igenstructure Assignment for Helicopter Flight Control

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Abstract

Traditional approaches to helicopter control law design involve the iterative application of single-input, single-output loop-at-a-time classical methods. Helicopters are typically highly cross-coupled systems, and such approaches become very laborious under these conditions. Modern multivariable techniques, despite their ability to solve control design problems very efficiently, have not been embraced by practitioners. One possible reason for this is the lack of design visibility provided by such techniques, in terms of performance and controller structure.

This thesis presents new observations and algorithms which address this problem. Eigenstructure assignment is introduced in the context of classical control in order to illustrate the extent to which the two methodologies share a common language of expression. The sources of the primary dynamics of a helicopter are identified, and a new ideal eigenstructure is derived which fulfills the UK Def.Stan.00-970 handling qualities specification.

Dynamic compensators are investigated in detail, to identify the distribution of the design freedom added by these structures and its possible uses in the context of eigenstructure assignment. It is found that the manner in which the freedom is expressed does not lend itself to eigenstructure assignment, and so other sources of design freedom are sought. This leads to the development of two novel algorithms, and several extensions, for the assignment of eigenstructure to systems with a direct transmission term and consequently to helicopters with acceleration feedback or proportional-plus-derivative control structures.

The use of design freedom remaining after eigenstructure assignment is considered, and an algorithm for using it to impose structure on the controller without affecting the assigned eigenstructure is developed. Finally all of the algorithms developed in the thesis, along with the ideal eigenstructure, are demonstrated by application to a linearised helicopter model.

Contents

List of Figures

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Declaration

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Pomfret, A. J. and Clarke, T. (2005), Using post-eigenstructure assignment design freedom for the imposition of controller structure, *in* P. Horacek, M. Simandl and P. Zitek, eds, 'Proceedings of the 16th IFAC World Congress', Prague.

Pomfret, A. J., Clarke, T. and Ensor, J. (2005), Eigenstructure assignment for semi-proper systems: Pseudo-state feedback, *in* P. Horacek, M. Simandl and P. Zitek, eds, 'Proceedings of the 16th IFAC World Congress', Prague.

Nomenclature

Classical Control (Chapter 2)

$E(s)$ Error signal

$G(s)$ System transfer function

$H(s)$ Controller transfer function

K Controller gain

$\mathfrak{L}[x]$ Laplace transform of x

$P(s)$,$P_c(s)$ System and controller denominator polynomials

$T(s)$ Closed-loop transfer function

$U(s)$ System input

$Y(s)$ System output

$Z(s)$,$Z_c(s)$ System and controller numerator polynomials

Helicopter Modelling (Chapter 3)

A Area of rotor disc

A_{1s}, B_{1s} Blade flapping angles

A_1, B_1 Cyclic pitch inputs

α_0 Angle of attack of rotor blade

α_i Angle of incidence of air on rotor disc

b Number of rotor blades

c Chord of blade element

C_{di} Induced drag

C_{dp} Parasitic drag

C_l,C_d Non-dimensional lift and drag aerofoil coefficients

dD Drag from blade element

dL Lift from blade element

dQ Torque from blade element

dr	Width of blade element
dT	Resultant force from blade element
dS	Area of blade element
dT	Thrust from blade element
η_{ls}	Lateral cyclic pitch input
λ_p	Eigenvalue associated with roll rate
λ_q	Eigenvalue associated with pitch rate
λ_r	Eigenvalue associated with yaw rate
λ_u	Eigenvalue associated with longitudinal velocity
λ_v	Eigenvalue associated with lateral velocity
λ_w	Eigenvalue associated with heave velocity
$\frac{m}{dt}$	Mass flow through rotor
μ	Advance ratio
Ω	Rotor speed (rads/sec)
P	Power
p	Roll rate
p_s	Normal atmospheric pressure
ϕ	Roll angle
P_i	Induced power
ψ	Yaw angle
p_{t1}, p_{t2}	Total pressure far above and far below rotor
p_t	Total stream pressure
q	Pitch rate
q_r	Airflow coefficient for blade element
ρ	Density of air
R	Length of a blade
r	Yaw rate
σ	Rotor disc solidity
T	Thrust generated by rotor disc
θ	Pitch angle
θ_0	Collective pitch input

θ_t Tail rotor pitch input

u Forward speed

V Air velocity

v Lateral speed

v_i Increase in stream velocity to rotor disc (induced velocity)

V_s Velocity of free stream (far above rotor)

v_∞ Final increase in stream velocity

w Vertical speed (heave velocity)

Eigenstructure Assignment (Chapters 2, 4, 5 and 6)

$\mathbf{A}$ State-space system matrix

$\bar{\mathbf{A}}$, $\bar{\mathbf{B}}$, $\bar{\mathbf{C}}$ Reduced system matrices for retro-assignment

$\mathbf{A}_c$ Compensator system matrix

$\mathbf{A}_{cl}$ Closed-loop system matrix

$\hat{\mathbf{A}}_{cl}$ Closed-loop augmented system matrix

$\hat{\mathbf{A}}$ Augmented system matrix

$\mathbf{B}$ State-space input matrix

$\mathbf{B}_c$ Compensator input matrix

$\mathbf{B}_{cl}$ Closed-loop input matrix

$\mathbf{B}_F$ Feedforward compensator term

$\hat{\mathbf{B}}_F$ Augmented feedforward input matrix

$\hat{\mathbf{B}}$ Augmented input matrix

$\tilde{\mathbf{B}}$ Augmented system external input matrix

$\mathbf{C}$ State-space output matrix

c Number of compensator states

$\mathbf{C}_c$ Compensator output matrix

$\mathbf{C}_{cl}$ Closed-loop output matrix

$\hat{\mathbf{C}}$ Augmented output matrix

$\tilde{\mathbf{C}}$ Augmented system external output matrix

Δ Transmission zero test matrix

$\mathbf{D}$ State-space direct transmission matrix

$\mathbf{D}_c$ Compensator direct transmission matrix

$\mathbf{D}_{cl}$	Closed-loop direct transmission matrix
$\mathbf{E}$	Input pre-filter matrix
$\mathbf{f}_i$	Right design vector
$\mathbf{g}_j$	Left design vector
$\mathbf{G}(s)$	MIMO transfer function matrix
$\mathbf{i}_j$	Input coupling vector
Υ	Matrix of input coupling vectors
$\bar{\mathbf{K}}$	Equivalent feedback compensator gain matrix
$\mathbf{K}_d$	Derivative gain matrix
$\hat{\mathbf{K}}$	Augmented gain matrix
λ_i	Individual eigenvalue
λ_{di}	Individual desired eigenvalue
$\mathbf{L}_j$	Allowable subspace for the j^{th} left eigenvector
$\hat{\Lambda}$	Augmented diagonal matrix of eigenvalues
Λ	Diagonal matrix of eigenvalues
Λ_d	Diagonal matrix of desired eigenvalues
$\mathbf{M}_j$	Subspace for the j^{th} left gain-eigenvector product
m	Number of outputs
$\mathbf{N}$	Gain term for semi-proper assignment
n	Number of states
$\mathbf{o}_i$	Output coupling vector
Ω	Matrix of output coupling vectors
$\mathbf{P}_i$	Allowable subspace for the i^{th} right eigenvector
$\mathbf{Q}_i$	Subspace for the i^{th} right gain-eigenvector product
r	Number of inputs
$\mathbf{S}'$	Product of partial right eigenvector set and gain matrix
$\mathbf{T}'$	Product of partial left eigenvector set and gain matrix
$\mathbf{u}$	Input vector
$\mathbf{V}_d$	Matrix of desired right eigenvectors
$\mathbf{v}_i$	i^{th} right eigenvector
$\mathbf{V}$	Matrix of right eigenvectors

v/w Number of assigned right/left eigenvectors

$\mathbf{v}_{ci}$,$\mathbf{w}_{cj}$ Compensator sub-eigenvectors

$\hat{\mathbf{V}}_c$,$\hat{\mathbf{W}}_c$ Augmented compensator sub-eigenvector sets

$\hat{\mathbf{V}}$,$\hat{\mathbf{W}}$ Augmented eigenvector sets

$\mathbf{v}_{pi}$,$\mathbf{w}_{pj}$ Plant sub-eigenvectors

$\hat{\mathbf{V}}_p$,$\hat{\mathbf{W}}_p$ Augmented plant sub-eigenvector sets

$\mathbf{W}_d$ Matrix of desired left eigenvectors

$\mathbf{w}_j$ j^{th} left eigenvector

$\mathbf{W}$ Matrix of left eigenvectors

$\mathbf{V}'$,$\mathbf{W}'$ Partial sets of eigenvectors

$\mathbf{K}$ Gain matrix

$\mathbf{x}$ State vector

$\mathbf{x}_c$ Compensator state vector

$\mathbf{y}$ Output vector

$\mathbf{Z}$ Free parameter matrix

Controller Structure (Chapter 6)

δ Number of structural constraints

Ξ Freedom matrix

$\mathbf{U}$ Structural permutation matrix

$\tilde{\mathbf{Z}}$ Free parameter matrix

Mathematical Notation (See Appendix C)

$\mathbf{A}^\dagger$ Moore-Penrose pseudo-inverse of $\mathbf{A}$

adj($\mathbf{A}$) Adjoint of $\mathbf{A}$

$\mathbf{A}_{p\cdot}$ Row selection operator

$\mathbf{A}_{pq}$ Element selection operator

$\mathbf{A}_{\cdot q}$ Column selection operator

$\mathbf{A}^\star$ Conjugate transpose of $\mathbf{A}$

$\mathbf{A}^T$ Transpose of $\mathbf{A}$

det($\mathbf{A}$) Determinant of $\mathbf{A}$

ker($\mathbf{A}$) Right nullspace of $\mathbf{A}$

$\otimes$ Kronecker product

vec(**A**) Column-stacked vector equivalent of **A**

$\bar{x}$ Complex conjugate of x

$\bar{\mathbf{x}}$ Element-wise complex conjugate of vector **x**

$\|\mathbf{a}\|$ Euclidean vector norm of **a**

List of Abbreviations

AFCS	Automatic Flight Control System.
ASE	Auto-Stabilisation Equipment.
Def.Stan.00-970	UK Ministry of Defence Defence Standard 00-970.
DoF	Degrees of Freedom.
EA	Eigenstructure Assignment.
IFR	Instrument Flight Rules.
IMU	Inertial Measurement Unit.
INS	Inertial Navigation System.
MIMO	Multi-Input, Multi-Output.
PD	Proportional-plus-Derivative.
PIO	Pilot-Induced Oscillation.
SISO	Single-Input, Single-Output.

Chapter 1

Introduction

Contents

The study of control engineering can be traced back to the ancient Greeks, and examples of mechanical control systems from the time of the Industrial Revolution are numerous (Åström, 1999). Modern control systems usually consist of electronic sensors, controllers and actuators, and increasingly the controllers themselves are implemented digitally. As the demands on physical systems increase, and the systems themselves become more and more complex, control systems become increasingly important.

Helicopters, being inherently complex systems, required the development of control system technology before they could become useful - despite having existed as a technology for almost as long as their fixed-wing counterparts. All high-performance aircraft, both rotary-wing and fixed-wing, employ control systems to achieve their desired levels of performance (Blight et al., 1994). Without feedback control, helicopters tend to be highly unstable and difficult to fly.

This thesis is the latest in a long line of D.Phil. and Ph.D theses covering aspects of Eigenstructure Assignment (EA) and rotorcraft control at the University of York (Young, 1989; Burrows, 1990; Lawes, 1994; Davies, 1994; Griffin, 1997; Ensor, 2000; Gee, 2000).

1.1 Control and Visibility

Traditionally, 'classical' control techniques have been used to approach helicopter control design problems (McLean, 1990), and these techniques are still the most commonly used today (Taylor, 2006). These involve forming single-input, single-output control loops, one at a time, and iteratively adjusting their parameters until the desired performance is achieved. Since the helicopter is a large system with many inputs and outputs, this process can be extremely time-consuming.

Multivariable control design methodologies, such as H_∞ (Kwakernaak, 1993), Linear Quadratic Gaussian with Loop Transfer Recovery (LQG/LTR) (Stein and Athens, 1987) and EA (Moore, 1976) use mathematical analysis of the problem to arrive at a solution for all variables quickly. However, classical approaches possess the quality that changes to the design parameters produce a predictable change in system response, placing the control engineer in direct contact with the system and allowing the design of a control system to become an art. This quality is known as *visibility*, and is not generally shared by multivariable approaches, wherein the design parameters are often abstracted quite considerably from the system response. It is reasonable to suggest that the lack of uptake of multivariable methods among practitioners (Blight et al., 1994; Griffin, 1997) is partly attributable to this lack of visibility.

The gap between current theory and practice is regrettable, since the application of modern multivariable control techniques could doubtless improve on the performance of current controller designs, while simultaneously increasing the speed of arrival at a control solution by several orders of magnitude. (Griffin, 1997) notes that EA has the potential to address this problem, since the manner in which the control problem is expressed has close links to classical control, but more work is still needed to produce algorithms which display the visibility and flexibility required. The aim of this thesis is to attack the practice-theory gap by promoting an understanding of EA in a classical context, and by developing algorithms which extend the capabilities of EA towards current practice.

1.2 Controller Structure

EA, in common with other multivariable design methodologies, is inclined to use all of the available design freedom to generate a control solution. When this design freedom is expressed as a matrix of gains linking every plant output to every plant input, as it typically is, this is likely to result in a fully interconnected controller. This does not fit well with classical control design practice, where the structure is designed a loop at a time and is hence generally sparse. While the visibility of the EA process is high during the design stage, it can be difficult to determine the roles of the individual gains in the final controller.

Two approaches to addressing these issues present themselves. The first is to attempt to use EA to generate (or optimise) a set of control laws within a given structure. For example, an existing set of helicopter controller gains and compensators could be subjected to an EA process. This presents a challenge, since the set of available gains would have to be represented in an unconventional form. Additionally it is restrictive in that it is possible that EA might, if unconstrained, lead to the discovery of a new, better controller structure. The second option therefore is that EA is used to generate a fully interconnected controller, but a portion of the design freedom is reserved and used to impose a structure on the controller. This imposition of structure could be guided so that the links that are removed are those which have the least effect on the remaining controller gains; in this way the EA process would have had an influence on the final chosen structure.

It is also important that dynamic compensators are investigated, since these could provide a source of the design freedom needed for structuring a controller. Accordingly dynamic compensators, other freedom-increasing methods, and structural imposition will form core themes in this thesis.

1.3 Robustness

Robustness is a term which takes on many meanings, but it generally refers to the ability of a closed-loop system to respond to changes in the open-loop plant in a way which conforms to a specification - often in a way which minimises the deviation of the closed-loop system from its nominal performance according to some measure. These changes could be the result of nonlinearities, perturbations or uncertainties. EA has no robustness guarantees, unlike for example H_∞; but it provides access to all the available degrees of freedom and hence '...may assign a robust solution as easily as a non-robust solution' (Griffin, 1997, p176). The necessary level of robustness must be introduced either by careful selection of the desired eigenstructure, or through an iterative design process where the robustness of an EA design is checked post-assignment and the design revised if necessary. Since robustness and performance are inherently different sets of requirements, improvement of the robustness of a controller must inevitably result in some performance degradation. However this trade-off can be performed by manipulating the assigned eigenstructure, so the effects on performance can be seen and controlled. 'Robustification' algorithms for achieving this are well-developed (Griffin, 1997; Ensor, 2000; Ensor and Davies, 2000), and can work with any EA algorithm. Therefore this thesis will not be concerned with issues of robustness, and will instead focus on EA for performance on the understanding that these techniques can be applied to improve robustness if required.

1.4 Thesis Overview

It is the author's thesis that EA can be used as part of a visible process, sharing much of the language of classical control, to design structured controllers for helicopter applications.

To produce an accessible, readable document, the organisation of this thesis is such that the chapters form self-contained units. In each case a summary of the contents of the chapter will be found at the start, and a summary and list of references at the end. A number of appendices follow the main body of the thesis, and contain supporting material which would otherwise break the flow of the main text.

Chapter 2 introduces EA from the perspective of traditional 'classical' control approaches. The fundamental concepts of classical control are introduced and discussed, and the difficulties introduced by cross-coupled multivariable systems demonstrated. A review of the development and types of EA algorithms demonstrates that these difficulties may be addressed by EA in a way which retains as much as possible of the terminology and approach of classical control, and that highly visible access to the available design freedom can be obtained.

In Chapter 3, the helicopter is introduced from a theoretical standpoint. The understanding of a plant is important to the generation of an effective control strategy, and several of the characteristic helicopter dynamics are derived. The specific issues related to the application of EA to the helicopter control problem are then considered, and a new ideal eigenstructure for a helicopter in forward flight is developed and shown to be kinematically correct.

The application of standard state- or output-feedback EA generates a set of fixed gains linking the outputs of a plant to its inputs. Chapter 4 introduces the dynamic compensator, an alternative controller structure in which the controller itself possesses dynamics. A simple representation for these compensators is shown which allows the use of standard EA techniques, by manipulating the representation of the compensator until all of its degrees of freedom are represented by one large matrix of fixed gains. A new analysis of the effect of this freedom on the eigenvectors is presented, and an alternative compensator structure is also presented which carries the potential for deciding on the eventual structure of a compensator after EA has taken place. However it is postulated that the distribution of design freedom in a compensated system does not lend itself readily to EA, and if additional design freedom is required, there are more suitable ways of obtaining it.

Chapter 5 contains two, separate, novel algorithms for EA. These algorithms are capable of acting upon semi-proper (proper but not strictly proper) systems. It is shown that the formation of semi-proper systems, by the addition of certain types of sensors or the implementation of Proportional-plus-Derivative (PD) control, is a valid approach to the generation of additional design freedom. A number of extensions are considered to the basic algorithms, to allow the assignment of modal coupling vectors in the face of changes to the input and output matrices, and to recover unused design freedom from both algorithms for use in a retro-assignment stage or for any other purpose.

Several EA algorithms, including those developed for Chapter 5, have the potential to leave a portion of the design freedom unused. Chapter 6 considers a new use for this freedom: the imposition of structure on the controller through selective elimination of gains or the establishment of known links between gain matrix entries. A novel algorithm for the achievement of this aim is developed, along with tools for assessing the impact of these structural constraints on the overall magnitude of entries in the gain matrix.

Chapter 7 puts the tools developed in the preceding chapters into context by using EA to control two linearised helicopter models. The ideal eigenstructure from Chapter 3 is verified by applying it to a helicopter in forward flight, which is shown to exhibit Level 1 handling qualities. The two EA algorithms of Chapter 5 and the structural algorithm from Chapter 6 are also used and shown to work well, as expected.

Finally, overall conclusions and a summary of suggested further work can be found in Chapter 8.

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Chapter 2

Control and Eigenstructure Assignment

Contents

2.1 Introduction

In this chapter, Eigenstructure Assignment (EA) will be introduced from the perspective of traditional 'classical' control approaches.

Firstly, the fundamental concepts of classical control will be introduced and discussed, before the difficulties introduced by cross-coupled multivariable systems are demonstrated. These difficulties, it will be shown, are addressed by good EA algorithms in a way which retains as much as possible of the terminology and approach of classical control, while providing access to all available design freedom in a visible way.

The review of EA algorithms bears some superficial similarity to those presented by White (1995), Griffin (1997) and Ensor (2000). However, the analysis of the algorithms presented here is new, and has been performed in the context of the aims of this thesis. Several assertions are made here about the nature of various existing algorithms and the manner in which they allow a control system designer to specify the performance of the system.

In summary, this chapter is intended both to demonstrate the clear links that exist between classical control and EA, and to provide the reader with an understanding of the language of control and the importance of a variety of control concepts.

2.2 Classical Control

Classical control generally involves the formation of a control law to affect the behaviour of a Single-Input, Single-Output (SISO) system whose dynamics can be described by an ordinary differential equation. This control law is specified in terms of the dynamic response of the system, not directly in terms of the coefficients of the differential equation. The characteristics of such a system will be considered first, along with the way in which the system is treated mathematically for the purposes of applying control. This will be followed by an overview of classical techniques for calculating the parameters of a feedback controller, which typically involve graphical tools and rules of thumb.

2.2.1 Problem Formulation

Before the approaches to control can be considered, it is important to understand the form of the systems to which control is to be applied.

Firstly, a mathematical model of a system is formed. This model is then cast into a useful form (the 'transfer function') for the purposes of applying control, and finally some function of the output is fed back to the input in order to affect the behaviour of the system.

2.2.1.1 Modelling

For the purposes of mathematical analysis, continuous-time linear systems with one input and one output are described by continuous-time linear differential equations (Jacobs, 1974; Nise, 1995) of the form

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \ldots + a_0 y(t) = \frac{d^m u(t)}{dt^m} + b_{m-1}\frac{d^{m-1}u(t)}{dt^{m-1}} + \ldots + b_0 u(t) \qquad (2.1)$$

where $u(t)$ is the system input, $y(t)$ is the system output and the system parameters a_i and b_j serve to describe the relationship between input and output.

No real system is completely linear (Schwarzenbach and Gill, 1984), and so in order to arrive at the form of Equation 2.1, simplifying assumptions are often made when describing the system (Nise, 1995; Jacobs, 1974). Furthermore, even complex nonlinear systems may be described using equations such as Equation 2.1, where such equations are approximations and are valid only for a small region of operation or for a short period of time (Schwarzenbach and Gill, 1984; Banks, 1986; Schoukens and Pintelon, 1991; Griffin, 1997; Ogata, 1997).

The complete solution to a linear differential equation is the sum of two parts: the particular integral (or 'forced response') and the complementary function (or 'natural response'). The forced response is due to the input $u(t)$, while the natural response follows only from the initial conditions of the equation. A system is described as *stable* if its response to any input decays to zero after the input is removed, and so the complementary function is of great importance. The complementary function is found by setting the right-hand side of Equation 2.1 to zero and solving the resulting homogeneous equation. The solution (Jacobs, 1974) will have the general form

$$y_{cf} = \sum_{k=1}^{n} c_k e^{s_k t} \qquad (2.2)$$

where $c_k \in \mathbb{R}$ are constants determined by the initial conditions and $s_k \in \mathbb{C}$ are the roots of the characteristic equation

$$s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 = 0 \qquad (2.3)$$

The roots s_k of the characteristic equation are known as the *system poles*, and play a large part in shaping the response of the system. Since the natural response is composed of the sum of exponential terms with s_k as the exponents, it is clear that the system is stable if and only if all system poles have negative real parts. Such poles are described as lying in the *left half plane*, referring to the fact that if plotted on an Argand diagram they would lie to the left of the imaginary axis. Also, if a pair of complex conjugate poles $a + j\omega_d$ and $a - j\omega_d$ form part of the set, their contribution to the response can be written as

$$e^{(a+j\omega_d)t} + e^{(a-j\omega_d)t} = e^{at}\left(e^{j\omega_d t} + e^{-j\omega_d t}\right) \tag{2.4}$$

$$= e^{at}\left(\cos\omega_d t + j\sin\omega_d t + \cos(-\omega_d t) + j\sin(-\omega_d t)\right) \tag{2.5}$$

$$= 2e^{at}\cos\omega_d t \tag{2.6}$$

If the system is stable, giving $a < 0$, the contribution of a complex pole pair is therefore a cosinusoid that decays in magnitude exponentially. The frequency of the cosinusoid ω_d is termed the *damped natural frequency.*

It is worth noting also that if a subset of the poles of a system have significantly smaller negative real parts then the rest, then the portions of the response associated with these poles will decay much more slowly. As a result, the overall shape of the response will be dominated by these poles. Such poles are therefore referred to as *dominant*; as will be seen, the characterisation of the dominant poles affords certain simplifications when designing a control system.

2.2.1.2 Transfer Functions

The Laplace transform (or unilateral Laplace transform) is an integral transform capable of converting differential equations into algebraic equations. It is defined as

$$\mathfrak{L}\left[f(t)\right] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt \tag{2.7}$$

Applying this to both sides of Equation 2.1, and assuming that all initial conditions are zero, yields

$$\left(s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0\right)Y(s) = \left(s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0\right)U(s) \tag{2.8}$$

Rearranging,

$$\frac{Y(s)}{U(s)} = \frac{s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0} = G(s) \tag{2.9}$$

Hence if the Laplace transform of the input signal is known, or can be found, Equation 2.9 can be used to find the Laplace transform of the output signal. The inverse Laplace transform (also known as the Bromwich integral or the Fourier-Mellin integral), given as

$$\mathfrak{L}^{-1}\left[F(s)\right] = \frac{1}{2\pi}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}ds \tag{2.10}$$

where c is chosen such that all discontinuities in $F(s)$ lie to the left of it in the complex plane, can then be used to recover the time-domain version of the output signal.

Comparing Equations 2.9 and 2.3 shows that the transfer function denominator is the same as the characteristic polynomial, and therefore that the system poles are the roots of the denominator of the transfer function. It is also clear that the system poles are those values $\{\lambda\}$ where $G(\lambda) = \infty$.

The numerator of the transfer function is also important. From Equation 2.8, it is clear that the numerator and denominator of the transfer function share a kind of duality of function; were one to drive the output of the system and observe the effect on its input, the transfer function of the observed response would be the reciprocal of that in Equation 2.9.

The numerator has no effect on the natural response however, as shown by Equation 2.3, and its influence must therefore be over the forced response. The roots of the numerator of the transfer function are known as *zeros*, and are naturally those values $\{z\}$ where $G(z) = 0$.

The location of the system zeros has no effect on stability. The effect of zeros on the system response can be understood by considering the following example (from Nise, 1995).

Let $Y(s)$ be the Laplace transform of the output response of a certain system, whose transfer function $G(s)$ has a simple constant term in the numerator. If a single zero at $s = a$ is added to the transfer function, it will become $(s - a)G(s)$; the corresponding output response will become

$$(s - a)Y(s) = sY(s) - aY(s) \tag{2.11}$$

The output response now consists of two terms: a scaled version of the original response, and the derivative of the original response. If a is large and negative, the derivative term will be largely swamped and the response shape will be almost unchanged. Smaller values of a will have larger effects. If $a = 0$, the zero is in fact a pure differentiator, and the response shape will change completely.

Interestingly, if $a > 0$, placing the zero in the right half plane, the scaled response and the derivative will have opposite signs. In response to a step input, the derivative of the output is typically positive initially; if a is small enough, this will lead to the output response starting off in the negative direction even though the final response will be positive. A system that exhibits this behaviour is known as a *nonminimum-phase* system.

Clearly then, the poles and zeros of the transfer function of a system define its dynamic response to any given input. Manipulation of the location of these poles and zeros is therefore the aim of classical feedback control system design.

2.2.1.3 Closing the Loop

Feedback can take many forms. In each case the system output is taken, modified and used to augment the input, leading to a modified system response. Figure 2.1 shows a simple form of feedback control known as *unity negative feedback*. The output is subtracted from the input, leaving an *error signal* which is fed to the system input. The overall input to the closed-loop system is now a demand for a particular output, and while the system output does not match the demand, the error signal will be nonzero.

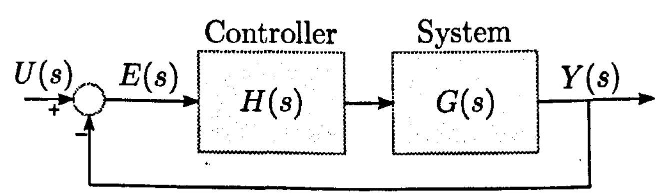

Figure 2.1: Unity Negative Feedback

Unity negative feedback, on its own, is not of much use since the effect of closing the loop is not adjustable. Instead, a controller term is introduced; this is shown as $H(s)$ in Figure 2.1. In the simplest case, $H(s)$ would be a simple gain..

The effect on the system transfer function is simple to deduce. Given a transfer function $G(s)$, external input $U(s)$, and output $Y(s)$, the error signal is given by

$$E(s) = U(s) - Y(s) \tag{2.12}$$

and the output is therefore given by

$$Y(s) = H(s)G(s)E(s) = H(s)G(s)U(s) - H(s)G(s)Y(s) \tag{2.13}$$

The transfer function $T(s)$ of the equivalent unity negative feedback closed-loop system can now be found:

$$(1 + G(s)H(s))\,Y(s) = G(s)H(s)U(s) \tag{2.14}$$

$$T(s) = \frac{Y(s)}{U(s)} = \frac{G(s)H(s)}{1 + G(s)H(s)} \tag{2.15}$$

The poles and zeros of this closed-loop system are interesting to investigate. If the open loop transfer function $G(s)$ is defined as

$$G(s) = \frac{Z(s)}{P(s)} \tag{2.16}$$

and the controller $H(s)$ as

$$H(s) = \frac{Z_c(s)}{P_c(s)} \tag{2.17}$$

then by substitution,

$$T(s) = \frac{\frac{Z_c(s)}{P_c(s)}\frac{Z(s)}{P(s)}}{1 + \frac{Z_c(s)}{P_c(s)}\frac{Z(s)}{P(s)}} \tag{2.18}$$

$$= \frac{Z(s)Z_c(s)}{P(s)P_c(s) + Z(s)Z_c(s)} \tag{2.19}$$

Examination of Equation 2.19 shows that any root of $Z(s)$ is a root of the numerator of the closed-loop transfer function $T(s)$. This shows that the zeros of $G(s)$ are invariant under feedback, though any zeros present in the controller $H(s)$ will manifest themselves as additional zeros in $T(s)$.

As described, the simplest type of unity negative feedback control uses a simple gain, so that $H(s) = K$. Then from Equations 2.15 and 2.19, the closed loop transfer function reduces to

$$T(s) = \frac{KG(s)}{1 + KG(s)} = \frac{KZ(s)}{P(s) + KZ(s)} \tag{2.20}$$

showing once more that the system zeros remain unchanged under feedback.

An alternative feedback structure is shown in Figure 2.2. Now the controller $H(s)$ is placed not in series with the plant, but in the feedback path. Note that in the absence of an external input $U(s)$, the structure is identical to that of the system in Figure 2.1. Consequently it would be expected that the natural response would be the same, and hence that the closed-loop system poles would also be the same.

The role of the external input $U(s)$ has changed, however. In the system of Figure 2.1, it

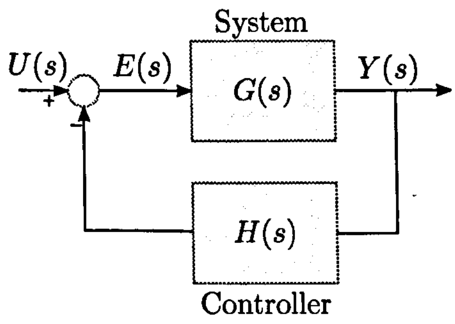

Figure 2.2: Alternative Negative Feedback

acted to demand a particular output response; this is no longer the case, since the summing junction appears *after* the feedback signal has been modified by the controller $H(s)$. For the same reason, it is no longer strictly accurate to refer to the signal $E(s)$ as the error signal, though the notation will be retained for convenience.

Analysis of Figure 2.2 allows the construction of the closed loop transfer function:

$$E(s) = U(s) - H(s)Y(s) \tag{2.21}$$

$$Y(s) = G(s)E(s) \tag{2.22}$$

$$= G(s)U(s) - G(s)H(s)Y(s) \tag{2.23}$$

$$(1 + G(s)H(s))\,Y(s) = G(s)U(s) \tag{2.24}$$

$$T(s) = \frac{Y(s)}{U(s)} = \frac{G(s)}{1 + G(s)H(s)} \tag{2.25}$$

Substitution of Equations 2.16 and 2.17 now gives

$$T(s) = \frac{\frac{Z(s)}{P(s)}}{1 + \frac{Z(s)}{P(s)}\frac{Z_c(s)}{P_c(s)}} \tag{2.26}$$

$$= \frac{Z(s)P_c(s)}{P(s)P_c(s) + Z(s)Z_c(s)} \tag{2.27}$$

Comparison with Equation 2.19 shows that the poles are indeed identical as predicted. The zeros of this alternative closed loop system, however, are composed not of the zeros of the plant and controller, but of the plant zeros and the controller *poles*.

Once again, a simple gain $H(s) = K$ is employed in the simplest case; the closed loop transfer function now reduces to

$$T(s) = \frac{G(s)}{1 + KG(s)} = \frac{Z(s)}{P(s) + KZ(s)} \tag{2.28}$$

The manipulation of the poles of the closed loop system, by adjustment of $H(s)$ or K, is the aim of the classical approach to the feedback control of a linear, single-input single-output system.

2.2.2 Gain Determination

Determining the gain, or controller transfer function, necessary to place the system poles at the required locations is not a trivial task. Even before this, however, comes the problem of determining suitable pole locations to satisfy a given set of design criteria.

The subject of dominance has been introduced, and this provides a mechanism for simplifying the problem of pole placement. If a system can be considered to be dominantly first or second order, or by the assignment of poles it can be *forced* to be so, then optimal locations for the dominant poles can be found relatively easily. The response of simple first-order poles and complex conjugate second order pole pairs was derived above, and these derivations are extended to allow the determination of ideal pole locations with respect to pseudo-quantitative time-domain measures such as settling time, rise time, time to peak overshoot, and percentage overshoot (for details of these measures, see Nise, 1995).

Once suitable locations have been found for the dominant poles, the problem of finding the appropriate controller transfer function can be addressed. For simplicity, this process will first be described in the context of finding a simple gain; more complex controllers are introduced in Section 2.2.3.

2.2.2.1 Root Locus Plots

One of the most commonly used tools for analysing the effect of changing the loop gain in a classical feedback control system is the root locus plot. Consider the simplified unity negative feedback system of Figure 2.3.

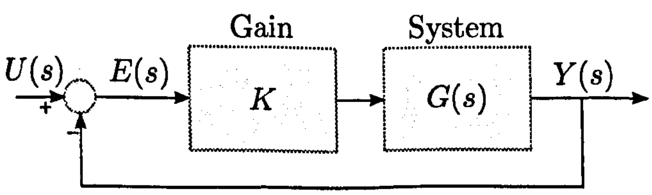

Figure 2.3: Simple Unity Negative Feedback

As the gain K is changed, the denominator of the closed-loop transfer function will change (via Equation 2.19) and hence so will the location of the poles. The root locus plot is a graphical method of showing the migration of the poles with respect to the changing gain. For example, consider the open-loop system

$$G(s) = \frac{s+4}{s^3 + 7s^2 + 16s + 10} \tag{2.29}$$

Substitution into Equation 2.20 shows that the closed loop transfer function is given by

$$T(s) = \frac{K(s+4)}{s^3 + 7s^2 + (16+K)s + 10 + 4K} \tag{2.30}$$

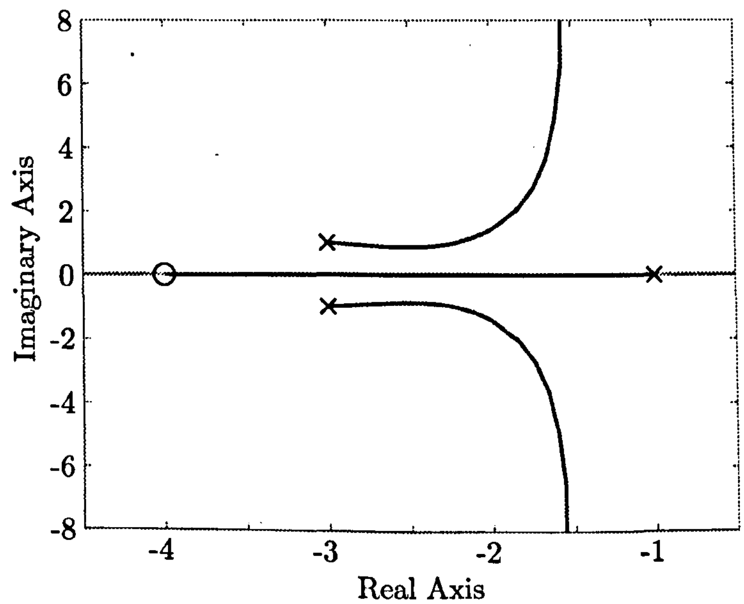

Figure 2.4: Root Locus Plot for the system of Equation 2.30

Figure 2.4 is the root locus plot for $T(s)$, showing the migration of the poles of $T(s)$ as the gain K is changed. The starting points of the loci, represented by crosses on the figure, are at $s = -1$, $s = -3 + j$ and $s = -3 - j$, corresponding to the poles of the open-loop system. The reason for this is obvious if one considers Equation 2.30; as K is reduced to zero, the denominators of the open- and closed-loop systems become identical. Similarly, two of the loci are asymptotic but the third ends at $s = -4$, corresponding to the position of the open loop zero. Again this may be deduced from Equation 2.30, by noting that if K is made very large, the denominator of $T(s)$ becomes a scalar multiple of the numerator of $G(s)$.

The shapes of the root loci are non-analytical in nature. Sketching the root locus plot by hand is a practical proposition in most cases, since the behaviour of the loci is summarised by a small set of rules (Nise, 1995); accurate drawing, for example by a computer program, requires a set of gains to be chosen and the loci to be plotted stepwise (The Mathworks, Inc., 2005). It is relatively easy, given a point on a root locus, to find the value of K necessary to place a pole there.

By definition, the root locus describes those points on the complex plane at which poles *can* lie; placing poles at any other point is impossible. Hence, if the system *requirements* can be expressed as graphical constraints, the intersection of the root locus with the constraints gives the solution.

For example, if it is desired that the closed-loop system should exhibit a 10% overshoot in its response to a unit step input, it can be shown (Nise, 1995; Jacobs, 1974) that the *damping ratio* (the ratio of real part to magnitude) of the dominant second order pole pair must be approximately 0.5912. Figure 2.5 shows the root locus again with these constraint lines added. By finding the necessary gain (around 5.3 in this case), then substituting into $T(s)$ and factorising the denominator, the locations of all three closed-loop poles may be found. In this case these are $s = -1.78 \pm 2.43j$ and $s = -3.45$, as indicated by the vertical crosses on the figure.

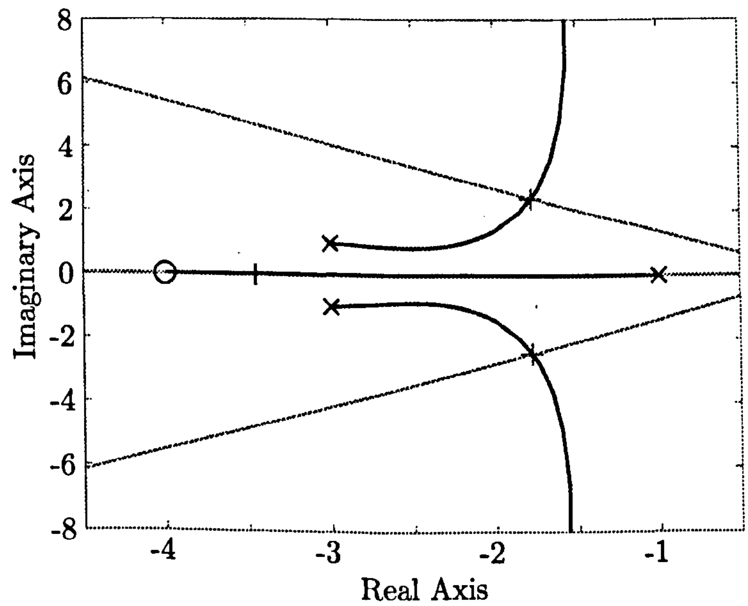

Figure 2.5: Root Locus Plot With Lines of 10% Overshoot

Figure 2.6 shows the (normalised) step responses of the open and closed loop systems. It may be seen that the closed loop system displays approximately the correct overshoot. It is not exact however, and this is due to the action of the third pole - this system is only marginally dominantly second order.

Clearly, for a 10% overshoot, there is only one solution for the gain K in this case. Thus if one constraint is imposed, the controller is fully specified and no further imposition of constraints

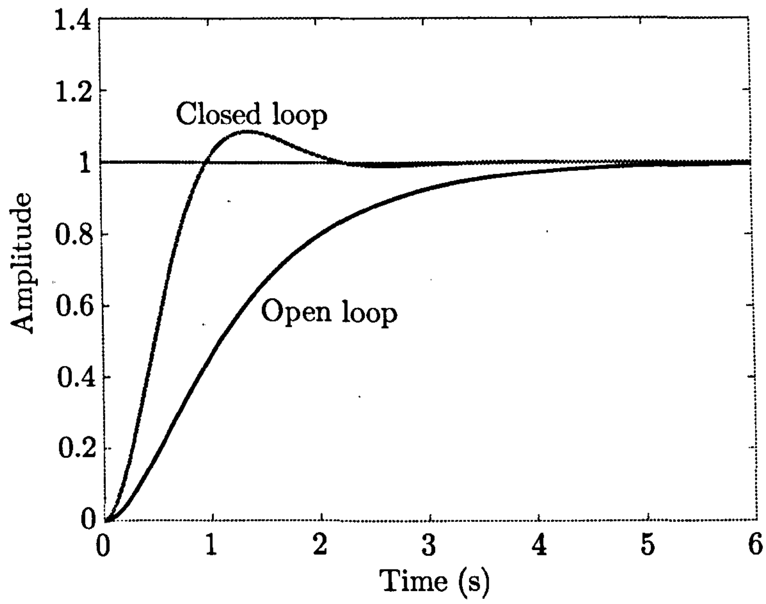

Figure 2.6: Normalised Step Response of Open and Closed Loop Systems

is possible. It can be seen from Figure 2.6 that the closed loop system in this example rises to its final value in approximately 0.8 seconds; this is the *only* rise time corresponding to a 10% overshoot for this system using a simple gain as a controller.

2.2.3 Dynamic Compensation

The lack of design freedom offered by the simple gain controllers introduced in Section 2.2.2 can be overcome by choosing a more complex transfer function for the controller. However, tools such as Bode, Nyquist and root locus plots are suitable only for designs involving a single variable parameter.

The solution is to assign all but one parameters of the controller *a priori*, and then to use the root locus to optimise the remaining parameter, usually a gain. Thus, designing more complex controllers requires a knowledge of how certain controller designs can be used to affect the shape of the root locus. As a result, small 'building blocks' are employed, each of which has a relatively predictable effect on the shape of the root locus. These blocks come in many forms, and are known as *compensators.*

An example of the use of a simple compensator is as follows. Let us assume that the response obtained in Figure 2.6 via the root locus of Figure 2.5 is unsatisfactory, and it is necessary to reduce even further the rise time of the system. By adding a compensator such that $H(s)$ takes the form

$$H(s) = K\frac{s+3.5}{s+4.5} \tag{2.31}$$

then the new root locus plot with respect to K is shown in Figure 2.7.

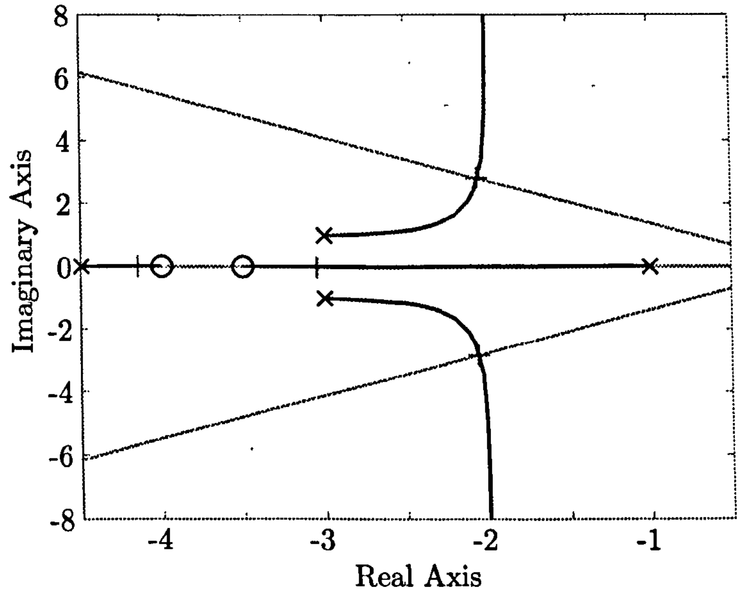

Figure 2.7: Root Locus Plot for Compensated System

As can be seen, the point at which the root loci for the dominant pole pair crosses the line of 10% overshoot has moved as a result of the addition of the compensator. The closed loop poles shown, achieved with a gain $K = 8.2849$, are at $s = -2.0563 \pm 2.8076j$, $s = -4.2862$ and $s = -3.1012$.

The effect of these modified pole locations on the normalised step response of the closed loop system is shown in Figure 2.8.

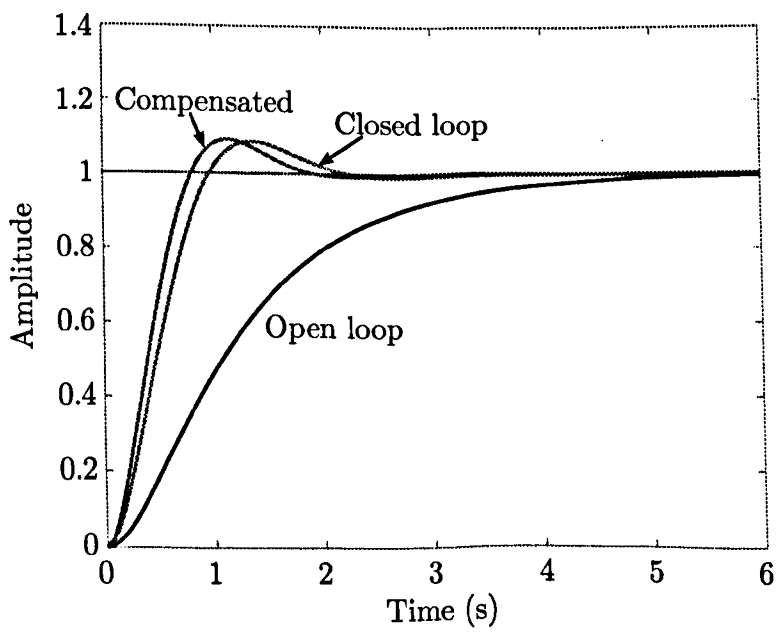

Figure 2.8: Normalised Step Response showing Compensated Closed Loop System

The compensator of Equation 2.31 is known as a *phase lead* compensator, but many different configurations exist. Their application and design is the choice of the control engineer, and although tools exist for assessing the likely impact of the use of a compensator, such design decisions must ultimately come from experience.

2.2.4 Multiple Loops

If a system has more than one output, but retains a single input, standard classical approaches can often be used to good effect. For example, Figure 2.9 shows a system wherein one output is a function of the other.

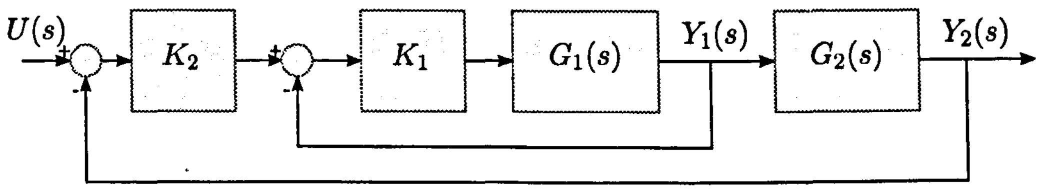

Figure 2.9: Concentric Two-Loop Control System

This is a common occurrence - typically one output might be the integral of another, such as in an actuator system whose outputs are velocity and position. The design procedure in this case is usually to close the 'innermost' loop first, and work outwards - so in this case, to find K_1 first and then K_2. Clearly the selection of K_1 will change the dynamics of the system from $U(s)$ to $Y_1(s)$, and consequently will change the shape of the root locus with respect to $Y_2(s)$. The design process is therefore usually iterative. An initial value of K_1 is selected, and the shape of the resulting outer root locus found. K_1 is then adjusted to change the shape of the outer root locus until it passes through the site of the desired closed-loop poles, and K_2 is then selected.

If the system to be controlled has multiple outputs *and* multiple inputs, the process becomes significantly more involved. Figure 2.10 shows just one possible configuration of such a system. The figure shows only the open-loop plant, with the controller gains and feedback loops omitted for clarity.

Clearly now there are four possible feedback paths, from $Y_1(s)$ or $Y_2(s)$ to $U_1(s)$ or $U_2(s)$. There is no longer a logical sequence to follow when constructing a controller. Choosing, for example, the gain K_{22} linking U_2 to Y_2 will affect not only the transfer function $\frac{Y_2(s)}{U_2(s)}$ but also all the others. Hence every gain chosen has an effect on the others which must all be recalculated to take account of the change. The process is iterative, non-intuitive

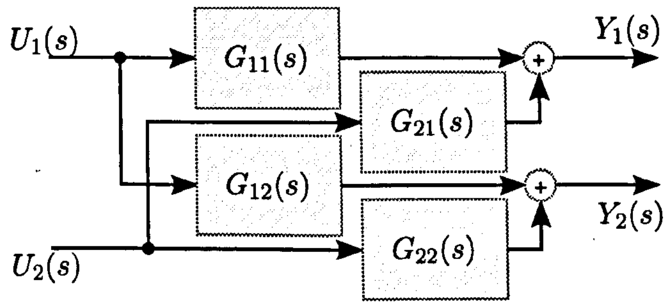

Figure 2.10: Multi-Input, Multi-Output Plant

and incredibly slow - but is still used for the control of rotorcraft today (Griffin, 1997). There is a need for such techniques to be replaced. The replacement must allow for the simultaneous selection of all gains in a Multi-Input, Multi-Output (MIMO) system, but must give the control engineer access to the design parameters of the system in such a way that the criteria applied to current designs (for example pole locations) may still be applied, without modification.

Many multivariable control techniques exist. Most do not concern themselves directly with the placement of poles. Attempts have been made to extend classical approaches to MIMO systems (Brockett and Byrnes, 1981), and to evaluate multivariable control techniques in terms of classical performance criteria (Doyle and Stein, 1981). However, neither approach is truly satisfactory for this purpose; it is necessary to find a true multivariable design methodology with clear links to classical control, rather than either a complex extension of existing classical techniques.

2.3 Eigenstructure Assignment

This section aims to introduce EA in the context of classical control, and then to outline its historical development. Since this thesis depends in part on showing that EA represents a suitable replacement for classical control methods in the design of rotorcraft control systems, the links between the two will be drawn out in some detail.

Sections 2.3.2 to 2.3.4 form, for the most part, a review of the literature. Inevitably, similarities exist between this review and those of others (White, 1995; Griffin, 1997; Ensor, 2000); the aim here however is specifically to develop a context within which the later chapters of this thesis may be set.

2.3.1 Background

This section will introduce the eigenstructure of a matrix, and a matrix representation for a multivariable linear system, before showing how the eigenstructure of one matrix in this representation can provide information about the system dynamics.

2.3.1.1 The Eigenstructure of a Matrix

Given a square matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$, a vector $\mathbf{v}_i \in \mathbb{C}^{n \times 1}$ is an *eigenvector* (strictly a *right eigenvector*) of $\mathbf{A}$ if and only if

$$\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i \tag{2.32}$$

where the constant λ_i is the corresponding *eigenvalue* (Wilkinson, 1965). (Note that eigenvectors must always be nonzero, so an eigenvector of $\mathbf{A}$ which corresponds to a zero eigenvalue is a vector from the right nullspace of $\mathbf{A}$.)

Any n-square matrix will always have n eigenvalues, but may have any number from 0 to n linearly independent eigenvectors (Weisstein, 2005). For the purposes of this section, we will assume that the matrix $\mathbf{A}$ has n linearly independent eigenvectors. The analysis below is more complex if this is not the case, but the definitions derived are still essentially valid. Therefore, if the matrix $\mathbf{A}$ has n eigenvalue-eigenvector pairs, we may write

$$\mathbf{A}\begin{bmatrix}\mathbf{v}_1 & \dots & \mathbf{v}_n\end{bmatrix} = \begin{bmatrix}\mathbf{v}_1 & \dots & \mathbf{v}_n\end{bmatrix}\begin{bmatrix}\lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n\end{bmatrix} \tag{2.33}$$

$$\mathbf{AV} = \mathbf{V\Lambda} \tag{2.34}$$

Since the set of right eigenvectors is linearly independent, it follows that $\mathbf{V}$ is full rank, and so we may also define

$$\mathbf{W} = \mathbf{V}^{-1} = \begin{bmatrix}\mathbf{w}_1 \\ \vdots \\ \mathbf{w}_n\end{bmatrix} \tag{2.35}$$

and write

$$\mathbf{A} = \mathbf{V\Lambda W} \tag{2.36}$$

$$\mathbf{WA} = \mathbf{\Lambda W} \tag{2.37}$$

From Equation 2.37 it can be seen that for a given eigenvalue λ_i, the eigenvalue-eigenvector relationship of Equation 2.32 has a dual:

$$\mathbf{w}_i\mathbf{A} = \lambda_i\mathbf{w}_i \tag{2.38}$$

The vector $\mathbf{w}_i \in \mathbb{C}^{1\times n}$ is therefore known as a *left eigenvector* of $\mathbf{A}$.

Note that for a real-valued matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, the eigenvalues $\{\lambda_i\}$ of $\mathbf{A}$ form a self-conjugate set and that $\lambda_i = \bar{\lambda}_j$ implies $\mathbf{v}_i = \bar{\mathbf{v}}_j$ and $\mathbf{w}_i = \bar{\mathbf{w}}_j$.

2.3.1.2 The State Variable Representation

The state variable representation of linear systems enables a large number of interconnected simultaneous differential equations to be formed into a single matrix equation. If, for a given system, enough information is known to permit the calculation of the unforced system output for all future time, then the system's *state* is known. The *state variable* approach assigns a set of independent variables to represent the system state, and considers the variation of the state with time as the system response.

Single input, single output systems such as those discussed above may be represented in state variable form. For example, a system in the general form of Equation 2.1 such as

$$\frac{d^3y}{dt^3} + a_2\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_0y = \frac{d^2u}{dt^2} + b_1\frac{du}{dt} + b_0u \tag{2.39}$$

may be written using the process of Section 2.2.1 in transfer function form as

$$\frac{Y(s)}{U(s)} = \frac{s^2 + b_1s + b_0}{s^3 + a_2s^2 + a_1s + a_0} \tag{2.40}$$

For clarity it will be assumed henceforth that all variables are in the Laplace domain and are functions of s.

The denominator of the transfer function is third order, and there are therefore three initial conditions (state variables) that must be known in order to develop a time response. Denoting these variables as X_0, X_1 and X_2, we may set

$$Y = X_0 + b_1 X_1 + b_0 X_2 \tag{2.41}$$

$$X_1 = \frac{1}{s} X_0 \tag{2.42}$$

$$X_2 = \frac{1}{s^2} X_0 \tag{2.43}$$

and hence we may write

$$Y = \left(1 + \frac{b_1}{s} + \frac{b_0}{s^2}\right) X_0 \tag{2.44}$$

$$\frac{X_0}{Y} = \frac{1}{1 + \frac{b_1}{s} + \frac{b_0}{s^2}} \tag{2.45}$$

$$= \frac{s^2}{s^2 + b_1 s + b_0} \tag{2.46}$$

$$\frac{X_0}{U} = \frac{X_0}{Y} \times \frac{Y}{U} = \frac{s^2}{s^3 + a_2 s^2 + a_1 s + a_0} \tag{2.47}$$

Rearranging,

$$s^2 U = \left(s^3 + a_2 s^2 + a_1 s + a_0\right) X_0 \tag{2.48}$$

$$U = \left(s + a_2 + \frac{a_1}{s} + \frac{a_0}{s^2}\right) X_0 \tag{2.49}$$

$$= s X_0 + a_2 X_0 + \frac{a_1}{s} X_0 + \frac{a_0}{s^2} X_0 \tag{2.50}$$

$$s X_0 = -a_2 X_0 - \frac{a_1}{s} X_0 - \frac{a_0}{s^2} X_0 + U \tag{2.51}$$

Substituting from Equations 2.42 and 2.43 gives

$$s X_0 = -a_2 X_0 - a_1 X_1 - a_0 X_2 + U \tag{2.52}$$

Equations 2.42, 2.43 and 2.52 may now be concatenated into a single matrix equation:

$$\begin{bmatrix} sX_0 \\ sX_1 \\ sX_2 \end{bmatrix} = \begin{bmatrix} -a_2 & -a_1 & -a_0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} U \tag{2.53}$$

It remains only to represent Equation 2.41 by writing

$$Y = \begin{bmatrix} 1 & b_1 & b_0 \end{bmatrix} \cdot \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix} \tag{2.54}$$

Taking the inverse Laplace transform of Equations 2.53 and 2.54 is trivial, and yields the following (where now all variables are functions of time):

$$\begin{bmatrix} \dot{x}_0 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -a_2 & -a_1 & -a_0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u \tag{2.55}$$

$$y = \begin{bmatrix} 1 & b_1 & b_0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} \tag{2.56}$$

or equivalently

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \tag{2.57}$$

$$y = \mathbf{C}\mathbf{x} \tag{2.58}$$

Equations 2.57 and 2.58 are the general form of the state variable representation of a single input, single output system providing that the denominator of the transfer function is of higher order than the numerator (if this is the case, the system is termed *strictly proper*). If the numerator and denominator are of equal order, there is a direct coupling from the input to the output, and this is represented by a separate term in the representation as will be seen below.

So far we have considered only single-input, single-output systems. The power of the state variable representation, however, lies in its ability to represent systems with multiple inputs and outputs. To do this it is simply necessary to represent the inputs and outputs as vectors, and to expand the matrices **B** and **C** in Equations 2.57 and 2.58 as necessary. Thus the general state variable form of a multi-input multi-output system is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{2.59}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \tag{2.60}$$

where $\mathbf{A} \in \mathbb{R}^{(n \times n)}$ is the *system matrix*, $\mathbf{B} \in \mathbb{R}^{(n \times r)}$ is the *input matrix*, $\mathbf{C} \in \mathbb{R}^{(m \times n)}$ is the *output matrix* and $\mathbf{D} \in \mathbb{R}^{(m \times r)}$ is the *direct transmission matrix.*

Throughout this thesis, and in the wider literature, a system might be referred to for example as 'a state-space system $(\mathbf{A}, \mathbf{B})$' meaning a state variable system whose direct transmission matrix is null and whose output matrix is the identity matrix (ie. all states are measurable).

Clearly, for a MIMO system, there can no longer be a single gain in a simple controller. Instead, typically, all system outputs are joined to all system inputs via a *matrix* of gains. For the remainder of this thesis, unless otherwise specified, the design of multivariable feedback systems will search for controllers that fit the design of Figure 2.11. The alternative arrangement, placing the gain matrix $\mathbf{K}$ in line with the system in a direct analogue of the arrangement of Figure 2.1, is usually more restrictive. This is because the gain matrix of Figure 2.11 has $(m \times r)$ entries for a system with r inputs and m outputs; the alternative, with the gain matrix in series with the system, has r^2 entries. Since in general $m > r$ (systems usually have more outputs than inputs), this latter situation is more restrictive.

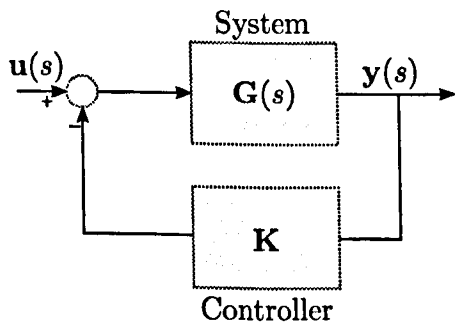

Figure 2.11: Multivariable Negative Feedback

Assuming the feedback structure of Figure 2.11, and further assuming that the direct transmission matrix is null, the closed-loop form of a state space system may now be found. Consider a state-space system under the influence of feedback such that

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{2.61}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} \tag{2.62}$$

$$\mathbf{u} = \mathbf{K}\mathbf{y} \tag{2.63}$$

By substitution,

$$\mathbf{u} = \mathbf{KCx} \tag{2.64}$$

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{BKCx} \tag{2.65}$$

$$= (\mathbf{A} + \mathbf{BKC})\mathbf{x} \tag{2.66}$$

Therefore Equations 2.62 and 2.66 together define the closed-loop system dynamics, and by comparison with Equation 2.61 we may define the closed loop system matrix $\mathbf{A}_{cl}$ as

$$\mathbf{A}_{cl} = \mathbf{A} + \mathbf{BKC} \tag{2.67}$$

2.3.1.3 Multivariable Transfer Functions

Given a system in the state variable form, it is possible to recover a transfer function representation even if the system has multiple inputs and outputs. In the Laplace domain, from Equation 2.59,

$$s\mathbf{x}(s) = \mathbf{Ax}(s) + \mathbf{Bu}(s) \tag{2.68}$$

$$(s\mathbf{I} - \mathbf{A})\,\mathbf{x}(s) = \mathbf{Bu}(s) \tag{2.69}$$

$$\mathbf{x}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{Bu}(s) \tag{2.70}$$

Substituting into Equation 2.58,

$$\mathbf{y}(s) = \mathbf{C}\,(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{Bu}(s) + \mathbf{Du}(s) \tag{2.71}$$

$$= \left(\mathbf{C}\,(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B} + \mathbf{D}\right)\mathbf{u}(s) \tag{2.72}$$

$$\triangleq \mathbf{G}(s)\mathbf{u}(s) \tag{2.73}$$

where $\mathbf{G}(s)$ is the *transfer function matrix*, containing a polynomial transfer function for each input-output pair.

Since the inverse of a matrix $\mathbf{X}$ may be written in terms of its adjoint matrix and its determinant as

$$\mathbf{X}^{-1} = \frac{\mathrm{adj}(\mathbf{X})}{\det(\mathbf{X})} \tag{2.74}$$

it is possible to write Equation 2.73 as

$$\mathbf{G}(s) = \frac{\mathbf{C}\,\mathrm{adj}\,(s\mathbf{I}-\mathbf{A})\,\mathbf{B}+\mathbf{D}}{\det\,(s\mathbf{I}-\mathbf{A})} \tag{2.75}$$

Equation 2.75 reveals that the denominator of each entry in the transfer function matrix is the same, while the numerators differ.

Interestingly, the transfer function matrix demonstrates that state variable systems are non-unique; it is possible to transform the state vector, and the system matrices, in such a way that the transfer function matrix remains unchanged.

Consider forming a transformed state vector $\hat{\mathbf{x}}$ such that

$$\mathbf{x} = \mathbf{E}\hat{\mathbf{x}} \tag{2.76}$$

where $\mathbf{E}$ is a square, nonsingular matrix. Now Equations 2.59 and 2.60 may be written as

$$\mathbf{E}\dot{\hat{\mathbf{x}}} = \mathbf{A}\mathbf{E}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} \tag{2.77}$$

$$\mathbf{y} = \mathbf{C}\mathbf{E}\hat{\mathbf{x}} + \mathbf{D}\mathbf{u} \tag{2.78}$$

and premultiplying Equation 2.77 by the inverse of $\mathbf{E}$ gives

$$\dot{\hat{\mathbf{x}}} = \left(\mathbf{E}^{-1}\mathbf{A}\mathbf{E}\right)\hat{\mathbf{x}} + \left(\mathbf{E}^{-1}\mathbf{B}\right)\mathbf{u} \tag{2.79}$$

$$\mathbf{y} = (\mathbf{C}\mathbf{E})\,\hat{\mathbf{x}} + \mathbf{D}\mathbf{u} \tag{2.80}$$

If we now define the transformed system matrices as

$$\hat{\mathbf{A}} \triangleq \mathbf{E}^{-1}\mathbf{A}\mathbf{E} \tag{2.81}$$

$$\hat{\mathbf{B}} \triangleq \mathbf{E}^{-1}\mathbf{B} \tag{2.82}$$

$$\hat{\mathbf{C}} \triangleq \mathbf{C}\mathbf{E} \tag{2.83}$$

$$\hat{\mathbf{D}} \triangleq \mathbf{D} \tag{2.84}$$

then the transformed system may be written as

$$\dot{\hat{\mathbf{x}}} = \hat{\mathbf{A}}\hat{\mathbf{x}} + \hat{\mathbf{B}}\mathbf{u} \tag{2.85}$$

$$\mathbf{y} = \hat{\mathbf{C}}\hat{\mathbf{x}} + \hat{\mathbf{D}}\mathbf{u} \tag{2.86}$$

The transfer function of the transformed system, from Equation 2.73, can now be seen to be

$$\hat{\mathbf{G}}(s) = \hat{\mathbf{C}}\left(s\mathbf{I} - \hat{\mathbf{A}}\right)^{-1}\hat{\mathbf{B}} + \hat{\mathbf{D}} \tag{2.87}$$

$$= (\mathbf{CE})\left(s\mathbf{I} - \mathbf{E}^{-1}\mathbf{AE}\right)^{-1}(\mathbf{E}^{-1}\mathbf{B}) + \mathbf{D} \tag{2.88}$$

$$= \mathbf{C}\left(\mathbf{E}\left(s\mathbf{I} - \mathbf{E}^{-1}\mathbf{AE}\right)\mathbf{E}^{-1}\right)^{-1}\mathbf{B} \tag{2.89}$$

$$= \mathbf{C}\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B} + \mathbf{D} \tag{2.90}$$

$$= \mathbf{G}(s) \tag{2.91}$$

as expected.

2.3.1.4 Eigenstructure and the State Space

There exists a simple form of time-domain solution for a state-variable system in the unforced case, a multivariable analogue of the response of Equation 2.2.

It may be shown (see Nise, 1995) that the unforced time response of a state variable system is given by

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) \tag{2.92}$$

where the term $e^{\mathbf{A}t}$, often denoted $\mathbf{\Phi}$, is known as the *state transition matrix*. If the state vector is considered as describing a direction in a space (or a hyperspace), then the state transition matrix describes the *trajectory* of the state vector through the *state space*. The term 'state space' has come to be employed to describe state variable systems, just as the word 'state' is used to mean 'state variable'. These terms will be used interchangeably for the remainder of this thesis.

The state transition matrix, although useful for determining a time response, is not particularly informative. It is not clear how changing the system matrix $\mathbf{A}$ will affect the time response of the resulting system. If instead a transformed system is constructed via Equations 2.81 to 2.84 using a transformation matrix $\mathbf{E} = \mathbf{V}$ (the matrix of right eigenvectors), then evaluating the time response of the transformed system is instructive. From Equation 2.81, the transformed system matrix is

$$\hat{\mathbf{A}} = \mathbf{V}^{-1}\mathbf{AV} = \mathbf{\Lambda} \tag{2.93}$$

via Equation 2.36, where $\mathbf{\Lambda}$ is the diagonal matrix of eigenvalues of $\mathbf{A}$. Using Equation 2.92, the unforced time response of the transposed system can be seen to be

$$\hat{\mathbf{x}}(t) = e^{\mathbf{\Lambda}t}\hat{\mathbf{x}}(0) \tag{2.94}$$

Applying Equation 2.76 yields

$$\mathbf{V}^{-1}\mathbf{x}(t) = e^{\mathbf{\Lambda}t}\mathbf{V}^{-1}\mathbf{x}(0) \tag{2.95}$$

$$\mathbf{x}(t) = \mathbf{V}e^{\mathbf{\Lambda}t}\mathbf{W}\mathbf{x}(0) \tag{2.96}$$

since $\mathbf{V}^{-1} = \mathbf{W}$ (from Equation 2.35).

Equation 2.96 is very informative, given that the transfer functions (and hence the poles) of the original and transposed systems are identical. Firstly it may be seen that the only time-variant portion of the unforced response is

$$e^{\mathbf{\Lambda}t} = \begin{bmatrix} e^{\lambda_1 t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n t} \end{bmatrix} \tag{2.97}$$

This is a diagonal matrix containing all of the exponential modes of the system (cf. Equation 2.2). Hence *the eigenvalues of* $\mathbf{A}$ *correspond directly to the system poles.* Secondly, the matrices $\mathbf{V}$ and $\mathbf{W}$ of the right and left eigenvectors respectively have a physical interpretation as well. The initial state vector $\mathbf{x}(0)$ acts to excite the system modes via $\mathbf{W}$, and the modal response manifests itself in the time-varying state vector $\mathbf{x}(t)$ via $\mathbf{V}$. Hence, for a given mode, the associated *left eigenvector describes the way in which state disturbances excite the system modes*, and the associated *right eigenvector describes the coupling of the system modes into the state vector.* These facts underpin the idea of EA: If a control system can be designed that affects the eigenstructure of $\mathbf{A}$ in a predictable manner, then the system poles may be placed and the coupling between the system modes and the states may be influenced.

2.3.1.5 Controllability and Observability

Controllability and observability, as properties of a state variable system, are important concepts in the context of control system design.

A system is said to be *controllable* if it is possible to generate a system input that will transfer the system from its initial state $\mathbf{x}(0)$ to any other state in a finite period of time. If this is not the case it implies that there are uncontrollable system modes which can not be affected by manipulation of the system input. It can be shown (Ogata, 1997) that the controllability of a state space system $(\mathbf{A}, \mathbf{B})$ is equivalent to the condition that

$$\operatorname{rank}\left(\begin{bmatrix}\mathbf{B} & \vdots & \mathbf{AB} & \vdots & \cdots & \vdots & \mathbf{A}^{n-1}\mathbf{B}\end{bmatrix}\right) = n \tag{2.98}$$

Similarly, a system is said to be *observable* if it is possible, by observation of the system output $\mathbf{y}(t)$ over a finite time, to determine the system state. If this is not the case, it implies that there are unobservable system modes which do not manifest themselves in the system output. It can again be shown (Ogata, 1997) that the observability of a state space system $(\mathbf{A}, \mathbf{C})$ is equivalent to the condition that

$$\operatorname{rank}\left(\begin{bmatrix}\mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1}\end{bmatrix}\right) = n \tag{2.99}$$

Controllability and observability are often given as preconditions for forming multivariable control systems, as will be seen. This is not as restrictive a condition as it would at first appear to be, however. If a system has uncontrollable modes, either these modes are important to the system response or they are not; if they are not, a reduced model may be formed that does not include these dynamics. Similarly, if unobservable modes are present then further sensors should be added to the system to allow detection of these modes, lest they become unstable under feedback.

Interestingly, it is possible for the controllability or observability test matrices to be full rank but ill-conditioned; this implies either a nearly-uncontrollable mode, requiring large control efforts to affect it, or a nearly-unobservable mode, whose effect is hard to detect in the output vector. For this reason Mehrmann and Xu (2000) define the concept of 'distance to controllability' and use it as a system analysis tool.

2.3.1.6 Modal Coupling Matrices

Taking the multivariable transfer function matrix from Equation 2.73

$$\mathbf{G}(s) = \mathbf{C}\,(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B} + \mathbf{D} \tag{2.100}$$

it may be seen that the eigenvalue-eigenvector decomposition of $\mathbf{A}$ from Equation 2.36 may be used to yield

$$\mathbf{G}(s) = \mathbf{C}\,(s\mathbf{I} - \mathbf{V\Lambda W})^{-1}\,\mathbf{B} + \mathbf{D} \tag{2.101}$$

$$= \mathbf{C}\,(\mathbf{V}\,(s\mathbf{I} - \mathbf{\Lambda})\,\mathbf{W})^{-1}\,\mathbf{B} + \mathbf{D} \tag{2.102}$$

$$= \mathbf{CV}\,(s\mathbf{I} - \mathbf{\Lambda})^{-1}\,\mathbf{WB} + \mathbf{D} \tag{2.103}$$

The term $(s\mathbf{I} - \mathbf{\Lambda})^{-1}$ is a diagonal matrix of terms which, in the Laplace domain, describe the exponential response of each mode. It is known that the matrices $\mathbf{W}$ and $\mathbf{V}$ describe the coupling from the states to the modes and the modes to the states respectively.

Hence, the terms $\mathbf{CV}$ and $\mathbf{WB}$ in Equation 2.103 describe the coupling from the modes to the outputs, and from the inputs to the modes. They are known respectively as the *output coupling matrix* and the *input coupling matrix*, and collectively as the *modal coupling matrices*.

Often, entries of the modal coupling matrices (particularly of the output coupling matrix) will be of more importance to the satisfaction of design criteria than the eigenvectors themselves. Indeed, for systems which have been obtained using system identification techniques rather than direct modelling, the states have no physical meaning and assignment of the modal coupling matrices is the only satisfactory solution (Griffin, 1997).

2.3.1.7 Multivariable Zeros

It has been seen that the poles and zeros of a SISO system are those values of s which cause the transfer function $G(s)$ to equal ∞ or 0, respectively. It is instructive to draw a parallel between this behaviour and that of their MIMO counterparts.

From Equation 2.73, it is clear that any eigenvalue $s = \lambda$ of $\mathbf{A}$ will cause the matrix $(s\mathbf{I} - \mathbf{A})$ to have a zero determinant. Hence the system poles are those values $\{\lambda\}$ for which $\mathbf{G}(\lambda) = \infty$. Such a definition is possible because all the entries in the transfer function matrix share a common denominator (from Equation 2.75).

Since the entries of the transfer function matrix do not share a common numerator, such a simple definition will not suffice for multivariable zeros. Instead, we may define any constant z such that $\mathbf{G}(z) = 0$ as a *blocking zero*. The response of a system with a blocking zero z to an input $\mathbf{u}(t)e^{zt}$ is zero for any $\mathbf{u}(t)$ (Tsui, 1996).

However, this definition tells us only when a value of s causes *every* element in $\mathbf{G}(s)$ to become zero. It is also useful to characterise those values of s which cause *any* element in $\mathbf{G}(s)$ to become zero, and hence represent a null in the response of a subset of the input-output paths. Such values are known as *transmission zeros*, and they may be identified as those values of z for which $\text{rank}(\mathbf{G}(s)) < \min(m, r)$ for a system with r inputs and m outputs.

Tsui (1996) states that the response of a system with a transmission zero z to an input $\mathbf{u}(t)e^{zt}$ is zero for any vector $\mathbf{u}(t)$ such that $\mathbf{G}(z)\mathbf{u}(t) = 0$, provided the system has more outputs than inputs. Clearly then, transmission zeros have *direction* as well as value; also, blocking zeros can be seen to be a special case of transmission zeros.

Finally, it is readily shown that the transmission zeros of a system are those values of z for which the test matrix

$$\mathbf{S} = \begin{bmatrix} s\mathbf{I} - \mathbf{A} & -\mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \tag{2.104}$$

loses rank. The following derivation is taken, modified, from Tsui (1996).

$$\begin{bmatrix} \mathbf{I} & 0 \\ -\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1} & \mathbf{I} \end{bmatrix} \begin{bmatrix} z\mathbf{I} - \mathbf{A} & -\mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} s\mathbf{I} - \mathbf{A} & -\mathbf{B} \\ 0 & \mathbf{G}(s) \end{bmatrix} \tag{2.105}$$

$$\text{rank}(\mathbf{S}) = \text{rank}\left(\begin{bmatrix} s\mathbf{I} - \mathbf{A} & -\mathbf{B} \\ 0 & \mathbf{G}(s) \end{bmatrix}\right). \tag{2.106}$$

$$= \text{rank}\,(s\mathbf{I} - \mathbf{A}) + \text{rank}\,(\mathbf{G}(s)) \tag{2.107}$$

Hence provided z is not an eigenvalue of $\mathbf{A}$, if the test matrix $\mathbf{S}$ as defined by Equation 2.104 loses rank then z must be a transmission zero of the system.

2.3.2 Multivariable Pole Placement

White (1995) traces the origins of EA to early 'eigenvalue shifting' (pole placement) algorithms of the 1960s, but Griffin (1997) identifies the work of Wonham (1967) as providing the impetus for the development of early algorithms by demonstrating that the controllability of the target system is necessary and sufficient for the arbitrary assignment of its poles assuming that all states were measurable and employed for control (a situation known as *state feedback*, where effectively $\mathbf{C} = \mathbf{I}$). State feedback is not a realistic representation of most systems however, and attention turned to addressing the problem of *output feedback.* Output feedback may be considered to be a constrained form of state feedback, wherein the outputs and not the states are available for measurement; however almost invariably the $\mathbf{D}$ matrix is still assumed to be null. Early results (Davison and Chatterjee, 1971) showed that incomplete pole placement was feasible, with at least $\max(m, r)$ poles being assignable.

In 1975 however, two papers (Kimura, 1975; Davison and Wang, 1975) were published on the subject of output-feedback pole placement. The result presented in these, and in a follow-up paper by Kimura (1977), is that for a system with m outputs and r inputs, it is possible to assign arbitrarily $(m + r - 1)$ eigenvalues, providing the system is both controllable and observable. Hence for a system with n states, if

$$m + r > n \tag{2.108}$$

a gain matrix $\mathbf{K}$ may be found such that all the closed-loop system poles may be assigned.

More recently, several authors (Wang, 1992; Rosenthal et al., 1995; Griffin, 1997) have demonstrated that for arbitrary placement of n poles, the condition

$$mr \geq n \tag{2.109}$$

is sufficient. However, the use of linear algebra to solve the pole placement or EA problem generally results in the need for the condition of Equation 2.108. Recently, a family of EA algorithms has been published (Tsui, 2005), several of which achieve pole assignment along with restricted eigenvector assignment under the less restrictive conditions of Equation 2.109. These algorithms will be considered in more detail later on.

There exists in the literature a fair amount of confusion between the terms 'eigenstructure assignment' and 'pole placement'. The opinion of the author is that if eigen*vector* assignment is not considered, or if the eigenvectors are implicitly assigned to meet a goal that does not form part of the design specification, such an algorithm constitutes pole placement and not EA. Attention will be drawn to such algorithms in the following discussion.

2.3.3 State Feedback EA

Eigenstructure assignment proper has its origins in a paper by Moore (1976). In this paper, it is shown that given a controllable state-feedback system (**A**,**B**) with r inputs and n states, it is possible to assign all n eigenvalues *and* all n right eigenvectors, where each eigenvector may be selected from a subspace of dimension r. A summary of his result follows for reference.

Given a state space system (**A**,**B**) under state feedback, the closed loop system matrix $\mathbf{A}_{cl}$ can be seen (from Equation 2.67) to be

$$\mathbf{A}_{cl} = \mathbf{A} + \mathbf{BK} \tag{2.110}$$

Now for any given closed-loop eigenvalue-eigenvector pair $(\lambda_i, \mathbf{v}_i)$, by definition

$$\mathbf{A}_{cl}\mathbf{v}_i = \lambda_i \mathbf{v}_i \tag{2.111}$$

$$(\mathbf{A} + \mathbf{BK})\mathbf{v}_i = \lambda_i \mathbf{v}_i \tag{2.112}$$

$$\mathbf{0} = (\mathbf{A} - \lambda_i \mathbf{I})\mathbf{v}_i + \mathbf{BK}\mathbf{v}_i \tag{2.113}$$

$$\mathbf{0} = \begin{bmatrix} \mathbf{A} - \lambda_i \mathbf{I} & \vdots & \mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{v}_i \\ \mathbf{K}\mathbf{v}_i \end{bmatrix} \tag{2.114}$$

Hence, for some $\mathbf{f}_i \in \mathbb{C}^{r \times 1}$,

$$\begin{bmatrix} \mathbf{v}_i \\ \mathbf{K}\mathbf{v}_i \end{bmatrix} = \begin{bmatrix} \mathbf{P}_i \\ \mathbf{Q}_i \end{bmatrix} \mathbf{f}_i \tag{2.115}$$

where

$$\text{range}\left(\begin{bmatrix} \mathbf{P}_i \\ \mathbf{Q}_i \end{bmatrix} \right) = \ker\left(\begin{bmatrix} \mathbf{A} - \lambda_i \mathbf{I} & \vdots & \mathbf{B} \end{bmatrix} \right) \tag{2.116}$$

and $\mathbf{P}_i \in \mathbb{C}^{n \times r}$ is the allowable subspace for the selection of the eigenvector $\mathbf{v}_i$.

Moore (1976) shows that for assignment to succeed, the selected eigenvectors $\{\mathbf{v}_i\}$ must be linearly independent and that $\lambda_i = \bar{\lambda}_i$ must imply $\mathbf{v}_i = \bar{\mathbf{v}}_i$. The condition that the selected eigenvalues should be distinct was also included, but this was later removed by Porter and D'Azzo (1978).

Once the design vectors $\{\mathbf{f}_i\}$ have been selected, the gain matrix **K** may be recovered trivially by finding

$$\mathbf{V} = \begin{bmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_n\end{bmatrix} = \begin{bmatrix}\mathbf{P}_1\mathbf{f}_1 & \cdots & \mathbf{P}_n\mathbf{f}_n\end{bmatrix} \tag{2.117}$$

$$\mathbf{KV} = \begin{bmatrix}\mathbf{Kv}_1 & \cdots & \mathbf{Kv}_n\end{bmatrix} = \begin{bmatrix}\mathbf{Q}_1\mathbf{f}_1 & \cdots & \mathbf{Q}_n\mathbf{f}_n\end{bmatrix} \tag{2.118}$$

$$\mathbf{K} = \mathbf{KV} \cdot \mathbf{V}^{-1} \tag{2.119}$$

The choice of EA as a design methodology in any given context implies that the control over modal coupling provided by the assignment of eigenvectors is of use in satisfying the design goals. Consequently, Equation 2.115 is most likely to be satisfied by the projection of a desired eigenvector into the allowable eigenvector subspace, and this projection will generally introduce a discrepancy between the desired and achieved eigenvectors. It may be seen from Equation 2.116 that the only variable upon which the right eigenvector subspace depends is the corresponding chosen eigenvalue, and for this reason Griffin (1997) describes a tradeoff algorithm wherein desired eigenvalues can be moved to align the eigenvector subspaces more closely to the desired eigenvectors, minimising the projection error.

2.3.4 Output Feedback EA

It has been mentioned that output feedback may be seen as a restricted case of state feedback. The restriction, via the matrix $\mathbf{C}$, does not at first seem to pose a significant problem. Following the method of Section 2.3.3 with a non-identity $\mathbf{C}$ matrix produces the following result (Andry et al., 1983):

$$\mathbf{A}_{cl} = \mathbf{A} + \mathbf{BKC} \tag{2.120}$$

$$\mathbf{A}_{cl}\mathbf{v}_i = \lambda_i\mathbf{v}_i \tag{2.121}$$

$$(\mathbf{A} + \mathbf{BKC})\mathbf{v}_i = \lambda_i\mathbf{v}_i \tag{2.122}$$

$$\mathbf{0} = (\mathbf{A} - \lambda_i\mathbf{I})\mathbf{v}_i + \mathbf{BKCv}_i \tag{2.123}$$

$$\mathbf{0} = \begin{bmatrix}\mathbf{A} - \lambda_i\mathbf{I} & \vdots & \mathbf{B}\end{bmatrix} \begin{bmatrix}\mathbf{v}_i \\ \mathbf{KCv}_i\end{bmatrix} \tag{2.124}$$

Hence, for some $\mathbf{f}_i \in \mathbb{C}^{r\times 1}$,

$$\begin{bmatrix}\mathbf{v}_i \\ \mathbf{Kv}_i\end{bmatrix} = \begin{bmatrix}\mathbf{P}_i \\ \mathbf{Q}_i\end{bmatrix}\mathbf{f}_i \tag{2.125}$$

where $\mathbf{P}_i$ and $\mathbf{Q}_i$ have meanings as before. The gain matrix may be recovered by finding

$$\mathbf{V} = \begin{bmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_p\end{bmatrix} = \begin{bmatrix}\mathbf{P}_1\mathbf{f}_1 & \cdots & \mathbf{P}_p\mathbf{f}_p\end{bmatrix} \tag{2.126}$$

$$\mathbf{KCV} = \begin{bmatrix}\mathbf{KCv}_1 & \cdots & \mathbf{KCv}_p\end{bmatrix} = \begin{bmatrix}\mathbf{Q}_1\mathbf{f}_1 & \cdots & \mathbf{Q}_p\mathbf{f}_p\end{bmatrix} \tag{2.127}$$

$$\mathbf{K} = \mathbf{KCV} \cdot (\mathbf{CV})^{\dagger} \tag{2.128}$$

where $\mathbf{X}^{\dagger}$ is the Moore-Penrose pseudo-inverse (Ben-Israel and Greville, 1974) of $\mathbf{X}$. However, Equation 2.128 holds only if $p \leq m$, and consequently a maximum of m poles may be assigned using this method. The remainder are left unassigned and may become unstable. The conditions under which a state-feedback solution may be transformed directly into an output-feedback one in this manner are extremely restrictive (Porter, 1977).

The output feedback problem is essentially a symmetrical one, with the matrices $\mathbf{B}$ and $\mathbf{C}$ being duals of one another in a functional sense. Hence, assigning *left* eigenvectors along with their associated eigenvalues by using subspaces generated as a function of the output matrix $\mathbf{C}$ is feasible, and no more restrictive than the right eigenvector assignment described above.

Given the definition of a left eigenvector $\mathbf{w}_j$,

$$\mathbf{A}_{cl} = \mathbf{A} + \mathbf{BKC} \tag{2.129}$$

$$\mathbf{w}_j\mathbf{A}_{cl} = \lambda_j\mathbf{w}_j \tag{2.130}$$

$$\mathbf{w}_j(\mathbf{A} + \mathbf{BKC}) = \lambda_j\mathbf{w}_j \tag{2.131}$$

$$\mathbf{0} = \mathbf{w}_j(\mathbf{A} - \lambda_i\mathbf{I}) + \mathbf{w}_j\mathbf{BKC} \tag{2.132}$$

$$\mathbf{0} = \begin{bmatrix}\mathbf{w}_j & \vdots & \mathbf{w}_j\mathbf{BK}\end{bmatrix}\begin{bmatrix}\mathbf{A} - \lambda_j\mathbf{I} \\ \mathbf{C}\end{bmatrix} \tag{2.133}$$

Hence, for some $\mathbf{g}_j \in \mathbb{C}^{1\times m}$,

$$\begin{bmatrix}\mathbf{w}_j & \vdots & \mathbf{w}_j\mathbf{BK}\end{bmatrix} = \mathbf{g}_j\begin{bmatrix}\mathbf{L}_j & \vdots & \mathbf{M}_j\end{bmatrix} \tag{2.134}$$

where

$$\text{range}\left(\begin{bmatrix}\mathbf{L}_j & \vdots & \mathbf{M}_j\end{bmatrix}^T\right) = \ker\left(\begin{bmatrix}\mathbf{A} - \lambda_j\mathbf{I} \\ \mathbf{C}\end{bmatrix}^T\right) \tag{2.135}$$

and $\mathbf{L}_j \in \mathbb{C}^{m\times n}$ is the allowable subspace for the selection of the eigenvector $\mathbf{w}_j$.

Once again, following the selection of q design vectors $\{\mathbf{g}_j\}$, the gain matrix may be recovered by finding

$$\mathbf{W} = \begin{bmatrix} \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_q \end{bmatrix} = \begin{bmatrix} \mathbf{g}_1\mathbf{L}_1 \\ \vdots \\ \mathbf{g}_q\mathbf{L}_q \end{bmatrix} \tag{2.136}$$

$$\mathbf{WBK} = \begin{bmatrix} \mathbf{w}_1\mathbf{BK} \\ \vdots \\ \mathbf{w}_q\mathbf{BK} \end{bmatrix} = \begin{bmatrix} \mathbf{g}_1\mathbf{M}_1 \\ \vdots \\ \mathbf{g}_q\mathbf{M}_q \end{bmatrix} \tag{2.137}$$

$$\mathbf{K} = (\mathbf{WB})^{\dagger} \cdot \mathbf{WBK} \tag{2.138}$$

where this time $\mathbf{K}$ may be recovered if $q \leq r$.

It is therefore possible, by considering either the right or left eigenvector subspaces, to specify $\max(m, r)$ eigenvalues and $\min(m, r)$ elements of each corresponding eigenvector, a result found by Srinathkumar (1978). However, by imposing more stringent constraints on the eigenvector subspaces, it is possible to assign both $p \leq m$ eigenvalues and right eigenvectors, and $q \leq r$ eigenvalues and left eigenvectors (though usually not if $p = m$ *and* $q = r$); hence, if $m + r > n$, it is possible to assign all n eigenvalues. The algorithms which follow attempt to do just this.

White (1995) identifies four distinct EA methods: Protection methods, parametric methods, projection methods and orthogonal eigenvector methods. However, the latter two may be considered to be special cases of the others, and so for clarity the only classification that will be made here is to divide parametric methods from protection methods.

2.3.4.1 Protection Methods

Protection methods (Davison and Wang, 1975; Srinathkumar, 1978; Fletcher, 1981*a*, *b*; Fletcher and Magni, 1987; Fahmy and O'Reilly, 1988) work by assigning a subset of the eigenvalues, then constructing a system with a reduced input or output vector such that the assigned modes become uncontrollable or unobservable. If right eigenvectors have been assigned, the reduced system has a shorter output vector than the original; if left eigenvectors were assigned, it is the input vector that is shortened. These modes are *protected* since uncontrollable or unobservable modes are invariant under output feedback. The remainder of the eigenvalues can then be assigned to the reduced system, and the gain matrix reconstituted by superposition at the end.

The algorithm of Fletcher (1981*a*, *b*) is interesting as it includes provision for a nonzero direct transmission matrix, something that very few EA algorithms do. His algorithm assigned eigenvalue-eigenvector pairs one at a time, each time constructing a reduced system before continuing.

Protection methods illustrate the requirement of most EA algorithms for the condition $m+r > n$ to be satisfied. Consider assigning $m-1$ eigenvalues to a system, together with $m-1$ right eigenvectors selected from subspaces of dimension r. The reduced system will have a single output, and so in the second stage one may assign r eigenvalues and r left eigenvectors, each chosen with no freedom from a subspace of dimension 1. Hence, for complete assignment, $(m-1)+r \geq n$, or equivalently

$$m + r > n \tag{2.139}$$

Although protection methods constitute a mathematically sound approach to the problem of output feedback EA they are not ideal replacements for classical approaches. The main problem is one of *visibility*, as described in Chapter 1. During the assignment process it is impossible to see the effect that assigning one eigenvector will have on the allowable subspace for the remainder, and this effectively precludes the iterative tradeoff of some design parameters for others during the assignment process. Such iterative processes are at the core of classical control techniques, and their use helps to reinforce an understanding of the role that the various design parameters have in the performance of the closed loop system.

Additionally, protection methods rely heavily on the numerical accuracy of the protection process. If the protection is not exact, assigned eigenvalues and eigenvectors can move during the later stages of assignment, or very large gains may be generated as the algorithm attempts to reassign almost-uncontrollable eigenvalues to unsuitable locations (Fahmy and O'Reilly, 1988; White, 1991).

Finally it should be noted that Fletcher's algorithms mention eigenvectors, but do not consider their importance to the design solution or how they should be chosen. Consequently these algorithms border on pole placement rather than EA.

2.3.4.2 Parametric Methods

Parametric methods constitute extensions to the original state feedback EA algorithm presented by Moore (1976), which parameterised the allowable subspace for the right eigenvectors as a means of generating a gain matrix. At the beginning of the Section it was shown that merely parameterising the right or left eigenvector subspaces alone was insufficient in the output feedback case, but the protection methods detailed above show that sufficient design freedom does exist for the placement of n poles and the selection of n eigenvectors given $m+r > n$. Hence, restrictions on the eigenvectors beyond the allowable subspace constraints must exist in the output feedback case. Many researchers (Fahmy and O'Reilly, 1983; White, 1991; Duan, 1993; Griffin, 1997; Clarke et al., 2003; Tsui, 2005) have continued the search for parametric EA methods on the basis that the design freedom is available in a more visible way than is the case with protection methods.

As an aside, White (1995) describes 'projection methods' as being those which project desired eigenvectors into their allowable subspaces. However, they can be viewed simply as parametric methods wherein the parameters are employed for the projection of desired eigenvectors, and consideration of these as distinct from parametric methods as a whole is not helpful. Similarly, 'orthogonal eigenvector methods' are those which seek to assign a near-orthogonal set of eigenvectors since analysis shows that the sensitivity of the closed-loop eigenvalues to perturbations in the set $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{K}\}$ is at a minimum when the eigenvectors are orthogonal (Wilkinson, 1965; Kautsky et al., 1982; Fletcher et al., 1985). These are not assignment methods as such, but employ complex methods to determine a set of assignable eigenvectors which are as near as possible to being mutually orthogonal prior to assignment. In general parametric methods are used for the actual assignment, often in an iterative process in which the eigenvector directions are adjusted gradually until the desired trade-off between nominal performance and eigenvalue sensitivity is achieved.

For simplicity of notation let us now define the left and right allowable eigenvector subspaces for a given eigenvalue s as

$$\mathfrak{L}(s) = \ker\left(\begin{bmatrix} \mathbf{A} - s\mathbf{I} \\ \mathbf{C} \end{bmatrix}^T\right) \tag{2.140}$$

$$\mathfrak{R}(s) = \ker\left(\begin{bmatrix} \mathbf{A} - s\mathbf{I} & \vdots & \mathbf{B} \end{bmatrix}\right) \tag{2.141}$$

Kimura (1977) identified the restriction incumbent upon the eigenvector subspaces as a result of the output feedback restrictions. His theorem is duplicated here for reference.

Theorem 2.3.1. (Kimura, 1977, Theorem 1) *A self-conjugate set $\{\lambda_i\}$ is pole assignable if and only if there exists $\mathbf{v}_i \in \mathfrak{R}(\lambda_i)$ and $\mathbf{w}_j \in \mathfrak{L}(\lambda_j)$ such that*

C1 $(\mathbf{v}_i, i = 1 \ldots n)$ *are linearly independent and* $\lambda_i = \bar{\lambda}_j$ *implies* $\mathbf{v}_i = \bar{\mathbf{v}}_j$*;*

C2 $(\mathbf{w}_i, i = 1 \ldots n)$ *are linearly independent and* $\lambda_i = \bar{\lambda}_j$ *implies* $\mathbf{w}_i = \bar{\mathbf{w}}_j$*;*

C3 $\mathbf{w}_j \mathbf{v}_i = 0$ *for all* $i \neq j$.

◊◊

Conditions C1 and C2 of Theorem 2.3.1 have already been identified in the context of state feedback EA and the simple output feedback extensions described at the start of this Section. Condition C3 is known as the 'orthogonality condition' and encapsulates the additional constraints on the eigenvector subspaces due to output feedback.

Once a set of eigenvectors satisfying Conditions C1 to C3 of Theorem 2.3.1 have been found, the gain matrix may be recovered by finding

$$\mathbf{K} = \mathbf{KCV}(\mathbf{CV})^{\dagger} = (\mathbf{WB})^{\dagger}\mathbf{WBK} \tag{2.142}$$

or

$$\mathbf{K} = \mathbf{B}^{\dagger}(\mathbf{A}_{cl} - \mathbf{A})\mathbf{C}^{\dagger} \tag{2.143}$$

where $\mathbf{A}_{cl} = \mathbf{V\Lambda W}$ (Fletcher et al., 1985; White, 1995; Griffin, 1997).

Finding a set of eigenvectors such that the conditions of Theorem 2.3.1 are satisfied is not trivial. A reduced orthogonality condition, introduced by Griffin (1997) and published by Clarke et al. (2003), assists greatly in this process. It is summarised in Theorem 2.3.2:

Theorem 2.3.2. (Clarke et al., 2003, Theorem 2) *A self-conjugate set* $\{\lambda_i\}$ *is pole assignable if and only if there exists* $\mathbf{v}_i \in \mathfrak{R}(\lambda_i)$ *and* $\mathbf{w}_j \in \mathfrak{L}(\lambda_j)$ *such that*

C1 $\text{rank}(\mathbf{C}[\mathbf{v}_1, \ldots, \mathbf{v}_v]) = v$ *and* $\lambda_i = \bar{\lambda}_k$ *implies* $\mathbf{v}_i = \bar{\mathbf{v}}_k$*;*

C2 $\text{rank}\left(\mathbf{B}^T\left[\mathbf{w}_{v+1}^T, \ldots, \mathbf{w}_n^T\right]\right) = n - v$ *and* $\lambda_j = \bar{\lambda}_k$ *implies* $\mathbf{w}_j = \bar{\mathbf{w}}_k$*;*

C3 $\mathbf{w}_j \mathbf{v}_i = 0$ *for all* $i = 1 \ldots v$, $j = (v+1) \ldots n$.

◊◊

The reduced orthogonality condition clearly shows that only n eigenvectors need be specified for the placement of n poles, a fact made obvious by the protection methods described above. In addition, the proof of Theorem 2.3.2 gives rise to a formula for the recovery of the gain matrix. If the right and left subspaces are defined by Equations 2.116 and 2.135 respectively and parameterised by design vectors $\{\mathbf{f}_i\}$ and $\{\mathbf{g}_j\}$ as in Equations 2.115 and 2.134, the following matrices may be found:

$$\mathbf{V}' = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_v \end{bmatrix} = \begin{bmatrix} \mathbf{P}_1\mathbf{f}_1 & \cdots & \mathbf{P}_v\mathbf{f}_v \end{bmatrix} \tag{2.144}$$

$$\mathbf{S}' = \begin{bmatrix} \mathbf{KCv}_1 & \cdots & \mathbf{KCv}_v \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_1\mathbf{f}_1 & \cdots & \mathbf{Q}_v\mathbf{f}_v \end{bmatrix} \tag{2.145}$$

$$\mathbf{W}' = \begin{bmatrix} \mathbf{w}_{(v+1)} \\ \vdots \\ \mathbf{w}_n \end{bmatrix} = \begin{bmatrix} \mathbf{g}_{(v+1)}\mathbf{L}_{(v+1)} \\ \vdots \\ \mathbf{g}_n\mathbf{L}_n \end{bmatrix} \tag{2.146}$$

$$\mathbf{T}' = \begin{bmatrix} \mathbf{w}_{(v+1)}\mathbf{BK} \\ \vdots \\ \mathbf{w}_n\mathbf{BK} \end{bmatrix} = \begin{bmatrix} \mathbf{g}_{(v+1)}\mathbf{M}_{(v+1)} \\ \vdots \\ \mathbf{g}_n\mathbf{M}_n \end{bmatrix} \tag{2.147}$$

$$\tag{2.148}$$

Finally, the gain matrix may be constructed as

$$\begin{aligned}\mathbf{K} = {} & (\mathbf{W}'\mathbf{B})^{\dagger}\mathbf{T}' + \mathbf{S}'(\mathbf{CV}')^{\dagger} - (\mathbf{W}'\mathbf{B})^{\dagger}\mathbf{T}'\mathbf{CV}'(\mathbf{CV}')^{\dagger} \\ & + \left(\mathbf{I} - (\mathbf{W}'\mathbf{B})^{\dagger}\mathbf{W}'\mathbf{B}\right)\mathbf{Z}\left(\mathbf{I} - \mathbf{CV}'(\mathbf{CV}')^{\dagger}\right)\end{aligned} \tag{2.149}$$

where $\mathbf{Z}$ is a matrix of free parameters characterising all possible solutions, which may be set to zero if not required. Clarke et al. (2003) show that the mapping of $\mathbf{Z}$ onto $\mathbf{K}$ is non-null if

$$(m-p)(r-n+v) > 0 \tag{2.150}$$

and go on to develop an algorithm to use any remaining design freedom in a *retro-assignment* phase which assigns, from a very restricted subspace, eigenvectors in the opposite sets to those assigned in the first instance (Clarke and Griffin, 2004).

Some recent work has concentrated on the problem of pole placement and EA under the condition that $mr \geq n$. Alexandridis and Paraskevopoulos (1996), for example, demonstrate that pole placement is feasible under these conditions, but do not consider any available freedom over the eigenvectors. Most recently, Tsui (2005) has generated a set of algorithms with the aim of exploiting the available design freedom to the greatest possible extent. These algorithms *are* EA, since eigenvectors are considered and explicitly assigned. The approach is essentially a two-stage one, where the eigenvectors in the first stage are chosen specifically to allow the assignment of the desired eigenvalues, together with eigenvectors which meet the orthogonality condition, in the second stage. While interesting, these algorithms concentrate on the exploitation of design freedom to a greater extent than will usually be necessary for the helicopter control problem, since such systems are generally well instrumented and have no problem meeting the more stringent condition that $m + r > n$.

2.3.5 Descriptor Systems

Descriptor systems (or *singular systems*) are those in which the state equation may be written

$$\mathbf{E}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{2.151}$$

where the matrix $\mathbf{E}$ is square but need not be nonsingular. Indeed, if $\mathbf{E}$ *is* nonsingular, it may be seen that premultiplication of Equation 2.151 by the inverse of $\mathbf{E}$ will result in the expression of the system in standard form. The assignment of eigenstructure to descriptor systems has received considerable attention in recent years (Duan, 1998, and references therein).

Inasmuch as models of helicopters can readily be generated in standard form (Griffin, 1997; Gee, 2000), the assignment of eigenstructure to descriptor systems - and, indeed, to linear systems of higher order (Duan and Liu, 2002; Duan, 2005) - is outside the scope of this thesis. However, algorithms for Proportional-plus-Derivative (PD) control of descriptor systems using EA exist (Jing, 1994; Duan and Patton, 1997, 1999; Owens and Askarpour, 2000) and these potentially solve a class of problems addressed by the algorithms developed by the author in Chapter 5. Hence it is fitting to mention such systems briefly here.

2.4 Conclusions

The link between EA and classical control is clear, and this fact makes EA appropriate for helicopter control since so much control law design is still conducted using classical techniques (Griffin, 1997). Additionally, as will be seen in Chapter 3, the specifications for the response of a helicopter are readily converted into restrictions on pole locations.

The difference between pole placement and EA has been discussed. In a plant so highly cross-coupled as a helicopter, direct access to the eigenvectors is very useful during the design process. Of the EA algorithms considered, the state feedback algorithm presented by Moore (1976) and the output feedback algorithm from Clarke et al. (2003) provide the most visibility - that is, they provide the control engineer with an obvious link between the choice of design parameters and the closed-loop system performance.

2.5 Chapter Bibliography

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Chapter 3

Helicopters and Eigenstructure Assignment

Contents

3.1 Introduction

In order to effect control of a system as complex as a helicopter, it must first be understood from a theoretical standpoint. Once the general characteristics have been identified by examining the physics of the system, a controller can be developed which will specifically address the problems found.

This chapter will introduce the helicopter as a mechanical system, and derive a sufficiently detailed description of its operation as is required in understanding the problems faced by the designer of a helicopter control system. Following that, the specific issues concerned with the application of Eigenstructure Assignment (EA) to helicopters will be considered, including the specification of closed-loop performance, and a new ideal eigenstructure for forward flight will be developed.

It should be noted that the vast majority of helicopters currently in service are in the same configuration - a main rotor on top of the aircraft and a tail rotor mounted vertically at the rear - and hence this is the configuration that will be considered here.

3.2 Helicopter Dynamics

As can be determined from observing its operation, the helicopter is supported while in flight by a set of rotor blades, which revolve about a central hub. These blades are essentially the same as aircraft wings, but instead of generating lift through linear motion of the aircraft through the air, they generate lift by means of their rotational speed relative to the air.

The analysis of how these blades act to lift and propel the helicopter, and of the dynamics of the resulting system, is complex but of enormous value. The development of a control system for any purpose cannot be undertaken without a reasonable knowledge of the plant to be controlled.

3.2.1 Modelling Helicopter Flight

Helicopters are inherently extremely complex machines, and detailed analysis of their operation is consequently very involved. However, various simplifications can be made to the flight model, allowing a variety of compromises between complexity and accuracy. There are many texts providing full descriptions of helicopter dynamics (Bramwell, 1976; Layton, 1984; Johnson, 1994; Padfield, 1996; Prouty, 1990; Stepniewski and Keys, 1984). In particular the work of Layton (1984) has been consulted extensively in preparing this section.

3.2.1.1 Momentum Theory

Momentum theory is the simplest of all helicopter analysis techniques. The spinning rotor blades are regarded as a solid disc, and the effect of this lifting disc on the helicopter is considered simply in terms of Newton's laws of motion and the conservation of energy, mass and momentum. Momentum theory is simplified by the following assumptions (Layton, 1984):

- Air is incompressible and frictionless.
- The rotor, being a solid disc, imparts no rotation or oscillation to the wake; flow through the disc is uniform and steady.
- Energy is added to the air by the rotor disc in the form of an instantaneous pressure increase. The pressure both above and below the disc is constant.
- The density of air does not change with altitude.

The above assumptions will introduce errors, but the simplifications afforded are enormous in comparison. In addition, most of the errors may be rendered negligible by the application of a suitable correction factor to the results. The derivations below rely on the following nomenclature:

V: Air velocity

V_s: Velocity of free stream (far above rotor)

v_i: Increase in stream velocity to rotor disc (induced velocity)

v_∞: Final increase in stream velocity

p_{t1}, p_{t2}: Total pressure far above and far below rotor

p_s: Normal atmospheric pressure

p_t: Total stream pressure

$\frac{m}{dt}$: Mass flow through rotor

ρ: Density of air

A: Area of rotor disc

T: Thrust generated by rotor disc

The aim of momentum theory is to determine simple equations for the thrust generated by the rotor and its efficiency, the velocity of air through the disc, and other aspects of the action of the disc that are important in understanding the basics of helicopter flight. Starting with Bernoulli's Equation, as the velocity of fluid in a stream tube increases, its static pressure falls:

$$p_t = p_s + \frac{1}{2}\rho V^2 \tag{3.1}$$

Invoking this far above the rotor gives

$$p_{t1} = p_s + \frac{1}{2}\rho V_s^2 \tag{3.2}$$

Far below the rotor, when the static pressure has returned to ambient levels and the stream has ceased to accelerate,

$$p_{t2} = p_s + \frac{1}{2}\rho\left(V_s + v_\infty\right)^2 \tag{3.3}$$

Subtracting Equation 3.2 from Equation 3.3 gives the difference in pressure induced by the rotor disc:

$$\Delta p = \rho v_\infty \left(V_s + \frac{1}{2}v_\infty\right) \tag{3.4}$$

Newton's second law states that F = ma, or equivalently in terms of thrust and mass flow,

$$T = \frac{m}{dt}dV \tag{3.5}$$

Mass flow is defined as the mass of air passing through the disc per unit time:

$$\frac{m}{dt} = \rho A\left(V_s + v_i\right) \tag{3.6}$$

The pressure change Δp acts on the area of the disc A, giving

$$T = A\Delta p \tag{3.7}$$

Hence, by equating Equation 3.5 and Equation 3.7, and substituting Equation 3.6:

$$A\Delta p = \rho A \left(V_s + v_i\right) dV \tag{3.8}$$

$$\Delta p = \rho \left(V_s + v_i\right) dV \tag{3.9}$$

Over the complete stream tube, $dV = v_\infty$, and equating with Equation 3.4:

$$\Delta p = \rho \left(V_s + v_i\right) v_\infty = \rho v_\infty \left(V_s + \frac{1}{2} v_\infty\right) \tag{3.10}$$

$$v_i = \frac{1}{2} v_\infty \tag{3.11}$$

The acceleration of air in the stream tube, then, occurs half above and half below the rotor, with the final added velocity being double that at the disc. Developing an expression for the thrust developed by the rotor is trivial:

$$T = A\Delta p \tag{3.12}$$

$$= \rho A \left(V_s + \frac{1}{2} v_\infty\right) v_\infty \tag{3.13}$$

$$= 2\rho A \left(V_s + v_i\right) v_i \tag{3.14}$$

In the hover, this simplifies to

$$T = 2\rho A v_i^2 \tag{3.15}$$

or equivalently

$$v_i = \sqrt{\frac{T}{2\rho A}} \tag{3.16}$$

Clearly, thrust is required (and therefore power is consumed) to support the helicopter in the hover. To measure efficiency, a figure of merit FM has been defined for helicopters. This differs from the standard definition of efficiency as the ratio of power input to power output, because for a general propeller the power output is taken to be the product of the thrust and the velocity, and therefore the efficiency for a stationary helicopter is undefined. Instead, FM describes the ratio of the hover power of an ideal rotor to a real rotor. The power output is taken as the product of the thrust and the inflow velocity, known as the *induced power*, and is compared to the power used to turn the rotor (P_{in}). So for the figure of merit:

$$FM = \frac{Tv_i}{P_{in}} \tag{3.17}$$

$$= \frac{T\left(\frac{T}{2\rho A}\right)^{\frac{1}{2}}}{P_{in}} \tag{3.18}$$

$$= \frac{T^{\frac{3}{2}}}{P_{in}\,(2\rho A)^{\frac{1}{2}}} \tag{3.19}$$

3.2.1.2 Blade Element Theory

The calculations of power above did not deduce the torque required to turn the rotor, and this was because torque effects are to do with the drag on each rotor blade as it passes through the air. Clearly, a slightly more detailed model is called for, and this need is fulfilled by blade element theory.

Consider a small element of a single rotor blade, as viewed from the hub. The element is located at a distance r from the hub; its width is dr and its chord c. The rotor is spinning at a rate of Ω radians per second, and so the linear speed of the element through the air is Ωr. Now as this element travels through the air it will generate a force dR which may be resolved into the thrust dT and torque dQ of the element. The torque vector lies parallel to the path of the element, while the thrust is generated perpendicular to it. However, these directions are not consistent with the lift and drag forces generated by a moving aerofoil, which are referenced not to the direction of travel, but to the direction of the inflow air, V_i. Figure 3.1 shows the geometry and also defines the lift and drag forces dL and dD in terms of the angle of attack α_0 and angle of incidence α_i.

From Figure 3.1, it can be seen that:

$$dT = dL\cos\alpha_i - dD\sin\alpha_i \tag{3.20}$$

$$dQ = dL\sin\alpha_i - dD\cos\alpha_i \tag{3.21}$$

It is standard to refer to the lift and drag of an aerofoil in terms of non-dimensional lift and

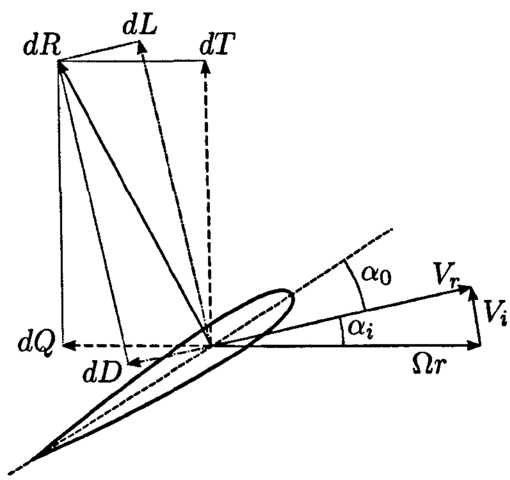

Figure 3.1: Blade Element Force Vectors (from Layton, 1984)

drag coefficients C_l and C_d:

$$dL = C_l q_r dS \tag{3.22}$$

$$dD = C_d q_r dS \tag{3.23}$$

where

$$dS = c \cdot dr \tag{3.24}$$

$$q_r = \frac{1}{2}\rho V_r^2 = \frac{1}{2}\rho\left(\Omega r \cos\alpha_i\right)^2 \tag{3.25}$$

Hence, concentrating on thrust, if $C_l \gg C_d$, $C_l \cos\alpha_i \gg C_d \sin\alpha_i$ and $\Omega r \gg V_i$ and the small angle approximations $\sin\alpha_i \approx \alpha_i$ and $\cos\alpha_i \approx 1$ are used:

$$dT = \frac{1}{2}\rho\left(\Omega r\right)^2 \cos^2\alpha_i c \cdot dr \left(C_l \cos\alpha_i - C_d \sin\alpha_i\right) \tag{3.26}$$

$$= \frac{1}{2}\rho\left(\Omega r\right)^2 c \cdot dr \cdot C_l \tag{3.27}$$

Thus integrating over a blade of length R:

$$T_{blade} = \int dT \tag{3.28}$$

$$= \int_0^R \frac{1}{2}\rho\left(\Omega r\right)^2 c \cdot dr \cdot C_l \tag{3.29}$$

$$= \frac{1}{2}\rho c C_l \int_0^R \left(\Omega r\right)^2 dr \tag{3.30}$$

$$= \frac{1}{6}\rho c\left(\Omega r\right)^2 R C_l \tag{3.31}$$

Hence for a complete rotor with b blades,

$$T = \frac{1}{6}\rho bc\left(\Omega r\right)^2 RC_l \tag{3.32}$$

For the torque, an equation similar Equation 3.27 may be formed:

$$dQ = \frac{1}{2}\rho\left(\Omega r\right)^2 \cos^2\alpha_i c \cdot dr\left(C_l \sin\alpha_i - C_d \cos\alpha_i\right) \tag{3.33}$$

$$= \frac{1}{2}\rho\left(\Omega r\right)^2 c \cdot dr\left(C_{di} + C_{dp}\right) \tag{3.34}$$

where C_{di} is the *induced drag*, due to the lift and thrust vectors not being parallel, and C_{dp} is the parasitic or profile drag.

The power required to move a blade element is the product of the torque required, the radius to the element, and the rotational velocity:

$$dP = \Omega dQ \cdot r \tag{3.35}$$

Integrating for all blades:

$$P = \frac{1}{8}\rho bc\left(C_{di} + C_{dp}\right)\Omega^3 R^4 \tag{3.36}$$

One further simplification which may be made is to define solidity, σ, as the ratio of the disc area which is occupied by blades. Now:

$$P = \frac{1}{8}\rho\sigma A\left(C_{di} + C_{dp}\right)\left(\Omega R\right)^3 \tag{3.37}$$

When calculating the total power required to hover, it is easiest to calculate the induced power using momentum theory and the profile power using blade element analysis. From Equations 3.37 and 3.15, and knowing that induced power $P_i = Tv_i$ and that the angle of incidence $\alpha_i = 0$ in the hover, the total power P_{total} can be found:

$$P_{total} = \frac{T^{\frac{3}{2}}}{\sqrt{2\pi\rho R^2}} + \frac{1}{8}\sigma C_{dp}\rho A\left(\Omega R\right)^3 \tag{3.38}$$

It has been seen therefore, through a combination of analysis techniques, that the rotor of a helicopter acts by accelerating a stream of air downwards in order to produce a thrust upwards. This results, even in the hover, in an induced air velocity through the rotor and in a power requirement that is dependant on the geometry of the rotor and its blades as well as on the thrust required.

3.2.1.3 Forward Flight and Blade Hinges

Up to this point, it has been assumed that the blades are rigidly attached to the central hub and that they therefore proceed with a fixed angular velocity and do not move vertically. This is not the case, however. Figure 3.2 shows the basis of the reason for this.

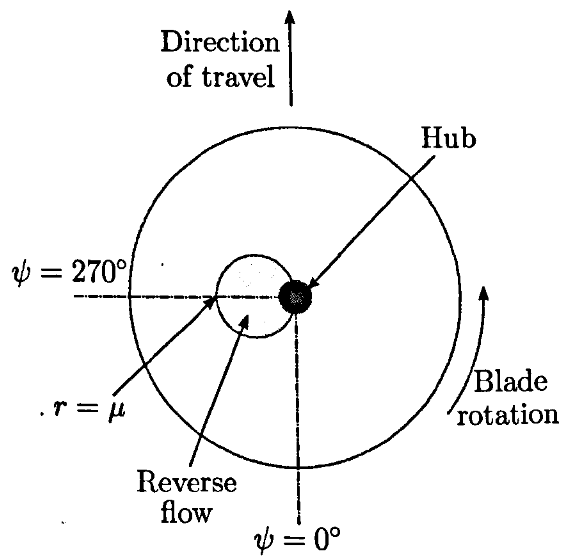

Figure 3.2: Reverse Flow Region

Some of the features of Figure 3.2 require further explanation. Firstly, as a blade sweeps out a circle, it is assigned an *azimuth angle*, ψ. This is the angle of the blade with respect to its downstream position. In the Western world, this angle is measured anticlockwise as viewed from above to be consistent with the direction of blade rotation. Secondly, for a given forward velocity V, we can define an *advance ratio*, μ:

$$\mu = \frac{V}{\Omega R} \tag{3.39}$$

The advance ratio can be seen to be the ratio of the velocity of the aircraft to the maximum tip velocity. The velocity of air over any part of any blade may be calculated with respect to the direction of blade travel:

$$u = \Omega r + V \sin \psi \tag{3.40}$$

$$= \Omega \left(r + \mu R \sin \psi \right) \tag{3.41}$$

It can be seen that the airflow drops to zero when $r = -\mu R \sin \psi$, which at $\psi = 270°$ is when $r = \mu$, as shown on the diagram. Any point where $r < -\mu R \sin \psi$ lies in the *reverse flow region*. Clearly, this region of reduced or reversed airflow, existent only on the retreating side of the rotor, will produce a large rolling moment on the airframe. In practice, this effect prevented early helicopters from flying at anything but the slowest of speeds. The gyrocopter pioneer Juan de la Cierva invented the solution to this problem, and his technique is still used today. He introduced *flapping hinges* to the blade roots to allow them freedom of movement vertically. This vertical freedom means that as a blade retreats, it generates less lift and hence flaps downwards instead of imparting a moment on the hub. This in turn increases the effective angle of attack of the blade and so damps the flapping movement, removing the need for extra mechanical damping.

The vertical flapping of the blades was found to introduce great stresses at the rotor hub. The reason for this was that the path taken by the blades caused them to be subject to a Coriolis force that acted laterally on the blades. In addition, the cyclic variations in lift caused by the flapping hinges induced cyclic variations in drag. Another hinge was therefore added, the lead-lag hinge, to allow the blades some freedom in this axis. Unlike flapping, however, the lead-lag motion is not aerodynamically damped and needs mechanical dampers and end stops.

It can be seen then that the motion of the blades through the air is a complex one. It may be modelled reasonably accurately if enough is known about the blades and hinges, and this modelling is important for the designer of a control system. It is useful to note that in general, the flapping and lead-lag modes are considerably faster than the body dynamics. For this reason, they are often omitted from helicopter models for the purposes of control system design. For example the helicopter model used in Chapter 7 of this thesis, and described in Appendix A, contains flapping modes; but these are removed using a modal approximation technique before control is attempted.

However, while this has obvious advantages for proving the design process, when a controller designed in this way is fitted to an aircraft it may display significantly degraded handling qualities (Ingle and Celi, 1994).

3.2.1.4 Rotor Flow Effects and Wake Analysis

Simply to state that the rotor imparts vertical movement to the air against which it works is clearly to oversimplify reality. The interaction between the rotor and the air is extremely complex, and an understanding of the exact nature of this interaction is not necessary for the understanding of the control problem. It is, however, important for producing reliable mathematical models of helicopters, such as that produced by Gee (2000). A simple summary of the most important of these complex effects will be given here for reference.

Through the torque imposed on the air by the drag forces, the rotor imparts a rotation to the air as well as a downward velocity, and this affects the angle of the airflow over each successive blade. In addition, the existence of a high-pressure region below each blade and a low-pressure region above causes an upwash of air behind the blade as it travels. This upwash, strongest at the tips due to their higher speed and consequently greater lift, affects the next blade by increasing the angle of attack. This increases the tip loading of each blade. The upwash is not the only effect produced by the pressure differential across the surfaces of the blade. At the tips, air tends to flow from the lower surface to the upper round the end of the blade. As the blade moves this produces a strong vortex behind the tip, reducing efficiency. Figure 3.3 shows this vortex action.

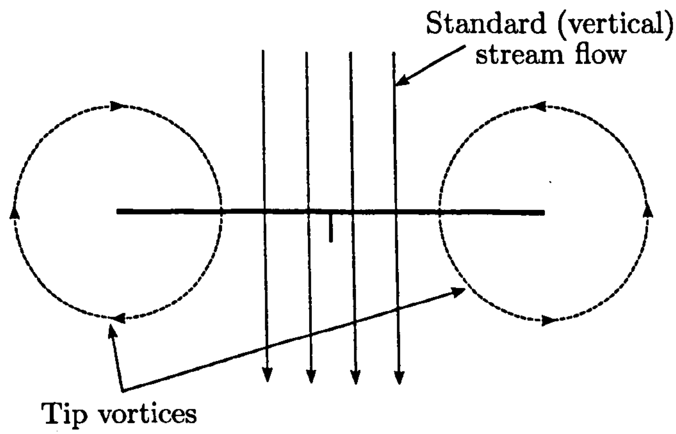

Figure 3.3: Tip Vortices

The wake from the main rotor also has an effect on the motion of the helicopter. The analysis of the progress of the wake is highly computationally intensive, though simplifications can be made (Gee, 2000). The most obvious wake effects are that the wake acts on the fuselage and tail; on the fuselage, the effect is mainly a loss of lift whereas in forward flight the effect of the wake on the tail changes the effective angle of attack of the tail rotor blades and also acts on the fin and tail plane.

3.2.2 Control and Stability

The main rotor is not the only component of a helicopter with control over its movement, and this section introduces the role of the tail. It also shows how the movable parts of a helicopter are used to fulfil the demands of the pilot.

3.2.2.1 The Tail

The tail typically consists of three principal components. Two of them, the horizontal and vertical stabilisers (tail plane and the tail fin), are static fixtures, much like the tail of an aeroplane. Unlike an aeroplane, however, they rarely have any movable control surfaces. The horizontal stabiliser gives pitch stability at speed, and the vertical stabiliser performs the same function for yaw. Without them the rotor and fuselage together would be very unstable in these axes.

The third component of the tail assembly is the tail rotor. With a few exceptions, the vast majority of single-rotor helicopter designs incorporate a tail rotor consisting of a vertically mounted rotor driven through a gearbox from the main rotor such that their speeds are related. The tail rotor produces a lateral thrust at the end of the tail, and this is arranged to counteract the torque supplied to the main rotor by the engine. Of course, this introduces a small lateral drift component as well as an anti-torque moment, and the main rotor is often mounted at a slight angle to counteract this.

3.2.2.2 Pilot Controls

The pilot must be given a large amount of control over the helicopter. S/he must be able to control its height, its velocity in both horizontal axes, and its yaw rate at the very least. The necessary degrees of mechanical freedom are built into the hubs of the two rotors. In both cases, they apply *feathering* to the blades.

The feathering hinge on a main rotor blade is supplied in addition to the flapping and lead-lag hinges already described, and it allows the blade to twist along its axis. This twisting is not in response to aerodynamic forces, but in direct response to the pilot's commands. The feathering angle of the blades is determined by the position of a *swash plate*. The swash plate is in fact two plates, coupled so as to lie parallel to each other. The upper plate revolves with the blades, while the lower one does not. The pilot, through the controls, has the ability to move the lower plate in various ways; the upper plate, coupled to the blades, causes the feathering angle of the blades to change.

The pilot must have control over the total amount of lift that is generated by the main rotor so that the rate of climb or descent may be controlled. A simple solution would be to control the engine throttle; however, due to the inherently high rotational inertia possessed by such large blades, the response would be too slow. The pilot's *collective pitch* control, normally a lever like a car handbrake to the left of the seat, moves the swash plate up and down and hence changes the pitch of all the blades simultaneously. This gives an increase or decrease in the thrust generated by the main rotor. Of course, it also affects the torque required to turn the rotor; this torque demand may be satisfied by a throttle control built into the collective lever as a twist grip, or it may be satisfied automatically by an engine speed governor.

The *cyclic pitch* control changes the angle of the swash plate with respect to the horizontal, in both directions. The pilot alters the cyclic pitch by means of a long joystick, extending from either the ceiling or the floor, capable of moving both laterally and longitudinally. If the joystick is pushed forwards, this raises the back of the swash plate and consequently gives the blades a greater feathering angle as they sweep round the back. This has the effect of tilting the rotor disc forwards and causing the thrust vector to deviate from vertical, thus propelling the helicopter forwards.

The tail rotor pitch is also variable. An arrangement similar to a swash plate exists at its hub, but it is not capable of imparting a cyclic pitch change to the blades. The collective pitch of the tail rotor is variable though, and this is controlled by a pair of pedals, linked so as to act against each other. The variable pitch gives rise to variable thrust and hence, because of the length of the tail, to a variable torque applied to the fuselage. In the hover, this effect gives the pilot control over yaw. In forward flight, the effect of the tail rotor is limited due to the presence of the tail fin; the yaw moment provided by the tail rotor is counteracted quickly by an opposing moment from the vertical stabiliser due to the resulting sideslip. Turns at speed are accomplished instead using the cyclic pitch control to roll the aircraft and then pitch its nose up into the turn, much in the same manner as a fixed-wing aircraft.

It can be seen, then, that the combination of collective and cyclic pitch controls, together with the tail rotor pitch, are sufficient to allow the pilot control over the helicopter in the four principal directions.

3.2.2.3 Stability Definitions

Helicopters are not generally inherently stable. There are two different forms of instability, which due to the difference in the required response of the pilot to each, need naming so that they may be referred to when defining performance limits for the aircraft.

Static instability is the result of the aircraft responding to an imbalance in such a way as to increase that imbalance. For example, if a helicopter is slipping sideways, the effect of the crosswind on the fuselage is to tilt the aircraft into the direction of travel, thereby increasing its speed.

Dynamic instability affects only modes that are statically stable, and is the result of an imbalance leading to an overcompensation that produces a larger imbalance in the opposite direction - with inevitable, increasingly oscillatory consequences. Often a dynamic instability is the result of the interaction between the pilot and the aircraft, with the pilot overcompensating for an error because the helicopter is too slow in reacting. In this case, it is referred to as Pilot-Induced Oscillation (PIO), and is a major problem when implementing control systems - a delay in the response of the controller due (for example) to the computational speed of the platform on which it is implemented would be likely to induce or strengthen a tendency towards PIO.

3.2.2.4 Auto-Stabilisation Equipment

Auto-Stabilisation Equipment (ASE) is the name given to the equipment which performs the most basic function of the Automatic Flight Control System (AFCS) on board a helicopter. It consists of an electronic controller, which, by using an array of sensors, modifies the pilot's inputs such that the helicopter's response is better matched to the pilot's expectations. The main application of the mathematical techniques described and developed in later chapters is in the development of ASE.

3.2.3 Control Problems

From a control perspective, the complex interaction between the various components of a helicopter while in flight provides some unique and difficult problems. Take as an example a helicopter with blades rotating anticlockwise as viewed from above. The tail rotor, designed to prevent the fuselage from yawing under the effects of the rotor torque, also introduces a lateral force and will tend to push the helicopter to the right. To counteract this effect, once the helicopter has left the ground it must be inclined slightly to the left, necessitating the application of lateral cyclic pitch.

As a result of the increased drag of the blades as they travel forwards with increased pitch, the helicopter will be pushed backwards and turned to the right. It will also fall as the added load due to the cyclic pitch slows the engine. An increase in engine power to counteract the dip will cause a greater torque reaction, and so on.

This is merely an example, but it serves to show that even in the hover, the dynamics of a helicopter are far from simple. From a pilot's point of view, and that of the control engineer, the biggest problem is the cross-coupling - it is not possible to alter anything without affecting everything else.

Additionally, helicopters are highly non-linear, and the case in the hover is vastly different from the case at even a low forward speed. It can be shown (Bramwell, 1976; Layton, 1984; Padfield, 1996; Stepniewski and Keys, 1984) that the efficiency of the rotor disc increases with the flow of air through it, so as soon as the helicopter starts to move it will gain translational lift.

The added oscillatory mode introduced by the motion of the blades around the flapping hinges is not in phase with the blade pitch change and tends to introduce a phase shift to the lift. Therefore the pilot must apply a coupled input to the swash plate in order to effect a simple shift in the lift vector. This phase shift changes with forward speed.

The effect of delays in control response has already been mentioned; a helicopter is heavy and the effect of its controls not immediate, so the potential for PIO is great even before any additional delays due to the controller are considered. In a digital controller, the sample rate must be high enough to avoid compounding these problems.

In summary, severe nonlinearities, time constraints and cross-couplings make for a plant which presents a unique and complex set of challenges to the designer of a control system.

3.3 Handling Qualities Specification

When designing any controller for any plant it is vitally important to have a detailed specification of the desired final performance of the system. In the case of a helicopter, however, these specifications are hard to characterise. The final performance of the helicopter in a mission role is dependent on both the machine and its pilot, and consequently the pilot's feel for the aircraft is vital. Defining this is very subjective, however, and so over the years a number of documents have emerged which have attempted to describe these qualitative characteristics in a quantitative way. The intended role of the helicopter will define which one of these applies. These documents are considered in addition to and in support of the aircraft-specific definitions of performance laid down when a new design is proposed.

This section draws on the work of Clarke and Taylor (1999) for details of the various handling qualities documents.

3.3.1 Rotorcraft Specification

The aircraft-specific handling qualities are laid out in the Rotorcraft Specification for the helicopter in question. Defined here are the operational conditions under which the aircraft must operate, along with the levels of performance, handling and failure-tolerance that must be met. Also defined are the roles and missions of the helicopter.

The Rotorcraft Specification is the most important document describing the desired performance of the aircraft. It is also the most detailed, as the other documents provide a much more general set of requirements. In the case that a conflict between documents does arise, however, the Rotorcraft Specification takes precedence.

3.3.2 ADS-33D

The ADS-33D is the document describing the handling qualities requirements for the military procurement of helicopters in the US (Padfield, 1996). It was introduced during the design process of the RAH-66 Comanche, where advancing technology was meeting outdated design documents at an impasse. The ADS-33D recognises the need for different handling qualities during different phases of the flight mission. Handling qualities are defined as Level 1, 2 or 3 where a Level 1 response is the best, in a similar way to the Def.Stan.970 (see below). Handling quality levels are also defined under degraded environmental conditions or failure states.

Unusually, the handling qualities requirements in the ADS-33D are largely specified in terms of frequency domain characteristics, principally bandwidth and phase delay. Since the frequency domain is a concept associated with linear systems, whereas the helicopter is highly nonlinear, this approach has precluded the universal and complete acceptance of the ADS-33D to date.

3.3.3 Defence Standard 00-970

The UK Ministry of Defence Defence Standard 00-970 (Def.Stan.00-970)(Pitkin, 1989) is the UK equivalent of the ADS-33D. Because it is the document most likely to be applicable to aircraft using control systems designed using techniques in this thesis, it will be covered in slightly more detail than the ADS-33D.

The Def.Stan.00-970 handling qualities specifications are organised according to the appropriate *operational phase.* One of the four phases refers to the operations of the helicopter on the ground, which is of little use in this context. The other three are summarised below.

Active Flight Phase: In the Active Flight Phase, pilot involvement is intensive. This phase may be further classified according to whether the manoeuvre is Aggressive or Moderate.

Attentive Flight Phase: The Attentive Flight Phase demands less of the pilot, and generally involves changes to the flight condition or flight under Instrument Flight Rules (IFR).

Passive Flight Phase: Pilot involvement during the Passive Flight Phase is low. Occasional re-trimming of the flight condition and monitoring of the AFCS is all that is generally required.

The Def.Stan.00-970 document also defines terms for the maximum operating envelopes in the various flight phases, as well as tolerance of environmental conditions. Handling qualities are rated according to a structure of Levels, which define the level of ease with which the mission may be carried out.

Level 1: Handling qualities are clearly sufficient to allow the mission to be completed with ease.

Level 2: The mission may be accomplished, but will require an increase in pilot workload and this may also lead to a decrease in mission effectiveness.

Level 3: The helicopter can be controlled, but the pilot workload will be excessive and mission effectiveness unsatisfactory.

It is the objective of the control system to ensure that the handling qualities do not fall below Level 1 while inside the operational envelope (the flight envelope required while carrying out a particular role) or Level 2 at other points within the service envelope (the normal operating range of the helicopter).

The short- and long-term modes that form the responses of the helicopter to the pilot inputs are also well defined by the Def.Stan.00-970. They are defined in terms of time-domain criteria, which are more intuitive than the frequency-domain criteria of the ADS-33D; qualitative descriptions of handling qualities tend to be described in the time domain, using words such as speed, overshoot and sensitivity. The derivation of an ideal eigenstructure in Section 3.4 uses these criteria directly.

The short term stability criteria defined by the Def.Stan.00-970 are summarised in template form in Figure 3.4. They are specified in terms of the helicopter's response to a one-second pulse at 10% maximum control deflection.

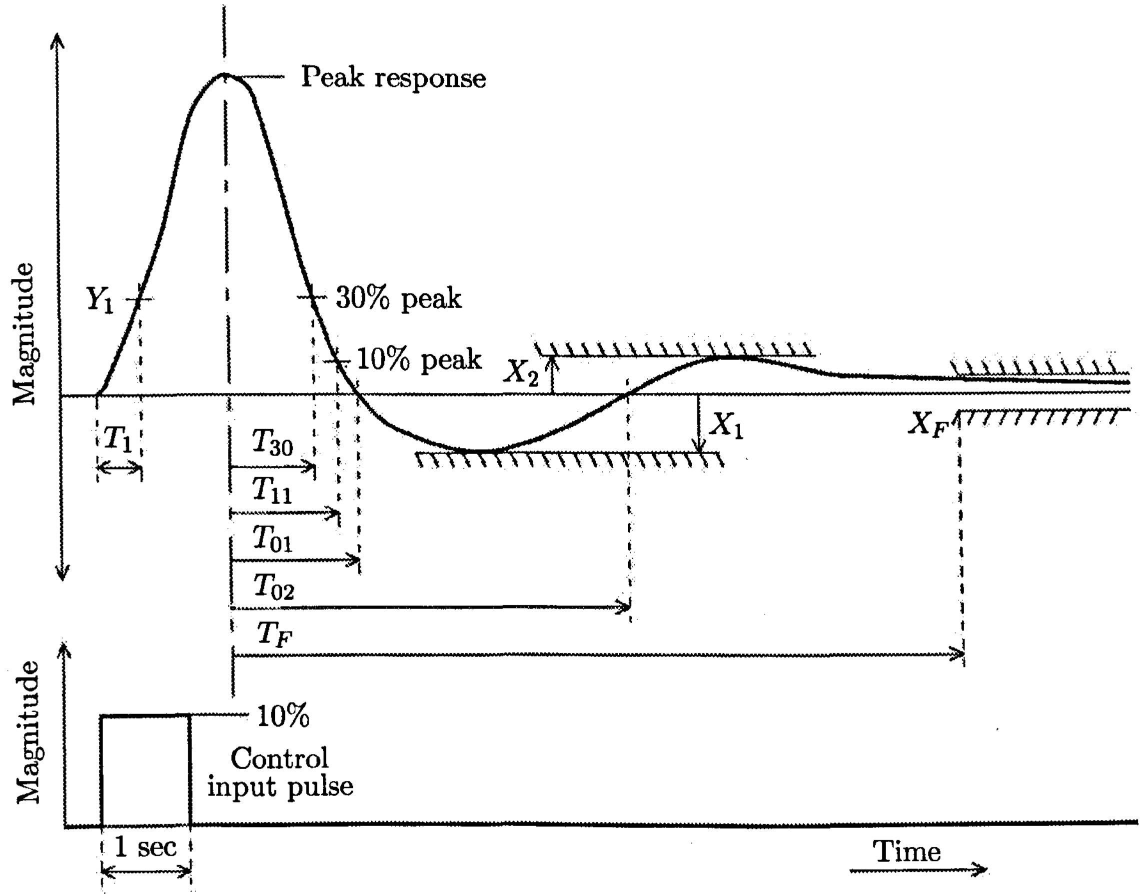

Figure 3.4: Def.Stan.00-970 transient response (from Griffin (1997))

The parameters T_1 and Y_1 confine the initial response delay, while T_F and X_F do the same for the final settling time. T_{30} and T_{11} help define the shape of the response decay after the initial peak, and T_{01}, T_{02}, X_1 and X_2 put bounds on the damping of the response. For full details of the parameters included in the figure, see Griffin (1997). This template will be used in Chapter 7 to evaluate the success of various control strategies by comparing their responses directly to the Def.Stan.00-970 time-domain criteria.

The Def.Stan.00-970 is a comprehensive document for the specification of all aspects of the pilot's interaction with the helicopter. As a result it covers a vast range of specifications beyond those listed above.

3.4 An Existing Ideal Eigenstructure

The handling problems inherent to helicopters (primarily instability and cross-coupling), combined with the close links explored in Chapter 2 between EA and the classical techniques still so often used for the control of helicopters, render EA an ideal methodology for the generation of controllers. However, before EA can be applied to any control problem, it is necessary that the problem be formulated in terms of the desired closed-loop pole locations and eigenvectors: the eigenstructure.

The formulation of an ideal eigenstructure was considered in detail by Griffin (1997) and Clarke et al. (2003), and considered the Def.Stan.00-970 as the basis for the eigenstructure. Eigenvalues and eigenvectors were treated separately and the eigenvectors were formulated for the case where the cyclic pitch inputs control the *attitude* of the helicopter. Level 1 handling qualities are achievable only in the hover using this type of control response.

It is at this point appropriate to introduce the naming convention for the states and inputs of a standard helicopter model, since these will be needed to understand the ideal eigenstructure.

u: Forward speed

v: Lateral speed

w: Vertical speed (heave velocity)

p: Roll rate

q: Pitch rate

r: Yaw rate

ϕ: Roll angle

θ: Pitch angle

ψ: Yaw angle

A_{1s}, B_{1s}: Blade flapping angles

θ_0: Collective pitch input

θ_t: Tail rotor pitch input

A_1, B_1: Cyclic pitch inputs

3.4.1 Eigenvalue Locations

Clarke et al. (2003) perform a detailed analysis of the Def.Stan.00-970 and the ADS-33 in order to determine a set of eigenvalue locations for the modes associated with longitudinal and lateral velocity (λ_u and λ_v), pitch and roll rates (λ_q and λ_p), heave velocity (λ_w) and yaw rate (λ_r). Their analysis holds for both the attitude command and rate command cases. Oscillatory modes are assumed to be 2^{nd} order in nature, since second-order modes are the simplest that can fulfil the short-term requirements of the Def.Stan.00-970.

The reader is referred to Clarke et al. (2003) for a full description of the methods used to derive a region in the s-plane within which the poles of a second-order system must lie if this system is to meet the Def.Stan.00-970 criteria. However, the important constraints, along with the allowed region generated by their intersection, are shown in Figure 3.5.

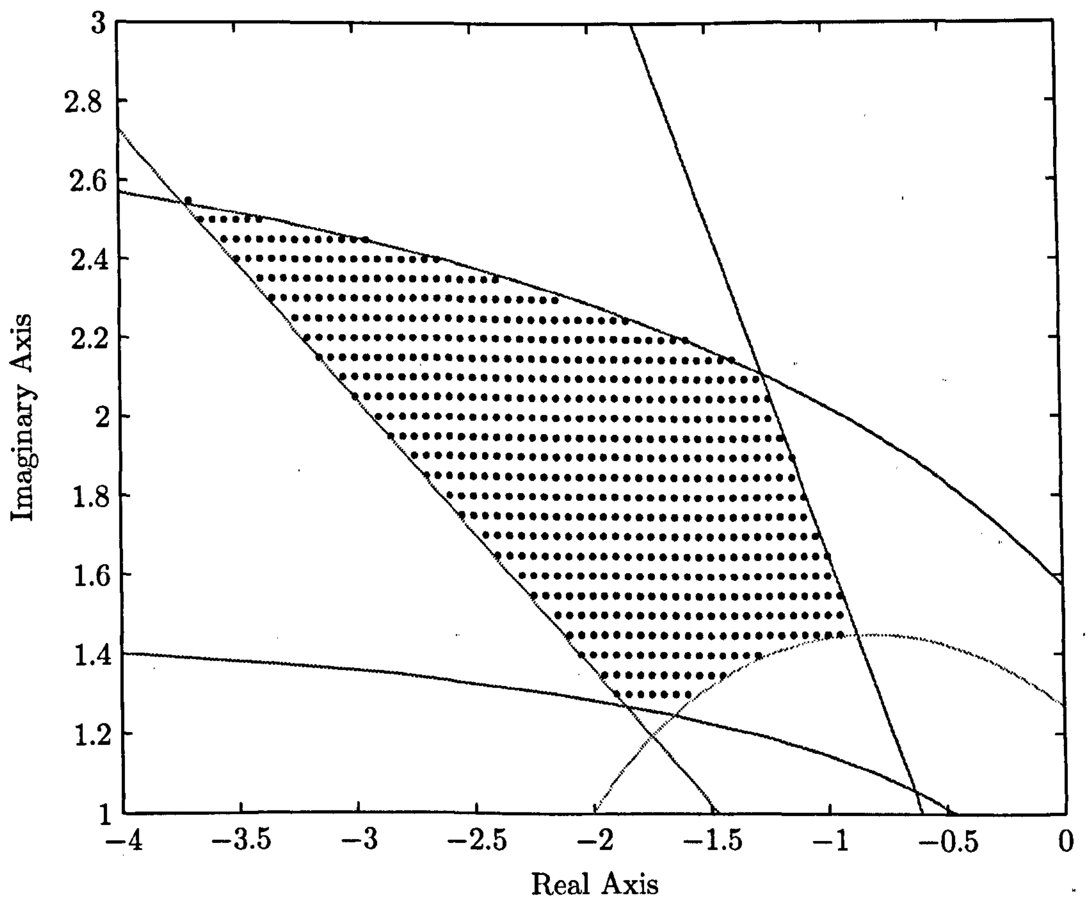

Figure 3.5: An s-plane interpretation of the Def.Stan.00-970 short-term stability criteria (after Clarke et al. (2003))

As can be seen, the region is well-defined and has its centroid at approximately $-1.8 \pm j1.75$. Clarke et al. also demonstrate that the requirements of the ADS-33 are also satisfied by this region for simple systems.

This allowable region is the same for all channels except heave, although Clarke et al. go on to develop a suitable location ($-2.2 < \lambda_r \leq -1.2$) for a single pole in the yaw channel. This is because if static gain feedback is used, this channel is likely to be associated with only one pole and hence will display first-order characteristics. In addition, a single pole location ($\lambda_w < -0.2$) is derived from the ADS-33 for the heave response, for which little guidance is available in the Def.Stan.00-970.

3.4.2 Attitude Command Eigenvectors

The desired eigenvector sets may be considered in small decoupled subsystems since a requirement of the Def.Stan.00-970 is that coupling between the responses in different axes should be minimal. Clarke et al. (2003) derive the ideal eigenvector set by using a transfer function approach. In this way kinematic constraints are introduced early on, ensuring the correct integral relationship between q and θ, for example. Combining these kinematic constraints with the mathematical constraint of orthogonality of the left and right eigenvector sets leads to the following result:

$$\mathbf{x} = \begin{bmatrix} u & q & \theta & v & p & \phi & w & r \end{bmatrix}^T$$

$$\mathbf{V}_d = \begin{bmatrix} \frac{1}{(\lambda_u-\lambda_q)} & \frac{1}{(\lambda_u-\lambda_q)} & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\lambda_q} & \frac{1}{\lambda_q} & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{(\lambda_v-\lambda_p)} & \frac{1}{(\lambda_v-\lambda_p)} & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\lambda_p} & \frac{1}{\lambda_p} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

3.5 Rate Command Eigenvectors

If it is desired to use EA to cause a helicopter to display Level 1 handling qualities, as defined by the Def.Stan.00-970, this can only be achieved in the 'Attentive' flight phase using the ideal eigenstructure described by Clarke et al. (2003). For the 'Active' flight phase, both

Aggressive and Moderate manoeuvres, an alternative eigenstructure designed to couple the 2^{nd} order lateral and longitudinal modes into the body rates instead of the body angles is required. Such an eigenstructure has been developed by the author, and is presented here.

3.5.1 Problem Definition

The derivation of this eigenstructure is very much more complex than in the attitude command case, since the kinematic coupling from the cyclic pitch inputs to the body angular rates is first-order. The ideal pole locations for the roll, pitch and yaw rate responses are the same as those for the attitude response derived by Clarke et al. (2003) (see Griffin, 1997). However, in the lateral and longitudinal subsystems, the pole associated with pure angular rate must form a complex conjugate pair in combination with another pole, in order that the rate response be second order as required by the Def.Stan.00-970 for Level 1 handling qualities in the Active Moderate flight phase or for Level 1 or Level 2 handling qualities in the Active Aggressive flight phase (Pitkin, 1989).

This element of the specification introduces the requirement that a compensator state be added to provide the extra pole. This fits well with current practice, since it is common to feed 'washed out' (high-pass filtered) versions of the body attitudes into the controller along with the original signals. The washout filters introduce new poles, the locations of which can be chosen by manipulation of the cutoff frequency of the filters. This arrangement is shown, simplified, in Figure 3.6. (The notation used in the Figure is defined by Griffin (1997).)

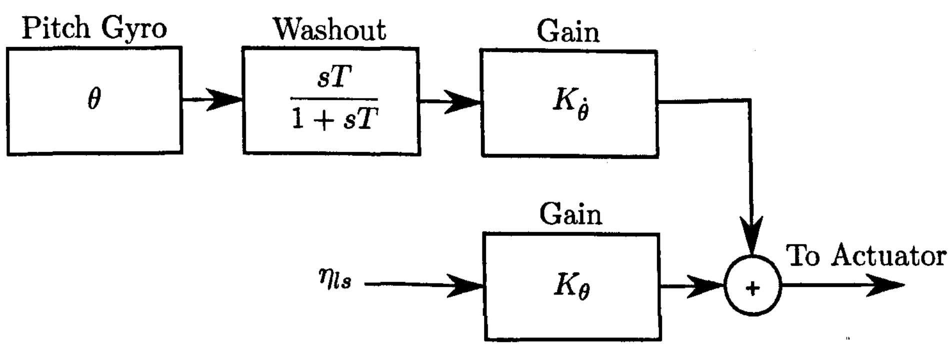

Figure 3.6: Simplified pitch channel control structure (after Griffin (1997))

The assignment of an eigenstructure to a system requires that all of the system modes be controllable, however, and so it is necessary to implement these extra poles as states in a feedback compensator. If the compensator is sufficiently decoupled from the outputs, either by the decoupled nature of the eigenvectors, or following the application of a gain suppression technique such as that described in Chapter 6, the feedback compensator may be expressed as a feedforward compensator post-assignment (see Chapter 4).

The preceding description leads to the following ideal transfer functions, which illustrate the longitudinal case $(u,\ q,\ \theta)$:

$$\frac{U(s)}{B_1(s)} = -\frac{s+z}{s(s+\lambda_1)(s+\bar{\lambda}_1)(s+\lambda_2)} \tag{3.42}$$

$$\frac{Q(s)}{B_1(s)} = \frac{s+z}{(s+\lambda_1)(s+\bar{\lambda}_1)} \tag{3.43}$$

$$\frac{\Theta(s)}{B_1(s)} = \frac{s+z}{s(s+\lambda_1)(s+\bar{\lambda}_1)} \tag{3.44}$$

where B_1 is longitudinal cyclic input.

The zero will exist because of the presence of the the compensator in the feedback path. It corresponds to the open loop pole of the compensator, which is of the form

$$G_{comp}(s) = \frac{s}{s+z} \tag{3.45}$$

In order to recover the desired closed loop response, it will be necessary to apply first order pole pre-compensation to the input signal to cancel out the zero. This is acceptable practice, since the zero is predetermined by the wash-out filter and is not prone to migration, unlike open loop system poles and zeros.

3.5.2 Modal Coupling Matrices

For the longitudinal case, the subsystem states are $\begin{bmatrix} q & comp_q & \theta & u \end{bmatrix}^T$. The output matrix may be defined as

$$\mathbf{C} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{3.46}$$

and the input matrix as

$$\mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{3.47}$$

The left and right eigenvectors are denoted $\mathbf{W}$ and $\mathbf{V}$ and eigenvalue matrix denoted $\mathbf{\Lambda}$.

The subsystem transfer function matrix ($\mathbf{G}(s)$) may be expressed as:

$$\mathbf{G}(s) = \mathbf{CV}\,(s\mathbf{I} - \mathbf{\Lambda})^{-1}\,\mathbf{WB} \tag{3.48}$$

For the longitudinal case, the above is expanded as follows:

$$\begin{bmatrix} u \\ q \\ \theta \end{bmatrix} = \begin{bmatrix} r_1 & \bar{r}_1 & r_4 & r_6 \\ r_2 & \bar{r}_2 & 0 & 0 \\ r_3 & \bar{r}_3 & r_5 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{(s-\lambda_1)} & 0 & 0 & 0 \\ 0 & \frac{1}{(s-\bar{\lambda}_1)} & 0 & 0 \\ 0 & 0 & \frac{1}{s} & 0 \\ 0 & 0 & 0 & \frac{1}{(s-\lambda_2)} \end{bmatrix} \begin{bmatrix} t_1 \\ \bar{t}_1 \\ t_2 \\ t_3 \end{bmatrix} B_1 \tag{3.49}$$

where r_m and t_n are free parameters.

The zeros in $\mathbf{CV}$ exist because the mode associated with velocity is not seen in the attitude and rate states; similarly, the attitude mode is not seen in the rate state. Equation 3.49 can be evaluated to give:

$$\mathbf{G}(s) = \begin{bmatrix} \frac{r_1 t_1}{s+\lambda_1} + \frac{\bar{r}_1 \bar{t}_1}{s+\bar{\lambda}_1} + \frac{r_4 t_2}{s} + \frac{r_6 t_3}{s+\lambda_2} \\ \frac{r_2 t_1}{s+\lambda_1} + \frac{\bar{r}_2 \bar{t}_1}{s+\bar{\lambda}_1} \\ \frac{r_3 t_1}{s+\lambda_1} + \frac{\bar{r}_3 \bar{t}_1}{s+\bar{\lambda}_1} + \frac{r_5 t_2}{s} \end{bmatrix} \tag{3.50}$$

Simplifying Equation 3.50 and equating numerator term leading coefficients yields the following constraints:

For $\frac{U(s)}{B_1(s)}$:

$$s^3 : 0 = (r_4 t_2 + r_1 t_1 + r_6 t_3 + \bar{r}_1 \bar{t}_1) \tag{3.51}$$

$$s^2 : 0 = (r_6 t_3 \bar{\lambda}_1 + \bar{r}_1 \bar{t}_1 \lambda_2 + r_4 t_2 \lambda_1 + r_4 t_2 \lambda_2 + r_4 t_2 \bar{\lambda}_1 + \cdots + r_1 t_1 \lambda_2 + r_1 t_1 \bar{\lambda}_1 + r_6 t_3 \lambda_1 + \bar{r}_1 \bar{t}_1 \lambda_1) \tag{3.52}$$

$$s^1 : -1 = (r_4 t_2 \lambda_1 \lambda_2 + r_4 t_2 \lambda_1 \bar{\lambda}_1 + r_1 t_1 \bar{\lambda}_1 \lambda_2 + r_6 t_3 \lambda_1 \bar{\lambda}_1 + \bar{r}_1 \bar{t}_1 \lambda_1 \lambda_2 + r_4 t_2 \bar{\lambda}_1 \lambda_2) \tag{3.53}$$

$$s^0 : z = (r_4 t_2 \lambda_1 \bar{\lambda}_1 \lambda_2) \tag{3.54}$$

For $\frac{Q(s)}{B_1(s)}$:

$$s^1 : 1 = (r_2 t_1 + \bar{r}_2 \bar{t}_1) \tag{3.55}$$

$$s^0 : z = (r_2 t_1 \bar{\lambda}_1 + \bar{r}_2 \bar{t}_1 \lambda_1) \tag{3.56}$$

And, finally, for $\frac{\Theta(s)}{B_1(s)}$:

$$s^2 : 0 = (r_3t_1 + r_5t_2 + \bar{r}_3\bar{t}_1) \tag{3.57}$$

$$s^1 : -1 = (r_3t_1\bar{\lambda}_1 + r_5t_2\lambda_1 + r_5t_2\bar{\lambda}_1 + \bar{r}_3\bar{t}_1\lambda_1) \tag{3.58}$$

$$s^0 : -z = (r_5t_2\lambda_1\bar{\lambda}_1) \tag{3.59}$$

A further important identity is

$$\mathbf{CVWB} = \mathbf{CB} \tag{3.60}$$

which translates to further constraints:

$$\begin{bmatrix} r_1t_1 + \bar{r}_1\bar{t}_1 + r_4t_2 + r_6t_3 \\ r_2t_1 + \bar{r}_2\bar{t}_1 \\ r_3t_1 + \bar{r}_3\bar{t}_1 + r_5t_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \tag{3.61}$$

Eigenvectors are unique up to scaling; therefore, for convenience, let us assume that r_2, r_4 and r_6 are equal to unity.

The above constraints, after considerable algebraic manipulation, yield the following results:

$$\mathbf{CV} = \begin{bmatrix} -\frac{1}{(\lambda_1-\lambda_2)\lambda_1} & -\frac{1}{(\bar{\lambda}_1-\lambda_2)\bar{\lambda}_1} & 1 & 1 \\ 1 & 1 & 0 & 0 \\ \frac{1}{\lambda_1} & \frac{1}{\bar{\lambda}_1} & \lambda_2 & 0 \end{bmatrix} \tag{3.62}$$

and

$$\mathbf{WB} = \begin{bmatrix} \frac{z-\lambda_1}{\lambda_1-\lambda_1} \\ \frac{z-\bar{\lambda}_1}{\lambda_1-\bar{\lambda}_1} \\ -\frac{z}{\lambda_1\bar{\lambda}_1\lambda_2} \\ \frac{z-\lambda_2}{\lambda_2(-\lambda_1\lambda_2+\lambda_2^2+\lambda_1\bar{\lambda}_1-\bar{\lambda}_1\lambda_2)} \end{bmatrix} \tag{3.63}$$

The correctness of this result may easily be verified by substitution of Equations 3.62 and 3.63 into Equations 3.42, 3.43, 3.44 and 3.60.

3.5.3 Eigenvectors

It is now necessary to extend the algebraic manipulations to calculate the full left and right eigenvector matrices (**W** & **V** respectively) from the input and output coupling matrices (**WB** & **CV** respectively).

Defining the full matrices as follows:

$$\mathbf{V} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ v_1 & \overline{v_1} & v_3 & v_4 \\ \frac{1}{\lambda_1} & \frac{1}{\bar{\lambda}_1} & \lambda_2 & 0 \\ -\frac{1}{(\lambda_1-\lambda_2)\lambda_1} & -\frac{1}{(\bar{\lambda}_1-\lambda_2)\bar{\lambda}_1} & 1 & 1 \end{bmatrix} \tag{3.64}$$

and

$$\mathbf{W} = \begin{bmatrix} \frac{z-\lambda_1}{\bar{\lambda}_1-\lambda_1} & w_{1,2} & w_{1,3} & w_{1,4} \\ \frac{z-\bar{\lambda}_1}{\lambda_1-\bar{\lambda}_1} & \overline{w_{1,2}} & \overline{w_{1,3}} & \overline{w_{1,}} \\ -\frac{z}{\lambda_1\bar{\lambda}_1\lambda_2} & w_{3,2} & w_{3,3} & w_{3,4} \\ \frac{z-\lambda_2}{\lambda_2(-\lambda_1\lambda_2+\lambda_1\bar{\lambda}_1+\lambda_2^2-\bar{\lambda}_1\lambda_2)} & w_{4,2} & w_{4,3} & w_{4,4} \end{bmatrix} \tag{3.65}$$

From the matrix identity

$$\mathbf{VW} = \mathbf{I} \tag{3.66}$$

some fundamental scalar identities ensue. These lead to some important properties of the unknowns defined in **W** and **V**.

Starting with

$$w_{1,2} + \overline{w_{1,2}} = 0 \tag{3.67}$$

$$w_{1,3} + \overline{w_{1,3}} = 0 \tag{3.68}$$

$$w_{1,4} + \overline{w_{1,4}} = 0 \tag{3.69}$$

These lead to the implication that the above unknowns are all purely imaginary values. Also, the structure of the compensator and the manner of its connection dictates that

$$v_3 = v_4 = 0 \tag{3.70}$$

Substituting these constraints leads, after considerable algebraic manipulation, to the property

$$z = \Re e(\lambda_1) \tag{3.71}$$

where $\mathfrak{Re}$ denotes the real part of a complex scalar. It follows that

$$\mathbf{W} = \begin{bmatrix} \frac{1}{2} & -\frac{\frac{1}{2}j}{x} & 0 & 0 \\ \frac{1}{2} & \frac{\frac{1}{2}j}{x} & 0 & 0 \\ -\frac{\mathfrak{Re}(\lambda_1)}{|\lambda_1|^2\lambda_2} & \frac{\mathfrak{Im}(\lambda_1)}{x|\lambda_1|^2\lambda_2} & \frac{1}{\lambda_2} & 0 \\ \frac{\lambda_2-\mathfrak{Re}(\lambda_1)}{(-\lambda_1+\lambda_2)(\lambda_1-\lambda_2)\lambda_2} & \frac{\mathfrak{Im}(\lambda_1)}{x(-\lambda_1+\lambda_2)(\lambda_1-\lambda_2)\lambda_2} & -\frac{1}{\lambda_2} & 1 \end{bmatrix} \tag{3.72}$$

where $\mathfrak{Im}$ denotes the imaginary part of a complex scalar, and finally that

$$\mathbf{V} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ xj & -xj & 0 & 0 \\ \frac{1}{\lambda_1} & \frac{1}{\lambda_1} & \lambda_2 & 0 \\ -\frac{1}{(\lambda_1-\lambda_2)\lambda_1} & -\frac{1}{(\lambda_1-\lambda_2)\lambda_1} & 1 & 1 \end{bmatrix} \tag{3.73}$$

The real scalar value x does not affect the transfer functions resulting from $\mathbf{V}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{W}$ and may be arbitrarily set. It may be interpreted as a gain block applied just prior to the wash out filter. A second inverse gain block occurs just after the filter. A good choice for x would be such that the modal matrix $\mathbf{V}$ is as well-conditioned as possible for inversion; alternatively x could be chosen such that the projection of the set of desired eigenvectors $\{\mathbf{v}_i\}$ into their allowable subspaces results in the least possible projection error.

Note that the eigenvector elements associated with the compensator state reflect its intended role as a washout filter, and its consequent links only to q and not θ or u. In fact, if these structural constraints are not important, these elements may be chosen arbitrary and the denominators of the resulting transfer functions for the states of interest will not change. The numerators on the other hand may change, leading to migration of the zero. In practice this is a likely outcome of the projection of the desired eigenvectors into their allowable subspaces - see Chapter 4 for more details.

3.5.3.1 A Numerical Example

Once again an algebraic check of the correctness of Equations 3.72 and 3.73 can be made by substitution into the original constraints.

However, in order to check that the derived eigenstructure represents an appropriate response for a rate-commanded helicopter, a quick example will now be derived and presented.

Assume, arbitrarily, that $\lambda_1 = -3 + 4j$, $\lambda_2 = -5$ and $x = 1$. Then using the above eigenstructure yields

$$\frac{Q(s)}{B1(s)} = \frac{s+3}{s^2+6s+25} \tag{3.74}$$

$$\frac{Comp(s)}{B1(s)} = -\frac{4}{s^2+5s+25} \tag{3.75}$$

$$\frac{\Theta(s)}{B1(s)} = \frac{s+3}{s(s^2+6s+25)} \tag{3.76}$$

$$\frac{U(s)}{B1(s)} = -\frac{s+3}{s(s^2+6s+25)(s+5)} \tag{3.77}$$

The zero at $s = -3$ may be safely cancelled by applying an appropriate first order input pre-filter to the closed loop system.

Applying the same arbitrary eigenvalues leads to the following state space system:

$$\mathbf{A} = \begin{bmatrix} -3 & 4 & 0 & 0 \\ -4 & -3 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

On adding the pre-filter, the response to a unit step input at $B_1(t)$ is shown in Figure 3.7. The Figure demonstrates the suitability of the ideal eigenstructure in several ways:

- The second-order nature of the rate response $q(t)/B_1(t)$ can be seen to match that required by Equation 3.43;
- The integral action between $q(t)$ and $\theta(t)$, required for kinematic consistency and for the satisfaction of Equation 3.44, can be seen;
- The observed relationship between $\theta(t)$ and $u(t)$ is not inconsistent with a first-order lag, as specified by Equation 3.42;
- In response to a backward step applied to the collective pitch lever, the 'helicopter' in this simple example can be seen to attain a steady nose-up pitch rate, and to lose forward velocity, as would be expected.

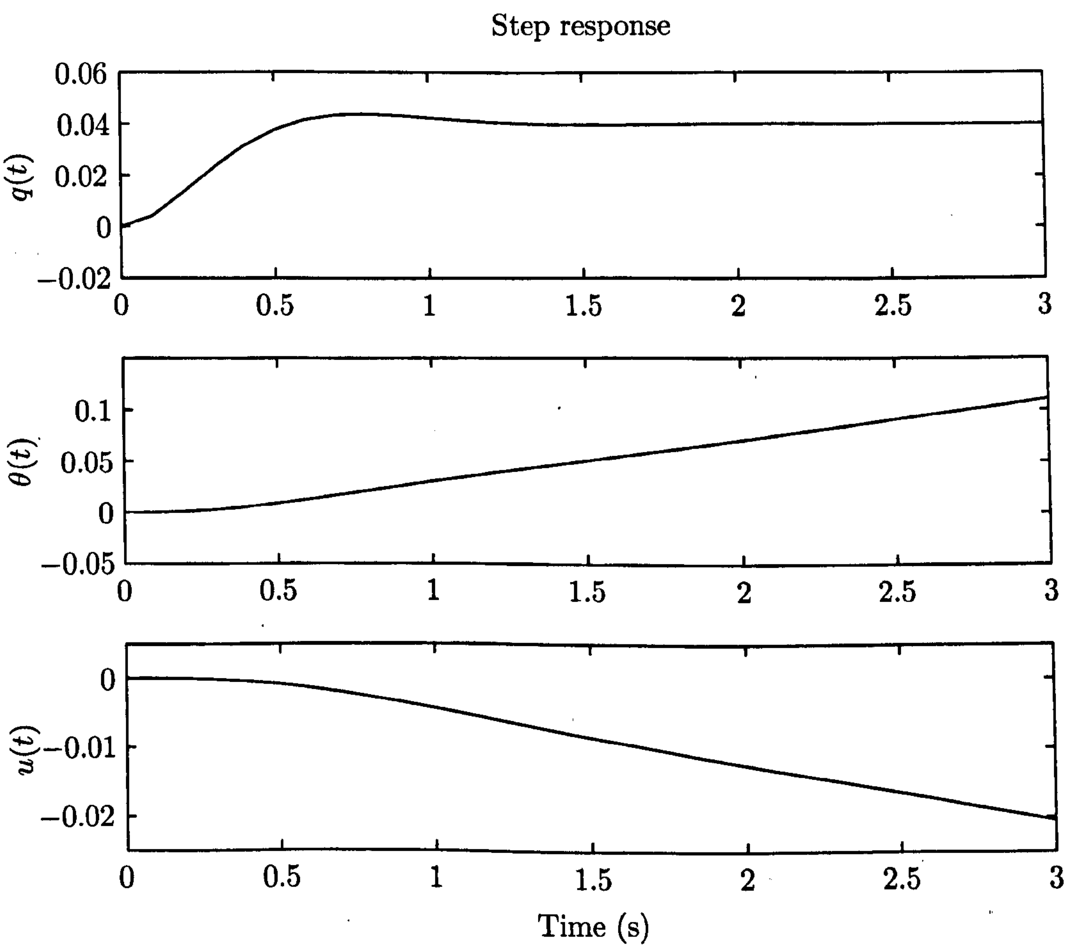

Figure 3.7: Example step response of ideal rate-command eigenstructure block

3.5.3.2 Second Order Tail Rotor Response

The Def.Stan.00-970 requirements for the yaw channel are specified using the same criteria as for the cyclic channels, and with the exception of the peak response (a scaling parameter), the requirements themselves are identical. Clarke et al. (2003) developed a first-order yaw rate response because only static feedback controller designs were to be considered. If this restriction is dropped, a compensator state can instead be added in order to generate the required second-order response in yaw.

Such a response carries no requirements on the elements of the eigenvectors associated with the compensator state. The eigenvector subset may therefore be described using

$$\mathbf{x} = \begin{bmatrix} r & comp_r \end{bmatrix}^T$$

$$\mathbf{V} = \begin{bmatrix} 1 & 1 \\ z & \bar{z} \end{bmatrix}$$

where z is an arbitrary complex constant. The mechanics of assigning an eigenstructure wherein a subset of the states have no specific requirements on their eigenvector elements is investigated in Chapter 4.

Note that this yaw response, which is a rate response, is valid for achieving Level 1 handling qualities in both the 'Active' and 'Attentive' flight phases.

3.5.4 Complete Eigenvector Set

The complete set of ideal eigenvectors, constructed using four decoupled blocks and including the first-order tail rotor subsystem, is shown below:

$$\mathbf{x} = \begin{bmatrix} v & p & comp_p & \phi & u & q & comp_q & \theta & w & r \end{bmatrix}^T$$

$$\mathbf{V}_d = \begin{bmatrix} -\frac{1}{(\lambda_p-\lambda_v)\lambda_p} & -\frac{1}{(\lambda_p-\lambda_v)\lambda_p} & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ j & -j & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\lambda_p} & \frac{1}{\lambda_p} & \lambda_v & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{(\lambda_q-\lambda_u)\lambda_q} & -\frac{1}{(\lambda_q-\lambda_u)\lambda_q} & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & j & -j & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\lambda_q} & \frac{1}{\lambda_q} & \lambda_u & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

This assumes that the arbitrary constants x for each of the cyclic pitch subsystems have been chosen to be unity. Chapter 7 contains design examples wherein this new eigenstructure is used to develop sets of control laws for a helicopter in forward flight, and these examples will serve as further evidence that the eigenstructure presented here is consistent both with the kinematics of a helicopter and with the requirements of the Def.Stan.00-970.

3.6 Conclusions

In this chapter, the complex workings of the helicopter and their effect on the manner of its flight have been outlined, along with the means by which the pilot may exert control over the movement of the aircraft. The sources of cross-coupling, nonlinearity and instability which render the control of a helicopter difficult have been identified. The documents that specify the handling qualities requirements for a helicopter have been introduced, and an ideal eigenstructure (both eigenvalues and eigenvectors) for a helicopter in the 'Attentive' flight phase, derived directly from the Def.Stan.00-970 by Clarke et al. (2003), has been described.

Following this review, an ideal set of eigenvectors for the achievement of Level 1 handling qualities in the Active flight phase has been developed. This new eigenstructure requires the use of a feedback compensator for its implementation. The eigenvector set has been demonstrated to be kinematically consistent with the expected motion of the airframe.

Chapter 7 contains design examples which use this new eigenstructure to achieve Level 1 Def.Stan.00-970 handling qualities for a helicopter in forward flight.

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Chapter 4

Dynamic Compensation

Contents

4.1 Introduction

Standard state- or output-feedback generates a matrix of fixed gains as a controller. On occasion, a more complex controller may be required; this could be to increase the order of the system response, or to increase the number of Degrees of Freedom (DoF) available to the designer.

A basic Multi-Input, Multi-Output (MIMO) system with a feedback dynamic compensator is shown in Figure 4.1.

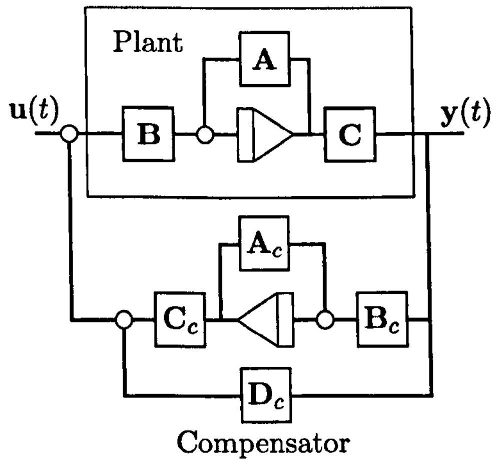

Figure 4.1: Dynamic Compensator

It may be thought that the massively increased complexity of the controller would introduce a similar increase in the complexity of the design process, but - superficially at least - this is not so. A well-known representation for dynamically compensated systems will be described that allows a control system to be designed as if the controller was merely a set of static gains.

This representation offers some insights into the way in which the added design freedom is distributed and into the effects of adding a compensator to a system. It will be shown that the addition of a dynamic compensator increases the effective system order, thereby increasing the number of poles that must be placed; it will also be shown that dynamic compensators add transmission zeros to the closed-loop system and that this has an impact on performance.

An alternative compensator structure is also considered, whereby an entire class of compensators yielding the same closed-loop eigenstructure can be generated, with the actual implementation chosen post-assignment.

4.2 Expression of Dynamic Compensators

4.2.1 Augmented System Description

The 'Augmented System' description for MIMO systems with dynamic compensation permits their representation as larger systems with static feedback. The compound system so generated may be controlled using any standard MIMO technique.

The Augmented System Method has 'proved an expedient and popular method of achieving dynamic compensation' (Griffin, 1997). Many researchers (Sobel and Shapiro, 1986*a*; Hippe and O'Reilly, 1987; Han, 1989; Pyburn and Owens, 1990; Magni, 1999; Tsui, 1999) have employed or developed the method, and others (Kimura, 1975; Askarpour and Owens, 1997) have cited its use as a way to circumvent the general condition that $m + r > n$ discussed in Chapter 2.

The derivation of the augmented system description is straightforward. Consider a plant under the influence of feedback via a dynamic compensator, as in Figure 4.1. The usual state equations apply:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{t} \tag{4.1}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} \tag{4.2}$$

$$\tag{4.3}$$

where now $\mathbf{t}$ is the plant input. An equivalent set of state equations apply to the compensator:

$$\dot{\mathbf{x}}_c = \mathbf{A}_c\mathbf{x}_c + \mathbf{B}_c\mathbf{y} \tag{4.4}$$

$$\mathbf{t} = \mathbf{C}_c\mathbf{x}_c + \mathbf{D}_c\mathbf{y} + \mathbf{u} \tag{4.5}$$

where $\mathbf{u}$ is the external input to the system. The subscripts indicate an association with the compensator; specifically $\mathbf{x}_c \in \mathbb{R}^c$ is the compensator state vector, so the compensator is of order c.

By substitution, we may obtain

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{C}_c\mathbf{x}_c + \mathbf{B}\mathbf{D}_c\mathbf{y} + \mathbf{B}\mathbf{u} \tag{4.6}$$

$$= (\mathbf{A} + \mathbf{B}\mathbf{D}_c\mathbf{C})\,\mathbf{x} + \mathbf{B}\mathbf{C}_c\mathbf{x}_c + \mathbf{B}\mathbf{u} \tag{4.7}$$

and

$$\dot{\mathbf{x}}_c = \mathbf{A}_c\mathbf{x}_c + \mathbf{B}_c\mathbf{C}\mathbf{x} \tag{4.8}$$

Combining Equations 4.7 and 4.8 gives

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{x}}_c \end{bmatrix} = \begin{bmatrix} \mathbf{A} + \mathbf{B}\mathbf{D}_c\mathbf{C} & \mathbf{B}\mathbf{C}_c \\ \mathbf{B}_c\mathbf{C} & \mathbf{A}_c \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_c \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \mathbf{u} \tag{4.9}$$

$$\mathbf{y} = \begin{bmatrix} \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_c \end{bmatrix} \tag{4.10}$$

Equation 4.9 may now be rearranged as

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{x}}_c \end{bmatrix} = \left(\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{D}_c & \mathbf{C}_c \\ \mathbf{B}_c & \mathbf{A}_c \end{bmatrix} \begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \right) \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_c \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \mathbf{u} \tag{4.11}$$

Comparing Equations 4.10 and 4.11 with Equations 2.62 and 2.66 from Section 2.3.1.2, it may be seen that the closed-loop compensated system can be regarded as an equivalent augmented system

$$\hat{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \qquad \hat{\mathbf{B}} = \begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_c \end{bmatrix} \qquad \hat{\mathbf{C}} = \begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_c \end{bmatrix} \tag{4.12}$$

subjected to static gain feedback

$$\hat{\mathbf{K}} = \begin{bmatrix} \mathbf{D}_c & \mathbf{C}_c \\ \mathbf{B}_c & \mathbf{A}_c \end{bmatrix} \tag{4.13}$$

such that

$$\dot{\hat{\mathbf{x}}} = \left(\hat{\mathbf{A}} + \hat{\mathbf{B}}\hat{\mathbf{K}}\hat{\mathbf{C}} \right) \hat{\mathbf{x}} + \tilde{\mathbf{B}}\mathbf{u} \tag{4.14}$$

$$\mathbf{y} = \tilde{\mathbf{C}}\hat{\mathbf{x}} \tag{4.15}$$

where

$$\hat{\mathbf{x}} = \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_c \end{bmatrix} \qquad \tilde{\mathbf{B}} = \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \qquad \tilde{\mathbf{C}} = \begin{bmatrix} \mathbf{C} & \mathbf{0} \end{bmatrix} \tag{4.16}$$

Equations 4.14 and 4.15 form the augmented system description of the plant and compensator. All of the available design freedom is encompassed by $\hat{\mathbf{K}} \in \mathbb{R}^{(m+c)\times(r+c)}$ in a form which renders standard MIMO control techniques, including Eigenstructure Assignment (EA), directly applicable.

A corollary of the augmented system description is that a closed-loop augmented system matrix $\hat{\mathbf{A}}_{cl}$ may be readily derived from Equation 4.9:

$$\hat{\mathbf{A}}_{cl} = \hat{\mathbf{A}} + \hat{\mathbf{B}}\hat{\mathbf{K}}\hat{\mathbf{C}} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{D}_c & \mathbf{C}_c \\ \mathbf{B}_c & \mathbf{A}_c \end{bmatrix} \begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{4.17}$$

$$= \begin{bmatrix} \mathbf{A} + \mathbf{B}\mathbf{D}_c\mathbf{C} & \mathbf{B}\mathbf{C}_c \\ \mathbf{B}_c\mathbf{C} & \mathbf{A}_c \end{bmatrix} \tag{4.18}$$

The augmented system description has been expanded a little here to include the matrices $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{C}}$, known hereinafter as the *external input matrix* and *external output matrix* respectively. In general, only the dynamic response is of interest when controlling a system, since the coupling of inputs and outputs to states is invariant under feedback, and such definitions are consequently superfluous. However when considering the effects of the inclusion of dynamic compensation upon the overall system response in Section 4.3.2, these reduced input and output matrices will prove important.

4.2.2 Sub-Eigenvectors

Let us subject the augmented system matrix to eigenvalue-eigenvector decomposition such that

$$\hat{\mathbf{A}} = \hat{\mathbf{V}}\hat{\mathbf{\Lambda}}\hat{\mathbf{W}} \tag{4.19}$$

If each eigenvector of the augmented system is now partitioned into *sub-eigenvectors* so that

$$\mathbf{v}_i = \begin{bmatrix} \mathbf{v}_{pi} \\ \mathbf{v}_{ci} \end{bmatrix} \tag{4.20}$$

$$\mathbf{w}_j = \begin{bmatrix} \mathbf{w}_{pj} & \mathbf{w}_{cj} \end{bmatrix} \tag{4.21}$$

where $\{\mathbf{v}_{pi}\}$ and $\{\mathbf{w}_{pj}\}$ are of length n and $\{\mathbf{v}_{ci}\}$ and $\{\mathbf{w}_{cj}\}$ are of length c, then we may write

$$\hat{\mathbf{V}} = \begin{bmatrix} \mathbf{V}_p \\ \mathbf{V}_c \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{p1} & \mathbf{v}_{p2} & \cdots & \mathbf{v}_{p(n+c)} \\ \mathbf{v}_{c1} & \mathbf{v}_{c2} & \cdots & \mathbf{v}_{c(n+c)} \end{bmatrix} \tag{4.22}$$

$$\hat{\mathbf{W}} = \begin{bmatrix} \mathbf{W}_p & \mathbf{W}_c \end{bmatrix} = \begin{bmatrix} \mathbf{w}_{p1} & \mathbf{w}_{c1} \\ \mathbf{w}_{p2} & \mathbf{w}_{c2} \\ \vdots & \vdots \\ \mathbf{w}_{p(n+c)} & \mathbf{w}_{c(n+c)} \end{bmatrix} \tag{4.23}$$

(4.24)

and hence

$$\hat{\mathbf{A}} = \begin{bmatrix} \mathbf{V}_p \\ \mathbf{V}_c \end{bmatrix} \hat{\Lambda} \begin{bmatrix} \mathbf{W}_p & \mathbf{W}_c \end{bmatrix} \tag{4.25}$$

The transfer function matrix of a state-space system, linking the inputs to the outputs, is given by Equation 2.73 as

$$\mathbf{G}(s) = \mathbf{C}\,(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \tag{4.26}$$

Substituting the matrices associated with an augmented system yields

$$\mathbf{G}(s) = \tilde{\mathbf{C}}\left(s\mathbf{I} - \hat{\mathbf{A}}\right)^{-1}\tilde{\mathbf{B}} \tag{4.27}$$

$$= \tilde{\mathbf{C}}\left(s\mathbf{I} - \hat{\mathbf{V}}\hat{\Lambda}\hat{\mathbf{W}}\right)^{-1}\tilde{\mathbf{B}} \tag{4.28}$$

$$= \tilde{\mathbf{C}}\hat{\mathbf{V}}\left(s\mathbf{I} - \hat{\Lambda}\right)^{-1}\hat{\mathbf{W}}\tilde{\mathbf{B}} \tag{4.29}$$

$$= \begin{bmatrix} \mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{V}_p \\ \mathbf{V}_c \end{bmatrix} \left(s\mathbf{I} - \hat{\Lambda}\right)^{-1} \begin{bmatrix} \mathbf{W}_p & \mathbf{W}_c \end{bmatrix} \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} \tag{4.30}$$

$$= \mathbf{C}\mathbf{V}_p\left(s\mathbf{I} - \hat{\Lambda}\right)^{-1}\mathbf{W}_p\mathbf{B} \tag{4.31}$$

It is clear from Equation 4.31 that only the partial eigenvectors $\{\mathbf{v}_{pi}\}$ and $\{\mathbf{w}_{pj}\}$ have a role in determining the transfer function of the closed-loop system. These will be designated the *plant sub-eigenvectors*, and by definition they describe the coupling of each mode into the portion of the augmented state vector that is associated with the plant. The other sets of sub-eigenvectors, $\{\mathbf{v}_{ci}\}$ and $\{\mathbf{w}_{cj}\}$, have no role in the formation of the closed-loop transfer function; they describe the coupling of each mode into the portion of the augmented state vector that is associated with the compensator, and will consequently be designated the *compensator sub-eigenvectors.*

The process of assigning eigenstructure to an augmented system is the same as that described in Section 2.3.4. However, it is instructive to investigate the structure of the augmented allowable subspaces for the eigenvectors. From Equations 2.116, 2.124 and 2.125, the following restriction applies to selected right eigenvectors:

$$\begin{bmatrix} \mathbf{v}_i \\ \mathbf{KCv}_i \end{bmatrix} = \begin{bmatrix} \mathbf{P}_i \\ \mathbf{Q}_i \end{bmatrix} \mathbf{f}_i \tag{4.32}$$

where

$$\begin{bmatrix} \mathbf{A} - \lambda_i \mathbf{I} & \vdots & \mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{P}_i \\ \mathbf{Q}_i \end{bmatrix} = \mathbf{0} \tag{4.33}$$

In the augmented system case, Equation 4.33 may be written as

$$\begin{bmatrix} \hat{\mathbf{A}} - \lambda_i \mathbf{I} & \vdots & \hat{\mathbf{B}} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{P}}_i \\ \hat{\mathbf{Q}}_i \end{bmatrix} = \mathbf{0} \tag{4.34}$$

Expansion and examination of the leftmost term in Equation 4.34 reveals that

$$\begin{bmatrix} \mathbf{A} - \lambda_i \mathbf{I} & \mathbf{0} & \mathbf{B} & \mathbf{0} \\ \mathbf{0} & -\lambda_i \mathbf{I} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{P}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \\ \mathbf{Q}_i & \mathbf{0} \\ \mathbf{0} & \lambda_i \mathbf{I} \end{bmatrix} = \mathbf{0} \tag{4.35}$$

Consequently, the allowable subspace for the right plant sub-eigenvectors is the same as in the un-augmented case, while the compensator sub-eigenvectors may be selected arbitrarily (Griffin, 1997; Ensor, 2000).

It is interesting now to note that if EA is to be employed for the purposes of affecting the performance (modal response) of the system, it is only necessary to assign the eigenvalues and the plant sub-eigenvectors. Indeed, Magni (1999) suggests that since the compensator sub-eigenvector directions do not affect the modal response of the system, they *must* be ignored, and redefines the concept of eigenvector assignment to reflect this. Ensor (2000) states that this 'appears a little extreme', but does not provide a justification for this view.

A sensible approach would be to develop an assignment process such that the compensator sub-eigenvectors remain unassigned until the primary performance design goals have been satisfied, whereupon the design freedom they represent could be employed for other purposes (such as gain suppression or robustness improvement).

4.3 Dynamic Compensation in Practice

Having seen how dynamic compensators may be expressed, and how the eigenstructure of an augmented system relates to its closed-loop response, it is important to consider the impact that these facts have on the development of dynamic control systems using EA.

If a system with n states, m outputs and r inputs is augmented with a compensator of order c, the augmented system has $m+c$ outputs, $r+c$ inputs and $n+c$ states. Consequently the condition for pole-assignability using standard linear techniques (see Chapter 2) is

$$m+c+r+c > n+c \tag{4.36}$$

$$(m+r+c) > n \tag{4.37}$$

Hence any system which is not pole-assignable due to its dimensionality may be rendered so by adding a feedback compensator. Since it was first proposed (Kimura, 1975), this result has been widely cited as justification for considering the restriction $m+r > n$ to be a weak one (Askarpour and Owens, 1997, for example).

However, to assert simply that a dynamic compensator can be used to render a system pole-assignable is to evade the considerable impact that such a compensator will have upon the form of the system's specification. The nature of EA is that the system specification is in terms of its modal response, and the increased order of the closed-loop system must be taken into account when specifying the required modal response.

4.3.1 Added Poles

Equation 4.31 shows that the closed-loop transfer function is dependent on the plant sub-eigenvectors and the entire augmented eigenvalue set $\hat{\mathbf{\Lambda}}$. The eigenvalue set *cannot* be split into plant and compensator eigenvalues, and *all* of the $n+c$ closed-loop system poles contribute to the response.

This fact has serious implications for the development of a specification for the eigenstructure of an augmented system. If, for example, both the original system and its ideal response are second-order, but the augmented system has three states, then three poles must be placed and the final response will be third-order.

Placing only a subset of the poles is unlikely to yield good results, since there is no guarantee that the unconstrained poles will be stable; even if they are, they may have a significant effect on performance. For example, Sobel and Shapiro (1986*a*) assign four poles to a sixth-order augmented system, and the unassigned poles take up a slower (and hence more dominant) position than the assigned poles. This has the effect of causing the response of the compensated system to be more sluggish than its uncompensated counterpart. This effect was noted in more detail by (Sobel and Shapiro, 1986*b*).

Assigning a subset of the poles such that they are an order of magnitude faster than the natural plant modes is not a sensible approach, since attempting to force a system to respond more quickly than its natural dynamics will allow will generally lead to large controller gains and a solution that lacks robustness.

One possibility for placing 'extra' poles is to place multiple poles in coincident or near-coincident locations, to ensure that all modes of the closed-loop system are close in frequency and damping to modes present in the specification. However, near-coincident eigenvalues will have nearly co-linear eigenvector subspaces, and this is likely to result in a poorly conditioned eigenvector set (Tsui, 1996). Systems with ill-conditioned sets of eigenvectors are sensitive to the variation of parameters in the open-loop plant (Ensor, 2000), and are also likely to generate large gains since the calculation of a gain matrix relies on calculating the inverse, or pseudo-inverse, of the assigned sets of eigenvectors.

Another possibility would be to attempt to ensure that a subset of the poles of the closed-loop system are assigned such that their associated right plant sub-eigenvectors are zero. The mechanism for this would be that the added transmission zeros of the closed-loop system would be assigned to the same locations as a subset of the poles. Were this possible, it is likely that the robustness of such a system would be poor; the closed-loop poles are subject to migration in respect of changes to the open-loop system, while the added transmission zeros (being the poles of the compensator) are not. However, this type of assignment is *not* possible, since the restrictions that are placed on the eigenvector directions make it impossible to meet both the orthogonality condition and the rank constraints on the modal coupling matrices. The reasons for this will be discussed more fully in Section 4.4.3.1.

In summary, attempts to hide or mask the additional system poles introduced by a dynamic compensator are unlikely to succeed. The locations of such poles must instead be included into the specification for the system.

4.3.2 Added Transmission Zeros

It was shown in Chapter 2, for Single-Input, Single-Output (SISO) systems, that the effect of placing a controller with a dynamic transfer function in the feedback path of a system is to add both poles and zeros to the resulting system. The new zeros are due entirely to the *poles* of the controller. A similar phenomenon is exhibited by MIMO systems, and the following theorem is presented as a proof.

Patel (1978) presents a similar theorem, but considers a compensator with no direct transmission term. The following theorem involves a full compensator as described above.

Theorem 4.3.1. *Poles of a feedback dynamic compensator manifest themselves as additional transmission zeros in the closed loop system.*

Proof. Given an augmented system under feedback, the closed-loop system matrix $\hat{\mathbf{A}}_{cl}$ is given by Equation 4.18, and the external input and output matrices $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{C}}$ by Equation 4.16.

Recalling Equation 2.104, transmission zero locations for the closed-loop system are given as those values of λ which cause test matrix

$$\Delta = \begin{bmatrix} \lambda\mathbf{I} - \hat{\mathbf{A}}_{cl} & -\tilde{\mathbf{B}} \\ \tilde{\mathbf{C}} & \mathbf{0} \end{bmatrix} \tag{4.38}$$

to lose rank.

If, as is usually the case, the number of plant outputs is at least equal to the number of inputs ($m \geq r$), then Δ is injective and it is sufficient to demonstrate the rank deficiency of a subset of its columns. If $m < r$, then Δ is surjective and it is sufficient instead to demonstrate the rank deficiency of a subset of its *rows*. The resulting proofs are substantively identical, so without loss of generality it will hereafter be assumed that $m \geq r$.

Substituting the closed-loop matrix definitions,

$$\Delta = \begin{bmatrix} \lambda\mathbf{I} - \mathbf{A} - \mathbf{B}\mathbf{D}_c\mathbf{C} & -\mathbf{B}\mathbf{C}_c & -\mathbf{B} \\ -\mathbf{B}_c\mathbf{C} & \lambda\mathbf{I} - \mathbf{A}_c & \mathbf{0} \\ \mathbf{C} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{4.39}$$

It is thus sufficient to show that setting λ equal to an eigenvalue of $\mathbf{A}_c$ causes Δ to lose rank.

Since it is assumed that $m \geq r$, Δ is injective and hence is rank deficient if

$$\begin{bmatrix} -\mathbf{BC}_c & -\mathbf{B} \\ \lambda\mathbf{I} - \mathbf{A}_c & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{4.40}$$

is rank deficient, or equivalently if

$$\text{rank}\left(\begin{bmatrix} -\mathbf{BC}_c & -\mathbf{B} \\ \lambda\mathbf{I} - \mathbf{A}_c & \mathbf{0} \end{bmatrix}\right) < r + c \tag{4.41}$$

If λ is an eigenvalue of $\mathbf{A}_c \in \mathbb{R}^{c \times c}$, then $\lambda\mathbf{I} - \mathbf{A}_c$ may be factorised as

$$\lambda\mathbf{I} - \mathbf{A}_c = \mathbf{XY} \tag{4.42}$$

where $\mathbf{X} \in \mathbb{R}^{c \times z}$, $\mathbf{Y} \in \mathbb{R}^{z \times c}$ and $z < c$. Equation 4.41 may then be written

$$\text{rank}\left(\begin{bmatrix} -\mathbf{BC}_c & -\mathbf{B} \\ \mathbf{XY} & \mathbf{0} \end{bmatrix}\right) < r + c \tag{4.43}$$

$$\text{rank}\left(\begin{bmatrix} -\mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{X} \end{bmatrix}\begin{bmatrix} \mathbf{C}_c & \mathbf{I} \\ \mathbf{Y} & \mathbf{0} \end{bmatrix}\right) < r + c \tag{4.44}$$

which is guaranteed to be satisfied due to the dimensionality of the matrices involved. ◊◊

Clearly the added transmission zeros introduced by the poles of the compensator will have an effect on the response of the closed-loop system. This effect will depend upon the placing of the zeros, but could include 'pole masking', where the effect of a mode is diminished by its proximity to a zero, or the creation of nonminimum-phase output responses.

Unfortunately, assignment techniques employing the augmented system method do not permit direct manipulation of the poles of the compensator. Indeed, Tsui (1999) claims that this fact is sufficient grounds for discarding the technique, since '[the compensator] is not even guaranteed to be stable'. However, the stability of the compensator is not a prerequisite for the stability (or even the performance) of the closed-loop system. If the performance and robustness of the closed-loop system is deemed adequate, then the stability of the compensator is of no consequence - just as the stability of the open-loop system is not a prerequisite for successful control.

It is interesting to note, however, that if the compensator is stable it is possible to mask the transmission zeros through the addition of an input pre-filter. The poles of the pre-filter and the transmission zeros they mask are all products of the controller and are therefore not prone to migration with changes in operating condition. If the compensator is *not* stable, however, attempting masking in this way would be unwise since it would involve an unstable pre-filter. If the poles of this pre-filter are not precisely coincident with those of the compensator, this will lead to unstable modes being present in the output. Moreover, even if the matching is precise, the condition wherein an unstable mode of the pre-filter is cancelled by a controller zero could lead to unbounded signals within the controller, and subsequent problems with implementation.

4.3.2.1 Transmission Zero Locations

The locations of the added transmission zeros are, as shown, given by the poles of the compensator. These are, in turn, given by the eigenvalues of the compensator matrix $\mathbf{A}_c$ which is a partition of the static gain matrix $\mathbf{K}$. Attempting to control the locations of the added transmission zeros during assignment is, then, a problem of controlling the eigenvalues of a square partition of the gain matrix. This is not trivial.

Alternatively, Equation 4.18 may be used to show that the matrix $\mathbf{A}_c$ is also a partition of the closed-loop augmented system matrix $\hat{\mathbf{A}}_{cl}$. Hence the zero-locations problem could be cast as one of assigning the eigenvalues of a partition of the system matrix. This approach was taken by Tsui (1999), who presented an algorithm for the assignment of both the transmission zeros and the system poles in a two-stage process. However, this algorithm was not EA but pole-placement, since the eigenvectors were not explicitly considered.

Magni (1999) chooses *a priori* the denominators of the compensator transfer function, hence fixing the locations of the transmission zeros. This is done by considering the desired physical structure of the controller. However the technique presented by Magni is not pure EA and the design process, though powerful, is somewhat convoluted.

Yet another approach is possible. The transfer function matrix (from Equation 2.73) can be written as

$$\mathbf{G}(s) = \mathbf{C}\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B} + \mathbf{D} \tag{4.45}$$

$$= \mathbf{C}\mathbf{V}\left(s\mathbf{I} - \mathbf{\Lambda}\right)^{-1}\mathbf{W}\mathbf{B} + \mathbf{D} \tag{4.46}$$

and hence the zero locations, which may be found from the numerators of the entries in the transfer function matrix, must be calculable given the eigenstructure of the closed-loop system. This was demonstrated in Chapter 3 where the ideal rate-command eigenstructure was derived. This eigenstructure, derived by considering the required mode-state couplings, introduced a zero in a known location.

It may be thought, then, that the locations of the added transmission zeros may be fixed by choosing the desired eigenstructure carefully. Unfortunately this is not the case. It is extremely unlikely that all of the desired eigenvectors will lie in their associated allowable subspaces, and hence the assigned eigenstructure will not match the desired eigenstructure precisely. It is likely therefore that the zeros will move also. This situation is, if anything, compounded if output-feedback EA is employed, since it is not possible to assign a complete set of either right or left eigenvectors.

The algorithm presented by Hippe and O'Reilly (1987) effectively allows the assignment of the compensator poles after assignment has taken place, but the assignment of eigenvectors is not explicitly considered, and the approach taken relies heavily on symbolic algebra and consequently does not lend itself to high-order systems such as helicopters.

Useful further work would be to develop a set of constraints on the assigned eigenstructure such that the added transmission zeros are assigned, or at least constrained to a region (for example, the left half s-plane). If these constraints were linear, they could be incorporated as additional constraints on the eigenvector subspaces.

4.4 Freedom over Eigenvectors

It has been seen that the distribution of the additional design freedom in an augmented system is such that the subspace from which the plant sub-eigenvectors may be selected is the same as in the un-augmented case, while the compensator sub-eigenvectors may be chosen arbitrarily. However, if the reduced orthogonality condition of Clarke et al. (2003) (Theorem 2.3.2) and its associated EA algorithm are being employed, the restrictions on the rank of the input and output coupling matrices (Conditions C1 and C2 of Theorem 2.3.2) and on the orthogonality of the selected eigenvectors (Condition C3 of Theorem 2.3.2) are expressed in terms of complete eigenvectors. Since the system specification is likely to be in terms only of the plant sub-eigenvectors, it is interesting to re-evaluate these conditions in this context.

4.4.1 Orthogonality Conditions

Condition C3 of Theorem 2.3.2, known as the orthogonality condition, requires that the sets of assigned left and right eigenvectors must be orthogonal to one another. Thus if

$$\hat{\mathbf{V}}' = \begin{bmatrix} \hat{\mathbf{v}}_1 & \cdots & \hat{\mathbf{v}}_v \end{bmatrix} \tag{4.47}$$

and

$$\hat{\mathbf{W}}' = \begin{bmatrix} \hat{\mathbf{w}}_{v+1} \\ \vdots \\ \hat{\mathbf{v}}_{n+c} \end{bmatrix} \tag{4.48}$$

then the orthogonality condition may be expressed as

$$\hat{\mathbf{W}}'\hat{\mathbf{V}}' = \mathbf{0} \tag{4.49}$$

If the sets of assigned eigenvectors are now partitioned into plant and compensator sub-eigenvectors, such that

$$\hat{\mathbf{V}}' = \begin{bmatrix} \mathbf{V}'_p \\ \mathbf{V}'_c \end{bmatrix} \tag{4.50}$$

and

$$\hat{\mathbf{W}}' = \begin{bmatrix} \mathbf{W}'_p & \mathbf{W}'_c \end{bmatrix} \tag{4.51}$$

then Equation 4.49 may be written

$$\begin{bmatrix} \mathbf{W}'_p & \mathbf{W}'_c \end{bmatrix} \begin{bmatrix} \mathbf{V}'_p \\ \mathbf{V}'_c \end{bmatrix} = \mathbf{0} \tag{4.52}$$

$$\mathbf{W}'_p\mathbf{V}'_p + \mathbf{W}'_c\mathbf{V}'_c = \mathbf{0} \tag{4.53}$$

$$\mathbf{W}'_p\mathbf{V}'_p = -\mathbf{W}'_c\mathbf{V}'_c \tag{4.54}$$

Since the compensator sub-eigenvectors may be selected arbitrarily, Equation 4.54 may be expressed as a rank condition:

$$\operatorname{rank}\left(\mathbf{W}'_p\mathbf{V}'_p\right) \leq c \tag{4.55}$$

Hence, in the augmented system case, the orthogonality condition is equivalent to a rank condition upon the product of the selected left and right plant sub-eigenvector sets; the greater the compensator order, the greater the extent to which the orthogonality condition may be 'broken' by the selected plant sub-eigenvector sets. Indeed, if the compensator is of very high order ($c \geq \max(v, n, n + c - v)$), the orthogonality condition may be dismissed entirely when constructing sets of plant sub-eigenvectors for assignment.

Assuming that plant sub-eigenvectors have been chosen according to Equation 4.55, with

$$\text{rank}\left(\mathbf{W}'_p\mathbf{V}'_p\right) = d \tag{4.56}$$

it remains necessary to find compensator sub-eigenvectors which fulfill Equation 4.54.

The following method for so doing assumes that the closed-loop system eigenvalues, and hence the partial eigenvector subsets, are real. A simple extension to the complex case exists, but is given in Appendix B to avoid obscuring this description.

If two matrices $\mathbf{W}_x \in \mathbb{R}^{(n+c-v)\times d}$ and $\mathbf{V}_x \in \mathbb{R}^{d\times v}$ can be found such that

$$\mathbf{W}_x\mathbf{V}_x = \mathbf{W}'_p\mathbf{V}'_p \tag{4.57}$$

then Equation 4.55 may be satisfied by choosing two arbitrary full-rank matrices $\mathbf{R}_1 \in \mathbb{R}^{d\times c}$ and $\mathbf{R}_2 \in \mathbb{R}^{c\times d}$ and setting

$$\mathbf{W}'_c = -\mathbf{W}_x\mathbf{R}_1 \tag{4.58}$$

$$\mathbf{V}'_c = \mathbf{R}_2\left(\mathbf{R}_1\mathbf{R}_2\right)^{-1}\mathbf{V}_x \tag{4.59}$$

The inverse is guaranteed to exist due to the constraint that $d \leq c$, and the matrices $\mathbf{W}_x$ and $\mathbf{V}_x$, defined by Equation 4.57, may be obtained by subjecting the product $\mathbf{W}'_p\mathbf{V}'_p$ to a singular-value decomposition (Golub and van Loan, 1996).

If the orthogonality of $\mathbf{W}'_p$ and $\mathbf{V}'_p$ is pushed to its limit, such that $c = d$, the matrices $\mathbf{R}_1$ and $\mathbf{R}_2$ will be square, and Equation 4.59 will simplify to

$$\mathbf{V}'_c = \mathbf{R}_2\mathbf{R}_2^{-1}\mathbf{R}_1^{-1}\mathbf{V}_x \tag{4.60}$$

$$= \mathbf{R}_1^{-1}\mathbf{V}_x \tag{4.61}$$

and so the matrix $\mathbf{R}_2$ no longer carries any design freedom. It is interesting to note that the matrix $\mathbf{R}_1$ now has c^2 DoF. In Chapter 2 it was shown that the state vector may be transformed without affecting the transfer function matrix. Such a transformation, applied to the compensator only, would require c^2 DoF. Thus if the product $\mathbf{W}'_p\mathbf{V}'_p$ has rank c, the freedom present in the matrix $\mathbf{R}_1$ acts only to transform the compensator and hence has no effect on the locations of the transmission zeros. Indeed in general, c^2 DoF have no effect on the final transfer function for this reason (Hippe and O'Reilly, 1987); the algorithm described by Hippe and O'Reilly explicitly isolates the design freedom which exists only to characterise a class of compensators.

4.4.2 Rank Conditions

Conditions C1 and C2 of Theorem 2.3.2 can also be interpreted in the context of an augmented system. The requirement is that the selected sets of eigenvectors must be linearly independent, ie. that

$$\text{rank}\left(\hat{\mathbf{C}}\hat{\mathbf{V}}'\right) = v \tag{4.62}$$

$$\text{rank}\left(\hat{\mathbf{W}}'\hat{\mathbf{B}}\right) = n + c - v \tag{4.63}$$

Most obviously, these conditions imply that

$$v \leq m + c \tag{4.64}$$

and

$$n + c - v \leq r + c \tag{4.65}$$

$$n - r \leq v \tag{4.66}$$

and therefore

$$n - r \leq v \leq m + c \tag{4.67}$$

which places lower and upper bounds upon the number of right eigenvectors that may be selected during the assignment process.

However, the rank conditions can also be expressed as a constraint only on the selected plant sub-eigenvectors, by re-writing Equation 4.62 as

$$\text{rank}\left(\begin{bmatrix}\mathbf{C}\mathbf{V}'_p \\ \mathbf{V}'_c\end{bmatrix}\right) = v \tag{4.68}$$

If the selection of the compensator sub-eigenvectors is considered arbitrary, it will always be possible to obtain a set of right plant sub-eigenvectors such that

$$\text{rank}\left(\mathbf{V}_c'\right) = \min(c, v) \tag{4.69}$$

Hence if $v \leq c$, there need be no rank conditions imposed upon the set of chosen plant sub-eigenvectors, and Equation 4.62 can be satisfied by careful choice of the compensator sub-eigenvectors.

If $v > c$ however, then in order to satisfy Equation 4.62, it is necessary that

$$\text{rank}\left(\mathbf{C}\mathbf{V}_p'\right) \geq v - c \tag{4.70}$$

The set of compensator sub-eigenvectors needs still to be full rank, but this alone is not sufficient to ensure that Equation 4.62 is satisfied. The rows of $\mathbf{V}_c'$ must also be linearly independent from the rows of $\mathbf{C}\mathbf{V}_p'$.

Similar constraints can be derived for the left eigenvectors, by writing

$$\text{rank}\left(\begin{bmatrix}\mathbf{W}_p'\mathbf{B} & \mathbf{W}_c'\end{bmatrix}\right) = n + c - v \tag{4.71}$$

The left compensator sub-eigenvector set may be chosen such that

$$\text{rank}\left(\mathbf{W}_c'\right) = \min(c, n + c - v) \tag{4.72}$$

If $n + c - v \leq c$ (ie. if $v > n$), the selected left plant sub-eigenvectors need not be subject to a rank condition, and $\mathbf{W}_c'$ may be chosen to ensure that the set of complete left eigenvectors is full rank. If $v \leq n$, the condition is that

$$\text{rank}\left(\mathbf{W}_p'\mathbf{B}\right) \geq n - v \tag{4.73}$$

and the columns of $\mathbf{W}_c'$ must be chosen to be linearly independent from those of $\mathbf{W}_p'\mathbf{B}$.

It is unlikely that the assignment of reduced-rank plant sub-eigenvector sets will form an important part of the realisation of a design specification. However, a certain amount of freedom over rank is necessary for assignment in many cases.

Consider the case where a compensator is employed to overcome the fact that $m + r < n$. In this case, it is necessary either to set $v > m$ or $n + c - v > r$ in order to allow complete assignment. Either the left or the right plant sub-eigenvector sets must therefore be of reduced rank with respect to m or r respectively.

4.4.3 Orthogonality vs. Rank

Equations 4.55, 4.70 and 4.73 re-express conditions C1, C2 and C3 of Theorem 2.3.2 as conditions on the selected plant sub-eigenvector sets. The derivation of each of these equations, however, contained the assumption that the compensator sub-eigenvector sets could be selected arbitrarily. It is unlikely that the restrictions imposed upon the compensator sub-eigenvectors by these equations are compatible, and a compromise will have to be sought between the rank of the selected plant sub-eigenvector sets and the rank of their product.

The interplay between these sets of constraints is extremely complex, and has thus far defied attempts to reduce them to a linear set of simultaneous constraints or generate a logical design procedure. Further work must be undertaken in this area to make the best use of the insights provided here into the distribution and potential use of the design freedom in a compensated system.

4.4.3.1 Masking Poles

In Section 4.3.1 it was claimed that placing system poles to be coincident with added transmission zeros by assigning zero entries into right plant sub-eigenvectors was not feasible. The reason for this is the conflict between the rank conditions and the orthogonality condition.

Consider a subset of the assigned right eigenvectors, $\hat{\mathbf{V}}_H$, wherein the plant sub-eigenvectors have been assigned zero entries. The orthogonality condition requires that every assigned left eigenvector must be orthogonal to these right eigenvectors, and hence that

$$\hat{\mathbf{W}}'\hat{\mathbf{V}}_H = \mathbf{0} \tag{4.74}$$

$$\begin{bmatrix} \mathbf{W}_p' & \mathbf{W}_c' \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{V}_{Hc} \end{bmatrix} = \mathbf{0} \tag{4.75}$$

$$\mathbf{W}_c'\mathbf{V}_{Hc} = \mathbf{0} \tag{4.76}$$

The rank conditions upon $\hat{\mathbf{V}}'$ effectively force $\mathbf{V}_{Hc}$ to be full rank. Hence if the number of right eigenvectors in $\mathbf{V}_H$ is given by h, a solution to Equation 4.76 for $\mathbf{V}_{Hc}$ exists if and only if $\mathbf{W}'_c$ has a right null-space of dimension h, so

$$c - \text{rank}(\mathbf{W}'_c) \geq h \tag{4.77}$$

$$\text{rank}(\mathbf{W}'_c) \leq c - h \tag{4.78}$$

If $\mathbf{W}'_c$ has zero rank (has all zero entries), then the rank conditions on $\hat{\mathbf{W}}'$ can be satisfied only if $\mathbf{W}'_p\mathbf{B}$ has full rank, which is only possible for $w \leq r$ where w is the number of assigned left eigenvectors. As the rank of $\mathbf{W}'_c$ increases, so the constraint on the size of $\mathbf{W}'_p$ is relaxed, and more left eigenvectors may be assigned.

This revised constraint on the left eigenvectors may be written as

$$w \leq r + c - h \tag{4.79}$$

and substituting into the constraint for pole-assignability,

$$m + c + r + c - h > n + c \tag{4.80}$$

$$m + r + c - h > n \tag{4.81}$$

Hence adding compensator states will not render a system pole-assignable if the added modes are 'hidden' in this way.

4.5 Dynamic Compensation for Performance

Using a dynamic compensator instead of a static feedback network increases the number of DoF available for the design of a control system. It has been seen that the distribution of this additional freedom is not simple, and its exploitation fraught with difficulties; the system order is increased, transmission zeros are added, and no additional freedom over the coupling of modes into the original system states is gained. Although the additional DoF can be exploited for other means (Hippe and O'Reilly, 1987), their use for EA is very limited.

One major exception to this situation exists, and that is when the specification on a system is already of higher order than the system itself. This is the case for a helicopter in forward flight; its lateral and longitudinal angular rates are required to be second-order functions of the cyclic stick position, while the open-loop dynamics are first-order. As seen in Chapter 3, the ideal eigenstructure for this flight condition requires the introduction of two compensator states. The eigenstructure includes specifications on the compensator sub-eigenvectors, and predicts the addition of the added zeros. Clearly, in terms of EA, this represents a far simpler situation than that discussed above.

4.6 Adding a Feedforward Path

The compensators considered thus far take information only from the plant outputs. If a compensator were given extra information, in the form of the plant inputs, the number of DoF available to the control system designer would be increased.

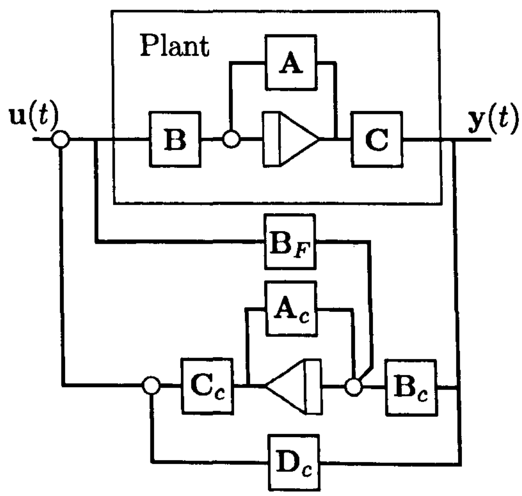

Figure 4.2: Dynamic Compensator with Feedforward Path

The structure thus created is illustrated in Figure 4.2. The effect of the new feedforward path is to add a term $\mathbf{B}_F \in \mathbb{R}^{c \times r}$ into the augmented input matrix, $\hat{\mathbf{B}}$, to form

$$\bar{\mathbf{B}} = \begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{B}_F & \mathbf{I} \end{bmatrix} \tag{4.82}$$

4.6.1 Implications of the Feedforward Term

Griffin (1997) states that since this modified compensator takes information from both the inputs and the outputs of the system, it may be expressed as a feedback compensator and input pre-compensator. This is partially true. Factorisation of $\bar{\mathbf{B}}$ yields

$$\bar{\mathbf{B}} = \begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{B}_F & \mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{B}_F & \mathbf{I} \end{bmatrix} \triangleq \hat{\mathbf{B}}\hat{\mathbf{B}}_F \tag{4.83}$$

Substitution into Equation 4.14 shows that the modified closed-loop system matrix is given by

$$\tilde{\mathbf{A}} = \hat{\mathbf{A}} + \bar{\mathbf{B}}\hat{\mathbf{K}}\hat{\mathbf{C}} \tag{4.84}$$

$$= \hat{\mathbf{A}} + \hat{\mathbf{B}}\hat{\mathbf{B}}_F\hat{\mathbf{K}}\hat{\mathbf{C}} \tag{4.85}$$

$$\triangleq \hat{\mathbf{A}} + \hat{\mathbf{B}}\bar{\mathbf{K}}\hat{\mathbf{C}} \tag{4.86}$$

In other words the dynamic response of the system ($\hat{\mathbf{A}}$,$\bar{\mathbf{B}}$,$\hat{\mathbf{C}}$), which includes a feedforward term, under the influence of feedback $\hat{\mathbf{K}}$ is the same as the dynamic response of the system ($\hat{\mathbf{A}}$,$\hat{\mathbf{B}}$,$\hat{\mathbf{C}}$) under the influence of feedback $\bar{\mathbf{K}}$, where

$$\bar{\mathbf{K}} = \hat{\mathbf{B}}_F\hat{\mathbf{K}} = \begin{bmatrix} \mathbf{D}_c & \mathbf{C}_c \\ \mathbf{B}_F\mathbf{D}_c + \mathbf{B}_c & \mathbf{B}_F\mathbf{C}_c + \mathbf{A}_c \end{bmatrix} \tag{4.87}$$

Since the matrix $\hat{\mathbf{B}}_F$ is full rank, the mapping between $\hat{\mathbf{K}}$ and $\bar{\mathbf{K}}$ is bijective; every compensator in the form of Figure 4.2 has an equivalent in the form of Figure 4.1 with the same dynamic response, and vice-versa.

If it is required to generate a compensator with a feedforward term from an existing feedback compensator, this can be done simply by noting that

$$\hat{\mathbf{B}}_F^{-1} = \begin{bmatrix} \mathbf{I} & 0 \\ \mathbf{B}_F & \mathbf{I} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{I} & 0 \\ -\mathbf{B}_F & \mathbf{I} \end{bmatrix} \tag{4.88}$$

Hence, to retain the same closed-loop eigenstructure, the augmented system gain matrix can simply be pre-multiplied by $\hat{\mathbf{B}}_F^{-1}$ once $\mathbf{B}_F$ has been (arbitrarily) chosen.

The input coupling is changed also. The external input matrix becomes

$$\tilde{\bar{\mathbf{B}}} = \begin{bmatrix} \mathbf{B} \\ \mathbf{B}_F \end{bmatrix} \tag{4.89}$$

and consequently the input coupling matrix becomes

$$\hat{\mathbf{W}}\tilde{\bar{\mathbf{B}}} = \mathbf{W}_p\mathbf{B} + \mathbf{W}_c\mathbf{B}_F \tag{4.90}$$

so the system response, from Equation 4.31, becomes

$$\mathbf{y}(s) = \mathbf{C}\mathbf{V}_p \left(s\mathbf{I} - \hat{\mathbf{\Lambda}}\right)^{-1} \left(\mathbf{W}_p\mathbf{B} + \mathbf{W}_c\mathbf{B}_F\right)\mathbf{u}(s) \tag{4.91}$$

This shows that the feedforward term has pre-filtering effect upon the closed-loop system by introducing a dependence on the compensator sub-eigenvectors that did not previously exist. However, it is not valid to state that this effect is the same as that of an input pre-compensator. In order to achieve this effect, a pre-compensation matrix $\mathbf{E}$ would have to satisfy

$$\mathbf{W}_p\mathbf{B} + \mathbf{W}_c\mathbf{B}_F = \mathbf{W}_p\mathbf{B}\mathbf{E} \tag{4.92}$$

A solution for $\mathbf{E}$ exists (Ben-Israel and Greville, 1974) only if

$$\left(\mathbf{W}_p\mathbf{B}\right)\left(\mathbf{W}_p\mathbf{B}\right)^{\dagger}\mathbf{W}_c\mathbf{B}_F = \mathbf{W}_c\mathbf{B}_F \tag{4.93}$$

For general $\mathbf{W}_p$, $\mathbf{W}_c$, $\mathbf{B}$ and $\mathbf{B}_F$, this requires

$$\left(\mathbf{W}_p\mathbf{B}\right)\left(\mathbf{W}_p\mathbf{B}\right)^{\dagger} = \mathbf{I} \tag{4.94}$$

which cannot be satisfied due to the dimensionality of the term $\mathbf{W}_p\mathbf{B}$. Similarly a solution for $\mathbf{B}_F$ from Equation 4.92 for general $\mathbf{W}_p$, $\mathbf{W}_c$, $\mathbf{B}$ and $\mathbf{E}$ would require

$$\mathbf{W}_c^{\dagger}\mathbf{W}_c = \mathbf{I} \tag{4.95}$$

which again cannot generally be satisfied. Hence the design freedom offered by a pre-compensator is not equivalent (but is potentially complementary) to that offered by the feedforward term.

4.6.2 Feedforward Terms and Observers

The *observer* is a well-known form of dynamic compensator which, in its usual form, seeks to estimate and synthesise the state vector through observation of the system input and output vectors. The observer has historically been used in attempts to solve the output-feedback problem by reconstituting the state vector and applying state-feedback control methodologies.

Observers are usually designed before a solution to the control problem is sought, due to the *separation theorem* (Brogan, 1991) which guarantees that the characteristics of an observer will not change when its outputs are used for feedback control. An observer can be designed such that the synthesised states are guaranteed to converge to their physical counterparts, and such that the convergence time is minimal and robust to changes in the open-loop plant characteristics.

Unfortunately, when used for control, the robustness properties of an observer are not inherited by the closed-loop system (Doyle and Stein, 1979). Ideally, the controller and observer would be designed simultaneously, such that the robust performance of the complete closed-loop system could be optimised at once.

The techniques described by Tsui (1996) provide an improvement upon this situation. The state-feedback controller is designed taking into account the performance of the observer, which is still designed first. For most systems[1] the robustness of the closed-loop system is guaranteed to be the same as that of the implementing observer. Tsui considers the use both of optimal quadratic techniques and of EA for the design of the controller stage.

The design process is cumbersome, however, and is a long way from exhibiting 'visibility' as described in Chapter 1. The design of the observer uses a mixture of techniques in order to achieve good results from an observer of minimal order, leading to a complex, decision-based design process. Despite being state-feedback techniques, the EA algorithms presented also suffer from a lack of visibility. They concentrate upon the assignment of eigenvectors with specific characteristics (maximal decoupling or maximal eigenvector orthogonality) and consequently border on pole placement; also, for partial-state observers, they employ 'generalised state feedback' rather than output feedback methodologies.

However, observers can be seen as compensators with feedforward terms in the form of Figure 4.2. From Griffin (1997), a controller with a full-state observer may be written in augmented system form as

$$\hat{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} \end{bmatrix} \qquad \hat{\mathbf{B}} = \begin{bmatrix} \mathbf{0} & \mathbf{B} \\ \mathbf{I} & \mathbf{B} \end{bmatrix} \qquad \hat{\mathbf{C}} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{C} & -\mathbf{C} \end{bmatrix} \qquad \hat{\mathbf{K}} = \begin{bmatrix} \mathbf{0} & \mathbf{K}_0 \\ \mathbf{K}_c & \mathbf{0} \end{bmatrix} \tag{4.96}$$

It is easily verified that the closed-loop system matrix $\hat{\mathbf{A}}_{cl} = \hat{\mathbf{A}} + \hat{\mathbf{B}}\hat{\mathbf{K}}\hat{\mathbf{C}}$ of this arrangement

[1]Most systems, in this context, means 'all open-loop systems with more outputs than inputs or with at least one stable transmission zero' (Tsui, 1996, p.vi)

is the same as that of the augmented system

$$\hat{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \quad \hat{B} = \begin{bmatrix} B & 0 \\ B & I \end{bmatrix} \quad \hat{C} = \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} \quad \hat{K} = \begin{bmatrix} 0 & K_c \\ K_0 & A - K_0 C \end{bmatrix} \tag{4.97}$$

which has the form of (a simplified) Figure 4.2.

Therefore, the observer and controller can be considered together, along with the open-loop plant, and the overall closed-loop system response considered. If techniques are used to improve the robustness of the closed-loop system while retaining dynamic performance (Griffin, 1997; Ensor, 2000), then this can be achieved without having ensured specifically that the observer itself is robust. This approach is similar to that of Tsui (1996) above, except that:

1. The design process is considered in a single stage, with EA being employed throughout.
2. The robustness (and performance) of the observer, on its own, is never considered.
3. All $n + c$ closed-loop eigenvalues are explicitly assigned, and the coupling of all modes into the states is considered.

Item 1 on the list above ensures that the design process is visible. Item 2 simplifies the process, although the fact that the controller performance is not explicitly considered also means that the added transmission zeros of the closed-loop system are not predictable. Item 3 ensures that the closed-loop response is known accurately and in full, but requires that a larger number of pole locations are determined from the specification than may be desirable.

4.6.3 Design Considerations

The design of a compensator with a feedforward term is not simple, and this complexity comes on top of the problems already identified with dynamic compensators in general. No algorithms for the use of these compensator structures have been generated. However, some general principles may be identified.

Since a compensator with a feedforward term always has an equivalent pure feedback counterpart with the same dynamic response, the compensator may be designed as a feedback type and the feedforward term decided later. It is not clear exactly what effect the choice of the feedforward term will have upon the resulting controller. However, the assigned eigenstructure will not change if the feedback portion of the compensator is modified appropriately through Equation 4.87. Some potential areas for research into the use of the feedforward term could be:

- The input coupling will be affected by the addition of a feedforward term, and this could be used to advantage to manipulate the input coupling. This could be especially useful in situations where direct manipulation of the system input signals, as required for a standard input pre-filter, is difficult.
- Since the lower right-hand partition of the augmented gain matrix (the compensator **A** matrix) is modified when compensating for the addition of a feedforward term, it is likely that the added transmission zeros will migrate. Hence these could be placed or constrained, post-assignment, by a suitable algorithm.
- The introduction of a feedforward term changes entries in the **B** and **A** matrices of the feedback compensator. This fact could potentially be exploited to reduce certain elements in these matrices to zero, thereby imposing structure upon the feedback portion of the compensator.
- The robustness of the closed-loop system to variations in the open-loop plant characteristics has not been studied, but it could be found that robustness improvements can be made post-assignment by careful choice of a feedforward term.

4.7 Conclusions and Further Work

In this chapter, dynamic compensators and the augmented system description have been introduced, and an analysis of the distribution of the design freedom added by a compensator has been performed in the context of EA. Although no new algorithms have been presented, it is hoped that this new exposition of the problem of applying EA to compensated systems will form a useful contribution to the existing literature on the subject.

The idea of adding feedforward terms to feedback compensators has been introduced, and it has been shown that it is always possible to add a feedforward term to a compensator without changing the closed-loop eigenstructure, by means of a corresponding manipulation of the feedback portion of the compensator. Section 4.6.3 contains several ideas for the potential exploitation of the design freedom offered post-assignment by this technique, and these would form an ideal starting point for a programme of further work on this topic.

Unless the addition of compensator states is required by the closed-loop specification, as in Chapter 3, such an addition does not usefully increase the freedom available for EA beyond rendering a system pole-assignable if it previously was not. The system order is increased; transmission zeros are added and, at present, there is no method for controlling their locations while retaining freedom over the eigenvectors; and the additional design freedom does nothing to increase the allowable eigenvector subspaces. A method for constraining the final locations of the added transmission zeros during assignment would be very useful further work however.

It is clear that dynamic compensation, without considerable further work, does not represent a viable solution to the problem of increasing the available DoF when using EA to generate a control system for a helicopter. An alternative must therefore be sought if an improvement in performance over the static output feedback case is required.

4.7.1 Alternatives to Dynamic Compensation

If the control designer has any influence over the configuration of the plant, it is always beneficial to increase the number of states that can be directly or indirectly measured. In a linear system model this equates to an increase in rank of the **C** matrix. This adds design freedom by increasing the size of the gain matrix, and could be seen as preferable to dynamic compensation in this respect since the added freedom allows more right eigenvectors to be assigned without increasing the system order. This benefit comes at a price - that of the expense of the sensors themselves, including their maintenance - but the simplicity of the resulting controller, free of dynamic compensation, may well compensate for this. It may even be possible to convert an output-feedback problem to a state-feedback problem by adding sensors. Unfortunately, many sensors - including accelerometers, commonly used on aircraft - introduce nonzero terms in the system **D** matrix, rendering standard approaches to EA inapplicable.

Similarly, Proportional-plus-Derivative (PD) control also has the potential to increase the rank of the **C** matrix and hence the number of available DoF. The implementation of PD controllers requires some care, because differentiators have a gain which increases with frequency and hence can reduce the signal-to-noise ratio of the signals in the controller. Bandwidth-limited differentiators (bandpass filters) may be employed instead, provided that their break frequencies are significantly higher than the fastest system modes and can be ignored for the purposes of modelling the controller. But again, PD control will introduce nonzero terms in the **D** matrix.

Chapter 5 will introduce extensions to some standard EA techniques which are capable of assigning eigenstructure to these types of systems, and demonstrate that the design freedom represented by these structures is more suitable for use with visible EA algorithms than that which is provided by dynamic compensation.

4.8 Chapter Bibliography

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Chapter 5

Eigenstructure Assignment in Semi-Proper Systems

Contents

5.1 Introduction

As discussed in Chapter 2, standard Eigenstructure Assignment (EA) algorithms can be divided roughly into two groups: state-feedback, where the output (**C**) matrix is assumed to be an identity, and output-feedback, where assumptions are made only about the number of inputs and outputs (White, 1995). However, both groups of algorithms generally rely on the direct transmission (**D**) matrix being null. This is valid only if the system is *strictly proper.* Given the multivariable transfer function matrix $\mathbf{G}(s)$, a system is said to be:

- *Strictly proper* if $\mathbf{G}(s) \rightarrow \mathbf{0}$ as $s \rightarrow \infty$;
- *Semi-proper* if $\mathbf{G}(s) \rightarrow \mathbf{D} \neq \mathbf{0}$ as $s \rightarrow \infty$;
- *Proper* if either strictly proper or semi-proper;
- *Improper* if $\mathbf{G}(s) \rightarrow \infty$ as $s \rightarrow \infty$.

Improper systems are not physically realisable Skogestad and Postlethwaite (1996).

Very few EA algorithms are compatible with semi-proper systems. A pair of papers by Fletcher (1981*a*, *b*), published during the formative years of EA, are concerned with output-feedback pole placement rather then EA specifically. Eigenvectors are selected but no mention is made of their importance to the design or solution, or how they should be chosen. Moreover, Fletcher's technique is essentially a protection method (White, 1995) and consequently suffers from the lack of design visibility offered by these approaches. The development of the method does not include a formulation of the complete closed-loop system, and therefore fails to show that the input and output matrices change when the loop is closed, a fact which can be important in the design process.

White (1997) does not assume a zero **D** matrix. The algorithm described by White was developed specifically for robust EA (more accurately for robust pole placement), and used graphical techniques with multivariable pole and zero loci to select gains in highly structured controllers. These techniques are effective but lack any kind of design visibility.

A number of EA algorithms for descriptor systems (Le, 1992, for example) also do not make the assumption that the open-loop system is strictly proper. Clearly any algorithm for a descriptor system may be applied to a system in standard form, but the operation of these algorithms in this context is in general obfuscated by their nature as algorithms for descriptor systems.

This chapter will introduce and develop several novel algorithms for the assignment of eigenstructure to semi-proper systems in standard form, firstly for systems with at least as many outputs as states, and then for those with fewer outputs than states. The first algorithm represents the generalisation of a standard state-feedback EA algorithm such as that of Moore (1976), and the second a generalisation of the output-feedback algorithm presented by Clarke et al. (2003).

5.2 Sources of Semi-Proper Systems

Although semi-proper systems are mathematically feasible, all physical systems are strictly proper (Skogestad and Postlethwaite, 1996). However, semi-proper systems are often useful approximations.

For example, an aircraft mathematical model will often contain several velocity states. Velocities are almost impossible to measure in the absence of a fixed reference frame and, therefore, accelerometers are used to obtain state information. Incorporating the measured accelerations into the model results in the addition of nonzero entries in the **D** matrix. Additionally, many control structures familiar to designers of classical control systems, including Proportional-plus-Derivative (PD) controllers and phase-advance networks, involve the effective differentiation of states and will have the same effect. These are all approximations since no accelerometer or controller has an infinite bandwidth. However, the realisation of these differentiations in the form of a semi-proper system formulation is convenient and, provided that the bandwidths of the approximated components is sufficiently high, fit for practical purposes. A further advantage of such a semi-proper approximation is that the system order is reduced when compared to an equivalent system with modelled sensor dynamics, meaning that fewer Degrees of Freedom (DoF) are required for pole-assignability.

Other sources of semi-proper system descriptions exist. The reduction of a linear system of high order to one of lower order (a process discussed in Appendix A) will often generate a reduced-order system that is semi-proper, due the approximation of fast system modes as direct input-output couplings. Conversion of a continuous-time model to a discrete-time form (or vice-versa) using the bilinear (Tustin) transform (Banks, 1986) will lead to nonzero terms in the **D** matrix.

5.2.1 Semi-Proper Systems for State Derivative Control

It should be noted that although the feedback of state derivatives is a common source of semi-proper systems, other methods exist for the assignment of eigenstructure to systems with state derivative feedback. In particular, given a strictly proper system under proportional-plus-derivative control, the closed-loop system equation may be written

$$(\mathbf{I} - \mathbf{B}\mathbf{K}_d\mathbf{C})\dot{\mathbf{x}} = (\mathbf{A} + \mathbf{B}\mathbf{K}\mathbf{C})\mathbf{x} + \mathbf{B}u \tag{5.1}$$

where $\mathbf{K}_d$ is the derivative feedback gain matrix and $\mathbf{K}$ the state feedback gain matrix. Equation 5.1 is in the form of a descriptor system, and as mentioned in Chapter 2, EA algorithms for PD control of descriptor systems exist (Duan and Patton, 1997; Duan and Wang, 2004; Duan and Patton, 1999; Owens and Askarpour, 2000).

However, if PD or state derivative control of a system in standard form is the aim, the algorithms to be presented here have several potential advantages over forming and controlling a descriptor system. The feedback gains from both states and state derivatives are contained in a single gain matrix, and found simultaneously; the available design freedom is presented in the same form as in standard state feedback (or output feedback) EA. Consequently the distribution of the design freedom is the same and this ensures that:

- The conditions for pole-assignability are simple to derive, and reflect those of a strictly proper system;
- The formation of a semi-proper system whose outputs are composed of an arbitrary combination of states and state derivatives is trivial, and this provides a highly flexible mechanism for controller design;
- The controller design process is substantially identical to that of standard state-feedback EA or of output feedback EA as proposed by Clarke et al. (2003), and consequently offers unprecedented levels of design visibility;
- Access to unused design freedom is available post-assignment, and may be used for retro-assignment (Clarke and Griffin, 2004) or gain suppression (Chapter 6). The best (simplest and most visible) of the current algorithms (Owens and Askarpour, 2000) considers only the minimisation of the Frobenius norm of the gain matrix.

5.2.1.1 Robustness Considerations

It is not the aim of this thesis to consider the robustness of control systems designed using EA. However, it is important to note that if partial or complete state derivative feedback is employed, the resulting open-loop **C** matrix will contain rows taken directly from the open-loop **A** matrix. Hence, if perturbations in the open-loop **A** matrix are to be considered for the purposes of calculating robustness measures, their effect on the open-loop **C** matrix must be borne in mind.

5.3 Problem Formulation

Consider a controllable, observable, minimal, semi-proper state-space system under the influence of feedback such that

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{t} \tag{5.2}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{t} \tag{5.3}$$

$$\mathbf{t} = \mathbf{K}\mathbf{y} + \mathbf{u} \tag{5.4}$$

where

$\mathbf{u}$ is an exogenous input, $\mathbf{t}$ is the plant input, $\mathbf{A} \in \mathbb{R}^{n \times n}$, $\mathbf{B} \in \mathbb{R}^{n \times r}$, $\mathbf{C} \in \mathbb{R}^{m \times n}$, $\mathbf{D} \in \mathbb{R}^{m \times r}$ and $\mathbf{K} \in \mathbb{R}^{r \times m}$.

By substitution we may readily obtain

$$\mathbf{t} = (\mathbf{I} - \mathbf{K}\mathbf{D})^{-1}\mathbf{u} + (\mathbf{I} - \mathbf{K}\mathbf{D})^{-1}\mathbf{K}\mathbf{C}\mathbf{x} \tag{5.5}$$

under the assumption that the term $\mathbf{I} - \mathbf{K}\mathbf{D}$ is nonsingular (the implications of this restriction are discussed in Section 5.3.2).

To simplify subsequent analysis, we define

$$\mathbf{N} \triangleq (\mathbf{I} - \mathbf{K}\mathbf{D})^{-1}\mathbf{K} \tag{5.6}$$

giving, after substitution:

$$\dot{\mathbf{x}} = (\mathbf{A} + \mathbf{BNC})\,\mathbf{x} + \mathbf{B}\,(\mathbf{I} - \mathbf{KD})^{-1}\,\mathbf{u} \tag{5.7}$$

$$\mathbf{y} = (\mathbf{C} + \mathbf{DNC})\,\mathbf{x} + \mathbf{D}\,(\mathbf{I} - \mathbf{KD})^{-1}\,\mathbf{u} \tag{5.8}$$

We may therefore define

$$\mathbf{A}_{cl} \triangleq \mathbf{A} + \mathbf{BNC} \tag{5.9}$$

$$\mathbf{B}_{cl} \triangleq \mathbf{B}\,(\mathbf{I} - \mathbf{KD})^{-1} \tag{5.10}$$

$$\mathbf{C}_{cl} \triangleq \mathbf{C} + \mathbf{DNC} \tag{5.11}$$

$$\mathbf{D}_{cl} \triangleq \mathbf{D}\,(\mathbf{I} - \mathbf{KD})^{-1} \tag{5.12}$$

The system structure derived above is accurate, but is not symmetrical in that the formulations of $\mathbf{B}_{cl}$ and $\mathbf{C}_{cl}$ appear quite different. The asymmetric nature of the structure does not appear to originate from any inherent asymmetry in the open-loop system or the feedback network. This observation forms the motivation for the following alternative formulation.

Consider again the system of Equations 5.2 to 5.4. By substitution once more, we may determine that

$$\mathbf{y} = (\mathbf{I} - \mathbf{DK})^{-1}\,\mathbf{Cx} + (\mathbf{I} - \mathbf{DK})^{-1}\,\mathbf{Du} \tag{5.13}$$

where, this time, it is assumed that the term $\mathbf{I} - \mathbf{DK}$ is nonsingular.

Substitution of Equation 5.13 into Equation 5.2 now gives:

$$\dot{\mathbf{x}} = \left(\mathbf{A} + \mathbf{BK}\,(\mathbf{I} - \mathbf{DK})^{-1}\,\mathbf{C}\right)\mathbf{x} + \left(\mathbf{B} + \mathbf{BK}\,(\mathbf{I} - \mathbf{DK})^{-1}\,\mathbf{D}\right)\mathbf{u} \tag{5.14}$$

Using the identity (Miller, 1987, 10),

$$\mathbf{X}\,(\mathbf{I} + \mathbf{YX})^{-1} = (\mathbf{I} + \mathbf{XY})^{-1}\,\mathbf{X} \tag{5.15}$$

Equation 5.14 becomes:

$$\dot{\mathbf{x}} = (\mathbf{A} + \mathbf{BNC})\,\mathbf{x} + (\mathbf{B} + \mathbf{BND})\,\mathbf{u} \tag{5.16}$$

The differences between Equations 5.13 and 5.16 and Equations 5.7 and 5.8 are due simply to a difference in the order of substitution. The following is a proof of their equivalence. From the formulation of Equation 5.16,

$$\begin{aligned}\mathbf{B}+\mathbf{BND} &= \mathbf{B}+\mathbf{B}\,(\mathbf{I}-\mathbf{KD})^{-1}\,\mathbf{KD} \\ &= \mathbf{B}\left(\mathbf{I}+(\mathbf{I}-\mathbf{KD})^{-1}\,\mathbf{KD}\right) \\ &= \mathbf{B}\Big((\mathbf{I}-\mathbf{KD})^{-1}\,(\mathbf{I}-\mathbf{KD})+ \\ &\qquad (\mathbf{I}-\mathbf{KD})^{-1}\,\mathbf{KD}\Big) \\ &= \mathbf{B}\,(\mathbf{I}-\mathbf{KD})^{-1}\,((\mathbf{I}-\mathbf{KD})+\mathbf{KD}) \\ &= \mathbf{B}\,(\mathbf{I}-\mathbf{KD})^{-1} \end{aligned} \tag{5.17}$$

which demonstrates the equivalence of Equations 5.7 and 5.16. This result could also be obtained using Woodbury's Formula (Miller, 1987, 11). The same logic may be applied to prove the equivalence of the Equations 5.8 and 5.13.

Consequently, we may define:

$$\mathbf{A}_{cl} \triangleq \mathbf{A}+\mathbf{BNC} \tag{5.18}$$

$$\mathbf{B}_{cl} \triangleq \mathbf{B}+\mathbf{BND} \equiv \mathbf{B}\,(\mathbf{I}-\mathbf{KD})^{-1} \tag{5.19}$$

$$\mathbf{C}_{cl} \triangleq \mathbf{C}+\mathbf{DNC} \equiv (\mathbf{I}-\mathbf{DK})^{-1}\,\mathbf{C} \tag{5.20}$$

$$\begin{aligned}\mathbf{D}_{cl} &\triangleq \mathbf{D}+\mathbf{DND} \equiv \mathbf{D}\,(\mathbf{I}-\mathbf{KD})^{-1} \\ &\equiv (\mathbf{I}-\mathbf{DK})^{-1}\,\mathbf{D}\end{aligned} \tag{5.21}$$

where

$$\dot{\mathbf{x}} = \mathbf{A}_{cl}\mathbf{x}+\mathbf{B}_{cl}\mathbf{u} \tag{5.22}$$

$$\mathbf{y} = \mathbf{C}_{cl}\mathbf{x}+\mathbf{D}_{cl}\mathbf{u} \tag{5.23}$$

It may be seen that the closed-loop matrices $\mathbf{A}_{cl}$, $\mathbf{B}_{cl}$, $\mathbf{C}_{cl}$, $\mathbf{D}_{cl}$ all differ from the open-loop matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ if $\mathbf{D} \neq \mathbf{0}$. Hence if the open-loop system is semi-proper, it is not only the system dynamics which change when the loop is closed but also the input-state, state-output and input-output couplings.

5.3.1 Closed Loop System Structure

In Fletcher's paper (Fletcher, 1981*a*), Equation 5.9 is stated (in expanded form) but it is not derived. The other three closed-loop matrices are not presented. However, the effect of loop closure on **B**, **C** and **D** is important.

Consider the case of right eigenvector assignment in order to control the coupling of modes into system outputs. The closed-loop output matrix $\mathbf{C}_{cl}$ (Equation 5.11) depends upon an inverse involving the gain matrix which, at the time of assignment, is unknown. Consequently the change from **C** to $\mathbf{C}_{cl}$, when the loop is closed, cannot be predicted. Therefore, the assignment of eigenvectors to determine mode-output coupling is not appropriate.

Whether or not the change in coupling between the states and outputs is of concern depends upon the nature of the assignment taking place. If it is necessary to ensure a specific coupling of modes into states, then assignment of eigenvectors is appropriate. If, instead, it is desired to control the appearance of the modes in the outputs, then techniques leading to the direct assignment of the output-coupling vectors are required.

A secondary benefit of assigning output coupling vectors *directly* is that the algorithm is immediately suitable for those systems in which the states themselves have no direct physical interpretation. Models derived using identification techniques are likely to fall into this category if the identification process can only approximate input-output relationships (Griffin, 1997).

5.3.2 Singularities in the Closed Loop System

Section 5.3 introduced the pre-conditions on the gain matrix that $\mathbf{I}-\mathbf{KD}$ and $\mathbf{I}-\mathbf{DK}$ must be nonsingular. Here, we consider the reasons for this, and its implications for control system design.

The constraint is not a curiosity of the exposition presented here. Rather, it represents a system singularity. The feedforward $\mathbf{D}_{cl}$ matrix and the feedback **K** matrix form direct forward and backward transmission paths, coupling the input and output through a pair of simultaneous equations. When the constraint is not satisfied, no instantaneous solution exists to these equations for **y** given **u** and **x**. Thus the constraint is somewhat pathological; it is reasonable to assume that a control system design approach based on meeting performance goals would never give rise to such a situation. Nevertheless, ensuring that this is the case is a simple matter in most cases.

Theorem 5.3.1. *If $m = n$, assigning a linearly independent set of output coupling vectors is a necessary and sufficient condition for ensuring the causality of the closed loop system.*

The following lemma assists with the proof.

Lemma 5.3.2. *The non-singularity of any one of the terms* $\mathbf{I} - \mathbf{KD}$, $\mathbf{I} - \mathbf{DK}$, $\mathbf{I} + \mathbf{ND}$ *and* $\mathbf{I} + \mathbf{DN}$ *(where* $\mathbf{N}$ *is defined by Equation 5.6) is necessary and sufficient to ensure the non-singularity of the remaining terms.*

Proof. Consider rearranging the defined structure of $\mathbf{N}$, given in Equation 5.6:

$$\mathbf{N} = (\mathbf{I} - \mathbf{KD})^{-1}\mathbf{K} \tag{5.24}$$

$$\mathbf{N} - \mathbf{KDN} = \mathbf{K} \tag{5.25}$$

$$\mathbf{N} = \mathbf{K} + \mathbf{KDN} \tag{5.26}$$

$$\mathbf{N}(\mathbf{I} + \mathbf{DN})^{-1} = \mathbf{K} \tag{5.27}$$

The transformation between $\mathbf{N}$ and $\mathbf{K}$ is bijective, and hence $\mathbf{I} - \mathbf{KD}$ is nonsingular if and only if $\mathbf{I} + \mathbf{DN}$ is nonsingular.

Finally, $\mathbf{I} + \mathbf{DN}$ is nonsingular if and only if $\mathbf{I} + \mathbf{ND}$ is nonsingular (see Miller, 1987, Corollary 3, p9); it is trivial to show that the same is true for $\mathbf{I} - \mathbf{KD}$ and $\mathbf{I} - \mathbf{DK}$. ◊◊

Proof of Theorem 5.3.1. From Equations 5.5 and 5.13, the causality of the closed-loop system is dependent upon the terms $\mathbf{I} - \mathbf{KD}$ and $\mathbf{I} - \mathbf{DK}$ being nonsingular. Lemma 5.3.2 gives a set of alternative necessary and sufficient conditions for this.

The term $\mathbf{I} + \mathbf{DN}$, implied in Equation 5.11 and listed in Lemma 5.3.2, is a factor that links the open-loop $\mathbf{C}$ and closed-loop $\mathbf{C}_{cl}$ matrices.

Therefore, given a set of mode-output coupling vectors

$$\Omega = \begin{bmatrix} o_1 & o_2 & \dots & o_n \end{bmatrix} \tag{5.28}$$

$$= \mathbf{C}_{cl} \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix} \tag{5.29}$$

$$= \mathbf{C}_{cl}\mathbf{V} \tag{5.30}$$

$$= (\mathbf{I} + \mathbf{DN})\mathbf{CV} \tag{5.31}$$

it is clear that

$$\operatorname{rank}(\Omega) = \operatorname{rank}((\mathbf{I} + \mathbf{DN})\mathbf{CV}) \tag{5.32}$$

and since by requirement $\text{rank}(\mathbf{C}) = n$ and $\text{rank}(\mathbf{V}) = n$,

$$\text{rank}(\Omega) = \min(n, \text{rank}(\mathbf{I} + \mathbf{DN})) \tag{5.33}$$

Since the term $\mathbf{I} + \mathbf{DN}$ is of size $m \times m$, if $m = n$ Equation 5.33 reduces to

$$\text{rank}(\Omega) = \text{rank}(\mathbf{I} + \mathbf{DN}) \tag{5.34}$$

Consequently the linear independence of the set of closed-loop output coupling vectors is necessary for term $\mathbf{I} - \mathbf{KD}$ to be full rank; if the number of system outputs is the same as the number of states, it is also sufficient. ◊◊

5.4 Pseudo-State Feedback

There follows an algorithm, developed by the author, that forms a simple extension to standard state-feedback EA. It appears in a simplified, less generic form in Pomfret and Clarke (2003), and in this form in Pomfret et al. (2005). The term 'pseudo-state feedback' is coined here to describe the application of output feedback to a controllable, observable system with the same number of independent outputs as states. *State* feedback implies that the states are measurable directly (ie. that $\mathbf{C} = \mathbf{I}$ and $\mathbf{D} = \mathbf{0}$), while *pseudo*-state feedback simply requires that $\text{rank}(\mathbf{C}) = m = n$ and otherwise carries no constraints beyond those of output feedback. Nevertheless it is not a misnomer, since the condition $\text{rank}(\mathbf{C}) = m = n$ allows for the placement of all the system poles by assigning only right-eigenvectors. This is the common characteristic of all state-feedback EA algorithms.

By definition, for any closed-loop eigenvalue-eigenvector pair $\{\lambda_i, \mathbf{v}_i\}$,

$$\mathbf{A}_{cl}\mathbf{v}_i = \mathbf{v}_i\lambda_i \tag{5.35}$$

and consequently

$$(\mathbf{A} + \mathbf{BNC})\,\mathbf{v}_i = \mathbf{v}_i\lambda_i \tag{5.36}$$

$$(\mathbf{A} - \lambda_i\mathbf{I})\,\mathbf{v}_i = \mathbf{BNCv}_i \tag{5.37}$$

$$\begin{bmatrix}\mathbf{A} - \lambda_i\mathbf{I} & \vdots & \mathbf{B}\end{bmatrix}\begin{bmatrix}\mathbf{v}_i \\ \mathbf{NCv}_i\end{bmatrix} = \mathbf{0} \tag{5.38}$$

Equation 5.38 may be parameterised by setting

$$\begin{bmatrix} \mathbf{v}_i \\ \mathbf{NCv}_i \end{bmatrix} = \begin{bmatrix} \mathbf{P}_i \\ \mathbf{Q}_i \end{bmatrix} \cdot \mathbf{f}_i \tag{5.39}$$

where

$$\text{range}\left(\begin{bmatrix} \mathbf{P}_i \\ \mathbf{Q}_i \end{bmatrix}\right) = \ker\left(\begin{bmatrix} \mathbf{A} - \lambda_i \mathbf{I} & \vdots & \mathbf{B} \end{bmatrix}\right) \tag{5.40}$$

The output-coupling vector $\mathbf{o}_i$ describes the distribution of a given mode into the outputs:

$$\mathbf{o}_i = \mathbf{C}_{cl}\mathbf{v}_i \tag{5.41}$$

$$= (\mathbf{C} + \mathbf{DNC})\,\mathbf{v}_i \tag{5.42}$$

$$= \begin{bmatrix} \mathbf{C} & \vdots & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{v}_i \\ \mathbf{NCv}_i \end{bmatrix} \tag{5.43}$$

$$\mathbf{o}_i = \begin{bmatrix} \mathbf{C} & \vdots & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{P}_i \\ \mathbf{Q}_i \end{bmatrix} \cdot \mathbf{f}_i \tag{5.44}$$

Consequently, the design vector, $\mathbf{f}_i$, for a given mode may used to select either an output-coupling vector $\mathbf{o}_i$, or an eigenvector $\mathbf{v}_i$ using, for example, a least-squares projection of a desired vector into the allowable subspace. Note that, since $\mathbf{C}_{cl}$ is square and also guaranteed to be full-rank (see Section 5.3.2), the condition that the closed-loop eigenvectors must be linearly independent can be satisfied by ensuring instead that the selected output-coupling vectors are linearly independent.

Having selected the design vectors $\{\mathbf{f}_i\}$, the matrix $\mathbf{N}$ may be recovered:

$$\mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix} \tag{5.45}$$

$$= \begin{bmatrix} \mathbf{P}_1\mathbf{f}_1 & \mathbf{P}_2\mathbf{f}_2 & \dots & \mathbf{P}_n\mathbf{f}_n \end{bmatrix} \tag{5.46}$$

$$\mathbf{S} = \mathbf{NCV} \tag{5.47}$$

$$= \begin{bmatrix} \mathbf{Q}_1\mathbf{f}_1 & \mathbf{Q}_2\mathbf{f}_2 & \dots & \mathbf{Q}_n\mathbf{f}_n \end{bmatrix} \tag{5.48}$$

$$\mathbf{N} = \mathbf{SV}^{-1}\mathbf{C}^{-1} \tag{5.49}$$

It only remains to rearrange $\mathbf{N}$ (defined at Equation 5.6) to find $\mathbf{K}$ (using Equation 5.27).

In Theorem 5.3.1 it was shown that a solution is guaranteed given a linearly independent set of output-coupling vectors. Consequently, assigning a linearly independent set of output vectors guarantees both a solution to Equation 5.49 *and* a solution to Equation 5.6.

5.4.1 Excess Freedom

If accelerometers, or other forms of derivative feedback, are used in order to increase the number of system outputs to the point where pseudo-state feedback is practical, it is also feasible that the number of linearly independent outputs may be made to exceed the number of states, ie. $m > n$. In this case, the gain matrix solution is not unique.

If the condition that rank($\mathbf{C}$) $= m = n$ is replaced by the new condition rank($\mathbf{C}$) $= n \leq m$, the procedure of Section 5.4 may be followed until Equation 5.48. At this point, the algorithm relies on the inversion of $\mathbf{C}$ and must therefore be modified.

From Equations 5.47 and 5.6, it is clear that

$$\mathbf{S} = \mathbf{NCV} \tag{5.50}$$

$$= (\mathbf{I} - \mathbf{KD})^{-1}\mathbf{KCV} \tag{5.51}$$

Simplifying,

$$(\mathbf{I} - \mathbf{KD})\,\mathbf{S} = \mathbf{KCV} \tag{5.52}$$

$$\mathbf{S} - \mathbf{KDS} = \mathbf{KCV} \tag{5.53}$$

$$\mathbf{S} = \mathbf{KCV} + \mathbf{KDS} \tag{5.54}$$

$$\mathbf{S} = \mathbf{K}\,(\mathbf{CV} + \mathbf{DS}) \tag{5.55}$$

where $\mathbf{C}$, $\mathbf{D}$, $\mathbf{S}$ and $\mathbf{V}$ are known. A direct solution for $\mathbf{K}$ by inversion from Equation 5.55 is not possible since by hypothesis the term $\mathbf{CV} + \mathbf{DS}$ is non-square. However, a solution for $\mathbf{K}$ can still be found (Ben-Israel and Greville, 1974, p39):

$$\mathbf{K} = \mathbf{S}\,(\mathbf{CV} + \mathbf{DS})^{\dagger} + \mathbf{Z}\left(\mathbf{I} - (\mathbf{CV} + \mathbf{DS})\,(\mathbf{CV} + \mathbf{DS})^{\dagger}\right) \tag{5.56}$$

where $\mathbf{A}^{\dagger}$ is the Moore-Penrose pseudo-inverse of $\mathbf{A}$ (see Appendix C) and $\mathbf{Z}$ is a matrix of free parameters. This parameter matrix is expressed in a similar way to the matrix of free parameters existing at the end of the output-feedback EA algorithm of Clarke et al. (2003).

Hence, for a system with $m > n$, it is possible, not only to assign n eigenvalues and right-eigenvectors, but also to recover the unused design freedom and employ it for another purpose, for example the imposition of controller structure (see Chapter 6).

Note that Theorem 5.3.1 no longer ensures that the resulting system is causal since $m \neq n$; however, there now exists the free parameter matrix $\mathbf{Z}$ which may be employed to avoid the pathological condition that the term $\mathbf{I} - \mathbf{KD}$ is not full-rank.

5.4.2 Design Procedure

A design procedure formalising the pseudo-state feedback EA process will now be described. It is assumed that the system $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ is controllable and observable, and that $\text{rank}(\mathbf{C}) = n$.

5.4.2.1 Choosing Eigenvectors

For each desired eigenvalue λ_i, $i = 1 \ldots n$, form the allowable subspace

$$\mathbf{0} = \begin{bmatrix} \mathbf{A} - \lambda_i \mathbf{I} & \mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{P}_i \\ \mathbf{Q}_i \end{bmatrix} \tag{5.57}$$

Choose a design vector $\mathbf{f}_i \in \mathbb{C}^{r \times 1}$ to select either the corresponding right eigenvector

$$\mathbf{v}_i = \mathbf{P}_i \mathbf{f}_i \tag{5.58}$$

or output coupling vector

$$\mathbf{o}_i = (\mathbf{C}\mathbf{P}_i + \mathbf{D}\mathbf{Q}_i)\,\mathbf{f}_i \tag{5.59}$$

Having selected $\{\mathbf{f}_i\}$, form the matrices

$$\mathbf{V} = [\mathbf{v}_1, \ldots, \mathbf{v}_n] = [\mathbf{P}_1 \mathbf{f}_1, \ldots, \mathbf{P}_n \mathbf{f}_n] \tag{5.60}$$

$$\mathbf{S} = [\mathbf{Q}_1 \mathbf{f}_1, \ldots, \mathbf{Q}_n \mathbf{f}_n] \tag{5.61}$$

$$\Omega = [\mathbf{o}_1, \ldots, \mathbf{o}_n] = \begin{bmatrix} \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{V} \\ \mathbf{S} \end{bmatrix} \tag{5.62}$$

and check that Ω is full rank, as required by Theorem 5.3.1.

5.4.2.2 Gain Matrix Recovery

The gain matrix formula of Equation 5.56 can be employed if there is an excess of outputs over states:

$$\mathbf{K} = \mathbf{S}(\mathbf{CV}+\mathbf{DS})^{\dagger} + \mathbf{Z}\left(\mathbf{I} - (\mathbf{CV}+\mathbf{DS})(\mathbf{CV}+\mathbf{DS})^{\dagger}\right) \tag{5.63}$$

If not, Equations 5.49 and 5.27 can be employed to recover first **N** and then **K**. Alternatively, Equation 5.56 can be modified to return **K** directly; if $m = n$, then the term $(\mathbf{CV}+\mathbf{DS})$ is square and full rank, so

$$(\mathbf{CV}+\mathbf{DS})^{\dagger} = (\mathbf{CV}+\mathbf{DS})^{-1} \tag{5.64}$$

Under these conditions, Equation 5.56 reduces to

$$\mathbf{K} = \mathbf{S}(\mathbf{CV}+\mathbf{DS})^{-1} \tag{5.65}$$

This concludes the design procedure.

5.5 Output Feedback

Due to the symmetrical form of the closed-loop system derived in Section 5.3, an output-feedback EA algorithm for systems with direct transmission terms is easily derived. The multi-stage EA algorithms of Clarke et al. (2003) aim to assign eigenvalues and eigenvectors to a closed-loop system matrix of the form

$$\mathbf{A}_{cl} = \mathbf{A} + \mathbf{BKC} \tag{5.66}$$

where the gain matrix, **K**, is free and the other matrices are fixed. From Equation 5.9, the closed-loop system matrix to which it is now desired to assign an eigenstructure is given by

$$\mathbf{A}_{cl} = \mathbf{A} + \mathbf{BNC} \tag{5.67}$$

Consequently it may be seen that the multi-stage EA algorithms may be used without modification to determine **N**, with Equation 5.27 used to recover **K**.

However, as shown by Equations 5.10 and 5.11, the coupling between the inputs and the states, and that between the states and the outputs, is subject to change when the loop is closed. The nature of the change is not known *a priori*, and if the system specification is in terms of, for example, output modal coupling, assignment of eigenvectors will not suffice. A technique for specifying modal coupling vectors is desirable.

In addition, applying the multi-stage algorithms directly would yield **N**, not **K** directly; although recovery of **K** via Equation 5.27 is trivial, the gain matrix equation is not returned in a form compatible with the imposition of controller structure (see Chapter 6).

5.5.1 Generalising Multi-Stage Eigenstructure Assignment

By definition, for any closed-loop eigenvalue-right eigenvector pair $\{\lambda_i, \mathbf{v}_i\}$ or eigenvalue-left eigenvector pair $\{\lambda_j, \mathbf{w}_j\}$,

$$\mathbf{A}_{cl}\mathbf{v}_i = \mathbf{v}_i\lambda_i \tag{5.68}$$

$$\mathbf{w}_j\mathbf{A}_{cl} = \mathbf{w}_j\lambda_j \tag{5.69}$$

and consequently

$$(\mathbf{A} + \mathbf{BNC})\,\mathbf{v}_i = \mathbf{v}_i\lambda_i \tag{5.70}$$

$$(\mathbf{A} - \lambda_i\mathbf{I})\,\mathbf{v}_i = \mathbf{BNC}\mathbf{v}_i \tag{5.71}$$

$$\begin{bmatrix}\mathbf{A} - \lambda_i\mathbf{I} & \vdots & \mathbf{B}\end{bmatrix}\begin{bmatrix}\mathbf{v}_i \\ \mathbf{NC}\mathbf{v}_i\end{bmatrix} = \mathbf{0} \tag{5.72}$$

and

$$\mathbf{w}_j\,(\mathbf{A} + \mathbf{BNC}) = \mathbf{w}_j\lambda_j \tag{5.73}$$

$$\mathbf{w}_j\,(\mathbf{A} - \lambda_j\mathbf{I}) = \mathbf{w}_j\mathbf{BNC} \tag{5.74}$$

$$\begin{bmatrix}\mathbf{w}_j & \vdots & \mathbf{w}_j\mathbf{BN}\end{bmatrix}\begin{bmatrix}\mathbf{A} - \lambda_j\mathbf{I} \\ \mathbf{C}\end{bmatrix} = \mathbf{0} \tag{5.75}$$

Equations 5.72 and 5.75 may be parameterised by setting

$$\begin{bmatrix}\mathbf{v}_i \\ \mathbf{NC}\mathbf{v}_i\end{bmatrix} = \begin{bmatrix}\mathbf{P}_i \\ \mathbf{Q}_i\end{bmatrix}\cdot\mathbf{f}_i \tag{5.76}$$

$$\begin{bmatrix}\mathbf{w}_j & \vdots & \mathbf{w}_j\mathbf{BN}\end{bmatrix} = \mathbf{g}_j\begin{bmatrix}\mathbf{L}_j & \vdots & \mathbf{M}_j\end{bmatrix} \tag{5.77}$$

where

$$\text{range}\left(\begin{bmatrix}\mathbf{P}_i\\ \mathbf{Q}_i\end{bmatrix}\right) = \ker\left(\begin{bmatrix}\mathbf{A}-\lambda_i\mathbf{I} & \vdots & \mathbf{B}\end{bmatrix}\right) \tag{5.78}$$

$$\text{range}\left(\begin{bmatrix}\mathbf{L}_j & \vdots & \mathbf{M}_j\end{bmatrix}^T\right) = \ker\left(\begin{bmatrix}\mathbf{A}-\lambda_j\mathbf{I}\\ \mathbf{C}\end{bmatrix}^T\right) \tag{5.79}$$

The right design vector, $\mathbf{f}_i$, or left design vector, $\mathbf{g}_j$, for a given mode may used to select an eigenvector $\mathbf{v}_i$ or $\mathbf{w}_j$ using, for example, a least-squares projection of a desired vector.

5.5.1.1 Modal Coupling Vector Assignment

The output-coupling vector $\mathbf{o}_i$, which describes the distribution of a given mode into the outputs, is given by Equation 5.44; consequently the right design vector, $\mathbf{f}_i$, for a given mode may used to select either an output-coupling vector $\mathbf{o}_i$ or an eigenvector $\mathbf{v}_i$. Similarly the input-coupling vector $\mathbf{i}_j$ describing the coupling of the inputs into a given mode is given by

$$\mathbf{i}_j = \mathbf{w}_j\mathbf{B}_{cl} \tag{5.80}$$

$$= \mathbf{w}_j\left(\mathbf{B}+\mathbf{BND}\right) \tag{5.81}$$

$$= \begin{bmatrix}\mathbf{w}_j & \vdots & \mathbf{w}_j\mathbf{BN}\end{bmatrix}\begin{bmatrix}\mathbf{B}\\ \mathbf{D}\end{bmatrix} \tag{5.82}$$

$$\mathbf{i}_j = \mathbf{g}_j \cdot \begin{bmatrix}\mathbf{L}_j & \vdots & \mathbf{M}_j\end{bmatrix}\begin{bmatrix}\mathbf{B}\\ \mathbf{D}\end{bmatrix} \tag{5.83}$$

and the left design vectors $\{\mathbf{g}_j\}$ may similarly be used for the selection of either eigenvectors or input-coupling vectors.

5.5.1.2 Gain Matrix Recovery

Once the right design vectors $\{\mathbf{f}_i\}$ and left design vectors $\{\mathbf{g}_j\}$ have been selected, where $i = 1 \ldots v$ and $j = v+1 \ldots n$, the following matrices may be computed:

$$\mathbf{V}' = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_v \end{bmatrix} = \begin{bmatrix} \mathbf{P}_1\mathbf{f}_1 & \mathbf{P}_2\mathbf{f}_2 & \ldots & \mathbf{P}_v\mathbf{f}_v \end{bmatrix} \tag{5.84}$$

$$\mathbf{S}' = \mathbf{NCV}' = \begin{bmatrix} \mathbf{Q}_1\mathbf{f}_1 & \mathbf{Q}_2\mathbf{f}_2 & \ldots & \mathbf{Q}_v\mathbf{f}_v \end{bmatrix} \tag{5.85}$$

$$\mathbf{W}' = \begin{bmatrix} \mathbf{w}_{v+1} \\ \mathbf{w}_{v+2} \\ \ldots \\ \mathbf{w}_n \end{bmatrix} = \begin{bmatrix} \mathbf{g}_{v+1}\mathbf{L}_{v+1} \\ \mathbf{g}_{v+2}\mathbf{L}_{v+2} \\ \ldots \\ \mathbf{g}_n\mathbf{L}_n \end{bmatrix} \tag{5.86}$$

$$\mathbf{T}' = \mathbf{W}'\mathbf{BN} = \begin{bmatrix} \mathbf{g}_{v+1}\mathbf{M}_{v+1} \\ \mathbf{g}_{v+2}\mathbf{M}_{v+2} \\ \ldots \\ \mathbf{g}_n\mathbf{M}_n \end{bmatrix} \tag{5.87}$$

Hence it remains to solve the equations

$$\mathbf{S}' = \mathbf{NCV}' \tag{5.88}$$

$$\mathbf{T}' = \mathbf{W}'\mathbf{BN} \tag{5.89}$$

to find $\mathbf{K}$.

From Equation 5.89,

$$\mathbf{T}' = \mathbf{W}'\mathbf{BN} \tag{5.90}$$

$$= \mathbf{W}'\mathbf{B}\left(\mathbf{I} - \mathbf{KD}\right)^{-1}\mathbf{K} \tag{5.91}$$

Invoking the identity of Equation 5.15,

$$\mathbf{T}' = \mathbf{W}'\mathbf{BK}\left(\mathbf{I} - \mathbf{KD}\right)^{-1} \tag{5.92}$$

$$\mathbf{T}' - \mathbf{T}'\mathbf{DK} = \mathbf{W}'\mathbf{BK} \tag{5.93}$$

$$\mathbf{T}' = \left(\mathbf{W}'\mathbf{B} + \mathbf{T}'\mathbf{D}\right)\mathbf{K} \tag{5.94}$$

From Equations 5.50 to 5.55,

$$\mathbf{S}' = \mathbf{K}\left(\mathbf{C}\mathbf{V}' + \mathbf{D}\mathbf{S}'\right) \tag{5.95}$$

A consistent solution to Equations 5.94 and 5.95 is now required.

The gain matrix may be obtained using the following lemma:

Lemma 5.5.1. (Clarke et al., 2003, Lemma 1) *Let* $\mathbf{K} \in \mathbb{C}^{r\times m}$, $\mathbf{X} \in \mathbb{C}^{m\times x}$, $\mathbf{S}_1 \in \mathbb{C}^{r\times x}$, $\mathbf{Y} \in \mathbb{C}^{y\times r}$ *and* $\mathbf{T}_1 \in \mathbb{C}^{y\times m}$, *where* $m \geq x$ *and* $r \geq y$. *Then the matrix equations*

$$\mathbf{K}\mathbf{X} = \mathbf{S}_1 \tag{5.96}$$

$$\mathbf{Y}\mathbf{K} = \mathbf{T}_2 \tag{5.97}$$

have a consistent solution for $\mathbf{K}$ *if all the following conditions hold:*

C1 $\text{rank}(\mathbf{X}) = x$

C2 $\text{rank}(\mathbf{Y}) = y$

C3 $\mathbf{T}_2\mathbf{X} = \mathbf{Y}\mathbf{S}_1$

The general solution for K *is then*

$$\mathbf{K} = \mathbf{Y}^{\dagger}\mathbf{T}_2 + \mathbf{S}_1\mathbf{X}^{\dagger} - \mathbf{Y}^{\dagger}\mathbf{Y}\mathbf{S}_1\mathbf{X}^{\dagger} + \left(\mathbf{I} - \mathbf{Y}^{\dagger}\mathbf{Y}\right)\mathbf{Z}\left(\mathbf{I} - \mathbf{X}\mathbf{X}^{\dagger}\right) \tag{5.98}$$

or equivalently

$$\mathbf{K} = \mathbf{Y}^{\dagger}\mathbf{T}_2 + \mathbf{S}_1\mathbf{X}^{\dagger} - \mathbf{Y}^{\dagger}\mathbf{T}_2\mathbf{X}\mathbf{X}^{\dagger} + \left(\mathbf{I} - \mathbf{Y}^{\dagger}\mathbf{Y}\right)\mathbf{Z}\left(\mathbf{I} - \mathbf{X}\mathbf{X}^{\dagger}\right) \tag{5.99}$$

Proof. See Clarke et al. (2003). ◊◊

The implications of the conditions in Lemma 5.5.1 are not immediately clear. They are discussed in detail by Clarke et al. (2003, Theorem 2) in the context of the strictly-proper EA algorithm presented therein, and the following theorem (and its proof) follow the above closely.

Theorem 5.5.2. *Given the controllable, observable state-space system* $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$, *the self-conjugate set* $\{\lambda_{d_i}\}$ *is pole assignable if there exists* $\mathbf{f}_i$ *and* $\mathbf{g}_j$ *such that*

C1 $\text{rank}\left(\begin{bmatrix}\mathbf{C} & \mathbf{D}\end{bmatrix}\begin{bmatrix}\mathbf{V}' \\ \mathbf{S}'\end{bmatrix}\right) = v$

C2 $\text{rank}\left(\begin{bmatrix}\mathbf{W}' & \mathbf{T}'\end{bmatrix}\begin{bmatrix}\mathbf{B} \\ \mathbf{D}\end{bmatrix}\right) = (n-v)$

C3 $\mathbf{w}_j\mathbf{v}_i = 0$ *for all* $i = 1 \ldots v$; $j = (v+1)\ldots n$

where

$$\mathbf{V}' = [\mathbf{v}_1, \ldots, \mathbf{v}_v] = [\mathbf{P}_1\mathbf{f}_1, \ldots, \mathbf{P}_v\mathbf{f}_v] \tag{5.100}$$

$$\mathbf{S}' = [\mathbf{s}_1, \ldots, \mathbf{s}_v] = [\mathbf{Q}_1\mathbf{f}_1, \ldots, \mathbf{Q}_v\mathbf{f}_v] \tag{5.101}$$

$$\mathbf{W}' = \begin{bmatrix}\mathbf{w}_{v+1} \\ \vdots \\ \mathbf{w}_n\end{bmatrix} = \begin{bmatrix}\mathbf{g}_{v+1}\mathbf{L}_{v+1} \\ \vdots \\ \mathbf{g}_n\mathbf{L}_n\end{bmatrix} \tag{5.102}$$

$$\mathbf{T}' = \begin{bmatrix}\mathbf{t}_{v+1} \\ \vdots \\ \mathbf{t}_n\end{bmatrix} = \begin{bmatrix}\mathbf{g}_{v+1}\mathbf{M}_{v+1} \\ \vdots \\ \mathbf{g}_n\mathbf{M}_n\end{bmatrix} \tag{5.103}$$

$$\ker\left(\begin{bmatrix}\mathbf{A} - \lambda_{d_i}\mathbf{I} & \mathbf{B}\end{bmatrix}\right) = \text{range}\left(\begin{bmatrix}\mathbf{P}_i \\ \mathbf{Q}_i\end{bmatrix}\right) \tag{5.104}$$

$$\ker\left(\begin{bmatrix}\mathbf{A} - \lambda_{d_j}\mathbf{I} \\ \mathbf{C}\end{bmatrix}^T\right) = \text{range}\left(\begin{bmatrix}\mathbf{L}_j & \mathbf{M}_j\end{bmatrix}^T\right) \tag{5.105}$$

Proof. Let $\mathbf{f}_i$ be chosen such that Condition C1 of Theorem 5.5.2 is satisfied. From Equation 5.104,

$$[(\mathbf{A} - \lambda_{d_1}\mathbf{I})\,\mathbf{P}_1\mathbf{f}_1, \ldots, (\mathbf{A} - \lambda_{d_v}\mathbf{I})\,\mathbf{P}_v\mathbf{f}_v] = -\,[\mathbf{B}\mathbf{Q}_1\mathbf{f}_1, \ldots, \mathbf{B}\mathbf{Q}_v\mathbf{f}_v] \tag{5.106}$$

$$\mathbf{V}'\mathbf{\Lambda}_{D_v} - \mathbf{A}\mathbf{V}' = \mathbf{B}\mathbf{S}' \tag{5.107}$$

where $\mathbf{\Lambda}_{D_v} = \text{diag}(\lambda_{d_1}, \ldots, \lambda_{d_v})$ is a diagonal matrix and the values along the main diagonal form a self-conjugate subset of the desired eigenvalues; $\mathbf{v}_i$ is an assigned right eigenvector, and $\mathbf{V}'$ is the concatenation of these eigenvectors.

Also let $\mathbf{g}_j$ be chosen such that Condition C2 of Theorem 5.5.2 is satisfied. From Equation 5.105,

$$\begin{bmatrix} \mathbf{g}_{v+1}\mathbf{L}_{v+1}\left(\mathbf{A}-\lambda_{d_{v+1}}\mathbf{I}\right) \\ \vdots \\ \mathbf{g}_n\mathbf{L}_n\left(\mathbf{A}-\lambda_{d_n}\mathbf{I}\right) \end{bmatrix} = -\begin{bmatrix} \mathbf{g}_{v+1}\mathbf{M}_{v+1}\mathbf{C} \\ \vdots \\ \mathbf{g}_n\mathbf{M}_n\mathbf{C} \end{bmatrix} \tag{5.108}$$

$$\mathbf{\Lambda}_{D_w}\mathbf{W}' - \mathbf{W}'\mathbf{A} = \mathbf{T}'\mathbf{C} \tag{5.109}$$

where $\mathbf{\Lambda}_{D_w} = \text{diag}(\lambda_{d_{v+1}}, \ldots, \lambda_{d_n})$ is a diagonal matrix and the values along the main diagonal are the remaining unassigned desired eigenvalues; $\mathbf{w}_i$ is an assigned left eigenvector, and $\mathbf{W}'$ is the concatenation of these eigenvectors.

A solution for $\mathbf{K}$ now requires a consistent solution to the following two equations:

$$\mathbf{K}\left(\mathbf{C}\mathbf{V}' + \mathbf{D}\mathbf{S}'\right) = \mathbf{S}' \tag{5.110}$$

$$\left(\mathbf{W}'\mathbf{B} + \mathbf{T}'\mathbf{D}\right)\mathbf{K} = \mathbf{T}' \tag{5.111}$$

Applying conditions C1 and C2 of Lemma 5.5.1 to the above equations implies that

$$\text{rank}\left(\mathbf{C}\mathbf{V}' + \mathbf{D}\mathbf{S}'\right) = v \tag{5.112}$$

$$\text{rank}\left(\mathbf{W}'\mathbf{B} + \mathbf{T}'\mathbf{D}\right) = (n - v) \tag{5.113}$$

which is a restatement of conditions C1 and C2 of Theorem 5.5.2 and therefore guaranteed to be satisfied.

Finally, condition C3 of Lemma 5.5.1 requires that

$$\mathbf{T}'\left(\mathbf{C}\mathbf{V}' + \mathbf{D}\mathbf{S}'\right) = \left(\mathbf{W}'\mathbf{B} + \mathbf{T}'\mathbf{D}\right)\mathbf{S}' \tag{5.114}$$

$$\mathbf{T}'\mathbf{C}\mathbf{V}' + \mathbf{T}'\mathbf{D}\mathbf{S}' = \mathbf{W}'\mathbf{B}\mathbf{S}' + \mathbf{T}'\mathbf{D}\mathbf{S}' \tag{5.115}$$

$$\mathbf{T}'\mathbf{C}\mathbf{V}' = \mathbf{W}'\mathbf{B}\mathbf{S}' \tag{5.116}$$

Substituting Equations 5.107 and 5.109,

$$\mathbf{\Lambda}_{D_w}\mathbf{W}'\mathbf{V}' - \mathbf{W}'\mathbf{A}\mathbf{V}' = \mathbf{W}'\mathbf{V}'\mathbf{\Lambda}_{D_v} - \mathbf{W}'\mathbf{A}\mathbf{V}' \tag{5.117}$$

$$\mathbf{\Lambda}_{D_w}\mathbf{W}'\mathbf{V}' = \mathbf{W}'\mathbf{V}'\mathbf{\Lambda}_{D_v} \tag{5.118}$$

For general $\mathbf{\Lambda}_{D_v}$ and $\mathbf{\Lambda}_{D_w}$, the only solution to Equation 5.118 is the trivial case $\mathbf{W}'\mathbf{V}' = \mathbf{0}$, or equivalently

$$\mathbf{w}_j\mathbf{v}_i = 0 \text{ for all } i = 1\ldots v,\ j = (v+1)\ldots n \tag{5.119}$$

as given by condition C3 of Theorem 5.5.2. ◊◊

Condition C3 of Theorem 5.5.2 is the same reduced orthogonality condition given by Clarke et al. (2003), and is easily satisfied by ensuring that any selected eigenvector is orthogonal to the set of complimentary eigenvectors already chosen.

Conditions C1 and C2, however, are somewhat different. In the strictly proper case, they serve to restrict the selected sets of eigenvectors to be linearly independent. A similar restriction is imposed by these revised, generalised conditions. Comparison of conditions C1 and C2 of Theorem 5.5.2 with Equations 5.44 and 5.83 reveals that they may be written as

$$\text{rank}\left([\mathbf{o}_1, \ldots, \mathbf{o}_v]\right) = v \tag{5.120}$$

$$\text{rank}\left(\begin{bmatrix}\mathbf{i}_{v+1}\\ \vdots \\ \mathbf{i}_n\end{bmatrix}\right) = (n-v) \tag{5.121}$$

Hence the linear independence of the selected sets of modal coupling vectors is a necessary and sufficient condition for the satisfaction of conditions C1 and C2 of Theorem 5.5.2.

Theorem 5.3.1 showed that recovery of the gain matrix is possible only if the selected output coupling vectors are linearly independent. However this is a *sufficient* condition for the recovery of $\mathbf{K}$ only in the case where $m = n$. This theorem will therefore not suffice in the output feedback case. The following theorem is a generalisation of Theorem 5.3.1 which renders it applicable to the output feedback case.

Theorem 5.5.3. *If $m = v$ or $r = n - v$, the closed-loop system is guaranteed to be causal.*

Proof. Lemma 5.3.2 showed that the causality of the closed loop system and the recovery of a gain matrix is guaranteed if either $\mathbf{I} + \mathbf{DN}$ or $\mathbf{I} + \mathbf{ND}$ can be shown to be nonsingular.

Let us assume that all n eigenvalues have been assigned, together with v right eigenvectors and $(n-v)$ left eigenvectors. Consider a matrix formed from the set of assigned output-coupling vectors:

$$\Omega' = \begin{bmatrix}\mathbf{o}_1 & \mathbf{o}_2 & \ldots & \mathbf{o}_v\end{bmatrix} \tag{5.122}$$

It may be expressed in terms of the set of implicitly assigned right eigenvectors

$$\mathbf{V}' = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_v \end{bmatrix} \tag{5.123}$$

as

$$\Omega' = \mathbf{C}_{cl}\mathbf{V}' \tag{5.124}$$

$$= (\mathbf{I}+\mathbf{DN})\,\mathbf{CV}' \tag{5.125}$$

If Ω' is full-rank, this implies not only that $\mathbf{V}'$ is full-rank but that $\text{rank}\,(\mathbf{I}+\mathbf{DN}) \geq v$. Note that this does not, in itself, guarantee that $(\mathbf{I}+\mathbf{DN})$ is non-singular.

A dual argument can be used to show that if the selected input-coupling vectors are linearly independent, the implication is that $\mathbf{W}'$ is full rank and that $\text{rank}\,(\mathbf{I}+\mathbf{ND}) \geq n-v$.

If now either $m=v$ or $r=n-v$, it is clear that the system is causal. ◊◊

A sufficient condition for the existence of a gain matrix in ALL circumstances has not yet been found. However, as stated, the situation that $\mathbf{K}$ cannot be recovered from $\mathbf{N}$ is a pathological one and very unlikely to arise given a well-posed problem and a kinematically consistent set of requirements. Additionally, if $m > v$ and $r > n-v$, there will exist a nonzero mapping of the free parameter matrix $\mathbf{Z}$ onto $\mathbf{K}$ through Equation 5.98 and hence there is yet more opportunity to avoid the pathological case that $\mathbf{I}-\mathbf{KD}$ is singular.

5.5.2 Design Procedure

The design procedure is very similar to that suggested by Clarke et al. (2003). First, a set of s_1 left or right eigenvectors or modal coupling vectors are selected from their allowable subspaces, and are chosen to meet condition C1 or C2 of Theorem 5.5.2. Next, a set of s_2 dual eigenvectors or modal coupling vectors are selected; these must also be selected from their allowable subspaces and chosen to meet condition C1 or C2 of Theorem 5.5.2, and are subject to further restrictions via condition C3. Finally the gain matrix is calculated using Lemma 5.5.1. In order to assign all n system poles, it is necessary that $s_1 + s_2 = n$.

The decision on whether to assign right or left eigenvectors in stage one depends upon the design requirements, since those eigenvectors assigned in stage two are subject to more stringent restrictions than those assigned in stage one. On occasion there may only be one suitable choice for stage one (Clarke et al., 2003).

The design procedure will now be described in detail, assuming that the condition $m+r > n$ is satisfied for the system under consideration.

5.5.2.1 Stage One

If right eigenvectors or output coupling vectors are to be assigned in stage one, first calculate the allowable eigenvector subspace

$$\mathbf{0} = \begin{bmatrix} \mathbf{A} - \lambda_i \mathbf{I} & \mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{P}_i \\ \mathbf{Q}_i \end{bmatrix} \tag{5.126}$$

for all $i = 1 \dots s_1$. Using design vectors $\mathbf{f}_i \in \mathbb{C}^{r \times 1}$, select eigenvectors

$$\mathbf{v}_i = \mathbf{P}_i \mathbf{f}_i \tag{5.127}$$

or output coupling vectors

$$\mathbf{o}_i = (\mathbf{C}\mathbf{P}_i + \mathbf{D}\mathbf{Q}_i)\mathbf{f}_i \tag{5.128}$$

Form the matrices

$$\mathbf{V}' = [\mathbf{P}_1 \mathbf{f}_1, \dots, \mathbf{P}_{s_1} \mathbf{f}_{s_1}] \tag{5.129}$$

$$\mathbf{S}' = [\mathbf{Q}_1 \mathbf{f}_1, \dots, \mathbf{Q}_{s_1} \mathbf{f}_{s_1}] \tag{5.130}$$

$$\Omega' = \mathbf{C}\mathbf{V}' + \mathbf{D}\mathbf{S}' \tag{5.131}$$

Check that the matrix of assigned output coupling vectors Ω' is full rank as required by condition C1 of Theorem 5.5.2.

If instead left eigenvectors or input coupling vectors are to be assigned in stage one, calculate the allowable eigenvector subspace

$$\mathbf{0} = \begin{bmatrix} \mathbf{L}_j & \mathbf{M}_j \end{bmatrix} \begin{bmatrix} \mathbf{A} - \lambda_j \mathbf{I} \\ \mathbf{C} \end{bmatrix} \tag{5.132}$$

for all $j = 1 \dots s_1$. Using design vectors $\mathbf{g}_j \in \mathbb{C}^{1 \times m}$, select eigenvectors

$$\mathbf{w}_j = \mathbf{g}_j \mathbf{L}_j \tag{5.133}$$

or input coupling vectors

$$\mathbf{i}_j = \mathbf{g}_j(\mathbf{L}_j\mathbf{B} + \mathbf{M}_j\mathbf{D}) \tag{5.134}$$

Form the matrices

$$\mathbf{W}' = \begin{bmatrix} \mathbf{g}_1\mathbf{L}_1 \\ \vdots \\ \mathbf{g}_{s_1}\mathbf{L}_{s_1} \end{bmatrix} \tag{5.135}$$

$$\mathbf{T}' = \begin{bmatrix} \mathbf{g}_1\mathbf{M}_1 \\ \vdots \\ \mathbf{g}_{s_1}\mathbf{M}_{s_1} \end{bmatrix} \tag{5.136}$$

$$\Upsilon' = \mathbf{W}'\mathbf{B} + \mathbf{T}'\mathbf{D} \tag{5.137}$$

Check that the matrix of assigned input coupling vectors Υ' is full rank as required by condition C2 of Theorem 5.5.2.

5.5.2.2 Stage Two

In order to meet the orthogonality condition (C3 of Theorem 5.5.2), the allowable subspace equations in stage two must be augmented as

$$\mathbf{0} = \begin{bmatrix} \mathbf{L}_j & \mathbf{M}_j \end{bmatrix} \begin{bmatrix} \mathbf{V}' & \mathbf{A} - \lambda_j\mathbf{I} \\ \mathbf{0} & \mathbf{C} \end{bmatrix} \tag{5.138}$$

if right eigenvectors were assigned in stage one, or as

$$\mathbf{0} = \begin{bmatrix} \mathbf{A} - \lambda_i\mathbf{I} & \mathbf{B} \\ \mathbf{W}' & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{P}_i \\ \mathbf{Q}_i \end{bmatrix} \tag{5.139}$$

if left eigenvectors were assigned in stage one.

Now use design vectors to select eigenvectors or modal coupling vectors and form the matrices given in Equations 5.129 to 5.131 or Equations 5.135 to 5.137, as appropriate. Check once again that the matrix of modal coupling vectors assigned in stage two, Ω' or Υ', is full rank as required by condition C1 or C2 of Theorem 5.5.2.

5.5.2.3 Gain Matrix Calculation

Finally, calculate the gain matrix using the formula

$$\mathbf{K} = (\Upsilon')^{\dagger}\mathbf{T}' + \mathbf{S}'(\Omega')^{\dagger} - (\Upsilon')^{\dagger}\Upsilon'\mathbf{S}'(\Omega')^{\dagger} + \left(\mathbf{I} - (\Upsilon')^{\dagger}\Upsilon'\right)\mathbf{Z}\left(\mathbf{I} - \Omega'(\Omega')^{\dagger}\right) \tag{5.140}$$

where **Z** is a matrix of free parameters, which may be set to zero if not required.

This completes the assignment process.

5.5.2.4 A Note on Unused Freedom

The gain matrix solution offered by Equation 5.140 yields any unused design freedom in a form that is directly compatible with the gain suppression algorithm to be given in Chapter 6. However, this form is not suitable for every application.

Clarke and Griffin (2004) present an algorithm for 'Retro-Assignment' in which unused design freedom may be employed for assigning complimentary eigenvectors to those assigned initially. If this is desirable, the following approach should be taken.

By using Lemma 5.5.1, a solution for **N** to Equations 5.88 and 5.89 may be found as

$$\begin{aligned}\mathbf{N} = &(\mathbf{W}'\mathbf{B})^{\dagger}\mathbf{T}' + \mathbf{S}'(\mathbf{C}\mathbf{V}')^{\dagger} - (\mathbf{W}'\mathbf{B})^{\dagger}(\mathbf{W}'\mathbf{B})\mathbf{S}'(\mathbf{C}\mathbf{V}')^{\dagger}\\ &+ \left(\mathbf{I} - (\mathbf{W}'\mathbf{B})^{\dagger}(\mathbf{W}'\mathbf{B})\right)\mathbf{Z}\left(\mathbf{I} - (\mathbf{C}\mathbf{V}')(\mathbf{C}\mathbf{V}')^{\dagger}\right)\end{aligned} \tag{5.141}$$

Following the method of Clarke and Griffin (2004), if now two full-rank matrices $\mathbf{H_{CV}}$ and $\mathbf{H_{WB}}$ are found such that

$$\text{range}\left(\mathbf{H_{CV}}^{T}\right) = \ker\left((\mathbf{C}\mathbf{V}')^{T}\right) \tag{5.142}$$

$$\text{range}\left(\mathbf{H_{WB}}\right) = \ker\left(\mathbf{W}'\mathbf{B}\right) \tag{5.143}$$

then Equation 5.141 may be written as

$$\mathbf{N} = \mathbf{N}_0 + \mathbf{H_{WB}}\mathbf{Z}\mathbf{H_{CV}} \tag{5.144}$$

where

$$\mathbf{N}_0 = (\mathbf{W}'\mathbf{B})^{\dagger}\mathbf{T}' + \mathbf{S}'(\mathbf{C}\mathbf{V}')^{\dagger} - (\mathbf{W}'\mathbf{B})^{\dagger}(\mathbf{W}'\mathbf{B})\mathbf{S}'(\mathbf{C}\mathbf{V}')^{\dagger} \tag{5.145}$$

It is now possible to write the closed-loop system matrix, from Equation 5.18, as

$$\mathbf{A}_{cl} = \mathbf{A} + \mathbf{BNC} \tag{5.146}$$

$$= (\mathbf{A} + \mathbf{BN_0C}) + (\mathbf{BH_{WB}})\,\mathbf{Z}\,(\mathbf{H_{CV}C}) \tag{5.147}$$

Retro-assignment may now be performed on the reduced, strictly proper system

$$\bar{\mathbf{A}} = \mathbf{A} + \mathbf{BN_0C} \tag{5.148}$$

$$\bar{\mathbf{B}} = \mathbf{BH_{WB}} \tag{5.149}$$

$$\bar{\mathbf{C}} = \mathbf{H_{CV}C} \tag{5.150}$$

using the algorithms presented by Clarke and Griffin (2004). It is interesting to note that the reduced system is strictly proper, and so the original algorithms may be used without modification. The allowable right eigenvector subspace in the retro stage can be parameterised as

$$\begin{bmatrix} \mathbf{v}_i \\ \mathbf{Z}\bar{\mathbf{C}}\mathbf{v}_i \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{\mathbf{N}i} \\ \mathbf{Q}_{\mathbf{N}i} \end{bmatrix} \cdot \mathbf{f}_{\mathbf{N}i} \tag{5.151}$$

and the allowable left eigenvector subspace as

$$\begin{bmatrix} \mathbf{w}_j & \vdots & \mathbf{w}_j\bar{\mathbf{B}}\mathbf{Z} \end{bmatrix} = \mathbf{g}_{\mathbf{N}j} \begin{bmatrix} \mathbf{L}_{\mathbf{N}j} & \vdots & \mathbf{M}_{\mathbf{N}j} \end{bmatrix} \tag{5.152}$$

where

$$\text{range}\left(\begin{bmatrix} \mathbf{P}_{\mathbf{N}i} \\ \mathbf{Q}_{\mathbf{N}i} \end{bmatrix}\right) = \ker\left(\begin{bmatrix} \bar{\mathbf{A}} - \lambda_i\mathbf{I} & \vdots & \bar{\mathbf{B}} \end{bmatrix}\right) \tag{5.153}$$

$$\text{range}\left(\begin{bmatrix} \mathbf{L}_{\mathbf{N}j} & \vdots & \mathbf{M}_{\mathbf{N}j} \end{bmatrix}^T\right) = \ker\left(\begin{bmatrix} \bar{\mathbf{A}} - \lambda_j\mathbf{I} \\ \bar{\mathbf{C}} \end{bmatrix}^T\right) \tag{5.154}$$

The final gain matrix **K** must, of course, be recovered from **N** post-assignment through Equation 5.27.

Although the reduced system is strictly proper, it must be remembered that the system to which eigenstructure is being assigned for the purpose of control is not. This is important if modal coupling vectors have been assigned during the main EA process, since it is likely that the assignment of further modal coupling vectors will be the aim of any retro-assignment.

Assignment of modal coupling vectors to the reduced system may be achieved as follows.

Recall from Equations 5.19 and 5.20 that the closed-loop modal coupling matrices may be written

$$\mathbf{B}_{cl} = \mathbf{B} + \mathbf{BND} \tag{5.155}$$
$$= \mathbf{B} + \mathbf{BN}_0\mathbf{D} + \mathbf{BH}_{\mathbf{WB}}\mathbf{ZH}_{\mathbf{CV}}\mathbf{D} \tag{5.156}$$
$$= \mathbf{B} + \mathbf{BN}_0\mathbf{D} + \bar{\mathbf{B}}\mathbf{H}_{\mathbf{CV}}\mathbf{D} \tag{5.157}$$

and

$$\mathbf{C}_{cl} = \mathbf{C} + \mathbf{DNC} \tag{5.158}$$
$$= \mathbf{C} + \mathbf{DN}_0\mathbf{C} + \mathbf{DH}_{\mathbf{WB}}\mathbf{ZH}_{\mathbf{CV}}\mathbf{C} \tag{5.159}$$
$$= \mathbf{C} + \mathbf{DN}_0\mathbf{C} + \mathbf{DH}_{\mathbf{WB}}\bar{\mathbf{C}} \tag{5.160}$$

Hence the output coupling vector $\mathbf{o}_i$ for a given eigenvalue λ_i is given in terms of its associated eigenvector $\mathbf{v}_i$ by

$$\mathbf{o}_i = \mathbf{C}_{cl}\mathbf{v}_i \tag{5.161}$$
$$= \mathbf{Cv}_i + \mathbf{DN}_0\mathbf{Cv}_i + \mathbf{DH}_{\mathbf{WB}}\bar{\mathbf{C}}\mathbf{v}_i \tag{5.162}$$
$$= \begin{bmatrix} \mathbf{C} + \mathbf{DN}_0\mathbf{C} & \vdots & \mathbf{DH}_{\mathbf{WB}} \end{bmatrix} \begin{bmatrix} \mathbf{v}_i \\ \bar{\mathbf{C}}\mathbf{v}_i \end{bmatrix} \tag{5.163}$$
$$= \begin{bmatrix} \mathbf{C} + \mathbf{DN}_0\mathbf{C} & \vdots & \mathbf{DH}_{\mathbf{WB}} \end{bmatrix} \begin{bmatrix} \mathbf{P}_{\mathbf{N}i} \\ \mathbf{Q}_{\mathbf{N}i} \end{bmatrix} \cdot \mathbf{f}_{\mathbf{N}i} \tag{5.164}$$

and the input coupling vector $\mathbf{i}_j$ in terms of its associated eigenvector $\mathbf{w}_j$ as

$$\mathbf{i}_j = \mathbf{w}_j\mathbf{B}_{cl} \tag{5.165}$$
$$= \mathbf{w}_j\mathbf{B} + \mathbf{w}_j\mathbf{BN}_0\mathbf{D} + \mathbf{w}_j\bar{\mathbf{B}}\mathbf{H}_{\mathbf{CV}}\mathbf{D} \tag{5.166}$$
$$= \begin{bmatrix} \mathbf{w}_j & \vdots & \mathbf{w}_j\bar{\mathbf{B}} \end{bmatrix} \begin{bmatrix} \mathbf{B} + \mathbf{BN}_0\mathbf{D} \\ \mathbf{H}_{\mathbf{CV}}\mathbf{D} \end{bmatrix} \tag{5.167}$$
$$= \mathbf{g}_{\mathbf{N}j} \begin{bmatrix} \mathbf{L}_{\mathbf{N}j} & \vdots & \mathbf{M}_{\mathbf{N}j} \end{bmatrix} \begin{bmatrix} \mathbf{B} + \mathbf{BN}_0\mathbf{D} \\ \mathbf{H}_{\mathbf{CV}}\mathbf{D} \end{bmatrix} \tag{5.168}$$

Hence once again the design vectors may be chosen to select either eigenvectors or modal coupling vectors of the closed-loop system.

5.6 Conclusions

Following a detailed analysis of the problem, two novel EA algorithms have been presented which are capable of operating upon semi-proper systems. Due attention has been paid to the changes in input-output coupling that occur when the loop is closed around a semi-proper system, and simple modal coupling vector assignment techniques have been developed which require no *a priori* knowledge of these changes. The output feedback assignment technique presented here forms a natural extension to the multi-stage algorithms of Clarke et al. (2003), and is consequently more visible and allows for faster design iterations than the techniques available thus far.

Straightforward necessary and sufficient conditions have been developed for the construction of a gain matrix in both cases.

Finally, the pseudo-state feedback algorithm has been modified to allow for the situation where there are more outputs than states and to encapsulate the excess design freedom in a usable form.

5.7 Chapter Bibliography

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Chapter 6

Imposition of Controller Structure

Contents

6.1 Introduction

The recent work of Clarke et al. (2003) detailed new algorithms for output-feedback eigenstructure assignment control for linear systems. The algorithms employ a multi-stage approach, in which the available design freedom is reduced stepwise by the assignment of eigenvectors and associated eigenvalues. Depending on the number of Degrees of Freedom (DoF) available and the manner in which they are distributed between the design stages, it is possible that some may remain unused after the assignment is complete. In other published work, Clarke and Griffin (2004) introduce an algorithm (the 'retro-assignment stage') that makes use of this post-assignment freedom to assign complementary eigenvectors to those assigned using the original algorithm.

It is likely, however, that further eigenvector assignment is not the most appropriate use for this design freedom. Typically, only a few right eigenvectors (corresponding to dominant modes) are crucial to the system specification, but the formation of the general nonlinear eigenstructure assignment problem into a problem with a linear solution requires that one eigenvector is assigned for every eigenvalue. The control of modal coupling is therefore likely to have been satisfied by the primary assignment algorithm and the design freedom could, instead, be employed to achieve some other objective.

One such objective is a defined controller structure. Eigenstructure Assignment (EA), in common with most multi-input multi-output control system design techniques, will in general generate a fully-populated matrix of feedback gains. The resulting complex, fully-interconnected controller bears little resemblance to the sparse, modular control systems achieved using classical approaches. In order to impose structure upon a controller, it is necessary to reduce a subset of the gains to zero, thereby reducing the complexity of the connections from plant outputs to plant inputs. This chapter presents a novel method by which, after eigenstructure assignment is completed, any remaining design freedom may be used for this purpose. It has previously appeared in published work (Pomfret and Clarke, 2005).

To the author's knowledge, no similar method has been developed before, although Sobel and Shapiro (1986) use a similar mathematical approach to the problem of assigning eigenstructure to a system with structured gains. However, their method constitutes part of the assignment process itself, and also potentially changes the assigned locations of poles.

Griffin (1997) develops a method whereby, using an iterative approach, gain matrix entries may be suppressed without affecting the closed-loop eigenvalues. The eigenvectors are subject to change, however, since his technique does not employ any design freedom in its operation. Ensor (2000) presents an algorithm which does not attempt to constrain either the eigenvalues or eigenvectors, but rather allows the effects of gain suppression on the eigenstructure to be seen clearly.

6.2 Problem Definition

The algorithms described by Clarke et al. (2003), and the output-feedback algorithms described in Chapter 5, yield a gain matrix $\mathbf{K}$ which is dependant upon a matrix of free parameters $\mathbf{Z}$. The matrix $\mathbf{Z}$ may be chosen arbitrarily, and any changes exhibited by the gain matrix $\mathbf{K}$ as a result will not affect the eigenvalues or assigned eigenvectors of the closed loop system. The gain matrix equation takes the following form (see Equations 2.149 and 5.140):

$$\mathbf{K} = \mathbf{K}_0 + (\mathbf{I} - \mathbf{X}^\dagger\mathbf{X})\mathbf{Z}(\mathbf{I} - \mathbf{Y}\mathbf{Y}^\dagger) \tag{6.1}$$

where $\mathbf{X} \in \mathbb{C}^{w \times r}$, $\mathbf{Y} \in \mathbb{C}^{m \times v}$, and $\mathbf{K}$, $\mathbf{K}_0$, $\mathbf{Z} \in \mathbb{R}^{r \times m}$. $\mathbf{A}^\dagger$ is the Moore-Penrose pseudo-inverse of $\mathbf{A}$ (see Appendix C).

The mapping of $\mathbf{Z}$ to $\mathbf{K}$ through Equation 6.1 is not bijective, so a multiplicity of values for $\mathbf{Z}$ can yield the same $\mathbf{K}$. Clarke and Griffin (2004) show that the number of DoF available at this stage is given by

$$f = (m - v)(r - w) \tag{6.2}$$

and that, if $f = 0$, the term involving $\mathbf{Z}$ in Equation 6.1 will evaluate to zero.

In order to reduce an arbitrary set of gain matrix entries to zero, it is necessary to find a simple mathematical representation of the desired constraints. This can be achieved by using a permutation matrix $\mathbf{U}_{\delta \times mr}$, which possesses exactly one unity element per row and is zero elsewhere, to select individual entries from a vector version of $\mathbf{K}$:

$$\mathbf{U}\,\mathrm{vec}\,\mathbf{K} = \mathbf{0} \tag{6.3}$$

The parameter δ, given in the definition of the permutation matrix $\mathbf{U}$, is equal to the number of gain matrix entries that are to be suppressed.

6.2.1 Extension to Pseudo-State Feedback

Section 5.4.1 in Chapter 5 introduced an algorithm for EA control of a semi-proper system with more independent outputs than states. It was shown that unused design freedom exists post-assignment, and this freedom was characterised by a gain matrix equation (Equation 5.56) of the form

$$\mathbf{K} = \mathbf{K}_0 + \mathbf{Z}(\mathbf{I} - \mathbf{Y}\mathbf{Y}^\dagger) \tag{6.4}$$

which is identical in form to Equation 6.1 but with $\mathbf{X} = \mathbf{0}_r$. Hence Equations 6.1 and 6.3 define the problem fully for the pseudo-state feedback case just as they do for the output feedback case.

6.3 Solution for Z and K

Theorem 6.3.1. *There exists a consistent solution to Equations 6.1 and 6.3 if and only if*

$$\mathbf{U}\,\mathrm{vec}\,\mathbf{K}_0 \in \mathrm{range}\,(\mathbf{U}\Xi) \tag{6.5}$$

The general solution for $\mathbf{K}$ *is then*

$$\mathrm{vec}\,\mathbf{K} = \left(\mathbf{I} - (\mathbf{U}\Xi)^\dagger\,\mathbf{U}\right)\left(\mathrm{vec}\,\mathbf{K}_0 + \Xi\,\mathrm{vec}\,\tilde{\mathbf{Z}}\right) \tag{6.6}$$

where $\tilde{\mathbf{Z}}$ *is a matrix of remaining free parameters and in each case*

$$\Xi = (\mathbf{I} - \mathbf{Y}\mathbf{Y}^\dagger) \otimes (\mathbf{I} - \mathbf{X}^\dagger\mathbf{X}) \tag{6.7}$$

Proof. Substituting Equation 6.1 into Equation 6.3,

$$\mathbf{U}\,\mathrm{vec}\left(\mathbf{K}_0 + (\mathbf{I} - \mathbf{X}^\dagger\mathbf{X})\mathbf{Z}(\mathbf{I} - \mathbf{Y}\mathbf{Y}^\dagger)\right) = \mathbf{0} \tag{6.8}$$

$$\mathbf{U}\,\mathrm{vec}\left((\mathbf{I} - \mathbf{X}^\dagger\mathbf{X})\mathbf{Z}(\mathbf{I} - \mathbf{Y}\mathbf{Y}^\dagger)\right) = -\mathbf{U}\,\mathrm{vec}\,\mathbf{K}_0 \tag{6.9}$$

The identity (Graham, 1981, p25)

$$\mathrm{vec}\,(\mathbf{ABC}) = \left(\mathbf{C}^T \otimes \mathbf{A}\right)\mathrm{vec}\,\mathbf{B} \tag{6.10}$$

can now be applied, yielding

$$\mathbf{U}\left((\mathbf{I} - \mathbf{Y}\mathbf{Y}^\dagger) \otimes (\mathbf{I} - \mathbf{X}^\dagger\mathbf{X})\right)\mathrm{vec}\,\mathbf{Z} = -\mathbf{U}\,\mathrm{vec}\,\mathbf{K}_0 \tag{6.11}$$

Note the lack of a transpose operator since the matrices forming the Kronecker product are symmetric. We may now define

$$\Xi \triangleq (\mathbf{I} - \mathbf{Y}\mathbf{Y}^\dagger) \otimes (\mathbf{I} - \mathbf{X}^\dagger\mathbf{X}) \tag{6.12}$$

so that

$$\mathbf{U}\Xi \operatorname{vec}\mathbf{Z} = -\mathbf{U}\operatorname{vec}\mathbf{K}_0 \tag{6.13}$$

The remainder of this proof will now concentrate upon finding a solution to Equation 6.13, which is of the form

$$\mathbf{Ax} = \mathbf{b} \tag{6.14}$$

and therefore (Ben-Israel and Greville, 1974, p40) has a solution if and only if

$$\mathbf{AA}^{\dagger}\mathbf{b} = \mathbf{b} \tag{6.15}$$

with the solution being given by

$$\mathbf{x} = \mathbf{A}^{\dagger}\mathbf{b} + \left(\mathbf{I} - \mathbf{A}^{\dagger}\mathbf{A}\right)\mathbf{y} \tag{6.16}$$

where $\mathbf{y}$ is a vector of free parameters.

Consequently, a solution to Equation 6.13 exists if and only if

$$\mathbf{U}\Xi\left(\mathbf{U}\Xi\right)^{\dagger}\mathbf{U}\operatorname{vec}\mathbf{K}_0 = \mathbf{U}\operatorname{vec}\mathbf{K}_0 \tag{6.17}$$

A matrix $\mathbf{E}$ is idempotent if $\mathbf{E}^2 = \mathbf{E}$; the term $\mathbf{U}\Xi\left(\mathbf{U}\Xi\right)^{\dagger}$ is idempotent (via Equations C.4 and C.5 in Appendix C), and it holds for an idempotent matrix $\mathbf{E}$ (Ben-Israel and Greville, 1974, p49) that

$$\mathbf{Ex} = \mathbf{x} \tag{6.18}$$

if and only if

$$\mathbf{x} \in \operatorname{range}\left(\mathbf{E}\right) \tag{6.19}$$

Therefore a solution exists for Equation 6.13 if and only if

$$\mathbf{U}\operatorname{vec}\mathbf{K}_0 \in \operatorname{range}\left(\mathbf{U}\Xi\left(\mathbf{U}\Xi\right)^{\dagger}\right) \tag{6.20}$$

$$\in \operatorname{range}\left(\mathbf{U}\Xi\right) \tag{6.21}$$

Assuming, then, that $\mathbf{U}$ has been selected to meet Equation 6.21, the solution to Equation 6.13 is given by substitution into Equation 6.16:

$$\operatorname{vec}\mathbf{Z} = -\left(\mathbf{U}\Xi\right)^{\dagger}\mathbf{U}\operatorname{vec}\mathbf{K}_0 + \left(\mathbf{I} - \left(\mathbf{U}\Xi\right)^{\dagger}\mathbf{U}\Xi\right)\operatorname{vec}\tilde{\mathbf{Z}} \tag{6.22}$$

where $\tilde{\mathbf{Z}}$ is a matrix characterising any remaining free parameters. A solution for $\mathbf{K}$ may now be found. From Equation 6.1,

$$\mathbf{K} = \mathbf{K}_0 + (\mathbf{I} - \mathbf{X}^\dagger\mathbf{X})\mathbf{Z}(\mathbf{I} - \mathbf{Y}\mathbf{Y}^\dagger) \tag{6.23}$$

$$\operatorname{vec}\mathbf{K} = \operatorname{vec}\mathbf{K}_0 + \Xi \operatorname{vec}\mathbf{Z} \tag{6.24}$$

Substituting Equation 6.22, we may obtain

$$\operatorname{vec}\mathbf{K} = \operatorname{vec}\mathbf{K}_0 - \Xi\,(\mathbf{U}\Xi)^\dagger\,\mathbf{U}\operatorname{vec}\mathbf{K}_0 + \Xi\left(\mathbf{I} - (\mathbf{U}\Xi)^\dagger\,\mathbf{U}\Xi\right)\operatorname{vec}\tilde{\mathbf{Z}} \tag{6.25}$$

$$= \operatorname{vec}\mathbf{K}_0 - \Xi\,(\mathbf{U}\Xi)^\dagger\,\mathbf{U}\operatorname{vec}\mathbf{K}_0 + \Xi\operatorname{vec}\tilde{\mathbf{Z}} - \Xi\,(\mathbf{U}\Xi)^\dagger\,\mathbf{U}\Xi\operatorname{vec}\tilde{\mathbf{Z}} \tag{6.26}$$

Much of the manipulation in the remainder of this proof relies on the following lemma.

Lemma 6.3.2. *The term Ξ, defined in Equation 6.12, is idempotent and symmetric.*

Proof. The idempotence of the component terms of Ξ is easily shown, and the product of two Kronecker products (Graham, 1981, p24) is given by

$$(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = \mathbf{AC} \otimes \mathbf{BD} \tag{6.27}$$

provided that the dimensions are such that the various matrices exist. Consequently, if both $\mathbf{E}$ and $\mathbf{F}$ below are idempotent, then

$$\begin{aligned}(\mathbf{E} \otimes \mathbf{F})^2 &= \mathbf{E}^2 \otimes \mathbf{F}^2 \\ &= \mathbf{E} \otimes \mathbf{F}\end{aligned} \tag{6.28}$$

and, therefore, Ξ is idempotent.

From Equations C.6 and C.7 in Appendix C, the expressions $\mathbf{X}^\dagger\mathbf{X}$ and $\mathbf{Y}\mathbf{Y}^\dagger$ can be seen to be symmetric and consequently so are $\left(\mathbf{I} - \mathbf{X}^\dagger\mathbf{X}\right)$ and $\left(\mathbf{I} - \mathbf{Y}\mathbf{Y}^\dagger\right)$. The transpose of a Kronecker product is given by Graham (1981, p24) as

$$(\mathbf{A} \otimes \mathbf{B})^\star = \mathbf{A}^\star \otimes \mathbf{B}^\star \tag{6.29}$$

and so it is clear that the term Ξ is symmetric. ◊◊

Now equation 6.26 may be simplified by noting (from Equations C.5 and C.7 in Appendix C) that

$$\mathbf{A}^{\dagger} = \mathbf{A}^{\dagger}\mathbf{A}\mathbf{A}^{\dagger} \tag{6.30}$$
$$= \left(\mathbf{A}^{\dagger}\mathbf{A}\right)^{*}\mathbf{A}^{\dagger} \tag{6.31}$$
$$= \mathbf{A}^{*}\left(\mathbf{A}^{\dagger}\right)^{*}\mathbf{A}^{\dagger} \tag{6.32}$$

and therefore that

$$\mathbf{A}^{\dagger} \in \text{range}\,(\mathbf{A}^{*}) \tag{6.33}$$

It may therefore be seen that

$$(\mathbf{U}\Xi)^{\dagger} \in \text{range}\,(\Xi^{*}\mathbf{U}^{*}) \tag{6.34}$$
$$\in \text{range}\,(\Xi) \tag{6.35}$$

and therefore that

$$\Xi\,(\mathbf{U}\Xi)^{\dagger} = (\mathbf{U}\Xi)^{\dagger} \tag{6.36}$$

since Ξ is idempotent.

So, using Equation 6.36, Equation 6.26 becomes

$$\operatorname{vec}\mathbf{K} = \operatorname{vec}\mathbf{K}_0 - (\mathbf{U}\Xi)^{\dagger}\,\mathbf{U}\operatorname{vec}\mathbf{K}_0 + \Xi\operatorname{vec}\tilde{\mathbf{Z}} - (\mathbf{U}\Xi)^{\dagger}\,\mathbf{U}\Xi\operatorname{vec}\tilde{\mathbf{Z}} \tag{6.37}$$
$$= \left(\mathbf{I} - (\mathbf{U}\Xi)^{\dagger}\,\mathbf{U}\right)\left(\operatorname{vec}\mathbf{K}_0 + \Xi\operatorname{vec}\tilde{\mathbf{Z}}\right) \tag{6.38}$$

as expected. ◊◊

It should be noted that it is not possible to recover a matrix formulation for **K** since the vec operator has no effective inverse. The gain matrix must instead be derived in vector form as above, and reconstituted numerically into a matrix afterwards.

The following corollary extracts a simple condition on the number of gain matrix entries that may be nulled.

Corollary 6.3.3. *In general, the number of gain matrix entries reduced to zero may not exceed the number of available DoF.*

Proof. Remembering that δ is the number of gain matrix entries reduced to zero, a simple sufficient condition for the satisfaction of Equation 6.21 is easily seen to be

$$\operatorname{rank}(\mathbf{U}\Xi) = \delta \tag{6.39}$$

This condition, although not strictly necessary, is necessary for *general* $\mathbf{K}_0$ and Ξ since otherwise there is no guarantee of the existence of a $\mathbf{U}$ which satisfies Equation 6.21.

A necessary (but not sufficient) condition for the fulfillment of Equation 6.39 is that

$$\begin{aligned}
\delta &\leq \operatorname{rank}(\Xi) && (6.40)\\
&\leq \operatorname{rank}\left(\left(\mathbf{I} - \mathbf{Y}\mathbf{Y}^{\dagger}\right) \otimes \left(\mathbf{I} - \mathbf{X}^{\dagger}\mathbf{X}\right)\right) && (6.41)\\
&\leq \operatorname{rank}\left(\mathbf{I} - \mathbf{Y}\mathbf{Y}^{\dagger}\right) \cdot \operatorname{rank}\left(\mathbf{I} - \mathbf{X}^{\dagger}\mathbf{X}\right) && (6.42)\\
&\leq \left(m - \operatorname{rank}\left(\mathbf{Y}\mathbf{Y}^{\dagger}\right)\right)\left(r - \operatorname{rank}\left(\mathbf{X}^{\dagger}\mathbf{X}\right)\right) && (6.43)\\
&\leq (m - v)(r - w) && (6.44)
\end{aligned}$$

Comparison with Equation 6.2 shows that Equation 6.44 may be written as

$$\delta \leq f \tag{6.45}$$

◊◊

Corollary 6.3.3 demonstrates that, for general $\mathbf{K}_0$ and Ξ, the number of gain matrix entries to be reduced to zero may not exceed the number of available degrees of freedom. Note that the satisfaction of Equation 6.45 is not sufficient for the existence of a solution to Equation 6.13, and that the satisfaction of Equation 6.21 is still required.

6.4 Sensitivity of the Gain Matrix

The Frobenius norm $|\mathbf{K}|_F$ of $\mathbf{K}$ is the square root of the sum of squared gain matrix entries. There exists a value of $\mathbf{Z}$ for which $|\mathbf{K}|_F$ is minimal. It is safe to assume that the value of $\mathbf{Z}$ found via Equation 6.6 will not equal this 'optimal' value. Therefore, nulling individual gain matrix entries will, in general, raise $|\mathbf{K}|_F$.

This is important because large gain matrix entries will result in proportionally greater inputs being applied to the system by the controller. These could be in response to deviations of the system from a required datum, leading to greater control effort; or they could be in response to noise on the system outputs, resulting in poor noise rejection.

The extent to which $|\mathbf{K}|_F$ is affected will depend upon the elements of the gain matrix selected by the permutation matrix $\mathbf{U}$. A mechanism for determining the increase in $|\mathbf{K}|_F$ would be a useful tool when attempting to determine which elements should be set to zero.

Since the formulation for $\mathbf{K}$ in Equation 6.6 contains a free parameter matrix, it is useful to find the value of the free parameter matrix that minimises the Frobenius norm of the final gain matrix.

6.4.1 Minimum Frobenius Norm

From the examination of the properties of Ξ in Lemma 6.3.2, we may now determine the value of $\tilde{\mathbf{Z}}$ that leads to the minimum $|\mathbf{K}|_F$. This is a complex procedure, and details may therefore be found in Appendix B. It is shown there that the appropriate value of $\tilde{\mathbf{Z}}$ is

$$\tilde{\mathbf{Z}} = \mathbf{0} \tag{6.46}$$

Therefore, following the suppression of a subset of the allowable gains, the gain matrix with the minimum Frobenius norm is found by setting the matrix of remaining free parameters to zero.

6.4.2 Increase in Minimum Norm

The minimum $|\mathbf{K}|_F$ may now be simply calculated by substituting into Equation 6.6:

$$\min |\mathbf{K}|_F = \min \|\operatorname{vec} \mathbf{K}\| = \left\| \left(\mathbf{I} - (\mathbf{U}\Xi)^\dagger \mathbf{U}\right) \left(\operatorname{vec} \mathbf{K}_0 + \Xi \operatorname{vec} \tilde{\mathbf{Z}}\right) \right\| \tag{6.47}$$

$$= \left\| \left(\mathbf{I} - (\mathbf{U}\Xi)^\dagger \mathbf{U}\right) \operatorname{vec} \mathbf{K}_0 \right\| \tag{6.48}$$

The minimum $|\mathbf{K}|_F$ achievable prior to nulling any gain matrix entries may be found by setting $\mathbf{U} = \mathbf{0}$:

$$\min |\mathbf{K}|_F = \min \|\operatorname{vec} \mathbf{K}\| = \|\operatorname{vec} \mathbf{K}_0\| = |\mathbf{K}_0|_F \tag{6.49}$$

Comparison of Equations 6.48 and 6.49 shows the effect of nulling a subset of the gains. This information can be used to determine which gains may most easily be reduced to zero whilst maintaining the lowest possible gains elsewhere.

6.5 Alternative Structural Constraints

It is interesting to note that the above derivation makes no demands upon the form of the matrix $\mathbf{U}$. It was defined in Section 6.2 as a permutation matrix, with the specific aim of gain suppression, but this need not be its structure.

Consider a gain matrix

$$\mathbf{K} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \end{bmatrix} \tag{6.50}$$

and assume, for the sake of example, that two DoF remain for the imposition of structure. It is now possible to construct the matrix $\mathbf{U}$ to achieve a variety of effects.

For simple gain suppression, a matrix $\mathbf{U}$ of the form

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{6.51}$$

will reduce two elements to zero through the constraint of Equation 6.3 (in this case elements k_{11} and k_{22}). However, by setting

$$\mathbf{U} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{6.52}$$

element k_{22} will be reduced to zero by the second row of $\mathbf{U}$, but elements k_{11} and k_{21} will be forced to be *equal* by the first. This can be seen by expanding Equation 6.3 to give

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_{11} \\ k_{21} \\ k_{12} \\ k_{22} \\ k_{13} \\ k_{23} \end{bmatrix} = \mathbf{0}$$

$$\begin{bmatrix} k_{11} - k_{21} \\ k_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This type of constraint could be taken further, using selection matrices such as

$$\mathbf{U} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}. \tag{6.53}$$

which by similar expansion gives

$$\begin{bmatrix} -k_{11} + k_{21} \\ -k_{11} + k_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{6.54}$$

causing k_{11}, k_{21} and k_{22} all to have the same value, or

$$\mathbf{U} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 \end{bmatrix} \tag{6.55}$$

which gives

$$\begin{bmatrix} -k_{11} + k_{21} \\ -k_{11} + 2k_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{6.56}$$

forcing $k_{11} = k_{12} = -\frac{1}{2}k_{22}$.

Such constraints are not frivolous use of the remaining design freedom. For example if two gains are forced to be equal, the summation of two signals may be implemented before a gain is applied to the result. This represents a simplification to the controller, and hence may be considered to have introduced structure. The techniques described above therefore provide a flexible interface onto the remaining design freedom for the purposes of imposing structure.

6.6 Design Example

The following design example is taken directly from Clarke et al. (2003), where it was used to demonstrate the multi-stage EA process. The example was contrived to leave one degree of freedom remaining after assignment, and consequently the same example was re-used by Clarke and Griffin (2004) to demonstrate their retro-assignment stage.

It will now be demonstrated that the same remaining degree of freedom may be employed to suppress one entry in the gain matrix while leaving the assigned eigenstructure unchanged.

6.6.1 Example Assignment

The example system was as follows:

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 2 & -3 & 5 \\ 0 & 3 & -1 & 7 \\ 5 & 8 & 1 & -9 \\ 2 & 6 & 3 & 8 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & 7 \\ 9 & -2 & 1 \\ 5 & 2 & 4 \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} 7 & 3 & 0 & 2 \\ 1 & -1 & 0 & 1 \\ 2 & 3 & 1 & 2 \end{bmatrix} \mathbf{x}$$

The desired eigenvalues were $\{-1, -2, -3, -4\}$.

In the original example, two right eigenvectors were assigned first, along with eigenvalues of -1 and -2. Simple design vectors were chosen arbitrarily, yielding eigenvectors of

$$\mathbf{V}' = \begin{bmatrix} -0.7869 & -0.0000 \\ 0.2676 & -0.1259 \\ -0.2613 & -0.2746 \\ 0.0850 & -0.0483 \end{bmatrix}$$

In the second stage, there was no design freedom available for the selection of left eigenvectors, so these emerged as

$$\mathbf{W}' = \begin{bmatrix} 0.0060 & 0.0782 & -0.0015 & -0.1952 \\ -0.0080 & -0.0762 & 0.0037 & 0.1773 \end{bmatrix}$$

corresponding to the remaining two eigenvalues, -3 and -4.

The gain matrix was found, setting the free parameter matrix to zero, as

$$\mathbf{K}_0 = \begin{bmatrix} 0.4221 & -1.4721 & -0.9173 \\ -0.0290 & 1.2579 & -0.9231 \\ -0.4864 & 1.6742 & 0.9507 \end{bmatrix}$$

and decomposition of the resulting closed-loop matrix as

$$\mathbf{A}_0 = \mathbf{A} + \mathbf{B}\mathbf{K}_0\mathbf{C} = \mathbf{V}_0\mathbf{\Lambda}_0\mathbf{W}_0$$

gave

$$\mathbf{V}_0 = \begin{bmatrix} -0.7869 & -0.0000 & 421.7856 & 295.0726 \\ 0.2676 & -0.1259 & -114.4962 & -143.5913 \\ -0.2613 & -0.2746 & 590.3538 & 430.0421 \\ 0.0850 & -0.0483 & -42.5685 & -51.7729 \end{bmatrix}$$

$$\mathbf{W}_0 = \begin{bmatrix} -1.0368 & 13.3750 & 0.5831 & -38.1610 \\ 1.4294 & 36.2016 & -1.6271 & -105.7731 \\ 0.0060 & 0.0782 & -0.0015 & -0.1952 \\ -0.0080 & -0.0762 & 0.0037 & 0.1773 \end{bmatrix}$$

$$\mathbf{\Lambda}_0 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

showing clearly that the required eigenvalues had been assigned along with all the expected eigenvectors.

6.6.2 Choosing Gains

Having obtained $\mathbf{K}_0$, it is now possible to suppress a single gain matrix entry. However, as discussed, reducing any entry to zero will have the effect of increasing the overall magnitude of the gains. This must be taken into consideration when choosing which entries are most suitable for this treatment.

If $\mathbf{K}$ is defined as Equation 6.1, we may further define

$$\mathbf{K}_{xy} \triangleq \mathbf{K} = [k_{ij}]\,,\; k_{xy} = 0 \tag{6.57}$$

$$\delta f_{xy} \triangleq \frac{|\mathbf{K}_{xy}|_F}{|\mathbf{K}_0|_F} \tag{6.58}$$

$$\Delta|\mathbf{K}|_F \triangleq [\delta f_{ij}] \tag{6.59}$$

so that $\Delta|\mathbf{K}|_F$ is a matrix giving the rise in $|\mathbf{K}|_F$ that will arise from suppressing any individual gain matrix entry. By applying Equation 6.48 repeatedly, this matrix was found to be:

$$\Delta|\mathbf{K}|_F = \begin{bmatrix} 14.8573 & 13.7576 & 31.4842 \\ 1.0009 & 1.1148 & 1.6646 \\ 1.6843 & 1.5915 & 2.7743 \end{bmatrix} \tag{6.60}$$

It is clear that suppressing a gain from the top row of the gain matrix would have the largest effect on the overall magnitude of the gains, while choosing element $(2,1)$ would have a negligible effect. This is valuable information.

6.6.3 Gain Suppression

It is now informative to suppress two different gains, both to demonstrate the operation of the algorithm and to show the difference in the magnitudes of the gain matrices obtained.

6.6.3.1 Element $(2,1)$

Equation 6.60 shows that suppressing this element will have the smallest effect on $|\mathbf{K}|_F$. From Equation 6.6, the new gain matrix is found to be:

$$\mathbf{K}_{21} = \begin{bmatrix} 0.4209 & -1.4675 & -0.9160 \\ 0 & 1.1488 & -0.9528 \\ -0.5018 & 1.7319 & 0.9664 \end{bmatrix}$$

Its Frobenius norm is $|\mathbf{K}_{21}|_F = 3.0956$ compared to $|\mathbf{K}_0|_F = 3.0928$, demonstrating the very small expected increase in overall gain matrix entry magnitude. Furthermore, forming and factorising $\mathbf{A}_{21} = \mathbf{A} + \mathbf{B}\mathbf{K}_{21}\mathbf{C} = \mathbf{V}_{21}\mathbf{\Lambda}_{21}\mathbf{W}_{21}$ gives

$$\mathbf{V}_{21} = \begin{bmatrix} -0.7869 & -0.0000 & 415.9577 & 290.5659 \\ 0.2676 & -0.1259 & -124.8504 & -149.2139 \\ -0.2613 & -0.2746 & 561.5162 & 412.9417 \\ 0.0850 & -0.0483 & -46.6739 & -54.0324 \end{bmatrix}$$

$$\mathbf{W}_{21} = \begin{bmatrix} -1.0357 & 13.2318 & 0.5730 & -37.7307 \\ 1.2930 & 32.8639 & -1.6901 & -96.7189 \\ 0.0060 & 0.0782 & -0.0015 & -0.1952 \\ -0.0080 & -0.0762 & 0.0037 & 0.1773 \end{bmatrix}$$

$$\mathbf{\Lambda}_{21} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

showing that the eigenvalues and explicitly assigned eigenvectors remain unchanged.

6.6.3.2 Element (1,3)

Suppressing this element will have a much larger effect on $|\mathbf{K}|_F$ according to Equation 6.60. From Equation 6.6, the new gain matrix is found to be:

$$\mathbf{K}_{13} = \begin{bmatrix} -0.4740 & 1.9040 & 0 \\ 21.2960 & -79.0894 & -22.7529 \\ -11.7801 & 44.2260 & 12.5118 \end{bmatrix}$$

Its Frobenius norm is $|\mathbf{K}_{13}|_F = 97.3728$ compared to $|\mathbf{K}_0|_F = 3.0928$, showing a very significant increase in overall gain matrix entry magnitude. However, once again, forming and factorising $\mathbf{A}_{13} = \mathbf{A} + \mathbf{B}\mathbf{K}_{13}\mathbf{C} = \mathbf{V}_{13}\mathbf{\Lambda}_{13}\mathbf{W}_{13}$ gives

$$\mathbf{V}_{13} = \begin{bmatrix} -0.7869 & -0.0000 & -3869.4581 & -3023.3241 \\ 0.2676 & -0.1259 & -7738.6648 & -4283.7425 \\ -0.2613 & -0.2746 & -20643.7118 & -12161.4692 \\ 0.0850 & -0.0483 & -3065.4598 & -1715.5575 \end{bmatrix}$$

$$\mathbf{W}_{13} = \begin{bmatrix} -0.2030 & -92.0349 & -6.8437 & 278.6835 \\ -98.9955 & -2421.4217 & -48.0097 & 6561.0803 \\ 0.0060 & 0.0782 & -0.0015 & -0.1952 \\ -0.0080 & -0.0762 & 0.0037 & 0.1773 \end{bmatrix}$$

$$\mathbf{\Lambda}_{13} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

demonstrating once again that the eigenvalues and explicitly assigned eigenvectors remain unchanged during the gain suppression process.

6.7 Conclusions and Further Work

The design freedom remaining at the end of multi-stage EA algorithms (Clarke et al., 2003) has use beyond the retro-assignment stage offered by Clarke and Griffin (2004). Specifically, this freedom may be used to reduce individual entries in the gain matrix to zero, thereby imposing a structure upon the resulting controller.

An algorithm for nulling a subset of gain matrix entries has been presented, and it has been demonstrated that the maximum number of entries that may be nulled is, in general, equal to the number of available DoF. In addition, the effect upon the remaining entries of the gain matrix has been considered, and an expression generated for the minimum Frobenius norm of the gain matrix both before and after the nulling of entries.

At present the algorithm utilises only the design freedom remaining after EA. A possible extension would be to attempt to use similar techniques to restrict the available set of gains *during* assignment, while still allowing freedom over eigenvector selection.

An example of the algorithm presented here at work can be found in Chapter 7.

6.8 Chapter Bibliography

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Chapter 7

Helicopter Design Examples

Contents

7.1 Introduction

In this chapter, the Eigenstructure Assignment (EA) algorithms developed in this thesis will be demonstrated in the context of helicopter control. In particular, the ideal eigenstructure for forward flight introduced in Chapter 3 will be verified against the UK Ministry of Defence Defence Standard 00-970 (Def.Stan.00-970) requirements; the EA algorithms for semi-proper systems from Chapter 5 will be employed and their efficacy demonstrated; and the gain suppression methods of Chapter 6 will be used to introduce structure to a controller without affecting its performance.

Inertial Navigation Systems (INSs) effect control of an aircraft by measuring inertial forces. Linear accelerations can be measured, along with rotational velocities, by using accelerometers and gyroscopes. Inertial navigation technology is mature (Lawrence, 1998; Siouris, 1993; Titterton and Weston, 1997) having been employed as early as 1944 in the German V-2 guided rocket programme.

An INS has at its heart an Inertial Measurement Unit (IMU), responsible for the measurement of acceleration and angular rate. Modern IMUs are of the 'strapdown' variety, wherein the accelerometers and gyroscopes are rigidly attached to the airframe. The resolution of the body-relative measurements into fixed-frame readings, and the integration of the resulting acceleration readings to generate velocity and position estimates, is performed electronically. Using highly accurate laser-ring gyroscopes, even following the double-integration required to estimate position, the accuracy of modern INSs can be better than 0.6 nautical miles per hour (Titterton and Weston, 1997).

In this chapter, depending upon the aim of each design example, various of the quantities available from the IMU will be deemed measurable. In some of the examples, it will also be assumed that earth-relative vertical velocity can also be measured. Typically this would be achieved using a radar altimeter (Cundy and Brown, 1997).

This chapter begins by replicating the state-feedback results of Griffin (1997) for a Lynx helicopter in the hover. It will then be demonstrated that the same performance is achievable using a controller which does not have access directly to state information, but instead has access to a combination of states and state derivatives, by using the methods derived in Chapter 5.

Forward flight is considered next, with a state-feedback example again employed to provide a benchmark against which other controllers can be compared, and to demonstrate the ideal eigenstructure of Chapter 3. A pseudo-state feedback example is considered and its performance verified. A controller using all available IMU signals and a radar altimeter is then derived and it is shown that the excess design freedom remaining after assignment may be employed to build a structured controller with the same performance again.

Finally, as a pedagogical exercise, control of the helicopter in hover using only the direct inertial measurements (accelerations and angular rates) is performed. This is to demonstrate the operation of the output-feedback algorithms from Chapter 5 and to highlight the performance problems suffered by output-feedback systems compared to their state-feedback (or pseudo-state feedback) counterparts.

7.2 Hover: A Pseudo-State Feedback Control Law

Griffin (1997) developed a state-feedback control law for an 8th-order model of a Lynx helicopter in hover. Since aircraft velocities are not measurable directly to the required precision, the example was intended to represent a benchmark against which more realistic output-feedback control laws could be compared. However, measurement of acceleration is relatively simple, and the design techniques of Chapter 5 can be employed to synthesise an equivalent pseudo-state feedback controller by feeding back state derivatives.

The eigenvalues assigned to the system are listed in Table 7.2.

λ_p	λ_v	λ_q	λ_u	λ_w	λ_r
$-1.5 \pm j1.6$	-0.004	$-1.5 \pm j1.6$	-0.002	-0.33	-1.75

Table 7.1: Desired Eigenvalue Locations

The desired eigenvectors (rounded to 2*dp*. for compactness) are:

$$\mathbf{x} = \begin{bmatrix} v & p & \phi & u & q & \theta & w & r \end{bmatrix}^T$$

$$\mathbf{\Lambda}_d = \mathrm{diag}\left(\begin{bmatrix} \lambda_p & \bar{\lambda}_p & \lambda_v & \lambda_q & \bar{\lambda}_q & \lambda_u & \lambda_w & \lambda_r \end{bmatrix}\right)$$

$$\mathbf{V}_d = \begin{bmatrix} -0.31+0.33j & -0.31-0.33j & -1 & 0 & 0 & 0 & 0 & 0 \\ -0.31-0.33j & -0.31+0.33j & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.31-0.33j & 0.31+0.33j & 1 & 0 & 0 \\ 0 & 0 & 0 & -0.31-0.33j & -0.31+0.33j & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.1}$$

7.2.1 Existing State-Feedback Control Law

For reference, and for comparison with the pseudo-state feedback technique, the results obtained are reproduced as Figures 7.1 to 7.3.

Figure 7.1 shows the response of the helicopter's roll attitude ϕ to a one-second lateral pulse input on the cyclic pitch stick, and of its pitch attitude θ to a one-second longitudinal pulse input on the cyclic pitch stick. In each case the response is superimposed upon a template

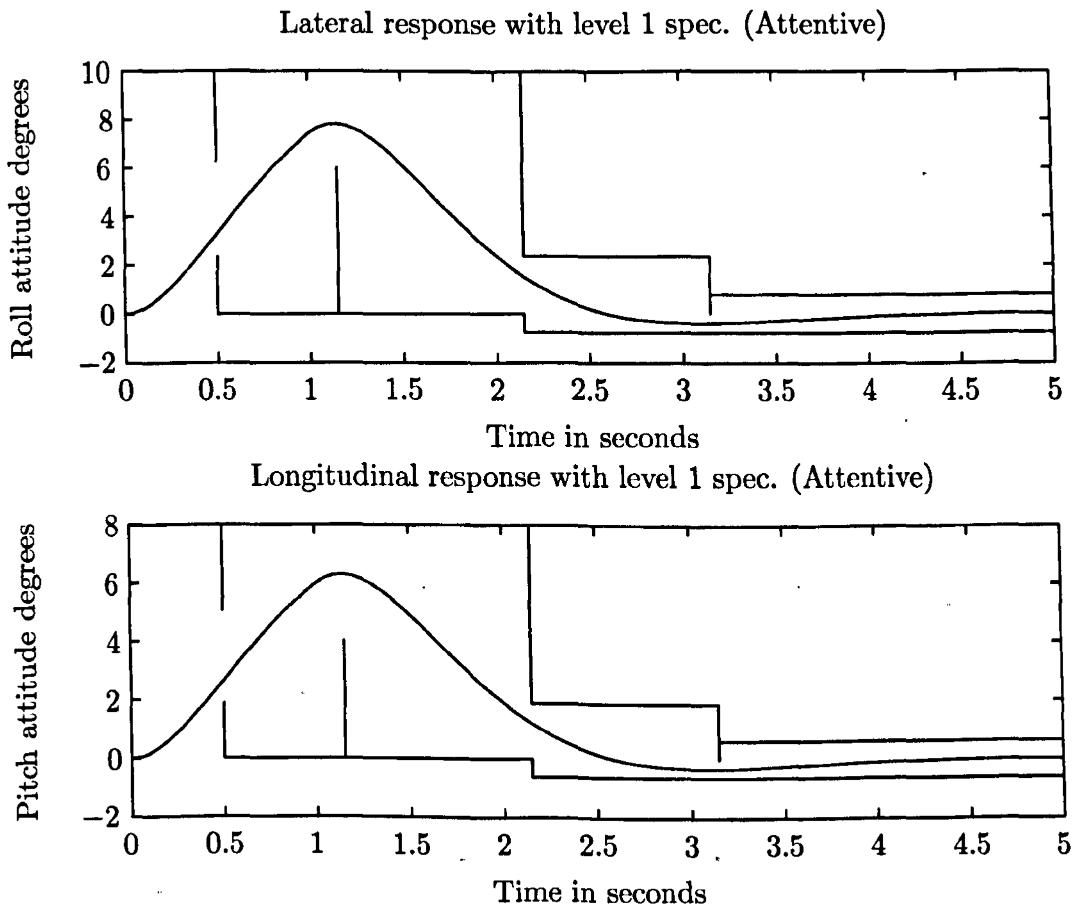

Figure 7.1: Longitudinal and lateral responses of the state feedback controller (from Griffin, 1997)

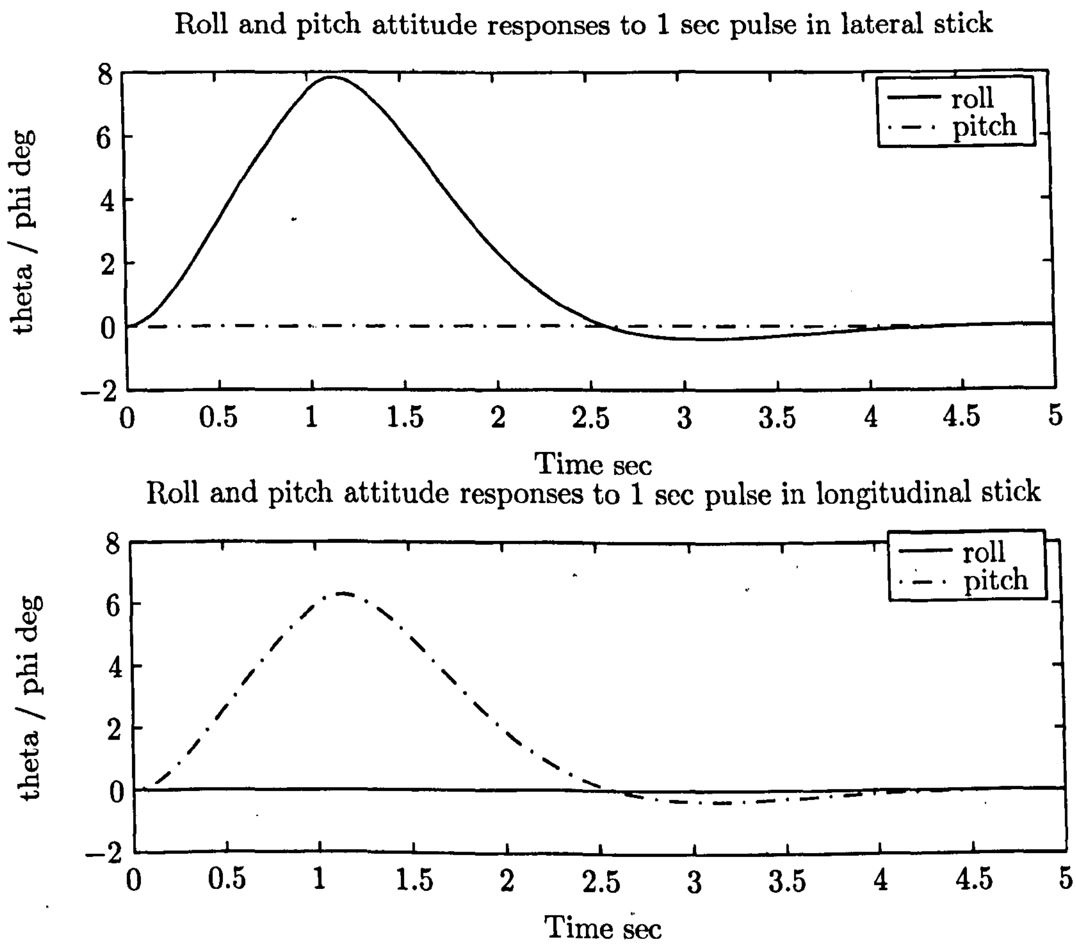

Figure 7.2: On- and off-axis attitude responses to lateral and longitudinal stick (from Griffin, 1997)

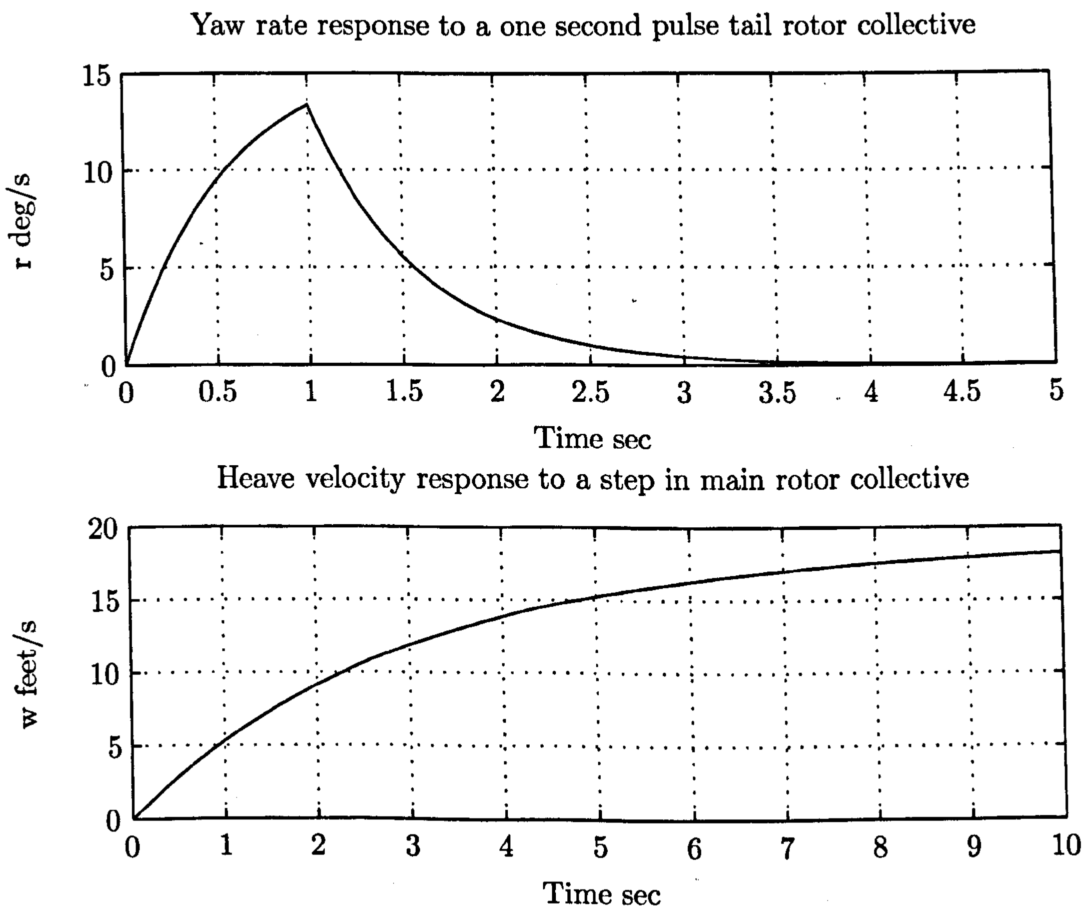

Figure 7.3: Yaw and heave responses for the state feedback controller (from Griffin, 1997)

representing the Def.Stan.00-970 requirements for Level 1 handling qualities in the attentive flight phase (derived from Figure 3.4).

Figure 7.2 shows the cross-coupling between the roll and pitch attitudes when performing the same manoeuvre. The level of cross-coupling is negligible, due to the assignment of all right-eigenvectors and the close alignment of the allowable subspaces with the desired eigenvectors (indicating that the design requirements match the kinematics of the plant).

Finally, Figure 7.3 shows the yaw rate response (r) to the application of a one-second pulse applied to the tail rotor pitch, and the heave velocity response (w) to a step in the main rotor collective pitch.

The gain matrix employed to generate these results was

$$\mathbf{x} = \begin{bmatrix} u & v & w & p & q & r & \phi & \theta \end{bmatrix}^T$$

$$\begin{bmatrix} A_1 \\ B_1 \\ \theta_0 \\ \theta_t \end{bmatrix} = \begin{bmatrix} -0.0013 & 0.0004 & -0.0001 & 0.0525 & -0.0361 & -0.0002 & -0.0383 & -0.0034 \\ 0.0008 & 0 & -0.0001 & -0.0398 & 0.0743 & 0.0058 & -0.0015 & 0.2450 \\ 0.0001 & 0 & 0 & 0 & 0.0007 & -0.0002 & 0.0062 & -0.0052 \\ 0.0001 & 0.0011 & 0.0004 & -0.0576 & 0.0154 & 0.1911 & -0.0807 & 0.0103 \end{bmatrix} \mathbf{x}$$

It can be seen that all the gains in this controller are small, with the largest (the yaw-damping gain linking yaw rate r to tail rotor pitch θ_t) less than 0.2.

The achieved eigenvectors are given below, and have the same state and mode order as the ideal eigenvectors in Equation 7.1 for ease of comparison. Again the entries have been rounded to $2dp.$ for compactness.

$$\mathbf{V}_a = \begin{bmatrix} -1.09+0.98\jmath & -1.09-0.98\jmath & -1 & -0.14+0.01\jmath & -0.14-0.01\jmath & 0 & 0 & -0.06 \\ -1.50-1.60\jmath & -1.50+1.60\jmath & 0 & 0.00+0.05\jmath & 0.00-0.05\jmath & 0 & 0 & 0.04 \\ 1 & 1 & 0 & -0.02-0.02\jmath & -0.02+0.02\jmath & 0 & 0 & -0.06 \\ -0.15+0.00\jmath & -0.15-0.00\jmath & 0 & 1.29-0.76\jmath & 1.29+0.76\jmath & 1 & 0 & -0.02 \\ -0.01-0.07\jmath & -0.01+0.07\jmath & 0 & -1.50-1.60\jmath & -1.50+1.60\jmath & 0 & 0 & -0.03 \\ 0.03+0.02\jmath & 0.03-0.02\jmath & 0 & 1 & 1 & 0 & 0 & -0.02 \\ -0.01+0.00\jmath & -0.01-0.00\jmath & 0 & 0.01+0.01\jmath & 0.01-0.01\jmath & 0 & 1 & 0 \\ -0.06-0.04\jmath & -0.06+0.04\jmath & 0 & -0.06-0.03\jmath & -0.06+0.03\jmath & 0 & 0 & 1 \end{bmatrix}$$

7.2.2 A Pseudo-State Feedback Equivalent

The states available in the original model (Griffin, 1997) are the longitudinal, lateral and vertical velocities u, v and w; the roll, pitch and yaw angular rates p, q and r; and the pitch and roll angles ϕ and θ. For the output-feedback control laws presented later, Griffin assumed that the velocity states were unmeasurable, but that vertical speed in the inertial frame ($\dot{h}$) could be measured. He also showed that $\dot{h}$ was equivalent in the hover to

$$\dot{h} \approx \begin{bmatrix} 0.057 & 0.057 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{7.2}$$

Although there is no need to do so, for the purposes of demonstrating the flexibility of the pseudo-state feedback technique it will be assumed that $\dot{h}$ is measured as above, and also that the longitudinal and lateral accelerations $\dot{u}$ and $\dot{v}$ are measured.

Since $\dot{u}$ and $\dot{v}$ are state derivatives, it is necessary to re-form the output matrix and the direct transmission matrix such that if

$$\mathbf{y} = \begin{bmatrix} \dot{u} & \dot{v} & \dot{h} & p & q & r & \phi & \theta \end{bmatrix}^T$$
$$\mathbf{x} = \begin{bmatrix} u & v & w & p & q & r & \phi & \theta \end{bmatrix}^T$$
$$\mathbf{u} = \begin{bmatrix} A_1 & B_1 & \theta_0 & \theta_t \end{bmatrix}^T$$

then

$$\mathbf{y} = \mathbf{Cx} + \mathbf{Du} \tag{7.3}$$

which is achieved by setting

$$\mathbf{C} = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} & a_{1,6} & a_{1,7} & a_{1,8} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} & a_{2,6} & a_{2,7} & a_{2,8} \\ 0.057 & 0.057 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.4}$$

$$\mathbf{D} = \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} \\ b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{7.5}$$

where $a_{i,j}$ and $b_{i,j}$ are the corresponding elements of the system matrix $\mathbf{A}$ and input matrix $\mathbf{B}$, respectively.

The gain matrix produced using the techniques from Chapter 5, and the same desired eigenstructure as the state-feedback case above, is

$$\mathbf{K} = \begin{bmatrix} -0.0027 & 0.0048 & 0.0089 & -0.0856 & 0.0591 & 0.0587 & -0.2486 & 0.1595 \\ -0.0013 & 0.0001 & 0.0052 & -0.0502 & 0.0797 & 0.0052 & -0.0689 & 0.3333 \\ -0.0002 & 0 & 0.0008 & -0.0021 & 0.0020 & 0 & -0.0043 & 0.0080 \\ -0.0144 & 0.0132 & 0.0475 & -0.4669 & 0.2889 & 0.3416 & -0.9449 & 0.8488 \end{bmatrix}$$

and it can be seen that the gains, although slightly larger, are still acceptably small, with none exceeding 1.0. The increase in the magnitudes of the gains reflects the fact that the new entries in the $\mathbf{C}$ matrix due to the feedback of state derivatives are rather smaller than those retained from the original matrix, and since the closed-loop $\mathbf{A}$ matrix is given as $\mathbf{A}_c = \mathbf{A} + \mathbf{BNC}$ it is natural that this is accompanied by correspondingly larger entries in $\mathbf{N}$, and hence in $\mathbf{K}$. It is interesting to note that the largest gain is now that which links roll angle to the tail rotor - it is not clear why this is the case.

Figures 7.4 to 7.6 show the responses of the helicopter and pseudo-state feedback controller, and mirror directly Figures 7.1 to 7.3.

The achieved eigenvector set is shown below, and can be seen to be identical to that obtained

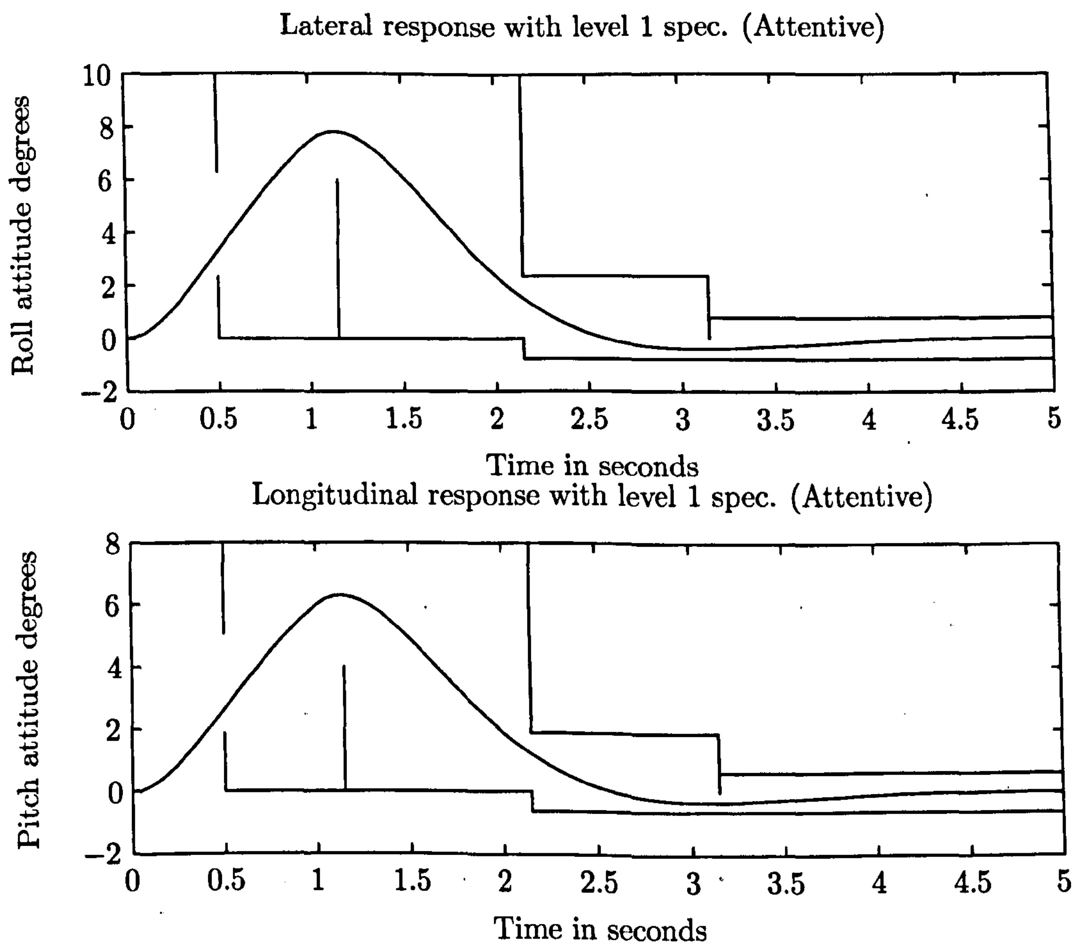

Figure 7.4: Longitudinal and lateral responses of the pseudo-state feedback controller

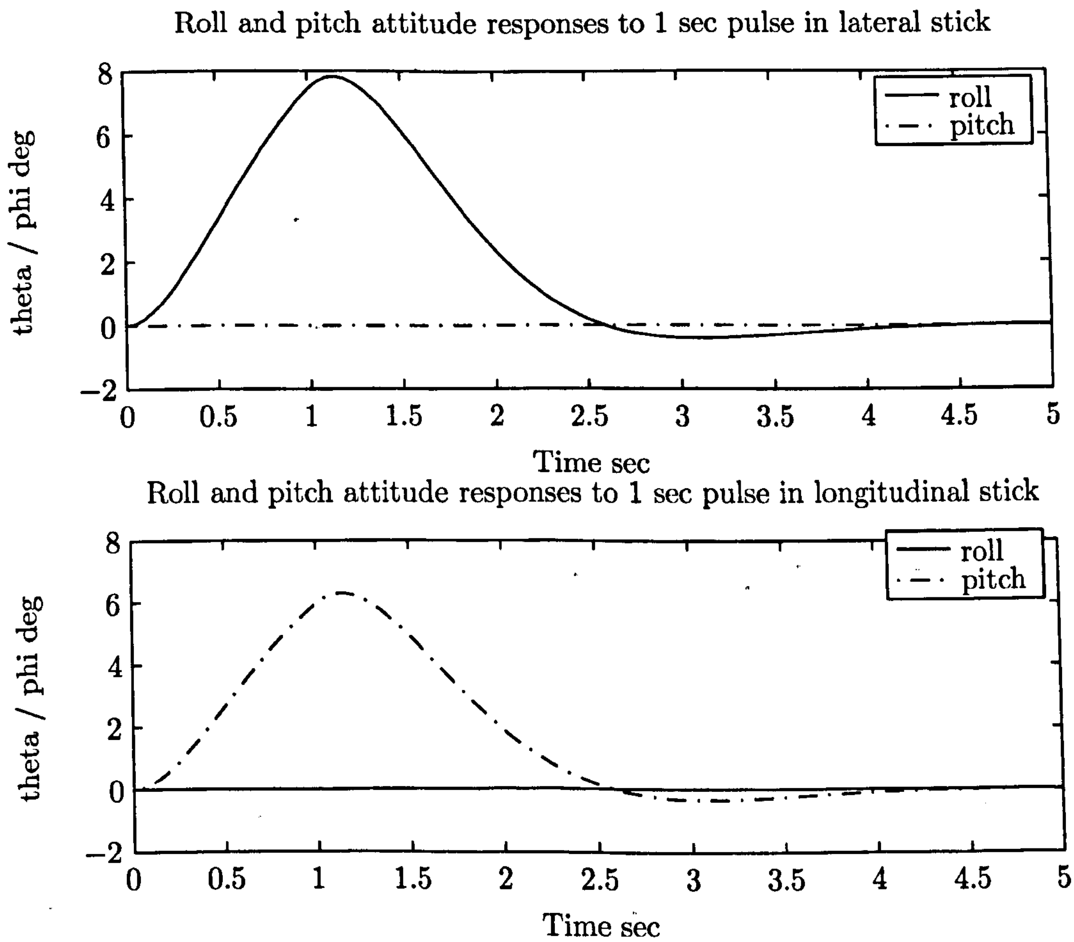

Figure 7.5: On- and off-axis attitude responses to lateral and longitudinal stick

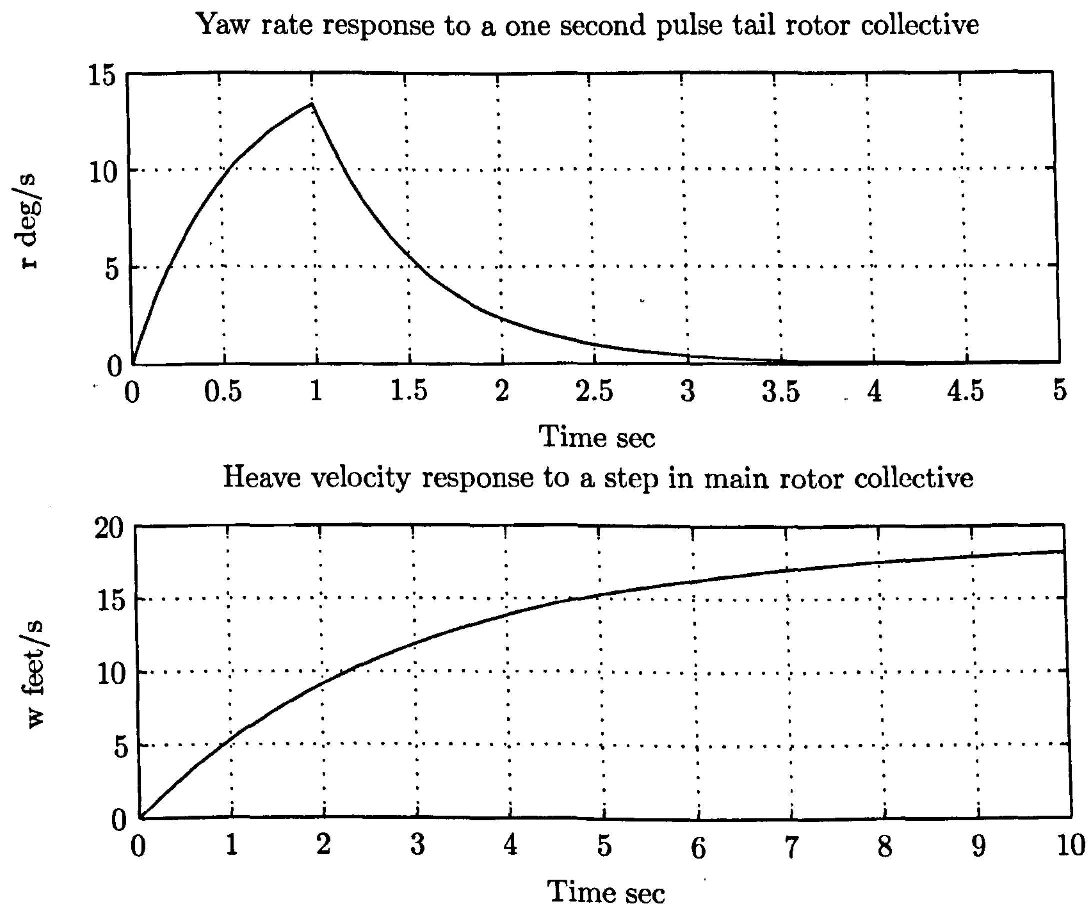

Figure 7.6: Yaw and heave responses for the pseudo-state feedback controller

using the state-feedback approach above.

$$\mathbf{V}_a = \begin{bmatrix} -1.09+0.98j & -1.09-0.98j & -1 & -0.14+0.01j & -0.14-0.01j & 0 & 0 & -0.06 \\ -1.50-1.60j & -1.50+1.60j & 0 & 0.00+0.05j & 0.00-0.05j & 0 & 0 & 0.04 \\ 1 & 1 & 0 & -0.02-0.02j & -0.02+0.02j & 0 & 0 & -0.06 \\ -0.15+0.00j & -0.15-0.00j & 0 & 1.29-0.76j & 1.29+0.76j & 1 & 0 & -0.02 \\ -0.01-0.07j & -0.01+0.07j & 0 & -1.50-1.60j & -1.50+1.60j & 0 & 0 & -0.03 \\ 0.03+0.02j & 0.03-0.02j & 0 & 1 & 1 & 0 & 0 & -0.02 \\ -0.01+0.00j & -0.01-0.00j & 0 & 0.01+0.01j & 0.01-0.01j & 0 & 1 & 0 \\ -0.06-0.04j & -0.06+0.04j & 0 & -0.06-0.03j & -0.06+0.03j & 0 & 0 & 1 \end{bmatrix}$$

The power of the pseudo-state feedback technique is clearly demonstrated. It may be seen that the performance of the state-feedback controller is retained by a controller which is practical and uses only measurable quantities.

7.3 Forward Flight

As discussed in Chapter 3, a standard-configuration helicopter will behave very differently in forward flight from in the hover. Higher airflow through the rotor, the effects of the main rotor wake on the fuselage and tail rotor, and the forces generated by the flow of air over the horizontal and vertical tail stabilisers all contribute to a significant change in performance.

Additionally, the Def.Stan.00-970 requirements for Level 1 handling qualities are significantly different from the hover case, and the ideal eigenstructure of Chapter 3 requires a second-order dynamic compensator for its implementation. The following control laws are consequently quite different from those derived for the helicopter in hover.

The augmented desired eigenvectors (again rounded for compactness) are:

$$\mathbf{x} = \begin{bmatrix} v & p & comp_\phi & \phi & u & q & comp_\theta & \theta & w & r \end{bmatrix}^T$$

$$\Lambda_d = \text{diag}\left(\begin{bmatrix} \lambda_p & \bar{\lambda}_p & 0 & \lambda_v & \lambda_q & \bar{\lambda}_q & 0 & \lambda_u & \lambda_w & \lambda_r \end{bmatrix}\right)$$

$$\mathbf{V}_d = \begin{bmatrix} 0.014+0.208j & 0.014-0.208j & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ j & -j & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.312+0.333j & -0.312-0.333j & -0.004 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.014+0.208j & 0.014-0.208j & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & j & -j & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.312+0.333j & -0.312-0.333j & -0.002 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (7.6)$$

7.3.1 A State Feedback Control Law

In order to establish a benchmark against which other control laws can be compared, a controller using full state feedback will now be described.

Since the desired eigenstructure includes two compensator states, the open-loop helicopter model (given in Appendix A) can be augmented with integrators as in Chapter 4, and the state-feedback controller developed as a static gain matrix using standard EA.

The resulting 2^{nd} order controller has the state vector

$$\mathbf{x_c} = \begin{bmatrix} c_1 & c_2 \end{bmatrix}^T \qquad (7.7)$$

and is described by the state-variable equations

$$\dot{\mathbf{x}}_\mathbf{c} = \mathbf{A_c x_c} + \mathbf{B_c y}$$

$$\mathbf{u} = \mathbf{C_c x_c} + \mathbf{D_c y}$$

The values of the controller matrices are shown below.

$$\mathbf{A}_c = \begin{bmatrix} -1.5227 & 0.0078 \\ -0.0449 & -1.4492 \end{bmatrix}$$

$$\mathbf{B}_c = \begin{bmatrix} 0 & 0 & 0 & 0.1631 & -0.0043 & -0.0350 & 0.0314 & 0.0005 \\ 0 & 0 & 0 & -0.0052 & 0.2008 & 0.0039 & 0.0026 & -0.0013 \end{bmatrix}$$

$$\mathbf{C}_c = \begin{bmatrix} -0.1258 & -0.0162 \\ 0.0183 & 0.6596 \\ -0.0009 & 0.0915 \\ -0.3646 & 0.1802 \end{bmatrix}$$

$$\mathbf{D}_c = \begin{bmatrix} 0.0003 & 0.0005 & -0.0008 & 0.0635 & -0.1056 & 0.0011 & -0.0026 & 0.0016 \\ 0.0003 & 0.0007 & 0.0002 & -0.0377 & 0.0434 & 0.0005 & 0.0027 & -0.0016 \\ -0.0004 & 0.0001 & -0.0008 & -0.0053 & 0.2029 & 0 & 0.0045 & -0.0052 \\ -0.0012 & 0.0016 & -0.0012 & -0.1431 & 0.2853 & 0.1892 & -0.1691 & -0.0082 \end{bmatrix}$$

Once again the controller gains can be seen to be small, with those in the system matrix being the largest in magnitude. The remainder do not exceed 0.4. The eigenvalues of the controller system matrix are -1.5175 and -1.4543, so the compensator is stable but the closed-loop system will possess transmission zeros at these locations. The locations of the transmission zeros also control the magnitude of the leading-diagonal entries in the system matrix, since the trace of a matrix is equal to the sum of its eigenvalues (Miller, 1987).

After applying a suitable pre-filter with poles coincident with these added zeros, the system response is shown in Figures 7.7 to 7.9.

Figure 7.7 shows that the closed-loop system response comfortably meets the Def.Stan.00-970 specifications for Level 1 handling qualities in the Active Aggressive flight phase. Figure 7.8 shows that the expected integration between the angular rates and the body angles is indeed taking place, and that the cross-coupling between the longitudinal and lateral channels is minimal.

Figure 7.9 shows the cross-coupling in angular rates. A large degree of cross-coupling may be observed between the lateral cyclic stick and yaw rate. This is to be expected, and is kinematic in nature. The desired steady-state offset in sideslip velocity following the lateral cyclic pulse, evident in Figure 7.8, causes a reaction from the vertical tail fin that leads to a yaw in the direction of the sideslip; as noted in Chapter 3, the tail rotor can not supply a

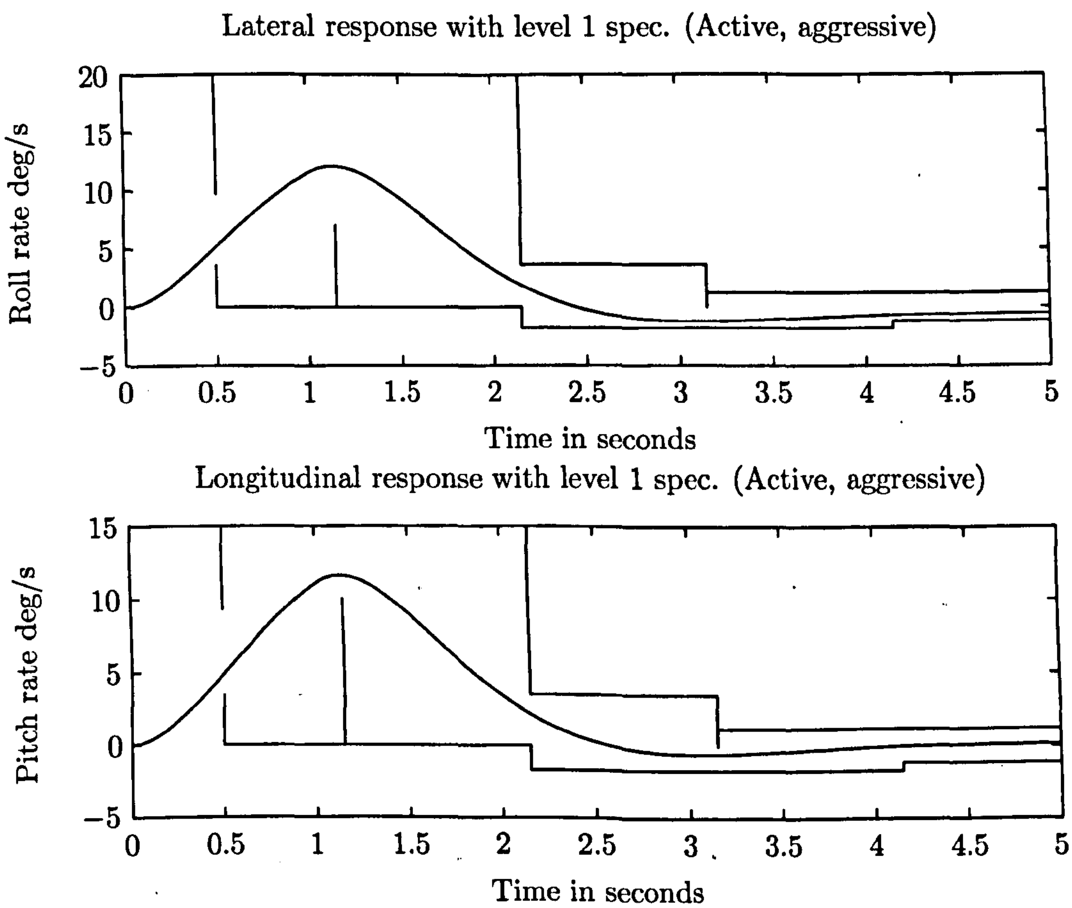

Figure 7.7: Longitudinal and lateral responses of the state feedback controller

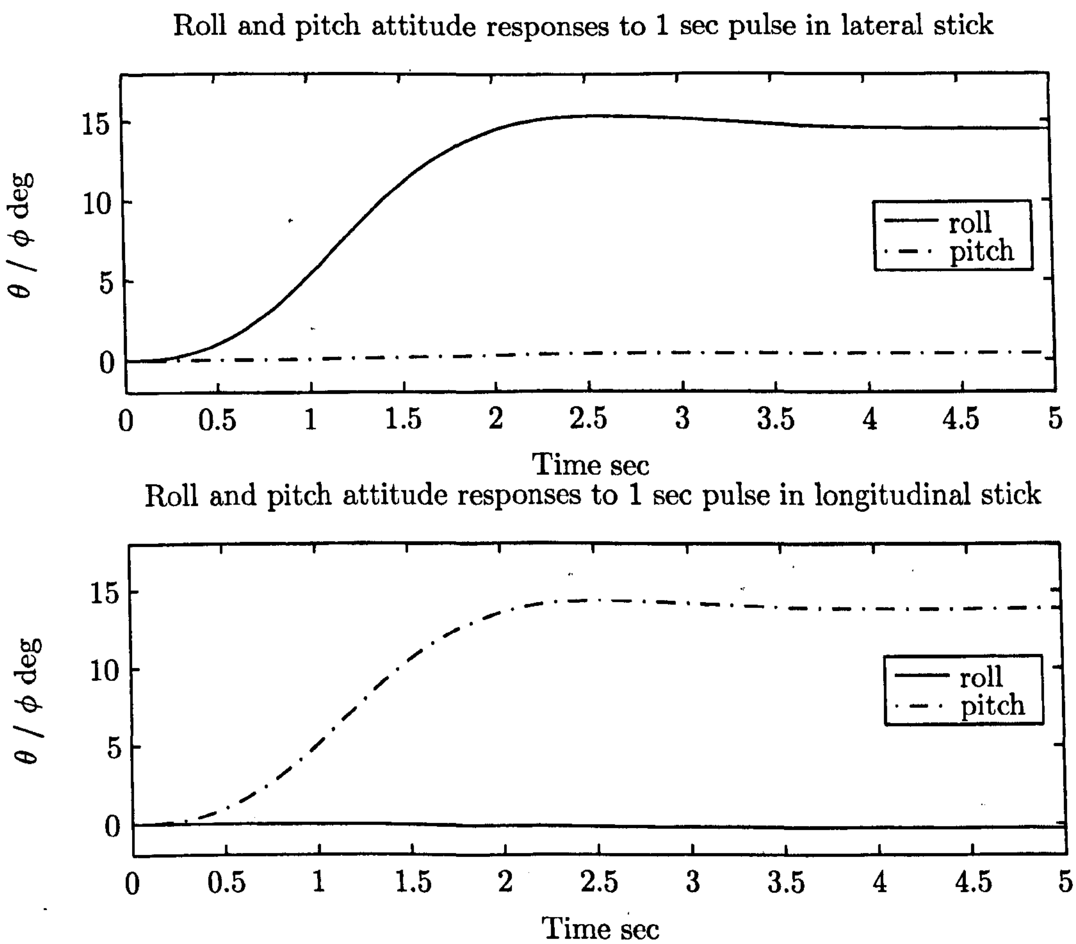

Figure 7.8: On- and off-axis attitude responses to lateral and longitudinal stick

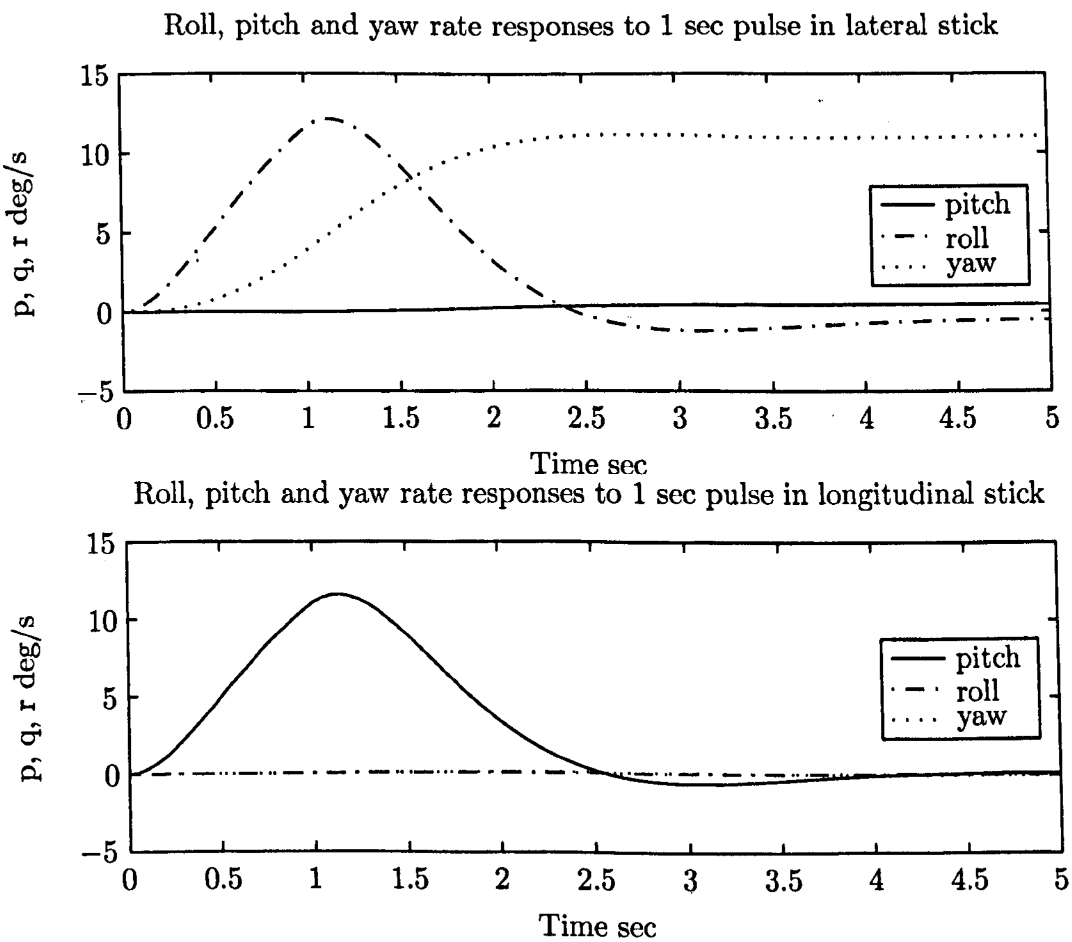

Figure 7.9: Rate responses to lateral and longitudinal stick

sufficiently great yaw moment to prevent this coupling from occurring.

The achieved eigenvectors are given below, in a format suitable for comparison with Equation 7.6.

$$\mathbf{V}_a = \begin{bmatrix} 0.01+0.21j & 0.01-0.21j & 1 & 1 & -0.01+0.00j & -0.01-0.00j & 0 & 0 & 0 & 3.65 \\ 1 & 1 & 0 & 0 & 0.02-0.01j & 0.02+0.01j & 0 & 0 & 0 & 0.26 \\ j & -j & 0.004 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.31+0.33j & -0.31-0.33j & 0 & 0 & -0.01+0.01j & -0.01-0.01j & 0 & 0 & 0 & -0.17 \\ 0.04-0.05j & 0.04+0.05j & 0 & 0 & -0.10+0.54j & -0.10-0.54j & 1 & 1 & 0 & 0 \\ 0.03+0.01j & 0.03-0.01j & 0 & 0 & 0.80-0.02j & 0.80+0.02j & 0 & 0 & 0 & -0.01 \\ 0 & 0 & 0 & 0 & j & -j & 0 & 0 & 0 & 0 \\ -0.02+0.00j & -0.02-0.00j & 0 & 0 & -0.24+0.27j & -0.24-0.27j & 0.003 & 0 & 0.01 & -0.02 \\ 0 & 0 & 0 & 0 & -0.01+0.02j & -0.01-0.02j & 0 & 0 & 1 & 0 \\ -0.21+0.22j & -0.21-0.22j & -0.01 & -0.01 & 0.01+0.01j & 0.01-0.01j & 0 & 0 & 0 & 1 \end{bmatrix}$$

The cross-coupling from λ_v to r, and the inverse coupling from λ_r to v, can clearly be seen.

This state-feedback example serves to support the validity of the ideal eigenstructure presented in Chapter 3, and forms a basis for comparison with the following pseudo-state feedback law.

7.3.2 A Pseudo-State Feedback Control Law

To demonstrate the pseudo-state feedback design procedure, it will once again be assumed that vertical speed in the inertial frame ($\dot{h}$) can be measured, along with $\dot{u}, \dot{v}, p, q, r, \phi, \theta$ and the two compensator states c_1 and c_2. Since the helicopter is no longer in the hover, $\dot{h}$ must be re-approximated. The vertical inertial velocity can be found to be

$$\dot{h} = \sin(\theta)u - \cos(\theta)\sin(\phi)v - \cos(\theta)\cos(\phi)w \tag{7.8}$$

In forward flight at 60 knots, the helicopter model trims to a point such that

$$\dot{h} \approx \begin{bmatrix} 0.048 & 0.039 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{7.9}$$

Once again it is necessary to re-form the output matrix and the direct transmission matrix to feed back the state derivatives $\dot{u}$ and $\dot{v}$. This is achieved just as in Section 7.2.2.

Again the desired eigenstructure includes two compensator states, but using the augmented system description, the controller can be developed as a static gain matrix using pseudo-state feedback EA.

The resulting 2^{nd} order controller has the state vector

$$\mathbf{x}_c = \begin{bmatrix} c_1 & c_2 \end{bmatrix}^T \tag{7.10}$$

and is described by the state-variable equations

$$\dot{\mathbf{x}}_c = \mathbf{A}_c\mathbf{x}_c + \mathbf{B}_c\mathbf{y}$$
$$\mathbf{u} = \mathbf{C}_c\mathbf{x}_c + \mathbf{D}_c\mathbf{y}$$

The values of the controller matrices are shown below.

$$\mathbf{A}_c = \begin{bmatrix} -1.5564 & -0.0043 \\ -0.0467 & -1.4502 \end{bmatrix}$$

$$\mathbf{B}_c = \begin{bmatrix} 0.0004 & -0.0006 & 0 & 0.1903 & -0.0081 & -0.0746 & 0.0553 & 0.0003 \\ 0 & 0 & 0 & -0.0038 & 0.2011 & 0.0018 & 0.0039 & -0.0013 \end{bmatrix}$$

$$\mathbf{C}_c = \begin{bmatrix} 0.1642 & 0.0510 \\ 0.4743 & 0.7785 \\ 0.1595 & 0.2628 \\ 1.1153 & 1.0566 \end{bmatrix}$$

$$\mathbf{D}_c = \begin{bmatrix} -0.0026 & 0.0050 & 0.0003 & -0.1702 & -0.0196 & 0.3453 & -0.2097 & 0.0022 \\ -0.0057 & 0.0079 & 0.0005 & -0.4053 & 0.1597 & 0.5405 & -0.3224 & -0.0001 \\ -0.0020 & 0.0026 & 0.0006 & -0.1349 & 0.0575 & 0.1772 & -0.1055 & -0.0001 \\ -0.0198 & 0.0248 & 0.0031 & -1.3374 & -0.0434 & 1.8934 & -1.2078 & 0.0139 \end{bmatrix}$$

There are now gains approaching a magnitude of 2. The largest gain is that linking yaw rate r to tail rotor pitch θ_t, which is performing a yaw-damping role. Figures 7.10 to 7.12 show the response of the pseudo-state feedback system. Once again, it is possible to attribute the increase in gain magnitudes to the fact that terms added to the $\mathbf{C}$ matrix for the feedback of the two state derivatives are small compared to the remainder of the entries.

Since the eigenvector assignment freedom is the same as in the state-feedback case, it would be expected that the two controllers would exhibit the same performance. This can be seen to be the case, and can be confirmed by inspection of the closed-loop eigenvectors:

$$
\mathbf{V}_a = \begin{bmatrix}
0.01+0.21j & 0.01-0.21j & 1 & 1 & -0.01+0.00j & -0.01-0.00j & 0 & 0 & 0 & 3.65 \\
1 & 1 & 0 & 0 & 0.02-0.01j & 0.02+0.01j & 0 & 0 & 0 & 0.26 \\
j & -j & 0.004 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.31+0.33j & -0.31-0.33j & 0 & 0 & -0.01+0.01j & -0.01-0.01j & 0 & 0 & 0 & -0.17 \\
0.04-0.05j & 0.04+0.05j & 0 & 0 & -0.10+0.54j & -0.10-0.54j & 1 & 1 & 0 & 0 \\
0.03+0.01j & 0.03-0.01j & 0 & 0 & 0.80-0.02j & 0.80+0.02j & 0 & 0 & 0 & -0.01 \\
0 & 0 & 0 & 0 & j & -j & 0 & 0 & 0 & 0 \\
-0.02+0.00j & -0.02-0.00j & 0 & 0 & -0.24+0.27j & -0.24-0.27j & 0.003 & 0 & 0.01 & -0.02 \\
0 & 0 & 0 & 0 & -0.01+0.02j & -0.01-0.02j & 0 & 0 & 1 & 0 \\
-0.21+0.22j & -0.21-0.22j & -0.01 & -0.01 & 0.01+0.01j & 0.01-0.01j & 0 & 0 & 0 & 1
\end{bmatrix}
$$

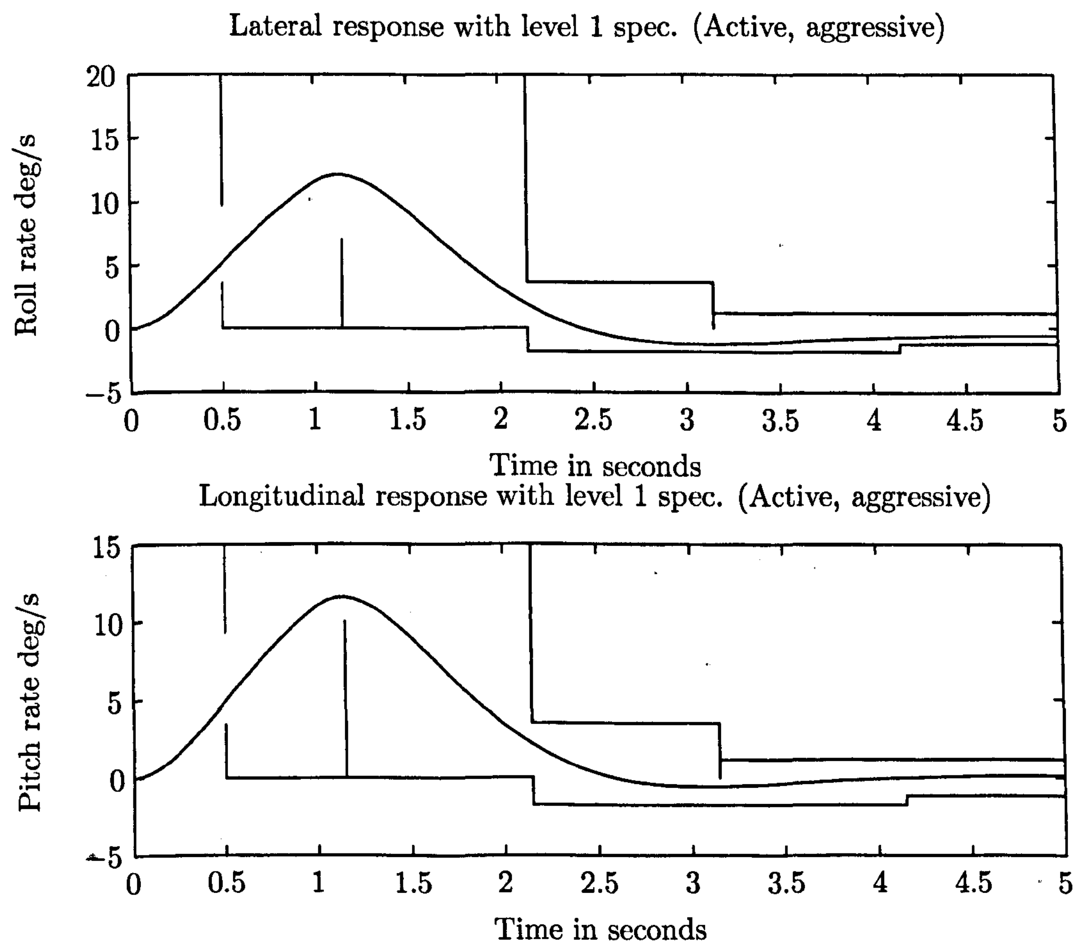

Figure 7.10: Longitudinal and lateral responses of the pseudo-state feedback controller

As an aside, although no attempt has been made to render the pseudo-state feedback controller robust to changes in the open-loop plant, it is reasonably robust to such changes. Figure 7.13 shows the effect on performance of using the pseudo-state feedback controller above to control the 10^{th} order un-reduced model with flapping modes as described in Appendix A. Although the lateral response is somewhat muted, the Level 1 handling qualities specifications are still met. This acts to confirm Griffin's (1997) assertion that despite having no explicit means of guaranteeing that a robust controller will be generated, EA '...[provides]

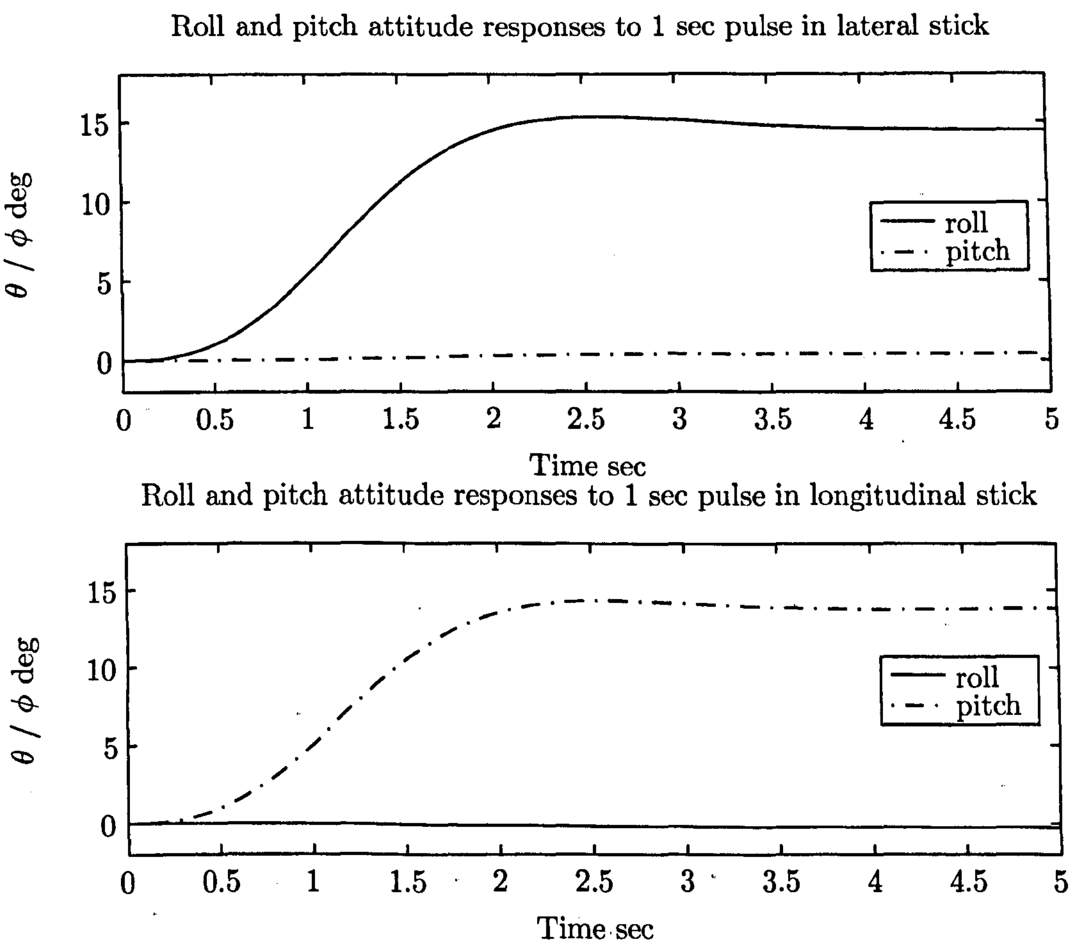

Figure 7.11: On- and off-axis attitude responses to lateral and longitudinal stick

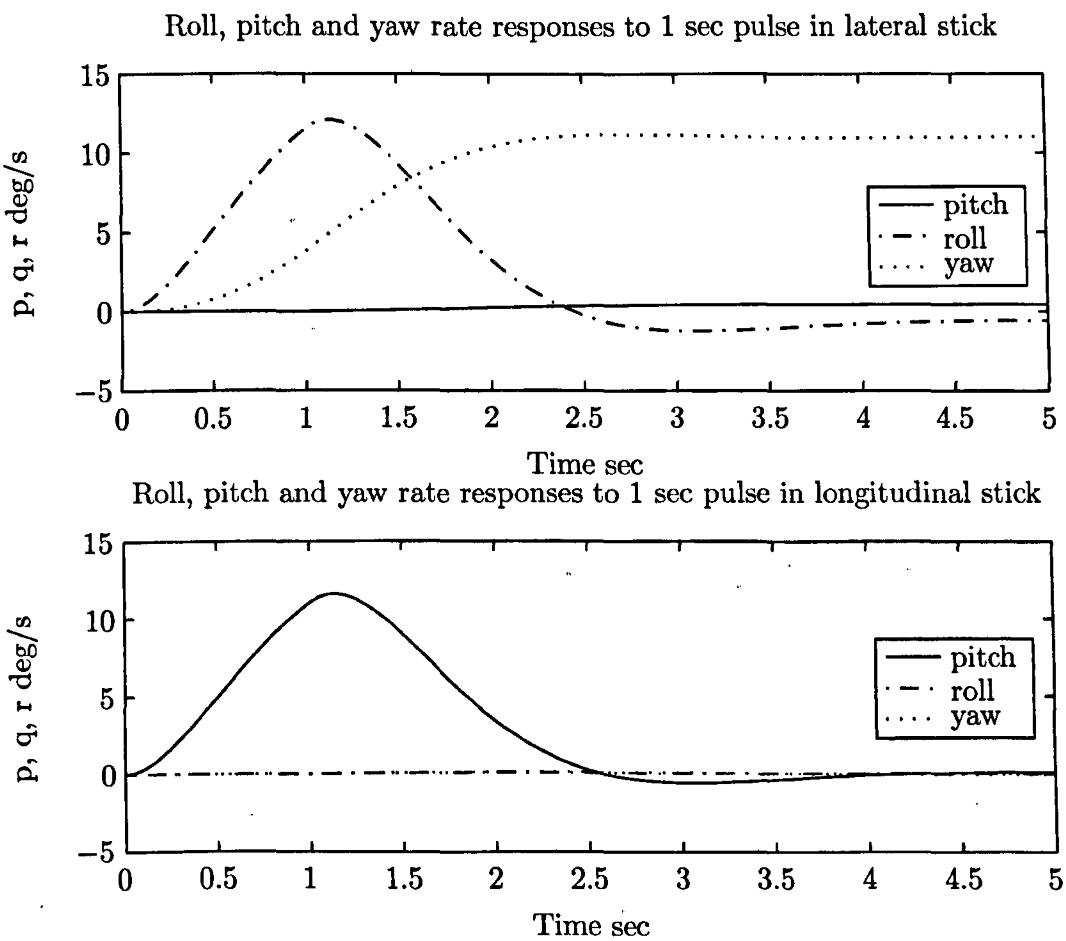

Figure 7.12: Rate responses to lateral and longitudinal stick

access to all the available design freedom and may assign a robust solution as easily as a non-robust solution' (Griffin, 1997, p176).

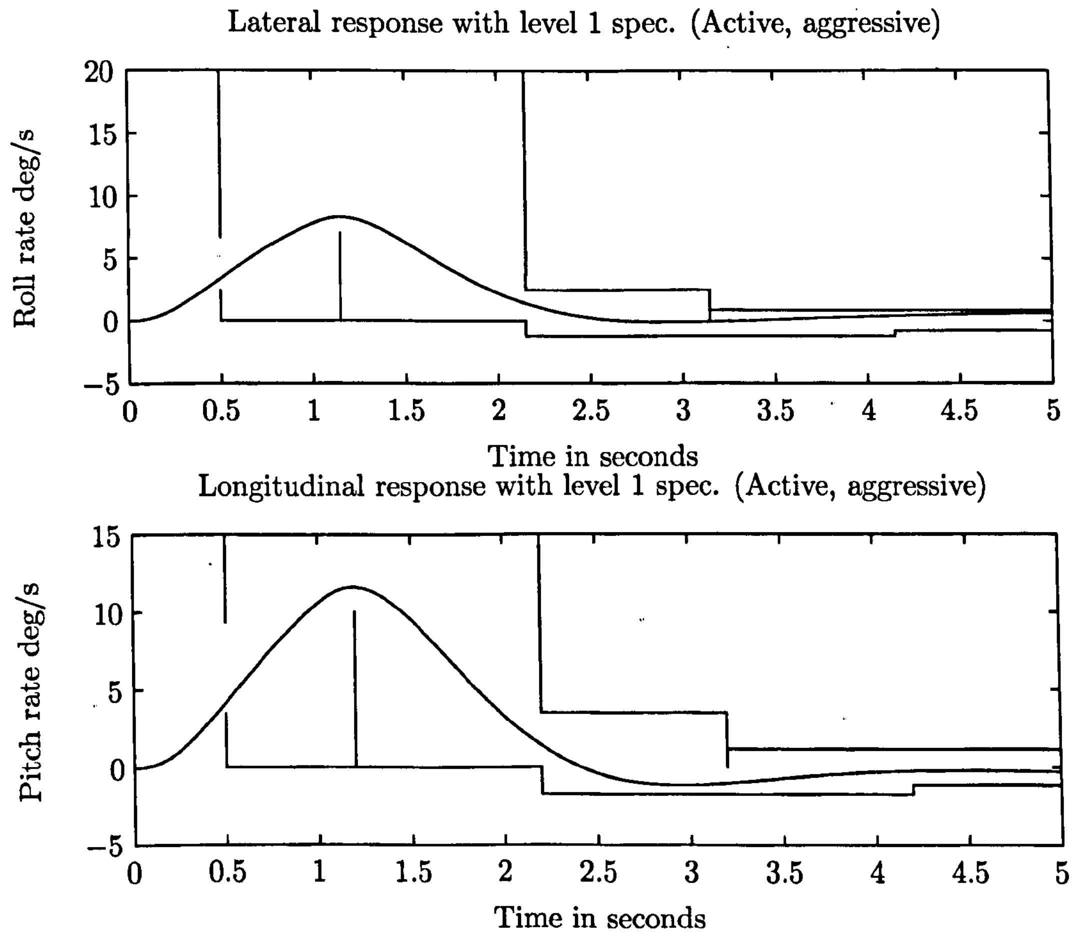

Figure 7.13: Longitudinal and lateral responses of the pseudo-state feedback controlller with main rotor dynamics

7.3.3 Pseudo-State Feedback and Gain Suppression

If it is assumed that a full, accurate INS is present, then the angular rates (p, q, r), body angles (ϕ, θ), accelerations $(\dot{u}, \dot{v}, \dot{w})$, *and* inertial velocities (u, v, w) can all be used for control. In this case, in forward flight, the complete augmented open-loop system will have six inputs, ten states and thirteen outputs. Enough design freedom exists to perform pseudo-state feedback EA and to leave 18 Degrees of Freedom (DoF) remaining post-assignment.

The output vector of such a system, without the compensator augmentation, is

$$\mathbf{y} = \begin{bmatrix} \dot{u} & \dot{v} & \dot{w} & u & v & w & p & q & r & \phi & \theta \end{bmatrix}^T \tag{7.11}$$

and the 2^{nd} order controller has the state vector

$$\mathbf{x}_c = \begin{bmatrix} c_1 & c_2 \end{bmatrix}^T \tag{7.12}$$

and is described by the state-variable equations

$$\dot{\mathbf{x}}_c = \mathbf{A}_c \mathbf{x}_c + \mathbf{B}_c \mathbf{y}$$
$$\mathbf{u} = \mathbf{C}_c \mathbf{x}_c + \mathbf{D}_c \mathbf{y}$$

Having assigned the same desired eigenstructure as in the state-feedback case, the values of the initial controller matrices are shown below.

$$\mathbf{A}_{c0} = \begin{bmatrix} -1.0368 & 0.0180 \\ -0.0176 & -0.0189 \end{bmatrix}$$

$$\mathbf{B}_{c0} = \begin{bmatrix} -0.0008 & 0.0083 & 0.0008 & -0.0004 & -0.0013 & 0.0065 & -0.2214 & -0.0320 & 0.5369 & -0.3164 & -0.0185 \\ 0.0044 & -0.0003 & 0.0006 & 0.0014 & 0.0018 & 0.0017 & -0.0703 & -0.0079 & -0.0178 & 0.0161 & 0.1433 \end{bmatrix}$$

$$\mathbf{C}_{c0} = \begin{bmatrix} -0.0789 & 0.0027 \\ 0.0005 & 0.0080 \\ 0.0027 & 0.0026 \\ -0.3860 & 0.0089 \end{bmatrix}$$

$$\mathbf{D}_{c0} = \begin{bmatrix} 0.0002 & 0.0009 & -0.0001 & 0 & 0.0004 & -0.0007 & 0.0236 & -0.0031 & 0.0606 & -0.0367 & 0.0071 \\ -0.0023 & 0 & 0 & 0 & -0.0001 & 0 & -0.0009 & 0.0039 & -0.0024 & 0.0016 & -0.0742 \\ -0.0008 & 0 & 0.0004 & 0 & 0 & 0.0002 & -0.0008 & 0.0014 & -0.0026 & 0.0017 & -0.0254 \\ -0.0013 & -0.0005 & 0.0006 & -0.0006 & 0.0014 & -0.0002 & -0.1140 & -0.0090 & 0.1551 & -0.1550 & -0.0413 \end{bmatrix}$$

Note that although the assigned eigenstructure is the same as in the state-feedback case, the added transmission zeros are not in the same place; the eigenvalues of $\mathbf{A}_{c0}$ are -1.0365 and

−0.0192. This demonstrates the difficulty of predicting the locations of added zeros.

The sum of squares of all the entries in the compensator matrices is 1.1711. This will be compared to the equivalent figure for the structured controller once the process has been completed.

It was decided to use the available DoF to effect a partial decoupling between the dynamic section of the compensator and the plant. This aim was chosen partly due for its illustrative effect, and partly because it would significantly simplify the structure of the resulting controller, leading to simpler implementation. To this end it was decided to suppress the gains linking the velocities and accelerations $(u, v, w, \dot{u}, \dot{v}, \dot{w})$ to the compensator states, and also those gains coupling the compensator states to the lateral and longitudinal cyclic pitch controls (A_1, B_1).

In practice it was discovered that forcing the suppression of the gains linking the acceleration outputs to the compensator states also suppressed the gains linking the velocity outputs to the compensator states. Hence only 10 DoF were used in the suppression process. The result of suppressing these gains is that the revised controller matrices are as follows:

$$\mathbf{A}_c = \begin{bmatrix} -1.5227 & 0.0078 \\ -0.0449 & -1.4492 \end{bmatrix}$$

$$\mathbf{B}_c = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0.1631 & -0.0043 & -0.0350 & 0.0314 & 0.0005 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.0052 & 0.2008 & 0.0039 & 0.0026 & -0.0013 \end{bmatrix}$$

$$\mathbf{C}_c = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -0.1513 & 0.0069 \\ -0.4285 & 0.0080 \end{bmatrix}$$

$$\mathbf{D}_c = \begin{bmatrix} 0.0001 & 0.0022 & 0 & -0.0001 & 0.0002 & 0.0004 & -0.0387 & -0.0091 & 0.1536 & -0.0932 & 0.0035 \\ -0.0023 & 0 & 0 & 0 & -0.0001 & 0 & -0.0003 & 0.0040 & -0.0027 & 0.0018 & -0.0750 \\ -0.0005 & -0.0027 & 0.0002 & 0.0001 & 0.0004 & -0.0019 & 0.1208 & 0.0132 & -0.1839 & 0.1120 & -0.0185 \\ -0.0012 & -0.0012 & 0.0005 & -0.0005 & 0.0015 & -0.0007 & -0.0804 & -0.0057 & 0.1052 & -0.1246 & -0.0396 \end{bmatrix}$$

The sum of squares of all of the entries in the compensator matrices has risen to 4.8323, but the individual gains are still very small, with the exception of those in the compensator system matrix $\mathbf{A}_c$ (which serve only to generate the required dynamic response in the compensator). The required decoupling is evident from the structure of the compensator input and output matrices $\mathbf{B}_c$ and $\mathbf{C}_c$.

Decoupling the compensator from the velocity states imposes the same structure on the compensator input matrix as was seen in the state-feedback case. Interestingly however, this generates compensator system and input matrices containing identical values to those generated in the state-feedback procedure. The reason for this is not clear, but it implies an interesting (if obscure) link between structure, individual gains and transmission zero locations. This is worthy of further investigation in the future.

Figures 7.14 to 7.16 show the performance of this structured controller. In each case the response can be seen to be identical to that of the original state-feedback controller for forward flight.

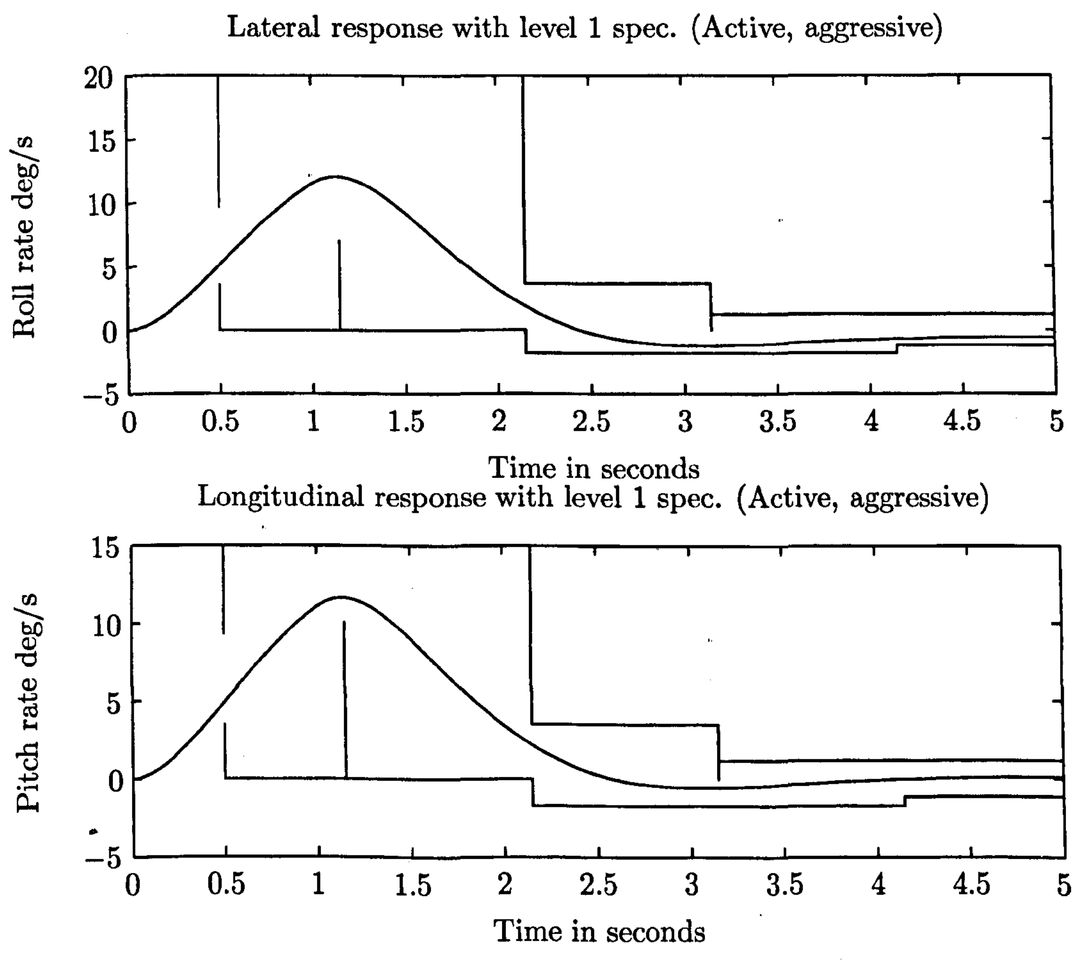

Figure 7.14: Longitudinal and lateral responses of the structured controller

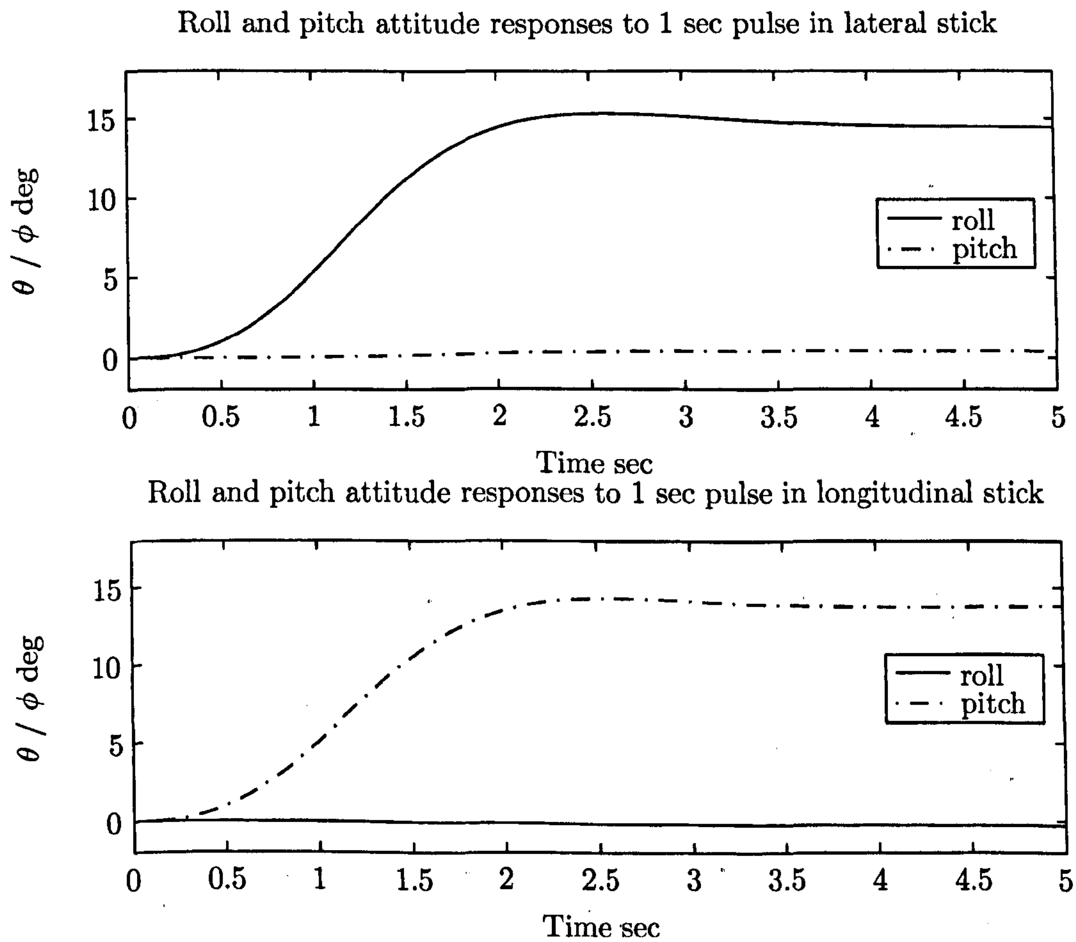

Figure 7.15: On- and off-axis attitude responses to lateral and longitudinal stick

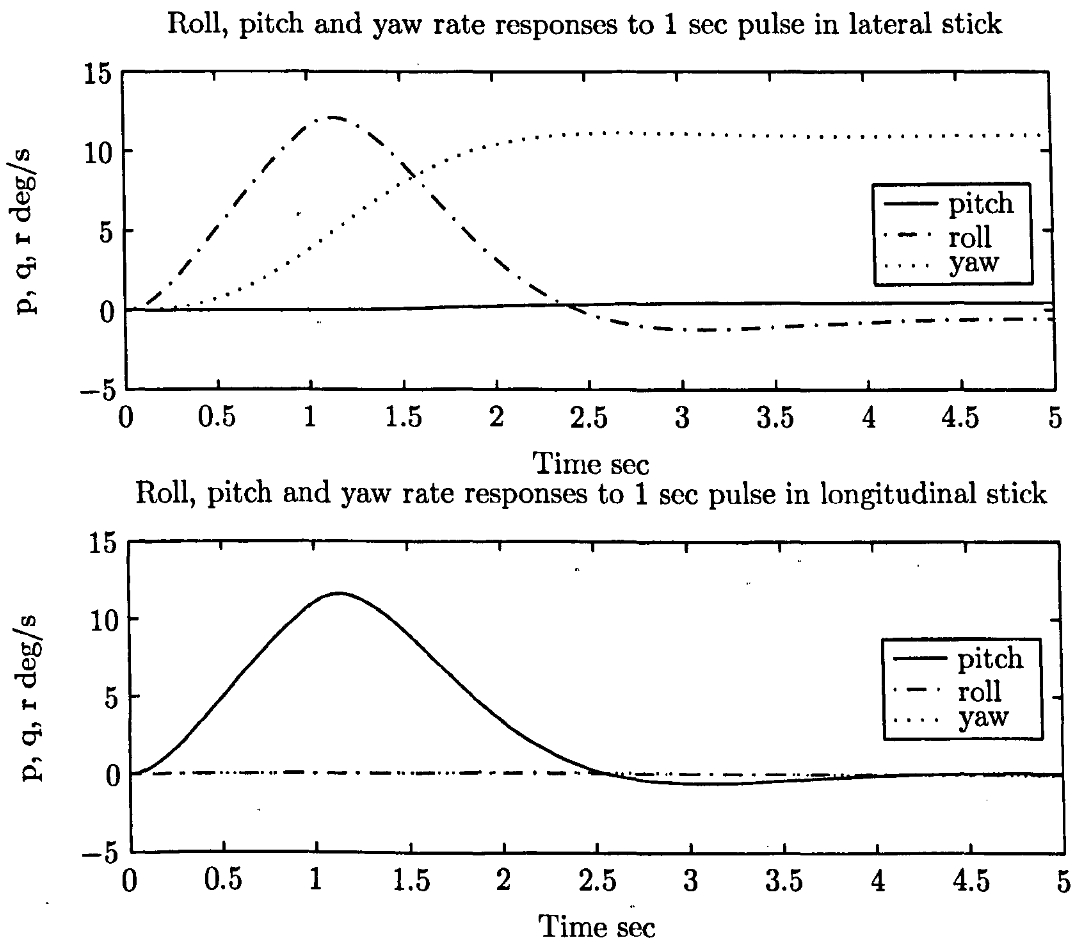

Figure 7.16: Rate responses to lateral and longitudinal stick

The achieved eigenvector set, shown below, can also be seen to be identical to that of the state-feedback and pseudo-state feedback designs above.

$$\mathbf{V}_a = \begin{bmatrix} 0.01+0.21j & 0.01-0.21j & 1 & 1 & -0.01 & -0.01 & 0 & 0 & 0 & 3.65 \\ 1 & 1 & 0 & 0 & 0.02-0.01j & 0.02+0.01j & 0 & 0 & 0 & 0.26 \\ j & -j & 0.004 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.31+0.33j & -0.31-0.33j & 0 & 0 & -0.01+0.01j & -0.01-0.01j & 0 & 0 & 0 & -0.17 \\ 0.04-0.05j & 0.04+0.05j & 0 & 0 & -0.10+0.54j & -0.10-0.54j & 1 & 1 & 0 & 0 \\ 0.03+0.01j & 0.03-0.01j & 0 & 0 & 0.80-0.02j & 0.80+0.02j & 0 & 0 & 0 & -0.01 \\ 0 & 0 & 0 & 0 & j & -j & 0 & 0 & 0 & 0 \\ -0.02+0.00j & -0.02-0.00j & 0 & 0 & -0.24+0.27j & -0.24-0.27j & 0.003 & 0 & 0.01 & -0.02 \\ 0 & 0 & 0 & 0 & -0.01+0.02j & -0.01-0.02j & 0 & 0 & 1 & 0 \\ -0.21+0.22j & -0.21-0.22j & -0.01 & -0.01 & 0.01+0.01j & 0.01-0.01j & 0 & 0 & 0 & 1 \end{bmatrix}$$

7.4 Semi-Proper Output Feedback: Control by IMU

For the purposes of demonstrating the output-feedback EA algorithms for semi-proper systems described in Chapter 5, the following design example for a helicopter in the hover will show how raw data from the IMU (body-relative accelerations and angular rates) may be employed for control. The aim of achieving performance through inertial measurement is a practical one, and since the Def.Stan.00-970 handing qualities specifications have already been analysed to obtain desired eigenstructures (Chapter 3 and Clarke and Taylor (1999)), these form a natural source for a specification.

It must be noted that using IMU signals alone to attempt to achieve Level 1 Def.Stan.00-970 handling qualities is of pedagogical interest only. The proliferation of integrators in the closed-loop system, and the lack of measurement or estimation of the integrated quantities (u, v, w, ϕ, θ), ensures that the steady-state error performance of the helicopter will be highly reliant on the lack of cross-coupling between these quantities. In turn this implies that the closed-loop system will be highly sensitive to eigenvector directions, and hence will not display good robustness characteristics. Nevertheless, this application of output feedback control to a semi-proper system serves to demonstrate the effectiveness of the algorithms presented in Chapter 5.

The longitudinal, lateral and vertical accelerations $(\dot{u}, \dot{v}, \dot{w})$ are derivatives of the velocity states and may be obtained in the output vector by adding rows from the **A** matrix into the **C** matrix and rows from the **B** matrix into the **D** matrix, as described in Section 7.2.2. The roll, pitch and yaw rates (p, q, r) are states and may be fed directly into the output vector.

This yields

$$
\begin{aligned}
\mathbf{y} &= \begin{bmatrix} \dot{v} & p & \dot{u} & q & \dot{w} & r \end{bmatrix}^T \\
\mathbf{x} &= \begin{bmatrix} v & p & \phi & u & q & \theta & w & r \end{bmatrix}^T \\
\mathbf{u} &= \begin{bmatrix} A_1 & B_1 & \theta_0 & \theta_t \end{bmatrix}^T
\end{aligned} .
$$

Again employing the same desired eigenstructure as in Section 7.2 (given by Equation 7.1), it is possible to assign eigenvectors in a variety of combinations. Following experimentation, a suitable order was found to be:

- Assign the right eigenvector associated with the yaw mode λ_r, with 4 DoF;
- Assign the left eigenvector associated with the heave mode λ_w, with 5 DoF;
- Assign right eigenvectors associated with the longitudinal and lateral rate modes λ_p and λ_q, with 3 DoF each;
- Assign the left eigenvectors associated with the longitudinal and lateral velocity modes λ_v and λ_w, with one DoF each.

The gain matrix, generated via Equation 5.140, is:

$$
\mathbf{K} = \begin{bmatrix}
0.00018 & 0.00237 & -0.00653 & 0.04604 & 0.00204 & 0.03325 \\
-0.00304 & -0.00015 & -0.00003 & -0.01492 & 0.03599 & 0.00032 \\
-0.00004 & -0.00002 & 0.00052 & 0.00033 & 0.00001 & -0.00076 \\
-0.00025 & 0.00196 & -0.01012 & -0.01612 & 0.04032 & 0.21922
\end{bmatrix}
$$

As can be observed, the gains are extremely small. The largest gain, less than 0.25, connects r to θ_t and provides yaw damping. The response of the resulting closed loop system is shown in Figures 7.17 to 7.19.

Figure 7.17 shows that the response does not meet the Def.Stan.00-970 Level 1 handling qualities specification, but does not fail by much. The response to the lateral stick results in insufficient undershoot, while the response to the longitudinal stick undershoots too far. Both of these problems appear to be caused by a cross-coupling into yaw, as shown in Figure 7.19. There is also a small coupling from longitudinal stick into roll attitude, as shown in Figure 7.18. The gradient discontinuities evident in Figure 7.19 are symptomatic of direct coupling of the input (a rectangular pulse) to the differential of the output.

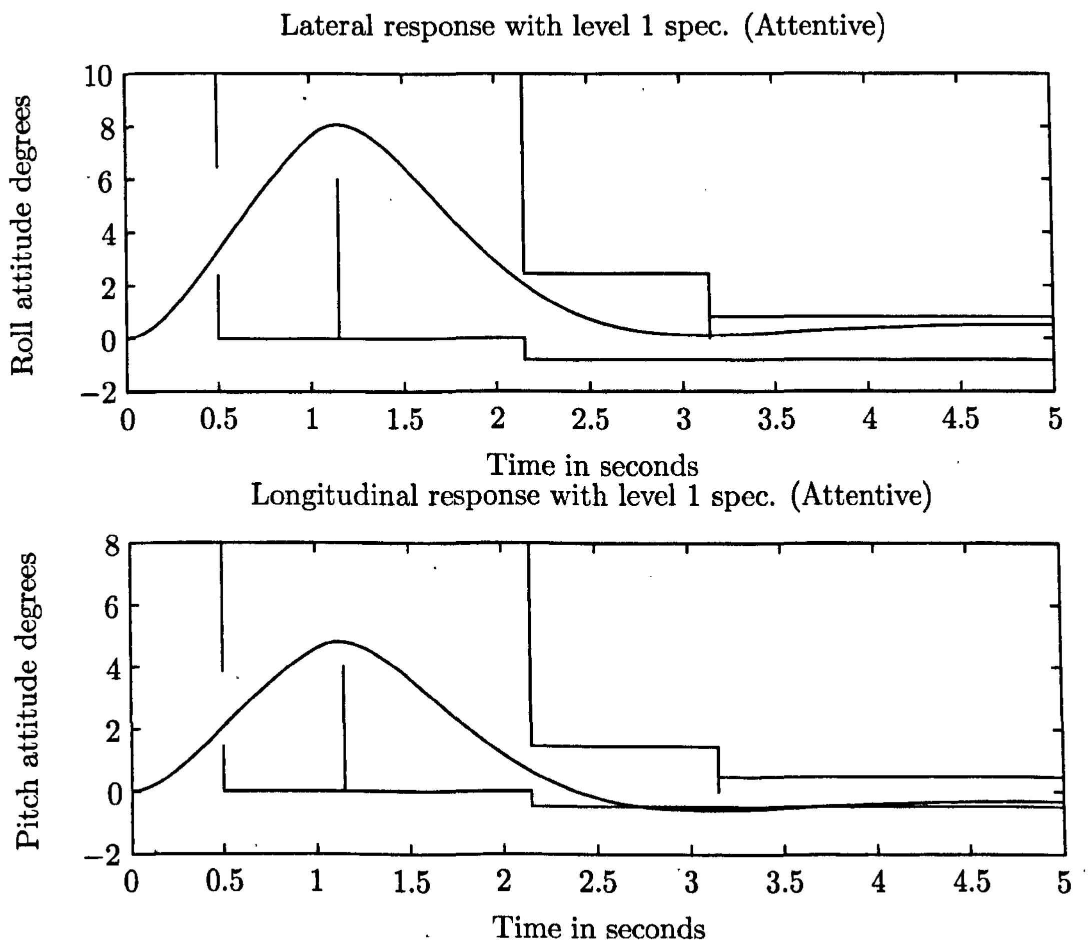

Figure 7.17: Longitudinal and lateral responses with Def.Stan.00-970 templates

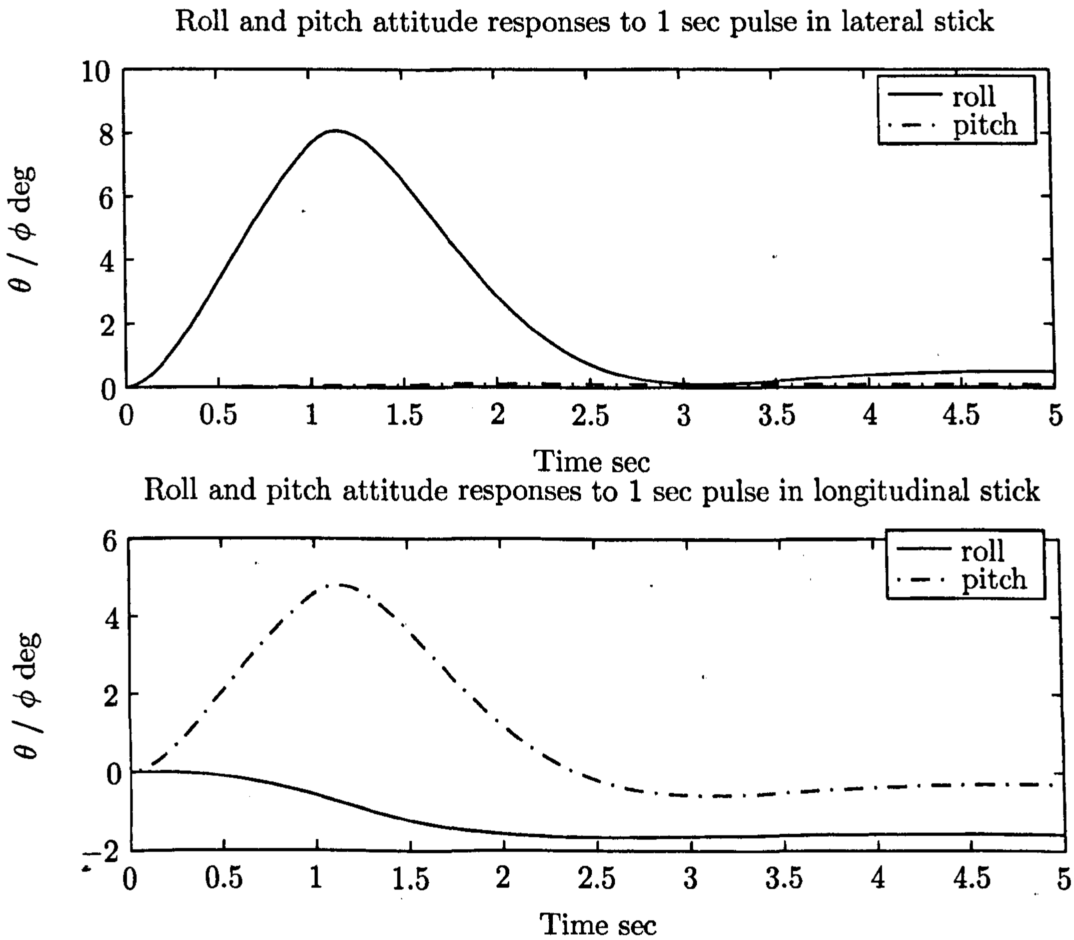

Figure 7.18: On- and off-axis attitude responses to lateral and longitudinal stick

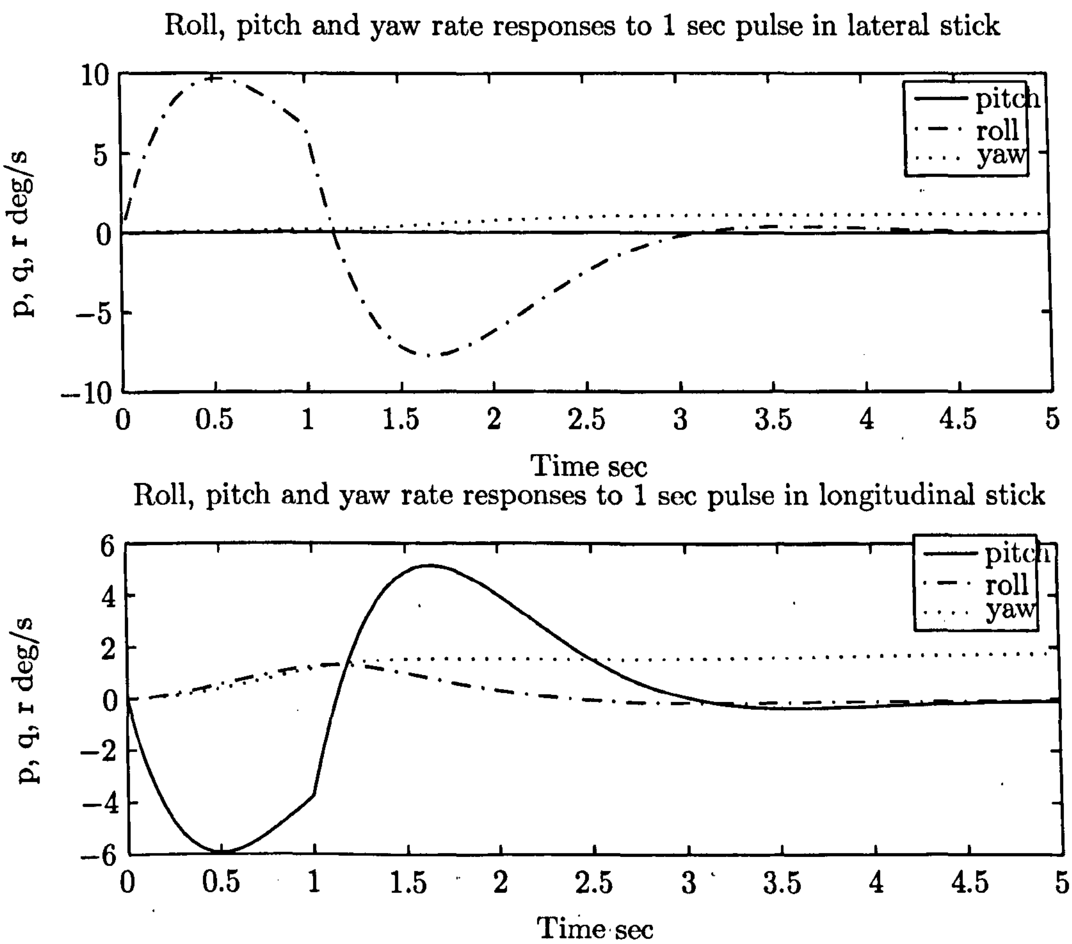

Figure 7.19: Rate responses to lateral and longitudinal stick

It is likely that all of these problems could be mitigated by the use of alternative projection schemes such as iterative projection (Griffin, 1997) or projection with eigenvalue tradeoff (Clarke et al., 2003); retro-assignment could even be employed (see Section 5.5.2.4), though with only one available degree of freedom, it is unlikely to offer a tangible benefit.

7.5 Conclusions

It has been seen that the algorithms presented in this thesis provide a suitable mechanism for the design of controllers for helicopters. In particular, by feeding back state derivatives via accelerometers, a helicopter has been controlled in the hover using pseudo-state feedback techniques; without this acceleration feedback, Griffin (1997) resorted to output feedback, with inevitably inferior results. The output-feedback techniques of Chapter 5 have also been verified, through the design of a control system to stabilise a hovering helicopter using only inertial measurements.

EA for forward flight has also been performed, utilising and verifying the ideal eigenstructure derived in Chapter 3. A state feedback solution, provided for reference, has been matched in performance by a pseudo-state feedback solution; a further pseudo-state feedback design, having more DoFthan required for complete pole placement, has been subjected to the structural imposition techniques of Chapter 6. These techniques have been found to perform well.

7.6 Chapter Bibliography

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Chapter 8

Conclusions

Contents

In Chapter 1 it was stated that the aim of this thesis was to attack the practice-theory gap by promoting an understanding of Eigenstructure Assignment (EA) in a classical context, and by developing algorithms which extend the capabilities of EA towards current practice. The author asserts that this aim has been met, although further work must be undertaken to close the gap completely.

8.1 Eigenstructure Assignment and Helicopters

The link between EA and classical control is clear and was drawn to the reader's attention in Chapter 2. This link is important because the techniques generally used by control engineers working on helicopters are still Single-Input, Single-Output (SISO) loop-at-a-time techniques (Taylor, 2006).

The difference between pole placement and EA was discussed, and is highly relevant. The helicopter is a highly cross-coupled plant, and the ability explicitly to control the coupling of modes into states by manipulation of the eigenvectors is of great importance. It was seen that true EA, along with sufficient design visibility, is provided only by a small number of the algorithms currently available.

In Chapter 3, the physical characteristics of the helicopter were investigated in sufficient detail to gain an understanding of the problems facing the designer of a control system. In particular, cross-coupling and instability were identified as important issues; the helicopter is also highly non-linear, but when working with linear models this is only of significance inasmuch as it indicates the degree to which the controller gains will need to be 'scheduled' across the operating envelope - a practice which, although problematic, is commonplace (and necessary, when the performance specification demands a change in the nature of the pilot control between hover and forward flight).

The specifications for the response of a helicopter, as given in the UK Ministry of Defence Defence Standard 00-970 (Def.Stan.00-970) (Pitkin, 1989) are readily converted into restrictions on pole locations, as demonstrated by Clarke et al. (2003*a*). This work also involved the creation of a set of ideal eigenvectors for a helicopter in the hover.

This thesis has introduced a new set of ideal eigenvectors for the case where the movement of the cyclic pitch stick is expected to generate a response in angular rate rather than attitude. This is not a trivial problem, and requires the use of a feedback compensator to raise the order of the open-loop system such that the coupling from the cyclic pitch inputs to the body rates is second-order. It has been shown that this process naturally introduces zeros into the transfer function matrix of the closed-loop system. The eigenvector set has been demonstrated to be kinematically consistent with the expected motion of the airframe.

8.2 Dynamic Compensators: The Problems

The original aim of this thesis was to investigate the practicability of using dynamic compensators to increase the design freedom available for the use of EA. It rapidly became obvious that the practical problems involved in the successful implementation of dynamic compensators, and the efficient exploitation of the design freedom they carry, are not trivial.

In Chapter 4, an analysis of the distribution of the design freedom added by a compensator was performed in the context of EA. It was shown that the introduction of a compensator increases the order of the system, adds transmission zeros in potentially unpredictable locations, and provides no additional freedom over eigenvector directions. The additional freedom is available for pole placement, if the system was not pole-assignable before the addition of the compensator. The remainder of the freedom is tied up in the compensator sub-eigenvectors, the directions of which have no effect on the plant outputs.

No available algorithms currently address the issues of dynamic compensation and EA efficiently and holistically. Algorithms which place the added transmission zeros (Magni, 1999; Tsui, 1999) exist, but while these ensure that the zero locations are known, the effective need for the determination of the poles of the open-loop compensator *a priori* seems overly restrictive. Hippe and O'Reilly's (1987) design methodology allows placement of these poles post-assignment, but does not consider eigenvector assignment explicitly and relies heavily on symbolic algebra.

In summary, unless the dynamic compensator is required and explicitly defined by the closed-loop system specification, such structures do not currently represent a flexible way of increasing the design freedom available for EA.

8.3 Alternatives to Dynamic Compensation

At the end of Chapter 4 it was argued that any technique which increases the rank of the open-loop output (**C**) matrix would increase the number of available degrees of freedom, and do so in a manner that was preferable to the way in which dynamic compensators do the same. This is because an increase in rank of the **C** matrix increases the number of explicitly assignable right eigenvectors without increasing the system order. This could render a system pole-assignable, or even allow state-feedback techniques to be used instead of output feedback techniques. Even if none of these advantages apply, the ability to assign more right eigenvectors is useful since the mode-input coupling can be influenced by the addition of a pre-filter if required, while the mode-output coupling cannot be similarly influenced and is controlled solely by the right eigenvectors; the allowable subspace for any assigned left eigenvectors is also inflated by an increased rank **C** matrix.

Increasing the rank of the **C** matrix can be achieved by adding more sensors to the plant. There is no reason to suppose the plant will not be well-instrumented already, but any sensor which measures a state derivative - accelerometers, for example - were heretofore generally not suited to EA because the appearance of state derivatives in the output vector introduces nonzero terms into the direct transmission (**D**) matrix. For example, the helicopter model considered by Griffin (1997) is controlled using output-feedback techniques because the linear velocity states were considered to be unmeasurable. If the derivatives of these states could be fed back, the problem would revert to one in which the numbers of outputs and states were the same.

Chapter 5 was therefore dedicated to the development and description of two novel algorithms for EA in semi-proper systems. Changes occur to both the input (**B**) and output (**C**) matrices in addition to the system (**A**) matrix when the loop is closed, and these have been dealt with in an elegant way. Techniques for assigning the eigenvectors or the modal coupling vectors have been developed, and the modal coupling vector assignment requires no *a priori* knowledge of the inevitable changes to the **B** and **C** matrices.

The first of the assignment techniques bears large similarities to the classic state-feedback EA algorithm of Moore (1976), but cannot be described as state-feedback because the output and state vectors are different and the **D** matrix is nonzero. Hence the term 'pseudo-state feedback' has been coined to describe this new algorithm. In common with state-feedback algorithms, it is possible using this technique to assign all the eigenvalues and all of the right eigenvectors of a system.

The second technique forms a natural extension to the output-feedback EA algorithms of Clarke et al. (2003*b*), and provides access to the eigenvalues and a subset of each of the left and right eigenvectors in a highly visible parametric form.

Each algorithm has the potential to leave design freedom unused post-assignment. In the case of the pseudo-state feedback algorithm, this is because the number of outputs may potentially exceed the number of states. The manner in which this unused freedom is expressed is essentially consistent between the two systems. In the case of the output-feedback algorithm, an extension has been presented to allow retro-assignment (Clarke and Griffin, 2004) to take place. However, in the opinion of the author, it is likely that the main performance goals will have been satisfied by the primary assignment of important right eigenvectors, and it is unlikely that any further manipulation of complimentary eigenvectors will improve the situation.

8.4 Imposing Structure

The freedom left by the two algorithms described in Chapter 5 has use beyond EA, and Chapter 6 presented a novel algorithm which used the freedom remaining after the application of either algorithm, or indeed that of Clarke et al. (2003*b*), for imposing structure upon the gain matrix. This could be by reducing gain matrix entries to zero, thus severing paths between the system outputs and inputs, or by forcing certain sets of entries to be equal or to be 'geared' in a given ratio.

The effect of the imposition of structure upon the overall magnitude of the entries in the gain matrix has been considered, and expressions for the minimum Frobenius norm of the gain matrix both before and after the process have been generated. These should assist in the choice of structural constraints. It is interesting to note that some structural constraints will have more effect on the overall gains than others, and so to some extent the final structure of the controller is governed (or at least influenced) by the initial EA process.

8.5 Applications

Chapter 7 provided design examples to support the algorithmic development described above. The hover case was considered first, and it was shown that by using feedback from accelerometers, the pseudo-state feedback algorithm of Chapter 5 was able to retain the performance of a state-feedback solution while using only measurable information. In the same context, Griffin (1997) had resorted to output feedback, and the results were inevitably inferior.

Forward flight required the use of the ideal eigenstructure derived in Chapter 3. Using this eigenstructure, Level 1 handling qualities were obtained at 60 knots using both state-feedback and pseudo-state feedback algorithms, demonstrating that the eigenstructure is correct both kinematically and in terms of the Def.Stan.00-970 specification.

A pseudo-state feedback controller was also designed for forward flight that had excessive Degrees of Freedom (DoF), and the remaining DoF after EA were used to impose structure on the controller. This demonstrated well the operation of the algorithm of Chapter 6.

In addition, a pedagogical case was presented wherein it was attempted to control a helicopter in forward flight using only the signals from an Inertial Measurement Unit (IMU) - that is, the roll, pitch and yaw rates and the body accelerations. The output feedback algorithm from Chapter 5 was used and the responses generated were not far from meeting Level 1 Def.Stan.00-970 specifications despite the lack of feedback from the body angle states and a consequent inability to control steady-state errors.

8.6 Contributions

To the best of the author's knowledge, the list below describes the novel contributions of this work:

- Derivation of a new ideal eigenstructure that meets the Def.Stan.00-970 response criteria for Level 1 handling qualities in forward flight, and demonstration that it is appropriate.
- Derivation of an ideal second-order yaw response eigenstructure.
- Characterisation of the design freedom offered by dynamic compensators in terms of the effect solely on the plant sub-eigenvectors, namely the rank and mutual orthogonality of the assigned eigenvector subsets.
- Analysis of the design freedom offered by a feedforward compensator term, and how this freedom may be exploited.
- Analysis of the effects on the modal coupling matrices of the change to the input and output matrices of a semi-proper system under feedback.
- Development of a novel algorithm to effect 'pseudo-state feedback' control for semi-proper systems, offering a substantially identical design procedure to that of standard state-feedback EA.
- Characterisation of the design freedom remaining after the assignment of eigenstructure to a system with more outputs than states.
- Development of a novel algorithm to allow output-feedback EA to be conducted on semi-proper systems.
- Characterisation of any remaining design freedom from the semi-proper output-feedback EA process in a form which allows a retro-assignment stage to be performed, and the development of this technique to reflect the change to the system input and output matrices in the closed loop.
- Implementation of Proportional-plus-Derivative (PD) controllers using semi-proper system descriptions, allowing EA to be conducted in a single, visible stage.
- A new analysis of the design freedom remaining after EA, and the resulting discovery of a method by which it may be used to impose structure upon the controller.

- Design of an algorithm to allow the freedom remaining after output-feedback EA, semi-proper output-feedback EA or pseudo-state feedback EA to be used for this purpose.
- Development of tools for the analysis of the effect of the imposition of structure on the overall magnitude of the gain matrix elements.

8.7 Further Work

Several areas have been identified in this thesis which warrant further work.

Chapter 4 revealed some problems with the available algorithms for performing EA on compensated systems. The ideal algorithm for the purposes of EA would constrain the transmission zeros, allow full control of eigenvalue locations and plant sub-eigenvector directions, and present the unused design freedom in a usable form. This would allow the EA process to inform the design of a controller structure, rather than vice-versa.

The idea of adding feedforward terms to feedback compensators was introduced. Section 4.6.3 contains several ideas for the potential exploitation of the design freedom offered post-assignment by this technique, and these would form an ideal starting point for a programme of further work on this topic. Structural constraints, transmission zero placement and robustness improvement are just a few of the potential applications for this extra freedom.

The algorithm for introducing structural constraints in Chapter 6, at present, utilises only the design freedom remaining after EA. If similar techniques could be used to restrict the available set of gains during assignment, while still allowing freedom over eigenvector selection, this could potentially provide an algorithm for using EA to optimise the gains of existing control structures. This would be an interesting new application area for EA.

8.8 Chapter Bibliography

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Appendix A

Helicopter Linearisations

Contents

For the design examples presented in Chapter 7, simplified linear approximations to a non-linear 11^{th} order model of a Lynx helicopter were employed. This Appendix provides an overview of the algorithms used to obtain the linear models, together with the state-variable descriptions of the models so obtained. For the sake of allowing direct comparison, the procedure used for generating the linear models was the same as that used by Griffin (1997), and is simply summarised here for reference.

A.1 Algorithm Descriptions

In order to extract a linear model from a non-linear one, it is first necessary to obtain an operating point about which a linear approximation can be formed. This process is known as *trimming*. In the case of the design examples of Chapter 7, two trim points (hover and forward flight at 60 knots) were chosen. Secondly, a linear approximation to the system at these points must be found; and thirdly, the model must be reduced in complexity to remove the near-uncontrollable blade flapping modes.

A.1.1 Trim

Before the model can be linearised, an operating point must be found where the system is in equilibrium.

In the case of the hover, this point is with zero body velocities and accelerations, zero angular rates and angular rate derivatives, and zero blade flapping angle derivatives. The body angles are not constrained. At 60 knots, the trim point is with zero angular rates and angular rate derivatives, zero body accelerations and heave velocity, and a zero roll angle with the pilot tolerating (and expecting) a small sideslip angle (Griffin, 1997). Once again the flapping angle derivatives are required to be zero, but the pitch angle is unconstrained, as is lateral velocity.

For the purposes of the design examples in this thesis, trimming was conducted using a Newton-Raphson nonlinear equation solver, after Griffin (1997) who achieved success with the same method on the same model.

A.1.2 Linearisation and Normalisation

The linearisation of a nonlinear function around a given point is a first-order Taylor approximation about that point. Such an approximation is easily calculated by perturbing each state and input in turn an observing its effect on the state derivatives to form entries in the **A** and **B** matrices. For the work in this thesis, an algorithm to adjust the perturbation sizes automatically (Clarke and Griffin, 2003) was employed in order to generate a model that was valid over as wide a range of operating conditions as possible.

Griffin (1997) also developed a set of normalisations for the model such that two eigenvector elements of equal magnitude would result in a *perceived* equal coupling of the mode into the two states. The normalisation values are given in Table A.1.2.

These normalisations have not been applied to the linear models as given below, but were instead applied (in inverse form) to the desired eigenstructure before assignment.

State	Value	State	Value	Input	Value
u	5 ft/s	θ	0.1 rad	θ_0	0.02 rad
v	5 ft/s	ϕ	0.1 rad	θ_t	0.035 rad
w	5 ft/s	ψ	0.1 rad	A_1	0.02 rad
p	0.1 rad/s	a_{1s}	0.025 rad	B_1	0.035 rad
q	0.1 rad/s	b_{1s}	0.025 rad		
r	0.1 rad/s				

Table A.1: Normalisation values (from Griffin, 1997)

A.1.3 Model Reduction

The dynamics of the helicopter are unaffected by heading, and so the heading state ψ affects no others and can be truncated from the model.

The flapping modes may also be removed if it is assumed that they are significantly faster than the body dynamics. This is achieved by assuming that the flapping modes are unexcited, and hence that the blade flapping state derivatives $\dot{a}_{1s}$ and $\dot{b}_{1s}$ are zero. These states may then be approximated as a combination of other states and inputs.

For the purposes of generating these example systems, model reduction was performed using MATLAB's `modred` function from the Control System Toolbox (The Mathworks, Inc., 2005); for an explanation of the mathematics involved, see Griffin (1997).

A.2 Linearisation at Hover

This is the model, produced using the techniques in Section A.1 from the nonlinear Lynx model in hover, that was used in Chapter 7.

A.2.1 State Space Description

The state and input vectors are defined as

$$\mathbf{x}^T = \begin{bmatrix} u & v & w & p & q & r & \phi & \theta & \psi & a_{1s} & b_{1s} \end{bmatrix}$$
$$\mathbf{u}^T = \begin{bmatrix} A_1 & B_1 & \theta_0 & \theta_t \end{bmatrix}$$

The linear velocities (u, v, w) are given in feet per second, and the angular rates (p, q, r) in radians per second. The body angles (ϕ, θ, ψ), the blade flapping angles (a_{1s}, b_{1s}) and the input angles $(A_1, B_1, \theta_0, \theta_t)$ are given in radians.

The system and input matrices are:

$$\mathbf{A} = \begin{bmatrix} -0.0043 & 0 & 0.0177 & 0 & 0.0089 & 0 & 0 & -32.1169 & 0 & -32.8194 & 0 \\ 0.0007 & -0.0169 & -0.0034 & -0.0697 & -0.0626 & 0.3541 & 32.0644 & 0.1096 & 0 & 0 & 32.8194 \\ 0.0314 & 0 & -0.3226 & 0 & -0.0537 & 0 & 1.8355 & -1.9141 & 0 & -2.2912 & 0 \\ 0.0001 & -0.0005 & -0.0001 & -0.0028 & 6.6266 & -0.0260 & 0 & 0 & 0 & 0 & 112.9618 \\ 0.0005 & 0 & -0.0021 & -1.0094 & -0.0153 & 0 & 0 & 0 & 0 & 17.2069 & 0 \\ -0.0004 & 0.0085 & 0.0027 & 0.0355 & 1.2457 & -0.2207 & 0 & 0 & 0 & 0 & 20.3546 \\ 0 & 0 & 0 & 1.0000 & -0.0034 & 0.0596 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.9984 & 0.0572 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.0573 & 1.0001 & 0 & 0 & 0 & 0 & 0 \\ 0.0158 & -0.0012 & 0.0011 & -0.0131 & -1.2725 & 0 & 0 & 0 & 0 & -14.0578 & -2.2016 \\ 0.0189 & -0.0070 & 0.0013 & -1.2357 & -0.0706 & 0 & 0 & 0 & 0 & 2.2016 & -14.0578 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 17.9002 & 0 \\ 0 & 0 & -1.4148 & 12.8888 \\ 0 & 0 & -299.3701 & 0 \\ 0 & 0 & 6.7227 & -0.9453 \\ 0 & 0 & -1.5234 & 0 \\ 0 & 0 & 14.2787 & -8.0313 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2.7396 & -16.0247 & 0 & 0 \\ 15.9324 & 2.7555 & 0 & 0 \end{bmatrix}$$

A.2.2 Reduced-Order Model

$$\mathbf{A} = \begin{bmatrix} -0.0336 & 0.0002 & 0.0157 & -0.4111 & 2.8834 & 0 & 0 & -32.1169 \\ 0.0494 & -0.0331 & 0 & -2.8901 & -0.6777 & 0.3541 & 32.0644 & 0.1096 \\ 0.0293 & 0 & -0.3228 & -0.0287 & 0.1470 & 0 & 1.8355 & -1.9141 \\ 0.1677 & -0.0565 & 0.0116 & -9.7105 & 4.5094 & -0.0260 & 0 & 0 \\ 0.0158 & -0.0001 & -0.0010 & -0.7938 & -1.5223 & 0 & 0 & 0 \\ 0.0298 & -0.0016 & 0.0048 & -1.7137 & 0.8642 & -0.2207 & 0 & 0 \\ 0 & 0 & 0 & 1.0000 & -0.0034 & 0.0596 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.9984 & 0.0572 & 0 & 0 \end{bmatrix}.$$

$$\mathbf{B} = \begin{bmatrix} -0.5570 & 37.4993 & 17.9002 & 0 \\ 37.2832 & 0.5602 & -1.4148 & 12.8888 \\ -0.0389 & 2.6179 & -299.3701 & 0 \\ 128.3257 & 1.9283 & 6.7227 & -0.9453 \\ 0.2920 & -19.6605 & -1.5234 & 0 \\ 23.1231 & 0.3475 & 14.2787 & -8.0313 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.0000 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.0000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.0000 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.0009 & 0 & 0.0001 & 0.0125 & -0.0876 & 0 & 0 & 0 \\ 0.0015 & -0.0005 & 0.0001 & -0.0859 & -0.0187 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.0170 & -1.1426 & 0 & 0 \\ 1.1360 & 0.0171 & 0 & 0 \end{bmatrix}$$

A.3 Linearisation at 60 Knots

This is the model, produced using the techniques in Section A.1 from the nonlinear Lynx model in forward flight at 60 Knots, that was used in Chapter 7.

A.3.1 State Space Description

The state and input vectors are again defined as

$$\mathbf{x}^T = \begin{bmatrix} u & v & w & p & q & r & \phi & \theta & \psi & a_{1s} & b_{1s} \end{bmatrix}$$
$$\mathbf{u}^T \doteq \begin{bmatrix} A_1 & B_1 & \theta_0 & \theta_t \end{bmatrix}$$

where the state variables have the same meanings and units as before. The system and input matrices are:

$$\mathbf{A} = \begin{bmatrix}
-0.0040 & 0.0010 & 0.0322 & 0.0429 & -2.9081 & 0 & 0 & -32.1373 & 0 & -32.1525 & 0 \\
0.0047 & -0.0806 & -0.0042 & 2.7750 & -0.0879 & -59.4753 & 32.1123 & 0.0607 & 0 & 0 & 32.1525 \\
-0.1435 & -0.0159 & -0.5577 & -0.6917 & 60.7602 & 0 & 1.2664 & -1.5379 & 0 & -2.2447 & 0 \\
-0.0007 & 0.0034 & -0.0032 & 0.0096 & 6.6152 & -0.0385 & 0 & 0 & 0 & -0.7849 & 112.5741 \\
-0.0001 & -0.0001 & -0.0013 & -1.0153 & -0.0383 & 0 & 0 & 0 & 0 & 17.1473 & 0 \\
-0.0042 & 0.0098 & -0.0020 & 0.0764 & 1.2644 & -0.3269 & 0 & 0 & 0 & -1.4968 & 20.2886 \\
0 & 0 & 0 & 1.0000 & -0.0019 & 0.0479 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.9992 & 0.0394 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.0395 & 1.0004 & 0 & 0 & 0 & 0 & 0 \\
0.0055 & 0.0105 & 0.0105 & 0.0399 & -1.2329 & 0 & 0 & 0 & 0 & -14.0402 & -2.2306 \\
-0.0032 & -0.0107 & 0.0170 & -1.2404 & 0.0160 & 0 & 0 & 0 & 0 & 2.1939 & -14.1237
\end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix}
-0.5892 & -2.3630 & 18.6706 & 0 \\
-0.0043 & -0.0172 & 0.1363 & 11.4212 \\
9.5059 & 38.1262 & -301.2394 & 0 \\
-0.1676 & -0.6721 & 5.3107 & -0.8376 \\
0.0398 & 0.1596 & -1.2612 & 0 \\
-0.3166 & -1.2699 & 10.0334 & -7.1168 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
2.7003 & -16.5120 & 5.2055 & 0 \\
16.0264 & 2.1585 & 3.9017 & 0
\end{bmatrix}$$

A.3.2 Reduced-Order Model

$$
\mathbf{A} = \begin{bmatrix}
-0.0175 & -0.0262 & 0.0147 & -0.4842 & -0.1471 & 0 & 0 & -32.1373 \\
-0.0004 & -0.1007 & 0.0373 & 0.0330 & -0.4803 & -59.4753 & 32.1123 & 0.0607 \\
-0.1444 & -0.0178 & -0.5589 & -0.7285 & 60.9529 & 0 & 1.2664 & -1.5379 \\
-0.0189 & -0.0675 & 0.1416 & -9.6035 & 5.3085 & -0.0385 & 0 & 0 \\
0.0071 & 0.0144 & 0.0081 & -0.7342 & -1.5108 & 0 & 0 & 0 \\
-0.0081 & -0.0041 & 0.0233 & -1.6783 & 1.1452 & -0.3269 & 0 & 0 \\
0 & 0 & 0 & 1.0000 & -0.0019 & 0.0479 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.9992 & 0.0394 & 0 & 0
\end{bmatrix}
$$

$$
\mathbf{B} = \begin{bmatrix}
-0.9672 & 35.3011 & 8.4142 & 0 \\
36.5384 & -0.9540 & 10.6117 & 11.4212 \\
9.4795 & 40.7556 & -301.9554 & 0 \\
127.7683 & -3.0326 & 41.7375 & -0.8376 \\
0.2414 & -19.9271 & 4.2087 & 0 \\
22.7247 & -0.1076 & 16.1660 & -7.1168 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\mathbf{C} = \begin{bmatrix}
1.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1.0000 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1.0000 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1.0000 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1.0000 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1.0000 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1.0000 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.0000 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.0004 & 0.0008 & 0.0005 & 0.0164 & -0.0859 & 0 & 0 & 0 \\
-0.0002 & -0.0006 & 0.0013 & -0.0853 & -0.0122 & 0 & 0 & 0
\end{bmatrix}
$$

$$\mathbf{D} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.0118 & -1.1714 & 0.3190 & 0 \\ 1.1365 & -0.0291 & 0.3258 & 0 \end{bmatrix}$$

A.4 Appendix Bibliography

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Appendix B

Supporting Mathematics

Contents

B.1 Introduction

This Appendix contains mathematical derivations which support the derivations in the main body of this thesis, but which would act as a distraction if placed therein.

B.2 Complex Eigenvector Extension to Section 4.4

The assigned eigenvector subsets in Equation 4.57 must form self-conjugate sets. Assuming that the rows of $\mathbf{W}'_p$ and the columns of $\mathbf{V}'_p$ are arranged such that conjugate pairs are next to each other, two matrices $\mathbf{\Gamma}_w \in \mathbb{C}^{(n+c-v)\times(n+c-v)}$ and $\mathbf{\Gamma}_v \in \mathbb{C}^{v\times v}$ may be formed with block diagonal structures. The blocks on the leading diagonal of $\mathbf{\Gamma}_w$ are:

- A '1' corresponding to any real row in $\mathbf{W}'_p$;
- A block $\begin{bmatrix} 1 & j \\ 1 & -j \end{bmatrix}$ corresponding to any conjugate pair of rows in $\mathbf{W}'_p$.

Similarly the blocks on the leading diagonal of $\mathbf{\Gamma}_v$ are:

- A '1' corresponding to any real column in $\mathbf{V}'_p$;
- A block $\begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix}$ corresponding to any conjugate pair of columns in $\mathbf{V}'_p$.

It is easily verified that it is therefore possible to write

$$\mathbf{W}'_p = \mathbf{\Gamma}_w \mathbf{W}'_{pr} \tag{B.1}$$

$$\mathbf{V}'_p = \mathbf{V}'_{pr} \mathbf{\Gamma}_v \tag{B.2}$$

where $\mathbf{W}'_{pr}$ and $\mathbf{V}'_{pr}$ are real-valued.

The nature and order of the conjugate rows and columns in $\mathbf{W}'_c$ and $\mathbf{V}'_c$ must be the same as those of $\mathbf{W}'_p$ and $\mathbf{V}'_p$. Hence the terms $\mathbf{W}_x$ and $\mathbf{V}_x$ can be formed by setting

$$\mathbf{W}_x \mathbf{V}_x = \mathbf{W}'_{pr} \mathbf{V}'_{pr} \tag{B.3}$$

in place of Equation 4.57. The compensator sub-eigenvector sets may then be chosen using

$$\mathbf{W}'_c = -\mathbf{\Gamma}_w \mathbf{W}_x \mathbf{R}_1 \tag{B.4}$$

$$\mathbf{V}'_c = \mathbf{R}_2 \left(\mathbf{R}_1 \mathbf{R}_2\right)^{-1} \mathbf{V}_x \mathbf{\Gamma}_v \tag{B.5}$$

where $\mathbf{R}_1$ and $\mathbf{R}_2$ are still real-valued.

B.3 Minimum Frobenius norm for Section 6.4.1

Because $\operatorname{vec}\mathbf{K}$ is simply a rearrangement of entries,

$$|\mathbf{K}|_F = \|\operatorname{vec}\mathbf{K}\| \tag{B.6}$$

and therefore

$$|\mathbf{K}|_F = \left\| \left(\mathbf{I} - (\mathbf{U}\Xi)^\dagger \mathbf{U}\right) \left(\operatorname{vec}\mathbf{K}_0 + \Xi \operatorname{vec}\tilde{\mathbf{Z}}\right) \right\| \tag{B.7}$$

$$= \left\| \left(\mathbf{I} - (\mathbf{U}\Xi)^\dagger \mathbf{U}\right) \operatorname{vec}\mathbf{K}_0 + \left(\mathbf{I} - (\mathbf{U}\Xi)^\dagger \mathbf{U}\right) \Xi \operatorname{vec}\tilde{\mathbf{Z}} \right\| \tag{B.8}$$

The problem is now one of solving an equation of the form

$$\frac{d}{d\mathbf{x}}\|\mathbf{Ax}+\mathbf{y}\| = 0 \tag{B.9}$$

to find a minimum. It may be shown (see Section B.4 of this Appendix) that

$$\frac{d}{d\mathbf{x}}\|\mathbf{Ax}+\mathbf{y}\| = \frac{\mathbf{A}^T(\mathbf{Ax}+\mathbf{y})}{\|\mathbf{Ax}+\mathbf{y}\|} \tag{B.10}$$

and therefore we must solve

$$\frac{d}{d\tilde{\mathbf{Z}}}\left\|\left(\mathbf{I}-(\mathbf{U}\Xi)^{\dagger}\mathbf{U}\right)\operatorname{vec}\mathbf{K}_0+\left(\mathbf{I}-(\mathbf{U}\Xi)^{\dagger}\mathbf{U}\right)\Xi\operatorname{vec}\tilde{\mathbf{Z}}\right\| = \mathbf{0} \tag{B.11}$$

i.e.

$$\left(\Xi-(\mathbf{U}\Xi)^{\dagger}\mathbf{U}\Xi\right)^T\left(\left(\mathbf{I}-(\mathbf{U}\Xi)^{\dagger}\mathbf{U}\right)\operatorname{vec}\mathbf{K}_0+\left(\mathbf{I}-(\mathbf{U}\Xi)^{\dagger}\mathbf{U}\right)\Xi\operatorname{vec}\tilde{\mathbf{Z}}\right) = \mathbf{0} \tag{B.12}$$

so,

$$\left(\Xi-(\mathbf{U}\Xi)^{\dagger}\mathbf{U}\Xi\right)\left(\left(\mathbf{I}-(\mathbf{U}\Xi)^{\dagger}\mathbf{U}\right)\operatorname{vec}\mathbf{K}_0+\left(\mathbf{I}-(\mathbf{U}\Xi)^{\dagger}\mathbf{U}\right)\Xi\operatorname{vec}\tilde{\mathbf{Z}}\right) = \mathbf{0} \tag{B.13}$$

since $\mathbf{A}^{\dagger}\mathbf{A}$ and Ξ are both symmetric.

Equation B.13 may be simplified further:

$$\left(\Xi-(\mathbf{U}\Xi)^{\dagger}\mathbf{U}\Xi\right)\left(\operatorname{vec}\mathbf{K}_0-(\mathbf{U}\Xi)^{\dagger}\mathbf{U}\operatorname{vec}\mathbf{K}_0+\Xi\operatorname{vec}\tilde{\mathbf{Z}}-(\mathbf{U}\Xi)^{\dagger}\mathbf{U}\Xi\operatorname{vec}\tilde{\mathbf{Z}}\right) = \mathbf{0} \tag{B.14}$$

$$\begin{aligned}\Xi\operatorname{vec}\mathbf{K}_0-\Xi(\mathbf{U}\Xi)^{\dagger}\mathbf{U}\operatorname{vec}\mathbf{K}_0+\Xi^2\operatorname{vec}\tilde{\mathbf{Z}}-\Xi(\mathbf{U}\Xi)^{\dagger}\mathbf{U}\Xi\operatorname{vec}\tilde{\mathbf{Z}}&\\ -(\mathbf{U}\Xi)^{\dagger}\mathbf{U}\Xi\operatorname{vec}\mathbf{K}_0+(\mathbf{U}\Xi)^{\dagger}\mathbf{U}\Xi(\mathbf{U}\Xi)^{\dagger}\mathbf{U}\operatorname{vec}\mathbf{K}_0&\\ +(\mathbf{U}\Xi)^{\dagger}\mathbf{U}\Xi^2\operatorname{vec}\tilde{\mathbf{Z}}-(\mathbf{U}\Xi)^{\dagger}\mathbf{U}\Xi\operatorname{vec}\tilde{\mathbf{Z}} &= \mathbf{0}\end{aligned} \tag{B.15}$$

$$\begin{aligned}\Xi\operatorname{vec}\mathbf{K}_0-(\mathbf{U}\Xi)^{\dagger}\mathbf{U}\operatorname{vec}\mathbf{K}_0+\Xi\operatorname{vec}\tilde{\mathbf{Z}}-(\mathbf{U}\Xi)^{\dagger}\mathbf{U}\Xi\operatorname{vec}\tilde{\mathbf{Z}}&\\ -(\mathbf{U}\Xi)^{\dagger}\mathbf{U}\Xi\operatorname{vec}\mathbf{K}_0+(\mathbf{U}\Xi)^{\dagger}\mathbf{U}\operatorname{vec}\mathbf{K}_0&\\ +(\mathbf{U}\Xi)^{\dagger}\mathbf{U}\Xi\operatorname{vec}\tilde{\mathbf{Z}}-(\mathbf{U}\Xi)^{\dagger}\mathbf{U}\Xi\operatorname{vec}\tilde{\mathbf{Z}} &= \mathbf{0}\end{aligned} \tag{B.16}$$

$$\Xi\operatorname{vec}\mathbf{K}_0-(\mathbf{U}\Xi)^{\dagger}\mathbf{U}\Xi\operatorname{vec}\mathbf{K}_0+\Xi\operatorname{vec}\tilde{\mathbf{Z}}-(\mathbf{U}\Xi)^{\dagger}\mathbf{U}\Xi\operatorname{vec}\tilde{\mathbf{Z}} = \mathbf{0} \tag{B.17}$$

since Ξ is idempotent (the identity of Equation 6.36 is employed). Hence

$$\left(\mathbf{I}-(\mathbf{U}\Xi)^{\dagger}\mathbf{U}\right)\Xi\operatorname{vec}\mathbf{K}_0+\left(\mathbf{I}-(\mathbf{U}\Xi)^{\dagger}\mathbf{U}\right)\Xi\operatorname{vec}\tilde{\mathbf{Z}} = \mathbf{0} \tag{B.18}$$

The term $\Xi \operatorname{vec} \mathbf{K}_0$ can be expressed (through Equations 6.10 and 6.12) as

$$\Xi \operatorname{vec} \mathbf{K}_0 = \left(\mathbf{I} - \mathbf{Y}^\dagger \mathbf{Y}\right) \otimes \left(\mathbf{I} - \mathbf{X}\mathbf{X}^\dagger\right) \operatorname{vec} \mathbf{K}_0 \tag{B.19}$$

$$= \operatorname{vec}\left(\left(\mathbf{I} - \mathbf{Y}^\dagger \mathbf{Y}\right) \mathbf{K}_0 \left(\mathbf{I} - \mathbf{X}\mathbf{X}^\dagger\right)\right) \tag{B.20}$$

$$= \operatorname{vec}\left(\mathbf{K}_0 - \mathbf{K}_0\mathbf{X}\mathbf{X}^\dagger - \mathbf{Y}^\dagger\mathbf{Y}\mathbf{K}_0 + \mathbf{Y}^\dagger\mathbf{Y}\mathbf{K}_0\mathbf{X}\mathbf{X}^\dagger\right) \tag{B.21}$$

and substituting $\mathbf{K}_0 = \mathbf{Y}^\dagger\mathbf{T} + \mathbf{S}\mathbf{X}^\dagger + \mathbf{Y}^\dagger\mathbf{Y}\mathbf{S}\mathbf{Y}^\dagger$ (Clarke et al., 2003),

$$\begin{aligned}\Xi \operatorname{vec} \mathbf{K}_0 = \operatorname{vec}\Big(&\mathbf{Y}^\dagger\mathbf{T} + \mathbf{S}\mathbf{X}^\dagger + \mathbf{Y}^\dagger\mathbf{Y}\mathbf{S}\mathbf{Y}^\dagger - \mathbf{Y}^\dagger\mathbf{T}\mathbf{X}\mathbf{X}^\dagger - \mathbf{S}\mathbf{X}^\dagger\mathbf{X}\mathbf{X}^\dagger - \mathbf{Y}^\dagger\mathbf{Y}\mathbf{S}\mathbf{Y}^\dagger\mathbf{X}\mathbf{X}^\dagger\\ &- \mathbf{Y}^\dagger\mathbf{Y}\mathbf{Y}^\dagger\mathbf{T} - \mathbf{Y}^\dagger\mathbf{Y}\mathbf{S}\mathbf{X}^\dagger - \mathbf{Y}^\dagger\mathbf{Y}\mathbf{Y}^\dagger\mathbf{Y}\mathbf{S}\mathbf{Y}^\dagger\\ &+\mathbf{Y}^\dagger\mathbf{Y}\mathbf{Y}^\dagger\mathbf{T}\mathbf{X}\mathbf{X}^\dagger + \mathbf{Y}^\dagger\mathbf{Y}\mathbf{S}\mathbf{X}^\dagger\mathbf{X}\mathbf{X}^\dagger + \mathbf{Y}^\dagger\mathbf{Y}\mathbf{Y}^\dagger\mathbf{Y}\mathbf{S}\mathbf{Y}^\dagger\mathbf{X}\mathbf{X}^\dagger\Big)\end{aligned} \tag{B.22}$$

$$\begin{aligned}= \operatorname{vec}\Big(&\mathbf{Y}^\dagger\mathbf{T} + \mathbf{S}\mathbf{X}^\dagger + \mathbf{Y}^\dagger\mathbf{Y}\mathbf{S}\mathbf{Y}^\dagger - \mathbf{Y}^\dagger\mathbf{T}\mathbf{X}\mathbf{X}^\dagger - \mathbf{S}\mathbf{X}^\dagger - \mathbf{Y}^\dagger\mathbf{Y}\mathbf{S}\mathbf{Y}^\dagger\mathbf{X}\mathbf{X}^\dagger\\ &- \mathbf{Y}^\dagger\mathbf{T} - \mathbf{Y}^\dagger\mathbf{Y}\mathbf{S}\mathbf{X}^\dagger - \mathbf{Y}^\dagger\mathbf{Y}\mathbf{S}\mathbf{Y}^\dagger\\ &+\mathbf{Y}^\dagger\mathbf{T}\mathbf{X}\mathbf{X}^\dagger + \mathbf{Y}^\dagger\mathbf{Y}\mathbf{S}\mathbf{X}^\dagger + \mathbf{Y}^\dagger\mathbf{Y}\mathbf{S}\mathbf{Y}^\dagger\mathbf{X}\mathbf{X}^\dagger\Big)\end{aligned} \tag{B.23}$$

$$= \operatorname{vec}(\mathbf{0}) \tag{B.24}$$

Therefore Equation B.18 reduces to

$$\left(\mathbf{I} - (\mathbf{U}\Xi)^\dagger \mathbf{U}\right) \Xi \operatorname{vec} \tilde{\mathbf{Z}} = \mathbf{0} \tag{B.25}$$

which has the simple solution

$$\tilde{\mathbf{Z}} = \mathbf{0} \tag{B.26}$$

B.4 Differentiation for Section B.3

The progress of the analysis in Section B.3 relies on the differentiation of a scalar with respect to a vector. The procedure for performing this differentiation is presented here for reference. The problem is of the form

$$\frac{d}{d\mathbf{x}} \|\mathbf{A}\mathbf{x} + \mathbf{y}\| = 0 \tag{B.27}$$

From Miller (1987), if $\mathbf{x}^T = \{x_1, \ldots, x_n\}$ is a vector and $s(\mathbf{x}) = s(x_1, \ldots, x_n)$ is a differentiable scalar function of $\mathbf{x}$,

$$\frac{d}{d\mathbf{x}}s(\mathbf{x}) \triangleq \begin{bmatrix} \frac{\partial s}{\partial x_1} \\ \vdots \\ \frac{\partial s}{\partial x_n} \end{bmatrix} \tag{B.28}$$

So:

$$\frac{d}{d\mathbf{x}}\|\mathbf{Ax}+\mathbf{y}\| = \frac{d}{d\mathbf{x}}\left(\sum_{i=1}^{n}(\mathbf{A}_{i.}\mathbf{x}+y_i)^2\right)^{\frac{1}{2}} \tag{B.29}$$

$$= \frac{d}{d\mathbf{x}}\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}(a_{ij}x_j)+y_i\right)^2\right)^{\frac{1}{2}} \tag{B.30}$$

For each element:

$$\frac{\partial}{\partial x_k}\|\mathbf{Ax}+\mathbf{y}\| = \frac{\partial}{\partial x_k}\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}(a_{ij}x_j)+y_i\right)^2\right)^{\frac{1}{2}} \tag{B.31}$$

$$= \frac{1}{2\|\mathbf{Ax}+\mathbf{y}\|}\cdot\frac{\partial}{\partial x_k}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}(a_{ij}x_j)+y_i\right)^2 \tag{B.32}$$

$$= \frac{1}{2\|\mathbf{Ax}+\mathbf{y}\|}\cdot\frac{\partial}{\partial x_k}\sum_{i=1}^{n}\left(a_{ik}x_k+\left(\sum_{j=1}^{n}(a_{ij}x_j)-a_{ik}x_k+y_i\right)\right)^2 \tag{B.33}$$

Evaluating the square gives

$$\frac{\partial}{\partial x_k}\|\mathbf{Ax}+\mathbf{y}\| = \frac{1}{2\|\mathbf{Ax}+\mathbf{y}\|}\cdot\frac{\partial}{\partial x_k}\sum_{i=1}^{n}\left\{(a_{ik}x_k)^2+\left(\sum_{j=1}^{n}(a_{ij}x_j)-a_{ik}x_k+y_i\right)^2\right.$$
$$\left. +2a_{ik}x_k\left(\sum_{j=1}^{n}(a_{ij}x_j)-a_{ik}x_k+y_i\right)\right\} \tag{B.34}$$

The terms in parentheses have no dependency on x_k. This is demonstrated by expanding the summations over j to give

$$\sum_{j=1}^{n}(a_{ij}x_j)-a_{ik}x_k+y_i = a_{i1}x_1+a_{i2}x_2+\ldots+a_{in}x_n-a_{ik}x_k+y_i \tag{B.35}$$

$$= a_{i1}x_1+a_{i2}x_2+\ldots+a_{i(k-1)}x_{(k-1)}+a_{i(k+1)}x_{(k+1)}+\ldots$$
$$+a_{in}x_n+y_i \tag{B.36}$$

Hence the expression in Equation B.34 can be written

$$\frac{\partial}{\partial x_k}\|\mathbf{Ax}+\mathbf{y}\| = \frac{1}{2\|\mathbf{Ax}+\mathbf{y}\|} \cdot \frac{\partial}{\partial x_k}\sum_{i=1}^{n}\left\{(a_{ik}x_k)^2 + 2a_{ik}x_k\left(\sum_{j=1}^{n}(a_{ij}x_j) - a_{ik}x_k + y_i\right)\right\} \tag{B.37}$$

$$= \frac{1}{2\|\mathbf{Ax}+\mathbf{y}\|} \cdot \sum_{i=1}^{n}\left\{\frac{\partial}{\partial x_k}(a_{ik}x_k)^2 + \frac{\partial}{\partial x_k}2a_{ik}x_k\left(\sum_{j=1}^{n}(a_{ij}x_j) - a_{ik}x_k + y_i\right)\right\} \tag{B.38}$$

$$= \frac{1}{2\|\mathbf{Ax}+\mathbf{y}\|} \cdot \left\{\sum_{i=1}^{n}2a_{ik}^2x_k + \sum_{i=1}^{n}2a_{ik}\left(\sum_{j=1}^{n}(a_{ij}x_j) - a_{ik}x_k + y_i\right)\right\} \tag{B.39}$$

$$= \frac{1}{\|\mathbf{Ax}+\mathbf{y}\|} \cdot \sum_{i=1}^{n}\left(a_{ik}^2x_k + a_{ik}\mathbf{A}_{i\cdot}\mathbf{x} - a_{ik}^2x_k + a_{ik}y_i\right) \tag{B.40}$$

Expanding the remaining summation gives

$$\frac{\partial}{\partial x_k}\|\mathbf{Ax}+\mathbf{y}\| = \frac{1}{\|\mathbf{Ax}+\mathbf{y}\|} \cdot \left(\left(a_{1k}\mathbf{A}_{1\cdot} + a_{2k}\mathbf{A}_{2\cdot} + \ldots\right)\mathbf{x} + \begin{bmatrix}a_{1k} & a_{2k} & \cdots\end{bmatrix}\mathbf{y}\right) \tag{B.41}$$

$$= \frac{(\mathbf{A}_{\cdot k})^T(\mathbf{Ax}+\mathbf{y})}{\|\mathbf{Ax}+\mathbf{y}\|} \tag{B.42}$$

Therefore

$$\frac{d}{d\mathbf{x}}\|\mathbf{Ax}+\mathbf{y}\| = \frac{\mathbf{A}^T(\mathbf{Ax}+\mathbf{y})}{\|\mathbf{Ax}+\mathbf{y}\|} \tag{B.43}$$

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B.5 Appendix Bibliography

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Appendix C

Matrix Notation

Contents

C.1 Introduction

The nature of this thesis dictates that much of the content is presented in mathematical form. This Appendix contains details, for reference, of some of the more important (and obscure) aspects of the matrix notation used.

C.2 Column, Row and Element Notation

Matrix subscripts identify individual elements, rows or columns.

If $\mathbf{A}^{m \times n} = [a_{ij}]$,

$$\mathbf{A}_{pq} = a_{pq} \tag{C.1}$$

$$\mathbf{A}_{p\cdot} = \begin{bmatrix} a_{p1} & a_{p2} & \cdots & a_{pn} \end{bmatrix} \tag{C.2}$$

$$\mathbf{A}_{\cdot q} = \begin{bmatrix} a_{1q} \\ a_{2q} \\ \vdots \\ a_{mq} \end{bmatrix} \tag{C.3}$$

C.3 Definitions of Operators

Moore-Penrose Pseudo-Inverse: The Moore-Penrose Pseudo-Inverse $\mathbf{A}^\dagger$ of $\mathbf{A}$ (see Ben-Israel and Greville, 1974) is a unique matrix, guaranteed to exist, that satisfies the following properties:

$$\mathbf{A}\mathbf{A}^\dagger\mathbf{A} = \mathbf{A} \tag{C.4}$$

$$\mathbf{A}^\dagger\mathbf{A}\mathbf{A}^\dagger = \mathbf{A}^\dagger \tag{C.5}$$

$$(\mathbf{A}\mathbf{A}^\dagger)^\star = \mathbf{A}\mathbf{A}^\dagger \tag{C.6}$$

$$(\mathbf{A}^\dagger\mathbf{A})^\star = \mathbf{A}^\dagger\mathbf{A} \tag{C.7}$$

where $\mathbf{A}^\star$ is the complex conjugate transpose of $\mathbf{A}$.

Kronecker Product: The Kronecker product (direct product, tensor product) of $\mathbf{A}^{m \times n} = [a_{ij}]$ and $\mathbf{B}^{r \times s} = [b_{ij}]$ is defined as the partitioned matrix

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \cdots & a_{2n}\mathbf{B} \\ \vdots & \vdots & & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix} \tag{C.8}$$

which can be seen to be of order $(mr \times ns)$.

Vec Operator: The vec operator converts a matrix of order $(m \times n)$ into a vector of length mn and is defined as

$$\text{vec}\,\mathbf{A} = \begin{bmatrix} \mathbf{A}_{\cdot 1} \\ \mathbf{A}_{\cdot 2} \\ \vdots \\ \mathbf{A}_{\cdot n} \end{bmatrix} \tag{C.9}$$

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