^t \int_Z f(s,z)\nu(dz)ds\Big] \\ &= \mathbb{E}\Big[\sum_{s\in(0,t]\cap\mathcal{D}(\pi)} f(s,\pi(s)) - \sum_{s\in(0,t]\cap\mathcal{D}(\pi)} f^n(s,\pi(s))\Big] + \mathbb{E}\Big[\int_0^t \int_Z f^n(s,z)\nu(dz)ds - \int_0^t \int_Z f(s,z)\nu(dz)ds\Big] \\ &\le \mathbb{E}\Big[\sum_{s\in(0,t]\cap\mathcal{D}(\pi)} \|f(s,\pi(s)) - f^n(s,\pi(s))\|\Big] + \mathbb{E}\Big[\int_0^t \int_Z \|f^n(s,z) - f(s,z)\|\nu(dz)ds\Big]. \end{aligned}$$

Letting $n \to \infty$ gives that

$$\mathbb{E}\Big[\sum_{s\in(0,t]\cap\mathcal{D}(\pi)} f(s,\pi(s)) - \int_0^t \int_Z f(s,z)\nu(dz)ds\Big] = 0,$$

which proves the equality (3.4.8). The equalty (3.4.9) can be done the same as in the proof of Proposition 3.4.5. Since we have already done the tedious work in the previous proofs, we shall not repeat it here. □

Remark 3.4.8. The integral $\int_0^t \int_Z f(s,z)\tilde{N}(ds,dz)$ on the left side of (3.4.9) is defined as the stochastic integral, since by assumption $f \in \mathcal{M}^p_{loc}(\hat{\mathcal{P}},\nu;E)$. However, in general, the stochastic integral could not be divided as the difference of two integrals $\int_0^t \int_Z f(s,z)N(ds,dz)$ and $\int_0^t \int_Z \xi(s,z)\nu(dz)ds$ as in (3.4.9) without the assumption $f \in \mathcal{M}^1_{loc}(\hat{\mathcal{P}},\nu;E)$. Because both integrals $\int_0^t \int_Z f(s,z)N(ds,dz)$ and $\int_0^t \int_Z \xi(s,z)\nu(dz)ds$ may have no meaning at all.

Theorem 3.4.9. *Let $f \in \mathcal{M}^p(\hat{\mathcal{P}};E)$. Let $\{D_n\}$ be a sequence of sets in $\mathcal{Z}$ such that $D_n \nearrow Z$ and $\mathbb{E}N(t,D_n) < \infty$, for all $n \in \mathbb{N}$ and $t \geq 0$. Let $f_n(t,\omega,z) = 1_{\|f\|\leq n}(t,\omega,z)1_{D_n}(z)f(t,\omega,z)$. Then*

$$\lim_{n\to\infty} \mathbb{E}\Big\| \int_0^t \int_Z f(s,z)\tilde{N}(ds,dz) - (B)\int_0^t \int_Z f_n(s,z)\tilde{N}(ds,dz)\Big\|^p = 0.$$

Proof. It is easy to see that

$$\|f_n(t,\omega,z) - f(t,\omega,z)\|^p$$

is monotonically decreasing to 0, as $n \to \infty$, for all $(t,\omega,z) \in \mathbb{R}_+ \times \Omega \times Z$. Applying the monotone convergence theorem yields that

$$\lim_{n\to\infty} \int_0^t \int_Z \|f_n(t,\cdot,z) - f(t,\cdot,z)\|^p \nu(dz)ds = 0.$$

Furthermore, on the basis of Proposition 3.4.5, for every $n \in \mathbb{N}$, we have

$$\mathbb{E}\| \int_0^t \int_Z f_n(s,\cdot,z)\tilde{N}(ds,dz) - (B)\int_0^t \int_Z f_n(s,\cdot,z)\tilde{N}(ds,dz)\|^p = 0.$$

It follows from inequality (3.3.2) that

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}\Big\| \int_0^t \int_Z f(s,\cdot,z)\tilde{N}(ds,dz) - (B)\int_0^t \int_Z f_n(s,z)\tilde{N}(ds,dz)\Big\|^p \\
&\leq 2^p \lim_{n\to\infty} \mathbb{E}\Big\| \int_0^t \int_Z f(s,\cdot,z)\tilde{N}(ds,dz) - \int_0^t \int_Z f_n(s,\cdot,z)\tilde{N}(ds,dz)\Big\|^p \\
&+ 2^p \lim_{n\to\infty} \mathbb{E}\Big\| \int_0^t \int_Z f_n(s,\cdot,z)\tilde{N}(ds,dz) - (B)\int_0^t \int_Z f_n(s,z)\tilde{N}(ds,dz)\Big\|^p \\
&\leq C_p \lim_{n\to\infty} \mathbb{E} \int_0^t \int_Z \|f(s,\cdot,z) - f_n(s,\cdot,z)\|^p \nu(dz)ds \\
&+ 2^p \lim_{n\to\infty} \mathbb{E}\Big\| \int_0^t \int_Z f_n(s,\cdot,z)\tilde{N}(ds,dz) - (B)\int_0^t \int_Z f_n(s,z)\tilde{N}(ds,dz)\Big\|^p \\
&= 0.
\end{aligned}$$

This completes the proof. □

Remark 3.4.10. Let $\mathcal{M}^2_T$ be the space of all real-valued square integrable martingales on $[0,T] \times \Omega$. Let $X \in \mathcal{M}^2_T$. Set $\|X\|_T = (\mathbb{E}\|X_T\|^2)^{1/2}$. Then $(\mathcal{M}^2_T, \|\cdot\|_T)$ is a Banach space. See [70] for a proof of this statement. Recall that in [40] and [70] the stochastic integral, which we shall call the Ikeda-Watanabe stochastic integral w.r.t. the c.P.r.m., is defined as a limit of a sequence of Bochner integrals

$$\int_0^t \int_Z f_n(s,\cdot,z)\tilde{N}(ds,dz) = \int_0^t \int_Z f_n(s,\cdot,z)N(ds,dz) - \int_0^t \int_Z f_n(s,\cdot,z)\nu(dz)ds$$

of functions f_n defined by $f_n = 1_{\|f\|\leq n}1_{D_n}(z)f$ in the space $(\mathcal{M}^2_T, \|\cdot\|_T)$.

The above Theorem 3.4.9 tells us that the stochastic integrals we defined in this thesis, which is approximating by a sequence of step functions with simple structures, is actually equal to the Ikeda-Watanabe stochastic integrals defined in [40]. The following Example 3.4.11 illustrates that the stochastic integrals agrees with the Bochner integrals if and only if the function f is predictable.

3.4.1 An Example

Example 3.4.11. Let N be a time homogenous Poisson random measure with intensity ν defined on $(\mathbb{R}_+ \times Z, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{Z})$. Let us take $A \in \mathcal{Z}$ with $\nu(A) < \infty$. We know from Lemma 3.1.13 that $N(t, A)$, $t \geq 0$ is a Poisson process with parameter $\nu(A)$. Set $N(t) := N(t, A)$, $t \geq 0$. Define the compensated Poisson process $\tilde{N}(t)$ by

$$\tilde{N}(t) = N(t) - \nu(A)t.$$

We are going to define two stochastic integrals

$$\int_0^t \int_Z N(s) 1_A(z) \tilde{N}(ds, dz)$$

and

$$\int_0^t \int_Z N(s-) 1_A(z) \tilde{N}(ds, dz).$$

Remark 3.4.12. Recall that the Poisson process $N(s)$, $s \geq 0$ is a càdlàg process which is optional, hence progressively measurable, but not predictable. However, the process $N(s-)$ is càglàd, so it is predictable.

Note that

$$\begin{aligned}
&\mathbb{E}(N_t) = \nu(A)t;\\
&\mathbb{E}(N_t|\mathcal{F}_s) = N_s + \nu(A)(t-s);\\
&\mathbb{E}(N_t^2) = \nu(A)^2 t^2 + \nu(A)t;\\
&\mathbb{E}(N_t^3) = (\nu(A)t)^3 + 3(\nu(A)t)^2 + \nu(A)t;\\
&\mathbb{E}(N_t)^4 = (\nu(A)t)^4 + 6(\nu(A)t)^3 + 7(\nu(A)t)^2 + \nu(A)t.
\end{aligned}$$

Let $0 = t_0^n < t_1^n < \cdots < t_n^n = t$ be a partition of a finite interval $[0, t]$, where $t_i^n = \frac{it}{n}$. Let us take the following simple functions

$$f^n(t) = \sum_{i=0}^{n-1} N(t_i^n) 1_{(t_i^n, t_{i+1}^n]}(t) 1_A, \ t \in [0, \infty).$$

Observe that

$$\begin{aligned}
\mathbb{E}\int_0^t \int_Z |N(s-)1_A(x) - f^n(s,z)|^2 d\nu(dz)ds &= \mathbb{E}\int_0^t \int_Z \left|N(s-) - \sum_{i=0}^{n-1} N(t_i) 1_{(t_i^n, t_{i+1}^n]}(s)\right|^2 1_A(z)\nu(dz)ds\\
&= \nu(A)\mathbb{E}\int_0^t \sum_{i=0}^{n-1} |N(s-) - N(t_i^n)|^2 1_{(t_i^n, t_{i+1}^n]}(s)ds\\
&= \nu(A)\sum_{i=0}^{n-1} \int_{t_i^n}^{t_{i+1}^n} \mathbb{E}|N(s-) - N(t_i^n)|^2 ds
\end{aligned}$$

$$= \nu(A) \sum_{i=0}^{n-1} \int_{t_i^n}^{t_{i+1}^n} (\nu(A)^2 (s - t_i^n)^2 + \nu(A)(s - t_i^n)) ds$$

$$= \nu(A) \sum_{i=0}^{n-1} \left(\frac{\nu(A)^2}{3} \frac{t^3}{n^3} + \frac{\nu(A)}{2} \frac{t^2}{n^2} \right)$$

$$= \frac{(\nu(A)t)^3}{3n^2} + \frac{(\nu(A)t)^2}{2n} \to 0, \qquad as\ n \to \infty.$$

Thus by the definition 3.2.20 we can compute the stochastic integral of f^n as follows

$$\begin{aligned} I_t(f^n) &= \sum_{i=0}^{n-1} N(t_i^n) \Big(\widetilde{N}(t_{i+1}^n, A) - \widetilde{N}(t_i^n, A) \Big) \\ &= \sum_{i=0}^{n-1} N(t_i^n) \Big(N(t_{i+1}^n, A) - N(t_i^n, A) - \nu(A)(t_{i+1}^n - t_i^n) \Big) \qquad (3.4.12) \\ &= \frac{1}{2} \sum_{i=0}^{n-1} (N(t_{i+1}^n)^2 - N(t_i^n)^2) - \frac{1}{2} \sum_{i=0}^{n-1} (N(t_{i+1}^n) - N(t_i^n))^2 - \sum_{i=0}^{n-1} \nu(A) N(t_i^n)(t_{i+1}^n - t_i^n) \\ &= \frac{1}{2} N(t)^2 - \frac{1}{2} \sum_{i=0}^{n-1} (N(t_{i+1}^n) - N(t_i^n))^2 - \sum_{i=0}^{n-1} \nu(A) N(t_i^n)(t_{i+1}^n - t_i^n). \end{aligned}$$

For the second summand on the right side of (3.4.12) , we will show that $\sum_{i=0}^{n-1} (N(t_{i+1}^n) - N(t_i^n))^2$ converges to $N(t)$ in $\mathcal{L}^2(\Omega)$. Indeed, we see that

$$\mathbb{E}\Big(\sum_{i=0}^{n-1} (N(t_{i+1}^n) - N(t_i^n))^2 - N(t) \Big)^2 = \mathbb{E}\Big(\sum_{i=0}^{n-1} ((N(t_{i+1}^n) - N(t_i^n))^2 - (N(t_{i+1}^n) - N(t_i^n))) \Big)^2. \qquad (3.4.13)$$

Set

$$Y_i = \big(N(t_{i+1}^n) - N(t_i^n) \big)^2 - (N(t_{i+1}^n) - N(t_i^n)).$$

Thus (3.4.13) becomes

$$\begin{aligned} \mathbb{E}(\sum_{i=0}^{n-1} Y_i)^2 &= \sum_{i=0}^{n-1} \mathbb{E}Y_i^2 + 2 \sum_{i<j} \mathbb{E}Y_i Y_j \\ &= \sum_{i=0}^{n-1} \mathbb{E}(N(t_{i+1}^n) - N(t_i^n))^4 - 2 \sum_{i=0}^{n-1} \mathbb{E}(N(t_{i+1}^n) - N(t_i^n))^3 \\ &\quad + \sum_{i=0}^{n-1} \mathbb{E}(N(t_{i+1}^n) - N(t_i^n))^2 + 2 \sum_{i<j} \mathbb{E}(\mathbb{E}(Y_i Y_j | Y_{t_j^n})) \\ &= \frac{(\nu(A)t)^4}{n^4} + 6 \frac{(\nu(A)t)^3}{n^3} + 7 \frac{(\nu(A)t)^2}{n^2} + \frac{\nu(A)t}{n} - 2 \frac{(\nu(A)t)^3}{n^3} - 6 \frac{(\nu(A)t)^2}{n^2} - 2 \frac{\nu(A)t}{n} \\ &\quad + \frac{(\nu(A)t)^2}{n^2} + \frac{\nu(A)t}{n} + 2 \sum_{i<j} \mathbb{E}Y_i \mathbb{E}Y_j \\ &= \frac{(\nu(A)t)^4}{n^4} + 4 \frac{(\nu(A)t)^3}{n^3} + 2 \frac{(\nu(A)t)^2}{n^2} + 2 \frac{(\nu(A)t)^4}{n^4} \to 0 \qquad as\ n \to \infty. \end{aligned}$$

This shows that $\sum_{i=0}^{n-1}(N(t_{i+1}^n) - N(t_i^n))^2$ converges to $N(t)$ in $\mathcal{L}^2(\Omega)$.
For the last summand of (3.4.12), by using the Cauchy-Schwartz inequality we have the following

$$\begin{aligned}
\mathbb{E}\left(\sum_{i=0}^{n-1} N(t_i^n)(t_{i+1}^n - t_i^n) - \int_0^t N(s)ds\right)^2 &= \mathbb{E}\left(\sum_{i=0}^{n-1}\int_{t_i^n}^{t_{i+1}^n}(N(t_i^n) - N(s))ds\right)^2 \\
&\le n\sum_{i=0}^{n-1}\mathbb{E}\left(\int_{t_i^n}^{t_{i+1}^n}(N(t_i^n) - N(s))ds\right)^2 \\
&\le n\sum_{i=0}^{n-1}(t_{i+1}^n - t_i^n)\mathbb{E}\left(\int_{t_i^n}^{t_{i+1}^n}(N(t_i^n) - N(s))^2 ds\right) \\
&= n\sum_{i=0}^{n-1}(t_{i+1}^n - t_i^n)\left(\frac{\nu(A)^2}{3}(t_{i+1}^n - t_i^n)^3 - \frac{\nu(A)}{2}(t_{i+1}^n - t_i^n)^2\right) \\
&= \frac{\nu(A)^2 t^4}{3n^2} - \frac{\nu(A)t^3}{2n} \to 0 \quad as\ n \to \infty.
\end{aligned}$$

Consequently,

$$I_t(f^n) \to \frac{1}{2}N(t)^2 - \frac{1}{2}N(t) - \int_0^t\int_Z N(s)1_A(z)\nu(dz)ds,$$

in $\mathcal{L}^2(\Omega)$ as $n \to \infty$.
Therefore, we have

$$\int_0^t\int_Z N(s-)\tilde{N}(ds, dz) = \frac{1}{2}N(t)^2 - \frac{1}{2}N(t) - \int_0^t\int_Z N(s)1_A(z)\nu(dz)ds. \tag{3.4.14}$$

Also, we observe that

$$\mathbb{E}\int_0^\infty\int_Z |N(s)1_A(z) - N(s-)1_A(z)|^2\nu(dz)ds = \nu(A)\int_0^\infty \mathbb{E}|N(s) - N(s-)|^2 ds = 0.$$

In view of Lemma 3.3.4, we have

$$\begin{aligned}
\int_0^t\int_Z N(s)1_A(z)\tilde{N}(ds, dz) &= \int_0^t\int_Z N(s-)1_A(z)\tilde{N}(ds, dz) \\
&= \frac{1}{2}N(t)^2 - \frac{1}{2}N(t) - \int_0^t\int_Z N(s)1_A(z)\nu(dz)ds. \qquad (3.4.15)
\end{aligned}$$

Meanwhile, the Lebesgue integral of functions $N(s)1_A$ and $N(s-)1_A(z)$ w.r.t. the Poisson random measure can be derived as follows,

$$\begin{aligned}
\int_0^t\int_Z N(s)1_A N(ds, dz) &= \sum_{s\in(0,t]\cap\mathcal{D}(\pi)} N(s)1_A(\pi(s)) = \sum_{s\le t} N(s)(\Delta N(s, A)) = \\
&= 1 + 2 + \cdots + N(t) = \frac{1}{2}N(t)^2 + \frac{1}{2}N(t); \\
\int_0^t\int_Z N(s-)1_A(z)N(ds, dz) &= \sum_{s\in(0,t]\cap\mathcal{D}(\pi)} N(s-)1_A(\pi(s)) = \sum_{s\le t} N(s-)(\Delta N(s, A)) \\
&= 1 + 2 + \cdots + (N(t) - 1) = \frac{1}{2}N(t)^2 - \frac{1}{2}N(t).
\end{aligned}$$

Therefore, we get the Lebesgue integral w.r.t. the compensated Poisson random measure

$$\begin{aligned}(L)\int_0^t\int_Z N(s)1_A\tilde{N}(ds,dz) &= \int_0^t\int_Z N(s)1_A N(ds,dz) - \int_0^t\int_Z N(s)1_A\nu(dz)ds\\ &= \frac{1}{2}N_t^2 + \frac{1}{2}N_t - \int_0^t\int_Z N(s)1_A\nu(dz)ds; \qquad (3.4.16)\\ (L)\int_0^t\int_Z N(s-)1_A\tilde{N}(ds,dz) &= \int_0^t\int_Z N(s-)1_A N(ds,dz) - \int_0^t\int_Z N(s-)1_A\nu(dz)ds\\ &= \frac{1}{2}N_t^2 - \frac{1}{2}N_t - \int_0^t\int_Z N(s)1_A\nu(dz)ds. \qquad (3.4.17)\end{aligned}$$

Here we notice that since for each fixed $\omega \in \Omega$, the function $t \mapsto N(t,\omega)$ is right continuous with left limits, the set $(\{t : N(t,\omega) \neq N(t-,\omega)\}) = 0$ is of Lebesgue measure zero, for each $\omega \in \Omega$. It follows that $\int_0^t\int_Z N(s-)1_A\nu(dz)ds = \int_0^t\int_Z N(s)1_A(z)\nu(dz)ds$. It can be seen that

$$\int_0^t\int_Z N(s-)1_A\tilde{N}(ds,dz) = (L)\int_0^t\int_Z N(s-)1_A\tilde{N}(ds,dz),$$

but

$$\int_0^t\int_Z N(s)1_A\tilde{N}(ds,dz) \neq (L)\int_0^t\int_Z N(s)1_A\tilde{N}(ds,dz),$$

which, simultaneously, verifies the Proposition 3.4.5 and shows that the $\mathfrak{F}$-predictable assumption is somehow essential.

Remark 3.4.13.

1. This example, on the other hand side, illustrates that the Poisson process is not predictable. Indeed, if the Poisson process is predictable, by Proposition 3.4.5, the stochastic integral $\int_0^t\int_Z N(s)1_A\tilde{N}(ds,dz)$ should equal to the the Bochner-Lebesgue integral $\int_0^t\int_Z N(s)1_A\tilde{N}(ds,dz)$. However, this is in fact not true.

2. As we have pointed out before, not every measurable function, or even $\mathfrak{F}$-progressively measurable function, is predictable. If a function $f(\cdot)$ has right continuous with left limits paths and it is progressively measurable, e.g. the function $N(t)1_A(z)$ defined in above Example, we usually take the function $f(t-,\cdot)$ to be the $\mathfrak{F}$-predictable version of the function $f(t,\cdot)$. It is seen that the function $f(t-,\cdot)$ is càglàd and adapted, and hence $\mathfrak{F}$-predictable. Furthermore, the stochastic integral of the function $f(t,\cdot)$ is indistinguishable with the stochastic integrable of the function $f(t-,\cdot)$, due to Theorem 3.3.4.

3.5 The Itô Formula

Let E be a martingale of type p, $1 < p \leq 2$, Banach space. In this section, we will study a version of the Itô formula for processes of the type

$$X_t = X_0 + \int_0^t a(s)ds + \int_0^t\int_Z f(s,z)\tilde{N}(ds,dz) + \int_0^t\int_Z g(s,z)N(ds,dz),\ t \geq 0. \qquad (3.5.1)$$

Here a is an E-valued progressively measurable process on the space $(\mathbb{R}_+ \times \Omega, \mathcal{B}(\mathbb{R}_+)\otimes\mathcal{F})$ such that for all $t \geq 0$, $\int_0^t \|a(s,\omega)\|ds < \infty$, $\mathbb{P}$-a.s., N is a Poisson random measure associated with a Poisson point process π, $f \in \mathcal{M}^p_{loc}(\hat{\mathcal{P}}, \nu; E)$ and $g \in \mathcal{M}_{loc}(\hat{\mathcal{P}}, N; E)$. Assume that $\|f(t,\omega,z)\|\|g(t,\omega,z)\| = 0$ for all $(t,\omega,z) \in \mathbb{R}_+ \times \Omega \times Z$. Without, loss of generality, by Theorem 3.3.2, we may assume that the process X is right-continuous with left-limits.

Remark 3.5.1. The assumption that $\|f\|\|g\| = 0$ on $\mathbb{R}_+ \times \Omega \times Z$ is somehow reasonable. For instance, if we set f to be the deterministic function $z1_{\|z\|<1}$ and $g = z1_{\|z\|\geq 1}$. Then we see that $\|f\|\|g\| = 0$ and the process X becomes

$$X_t = X_0 + \int_0^t a(s)ds + \int_0^t \int_{\|z\|<1} z\tilde{N}(ds,dz) + \int_0^t \int_{\|z\|\geq 1} zN(ds,dz),$$

which is a pure jump Lévy process. Hence the Itô formula for the process of the form 3.5.1 can be applied to Lévy processes without Gaussian components.

Before formulating the Itô formula, we will establish an auxiliary lemma whose proof is based on the ideas from [3] and [36].

Lemma 3.5.2. *Let E and G be separable Banach spaces and $\phi : E \to G$ be a Fréchet differentiable function such that the first Fréchet derivative $\phi' : E \to L(E;G)$ is $(p-1)$-Hölder continuous. In other words, for all $r > 0$, there exists $H = H(r) < \infty$ such that*

$$\sup_{\|x\|,\|y\|\leq r, x\neq y} \frac{\|\phi'(x) - \phi'(y)\|_{L(E;G)}}{\|x-y\|^{p-1}} \leq H(r). \tag{3.5.2}$$

Define, for every $x, y \in E$,

$$R(x,y) = \int_0^1 (1-\alpha)(\phi'(x+\alpha(y-x))(y-x) - \phi'(x)(y-x))d\alpha. \tag{3.5.3}$$

Then we have that

$$\phi(y) - \phi(x) - \phi'(x)(y-x) = R(x,y), \tag{3.5.4}$$

and for every $r > 0$, there exsits $C = C(r) > 0$ such that $\|R(x,y)\| \leq C\|y-x\|^p$, for all $\|x\|, \|y\| \leq r$.

Proof. Let us take $w \in E$ such that $\|w\| \leq 1$. Let $\phi^* \in G^*$, where G^* is the dual space of G. Define a function

$$F : \mathbb{R} \ni \theta \mapsto \langle \phi(x+\theta w), \phi^* \rangle \in \mathbb{R}.$$

Notice that the real-valued function F' is $(p-1)$-Hölder continuous. To see this, we find out that for $|\theta_1|, |\theta_2| \leq r$ and $\|x\| \leq r$, $\|x+\theta_1 w\|, \|x+\theta_2 w\| \leq 2r$, so by assumption, there exists $H(r) > 0$ such that

$$\begin{aligned} |F'(\theta_1) - F'(\theta_2)| &= |\langle \phi'(x+\theta_1 w)w, \phi^* \rangle - \langle \phi'(x+\theta_2 w)w, \phi^* \rangle| \\ &\leq \|\phi'(x+\theta_1 w) - \phi'(x+\theta_2 w)\|\|w\|\|\phi^*\| \\ &\leq H(r)|\theta_1 - \theta_2|^{p-1}\|w\|^p\|\phi^*\|. \end{aligned}$$

Hence it follows that

$$\sup_{\|\theta_1\|,\|\theta\|_2\leq R, \theta_1\neq\theta_2} \frac{|F'(\theta_1) - F'(\theta_2)|}{|\theta_1-\theta_2|^{p-1}} \leq H(r)\|\phi^*\|,$$

which shows the $(p-1)$-Hölder continuity of F'. Now applying the Taylor formula and the Theorem 3 in [3] to the function F yields that for $|t| \le 2r$

$$F(t) - F(0) = F'(0)t + R_F(0,t),$$

where $R_F(0,t) = \int_0^1 (1-\alpha)(F'(\alpha t) - F'(0))t \, d\alpha$ and

$$|R_F(0,t)| \le \frac{H(2r)\|\phi^*\|}{p}|t|^p, \quad |t| \le 2r.$$

Let $x, y \in E$ with $\|x\|, \|y\| \le r$. Set $t = \|y-x\|$ and $w = \frac{y-x}{\|y-x\|}$. Then $|t| \le 2r$ and $|w| \le 1$. It follows that

$$\begin{aligned} F(t) - F(0) - F'(0)t &= \langle \phi(x+tu), \phi^* \rangle - \langle \phi(x), \phi^* \rangle - \langle \phi('x)u, \phi^* \rangle t \\ &= \langle \phi(y), \phi^* \rangle - \langle \phi(x), \phi^* \rangle - \langle \phi'(x)(y-x), \phi^* \rangle \end{aligned}$$

and

$$|R_F(0,t)| = \left| \int_0^1 (1-\alpha)\langle \phi'(x+\alpha(y-x))(y-x) - \phi'(x)(y-x), \phi^* \rangle d\alpha \right| \le \frac{H(2r)\|\phi^*\|}{p}\|y-x\|^p.$$

Hence, we infer

$$\langle \phi(y), \phi^* \rangle - \langle \phi(x), \phi^* \rangle - \langle \phi'(x)(y-x), \phi^* \rangle = \int_0^1 (1-\alpha)\langle \phi'(x+\alpha(y-x))(y-x) - \phi'(x)(y-x), \phi^* \rangle d\alpha.$$

which holds for all $\phi^* \in G^*$.
In conclusion, we have that if $R(x,y)$ is defined by

$$\int_0^1 (1-\alpha)(\phi'(x+\alpha(y-x))(y-x) - \phi'(x)(y-x))d\alpha, \tag{3.5.5}$$

for $x, y \in E$, then

$$\phi(y) - \phi(x) - \phi'(x)(y-x) = R(x,y)$$

and $\|R(x,y)\|| \le \frac{H(2r)\|\phi^*\|}{p}\|y-x\|^p$, for all $\|x\|, \|y\| \le r$. □

Theorem 3.5.3. *Assume that E is a martingale type p Banach space, $p \in (1,2]$. Let X be a process given by* (3.5.1). *Assume that $\|f\|\|g\| = 0$ on $\mathbb{R}_+ \times \Omega \times Z$. Let G be a separable Banach space. Let $\phi : E \to G$ be a function of class C^1 such that the first derivative $\phi' : E \to L(E;G)$ is $(p-1)$-Hölder continuous. Then for every $t > 0$, we have $\mathbb{P}$-a.s.*

$$\begin{aligned} \phi(X_t) = \phi(X_0) &+ \int_0^t \phi'(X_s)(a(s))ds + \int_0^t \int_Z \Big[\phi(X_{s-} + g(s,z)) - \phi(X_{s-})\Big] N(ds,dz) \\ &+ \int_0^t \int_Z \Big[\phi(X_{s-} + f(s,z)) - \phi(X_{s-})\Big] \tilde{N}(ds,dz) \\ &+ \int_0^t \int_Z \Big[\phi(X_{s-} + f(s,z)) - \phi(X_{s-}) - \phi'(X_{s-})(f(s,z))\Big] \nu(dz)ds. \end{aligned} \tag{3.5.6}$$

Remark 3.5.4. (1) We may rewrite the Itô formula (3.5.6) in the following equivalent form. For every $t > 0$, $\mathbb{P}$-a.s.

$$\begin{aligned}\phi(X_t) = \phi(X_0) &+ \int_0^t \phi'(X_s)(a(s))ds + \int_0^t \int_Z \Big[\phi(X_{s-} + g(s,z)) - \phi(X_{s-})\Big] N(ds,dz) \\ &+ \int_0^t \int_Z \phi'(X_{s-})(f(s,z))\tilde{N}(ds,dz) \\ &+ \int_0^t \int_Z \Big[\phi(X_{s-} + f(s,z)) - \phi(X_{s-}) - \phi'(X_{s-})(f(s,z))\Big] N(ds,dz).\end{aligned} \tag{3.5.7}$$

This is because the integral

$$\int_0^t \int_Z \Big[\phi(X_{s-} + f(s,z)) - \phi(X_{s-}) - \phi'(X_{s-})(f(s,z))\Big] \nu(dz)ds$$

is well defined as the Bochner integral, see the proof of Theorem 3.5.3. Let us put this in other words, the function $\phi(X_{\cdot-} + f(\cdot,\cdot)) - \phi(X_{\cdot-}) - \phi'(X_{\cdot-})(f(\cdot,\cdot))$ is in $\mathcal{M}^1_{loc}(\hat{\mathcal{P}}, \nu; E)$. Hence, on the basis of Proposition 3.4.7, we infer that $\phi(X_{\cdot-} + f(\cdot,\cdot)) - \phi(X_{\cdot-}) - \phi'(X_{\cdot-})(f(\cdot,\cdot))$ is in $\mathcal{M}_{loc}(\hat{\mathcal{P}}, N; E)$ and

$$\begin{aligned}&\int_0^t \int_Z \Big[\phi(X_{s-} + f(s,z)) - \phi(X_{s-}) - \phi'(X_{s-})(f(s,z))\Big] \nu(dz)ds \\ &= \int_0^t \int_Z \Big[\phi(X_{s-} + f(s,z)) - \phi(X_{s-}) - \phi'(X_{s-})(f(s,z))\Big] N(ds,dz) \\ &\quad - (B)\int_0^t \int_Z \Big[\phi(X_{s-} + f(s,z)) - \phi(X_{s-}) - \phi'(X_{s-})(f(s,z))\Big] \tilde{N}(ds,dz) \\ &= \int_0^t \int_Z \Big[\phi(X_{s-} + f(s,z)) - \phi(X_{s-}) - \phi'(X_{s-})(f(s,z))\Big] N(ds,dz) \\ &\quad - \int_0^t \int_Z \Big[\phi(X_{s-} + f(s,z)) - \phi(X_{s-}) - \phi'(X_{s-})(f(s,z))\Big] \tilde{N}(ds,dz).\end{aligned}$$

(2) In view of the continuity of functions $\phi(x)$ and $\phi'(x)$ and the continuity property of the integration w.r.t. the measure ds, the Itô formula (3.5.6) can also be written as follows. For all $t > 0$, $\mathbb{P}$-a.s.

$$\begin{aligned}\phi(X_t) = \phi(X_0) &+ \int_0^t \phi'(X_s)(a(s))ds + \int_0^t \int_Z \Big[\phi(X_{s-} + g(s,z)) - \phi(X_{s-})\Big] N(ds,dz) \\ &+ \int_0^t \int_Z \Big[\phi(X_{s-} + f(s,z)) - \phi(X_{s-})\Big] \tilde{N}(ds,dz) \\ &+ \int_0^t \int_Z \Big[\phi(X_s + f(s,z)) - \phi(X_s) - \phi'(X_s)(f(s,z))\Big] \nu(dz)ds.\end{aligned} \tag{3.5.8}$$

(3) It is to be emphasized that the function ϕ in the Itô formula (3.5.6) can also be time dependent, even be random. In our current working paper, we are trying to extend the Itô formula (3.5.6) to the generalized Itô formula, or Itô-Wentzell formula, for a process $F(t, X_t)$, where $F(t, x, \omega)$, $t \geq 0$, $x \in E$ is a random variable with double parameters x and t, in other words $F(\cdot, t)$, $t \geq 0$ is a stochastic process with values in $C^2(E)$ and X_t is an Itô process given by (3.5.1).

Proof of Theorem 3.5.3. Without loss of generality, we may assume that the process X is bounded, namely, there exists $r > 0$ such that

$$\sup_{0 \le s \le t} \|X_s\| \le r. \tag{3.5.9}$$

Then we can relax the boundedness assumption (3.5.9) by the usual localization argument. Indeed, we can define a sequence of stopping times by

$$\sigma_n = \inf\{t \ge 0 : \|X_t\| > n\},$$

where $\inf \emptyset = \infty$ as usual. Since the process X is càdlàg, by Proposition 2.2.3 and right-continuity assumption of the filtration $\mathfrak{F}$, the random time σ_n is indeed a stopping time. Then the stopped process $X_t^{\sigma_n -} := X_{t\wedge\sigma_n -}$ defined by the formula (2.2.2) is bounded by n, that is $\|X_{t\wedge\sigma_n -}\| \le n$, for all $t \ge 0$. Since càdlàg functions are locally bounded, it follows that $\sigma_n \nearrow \infty$. Further, the process X agrees with $X^{\sigma_n -}$ on $[0, \sigma_n)$. If we can establish an Itô formula (3.5.6) for a bounded process $X^{\sigma_n -}$, the formula (3.5.6) would hold by letting $n \to \infty$.

Since the Poisson point process π is σ-finite, see Subsection 3.1.1, there exists a sequence of sets $\{D_n\}_{n\in\mathbb{N}}$ such that $\cup_{n\in\mathbb{N}} D_n = Z$ and $\mathbb{E}N(t, D_n) < \infty$ for every $0 < t < \infty$ and $n \in \mathbb{N}$. Define a sequence $\{f^n\}_{n\in\mathbb{N}}$ of functions by

$$f^n(s,\omega,z) := f(s,\omega,z)1_{D_n}(z), \ (s,\omega,z) \in \mathbb{R}_+ \times \Omega \times Z, \ n \in \mathbb{N}.$$

Since $\|f^n\| \le \|f\|$ and by the assumption, $f \in \mathcal{M}^p(\hat{\mathcal{P}}; E)$, we infer that $f^n \in \mathcal{M}^p(\hat{\mathcal{P}}; E)$. By the definition of stochastic integrals, see Section 3.3, we have

$$\int_0^t \int_Z f^n(s,z)\tilde{N}(ds,dz) = \int_0^t \int_Z 1_{D_n} f(s,z)\tilde{N}(ds,dz) = \int_0^t \int_{D_n} f(s,z)\tilde{N}(ds,dz)$$

Now applying Proposition 3.4.5 yields that

$$\begin{aligned}\int_0^t \int_Z f^n(s,z)\tilde{N}(ds,dz) &= (B)\int_0^t \int_{D_n} f(s,z)\tilde{N}(ds,dz) \\ &= \sum_{s\in(0,t]\cap\mathcal{D}(\pi)} f(s,\pi(s))1_{D_n}(\pi(s)) - \int_0^t \int_{D_n} f(s,z)\nu(dz)ds.\end{aligned} \tag{3.5.10}$$

Similarly, we define a sequence $\{g^n\}_{n\in\mathbb{N}}$ of functions by

$$g^n(s,\omega,z) = g(s,\omega,z)1_{D_n}(z), \quad (s,\omega,z) \in \mathbb{R}_+ \times \Omega \times Z, \ n \in \mathbb{N}.$$

Since $\|g^n\| \le \|g\|$ and $g \in \mathcal{M}^p(\hat{\mathcal{P}}, N; E)$, $g^n \in \mathcal{M}^p(\hat{\mathcal{P}}, N; E)$. Hence it follows from the definition of Bochner integral that

$$\int_0^t \int_Z g^n(s,z)N(ds,dz) = \sum_{s\in(0,t]\cap\mathcal{D}(\pi)} g(s,\pi(s))1_{D_n}(\pi(s)). \tag{3.5.11}$$

Let us fix $t > 0$. Since $\mathbb{E}N(t, D_n) < \infty$, $t \ge 0$, we see that for almost every $\omega \in \Omega$, the set $\{s \le t : \pi(s,\omega) \in D_n \cap \mathcal{D}(\pi)\}$ contains only finitely many points in each time interval $(0, t]$, for

$t > 0$. Hence we may denote these points according to their magnitude by $0 = \tau_0(\omega) < \tau_1(\omega) < \tau_2(\omega) < \cdots < \tau_m(\omega) < \cdots$. In other words, we put

$$\tau_0 = 0;$$
$$\tau_m = \inf\{s \in (0,t] \cap \mathcal{D}(\pi) : \pi(s) \in D_n; s > \tau_{m-1}\}, \ m \geq 1.$$

The random times $\tau_1, \tau_2, \cdots$ form a random configuration of points in $(0,t]$ with $\pi(\tau_i) \in D_n$ and for each m, the random time τ_m is a stopping time. Indeed, for every $u > 0$, we find out that

$$\{\tau_m \leq u\} = \{N(u, D_n) \geq m\} \in \mathcal{F}_u.$$

Let us define a sequence $\{X^n\}_{n\in\mathbb{N}}$ of process $X^n := (X_t^n)_{t\geq 0}$ by

$$X_t^n = X_0^t + \int_0^t a(s)ds + \int_0^t \int_Z f^n(s,z)\tilde{N}(ds,dz) + \int_0^t \int_Z g^n(s,z)N(ds,dz), \ t \geq 0, \ n \in \mathbb{N}.$$

It follows from (3.5.10) and (3.5.11) that for every $n \in \mathbb{N}$ and all $t \geq 0$, $\mathbb{P}$-a.s.

$$\begin{aligned} X_t^n &= X_0 + \int_0^t a(s)ds + \sum_{s\in(0,t]\cap\mathcal{D}(\pi)} f(s,\pi(s))1_{D_n}(\pi(s)) - \int_0^t \int_{D_n} f(s,z)\nu(dz)ds \\ &\quad + \sum_{s\in(0,t]\cap\mathcal{D}(\pi)} g(s,\pi(s))1_{D_n}(\pi(s)) \\ &= X_0 + \int_0^t a(s)ds - \int_0^t \int_{D_n} f(s,z)\nu(dz)ds + \sum_m f^n(\tau_m, \pi(\tau_m), \cdot)1_{\{\tau_m\leq t\}} \\ &\quad + \sum_m g^n(\tau_m, \pi(\tau_m), \cdot)1_{\{\tau_m \leq t\}}. \end{aligned}$$

Note that

$$\begin{aligned} \phi(X_t^n) - \phi(X_0) &= \sum_m \left[\phi(X^n_{t\wedge\tau_m}) - \phi(X^n_{t\wedge\tau_{m-1}})\right] \\ &= \sum_m \left[\phi(X^n_{t\wedge\tau_m}) - \phi(X^n_{t\wedge\tau_m -})\right] + \sum_m \left[\phi(X^n_{t\wedge\tau_m -}) - \phi(X^n_{t\wedge\tau_{m-1}})\right] \\ &=: I_1 + I_2. \end{aligned}$$

Here $X^n_{t\wedge\tau_m} = (X^n)^{\tau_m}_t$, $t \geq 0$ is the process X^n stopped at time τ_m, see formula (2.2.1), and $X^n_{t\wedge\tau_m -} = (\bar{X}^n)^{\tau_m}_t$, $t \geq 0$ is the process X^n stopped strictly before time τ_m, see formula (2.2.2). Namely,

$$X^n_{t\wedge\tau_m}(\omega) = (X^n)^{\tau_m}_t(\omega) = \begin{cases} X_t^n(\omega), & if\ t \leq \tau_m(\omega), \\ X^n_{\tau_m(\omega)}(\omega), & if\ t \geq \tau_m(\omega). \end{cases}$$

and

$$X^n_{t\wedge\tau_m -}(\omega) = (\bar{X}^n)^{\tau_m}_t(\omega) = \begin{cases} X_t^n(\omega), & if\ t < \tau_m(\omega) \\ X^n_{\tau_m(\omega)-}(\omega), & if\ t \geq \tau_m(\omega). \end{cases}$$

Step 1. We claim that for every $t \geq 0$, $\mathbb{P}$-a.s.

$$\sum_m \left[\phi(X^n_{t\wedge\tau_m -}) - \phi(X^n_{t\wedge\tau_{m-1}})\right] = \int_0^t \phi'(X_s)(a(s))ds + \int_0^t \phi'(X_s)(f^n(s,z))\nu(dz)ds. \qquad (3.5.12)$$

To prove equality (3.5.12), it suffices to show that each term of the sum in the equality (3.5.12) satisfies the following

$$\phi(X^n_{t\wedge\tau_m-}) - \phi(X^n_{t\wedge\tau_{m-1}}) = \int_{t\wedge\tau_{m-1}}^{t\wedge\tau_m} \int_Z \phi'(X_s)(a(s))ds + \int_{t\wedge\tau_{m-1}}^{t\wedge\tau_m} \int_Z \phi'(X_s)(f^n(s,z))\nu(dz)ds.$$

Define a sequence of partitions $\{v(k,i)\}_{i=0}^k$, $k \in \mathbb{N}$ of the random time interval $[t\wedge\tau_{m-1}, t\wedge\tau_m)$ by

$$\begin{aligned} v(k,0) &= t\wedge\tau_{m-1} \\ v(k,i) &= \frac{i(t\wedge\tau_m - t\wedge\tau_{m-1})}{k}, \quad i = 1,\cdots,k-1, \\ v(k,k) &= t\wedge\tau_m - . \end{aligned}$$

In other words we divide the random time interval $[t\wedge\tau_{m-1}, t\wedge\tau_m)$ into k equal parts. Hence, on the basis of the Lemma 3.5.2, we have

$$\begin{aligned} \phi(X^n_{t\wedge\tau_m-}) - \phi(X^n_{t\wedge\tau_{m-1}}) &= \sum_{i=0}^k \left[\phi(X^n_{v(k,i+1)}) - \phi(X^n_{v(k,i)})\right] \\ &= \sum_{i=0}^k \phi'(X^n_{v(k,i)})(X^n_{v(k,i+1)} - X^n_{v(k,i)}) + \sum_{i=0}^k R(X^n_{v(k,i)}, X^n_{v(k,i+1)}), \end{aligned}$$

where R was defined by (3.5.5) i.e. $R(x,y) = \int_0^1 (1-\alpha)(\phi'(x+\alpha(y-x))(y-x)) - \phi'(x)(y-x))d\alpha$, for $x, y \in E$. Recall that $\|R(x,y)\| \le C(t)\|y-x\|^p$ for $\|x\|, \|y\| \le r$.

Claim 1.1

$$\limsup_{k\to\infty} \sum_{i=0}^k R(X^n_{v(k,i)}, X^n_{v(k,i+1)})(\omega) = 0, \quad \mathbb{P}\text{-a.s.} \tag{3.5.13}$$

By assumption (3.5.9), we know that X is bounded on the interval $[0,t]$, that is $\sup_{s\in[0,t]}\|X_s\| \le r$. Since by assumption, the function ϕ' is $(p-1)$-Hölder continuous on bounded subsets of E, by Lemma 3.5.2 we infer that there exits a constant C depending on r such that

$$\left|R(X^n_{v(k,i)}, X^n_{v(k,i+1)})(\omega)\right| \le C\|X^n_{v(k,i+1)}(\omega) - X^n_{v(k,i)}(\omega)\|^p$$

Notice that there is no jumps of X in the random time interval $[t\wedge\tau_{m-1}, t\wedge\tau_m)$, in other words X contains only the continuous components. Hence we infer that

$$X^n_{v(k,i+1)}(\omega) - X^n_{v(k,i)}(\omega) = \int_{v(k,i)(\omega)}^{v(k,i+1)(\omega)} a(s,\omega)ds + \int_{v(k,i)(\omega)}^{v(k,i+1)(\omega)} \int_Z f^n(s,\omega,z)\nu(dz)ds.$$

By using the inequality $\|a+b\|^p \le 2^p\|a\|^p + 2^p\|b\|^p \le 4\|a\|^p + 4\|b\|^p$, we have

$$\left|R(X^n_{v(k,i)}, X^n_{v(k,i+1)})(\omega)\right| \le 4C\left\|\int_{v(k,i)(\omega)}^{v(k,i+1)(\omega)} a(s,\omega)ds\right\|^p + 4C\left\|\int_{v(k,i)(\omega)}^{v(k,i+1)(\omega)} \int_Z f^n(s,\omega,z)\nu(dz)ds\right\|^p,$$

Since $p \in (1,2]$, we infer that

$$\begin{aligned}
&\sum_{i=0}^{k} \left\| R(X^n_{v(k,i)}, X^n_{v(k,i+1)})(\omega) \right\| \\
&\le 4C \sup_i \left\| \int_{v(k,i)(\omega)}^{v(k,i+1)(\omega)} a(s,\omega) ds \right\|^{p-1} \times \sum_{i=0}^{k} \left\| \int_{v(k,i)(\omega)}^{v(k,i+1)(\omega)} a(s,\omega) ds \right\| \\
&+ 4C \sup_i \left\| \int_{v(k,i)(\omega)}^{v(k,i+1)(\omega)} \int_Z f^n(s,\omega,z)\nu(dz) ds \right\|^{p-1} \times \sum_{i=0}^{k} \left\| \int_{v(k,i)(\omega)}^{v(k,i+1)(\omega)} \int_Z f^n(s,\omega,z)\nu(dz) ds \right\| \\
&\le 4C \sup_i \left\| \int_{v(k,i)(\omega)}^{v(k,i+1)(\omega)} a(s,\omega) ds \right\|^{p-1} \times \int_{v(k,0)(\omega)}^{v(k,k)(\omega)} \|a(s,\omega)\| ds \\
&+ 4C \sup_i \left\| \int_{v(k,i)(\omega)}^{v(k,i+1)(\omega)} \int_Z f^n(s,\omega,z)\nu(dz) ds \right\|^{p-1} \times \int_{v(k,0)(\omega)}^{v(k,k)(\omega)} \int_Z \|f^n(s,\omega,z)\|\nu(dz) ds.
\end{aligned}$$

Here we used the fact that by the definition of the function f^n and the property $\mathbb{E}N(t, D_n) < \infty$, the function f^n is $\mathbb{P}$-a.e. Bochner integrable with respect ν. Clearly, the two integrals $\int_0^\cdot a(s,\omega)ds$ and $\int_0^\cdot \int_Z f^n \nu(dz)ds$ are continuous w.r.t. the time variable. Hence by letting $k \to \infty$, we have $\max_i\{v(k,i+1) - v(k,i)\} \to 0$. Therefore, we obtain

$$\limsup_{k\to\infty} \sum_{i=0}^{k} \left| R(X^n_{v(k,i)}, X^n_{v(k,i+1)})(\omega) \right| = 0, \ \mathbb{P}\text{-a.s.}$$

This shows the claim (3.5.13).

Claim 1.2

$$\begin{aligned}
&\lim_{k\to\infty} \sum_{i=0}^{k} \phi'(X^n_{v(k,i)}(\omega))(X^n_{v(k,i+1)}(\omega) - X^n_{v(k,i)}(\omega)) \qquad (3.5.14)\\
&= \int_{t\wedge\tau_{m-1}(\omega)}^{t\wedge\tau_m(\omega)} \phi'(X^n_s(\omega))(a(s,\omega))ds - \int_{t\wedge\tau_{m-1}(\omega)}^{t\wedge\tau_m(\omega)} \int_Z \phi'(X^n_s(\omega))(f^n(s,\omega,z))\nu(dz)ds, \ \mathbb{P}\text{-a.e.}
\end{aligned}$$

For this, we only need to show that the following two indentities

$$\lim_{k\to\infty} \left\| \sum_{i=0}^{k} \phi'(X^n_{v(k,i)}(\omega)) \Big(\int_{v(k,i)(\omega)}^{v(k,i+1)(\omega)} a(s,\omega)ds \Big) - \int_{t\wedge\tau_{m-1}(\omega)}^{t\wedge\tau_m(\omega)} \phi'(X^n_s(\omega))(a(s,\omega))ds \right\| = 0, \ \mathbb{P}\text{-a.s.} \qquad (3.5.15)$$

and

$$\begin{aligned}
&\lim_{k\to\infty} \Big\| \sum_{i=0}^{k} \phi'(X^n_{v(k,i)}(\omega)) \Big(\int_{v(k,i)(\omega)}^{v(k,i+1)(\omega)} f^n(s,\omega,z)\nu(dz)ds \Big) \\
&\qquad - \int_{t\wedge\tau_{m-1}(\omega)}^{t\wedge\tau_m(\omega)} \int_Z \phi'(X^n_s(\omega))(f^n(s,\omega,z))\nu(dz)ds \Big\| = 0 \ \mathbb{P}\text{-a.s.} \qquad (3.5.16)
\end{aligned}$$

By using the $(p-1)$-Hölder continuity of ϕ', one can see that

$$\begin{aligned}
&\lim_{k\to\infty}\left\|\sum_{i=0}^{k}\phi'(X^n_{v(k,i)}(\omega))\Big(\int_{v(k,i)(\omega)}^{v(k,i+1)(\omega)}a(s,\omega)ds\Big)-\int_{t\wedge\tau_{m-1}(\omega)}^{t\wedge\tau_m(\omega)}\phi'(X^n_s(\omega))(a(s,\omega))ds\right\|\\
&\leq \lim_{k\to\infty}\sum_{i=0}^{k}\left\|\int_{v(k,i)(\omega)}^{v(k,i+1)(\omega)}\Big(\phi'(X^n_{v(k,i)}(\omega))-\phi'(X^n_s(\omega))\Big)(a(s,\omega))ds\right\|\\
&\leq C\lim_{k\to\infty}\sum_{i=0}^{k}\int_{v(k,i)(\omega)}^{v(k,i+1)(\omega)}\left\|X^n_{v(k,i)}(\omega)-X^n_s(\omega)\right\|^{p-1}\|(a(s,\omega))\|ds\\
&\leq 2C\lim_{k\to\infty}\sup_{0\leq i\leq k-1}\Big(\int_{v(k,i)(\omega)}^{v(k,i+1)(\omega)}\left\|\int_{v(k,i)(\omega)}^{s}a(r,\omega)dr\right\|^{p-1}ds\\
&+\sup_{0\leq i\leq k-1}\int_{v(k,i)(\omega)}^{v(k,i+1)(\omega)}\left\|\int_{v(k,i)(\omega)}^{s}f^n(r,\omega,z)\nu(dz)dr\right\|^{p-1}ds\Big)\times\Big(\sum_{i=0}^{k}\int_{v(k,i)(\omega)}^{v(k,i+1)(\omega)}\|(a(s,\omega))\|ds\Big)\\
&=0,
\end{aligned}$$

This shows the assertion (3.5.15). The assertion (3.5.16) can be proved by a similar argument.

So far we have shown that

$$\begin{aligned}
\phi(X^n_{t\wedge\tau_m-})-\phi(X^n_{t\wedge\tau_{m-1}})&=\int_{t\wedge\tau_{m-1}(\omega)}^{t\wedge\tau_m(\omega)}\phi'(X^n_s(\omega))(a(s,\omega))ds\\
&-\int_{t\wedge\tau_{m-1}(\omega)}^{t\wedge\tau_m(\omega)}\int_Z\phi'(X^n_s(\omega))(f^n(s,\omega,z))\nu(dz)ds,\quad \mathbb{P}\text{-a.e.}
\end{aligned}$$

Hence adding these up, we obtain

$$\begin{aligned}
I_2&=\sum_m\phi(X^n_{t\wedge\tau_m-})-\phi(X^n_{t\wedge\tau_{m-1}})\\
&=\int_0^t\phi'(X^n_s)(a(s))ds-\int_0^t\phi'(X^n_s)(f^n(s,z))\nu(dz)ds,\quad \mathbb{P}\text{-a.s.}
\end{aligned}$$

Note that the jumps of X^n occur only at times $\{\tau_m\}$. So $X^n_{t\wedge\tau_m}\neq X^n_{t\wedge\tau_m-}$ if and only if

$$f^n(\tau_m,\pi(\tau_m),\cdot)1_{\{\tau_m\leq t\}}+g^n(\tau_m,\pi(\tau_m),\cdot)1_{\{\tau_m\leq t\}}\neq 0.$$

Since by assumption $\|f\|\|g\|=0$, we infer that $X^n_{t\wedge\tau_m}\neq X^n_{t\wedge\tau_m-}$ if and only if

$$f^n(\tau_m,\pi(\tau_m),\cdot)1_{\{\tau_m\leq t\}}\neq 0 \text{ and } g^n(\tau_m,\pi(\tau_m),\cdot)1_{\{\tau_m\leq t\}}=0$$

or

$$f^n(\tau_m,\pi(\tau_m),\cdot)1_{\{\tau_m\leq t\}}=0 \text{ and } g^n(\tau_m,\pi(\tau_m),\cdot)1_{\{\tau_m\leq t\}}\neq 0$$

Hence

$$\begin{aligned}
X^n_{t\wedge\tau_m}&=X^n_{t\wedge\tau_m-}+f^n(\tau_m,\pi(\tau_m))1_{\{\tau_m\leq t\}}+g^n(\tau_m,\pi(\tau_m))1_{\{\tau_m\leq t\}}\\
&=\begin{cases}X^n_{t\wedge\tau_m-}+f^n(\tau_m,\pi(\tau_m))1_{\{\tau_m\leq t\}}, & if\ f^n(\tau_m,\pi(\tau_m))\neq 0,\ g^n(\tau_m,\pi(\tau_m))=0\\ X^n_{t\wedge\tau_m-}+g^n(\tau_m,\pi(\tau_m))1_{\{\tau_m\leq t\}}, & if\ f^n(\tau_m,\pi(\tau_m))=0,\ g^n(\tau_m,\pi(\tau_m))\neq 0,\end{cases}
\end{aligned}$$

It follows that

$$
\begin{aligned}
I_1 &= \sum_m \Big[\phi(X^n_{t\wedge\tau_m}) - \phi(X^n_{t\wedge\tau_m-})\Big] \\
&= \sum_m \Big[\phi(X^n_{t\wedge\tau_m}) - \phi(X^n_{t\wedge\tau_m-})\Big] 1_{\{\|f(\tau_m,\pi(\tau_m))\|\neq 0\}\cap\{\|g(\tau_m,\pi(\tau_m))\|=0\}} 1_{\{\tau_m\le t\}} \\
&+ \sum_m \Big[\phi(X^n_{t\wedge\tau_m}) - \phi(X^n_{t\wedge\tau_m-})\Big] 1_{\{\|f(\tau_m,\pi(\tau_m))\|=0\}\cap\{\|g(\tau_m,\pi(\tau_m))\|\neq 0\}} 1_{\{\tau_m\le t\}} \\
&= \sum_m \Big[\phi(X^n_{t\wedge\tau_m-} + f^n(\tau_m,\pi(\tau_m))) - \phi(X^n_{t\wedge\tau_m-})\Big] 1_{\{\|f(\tau_m,\pi(\tau_m))\|\neq 0\}\cap\{\|g(\tau_m,\pi(\tau_m))\|=0\}} 1_{\{\tau_m\le t\}} \\
&+ \sum_m \Big[\phi(X^n_{t\wedge\tau_m-} + g^n(\tau_m,\pi(\tau_m))) - \phi(X^n_{t\wedge\tau_m-})\Big] 1_{\{\|f(\tau_m,\pi(\tau_m))\|=0\}\cap\{\|g(\tau_m,\pi(\tau_m))\|\neq 0\}} 1_{\{\tau_m\le t\}} \\
&= \int_0^t\int_Z \Big[\phi(X^n_{s-} + f^n(s,z,\omega)) - \phi(X^n_{s-})\Big] N(ds,dz) \\
&+ \int_0^t\int_Z \Big[\phi(X^n_{s-} + g^n(s,z,\omega)) - \phi(X^n_{s-})\Big] N(ds,dz) \\
&= \int_0^t\int_Z \Big[\phi(X^n_{s-} + f^n(s,z,\omega)) - \phi(X^n_{s-})\Big] \tilde{N}(ds,dz) \\
&+ \int_0^t\int_Z \Big[\phi(X^n_{s-} + g^n(s,z,\omega)) - \phi(X^n_{s-})\Big] N(ds,dz) \\
&+ \int_0^t\int_Z \Big[\phi(X^n_{s-} + f^n(s,z,\omega)) - \phi(X^n_{s-})\Big] \nu(dz)ds
\end{aligned}
$$

Combining I_1 and I_2 together yields that

$$
\begin{aligned}
\phi(X^n_t) - \phi(X_0) &= \int_0^t \phi'(X^n_s)(a(s))ds - \int_0^t \phi'(X^n_s)(f^n(s,z))\nu(dz)ds \\
&+ \int_0^t\int_Z \Big[\phi(X^n_{s-} + f^n(s,z,\omega)) - \phi(X^n_{s-})\Big] \tilde{N}(ds,dz) \\
&+ \int_0^t\int_Z \Big[\phi(X^n_{s-} + g^n(s,z,\omega)) - \phi(X^n_{s-})\Big] N(ds,dz) \\
&+ \int_0^t\int_Z \Big[\phi(X^n_{s-} + f^n(s,z,\omega)) - \phi(X^n_{s-})\Big] \nu(dz)ds \\
&= \int_0^t \phi'(X_s)(a(s))ds + \int_0^t\int_Z \Big[\phi(X^n_{s-} + f^n(s,z,\omega)) - \phi(X^n_{s-})\Big] \tilde{N}(ds,dz) \\
&+ \int_0^t\int_Z \Big[\phi(X^n_{s-} + g^n(s,z,\omega)) - \phi(X^n_{s-})\Big] N(ds,dz) \\
&+ \int_0^t\int_Z \Big[\phi(X^n_{s-} + f^n(s,z,\omega)) - \phi(X^n_{s-}) - \phi'(X^n_{s-})(f^n(s,z))\Big] \nu(dz)ds
\end{aligned}
$$

This shows that Itô formula (3.5.6) holds for the process X^n. Now let us consider the general case. Note that $f^n(s,\omega,z)$ converges to $f(s,\omega,z)$, as $n\to\infty$ and $\|f^n(s,\omega,z)\| \le \|f(s,\omega,z)\|$, for all $(s,\omega,z)\in\mathbb{R}_+\times\Omega\times Z$. On the basis of the inequality (3.3.2) and the Lebesgue Dominated

Convergence Theorem, we infer

$$\lim_{n\to\infty} \mathbb{E}\left\|\int_0^t \int_Z f^n(s,z)\tilde{N}(ds,dz) - \int_0^t \int_Z f(s,z)\tilde{N}(ds,dz)\right\|^p$$
$$\leq C \lim_{n\to\infty} \mathbb{E} \int_0^t \int_Z \|f^n(s,z) - f(s,z)\|^p \nu(dz)ds = 0.$$

By using a similar argument as in the proof of Theorem 3.3.2, this allows us to find a subsequence such that $\int_0^t \int_Z f^n(s,z)\tilde{N}(ds,dz)$ uniformly converges to $\int_0^t \int_Z f(s,z)\tilde{N}(ds,dz)$ on any finite interval $[0,T]$ $\mathbb{P}$-a.s. $0 < T < \infty$. Similarly, we can prove that $\int_0^t \int_Z g^n(s,\omega,z)N(ds,dz)$ uniformly converges to $\int_0^t \int_Z g(s,\omega,z)N(ds,dz)$ on $[0,T]$ $\mathbb{P}$-a.s. as well. Hence we infer that X_s^n converges uniformly to X_s as $n \to \infty$, $\mathbb{P}$-a.s. Also, X_{s-}^n converges uniformly to X_{s-}, as $n \to \infty$, $\mathbb{P}$-a.s. Hence by the continuity of ϕ and ϕ', we infer

$$\phi(X_{s-}^n(\omega) + f^n(s,z,\omega)) - \phi(X_{s-}^n(\omega)) - \phi'(X_{s-}^n(\omega))(f^n(s,\omega,z))$$

converges uniformly for almost all $\omega \in \Omega$ on $[0,t]$ to

$$\phi(X_{s-}(\omega) + f(s,z,\omega)) - \phi(X_{s-}(\omega)) - \phi'(X_{s-}(\omega))(f(s,\omega,z)), \text{ as } n \to \infty.$$

Since by assumption $\sup_{s\in[0,t]} \|X_s^n\| \leq R$, on the basis of Lemma 3.5.2, we observe that

$$\left\|\phi(X_{s-}^n(\omega) + f^n(s,z,\omega)) - \phi(X_{s-}^n(\omega)) - \phi'(X_{s-}^n(\omega))(f^n(s,\omega,z))\right\| \leq C\|f^n(s,\omega,z)\|^p$$
$$\leq C\|f(s,\omega,z)\|^p$$

and

$$\left\|\phi(X_{s-}(\omega) + f(s,z,\omega)) - \phi(X_{s-}(\omega)) - \phi'(X_{s-}(\omega))(f(s,\omega,z))\right\| \leq C\|f(s,\omega,z)\|^p.$$

Since the function $f \in \mathcal{M}^p(\hat{\mathcal{P}};E)$, it follows that $\|f\|^p \in \mathcal{M}^1(\hat{\mathcal{P}};\mathbb{R})$, so $\int_0^t \int_Z \|f(s,z)\|^p \nu(dz)ds < \infty$, $\mathbb{P}$-a.s.

Now applying the Lebesgue Dominate Convergence Theorem yields that

$$\int_0^t \int_Z \left[\phi(X_{s-}^n(\omega) + f^n(s,z,\omega)) - \phi(X_{s-}^n(\omega)) - \phi'(X_{s-}^n(\omega))(f^n(s,\omega,z))\right]\nu(dz)ds$$

converges to

$$\int_0^t \int_Z \phi(X_{s-}(\omega) + f(s,z,\omega)) - \phi(X_{s-}(\omega)) - \phi'(X_{s-}(\omega))(f(s,\omega,z))\nu(dz)ds$$

as $n \to \infty$, $\mathbb{P}$-a.s.

Similarly, in view of the $(p-1)$ Hölder continuity of ϕ' and the uniformly boundedness of X^n on $[0,t]$, we have

$$\sum_{s\in\mathcal{D}(\pi)\cap(0,t]} \left\|\phi(X_{s-}^n + g^n(s,\omega,\pi(s)) - \phi(X_{s-}^n)\right\|$$
$$\leq \sum_{s\in\mathcal{D}(\pi)\cap(0,t]} \|\phi'(X_{s-}^n + \theta g^n(s,\omega,\pi(s)))\|_{\mathcal{L}(E)} \|g^n(s,\omega,\pi(s))\|$$
$$\leq C \sum_{s\in\mathcal{D}(\pi)\cap(0,t]} \|g^n(s,\omega,\pi(s))\|$$

$$\leq C \sum_{s\in\mathcal{D}(\pi)\cap(0,t]} \|g(s,\omega,\pi(s))\|$$
$$= C\int_0^t\int_Z g(s,\omega,z)N(ds,dz)(\omega) < \infty.$$

Meanwhile, we also have

$$\sum_{s\in\mathcal{D}(\pi)\cap(0,t]} \Big\|\phi(X_{s-}+g(s,\omega,\pi(s)) - \phi(X_{s-})\Big\|$$
$$\leq C\int_0^t\int_Z g(s,\omega,z)N(ds,dz)(\omega) < \infty.$$

Again, by the Lebesgue's dominated convergence theorem, we infer that

$$\int_0^t\int_Z \phi(X^n_{s-}+g^n(s,z)) - \phi(X^n_{s-})N(ds,dz)$$
$$\to \int_0^t\int_Z \phi(X_{s-}+g(s,z)) - \phi(X_{s-})N(ds,dz), \quad \text{as } n\to\infty, \ \mathbb{P}\text{-a.s.}$$

For the convergence of the stochastic integrals, we shall apply the Theorem 3.3.2 to get

$$\begin{aligned}\mathbb{E}\| &\int_0^t\int_Z \Big[\phi(X^n_{s-}+f^n(s,z)) - \phi(X^n(s-))\Big]\tilde{N}(ds,dz)\\ &- \int_0^t\int_Z \Big[\phi(X_{s-}+f(s,z)) - \phi(X_{s-})\Big]\tilde{N}ds,dz\|^p\\ &\leq C\mathbb{E}\int_0^t\int_Z \Big\|\phi(X^n_{s-}+f^n(s,z)) - \phi(X^n_{s-}) - \phi(X_{s-}+f(s,z)) + \phi(X_{s-})\Big\|^p\nu(dz)ds\end{aligned} \tag{3.5.17}$$

Note that $\phi(X^n_{s-}+f^n(s,z))-\phi(X^n_{s-})$ converges uniformly path by path to $\phi(X_{s-}+f(s,z))+\phi(X_{s-})$ as $n\to\infty$, $\mathbb{P}$-a.s. and

$$\Big\|\phi(X^n_{s-}+f^n(s,z)) - \phi(X^n_{s-}) - \phi(X_{s-}+f(s,z)) + \phi(X_{s-})\Big\|^p \leq C_1\|f(s,\omega,z)\|^p.$$

Therefore, by the Lebesgue Dominated Convergence Theorem, we find out that the right side of (3.5.17) converges to 0 as $n\to\infty$. This means that

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}\| &\int_0^t\int_Z \Big[\phi(X^n_{s-}+f^n(s,z)) - \phi(X^n_{s-})\Big]\tilde{N}(ds,dz)\\ &- \int_0^t\int_Z \Big[\phi(X_{s-}+f(s,z)) - \phi(X_{s-})\Big]\tilde{N}ds,dz\|^p = 0\end{aligned}$$

This L^p convergence allows us to have convergence a.s. by taking a subsequence. □

3.6 The Stochastic Fubini Theorem

Lemma 3.6.1. *Let $(A_1,\mathcal{A}_1)$ and $(A_2,\mathcal{A}_2)$ be two measurable space. Suppose that $g: A_1\times A_2\to\mathcal{H}$ is an $\mathcal{A}_1\otimes\mathcal{A}_2\backslash\mathcal{B}(\mathcal{H})$-measurable function. Then g is measurable in each variable separately, that is for each $x_1\in A_1$, the function $x_2\mapsto g(x_1,x_2)$ is $\mathcal{A}_2$-measurable and for each $x_2\in A_2$, the function $x_1\mapsto g(x_1,x_2)$ is $\mathcal{A}_1$-measurable.*

Proof. First, we show that for each $x_1 \in A_1$, the function $x_2 \mapsto g(x_1, x_2)$ is $\mathcal{A}_2$-measurable. A similar argument will yield that for each $x_2 \in A_2$, the function $x_1 \mapsto g(x_1, x_2)$ is $\mathcal{A}_1$-measurable. Let $x_1 \in A_1$ be fixed. Take $B \in \mathcal{B}(\mathcal{H})$. We need to show that the set

$$\{x_2 \in A_2 : g(x_1, x_2) \in B\} \in \mathcal{A}_2.$$

Define a set $\mathcal{Q}(x_1) \subset \mathcal{A}_1 \otimes \mathcal{A}_2$ by

$$\mathcal{Q}(x_1) := \{C \in \mathcal{A}_1 \otimes \mathcal{A}_2 : C(x_1) \in \mathcal{A}_2\},$$

where $C(x_1) = \{x_2 \in A : (x_1, x_2) \in C\}$. We will show that $\mathcal{Q}(x)$ is a σ-field. To see this, we have to verify the three conditions in the definition of a σ-field.

(i) Observe that $(A_1 \times A_2)(x_1) = \{x_2 \in A_2 : (x_1, x_2) \in A_1 \times A_2\} = A_2$. Since $A_2 \in \mathcal{A}_2$, $(A_1 \times A_2)(x_1) \in \mathcal{A}_2$, so $A_1 \times A_2 \in \mathcal{Q}(x_1)$.

(ii) Take $C \in \mathcal{Q}(x_1)$. Then we have $C(x_1) \in \mathcal{A}_2$. We need to show that $C^c \in \mathcal{Q}(x_1)$. Observe that

$$C(x_1)^c = \{x_2 \in A_2 : (x_1, x_2) \in C\}^c = \{x_2 \in A_2 : (x_1, x_2) \in C^c\} = C^c(x_1)$$

Hence $C^c(x_1) = C(x_1)^c \in \mathcal{A}_2$. By the definition of $\mathcal{Q}(x_1)$, we infer that $C^c \in \mathcal{Q}(x_1)$.

(iii) Let $C_1, C_2, \cdots \in \mathcal{Q}(x_1)$. Then $C_1(x_1), C_2(x_1), \cdots \in \mathcal{A}_2$ and

$$\bigcup_{n=1}^{\infty}(C_n(x_1)) = \bigcup_{n=1}^{\infty}\{x_2 \in A_2, (x_1, x_2) \in C_n\} = \{x_2 \in A_2 : (x_1, x_2) \in \bigcup_{n=1}^{\infty} C_n\} = (\bigcup_{n=1}^{\infty} C_n)(x_1).$$

Since $\mathcal{A}_2$ is a σ-filed, $\bigcup_{n=1}^{\infty}(C_n(x_1)) \in \mathcal{A}_2$. It follows that $(\bigcup_{n=1}^{\infty} C_n)(x_1) \in \mathcal{A}_2$. Hence $\bigcup_{n=1}^{\infty} C_n \in \mathcal{Q}(x_1)$.

Therefore, $\mathcal{Q}(x_1)$ is a σ-field. On the other hand, if $C \in \mathcal{A}_1$ and $D \in \mathcal{A}_2$, then

$$(E \times F)(x_1) = \{x_2 \in A_2 : (x_1, x_2) \in E \times F\} = \begin{cases} F \in \mathcal{A}_2 & \text{if } x_1 \in A_1 \\ \emptyset \in \mathcal{A}_2 & \text{if } x_1 \notin A_1. \end{cases}$$

It follows that $E \times F \in \mathcal{Q}(x_1)$. Since $\mathcal{A}_1 \otimes \mathcal{A}_2$ is the smallest σ-field containing all Cartesian products $E \times F$ of sets $E \in \mathcal{A}_1$ and $F \in \mathcal{A}_2$, $\mathcal{A}_1 \otimes \mathcal{A}_2 \subset \mathcal{Q}(x_1)$. Therefore, we have $\mathcal{Q}(x_1) = \mathcal{A}_1 \otimes \mathcal{A}_2$. Recall that $\mathcal{Q}(x_1) = \{C \in \mathcal{A}_1 \otimes \mathcal{A}_2 : C(x_1) \in \mathcal{A}_2\}$. Therefore, we infer that $C(x_1) \in \mathcal{A}_2$, for any set $C \in \mathcal{A}_1 \otimes \mathcal{A}_2$. Finally, we have

$$\{x_2 \in A_2 : g(x_1, x_2) \in B\} = \{x_2 \in A_2 : (x_1, x_2) \in g^{-1}(B)\} = (g^{-1}(B))(x_1) \in \mathcal{A}_2,$$

which shows that the function $x_2 \mapsto g(x_1, x_2)$ is $\mathcal{A}_2$-measurable. □

Theorem 3.6.2. *Let E be a martingale type p Banach space, $1 \le p \le 2$. Let $(O, \mathcal{O}, \mu)$ be a σ-finite measure space. Suppose that $f : O \times [0,T] \times \Omega \times Z \to E$ be a $\mathcal{O} \otimes \mathcal{BF} \otimes \mathcal{Z}$-measurable process and*

$$f \in L^1(O, \mathcal{O}, \mu; \mathcal{M}_T^p(\mathcal{BF} \otimes \mathcal{Z}; E)) \cap L^p(O, \mathcal{O}, \mu; \mathcal{M}_T^p(\mathcal{BF} \otimes \mathcal{Z}; E))$$

That is

$$\int_O \|f(y, \cdot, \cdot, \cdot)\|_{\mathcal{M}_T^p} \mu(dy) < \infty \quad \text{and} \quad \int_O \|f(y, \cdot, \cdot, \cdot)\|_{\mathcal{M}_T}^p \mu(dy) < \infty,$$

where $\|f\|_{\mathcal{M}_T^p} = \left(\mathbb{E}\left(\int_0^T \int_Z \|f(t, \cdot, z)\|^p \nu(dz)dt\right)\right)^{1/p}$. *Then*

(1) for μ-almost all $y \in O$, $f(y,\cdot,\cdot,\cdot)$ is stochastic integrable with respect to Poisson random measure $\tilde{N}$.

(2) for all $z \in Z$, the process $[0,T]\times\Omega \ni (t,\omega) \mapsto \int_E f(y,t,\omega.z)\mu(dy)$ is progressively measurable.

(3) $\mathbb{P}$-a.s.

$$\int_E \left(\int_0^T \int_Z f(y,t,\omega,z)\tilde{N}(dt,dz)\right)\mu(dy) = \int_0^T \int_Z \left(\int_E f(y,t,\omega,z)\mu(dy)\right)\tilde{N}(dt,dz). \quad (3.6.1)$$

Proof. (1) Let us fix $y \in O$. Since $f(\cdot,\cdot,\cdot,\cdot)$ is $\mathcal{O}\otimes\mathcal{BF}\otimes\mathcal{Z}$-measurable, by the basic Fubini theorem, the function $f(y,\cdot,\cdot,\cdot)$ is $\mathcal{BF}\otimes\mathcal{Z}$-measurable. Also, we can show that $\|f(y,\cdot,\cdot,\cdot)\|_{\mathcal{M}_T^p} < \infty$ μ-a.e.. To see this, set $N := \{y : \|f(y,\cdot,\cdot,\cdot)\|_{\mathcal{M}_T^p} = \infty\}$. Then

$$N = \bigcap_{j=k}^{\infty}\{\|f(y,\cdot,\cdot,\cdot)\|_{\mathcal{M}_T^p} \geq k\}.$$

It follows from the Chebyshev inequality that

$$\mu(N) = \lim_{k\to\infty}\mu(\{\|f(y,\cdot,\cdot,\cdot)\|_{\mathcal{M}_T^p} \geq k\}) \leq \lim_{k\to\infty}\frac{\int_E \|f(y,\cdot,\cdot,\cdot)\|_{\mathcal{M}_T^p}\mu(dy)}{k} = 0,$$

as $\int_E \|f(y,\cdot,\cdot,\cdot)\|_{\mathcal{M}_T^p}\mu(dy) < \infty$. This gives that $\|f(y,\cdot,\cdot,\cdot)\|_{\mathcal{M}_T^p} < \infty$ μ-a.e.. Hence the stochastic integral $\int_0^T \int_Z f(y,t,\omega,z)\tilde{N}(dt,dz)$ is well defined for μ-almost all $y \in E$.

(2) Note that the spaces $(O,\mathcal{O},\mu)$ and $([0,T]\times\Omega\times Z, \mathcal{BF}\otimes\mathcal{Z}, \lambda\otimes\mathbb{P}\otimes\nu)$ are both σ-finite measure spaces. Since $f \in L^1(O,\mathcal{O},\mu;\mathcal{M}_T^p(\mathcal{BF}\otimes\mathcal{Z};E))$, applying Minkowski's inequality for integrals (see Theorem 202 in [35]) yields that

$$\left(\mathbb{E}\int_0^T \int_Z \left(\int_O \|f(y,t,\cdot,z)\|\mu(dy)\right)^p \nu(dz)dt\right)^{\frac{1}{p}}$$
$$\leq \int_O \left(\mathbb{E}\int_0^T \int_Z \|f(y,t,\cdot,z)\|^p\nu(dz)dt\right)^{\frac{1}{p}}\mu(dy) < \infty.$$

Hence $\mathbb{E}\int_0^T \int_Z \left(\int_O \|f(y,t,\cdot,z)\|\mu(dy)\right)^p \nu(dz)dt < \infty$. Using a similar argument as (1) gives that $\int_O \|f(y,t,\omega,z)\|\mu(dy) < \infty$, for $\lambda\otimes\mathbb{P}\otimes\nu$-almost all $(t,\omega,z) \in [0,T]\times\Omega\times Z$. This means that $\int_O f(y,t,\omega,z)\mu(dy)$ is well defined for $\lambda\otimes\mathbb{P}\otimes\nu$-almost all $(t,\omega,z) \in [0,T]\times\Omega\times Z$. Since function f is $\mathcal{O}\otimes\mathcal{BF}\otimes\mathcal{Z}$-measurable, again by Fubini theorem that the function $(t,\omega,z) \mapsto \int_O(y,t,\omega,z)\mu(dy)$ is $\mathcal{BF}\otimes\mathcal{Z}$-measurable. Furthermore, we have

$$\left\|\int_O f(y,\cdot,\cdot,\cdot)\mu(dy)\right\|_{\mathcal{M}_T^p} = \left(\mathbb{E}\int_0^T \int_Z \left\|\int_O f(y,t,\cdot,z)\mu(dy)\right\|^p \nu(dz)dt\right)^{\frac{1}{p}}$$
$$\leq \left(\mathbb{E}\int_0^T \int_Z \left(\int_O \|f(y,t,\cdot,z)\|\mu(dy)\right)^p \nu(dz)dt\right)^{\frac{1}{p}} < \infty.$$

This together with $\mathcal{BF}\otimes\mathcal{Z}$-measurability shows that the integral

$$\int_0^T \int_Z \left(\int_O f(y,t,\omega,z)\mu(dy)\right)\tilde{N}(dt,dz)$$

is well defined.

(3) In order to show equality (3.6.1), first we need to verify that the integral on the left side of (3.6.1) i.e.

$$\int_O \left(\int_0^T \int_Z f(y,t,\omega,z) \tilde{N}(dt,dz) \right) \mu(dy)$$

is well defined. For this, we have to show that $\Omega \times Z \ni (\omega, z) \mapsto \int_0^T \int_Z f(y,t,\omega,z)\tilde{N}(dt,dz)$ is $\mathcal{F}_T \otimes \mathcal{O}$-measurable. Since by assumption

$$\mathbb{E} \int_O \int_0^T \int_Z \|f^n(y,t,\cdot,z) - f(y,t\cdot,z)\|^p \nu(dz)dt\mu(dy) < \infty,$$

a similar argument as in the proof of Theorem 3.2.24 shows that there exists a sequence $\{f^n\}_{n=1}^\infty$ of step functions of the form

$$f^n(y,t,\omega,z) = \sum_{j=1}^n \sum_{i=1}^m \sum_{k=1}^{m_i} x_i 1_{F^i_{k,j-1}}(\omega) 1_{(t_{j-1},t_j]}(t) 1_{A^i_{k,j-1}}(z) 1_{E^i_{k,j-1}}(y) \tag{3.6.2}$$

such that

$$\lim_{n\to\infty} \int_O \int_0^T \int_Z \mathbb{E}\|f^n(y,t,\cdot,z) - f(y,t,\cdot,z)\|^p \nu(dz)dt\mu(dy) = 0$$

By passing to a subsequence, still denoted by $\{f^n\}$ for simplicity of notation, we may assume that $\{f^n(y,\cdot,\cdot,\cdot)\}$ converges to $f(y,\cdot,\cdot,\cdot)$ in $\mathcal{M}^p_T(\mathcal{BF}\otimes\mathcal{Z};E)$ for μ-almost all $y \in \mathcal{O}$. In the proof of (1), we showed that the function $(t,\omega,z) \mapsto f(y,t,\omega,z)$ belongs to $\mathcal{M}^p_T(\mathcal{BF}\otimes\mathcal{Z};E)$ for μ-almost all $y \in O$. Let $\hat{O}$ be the set such that for all $y \in \hat{O}$, $f(y,\cdot,\cdot,\cdot) \in \mathcal{M}^p_T(\mathcal{BF}\otimes\mathcal{Z};E)$ and $\{f^n(y,\cdot,\cdot,\cdot)\}$ converges to $f(y,\cdot,\cdot,\cdot)$ in $\mathcal{M}^p_T(\mathcal{BF}\otimes\mathcal{Z};E)$. Then $\mu(O\backslash\hat{O}) = 0$.

On the other hand, for each $y \in \hat{O}$ we can find a sequence of natural number $\{n_k\}_{k=1}^\infty$ such that

$$\|f^{n_k}(y,\cdot,\cdot,\cdot) - f(y,\cdot,\cdot,\cdot)\|^p_{\mathcal{M}^p_T} = \mathbb{E}\int_0^T \int_Z \|f^{n_k}(y,t,\cdot,z) - f(y,t,\cdot,z)\|^p \nu(dz)dt < \frac{1}{10^k}.$$

We write k instead of n_k for brevity, so we have

$$\mathbb{E}\int_0^T \int_Z \|f^k(y,t,\cdot,z) - f(y,t,\cdot,z)\|^p \nu(dz)dt < \frac{1}{10^k}.$$

It follows that for $y \in \hat{O}$,

$$\mathbb{E}\int_0^T \int_Z \|f^{k+1}(y,t,\cdot,z) - f^k(y,t,\cdot,z)\|^p \nu(dz)dt < \frac{2}{10^k}.$$

By the Chebyshev inequality, we have

$$\begin{aligned}
&\mathbb{P}\Big\{\omega \in \Omega : \Big\|\Big(\int_0^T \int_Z \Big[f^{k+1}(y,t,z) - f^k(y,t,z)\Big] \tilde{N}(dt,dz)\Big)(\omega)\Big\| \geq \frac{1}{2^n}\Big\} \\
&\leq \frac{C}{(1/2^k)^p}\mathbb{E}\int_0^T \int_Z \|f^{k+1}(y,t,\cdot,z) - f^k(y,t,\cdot,z)\|^p \nu(dz)dt \\
&\leq \frac{2C/10^k}{(1/2^k)^p} < \frac{C}{2^{k-3}}.
\end{aligned}$$

Since the series $\sum_{k=1}^{\infty} \frac{C}{2^{k-3}}$ is convergent, we infer that

$$\sum_{k=1}^{\infty} \mathbb{P}\Big\{\omega \in \Omega : \Big\|\Big(\int_0^T \int_Z \Big[f^{k+1}(y,t,z) - f^k(y,t,z)\Big] \tilde{N}(dt,dz)\Big)(\omega)\Big\| \geq \frac{1}{2^n}\Big\} < \infty.$$

Hence, by the first Borel-Cantelli lemma, see Theorem 4.3 of [9], it follows that

$$\mathbb{P}\left(\limsup_n \left\{\omega \in \Omega : \left\|\left(\int_0^T \int_Z [f^{k+1}(y,t,z) - f^k(y,t,z)]\tilde{N}(dt,dz)\right)(\omega)\right\| \geq \frac{1}{2^n}\right\}\right) = 0.$$

This implies that for each $y \in \hat{O}$ there exists a set $\hat{\Omega} \subset \Omega$ such that $\mathbb{P}(\hat{\Omega}) = 1$ and there exists some $j \in \mathbb{N}$ such that for all $k \geq j$ we have $\left\|\int_0^T \int_Z [f^{k+1}(y,t,z) - f^k(y,t,z)]\tilde{N}(dt,dz)\right\| < \frac{1}{2^k}$ $\omega \in \hat{\Omega}$. Hence

$$\sum_{n=j}^{\infty} \left\|\int_0^T \int_Z [f^{k+1}(y,t,z) - f^k(y,t,z)\tilde{N}(dt,dz)\right\|(\omega) < \infty, \quad \omega \in \hat{\Omega},\ y \in \hat{O}.$$

Consider the series

$$\sum_{k=0}^{\infty} \left(\int_0^T \int_Z [f^{k+1}(y,t,z) - f^k(y,t,z)]\tilde{N}(dt,dz)\right)(\omega),$$

where $f^0(y,s,z) \equiv 0$. Since for $k = 0, \cdots, j-1$

$$\begin{aligned}
&\mathbb{E}\left\|\int_0^T \int_Z [f^{k+1}(y,t,z) - f^k(y,t,z)]\tilde{N}(dt,dz)\right\|^p \\
&\leq C\mathbb{E}\int_0^T \int_Z \|f^{k+1}(y,t,z) - f^k(y,t,z)\|^p \nu(dz)dt \\
&< \infty,
\end{aligned}$$

we infer that $\| \int_0^T \int_Z [f^{(k+1)}(y,t,z) - f^k(y,t,z)\tilde{N}(dt,dz))\| < \infty$ on $\hat{\Omega}$ for $y \in \hat{O}$, $n = 0, \cdots, j-1$. Therefore we conclude that $\sum_{k=0}^{\infty} \left\|\int_0^T \int_Z [f^{k+1}(t,z) - f^k(t,z)]\tilde{N}(dt,dz)\right\| < \infty$ on $\hat{\Omega}$. This gives that $\sum_{k=0}^{\infty} \int_0^T \int_Z (f^{k+1}(y,t,z) - f^k(y,t,z))\tilde{N}(dt,dz)$ is convergent for each $\omega \in \hat{\Omega}$, $y \in \hat{O}$. Now we define $\xi_T(y,\omega) = \sum_{k=0}^{\infty} \int_0^T \int_Z (f^{k+1}(y,t,z) - f^k(y,t,z))\tilde{N}(dt,dz)$ when the sum converges, i.e.

$$\xi_T(y,\omega) = \lim_{k\to\infty} \int_0^T \int_Z f^k(y,t,z)\tilde{N}(dt,dz),$$

and if this limit diverges, we put $\xi_T(\cdot) = 0$. Note that for each k, the integral $\int_0^T \int_Z f^k(y,t,z)\tilde{N}(dt,dz)$ is of the form

$$\sum_{j=1}^{n} \sum_{i=1}^{m} \sum_{k=1}^{m_i} x_i 1_{F^i_{k,j-1}}(\omega)\tilde{N}((t_{j-1},t_j] \times A^i_{k,j-1}) 1_{E^i_{k,j-1}}(y)$$

which is $\mathcal{O} \otimes \mathcal{F}_T$-measurable. Therefore, the limit ξ_T is also $\mathcal{O} \otimes \mathcal{F}_T$-measurable. Since for each $y \in \hat{O}$, $\{f^k(y)\}_{k\in\mathbb{N}}$ is dense in $\mathcal{M}^p_T(\mathcal{BF} \otimes \mathcal{Z}; E)$, we have

$$\mathbb{E}\left\|\int_0^T \int_Z [f^k(y,t,z)\tilde{N}(dt,dz) - \int_0^T \int_Z f(y,t,z)\tilde{N}(dt,dz)\right\|^p \to 0, \quad as\ n \to \infty,$$

Then for each $y \in \hat{O}$ we can find a subsequence $\{f^{k_j}\}_{j\in\mathbb{N}}$ such that

$$\lim_{j\to\infty}\int_0^T\int_Z f^{k_j}(y,s,z)\tilde{N}(dt,dz) = \int_0^T\int_Z f(y,s,z)\tilde{N}(dt,dz) \quad \text{on } \tilde{\Omega},$$

with $\mathbb{P}(\tilde{\Omega}) = 1$. Hence in view of the definition of ξ_T and the uniqueness of the limit, we see that for each $y \in \hat{O}$, $\xi_T(y,\omega) = \left(\int_0^T\int_Z f(y,s,z)\tilde{N}(dt,dz)\right)(\omega)$ for $\omega \in \tilde{\Omega}\cap\hat{\Omega}$. Hence we constructed a $\mathcal{O}\otimes\mathcal{F}_T$-measurable version of $\int_0^T\int_Z f(y,s,z)\tilde{N}(dt,dz)$. So the integral

$$\int_O\left(\int_0^T\int_Z f(y,t,\omega,z)\tilde{N}(dt,dz)\right)\mu(dy)$$

is well defined. Now we are going to show equality (3.6.1). First we verify (3.6.1) for step function of the form

$$f^n(y,t,\omega,z) = \sum_{j=1}^n\sum_{i=1}^m\sum_{k=1}^{m_i} x_i 1_{F^i_{k,j-1}}(\omega)1_{(t_{j-1},t_j]}(t)1_{A^i_{k,j-1}}(z)1_{E^i_{k,j-1}}(y).$$

In this case the left side of (3.6.1) becomes

$$\begin{aligned}&\int_O\left(\int_0^T\int_Z f^n(y,t,\omega,z)\tilde{N}(dt,dz)\right)\mu(dy)\\ &= \int_O\left(\sum_{j=1}^n\sum_{i=1}^m\sum_{k=1}^{m_i} x_i 1_{F^i_{k,j-1}}(\omega)\tilde{N}((t_{j-1},t_j]\times A^i_k)1_{E^i_{k,j-1}}(y)\right)\mu(dy)\\ &= \sum_{j=1}^n\sum_{i=1}^m\sum_{k=1}^{m_i} x_i 1_{F^i_{k,j-1}}(\omega)\tilde{N}((t_{j-1},t_j]\times A^i_{k,j-1})\mu(E^i_{k,j-1}).\end{aligned}$$

Consider next the right side of (3.6.1) with f^n. We have

$$\begin{aligned}&\int_0^T\int_Z\left(\int_O f^n(y,t,\omega,z)\mu(dy)\right)\tilde{N}(dt,dz)\\ &\quad= \int_0^T\int_Z\sum_{j=1}^n\sum_{i=1}^m\sum_{k=1}^{m_i} x_i 1_{F^i_{k,j-1}}(\omega)1_{(t_{j-1},t_j]}(t)1_{A^i_{k,j-1}}(x)\mu(E^i_{k,j-1})\tilde{N}(dt,dz)\\ &\quad= \sum_{j=1}^n\sum_{i=1}^m\sum_{k=1}^{m_i} x_i 1_{F^i_{k,j-1}}(\omega)\tilde{N}((t_{j-1},t_j]\times A^i_{k,j-1})\mu(E^i_{k,j-1}).\end{aligned}$$

Hence we infer that

$$\int_O\left(\int_0^T\int_Z f^n(y,t,\omega,z)\tilde{N}(dt,dz)\right)\mu(dy) = \int_0^T\int_Z\left(\int_O f^n(y,t,\omega,z)\mu(dy)\right)\tilde{N}(dt,dz), \tag{3.6.3}$$

which means that equality (3.6.1) holds for every step function of the form (3.6.2). Now define

$$\tilde{f}^n(y,t,\omega,z) = f^n(y,t,\omega,z)1_{\{y:\|f^n\|_{\mathcal{M}^p_T}\le\|f\|_{\mathcal{M}^p_T}\}},\ (y,t,\omega,z)\in O\times[0,T]\times\Omega\times Z.$$

Since $f^n(y,\cdot,\cdot,\cdot)$ converges to $f(y,\cdot,\cdot,\cdot)$ in $\mathcal{M}^p_T$ for all $y \in \hat{O}$, it is clear that

$$\lim_{n\to\infty}\|\tilde{f}^n(y,\cdot,\cdot,\cdot) - f(y,\cdot,\cdot,\cdot)\|_{\mathcal{M}^p_T} = 0 \text{ and } \|\tilde{f}^n(y,\cdot,\cdot,\cdot)\|_{\mathcal{M}^p_T} \le \|f(y,\cdot,\cdot,\cdot)\|_{\mathcal{M}^p_T}$$

for all $y \in \hat{O}$. Since $f \in L^1(O, \mathcal{O}, \mu; \mathcal{M}^p_T(\mathcal{BF} \otimes \mathcal{Z}; E))$, the Lebesgue dominated convergence theorem tells us that

$$\lim_{n\to\infty} \int_O \|\tilde{f}^n(y, \cdot, \cdot, \cdot) - f(y, \cdot, \cdot, \cdot)\|_{\mathcal{M}^p_T}\mu(dy) = 0. \tag{3.6.4}$$

Clearly, by (3.6.3) we have

$$\int_O \left(\int_0^T \int_Z \tilde{f}^n(y, t, \omega, z)\tilde{N}(dt, dz)\right) \mu(dy) = \int_0^T \int_Z \left(\int_O \tilde{f}^n(y, t, \omega, z)\mu(dy)\right) \tilde{N}(dt, dz).$$

Observe that for every $f \in L^1(O, \mathcal{O}, \mu; \mathcal{M}^p_T(\mathcal{BF} \otimes \mathcal{Z}; E))$,

$$\begin{aligned}
&\mathbb{E}\left\|\int_O \left(\int_0^T \int_Z f(y, t, \cdot, z)\tilde{N}(dt, dz)\right) \mu(dy) - \int_0^T \int_Z \left(\int_O f(y, t, \cdot, z)\mu(dy)\right) \tilde{N}(dt, dz)\right\| \\
&\le \mathbb{E}\left\|\int_O \left(\int_0^T \int_Z f(y, t, \cdot, z)\tilde{N}(dt, dz)\right) \mu(dy) - \int_O \left(\int_0^T \int_Z \tilde{f}^n(y, t, \cdot, z)\tilde{N}(dt, dz)\right) \mu(dy)\right\| \\
&\quad + \mathbb{E}\left\|\int_0^T \int_Z \left(\int_O \tilde{f}^n(y, t, \cdot, z)\mu(dy)\right) \tilde{N}(dt, dz) - \int_0^T \int_Z \left(\int_O f(y, t, \cdot, z)\mu(dy)\right) \tilde{N}(dt, dz)\right\| \\
&= \mathbb{E}\left\|\int_O \left(\int_0^T \int_Z [f(y, t, \cdot, z) - \tilde{f}^n(y, t, \cdot, z)]\tilde{N}(dt, dz)\right) \mu(dy)\right\| \\
&\qquad + \mathbb{E}\left\|\int_0^T \int_Z \left(\int_O [\tilde{f}^n(y, t, \cdot, z) - f(y, t, \cdot, z)]\mu(dy)\right) \tilde{N}(dt, dz)\right\| \\
&\le \int_O \mathbb{E}\left\|\int_0^T \int_Z [f(y, t, \cdot, z) - \tilde{f}^n(y, t, \cdot, z)]\tilde{N}(dt, dz)\right\| \mu(dy) \\
&\qquad + \left(\mathbb{E}\left\|\int_0^T \int_Z \left(\int_O [\tilde{f}^n(y, t, \cdot, z) - f(y, t, \cdot, z)]\mu(dy)\right) \tilde{N}(dt, dz)\right\|^p\right)^{\frac{1}{p}} \\
&\le \int_O \left(\mathbb{E}\left\|\int_0^T \int_Z [f(y, t, \cdot, z) - \tilde{f}^n(y, t, \cdot, z)]\tilde{N}(dt, dz)\right\|^p\right)^{\frac{1}{p}} \mu(dy) \\
&\qquad + C\left(\mathbb{E}\int_0^T \int_Z \left\|\int_O [\tilde{f}^n(y, t, \cdot, z) - f(y, t, \cdot, z)]\mu(dy)\right\|^p \nu(dz)dt\right)^{\frac{1}{p}} \\
&\le C\int_O \left(\mathbb{E}\int_0^T \int_Z \|f(y, t, \cdot, z) - \tilde{f}^n(y, t, \cdot, z)\|^p \nu(dz)dt\right)^{\frac{1}{p}} \mu(dy) \\
&\qquad + C\left\|\int_O [f(y, \cdot, \cdot, \cdot) - \tilde{f}^n(y, \cdot, \cdot, \cdot)]\mu(dy)\right\|_{\mathcal{M}^p_T} \\
&\le C\int_O \|f(y, \cdot, \cdot, \cdot) - \tilde{f}^n(y, \cdot, \cdot, \cdot)\|_{\mathcal{M}^p_T}\mu(dy) + C\int_O \|f(y, \cdot, \cdot, \cdot) - \tilde{f}^n(y, \cdot, \cdot, \cdot)\|_{\mathcal{M}^p_T}\mu(dy) \\
&= 2C\int_O \|f(y, \cdot, \cdot, \cdot) - \tilde{f}^n(y, \cdot, \cdot, \cdot)\|_{\mathcal{M}^p_T}\mu(dy),
\end{aligned}$$

where we used Cauchy-Schwarz inequality and the inequality (3.3.2). By letting $n \to \infty$, it follows from (3.6.4) that

$$\mathbb{E}\left\|\int_O \left(\int_0^T \int_Z f(y, t, \cdot, z)\tilde{N}(dt, dz)\right) \mu(dy) - \int_0^T \int_Z \left(\int_O f(y, t, \cdot, z)\mu(dy)\right) \tilde{N}(dt, dz)\right\| = 0.$$

Therefore, we infer that

$$\left(\int_0^T \int_Z f(y,t,\omega,z)\tilde{N}(dt,dz)\right)\mu(dy) = \int_0^T \int_Z \left(\int_O f(y,t,\omega,z)\mu(dy)\right)\tilde{N}(dt,dz) \quad \mathbb{P}\text{-a.s.}$$

which completes our proof.

□

3.7 Maximal Inequlities

Let $T > 0$ be fixed. Let $\mathcal{M}_T^p(\hat{\mathcal{P}},\nu;E)$ be the space of all $\mathfrak{F}$-predictable E-valued functions such that

$$\mathbb{E}\int_0^T \int_Z \|f(s,\cdot,z)\|^p \nu(dz)ds < \infty. \tag{3.7.1}$$

From now on, while considering the stochastic process $\int_0^t \int_Z f(s,z)\tilde{N}(ds,dz)$, $0 \le t \le T$, $f \in \mathcal{M}_T^p(\hat{\mathcal{P}},\nu;E)$, it will be assumed that the process $\int_0^t \int_Z f(s,z)\tilde{N}(ds,dz)$, $0 \le t \le T$, has càdlàg trajectories.

3.7.1 The Stochastic Convolution

Let $(S(t))_{t\ge 0}$ be a contraction C_0-semigroup on E. Suppose that A is the infinitesimal generator of the C_0-semigroup $(S(t))_{t\ge0}$. If $\{A_\lambda : \lambda > 0\}$ is the Yosida approximation of A, then for each λ, A_λ is a bounded operator in E and $|A_\lambda x - Ax|_E$ converges to 0 as $\lambda \to \infty$ for all $x \in \mathcal{D}(A)$. Let $R(\lambda, A) = (\lambda I - A)^{-1}$. By the use of Hille-Yosida Theorem (see [62]), it is easy to establish that $\lim_{\lambda\to\infty} \lambda R(\lambda,A)x = x$ and $\lambda R(\lambda,A)x \in \mathcal{D}(A)$, for all $x \in X$.

Let $\xi \in \mathcal{M}_T^p(\hat{\mathcal{P}},\nu;E)$. We are going to consider the following stochastic convolution process

$$u(t) = \int_0^t \int_Z S(t-s)\xi(s,z)\tilde{N}(ds,dz), \; 0 \le t \le T, \tag{3.7.2}$$

where $\tilde{N}$ is a compensated Poisson random measure associated with the Poisson point process π.

We will first investigate the measurability of the process u.

Lemma 3.7.1. *The process $u(t)$, $0 \le t \le T$ given by (3.7.2) has a predictable version.*

Proof. Let $t \in [0,T]$ be fixed. We first show that a process X defined by $X(s) = 1_{(0,t]}(s)S(t-s)\xi(s,z)$, $0 \le s \le T$ is predictable. Define a function $F : [0,t] \times E \ni (s,x) \mapsto S(t-s)x \in E$. Since $S(t)$, $t \ge 0$ is a C_0-semigroup, so for every $x \in E$, $F(\cdot,x)$ is continuous on $[0,t]$. Also, for every $s \ge 0$, $F(s,\cdot)$ is continuous. Indeed, let us fix $x_0 \in E$. Then for every $x \in E$, and $0 \le t \le T$,

$$|F(t,x) - F(t,x_0)|_E = |S(t-s)(x-x_0)|_E \le |x-x_0|_E,$$

as $\|S(t)\|_{\mathcal{L}(E)} \leq 1$. This part shows that the function F is separably continuous. Since by assumption the process ξ is $\mathfrak{F}$-predictable, one can see that the mapping

$$(s, \omega, z) \mapsto (s, \xi(s, \omega, z))$$

of $[0, T] \times \Omega \times Z$ into $[0, T] \times E$ is $\mathfrak{F}$-predictable. Moreover, since the process $1_{(0,t]}$ is $\mathfrak{F}$-predictable and we showed that the function F is separably continuous, so the composition mapping

$$(s, \omega, z) \mapsto (s, \xi(s, \omega, z)) \mapsto F(s, \xi(s, \omega, z)) \mapsto 1_{(0,t]}(s)F(s, \xi(s, \omega, z))$$

is $\mathfrak{F}$-predictable as well. Therefore, process $X(s) = 1_{(0,t]}(s)F(s, \xi(s, z))$, $s \in [0, T]$ is $\mathfrak{F}$-predictable. On the other hand, since $S(t)$, $t \geq 0$ is a C_0-semigroup of contractions and ξ is in $\mathcal{M}^p([0, T] \times \Omega \times Z, \hat{\mathcal{P}}, \lambda \otimes \mathbb{P} \times \nu; E)$, we have

$$\mathbb{E} \int_0^T |1_{(0,t]}(s)S(t-s)\xi(s, z)|_E^p \nu(dz)\, ds \leq \mathbb{E} \int_0^T |\xi(s, z)|_E^p \nu(dz)\, ds < \infty.$$

Therefore, the process $1_{(0,t]}(s)S(t-s)\xi(s, z)$ is of class $\mathcal{M}^p([0, T] \times \Omega \times Z, \hat{\mathcal{P}}, \lambda \otimes \mathbb{P} \times \nu; E)$. Hence, when the number t is fixed, the integrals

$$\int_0^r \int_Z 1_{(0,t]}(s)S(t-s)\xi(s, z)\tilde{N}(ds, dz), \quad r \in [0, T]$$

are well defined and by Theorem 3.3.2, this process is a martingale. In particular, for each $r \in [0, T]$, the integral $\int_0^r \int_Z 1_{(0,t]}(s)S(t-s)\xi(s, z)\tilde{N}(ds, dz)$ is $\mathcal{F}_r$-measurable. Take $r = t$. This gives that $\int_0^t \int_Z 1_{(0,t]}(s)S(t-s)\xi(s, z)\tilde{N}(ds, dz)$ is $\mathcal{F}_t$-measurable.

Now we show that the process u is continuous in p-mean. On the basis of the inequality $|a+b|^p \leq 2^p|a|^p + 2^p|b|^p$, inequality (3.3.2) and the contraction property of the semigroup $S(t)$, $t \geq 0$, we have, for $0 \leq r < t \leq T$,

$$\begin{aligned}
\mathbb{E}|u(t) - u(r)|_E^p &= \mathbb{E}\left|\int_0^t \int_Z S(t-s)\xi(s, z)\tilde{N}(ds, dz) - \int_0^r \int_Z S(r-s)\xi(s, z)\tilde{N}(ds, dz)\right|_E^p \\
&\leq 2^p\mathbb{E}\left|\int_r^t \int_Z S(t-s)\xi(s, z)\tilde{N}(ds, dz)\right|_E^p \\
&\qquad + 2^p\mathbb{E}\left|\int_0^r \int_Z \Big(S(t-s) - S(r-s)\Big)\xi(s, z)\tilde{N}(ds, dz)\right|_E^p \\
&\leq 2^pC_p\mathbb{E}\int_r^t \int_Z |S(t-s)\xi(s, z)|_E^p \nu(dz)\, ds \\
&\qquad + 2^pC_p\mathbb{E}\int_0^r \int_Z \left|\Big(S(t-s) - S(r-s)\Big)\xi(s, z)\right|_E^p \nu(dz)\, ds \\
&\leq 2^pC_p\mathbb{E}\int_r^t \int_Z |\xi(s, z)|_E^p \nu(dz)\, ds \\
&\qquad + 2^pC_p\mathbb{E}\int_0^r \int_Z \left|\Big(S(t-s) - S(r-s)\Big)\xi(s, z)\right|_E^p \nu(dz)\, ds
\end{aligned}$$

$$\begin{aligned}
&= 2^pC_p\mathbb{E}\int_0^T \int_Z 1_{(r,t]}(s)|\xi(s, z)|_E^p \nu(dz)\, ds \\
&\qquad + 2^pC_p\mathbb{E}\int_0^T \int_Z \left|1_{(0,r]}(s)\Big(S(t-s) - S(r-s)\Big)\xi(s, z)\right|_E^p \nu(dz)\, ds.
\end{aligned}$$

Here we note that $1_{(r,t]}(s)|\xi(s,z)|_E^p$ converges to 0 for all $(s,\omega,z) \in [0,T] \times \Omega \times Z$, as $t \downarrow r$ or $r \uparrow t$. So by the Lebesgue Dominated Converges Theorem, the first term on the right side of the above inequality converges to 0 as $t \downarrow r$ or $r \uparrow t$. For the second term, by the continuity of C_0-semigroup $S(t)$, $t \geq 0$, the integrand $1_{(0,r]}\Big(S(t-s) - S(r-s)\Big)\xi(s,z)$ converges to 0 pointwise on $[0,T] \times \Omega \times Z$. Moreover we see that

$$|1_{(0,r]}(s)S(t-s) - S(r-s)\Big)\xi(s,z)|_E \leq |2\xi(s,z)|_E$$

So, again by the Lebesgue Dominated Convergence Theorem, the second term also converges to 0 as $t \downarrow r$ or $r \uparrow t$. Therefore, the process u is continuous in the p-mean. Since by Proposition 3.6 in [26], every adapted and stochastically continuous process on an interval $[0,T]$ has a predictable version, we conclude that the process $u(t)$, $0 \leq t \leq T$ has a predictable version.

□

Assume that A is the infinitesimal generator of a C_0-semigroup $S(t)$, $t \geq 0$ of contractions on the martingale type p, $1 < p \leq 2$ Banach space E and that ξ is a function belonging to $\mathcal{M}_T^p(\hat{\mathcal{P}}, \nu; E)$. We will consider the problem

$$\begin{aligned} du(t) &= Au(t)dt + \int_Z \xi(t,z)\tilde{N}(dt,dz), \ t \geq 0, \\ u(0) &= 0. \end{aligned} \tag{3.7.3}$$

Definition 3.7.2. Suppose that $\mathbb{E}\int_0^T \int_Z |\xi(s,z)|_E^p \nu(dz)dt < \infty$. A strong solution to Problem (3.7.3) is a $\mathcal{D}(A)$-valued predictable stochastic process $(u(t))_{0 \leq t \leq T}$ such that

(1) $u(0) = 0$ a.s.

(2) For any $t \in [0,T]$ the equality

$$u(t) = \int_0^t Au(s)\,ds + \int_0^t \int_Z \xi(s,z)\tilde{N}(ds,dz) \tag{3.7.4}$$

holds $\mathbb{P}$-a.s.

Lemma 3.7.3. *Let $\xi \in \mathcal{M}_T^p(\hat{\mathcal{P}}, \nu; \mathcal{D}(A))$. Then the process u defined by*

$$u(t) = \int_0^t \int_Z S(t-s)\xi(s,z)\tilde{N}(ds,dz), \ t \in [0,T], \tag{3.7.5}$$

is a unique strong solution of equation (3.7.3).

Proof. Let us us fix $t \in [0,T]$. First we need to show that $u(t) \in \mathcal{D}(A)$. For this, Let $R(\lambda, A) = (\lambda I - A)^{-1}$, $\lambda > 0$, be the resolvent of A. Since $AR(\lambda,A) = \lambda R(\lambda, A) - I_E$, $AR(\lambda, A)$ is bounded. Hence, since $\xi \in \mathcal{M}_T^p(\hat{\mathcal{P}}, \nu; \mathcal{D}(A))$, we obtain

$$\begin{aligned} R(\lambda, A)\int_0^t \int_Z AS(t-s)\xi(s,z)\tilde{N}(ds,dz) &= \int_0^t \int_Z R(\lambda,A)AS(t-s)\xi(s,z)\tilde{N}(ds,dz) \\ &= \lambda R(\lambda, A)\int_0^t \int_Z S(t-s)\xi(s,z)\tilde{N}(ds,dz) \\ &\quad - \int_0^t \int_Z S(t-s)\xi(s,z)\tilde{N}(ds,dz). \end{aligned}$$

Thus, it follows that

$$\int_0^t \int_Z S(t-s)\xi(s,z)\tilde{N}(ds,dz)$$
$$= R(\lambda, A)\left[\lambda \int_0^t \int_Z S(t-s)\xi(s,z)\tilde{N}(ds,dz) - \int_0^t \int_Z AS(t-s)\xi(s,z)\tilde{N}(ds,dz)\right].$$

Since $Rng(R(\lambda, A)) = \mathcal{D}(A)$, we infer that $\int_0^t \int_Z S(t-s)\xi(s,z)\tilde{N}(ds,dz) \in \mathcal{D}(A)$. On the other hand, let us take $h \in (0,t)$ and observe that since $\frac{S(h)-I}{h}$ is bounded, we get the following equality

$$\frac{S(h)-I}{h}\int_0^t \int_Z S(t-s)\xi(s,z)\tilde{N}(ds,dz)$$
$$= \int_0^t \int_Z \frac{S(h)-I}{h} S(t-s)\xi(s,z)\tilde{N}(ds,dz).$$

So by applying the triangle inequality and inequality (3.3.2), we find out that

$$\begin{aligned}
&\mathbb{E}\left|A\int_0^t \int_Z S(t-s)\xi(s,z)\tilde{N}(ds,dz) - \int_0^t \int_Z AS(t-s)\xi(s,z)\tilde{N}(ds,dz)\right|^p \\
&\quad \le 2^p\mathbb{E}\left|A\int_0^t \int_Z S(t-s)\xi(s,z)\tilde{N}(ds,dz) - \frac{S(h)-I}{h}\int_0^t \int_Z S(t-s)\xi(s,z)\tilde{N}(ds,dz)\right|^p \\
&\qquad + 2^p\mathbb{E}\left|\int_0^t \int_Z AS(t-s)\xi(s,z)\tilde{N}(ds,dz) - \int_0^t \int_Z \frac{S(h)-I}{h} S(t-s)\xi(s,z)\tilde{N}(ds,dz)\right|^p \\
&\quad \le 2^p\mathbb{E}\left|\left(A - \frac{S(h)-I}{h}\right)\int_0^t \int_Z S(t-s)\xi(s,z)\tilde{N}(ds,dz)\right|^p \\
&\qquad + C_p\mathbb{E}\int_0^t \int_Z \left|AS(t-s)\xi(s,z) - \frac{1}{h}\Big(S(h)-I\Big)S(t-s)\xi(s,z)\right|_E^p \nu(dz)\,ds \\
&\quad := \mathrm{I}(h) + \mathrm{II}(h).
\end{aligned} \tag{3.7.6}$$

For the integrand of $\mathrm{II}(h)$, since $\xi(s,z) \in \mathcal{D}(A)$, we observe that

$$\frac{S(h)-I}{h} S(t-s)\xi(s,z) = \frac{1}{h}\int_0^h S(r)AS(t-s)\xi(s,z)dr,$$

so we have $\left|\frac{S(h)-I}{h} S(t-s)\xi(s,z)\right|_E^p \le |A\xi(s,z)|_E^p$. Hence we infer that the integrand

$$\left|AS(t-s)\xi(s,z) - \frac{1}{h}\Big(S(h)-I\Big)S(t-s)\xi(s,z)\right|_E^p$$

of $\mathrm{I}(h)$ is bounded by a function $C_1|A\xi(s,z)|_E^p$ which is in $\mathcal{M}_T^p(\hat{\mathcal{P}}, \nu; E)$ by assumption. Since A is the infinitesimal generator of the C_0-semigroup $S(t)$, $t \ge 0$, the integrand

$$\left|AS(t-s)\xi(s,z) - \frac{1}{h}\Big(S(h)-I\Big)S(t-s)\xi(s,z)\right|_E^p$$

converges to 0 pointwisely on $[0,t] \times \Omega \times Z$. Therefore, by the Lebesgue Dominated convergence theorem, the term $\mathrm{II}(h)$ of above inequality (3.7.6) converges to 0 as $h \downarrow 0$.
Since we have already shown that $\int_0^t \int_Z S(t-s)\xi(s,z)\tilde{N}(ds,dz) \in \mathcal{D}(A)$, it is easy to see that the term $\mathrm{I}(h)$ of (3.7.6) converges to 0 as $h \downarrow 0$ as well. Hence by inequality (3.7.6) we conclude that

$$A\int_0^t \int_Z S(t-s)\xi(s,z)\tilde{N}(ds,dz) = \int_0^t \int_Z AS(t-s)\xi(s,z)\tilde{N}(ds,dz), \quad \mathbb{P}\text{-a.s.} \tag{3.7.7}$$

In order to verify equality (3.7.4), by the Fubini's theorem 3.6.2 and equality (3.7.7) we find out that

$$\begin{aligned}
\int_0^t Au(s)\,ds &= \int_0^t \int_0^s \int_Z AS(s-r)\xi(r,z)\tilde{N}(dr,dz)\,ds \\
&= \int_0^t \int_Z \int_r^t AS(s-r)\xi(r,z)\,ds\tilde{N}(dr,dz) \\
&= \int_0^t \int_Z \int_r^t \frac{dS(s-r)\xi(r,z)}{ds} ds\tilde{N}(dr,dz) \\
&= \int_0^t \int_Z (S(t-r)\xi(r,z) - \xi(r,z))\,\tilde{N}(dr,dz) \\
&= \int_0^t \int_Z S(t-r)\xi(r,z)\tilde{N}(dr,dz) - \int_0^t \int_Z \xi(r,z)\tilde{N}(dr,dz) \\
&= u(t) - \int_0^t \int_Z \xi(r,z)\tilde{N}(dr,dz), \quad \mathbb{P}\text{-a.s.}
\end{aligned}$$

which shows equality (4.1.1).

For the uniqueness, suppose that u^1 and u^2 are two strong solutions of Problem (3.7.3). Let $w = u^1 - u^2$. Then we infer that

$$w(t) = u^1(t) - u^2(t) = \int_0^t A(u^1(s) - u^2(s))\,ds = A\int_0^t w(s)\,ds.$$

Put $v(t) = \int_0^t w(s)\,ds$. Then $v(t)$ is continuously differentiable on $[0,T]$ and $v(t) \in \mathcal{D}(A)$. Now applying the integration by parts formula to the function $f(s) = S(t-s)v(s)$ yields

$$\begin{aligned}
\frac{df(s)}{ds} &= -AS(t-s)v(s) + S(t-s)\frac{dv(s)}{ds} \\
&= -AS(t-s)v(s) + S(t-s)w(s) = -AS(t-s)v(s) + S(t-s)Av(s) = 0.
\end{aligned}$$

So we infer $v(t) = f(t) = f(0) = S(t)v(0) = 0$ a.s. Therefore, $w(s) = 0$ a.s. That is $u^1(t) = u^2(t)$ a.s. $t \in [0,T]$. □

3.7.2 Maximal Inequalities for Stochastic Convolutions

Assumption 3.7.4. *Suppose that E is a real separable Banach space of martingale type p, $1 < p \leq 2$. In addition we assume that the Banach space E satisfies the following condition:*
(Cond. 1) *There exists an equivalent norm $|\cdot|_E$ on E and $q \in [p,\infty)$ such that the function $\phi : E \ni x \mapsto |x|_E^q \in \mathbb{R}$, is of class C^2 and there exists constant k_1, k_2 such that for every $x \in E$, $|\phi'(x)| \leq k_1|x|_E^{q-1}$ and $|\phi''(x)| \leq k_2|x|_E^{q-2}$.*

Remark 3.7.5. It can be proved that if E satisfies condition **(Cond. 1)** for some q and $q_2 > q$, then E satisfies condition **(Cond. 1)** for q_2.

Remark 3.7.6. Notice that the Sobolev space $H^{s,r}$ with $r \in [q,\infty)$ and $s \in \mathbb{R}$ satisfies above condition Cond. 1 and L^r-spaces with $r \geq q$ also satisfies condition Cond. 1.

Now we proceed with the study of the stochastic convolution

$$u(t) = \int_0^t \int_Z S(t-s)\xi(s,z)\tilde{N}(ds,dz), \quad t \in [0,T]. \tag{3.7.8}$$

Before proving the main theorem, we first need the following Lemmas.

Lemma 3.7.7. *For all $x \in D(A)$, $\phi'(x)(Ax) \leq 0$.*

Proof. Take $0 \leq r < t < \infty$. We have

$$\begin{aligned}|S(t)x|_E^q - |S(r)x|_E^q &= |S(t-r)S(r)x|_E^q - |S(r)x|_E^q \\ &\leq |S(t-r)|_{\mathcal{L}(E)}^q |S(r)x|_E^q - |S(r)x|_E^q \\ &\leq |S(r)x|_E^q - |S(r)x|_E^q = 0, \quad \text{for all } x \in E.\end{aligned}$$

Thus the function $t \mapsto \phi(x)(S(t)x)$ is decreasing. Also, observe that for $x \in D(A)$,

$$\left.\frac{d\phi(S(t)x)}{dt}\right|_{t=0} = \phi'(S(0)x)(Ax) = \phi'(x)(Ax).$$

Hence $\phi'(x)(Ax) = \left.\frac{d\phi(S(t)x)}{dt}\right|_{t=0} \leq 0$ which shows the Lemma. □

Lemma 3.7.8. *There exists a version $\bar{u}$ of u such that the function* $\sup_{0\leq t\leq T} |\bar{u}(t)|$ *is measurable.*

Proof. Suppose that $\xi \in \mathcal{M}_T^p(\hat{\mathcal{P}}, \nu; \mathcal{D}(A))$. It then follows from Lemma 3.7.3 that u can also be written in the following form

$$u(t) = \int_0^t Au(s)\, ds + \int_0^t \int_Z \xi(s,z)\tilde{N}(ds,dz), \ t \in [0,T]. \tag{3.7.9}$$

Set $w(t) := \int_0^t \int_Z \xi(s,z)\tilde{N}(ds,dz)$, $0 \leq t \leq T$. Recall that we showed the stochastically continuity of process u when $\xi \in \mathcal{M}^p([0,T] \times \Omega \times Z, \hat{\mathcal{P}} \otimes \mathcal{Z}, \lambda \otimes \mathbb{P} \times \nu; \mathcal{D}(A))$ in the proof of Lemma 3.7.1. Applying Theorem 5.3 in [80], we can find a version $\tilde{u}$ of u which is separable. That is there exists a countable subset T_0 which is everywhere dense in $[0,T]$ such that $\tilde{u}(t)$ belongs to the set of partial limits $\lim_{s\in T_0, s\to t} \tilde{u}(s)$ with probability 1 for all $t \in [0,T]\backslash T_0$. Hence

$$\sup_{t\in[0,T]} |\tilde{u}(t)| = \sup_{t\in[0,T]} \lim_{s_n\to t, s_n\in T_0} |\tilde{u}(s_n)| = \sup_{s_n\in T_0} |\tilde{u}(s_n)|,$$

where $\sup_{s_n\in T_0} |\tilde{u}(s_n)|$ is measurable. Therefore, the function $\sup_{t\in[0,T]} |\tilde{u}(t)|$ is measurable, i.e. it is a random variable.

□

Henceforth, when we study the stochastic convolution process u, we refer to the version of u such that it is predictable and its supremum over $[0,T]$ is measurable.

Theorem 3.7.9. *Suppose that E is an martingale type p, $1 < p \leq 2$ Banach space satisfying Assumption 3.7.4. Suppose $q' \geq q$, where q is the number from Assumption 3.7.4. If $\xi \in \mathcal{M}_T^p(\hat{\mathcal{P}}; E)$ such that*

$$\mathbb{E}\left(\int_0^t \int_Z |\xi(s,z)|_E^p N(ds,dz)\right)^{\frac{q'}{p}} < \infty, \tag{3.7.10}$$

then there exists a separable and càdlàg modification $\tilde{u}$ of u and a constant C such that for every $0 < t \leq T$,

$$\mathbb{E} \sup_{0\leq s\leq t} |\tilde{u}(s)|_E^{q'} \leq C\, \mathbb{E}\left(\int_0^t \int_Z |\xi(s,z)|_E^p N(ds,dz)\right)^{\frac{q'}{p}}. \tag{3.7.11}$$

Proof. **Case I.** First suppose that $\xi \in \mathcal{M}^p_T(\hat{\mathcal{P}};\mathcal{D}(A))$. We will prove

$$\mathbb{E}\sup_{0\le s\le t}|u(s)|^q_E \le C\,\mathbb{E}\left(\int_0^t\int_Z|\xi(s,z)|^p_E N(ds,dz)\right)^{\frac{q}{p}}, \tag{3.7.12}$$

We have shown in Lemma 3.7.3 that the process u is a unique strong solution to the following problem

$$\begin{aligned} du(t) &= Au(t)dt + \int_Z \xi(t,z)\tilde{N}(dt,dz), \quad t\in[0,T],\\ u(0) &= 0. \end{aligned} \tag{3.7.13}$$

Moreover, it can be written as

$$u(t) = \int_0^t Au(s)\,ds + \int_0^t\int_Z \xi(s,z)\tilde{N}(ds,dz),\ t\in[0,T]. \tag{3.7.14}$$

We shall note here that in view of the càdlàg property of the right side of (3.7.14), see Theorem 3.3.2, the càdlàg property of the function $u(t)$, $0\le t\le T$ follows immediately. Notice that the function $\phi: E\ni x\mapsto |x|^q_E$ is of C^2 class by assumption. Thus, one may apply the Itô formula, see Theorem 3.5.3, to the process u and get for $t\in[0,T]$,

$$\begin{aligned} \phi(u(t)) = &\int_0^t \phi'(u(s))(Au(s))\,ds + \int_0^t\int_Z \phi'(u(s-))(\xi(s,z))\tilde{N}(ds,dz)\\ &+\int_0^t\int_Z\Big[\phi(u(s-)+\xi(s,z)) - \phi(u(s-) - \phi'(u(s-))(\xi(s,z))\Big]N(ds,dz)\ \ \mathbb{P}\text{-a.s.} \end{aligned} \tag{3.7.15}$$

Since by Lemma 3.7.7, $\phi'(x)(Ax)\le 0$, for all $x\in D(A)$, we infer that for $t\in[0,T]$,

$$\begin{aligned} \phi(u(t)) \le &\int_0^t\int_Z \phi'(u(s-))(\xi(s,z))\tilde{N}(ds,dz)\\ &+\int_0^t\int_Z\Big[\phi(u(s-)+\xi(s,z)) - \phi(u(s-)) - \phi'(u(s-))(\xi(s,z))\Big]N(ds,dz)\ \ \mathbb{P}\text{-a.s.} \end{aligned} \tag{3.7.16}$$

Taking the supremum over the set $[0,t]$ and then the expectation to both sides of above inequality yields

$$\begin{aligned} \mathbb{E}\sup_{0\le s\le t}\phi(u(s)) \le\ &\mathbb{E}\sup_{0\le s\le t}\int_0^s\int_Z \phi'(u(r))(\xi(r,z))\tilde{N}(dr,dz)\\ &+\mathbb{E}\sup_{0\le s\le t}\int_0^s\int_Z\Big[\phi(u(r-)+\xi(r,z)) - \phi(u(r-)) - \phi'(u(r-))(\xi(r,z))\Big]N(dr,dz)\\ =:\ &I_1(t)+I_2(t). \end{aligned}$$

Applying the Davis inequality, see Corollary C.2 in [16], to I_1 we obtain for some constant C that

$$\begin{aligned} I_1(t) &\le C\mathbb{E}\left(\int_0^t\int_Z|\phi'(u(s-))(\xi(s,z))|^p N(ds,dz)\right)^{\frac{1}{p}}\\ &\le k_1 C\mathbb{E}\sup_{0\le s\le t}|u(s)|^{q-1}\left(\int_0^t\int_Z|\xi(s,z)|^p N(ds,dz)\right)^{\frac{1}{p}}. \end{aligned}$$

First we will estimate the integral $I_2(t)$. Note that for every $s \in [0,t]$,

$$\int_0^s \int_Z \Big|\phi(u(r)+\xi(r,z)) - \phi(u(r)) - \phi'(u(r-))(\xi(r,z))\Big|_E N(dr,dz)$$
$$= \sum_{r\in(0,s]\cap\mathcal{D}(\pi)} \Big|\phi(u(r-)+\xi(r,\pi(r))) - \phi(u(r-)) - \phi'(u(r-))(\xi(r,\pi(r)))\Big|_E, \quad \mathbb{P}\text{-a.s.}$$

Let us recall that by the assumption the function ϕ is of C^2 class. Applying the mean value Theorem, see [49], to the function ϕ, for each $r \in [0,s]$ we can find $0<\theta<1$ such that

$$\Big|\phi(u(r-)+\xi(r,\pi(r))) - \phi(u(r-))\Big|_E = |\xi(r,\pi(r))|_E \Big|\phi'(u(r-)+\theta\xi(r,\pi(r)))\Big|_{\mathcal{L}(E)}.$$

By the assumptions $|\phi'(x)| \le k_1 |x|_E^{q-1}$, $x \in E$ and the fact that $|x+\theta y|_E \le \max\{|x|_E, |x+y|_E\}$ for all $x, y \in E$, we obtain

$$\begin{aligned}\Big|\phi'(u(r-)+\theta\xi(r,\pi(r)))\Big|_{\mathcal{L}(E)} &\le k_1 |u(r-)+\theta\xi(r,\pi(s))|_E^{q-1} \\ &\le k_1 \max\big\{|u(r-)|_E^{q-1}, |u(r-)+\xi(r,\pi(r))|_E^{q-1}\big\}.\end{aligned}$$

Observe that for all $0 \le r \le s \le t$,

$$|u(r-)|_E^{q-1} \le \sup_{0\le\rho\le s} |u(\rho-)|_E^{q-1} \le \sup_{0\le\rho\le t} |u(\rho-)|_E^{q-1} \le \sup_{0\le\rho\le t} |u(\rho)|_E^{q-1}.$$

Moreover, since $u(r-)+\xi(r,\phi(r)) = u(r)$, we get

$$|u(r-)+\xi(r,\pi(r))|_E^{q-1} \le \sup_{0\le r\le s} |u(r)|_E^{q-1} \le \sup_{0\le s\le t} |u(s)|_E^{q-1}.$$

Therefore, we infer that for each $r \in [0,s]$,

$$\Big|\phi(u(r-)+\xi(r,\pi(r))) - \phi(u(r-))\Big|_E \le k_1 |\xi(r,\pi(r))|_E \sup_{0\le s\le t} |u(s)|_E^{q-1}.$$

It follows that

$$\begin{aligned}&\Big|\phi(u(r-)+\xi(r,\pi(r))) - \phi(u(r-)) - \phi'(u(r-))(\xi(r,\pi(r)))\Big|_E \\ &\qquad \le \Big|\phi(u(r)+\xi(r,\pi(r))) - \phi(u(r))\Big|_E + \Big|\phi'(u(r-))(\xi(r,\pi(r)))\Big|_E \\ &\qquad \le 2k_1 |\xi(r,\pi(r))|_E \sup_{0\le s\le t} |u(s)|_E^{q-1}.\end{aligned}$$

On the other side, we can also find some $0<\delta<1$ such that

$$\begin{aligned}\Big|\phi(u(r-)+\xi(r,\pi(r))) - \phi(u(r-)) - \phi'(u(r-))(\xi(r,\pi(r)))\Big|_E &= \frac{1}{2}|\xi(r,\pi(r))|_E^2 |\phi''(u(r-)+\theta\xi(r,\pi(r)))| \\ &\le \frac{k_2}{2}|\xi(r,\pi(r))|_E^2 |u(r-)+\theta\xi(r,\pi(r))|_E^{q-2}.\end{aligned}$$

By a similar argument as above, we obtain

$$\Big|\phi(u(r-)+\xi(r,\pi(r))) - \phi(u(r-)) - \phi'(u(r-))(\xi(r,\pi(r)))\Big|_E \le \frac{k_2}{2}|\xi(r,\pi(r))|_E^2 \sup_{0\le s\le t} |u(s)|_E^{q-2}.$$

Summing up, we have

$$\begin{aligned}
&\Big|\phi(u(r-)+\xi(r,\pi(r)))-\phi(u(r-))-\phi'(u(r-))(\xi(r,\pi(r)))\Big|_E \\
&\qquad = \Big|\phi(u(r-)+\xi(r,\pi(r)))-\phi(u(r-))-\phi'(u(r-))(\xi(r,\pi(r)))\Big|_E^{(2-p)+(p-1)} \\
&\qquad \le \left(2k_1|\xi(r,\pi(r))|_E \sup_{0\le s\le t}|u(s)|_E^{q-1}\right)^{2-p}\left(\frac{k_2}{2}|\xi(r,\pi(r))|_E^2 \sup_{0\le s\le t}|u(s)|_E^{q-2}\right)^{p-1} \\
&\qquad \le K|\xi(r,\pi(r))|_E^p \sup_{0\le s\le t}|u(s)|_E^{q-p},
\end{aligned}$$

where $K=(2k_1)^{2-p}(k_1/2)^{p-1}$.
Hence,

$$\begin{aligned}
&\sum_{r\in(0,s]\cap\mathcal{D}(\pi)}\Big|\phi(u(r-)+\xi(r,\pi(r)))-\phi(u(r-))-\phi'(u(r-))(\xi(r,\pi(r)))\Big|_E \\
&\qquad \le K\sup_{0\le s\le t}|u(s)|_E^{q-p}\sum_{r\in(0,t]\cap\mathcal{D}(\pi)}|\xi(r,\pi(r))|_E^p \\
&\qquad = K\sup_{0\le s\le t}|u(s)|_E^{q-p}\int_0^s\int_Z|\xi(r,z)|_E^p\,N(dr,dz),
\end{aligned}$$

which also shows that the integral $\int_0^t\int_Z\Big[\phi(u(s-)+\xi(s,z))-\phi(u(s-))-\phi'(u(s-))(\xi(s,z))\Big]N(ds,dz)$ is well defined since $\xi\in\mathcal{M}^p([0,T]\times\Omega\times Z,\hat{\mathcal{P}},\lambda\otimes\mathbb{P}\times\nu;\mathcal{D}(A))$. Therefore, we infer

$$\begin{aligned}
&\int_0^s\int_Z\Big|\phi(u(r-)+\xi(r,z))-\phi(u(r-))-\phi'(u(r-))(\xi(r,z))\Big|_E N(dr,dz) \\
&\qquad \le K\sup_{0\le s\le t}|u(s)|_E^{q-p}\int_0^s\int_Z|\xi(r,z)|_E^p\,N(dr,dz).
\end{aligned}$$

Hence, we get the following estimate for $I_2(t)$

$$I_2(t)\le K\mathbb{E}\sup_{0\le s\le t}|u(s)|_E^{q-p}\int_0^t\int_Z|\xi(r,z)|_E^p\,N(dr,dz),\quad t\in[0,T],$$

where the constant K only depends on k_1, k_2, p and q. Now applying Hölder's and Young's inequalities to $I_1(t)$ yields

$$\begin{aligned}
I_1(t) &\le k_1C\Big[\left(\mathbb{E}\Big(\sup_{0\le s\le t}|u(s)|_E^{q-1}\Big)^{\frac{q}{q-1}}\right)^{\frac{q-1}{q}}\left(\mathbb{E}\left(\int_0^t\int_Z|\xi(s,z)|_E^pN(ds,dz)\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} \\
&\le k_1C\Big[\left(\mathbb{E}\sup_{0\le s\le t}|u(s)|_E^q\right)^{\frac{q-1}{q}}\left(\mathbb{E}\left(\int_0^t\int_Z|\xi(s,z)|_E^pN(ds,dz)\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} \\
&= k_1C\Big[\left(\mathbb{E}\sup_{0\le s\le t}|u(s)|_E^q\,\varepsilon\right)^{\frac{q-1}{q}}\left(\mathbb{E}\left(\int_0^t\int_Z|\xi(s,z)|_E^pN(ds,dz)\right)^{\frac{q}{p}}\left(\frac{1}{\varepsilon}\right)^{q-1}\right)^{\frac{1}{q}} \\
&\le k_1C\left[\frac{q-1}{q}\,\varepsilon\,\mathbb{E}\sup_{0\le s\le t}|u(s)|_E^q+\frac{1}{\varepsilon^{q-1}q}\mathbb{E}\left(\int_0^t\int_Z|\xi(s,z)|_E^pN(ds,dz)\right)^{\frac{q}{p}}\right] \\
&= k_1C\frac{q-1}{q}\,\varepsilon\,\mathbb{E}\sup_{0\le s\le t}|u(s)|_E^q+k_1C\frac{1}{\varepsilon^{q-1}q}\mathbb{E}\left(\int_0^t\int_Z|\xi(s,z)|_E^pN(ds,dz)\right)^{\frac{q}{p}}.
\end{aligned}$$

In the same manner for the integral $I_2(t)$ we can see that

$$
\begin{aligned}
I_2(t) &\leq K\mathbb{E}\sup_{0\leq s\leq t}|u(s)|_E^{q-p}\int_0^t\int_Z|\xi(s,z)|_E^pN(ds,dz) \\
&\leq K\left(\mathbb{E}\sup_{0\leq s\leq t}|u(s)|_E^{(q-p)\frac{q}{q-p}}\right)^{\frac{q-p}{q}}\left(\mathbb{E}\left(\int_0^t\int_Z|\xi(s,z)|_E^pN(ds,dz)\right)^{\frac{q}{p}}\right)^{\frac{p}{q}} \\
&\leq K\left(\mathbb{E}\sup_{0\leq s\leq t}|u(s)|_E^{q}\right)^{\frac{q-p}{q}}\left(\mathbb{E}\left(\int_0^t\int_Z|\xi(s,z)|_E^pN(ds,dz)\right)^{\frac{q}{p}}\right)^{\frac{p}{q}} \\
&= K\left(\mathbb{E}\sup_{0\leq s\leq t}|u(s)|_E^{q}\,\varepsilon\right)^{\frac{q-p}{q}}\left(\mathbb{E}\left(\int_0^t\int_Z|\xi(s,z)|_EN(ds,dz)\right)^{q}\left(\frac{1}{\varepsilon}\right)^{\frac{q-p}{p}}\right)^{\frac{p}{q}} \\
&\leq K\frac{q-p}{q}\varepsilon\mathbb{E}\sup_{0\leq s\leq t}|u(s)|_E^q+K\frac{p}{q}\frac{1}{\varepsilon^{\frac{q-p}{q}}}\mathbb{E}\left(\int_0^t\int_Z|\xi(s,z)|_EN(ds,dz)\right)^q
\end{aligned}
$$

where we used the Hölder's inequality in the first and fourth inequalities and the Young's inequality in the third inequality.
It then follows that

$$
\begin{aligned}
\mathbb{E}\sup_{0\leq s\leq t}|u(s)|_E^q &\leq k_1C\frac{q-1}{q}\,\varepsilon\,\mathbb{E}\sup_{0\leq s\leq t}|u(s)|_E^q+k_1C\frac{1}{\varepsilon^{q-1}q}\mathbb{E}\left(\int_0^t\int_Z|\xi(s,z)|_E^pN(ds,dz)\right)^{\frac{q}{p}} \\
&\quad +K\frac{q-p}{q}\varepsilon\mathbb{E}\sup_{0\leq s\leq t}|u(s)|_E^q+K\frac{p}{q}\frac{1}{\varepsilon^{\frac{q-p}{q}}}\mathbb{E}\left(\int_0^t\int_Z|\xi(s,z)|_EN(ds,dz)\right)^q \\
&= \left(k_1C\frac{q-1}{q}+K\frac{q-p}{q}\right)\varepsilon\mathbb{E}\sup_{0\leq s\leq t}|u(s)|_E^q \\
&\quad +\left(k_1C\frac{1}{\varepsilon^{q-1}q}+K\frac{p}{q}\frac{1}{\varepsilon^{\frac{q-p}{q}}}\right)\mathbb{E}\left(\int_0^t\int_Z|\xi(s,z)|_E^pN(ds,dz)\right)^{\frac{q}{p}}.
\end{aligned}
$$

Now we can choose a suitable number ε such that

$$\left(k_1C\frac{q-1}{q}+K\frac{q-p}{q}\right)\varepsilon=\frac{1}{2}.$$

Consequently, there exists C which is independent of A such that

$$\mathbb{E}\sup_{0\leq s\leq t}|u(s)|_E^q\leq C\mathbb{E}\left(\int_0^t\int_Z|\xi(s,z)|_E^pN(ds,dz)\right)^{\frac{q}{p}}. \tag{3.7.17}$$

Case II. Suppose $\xi\in\mathcal{M}^p(\hat{\mathcal{P}};E)$. Set $R(n,A)=(nI-A)^{-1}$, $n\in\mathbb{N}$. Then we put $\xi^n(t,\omega)=nR(n,A)\xi(t,\omega)$ on $[0,T]\times\Omega$. Since A is the infinitesimal generator of the C_0-semigroup $S(t)$, $t\geq 0$ of contractions, by the Hille-Yosida Theorem, $\|R(n,A)\|\leq\frac{1}{n}$ and $\xi^n(t,\omega)\in\mathcal{D}(A)$, for every $(t,\omega)\in[0,T]\times\Omega$. Moreover, $\xi^n(t,\omega)\to\xi(t,\omega)$ pointwise on $[0,T]\times\Omega$. Also, we observe that $|\xi^n-\xi|=|nR(n,A)\xi-\xi|\leq 2|\xi|$. Therefore, it follows by applying the Lebesgue Dominated Convergence Theorem that

$$\int_0^T\int_Z|\xi^n(t,z)-\xi(t,z)|^p\nu(dz)dt$$

converges to 0 as $n \to \infty$, $\mathbb{P}$-a.s. Since the poisson random measure N is a $\mathbb{P}$-a.s. positive measure and we have

$$\mathbb{E}\int_0^T \int_Z |\xi^n(t,z) - \xi(t,z)|^p N(dt,dz) = \mathbb{E}\int_0^T \int_Z |\xi^n(t,z) - \xi(t,z)|^p \nu(dz)dt.$$

By taking a subsequence, still denoted by $\{\xi^n\}_n$, we infer that

$$\int_0^T \int_Z |\xi^n(t,z) - \xi(t,z)|^p N(dt,dz) \to 0, \quad \text{as } n \to \infty \ \mathbb{P}\text{-a.s.}.$$

One can also easily show that $\xi^n \in \mathcal{M}^p([0,T] \times \Omega \times Z, \hat{\mathcal{P}}, \lambda \otimes \mathbb{P} \times \nu; \mathcal{D}(A))$.

Define, for each $n \in \mathbb{N}$, a process u^n by

$$u^n(t) = \int_0^t S(t-s)\xi^n(s,z)\tilde{N}(ds,dz), \ t \in [0,T].$$

As we have already noted in case I, the function $u_n(t)$ can also be formulated in a way of strong solutions so that $u_n(t)$ is càdlàg for each $n \in \mathbb{N}$. By the discussion in case 1, for each $n \in \mathbb{N}$, $u^n(t)$, $0 \le t \le T$ satisfies the following

$$\mathbb{E} \sup_{0 \le t \le T} |u^n(t)|^q \le C\mathbb{E}\left(\int_0^t \int_Z |\xi^n(s,z)|_E^p N(ds,dz)\right)^{\frac{q}{p}}.$$

On the other hand, since by Theorem 3.3.2, we have

$$\begin{aligned}
\mathbb{E}|u^n(t) - u(t)|_E^p &= \mathbb{E}|u^n(t) - u(t)|_E^p \\
&= \mathbb{E}\left|\int_0^t \int_Z \Big(S(t-s)\xi^n(s,z) - S(t-s)\xi(s,z)\Big)\tilde{N}(ds,dz)\right|_E^p \\
&\le C_p\mathbb{E}\int_0^T \int_Z |\xi^n(s,z) - \xi(s,z)|^p \nu(dz)\,ds,
\end{aligned}$$

we infer that $u^n(t)$ converges to $u(t)$ in $L^p(\Omega)$ for every $t \in [0,T]$. Moreover, from case 1, we know that

$$\mathbb{E} \sup_{0 \le s \le t} |u^n(s) - u^m(s)|_E^q \le C\mathbb{E}\left(\int_0^t \int_Z |\xi^n(s,z) - \xi^m(s,z)|_E^p N(ds,dz)\right)^{\frac{q}{p}}.$$

From the above discussion, we know that the right hand-side of the above inequality converges to 0 as $n, m \to \infty$ if (3.7.10) holds. In this case, it is possible to construct a sequence $\{n_k\}_{k=1}^\infty$ of $\{n\}_{n=1}^\infty$ for which the following is satisfied

$$\mathbb{E} \sup_{0 \le s \le T} |u^{n_{k+1}}(s) - u^{n_k}(s)|^q < \frac{1}{k^{2q+2}}.$$

Hence, on the basis of the Chebyshev inequality, we obtain

$$\mathbb{P}\left\{\sup_{0 \le s \le T} |u^{n_{k+1}}(s) - u^{n_k}(s)| > \frac{1}{k^2}\right\} \le k^{2q}\mathbb{E} \sup_{0 \le s \le T} |u^{n_{k+1}}(s) - u^{n_k}(s)|^q < \frac{1}{k^2}.$$

Then the series $\sum_{k=1}^{\infty} \mathbb{P}\left\{\sup_{0\le s\le T} |u^{n_{k+1}}(s) - u^{n_k}(s)| > \frac{1}{k}\right\}$ will converges. It follows from the Borel-Cantelli Lemma that with probability 1 there exists an integer beyond which the inequality

$$\sup_{0\le s\le T} |u^{n_{k+1}}(s) - u^{n_k}(s)| \le \frac{1}{k^2}$$

holds. Consequently, the series of càdlàg functions

$$\sum_{k=1}^{\infty} [u^{n_{k+1}}(s) - u^{n_k}(s)]$$

converges uniformly on $[0,T]$ with probability 1 to a càdlàg function which we shall define by $\tilde{u} = (\tilde{u}(t))_{t\in[0,T]}$. In view of Lemma 3.7.8, it is possible to assume that the function $\tilde{u}$ is separable. In such a case, the function $\sup_{0\le t\le T} |\tilde{u}(t)|^q$ is measurable. Moreover, we have

$$\mathbb{E} \sup_{0\le t\le T} |u^n(t) - \tilde{u}(t)|^q \to 0, \quad \text{as } n \to \infty.$$

Therefore, by the Minkowski Inequality we have

$$\begin{aligned}\left[\mathbb{E} \sup_{0\le s\le t} |\tilde{u}(s)|^q\right]^{\frac{1}{q}} &\le \left[\mathbb{E} \sup_{0\le s\le t} |\tilde{u}(s) - u^n(s)|^q\right]^{\frac{1}{q}} + \left[\mathbb{E} \sup_{0\le s\le t} |u^n(s)|^q\right]^{\frac{1}{q}} \\ &\le \left[\mathbb{E} \sup_{0\le s\le t} |\tilde{u}(s) - u^n(s)|^q\right]^{\frac{1}{q}} + \left[C\mathbb{E}\left(\int_0^t \int_Z |\xi^n(s,z)|_E^p N(ds,dz)\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}.\end{aligned}$$

Note that the constant C on the right side of above inequality does not depend on operator A. So the constant C remains the same for every n. It follows by letting $n \to \infty$ in above inequality that

$$\mathbb{E} \sup_{0\le s\le t} |\tilde{u}(s)|^q \le C\mathbb{E}\left(\int_0^t \int_Z |\xi(s,z)|_E^p N(ds,dz)\right)^{\frac{q}{p}}$$

Also, we have for every $t \in [0,T]$, by Minkowski inequality that

$$\begin{aligned}\left(\mathbb{E}|\tilde{u}(t) - u(t)|_E^p\right)^{\frac{1}{p}} &\le \left(\mathbb{E}|\tilde{u}(t) - u_n(t)|_E^p\right)^{\frac{1}{p}} + \left(\mathbb{E}|u(t) - u_n(t)|_E^p\right)^{\frac{1}{p}} \\ &\le \left(\mathbb{E}|\tilde{u}(t) - u_n(t)|_E^q\right)^{\frac{1}{q}} + \left(\mathbb{E}|u(t) - u_n(t)|_E^p\right)^{\frac{1}{p}} \\ &\le \left(\mathbb{E} \sup_{0\le t\le T} |\tilde{u}(t) - u_n(t)|_E^q\right)^{\frac{1}{q}} + \left(\mathbb{E}|u(t) - u_n(t)|_E^p\right)^{\frac{1}{p}}.\end{aligned}$$

Letting $n \to \infty$, it follows that $u(t) = \tilde{u}(t)$ in $L^p(\Omega)$ for any $t \in [0,T]$. This shows the inequailty (3.7.11) for $q' = q$. The case $q' > q$ follows from the fact that if the martingale type p Banach space E satisfies Assumption 3.7.4 for some q, then Condition 1 is also satisfied with $q' > q$.

□

Corollary 3.7.10. *Let E be a martingale type p Banach space, $1 < p \le 2$. There exists a separable and càdlàg modification $\tilde{u}$ of u such that for some constant C and every stopping time $\tau > 0$ and $t > 0$,*

$$\mathbb{E} \sup_{0\le s\le t\wedge\tau} |\tilde{u}(s)|_E^q \le C\, \mathbb{E}\left(\int_0^{t\wedge\tau} \int_Z |\xi(s,z)|_E^p N(ds,dz)\right)^{\frac{q}{p}}, \tag{3.7.18}$$

provided the right hand-side of (3.7.18) is finite.

Proof. Let us first consider the case when $\xi \in \mathcal{M}^p([0,T]\times\Omega\times Z,\hat{\mathcal{P}},\lambda\otimes\mathbb{P}\times\nu;\mathcal{D}(A))$. A similar argument as in Theorem 3.7.9 gives the following

$$\begin{aligned}\phi(u(t)) &= \int_0^t \phi'(u(s))(Au(s))\,ds + \int_0^t\int_Z \phi'(u(s-))(\xi(s,z))\tilde{N}(ds,dz)\\ &\quad + \int_0^t\int_Z \Big[\phi(u(s-)+\xi(s,z)) - \phi(u(s-) - \phi'(u(s-))(\xi(s,z))\Big]N(ds,dz)\\ &\le \int_0^t\int_Z \phi'(u(s-))(\xi(s,z))\tilde{N}(ds,dz)\\ &\quad + \int_0^t\int_Z \Big[\phi(u(s-)+\xi(s,z)) - \phi(u(s-) - \phi'(u(s-))(\xi(s,z))\Big]N(ds,dz)\ \ \mathbb{P}\text{-a.s.}\end{aligned}$$

It follows that

$$\begin{aligned}\mathbb{E}\sup_{0\le s\le t\wedge\tau}\phi(u(s)) &= \mathbb{E}\sup_{0\le s\le t\wedge\tau}|u(s\wedge\tau)|_E^q\\ &\le \mathbb{E}\sup_{0\le s\le t\wedge\tau}\int_0^{s\wedge\tau}\int_Z \phi'(u(r-))(\xi(r,z))\tilde{N}(dr,dz)\\ &\quad + \mathbb{E}\sup_{0\le s\le t\wedge\tau}\int_0^{s\wedge\tau}\Big[\phi(u(r-)+\xi(r,z)) - \phi(u(r-) - \phi'(u(r-))(\xi(r,z))\Big]N(dr,dz)\\ &= \mathbb{E}\sup_{0\le s\le t\wedge\tau}\int_0^s\int_Z 1_{(0,\tau]}(r)\phi'(u(r-))(\xi(r,z))\tilde{N}(dr,dz)\\ &\quad + \mathbb{E}\sup_{0\le s\le t\wedge\tau}\int_0^s\int_Z 1_{(0,\tau]}(r)\Big|\phi(u(r-)+\xi(r,z))\\ &\qquad\qquad - \phi(u(r-) - \phi'(u(r-))(\xi(r,z))\Big|_E N(ds,dz)\\ &= I_1 + I_2.\end{aligned}$$

Now we consider integral I_2. By the definition of the Lebesgue-Stieltges inegral, we have

$$\begin{aligned}&\int_0^s\int_Z \Big|\phi(u(r-)+\xi(r,z)) - \phi(u(r-) - \phi'(u(r-))(\xi(r,z))\Big|_E 1_{(0,\tau]}(r)N(dr,dz)\\ &\qquad = \sum_{0<r\le s}\Big|\phi(u(r-)+\xi(r,\xi(r))) - \phi(u(r-) - \phi'(u(r-))(\xi(r,\xi(r)))\Big|_E 1_{(0,\tau]}(r),\end{aligned}$$

Notice that the function $\phi(\cdot) = |\cdot|^q$ is of class C^2. Applying Taylor formula to function ϕ we get for some $0<\theta,\delta<1$,

$$\begin{aligned}&\Big|\phi(u(r-)+\xi(r,\pi(r))) - \phi(u(r-))\Big|_E 1_{(0,\tau]}(r)\\ &\qquad\qquad \le |\xi(r,\pi(r))|_E\Big|\phi'(u(r-)+\theta\xi(r,\pi(r)))\Big|1_{(0,\tau]}(r),\\ &\Big|\phi(u(r-)+\xi(r,\pi(r))) - \phi(u(r-) - \phi'(u(r-))(\xi(r,\pi(r)))\Big|_E 1_{(0,\tau]}(r)\\ &\qquad\qquad \le \frac{1}{2}|\xi(r,\pi(r))|_E^2|\phi''(u(r-)) + \delta\xi(r,\pi(r))|1_{(0,\tau]}(r)\end{aligned}$$

Moreover we know that $|\phi'(x)|_{\mathcal{L}(E)} \le k_1|x|_E^{q-1}$, so we obtain

$$\begin{aligned}\Big|\phi'(u(r-)+\theta\xi(r,\pi(r)))\Big|_E 1_{(0,\tau]}(r) &\le k_1\big|u(r-)+\theta\xi(r,\pi(r))\big|_E^{q-1}1_{(0,\tau]}(r)\\ &\le k_1\max\big\{|u(r-)|_E^{q-1}1_{(0,\tau]}(r), \big|u(r-)+\xi(r,\pi(r))\big|_E^{q-1}1_{(0,\tau]}(r)\big\}.\end{aligned}$$

Observe that

$$|u(r-)|_E^{q-1}1_{(0,\tau]}(r) \le \sup_{0\le r\le s} |u(r-)|_E^{q-1}1_{(0,\tau]}(r) \le \sup_{0\le s\le t\wedge\tau} |u(s-)|_E^{q-1} \le \sup_{0\le s\le t\wedge\tau} |u(s)|_E^{q-1},$$

and

$$|u(r-)+\xi(r,\pi(r))|_E^{q-1}1_{(0,\tau]}(r) \le \sup_{0\le r\le s} |u(r)|_E^{q-1}1_{(0,\tau]}(r) \le \sup_{0\le s\le t\wedge\tau} |u(s)|_E^{q-1},$$

where $q\ge 2$. Therefore, we infer

$$\begin{aligned}\Big|\phi(u(r-)+\xi(r,\pi(r)))-\phi(u(r-))\Big|_E 1_{(0,\tau]}(r) &\le |\xi(r,\pi(r))|_E 1_{(0,\tau]}(r)\Big|\phi'(u(r-)+\theta\xi(r,\pi(r)))\Big|_{\mathcal{L}(E)}\\ &\le k_1|\xi(r,\pi(r))|_E 1_{(0,\tau]}(r)\sup_{0\le s\le t\wedge\tau}|u(s)|_E^{q-1}.\end{aligned}$$

Similarly, from the assumption $|\phi''(x)|\le k_2|x|_E^{q-2}$ we obtain

$$|\phi''(u(r-))+\delta\xi(r,\pi(r))|_E 1_{(0,\tau]}(r) \le k_2 \sup_{0\le s\le t\wedge\tau}|u(s)|_E^{q-2}1_{(0,\tau]}(r).$$

It then follows that

$$\begin{aligned}&\sum_{0<r\le s}\Big|\phi(u(r-)+\xi(r,\xi(r)))-\phi(u(r-)-\phi'(u(r-))(\xi(r,\xi(r)))\Big|_E 1_{(0,\tau]}(r)\\
&=\sum_{0<r\le s}\Big|\phi(u(r-)+\xi(r,\xi(r)))-\phi(u(r-)-\phi'(u(r-))(\xi(r,\xi(r)))\Big|_E^{(2-p)+(p-1)} 1_{(0,\tau]}(r)\\
&\le\sum_{0<r\le s}\Big(2k_1|\xi(r,\pi(r))|_E\sup_{0\le s\le t\wedge\tau}|u(s)|_E^{q-1}1_{(0,\tau]}(r)\Big)^{2-p}\Big(k_2|\xi(r,\pi(r))|^2\sup_{0\le s\le t\wedge\tau}|u(s)|_E^{q-2}1_{(0,\tau]}(r)\Big)^{p-1}\\
&=K\sup_{0\le s\le t\wedge\tau}|u(s)|_E^{q-p}\sum_{0<r\le s}|\xi(r,\pi(r))|_E^p 1_{(0,\tau]}(r).\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_0^s\int_Z\Big|\phi(u(r-)+\xi(r,z))-\phi(u(r-))-\phi'(u(r-))(\xi(r,z))\Big|_E 1_{(0,\tau]}(r)N(dr,dz)\\
&\qquad\le K\sup_{0\le s\le t\wedge\tau}|u(s)|_E^{q-p}\int_0^s\int_Z|\xi(r,z)|_E^p 1_{(0,\tau]}(r)\ N(dr,dz).\end{aligned}$$

Hence, for integral I_2, we can estimate as follows

$$I_2\le K\mathbb{E}\sup_{0\le s\le t\wedge\tau}|u(s)|_E^{q-p}\int_0^s\int_Z|\xi(r,z)|_E^p 1_{(0,\tau]}(r)\ N(dr,dz).$$

For integral I_1, applying the stopped Davis' inequality, see Corollary C.2 in [16], yields the following

$$\begin{aligned}I_1&\le C\mathbb{E}\Big(\int_0^s\int_Z|\phi'(u(r-))(\xi(r,z))|_E^p 1_{(0,\tau]}(r)N(dr,dz)\Big)^{\frac{1}{p}}\\
&\le k_1C\mathbb{E}\sup_{0\le s\le t\wedge\tau}|u(s)|_E^{q-1}\Big(\int_0^{t\wedge\tau}\int_Z|\xi(r,z)|_E^p N(ds,dz)\Big)^{\frac{1}{p}}.\end{aligned}$$

The rest argument goes without any difference with the proof of Theorem 3.7.9.

□

Theorem 3.7.11. *Let E be an martingale type p Banach space, $1 < p \leq 2$, satisfying Assumption 3.7.4. Suppose $0 < q' < \infty$, where q is the number from Assumption 3.7.4. If $\xi \in \mathcal{M}_T^p(\hat{\mathcal{P}}; E)$ such that*

$$\mathbb{E}\left(\int_0^t \int_Z |\xi(s,z)|_E^p N(ds,dz)\right)^{\frac{q'}{p}} < \infty, \tag{3.7.19}$$

then there exists a separable and càdlàg modification $\tilde{u}$ of u such that for all $0 \leq t \leq T$,

$$\mathbb{E} \sup_{0\leq s\leq t} |\tilde{u}(s)|_E^{q'} \leq C\, \mathbb{E}\left(\int_0^t \int_Z |\xi(s,z)|_E^p N(ds,dz)\right)^{\frac{q'}{p}}, \tag{3.7.20}$$

Proof. The inequality (3.7.20) has already been shown for $q' \geq q$ in Theorem 3.7.9. Now we are in a position to show it for $0 < q' < q$. Let us fix q' such that $0 < q' < q$. Take $\lambda > 0$. Define a stopping time

$$\tau := \inf\left\{t : \left(\int_0^t \int_Z |\xi(s,z)|_E^p N(ds,dz)\right)^{\frac{1}{p}} > \lambda\right\}.$$

Since the process $\int_0^t \int_Z |\xi(s,z)|_E^p N(ds,dz)$, $0 < t \leq T$ is right continuous, the random time τ is indeed a $\mathcal{F}_{t+}$-stopping time. Moreover, we find out that $\int_0^t \int_Z |\xi(s,z)|_E^p N(ds,dz) \leq \lambda$, for $0 < t < \tau$, and $\int_0^\tau \int_Z |\xi(s,z)|_E^p N(ds,dz) \geq \lambda$ when $\tau < \infty$. Also, we observe that for every $0 < t \leq T$,

$$\mathbb{E}\int_0^t \int_Z f(s,z)\tilde{N}(ds,dz) = \mathbb{E}\int_0^{t-} \int_Z f(s,z)\tilde{N}(ds,dz). \tag{3.7.21}$$

This equality can be verified first for step functions, then for every function f in $\xi \in \mathcal{M}^p(\hat{\mathcal{P}}; E)$ we can approximate it by step functions in $\mathcal{M}^p_{step}(\hat{\mathcal{P}}; E)$, so the equality (3.7.21) holds for every $f \in \mathcal{M}_T^p(\hat{\mathcal{P}}; E)$. Therefore, by using Chebyshev's inequaliy and Corollary 3.7.10 to Theorem 3.7.9, we obtain

$$\begin{aligned}
\mathbb{P}\left(\sup_{0\leq s\leq t\wedge\tau} |u(s)| > \lambda\right) &\leq \frac{1}{\lambda^q}\mathbb{E} \sup_{0\leq s\leq t\wedge\tau} |u(s)|^q \\
&\leq \frac{C}{\lambda^q}\mathbb{E}\left(\int_0^{t\wedge\tau} \int_Z |\xi(s,z)|^p N(ds,dz)\right)^{\frac{q}{p}} \\
&= \frac{C}{\lambda^q}\mathbb{E}\left(\int_0^{(t\wedge\tau)-} \int_Z |\xi(s,z)|^p N(ds,dz)\right)^{\frac{q}{p}} \\
&\leq \frac{C}{\lambda^q}\mathbb{E}\left[\left(\int_0^t \int_Z |\xi(s,z)|^p N(ds,dz)\right)^{\frac{q}{p}} \wedge \lambda^q\right].
\end{aligned} \tag{3.7.22}$$

On the other hand, since $\{\sup_{0\leq s\leq t} |u(s)| > \lambda, \tau \geq t\} \subset \{\sup_{0\leq s\leq t\wedge\tau} |u(s)| > \lambda\}$, we have

$$\begin{aligned}
\mathbb{P}(\sup_{0\leq s\leq t} |u(s)| > \lambda) &= \mathbb{P}(\sup_{0\leq s\leq t} |u(s)| > \lambda, \tau \geq t) + \mathbb{P}(\sup_{0\leq s\leq t} |u(s)| > \lambda, \tau < t) \\
&\leq \mathbb{P}(\sup_{0\leq s\leq t} |u(s)| > \lambda, \tau \geq t) + \mathbb{P}(\tau < t) \\
&\leq \mathbb{P}(\sup_{0\leq s\leq t\wedge\tau} |u(s)| > \lambda) + \mathbb{P}(\tau < t).
\end{aligned} \tag{3.7.23}$$

Substituting (3.7.22) into (3.7.23) results in

$$\mathbb{P}(\sup_{0\le s\le t}|u(s)|>\lambda)\le \frac{C}{\lambda^q}\mathbb{E}\left[\left(\int_0^t\int_Z|\xi(s,z)|^pN(ds,dz)\right)^{\frac{q}{p}}\wedge\lambda^q\right] + \mathbb{P}\left[\left(\int_0^t\int_Z|\xi(s,z)|^pN(ds,dz)\right)^{\frac{1}{p}}>\lambda\right].$$

Integrating both sides of the last inequality with respect to measure $q'\lambda^{q'-1}d\lambda$ and applying the equality $\mathbb{E}|X|^{q'}=\int_0^\infty q'\lambda^{q'-1}\mathbb{P}(|X|>\lambda)d\lambda$, see [39], we infer that

$$\begin{aligned}\mathbb{E}\sup_{0\le s\le t}|u(s)|^{q'} &= \int_0^\infty \mathbb{P}(\sup_{0\le s\le t}|u(s)|>\lambda)q'\lambda^{q'-1}d\lambda\\ &\le \int_0^\infty \frac{C}{\lambda^q}\mathbb{E}\left[\left(\int_0^t\int_Z|\xi(s,z)|^pN(ds,dz)\right)^{\frac{q}{p}}\wedge\lambda^q\right]q'\lambda^{q'-1}d\lambda\\ &\quad+\int_0^\infty \mathbb{P}\left[\left(\int_0^t\int_Z|\xi(s,z)|^pN(ds,dz)\right)^{\frac{1}{p}}>\lambda\right]q'\lambda^{q'-1}d\lambda \qquad (3.7.24)\\ &= \int_0^\infty \frac{C}{\lambda^q}\mathbb{E}\left[\left(\int_0^t\int_Z|\xi(s,z)|^pN(ds,dz)\right)^{\frac{q}{p}}\wedge\lambda^q\right]q'\lambda^{q'-1}d\lambda\\ &\quad+\mathbb{E}\left(\int_0^t\int_Z|\xi(s,z)|^pN(ds,dz)\right)^{\frac{q'}{p}}.\end{aligned}$$

Let us denote $\left(\int_0^t\int_Z|\xi(s,z)|^pN(ds,dz)\right)^{\frac{1}{p}}$ by X. The first term on the right side of (3.7.24) becomes

$$\begin{aligned}&\frac{C}{\lambda^q}\int_0^\infty \mathbb{E}\left[\left(\int_0^t\int_Z|\xi(s,z)|^pN(ds,dz)\right)^{\frac{q}{p}}\wedge\lambda^q\right]q'\lambda^{q'-1}d\lambda\\ &= C\int_0^\infty \mathbb{E}(X^q\wedge\lambda^q)q'\lambda^{q'-q-1}d\lambda\\ &= C\mathbb{E}\int_0^\infty (X^q\wedge\lambda^q)q'\lambda^{q'-q-1}d\lambda\\ &= C\mathbb{E}\int_0^X \lambda^q q'\lambda^{q'-q-1}d\lambda + C\mathbb{E}\int_X^\infty |X|^q q'\lambda^{q'-q-1}d\lambda\\ &= C\mathbb{E}X^{q'}+C\mathbb{E}X^q\int_X^\infty q'\lambda^{q'-q-1}d\lambda\\ &= C(1+\frac{q'}{q-q'})\mathbb{E}X^{q'}\\ &= \frac{Cq}{q-q'}\mathbb{E}X^{q'}\\ &= \frac{Cq}{q-q'}\mathbb{E}\left(\int_0^t\int_Z|\xi(s,z)|^pN(ds,dz)\right)^{\frac{q'}{p}}.\end{aligned}$$

Therefore, we conclude that

$$\mathbb{E}\sup_{0\le s\le t}|u(s)|^{q'} \le \frac{Cq}{q-q'}\mathbb{E}\left(\int_0^t\int_Z |\xi(s,z)|^p N(ds,dz)\right)^{\frac{q'}{p}} + \mathbb{E}\left(\int_0^t\int_Z |\xi(s,z)|^p N(ds,dz)\right)^{\frac{q'}{p}}$$
$$= \left(1+\frac{Cq}{q-q'}\right)\mathbb{E}\left(\int_0^t\int_Z |\xi(s,z)|^p N(ds,dz)\right)^{\frac{q'}{p}},$$

which completes the proof.

□

Corollary 3.7.12. *Let E be an martingale type p Banach space, $1 < p \le 2$ satisfying Assumption 3.7.4. Suppose $0 < q' \le p$. If $\xi \in \mathcal{M}^p_T(\hat{\mathcal{P}};E)$, then there exists a separable and càdlàg modification $\tilde{u}$ of u such that for some constant $C > 0$, independent of u, all $t \in [0,T]$,*

$$\mathbb{E}\sup_{0\le s\le t}|\tilde{u}(s)|_E^{q'} \le C\,\mathbb{E}\left(\int_0^t\int_Z |\xi(s,z)|_E^p \nu(dz)\,ds\right)^{\frac{q'}{p}} \tag{3.7.25}$$

Proof of Corollary 3.7.12. First, we consider the case $q' = p$. Since $\xi \in \mathcal{M}^p(\hat{\mathcal{P}};E)$, so both integrals $\int_0^t\int_Z |\xi(s,z)|_E^p\nu(dz)ds$ and $\int_0^t\int_Z |\xi(s,z)|_E^p N(ds,dz)$ are well defined as the Lebesgue-Stieltjes integrals. We can obtain from Theorem 3.7.11 with $q' = p$ that there exists a separable and càdlàg modification $\tilde{u}$ of u (3.7.25) such that

$$\mathbb{E}\sup_{0\le s\le t}|u(s)|_E^p \le C\,\mathbb{E}\left(\int_0^t\int_Z |\xi(s,z)|_E^p N(ds,dz)\right)$$
$$= C\,\mathbb{E}\left(\int_0^t\int_Z |\xi(s,z)|_E^p \nu(dz)ds\right) < \infty.$$

This shows (3.7.25) for $q' = p$. Now we are in a position to show Inequality (3.7.25) for $0 < q' < p$. Let q' be fixed. Take $\lambda > 0$. Define stopping time

$$\tau = \inf\{t \in [0,T] : \left(\int_0^t\int_Z |\xi(s,z)|^p\nu(dz)\,ds\right)^{\frac{1}{p}} > \lambda\}.$$

The random variable τ is a stopping time. Indeed the process $\int_0^t\int_Z |\xi(s,z)|^p\nu(dz)\,ds$, $0 \le t \le T$ is a continuous process and so the claim follows immediately. It follows from Chebyshev's inequaliy and Corollary 3.7.10 that

$$\mathbb{P}\left(\sup_{0\le s\le t\wedge\tau}|u(s)| > \lambda\right) = \mathbb{E}1_{\{\sup_{0\le s<t\wedge\tau}|u(s)|>\lambda\}} \tag{3.7.26}$$
$$\le \frac{1}{\lambda^q}\mathbb{E}\sup_{0\le s<t\wedge\tau}|u(s)|^q$$
$$\le \frac{C}{\lambda^q}\mathbb{E}\left(\int_0^{t\wedge\tau}\int_Z |\xi(s,z)|^p\nu(dz)\,ds\right)^{\frac{q}{p}}$$
$$\le \frac{C}{\lambda^q}\mathbb{E}\left[\left(\int_0^t\int_Z |\xi(s,z)|^p\nu(dz)\,ds\right)^{\frac{q}{p}} \wedge \lambda^q\right],$$

where we used the definition of stopping time τ and the increasing property of process $\int_0^t\int_Z |\xi(s,z)|^p\nu(dz)\,ds$, $0 \le t \le T$. The rest of the proof can be done exactly in the same manner as in the proof of Theorem 3.7.11. □

Corollary 3.7.13. *Let E be an martingale type p Banach space, $1 < p \le 2$ satisfying Assumption 3.7.4. Then for any $n \in \mathbb{N}$ there exists a constant $C = C(n)$ such that for every every $\xi \in \bigcap_{k=1}^n \mathcal{M}^{p^k}([0,T] \times \Omega \times Z, \hat{\mathcal{P}}, \lambda \otimes \mathbb{P} \times \nu; E)$ and $t \in [0,T]$ we have*

$$\mathbb{E} \sup_{0 \le s \le t} |\tilde{u}(s)|_E^{p^n} \le C \sum_{k=1}^n \mathbb{E} \left(\int_0^t \int_Z |\xi(s,z)|_E^{p^k} \nu(dz) ds \right)^{p^{n-k}} . \tag{3.7.27}$$

where $\tilde{u}$ is the càdlàg modification of u as before.

The proof of Corollary 3.7.13 is similar to the proof Lemma 5.2 in Bass and Cranston [6] or of Lemma 4.1 in Protter and Talay [67]. Essential ingredients of that proof are the following two results. The second of them being about integration of real valued processes.

Lemma 3.7.14. *Let E be an martingale type p Banach space, $1 < p \le 2$, satisfying Assumption 3.7.4. For any $0 < q' < \infty$, there exists a constant C such that for all $\xi \in \mathcal{M}^p(\hat{\mathcal{P}}; E)$ we have*

$$\mathbb{E} \sup_{0 \le s \le t} \left| \int_0^s \int_Z \xi(r,z) \tilde{N}(dr,dz) \right|_E^{q'} \le C \mathbb{E} \left(\int_0^t \int_Z |\xi(s,z)|_E^p N(ds,dz) \right)^{\frac{q'}{p}}, \quad t \in [0,T]. \tag{3.7.28}$$

Proof of Lemma 3.7.14. This result is a special case of Theorem 3.7.11 when $S(t) = I$, $0 \le t \le T$. □

Lemma 3.7.15. *For any $n \in \mathbb{N}$ there exists a constant $D_n > 0$ such for any process*

$$f \in \bigcap_{k=1}^n \mathcal{M}^{p^k}(\hat{\mathcal{P}}; \mathbb{R})$$

and $t \in [0,T]$, the following inequality

$$\mathbb{E} \sup_{0 \le s \le t} \left| \int_0^s \int_Z f(r,z) \tilde{N}(dr,dz) \right|^{p^n} \le D_n \sum_{k=1}^n \mathbb{E} \left(\int_0^t \int_Z |f(s,z)|^{p^k} \nu(dz) ds \right)^{p^{n-k}} \tag{3.7.29}$$

holds.

Proof of Lemma 3.7.15. We shall show this Lemma by induction. The case $n = 1$. This follows from [16]. Now we assume that the assertion in the Claim is true for $n-1$, where $n \in \mathbb{N}$ and $n \ge 2$. We will show that it is true for n. Since by assumption $f \in \mathcal{M}^p(\hat{\mathcal{P}}; \mathbb{R})$, so both integrals $\int_0^t \int_Z |f(s,z)|^p N(ds,dz)$ and $\int_0^t \int_Z |f(s,z)|^p \nu(dz) ds$ are well defined as Lebesgue-Stieltjes integrals. Moreover, we have

$$\int_0^t \int_Z |f(s,z)|^p \tilde{N}(ds,dz) = \int_0^t \int_Z |f(s,z)|^p N(ds,dz) - \int_0^t \int_Z |f(s,z)|^p \nu(dz) ds. \tag{3.7.30}$$

Hence by applying first inequality (3.7.28) and next the equality (3.7.30) we infer that

$$\mathbb{E} \sup_{0 \le s \le t} \left| \int_0^s \int_Z f(r,z) \tilde{N}(dr,dz) \right|^{p^n} \le C \mathbb{E} \left| \int_0^t \int_Z |f(s,z)|^p N(ds,dz) \right|^{p^{n-1}} \tag{3.7.31}$$

$$\le 2^{p^{n-1}} C \left\{ \mathbb{E} \left(\int_0^t \int_Z |f(s,z)|^p \, \tilde{N}(ds,dz) \right)^{p^{n-1}} + \mathbb{E} \left(\int_0^t \int_Z |f(s,z)|^p \, \nu(dz) \, ds \right)^{p^{n-1}} \right\}. \tag{3.7.32}$$

Next, by the inductive assumption applied to the real valued process $|f|^p \in \bigcap_{k=1}^{n-1} \mathcal{M}^{p^k}(\hat{\mathcal{P}};\mathbb{R})$, we have

$$\begin{aligned}
&\mathbb{E}\left|\int_0^t \int_Z f(s,z)\tilde{N}(ds,dz)\right|^{p^n} \\
&\leq 2^{p^{n-1}} C \left(D_{n-1} \sum_{i=1}^{n-1} \mathbb{E}\left(\int_0^t \int_Z |f(s,z)|^{p^{i+1}}\, \nu(dz)\, ds \right)^{p^{n-1-i}} + \mathbb{E}\left(\int_0^t \int_Z |f(s,z)|^{p}\, \nu(dz)\, ds \right)^{p^{n-1}} \right) \\
&\leq D_n \sum_{k=1}^{n} \mathbb{E}\Big(\int_0^t \int_Z |f(s,z)|^{p^k}\, \nu(dz)\, ds \Big)^{p^{n-k}}, \qquad (3.7.33)
\end{aligned}$$

This proves the validity of the assertion in the Lemma for n what completes the whole proof. □

Proof of Corollary 3.7.13. Let us take $n \in \mathbb{N}$. By applying first Theorem 3.7.11 and next the equality (3.7.30) when $\xi \in \mathcal{M}^p(\hat{\mathcal{P}}; E)$, we infer that for all $t \in [0,T]$,

$$\begin{aligned}
\mathbb{E} \sup_{0\leq s\leq t} |\tilde{u}(s)|_E^{p^n} &\leq C\, \mathbb{E}\left(\int_0^t \int_Z |\xi(s,z)|_E^p\, N(ds,dz) \right)^{p^{n-1}} \\
&\leq 2^{p^{n-1}} C\, \mathbb{E}\left(\int_0^t \int_Z |\xi(s,z)|_E^p\, \tilde{N}(ds,dz) \right)^{p^{n-1}} \\
&\quad + 2^{p^{n-1}} C\, \mathbb{E}\left(\int_0^t \int_Z |\xi(s,z)|_E^p\, \nu(dz)\, ds \right)^{p^{n-1}} \\
&\leq 2^{p^{n-1}}\, C\, D_{n-1} \sum_{k=1}^{n-1} \mathbb{E}\left(\int_0^t \int_Z |\xi(s,z)|_E^{p^{k+1}} \nu(dz) ds \right)^{p^{n-1-k}} \\
&\quad + 2^{p^{n-1}}\, C\, \mathbb{E}\left(\int_0^t \int_Z |\xi(s,z)|_E^p\, \nu(dz)\, ds \right)^{p^{n-1}} \\
&\leq C(n) \sum_{k=1}^{n} \mathbb{E}\left(\int_0^t \int_Z |\xi(s,z)|_E^{p^k} \nu(dz) ds \right)^{p^{n-k}},
\end{aligned}$$

where we used in the third inequality Lemma 3.7.14 with f replaced by real-valued process $|\xi|_E^p \in \bigcap_{k=1}^{n-1} \mathcal{M}^{p^k}(\hat{\mathcal{P}};\mathbb{R})$. This completes the proof of Corollary 3.7.13. □

It is possible to derive inequality (3.7.28) by the method used by Hausenblas and Seidler in [38], see as inequality (4) therein. These authors used the Szeköfalvi-Nagy's Theorem on unitary dilations in Hilbert spaces. The latter result has recently been extended by Fröhlich and Weis [32] to Banach spaces of finite cotype. However, this method works only for analytic semigroups of contraction type while the results from the current paper are valid for all C_0 semigroups of contraction type. Let us now formulate the following result whose proof is a clear combination of the proofs from [38] and [32]. For the explanation of the terms used we refer the reader to the latter work. Similar observation for processes driven by a Wiener process was made independently by Seidler [74].

Theorem 3.7.16. *Let E be an martingale type p Banach space, $1 < p \leq 2$. Let $-A$ be a generator of a bounded analytic semigroup in E such that for some $\theta < \frac{1}{2}\pi$, the operator A has a bounded $H^\infty(S_\theta)$ calculus. Then, for any $0 < q' < \infty$, there exists a constant C such that for all $\xi \in \mathcal{M}_T^p(\hat{\mathcal{P}}; E)$ we have*

$$\mathbb{E} \sup_{0 \leq s \leq t} \left(\int_0^s \int_Z S(s-r)\xi(r,z)\tilde{N}(dr,dz) \right)_E^{q'} \leq C\mathbb{E} \left(\int_0^t \int_Z |\xi(r,z)|_E^p \nu(dz)dr \right)^{\frac{q'}{p}} \quad t \in [0,T].$$

The following result could be derived immediately from the proof of above theorem.

Corollary 3.7.17. *Let E be a martingale type p Banach space, $1 < p \leq 2$. Let $-A$ be a generator of a bounded analytic semigroup in E such that for some $\theta < \frac{1}{2}\pi$ the operator A has a bounded $H^\infty(S_\theta)$ calculus. Then, the stochastic convolution process u defined by* (3.7.8) *has càdlàg modification.*

Chapter 4

Stochastic Nonlinear Beam Equations w.r.t. Compensated Poisson Random Measures

Throughout the whole chapter we assume that H is a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and corresponding norm $\| \cdot \|_H$. By $\mathcal{B}(H)$ we denote the Borel σ-field on H, i.e. the σ-field generated by the family of all open subsets of H. Let $B : \mathcal{D}(B) \to H$, $\mathcal{D}(B) \subset H$, be a self-adjoint operator. Suppose that $A : \mathcal{D}(A) \to H$, where $\mathcal{D}(A) \subset \mathcal{D}(B)$, is a self-adjoint (unbounded) operator and $A \geq \mu I$ for some $\mu > 0$. Moreover, we assume that $B \in \mathcal{L}(\mathcal{D}(A), H)$. Here $\mathcal{D}(A)$ is the domain of A endowed with the graph norm $\|x\|_{\mathcal{D}(A)} := \|Ax\|$. Let m be a nonnegative function of class C^1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with the filtration $\mathfrak{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfying the usual hypotheses and $(Z, \mathcal{Z}, \nu)$ be a measure space, where ν is a σ-finite measure. Let

$$\tilde{N}((0,t] \times B) = N((0,t] \times B) - t\nu(B), \; t \geq 0, \; B \in \mathcal{Z},$$

be a compensated Poisson random measure on $[0,T] \times \Omega \times Z$ with its compensator $\nu(\cdot)$.
Let $\mathcal{M}^2_{loc}(\mathcal{BF})$ be the space of all H-valued progressively measurable processes $\phi : \mathbb{R}_+ \times \Omega \to H$ such that for all $T \geq 0$,

$$\mathbb{E} \int_0^T \|\phi(t)\|^2 dt < \infty.$$

Let $\mathcal{M}^2_{loc}(\hat{\mathcal{P}})$ be the space of all H-valued $\mathfrak{F}$-predictable processes $\varphi : \mathbb{R}_+ \times \Omega \times Z \to H$ such that for all $T \geq 0$,

$$\mathbb{E} \int_0^T \int_Z \|\varphi(t,z)\|^2 \nu(dz) dt < \infty.$$

Our main aim is to consider the following stochastic evolution equation

$$\begin{aligned} u_{tt} &= -A^2u - f(t,u,u_t) - m(\|B^{\frac{1}{2}}u\|^2)Bu + \int_Z g(t,u(t-),u_t(t-),z)\tilde{N}(t,dz), \\ u(0) &= u_0,\ u_t(0) = u_1. \end{aligned} \tag{4.0.1}$$

Here $f : \mathbb{R}_+ \times \mathcal{D}(A) \times H \ni (t,\xi,\eta) \mapsto f(t,\xi,\eta) \in H$, is a $\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{B}(\mathcal{D}(A))\otimes\mathcal{B}(H)/\mathcal{B}(H)$-measurable function and $g : \mathbb{R}_+ \times \mathcal{D}(A) \times H \times Z \ni (t,\xi,\eta,z) \mapsto g(t,\xi,\eta,z) \in H$, is a $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathcal{D}(A)) \otimes \mathcal{B}(H) \otimes \mathcal{Z}/\mathcal{B}(H)$-measurable function. One can transform Equation (4.0.1) into the following first order system

$$\begin{aligned} du &= u_t dt \\ du_t &= -A^2udt - f(u,u_t)dt - m(\|B^{\frac{1}{2}}u\|^2)Budt + \int_Z g(t,u(t-),u_t(t-),z)\tilde{N}(dt,dz). \end{aligned} \tag{4.0.2}$$

Or equivalently, we can rewrite it in the form

$$\begin{aligned} \begin{pmatrix} du \\ du_t \end{pmatrix} = \begin{pmatrix} 0 & I \\ -A^2 & 0 \end{pmatrix} \begin{pmatrix} u \\ u_t \end{pmatrix} dt + \begin{pmatrix} 0 \\ -f(t,u,u_t) - m(\|B^{\frac{1}{2}}u\|)Bu \end{pmatrix} dt \\ + \begin{pmatrix} 0 \\ \int_Z g(t,u(t-),u_t(t-),z)\tilde{N}(dt,dz) \end{pmatrix}. \end{aligned}$$

Now we introduce a new space $\mathcal{H} := \mathcal{D}(A) \times H$ with the product norm

$$\left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|_{\mathcal{H}}^2 := \|Ax\|_H^2 + \|y\|_H^2.$$

It is easy to see that $\mathcal{H}$ is a Hilbert space with norm $\|\cdot\|_{\mathcal{H}}$. We also define functions

$$F : \mathbb{R}_+ \times \mathcal{D}(A) \times H \ni (t,\xi,\eta) \mapsto \begin{pmatrix} 0 \\ -f(t,\xi,\eta) - m(\|B^{\frac{1}{2}}\xi\|^2)B\xi \end{pmatrix} \in \mathcal{H} \tag{4.0.3}$$

$$G : \mathbb{R}_+ \times \mathcal{D}(A) \times H \times Z \ni (t,\xi,\eta,z) \mapsto \begin{pmatrix} 0 \\ g(t,\xi,\eta,z) \end{pmatrix} \in \mathcal{H}. \tag{4.0.4}$$

Put

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ -A^2 & 0 \end{pmatrix}, \ \mathcal{D}(\mathcal{A}) = \mathcal{D}(A^2) \times H.$$

Set $\mathfrak{u} = (u,u_t)^\top$ and $\mathfrak{u}_0 = (u_0,u_1)^\top$. Then Equation (4.0.1) allows the following form

$$\begin{aligned} d\mathfrak{u} &= \mathcal{A}\mathfrak{u}dt + F(t,\mathfrak{u}(t))dt + \int_Z G(t,\mathfrak{u}(t-),z)\tilde{N}(dt,dz), \quad t \geq 0 \\ \mathfrak{u}(0) &= \mathfrak{u}_0. \end{aligned} \tag{4.0.5}$$

Remark 4.0.18. See also Chapter V in [54]. The operator $\mathcal{A}$ generates a C_0-unitary group on $\mathcal{H}$. To prove this one needs to prove that both $\mathcal{A}$ and $-\mathcal{A}$ generate contraction C_0-semigroups on $\mathcal{H}$.

For this it is sufficient to apply the Lumer-Phillips theorem. Hence we only need to show that $\mathcal{A}$ is dissipative and $\mathcal{R}(I - A) = \mathcal{H}$ and the same for $-\mathcal{A}$.
The operator $\mathcal{A}$ is dissipative. To see this, we first observe that for every $x = (x_1, x_2)^\top \in \mathcal{H}$,

$$\mathcal{A}x = \begin{pmatrix} 0 & I \\ -A^2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -A^2 x_1 \end{pmatrix}.$$

So

$$\begin{aligned} \langle \mathcal{A}x, x \rangle &= \langle Ax_2, Ax_1 \rangle + \langle -A^2 x_1, x_2 \rangle \\ &= \langle Ax_2, Ax_1 \rangle - \langle Ax_1, Ax_2 \rangle = 0, \end{aligned}$$

which shows that $\mathcal{A}$ is dissipative. In order to apply Lummer-Phillips Theorem, we also need to verify that $\mathcal{R}(I - \mathcal{A}) = \mathcal{H}$. The inclusion "$\subset$" is clear. For the opposite part $\mathcal{R}(I - \mathcal{A}) \supset \mathcal{H}$, we take $y = (y_1, y_2)^\top \in \mathcal{H}$. We need to find $x = (x_1, x_2)^\top \in \mathcal{D}(\mathcal{A})$ such that $(I - A)x = y$. So

$$\begin{pmatrix} I & -I \\ A^2 & I \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

That is $x_1 - x_2 = y_1$ and $A^2 x_1 + x_2 = y_2$. By some simple operation, this system is equivalent to

$$\begin{aligned} (I + A^2)x_1 &= y_1 + y_2 \\ x_2 &= x_1 - y_1. \end{aligned}$$

This system has a unique solution if and only if $(I + A^2)$ is invertible. This is true by the following reasons. Since $A \geq \mu I$, so $A^2 \geq \mu^2 I$. Then $\sigma(A^2 - \mu^2 I) \subset \mathbb{R}_+$. Here $\sigma(A^2 - \mu^2 I)$ is the spectrum set of $A^2 - \mu^2 I$. Set $\lambda = -\mu^2 - 1 < 0$. So $\lambda \in \rho(A^2 - \mu^2 I)$. It follows that $\lambda I - (A^2 - \mu^2 I) = -I - A^2$ is invertible. Thus $I + A^2$ is invertible. Therefore, this system has a unique solution $x = (x_1, x_2)^\top$. This means that $y \in \mathcal{R}(I - \mathcal{A})$.
Now by applying the Lumer-Phillips Theorem [62], we find out that the operator $\mathcal{A}$ is the infinitesimal generator of a contraction C_0-semigroup, denoted by $(T_+(t))_{t \geq 0}$, in $\mathcal{H}$.
In the same way, one can show that the operator $-\mathcal{A}$ is dissipative and $\mathcal{R}(I + \mathcal{A}) = \mathcal{H}$. On the basis of the Lumer-Phillips Theorem mentioned above, we see that the operator $-\mathcal{A}$ is the infinitesimal generator of a contraction C_0-semigroup, denoted by $(T_-(t))_{t \geq 0}$, in $\mathcal{H}$.
Then one can see that $\mathcal{A}$ is the infinitesimal generator of a contraction C_0-group $e^{t\mathcal{A}}$, $-\infty < t < \infty$ given by

$$e^{t\mathcal{A}} = \begin{cases} T_+(t) & t \geq 0 \\ T_-(-t) & t < 0 \end{cases}.$$

4.1 Existence of Mild Solutions to the Stochastic Nonlinear Beam Equations

Definition 4.1.1. A **strong solution** to Equation (4.0.5) is a $\mathcal{D}(\mathcal{A})$-valued adapted stochastic process $(X(t))_{t \geq 0}$ with càdlàg paths such that

(1) $X(0) = u_0$ a.s.,

(2) the processes ϕ, φ defined by

$$\begin{aligned} \phi(t, \omega) &= F(t, X(t, \omega)) \quad (t, \omega) \in \mathbb{R}_+ \times \Omega; \\ \varphi(t, \omega, z) &= G(t, X(t-, \omega), z) \quad (t, \omega, z) \in \mathbb{R}_+ \times \Omega \times Z \end{aligned}$$

belong to the spaces $\mathcal{M}^2_{loc}(\mathcal{BF})$ and $\mathcal{M}^2_{loc}(\hat{\mathcal{P}})$ respectively.

(3) for any $t \geq 0$, the equality

$$X(t) = u_0 + \int_0^t \mathcal{A}X(s)ds + \int_0^t F(s, X(s))ds + \int_0^t \int_Z G(s, X(s-), z)\tilde{N}(ds, dz) \qquad (4.1.1)$$

holds $\mathbb{P}$-a.s.

Definition 4.1.2. A **mild solution** to Equation (4.0.5) is an $\mathcal{H}$-valued predictable stochastic process $(X(t))_{t\geq 0}$ with càdlàg paths defined on $(\Omega, \mathcal{F}, \mathfrak{F}, \mathbb{P})$ such that the conditions (1) and (2) in the definition of 4.1.1 are satisfied and for any $t \geq 0$, the equality

$$X(t) = e^{t\mathcal{A}}u_0 + \int_0^t e^{(t-s)\mathcal{A}}F(s, X(s))ds + \int_0^t \int_Z e^{(t-s)\mathcal{A}}G(s, X(s-), z)\tilde{N}(ds, dz) \qquad (4.1.2)$$

holds a.s.

We say that a solution $(X(t))_{t\geq 0}$ to the Equation (4.0.5) is **pathwise unique** (or **up to distinguishable**) if and only if for any other solution $(Y(t))_{t\geq 0}$, we have

$$\mathbb{P}(X(t) = Y(t), \text{ for all } t \geq 0) = 1.$$

Remark 4.1.3. (1) Note that the strong solution has to take values in $\mathcal{D}(\mathcal{A})$ while the mild solution takes values in the whole space $\mathcal{H}$. Besides, not every mild solution is a strong solution. But if a strong solution exists for Equation (4.0.5), then it should be of the form (4.1.2).

(2) Notice that since processes appearing on both sides of equality (2.1.7) are càdàg, so from the fact that if two processes are modifications of each other and they have a.s. right continuous paths, then they are indistinguishable, we infer that the order of the quantifiers "for all $t \geq 0$ " and "$\mathbb{P}$-a.s." can be interchanged.

Definition 4.1.4. We say that X is a mild solution on a closed stochastic interval $[0, \sigma]$ if the integral on the right of (4.1.2) is defined on $[0, \sigma]$ and it equals to X on $[0, \sigma]$, $\mathbb{P}$-a.s., namely

$$X(t) = e^{t\mathcal{A}}u_0 + \int_0^t e^{(t-s)\mathcal{A}}F(s, X(s))ds + \int_0^t \int_Z e^{(t-s)\mathcal{A}}G(s, X(s-), z)\tilde{N}(ds, dz) \text{ on } [0, \sigma], \ \mathbb{P}\text{-a.s.} \qquad (4.1.3)$$

Remark 4.1.5. Alternatively, we may rewrite (4.1.3) in the following equivalent form

$$X(t \wedge \tau) = e^{t\mathcal{A}}u_0 + \int_0^{t\wedge\tau} e^{(t\wedge\tau - s)\mathcal{A}}F(s, X(s))ds + I_\tau(G(X))(t \wedge \tau) \quad t \geq 0, \ \mathbb{P}\text{-a.s.}, \qquad (4.1.4)$$

where $I_\tau(G(X))$ is a process defined by

$$I_\tau(G(X))(t) = \int_0^t \int_Z 1_{[0,\tau]}(s)e^{(t-s)\mathcal{A}}G(s, X(s-), z)\tilde{N}(ds, dz), \ t \geq 0.$$

Remark 4.1.6. According to Corollary 13.7 in the monograph [58] every predictable and right-continuous martingale is continuous, so if we impose both properties on a process, it turns out that we are assuming nothing but the continuity of the process. In our definition, the reason why we need the predictability of the process X is to get the predictability of the integrand $e^{(t-s)\mathcal{A}}G(t, X(s), z)$. But since we assume that the process is càdlàg, we can get around this difficulty by taking the left-limit process.

In this section we are going to consider the simple case where the function F is given by

$$F : \mathbb{R}_+ \times \Omega \times H \times H \to \mathcal{H},\ (t, \omega, \xi, \eta) \mapsto \begin{pmatrix} 0 \\ -f(t, \omega, \xi, \eta) \end{pmatrix}. \tag{4.1.5}$$

To order to show the existence and the uniqueness of our mild solution, we impose certain growth conditions and the global Lipschitz conditions on f and g.

Assumption 4.1.7. *There exist constants K_f and K_g such that for all $t \geq 0$ and all $x = (x_1, x_2)^\top \in \mathcal{H}$,*

$$\|f(t, x_1, x_2)\|_H^2 \leq K_f(1 + \|x\|_{\mathcal{H}}^2) \tag{4.1.6}$$

$$\int_Z \|g(t, x_1, x_2, z)\|_H^2 \nu(dz) \leq K_g(1 + \|x\|_{\mathcal{H}}^2). \tag{4.1.7}$$

Assumption 4.1.8. *There exist constant L_f such that for all $t \geq 0$ and all $x = (x_1, x_2)^\top \in \mathcal{H}$, $y = (y_1, y_2)^\top \in \mathcal{H}$,*

$$\|f(t, x_1, x_2) - f(t, y_1, y_2)\|_H \leq L_f \|x - y\|_{\mathcal{H}}, \tag{4.1.8}$$

Assumption 4.1.9. *There exist constant L_g such that for all $t \geq 0$ and all $x = (x_1, x_2)^\top \in \mathcal{H}$, $y = (y_1, y_2)^\top \in \mathcal{H}$,*

$$\int_Z \|g(t, x_1, x_2, z) - g(t, y_1, y_2, z)\|_H^2 \nu(dz) \leq L_g \|x - y\|_{\mathcal{H}}^2. \tag{4.1.9}$$

Now let's start our main theorem of this section. The proof of existence of mild solution is based on Banach fixed point theorem.

Theorem 4.1.10. *Suppose that functions f, g satisfy Assumptions 4.1.7, 4.1.8 and 4.1.9. Then there exists a unique (up to distinguishable) mild solution of Equation* (4.0.5). *In particular, if*

$$\mathrm{u}_0 \in \mathcal{D}(\mathcal{A}),\ F(\cdot, \mathrm{u}(\cdot)) \in \mathcal{M}^2_{loc}(\mathcal{BF}; \mathcal{D}(\mathcal{A})) \text{ and } G(\cdot, \mathrm{u}(\cdot)) \in \mathcal{M}^2_{loc}(\hat{\mathcal{P}}; \mathcal{D}(\mathcal{A})),$$

then the mild solution coincides with probability 1 *with a strong solution at all the points over $\mathbb{R}_+$. More precisely, the mild solution satisfying* (4.1.2) *is $\mathbb{P}$-equivalent to*

$$\mathrm{u}(t) = \mathrm{u}_0 + \int_0^t \mathcal{A}\mathrm{u}(s)ds + \int_0^t F(s, \mathrm{u}(s))ds + \int_0^t \int_Z G(s, \mathrm{u}(s-), z)\tilde{N}(ds, dz)\ \mathbb{P}\text{-a.s. } t \geq 0.$$

In order to prove Theorem 4.1.10, we willl first establish several auxiliary results.

Proposition 4.1.11. *Suppose that $Z : \mathbb{R}_+ \to \mathcal{H}$ is a progressively measurable process. Let $X(t) = e^{t\mathcal{A}}Z(t)$, $t \geq 0$ and $Y(t) = e^{-t\mathcal{A}}Z(t)$ Then $X(t)$ and $Y(t)$, $t \geq 0$ are progressively measurable processes.*

Proof of Proposition 4.1.11. Define a function $\alpha : \mathbb{R}_+ \times \mathcal{H} \ni (t, x) \mapsto e^{t\mathcal{A}}x \in \mathcal{H}$. Since $e^{t\mathcal{A}}$, $t \geq 0$ is a contraction C_0-semigroup, so $\|e^{t\mathcal{A}}\|_{\mathcal{L}(\mathcal{H})} \leq 1$ and for every $x \in \mathcal{H}$, $\alpha(\cdot, x)$ is continuous. Also, for every $t \geq 0$, $\alpha(t, \cdot)$ is continuous. Indeed, let us fix $x_0 \in \mathcal{H}$. Then for every $x \in \mathcal{A}$

$$\|\alpha(t, x) - \alpha(t, x_0)\|_{\mathcal{H}} = \|e^{t\mathcal{A}}(x - x_0)\|_{\mathcal{H}} \leq \|x - x_0\|_{\mathcal{H}}.$$

Thus $\alpha(t, \cdot)$ is continuous. This shows that the function α is separably continuous. Since by the assumption the process Z is progressively measurable, one can see that the mapping

$$\mathbb{R}_+ \times \Omega \ni (s, \omega) \mapsto (s, Z(s, \omega)) \in \mathbb{R}_+ \times \mathcal{H}$$

is progressively measurable as well. So the composition mapping

$$\mathbb{R}_+ \times \Omega \ni (s, \omega) \mapsto (s, Z(s, \omega)) \mapsto \alpha(s, Z(s, \omega)) \in \mathcal{H}$$

is progressively measurable, and hence, the process $X(t)$, $t \geq 0$ is progressively measurable. The progressively measurability of process $Y(t)$, $t \geq 0$ follows from the above proof with $\mathcal{A}$ replaced by $-\mathcal{A}$. □

Proof of Theorem 4.1.10. Given $T \geq 0$. First we denote by $\mathcal{M}_T^2$ the set of all $\mathcal{H}$-valued progressively measurable processes $X : \mathbb{R}_+ \times \Omega \to \mathcal{H}$ such that

$$\|X\|_T := \sup_{0 \leq t \leq T} (\mathbb{E}\|X(t)\|_{\mathcal{H}}^2)^{\frac{1}{2}} < \infty.$$

Then the space $\mathcal{M}_T^2$ endowed with the norm $\|X\|_\lambda := \sup_{0 \leq t \leq T} e^{-\lambda t} (\mathbb{E}\|X(t)\|_{\mathcal{H}}^2)^{\frac{1}{2}}$, $\lambda > 0$, is a Banach space. Note that the norms $\|\cdot\|_\lambda$, $\lambda \geq 0$, are equivalent. Let us define a map $\Phi_T : \mathcal{M}_T^2 \to \mathcal{M}_T^2$ by

$$(\Phi_T X)(t) = e^{t\mathcal{A}} u_0 + \int_0^t e^{(t-s)\mathcal{A}} F(s, X(s)) ds + \int_0^t \int_Z e^{(t-s)\mathcal{A}} G(s, X(s), z) \tilde{N}(ds, dz).$$

We shall show that the operator Φ_T is a contraction operator on $\mathcal{M}_T^2$ for sufficiently large values of λ. We first verify that if $X \in \mathcal{M}_T^2$, then $\Phi_T X \in \mathcal{M}_T^2$.

Claim 1. The process $\int_0^t e^{(t-s)\mathcal{A}} F(s, X(s))\, ds$, $t \in [0, T]$, is progressively measurable.
Proof of Claim 1: Since F is $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathcal{H}) / \mathcal{B}(\mathcal{H})$-measurable and the process $X(t)$, $t \in [0, T]$ is progressively measurable, so the mapping

$$[0, T] \times \Omega \ni (t, \omega) \mapsto (t, X(t, \omega)) \mapsto F(t, X(t, \omega)) \in \mathcal{H}$$

is progressively measurable as well.

By Lemma 4.1.11 we find out that $e^{(-s)\mathcal{A}} F(s, X(s))$ is also progressively measurable. It then follows from the Fubini Theorem that the integral $\int_0^t e^{(-s)\mathcal{A}} F(s, X(s))\, ds$ is $\mathcal{F}_t$-measurable.

Since the process $[0, T] \ni t \mapsto \int_0^t e^{(-s)\mathcal{A}} F(s, X(s))\, ds \in \mathcal{H}$ is continuous in t, this together with the adaptedness assert the progressively measurability of the process $\int_0^t e^{(-s)\mathcal{A}} F(s, X(s))\, ds$, $t \in [0, T]$. Again, by Proposition 4.1.11, we infer that the process

$$\int_0^t e^{(t-s)\mathcal{A}} F(s, X(s))\, ds = e^{t\mathcal{A}} \int_0^t e^{-s\mathcal{A}} F(s, X(s))\, ds, \quad t \in [0, T],$$

is also progressively measurable.
Claim 2. The process $\int_0^t \int_Z e^{(t-s)\mathcal{A}} G(s, X(s), z) \tilde{N}(ds, dz)$, $t \in [0, T]$ has a progressively measurable version.
Proof of Claim 2: First of all, we show that the process $\int_0^t \int_Z e^{(t-s)\mathcal{A}} G(s, X(s), z) \tilde{N}(ds, dz)$, $0 \leq t \leq T$ is $\mathfrak{F}$-adapted. Let us fix $t \in [0, T]$. Since by assumption the process X is progressively measurable,

a similar argument as in the proof of claim 1 shows that the integrand function $e^{(t-s)\mathcal{A}}G(s, X(s), z)$ is progressively measurable. Hence by assumption 4.1.7, the integral process

$$\int_0^r \int_Z 1_{(0,t]} e^{(t-s)\mathcal{A}} G(s, X(s), z) \tilde{N}(ds, dz), \quad r \in [0, T]$$

is well defined. Moreover, we know from Theorem 3.3.2 that this process is none but a martingale. In particular, for each $r \in [0, T]$, the integral $\int_0^r \int_Z 1_{(0,t]} e^{(t-s)\mathcal{A}} G(s, X(s), z) \tilde{N}(ds, dz)$ is $\mathcal{F}_r$-measurable. By taking $r = t$, we infer that $\int_0^t \int_Z 1_{(0,t]} e^{(t-s)\mathcal{A}} G(s, X(s), z) \tilde{N}(ds, dz)$ is $\mathcal{F}_t$-measurable.

Also, In view of Theorem 3.7.9, the stochastic convolution process $\int_0^t \int_Z e^{(t-s)\mathcal{A}} G(s, X(s), z) \tilde{N}(ds, dz)$, $t \in [0, T]$ has a càdlàg modification. Therefore, we infer that the process

$$\int_0^t \int_Z e^{(t-s)\mathcal{A}} G(s, X(s), z) \tilde{N}(ds, dz), \ t \in [0, T]$$

has a progressively measurable version.

In conclusion, the process $(\Phi_T X)(t)$, $t \geq 0$ is progressively measurable. So it remains to show that $\|\Phi_T X\|_\lambda^2 < \infty$.

First, we find out that

$$\begin{aligned} \|\Phi_T X\|_\lambda &\leq \|e^{\cdot\mathcal{A}} u_0\|_\lambda + \left\| \int_0^\cdot e^{(\cdot-s)\mathcal{A}} F(s, X(s)) ds \right\|_\lambda \\ &\quad + \left\| \int_0^\cdot \int_Z e^{(\cdot-s)\mathcal{A}} G(s, X(s), z) \tilde{N}(ds, dz) \right\|_\lambda \\ &= I_1 + I_2 + I_3. \end{aligned}$$

For the first term I_1, by the definition of the norm $\|\cdot\|_\lambda$, we have

$$I_1 = \|e^{\cdot\mathcal{A}} u_0\|_\lambda = \sup_{0\leq t\leq T} e^{-\lambda t} \left(\mathbb{E} \left\| e^{t\mathcal{A}} u_0 \right\|_{\mathcal{H}}^2 \right)^{\frac{1}{2}} \leq \|u_0\|_{\mathcal{H}}.$$

where we used the fact that $e^{t\mathcal{A}}$ is a contraction C_0-semigroup. Also, by using the Cauchy-Schwartz inequality and the growth conditions (4.1.6) and (4.1.7), for the second term I_2, we obtain

$$\begin{aligned} I_2 &= \sup_{0\leq t\leq T} e^{-\lambda t} \left(\mathbb{E} \left\| \int_0^t e^{(t-s)\mathcal{A}} F(s, X(s)) ds \right\|_{\mathcal{H}}^2 \right)^{\frac{1}{2}} \\ &\leq \sup_{0\leq t\leq T} e^{-\lambda t} T^{\frac{1}{2}} \left(\mathbb{E} \int_0^t \|F(s, X(s))\|_{\mathcal{H}}^2 ds \right)^{\frac{1}{2}} \\ &\leq \sup_{0\leq t\leq T} e^{-\lambda t} T^{\frac{1}{2}} K_f^{\frac{1}{2}} \left(\mathbb{E} \int_0^t (1 + \|X(s)\|_{\mathcal{H}}^2) ds \right)^{\frac{1}{2}} \\ &\leq T K_f^{\frac{1}{2}} + \sup_{0\leq t\leq T} T^{\frac{1}{2}} K_f^{\frac{1}{2}} \left(\int_0^t e^{-2\lambda(t-s)} ds \sup_{0\leq s\leq T} \mathbb{E}\, e^{-2\lambda s} \|X(s)\|_{\mathcal{H}}^2 \right)^{\frac{1}{2}} \\ &\leq T K_f^{\frac{1}{2}} + \frac{1}{2\lambda} T^{\frac{1}{2}} K_f^{\frac{1}{2}} \sup_{0\leq s\leq T} e^{-\lambda s} \left(\mathbb{E} \|X(s)\|_{\mathcal{H}}^2 \right)^{\frac{1}{2}} \\ &\leq T K_f^{\frac{1}{2}} + \frac{1}{2\lambda} T^{\frac{1}{2}} K_f^{\frac{1}{2}} \sup_{0\leq s\leq T} e^{-\lambda s} \left(\mathbb{E} \|X(s)\|_{\mathcal{H}}^2 \right)^{\frac{1}{2}} \end{aligned}$$

In the same way, we have

$$\begin{aligned}
I_3 &= \sup_{0\le t\le T} e^{-\lambda t}\left(\mathbb{E}\left\|\int_0^t\int_Z e^{(t-s)\mathcal{A}}G(s,X(s),z)\tilde{N}(ds,dz)\right\|_{\mathcal{H}}^2\right)^{\frac12}\\
&= \sup_{0\le t\le T} e^{-\lambda t}\left(\mathbb{E}\int_0^t\int_Z \|e^{(t-s)\mathcal{A}}G(s,X(s),z)\|_{\mathcal{H}}^2\nu(dz)ds\right)^{\frac12}\\
&\le K_g^{\frac12}\sup_{0\le t\le T} e^{-\lambda t}\left(\mathbb{E}\int_0^t(1+\|X(s)\|_{\mathcal{H}}^2)ds\right)^{\frac12}\\
&\le K_g^{\frac12}T^{\frac12}+K_g^{\frac12}\left(\int_0^t e^{-2\lambda(t-s)}ds\sup_{0\le s\le T}\mathbb{E}\,e^{-2\lambda s}\|X(s)\|_{\mathcal{H}}^2\right)^{\frac12}\\
&\le K_g^{\frac12}T^{\frac12}+\frac{1}{2\lambda}K_g^{\frac12}\|X(s)\|_\lambda,
\end{aligned}$$

where the second equality follows from the isometry property of Itô integral w.r.t. compensated Poisson random measures and the second inequality follows from the growth condition (4.1.7) of the function g. Combining the above three estimates, we get

$$\|\Phi_T(X)\|_\lambda^2 \le I_1+I_2+I_3\le \|u_0\|_{\mathcal{H}}^2+TK_f^{\frac12}+K_g^{\frac12}T^{\frac12}+\frac{1}{2\lambda}(T^{\frac12}K_f^{\frac12}+K_g^{\frac12})\|X(s)\|_\lambda<\infty, \quad (4.1.10)$$

which implies that $\Phi_T(X)\in\mathcal{M}_T^2$.
Now we shall show that Φ_T is a contraction provided λ is chosen to be large enough. For this we take $X_1,X_2\in\mathcal{M}_T^2$. Then we obtain the following inequality

$$\begin{aligned}
\|\Phi_T(X_1)-\Phi_T(X_2)\|_\lambda &= \Big\|\int_0^\cdot e^{(\cdot-s)\mathcal{A}}\Big(F(s,X_1(s))-F(s,X_2(s))\Big)ds\\
&\qquad+\int_0^\cdot\int_Z e^{(\cdot-s)\mathcal{A}}\Big(G(s,X_1(s),z)-G(s,X_2(s),z)\Big)\tilde{N}(ds,dz)\Big\|_\lambda^2\\
&\le\left\|\int_0^\cdot e^{(\cdot-s)\mathcal{A}}\Big(F(s,X_1(s))-F(s,X_2(s))\Big)ds\right\|_\lambda\\
&\qquad+\left\|\int_0^\cdot\int_Z e^{(\cdot-s)\mathcal{A}}\Big(G(s,X_1(s),z)-G(s,X_2(s),z)\Big)\tilde{N}(ds,dz)\right\|_\lambda\\
&= I_4+I_5. \qquad (4.1.11)
\end{aligned}$$

Observe first that, similarly to the estimates on I_2 before, we have

$$\begin{aligned}
I_4 &= \sup_{0\le t\le T} e^{-\lambda t}\left(\mathbb{E}\left\|\int_0^t e^{(t-s)\mathcal{A}}\Big(F(s,X_1(s))-F(s,X_2(s))\Big)ds\right\|_{\mathcal{H}}^2\right)^{\frac12}\\
&\le T^{\frac12}\sup_{0\le t\le T} e^{-\lambda t}\left(\mathbb{E}\int_0^t\left\|e^{(t-s)\mathcal{A}}\Big(F(s,X_1(s))-F(s,X_2(s))\Big)\right\|_{\mathcal{H}}^2ds\right)^{\frac12}\\
&\le T^{\frac12}L_f\sup_{0\le t\le T} e^{-\lambda t}\left(\mathbb{E}\int_0^t\|X_1(s)-X_2(s)\|_{\mathcal{H}}^2ds\right)^{\frac12}\\
&\le T^{\frac12}L_f\left(\int_0^T e^{-2\lambda(t-s)}ds\right)^{\frac12}\left(\sup_{0\le s\le T}\mathbb{E}e^{-2\lambda s}\|X_1(s)-X_2(s)\|_{\mathcal{H}}^2\right)^{\frac12}\\
&\le\frac{T^{\frac12}L_f}{2\lambda}\|X_1(s)-X_2(s)\|_\lambda,
\end{aligned}$$

where we used the Cauchy-Schwartz inequality and the globally Lipschitz assumption (4.1.6) on f. Also on the basis of the Itô isometry property (see Theorem (3.3.2)) and the global Lipschitz assumption (4.1.7) on g, we find out that

$$
\begin{aligned}
I_5 &= \sup_{0\le t\le T} e^{-\lambda t}\left(\mathbb{E}\left\|\int_0^t\int_Z e^{(t-s)\mathcal{A}}\Big(G(s,X_1(s),z)-G(s,X_2(s),z)\Big)\tilde{N}(ds,dz)\right\|_{\mathcal{H}}^2\right)^{\frac{1}{2}} \\
&= \sup_{0\le t\le T} e^{-\lambda t}\left(\mathbb{E}\int_0^t\int_Z \left\|e^{(t-s)\mathcal{A}}\Big(G(s,X_1(s),z)-G(s,X_2(s),z)\Big)\right\|_{\mathcal{H}}^2 \nu(dz)ds\right)^{\frac{1}{2}} \\
&\le \mathbb{E}\int_0^T \left(e^{-\lambda t}\int_0^t\int_Z \|G(s,X_1(s),z)-G(s,X_2(s),z)\|_{\mathcal{H}}^2\nu(dz)ds\right)dt \\
&\le L_g \sup_{0\le t\le T} e^{-\lambda t}\left(\mathbb{E}\int_0^t \|X_1(s)-X_2(s)\|_{\mathcal{H}}^2\, ds\right)^{\frac{1}{2}} \\
&\le L_g\left(\int_0^T e^{-2\lambda(t-s)}ds\right)^{\frac{1}{2}}\left(\sup_{0\le s\le T}\mathbb{E}e^{-2\lambda s}\|X_1(s)-X_2(s)\|_{\mathcal{H}}^2\right)^{\frac{1}{2}} \\
&\le \frac{L_g}{2\lambda}\|X_1(s)-X_2(s)\|_{\lambda}.
\end{aligned}
$$

By substituting above estimates into the right-side of inequality (4.1.11), we get that

$$
\|\Phi_T(X_1)-\Phi_T(X_2)\|_{\lambda}^2 \le \frac{T^{\frac{1}{2}}L_f+L_g}{2\lambda}\|X_1-X_2\|_{\lambda}^2. \tag{4.1.12}
$$

Therefore, if $\frac{T^{\frac{1}{2}}L_f+L_g}{2\lambda}\le\frac{1}{2}$, then Φ_T is a strict contraction in $\mathcal{M}_T^2$. We then apply the Banach Fixed Point Theorem to infer that Φ_T has a unique fixed point in $\mathcal{M}_T^2$. This implies that for any $0<T<\infty$, there exists a unique (up to modification) process $(\bar{\mathfrak{u}}(t))_{0\le t\le T}\in\mathcal{M}_T^2$ such that $\bar{\mathfrak{u}}=\Phi_T(\bar{\mathfrak{u}})$ in $\mathcal{M}_T^2$.
Notice that we can always find a càdàg version satisfying (4.1.2). Indeed, we know that the uniqueness holds in the sense that if there exists another process $\mathfrak{v}\in\mathcal{M}_T^2$ satisfying $\mathfrak{v}=\Phi_T\mathfrak{v}$, then for every $t\in[0,T]$, $\bar{\mathfrak{u}}(t)=\mathfrak{v}(t)$, $\mathbb{P}$-a.s. Let $\mathcal{N}:=\{X\in\mathcal{M}_T^2 : X=\Phi_TX\}$. By the uniqueness, the set $\mathcal{N}$ contains all stochastically equivalent processes of the process $\bar{\mathfrak{u}}$. Among those stochastically equivalent processes in $\mathcal{N}$, we are trying to find a version $(\mathfrak{u}(t))$ of $(\bar{\mathfrak{u}}(t))$ such that $(\mathfrak{u}(t))$ is càdlàg and $(\mathfrak{u}(t))$ satisfies (4.1.2). For this, we define

$$
\begin{aligned}
\mathfrak{u}(t) &= (\Phi_T\bar{\mathfrak{u}})(t) \\
&= e^{t\mathcal{A}}\mathfrak{u}_0+\int_0^t e^{(t-s)\mathcal{A}}F(s,\bar{\mathfrak{u}}(s))ds+\int_0^t\int_Z e^{(t-s)\mathcal{A}}G(s,\bar{\mathfrak{u}}(s),z)\tilde{N}(ds,dz),\ t\in[0,T],
\end{aligned}
$$

Note that the process $\mathfrak{u}$ is càdlàg, see Theorem 3.7.9. Hence, we may define

$$
\begin{aligned}
\mathfrak{v}(t) &= (\Phi_T\mathfrak{u})(t) \\
&= e^{t\mathcal{A}}\mathfrak{u}_0+\int_0^t e^{(t-s)\mathcal{A}}F(s,\mathfrak{u}(s))ds+\int_0^t\int_Z e^{(t-s)\mathcal{A}}G(s,\mathfrak{u}(s-),z)\tilde{N}(ds,dz),\ t\in[0,T].
\end{aligned}
$$

We observe by the definition of two processes $\mathfrak{u}$ and $\bar{\mathfrak{u}}$ that for all $t\in[0,T]$, $\mathbb{E}\|\mathfrak{u}(t)-\bar{\mathfrak{u}}(t)\|_{\mathcal{H}}^2=0$. This implies that $\mathfrak{u}$ is a càdlàg version of $\bar{\mathfrak{u}}$. From this, we also find out that $\mathbb{E}\int_0^T\|\mathfrak{u}(t)-\bar{\mathfrak{u}}(t)\|_{\mathcal{H}}^2dt=0$. It follows form the continuity of functions $F(t,x)$ and $G(t,x,z)$ in the variable x that for all $t \in [0,T]$,

$$\begin{aligned}\mathbb{E}\|u(t)-\mathfrak{v}(t)\|^2_{\mathcal{H}} \leq & 2\mathbb{E}\left\|\int_0^t e^{(t-s)\mathcal{A}}\Big(F(s,\bar{u}(s))-F(s,u(s))\Big)ds\right\|^2_{\mathcal{H}} \\ & + 2\mathbb{E}\left\|\int_0^t\int_Z G(s,\bar{u}(s),z)-G(s,u(s-),z)\tilde{N}(ds,dz)\right\|^2_{\mathcal{H}} \\ = & 2\mathbb{E}\left\|\int_0^t e^{(t-s)\mathcal{A}}\Big(F(s,\bar{u}(s))-F(s,u(s))\Big)ds\right\|^2_{\mathcal{H}} \\ & + 2\mathbb{E}\int_0^t\int_Z \Big\|G(s,\bar{u}(s),z)-G(s,u(s),z)\Big\|^2_{\mathcal{H}}\nu(dz)ds = 0.\end{aligned}$$

Hence, we infer that for all $t \in [0,T]$,

$$u(t)=\mathfrak{v}(t)=e^{t\mathcal{A}}u_0+\int_0^t e^{(t-s)\mathcal{A}}F(s,u(s))ds+\int_0^t\int_Z e^{(t-s)\mathcal{A}}G(s,u(s-),z)\tilde{N}(ds,dz),\ \mathbb{P}\text{-a.s.,} \tag{4.1.13}$$

which shows that u satisfies (4.1.2). Since both sides of above equality are càdàg, the stochastically equivalence becomes $\mathbb{P}$-equivalence. More precisely, we obtain a pathwise uniqueness càdlàg process in $\mathcal{M}^2_T$ such that for all $t \in [0,T]$, the equality (4.1.2) holds. However, if we release the càdlàg property, the pathwise uniqueness no longer holds and we could only have stochastically uniqueness instead.

Now the uniqueness feature of a solution on any given priori time interval $[0,T]$ allows us to amalgamate them into a solution $(u(t))_{t\geq 0}$ to problem (4.0.5) on the positive real half-line. Moreover, this solution $(u(t))_{t\geq 0}$ to problem (4.0.5) is unique up to distinguishable.

In other words, for $t \geq 0$,

$$u(t)=e^{t\mathcal{A}}u_0+\int_0^t e^{(t-s)\mathcal{A}}F(s,u(s))ds+\int_0^t\int_Z e^{(t-s)\mathcal{A}}G(s,u(s-),z)\tilde{N}(ds,dz)\ \ \mathbb{P}\text{-a.s..} \tag{4.1.14}$$

Note also that since $u \in \mathcal{M}^2_T$, for every $T>0$,

$$\begin{aligned}&\mathbb{E}\int_0^T \|F(s,u(s))\|^2_{\mathcal{H}}ds \leq L_f^2\mathbb{E}\int_0^T(1+\|u(s)\|^2_{\mathcal{H}})ds \leq L_f^2T(1+\|u\|^2_T)<\infty;\\ &\mathbb{E}\int_0^T\int_Z \|G(s,u(s),z)\|^2_{\mathcal{H}}\nu(dz)ds \leq L_g^2\mathbb{E}\int_0^T(1+\|u(s)\|^2_{\mathcal{H}})ds \leq L_g^2T(1+\|u\|^2_T)<\infty;\end{aligned}$$

which shows that $F(\cdot,u(\cdot)) \in \mathcal{M}^2_{loc}(\mathcal{BF})$ and $G(\cdot,u(\cdot),z) \in \mathcal{M}^2_{loc}(\hat{\mathcal{P}})$. In conclusion, Problem (4.0.5) has a unique mild solution.

Now let us suppose that $u_0 \in \mathcal{D}(\mathcal{A})$, $F(\cdot,u(\cdot)) \in \mathcal{M}^2_{loc}(\mathcal{BF};\mathcal{D}(\mathcal{A}))$ and $G(\cdot,u(\cdot)) \in \mathcal{M}^2_{loc}(\hat{\mathcal{P}};\mathcal{D}(\mathcal{A}))$, where $\mathcal{D}(\mathcal{A})$ is endowed with the graph norm. We observe that $u(t) \in \mathcal{D}(\mathcal{A})$ for every $t \geq 0$. To see this, let us us fix $t \geq 0$. Let $R(\lambda,\mathcal{A})=(\lambda I-\mathcal{A})^{-1}$, $\lambda>0$, be the resolvent of $\mathcal{A}$. Since $\mathcal{A}R(\lambda,\mathcal{A})=\lambda R(\lambda,\mathcal{A})-I_E$, $\mathcal{A}R(\lambda,\mathcal{A})$ is bounded. Hence, since $G(\cdot,u(\cdot)) \in \mathcal{M}^2_{loc}(\hat{\mathcal{P}};\mathcal{D}(\mathcal{A}))$, we obtain

$$\begin{aligned}
R(\lambda,\mathcal{A})\int_0^t\int_Z \mathcal{A}e^{(t-s)\mathcal{A}}G(s,\mathrm{u}(s-),z)\tilde{N}(ds,dz)
&= \int_0^t\int_Z R(\lambda,\mathcal{A})\mathcal{A}e^{(t-s)\mathcal{A}}G(s,\mathrm{u}(s-),z)\tilde{N}(ds,dz)\\
&= \lambda R(\lambda,A)\int_0^t\int_Z e^{(t-s)\mathcal{A}}G(s,\mathrm{u}(s-),z)\tilde{N}(ds,dz)\\
&\quad -\int_0^t\int_Z e^{(t-s)\mathcal{A}}G(s,\mathrm{u}(s-),z)\tilde{N}(ds,dz).
\end{aligned}$$

Thus, it follows that

$$\begin{aligned}
&\int_0^t\int_Z e^{(t-s)\mathcal{A}}G(s,\mathrm{u}(s-),z)\tilde{N}(ds,dz)\\
&= R(\lambda,A)\left[\lambda\int_0^t\int_Z e^{(t-s)\mathcal{A}}G(s,\mathrm{u}(s-),z)\tilde{N}(ds,dz)-\int_0^t\int_Z Ae^{(t-s)\mathcal{A}}G(s,\mathrm{u}(s-),z)\tilde{N}(ds,dz)\right].
\end{aligned}$$

Since $Rng(R(\lambda,A))=\mathcal{D}(A)$, we infer that $\int_0^t\int_Z e^{(t-s)\mathcal{A}}G(s,\mathrm{u}(s-),z)\tilde{N}(ds,dz)\in\mathcal{D}(\mathcal{A})$. Here Rng denotes the range. In a similar manner, we can show that $\int_0^t e^{(t-s)\mathcal{A}}F(s,\mathrm{u}(s))ds\in\mathcal{D}(\mathcal{A})$. Hence, $\mathrm{u}(t)\in\mathcal{D}(\mathcal{A})$.

Now we are in a position to show that

$$\begin{aligned}
&\mathcal{A}\int_0^t\int_Z e^{(t-s)\mathcal{A}}G(s,\mathrm{u}(s-),z)\tilde{N}(ds,dz)=\int_0^t\int_Z \mathcal{A}e^{(t-s)\mathcal{A}}G(s,\mathrm{u}(s-),z)\tilde{N}(ds,dz),\quad \mathbb{P}\text{-a.s. } t\geq 0;\\
&\mathcal{A}\int_0^t e^{(t-s)\mathcal{A}}F(s,\mathrm{u}(s))ds=\int_0^t \mathcal{A}e^{(t-s)\mathcal{A}}F(s,\mathrm{u}(s))ds,\quad \mathbb{P}\text{-a.s. } t\geq 0.
\end{aligned}\tag{4.1.15}$$

For this, let us take $h\in(0,t)$. Since $\frac{e^{h\mathcal{A}}-I}{h}$ is a bounded operator, we have the following

$$\begin{aligned}
&\mathbb{E}\left\|\mathcal{A}\int_0^t\int_Z e^{(t-s)\mathcal{A}}G(s,\mathrm{u}(s-),z)\tilde{N}(ds,dz)-\int_0^t\int_Z \mathcal{A}e^{(t-s)\mathcal{A}}G(s,\mathrm{u}(s-),z)\tilde{N}(ds,dz)\right\|^2\\
&\leq 2\mathbb{E}\left\|\left(\frac{e^{h\mathcal{A}}-I}{h}-\mathcal{A}\right)\int_0^t\int_Z e^{(t-s)\mathcal{A}}G(s,\mathrm{u}(s-),z)\tilde{N}(ds,dz)\right\|^2\\
&\quad +2\mathbb{E}\left\|\int_0^t\int_Z\left(\frac{e^{h\mathcal{A}}-I}{h}-\mathcal{A}\right)e^{(t-s)\mathcal{A}}G(s,\mathrm{u}(s-),z)\tilde{N}(ds,dz)\right\|^2\\
&= 2\mathbb{E}\left\|\left(\frac{e^{h\mathcal{A}}-I}{h}-\mathcal{A}\right)\int_0^t\int_Z e^{(t-s)\mathcal{A}}G(s,\mathrm{u}(s-),z)\tilde{N}(ds,dz)\right\|^2\\
&\quad +2\mathbb{E}\int_0^t\int_Z\left\|\left(\frac{e^{h\mathcal{A}}-I}{h}-\mathcal{A}\right)e^{(t-s)\mathcal{A}}G(s,\mathrm{u}(s),z)\right\|^2\nu(dz)ds\\
&:= I(h)+II(h).
\end{aligned}$$

Since we showed that $\int_0^t\int_Z e^{(t-s)\mathcal{A}}G(s,\mathrm{u}(s-),z)\tilde{N}(ds,dz)\in\mathcal{D}(\mathcal{A})$, we infer that the term $I(h)$ converges to 0 a.s. as $h\downarrow 0$.

It is easy to see that the integrand

$$\left\|\left(\frac{e^{h\mathcal{A}}-I}{h}-\mathcal{A}\right)e^{(t-s)\mathcal{A}}G(s,\mathrm{u}(s),z)\right\|^2$$

is bounded by a function $C_1|\mathcal{A}G(s,\mathrm{u}(s),z)|^2$ which satisfies $\mathbb{E}\int_0^t\int_Z|\mathcal{A}G(s,\mathrm{u}(s),z)|^2\nu(dz)ds<\infty$ for every $t\geq 0$ by the assumptions. Since $\mathcal{A}$ is the infinitesimal generator of the C_0-semigroup $(e^{t\mathcal{A}})_{t\geq 0}$, the integrand converges to 0 pointwise on $[0,t]\times\Omega\times Z$. Therefore, the Lebesgue Dominated Convergence Theorem on interchanging a limit and an integral is applicable. So the second term $II(h)$ converges to 0 as $h\downarrow 0$ as well. Therefore, we have

$$\mathbb{E}\left\|\mathcal{A}\int_0^t\int_Z e^{(t-s)\mathcal{A}}G(s,\mathrm{u}(s-),z)\tilde{N}(ds,dz)-\int_0^t\int_Z \mathcal{A}e^{(t-s)\mathcal{A}}G(s,\mathrm{u}(s-),z)\tilde{N}(ds,dz)\right\|^2=0,$$

which gives that

$$\mathcal{A}\int_0^t\int_Z e^{(t-s)\mathcal{A}}G(s,\mathrm{u}(s-),z)\tilde{N}(ds,dz)=\int_0^t\int_Z \mathcal{A}e^{(t-s)\mathcal{A}}G(s,\mathrm{u}(s-),z)\tilde{N}(ds,dz),\quad \mathbb{P}\text{-a.s. } t\geq 0.$$

Similarly, one can show that

$$\mathcal{A}\int_0^t e^{(t-s)\mathcal{A}}F(s,\mathrm{u}(s))ds=\int_0^t \mathcal{A}e^{(t-s)\mathcal{A}}F(s,\mathrm{u}(s))ds,\quad \mathbb{P}\text{-a.s. } t\geq 0.$$

On the other hand, we have, for every $0<T<\infty$,

$$\mathbb{E}\int_0^T\int_0^t\|\mathcal{A}e^{(t-s)\mathcal{A}}F(s,\mathrm{u}(s))\|^2_{\mathcal{H}}dsdt\leq\mathbb{E}\int_0^T\int_0^t\|F(s,\mathrm{u}(s))\|^2_{\mathcal{D}(\mathcal{A})}dsdt<\infty.$$

It follows that for every $t\in[0,T]$,

$$\int_0^T\int_0^t\|\mathcal{A}e^{(t-s)\mathcal{A}}F(s,\mathrm{u}(s))\|^2_{\mathcal{H}}dsdt<\infty,\ \mathbb{P}\text{-a.s.}$$

Similarly, we also find out that for every $0<t<T<\infty$,

$$\begin{aligned}\mathbb{E}\int_0^T\int_0^t\int_Z&\|\mathcal{A}e^{(t-s)\mathcal{A}}G(s,\mathrm{u}(s),z)\|^2_{\mathcal{H}}\nu(dz)dsdt\\&\leq\mathbb{E}\int_0^T\int_0^t\int_Z\|G(s,\mathrm{u}(s),z)\|^2_{\mathcal{D}(\mathcal{A})}\nu(dz)dsdt<\infty.\end{aligned}$$

Now one can apply the general Fubini's Theorem and the stochastic Fubini's theorem to obtain for every $0<s<t<\infty$

$$\begin{aligned}\int_0^t\int_0^s&\mathcal{A}e^{(s-r)\mathcal{A}}F(r,\mathrm{u}(r))drds\\&=\int_0^t\int_r^t\mathcal{A}e^{(s-r)\mathcal{A}}F(r,\mathrm{u}(r))dsdr\\&=\int_0^t\left(e^{(t-r)\mathcal{A}}-I\right)F(r,\mathrm{u}(r))dr\\&=\int_0^t e^{(t-r)\mathcal{A}}F(r,\mathrm{u}(r))dr-\int_0^t F(r,\mathrm{u}(r))dr,\quad\mathbb{P}\text{-a.s.,}\end{aligned}\tag{4.1.16}$$

and

$$\int_0^t\int_0^s\int_Z \mathcal{A}e^{(s-r)\mathcal{A}}G(r,\mathrm{u}(r-),z)\tilde{N}(dr,dz)ds$$
$$= \int_0^t\int_Z\left(\int_r^t \mathcal{A}e^{(s-r)\mathcal{A}}G(r,\mathrm{u}(r-),z)ds\right)\tilde{N}(dr,dz)$$
$$= \int_0^t\int_Z\left(\int_r^t \mathcal{A}e^{(s-r)\mathcal{A}}ds\right)G(r,\mathrm{u}(r-),z)\tilde{N}(dr,dz)$$
$$= \int_0^t\int_Z\left(e^{(t-r)\mathcal{A}}-I\right)G(r,\mathrm{u}(r-),z)\tilde{N}(dr,dz)$$
$$= \int_0^t\int_Z e^{(t-r)\mathcal{A}}G(r,\mathrm{u}(r-),z)\tilde{N}(dr,dz) - \int_0^t\int_Z G(r,\mathrm{u}(r-),z)\tilde{N}(dr,dz),\quad \mathbb{P}\text{-a.s.,} \tag{4.1.17}$$

where we used the fact that since the semigroup $e^{t\mathcal{A}}$, $t \geq 0$ is strongly continuous, $t \mapsto e^{t\mathcal{A}}x$ is differentiable for every $x \in \mathcal{D}(\mathcal{A})$. From what we have proved in the preceding part, we know that Problem (4.0.5) has a unique mild solution which satisfies

$$\mathrm{u}(t) = e^{t\mathcal{A}}\mathrm{u}_0 + \int_0^t e^{(t-s)\mathcal{A}}F(s,\mathrm{u}(s))ds + \int_0^t\int_Z e^{(t-s)\mathcal{A}}G(s,\mathrm{u}(s-),z)\tilde{N}(ds,dz)\quad \mathbb{P}\text{-a.s.}\quad t \geq 0.$$

Hence first by (4.1.15) we conclude that $\mathcal{A}\mathrm{u}$ is integrable $\mathbb{P}$-a.s. and then by using (4.1.16) and (4.1.17) we obtain

$$\int_0^t \mathcal{A}\mathrm{u}(s)ds = \int_0^t \mathcal{A}e^{t\mathcal{A}}\mathrm{u}_0 + \int_0^t\int_0^s \mathcal{A}e^{(s-r)\mathcal{A}}F(r,\mathrm{u}(r))dr + \int_0^t\int_0^s\int_Z e^{(s-r)\mathcal{A}}G(r,\mathrm{u}(r-),z)\tilde{N}(dr,dz)$$
$$= e^{t\mathcal{A}}\mathrm{u}_0 - \mathrm{u}_0 + \int_0^t e^{(t-r)\mathcal{A}}F(r,\mathrm{u}(r))dr - \int_0^t F(r,\mathrm{u}(r))dr$$
$$+ \int_0^t\int_Z e^{(t-r)\mathcal{A}}G(r,\mathrm{u}(r),z)\tilde{N}(dr,dz) - \int_0^t\int_Z G(r,\mathrm{u}(r-),z)\tilde{N}(dr,dz)$$
$$= \mathrm{u}(t) - \mathrm{u}_0 - \int_0^t F(r,\mathrm{u}(r))dr - \int_0^t\int_Z G(r,\mathrm{u}(r-),z)\tilde{N}(dr,dz)$$

which shows that the mild solution is also a strong solution.

Conversely, let u be a strong solution. By making use of the Itô formula (3.5.6) to the function $\psi(s,y) = e^{(t-s)\mathcal{A}}y$ and process $\mathrm{u}_\lambda(s) = R(\lambda,\mathcal{A})\mathrm{u}(s)$, where $R(\lambda,\mathcal{A})$ is the resolvent of $\mathcal{A}$, we infer for every $t \geq 0$

$$e^{(t-s)\mathcal{A}}R(\lambda,\mathcal{A})\mathrm{u}(s) - R(\lambda,\mathcal{A})\mathrm{u}_0 = -\int_0^t e^{(t-s)\mathcal{A}}\mathcal{A}R(\lambda,\mathcal{A})\mathrm{u}(s)ds + \int_0^t e^{(t-s)\mathcal{A}}R(\lambda,\mathcal{A})\mathcal{A}\mathrm{u}(s)ds$$
$$+ \int_0^t e^{(t-s)\mathcal{A}}R(\lambda,\mathcal{A})F(s,\mathrm{u}(s))ds$$
$$+ \int_0^t\int_Z e^{(t-s)\mathcal{A}}R(\lambda,\mathcal{A})G(s,\mathrm{u}(s-),z)\tilde{N}(ds,dz),\ \mathbb{P}\text{-a.s.}$$

It follows that for every $t \geq 0$

$$R(\lambda,\mathcal{A})e^{(t-s)\mathcal{A}}\mathrm{u}(s) = R(\lambda,\mathcal{A})\Big(\mathrm{u}_0 + \int_0^t e^{(t-s)\mathcal{A}}F(s,\mathrm{u}(s))ds$$
$$+ \int_0^t\int_Z e^{(t-s)\mathcal{A}}G(s,\mathrm{u}(s-),z)\tilde{N}(ds,dz)\Big)\ \mathbb{P}\text{-a.s.}$$

Hence we have for every $t \geq 0$,

$$u(t) = e^{t\mathcal{A}}u_0 + \int_0^t e^{t-s\mathcal{A}}F(s, u(s))ds + \int_0^t \int_Z e^{t-s\mathcal{A}}G(s, u(s-), z)\tilde{N}(ds, dz), \quad \mathbb{P}\text{-a.s..}$$

Thus, we infer that u is of the form (4.1.1). Furthermore, the stochastic equivalence becomes $\mathbb{P}$-equivalence in view of the càdlàg property of the strong solution and the mild solution. Therefore, mild solution and strong solution are $\mathbb{P}$-equivalent or in other word, uniqueness of strong solution holds.

□

Remark 4.1.12. The following equation is a special form of Equation (4.0.5),

$$\begin{aligned} du &= \mathcal{A}udt + F(t)dt + \int_Z G(t, z)\tilde{N}(dt, dz) \\ u(0) &= u_0, \end{aligned} \tag{4.1.18}$$

and hence in view of Theorem 4.1.10 it has a unique mild solution. Therefore, u satisfies

$$u(t) = e^{t\mathcal{A}}u_0 + \int_0^t e^{(t-s)\mathcal{A}}F(s)ds + \int_0^t \int_Z e^{(t-s)}G(s, z)\tilde{N}(ds, dz) \ \mathbb{P}\text{-a.s.} \ \ t \geq 0.$$

In such a case, if $u_0 \in \mathcal{D}(\mathcal{A})$, $F(s) \in \mathcal{D}(\mathcal{A})$ and $G(s, z) \in \mathcal{D}(\mathcal{A})$, for every $s \geq 0$ and $z \in Z$, Equation (4.1.18) has a unique strong solution which is stochastically equivalent to a mild solution. Hence by Definition 4.1.1 u satisfies satisfies

$$u(t) = u_0 + \int_0^t \mathcal{A}u(s)ds + \int_0^t F(s)ds + \int_0^t \int_Z G(s, z)\tilde{N}(ds, dz) \ \ \mathbb{P}\text{-a.s.} \ \ t \geq 0.$$

4.2 Local Mild Solutions

Now we turn to consider the case where f is locally Lipschitz continuous, g is globally Lipschitz continuous and f, g satisfy the Assumption 4.1.7.

Assumption 4.2.1. *Assume that for every $R > 0$, there exists $L_R > 0$ such that for all $t \geq 0$ and for every $x = (x_1, x_2)^\top$, $y = (y_1, y_2)^\top \in \mathcal{H}$ satisfying $\|x\|_{\mathcal{H}}, \|y\|_{\mathcal{H}} \leq R$,*

$$\|f(t, x_1, x_2) - f(t, y_1, y_2)\|_H \leq L_R\|x - y\|_{\mathcal{H}}. \tag{4.2.1}$$

Now we shall examine stochastic equation (4.0.5) of a more general type than the equation with F defined by (4.1.5) in the preceding Theorem. Note that the function $\mathcal{H} \ni x = (x_1, x_2) \mapsto m(\|B^{\frac{1}{2}}x_1\|^2)Bx_1 \in H$, is locally Lipschitz continuous. Hence if we suppose that f satisfies Assumption (4.2.1), then the function F given by (4.0.3) satisfies the locally Lipschitz condition as well.

Remark 4.2.2. Note that since $m \in C^1(\mathbb{R}_+)$, the function $x \to m(\|B^{\frac{1}{2}}x_1\|^2)$ is locally Lipschitz continuous, hence $m(\|B^{\frac{1}{2}}x_1\|^2)Bx_1$ is also locally Lipschitz continuous.

For future reference we specifically state the following important observations.

Remark 4.2.3. (1) Because of the continuity of the function $F(t,x)$ in x and the integration of F is defined with respect to dt, the equation (4.0.5) can be rewritten in the following equivalent form

$$du = \mathcal{A}udt + F(t,u(t-))dt + \int_Z G(t,u(t-),z)\tilde{N}(dt,dz), \quad t \geq 0.$$

(2) Suppose that X and Y are two càdlàg processes and τ is a stopping time. If X and Y coincide on the open interval $[0,\tau)$, i.e.

$$X(s,\omega)1_{[0,\tau)}(s) = Y(s,\omega)1_{[0,\tau)}(s), \ (s,\omega) \in \mathbb{R}_+ \times \Omega,$$

then we have

$$G(s,X(s-),z)1_{[0,\tau]} = G(s,Y(s-),z)1_{[0,\tau]}.$$

This is because, the function $G(s,X(s-),z)$ depends only on the values of X on $[0,\tau)$. However, if $G(t,\omega,x,z)$ itself is a stochastic process rather than a deterministic function, the above fact may no longer hold.

Definition 4.2.4. A stopping time τ is called accessible if there exists an increasing sequence $\{\tau_n\}_{n\in\mathbb{N}}$ of stopping times such that $\tau_n < \tau$ and $\lim_{n\to\infty}\tau_n = \tau$ a.s. We call such sequence $\{\tau_n\}_{n\in\mathbb{N}}$ the approximating sequence for τ. A local mild solution to (4.0.5) is an $\mathcal{H}$-valued, predictable, càdlàg local process $X = (X(t))_{0\leq t<\tau}$, where τ is an accessible stopping time with an approximating sequence $\{\tau_n\}_{n\in\mathbb{N}}$ such that for any $n \in \mathbb{N}$ and $t > 0$, the stopped process $X_t^{\tau_n} := X(t\wedge\tau_n)$, $t \geq 0$ satisfies,

$$X(t\wedge\tau_n) = e^{t\mathcal{A}}u_0 + \int_0^{t\wedge\tau_n} e^{(t\wedge\tau_n - s)\mathcal{A}}F(s,X(s))ds + I_{\tau_n}(G(X))(t\wedge\tau_n) \ \ t \geq 0, \ \mathbb{P}\text{-a.s.}, \qquad (4.2.2)$$

where $I_{\tau_n}(G(X))$ is a process defined by

$$I_{\tau_n}(G(X))(t) = \int_0^t \int_Z 1_{[0,\tau_n]}(s)e^{(t-s)\mathcal{A}}G(s,X(s-),z)\tilde{N}(ds,dz), \ t \geq 0.$$

Here we call τ a life span of the local mild solution X. A local mild solution $X = (X(t))_{0\leq t\leq\tau}$ to equation (4.0.5) is pathwise unique if for any other local mild solution $\tilde{X} = \{\tilde{X}_{0\leq t<\tilde{\tau}}\}$ to equation (4.0.5),

$$X(t,\omega) = \tilde{X}(t,\omega), \quad (t,\omega) \in [0,\tau\wedge\tilde{\tau}) \times \Omega.$$

A local mild solution $X = (X(t))_{0\leq t<\tau}$ is called a maximal mild solution if for any other local mild solution $\tilde{X} = (\tilde{X}(t))_{0\leq t<\tilde{\tau}}$ satisfying $\tilde{\tau} \geq \tau$ a.s. and $\tilde{X}|_{[0,\tau)\times\Omega} \sim X$, then $\tilde{X} = X$. Furthermore, if $\mathbb{P}(\tau < \infty) > 0$, the stopping time τ is called an **explosion time** and if $\mathbb{P}(\tau = +\infty) = 1$, the local mild solution X have no explosion and it is called a global mild solution to Equation (4.0.5).

Remark 4.2.5. (1) There is an alternative way to define a local mild solution. We say that an $\mathcal{H}$-valued càdlàg process X defined on an open interval $[0,\tau)$ is a local mild solution if there exists an increasing sequence $\{\tau_n\}$ of stopping times such that $\tau_n \nearrow \tau$, or in other words $[0,\tau) = \cup_n[0,\tau_n]$, and X is a mild solution to problem (4.0.5) on every closed interval $[0,\tau_n]$, $n \in \mathbb{N}$ (see Remark 4.1.5).

(2) If Equation (4.0.5) has the property of uniqueness for local solutions, then the uniqueness of local maximal solution holds as well.

Lemma 4.2.6. *If a function $h : \mathcal{H} \to \mathcal{H}$ is locally Lipschitz on a closed ball $B(0,R) \subset \mathcal{H}$, then the function $\tilde{h} : \mathcal{H} \to \mathcal{H}$ defined by*

$$\tilde{h}(x) := \begin{cases} h(x), & \text{if } \|x\|_{\mathcal{H}} \leq R, \\ h(\frac{Rx}{\|x\|_{\mathcal{H}}}), & \text{otherwise.} \end{cases}$$

is globally Lipschitz.

Proof. Since h is locally Lipschitz, we assume that there is a constant K such that for all $\|x\|_{\mathcal{H}}, \|y\|_{\mathcal{H}} \leq R$,

$$\|h(x) - h(y)\|^2_{\mathcal{H}} \leq K\|x - y\|^2_{\mathcal{H}}.$$

Let's consider the function $\tilde{h}$ in three cases.
If $\|x\|_{\mathcal{H}}, \|y\| \leq R$, then by the definition of $\tilde{b}$ and local Lipschitz property of h we find out that

$$\|\tilde{h}(x) - \tilde{h}(y)\|^2_{\mathcal{H}} = \|h(x) - h(y)\|^2_{\mathcal{H}} \leq K\|x - y\|^2_{\mathcal{H}}$$

If $\|x\|_{\mathcal{H}} \leq R$ and $\|y\|_{\mathcal{H}} > R$, then

$$\inf_{z \in B(0,R)} \|z - y\|_{\mathcal{H}} = \left\| \frac{Ry}{\|y\|_{\mathcal{H}}} - y \right\|_{\mathcal{H}},$$

so we we deduce that

$$\begin{aligned} \|\tilde{h}(x) - \tilde{h}(y)\|^2_{\mathcal{H}} &= \left\| h(x) - h(\frac{Ry}{|y|}) \right\|^2_{\mathcal{H}} \leq K \left\| x - \frac{Ry}{\|y\|_{\mathcal{H}}} \right\|^2_{\mathcal{H}} \\ &\leq 2K\|x - y\|^2_{\mathcal{H}} + 2K \left\| y - \frac{Ry}{\|y\|_{\mathcal{H}}} \right\|^2_{\mathcal{H}} \\ &\leq 2K\|x - y\|^2_{\mathcal{H}} + 2K\|x - y\|^2_{\mathcal{H}} = 4K\|x - y\|^2_{\mathcal{H}}. \end{aligned}$$

Similarly, for all $\|x\|_{\mathcal{H}} > R$ and $\|y\|_{\mathcal{H}} \leq R$, we have

$$\|\tilde{h}(x) - \tilde{h}(y)\|^2_{\mathcal{H}} \leq 4K\|x - y\|^2_{\mathcal{H}}.$$

If $\|x\|_{\mathcal{H}}, \|y\|_{\mathcal{H}} > R$, then

$$\begin{aligned} \|\tilde{h}(x) - \tilde{h}(y)\|^2_{\mathcal{H}} &= \left\| h(\frac{Rx}{\|x\|_{\mathcal{H}}}) - h(\frac{Ry}{\|y\|_{\mathcal{H}}}) \right\|^2_{\mathcal{H}} \leq K \left\| \frac{Rx}{\|x\|_{\mathcal{H}}} - \frac{Ry}{\|y\|_{\mathcal{H}}} \right\|^2_{\mathcal{H}} \\ &= KR^2 \left| \frac{x}{\|x\|_{\mathcal{H}}} - \frac{y}{\|y\|_{\mathcal{H}}} \right|^2 \leq KR^2\|x - y\|^2_{\mathcal{H}}. \end{aligned}$$

Take $K' = \max\{4K, KR^2\}$. Therefore,

$$\|\tilde{h}(x) - \tilde{h}(y)\|_{\mathcal{H}} \leq K'\|x - y\|^2_{\mathcal{H}} \quad \forall x, y \in \mathcal{H},$$

as required. □

Theorem 4.2.7. *Suppose that Assumptions 4.1.7, 4.1.9 and 4.2.1 are satisfied. Then there exists a unique maximal local mild solution to Equation* (4.0.5).

Proof of Theorem 4.2.7. Set $\tilde{f}(x) = -f(t,\omega,x_1,x_2) - m(\|B^{\frac{1}{2}}x_1\|^2)Bx_1$. Since $\tilde{f}$ is locally Lipschitz continuous, for every $n \in \mathbb{N}$ we may define the following mapping

$$\tilde{f}_n(x) = \begin{cases} \tilde{f}(x) & \text{if } \|x\|_{\mathcal{H}} \le n \\ \tilde{f}(\frac{nx}{\|x\|_{\mathcal{H}}}) & \text{if } \|x\|_{\mathcal{H}} > n, \end{cases}$$

where $x \in \mathcal{H}$. Then $\tilde{f}_n$ is globally Lipschitz continuous by Lemma 4.2.6. Set $F_n(x) = \left(0, \tilde{f}_n(x)\right)^{\mathrm{T}}$ for every $x \in \mathcal{H}$. Therefore, by Theorem 4.1.10 for every $n \in \mathbb{N}$ there exists a unique mild solution $(X_n(t))_{t\ge 0}$ to Problem (4.0.5) with F substituted for F_n which is given by

$$X_n(t) = e^{t\mathcal{A}}u_0 + \int_0^t e^{(t-s)\mathcal{A}}F_n(s,X_n(s))ds + \int_0^t\int_Z e^{(t-s)\mathcal{A}}G(s,X_n(s),z)\tilde{N}(ds,dz),\ t \ge 0. \quad (4.2.3)$$

Define a sequence of stopping times $\{\tau_n\}_{n=1}^{\infty}$ by

$$\tau_n := \inf\{t \ge 0 :\ \|X_n(t)\|_{\mathcal{H}} > n\}.$$

By the càdlàg property of the solution X_n, we know, in view of the Propersition 2.2.3, that τ_n is indeed a stopping time. First let us note that for every $n < m$, we have $F_n(x) = F_m(x) = F(x)$ for all $\|x\|_{\mathcal{H}} \le n$. Since $\|X_n(t)\|_{\mathcal{H}} \le n$ for all $t < \tau_n$, so by (4.2.3) we obtain

$$\begin{aligned} X_n(t) &= e^{t\mathcal{A}}u_0 + \int_0^t e^{(t-s)\mathcal{A}}F_n(s,X_n(s))ds + \int_0^t\int_Z e^{(t-s)\mathcal{A}}G(s,X_n(s-),z)\tilde{N}(ds,dz) \\ &= e^{t\mathcal{A}}u_0 + \int_0^t e^{(t-s)\mathcal{A}}F(s,X_n(s))ds + \int_0^t\int_Z e^{(t-s)\mathcal{A}}G(s,X_n(s-),z)\tilde{N}(ds,dz),\ t \in [0,\tau_n). \end{aligned} \quad (4.2.4)$$

Set

$$\Phi(X_n) := e^{t\mathcal{A}}u_0 + \int_0^t e^{(t-s)\mathcal{A}}F(s,X_n(s))ds + \int_0^t\int_Z e^{(t-s)\mathcal{A}}G(s,X_n(s-),z)\tilde{N}(ds,dz).$$

Note that

$$\Delta\Phi(X_n)(\tau_n) = \int_Z G(\tau_n, X_n(\tau_n-),z)\tilde{N}(\{\tau_n\},dz).$$

which means that the value of $\Phi(X_n)$ at τ_n depends only on the values of X_n on $[0,\tau_n)$. Hence we may extend the solution X_n on $[0,\tau_n)$ to X_n on $[0,\tau_n]$ by setting (see Appendix)

$$X_n(\tau_n) = \Phi(X_n)(\tau_n) = e^{\tau_n\mathcal{A}}u_0 + \int_0^{\tau_n} e^{(\tau_n-s)\mathcal{A}}F(s,X_n(s))ds + I_{\tau_n}(G(X_n))(\tau_n) \quad (4.2.5)$$

where

$$I_{\tau_n}(G(X_n))(t) = \int_0^t\int_Z 1_{[0,\tau_n]}e^{(t-s)\mathcal{A}}G(s,X_n(s-),z)\tilde{N}(ds,dz),\ t \ge 0.$$

In such a case, combining (4.2.4) together with (4.2.5), we deduce that the stopped process $X(\cdot\wedge\tau_n)$ satisfies

$$X_n(t\wedge\tau_n) = e^{(t\wedge\tau_n)\mathcal{A}}u_0 + \int_0^{t\wedge\tau_n} e^{(t\wedge\tau_n-s)\mathcal{A}}F(s,X_n(s))ds + I_{\tau_n}(G(X_n))(t\wedge\tau_n),\ t \in \mathbb{R}_+ \quad (4.2.6)$$

In a similar way, we have

$$X_m(t\wedge\tau_m) = e^{(t\wedge\tau_n)\mathcal{A}}u_0 + \int_0^{t\wedge\tau_n} e^{(t\wedge\tau_m - s)\mathcal{A}}F(s, X_m(s))ds + I_{\tau_n}(G(X_m))(t\wedge\tau_n),\ \ t\in\mathbb{R}_+$$

Set $\tau_{n,m} = \tau_n\wedge\tau_m$. It follows that $\|X_n(t)\| \le n < m$ and $\|X_m(t)\| \le m$ for $t\in[0,\tau_{n,m})$. So $F_n(s, X_n(s)) = F(s, X_n(s))$ and $F_m(s, X_m(s)) = F(s, X_m(s))$. Therefore, X_n and X_m both satisfy the same Equation

$$X(t) = e^{t\mathcal{A}}u_0 + \int_0^t e^{(t-s)\mathcal{A}}F(s, X(s))ds + \int_0^t\int_Z e^{(t-s)\mathcal{A}}G(s, X(s-), z)\tilde{N}(ds, dz),\ \text{ on } [0,\tau_{n,m}).$$

Hence by the uniqueness of mild solution proved in the theorem 4.1.10, we have

$$X_n(t) = X_m(t),\ \ \text{on } [0,\tau_{n,m})\ a.s.$$

Since

$$\triangle X_n(\tau_{n,m}) = \int_Z G(\tau_{n,m}, X_n(\tau_{n,m}-), z)\tilde{N}(\{\tau_n\}, dz),$$

and the Remark 4.2.3 tells us that $G(s, X_n, z)$ and $G(s, X_m, z)$ coincide on $[0,\tau_{n,m}]$, we infer

$$X_n = X_m \text{ on } [0,\tau_{n,m}].$$

It follows that

$$\tau_n \le \tau_m \text{ if } n < m.$$

We will show this assertion by contradiction. Let us fix $n < m$. Suppose that $\mathbb{P}(\tau_n > \tau_m) > 0$. Set $A = \{\tau_n > \tau_m\}$. By the definition of the stopping time τ_n, we have $\|X_n(t)\|_{\mathcal{H}} \le n$ for $t\in[0,\tau_n)$ and $\|X_m(\tau_m)\|_{\mathcal{H}} \ge m > n$. Since X_n coincides with X_m on $[0,\tau_{n,m}]$, we find $\|X_n(\tau_m)\|_{\mathcal{H}} = \|X_m(\tau_m)\|_{\mathcal{H}} > n$ on A which would contradict the fact that $\|X_n(t)\|_{\mathcal{H}} \le n$ for $t\in[0,\tau_n)$. Therefore, we conclude that $\tau_n \le \tau_m$ a.s. for $n < m$. This means that $\{\tau_n\}_{n=1}^\infty$ is an increasing sequence. So the limit $\lim_{n\to\infty}\tau_n$ exists a.s. Let us denote this limit by τ_∞. Let $\Omega_0 = \{\omega : \lim_{n\to\infty}\tau_n = \tau_\infty\}$. Note that $\mathbb{P}(\Omega_0) = 1$.

Now define a local process $(X_t)_{0\le t<\tau_\infty}$ as follows. If $\omega\notin\Omega_0$, set $X(t,\omega) = 0$ for all $0\le t<\tau_\infty$. If $\omega\in\Omega_0$, then there exists a number $n\in\mathbb{N}$ such that $t\le\tau_n(\omega)$ and we set $X(t,\omega) = X_n(t,\omega)$. The process is well defined since $X_n(t)$ exists uniquely on $\{t\le\tau_n\}$. Indeed, for every $t\in\mathbb{R}_+$ by (4.2.6) we have

$$X_n(t\wedge\tau_n) = e^{(t\wedge\tau_n)\mathcal{A}}u_0 + \int_0^{t\wedge\tau_n} e^{(t\wedge\tau_n - s)\mathcal{A}}F(s, X_n(s))ds + I_{\tau_n}(G(X_n))(t\wedge\tau_n)$$

Since $X(t) = X_n(t)$ for $t\le\tau_n$, we infer that

$$X(t\wedge\tau_n) = e^{(t\wedge\tau_n)\mathcal{A}}u_0 + \int_0^{t\wedge\tau_n} e^{(t\wedge\tau_n - s)\mathcal{A}}F(s, X(s))ds + I_{\tau_n}(G(X))(t\wedge\tau_n)$$

where we used the fact that for all $t\ge 0$,

$$\begin{aligned}
I_{\tau_n}(G(X_n))(t) &= \int_0^t\int_Z 1_{[0,\tau_n]}(s)e^{(t-s)\mathcal{A}}G(s, X_n(s-), z)\tilde{N}(ds, dz)\\
&= \int_0^t\int_Z 1_{[0,\tau_n]}(s)e^{(t-s)\mathcal{A}}G(s\wedge\tau_n, X_n(s\wedge\tau_n-), z)\tilde{N}(ds, dz)\\
&= \int_0^t\int_Z 1_{[0,\tau_n]}(s)e^{(t-s)\mathcal{A}}G(s\wedge\tau_n, X(s\wedge\tau_n-), z)\tilde{N}(ds, dz)\\
&= I_{\tau_n}(G(X))(t).
\end{aligned}$$

Furthermore, by the definition of the sequence $\{\tau_n\}_{n=1}^{\infty}$ we obtain

$$\lim_{t\nearrow\tau_\infty(\omega)} \|X(t,\omega)\|_{\mathcal{H}} = \lim_n \|X(\tau_n(\omega),\omega)\|_{\mathcal{H}} \geq \lim_n n = \infty \text{ a.s..} \tag{4.2.7}$$

To show that the process $X(t)$, $0 \leq t < \tau_\infty$ is a maximal local mild solution to Problem (4.0.5). Let us suppose that $\tilde{X} = (\tilde{X}(t))_{0\leq t<\tilde{\tau}}$ is another local mild solution to Problem (4.0.5) such that $\tilde{\tau} \geq \tau_\infty$ a.s. and $\tilde{X}|_{[0,\tau_\infty)\times\Omega} \sim X$. It follows from (4.2.7) and the $\mathbb{P}$-equivalence of X and $\tilde{X}$ on $[0,\tau_\infty)$ that

$$\lim_{t\nearrow\tau_\infty(\omega)} \|\tilde{X}(t,\omega)\|_{\mathcal{H}} = \lim_{t\nearrow\tau_\infty(\omega)} \|X(t,\omega)\|_{\mathcal{H}} = \infty. \tag{4.2.8}$$

In order to get the maximality of X, we need to show that $\mathbb{P}(\tilde{\tau} > \tau_\infty) = 0$. To prove this, assume the contrary, namely $\mathbb{P}(\tilde{\tau} > \tau_\infty) > 0$. Since $\tilde{X}$ is a local mild solution, there exists a sequence $\{\tilde{\tau}_n\}$ of increasing stopping times such that $\tilde{X}$ is a mild solution on the interval $[0,\tilde{\tau}_n]$, i.e. the equation (4.2.2) is satisfied. Define a new family of stopping times by

$$\sigma_{n,k} := \tilde{\tau}_n \wedge \inf\{t : \|\tilde{X}(t)\| > k\}; \sigma_k := \sup \sigma_{n,k}.$$

Since $\sigma_{n,k} \leq \tilde{\tau}_n$, $\sigma_k \leq \tilde{\tau}_n$. Also, observe that $\lim_k \sigma_k = \tilde{\tau}$. This is because

$$\lim_k \sigma_k = \lim_k \sup_n \sigma_{n,k} = \sup_n \lim_k \sigma_{n,k} = \sup_n \tilde{\tau}_n = \tilde{\tau}.$$

Therefore, since $\sigma_k \nearrow \tilde{\tau}$ and $\mathbb{P}(\tilde{\tau} > \tau_\infty) > 0$, there exists a number k such that $\mathbb{P}(\sigma_k > \tau_\infty) > 0$. Hence, we have $\|\tilde{X}(t,\omega)\|_{\mathcal{H}} \leq k$ for $t \in [\tau_\infty(\omega), \sigma_k(\omega))$ contradicting the earlier observation (4.2.8).

Now we continue to show the uniqueness of the solution. Actually, the uniqueness of the solution has already shown in above construction of solution X. Alternatively, we may prove it in another way. Let X and Y be two mild solution to Problem (4.0.5) on the stochastic intervals $[0,\tau]$ and $[0,\sigma]$. respectively. We shall show that $X = Y$, $\mathbb{P}$-a.s. on $[0,\tau\wedge\sigma]$.
For each $n \in \mathbb{N}$, define

$$\sigma_n = \inf\{t \geq 0 : \|Y_n(t)\|_{\mathcal{H}} > n \text{ or } \|X(t)\|_{\mathcal{H}} > n\} \wedge \tau \wedge \sigma \wedge n.$$

Then $\|Y(t)\|_{\mathcal{H}} \leq n$ and $\|X(t)\|_{\mathcal{H}} \leq n$ on $[0,\sigma_n)$. Further, we find out that $\lim_{n\to\infty} \mathbb{P}(\sigma_n < \sigma\wedge\tau) = 0$. Hence we only need to verify that $X = Y$ on $[0,\sigma_n]$, $\mathbb{P}$-a.s. Since $X(t)$, $t \in [0,\tau]$ and $Y(t)$, $t \in [0,\sigma]$ are both mild solutions to Problem (4.0.5), we infer for that

$$X(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}F(s,X(s))ds + \int_0^t \int_Z e^{(t-s)A}G(s,X(s-),z)\tilde{N}(ds,dz) \text{ } \mathbb{P}\text{-a.s., on } [0,\sigma_n)$$
$$Y(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}F(s,Y(s))ds + \int_0^t \int_Z e^{(t-s)A}G(s,Y(s-),z)\tilde{N}(ds,dz) \text{ } \mathbb{P}\text{-a.s. on } [0,\sigma_n).$$

Therefore, by using the Cauchy-Schwarz and Burkholder-Davis inequalities (see Corollary 3.7.12) that

$$\mathbb{E}\left(\sup_{0\le s<\sigma_n}\left\|X(s)-Y(s)\right\|_{\mathcal{H}}^2\right)$$
$$\le 2\mathbb{E}\left(\sup_{0\le s<\sigma_n}\left\|\int_0^s e^{(s-r)\mathcal{A}}\Big(F(X(r))-F(Y(r))\Big)dr\right\|_{\mathcal{H}}^2\right)$$
$$+2\mathbb{E}\left(\sup_{0\le s<\sigma_n}\left\|\int_0^s\int_Z e^{(s-r)\mathcal{A}}\Big(G(r,X(r-),z)-G(r,Y(r-),z)\Big)\tilde{N}(dr,dz)\right\|_{\mathcal{H}}^2\right)$$
$$\le 2nL_n^2\mathbb{E}\int_0^{\sigma_n}\|X(s)-Y(s)\|_{\mathcal{H}}^2 ds$$
$$+2CL_g\mathbb{E}\int_0^{\sigma_n}\|X(s)-Y(s)\|_{\mathcal{H}}^2 ds$$
$$\le C(n)\mathbb{E}\int_0^{t\wedge\sigma_n}\sup_{0\le u\le\sigma_n}\|X(u)-Y(u)\|_{\mathcal{H}}^2 ds,$$

where $C(n)=2(nL_n+CL_g)$. By applying the Gronwall Lemma, we obtain for every $t\ge 0$,

$$\mathbb{E}\left(\sup_{0\le s<\sigma_n}\left\|X(s)-Y(s)\right\|_{\mathcal{H}}^2\right)=0.$$

This implies that for every $X|_{[0,\sigma_n)}$ and $Y|_{[0,\sigma_n}$ are indistinguishable. By Remark 4.2.3, we infer that $X=Y$ on $[0,\sigma_n]$ $\mathbb{P}$-a.s. □

4.3 Existence of Global Solutions

Suppose that Assumptions 4.1.7 and 4.2.1 are satisfied. By Theorem 4.2.7, there exists a unique maximal local mild solution to Equation (4.0.5) given by

$$\mathrm{u}(t\wedge\tau_n)=e^{t\mathcal{A}}\mathrm{u}_0+\int_0^{t\wedge\tau_n}e^{(t\wedge\tau_n-s)\mathcal{A}}F(s,\mathrm{u}(s))ds+I_{\tau_n}(G(X))(t\wedge\tau_n)\ \text{ a.s.,}\tag{4.3.1}$$

where $\tau_n=\inf\{t\ge 0:\|\mathrm{u}(t)\|_{\mathcal{H}}>n\}$, $\lim_{n\to\infty}\tau_n=\tau_\infty$ and

$$I_{\tau_n}(G(X))(t)=\int_0^{t+}\int_Z e^{(t-s)\mathcal{A}}G(s,\mathrm{u}(s-),z)\tilde{N}(ds,dz).$$

We call τ_∞ the explosion time of (4.3.1). Now we shall apply Khas'minski's test to show that $\tau_\infty=+\infty$ a.s. That is u is a unique global mild solution.

Lemma 4.3.1. *(Khas'minskii's test for nonexplosions) Let* $\mathrm{u}(t)$, $0\le t<\tau_\infty$ *be a maximal local mild solution to Equation* (4.0.5) *with an approximating sequence* $\{\tau_n\}_{n\in\mathbb{N}}$. *Suppose that there exists a function* $V:\mathcal{H}\to\mathbb{R}$ *such that*

1. $V\ge 0$ *on* $\mathcal{H}$,
2. $q_R=\inf_{\|x\|_{\mathcal{H}}\ge R}V(x)\to+\infty$

3. $\mathbb{E}V(\mathrm{u}(t\wedge\tau_n)) \le \mathbb{E}V(\mathrm{u}_0) + C\int_0^t \Big(1+\mathbb{E}(V(\mathrm{u}(s\wedge\tau_n)))\Big)ds$ *for each* $n\in\mathbb{N}$,

4. $\mathbb{E}V(\mathrm{u}_0) < \infty$.

Then $\tau_\infty = +\infty$ $\mathbb{P}$*-a.s. We call* V *a Lyapunov function for* (4.0.5).

Proof. Since by the assumptions

$$\mathbb{E}V(\mathrm{u}(t\wedge\tau_n)) \le \mathbb{E}V(\mathrm{u}_0) + C\int_0^t \Big(1+\mathbb{E}(V(\mathrm{u}(s\wedge\tau_n)))\Big)ds,\ t\ge 0,$$

we infer that

$$1+\mathbb{E}V(\mathrm{u}(t\wedge\tau_n)) \le 1+\mathbb{E}V(\mathrm{u}_0 + C\int_0^t \Big(1+\mathbb{E}(V(\mathrm{u}(s\wedge\tau_n)))\Big)ds, t\ge 0.$$

Hence, by applying the Gronwall Lemma, we obtain

$$1+\mathbb{E}V(\mathrm{u}(t\wedge\tau_n)) \le \Big(1+\mathbb{E}V(\mathrm{u}_0)\Big)e^{Ct}.$$

Thus we have for each $n\in\mathbb{N}$ that

$$\mathbb{E}V(\mathrm{u}(t\wedge\tau_n)) \le \Big(1+\mathbb{E}V(\mathrm{u}_0)\Big)e^{Ct}-1,\quad t\ge 0.$$

It follows that for each $n\in\mathbb{N}$,

$$\begin{aligned}\mathbb{P}(\{\tau_n<t\}) &= \mathbb{E}1_{\{\tau_n<t\}} = \int_\Omega 1_{\{\tau_n<t\}}d\mathbb{P} = \int_\Omega \frac{q_n}{q_n}1_{\{\tau_n<t\}}d\mathbb{P} \le \frac{1}{q_n}\int_\Omega V(\mathrm{u}(t\wedge\tau_n))1_{\{\tau_n<t\}}d\mathbb{P}\\ &\le \frac{1}{q_n}\int_\Omega V(\mathrm{u}(t\wedge\tau_n))d\mathbb{P} = \frac{1}{q_n}\mathbb{E}V(\mathrm{u}(t\wedge\tau_n)) \le \frac{1}{q_n}\left[\Big(1+\mathbb{E}V(\mathrm{u}_0)\Big)e^{Ct}-1\right].\end{aligned}$$

Since $\mathbb{E}V(\mathrm{u}_0)<\infty$ and $q_n\to\infty$ as $n\to\infty$, so $\lim_{n\to\infty}\mathbb{P}(\{\tau_n<t\})=0$. Since the sequence τ_n is increasing, the sets $\{\{\tau_n<t\}\}_n$ are decreasing. Thus we infer that for every $t\ge 0$,

$$\mathbb{P}(\{\tau_\infty<t\}) = \mathbb{P}(\{\lim_{n\to\infty}\tau_n<t\}) = \mathbb{P}\left(\bigcap_{n\in\mathbf{N}}\{\tau_n<t\}\right) = \lim_{n\to\infty}\mathbb{P}(\{\tau_n<t\}) = 0.$$

Hence $\tau_\infty=+\infty$, $\mathbb{P}$-a.s. □

Theorem 4.3.2. *Suppose that Assumptions 4.1.7 and 4.2.1 are satisfied and* u_0 *is* $\mathcal{F}_0$*-measurable. Let* u *be the unique maximal local mild solution to Equation* (4.0.5) *with life span* τ_∞*. Then* $\tau_\infty = +\infty$ $\mathbb{P}$*-a.s.*

Proof of Theorem 4.3.2. Let $\mathrm{u}(t)$, $0\le t<\tau_\infty$ be a maximal local mild solution to problem (4.0.5). Define a sequence of stopping times by

$$\tau_n = \inf\{t\ge 0: \|\mathrm{u}(t)\|_{\mathcal{H}}\ge n\},\quad n\in\mathbb{N}.$$

Then in the proof of Theorem 4.2.7, we showed that $\{\tau_n\}_{n\in\mathbf{N}}$ is an approximating sequence of the accessible stopping time τ_∞. In order to apply Khas'minskii's test, we need to find a Lyapunov function. Define a function $V:\mathcal{H}\to\mathbb{R}^+$ by

$$V(x) = \frac{1}{2}\|x\|^2_{\mathcal{H}} + \frac{1}{2}M(\|B^{\frac{1}{2}}x_1\|^2_H),$$

where $x = (x_1, x_2)^\top \in \mathcal{H}$ and $M(s) = \int_0^s m(r)dr$, $s \geq 0$. It is clear that for every $x \in \mathcal{H}$,

$$V(x) \geq 0.$$

Observe that

$$\begin{aligned} q_R = \inf_{\|x\|_{\mathcal{H}} \geq R} V(x) &= \frac{1}{2} \inf_{\|x\|_{\mathcal{H}} \geq R} \|x\|^2_{\mathcal{H}} + \frac{1}{2} \inf_{\|x\|_{\mathcal{H}} \geq R} M(\|B^{\frac{1}{2}} x_1\|^2) \\ &= \frac{1}{2} R^2 + \frac{1}{2} \inf_{\|x\|_{\mathcal{H}} \geq R} M(\|B^{\frac{1}{2}} x_1\|^2) \\ &= \frac{1}{2} R^2 + \frac{1}{2} \inf_{\|x\|_{\mathcal{H}} \geq R} \int_0^{\|B^{\frac{1}{2}} x_1\|^2} m(r)dr. \end{aligned}$$

Taking the limit in this equality as $R \to \infty$, we obtain that $q_R \to +\infty$. Meanwhile, we have

$$\mathbb{E}(V(\mathrm{u}_0)) = \frac{1}{2}\mathbb{E}\|\mathrm{u}_0\|^2_{\mathcal{H}} + \frac{1}{2}\mathbb{E}M(\|B^{\frac{1}{2}}\mathrm{u}_0\|_H) < \infty.$$

Thus conditions 1,2,4 in the definition of Lyapunov function are satisfied. It remains to verify condition 3 from Lemma 4.3.1, namely,

$$\mathbb{E}V(\mathrm{u}(t \wedge \tau_n)) \leq \mathbb{E}V(\mathrm{u}_0) + C \int_0^t \Big(1 + \mathbb{E}(V(\mathrm{u}(s \wedge \tau_n)))\Big) ds, \; t \geq 0. \tag{4.3.2}$$

The idea is to prove (4.3.2) first for global strong solution and then extend to the case when u is a local mild solution.

Step 1: Inequality* (4.3.2) *holds for global strong solutions. Suppose that u is a global strong solution to Problem (4.0.5) satisfying

$$\mathrm{u}(t) = \mathrm{u}_0 + \int_0^t \Big[\mathcal{A}\mathrm{u}(s) + F(s, u(s), u_t(s))\Big] ds + \int_0^t \int_Z G(s, \mathrm{u}(s-), z)\tilde{N}(ds, dz), \; \mathbb{P}\text{-a.s. } t \geq 0.$$

Applying the Itô formula, see Theorem 3.5.3, to the process $\mathrm{u}(\cdot \wedge \tau_n)$ and function $V(x) = \frac{1}{2}\|x\|^2_{\mathcal{H}} + \frac{1}{2}M(\|B^{\frac{1}{2}}x_1\|^2_H)$, we obtain

$$\begin{aligned} V(\mathrm{u}(t \wedge \tau_n)) - V(\mathrm{u}_0) = &\int_0^{t\wedge\tau_n} \langle DV(\mathrm{u}(s), \mathcal{A}\mathrm{u}(s) + F(s, \mathrm{u}(s)))\rangle_{\mathcal{H}} ds \\ &+ \int_0^{t\wedge\tau_n} \int_Z \Big[V(\mathrm{u}(s) + G(s, \mathrm{u}(s), z)) - V(\mathrm{u}(s)) \qquad (4.3.3) \\ &\qquad - \langle DV(\mathrm{u}(s)), G(s, \mathrm{u}(s), z)\rangle\Big]\nu(dz)ds \\ &+ \int_0^{t\wedge\tau_n} \int_Z \Big[V\big(\mathrm{u}(s-) + G(s, \mathrm{u}(s-), z)\big) - V(\mathrm{u}(s-))\Big]\tilde{N}(ds, dz), \; t \geq 0. \end{aligned} \tag{4.3.4}$$

Note that for any $x = (x_1, x_2)^\top$ and $h = (h_1, h_2)^\top$,

$$\begin{aligned} DV(x)h &= \langle x, h\rangle_{\mathcal{H}} + m(\|B^{\frac{1}{2}x_1}\|^2)\langle B^{\frac{1}{2}}x_1, B^{\frac{1}{2}}h_1\rangle \\ &= \langle x, h\rangle_{\mathcal{H}} + m(\|B^{\frac{1}{2}}x_1\|^2_H)\langle Bx_1, h_1\rangle \\ &= \langle x, h\rangle_{\mathcal{H}} + m(\|B^{\frac{1}{2}}x_1\|^2_H)\langle AA^{-2}Bx_1, Ah_1\rangle \\ &= \langle x, h\rangle_{\mathcal{H}} + m(\|B^{\frac{1}{2}}x_1\|^2_H)\langle \begin{pmatrix} A^{-2}Bx_1 \\ 0 \end{pmatrix}, \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}\rangle_{\mathcal{H}}. \end{aligned}$$

Hence for any $x = (x_1, x_2)^\top \in \mathcal{H}$,

$$DV(x) = x + m(\|B^{\frac{1}{2}}x_1\|_H^2)\begin{pmatrix} A^{-2}Bx_1 \\ 0 \end{pmatrix}.$$

It follows that for all $x \in \mathcal{D}(\mathcal{A})$,

$$\begin{aligned}\langle DV(x), \mathcal{A}x\rangle_{\mathcal{H}} &= \langle x, \mathcal{A}x\rangle_{\mathcal{H}} + m(\|B^{\frac{1}{2}}x_1\|_H^2)\langle\begin{pmatrix} A^{-2}Bx_1 \\ 0 \end{pmatrix}, \mathcal{A}x\rangle_{\mathcal{H}} \\ &= \langle\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_2 \\ -A^2x_1 \end{pmatrix}\rangle_{\mathcal{H}} + m(\|B^{\frac{1}{2}}x_1\|_H^2)\langle\begin{pmatrix} A^{-2}Bx_1 \\ 0 \end{pmatrix}, \begin{pmatrix} x_2 \\ -A^{-2}x_1 \end{pmatrix}\rangle_{\mathcal{H}} \\ &= \langle Ax_1, Ax_2\rangle_H + \langle x_2, -A^{-2}x_1\rangle_H + m(\|B^{\frac{1}{2}}x_1\|_H^2)\langle AA^{-2}Bx_1, Ax_2\rangle + 0 \\ &= m(\|B^{\frac{1}{2}}x_1\|_H^2)\langle Bx_1, x_2\rangle_H.\end{aligned}$$

Moreover,

$$\begin{aligned}\langle DV(x), F(x)\rangle_{\mathcal{H}} &= \langle x, F(x)\rangle_{\mathcal{H}} + m(\|B^{\frac{1}{2}}x_1\|_H^2)\langle\begin{pmatrix} A^{-2}Bx_1 \\ 0 \end{pmatrix}, F(x)\rangle_{\mathcal{H}} \\ &= \langle\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} 0 \\ -m(\|B^{\frac{1}{2}}x_1\|_H^2)Bx_1 - f(x_1, x_2) \end{pmatrix}\rangle_{\mathcal{H}} \\ &\quad + m(\|B^{\frac{1}{2}}x_1\|_H^2)\langle\begin{pmatrix} A^{-2}Bx_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -m(\|B^{\frac{1}{2}}x_1\|_H^2)Bx_1 - f(x_1, x_2) \end{pmatrix}\rangle_{\mathcal{H}} \\ &= \langle x_2, -m(\|B^{\frac{1}{2}}x_1\|_H^2)Bx_1 - f(x_1, x_2)\rangle_H + 0 \\ &= -m(\|B^{\frac{1}{2}}x_1\|_H^2)\langle x_2, Bx_1\rangle_H - \langle x_2, f(x_1, x_2)\rangle_H, \ x \in \mathcal{H}.\end{aligned}$$

Combining the above equalities, we infer that

$$\langle DV(x), \mathcal{A}x + F(x)\rangle_{\mathcal{H}} = -\langle x_2, f(x_1, x_2)\rangle_H \quad \text{for all } x \in \mathcal{D}(\mathcal{A}).$$

On the other hand, we find

$$\begin{aligned}\langle DV(x), G(x, z)\rangle_{\mathcal{H}} &= \langle x, G(x, z)\rangle_{\mathcal{H}} + m(\|B^{\frac{1}{2}}x_1\|_H^2)\langle\begin{pmatrix} A^{-2}Bx_1 \\ 0 \end{pmatrix}, G(x, z)\rangle_{\mathcal{H}} \\ &= \langle\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} 0 \\ g(x_1, x_2, z) \end{pmatrix}\rangle_{\mathcal{H}} + m(\|B^{\frac{1}{2}}x_1\|_H^2)\langle\begin{pmatrix} A^{-2}Bx_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ g(x_1, x_2, z) \end{pmatrix}\rangle_{\mathcal{H}} \\ &= \langle x_2, g(x_1, x_2, z)\rangle_H.\end{aligned}$$

and

$$\begin{aligned}V(x + G(x, z)) - V(x) &= \frac{1}{2}\|x + G(x, z)\|_{\mathcal{H}}^2 + \frac{1}{2}M(\|B^{\frac{1}{2}}x_1\|_H^2) - \frac{1}{2}\|x\|_{\mathcal{H}}^2 - \frac{1}{2}M(\|B^{\frac{1}{2}}x_1\|_H^2) \\ &= \frac{1}{2}\|x\|_{\mathcal{H}}^2 + \langle x, G(x, z)\rangle_{\mathcal{H}} + \frac{1}{2}\|G(x, z)\|_{\mathcal{H}}^2 - \frac{1}{2}\|x\|_{\mathcal{H}}^2 \\ &= \langle x_2, g(x_1, x_2, z)\rangle_H + \frac{1}{2}\|g(x_1, x_2, z)\|_H^2.\end{aligned}$$

From these relations we obtain

$$
\begin{aligned}
V(\mathrm{u}(t\wedge\tau_n)) - V(\mathrm{u}_0) &= \int_0^{t\wedge\tau_n} \langle DV(\mathrm{u}(s)), \mathcal{A}\mathrm{u}(s) + F(s,\mathrm{u}(s)))\rangle_{\mathcal{H}} ds \\
&\quad + \int_0^{t\wedge\tau_n}\int_Z \Big[V(\mathrm{u}(s)+G(s,\mathrm{u}(s),z)) - V(\mathrm{u}(s)) \\
&\qquad\qquad - \langle DV(\mathrm{u}(s)), G(s,\mathrm{u}(s),z)\rangle\Big]\nu(dz)ds \\
&\quad + \int_0^{t\wedge\tau_n}\int_Z \Big[V(\mathrm{u}(s-)+G(s,\mathrm{u}(s-),z)) - V(\mathrm{u}(s-))\Big]\tilde{N}(ds,dz) \\
&= -\int_0^{t\wedge\tau_n} \langle u_t(s), f(s,u(s),u_t(s))\rangle_H ds \\
&\quad + \int_0^{t\wedge\tau_n}\int_Z \Big[\langle u_t(s), g(s,u(s),u_t(s),z)\rangle_{\mathcal{H}} + \frac{1}{2}\|g(s,u(s),u_t(s),z)\|_H^2 \\
&\qquad - \langle u_t(s), g(u(s),u_t(s),z)\rangle_H\Big]\nu(dz)ds \\
&\quad + \int_0^{t\wedge\tau_n}\int_Z \Big[\langle \mathrm{u}_t(s-), g(s,\mathrm{u}(s-),z)\rangle_{\mathcal{H}} + \frac{1}{2}\|g(s,\mathrm{u}(s-),z)\|_{\mathcal{H}}^2\Big]\tilde{N}(ds,dz) \\
&= -\int_0^{t\wedge\tau_n} \langle u_t(s), f(s,u(s),u_t(s))\rangle_H ds + \frac{1}{2}\int_0^{t\wedge\tau_n}\int_Z \|g(s,\mathrm{u}(s),z)\|_{\mathcal{H}}^2\nu(dz)ds \\
&\quad + \int_0^{t\wedge\tau_n}\int_Z \Big[\langle \mathrm{u}_t(s-), g(s,\mathrm{u}(s-),z)\rangle_{\mathcal{H}} + \frac{1}{2}\|g(s,\mathrm{u}(s-),z)\|_{\mathcal{H}}^2\Big]\tilde{N}(ds,dz).
\end{aligned}
$$

Taking the expectation to both sides of the above equalities we infer that

$$
\begin{aligned}
\mathbb{E}V(\mathrm{u}(t\wedge\tau_n)) &= \mathbb{E}V(\mathrm{u}_0) - \mathbb{E}\int_0^{t\wedge\tau_n} \langle u_t(s), f(s,u(s),u_t(s))\rangle_H ds + \frac{1}{2}\mathbb{E}\int_0^{t\wedge\tau_n}\int_Z \|g(s,\mathrm{u}(s),z)\|_{\mathcal{H}}^2\nu(dz)ds \\
&= \mathbb{E}V(\mathrm{u}_0) - \mathbb{E}\int_0^t \langle u_t(s), f(s,u(s),u_t(s))\rangle_H 1_{(0,t\wedge\tau_n]}(s)ds \\
&\quad + \frac{1}{2}\mathbb{E}\int_0^t\int_Z \|g(s,\mathrm{u}(s),z)\|_{\mathcal{H}}^2 1_{(0,t\wedge\tau_n]}(s)\nu(dz)ds \\
&\leq \mathbb{E}V(\mathrm{u}_0) + \frac{1}{2}(1+K_f)\mathbb{E}\int_0^t (1+\|\mathrm{u}(s\wedge\tau_n)\|_{\mathcal{H}}^2)ds + \frac{1}{2}K_g\mathbb{E}\int_0^{t\wedge\tau_n}(1+\|\mathrm{u}(s\wedge\tau_n)\|_{\mathcal{H}}^2)ds \\
&= \mathbb{E}V(\mathrm{u}_0) + \frac{1}{2}(1+K_f+K_g)\int_0^t (1+\mathbb{E}\|\mathrm{u}(s\wedge\tau_n)\|_{\mathcal{H}}^2)ds, \; t\geq 0.
\end{aligned}
$$

Above we used the growth conditions (4.1.6)-(4.1.7) of functions f and g. Therefore, inequality (4.3.2) holds if we set $C = \frac{1}{2}(1+K_f+K_g)$.

Step 2: Inequality* (4.3.2) *holds for a local mild solution.

In this case, one of the main obstacles is that the solution u to the Problem (4.0.5) under Assumptions 4.1.7 and 4.2.1 is a local mild solution, so the lifespan of solution τ_∞ may be finite. For this, we fix $n \in \mathbb{N}$ and introduce the following functions

$$
\begin{aligned}
\tilde{f}(t) &= 1_{[0,\tau_n)}(t) f(t, \mathrm{u}(t\wedge\tau_n)), \; t\geq 0, \\
\tilde{g}(t,z) &= 1_{[0,\tau_n]}(t) g(t, \mathrm{u}(t\wedge\tau_n -), z), \; t\geq 0 \text{ and } z\in Z.
\end{aligned}
$$

Here $\mathrm{u}(t)$, $0\leq t<\tau_\infty$, with $\tau_\infty = \lim_{n\to\infty}\tau_n$, is the unique local mild solution of the Problem

(4.0.5) under the Assumptions 4.1.7 and 4.2.1. Denote

$$\tilde{F}(t) = \begin{pmatrix} 0 \\ -\tilde{f}(t) - m(\|B^{\frac{1}{2}}u(t\wedge\tau_n)\|_H^2)Bu(t\wedge\tau_n)1_{[0,\tau_n)}(t) \end{pmatrix} \text{ and } \tilde{G}(t,z) = \begin{pmatrix} 0 \\ \tilde{g}(t,z) \end{pmatrix}.$$

One can see that the process $\tilde{F}$ and $\tilde{G}$ are bounded. So Consider the following linear non-homogeneous stochastic equation

$$dv(t) = \mathcal{A}v(t)dt + \tilde{F}(t)dt + \int_Z \tilde{G}(t,z)\tilde{N}(dt,dz), \ t \geq 0, \tag{4.3.5}$$

$$v(0) = \mathrm{u}(0). \tag{4.3.6}$$

By Theorem 4.1.10, there exists a unique global mild solution of this equation which is given by

$$v(t) = e^{t\mathcal{A}}\mathrm{u}(0) + \int_0^t e^{(t-s)\mathcal{A}}\tilde{F}(s)ds + \int_0^t \int_Z e^{(t-s)\mathcal{A}}\tilde{G}(s,z)\tilde{N}(ds,dz), \ t \geq 0. \tag{4.3.7}$$

Hence the stopped process $v(\cdot \wedge \tau_n)$ satisfies

$$v(t\wedge\tau_n) = e^{(t\wedge\tau_n)\mathcal{A}}\mathrm{u}(0) + \int_0^{t\wedge\tau_n} e^{(t\wedge\tau_n - s)\mathcal{A}}\tilde{F}(s)ds + I_{\tau_n}(\tilde{G})(t\wedge\tau_n), \ t \geq 0,$$

where as usual

$$I_{\tau_n}(\tilde{G})(t) = \int_0^t \int_Z 1_{[0,\tau_n]}(s)e^{(t-s)\mathcal{A}}\tilde{G}(s,z)\tilde{N}(ds,dz).$$

One can observe that

$$\begin{aligned} I_{\tau_n}(\tilde{G})(t) &= \int_0^t \int_Z 1_{[0,\tau_n]}(s)e^{(t-s)\mathcal{A}}\tilde{G}(s,z)\tilde{N}(ds,dz) \\ &= \int_0^t \int_Z 1_{[0,\tau_n]}(s)e^{(t-s)\mathcal{A}}G(s,\mathrm{u}(s\wedge\tau_n-),z)\tilde{N}(ds,dz) \\ &= \int_0^t \int_Z 1_{[0,\tau_n]}(s)e^{(t-s)\mathcal{A}}G(s,\mathrm{u}(s-),z)\tilde{N}(ds,dz) \\ &= I_{\tau_n}(G(\mathrm{u}))(t), \ t \geq 0. \end{aligned}$$

Therefore, on the basis of Lemma 4.7.1, we find out that for each $n \in \mathbb{N}$

$$\begin{aligned} v(t\wedge\tau_n) &= e^{(t\wedge\tau_n)\mathcal{A}}\mathrm{u}(0) + \int_0^{t\wedge\tau_n} e^{(t\wedge\tau_n-s)\mathcal{A}}\tilde{F}(s)ds + I_{\tau_n}(\tilde{G})(t\wedge\tau_n) \\ &= e^{(t\wedge\tau_n)\mathcal{A}}\mathrm{u}(0) + \int_0^{t\wedge\tau_n} 1_{(0,\tau_n]}e^{(t\wedge\tau_n-s)\mathcal{A}}\tilde{F}(s)ds + I_{\tau_n}(G(\mathrm{u}))(t\wedge\tau_n) \\ &= e^{(t\wedge\tau_n)\mathcal{A}}\mathrm{u}(0) + \int_0^{t\wedge\tau_n} e^{(t\wedge\tau_n-s)\mathcal{A}}1_{[0,\tau_n]}(s)F(s,\mathrm{u}(s\wedge\tau_n))ds + I_{\tau_n}(G(\mathrm{u}))(t\wedge\tau_n) \\ &= \mathrm{u}(t\wedge\tau_n) \quad \mathbb{P}\text{-a.s. } t \geq 0. \end{aligned}$$

The second difficulty here is that the Itô formula is only applicable to the strong solution. So our next step is to find a sequence of global strong solutions which converges to the global mild solution v uniformly. To do this, let us set, with $R(m;\mathcal{A}) = (mI - \mathcal{A})^{-1}$,

$$\begin{aligned} \mathrm{u}_m(0) &= mR(m;\mathcal{A})\mathrm{u}(0); \\ \tilde{F}_m(t,\omega) &= mR(m;\mathcal{A})\tilde{F}(t,\omega) \ \text{ for } (t,\omega) \in \mathbb{R}_+ \times \Omega; \\ \tilde{G}_m(t,\omega,z) &= mR(m;\mathcal{A})\tilde{G}(t,\omega,z) \quad \text{for } (t,\omega,z) \in \mathbb{R}_+ \times \Omega \times Z. \end{aligned}$$

Since $\mathcal{A}$ is the infinitesimal generator of a contraction C_0-semigroup $(e^{t\mathcal{A}})_{t\geq 0}$, by the Hille-Yosida Theorem, $\|R(m;\mathcal{A})\| \leq \frac{1}{m}$, $\tilde{F}_m(t,\omega) \in \mathcal{D}(\mathcal{A})$, for every $(t,\omega) \in \mathbb{R}_+ \times \Omega$ and $\tilde{G}_m(t,\omega,z) \in \mathcal{D}(\mathcal{A})$ for every $(t,\omega,z) \in \mathbb{R}_+ \times \Omega \times Z$. Moreover, $\tilde{F}_m(t,\omega) \to \tilde{F}(t,\omega)$ pointwise on $\mathbb{R}_+ \times \Omega$ and $\tilde{G}_m(t,\omega,z) \to \tilde{G}(t,\omega,z)$ pointwise on $\mathbb{R}_+ \times \Omega \times Z$. Next, we note that $\|\tilde{F}_m\|_{\mathcal{H}}$ and $\|\tilde{F}_m - \tilde{F}\|_{\mathcal{H}}$ is bounded from above by a function $2\|F\|_{\mathcal{H}}$ belonging to $\mathcal{M}^2_{loc}(\mathcal{BF};\mathbb{R})$, so the Lebesgue Dominated Convergence Theorem tells us that for every $T > 0$,

$$\lim_{m\to\infty} \mathbb{E}\int_0^T \|\tilde{F}_m(t) - \tilde{F}(t)\|^2_{\mathcal{H}}dt = 0 \tag{4.3.8}$$

Analogously, we know that $\|\tilde{G}_m\|_{\mathcal{H}}$ and $\|\tilde{G}_m - \tilde{G}\|_{\mathcal{H}}$ are bounded by functions $\|G\|_{\mathcal{H}}$ and $2\|G\|_{\mathcal{H}}$, respectively, which are both belonging to $\mathcal{M}^2_{loc}(\hat{\mathcal{P}};\mathbb{R})$. So again we can apply the Lebesgue Dominated Convergence Theorem to find out that for all $T > 0$,

$$\lim_{m\to\infty} \mathbb{E}\int_0^T \int_Z |\tilde{G}_m(t,z) - \tilde{G}(t,z)|^2_{\mathcal{H}}\nu(dz)dt = 0. \tag{4.3.9}$$

Clearly, by the definition, $\tilde{F}_m(t,\omega) \in \mathcal{D}(\mathcal{A})$, for all $(t,\omega) \in \mathbb{R}_+ \times \Omega$ and $\tilde{G}_m(t,\omega,z) \in \mathcal{D}(\mathcal{A})$, for all $(t,\omega,z) \in \mathbb{R}_+ \times \Omega \times Z$. Hence by the boundedness discussed before, we infer $\tilde{F}_m \in \mathcal{M}^2_{loc}(\mathcal{BF};\mathcal{D}(\mathcal{A}))$ and $\tilde{G}_m \in \mathcal{M}^2_{loc}(\hat{\mathcal{P}};\mathcal{D}(\mathcal{A}))$, $m \in \mathbb{N}$.
From Theorem 4.1.10, it follows that the following Equation

$$\begin{aligned} dv_m(t) &= \mathcal{A}v_m(t)dt + \tilde{F}_m(t)dt + \int_Z \tilde{G}_m(t,z)\tilde{N}(dt,dz), \ t \geq 0 \\ v_m(0) &= u_m(0) \end{aligned}$$

has a unique global strong solution which satisfies that $\mathbb{P}$-a.s. for all $t \geq 0$,

$$v_m(t) = e^{t\mathcal{A}_m}u_m(0) + \int_0^t e^{(t-s)\mathcal{A}_m}\tilde{F}_m(s)ds + \int_0^t \int_Z e^{(t-s)\mathcal{A}}\tilde{G}_m(s,z)\tilde{N}(ds,dz), \tag{4.3.10}$$

Note that we can rewrite this global strong solution in the following form

$$v_m(t) = u_m(0) + \int_0^t \left[\mathcal{A}v_m(s) + \tilde{F}_m(s)\right]ds + \int_0^t \int_Z \tilde{G}_m(s,z)\tilde{N}(ds,dz), \ t \geq 0. \tag{4.3.11}$$

Let σ be a stopping time. Now we can apply Itô formula, see Theorem 3.5.3, to the process v_m of the form (4.3.11) and the function V to get

$$\begin{aligned} V(v_m(\sigma)) - V(u_m(0)) &= \int_0^\sigma \langle DV(v_m(s), \mathcal{A}v_m(s) + \tilde{F}_m(s))\rangle_{\mathcal{H}}ds \\ &+ \int_0^\sigma \int_Z \left[V(v_m(s) + \tilde{G}_m(s,z)) - V(v_m(s)) - \langle DV(v_m(s)), \tilde{G}_m(s,z)\rangle\right]\nu(dz)ds \\ &+ \int_0^\sigma \int_Z \left[V\left(v_m(s-) + \tilde{G}_m(s,z)\right) - V(v_m(s-))\right]\tilde{N}(ds,dz). \end{aligned} \tag{4.3.12}$$

We next observe that for every $T > 0$,

$$\lim_{m\to\infty} \mathbb{E}\sup_{0\leq t\leq T} \|v_m(t) - v(t)\|^2_{\mathcal{H}} = 0. \tag{4.3.13}$$

Indeed, from (4.3.7) and (4.3.10) we find out that

$$v_m(t) - v(t) = \int_0^t e^{(t-s)\mathcal{A}}\left(\tilde{F}(s) - \tilde{F}_m(s)\right)ds + \int_0^t \int_Z e^{(t-s)\mathcal{A}}\left(\tilde{G}(s,z) - \tilde{G}_m(s,z)\right)\tilde{N}(ds,dz), \ t \geq 0.$$

Using the Cauchy-Swartz inequality, we obtain

$$\mathbb{E}\sup_{0\le t\le T}\left\|\int_0^t e^{(t-s)\mathcal{A}}\left(\tilde{F}(s)-\tilde{F}_m(s)\right)ds\right\|_{\mathcal{H}}^2 \le T\mathbb{E}\sup_{0\le t\le T}\int_0^t\left\|e^{(t-s)\mathcal{A}}\left(\tilde{F}(s)-\tilde{F}_m(s)\right)\right\|_{\mathcal{H}}^2 ds$$
$$\le T\mathbb{E}\int_0^T\left\|\tilde{F}(s)-\tilde{F}_m(s)\right\|_{\mathcal{H}}^2 ds.$$

The right side of above inequality converges to 0, as $m\to\infty$, as we have already shown before in (4.3.8). Therefore, we obtain

$$\lim_{m\to\infty}\mathbb{E}\sup_{0\le t\le T}\left\|\int_0^t e^{(t-s)\mathcal{A}}\left(\tilde{F}(s)-\tilde{F}_m(s)\right)ds\right\|_{\mathcal{H}}^2 = 0.$$

Meanwhile, we can use the Davis inequality for stochastic convolution processes, see Section 3.7.2, to deduce that

$$\mathbb{E}\sup_{0\le t\le T}\left\|\int_0^t\int_Z e^{(t-s)\mathcal{A}}\left(\tilde{G}(s,z)-\tilde{G}_m(s,z)\right)\tilde{N}(ds,dz)\right\|_{\mathcal{H}}^2$$
$$\le C\mathbb{E}\int_0^T\int_Z\|\tilde{G}(t,z)-\tilde{G}_m(t,z)\|_{\mathcal{H}}^2\nu(dz)dt. \tag{4.3.14}$$

Note that the right side of (4.3.14) converges to 0 as $m\to\infty$ by (4.3.9). Hence we have

$$\lim_{m\to\infty}\mathbb{E}\sup_{0\le t\le T}\left\|\int_0^t\int_Z e^{(t-s)\mathcal{A}}\left(\tilde{G}(s,z)-\tilde{G}_m(s,z)\right)\tilde{N}(ds,dz)\right\|_{\mathcal{H}}^2 = 0,$$

which proves equality (4.3.13).
Therefore, we conclude that $v_m(t)$ converges to $v(t)$ uniformly on any closed interval $[0,T]$, $0<T<\infty$, $\mathbb{P}$-a.s. Hence, by taking a subsequence if necessary we may assume that $v_m(t)\to v(t)$, uniformly and $\tilde{F}_m(s)\to\tilde{F}(s)$ and $\tilde{G}_m(s,z)\to\tilde{G}(s,z)$ on $[0,\sigma(\omega)]$, as $m\to\infty$, for almost all ω in Ω.

We introduce the following canonical linear projection mappings

$$\pi_1:\mathcal{H}\ni\begin{pmatrix}x\\y\end{pmatrix}\mapsto x\in\mathcal{D}(A)$$
$$\pi_2:\mathcal{H}\ni\begin{pmatrix}x\\y\end{pmatrix}\mapsto y\in\mathcal{H}.$$

Calculations similar to those performed in Step 1 yield

$$\begin{aligned}
&\langle DV(v_m(s),\mathcal{A}v_m(s)+\tilde{F}_m(s))\rangle_{\mathcal{H}}\\
&=\langle v_m(s),\mathcal{A}v_m(s)+\tilde{F}_m(s))\rangle_{\mathcal{H}}\\
&\qquad+m(\|B^{\frac12}\pi_1v_m(s)\|_H^2)\langle\begin{pmatrix}A^{-2}B\pi_1v_m(s)\\0\end{pmatrix},\mathcal{A}v_m(s)+\tilde{F}_m(s))\rangle_{\mathcal{H}}\\
&=\langle v_m(s),\mathcal{A}v_m\rangle_{\mathcal{H}}+\langle v_m(s),\tilde{F}_m(s)\rangle_{\mathcal{H}}+m(\|B^{\frac12}\pi_1v_m(s)\|_H^2)\langle B\pi_1v_m(s),\pi_1\mathcal{A}v_m(s)+\pi_1\tilde{F}_m(s)\rangle_H\\
&\le\langle v_m(s),\tilde{F}_m(s)\rangle_{\mathcal{H}}+m(\|B^{\frac12}\pi_1v_m(s)\|_H^2)\langle B\pi_1v_m(s),\pi_2v_m(s)+\pi_1\tilde{F}_m(s)\rangle_H,\ s\ge0,
\end{aligned}$$

where we used the fact that $\pi_1\mathcal{A}v_m(t)=\pi_2v_m(t)$ on $[0,T]$ and $\langle v(s),\mathcal{A}v\rangle_{\mathcal{H}}\le0$, since the operator $\mathcal{A}$ is dissipative. Moreover, since $\pi_1\tilde{G}_m(s,z)=m(m^2I+A^2)^{-1}\tilde{g}(s,z)$ and $\pi_2\tilde{G}_m(s,z)=m^2(m^2I+$

$A^2)^{-1}\tilde{g}(s,z)$, we infer

$$\begin{aligned}
&\langle DV(v_m(s)), \tilde{G}_m(s,z)\rangle_{\mathcal{H}} \qquad\qquad (4.3.15)\\
&= \langle v_m(s), \tilde{G}_m(s,z)\rangle_{\mathcal{H}} + m(\|B^{\frac{1}{2}}\pi_1 v_m(s)\|_H^2)\langle \begin{pmatrix} A^{-2}B\pi_1 v_m(s) \\ 0 \end{pmatrix}, \tilde{G}_m(s,z)\rangle_{\mathcal{H}}\\
&= \langle v_m(s), \tilde{G}_m(s,z)\rangle_{\mathcal{H}} + m(\|B^{\frac{1}{2}}\pi_1 v_m(s)\|_H^2)\langle \begin{pmatrix} A^{-2}B\pi_1 v_m(s) \\ 0 \end{pmatrix}, \begin{pmatrix} \pi_1\tilde{G}_m(s,z) \\ \pi_1\tilde{G}_m(s,z) \end{pmatrix}\rangle_{\mathcal{H}}\\
&= \langle v_m(s), \tilde{G}_m(s,z)\rangle_{\mathcal{H}} + m(\|B^{\frac{1}{2}}\pi_1 v_m(s)\|_H^2)\langle A^{-2}B\pi_1 v_m(s), \pi_1\tilde{G}_m(s,z)\rangle_H, \ s \geq 0.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
&V(v_m(s) + \tilde{G}_m(s,z)) - V(v_m(s))\\
&= \frac{1}{2}\|v_m(s) + \tilde{G}_m(s,z)\|_{\mathcal{H}}^2 + \frac{1}{2}M(\|B^{\frac{1}{2}}\pi_1(v_m(s) + \tilde{G}_m(s,z))\|_H^2)\\
&\quad - \frac{1}{2}\|v_m(s)\|_{\mathcal{H}}^2 - \frac{1}{2}M(\|B^{\frac{1}{2}}\pi_1 v_m(s)\|_H^2) \qquad\qquad (4.3.16)\\
&= \langle v_m(s), \tilde{G}_m(s,z)\rangle_{\mathcal{H}} + \frac{1}{2}\|\tilde{G}_m(s,z)\|_{\mathcal{H}}^2 + \frac{1}{2}M(\|B^{\frac{1}{2}}\pi_1(v_m(s) + \tilde{G}_m(s,z))\|_H^2)\\
&\quad - \frac{1}{2}M(\|B^{\frac{1}{2}}\pi_1 v_m(s)\|_H^2).
\end{aligned}$$

Hence Equality (4.3.12) becomes

$$\begin{aligned}
&V(v_m(\sigma)) - V(u_m(0)) \qquad\qquad (4.3.17)\\
&\leq \int_0^\sigma \Big[\langle v_m(s), \tilde{F}_m(s)\rangle_{\mathcal{H}} + m(\|B^{\frac{1}{2}}\pi_1 v_m(s)\|_H^2)\langle B\pi_1 v_m(s), \pi_2 v_m(s) + \pi_1\tilde{F}_m(s)\rangle_H\Big]ds\\
&\quad + \int_0^\sigma\int_Z \Big[V(v_m(s) + \tilde{G}_m(s,z)) - V(v_m(s)) - \langle DV(v_m(s)), \tilde{G}_m(s,z)\rangle\Big]\nu(dz)ds\\
&\quad + \int_0^\sigma\int_Z \Big[V\big(v_m(s-) + \tilde{G}_m(s,z)\big) - V(v_m(s-))\Big]\tilde{N}(ds,dz).
\end{aligned}$$

Note that $\pi_1\tilde{F}(s,\omega) = 0$ on $\mathbb{R}_+ \times \Omega$ and $\pi_1\tilde{G}(s,\omega,z) = 0$ on $\mathbb{R}_+ \times \Omega \times Z$. Since the functions $m(\cdot)$ and $M(\cdot)$ are continuous and the operator $B \in \mathcal{L}(\mathcal{D}(\mathcal{A}), H)$, we have $\mathbb{P}$-a.s.

$$\begin{aligned}
\pi_1 v_m(s) &\to \pi_1 v(s),\\
m(\|B^{\frac{1}{2}}\pi_1 v_m(s)\|_H^2) &\to m(\|B^{\frac{1}{2}}\pi_1 v(s)\|_H^2),\\
\pi_2 v_m(s) &\to \pi_2 v(s),\\
B\pi_1 v_m(s) &\to B\pi_1 v_m(s),
\end{aligned}$$

uniformly on $[0,\sigma(\omega)]$, as $m \to \infty$ and

$$\begin{aligned}
\pi_1\tilde{G}_m(s,z) &\to 0,\\
M(\|B^{\frac{1}{2}}\pi_1(v_m(s) + \tilde{G}_m(s,z))\|_H^2) &\to M(\|B^{\frac{1}{2}}\pi_1 v(s)\|_H^2)
\end{aligned}$$

on $[0,\sigma(\omega)]$ for all most all $\omega \in \Omega$, as $m \to \infty$. We also notice that for every $m \in \mathbb{N}$, the set $\{v_m(t,\omega) : 0 \leq t \leq T\}$ is relatively compact for almost all ω. We will formulate this in the following lemma.

Lemma 4.3.3. *Let $f : [0,T] \to H$ be a càdlàg function. Then the set $\{f(t) : t \in [0,T]\}$ is a relatively compact subset of H.*

Proof. We argue by contradiction. We assume that the closure of the set in question is not compact. Then there exists a sequence $\{t_m\}_{m\in\mathbb{N}} \subset [0,T]$ such that the sequence $\{f(t_m)\}_{m\in\mathbb{N}}$ has no convergent subsequence in H. Since the interval $[0,T]$ is compact, the sequence $\{t_m\}_{m\in\mathbb{N}} \subset [0,T]$ has a convergent subsequence in $[0,T]$. For simplicity, we again use $\{t_m\}_{m\in\mathbb{N}}$ to denote this convergent subsequence. So we can assume that $\{t_m\}_{n\in\mathbb{N}}$ converges to some point $t^* \in [0,T]$. Since v is càdlàg and $f(T-)$ exists, we infer that $0 < t^* < T$. We have two possibilities. The sequence t_m has a subsequence $\{t_{m_{1,k}}\}_{k\in\mathbb{N}}$ convergent to t^* from the right side or it has a subsequence $\{t_{m_{2,k}}\}_{k\in\mathbb{N}}$ convergent to t^* from the left side.

If $t_{m_{1,k}} \searrow t^*$ as $k \to \infty$, then by the right continuity of f, $f(t_{m_{1,k}}) \to f(t^*)$ as $k \to \infty$. If $t_{m_{2,k}} \nearrow t^*$ as $k \to \infty$, then by the existence of left limits, $f(t_{m_{2,k}}) \to f(t^*-)$ as $k \to \infty$. In both cases, the subsequence of $\{f(t_m)\}_{m\in\mathbb{N}}$ is convergent. This leads to a contradiction with the assertion that $\{f(t_m)\}_{m\in\mathbb{N}}$ has no convergent subsequence. Therefore, the set $\{f(t) : 0 \le t \le T\}$ is compact. □

Note that since for every $m \in \mathbb{N}$, the set $\{v_m(s), s \in [0,T]\}$ is relatively compact $\mathbb{P}$-a.s. and the sequence $\{v_m\}_{m\in\mathbb{N}}$ converges uniformly to v, $\mathbb{P}$-a.s., the set $\{v_m(s), s \in [0,T], m \in \mathbb{N}\}$ is bounded in $\mathcal{H}$, $\mathbb{P}$-a.s. It follows that

$$\langle v_m(s), \tilde{F}_m(s)\rangle_{\mathcal{H}} \le \|v_m(s)\|_{\mathcal{H}}\|\tilde{F}_m(s)\|_{\mathcal{H}} \le \|\tilde{F}_m(s)\|_{\mathcal{H}} \sup_{0\le s\le T} \|v_m(s)\|_{\mathcal{H}} \le C\|\tilde{F}(s)\|_{\mathcal{H}}.$$

Therefore, on the basis of the Lebesgue Dominated convergence theorem, we conclude that

$$\int_0^\sigma \langle v_m(s), \tilde{F}_m(s)\rangle_{\mathcal{H}} ds \to \int_0^\sigma \langle v_m(s), \tilde{F}_m(s)\rangle_{\mathcal{H}} ds \quad \mathbb{P}\text{-a.s.}$$

Analogously, by the continuity of the function m and the fact that $B \in \mathcal{L}(\mathcal{D}(A), H)$, we infer for some constants C_1, C_2,

$$m(\|B^{\frac{1}{2}}\pi_1 v_m(s)\|_H^2)\langle B\pi_1 v_m(s), \pi_2 v_m(s) + \pi_1 \tilde{F}_m(s)\rangle_H \le C_1 + C_2\|\tilde{F}(s)\|_{\mathcal{H}}.$$

Moreover, we know that for almost all $\omega \in \Omega$

$$m(\|B^{\frac{1}{2}}\pi_1 v_m(s)\|_H^2)\langle B\pi_1 v_m(s), \pi_2 v_m(s) + \pi_1 \tilde{F}_m(s)\rangle_H$$

converges on $[0, \sigma(\omega)]$ as $m \to \infty$ to

$$m(\|B^{\frac{1}{2}}\pi_1 v(s)\|_H^2)\langle B\pi_1 v(s), \pi_2 v(s)\rangle_H.$$

Again, it follows from the Lebesgue Dominated convergence theorem that $\mathbb{P}$-a.s.

$$\int_0^\sigma \Big[m(\|B^{\frac{1}{2}}\pi_1 v_m(s)\|_H^2)\langle B\pi_1 v_m(s), \pi_2 v_m(s) + \pi_1 \tilde{F}_m(s)\rangle_H\rangle_{\mathcal{H}}\Big] ds$$

converges as $m \to \infty$ to

$$\int_0^\sigma m(\|B^{\frac{1}{2}}\pi_1 v(s)\|_H^2)\langle B\pi_1 v(s), \pi_2 v(s)\rangle_H ds.$$

In conclusion, $\mathbb{P}$-a.s. the first term on the right side of inequality (4.3.17) converges as $m \to \infty$ to

$$\int_0^\sigma \Big[\langle v(s), \mathcal{A}v\rangle_{\mathcal{H}} + \langle v(s), \tilde{F}(s)\rangle_{\mathcal{H}} + m(\|B^{\frac{1}{2}}\pi_1 v(s)\|_H^2)\langle B\pi_1 v(s), \pi_2 v(s)\rangle_H\Big] ds.$$

Also, we know from (4.3.15) and (4.3.16) that as $m \to \infty$ for all $t \in [0,T]$, $z \in Z$, $\mathbb{P}$-a.s.

$$V(v_m(s) + \tilde{G}_m(s,z)) - V(v_m(s)) - \langle DV(v_m(s)), \tilde{G}_m(s,z)\rangle_{\mathcal{H}} \to \frac{1}{2}\|\tilde{G}(s,z)\|_H^2$$

$$V\big(v_m(s) + \tilde{G}_m(s,z)\big) - V(v_m(s)) \to \langle v(s), \tilde{G}(s,z)\rangle_{\mathcal{H}} + \frac{1}{2}\|\tilde{G}(s,z)\|_{\mathcal{H}}^2.$$

Set $X(\omega) = \{v_m(t,\omega) : 0 \le t \le T, m \in \mathbb{N}\}$, for $\omega \in \Omega$. As we have noticed before, $X(\omega)$ is a bounded subset of $\mathcal{H}$ for almost all $\omega \in \Omega$. Since the functions DV and D^2V are uniformly continuous on bounded subsets of $\mathcal{H}$, so $\sup_{x\in X}|DV(x)| < \infty$ and $\sup_{x\in X}|D^2V(x)| < \infty$, $\mathbb{P}$-a.s. Hence by the Taylor formula, one have

$$\begin{aligned}
V(v_m(s) + \tilde{G}_m(s,z)) - V(v_m(s)) &- \langle DV(v_m(s)), \tilde{G}_m(s,z)\rangle_{\mathcal{H}} \\
&\le \frac{1}{2}\|D^2V(v_m(s))\|\|\tilde{G}_m(s,z)\|_{\mathcal{H}}^2 \\
&\le \frac{1}{2}\sup_{x\in X}\|D^2V(x)\|\|\tilde{G}(s,z)\|_{\mathcal{H}}^2.
\end{aligned}$$

We also observe that since $\tilde{G} \in \mathcal{M}^2_{loc}(\hat{\mathcal{P}};\mathcal{H})$, for every $0 < T < \infty$, we have

$$\int_0^T \int_Z \|\tilde{G}(s,z)\|^2 \nu(dz)ds < \infty, \ \mathbb{P}\text{-a.s.}$$

By using the above result, along with the Lebesgue Dominated Convergence Theorem, we obtain that

$$\int_0^\sigma \int_Z V(v_m(s) + \tilde{G}_m(s,z)) - V(v_m(s)) - \langle DV(v_m(s)), \tilde{G}_m(s,z)\rangle_{\mathcal{H}}\nu(dz)ds$$

converges to

$$\int_0^\sigma \int_Z \frac{1}{2}\tilde{G}(s,z)\nu(dz)ds, \ \mathbb{P}\text{-a.s. as } m \to \infty.$$

On the other hand, by the identity (3.3.3), we have

$$\begin{aligned}
\mathbb{E}\Big\| \int_0^\sigma \int_Z &\Big[V\big(v_m(s-) + \tilde{G}_m(s,z)\big) - V\big(v_m(s-)\big)\Big]\tilde{N}(ds,dz) \\
&- \int_0^\sigma \int_Z \Big[\langle v(s-), \tilde{G}(s,z)\rangle_{\mathcal{H}} + \frac{1}{2}\|\tilde{G}(s,z)\|_{\mathcal{H}}^2\Big]\tilde{N}(ds,dz)\Big\|_{\mathcal{H}}^2 \\
= \mathbb{E}\int_0^\sigma \int_Z &\Big|V\big(v_m(s) + \tilde{G}_m(s,z)\big) - V(v_m(s)) - \langle v(s), \tilde{G}(s,z)\rangle_{\mathcal{H}} + \frac{1}{2}\|\tilde{G}(s,z)\|_{\mathcal{H}}^2\Big|^2 \nu(dz)ds.
\end{aligned}$$

Moreover, we note that the integrand

$$\Big|V\big(v_m(s) + \tilde{G}_m(s,z)\big) - V(v_m(s)) - \langle v(s), \tilde{G}(s,z)\rangle_{\mathcal{H}} + \frac{1}{2}\|\tilde{G}(s,z)\|_{\mathcal{H}}^2\Big|^2$$

is bounded by $2\sup_{x\in X}\|DV(x)\|^2\|\tilde{G}(s,z)\|_{\mathcal{H}}^2 \le C\|\tilde{G}(s,z)\|_{\mathcal{H}}^2$. Since $G \in \mathcal{M}^2(\hat{\mathcal{P}};\mathcal{H})$, for every $0 < T < \infty$, $\mathbb{E}\int_0^T\int_Z \|\tilde{G}(s,z)\|_{\mathcal{H}}^2\nu(dz)ds < \infty$. So again, we can apply the Lebesgue Dominated Converges Theorem to get

$$\begin{aligned}
\lim_{m\to\infty}\mathbb{E}\Big\| \int_0^\sigma \int_Z &\Big[V\big(v_m(s-) + \tilde{G}_m(s,z)\big) - V\big(v_m(s-)\big)\Big]\tilde{N}(ds,dz) \\
&- \int_0^\sigma \int_Z \Big[\langle v(s-), \tilde{G}(s,z)\rangle_{\mathcal{H}} + \frac{1}{2}\|\tilde{G}(s,z)\|_{\mathcal{H}}^2\Big]\tilde{N}(ds,dz)\Big\|_{\mathcal{H}}^2 = 0.
\end{aligned}$$

Hence by taking a subsequence, we infer that

$$\int_0^\sigma \int_Z \Big[V\big(v_m(s-) + \tilde{G}_m(s,z)\big) - V(v_m(s-))\Big]\tilde{N}(ds,dz)$$

converges $\mathbb{P}$-a.s. to

$$\int_0^\sigma \int_Z \Big[\langle v(s-), \tilde{G}(s,z)\rangle_{\mathcal{H}} + \frac{1}{2}\|\tilde{G}(s,z)\|_{\mathcal{H}}^2\Big]\tilde{N}(ds,dz).$$

Also, it is not hard to see that

$$\begin{aligned}\lim_{n\to\infty} V(v_m(\sigma)) &= \frac{1}{2}\lim_{n\to\infty}\|v_m(\sigma)\|_H^2 + \frac{1}{2}\lim_{n\to\infty} M(\|B^{\frac{1}{2}}\pi_1 v_m(\sigma)\|_H^2)\\ &= \frac{1}{2}\|v(\sigma)\|_H^2 + \frac{1}{2}M(\|B^{\frac{1}{2}}\pi_1 v(\sigma)\|_H^2)\\ &= V(v(\sigma)).\end{aligned} \tag{4.3.18}$$

From the above observation, by letting $m\to\infty$ in (4.3.17), one easily deduces that

$$\begin{aligned}V(v(\sigma)) - V(\mathrm{u}_0) &= \int_0^\sigma \Big[\langle v(s), \mathcal{A}v\rangle_{\mathcal{H}} + \langle v(s), \tilde{F}(s)\rangle_{\mathcal{H}} + m(\|B^{\frac{1}{2}}\pi_1 v(s)\|_H^2)\langle B\pi_1 v(s), \pi_1\mathcal{A}v(s)\rangle_H\Big]ds\\ &\quad + \int_0^\sigma\int_Z \frac{1}{2}\tilde{G}(s,z)\nu(dz)ds\\ &\quad + \int_0^\sigma\int_Z \Big[\langle v(s-), \tilde{G}(s,z)\rangle_{\mathcal{H}} + \frac{1}{2}\|\tilde{G}(s,z)\|_{\mathcal{H}}^2\Big]\tilde{N}(ds,dz)\\ &\le \int_0^\sigma \Big[\langle v(s), \tilde{F}(s)\rangle_{\mathcal{H}} + m(\|B^{\frac{1}{2}}\pi_1 v(s)\|_H^2)\langle B\pi_1 v(s), \pi_1\mathcal{A}v(s)\rangle_H\Big]ds\\ &\quad + \int_0^\sigma\int_Z \frac{1}{2}\tilde{G}(s,z)\nu(dz)ds\\ &\quad + \int_0^\sigma\int_Z \Big[\langle v(s-), \tilde{G}(s,z)\rangle_{\mathcal{H}} + \frac{1}{2}\|\tilde{G}(s,z)\|_{\mathcal{H}}^2\Big]\tilde{N}(ds,dz), \ \mathbb{P}\text{-a.s.}\end{aligned} \tag{4.3.19}$$

Therefore, $\mathbb{P}$-a.s.

$$\begin{aligned}V(v(\sigma)) - V(\mathrm{u}_0) \le \int_0^\sigma &\Big[\langle \pi_2 v(s), \tilde{f}(s)\rangle_{\mathcal{H}} + m(\|B^{\frac{1}{2}}\pi_1 v(s)\|_H^2)\langle B\pi_1 v(s), \pi_1\mathcal{A}v(s)\rangle\Big]ds\\ &+ \frac{1}{2}\int_0^\sigma\int_Z \|\tilde{g}(s,z)\|_H^2\nu(dz)ds\\ &+ \int_0^\sigma\int_Z \Big[\langle \pi_2 v(s-), \tilde{g}(s,z)\rangle_H + \frac{1}{2}\|\tilde{g}(s,z)\|_H^2\Big]\tilde{N}(ds,dz).\end{aligned}$$

Taking the expectation to both sides, we have

$$\begin{aligned}\mathbb{E}V(v(\sigma)) \le \mathbb{E}V(\mathrm{u}_0) + \mathbb{E}\int_0^\sigma &\Big[\langle \pi_2 v(s), \tilde{f}(s)\rangle_{\mathcal{H}} + m(\|B^{\frac{1}{2}}\pi_1 v(s)\|_H^2)\langle B\pi_1 v(s), \pi_1\mathcal{A}v(s)\rangle\Big]ds\\ &+ \frac{1}{2}\mathbb{E}\int_0^\sigma\int_Z \|\tilde{g}(s,z)\|_H^2\nu(dz)ds.\end{aligned}$$

Now let us recall that $v(t\wedge\tau_n) = \mathrm{u}(t\wedge\tau_n)$, $\tilde{F}(t) = 1_{(0,\tau_n]}(t)F(t,\mathrm{u}(t\wedge\tau_n))$ and $\tilde{G}(t) = 1_{(0,\tau_n]}(t)G(t,\mathrm{u}(t\wedge$

$\tau_n-), z)$ for $t \geq 0$. Thus by setting $\sigma = t \wedge \tau_n$ and using the results achieved in step 1, we infer that

$$
\begin{aligned}
\mathbb{E}V(\mathrm{u}(t\wedge\tau_n)) &\leq \mathbb{E}V(\mathrm{u}_0) + \mathbb{E}\int_0^{t\wedge\tau_n} \Big[\langle \pi_2\mathrm{u}(s), \pi_2\tilde{F}(s)\rangle_{\mathcal{H}} + m(\|B^{\frac{1}{2}}\pi_1\mathrm{u}(s)\|_H^2)\langle B\pi_1\mathrm{u}(s), \pi_1\mathcal{A}u(s)\rangle\Big] ds \\
&\quad + \frac{1}{2}\mathbb{E}\int_0^{\sigma}\int_Z \|\tilde{g}(s,z)\|_H^2 \nu(dz)ds \\
&= \mathbb{E}V(\mathrm{u}_0) + \mathbb{E}\int_0^{t\wedge\tau_n}\Big[- m(\|B^{\frac{1}{2}}u(s\wedge\tau_n)\|_H^2)\langle u_t(s), Bu(s\wedge\tau_n)\rangle 1_{(0,\tau_n]}(s) \\
&\qquad - \langle u_t(s), f(\mathrm{u}(s\wedge\tau_n))\rangle_{\mathcal{H}} 1_{(0,\tau_n]}(s) + m(\|B^{\frac{1}{2}}u(s)\|_H^2)\langle Bu(s), u_t(s)\rangle\Big] ds \\
&\quad + \frac{1}{2}\mathbb{E}\int_0^{t\wedge\tau_n}\int_Z \|g(s,\mathrm{u}(s\wedge\tau_n-),z)\|_H^2 1_{(0,\tau_n]}(t)\nu(dz)ds \\
&= \mathbb{E}V(\mathrm{u}_0) - \mathbb{E}\int_0^{t\wedge\tau_n}\langle u_t(s), f(\mathrm{u}(s\wedge\tau_n))\rangle_{\mathcal{H}} ds + \frac{1}{2}\mathbb{E}\int_0^{t\wedge\tau_n}\int_Z \|g(s,\mathrm{u}(s-),z)\|_H^2\nu(dz)ds \\
&= \mathbb{E}V(\mathrm{u}_0) - \mathbb{E}\int_0^{t}\langle u_t(s), f(\mathrm{u}(s))\rangle_H 1_{(0,t\wedge\tau_n]}(s)ds \\
&\qquad + \frac{1}{2}\mathbb{E}\int_0^{t}\int_Z \|g(s,\mathrm{u}(s-),z)\|_{\mathcal{H}}^2 1_{(0,t\wedge\tau_n]}(s)\nu(dz)ds \\
&\leq \mathbb{E}V(\mathrm{u}_0) + \frac{1}{2}(1+K_f)\mathbb{E}\int_0^t (1+\|\mathrm{u}(s\wedge\tau_n)\|_{\mathcal{H}}^2)ds + \frac{1}{2}K_g\mathbb{E}\int_0^{t\wedge\tau_n}(1+\|\mathrm{u}(s\wedge\tau_n)\|_{\mathcal{H}}^2)ds \\
&= \mathbb{E}V(\mathrm{u}_0) + \frac{1}{2}(1+K_f+K_g)\int_0^t (1+\mathbb{E}\|\mathrm{u}(s\wedge\tau_n)\|_{\mathcal{H}}^2)ds.
\end{aligned}
$$

This finally proves inequality (4.3.2). In conclusion, we proved that V is indeed a Lyapunov function and hence we can apply Lemma 4.3.1 to deduce that $\tau_\infty = \infty$. □

4.4 The Stability of the Solution

In this section we shall consider the stability of the solution to Equation (4.0.5). To simplify our problem, we will impose the following additional assumptions.

Assumption 4.4.1. *1). Suppose that function f is given by $f(x) = \beta x_1$ for some $\beta \geq 0$, where $x = (x_1, x_2)^\top \in \mathcal{H}$;*

2). Assumptions (4.1.7) and (4.2.1) hold;

3). There exist nonnegative constants R_g and K such that

$$\int_Z \|g(x,z)\|_H^2 \nu(dz) \leq R_g^2\|x\|_{\mathcal{H}}^2 + K.$$

4). There exists $\alpha > 0$ such that for all nonnegative real number y

$$ym(y) \geq \alpha M(y).$$

Lemma 4.4.2. *Define an operator $P : \mathcal{H} \to \mathcal{H}$ by*

$$P := \begin{pmatrix} \beta^2 A^{-2} + 2I & \beta A^{-2} \\ \beta I & 2I \end{pmatrix}.$$

Then P is self-adjoint isomorphism of $\mathcal{H}$ and satisfies the following

$$\text{(1)}\quad \|P\|^{-1}_{\mathcal{L}(\mathcal{H})}\langle Px,x\rangle_{\mathcal{H}} \le \|x\|^2_{\mathcal{H}} \le \langle Px,x\rangle_{\mathcal{H}}, \qquad x\in\mathcal{H};$$

$$\text{(2)}\quad \langle\begin{pmatrix} 0 \\ -\beta x_2 \end{pmatrix}, Px\rangle_{\mathcal{H}} = -\beta^2\langle x_1,x_2\rangle = -2\beta\|x_2\|^2 \qquad x=(x_1,x_2)^\top\in\mathcal{H};$$

$$\text{(3)}\quad \langle \mathcal{A}x, Px\rangle_{\mathcal{H}} = -\beta\|Ax_1\|^2_H + \beta^2\langle x_1,x_2\rangle + \beta\|x_2\|^2.$$

Proof. First note that $P\in\mathcal{L}(\mathcal{H})$. Indeed, we find out that for every $x\in\mathcal{H}$

$$\begin{aligned}
\|Px\|^2_{\mathcal{H}} &= \|\beta A^{-2}Ax_1 + 2Ax_1 + \beta A^{-2}Ax_2\|^2_H + \|\beta x_1 + 2x + 2\|^2_H \\
&= 3\|\beta A^{-2}\|^2_{\mathcal{L}(H)}\|Ax_1\|^2_H + 6\|Ax_1\|^2_H + 3\beta^2\|A^{-1}\|^2_{\mathcal{L}(H)}\|x_2\|^2_H + 2\beta^2\|x_1\|^2_{\mathcal{H}} + 4\|x_2\|^2_H \\
&\le (3\|\beta A^{-2}\|^2_{\mathcal{L}(H)} + 6)\|Ax_1\|^2_H + 2\beta^2\|A^{-1}\|^2_{\mathcal{L}(H)}\|Ax_1\|^2_H + (3\beta^2\|A^{-1}\|_{\mathcal{L}(H)} + 4)\|x_2\|^2_{\mathcal{H}} \\
&\le C\left(\|Ax_1\|^2_H + \|x_2\|^2_H\right) \\
&= C\|x\|^2_{\mathcal{H}},
\end{aligned}$$

where $C=\max\{5\|\beta A^{-2}\|^2_{\mathcal{L}(H)}+6, 3\|\beta A^{-2}\|^2_{\mathcal{L}(H)}+4\}$. Further observe that

$$\begin{aligned}
\langle Px,y\rangle_{\mathcal{H}} &= \langle\begin{pmatrix} \beta^2A^{-2}x_1 + 2x_1 + \beta A^{-2}x_2 \\ \beta x - 1 + 2x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\rangle_{\mathcal{H}} \\
&= \langle \beta^2A^{-1}x_1 + 2x_1 + \beta A^{-1}x_2, Ay_1\rangle_H + \langle \beta x_1 + x_2, y_2\langle_H \\
&= \beta^2\langle x_1,y_2\rangle_H + 2\langle Ax_1,Ay_1\rangle_H + \beta\langle x_2,y_1\rangle_H + \beta\langle x_1,y_2\rangle_H + 2\langle x_2,y_2\rangle_H
\end{aligned}$$

and

$$\begin{aligned}
\langle x,Py\rangle_{\mathcal{H}} &= \langle\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} \beta^2A^{-2}x_1 + 2x_1 + \beta A^{-2}x_2 \\ \beta x - 1 + 2x_2 \end{pmatrix}\rangle_{\mathcal{H}} \\
&= \langle \beta^2A^{-1}x_1 + 2Ax_1 + \beta A^{-1}x_2, Ay_1\rangle_H + \langle \beta x_1 + 2x_2, y_1\rangle_H \\
&= \beta^2\langle x_1,y_1\rangle_H + 2\langle Ax_1,Ay_1\rangle_H + \beta\langle x_2,y_1\rangle_H + \beta\langle x_1,y_2\rangle_H + 2\langle x_2,y_2\rangle.
\end{aligned}$$

Thus $\langle Px,y\rangle = \langle x,Py\rangle$ for any $x,y\in\mathcal{H}$, which shows that P is self-adjoint. Replacing y by x in above derived formula, we get for $x\in\mathcal{H}$,

$$\begin{aligned}
\langle Px,x\rangle_{\mathcal{H}} &= \beta^2\langle x_1,x_1\rangle_H + 2\langle Ax_1,Ax_1\rangle_H + \beta\langle x_2,x_1\rangle_H + \beta\langle x_1,x_2\rangle_H + 2\langle x_2,x_2\rangle_H \\
&= \langle \beta x_1 + x_2, \beta x_1 + x_2\rangle + 2\|Ax_1\|^2_H + \|x_2\|^2_H \\
&= 2\|Ax_1\|^2_H + \|x_2\|^2_H + \|\beta x_1 + x_2\|^2_H \\
&\ge \|Ax_1\|^2_H + \|x_2\|^2_H \\
&= \|x\|^2_{\mathcal{H}},
\end{aligned}$$

which shows the second inequality of part (1). To see the other inequality of part (2), we use Cauchy-swartz inequality to obtain

$$\langle Px,x\rangle_{\mathcal{H}} \le \|Px\|_{\mathcal{H}}\|x\|_{\mathcal{H}} \le \|P\|_{\mathcal{L}(\mathcal{H})}\|x\|^2_{\mathcal{H}}.$$

Therefore, we infer

$$\|P\|^{-1}_{\mathcal{L}(\mathcal{H})}\langle Px,x\rangle_{\mathcal{H}} \le \|x\|^2_{\mathcal{H}} \le \langle Px,x\rangle_{\mathcal{H}}.$$

The second assertion can be achieved as follows

$$\begin{aligned}\langle\begin{pmatrix}0\\-\beta x_2\end{pmatrix}, Px\rangle_{\mathcal{H}} &= \langle\begin{pmatrix}0\\-\beta x_2\end{pmatrix}, \begin{pmatrix}\beta A^{-2}x_1+2x_1+\beta A^{-2}x_2\\ \beta x_1+2x_2\end{pmatrix}\rangle_{\mathcal{H}}\\
&= \langle -\beta x_2, \beta x_1 + 2x_2\rangle_H\\
&= -\beta^2\langle x_2, x_1\rangle_H - 2\beta\|x_2\|_H^2\\
&= -\beta^2\langle x_1, x_2\rangle_H - 2\beta\|x_2\|_H^2,\end{aligned}$$

where $x = (x_1, x_2)^\top \in \mathcal{H}$. Further, for the third assertion we have

$$\begin{aligned}\langle \mathcal{A}x, Px\rangle &= \langle\begin{pmatrix}x_2\\-A^{-2}x_1\end{pmatrix}, \begin{pmatrix}\beta A^{-2}x_1+2x_1+\beta A^{-2}x_2\\ \beta x_1+2x_2\end{pmatrix}\rangle_{\mathcal{H}}\\
&= \langle Ax_2, \beta^2A^{-1}x_1 + 2Ax_1 + \beta A^{-1}x_2\rangle_H + \langle -A^2x_1, \beta x_1 + 2x_2\rangle_H\\
&= \beta^2\langle x_2, x_1\rangle_H + 2\langle Ax_2, Ax_1\rangle_H + \beta\langle x_2, x_1\rangle_H - \beta\rangle Ax_1, Ax_1\rangle_H - 2\langle Ax_1, Ax_2\rangle_H\\
&= \beta\|x_2\|_H^2 + \beta^2\langle x_2, x_1\rangle_H - \beta\|Ax_1\|_H^2,\end{aligned}$$

which completes the lemma. □

We define for $x = (x_1, x_2)^\top \in \mathcal{H}$

$$\mathscr{E}(x) = \mathbb{E}\big[\|x\|_{\mathcal{H}}^2 + M(\|B^{\frac{1}{2}}x_1\|_H^2)\big].$$

Theorem 4.4.3. *Suppose that Assumption* (4.4.1) *is satisfied and* $\mathscr{E}(\mathrm{u}_0) < \infty$. *Let* u *be the unique mild global solution to Equation* (4.0.5). *Let* K *be the constant given in Part* (3) *of Assumption* (4.4.1). *If* $K = 0$, *then the solution is exponentially mean-square stable, that is for there exist constant* $0 < C < \infty$ *and* $\lambda > 0$ *such that for all* $t \geq 0$,

$$\mathbb{E}\|\mathrm{u}(t)\|_{\mathcal{H}}^2 \leq Ce^{-\lambda t}\mathscr{E}(\mathrm{u}_0).$$

If $K > 0$, *then*

$$\sup_{t\geq 0}\mathbb{E}\|\mathrm{u}(t)\|_{\mathcal{H}}^2 < \infty.$$

Proof of Theorem 4.4.3. Define a new Lyapunov function in terms of operator P by

$$\Phi(x) = \frac{1}{2}\langle Px, x\rangle_{\mathcal{H}} + M(\|B^{\frac{1}{2}}x_1\|_H^2),\ x \in \mathcal{H}.$$

Since $m \in C^1$ and $P \in \mathcal{L}(\mathcal{H})$, $\Phi \in C^2(\mathcal{H})$. Under the Assumptions (4.1.7) and (4.2.1), Theorems 4.2.7 and 4.3.2 imply that the Equation (4.0.5) has a unique global mild solution $\mathrm{u}(t)$, $t \geq 0$ given by

$$\mathrm{u}(t\wedge\tau_n) = e^{t\mathcal{A}}\mathrm{u}_0 + \int_0^{t\wedge\tau_n} e^{(t\wedge\tau_n - s)\mathcal{A}}F(s, \mathrm{u}(s))ds + I_{\tau_n}(G(\mathrm{u}))(t\wedge\tau_n) \text{ a.s., } t \geq 0. \tag{4.4.1}$$

where

$$I_{\tau_n}(G(\mathrm{u}))(t) = \int_0^t\int_Z 1_{(0,\tau_n]}e^{(t-s)\mathcal{A}}G(s, \mathrm{u}(s-), z)\tilde{N}(ds, dz),\ t \geq 0$$

and $\{\tau_n\}_{n\in\mathbb{N}}$ is an accessible sequence and $\lim_{n\to\infty}\tau_n = \tau_\infty = \infty$. We have already seen in the proof of Theorem 4.3.2 that the idea of getting an estimate of our Lyapunov function with mild solution is to approximate the mild solution by a sequence of strong solutions, to which we can apply Itô formula. We shall examine the new Lyapunov function Φ in the same way as for V. Let n be fixed. We first define functions $\tilde{F}$ and $\tilde{G}$ by,

$$\tilde{F}(t) = 1_{(0,\tau_n]}(t)F(\mathrm{u}(t\wedge\tau_n)) = \begin{pmatrix} 0 \\ -\tilde{f}(t) - m(\|B^{\frac{1}{2}}u(t\wedge\tau_n)\|_H^2)Bu(t\wedge\tau_n)1_{(0,\tau_n]}(t) \end{pmatrix}, \; t\in[0,T],$$

$$\tilde{G}(t) = 1_{(0,\tau_n]}(t)G(t,\mathrm{u}(t\wedge\tau_n-),z) = \begin{pmatrix} 0 \\ \tilde{g}(t,z) \end{pmatrix}, \; t\in[0,T].$$

Here $\tilde{f}(t) = -1_{(0,\tau_n]}(t)\beta u(t\wedge\tau_n)$, $t\geq 0$ and $\tilde{g}(t,z) = 1_{(0,\tau_n]}(t)g(t,\mathrm{u}(t\wedge\tau_n-),z)$, $t\geq 0$. Then the following Equation

$$\begin{aligned} dv(t) &= \mathcal{A}v(t)dt + \tilde{F}(t)dt + \int_Z \tilde{G}(t,z)\tilde{N}(dt,dz) \\ v(0) &= \mathrm{u}(0) \end{aligned} \tag{4.4.2}$$

has a unique global mild solution which satisfies

$$v(t) = e^{t\mathcal{A}}\mathrm{u}(0) + \int_0^t e^{(t-s)\mathcal{A}}\tilde{F}(s)ds + \int_0^t\int_Z e^{(t-s)\mathcal{A}}\tilde{G}(s,z)\tilde{N}(ds,dz), \;\mathbb{P}\text{-a.s.}, t\geq 0. \tag{4.4.3}$$

Since u is the local mild solution, so u satisfies (4.2.2), a similar argument used in the proof of Theorem 4.3.2 yields that for each $n\in\mathbb{N}$

$$v(t\wedge\tau_n) = \mathrm{u}(t\wedge\tau_n) \;\; \mathbb{P}\text{-a.s. } t\geq 0.$$

Set

$$\begin{aligned} \mathrm{u}_m(0) &= mR(m;\mathcal{A})\mathrm{u}(0) \\ \tilde{F}_m(t,\omega) &= mR(m;\mathcal{A})\tilde{F}(t,\omega) \text{ for } (t,\omega)\in\mathbb{R}_+\times\Omega; \\ \tilde{G}_m(t,\omega,z) &= mR(m;\mathcal{A})\tilde{G}(t,\omega,z) \text{ for } (t,\omega,z)\in\mathbb{R}_+\times\Omega\times Z. \end{aligned}$$

In exactly the same manner as in the proof of Theorem 4.3.2 we infer that $\mathbb{P}$-a.s.

$$\lim_{m\to\infty}\int_0^T \|\tilde{F}_m(t) - \tilde{F}(t)\|_{\mathcal{H}}^2 dt = 0 \tag{4.4.4}$$

$$\lim_{m\to\infty}\int_0^T\int_Z |\tilde{G}_m(t,z) - \tilde{G}(t,z)|_{\mathcal{H}}^2\nu(dz)dt = 0. \tag{4.4.5}$$

Also, we find out that $\tilde{G}_m\in\mathcal{M}^2([0,T]\times\Omega\times Z,\hat{\mathcal{P}},\lambda\otimes\mathbb{P}\times\nu;\mathcal{D}(\mathcal{A}))$. By using the Theorem 4.1.10, one can see that the equation

$$\begin{aligned} dv_m(t) &= \mathcal{A}v_m(t)dt + \tilde{F}_m(t)dt + \int_Z \tilde{G}_m(t,z)\tilde{N}(dt,dz) \\ v_m(0) &= \mathrm{u}(0) \end{aligned}$$

has a unique strong solution given by

$$v_m(t) = \mathrm{u}_m(0) + \int_0^t \Big[\mathcal{A}v_m(s) + \tilde{F}_m(s)\Big]ds + \int_0^t\int_Z \tilde{G}_m(ss,z)\tilde{N}(ds,dz) \;\mathbb{P}\text{-a.s.}, t\geq 0. \tag{4.4.6}$$

Equivalently, we can also write the solution in the mild form

$$v_m(t) = e^{t\mathcal{A}}\mathrm{u}(0) + \int_0^t e^{(t-s)\mathcal{A}}\tilde{F}_m(s)ds + \int_0^t \int_Z e^{(t-s)\mathcal{A}}\tilde{G}_m(s,z)\tilde{N}(ds,dz), \ \mathbb{P}\text{-a.s.}, t \geq 0. \tag{4.4.7}$$

Now applying the Itô Formula, see Theorem 3.5.3, to function $\Phi(x)e^{\lambda t}$ and the strong solution v_m yields

$$\begin{aligned}\Phi(v_m(t))e^{\lambda t} &= \Phi(v_m(s))e^{\lambda s} + \int_s^t e^{\lambda r}\Big[\lambda\Phi(v_m(r)) + \langle D\Phi(v_m(r)), \mathcal{A}v_m(r) + \tilde{F}_m(r)\rangle_{\mathcal{H}}\Big]dr \\ &+ \int_s^t \int_Z e^{\lambda r}\Big[\Phi(v_m(r) + \tilde{G}_m(r,z)) - \Phi(v_m(r)) - \langle D\Phi(v_m(s), \tilde{G}_m(r,z))\rangle_{\mathcal{H}}\Big]\nu(dz)dr \\ &+ \int_s^t \int_Z e^{\lambda t}\Big[\Phi(v_m(r-) + \tilde{G}_m(r,z)) - \Phi(v_m(r-))\Big]\tilde{N}(dr,dz).\end{aligned} \tag{4.4.8}$$

We first find the following facts

$$\begin{aligned}D\Phi(x)h &= \langle Ph, x\rangle_{\mathcal{H}} + 2m(\|B^{\frac{1}{2}}x_1\|_H^2)\langle B^{\frac{1}{2}}x_1, B^{\frac{1}{2}}h_1\rangle \\ &= \langle Ph, x\rangle_{\mathcal{H}} + 2m(\|B^{\frac{1}{2}}x_1\|_H^2)\langle \begin{pmatrix} A^{-2}Bx_1 \\ 0 \end{pmatrix}, \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}\rangle_{\mathcal{H}},\end{aligned}$$

where $x = (x_1, x_2)^\top$, $h = (h_1, h_2)^\top$ and $k = (k_1, k_2)^\top$ are all in $\mathcal{H}$. One can also rewrite the derivative $D\Phi$ as follows

$$D\Phi(x) = Px + 2m(\|B^{\frac{1}{2}}x_1\|_H^2)\begin{pmatrix} A^{-2}Bx_1 \\ 0 \end{pmatrix} \quad x \in \mathcal{H}.$$

We adopt the projections π_1 and π_2 which are defined in the proof of Theorem 4.3.2. Therefore, by using above derivative formula we get

$$\begin{aligned}&\langle D\Phi(v_m(r)), \mathcal{A}v_m(r) + \tilde{F}_m(r)\rangle_{\mathcal{H}} \\ &= \langle D\Phi(v_m(r)), \mathcal{A}v_m(r)\rangle_{\mathcal{H}} + \langle D\Phi(v_m(r)), \tilde{F}_m(r)\rangle_{\mathcal{H}} \\ &= \langle Pv_m(r) + 2m(\|B^{\frac{1}{2}}\pi_1 v_m(r)\|_H^2)\begin{pmatrix} A^{-2}B\pi_1 v_m(r) \\ 0 \end{pmatrix}, \mathcal{A}v_m(r)\rangle_{\mathcal{H}} \\ &\quad + \langle Pv_m(r) + 2m(\|B^{\frac{1}{2}}\pi_1 v_m(r)\|_H^2)\begin{pmatrix} A^{-2}B\pi_1 v_m(r) \\ 0 \end{pmatrix}, \tilde{F}_m(r)\rangle_{\mathcal{H}} \\ &= \langle Pv_m(r), \mathcal{A}v_m(r)\rangle_{\mathcal{H}} + 2m(\|B^{\frac{1}{2}}\pi_1 v_m(r)\|_H^2)\langle B\pi_1 v_m(r), \pi_2 v_m(r)\rangle_H \\ &\quad + \langle Pv_m(r), \tilde{F}_m(r)(r)\rangle_{\mathcal{H}} + 2m(\|B^{\frac{1}{2}}\pi_1 v_m(r)\|_H^2)\langle B\pi_1 v_m(r), \pi_1\tilde{F}_m(r)\rangle_H, \ r \in [0,T].\end{aligned} \tag{4.4.9}$$

From Lemma 4.4.2 and the fact that $A \geq \mu I$ for some $\mu > 0$, we have

$$\begin{aligned}\langle Pv_m(r), \mathcal{A}v_m(r)\rangle_{\mathcal{H}} &= -\beta\|A\pi_1 v_m(r)\|_H^2 + \beta^2\langle \pi_1 v_m(r), \pi_2 v_m(r)\rangle + \beta\|\pi_2 v_m(r)\|_H^2 \\ &\leq -\beta\|A\pi_1 v_m(r)\|_H^2 + \beta^2\|\pi_1 v_m(r)\|_H\|\pi_2 v_m(r)\|_H + \beta\|\pi_2 v_m(r)\|_H^2 \\ &\leq -\beta\|A\pi_1 v_m(r)\|_H^2 + \frac{\beta^2}{2}(\|\pi_1 v_m(r)\|_H^2 + \|\pi_2 v_m(r)\|_H^2) + \beta\|\pi_2 v_m(r)\|_H^2 \\ &\leq -\beta\|A\pi_1 v_m(r)\|_H^2 + \frac{\beta^2}{2\mu^2}\|A\pi_1 v_m(r)\|_H^2 + \frac{\beta^2}{2}\|\pi_2 v_m(r)\|_H^2 + \beta\|\pi_2 v_m(r)\|_H^2 \\ &= \Big(\frac{\beta^2}{2\mu^2} - \beta\Big)\|A\pi_1 v_m(r)\|_H^2 + \Big(\frac{\beta^2}{2} + \beta\Big)\|\pi_2 v_m(r)\|_H^2, \ r \geq 0.\end{aligned}$$

Recall that in the proof of Theorem 4.3.2 we have shown that for every $0 < T < \infty$,

$$\lim_{m\to\infty} \mathbb{E} \sup_{0\le t\le T} \|v_m(t) - v(t)\|_{\mathcal{H}}^2 = 0, \tag{4.4.10}$$

So there exists a subsequence, denoted also by $\{v_m(t)\}_{m\in\mathbb{N}}$ for simplicity, such that $v_m(t) \to v(t)$ uniformly on $[s,t]$ as $k \to \infty$ a.s.
Therefore,

$$\begin{aligned}\limsup_{m\to\infty} \langle Pv_m(r), \mathcal{A}v_m(r)\rangle_{\mathcal{H}} &\le \limsup_{m\to\infty} \left(\frac{\beta^2}{2\mu^2} - \beta\right) \|A\pi_1 v_m(r)\|_H^2 + \left(\frac{\beta^2}{2} + \beta\right) \|\pi_2 v_m(r)\|_H^2 \\ &= \left(\frac{\beta^2}{2\mu^2} - \beta\right) \|A\pi_1 v(r)\|_H^2 + \left(\frac{\beta^2}{2} + \beta\right) \|\pi_2 v(r)\|_H^2, \ r \in [s,t].\end{aligned}$$

Now by applying the Fatou Lemma we infer

$$\begin{aligned}\limsup_{m\to\infty} \int_s^t e^{\lambda r} \langle Pv_m(r), \mathcal{A}v_m(r)\rangle_{\mathcal{H}} dr &\le \int_s^t e^{\lambda r} \limsup_{m\to\infty} \langle \mathcal{A}v_m(r), Pv_m(r)\rangle_{\mathcal{H}} \\ &\le \int_s^t \Big[\Big(\frac{\beta^2}{2\mu^2} - \beta\Big)\|A\pi_1 v(r)\|_H^2 + \Big(\frac{\beta^2}{2} + \beta\Big)\|\pi_2 v(r)\|_H^2\Big] dr.\end{aligned}$$

Further, by the above derivative formula of $D\Phi$ and definition of Lyapunov function of Φ we get

$$\begin{aligned}&\langle D\Phi(v_m(r)), \tilde{G}_m(r,z)\rangle_{\mathcal{H}} \\ &\quad = \langle Pv_m(r), \tilde{G}_m(r,z)\rangle_{\mathcal{H}} + 2m(\|B^{\frac{1}{2}}\pi_1 v_m(r)\|_H^2)\langle B\pi_1 v_m(r), \pi_1 \tilde{G}_m(r,z)\rangle_{\mathcal{H}}, \ r \ge 0. \end{aligned} \tag{4.4.11}$$

and

$$\begin{aligned}&\Phi(v_m(r) + \tilde{G}_m(r,z)) - \Phi(v_m(r)) \\ &\quad = \frac{1}{2}\langle P\big(v_m(r) + \tilde{G}_m(r,z)\big), v_m(r) + \tilde{G}_m(r,z)\rangle_{\mathcal{H}} + M\big(\|B^{\frac{1}{2}}\pi_1\big(v_m(r) + \tilde{G}_m(r,z)\big)\|\big) \\ &\qquad - \frac{1}{2}\langle Pv_m(r), v_m(r)\rangle_{\mathcal{H}} - M(\|B^{\frac{1}{2}}\pi_1 v_m(r)\|_H^2) \\ &\quad = \frac{1}{2}\langle Pv_m(r), \tilde{G}_m(r,z)\rangle_{\mathcal{H}} + \frac{1}{2}\langle P\tilde{G}_m(r,z), v_m(r)\rangle_{\mathcal{H}} \\ &\qquad + \frac{1}{2}\langle P\tilde{G}_m(r,z), \tilde{G}_m(r,z)\rangle_{\mathcal{H}} + M(\|B^{\frac{1}{2}}\pi_1(v_m(r) + \tilde{G}_m(r,z))\|_H^2) \\ &\qquad - M(\|B^{\frac{1}{2}}\pi_1 v_m(r)\|_H^2) \\ &\quad = \langle Pv_m(r), \tilde{G}_m(r,z)\rangle_{\mathcal{H}} + \frac{1}{2}\langle P\tilde{G}_m(r,z), \tilde{G}_m(r,z)\rangle_{\mathcal{H}} \\ &\qquad + M(\|B^{\frac{1}{2}}\pi_1(v_m(r) + \tilde{G}_m(r,z))\|_H^2) - M(\|B^{\frac{1}{2}}\pi_1 v_m(r)\|_H^2), \ r \ge 0. \end{aligned} \tag{4.4.12}$$

Combining the above two equalities, we find that

$$\begin{aligned}&\Phi(v_m(r) + \tilde{G}_m(r,z)) - \Phi(v_m(r)) - \langle D\Phi(v_m(r)), \tilde{G}_m(r,z)\rangle_{\mathcal{H}} \\ &\quad = \frac{1}{2}\langle P\tilde{G}_m(r,z), \tilde{G}_m(r,z)\rangle_{\mathcal{H}} + M(\|B^{\frac{1}{2}}\pi_1(v_m(r) + \tilde{G}_m(r,z))\|_H^2) \\ &\qquad - M(\|B^{\frac{1}{2}}\pi_1 v_m(r)\|_H^2) - 2m(\|B^{\frac{1}{2}}\pi_1 v_m(r)\|_H^2)\langle B\pi_1 v_m(r), \pi_1 \tilde{G}_m(r,z)\rangle_{\mathcal{H}},\end{aligned}$$

which converges $\mathbb{P}$-a.s. to

$$\Phi(v(r) + \tilde{G}(r,z)) - \Phi(v(r)) - \langle D\Phi(v(r)), \tilde{G}(r,z)\rangle_{\mathcal{H}} = \frac{1}{2}\langle P\tilde{G}(r,z), \tilde{G}(r,z)\rangle_{\mathcal{H}}, \text{ as } m \to \infty, \ r \ge 0.$$

Also, we find

$$\begin{aligned}&\Phi(v_m(r)+\tilde{G}_m(r,z))-\Phi(v_m(r))\\&\quad=\langle Pv_m(r),\tilde{G}_m(r,z)\rangle_{\mathcal{H}}+\frac{1}{2}\langle P\tilde{G}_m(r,z),\tilde{G}_m(r,z)\rangle_{\mathcal{H}}\\&\qquad+M(\|B^{\frac{1}{2}}\pi_1(v_m(r)+\tilde{G}_m(r,z))\|_H^2)-M(\|B^{\frac{1}{2}}\pi_1 v_m(r)\|_H^2),\ r\geq 0.\end{aligned}$$

This converges $\mathbb{P}$-a.s. to

$$\Phi(v(r)+\tilde{G}(r,z))-\Phi(v(r))=\langle Pv(r),\tilde{G}(r,z)\rangle_{\mathcal{H}}+\frac{1}{2}\langle P\tilde{G}(r,z),\tilde{G}(r,z)\rangle_{\mathcal{H}}\text{ as }m\to\infty,\ r\geq 0.$$

On the other hand, since the function Φ is in $C^2(\mathcal{H})$, by the Taylor formula we infer that

$$\Phi(v_m(r)+\tilde{G}_m(r,z))-\Phi(v_m(r))\leq\sup_{x\in X}\|D\Phi(x)\|\|\tilde{G}_m(r,z)\|_{\mathcal{H}},\ r\in[s,t]$$

and

$$\begin{aligned}\Phi(v_m(r)+\tilde{G}_m(r,z))-\Phi(v_m(r))-\langle D\Phi(v_m(r)),\tilde{G}_m(r,z)\rangle_{\mathcal{H}}&\\\leq\frac{1}{2}\|D^2\Phi(v_m(r))\|\|\tilde{G}_m(r,z)\|_{\mathcal{H}}^2&\\\leq\frac{1}{2}\sup_{x\in X}\|D^2\Phi(x)\|\|\tilde{G}_m(r,z)\|_{\mathcal{H}}^2,&\end{aligned}$$

where we used the uniformly boundedness of $\{v_m\}_{m\in\mathbb{N}}$ on $[s,t]$. Hence it follows from the Lebesgue Dominated Convergence Theorem that for $0\leq s\leq t\leq\infty$,

$$\int_s^t\int_Z e^{\lambda r}\Big[\Phi(v_m(r)+\tilde{G}_m(r,z))-\Phi(v_m(r))-\langle D\Phi(v_m(s),\tilde{G}_m(r,z))\rangle_{\mathcal{H}}\Big]\nu(dz)dr$$

converges $\mathbb{P}$-a.s. to

$$\int_s^t\int_Z\frac{e^{\lambda r}}{2}\langle P\tilde{G}(r,z),\tilde{G}(r,z)\rangle_{\mathcal{H}}\nu(dz)dr.$$

On the basis of the Itô isometry for stochastic integral w.r.t. the compensated Poisson random measure (see Theorem 3.3.2), we obtain

$$\begin{aligned}&\mathbb{E}\Big\|\int_s^t\int_Z e^{\lambda t}\Big[\Phi(v_m(r-)+\tilde{G}_m(r,z))-\Phi(v_m(r-))\Big]\tilde{N}(dr,dz)\\&\quad-\int_s^t\int_Z e^{\lambda t}\Big[\langle Pv(r-),\tilde{G}(r,z)\rangle_{\mathcal{H}}+\frac{1}{2}\langle P\tilde{G}(r,z),\tilde{G}(r,z)\rangle_{\mathcal{H}}\Big]\tilde{N}(dr,dz)\Big\|^2\\&\quad\leq\mathbb{E}\int_s^t\int_Z e^{2\lambda}\Big\|\Phi(v_m(r)+\tilde{G}_m(r,z))-\Phi(v_m(r))-\langle Pv(r),\tilde{G}(r,z)\rangle_{\mathcal{H}}\\&\quad-\frac{1}{2}\langle P\tilde{G}(r,z),\tilde{G}(r,z)\rangle_{\mathcal{H}}\Big\|_{\mathcal{H}}^2\nu(dz)dr.\end{aligned}$$

Note that the integrand on the right side of above equality is dominated by

$$2\sup_{x\in X}\|D\Phi(x)\|^2\|\tilde{G}(s,z)\|^2,$$

where X is a compact set on $\mathcal{H}$. Again, by passing to the limit as $m \to \infty$, the Lebesgue Dominated Convergence Theorem tells us that the right-side of above equality converges to 0. Hence, by taking a subsequence we infer that

$$\int_s^t \int_Z e^{\lambda t}\Big[\Phi(v_m(r-) + \tilde{G}_m(r-, z)) - \Phi(v_m(r-))\Big]\tilde{N}(dr, dz)$$

converges $\mathbb{P}$-a.s. to

$$\int_s^t \int_Z e^{\lambda t}\Big[\langle Pv(r-), \tilde{G}(r-, z)\rangle_{\mathcal{H}} + \frac{1}{2}\langle P\tilde{G}(r-, z), \tilde{G}(r-, z)\rangle_{\mathcal{H}}\Big]\tilde{N}(dr, dz) \text{ as } m \to \infty.$$

Combining all the observations together and letting $m \to \infty$ yields that for $0 \le s \le t < \infty$

$$\begin{aligned}\Phi(v(t))e^{\lambda t} \le \Phi(v(s))e^{\lambda s} + \int_s^t e^{\lambda r}\Big[&\lambda\Phi(v(r)) + \Big(\frac{\beta^2}{2\mu^2} - \beta\Big)\|A\pi_1 v(r)\|_H^2 + \Big(\frac{\beta^2}{2} + \beta\Big)\|\pi_2 v(r)\|_H^2 \\ &+ 2m(\|B^{\frac{1}{2}}\pi v(r)\|_H^2)\langle B\pi_1 v(r), \pi_2 v(r)\rangle_H - \langle \beta\pi_1 v(r) + 2\pi_2 v(r), \pi_2\tilde{F}(r)\rangle_H\Big]dr \\ &+ \int_s^t \int_Z e^{\lambda r}\|\tilde{g}(r, z)\|_H^2 \nu dz dr \\ &+ \int_s^{t+} \int_Z e^{\lambda r}\Big[\langle \beta\pi_1 v(r-) + 2\pi_2 v(r-), \tilde{g}(r, z)\rangle_{\mathcal{H}} + \|\tilde{g}(r, z)\|_H^2\Big]\tilde{N}(dr, dz), \ \mathbb{P}\text{-a.s.}\end{aligned}$$

Recall that for every $n \in \mathbb{N}$, $v(t \wedge \tau_n) = \mathrm{u}(t \wedge \tau_n)$ $\mathbb{P}$-a.s., by replacing t by $t \wedge \tau_n$ in the above inequality we have, for every $0 \le s \le t < \infty$,

$$\begin{aligned}&\Phi(\mathrm{u}(t \wedge \tau_n))e^{\lambda(t\wedge\tau_n)} \\ &\le \Phi(\mathrm{u}(s)) + \int_s^{t\wedge\tau_n} e^{\lambda r}\Big[\lambda\Phi(\mathrm{u}(r)) + \Big(\frac{\beta^2}{2\mu^2} - \beta\Big)\|Au(r)\|_H^2 + \Big(\frac{\beta^2}{2} + \beta\Big)\|u_t(r)\|_H^2 \\ &\qquad + 2m(\|B^{\frac{1}{2}}u(r)\|_H^2)\langle Bu(r), u_t(r)\rangle_H - \langle \beta u(r) + 2u_t(r), \beta u_t(r)\rangle_H \\ &\qquad - m(\|B^{\frac{1}{2}}u(r)\|_H^2)\langle \beta u(r) + 2u_t(r), Bu(r)\rangle_H\Big]dr \\ &\quad + \int_s^{t\wedge\tau_n} \int_Z e^{\lambda r}\|g(r, \mathrm{u}(r), z)\|_H^2 \nu(dz)dr \\ &\quad + \int_s^{t\wedge\tau_n +} \int_Z e^{\lambda r}\Big[\langle \beta\pi_1\mathrm{u}(r-) + 2\pi_2\mathrm{u}(r-), g(r, \mathrm{u}(r), z)\rangle_{\mathcal{H}} + \|g(r, \mathrm{u}(r), z)\|_H^2\Big]\tilde{N}(dr, dz) \\ &= \Phi(\mathrm{u}(s)) + \int_s^{t\wedge\tau_n} e^{\lambda r}\Big[\lambda\Phi(\mathrm{u}(r)) + \Big(\frac{\beta^2}{2\mu^2} - \beta\Big)\|Au(r)\|_H^2 + \Big(\frac{\beta^2}{2} - \beta\Big)\|u_t(r)\|_H^2 \\ &\qquad - \beta^2\langle u(r), u_t(r)\rangle_H - m(\|B^{\frac{1}{2}}u(r)\|_H^2)\langle \beta u(r), Bu(r)\rangle_H\Big]dr \\ &\quad + \int_s^{t\wedge\tau_n} \int_Z e^{\lambda r}\|g(r, \mathrm{u}(r), z)\|_H^2 \nu(dz)dr \\ &\quad + \int_s^{t\wedge\tau_n +} \int_Z e^{\lambda r}\Big[\langle \beta\pi_1\mathrm{u}(r-) + 2\pi_2\mathrm{u}(r-), g(r, \mathrm{u}(r), z)\rangle_{\mathcal{H}} + \|\tilde{g}(r, z)\|_H^2\Big]\tilde{N}(dr, dz) \\ &\le \Phi(\mathrm{u}(s)) + \int_s^{t\wedge\tau_n} e^{\lambda r}\Big[\lambda\Phi(\mathrm{u}(r)) + (2C\beta^2 - \beta)\|\mathrm{u}(r)\|_{\mathcal{H}}^2 - \beta m(\|B^{\frac{1}{2}}u(r)\|_H^2)\|B^{\frac{1}{2}}u(r)\|_H^2\Big]dr \\ &\quad + \int_s^{t\wedge\tau_n} \int_Z e^{\lambda r}\|g(r, \mathrm{u}(r), z)\|_H^2 \nu dz dr \\ &\quad + \int_s^{t\wedge\tau_n +} \int_Z e^{\lambda r}\Big[\langle \beta\pi_1\mathrm{u}(r-) + 2\pi_2\mathrm{u}(r-), g(r, \mathrm{u}(r), z)\rangle_{\mathcal{H}} + \|g(r, \mathrm{u}(r), z)\|_H^2\Big]\tilde{N}(dr, dz),\end{aligned}$$

where $C = \max\{\frac{1}{2\mu^2}, \frac{1}{2}\}$. Now applying part (3) of the Assumption (4.4.1) and the definition of the function Φ yields that for $0 \le s \le t < \infty$,

$$\begin{aligned}
&\Phi(\mathrm{u}(t \wedge \tau_n))e^{\lambda(t\wedge\tau_n)} \\
&\quad \le \Phi(\mathrm{u}(s)) + \int_s^{t\wedge\tau_n} e^{\lambda r}\Big[\lambda\Phi(\mathrm{u}(r)) + (2C\beta^2 - \beta)\|\mathrm{u}(r)\|_{\mathcal{H}}^2 - \beta m(\|B^{\frac{1}{2}}u(r)\|_H^2)\|B^{\frac{1}{2}}u(r)\|_H^2 \\
&\qquad\qquad + R_g^2\|\mathrm{u}(r)\|_{\mathcal{H}}^2 + K\Big]dr \\
&\qquad + \int_s^{t\wedge\tau_n}\int_Z e^{\lambda r}\Big[\langle\beta\pi_1\mathrm{u}(r-) + 2\pi_2\mathrm{u}(r-), g(r,\mathrm{u}(r),z)\rangle_{\mathcal{H}} + \|g(r,\mathrm{u}(r),z)\|_H^2\Big]\tilde{N}(dr,dz) \\
&\quad = \Phi(\mathrm{u}(s)) + \int_s^{t\wedge\tau_n} e^{\lambda r}\Big[\frac{\lambda}{2}\langle P\mathrm{u}(r),\mathrm{u}(r)\rangle_{\mathcal{H}} + \lambda M(\|B^{\frac{1}{2}}u(r)\|_H^2) + (R_g^2 + 2C\beta^2 - \beta)\|\mathrm{u}(r)\|_{\mathcal{H}}^2 \\
&\qquad\qquad - \beta m(\|B^{\frac{1}{2}}u(r)\|_H^2)\|B^{\frac{1}{2}}u(r)\|_H^2 + K\Big]dr \\
&\qquad + \int_s^{t\wedge\tau_n}\int_Z e^{\lambda r}\Big[\langle\beta\pi_1\mathrm{u}(r-) + 2\pi_2\mathrm{u}(r-), g(r,\mathrm{u}(r),z)\rangle_{\mathcal{H}} + \|g(r,\mathrm{u}(r),z)\|_H^2\Big]\tilde{N}(dr,dz) \\
&\quad \le \Phi(\mathrm{u}(s)) + \int_s^{t\wedge\tau_n} e^{\lambda r}\Big[\Big(\frac{\lambda}{2}\|P\|_{\mathcal{L}(H)} + R_g^2 + 2C\beta^2 - \beta\Big)\|\mathrm{u}(r)\|_{\mathcal{H}}^2 \\
&\qquad\qquad + (\frac{\lambda}{\alpha} - \beta)m(\|B^{\frac{1}{2}}u(r)\|_H^2)\|B^{\frac{1}{2}}u(r)\|_H^2 + K\Big]dr \\
&\qquad + \int_s^{t\wedge\tau_n}\int_Z e^{\lambda r}\Big[\langle\beta\pi_1\mathrm{u}(r-) + 2\pi_2\mathrm{u}(r-), g(r,\mathrm{u}(r),z)\rangle_{\mathcal{H}} + \|g(r,\mathrm{u}(r),z)\|_H^2\Big]\tilde{N}(dr,dz),
\end{aligned}$$

where we used the inequality $\langle Px, x\rangle_H \le \|Px\|_{\mathcal{L}(H)}\|x\|_H^2$ in the last inequality.
Now let $n \to \infty$. Since in Theorem 4.3.2, $\tau_\infty = \infty$, we have for $0 \le s \le t < \infty$,

$$\begin{aligned}
\Phi(\mathrm{u}(t))e^{\lambda t} \le \Phi(\mathrm{u}(s)) &+ \int_s^t e^{\lambda r}\Big[\Big(\frac{\lambda}{2}\|P\|_{\mathcal{L}(H)} + R_g^2 + 2C\beta^2 - \beta\Big)\|\mathrm{u}(r)\|_{\mathcal{H}}^2 \\
&\qquad + (\frac{\lambda}{\alpha} - \beta)m(\|B^{\frac{1}{2}}u(r)\|_H^2)\|B^{\frac{1}{2}}u(r)\|_H^2 + K\Big]dr \\
&+ \int_s^{t+}\int_Z e^{\lambda r}\Big[\langle\beta\pi_1 v(r-) + 2\pi_2 v(r-), \tilde{g}(r,z)\rangle_{\mathcal{H}} + \|\tilde{g}(r,z)\|_H^2\Big]\tilde{N}(dr,dz).
\end{aligned}$$

Choose λ such that $0 < \lambda < 2\|P\|_{\mathcal{L}(H)}^{-1}(\beta - 2C\beta^2 - R_g^2) \wedge \alpha\beta$. It follows that

$$\frac{\lambda}{2}\|P\|_{\mathcal{L}(H)} + R_g^2 + 2C\beta^2 - \beta < 0 \text{ and } \frac{\lambda}{2} - \beta < 0.$$

Therefore, we infer that for $0 \le s \le t < \infty$,

$$\begin{aligned}
\Phi(\mathrm{u}(t))e^{\lambda t} \le \Phi(\mathrm{u}(s)) &+ \int_s^t e^{\lambda r}K\,dr \\
&+ \int_s^t\int_Z e^{\lambda r}\Big[\langle\beta\pi_1 v(r-) + 2\pi_2 v(r-), \tilde{g}(r,z)\rangle_{\mathcal{H}} + \|\tilde{g}(r,z)\|_H^2\Big]\tilde{N}(dr,dz).
\end{aligned} \tag{4.4.13}$$

First consider the case when $K = 0$. Then equality (4.4.13) becomes, $0 \le s \le t < \infty$,

$$\Phi(\mathrm{u}(t))e^{\lambda t} \le \Phi(\mathrm{u}(s)) + \int_s^t\int_Z e^{\lambda r}\Big[\langle\beta\pi_1 v(r-) + 2\pi_2 v(r-), \tilde{g}(r,z)\rangle_{\mathcal{H}} + \|\tilde{g}(r,z)\|_H^2\Big]\tilde{N}(dr,dz). \tag{4.4.14}$$

Taking the conditional expectation with respect to $\mathcal{F}_s$ to both sides yields

$$\begin{aligned}
\mathbb{E}\big(\Phi(\mathrm{u}(t))e^{\lambda t}\big|\mathcal{F}_s\big) &\le \mathbb{E}\big(\Phi(\mathrm{u}(s))\big|\mathcal{F}_s\big) \\
&\quad + \mathbb{E}\Big(\int_s^t \int_Z e^{\lambda r}\Big[\langle \beta\pi_1 v(r-) + 2\pi_2 v(r-), \tilde{g}(r,z)\rangle_{\mathcal{H}} + \|\tilde{g}(r,z)\|_H^2\Big]\tilde{N}(dr,dz)\Big|\mathcal{F}_s\Big) \\
&= \Phi(\mathrm{u}(s)),\ 0 \le s \le t < \infty,
\end{aligned}$$

where the equality follows from the measurability of $\Phi(\mathrm{u}(s))$ with respect to $\mathcal{F}_s$ and independence of the integrals with respect to $\mathcal{F}_s$. This means that the process $\Phi(\mathrm{u}(t))e^{\lambda t}$ is a supermartingale. Take $\lambda^* \in (0,\lambda)$. We observe that for every $k = 0, 1, 2\cdots$,

$$\sup_{t\in[k,k+1]} e^{\lambda^* t}\Phi(\mathrm{u}(t)) = \sup_{t\in[k,k+1]} e^{(\lambda^*-\lambda)t}e^{\lambda t}\Phi(\mathrm{u}(t)) \le e^{(\lambda^*-\lambda)k}\sup_{t\in[k,k+1]} e^{\lambda t}\Phi(\mathrm{u}(t)).$$

Therefore,

$$\begin{aligned}
\mathbb{P}\Big\{\sup_{t\in[k,k+1]} e^{\lambda^* t}\Phi(\mathrm{u}(t)) \ge \mathbb{E}\Phi(\mathrm{u}(0))\Big\} &\le \mathbb{P}\Big\{\sup_{t\in[k,k+1]} e^{\lambda t}\Phi(\mathrm{u}(t)) \ge e^{(\lambda-\lambda^*)k}\mathbb{E}\Phi(\mathrm{u}(0))\Big\} \\
&\le \frac{\mathbb{E}\big(e^{\lambda k}\Phi(\mathrm{u}(k))\big)}{e^{(\lambda-\lambda^*)k}\mathbb{E}\Phi(\mathrm{u}(0))} \\
&\le \frac{\mathbb{E}\Phi(\mathrm{u}(0))}{e^{(\lambda-\lambda^*)k}\mathbb{E}\Phi(\mathrm{u}(0))} = e^{-(\lambda-\lambda^*)k}
\end{aligned}$$

By the ratio test, we know that the series $\sum_{k=1}^{\infty} e^{-(\lambda-\lambda^*)k}$ is convergent. Thus

$$\sum_{k=1}^{\infty}\mathbb{P}\Big\{\sup_{t\in[k,k+1]} e^{\lambda^* t}\Phi(\mathrm{u}(t)) \ge \mathbb{E}\Phi(\mathrm{u}(0))\Big\} \le \sum_{k=1}^{\infty} e^{-(\lambda-\lambda^*)}k < \infty.$$

Now by applying Borel-Cantelli Theorem, we have

$$\mathbb{P}\Big(\bigcap_{j=1}^{\infty}\bigcup_{k\ge j}\Big\{\sup_{t\in[k,k+1]} e^{\lambda^* t}\Phi(\mathrm{u}(t)) \ge \mathbb{E}\Phi(\mathrm{u}(0))\Big\}\Big) = 0.$$

It follows that

$$\mathbb{P}\Big(\bigcup_{j=1}^{\infty}\bigcap_{k\ge j}\Big\{\sup_{t\in[k,k+1]} e^{\lambda^* t}\Phi(\mathrm{u}(t)) \ge \mathbb{E}\Phi(\mathrm{u}(0))\Big\}\Big) = 1.$$

Therefore, there exists $j \in \mathbb{N}$ such that for every $k \ge j$,

$$\sup_{t\in[k,k+1]} e^{\lambda^*}\Phi(\mathrm{u}(t)) \le \mathbb{E}\Phi(\mathrm{u}(0)) \ \ \mathbb{P}\text{-a.s.}$$

Then we can infer that for every $t \ge j$

$$e^{\lambda^* t}\Phi(\mathrm{u}(t)) \le \mathbb{E}\Phi(\mathrm{u}(0)) \ \ \mathbb{P}\text{-a.s.}.$$

It then follows that

$$\begin{aligned}
\mathbb{E}\|\mathrm{u}(t)\|_H^2 &\le \mathbb{E}\langle P\mathrm{u}(t), \mathrm{u}(t)\rangle_{\mathcal{H}} \\
&\le 2\mathbb{E}\Big[\frac{1}{2}\langle P\mathrm{u}(t), \mathrm{u}(t)\rangle_{\mathcal{H}} + M\big(\|B^{\frac{1}{2}}u(t)\|_H^2\big)\Big] \\
&= \mathbb{E}\Phi(\mathrm{u}(r)) \\
&\le 2e^{-\lambda^* t}\mathbb{E}\Phi(\mathrm{u}(0)),
\end{aligned}$$

where the first inequality follows from part (1) of Lemma 4.4.2, the last inequality follows from above result. Also, note that

$$
\begin{aligned}
\mathbb{E}\Phi(\mathrm{u}(0)) &= \mathbb{E}\left[\frac{1}{2}\langle P\mathrm{u}(0), \mathrm{u}(0)\rangle_{\mathcal{H}} + M(\|B^{\frac{1}{2}}u(0)\|)\right] \\
&= \mathbb{E}\Big[\frac{1}{2}\|P\|_{\mathcal{L}(\mathcal{H})}\|\mathrm{u}(0)\|_{\mathcal{H}}^2 + M(\|B^{\frac{1}{2}}u(0)\|)\Big] \\
&\leq \Big(\frac{1}{2}\|P\|_{\mathcal{L}(\mathcal{H})} + 1\Big)\mathbb{E}\left[\|\mathrm{u}(0)\|_{\mathcal{H}}^2 + M(\|B^{\frac{1}{2}}u(0)\|)\right] \\
&= \Big(\frac{1}{2}\|P\|_{\mathcal{L}(\mathcal{H})} + 1\Big)\mathscr{E}(\mathrm{u}(0)). \qquad (4.4.15)
\end{aligned}
$$

Therefore, we conclude that

$$\mathbb{E}\|\mathrm{u}(t)\|_H^2 \leq 2\Big(\frac{1}{2}\|P\|_{\mathcal{L}(\mathcal{H})} + 1\Big)e^{-\lambda^* t}\mathscr{E}(\mathrm{u}(0)),\ t \geq 0.$$

Set $C = \|P\|_{\mathcal{L}(\mathcal{H})} + 2$. In conclusion, we find out that

$$\mathbb{E}\|\mathrm{u}(t)\|_H^2 \leq Ce^{-\lambda^* t}\mathscr{E}(\mathrm{u}(0)),\ t \geq j,$$

which shows the exponentially mean-square stable of our mild solution.
For the case $K \neq 0$, first taking expectation to both side of (4.4.13) and setting $s = 0$ gives

$$\mathbb{E}\left(\Phi(\mathrm{u}(t))e^{\lambda t}\right) \leq \mathbb{E}\Phi(\mathrm{u}(s)) + \frac{K}{\lambda}(e^{\lambda t} - 1),\ 0 \leq s \leq t < \infty.$$

Thus

$$\mathbb{E}\Phi(\mathrm{u}(t)) \leq e^{-\lambda t}\mathbb{E}\Phi(\mathrm{u}(s)) + \frac{K}{\lambda}(1 - e^{-\lambda t}),\ 0 \leq s \leq t < \infty.$$

By the definition of function Φ, we obtain

$$\mathbb{E}\Big(\frac{1}{2}\langle P\mathrm{u}(t), \mathrm{u}(r)\rangle_{\mathcal{H}}\Big) + \mathbb{E}(M(\|B^{\frac{1}{2}}u(r)\|_H^2)) = \mathbb{E}\Phi(\mathrm{u}(t)) \leq e^{-\lambda t}\mathbb{E}\Phi(\mathrm{u}(0)) + \frac{K}{\lambda}(1 - e^{-\lambda t}).$$

Thus applying the inequality $\|x\|_H^2 \leq \langle x, Px\rangle_H$ from Lemma 4.4.2 gives that

$$
\begin{aligned}
\mathbb{E}\|\mathrm{u}(t)\|_H^2 \leq \mathbb{E}\langle \mathrm{u}(t), P\mathrm{u}(t)\rangle_{\mathcal{H}} &\leq 2e^{-\lambda t}\mathbb{E}\Phi(\mathrm{u}(0)) + \frac{2K}{\lambda}(1 - e^{-\lambda t}) \\
&\leq 2e^{-\lambda t}\mathbb{E}\Phi(\mathrm{u}(0)) + \frac{2K}{\lambda},\ t \geq 0.
\end{aligned}
$$

It then follows from (4.4.15) that

$$\mathbb{E}\|\mathrm{u}(t)\|_H^2 \leq 2e^{-\lambda t}\Big(\frac{1}{2}\|P\|_{\mathcal{L}(\mathcal{H})} + 1\Big)\mathscr{E}(\mathrm{u}(0)) + \frac{2K}{\lambda},\ t \geq 0.$$

Therefore,

$$\sup_{t\geq 0} E\|\mathrm{u}(t)\|_H^2 \leq \left(\|P\|_{\mathcal{L}(\mathcal{H})} + 2\right)\mathscr{E}(\mathrm{u}(0)) + \frac{2K}{\lambda} < \infty,$$

which completes our proof. □

4.5 Stochastic nonlinear beam equations

In this section we will examine that all the results achieved in the preceding section can be applied to the following problem

$$\frac{\partial^2 u}{\partial t^2} - m\left(\int_D |\nabla u|^2 dx\right)\Delta u + \gamma\Delta^2 u + \Upsilon\left(t, x, u, \frac{\partial u}{\partial t}, \nabla u\right) = \int_Z \Pi(t, x, u, \frac{\partial u}{\partial t}, \nabla u, z)\tilde{N}(t, dz) \tag{4.5.1}$$

with the hinged boundary condition

$$u = \Delta u = 0 \text{ on } \partial D. \tag{4.5.2}$$

Here $\Upsilon, \Pi : [0, T] \times D \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ are Borel functions, $m \in C^1(\mathbb{R}_+)$ is a nonnegative function, $\gamma > 0$ and $D \subset \mathbb{R}^n$ is a bounded domain with a $\mathcal{C}^\infty$- boundary ∂D.

We shall also make the following standing assumptions on the functions Υ and Π under considerations.

1. For every $n \in \mathbb{N}$, there exist constants L_N and L such that for all $t \in [0, T]$, $x \in D$, $c_1, c_2 \in \mathbb{R}$ and for all $a_1, a_2 \in \mathbb{R}$, $b_1, b_2 \in \mathbb{R}^n$ satisfying $|a_1|, |a_2| \leq N$ and $|b_1|, |b_2| \leq N$,

$$|\Upsilon(t, x, a_1, b_1, c_1) - \Upsilon(t, x, a_2, b_2, c_2)| \leq L_N|a_1 - a_2| + L_N|b_1 - b_2| + L|c_1 - c_2|. \tag{4.5.3}$$

2. There exist constant L_Υ such that for all $t \in [0, T]$, $x \in D$, $a \in \mathbb{R}$, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$,

$$|\Upsilon(t, x, a, b, c)|^2 \leq L_\Upsilon(1 + |c|^2). \tag{4.5.4}$$

3. There exist constant L' such that for all $t \in [0, T]$, $x \in D$, $c_1, c_2 \in \mathbb{R}$ $a_1, a_2 \in \mathbb{R}$ and $b_1, b_2 \in \mathbb{R}^n$,

$$\int_Z |\Pi(t, x, a_1, b_1, c_1, z) - \Pi(t, x, a_2, b_2, c_2, z)|^2 \nu(dz) \leq L'|a_1 - a_2|^2 + L'|b_1 - b_2|^2 + L'|c_1 - c_2|^2. \tag{4.5.5}$$

4. There exist constant L_Π such that for all $t \in [0, T]$, $x \in D$, $a \in \mathbb{R}$, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$,

$$\int_Z |\Pi(t, x, a, b, c, z)|^2 \nu(dz) \leq L_\Pi(1 + |c|^2). \tag{4.5.6}$$

Let $H = L^2(D)$. Let A and B be both the Laplacian with Dirichlet boundary conditions. That is

$$A\psi = -\Delta\psi, \quad \psi \in \mathcal{D}(A),$$
$$\mathcal{D}(A) = H^2(D) \cap H_0^1(D).$$

Then $A \geq \mu I$, for some $\mu > 0$. To see this, since $\mathcal{D}(A) \subset H_0^1(\Omega)$, on the basis of Poincaré inequality, we have

$$\langle A\psi, \psi\rangle_{L^2(D)} = -\int_D \Delta\psi \cdot \psi dx = \int_D |\nabla\psi|^2 dx \geq C|\psi|_{L^2(D)}, \quad \text{for } \psi \in \mathcal{D}(A).$$

Note that our results are thus valid also for unbounded domains satisfying Poincaré inequality. Let us set

$$f : [0,T] \times \mathcal{D}(A) \times L^2(D) \ni (t, \psi, \phi) \mapsto \Upsilon(t, \cdot, \psi(\cdot), \nabla\psi(\cdot), \phi(\cdot)) \in L^2(D) \tag{4.5.7}$$

and

$$g : [0,T] \times \mathcal{D}(A) \times L^2(D) \ni (t, \psi, \phi) \mapsto \Pi(t, \cdot, \psi(\cdot), \nabla\psi(\cdot), \phi(\cdot)) \in L^2(D). \tag{4.5.8}$$

In such case, one can easily see that equation (4.5.1) is a particular case of equation (4.0.1). In order to make use of the results presented in the preceding section, one also need to verify that all the assumptions 4.1.6, 4.1.7, 4.1.9 and 4.2.1 given in the preceding section on the functions f and g are fulfilled. To prove the local lipschitz continuity of the function f, we first notice first that $\mathcal{D}(A) \subset H^2(D)$. Hence by the Sobolev embedding theorem, when $n = 1$, we have $H^2(D) \hookrightarrow C^1(D)$, so there exists a constant M such that $|\psi|_{L^\infty(D)} + |\nabla\psi|_{L^\infty(D)} \leq M|\psi|_{H^2(D)}$. Take $\phi_i \in H$ and $\psi_i \in \mathcal{D}(A) \subset H^2(D)$, $i = 1, 2$ such that $|\psi_i|_{H^2(D)} \leq N$. It follows that $|\psi|_{L^\infty(D)} \leq MN$ and $|\nabla\psi|_{L^\infty(D)} \leq MN$ which gives that $|\psi(x)| \leq MN$ and $|\nabla\psi(x)| \leq MN$ for almost all $x \in D$. We obtain on the basis of the first assumption 4.5.3 and the boundedness assumption of the domain D

that

$$
\begin{aligned}
|f(t,\psi_1,\phi_1)-&f(t,\psi_2,\phi_2)|_{L^2(D)} \\
&= \int_D |\Upsilon(t,\psi_1(x),\nabla\psi_1(x),\phi_1(x)) - \Upsilon(t,\psi_2(x),\nabla\psi_2(x),\phi_2(x))|^2 dx \\
&\le \int_D L_{MN}|\psi_1(x)-\psi_2(x)|^2 + L_{MN}|\nabla\psi_1(x)-\nabla\psi_2(x)|^2 + L|\phi_1(x)-\phi_2(x)|^2 dx \\
&\le L_{MN}|D||\psi_1-\psi_2|^2_{L^\infty(D)} + L_{MN}|D||\nabla\psi_1-\nabla\psi_2|^2_{L^\infty(D)} + L|\phi_1-\phi_2|^2_{L^2(D)} \\
&\le M^2|D|L_{MN}|\psi_1-\psi_2|^2_{H^2(D)} + L|\phi_1-\phi_2|^2_{L^2(D)},
\end{aligned}
\tag{4.5.9}
$$

In particular, if the function Υ doesn't depend on the third variable, that is $f(t,\psi,\phi)=\Upsilon(t,\cdot,\psi(\cdot),\phi(\cdot))$. The Sobolev embedding theorem tells us that $H^2(D)\hookrightarrow \mathcal{C}(D)$, for $n\le 3$, which implies that there exists K such that $|\psi|_{L^\infty(D)}\le K|\psi|_{H^2(D)}$. Take $\phi_i\in H$ and $\psi_i\in\mathcal{D}(A)\subset H^2(D)$, $i=1,2$ such that $|\psi_i|_{H^2(D)}\le N$. Again, in view of the assumption 4.5.3, we infer that

$$
\begin{aligned}
|f(t,\psi_1,\phi_1)-&f(t,\psi_2,\phi_2)|_{L^2(D)} \\
&= \int_D |\Upsilon(t,\psi_1(x),\phi_1(x)) - \Upsilon(t,\psi_2(x),\phi_2(x))|^2 dx \\
&\le \int_D L_{KN}|\psi_1(x)-\psi_2(x)|^2 + L|\phi_1(x)-\phi_2(x)|^2 dx \\
&\le K^2|D|L_{KN}|\psi_1-\psi_2|^2_{H^2(D)} + L|\phi_1-\phi_2|^2_{L^2(D)}.
\end{aligned}
\tag{4.5.10}
$$

From (4.5.9) and (4.5.10), we see that the function f defined by 4.5.7 is locally Lipschitz continuous which verifies Assumption 4.2.1.

For the growth condition 4.1.6 of f, by making use of Assumption 4.5.4, it can be easily achieved as follows

$$
\begin{aligned}
|f(t,\psi,\phi)|^2_{L^2(D)} &= \int_D |\Upsilon(t,x,\psi(x),\nabla\psi(x),\phi(x))|^2 dx \\
&\le \int_D L_\Upsilon(1+|\phi(x)|^2)dx \\
&\le L_\Upsilon|D|(1+|\phi|^2_{L^2(D)}) \\
&\le L_\Upsilon|D|(1+|\psi|^2_{H^2(D)}+|\phi|^2_{L^2(D)}).
\end{aligned}
$$

Let us now show that the global Lipschitz condition (4.1.8) are satisfied for the function g defined by (4.5.8). Take $\phi_i\in L^2(D)$ and $\psi_i\in\mathcal{D}(A)$. By using Assumption 4.5.5, an analogous calculation as verifying the Lipschitz continuity of f before, shows that if $n = 1$, then

$$\begin{aligned}
&\int_Z |g(t,\psi_1,\phi_1) - g(t,\psi_2,\phi_2)|^2_{L^2(D)}\nu(dz) \\
&\qquad = \int_Z\int_D |\Pi(t,x,\psi_1(x),\nabla\psi_1(x),\phi_1(x)) - \Pi(t,x,\psi_2(x),\nabla\psi_2(x),\phi(x))|^2 dx\nu(dz) \\
&\qquad = \int_D\int_Z |\Pi(t,x,\psi_1(x),\nabla\psi_1(x),\phi_1(x)) - \Pi(t,x,\psi_2(x),\nabla\psi_2(x),\phi(x))|^2 \nu(dz)dx \\
&\qquad \le \int_D L'|\psi_1(x) - \psi_2(x)|^2 + L'|\nabla\psi_1(x) - \nabla\psi_2(x)|^2 + L'|\phi_1(x) - \phi_2(x)|^2 dx \\
&\qquad = L'|\psi_1 - \psi_2|^2_{L^2(D)} + L'|\nabla\psi_1 - \nabla\psi_2|^2_{L^2(D)} + L'|\phi_1 - \phi_2|^2_{L^2(D)} \\
&\qquad \le L'|D||\psi_1 - \psi_2|^2_{L^\infty(D)} + L'|D||\nabla\psi_1 - \nabla\psi_2|^2_{L^\infty(D)} + L'|\phi_1 - \phi_2|^2_{L^2(D)} \\
&\qquad \le L'|D|M^2|\psi_1 - \psi_2|^2_{H^2(D)} + L'|\phi_1 - \phi_2|^2_{L^2(D)},
\end{aligned}$$

and if $n \le 3$ and Π does depends on the third variable, then

$$\begin{aligned}
&\int_Z |g(t,\psi_1,\phi_1) - g(t,\psi_2,\phi_2)|^2_{L^2(D)}\nu(dz) \\
&\qquad = \int_Z\int_D |\Pi(t,x,\psi_1(x),\phi_1(x)) - \Pi(t,x,\psi_2(x),\phi(x))|^2 dx\nu(dz) \\
&\qquad \le \int_D L'|\psi_1(x) - \psi_2(x)|^2 + L'|\phi_1(x) - \phi_2(x)|^2 dx \\
&\qquad = L'|\psi_1 - \psi_2|^2_{L^2(D)} + L'|\nabla\psi_1 - \nabla\psi_2|^2_{L^2(D)} + L'|\phi_1 - \phi_2|^2_{L^2(D)} \\
&\qquad \le L'|D|K^2|\psi_1 - \psi_2|^2_{H^2(D)} + L'|\phi_1 - \phi_2|^2_{L^2(D)},
\end{aligned}$$

which verifies the global Lipschitz condition (4.1.8) of the function g. In exactly the same manner, we have

$$\begin{aligned}
\int_Z |g(t,\psi,\phi)|^2_{L^2(D)}\nu(dz) &= \int_Z\int_D |\Pi(t,x,\psi(x),\nabla\psi(x),\phi(x))|^2 dx\nu(dz) \\
&= \int_D\int_Z |\Pi(t,x,\psi(x),\nabla\psi(x),\phi(x))|^2 \nu(dz)dx \\
&\le L_\Pi \int_D (1 + |\phi(x)|^2)dx \\
&\le L_\Pi|D|(1 + |\psi|^2_{H^2(D)} + |\phi|^2_{L^2(D)}).
\end{aligned}$$

To deal with the Equation (4.5.1) with the clamped boundary condition

$$u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial D,$$

we define an operator C by

$$\mathcal{D}(C) = \{\varphi \in H^4(D) : \varphi = \frac{\partial \varphi}{\partial n} = 0 \text{ on } \partial D\}$$
$$C\varphi = \Delta^2 \varphi, \text{ for } \varphi \in \mathcal{D}(C).$$

It is easy to observe that the operator C is positive. To see this, take $\varphi \in \mathcal{D}(C)$. Then the Green formula tells us that

$$\langle C\varphi, \varphi \rangle_H = \int_D \Delta^2 \varphi \cdot \varphi \, dx = \int_D (\Delta \varphi, \Delta \varphi) dx = \|\Delta \varphi\|_H^2 \geq 0.$$

Further, by Lemma 9.17 in [33] , since $\mathcal{D}(C) \subset H^2(D) \cap H_0^2(D)$, we have

$$\langle C\varphi, \varphi \rangle_H = \|\Delta \varphi\|_H^2 \geq \frac{1}{K} \|u\|_H^2, \quad \phi \in \mathcal{D}(C),$$

where the constant K is independent of φ. This part also shows that the operator C is uniformly positive with $C \geq \frac{1}{K}$. In this case, we set

$$A = C^{\frac{1}{2}}.$$

Then by the uniqueness of positive square root operator, we find out that $A = -\Delta$ and $\mathcal{D}(A) = \{\varphi \in H^2(D) : \varphi = \frac{\partial \phi}{\partial n} = 0 \text{ on } \partial D\}$. Since $\mathcal{D}(A) \subset H_0^1(D)$, by the Poincaré inequality, we infer that $A \geq \mu I$, for some $\mu > 0$. Analogously, we define

$$B\psi = -\Delta \psi, \quad \psi \in \mathcal{D}(A),$$
$$\mathcal{D}(B) = H^2(D) \cap H_0^1(D).$$

By adapting the definitions (4.5.7), (4.5.8) of the functions f and g and assumptions (4.5.3)-(4.5.6) of the functions Υ and Π, all the requirements on the functions f and g are fulfilled in the same way as above.

4.6 The Markov Property

Suppose Assumption 4.2.1 and 4.1.7 hold. From Theorems 4.2.7 and 4.3.2, we know that Problem (4.0.5) has a unique global mild solution satisfying

$$u(t,x) = e^{tA}x + \int_0^t e^{(t-r)A} F(r, u(r,x)) dr + \int_0^t e^{(t-r)A} G(r, u(r-,x), z) \tilde{N}(dr, dz). \tag{4.6.1}$$

Here we assume that the functions F and G both don't depend on the first variable. Suppose that $\mathrm{u}(t,s,x)$ denotes the value of the solution to Equation (4.0.5) at time t which starts at time s from value x and P_t instead of $P_{s,t}$. Define

$$(P_{s,t}\varphi)(x) := \mathbb{E}\big[\varphi(\mathrm{u}(t,s,x))\big] \quad 0 \leq s \leq t \leq T,\ \varphi \in C_b(\mathcal{H})$$

where $B_b(\mathcal{H})$ is the set of all real-valued bounded Borel function on $\mathcal{H}$. For simplicity, we denote $\mathrm{u}(t,x)$ instead of $\mathrm{u}(t,0,x)$.

Proposition 4.6.1. *The transition semigroup P_t is Feller. That is for every $\varphi \in C_b(\mathcal{H})$, $P_t\varphi \in C_b(\mathcal{H})$, $t \geq 0$.*

Proof. Let us first take $x_1, x_2 \in \mathcal{H}$. Define $\tau_n^{x_i} := \inf\{0 \leq t \leq T : \|\mathrm{u}(t,x_i)\|_{\mathcal{H}} > n\}$, $i = 1,2$. Let $\tau_n = \tau_n^{x_1} \wedge \tau_n^{x_2}$. Then $\|\mathrm{u}(t,x_1)\|, \|\mathrm{u}(t,x_2)\| \leq n$ on $[0,\tau_n)$. Let $\varepsilon > 0$ be fixed. Then we obtain

$$\begin{aligned}
&\|P_t\varphi(x_1) - P_t\varphi(x_2)\| = \|\mathbb{E}(\varphi(\mathrm{u}(t,x_1))) - \mathbb{E}(\varphi(\mathrm{u}(t,x_2)))\| \\
&= \Big\|\mathbb{E}\big(\varphi(\mathrm{u}(t,x_1)) - \varphi(\mathrm{u}(t,x_2))1_{\{t<\tau_n\}}\big) + \mathbb{E}\big(\varphi(\mathrm{u}(t,x_1)) - \varphi(\mathrm{u}(t,x_2))1_{\{\tau_n \leq t\}}\big)\Big\| \\
&\leq \Big\|\mathbb{E}\big(\varphi(\mathrm{u}(t,x_1)) - \varphi(\mathrm{u}(t,x_2))1_{\{t<\tau_n\}}\big)\Big\| + \Big\|\mathbb{E}\big(\varphi(\mathrm{u}(t,x_1)) - \varphi(\mathrm{u}(t,x_2))1_{\{\tau_n^{x_1} \leq t\}\cup\{\tau_n^{x_2} \leq t\}}\big)\Big\| \\
&\leq \Big\|\mathbb{E}\big(\varphi(\mathrm{u}(t,x_1)) - \varphi(\mathrm{u}(t,x_2))1_{\{t<\tau_n\}}\big)\Big\| + 2\|\varphi\|\big(\mathbb{P}(\{\tau_n^{x_1} \leq t\}) + \mathbb{P}(\{\tau_n^{x_2} \leq t\})\big),
\end{aligned}$$

where $\|\varphi\| = \sup_{x\in\mathcal{H}} \|\varphi(x)\| < \infty$ by the boundedness of φ. By Lemma 4.3.1 and Theorem 4.2.7 we have

$$\lim_{n\to\infty} \mathbb{P}(\{\tau_n^{x_1} \leq t\}) = 0 \quad \text{and} \quad \lim_{n\to\infty} \mathbb{P}(\{\tau_n^{x_2} \leq t\}) = 0.$$

Then there exists $N \in \mathbb{N}$ such that for all $n > N$ we have

$$\mathbb{P}(\{\tau_n \leq t\}) < \frac{\varepsilon}{8\|\varphi\|}$$
$$\mathbb{P}(\{\tau_n \leq t\}) < \frac{\varepsilon}{8\|\varphi\|}.$$

Thus above estimate becomes

$$\begin{aligned}
\|P_t\varphi(x_1) - P_t\varphi(x_2)\| &\leq \Big\|\mathbb{E}\big(\varphi(\mathrm{u}(t,x_1)) - \varphi(\mathrm{u}(t,x_2))1_{\{t<\tau_n\}}\big)\Big\| + 2\|\varphi\|\Big(\frac{\varepsilon}{8\|\varphi\|} + \frac{\varepsilon}{8\|\varphi\|}\Big) \\
&= \Big\|\mathbb{E}\big(\varphi(\mathrm{u}(t\wedge\tau_n,x_1)) - \varphi(\mathrm{u}(t\wedge\tau_n,x_2))1_{\{t<\tau_n\}}\big)\Big\| + \frac{\varepsilon}{2}
\end{aligned}$$

Now we observe that

$$\begin{aligned}
\|\mathrm{u}(t\wedge\tau_n-,x_1)) - \mathrm{u}(t\wedge\tau_n-,x_2))\|^2 &\leq 3\|e^{t\wedge\tau_n}(x_1 - x_2)1_{t<\tau_n}\|^2 \\
&+ 3\left\|1_{t<\tau_n}\int_0^{t\wedge\tau_n} e^{(t\wedge\tau_n - s)\mathcal{A}}\Big[F(\mathrm{u}(s,x_1)) - F(\mathrm{u}(s,x_2))\Big]ds\right\|^2 \\
&+ 3\|1_{t<\tau_n}I_{\tau_n}(G)(t\wedge\tau_m)\|^2,
\end{aligned}$$

where

$$I_{\tau_n}(G)(t) = \int_0^t 1_{[0,\tau_n]}(s)e^{(t-s)\mathcal{A}}\Big[G(\mathrm{u}(s-,x_1),z) - G(\mathrm{u}(s-,x_2),z)\Big]\tilde{N}(ds,dz).$$

Observe that if $s \le t$ and $t < \tau_n$, then $s < \tau_n$. Hence by Lemma 4.7.1, we find that

$$\begin{aligned}
&\mathbb{E}\|1_{t<\tau_n}I_{\tau_n}(G)(t\wedge\tau_n)\|^2 \\
&= \mathbb{E}\left\|1_{t<\tau_n}\int_0^t 1_{[0,\tau_n]}(s)e^{(t-s)\mathcal{A}}\Big[G(\mathrm{u}(s\wedge\tau_n-,x_1),z) - G(\mathrm{u}(s\wedge\tau_n-,x_2),z)\Big]\tilde{N}(ds,dz)\right\|^2 \\
&\le \mathbb{E}\left\|\int_0^t 1_{[0,\tau_n)}(s)e^{(t-s)\mathcal{A}}\Big[G(\mathrm{u}(s\wedge\tau_n-,x_1),z) - G(\mathrm{u}(s\wedge\tau_n-,x_2),z)\Big]\tilde{N}(ds,dz)\right\|^2 \\
&= \mathbb{E}\int_0^t \left\|1_{[0,\tau_n)}(s)e^{(t-s)\mathcal{A}}\Big[G(\mathrm{u}(s\wedge\tau_n-,x_1),z) - G(\mathrm{u}(s\wedge\tau_n-,x_2),z)\Big]\right\|^2 \nu(dz)ds \\
&\le C_T^2\hat{C}_T^2L_g\mathbb{E}\int_0^t \|\mathrm{u}(s\wedge\tau_n-,x_1) - \mathrm{u}(s\wedge\tau_n-,x_2)\|^2\,ds.
\end{aligned}$$

Here $C_T = \sup_{0\le s\le T}\|e^{s\mathcal{A}}\|$ and $\hat{C}_T = \sup_{0\le s\le T}\|e^{-s\mathcal{A}}\|$. It follows from Cauchy-Swartz inequality that

$$\begin{aligned}
&\mathbb{E}\,\|\mathrm{u}(t\wedge\tau_n-,x_1)) - \mathrm{u}(t\wedge\tau_n-,x_2))\|^2 \\
&\le 3C_T^2\|x_1-x_2\|^2 + 3t\mathbb{E}\int_0^t \left\|1_{[0,\tau_n)}(s)e^{(t-s)\mathcal{A}}\Big[F(\mathrm{u}(s\wedge\tau_n-,x_1)) - F(\mathrm{u}(s\wedge\tau_n-,x_2))\Big]\right\|^2 ds \\
&\qquad + 3C_T^2\hat{C}_T^2L_g\mathbb{E}\int_0^t \|\mathrm{u}(s\wedge\tau_n-,x_1) - \mathrm{u}(s\wedge\tau_n-,x_2)\|^2\,ds \\
&\le 3C_T^2\|x_1-x_2\|^2 + 3TC_T^2\hat{C}_t^2L_n^2\mathbb{E}\int_0^t \|\mathrm{u}(s\wedge\tau_n-,x_1) - \mathrm{u}(s\wedge\tau_n-,x_2)\|^2\,ds \\
&\qquad + 3C_t^2\hat{C}_t^2L_g\mathbb{E}\int_0^t \|\mathrm{u}(s\wedge\tau_n-,x_1) - \mathrm{u}(s\wedge\tau_n-,x_2)\|^2\,ds \\
&= 3C_T^2\|x_1-x_2\|^2 + c(T)\mathbb{E}\int_0^t \|\mathrm{u}(s\wedge\tau_n-,x_1) - \mathrm{u}(s\wedge\tau_n-,x_2)\|^2\,ds,
\end{aligned}$$

where $c(T) = 3tC_T^2\hat{C}_T^2L_n^2 + 3C_T^2\hat{C}_T^2L_g$. Applying Gronwall's Lemma yields

$$\mathbb{E}\,\|\mathrm{u}(t\wedge\tau_n-,x_1) - \mathrm{u}(t\wedge\tau_n-,x_2)\|^2 \le 3C_T^2\|x_1-x_2\|^2e^{c(T)t} \le K\|x_1-x_2\|^2, \tag{4.6.2}$$

where $K = 3C_T^2e^{c(T)T}$.

Now, let us take an element $x \in \mathcal{H}$. Let $\{x_m\}_{m\in\mathbb{N}}$ be any sequence in $\mathcal{H}$ convergent to x. We need to show that $(P_t\varphi)(x_m) \to (P_t\varphi)(x)$, as $m\to\infty$. Given $\varepsilon > 0$, by the continuity of the function φ, there exists $\eta > 0$ such that if $\|\mathrm{u}(t\wedge\tau_n-,x_m) - \mathrm{u}(t\wedge\tau_n-,x)\| < \eta$, then we have

$$|\varphi(\mathrm{u}(t\wedge\tau_n-,x_m)) - \varphi(\mathrm{u}(t\wedge\tau_n-,x))| < \frac{\varepsilon}{4}.$$

Meanwhile, by Chebyshev inequality and inequality (4.6.2) we obtain

$$\begin{aligned}
\mathbb{P}\Big\{\|\mathrm{u}(t\wedge\tau_n-,x_m) - \mathrm{u}(t\wedge\tau_n-,x)\| \ge \eta\Big\} &\le \frac{\mathbb{E}\,\|\mathrm{u}(t\wedge\tau_n-,x_m) - \mathrm{u}(t\wedge\tau_n-,x)\|^2}{\eta^2} \\
&\le \frac{K\|x_m-x\|^2}{\eta^2}
\end{aligned}$$

Take $\delta^2 = \frac{\eta^2}{8\|\varphi\|K}\varepsilon$. Then there exists $N \in \mathbb{N}$ depending on ε such that for all $m > N$, we have $\|x_m - x\| < \delta$. Hence we infer that

$$\mathbb{P}\Big\{ \|u(t\wedge\tau_n-, x_1) - u(t\wedge\tau_n-, x_2)\| \geq \eta \Big\} < \frac{\varepsilon}{8\|\varphi\|}.$$

Combining all the above estimates, we find that for all $m > N$,

$$\begin{aligned}
\|P_t\varphi(x_m) - P_t\varphi(x)\| &\leq \Big\|\mathbb{E}\big(\varphi(u(t\wedge\tau_n-, x_m)) - \varphi(u(t\wedge\tau_n-, x))\big)\Big\| + \frac{\varepsilon}{2} \\
&\leq \Big\|\mathbb{E}\big(\varphi(u(t\wedge\tau_n-, x_m)) - \varphi(u(t\wedge\tau_n-, x))\big)1_{\{\|u(t\wedge\tau_n-,x_m)-u(t\wedge\tau_n-,x)\|\geq\eta\}}\Big\| \\
&\qquad + \Big\|\mathbb{E}\big(\varphi(u(t\wedge\tau_n-, x)) - \varphi(u(t\wedge\tau_n-, x))\big)1_{\{\|u(t\wedge\tau_n-,x_m)-u(t\wedge\tau_n-,x)\|<\eta\}}\Big\| + \frac{\varepsilon}{2} \\
&\leq 2\|\varphi\|\mathbb{P}\big\{\|u(t\wedge\tau_n-, x_m) - u(t\wedge\tau_n-, x)\| \geq \eta\big\} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} \\
&< 2\|\varphi\|\frac{\varepsilon}{8\|\varphi\|} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

This completes the proof. □

Theorem 4.6.2. *For every $0 \leq s \leq t \leq T$, we have*

$$\mathbb{E}\big[\varphi(u(t+s,x))|\mathcal{F}_s\big] = K_{s,s+t,\varphi}(u(s,x)), \tag{4.6.3}$$

where $K_{s,s+t,\varphi}(y) = \mathbb{E}\big[\varphi(u(t+s,s,y))\big]$, $y \in \mathcal{H}$. In particular, $P_{t+s} = P_tP_s$.

Proof. Since Problem (4.0.5) has a unique mild solution satisfying

$$u(t,x) = e^{tA}x + \int_0^t e^{(t-r)A}F(u(r,x))dr + \int_0^t e^{(t-r)A}G(u(r-,x),z)\tilde{N}(dr,dz),$$

byy shifting time t by $t+s$, we have

$$\begin{aligned}
u(t+s,x) &= e^{(t+s)A}x + \int_0^{t+s} e^{(t+s-r)A}F(u(r,x))dr + \int_0^{t+s} e^{(t+s-r)A}G(u(r-,x),z)\tilde{N}(dr,dz) \\
&= e^{tA}e^{sA}x + \int_0^{s} e^{(t+s-r)A}F(u(r,x))dr + \int_0^{s+} e^{(t+s-r)A}G(u(r-,x),z)\tilde{N}(ds,dz) \\
&\qquad + \int_s^{t+s} e^{(t+s-r)A}F(u(r,x))dr + \int_s^{t+s} e^{(t+s-r)A}G(u(r-,x),z)\tilde{N}(dr,dz) \\
&= e^{tA}\left[e^{sA}x + \int_0^{s} e^{(s-r)A}F(u(r,x))dr + \int_0^{s} e^{(s-r)A}G(u(r-,x),z)\tilde{N}(dr,dz)\right] \\
&\qquad + \int_s^{t+s} e^{(t+s-r)A}F(u(r,x))dr + \int_s^{t+s} e^{(t+s-r)A}G(u(r-,x),z)\tilde{N}(ds,dz) \\
&= e^{tA}u(s,x) + \int_s^{t+s} e^{(t+s-r)A}F(u(r,x))dr + \int_s^{t+s} e^{(t+s-r)A}G(u(r-,x),z)\tilde{N}(dr,dz).
\end{aligned}$$

Changing variable and denoting $\tilde{N}^s(r,z) = \tilde{N}(s+r,z) - \tilde{N}(s,z)$ for $0 \leq r \leq T$ and $z \in Z$ yields

$$u(t+s,x) = e^{tA}u(s,x) + \int_0^t e^{(t-r)A}F(u(r+s,x))dr + \int_0^t e^{(t-r)A}G(u((r+s)-,x),z)\tilde{N}^s(dr,dz).$$

Meanwhile by shifting the start time by s we get

$$\begin{aligned}\mathrm{u}(t+s,s,\mathrm{u}(s,x)) =& e^{t\mathcal{A}}\mathrm{u}(s,x) + \int_s^{t+s} e^{(t+s-r)\mathcal{A}} F(\mathrm{u}(r,s,x))dr \\ &+ \int_s^{t+s} e^{(t+s-r)\mathcal{A}} G(\mathrm{u}(r-,s,x),z)\tilde{N}(dr,dz) \\ =& e^{t\mathcal{A}}\mathrm{u}(s,x) + \int_0^t e^{(t-r)\mathcal{A}} F(\mathrm{u}(r+s,s,x))dr \\ &+ \int_0^{t+} e^{(t-r)\mathcal{A}} G(\mathrm{u}((r+s)-,s,x),z)\tilde{N}^s(dr,dz).\end{aligned}$$

Note that $\mathrm{u}(s,x)$ is an $\mathcal{F}_s$-measurable process and the new Poisson random measure $\tilde{N}^s$ is independent with respect to $\mathcal{F}_s$ and by the definition and stationarity of $\tilde{N}$ we find out that it has the same distribution with $\tilde{N}$. Let $\mathcal{F}^s$ be the completion of the σ-field generated by $\mathrm{u}(s,x)$ and $\tilde{N}^s$. Let $\mathcal{F}_t^s$ be the σ-filed generated by $\mathrm{u}(s,x)$ and $\{\tilde{N}^s(r,z), 0 \le r \le T\}$ which satisfies the usual hypotheses. Then $\mathrm{u}(t+s,x)$ and $\mathrm{u}(t+s,s,\mathrm{u}(s,x))$ are both solutions to the following stochastic differential equation on the probability space $(\Omega, \mathbb{P}, \mathcal{F}_t^s, \mathcal{F}^s)$

$$\begin{aligned}&d\mathrm{u}^s(t) = (\mathcal{A}\mathrm{u}^s(t) + F(\mathrm{u}^s(t)))dt + \int_Z G(\mathrm{u}^s(t-),z)\tilde{N}^s(dt,dz) \\ &\mathrm{u}^s(0) = \mathrm{u}(s,x).\end{aligned}$$

By the uniqueness of the solution, we infer that $\mathrm{u}(t+s,x) = \mathrm{u}(t+s,s,\mathrm{u}(s,x))$ $\mathbb{P}$-a.s. In the proof of Theorem (4.2.7), we construct functions $F_n, n \in \mathbb{N}$ which are globally Lipschitz continuous such that for each $n \in \mathbb{N}$ Equation (4.0.5) with drift F replaced by F_m has a unique mild solution satisfying

$$\mathrm{u}_m(t,x) = e^{t\mathcal{A}}x + \int_0^t e^{(t-r)\mathcal{A}} F_m(\mathrm{u}_m(r,x))dr + \int_0^t e^{(t-r)\mathcal{A}} G(\mathrm{u}_m(r-,x),z)\tilde{N}(dr,dz).$$

Moreover, we have for all $0 \le t \le T$, $\lim_{m\to} \mathrm{u}_m(t,x) = \mathrm{u}(t,x)$ $\mathbb{P}$-a.s. Define a map

$$\Phi^s(X) = e^{t\mathcal{A}}\mathrm{u}(s,x) + \int_0^t F_m(X(r))dr + \int_0^t \int_Z G(X(r-),z)\tilde{N}^s(dr,dz)$$

Now define $\mathrm{u}_{0,m}(t+s,s,\mathrm{u}(s,x)) = \mathrm{u}(s,x)$ and define recursively for $n \ge 1$

$$\begin{aligned}\mathrm{u}_{n,m}(t+s,s,\mathrm{u}(s,x)) &= \Phi^s(\mathrm{u}_{n+1,m}(t+s,s,\mathrm{u}(s,x))) \\ &= e^{t\mathcal{A}}\mathrm{u}(s,x) + \int_0^t F_m(\mathrm{u}_{n+1,m}(r+s,s,\mathrm{u}(s,x)))dr \\ &\quad + \int_0^t \int_Z G(\mathrm{u}_{n+1,m}((r+s)-,s,\mathrm{u}(s,x)),z)\tilde{N}^s(dr,dz).\end{aligned}$$

Note that the law of $\mathrm{u}_{0,m}(t+s,s,\mathrm{u}(s,x))$ is uniquely determined by the law of $\mathrm{u}(s,x)$ and

$$\begin{aligned}\mathrm{u}_{1,m}(t+s,s,\mathrm{u}(s,x)) = & e^{t\mathcal{A}}\mathrm{u}(s,x) + \int_0^t F_m(\mathrm{u}_{0,m}(r+s,s,\mathrm{u}(s,x)))dr \\ &+ \int_0^t \int_Z G(\mathrm{u}_{0,m}((r+s)-,s,\mathrm{u}(s,x)),z)\tilde{N}^s(dr,dz).\end{aligned}$$

So the law of $\mathrm{u}_{1,m}(t+s,s,\mathrm{u}(s,x))$ is uniquely determined by the law of $\mathrm{u}(s,x)$ and the law of $\tilde{N}^s$, where $\mathrm{u}(s,x)$ and $\tilde{N}^s$ are independent. By induction, we infer that the law of $\mathrm{u}_{n,m}(t+s,s,\mathrm{u}(s,x))$

is uniquely determined by the law of $u(s,x)$ and the law of $\tilde{N}^s$, for each $n \in \mathbb{N}$. Note that $\tilde{N}^s$ is independent of $\mathcal{F}_s$. Since the map Φ^s is a contradiction, it follows from Banach fixed point theorem, we infer that

$$\lim_{n\to\infty} u_{n,m}(t+s,s,u(s,x)) = u_m(t+s,s,u(s,x)) \quad \mathbb{P}-a.s.$$

We first consider the case where $u(s,x)$ is a simple function of the form

$$u(s,x) = \sum_{j=1}^{k} x_j 1_{A_j},$$

where A_j is $\mathcal{F}_s$-measurable. It then follows that

$$\begin{aligned}
\mathbb{E}\big[\varphi(u(t+s,x))|\mathcal{F}_s\big] &= \mathbb{E}\big[\varphi(u(t+s,s,u(s,x)))|\mathcal{F}_s\big] \\
&= \lim_{m\to\infty}\lim_{n\to\infty} \mathbb{E}\big[\varphi(u_{n,m}(t+s,s,u(s,x)))|\mathcal{F}_s\big] \\
&= \lim_{m\to\infty}\lim_{n\to\infty} \mathbb{E}\big[\sum_{j=1}^{m} \varphi(u_{n,m}(t+s,s,x_j))1_{A_j}|\mathcal{F}_s\big] \\
&= \lim_{m\to\infty}\lim_{n\to\infty} \sum_{j=1}^{m} \mathbb{E}\big[\varphi(u_{n,m}(t+s,s,x_j))|\mathcal{F}_s\big]1_{A_j}
\end{aligned}$$

$$\begin{aligned}
&= \lim_{m\to\infty} \sum_{j=1}^{k} \lim_{n\to\infty} \mathbb{E}\big[\varphi(u_{n,m}(t+s,s,x_j))\big]1_{A_j} \\
&= \sum_{j=1}^{k} \lim_{m\to\infty} \mathbb{E}\big[\varphi(u_m(t+s,s,x_j))\big]1_{A_j} \\
&= \mathbb{E}\big[u(t+s,s,\cdot)\big](u(s,x))
\end{aligned}$$

where we used the $\mathcal{F}_s$-measurability of 1_{A_j} and independence of $u_{n,m}(t+s,s,x_j)$ with respect to $\mathcal{F}_s$. Now for arbitrary $u(s,x) \in L^2(\Omega,\mathcal{F}_s)$, we can find a sequence of simple functions $\{u^n(s,x)\}_{n\in\mathbb{N}}$ in $L^2(\Omega,\mathcal{F}_s)$ such that

$$\lim_{n\to\infty} \mathbb{E}\|u^n(s,x) - u(s,x)\|^2 = 0 \text{ and } \lim_{n\to\infty} u^n(s,x) = u(s,x) \text{ } \mathbb{P}\text{-a.s.}$$

In the proof of Proposition 4.6.1 we show that the map $x \mapsto u(t,x)$ is continous. Then we conclude that $u(t+s,s,u^n(s,x))$ converges to $u(t+s,s,u(s,x))$ $\mathbb{P}$-a.s. Thus for $\varphi \in C_b(\mathcal{H})$, we infer

$$\lim_{n\to} \varphi(u(t+s,s,u^n(s,x))) = \varphi(u(t+s,s,u(s,x)))$$

It then follows from the Lebesgue dominated convergence theorem that

$$\lim_{n\to} \mathbb{E}\big[\varphi(u(t+s,s,u^n(s,x)))|\mathcal{F}_s\big] = \mathbb{E}\big[\varphi(u(t+s,s,u(s,x)))|\mathcal{F}_s\big]$$

Also by Feller property of P_t in Proposition 4.6.1, we have

$$\lim_{n\to\infty} \mathbb{E}\big[u(t+s,s,\cdot)\big](u^n(s,x)) = \mathbb{E}\big[u(t+s,s,\cdot)\big](u(s,x)).$$

Moreover, for each n, the simple function $\mathrm{u}^n(s,x)$ satisfies

$$\mathbb{E}\big[\varphi(\mathrm{u}(t+s,s,\mathrm{u}^n(s,x)))|\mathcal{F}_s\big] = \mathbb{E}\big[\mathrm{u}(t+s,s,\cdot)\big](\mathrm{u}^n(s,x))$$

Letting $n \to \infty$ in both sides of above equality, we conclude that

$$\mathbb{E}\big[\varphi(\mathrm{u}(t+s,s,\mathrm{u}(s,x)))|\mathcal{F}_s\big] = \mathbb{E}\big[\mathrm{u}(t+s,s,\cdot)\big](\mathrm{u}(s,x)),$$

which shows equality (4.6.3).
Since $\tilde{N}^s$ and $\tilde{N}$ have the same law, by above discussion the law of $\mathrm{u}(t+s,s,\mathrm{u}(s,x))$ is uniquely determined by the law of $\mathrm{u}(s,x)$ and the law of $\tilde{N}$. Consider the following equation

$$\begin{aligned} d\mathrm{u}(t) &= (\mathcal{A}\mathrm{u}(t) + F(\mathrm{u}(t)))dt + \int_Z G(\mathrm{u}(t-),z)\tilde{N}(dt,dz) \\ \mathrm{u}(0) &= \mathrm{u}(s,x). \end{aligned}$$

Then this equation has a unique mild solution which satisfies

$$\mathrm{u}(t,\mathrm{u}(s,x)) = e^{t\mathcal{A}}\mathrm{u}(s,x) + \int_0^t e^{(t-r)\mathcal{A}}F(\mathrm{u}(r,\mathrm{u}(s,x)))dr + \int_0^t\int_Z G(\mathrm{u}(r-,\mathrm{u}(s,x)),z)\tilde{N}(dr,dz).$$

By a similar argument as above, we see that the law of $\mathrm{u}(t,\mathrm{u}(s,x))$ is uniquely determined by the law of $\mathrm{u}(s,x)$ and the law of $\tilde{N}$. Thus we infer that

$$\mathbb{E}\big[\mathrm{u}(t+s,s,\mathrm{u}(s,x))\big] = \mathbb{E}\big[\mathrm{u}(t,\mathrm{u}(s,x))\big].$$

Finally, we obtain for every $\varphi \in C_b(\mathcal{H})$,

$$\begin{aligned} P_{t+s}\varphi(x) = \mathbb{E}\big[\varphi(\mathrm{u}(t+s,x))\big] &= \mathbb{E}\left[\mathbb{E}\big(\varphi(\mathrm{u}(t+s,x))|\mathcal{F}_s\big)\right] \\ &= \mathbb{E}\left[\mathbb{E}\big(\varphi(\mathrm{u}(t+s,s,\mathrm{u}(s,x)))\big)\right] \\ &= \mathbb{E}\left[\mathbb{E}\big(\varphi(\mathrm{u}(t,\mathrm{u}(s,x)))\big)\right] \\ &= \mathbb{E}\left[(P_t\circ\varphi)(\mathrm{u}(s,x))\right] \\ &= P_s(P_t\circ\varphi(x)), \end{aligned}$$

which completes our proof. □

Remark 4.6.3. The Feller property proved in this section makes it possible to define an invariant probability measure for the process (4.6.1). At this stage, let us recall the definition of an invariant measure. We say that a probability measure μ is an invariant measure for (4.6.1) if and only if for any function $\varphi \in B_b(\mathcal{H})$, we have

$$\langle P_t\varphi,\mu\rangle = \langle\varphi,\mu\rangle, \ t \geq 0.$$

Here $\langle P_t\varphi,\mu\rangle = \int_{\mathcal{H}} P_t\varphi(x)\mu(dx)$ and $\langle\varphi,\mu\rangle = \int_{\mathcal{H}}\varphi(x)\mu(dx)$. The existence of invariant measure for (4.6.1), in contrary to the finite dimensional case e.g. in [2], is still an open problem.

4.7 Appendix

Let $X = (X(t))_{t\geq 0}$ be an $\mathcal{H}$-valued process. Let $(e^{t\mathcal{A}})_{t\in\mathbb{R}}$ be a contraction C_0-group.Let φ be an $\mathcal{H}$-valued process belonging to $\mathcal{M}^2_{loc}(\hat{\mathcal{P}};\mathcal{H})$. Set

$$I(t) = \int_0^t \int_Z e^{(t-s)\mathcal{A}} \varphi(s,z)\tilde{N}(ds,dz),\ t \geq 0,$$
$$I_\tau(t) = \int_0^t \int_Z 1_{[0,\tau]}(s) e^{(t-s)\mathcal{A}} \varphi(s,z)\tilde{N}(ds,dz),\ t \geq 0.$$

By the choice of process φ, Proposition 4.1.11 and the assumption about $(e^{t\mathcal{A}})_{t\in\mathbb{R}}$, the stochastic convolution process $I(t)$, $t \geq 0$, is well defined. Also for any stopping time τ, the process $1_{[0,\tau]}(t,\omega)$ is predictable. In fact, the predictable σ-field is generated by the family of closed stochastic intervals $\{[0,T] : T \text{ is a stopping time}\}$, see Corollary 2.2.13. This together with the predictability of φ and Proposition 4.1.11 implies that integrand of $I_\tau(t)$ is predictable. Thus the stochastic convolution $I_\tau(t)$ is well defined as well. Moreover, Theorem 3.7.9 allows us to assume that the stochastic convolution process $I(t)$, $t \geq 0$ is càdlàg. The following lemma, which was first explicitly stated by Carroll in his Ph.D thesis [20], verifies the definition (4.2.2) of a local mild solution. The proof is mainly based on [15] and [14].

Lemma 4.7.1. *For any stopping time τ,*

$$e^{(t-t\wedge\tau)\mathcal{A}} I(t\wedge\tau) = I_\tau(t) \tag{4.7.1}$$

holds for all $t \geq 0$, $\mathbb{P}$-a.s.

Proof. We first verify it for deterministic time. Let $\tau = a$. If $t < a$, then

$$\begin{aligned} e^{(t-t\wedge a)\mathcal{A}} I(t\wedge a) = e^{(t-t)\mathcal{A}} I(t) = I(t) &= \int_0^T \int_Z 1_{[0,t]} e^{(t-s)\mathcal{A}} \varphi(s,z)\tilde{N}(ds,dz) \\ &= \int_0^T \int_Z 1_{[0,t]} 1_{[0,a]} e^{(t-s)\mathcal{A}} \varphi(s,z)\tilde{N}(ds,dz) \\ &= \int_0^t \int_Z 1_{[0,a]} e^{(t-s)\mathcal{A}} \varphi(s\wedge a,z)\tilde{N}(ds,dz) = I_a(t), \end{aligned}$$

where we used in the equality the fact that $1_{[0,a]}(s)\varphi(s,z) = 1_{[0,a]}(s)\varphi(s\wedge a, z)$. If $t \geq a$, then

$$\begin{aligned}
e^{(t-t\wedge a)\mathcal{A}}I(t\wedge a) &= e^{(t-a)\mathcal{A}}I(a) = e^{(t-a)\mathcal{A}}\int_0^a\int_Z e^{(a-s)\mathcal{A}}\varphi(s,z)\tilde{N}(ds,dz)\\
&= e^{(t-a)\mathcal{A}}\int_0^T\int_Z 1_{[0,a]}(s)e^{(a-s)\mathcal{A}}\varphi(s,z)\tilde{N}(ds,dz)\\
&\qquad + e^{(t-a)\mathcal{A}}\int_0^T\int_Z 1_{(a,t]}(s)1_{[0,a]}(s)e^{(a-s)\mathcal{A}}\varphi(s,z)\tilde{N}(ds,dz)\\
&= e^{(t-a)\mathcal{A}}\int_0^a\int_Z 1_{[0,a]}(s)e^{(a-s)\mathcal{A}}\varphi(s,z)\tilde{N}(ds,dz)\\
&\qquad + e^{(t-a)\mathcal{A}}\int_a^t\int_Z 1_{[0,a]}(s)e^{(a-s)\mathcal{A}}\varphi(s,z)\tilde{N}(ds,dz)\\
&= e^{(t-a)\mathcal{A}}\int_0^t\int_Z 1_{[0,a]}(s)e^{(a-s)\mathcal{A}}\varphi(s\wedge a,z)\tilde{N}(ds,dz)\\
&= \int_0^t\int_Z 1_{[0,a]}(s)e^{(t-s)\mathcal{A}}\varphi(s,z)\tilde{N}(ds,dz) = I_a(t).
\end{aligned}$$

Thus equality (4.7.1) holds for any deterministic time. Now let τ be an arbitrary stopping time. Define $\tau_n := 2^{-n}([2^n\tau]+1)$, for each $n \in \mathbb{N}$. That is $\tau_n = \frac{k+1}{2^n}$ if $\frac{k}{2^n} \leq \tau < \frac{k+1}{2^n}$. Then τ_n converges down to τ as $n \to \infty$ pointwisely. Note that the equality (4.7.1) proved above holds for each deterministic time $k2^{-n}$. It follows that

$$\begin{aligned}
e^{(t-t\wedge\tau_n)\mathcal{A}}I(t\wedge\tau_n) &= \sum_{k=0}^{\infty} 1_{\{k2^{-n}\leq\tau<(k+1)2^{-n}\}}e^{(t-t\wedge(k+1)2^{-n})\mathcal{A}}I(t\wedge(k+1)2^{-n})\\
&= \sum_{k=0}^{\infty} 1_{\{k2^{-n}\leq\tau<(k+1)2^{-n}\}}I_{(k+1)2^{-n}}(t) = I_{\tau_n}(t). \qquad (4.7.2)
\end{aligned}$$

Since τ_n converges down to τ, so by the $\mathbb{P}$-a.s. right-continuity of $I(t)$, $I(t\wedge\tau_n)$ converges pointwise on Ω to $I(t\wedge\tau)$ as $n\to\infty$ for every $t \geq 0$ $\mathbb{P}$-a.s. Also, observe that

$$\begin{aligned}
&\left\|e^{(t-t\wedge\tau_n)\mathcal{A}}I(t\wedge\tau_n) - e^{(t-t\wedge\tau)\mathcal{A}}I(t\wedge\tau)\right\|\\
&\quad\leq \left\|e^{(t-t\wedge\tau_n)\mathcal{A}}\left(I(t\wedge\tau_n) - I(t\wedge\tau)\right)\right\| + \left\|\left(e^{(t-t\wedge\tau_n)\mathcal{A}} - e^{(t-t\wedge\tau)\mathcal{A}}\right)I(t\wedge\tau)\right\|\\
&\quad\leq \|I(t\wedge\tau_n) - I(t\wedge\tau)\| + \left\|\left(e^{(t-t\wedge\tau_n)\mathcal{A}} - e^{(t-t\wedge\tau)\mathcal{A}}\right)I(t\wedge\tau)\right\|.
\end{aligned}$$

converges to 0 as $n\to\infty$. Thus we conclude that $e^{(t-t\wedge\tau_n)\mathcal{A}}I(t\wedge\tau_n)$ converges to $e^{(t-t\wedge\tau)\mathcal{A}}I(t\wedge\tau)$, for each $t \geq 0$, $\mathbb{P}$-a.s. For the term $I_{\tau_n}(t)$, by the isometry we find out that

$$\begin{aligned}
\mathbb{E}\|I_{\tau_n}(t) - I_\tau(t)\|^2 &= \mathbb{E}\left\|\int_0^t\int_Z \left(1_{[0,\tau_n]}(s) - 1_{[0,\tau]}(s)\right)e^{(t-s)\mathcal{A}}\varphi(s,z)\tilde{N}(ds,dz)\right\|^2\\
&= \mathbb{E}\int_0^t\int_Z \left\|\left(1_{[0,\tau_n]}(s) - 1_{[0,\tau]}(s)\right)e^{(t-s)\mathcal{A}}\varphi(s,z)\right\|^2\nu(dz)ds.
\end{aligned}$$

Recall that that $\tau_n \downarrow \tau$ as $n\to\infty$. So $1_{[0,\tau_n]}$ converges to $1_{[0,\tau]}$ as $n\to\infty$. Obviously, the integrand is bounded by $\|\varphi(s,z)\|^2$ for all n. It then follows from dominated convergence theorem that

$$\lim_{n\to\infty}\mathbb{E}\|I_{\tau_n}(t) - I_\tau(t)\|^2 \to 0.$$

Hence we can always find a subsequence which is convergent a.s. Finally, Letting $n \to \infty$ in both sides of (4.7.2) yields

$$e^{(t - t\wedge\tau)\mathcal{A}} I(t \wedge \tau) = I_\tau(t)$$

which completes our proof.

□

Remark 4.7.2. Note in particular that if we replace t by $t \wedge \tau$ in (4.7.1), we obtain

$$I(t \wedge \tau) = I_\tau(t \wedge \tau).$$

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