# Some Representation Theory Of Decorated Partial Brauer Algebra

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Submitted in accordance with the requirements for the degree of
Doctor of Philosophy

University of Leeds
School of Mathematics

July 2018

The candidate confirms that the work submitted is her own and that appropriate credit has been given where reference has been made to the work of others.

# *Acknowledgements*

First and foremost, I would like to express my sincerest gratitude and thanks to my supervisor Dr. Alison Parker for her continuous support, patient, kindness and feedback provided all along my PhD study and research. Her guidance and encouragement motivated me all around the time of research. Without her assistance and advices this thesis would have never been accomplished. I would also like to express my appreciation and thanks to my co-supervisor prof. Paul Martin, for proposing the project, for his advice, support and cooperation.

Moreover, my thanks and appreciations are given to all my friends in the school of mathematics for their cooperation, and encouragement.

I am greatly indebted to my dear mother for her encouragement, her patience and her alienation from her homeland in order to be with me. Words cannot express my deep gratitude for her. I also would like to thanks my family and friend who supported me in achieving my goal.

Additionally, I gratefully acknowledge the Ministry of Education and Umm Al-Qura University for the financial support provided.

# *Abstract*

In this thesis we introduce a new family of finite dimensional diagram algebras over a commutative ring with identity, the decorated partial Brauer algebras, denoted by $D\mathcal{P}\mathfrak{B}_n(\delta, \delta_\circ, \delta', \mu, \mu')$. These algebras are unital, associative and have a basis consisting of decorated partial Brauer diagrams which are partial Brauer diagrams with possibly decorated edges and decorated isolated vertices.

We show that this algebra is a cellular algebra by applying Theorem of Green and Paget to iterated construction . Subsequently, we give an indexing set for the simple modules. Over a field of characteristic different from 2, we determine when the decorated partial Brauer algebra is quasi-hereditary. Finally, we give a complete description of the restriction rule for the cell modules over $\mathbb{C}$.

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# Chapter 1

# Introduction

The representation theory over $\mathbb{C}$ of the general linear groups $GL_n(\mathbb{C})$ and the symmetric groups $\mathfrak{S}_n$ are related by Schur-Weyl duality via the mutually centralising actions of the two groups on the $r^{th}$ tensor product $V^{\otimes r}$, where $V$ is a complex vector space of dimension $n$. In [2], Richard Brauer introduced a class of finite dimensional algebras which are called *Brauer algebras* to provide a corresponding result when replacing the general linear groups by either orthogonal or symplectic groups and replacing the group algebra of the symmetric group by a Brauer algebra.

For an arbitrary ring $R$, $n \in \mathbb{N}$, $\delta \in R$, the Brauer algebra, denoted $\mathfrak{B}_n(\delta)$, has a basis consisting of Brauer diagrams which consist of $2n$ vertices in a rectangle frame with $n$ of them in the top row numbered 1 to $n$ from left to right and others in the bottom row numbered $1'$ to $n'$ from left to right, where each vertex is connected to precisely one other by an edge. The product of two diagrams is given by concatenation, that is by placing one diagram above the other, and identifying the vertices in the middle row. This produces a new diagram possibly with some number ($r$ say) of closed loops which are removed and we record this by multiplying by $\delta^r$.

Brauer algebras have been studied by Hanlon and Wales [9], they conjectured that $\mathfrak{B}_n(\delta)$ is semisimple over $\mathbb{C}$ if $\delta \notin \mathbb{Z}$. This was proved by Wenzl in [20].

Cellular algebras were first introduced by Graham and Lehrer in [6]. For these algebra, they defined cell representations. Also they obtained a general description of the irreducible representations of cellular algebras together with a criterion for the cellular algebra to be semi-simple. Graham and Lehrer proved that the Brauer algebra is a cellular algebra. Moreover, over a field of characteristic $p$ (possibly $p = 0$) they showed that the set of irreducible modules of Brauer algebra is indexed by the set $p$-regular partitions of $n$, $n-2$, ..., 0 or 1.

In [14] König and Xi described the Brauer algebra as an iterated inflation of symmetric group algebras. Using their result, in [12], which is that an iterated inflation of cellular algebras is cellular, they proved cellularity of the Brauer algebra. Also, in [13] they determined for which parameters the Brauer algebra is quasi-hereditary.

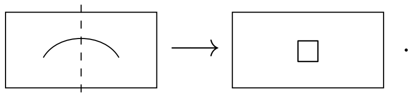
As a generalization of Brauer algebras, a class of algebras called the *Partial Brauer algebra* was introduced by Martin and Mazorchuk [15]. This algebra, denoted $\mathcal{P}\mathfrak{B}_n(\delta, \delta')$, $\delta, \delta' \in R$, is an associative algebra with 1 which has a basis given by partial Brauer diagrams, these are Brauer diagrams that allow for the possibility of removing edges. The representations of the partial Brauer algebra are studied by Martin and Mazorchuk [15]. They showed that this algebra is generically semisimple over $\mathbb{C}$. Furthermore, they constructed the Specht modules and determined a restriction rule for the Specht modules.

Motivated by Brauer and partial Brauer algebras, in this thesis, we define a new class of finite dimensional algebras which has a basis consisting of the decorated partial Brauer diagrams, these are the partial Brauer diagrams where the edges and isolated vertices may be decorated. We call them *the decorated partial Brauer algebra* and denote them $D\mathcal{P}\mathfrak{B}_n(\delta, \delta_\circ, \delta', \mu, \mu')$.
These algebras are non-trivial 5-parameter deformation of the left-right symmetric partial Brauer algebras, where a left-right symmetric partial Brauer algebra is a subalgebra of the partial Brauer algebra spanned by the partial Brauer diagrams with $2n$ northern nodes and $2n$ southern nodes that are symmetric under reflection about the vertical axis. The decorated partial Brauer diagrams are constructed from the left-hand halves of the left-right symmetric partial Brauer diagrams (after cutting along the axis of symmetry) as follows: Each pair of lines that intersect on the vertical axis of symmetry are joined by a decorated line with a " $\circ$ " decoration. Individual

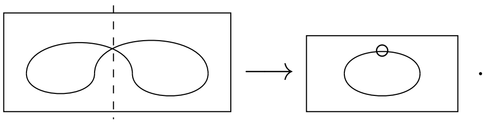
lines which cross the axis of symmetry are contracted to decorated labelled isolated vertices with a "□" decoration.

The deformation of the isolated components, which form during the product of two symmetric partial Brauer diagrams, introduce the new parameters of the decorated partial Brauer algebra. We describe this deformation as follows:

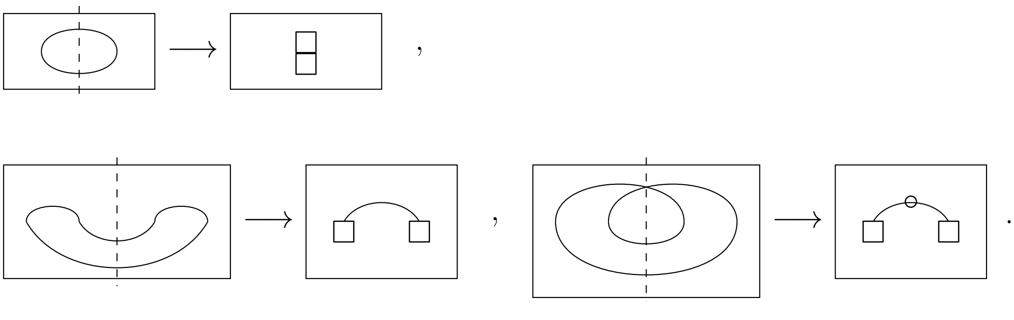
1. an open string that crosses the axis of symmetry which does not cross other open stings is contracted to a decorated isolated vertex with a "□" decoration and this is replaced with a parameter "$\mu$".



2. a loop which crosses the axis of symmetry at one point introduces a decorated loop and this is replaced with a parameter "$\delta_\circ$".



3. a loop which crosses the axis of symmetry at more than one point introduces two meeting squares or a decorated (or undecorated) open string with square in both of its endpoints, all these are replaced with a parameter "$\mu'$".



The structure of this thesis is as follows:

In chapter two we recall some definitions and results which are useful for the later chapters. In the first section of this chapter we give a brief review of the partial

Brauer algebra. In the next section we recall the definition of cellular algebra in sense of Graham and Lehrer and its equivalent version given by König and Xi. In section three we recall the definition of the group $\mathbb{Z}_2 \wr S_n$, describe some combinatorics, as well as the cellularity of the Hecke algebra of type $B$.

Chapter three is devoted to defining our main object of study, the decorated partial Brauer algebra, which has basis the decorated partial Brauer diagrams. In the first section we give a definition of the set of the *decorated partial Brauer partition* and its size. In section two we define decorated partial Brauer diagrams and we show that the set of these diagrams are equivalent to the set of the decorated partial Brauer partitions. The multiplication of these diagrams is defined in the third section which is as the multiplication of the partial Brauer diagrams with additional rules that handle the decorated lines and decorated vertices. Also, we show that this operation is associative. Finally, we state the definition of the algebra with its dimension. In the last section, we define the *Symmetric partial Brauer algebra* and we show that there is a bijection between the set of symmetric partial Brauer diagrams and the set of decorated partial Brauer diagrams.

Chapter four is devoted to describing the cellular structure of the decorated partial Brauer algebra. In particular it proves the following

**Theorem 1.0.1** (Theorem 4.5.1)**.** *Let $K$ be a field, $\delta$, $\delta_\circ$, $\delta'$, $\mu$, $\mu' \in K$. Then the decorated partial Brauer algebra, $D\mathcal{P}\mathfrak{B}_n(\delta, \delta_\circ, \delta', \mu, \mu')$ is a cellular algebra over $K$.*

We prove this result by using Theorem 4.1.2 introduced by Green and Paget, which establish that an algebra is an iterated inflation of cellular algebra and hence is cellular, [7, Theorem 1]. Firstly, we identify the group algebra $K\widetilde{S_n}$ as a subalgebra of $D\mathcal{P}\mathfrak{B}_n(\delta, \delta_\circ, \delta', \mu, \mu')$, where $\widetilde{S_n}$ is the set of all decorated partial Brauer diagrams that only have propagating lines. For $l = 0, \ldots, n$, we define a $K$-vector space $J_l$ which is spanned by all decorated partial Brauer diagrams with at most $l$ propagating lines and we show that $J_l$ is a two-sided ideal of $D\mathcal{P}\mathfrak{B}_n(\delta, \delta_\circ, \delta', \mu, \mu')$, this gives a filtration of $D\mathcal{P}\mathfrak{B}_n(\delta, \delta_\circ, \delta', \mu, \mu')$ ,

$$0 \subset J_0 \subset J_1 \subset \cdots \subset J_{n-1} \subset J_n = D\mathcal{P}\mathfrak{B}_n(\delta, \delta_\circ, \delta', \mu, \mu') \qquad (*)$$

Then we show that each $J_l/J_{l-1}$ in $(*)$ is an inflation of $K\widetilde{S}_l$ along $V_l$ ( i.e. $J_l/J_{l-1} \cong i(V_l) \otimes V_l \otimes K\widetilde{S}_l$ ), where $V_l$ is a $K$-vector space spanned by the set of decorated partial Brauer lower half diagrams with $l$ non-crossing undecorated propagating lines, (Lemma 4.4.6 and 4.4.9). In addition we define the required $K$-bilinear form $\varphi_l : i(V_l) \otimes V_l \to K\widetilde{S}_l$ to give a multiplication structure on this inflation (Definition 4.4.7). Then we define an involution on $i(V_l) \otimes V_l \otimes K\widetilde{S}_l$. In particular we show that the map $\iota$ given by

$$\iota\big(i(x) \otimes y \otimes \pi\big) = i(y) \otimes x \otimes \pi^{-1}$$

is an anti-involution on $i(V_l) \otimes V_l \otimes K\widetilde{S}_l$ (Lemma 4.4.12). Finally, we show that, for each $0 \leq l \leq n$ and any $u, v \in h_l(DPB(n))$ and $\pi \in \widetilde{S}_l$, we have for any $d \in DPB(n)$ that

$$d.(i(u) \otimes v \otimes \pi) \equiv \phi_l(d, i(u)) \otimes v \otimes \theta_l(d, i(u))\pi \pmod{J_{l-1}}$$

where $J_{l-1} = \bigoplus_{k=0}^{l-1} i(V_k) \otimes V_k \otimes K\widetilde{S_k}$ and $\phi_l(d, i(u)) \in i(V_l)$, $\theta_l(d, i(u)) \in K\widetilde{S}_l$ depend only on $d$ and $i(u)$ (Lemma 4.4.14). These results satisfy all conditions of Theorem 4.1.2 of Green and Paget, which establishes the cellularity of the decorated partial Brauer algebra.

Applying the cellularity of the decorated partial Brauer algebra and a result due to Dipper and James for simple modules of $K\widetilde{S_n}$, we give an indexing set of the simple modules of $D\mathcal{P}\mathfrak{B}_n(\delta, \delta_\circ, \delta', \mu, \mu')$ over a field $K$ of characteristic $p$, $p \neq 2$. In particular we show that: If $k$ is a field of characteristic $p$, $p \neq 2$, and at least one of the elements $\delta'$, $\mu$ or $\mu'$ is non-zero, then the simple modules of $D\mathcal{P}\mathfrak{B}_n(\delta, \delta_\circ, \delta', \mu, \mu')$ are indexed by $\{(l, \lambda) \mid 0 \leq l \leq n,\ \lambda \text{ is a } p-\text{restricted bipartition of } l\}$ (Theorem 4.7.1).

In chapter five we give a necessary and sufficient condition for the decorated partial Brauer algebra over a field of characteristic $p$, $p \neq 2$ to be quasi-hereditary (Theorem 5.2.1).

Chapter six is dedicated to proving the restriction rule for the cell modules of the decorated partial Brauer algebra.

# Chapter 2

# Background

## 2.1 The Partial Brauer algebra

Let $R$ be a commutative ring, $\delta, \delta' \in R$ and $n$ a natural number. *The partial Brauer algebra* denoted $\mathcal{P}\mathfrak{B}_n(\delta, \delta')$, introduced by Martin and Mazorchuk (*see [15]*), is a unital associative finite dimensional algebra with the basis the so-called *partial Brauer partitions*. In this section we briefly recall the definition of the partial Brauer algebra.

### Partial partition

**Definition 2.1.1.** (**Partial partition**)[15]

For a finite set $T$, a partition of a set $T$ is a collection $\{X_1, X_2, X_3, \ldots\}$ of nonempty subsets of $T$ such that $\cup_i X_i = T$ and $X_i \cap X_j = \emptyset$ $(i \neq j)$.
We call each subset in a partition of $T$ a part or a block.

A partition of a set $T$ in which each part (each subset) has exactly two elements is called a *pair partition* or *Brauer partition*.

A partition of a set $T$ in which each part has at most two elements is called a *partial (Brauer) partition*. In other words a *partial (Brauer) partition* is a partition of a set $T$ into pairs and singletons.

The set of all pair partitions of $T$ is denoted by $B_T$, and the set of all partial (Brauer) partitions is denoted by $PB_T$.

For $n \in \mathbb{N}$, let $\underline{n} = \{1, \ldots, n\}$, $\underline{m'} = \{1', \ldots, m'\}$.

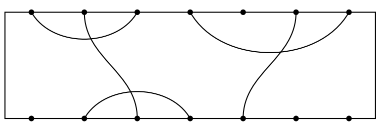
Let $T = \underline{n} \cup \underline{m'}$, a partial partition of a finite set $T = \underline{n} \cup \underline{m'}$ may be represented in the plane by a diagram, the so-called *(n,m)-partial Brauer diagram*.

## Partial Brauer diagrams

**Definition 2.1.2.** [8] An *(n,m)-partial Brauer diagram* is a rectangle with $n$ vertices labelled $1, ..., n$ on the top row and $m$ vertices labelled $1', ..., m'$ on the bottom row such that each vertex is connected to at most one other vertex by an edge.

In this diagram two vertices form an edge (are joined together) if and only if they are in the same part of the partial partition.

**Example 2.1.3.** The set $\{\{1,3\},\{2,3'\},\{4,7\},\{5\},\{6,5'\},\{1'\},\{2',4'\},\{6'\},\{7'\}\}$ is represented by the following partial Brauer diagram:



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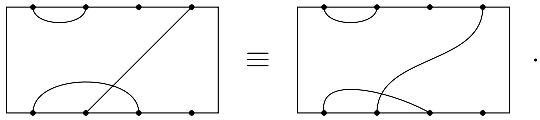
Note that the partial Brauer diagrams consist of vertical edges (propagating lines) which connect a vertex in the top row to a vertex in the bottom row, horizontal edges (arcs), which connect vertices in same row, and isolated (singleton) vertices which are not incident to an edge.

**Definition 2.1.4.** An *isolated (singleton) vertex* in a partial Brauer diagram is a vertex on the top row or bottom row of the rectangle frame which is not incident to an edge.

The diagram representing a partial partition is not unique. We say that two diagrams are equivalent if they represent the same partial partition. Thus we identify diagrams if they are equivalent. Since we are interested in equivalent diagrams so we will use the term *partial Brauer diagram* to mean the equivalence class of the given diagram.



Let $PB(n,m)$ denoted to the set of all partial Brauer diagrams where $n$ is the number of labelled vertices in the top row and $m$ is the number of the labelled vertices in the bottom row. The following gives the number of its elements. (Note that the number $n+m$ of the labelled vertices could be even or odd.)

**Proposition 2.1.5.** *(see section (2) in [8] for the special case when $m = n$)*

$$|PB(n,m)| = \sum_{l=0}^{\lfloor \frac{n+m}{2} \rfloor} \binom{n+m}{2l} (2l-1)!!$$

*where $l$ is the number of edges in the diagram , $(2l-1)!! = (2l-1)(2l-3)\cdots 3.1$ and $(-1)!! = 1$.*

*Proof.* Let $PB^l(n,m)$ denote the set of partial Brauer diagrams which have $l$ edges. To count the size of this set, firstly we choose $2l$ vertices of $n+m$ to be in pairs. This gives $\binom{n+m}{2l}$ ways for a fixed $l$. We then choose two vertices of $2l$ to be an edge. For each choice we get two vertices less to choose from. So there are $(2l-1)$ choices for the first edge, $(2l-3)$ for the second edge. Continuing in this manner we get the number of ways to draw a diagram with $l$ edges which is $(2l-1)!!$. Therefore for a fixed number $l$ of edges we have

$$|PB^l(n,m)| = \binom{n+m}{2l} (2l-1)!!.$$

Take a sum $\Sigma_{l=0}^{n} \binom{n+m}{2l} (2l-1)!!$ over $l$ to get all possible elements of the set $PB(n,m)$. $\square$

Note that since the set of partial Brauer diagrams on $n+m$ vertices correspond to the set of partial Brauer partitions the above formula gives the number of partial Brauer partitions of $\underline{n} \cup \underline{m'}$.

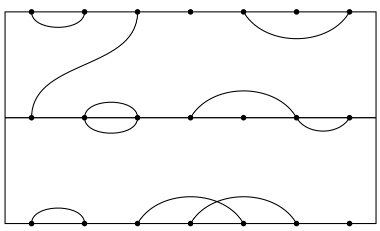
Note that we are mainly interested in the case $T = \{1, \ldots, n, 1', \ldots, n'\} = \underline{n} \cup \underline{n'}$, we define $PB_n = PB_{\underline{n} \cup \underline{n'}}$ and $PB(n) = PB(n, n)$

## Multiplication of partial Brauer diagrams

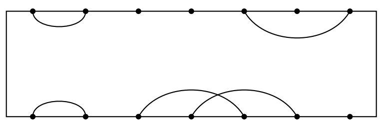
The multiplication of two partial Brauer diagrams $d_1$ and $d_2$ is given by concatenation, that is, by identifying the bottom row vertices in $d_1$ with corresponding top row vertices in $d_2$. We call the set of vertices formed by top row of $d_2$ and the bottom row of $d_1$ *the middle row* of $d_1 d_2$ or the *equator*.
In the multiplication of partial Brauer diagrams there are two different connected components which can be formed in the middle row, namely, loops and open strings (which are not closed pathes of connected lines in the middle row). These connected components will be removed with the middle row and replace them with parameters as follows: Let a factor $\delta \in R$ be associated to each removed loop and a factor $\delta' \in R$ be associated to each removed open string. This gives a 2-parameter version of the partial Brauer algebra (*see section 1 in [15]*). So the multiplication of $d_1$ and $d_2$ is

$$d_1 d_2 = \delta^l (\delta')^m d_3.$$

where $d_3$ is the resulting partial Brauer diagram after removing the middle row with connected components, $l$ denotes the number of loops that are removed from the middle row, and $m$ is the number of open strings or isolated vertices that are removed from the middle row (*see [8]*). For example,



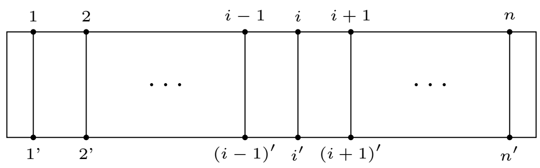
$= \delta(\delta')^2$



.

This diagram multiplication is associative with identity element $\mathbb{I}$ (see [8]):

$\mathbb{I} =$




We may now define the partial Brauer algebra.

**Definition 2.1.6.** [8] Let $R$ be a commutative ring, $\delta, \delta' \in R$, $n$ a natural number. *The partial Brauer algebra* $\mathcal{P}\mathfrak{B}_n(\delta, \delta')$, is an associative unitial algebra with basis the set of partial Brauer diagrams and multiplication as defined above.

The dimension of $\mathcal{P}\mathfrak{B}_n(\delta, \delta')$ is

$$\dim(\mathcal{P}\mathfrak{B}_n(\delta, \delta')) = \sum_{l=0}^{n} \binom{2n}{2l} (2l-1)!!$$

where $l$ is the number of edges in the diagram.

## 2.2 Cellular algebras

In this section we recall the original definition of cellular algebras in the sense of Graham and Lehrer and an equivalent definition given by König and Xi.

**Definition 2.2.1.** (Graham and Lehrer, [6]). Let $R$ be a commutative ring with identity. A *cellular algebra* over $R$ is an associative (unital) algebra $A$ together with *cell datum* $(\Lambda, M, C, i)$ where

$(C_1)$ $\Lambda$ is a partially ordered set (poset) and for each $\lambda \in \Lambda$, $M(\lambda)$ is a finite set such that the algebra $A$ has an $R$-basis $C^{\lambda}_{S,T}$, where $(S, T)$ runs through all elements of $M(\lambda) \times M(\lambda)$ for all $\lambda \in \Lambda$.

$(C_2)$ Let $\lambda \in \Lambda$ and $S, T \in M(\lambda)$. Then the map $i$ is an $R$-linear anti-involution of $A$ such that $i(C^{\lambda}_{S,T}) = C^{\lambda}_{T,S}$.

$(C_3)$ For each $\lambda \in \Lambda$, $S, T \in M(\lambda)$ and for any element $a \in A$ we have

$$aC^{\lambda}_{S,T} \equiv \sum_{U \in M(\lambda)} r_a(U,S) C^{\lambda}_{U,T} \qquad (\text{mod } A(< \lambda))$$

where $r_a(U,S) \in R$ is independent of $T$, and $A(< \lambda)$ is the $R$-submodule of $A$ generated by $\{C^{\mu}_{S',T'} \mid \mu < \lambda, S', T' \in M(\mu)\}$.
If we apply $i$ to the equation in $(C_3)$, we obtain

$$C^{\lambda}_{T,S}\ i(a) \equiv \sum_{U \in M(\lambda)} r_a(U,S) C^{\lambda}_{T,U} \qquad (\text{mod } A(< \lambda))$$

Thus, the R-module $A(\leq \lambda)$ generated by the set $\{C^{\mu}_{S'',T''} \mid \mu \leq \lambda,\ S'', T'' \in M(\mu)\}$ is a two-sided ideal of $A$ fixed by $i$.

Examples of cellular algebras are the following: [6]

**(a)** Ariki-Koike algebras;

**(b)** Brauer's algebras;

**(c)** Temperley-Lieb algebras;

**Definition 2.2.2.** [6]. For each $\lambda \in \Lambda$, let $W(\lambda)$ be the free $R$-module with basis $\{C_s \mid S \in M(\lambda)\}$ and left $A$-action defined by

$$aC_S = \sum_{S' \in M(\lambda)} r_a(S',S) C_{S'} \qquad (a \in A, S \in M(\lambda))$$

$W(\lambda)$ is called the *cell module* of $A$ corresponding to $\lambda$.
Also, $W(\lambda)$ may be thought of as a right $A$-module with action

$$C_S a = \sum_{S' \in M(\lambda)} r_{i(a)}(S',S) C_{S'}.$$

In [6], Graham and Lehrer defined a bilinear form $\phi_\lambda : W(\lambda) \times W(\lambda) \to R$ such that $\phi_\lambda(C_T, C_U)$, for $T$, $U \in M(\lambda)$, is given by

$$C^\lambda_{S,T} C^\lambda_{U,V} = \phi_\lambda(C_T, C_U) C^\lambda_{S,V} \qquad (\text{mod } A < \lambda)$$

where $S$, $V$ are any elements in $M(\lambda)$.

When $R$ is a field, they proved in [6, Theorem 3.4], that the isomorphic classes of simple modules are indexed by the set

$$\Lambda_0 = \{\lambda \in \Lambda \ \mid \ \phi_\lambda \neq 0\}.$$

**Definition 2.2.3.** [6] Let $R$ be a field. The radical of the cell module $W(\lambda)$ is given by

$$\text{rad}W(\lambda) = \{x \in W(\lambda) \mid \phi_\lambda(x, y) = 0 \text{ for all } y \in W(\lambda).$$

**Proposition 2.2.4.** *[6] Let $R$ be a field. Then*

**(i)** *rad $W(\lambda)$ is a submodule of $W(\lambda)$.*

**(ii)** *If $\phi_\lambda \neq 0$, the quotient $W(\lambda)/$rad $W(\lambda)$ is irreducible.*

The following is a basis-free definition of cellular algebra which is equivalent to the given by Graham and Lehrer.

**Definition 2.2.5.** (König and Xi, [11]). Let $A$ be an $R$-algebra where $R$ is a commutative Noetherian integral domain. Assume there is an anti-involution $i$ in $A$ with $i^2 = \text{Id}$. A two-sided ideal $J$ in $A$ is called *cell ideal* if and only if $i(J) = J$ and there exists a left ideal $\Delta \subset J$ such that $\Delta$ is finitely generated and free over $R$ and such that there is an isomorphism of $A$-bimodules $\alpha : J \simeq \Delta \otimes_R i(\Delta)$ (where $i(\Delta) \subset J$ is the $i$-image of $\Delta$) making the following diagram commutative:

$$\begin{array}{ccc} J & \xrightarrow{\alpha} & \Delta \otimes_R i(\Delta) \\ \Big\downarrow i & & \Big\downarrow x\otimes y \mapsto i(y) \otimes i(x) \\ J & \xrightarrow{\alpha} & \Delta \otimes_R i(\Delta) \end{array}$$

The algebra $A$ with the involution $i$ is called *cellular* if and only if there is an $R-$module decomposition $A = J'_1 \oplus J'_2 \oplus \ldots \oplus J'_n$ (for some $n$) with $i(J'_j) = J'_j$ for each $j$ and such that setting $J_j = \oplus_{l=1}^{j} J'_l$ gives a chain of two-sided ideals of $A : 0 = J_0 \subset J_1 \subset J_2 \subset \ldots \subset J_n = A$ (each of them fixed by $i$) and for each $j$ ($j = 1, \ldots, n$) the quotient $J'_j = J_j/J_{j-1}$ is a cell ideal (with respect to the involution induced by $i$ on the quotient) of $A/J_{j-1}$.
The $\Delta$'s obtained from each section $J_j/J_{j-1}$ are called *cell modules* of the cellular algebra $A$, and the above chain of ideals in $A$ is called a cell chain of $A$.

In [11] it is proved that the two above definitions of cellular algebra are equivalent.

## 2.3 The wreath product

**Definition 2.3.1.** [10]

Put $\mathbb{Z}_2^n = \{f \mid f : \{1, \ldots, n\} \to \mathbb{Z}_2\}$, the set of all mappings from $\{1, \cdots, n\}$ into $\mathbb{Z}_2$. Define $\mathbb{Z}_2 \wr S_n = \mathbb{Z}_2^n \times S_n = \{(f, \pi) \mid f : \{1, \ldots, n\} \to \mathbb{Z}_2, \pi \in S_n\}$, where $S_n$ is the symmetric group on $n$ symbols, with multiplication in $\mathbb{Z}_2 \wr S_n$ defined as

$$(f, \pi)(f', \pi') = (f + {}_{\pi}f', \pi\pi')$$

where

$$(i)\pi\pi' = ((i)\pi)\pi', \qquad (f + f')(i) = f(i) + f'(i), \qquad \text{for all} \;\; i \in \{1, \ldots, n\}$$

and ${}_{\pi}f \in \mathbb{Z}_2^n$, defined by

$${}_{\pi}f(i) = f(i\pi), \;\; \text{for all} \;\; i \in \{1, \ldots, n\}.$$

$\mathbb{Z}_2 \wr S_n$ is called the *wreath product* of $\mathbb{Z}_2$ by $S_n$ and its order is $|\mathbb{Z}_2|^n|S_n| = 2^n n!$.

**Theorem 2.3.2.** *The set $\mathbb{Z}_2 \wr S_n$ which is defined in 2.3.1 is a group called the* wreath product group.

*Proof.* (1) The identity element in $\mathbb{Z}_2 \wr S_n$ is $(0, \mathrm{Id})$, where

$$(0, \mathrm{Id})(f, \pi) = (0 + {}_{\mathrm{Id}}f, \mathrm{Id}\pi) = (f, \pi)$$

and

$$(f, \pi)(0, \mathrm{Id}) = (f + {}_{\pi}0, \pi\ \mathrm{Id}) = (f, \pi).$$

(2) The inverse of an element $(f, \pi)$ in $\mathbb{Z}_2 \wr S_n$ is $(f, \pi)^{-1} = ({}_{\pi^{-1}}f, \pi^{-1})$, where

$$(f, \pi)({}_{\pi^{-1}}f, \pi^{-1}) = (f + {}_{\pi}({}_{\pi^{-1}}f), \pi\pi^{-1}) = (f + f, \mathrm{Id}) = (2f, \mathrm{Id}) = (0, \mathrm{Id})$$

and

$$({}_{\pi^{-1}}f, \pi^{-1})(f, \pi) = ({}_{\pi^{-1}}f + {}_{\pi^{-1}}f, \pi^{-1}\pi) = (2\,{}_{\pi^{-1}}f, \mathrm{Id}) = (0, \mathrm{Id}).$$

(3) The associativity:

Let $(f_1, \pi_1), (f_2, \pi_2), (f_3, \pi_3) \in \mathbb{Z}_2 \wr S_n$. Then

$$\begin{aligned}[(f_1, \pi_1)(f_2, \pi_2)](f_3, \pi_3) &= (f_1 + {}_{\pi_1}f_2, \pi_1\pi_2)(f_3, \pi_3) \\ &= ((f_1 + {}_{\pi_1}f_2) + {}_{\pi_1\pi_2}f_3, \pi_1\pi_2\pi_3).\end{aligned}$$

Also

$$\begin{aligned}(f_1, \pi_1)[(f_2, \pi_2)(f_3, \pi_3)] &= (f_1, \pi_1)(f_2 + {}_{\pi_2}f_3, \pi_2\pi_3) \\ &= (f_1 + {}_{\pi_1}(f_2 + {}_{\pi_2}f_3), \pi_1\pi_2\pi_3) \\ &= (f_1 + {}_{\pi_1}f_2 + {}_{\pi_1}({}_{\pi_2}f_3), \pi_1\pi_2\pi_3) \\ &= (f_1 + {}_{\pi_1}f_2 + {}_{\pi_1\pi_2}f_3, \pi_1\pi_2\pi_3).\end{aligned}$$

then the multiplication is associative. □

### The Hyperoctahedral group (Coxeter group of type $B_n$)

**Definition 2.3.3.** Let $S$ be a set. A matrix $m : S \times S \longrightarrow \{1, 2, \ldots, \infty\}$ is called a *Coxeter matrix* if it satisfies

$$m(s, s') = m(s', s) \text{ for all } \; s, s' \in S;$$
$$m(s, s') = 1 \Longleftrightarrow s = s'.$$

Let $S^2_{fin} = \{(s, s') \in S^2 \mid m(s, s') \neq \infty\}$. A Coxeter matrix $m$ determines a group $W$ with the presentation

$$\begin{cases} \text{Generators : S;} \\ \text{Relations : } (ss')^{m(s,s')} = e, \qquad \text{for all } \; (s, s') \in S^2_{fin} \end{cases}$$

Here, "$e$" denotes the identity element of any group under consideration. Since $m(s, s) = 1$, we have that

$$s^2 = e \quad \text{ for all } \; s \in S$$

so the relation $(ss')^{m(s,s')} = e$ is equivalent to

$$\underbrace{ss'ss's\cdots}_{m(s,s')} = \underbrace{s'ss'ss'\cdots}_{m(s,s')}$$

The group $W$ is called a *Coxeter group*, $S$ is the set of *Coxeter generators* and the pair $(W, S)$ is called a *Coxeter system*.

**Definition 2.3.4.** [3] The *Coxeter group of type $B_n$ (or the hyperoctahedral group)*, denoted by $W_n$, is a group of signed permutations of $1, \ldots, n$.
Let $n \geq 1$, consider the set $I_n = I^+_n \cup I^-_n$ where $I^+_n = \{1, \ldots, n\}$ , $I^-_n = \{-1, \ldots, -n\}$. Let $S(I_n)$ denoted the group of permutations of the set $I_n$, then $W_n$ is (a subgroup of $S(I_n)$) defined by

$$W_n := \{\pi \in S(I_n) \mid \pi(-i) = -\pi(i) \quad \text{ for all } \; i \in I_n\}.$$

The Coxeter group $W_n$ is generated by the set $S = \{s_0, s_1, \ldots, s_{n-1}\}$ subject to the following relations:

$$s_i^2 = 1 \quad \text{for all} \quad i$$
$$s_i s_j = s_j s_i \quad \text{if} \quad |i-j| \neq 1$$
$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad \text{for} \quad i \geq 1$$
$$s_1 s_0 s_1 s_0 = s_0 s_1 s_0 s_1.$$

where

$$s_0 = (1, -1)$$

and

$$s_i = (i, i+1)(-i, -i-1) \quad \text{for} \quad 1 \leq i \leq n-1$$

Note that, the elements

$$\begin{aligned} &s_1 := (1,2)(-1,-2) \\ &s_2 := (2,3)(-2,-3) \\ &\vdots \\ &s_{n-1} := ((n-1), n)(-(n-1), -n) \end{aligned}$$

generate a subgroup $\overline{W_n} \subseteq W_n$ which isomorphic to $S_n$, the symmetric group of degree $n$. Also, if we put $t_1 = s_0 = (1, -1)$ and $t_i = s_{i-1} t_{i-1} s_{i-1}$ for $2 \leq i \leq n$, means $t_i = (-i, i)$ for $1 \leq i \leq n$, then $t_i^2 = 1$, $t_i t_j = t_j t_i$ and the subgroup $C \subseteq W_n$ generated by $\{t_1, \ldots, t_n\}$ is a subgroup of $W_n$ isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n = \mathbb{Z}_2^n$.
In fact $W_n$ is isomorphic to the wreath product $\mathbb{Z}_2 \wr S_n$ of $\mathbb{Z}_2$ with $S_n$ , where $\mathbb{Z}_2$ is a cyclic group of order 2, and then $|W_n| = 2^n n!$. (See [3]).

**Definition 2.3.5.** Let $w \in W_n$. An expression $w = v_1 v_2 \ldots v_k$, $v_i \in \{s_0, s_1, \cdots, s_n\}$ in which $k$ is minimal is called a *reduced* expression for $w$ and $l(w) = k$ is the length of $w$.

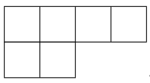
## 2.4 Bipartitions and Bitableaux

**Partitions and tableaux**

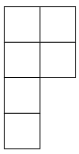
**Definition 2.4.1.** Let $n$ be a non-negative integer. A *composition of* $n$ is a finite sequence of non-negative integers $\lambda = (\lambda_1, \lambda_2, \ldots)$ such that $|\lambda| = \sum_i \lambda_i = n$.
The integer $\lambda_i$ for all $i \geq 1$ is called a *part* of $\lambda$.
A composition $\lambda$ of $n$ is called a *partition* if $\lambda_i \geq \lambda_{i+1}$ for all $i \geq 1$ and we write $\lambda \vdash n$.

A partition $\lambda$ can be illustrated graphically by a diagram called a *Young diagram.*

**Definition 2.4.2.** A *Young diagram* $[\lambda]$ of $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash n$ is $\{(i,j) \mid 1 \leq i, 1 \leq j \leq \lambda_i\}$, which is array of $n$ boxes placed in rows. The *ith* row of $[\lambda]$ consists of $\lambda_i$ boxes, $1 \leq i \leq l$. For example:

if $\lambda = (4, 2)$, then $[\lambda]$=  .

**Definition 2.4.3.** Let $\lambda$ be a partition of $n$. Then the conjugate partition $\lambda'$ of $\lambda$ is a partition of $n$ whose Young diagram $[\lambda']$ is obtained from the Young diagram $[\lambda]$ of $\lambda$ by exchanging the rows and columns in $[\lambda]$. For example:

Let $\lambda = (4, 2)$, so $[\lambda']$=  and then $\lambda' = (2^2, 1^2)$.

**Definition 2.4.4.** (1) A $\lambda$-tableau $\mathbf{t}$ is obtained from $[\lambda]$ by filling in the boxes of $[\lambda]$ with the non-repeated numbers $1, \ldots, n$. We say that $\mathbf{t}$ has shape $\lambda$ and write Shape($\mathbf{t}$) = $\lambda$. for example:

| | | | |
|---|---|---|---|
| 5 | 2 | 1 | 3 |
| 4 | 6 | | |

.

(2) A $\lambda$-tableau $\mathbf{t}$ is called *row standard* if the entries in $\mathbf{t}$ increase from left to right in each row and $\mathbf{t}$ is called *standard* if it is row standard and the entries increase from top to bottom in each column. For example:

| | | | |
|---|---|---|---|
| 2 | 3 | 4 | 6 |
| 1 | 5 | | |

is a row standard tableau but not standard, while

| | | | |
|---|---|---|---|
| 1 | 3 | 5 | 6 |
| 2 | 4 | | |

is standard.

(3) The *initial tableau* $\mathbf{t}^\lambda$ is a $\lambda$-tableau in which the numbers $1, \ldots, n$ appear in order along successive rows. For example:

if $\lambda = (4,2)$, then $\mathbf{t}^\lambda = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 \\ \cline{1-2}\end{array}$ .

**Definition 2.4.5.** The *dominance order* is a partial order, denoted by $\trianglerighteq$, defined on the set of partitions of $n$ as follows:

$$\lambda \trianglerighteq \mu \quad \text{if and only if} \quad \Sigma_{i=1}^{k} \lambda_i \geq \Sigma_{i=1}^{k} \mu_i \quad \text{for all} \ \ k.$$

The symmetric group $S_n$ acts on the set of $\lambda$-tableaux by permuting the entries of $[\lambda]$. For example:

let $\mathbf{t} = \begin{array}{|c|c|c|c|}\hline 5 & 6 & 1 & 3 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}$ , then $\mathbf{t}(2,6)(1,4,5) = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 3 \\ \hline 6 & 5 \\ \cline{1-2}\end{array}$ .

**Definition 2.4.6.** Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of $n$. The Young subgroup $S_\lambda$ of $S_n$ is the subgroup

$$S_\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_k}$$

which is also the row stabiliser of $\mathbf{t}^\lambda$.

Let $\mathbf{t}$ be a row-standard $\lambda$-tableau, we define the element $d(\mathbf{t})$ to be a permutation of $S_n$ such that

$$\mathbf{t} = \mathbf{t}^\lambda d(\mathbf{t}).$$

**Bipartitions and Bitableaux**

**Definition 2.4.7.** A *bicomposition* $\lambda$ of $n$ is an ordered pair $(\lambda^{(1)}, \lambda^{(2)})$ of compositions such that $|\lambda| = |\lambda^{(1)}| + |\lambda^{(2)}| = n$. We call $\lambda^{(i)}$ the *ith* component of $\lambda$. If both $\lambda^{(1)}$ and $\lambda^{(2)}$ are partitions, then $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ is called a *bipartition* of $n$.

**Definition 2.4.8.** A *Young diagram* of a bipartition $\lambda$ is:

$$[\lambda] = \{(i,j,k) \mid 1 \leq j \leq \lambda_i^{(k)} \text{ for } i \geq 1 \text{ and } k = 1,2\}$$

which is the ordered pair of Young diagrams of its components. Note that the triple $(i,j,k)$ refers to the row, column and component in which that node appears. For example:

if $\lambda = ((4,3,2),(2,1))$, then $[\lambda] = \left( \begin{array}{|c|c|c|c|} \hline \ & \ & \ & \ \\ \hline \ & \ & \ & \multicolumn{1}{|c}{} \\ \cline{1-3} \ & \ & \multicolumn{2}{|c}{} \\ \cline{1-2} \end{array} , \begin{array}{|c|c|} \hline \ & \ \\ \hline \ & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array} \right)$.

**Definition 2.4.9.** Let $\lambda$ be a bipartition.

(1) A $\lambda$-*bitableau* $\mathbf{t} = (\mathbf{t}^{(1)}, \mathbf{t}^{(2)})$ is obtained from $[\lambda]$ by filling in each box in $[\lambda]$ with one of the numbers $1, 2, \ldots, n$, allowing no repeats. We say that $\mathbf{t}$ has shape $\lambda$ and write Shape$(\mathbf{t}) = \lambda$. For example:

$\left( \begin{array}{|c|c|c|} \hline 2 & 4 & 7 \\ \hline 8 & 3 & \multicolumn{1}{|c}{} \\ \cline{1-2} \end{array} , \begin{array}{|c|c|} \hline 1 & 6 \\ \hline 5 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array} \right)$ is a ((3,2),(2,1))-bitableau.

(2) A $\lambda$-bitabeau $\mathbf{t} = (\mathbf{t}^{(1)}, \mathbf{t}^{(2)})$ is called *row standard* if the entries increase from left to right in each row of $\mathbf{t}^{(1)}$ and in each row of $\mathbf{t}^{(2)}$ and $\mathbf{t} = (\mathbf{t}^{(1)}, \mathbf{t}^{(2)})$ is called *standard* if it is row standard, and all the entries increase from top to bottom in each column of $\mathbf{t}^{(1)}$ and in each column of $\mathbf{t}^{(2)}$. For example:

$\left( \begin{array}{|c|c|c|} \hline 4 & 7 & 8 \\ \hline 2 & 3 & \multicolumn{1}{|c}{} \\ \cline{1-2} \end{array} , \begin{array}{|c|c|} \hline 5 & 6 \\ \hline 1 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array} \right)$ is a row standard bitableau,

$\left( \begin{array}{|c|c|c|} \hline 2 & 7 & 8 \\ \hline 3 & 4 & \multicolumn{1}{|c}{} \\ \cline{1-2} \end{array} , \begin{array}{|c|c|} \hline 1 & 6 \\ \hline 5 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array} \right)$ is a standard bitableau.

The set of all standard $\lambda$-bitabeaux is denoted by Std$(\lambda)$.

(3) We define $\mathbf{t}^{\lambda} = (\mathbf{t}^{\lambda^{(1)}}, \mathbf{t}^{\lambda^{(2)}})$ to be the standard $\lambda$-bitabeau in which the numbers $1, 2, \ldots, n$ appear in order along the rows of first component $\mathbf{t}^{\lambda^{(1)}}$ and then along the rows of second component $\mathbf{t}^{\lambda^{(2)}}$. For example:

let $\lambda = ((3,2),(2,1))$, then $\mathbf{t}^{\lambda} = \left( \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \multicolumn{1}{|c}{} \\ \cline{1-2} \end{array} , \begin{array}{|c|c|} \hline 6 & 7 \\ \hline 8 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array} \right)$.

**Definition 2.4.10.** The row stabilizer of $\mathbf{t}^{\lambda}$ is the Young subgroup $S_{\lambda} = S_{\lambda^{(1)}} \times S_{\lambda^{(2)}}$ of $S_n$.

(2) For a row standard $\lambda$-bitabeau $\mathbf{t}$ we define $d(\mathbf{t}) \in S_n$ to be the element of $S_n$ such that

$$\mathbf{t} = \mathbf{t}^{\lambda} d(\mathbf{t}).$$

For example, let $\lambda = ((3,2),(2,1))$, $\mathbf{t}= \left( \begin{array}{|c|c|c|}\hline 2 & 7 & 8 \\ \hline 3 & 4 \\ \cline{1-2} \end{array}, \begin{array}{|c|c|}\hline 1 & 6 \\ \hline 5 \\ \cline{1-1} \end{array} \right)$ then $d(\mathbf{t}) = (1276)(3854)$.

**Definition 2.4.11.** Let $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ and $\mu = (\mu^{(1)}, \mu^{(2)})$ be bipartitions of $n$. The set of bipartitions is a poset with partial order $\trianglerighteq$ where $\lambda \trianglerighteq \mu$, if

$$\Sigma_{i=1}^{j} \lambda_i^{(1)} \geq \Sigma_{i=1}^{j} \mu_i^{(1)} \qquad \text{for all } \ j$$

and

$$|\lambda^{(1)}| + \Sigma_{i=1}^{k} \lambda_i^{(2)} \geq |\mu^{(1)}| + \Sigma_{i=1}^{k} \mu_i^{(2)} \qquad \text{for all } \ k.$$

If $\lambda \trianglerighteq \mu$ we say that $\lambda$ dominates $\mu$. If $\lambda \trianglerighteq \mu$ and $\lambda \neq \mu$ we write $\lambda \triangleright \mu$.

For example, let $n = 2$, then

$$((2),(0)) \triangleright ((1,1),(0)) \triangleright ((1),(1)) \triangleright ((0),(2)) \triangleright ((0),(1,1)).$$

## 2.5 A cellular basis of the group algebra $RW_n$

Let $R$ be a commutative ring with identity. In this section we are going to recall the cellular basis of the group algebra $RW_n$, where $W_n \cong \mathbb{Z}_2 \wr S_n$ (defined in Definition 2.3.4) is the hyperoctahedral group (wreath product of $\mathbb{Z}_2$ with $S_n$) with generators $t_1 = s_0, s_1, \cdots, s_n$ subject to the relations in Definition 2.3.4.

Note that the references, which are used in this section, are about the Hecke algebras of type $B$ where these algebras are the deformation of the group algebra of $W_n \cong \mathbb{Z}_2 \wr S_n$, (in other words, the group algebra of $W_n \cong \mathbb{Z}_2 \wr S_n$ is a special case of the Hecke algebra of type $B$). (See section 3 in [3].)

**Theorem 2.5.1.** *[4], [5]. The algebra $RW_n$ is a free $R$-module with basis $\{w \mid w \in W_n\}$.*

**Lemma 2.5.2.** *[4], [5]. Let $*$ be the R-linear antiautomorphism of $RW_n$ determined by $s_i^* = s_i$ for all $i$ with $0 \leq i \leq n-1$. Then $w^* = w^{-1}$ for all $w \in W_n$.*

*Proof.* Let $w = v_1 v_2 \cdots v_l$, $v_i \in S = \{s_0, s_1, \cdots, s_{n-1}\}$, then $w^{-1} = v_l v_{l-1} \cdots v_1$ as $v_i^{-1} = v_i$, then

$$\begin{aligned} w^* &= (v_1 v_2 \cdots v_l)^* \\ &= v_l^* v_{l-1}^* \cdots v_1^* \\ &= v_l v_{l-1} \cdots v_1 = w^{-1}. \end{aligned}$$ □

For each pair $i, j$ of positive integers, define $s_{i,j} \in W_n$ inductively by $s_{i,i} = 1$ (for all $i$) and

$$s_{i,j} = \begin{cases} s_i s_{i+1} \cdots s_{j-1} & \text{if } i < j, \\ s_{i-1} s_{i-2} s_j & \text{if } i > j. \end{cases}$$

**Proposition 2.5.3** ([3],(2.1))**.** *Let $a, b$ be any positive integers then*
*(i) $t_a$ commutes with $t_b$ ;*
*(ii) $t_a$ commutes with $s_b$ unless $b = a-1$ or $b = a$.*

Note that from Definition 2.3.4, we have $t_1 = s_0$, $t_i = s_{i-1} t_{i-1} s_{i-1}$ for $2 \leq i \leq n$ so $t_a = s_{a,1} t_1 s_{1,a}$ for any positive integer $a$.

**Definition 2.5.4** ([3],[5])**.** For $0 \leq a \leq n$, $a = |\lambda^{(1)}|$, let the element $u_a^+$ of $RW_n$ be given by

$$u_0^+ = 1, \qquad u_a^+ = \prod_{i=1}^{a} (1 + s_{i,1} t_1 s_{1,i}) = \prod_{i=1}^{a} (1 + t_i).$$

where $t_1 = s_0$.

For example,

$$u_1^+ = (1 + t_1)$$

$$u_2^+ = (1 + t_1)(1 + t_2).$$

**Proposition 2.5.5** ([3](3.4) and [5](2.4))**.** *Let* $0 \leq a \leq n$.
*(i) if* $a \geq 1$*, then* $u_a^+ t_1 = t_1 u_a^+$.
*(ii) if* $a \geq 0$*,* $b \geq 1$ *are distinct integers, then* $u_a^+$ *commutes with* $s_b$.

Note that from lemma (2.5.2), we can show that $s_{a,1}^* = s_{1,a}$ as follows:

$$\begin{aligned} s_{a,1}^* &= (s_{a-1}s_{a-2}\cdots s_2 s_1)^* \\ &= s_1^* s_2^* \cdots s_{a-2}^* s_{a-1}^* \\ &= s_1 s_2 \cdots s_{a-2} s_{a-1} = s_{1,a}. \end{aligned}$$

and then $t_a^* = (s_{a,1}t_1 s_{1,a})^* = s_{1,a}^* t_1^* s_{a,1}^* = s_{a,1}t_1 s_{1,a} = t_a$.

Also, from proposition 2.5.3, we can show that $(1+t_a)$ commutes with $(1+t_b)$ for any positive integers $a, b$ as follows

$$\begin{aligned} &(1+t_a)(1+t_b) \\ &= 1 + t_a + t_b + t_a t_b \\ &= 1 + t_a + t_b + t_b t_a \\ &= 1 + t_a + t_b(1+t_a) \\ &= (1+t_b)(1+t_a). \end{aligned}$$

So from the above relation we can show that $(u_a^+)^* = u_a^+$ as follows:

$$\begin{aligned} (u_a^+)^* &= (\Pi_{i=1}^a (1+t_i))^* \\ &= (1+t_a)^* \cdots (1+t_2)^* (1+t_1)^* \\ &= (1+t_a^*) \cdots (1+t_2^*)(1+t_1^*) \\ &= (1+t_a) \cdots (1+t_2)(1+t_1) \\ &= (1+t_1)(1+t_2) \cdots (1+t_a) \\ &= \Pi_{i=1}^a (1+t_i) = u_a^+. \end{aligned}$$

**Definition 2.5.6.** [4],[17] Let $\lambda$ be a bipartition of $n$. The elements $x_\lambda$, $m_\lambda$ are defined as follows:
(i) $x_\lambda = \Sigma_{w \in S_\lambda} w$ where $_\lambda = S_{\lambda^{(1)}} \times S_{\lambda^{(2)}}$ is the row stabilizer of $\mathbf{t}^\lambda$.
(ii) $m_\lambda = u_a^+ x_\lambda$ where $a = |\lambda^{(1)}|$.

Note that $(x_\lambda)^* = x_\lambda$ and from proposition 2.5.5 we have $m_\lambda = u_a^+ x_\lambda = x_\lambda u_a^+$. Hence $m_\lambda^* = m_\lambda$.

**Definition 2.5.7.** [4],[17] Let $\lambda$ be a bipartition of $n$ and $\mathbf{s}$, $\mathbf{t}$ row standard $\lambda$-bitableax. Let $C_{\mathbf{st}} = d(\mathbf{s})^* m_\lambda d(\mathbf{t})$, where $d(s)$, $d(t) \in S_n$ as is defined in Definition 2.4.10.

Note that $(C_{\mathbf{st}})^* = d(\mathbf{t})^* m_\lambda^* d(\mathbf{s}) = d(\mathbf{t})^* m_\lambda d(\mathbf{s}) = C_{\mathbf{ts}}$.

**Definition 2.5.8.** [4] Suppose $\lambda$ is a bipartition of $n$.
(i) Let $A^\lambda$ be the $R$-module spanned by

$$\{C^\mu_{\mathbf{st}} \mid \ \mathbf{s} \text{ and } \mathbf{t} \text{ are standard } \mu - \text{bitableaux for some bipartition } \mu \text{ of } n \text{ with } \mu \trianglerighteq \lambda\}.$$

(ii) Let $\overline{A^\lambda}$ be the $R$-module spanned by

$$\{C^\mu_{\mathbf{st}} \mid \ \mathbf{s} \text{ and } \mathbf{t} \text{ are standard } \mu - \text{bitableaux for some bipartition } \mu \text{ of } n \text{ with } \mu \triangleright \lambda\}.$$

**Proposition 2.5.9.** *[4] Let $\lambda$ be a bipartition of $n$. Then $A^\lambda$ and $\overline{A^\lambda}$ are two sided ideals of $RW_n$.*

The following theorem shows that $RW_n$ is a cellular algebra in sense of Graham and Lehrer.

**Theorem 2.5.10.** *[4],[17] The group algebra $RW_n$ is a free $R$-module with basis*

$$\mathfrak{M} = \{C^\lambda_{\boldsymbol{st}} \mid \ \boldsymbol{s} \textit{ and } \boldsymbol{t} \textit{ are standard } \lambda - \textit{bitableax for some bipartition } \lambda \ \textit{ of } n\}$$

*Moreover the following statements hold*
*(1) The $R$-linear antiautomorphism $*$ satisfies $* : C^\lambda_{\boldsymbol{st}} \mapsto C^\lambda_{\boldsymbol{ts}}$ for all $\boldsymbol{s}, \boldsymbol{t} \in Std(\lambda)$.*
*(2) Let $\lambda$ be a bipartition of $n$ and $\boldsymbol{t}$ a standard $\lambda$-bitableau. Let $h \in RW_n$. Then for each standard $\lambda$-bitableau $\boldsymbol{b}$ there exists $r_{\boldsymbol{b}} \in R$ such that, for all standard $\lambda$-tableau $\boldsymbol{s}$, we have*

$$hC^\lambda_{\boldsymbol{st}} \equiv \sum_{\boldsymbol{b} \in Std(\lambda)} r_{\boldsymbol{b}} C^\lambda_{\boldsymbol{bt}} \pmod{\overline{A^\lambda}}.$$

*So the basis $\mathfrak{M}$ is a cellular basis of $RW_n$ and we call $\mathfrak{M}$ the standard basis of $RW_n$.*

The following definition describes the cell (Specht) modules of $RW_n$.

**Definition 2.5.11.** [4],[17] Let $\lambda$ be a bipartition of $n$. Let $C_{\mathbf{t}}^{\lambda} = C_{\mathbf{t}\mathbf{t}^{\lambda}} + \overline{A^{\lambda}} = d(\mathbf{t})m_{\lambda} + \overline{A^{\lambda}}$. The *Specht module* $S^{\lambda}$ is a free $R$-module with basis

$$\{C_{\mathbf{t}}^{\lambda} \mid \mathbf{t} \text{ a standard } \lambda - \text{bitableau}\}.$$

The action of $RW_n$ on this basis is given by

$$hC_{\mathbf{t}}^{\lambda} = \sum_{\mathbf{b}\in Std(\lambda)} r_{\mathbf{b}} C_{\mathbf{b}}^{\lambda}$$

The bilinear form $\langle\ ,\ \rangle$ on $S^{\lambda}$ is a symmetric map from $S^{\lambda} \times S^{\lambda}$ to $R$ defined by

$$C_{\mathbf{us}}^{\lambda} C_{\mathbf{tb}}^{\lambda} \equiv \langle C_{\mathbf{s}}^{\lambda}, C_{\mathbf{t}}^{\lambda}\rangle C_{\mathbf{ub}}^{\lambda} \pmod{\overline{A^{\lambda}}}$$

for all standard $\lambda$-bitableaux $\mathbf{s}$, $\mathbf{t}$. This form satisfies $\langle hu, v\rangle = \langle u, h^{*}v\rangle$ for all $u, v \in S^{\lambda}$ and $h \in RW_n$.

The following example will illustrate the cellular basis and Specht modules of $R(\mathbb{Z}_2 \wr S_2)$.

**Example 2.5.12.** The cell datum of $R(\mathbb{Z}_2 \wr S_2)$ is $(\Lambda, M, \mathfrak{M}, *)$ where

- $\Lambda = \{\lambda_i \mid \lambda_i \text{ is a bipartition of } 2\}$ is a dominance ordered set which is a partially ordered set.
- $M(\lambda)$, $\lambda \in \Lambda$ is a set of standard tableaux of shape $\lambda$.
- Basis $\{C_{\mathbf{st}}^{\lambda} \mid \mathbf{s}, \mathbf{t} \in M(\lambda)\}$.
- The anti-involution map $*$, where $(C_{\mathbf{st}}^{\lambda})^{*} = C_{\mathbf{ts}}^{\lambda}$.

*The bipartitions of* $n = 2$ *are:*

$\lambda_1 = ((2), (0))$ so $|\lambda_1^{(1)}| = 2$;

$\lambda_2 = ((1,1), (0))$ so $|\lambda_2^{(1)}| = 2$;

$\lambda_3 = ((1),(1))$ so $|\lambda_3^{(1)}| = 1$;

$\lambda_4 = ((0),(2))$ so $|\lambda_4^{(1)}| = 0$;

$\lambda_5 = ((0),(1,1))$ so $|\lambda_5^{(1)}| = 0$.

So $\Lambda = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$ and the dominance order on the elements of $\Lambda$ is as follows:

$$\lambda_1 \rhd \lambda_2 \rhd \lambda_3 \rhd \lambda_4 \rhd \lambda_5.$$

*The standard $\lambda$- tableaux are:*

$\lambda_1$-bitableau $\mathbf{t}_1 = \left(\boxed{1}\boxed{2}, \emptyset\right)$;

$\lambda_2$-bitableau $\mathbf{t}_2 = \left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}, \emptyset\right)$;

$\lambda_3$-bitableax $\mathbf{t}_3 = \left(\boxed{1}, \boxed{2}\right)$,and $\mathbf{s}_3 = \left(\boxed{2}, \boxed{1}\right)$;

$\lambda_4$-bitableau $\mathbf{t}_4 = \left(\emptyset, \boxed{1}\boxed{2}\right)$;

$\lambda_5$-bitableau $\mathbf{t}_5 = \left(\emptyset, \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}\right)$.

So we have: $M(\lambda_1) = \{\mathbf{t}_1\}$, $M(\lambda_2) = \{\mathbf{t}_2\}$, $M(\lambda_3) = \{\mathbf{t}_3, \mathbf{s}_3\}$, $M(\lambda_4) = \{\mathbf{t}_4\}$, $M(\lambda_5) = \{\mathbf{t}_5\}$.

Now we will construct the elements $C^{\lambda}_{st} = d^*(s)m_\lambda d(t)$ of the basis, where
$m_\lambda = u_a^+ x_\lambda$, $\quad u_a^+ = \prod_{i=1}^{a}(1 + s_{i,1}s_0s_{1,i})$, $\quad x_\lambda = \Sigma_{w\in\mathfrak{S}_\lambda} w$, $a = |\lambda^{(1)}|$.

We have: $u_0^+ = 1$, $\quad u_1^+ = 1 + s_0$, $\quad u_2^+ = (1+s_0)(1+s_1s_0s_1)$.
$x_{\lambda_1} = 1 + s_1$, $\quad x_{\lambda_2} = 1$, $\quad x_{\lambda_3} = 1$, $\quad x_{\lambda_4} = 1 + s_1$, $\quad x_{\lambda_5} = 1$. Therefore

$m_{\lambda_1} = u_2^+ x_{\lambda_1} = (1+s_0)(1+s_1s_0s_1)(1+s_1)$;

$m_{\lambda_2} = u_2^+ x_{\lambda_2} = (1+s_0)(1+s_1s_0s_1)$;

$m_{\lambda_3} = u_1^+ x_{\lambda_3} = (1+s_0)$;

$m_{\lambda_4} = u_0^+ x_{\lambda_4} = 1 + s_1$;

$m_{\lambda_5} = u_0^+ x_{\lambda_5} = 1$, and then

$C^{\lambda_1}_{\mathbf{t}_1\mathbf{t}_1} = d^*(\mathbf{t}_1)m_{\lambda_1}d(\mathbf{t}_1) = 1.m_{\lambda_1} = m_{\lambda_1}$;

$C^{\lambda_2}_{\mathbf{t}_2\mathbf{t}_2} = d^*(\mathbf{t}_2)m_{\lambda_2}d(\mathbf{t}_2) = m_{\lambda_2}$;

$C^{\lambda_3}_{\mathbf{t}_3\mathbf{t}_3} = m_{\lambda_3}$;

$C^{\lambda_3}_{\mathbf{t}_3\mathbf{s}_3} = 1.m_{\lambda_3}s_1 = (1+s_0)s_1;$
$C^{\lambda_3}_{\mathbf{s}_3\mathbf{t}_3} = s_1.m_{\lambda_3}.1 = s_1(1+s_0);$
$C^{\lambda_3}_{\mathbf{s}_3\mathbf{s}_3} = s_1.m_{\lambda_3}.s_1 = s_1(1+s_0)s_1;$
$C^{\lambda_4}_{\mathbf{t}_4\mathbf{t}_4} = m_{\lambda_4};$
$C^{\lambda_5}_{\mathbf{t}_5\mathbf{t}_5} = m_{\lambda_5} = 1.$

Thus the cellular basis of $R(\mathbb{Z}_2 \wr S_2)$ is

$\mathfrak{M} = \{C^{\lambda_1}_{\mathbf{t}_1\mathbf{t}_1}, C^{\lambda_2}_{\mathbf{t}_2\mathbf{t}_2}, C^{\lambda_3}_{\mathbf{t}_3\mathbf{t}_3}, C^{\lambda_3}_{\mathbf{t}_3\mathbf{s}_3}, C^{\lambda_3}_{\mathbf{s}_3\mathbf{t}_3}, C^{\lambda_3}_{\mathbf{s}_3\mathbf{s}_3}, C^{\lambda_4}_{\mathbf{t}_4\mathbf{t}_4}, C^{\lambda_5}_{\mathbf{t}_5\mathbf{t}_5}\}.$

Now we will find the ideals
$A^\lambda = \mathrm{span}\{C^{\mu}_{\mathbf{st}} \mid \mathbf{s},\mathbf{t} \in M(\mu), \mu \text{ is a bipartition of } 2, \mu \trianglerighteq \lambda\}$:
$A^{\lambda_1} = \mathrm{span}\{C^{\lambda_1}_{\mathbf{t}_1\mathbf{t}_1}\};$
$A^{\lambda_2} = \mathrm{span}\{C^{\lambda_1}_{\mathbf{t}_1\mathbf{t}_1}, C^{\lambda_2}_{\mathbf{t}_2\mathbf{t}_2}\};$
$A^{\lambda_3} = \mathrm{span}\{C^{\lambda_1}_{\mathbf{t}_1\mathbf{t}_1}, C^{\lambda_2}_{\mathbf{t}_2\mathbf{t}_2}, C^{\lambda_3}_{\mathbf{t}_3\mathbf{t}_3}, C^{\lambda_3}_{\mathbf{t}_3\mathbf{s}_3}, C^{\lambda_3}_{\mathbf{s}_3\mathbf{t}_3}, C^{\lambda_3}_{\mathbf{s}_3\mathbf{s}_3}\};$
$A^{\lambda_4} = \mathrm{span}\{C^{\lambda_1}_{\mathbf{t}_1\mathbf{t}_1}, C^{\lambda_2}_{\mathbf{t}_2\mathbf{t}_2}, C^{\lambda_3}_{\mathbf{t}_3\mathbf{t}_3}, C^{\lambda_3}_{\mathbf{t}_3\mathbf{s}_3}, C^{\lambda_3}_{\mathbf{s}_3\mathbf{t}_3}, C^{\lambda_3}_{\mathrm{s}_3\mathbf{s}_3}, C^{\lambda_4}_{\mathbf{t}_4\mathbf{t}_4}\};$
$A^{\lambda_5} = \mathrm{span}\{C^{\lambda_1}_{\mathbf{t}_1\mathbf{t}_1}, C^{\lambda_2}_{\mathbf{t}_2\mathbf{t}_2}, C^{\lambda_3}_{\mathbf{t}_3\mathbf{t}_3}, C^{\lambda_3}_{\mathbf{t}_3\mathbf{s}_3}, C^{\lambda_3}_{\mathbf{s}_3\mathbf{t}_3}, C^{\lambda_3}_{\mathbf{s}_3\mathbf{s}_3}, C^{\lambda_4}_{\mathbf{t}_4,\mathbf{t}_4}, C^{\lambda_5}_{\mathbf{t}_5\mathbf{t}_5}\}.$

Note that $A^\lambda$ is an ideal in $R(\mathbb{Z}_2 \wr S_2)$ for all $\lambda \in \Lambda$ and we have the following chain of ideals:

$$A^{\lambda_1} \subset A^{\lambda_2} \subset A^{\lambda_3} \subset A^{\lambda_4} \subset A^{\lambda_5} = R(\mathbb{Z}_2 \wr S_2).$$

Now we will find the cell (Specht) modules $S^\lambda$ of $R(\mathbb{Z}_2 \wr S_2)$, where

$$S^\lambda = \mathrm{span}\{C^{\lambda}_{\mathbf{t}} \mid \mathbf{t} \text{ is a standard } \lambda - \text{bitableau}\}.$$

$S^{\lambda_1} = \mathrm{span}\{C_{\mathbf{t}_1}\}$, where $C_{\mathbf{t}_1} = d(\mathbf{t}_1)m_{\lambda_1} + \overline{A^{\lambda_1}} = m_{\lambda_1};$
$S^{\lambda_2} = \mathrm{span}\{C_{\mathbf{t}_2}\}$ where $C_{\mathbf{t}_2} = d(\mathbf{t}_2)m_{\lambda_2} + \overline{A^{\lambda_2}} = m_{\lambda_2} + \overline{A^{\lambda_2}};$
$S^{\lambda_3} = \mathrm{span}\{C_{\mathbf{t}_3}, C_{\mathbf{s}_3}\}$ where
$C_{\mathbf{t}_3} = d(\mathbf{t}_3)m_{\lambda_3} + \overline{A^{\lambda_3}} = m_{\lambda_3} + \overline{A^{\lambda_3}}$, $C_{\mathbf{s}_3} = d(\mathbf{s}_3)m_{\lambda_3} + \overline{A^{\lambda_3}} = m_{\lambda_3}s_1 + \overline{A^{\lambda_3}};$
$S^{\lambda_4} = \mathrm{span}\{C_{\mathbf{t}_4}\}$ where $C_{\mathbf{t}_4} = d(\mathbf{t}_4)m_{\lambda_4} + \overline{A^{\lambda_4}} = m_{\lambda_4} + \overline{A^{\lambda_4}};$
$S^{\lambda_5} = \mathrm{span}\{C_{\mathbf{t}_5}\}$ where $C_{\mathbf{t}_5} = d(\mathbf{t}_5)m_{\lambda_5} + \overline{A^{\lambda_5}} = 1 + \overline{A^{\lambda_5}}.$

# Chapter 3

# The Decorated partial Brauer algebra

The purpose of this chapter is to define a new algebra called the *decorated partial Brauer algebra*, which is a unital associative algebra over a commutative ring $R$ with a basis of diagrams. These diagrams, which are called the *decorated partial Brauer diagrams*, are the partial Brauer diagrams where each line and each isolated vertex can be decorated.

We begin with defining *the set of decorated partial Brauer partitions*, which can be represented by the decorated partial Brauer diagrams and find its size. Then in the second section we define the decorated partial Brauer diagrams, identify them with the decorated partial Brauer partitions, describe their multiplication and determine the dimension of the algebra.

In the last section we define the symmetric partial Brauer algebra, which is a subalgebra of the partial Brauer algebra and then show a correspondence between the decorated partial Brauer diagrams and the symmetric partial Brauer diagrams.

## 3.1 Decorated partial (Brauer) partitions

From the set of partial partitions, which are defined in Definition 2.1.1, we will construct a new set called a *decorated partial (Brauer) partition*.

Given $P = \{P_1, P_2, \ldots, P_l\}$ a partition of a finite set $T$, we put

$$P^d := \{P_i \in P \mid \ |P_i| = d\}.$$

Note that for $P$ a partial partition $P = P^1 \cup P^2$.

**Definition 3.1.1.** For a finite set $T$, a *decorated partial (Brauer) partition* of $T$ is a triple $(P, F, G)$ with

**(i)** $P = \{P_1, P_2, \ldots, P_l\}$ is a partial (Brauer) partition of $T$.

**(ii)** $F$ is an element of $\mathcal{P}(P^2)$ (the power set of the set $P^2$).

**(iii)** $G$ is an element of $\mathcal{P}(P^1)$ (the power set of the set $P^1$).

The set of all decorated partial (Brauer) partitions of $T$ is denoted $DPB_T$, so

$$DPB_T = \{(P, F, G) \mid \ P \in PB_T,\ F \in \mathcal{P}(P^2),\ G \in \mathcal{P}(P^1)\}.$$

**Example 3.1.2.** Let $T = \{1, 2, 3, 4, 5, 6\}$. Then $P = \{\{1, 3\}, \{2, 5\}, \{4\}, \{6\}\} \in PB_T$.
Note that $P = P^2 \cup P^1$, where $P^2 = \{\{1, 3\}, \{2, 5\}\}$ and $P^1 = \{\{4\}, \{6\}\}$.
Consider $F = \{\{1, 3\}\} \in \mathcal{P}(P^2)$ and $G = \{\{4\}, \{6\}\} \in \mathcal{P}(P^1)$.
Then $(P, F, G) \in DPB_T$.

It will be helpful to illustrate the decorated partial Brauer partitions with a picture, see Example 3.2.4. In the next section the decorated partial Brauer partitions will be identified with diagrams.

**Lemma 3.1.3.** *For $T$ finite,*

$$|DPB_T| = \sum_{k=0}^{\lfloor \frac{|T|}{2} \rfloor} \binom{|T|}{2k} \; 2^{|T|-2k} \; 2^k \; (2k-1)!!.$$

*where $(2k-1)!! = (2k-1).(2k-3)\cdots 3.1$ and $k$ is the number of blocks of size two in $P \in DPB_T$.*

*Proof.* Let $PB_T^k := \{P \in PB_T \mid \; |P^2| = k\}$. So $PB_T = \bigsqcup_{k=0}^{\lfloor \frac{|T|}{2} \rfloor} PB_T^k$ and,

$$\begin{aligned} DPB_T &= \bigsqcup_{k=0}^{\lfloor \frac{|T|}{2} \rfloor} \bigsqcup_{P \in PB_T^k} \{(P, F, G) \mid \; F \in \mathcal{P}(P^2), G \in \mathcal{P}(P^1)\} \\ &= \bigsqcup_{k=0}^{\lfloor \frac{|T|}{2} \rfloor} \bigsqcup_{P \in PB_T^k} \{(P, f) \mid \; f \in \mathcal{P}(P^2) \times \mathcal{P}(P^1)\} \\ &= \bigsqcup_{k=0}^{\lfloor \frac{|T|}{2} \rfloor} \{(P, f) \mid \; P \in PB_T^k, \; f \in \mathcal{P}(P^2) \times \mathcal{P}(P^1)\}. \end{aligned}$$

Note that, since $|P^2| = k$ then $|P^1| = |T| - 2k$ and therefore $|\mathcal{P}(P^2)| = 2^k$ and $|\mathcal{P}(P^1)| = 2^{|T|-2k}$. Also, from Proposition 2.1.5, we have $|PB_T^k| = \binom{|T|}{2k}(2k-1)!!$. So,

$$\begin{aligned} |DPB_T| &= |\bigsqcup_{k=0}^{\lfloor \frac{|T|}{2} \rfloor} \{(P, f) \mid \; P \in PB_T^k, \; f \in \mathcal{P}(P^2) \times \mathcal{P}(P^1)| \\ &= \sum_{k=0}^{\lfloor \frac{|T|}{2} \rfloor} |PB_T^k|.2^k.2^{|T|-2k} \\ &= \sum_{k=0}^{\lfloor \frac{|T|}{2} \rfloor} \binom{|T|}{2k} \; 2^k \; 2^{|T|-2k} \; (2k-1)!!. \end{aligned}$$

□

Note that $|DPB_T|$ depends on $|T|$, for example the first few values are:

| $\lvert T\rvert$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $\lvert DPB_T\rvert$ | 1 | 2 | 6 | 20 | 76 | 312 | 1348 | 6512 | 32400 |

We are mainly interested in the case $T = \{1, \ldots, n, 1', \ldots, n'\} = \underline{n} \cup \underline{n'}$ and we define $DPB_n = DPB_{\underline{n} \cup \underline{n'}}$ where

$DPB_n = \{(P, F, G) \mid P \in PB_n, F \in \mathcal{P}(P^2)$ where $P^2 = \{P_i \in P \mid |P_i| = 2\} \subseteq P$, and $G \in \mathcal{P}(P^1)$ where $P^1 = \{P_j \in P \mid |P_j| = 1\} \subseteq P\}$.

## 3.2 Decorated partial Brauer diagrams

**Definition 3.2.1.** For $n, m \in \mathbb{N}$, let $\underline{n} := \{1, \ldots, n\}$ and $\underline{m'} := \{1', \ldots, m'\}$.
An $(n, m)$-*rectangle* $R$ is $[0, 1] \times [0, 1]$, together with $n + m$ labelled vertices which is divided into two sets $\{1, \ldots, n\}$ and $\{1', \ldots, m'\}$ of vertices, arranged (from left to right) on the top row and the bottom row (respectively) in the frame $\partial R$ of a rectangle $R$ (by a frame we mean the boundary of $R$). We also allow any number of (unlabelled) isolated vertices in the interior $(R)$ of a rectangle $R$, (by isolated vertex we mean a distinguished vertex in $R$ which does not lie on any edge), where $(R) = R \backslash \partial R$.

Note that we use a dash to indicate a vertex in the bottom row.

**Definition 3.2.2.** An $(n, m)$-*decorated partial Brauer pseudo-diagram* is an $(n, m)$-rectangle $R$ together with $l$ edges which are embeddings $f_i : [0, 1] \longrightarrow R$, $i \in \{1, \ldots, l\}$ such that

$(L_1)$ $f_i(x) \in (R)$ where $0 < x < 1$.

$(L_2)$ If $f_i(x) \in \partial R$, $x \in \{0, 1\}$ then $f_i(x) \in \underline{n} \cup \underline{m'}$.

$(L_3)$ If $f_i(x) = f_j(y)$, $x, y \in \{0, 1\}$ and $i \neq j$ then $f_i(x) \in (R)$.

$(L_4)$ If $f_i(x) = f_j(y) = f_k(z)$, $x, y, z \in \{0, 1\}$ then at least two of $i, j, k$ coincide.

$(L_5)$ $f_i, f_j$ are pairwise transversal (i.e. their tangent lines at their intersection point are distinct).

With potential decorations, " $\circ$ " and "$\square$" as follows:

**(i)** Any number of the decorations " $\circ$ " can appear anywhere on the edges but not at their endpoints, or on any isolated vertex.

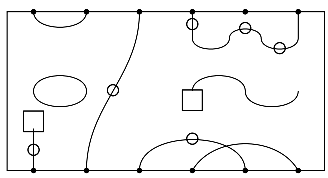
**(ii)** The decoration "□" can only appear in the following cases:

($D_1$) On isolated vertices which do not lie on any edges such that if an isolated vertex is on $\partial R$ then there is at most one decoration.

($D_2$) The non-concurrent endpoints of edges which lie in the interior ($R$) can have at most one decoration.

We write $C(n, m)$ for the set of decorated partial Brauer pseudo-diagrams, where $n$ is the number of labelled vertices in the top row and $m$ is the number of labelled vertices in the bottom row.

An example of a decorated partial Brauer pseudo-diagram:

 .

Note that many connected components which can be formed by some of the embedded edges $f_i$, appear in the interior ($R$) of the $(n, m)$-pseudo-diagram and they do not connect to the top and the bottom row of the rectangle frame. These connected components will be called *isolated components.*

Now we will reduce a decorated partial Brauer pseudo-diagram to define a decorated partial Brauer diagram as follows:

**Definition 3.2.3.** Consider a diagram $d \in C(n, m)$ such that it has no isolated components and in addition

**(i)** Each edge that connects two labelled vertices has at most one decoration " $\circ$ ".

**(ii)** There is no edge with an interior endpoint.

Such a diagram is called a *decorated partial Brauer diagram.* (See Figure 3.1). In such a diagram undecorated (resp. decorated) lines which connect a vertex in the top row with a vertex in the bottom row are called undecorated (resp. decorated) propagating

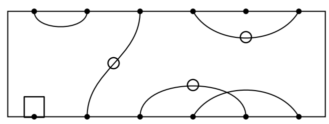


FIGURE 3.1: Example of a decorated partial Brauer diagram.

lines. Undecorated (resp. decorated) lines which connect vertices in the same row are called undecorated (resp. decorated) arcs. Undecorated (resp. decorated) vertices on the top row or on the bottom row on the rectangle frame $\partial R$ which are not incident on an edge are called undecorated (resp. decorated) isolated vertices.

Note that the underlying diagram of a decorated partial Brauer diagram, by which we mean the same diagram but with all decorations removed, is a partial Brauer diagram.

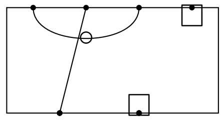
In the following we will identify a decorated partial Brauer partition $(P, F, G)$ (which is defined in Definition 3.1.1) with a decorated partial Brauer diagram.

**(I)** A decorated partial Brauer partition $(P, F, G)$ can be represented by a decorated partial Brauer diagram as follows:

**(a)** Any part of size two $\{i, j\} \in P$ containing vertices $i$ and $j$ is represented by an edge joining the corresponding vertices labelled $i$ and $j$. If $\{i, j\} \in F$ then put one " $\circ$ " decoration on this edge.

**(b)** Each part of size one $\{i\} \in P$ is represented by an isolated vertex which coincides with the vertex $i$ on $\partial R$. If $\{i\} \in G$ then put "$\Box$" on this vertex.

**Example 3.2.4.** Let $P = \{\{1, 3\}, \{2, 1'\}, \{4\}, \{2'\}\}$, $F = \{\{1, 3\}\}$ and $G = \{\{4\}, \{2'\}\}$. So the decorated partial Brauer partition $(P, F, G) \in DPB_{\underline{4} \cup \underline{2'}}$ can be represented by the following diagram:

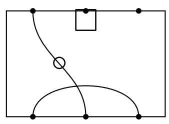


**(II)** Each decorated partial Brauer diagram $d$ represents a decorated partial Brauer partition $(P, F, G) \in DPB_{\underline{n} \cup \underline{m'}}$ as follows:

**(a)** If two labelled vertices $i$ and $j$ in a diagram $d$ is connected by an edge then $i$ and $j$ belong to the same part partition i.e. $\{i, j\} \in P$. If this edge which connects $i$ and $j$ is decorated with " $\circ$ " decoration then $\{i, j\} \in P \cap F$.

**(b)** If $i$ is any labelled isolated vertex in $\underline{n} \cup \underline{m'}$ which does not lie on any edge then we have $\{i\} \in P$. If this labelled isolated vertex is decorated with "$\square$" then $\{i\} \in P \cap G$.

**Example 3.2.5.** Let

$d =$ 

then $d$ represents $(P, F, G) = \Big(\{\{1, 2'\}, \{1', 3'\}, \{2\}, \{3\}\}, \{\{1, 2'\}\}, \{\{2\}\}\Big).$

Note that from the above discussion the following can be deduced:

**(i)** A part of size two $\{i, j\} \in P$ (resp. in $P$ and $F$) **if and only if** the labelled vertices $i$ and $j$ are connected by an edge (resp. decorated edge with one decoration " $\circ$ ").

**(ii)** A part of size one $\{i\} \in P$ (resp. in $P$ and $G$) **if and only if** $i$ is a labelled isolated vertex (resp. a labelled decorated isolated vertex with a single decoration "$\square$") in $\underline{n} \cup \underline{m'}$.

Therefore we have the following.

**Definition 3.2.6.** We consider two diagrams $d_1$, $d_2$ which represent decorated partial Brauer partition *equivalent* and we write $d_1 \sim d_2$ if they represent the same decorated partial Brauer partition.

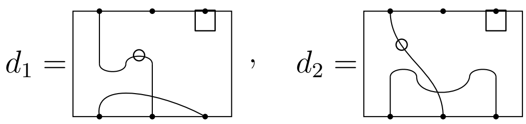
So by decorated partition Brauer diagram we mean the equivalence class of a given diagram.

**Definition 3.2.7.** The *set of decorated partial Brauer diagrams* denoted by $DPB(n, m)$ is the set of all $\sim$ equivalence classes of decorated partial Brauer diagrams.

Since a diagram is identified with its partition that means that there is a map from the set of decorated partial Brauer diagrams to the set of decorated partial Brauer partitions:

$$\begin{aligned} \pi : DPB(n, m) &\xrightarrow{\sim} DPB_{\underline{n} \cup \underline{m'}} \\ d &\mapsto \pi(d) \end{aligned} \tag{3.1}$$

where $d$ is a decorated partial Brauer diagram and $\pi(d)$ is the decorated partial partition it represents.
For example:
Let



So, $\pi(d_1) = \pi(d_2) = \big(\{\{1, 2'\}, \{2\}, \{3\}, \{1', 3'\}\}, \{\{1, 2'\}\}, \{\{3\}\}\big)$, and then $d_1$, $d_2$ are equivalent.

## 3.3 Multiplication of decorated partial Brauer diagrams

The method for multiplying decorated partial Brauer diagrams is given by concatenation. This is like multiplying partial Brauer diagrams with some additional rules which handle the joining of decorated lines and decorated vertices (see Figure 3.2 which illustrate these rules).

Let $R$ be a commutative ring with identity, and $\delta, \delta_\circ, \delta', \mu, \mu'$ elements in $R$. We define a multiplication of decorated partial Brauer diagrams as the map

$$\mathcal{P} : DPB(n,m) \times DPB(m,l) \to RDPB(n,l)$$

as follows. Let $d_1 \in DPB(n,m)$, $d_2 \in DPB(m,l)$ be diagrams. We define their product $d_1d_2$ as follows:

Place $d_1$ above $d_2$ and then identify the bottom of $d_1$ with the top of $d_2$ in such a way so that the vertex labelled $x'$ in $d_1$ is identified with the vertex labelled $x$ in $d_2$. These vertices will be referred to as a *middle row* of $d_1d_2$. Note that this concatenation of $d_1$ and $d_2$ gives us $d_1d_2$ as an element of $C(n,l)$. To obtain $d_1d_2$ as an element of $RDPB(n,l)$ we use the following rules:

**(1)** If more than one decoration " $\circ$ " appear on the same edge then they should be cancelled in pairs according to the local cancellation (1) in Figure 3.2.

**(2)** Any undecorated or decorated edge with " $\circ$ " that does not join two labelled vertices from the top row or the bottom row of $d_1d_2$ may contract to a (possibly decorated) vertex. (See (2) in Figure 3.2.)

**(3)** Isolated components, which can appear in this multiplication, (see Figure 3.3 which illustrates such components) should be removed and replaced with parameters as follows:

($C_1$) Chains of edges in the middle row which do not connect to the top and the bottom row of $d_1d_2$ may form the following:

(i) undecorated (resp. decorated with a single decoration " $\circ$ ") closed loop, which is then replaced with parameter $\delta$, (resp. $\delta_\circ$).

(ii) undecorated (or decorated with a single decoration " $\circ$ ") open string, which is then replaced with a parameter $\delta'$.

(iii) undecorated (or decorated with a single decoration " $\circ$ ") open string with one side (resp. both sides) of its endpoints is decorated with a single decoration "□", which is then replaced with a parameter $\mu$ (resp. $\mu'$).

$(C_2)$ If two undecorated isolated vertices meet in the middle row so an undecorated isolated vertex is formed in the middle row which is then replaced with a parameter $\delta'$.

$(C_3)$ If an undecorated isolated vertex meets a decorated isolated vertex with "□" in the middle row then a decorated isolated vertex with "□" is formed which is then replaced with a parameter $\mu$.

$(C_4)$ If two decorated isolated vertices each of them is decorated with "□" meet in the middle row, they are replaced with a parameter $\mu'$ (we call such a feature two meeting squares).

(See Figure 3.3 which illustrates these isolated components with their parameters.)

So the product of $d_1$ and $d_2$ is

$$d_1 d_2 = \delta^l \ \delta_\circ^m \ (\delta')^n \ \mu^k \ (\mu')^t \ d_3$$

where $d_3$ is a decorated partial Brauer diagram obtained from the concatenation with the isolated components deleted. Here, $l$ is the number of undecorated loops, the number $m$ is the number of decorated loops with "$\circ$", the number $n$ is the number of undecorated open strings, decorated open strings with "$\circ$" or undecorated isolated vertices, the number $k$ is the number of squares, decorated or undecorated open strings with square in one side of their endpoints and $t$ is the number of two meeting squares, decorated or undecorated open strings with square in both sides of their endpoints, that arise in the middle row.
(See Figure 3.4 for an illustrative example of the multiplication of two decorated partial Brauer diagrams.)

Note that the multiplication of decorated partial Brauer diagrams produces a decorated partial Brauer pseudo-diagram, which is then reduced to a decorated partial Brauer diagram that by using the rules in Figure 3.2 and then removing the induced isolated components.

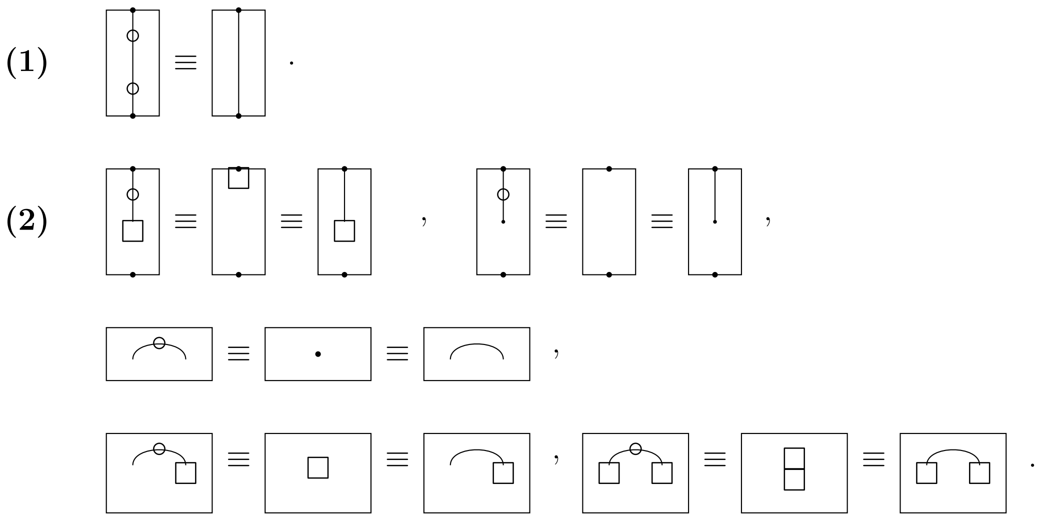



FIGURE 3.2: Rules in a product of two decorated partial Brauer diagrams.

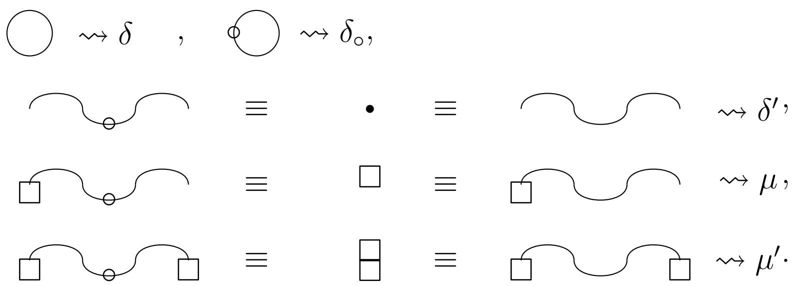


FIGURE 3.3: isolated components that may appear in the decorated partial Brauer pseudo-diagram or in the middle row during the product of two decorated partial Brauer diagrams.

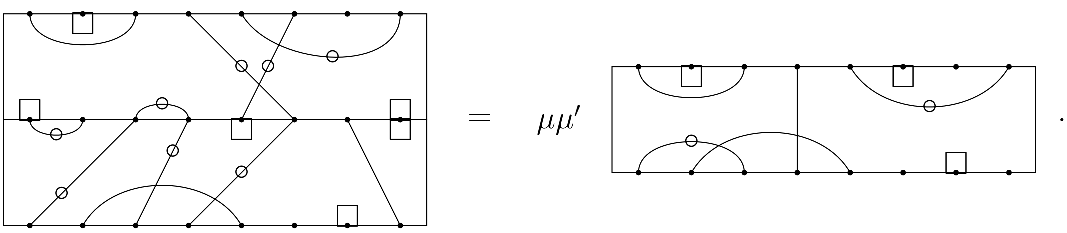


FIGURE 3.4: An example of the multiplication of decorated partial Brauer diagrams.

In next step we show that this reduction is consistent (i.e. satisfies the diamond condition [1]). First we will recall the diamond condition.

We consider the algebra $A$ generated by indeterminate $x_1, \dots, x_n$, subject to relations

$$w_j = s_j \quad (1 \leq j \leq m),$$

where each $w_j$ is a word in $x_1, \dots, x_n$ and $s_j \in K\langle x_1, \dots, x_n \rangle$.

**Definition 3.3.1.** Given words $u, v$ and a relation $w_j = s_j$ we consider the linear map

$$K\langle x_1, \dots, x_n \rangle \to K\langle x_1, \dots, x_n \rangle$$

sending $f = \lambda u w_j v + f'$ where $\lambda \in K$, and $f'$ is a linear combination of other words different from $u w_j v$, to $g = \lambda u s_j v + f'$.

We call $g$ the *reduction of $f$ with respect to $u, v$ and the relation $w_j = s_j$.*

We write $f \rightsquigarrow g$ to indicate that $g$ is a reduction of $f$ for some $u, v$ and $w_j = s_j$.

**Definition 3.3.2.** We say that two reductions of $f$, say $f \rightsquigarrow g$ and $f \rightsquigarrow h$, satisfy the *diamond condition* if there exist sequences of reductions starting with $g$ and $h$, which lead to the same element.

Pictorially.

$f$

$g \quad h$

$k$

In particular we are interested in this in the following two cases:

An *overlap ambiguity* is a word which can be written as $w_i v$ and also as $u w_j$ for some $i, j$ and some words $u, v \neq 1$, so that $w_i$ and $w_j$ overlap. There are reductions $f \rightsquigarrow s_i v$ and $f \rightsquigarrow u s_j$.

An *inclusion ambiguity* is a word which can be written as $w_i$ and as $uw_jv$ for some $i \neq j$ and some $u, v$. There are reductions $f \rightsquigarrow s_i$ and $f \rightsquigarrow us_jw$.

**Example 3.3.3.** For the relations $x^2 = x$, $y^2 = 1$, $yx = 1 - xy$ the ambiguities are:

$$(xx)x = x(xx), \quad (yy)y = y(yy), \quad (yy)x = y(yx), \quad (yx)x = y(xx).$$

Does the diamond condition hold for these?

$(xx)x \rightsquigarrow xx \rightsquigarrow x$ and $x(xx) \rightsquigarrow xx \rightsquigarrow x$. Yes.

$(yy)y \rightsquigarrow 1y = y$ and $y(yy) \rightsquigarrow y1 = y$. Yes.

$(yy)x \rightsquigarrow 1x = x$ and $y(yx) \rightsquigarrow y(1 - xy) = y - yxy = y - (yx)y \rightsquigarrow y - (1 - xy)y = xyy = x(yy) \rightsquigarrow x1 = x$. Yes.

$(yx)x \rightsquigarrow (1-xy)x = x-xyx \rightsquigarrow x-x(1-xy) = xxy \rightsquigarrow xy$ and $y(xx) \rightsquigarrow yx \rightsquigarrow 1-xy$. No.

In the following Lemma we will show that the rules which are used in the product of the decorated partial Brauer diagrams satisfy the diamond condition.

**Lemma 3.3.4.** *Let $d \in C(n, m)$. Let $x$ be any decorated edge with single decoration "$\circ$" in $d$, $y$ and $z$ be any decorated with single decoration "$\square$" (resp. undecorated) isolated vertex in $d$. Pictorially:*

$x =$ , $\quad y = \square$, $\quad z = \bullet$

*We have the relations:*

$x^2 = 1$,

$yx = y = xy$ *(which is defined if $x$ and $y$ are on the same edge),*

$zx = z = xz$ *(which is defined if $x$ and $z$ are on the same edge),*

$yz = y = zy$ *(which is defined if $y$ and $z$ are on the same vertex),*

$z^2 = z$

*and their ambiguities are:*

$$(xx)x = x(xx), \quad (yx)x = y(xx), \quad (zx)x = z(xx)$$
$$(xx)y = x(xy), \quad (yx)y = y(xy), \quad (zx)y = z(xy)$$
$$(xx)z = x(xz), \quad (yx)z = y(xz), \quad (zx)z = z(xz).$$

*Then the diamond condition holds for them.*

*Proof.*

$$(xx)x \rightsquigarrow 1.x \rightsquigarrow x \quad \text{and} \quad x(xx) \rightsquigarrow x.1 \rightsquigarrow x$$
$$(xx)y \rightsquigarrow 1.y \rightsquigarrow y \quad \text{and} \quad x(xy) \rightsquigarrow xy \rightsquigarrow y$$
$$(xx)z \rightsquigarrow 1.z \rightsquigarrow z \quad \text{and} \quad x(xz) \rightsquigarrow xz \rightsquigarrow z$$
$$(yx)x \rightsquigarrow yx \rightsquigarrow y \quad \text{and} \quad y(xx) \rightsquigarrow y.1 \rightsquigarrow y$$
$$(yx)y \rightsquigarrow yy = y^2 \quad \text{and} \quad y(xy) \rightsquigarrow yy = y^2$$
$$(yx)z \rightsquigarrow yz \rightsquigarrow y \quad \text{and} \quad y(xz) \rightsquigarrow yz \rightsquigarrow y$$
$$(zx)x \rightsquigarrow zx \rightsquigarrow z \quad \text{and} \quad z(xx) \rightsquigarrow z.1 \rightsquigarrow z$$
$$(zx)y \rightsquigarrow zy \rightsquigarrow y \quad \text{and} \quad z(xy) \rightsquigarrow zy \rightsquigarrow y$$
$$(zx)z \rightsquigarrow zz = z^2 = z \quad \text{and} \quad z(xz) \rightsquigarrow zz = z^2 = z. \quad \square$$

Now let *pd* be a decorated partial Brauer pseudo-diagram. Write $pd \rightsquigarrow d'$ to mean $d'$ is obtained from *pd* by applying one of the rules in Figure 3.2 or by removing one of the isolated components in Figure 3.3. We have the following.

**Proposition 3.3.5.** *For any pd there is a chain of relations "* $\rightsquigarrow$ *" starting from pd and ending in a (reduced) diagram with no isolated components and which has at most one decoration on each line and on each isolated vertex. If there are multiple such chains from pd, then every one ends in the same reduced diagram.*

*Proof.* For any *pd*, there are three ways of reductions, that are: reduced the number of the decorations "$\circ$" on lines (lines which have more than one decoration), contract a line with one decorated (resp. undecorated) interior endpoint to a decorated (resp.

undecorated) vertex, and remove an isolated component. Note that all these ways of reductions are local in the sense that each single relation occurs on individual line, also each isolated component is removed individually from its location. Therefore the reduction of any *pd* diagram can be considered as the reduction of each of its individual lines. From Lemma 3.3.4 we have that the Diamond condition [1] is satisfied for all sequences of relations on each line. Then after finishing the reduction on lines it is easy to remove any isolated component. Note that having the Diamond condition for our reduction ensures that any different such chain of the same *pd* should lead to the same reduced diagram. □

In the following we will show that the multiplication of decorated partial Brauer diagrams is associative.
We first define some notations.

**Notation.**
Since we mainly interested in the set $\underline{n} \cup \underline{n'}$, we write $DPB(n, n) = DPB(n)$.
Let $i, j \in \underline{n} \cup \underline{n'}$, $d \in DPB(n)$, we write $i \sim_d j$ if there is an edge that joins $i$ to $j$ in the diagram $d$.

Let $d_1, d_2, d_3 \in DPB(n)$.
We use $\widehat{d_1 d_2}$ to denote the graph obtained by placing the diagram $d_1$ above the diagram $d_2$ and then $d_1 d_2$ is $\widehat{d_1 d_2}$ after multiplying $d_1$ with $d_2$ (i.e. the result of the multiplication). Vertices on the top row of $\widehat{d_1 d_2}$ are labelled by $1, \ldots, n$, vertices in the middle row of $\widehat{d_1 d_2}$ are labelled by $1', \ldots, n'$ and the vertices in the bottom row of $\widehat{d_1 d_2}$ are labelled by $1'', \ldots, n''$.

The graph $\widehat{d_1 d_2 d_3}$ means that the diagram $d_1$ is stacked on top of the diagram $d_2$ stacked on top of the diagram $d_3$.

**Definition 3.3.6.** Let $i, j \in \underline{n} \cup \underline{n'} \cup \underline{n''}$, $P$ be a path (chain of edges) in $\widehat{d_1 d_2}$ that joins $i$ to $j$.
We say that the path $P$ in $\widehat{d_1 d_2}$ is a *lift* of the edge $i \sim_{d_1 d_2} j$ and the edge $i \sim_{d_1 d_2} j$ is a *contraction* of the path $P$. (NB: the edge $i \sim_{d_1 d_2} j$ is considered to be a path that is a contraction and a lift of itself.)

From the multiplication of decorated partial Brauer diagrams (which is by concatenation) the following can be deduced:

**(I)** For $i, j \in \underline{n} \cup \underline{n''}$, we have $i \sim_{d_1 d_2} j$ **if and only if** the edge $i \sim_{d_1 d_2} j$ is a contraction of a path in $\widehat{d_1 d_2}$.

**(II)** For $i \in \underline{n}$, $i$ is an undecorated (resp. a decorated) isolated vertex in the top row of $d_1 d_2$ **if and only if** $i$ is an undecorated (resp. a decorated) isolated vertex in the top row of $d_1$ **or**
there exists a path in $\widehat{d_1 d_2}$ that joins $i$ to an undecorated (resp. a decorated) vertex in the middle row of $\widehat{d_1 d_2}$.

**Similarly**, for $i \in \underline{n'}$, $i$ is an undecorated (resp. a decorated) isolated vertex in the bottom row of $d_1 d_2$ **if and only if** $i$ is an undecorated (resp. a decorated) isolated vertex in the bottom row of $d_2$ **or**
there exists a path in $\widehat{d_1 d_2}$ that joins $i$ to an undecorated (resp. a decorated) vertex in the middle row of $\widehat{d_1 d_2}$.

**Proposition 3.3.7.** *The multiplication on the set $DPB(n)$ is associative.*

*Proof.* Let $d_1, d_2, d_3 \in DPB(n)$. We want to show the following:
**(A)** For $i, j \in \underline{n} \cup \underline{n''}$ , $i \sim_{(d_1 d_2) d_3} j$ if and only if $i \sim_{d_1 (d_2 d_3)} j$.
**(B)** If $i$ is an undecorated (resp. a decorated) isolated vertex in the top row of $(d_1 d_2) d_3$ then it is also in the top row of $d_1(d_2 d_3)$ and vice versa.
**(C)** If $i$ is an undecorated (resp. a decorated) isolated vertex in the bottom row of $(d_1 d_2) d_3$ then it is also in the bottom row of $d_1(d_2 d_3)$ and vice versa.
**(D)** $(d_1 d_2) d_3$ and $d_1(d_2 d_3)$ have the same parameters.

Proof:
**(A)** Let $i \sim_{(d_1 d_2) d_3} j$ then from **(I)** there is a path $P$ in $\widehat{(d_1 d_2) d_3}$ which is a lift of the edge $i \sim_{(d_1 d_2) d_3} j$. Each edge in the path $P$ that lies in $d_1 d_2$ is in turn a contraction of a path in $\widehat{d_1 d_2}$.
So we lift each such edge to a path in $\widehat{d_1 d_2}$ to obtain a path $Q$ which joins $i$ to $j$ in the graph $\widehat{d_1 d_2 d_3}$. Now we contract each subpath of $Q$ that lies wholly in $\widehat{d_2 d_3}$ to an

edge in $d_2d_3$. This gives a path $R$ in $\widehat{d_1(d_2d_3)}$ that joins $i$ to $j$. This implies that (by using **(I)**) $i \sim_{d_1(d_2d_3)} j$ which is a contraction of a path $R$.

**Similarly** (by using the same process) we can show that if $i \sim_{d_1(d_2d_3)} j$ then we get $i \sim_{(d_1d_2)d_3}$.

It remains to show that $i \sim_{(d_1d_2)d_3} j$ and $i \sim_{d_1(d_2d_3)} j$ are both decorated or both undecorated (i.e. $i \sim_{(d_1d_2)d_3} j$ is decorated if and only if $i \sim_{d_1(d_2d_3)} j$ is decorated).

Let $P$ be a path that joins $i$ to $j$ in the graph $\widehat{d_1d_2d_3}$.

Let $x$, $y$ and $z$ be the number of decorated edges which are in the path $P$ and lie in the diagram $d_1$, $d_2$, $d_3$ respectively.

Let $h$ (resp. $\tilde{h}$) be the number of subpaths of the path $P$ that lie wholly in $\widehat{d_1d_2}$ (resp. $\widehat{d_2d_3}$).

Let $r_i$, $1 \leq i \leq h$ (resp. $\tilde{r_i}$, $1 \leq i \leq \tilde{h}$) be the number of decorated edges in each subpath $p_i$ (resp. $\tilde{p_i}$) of the path $P$ that lies wholly in $\widehat{d_1d_2}$ (resp. $\widehat{d_2d_3}$).

Let $Q$ be a path in the graph $\widehat{(d_1d_2)d_3}$ (where $\widehat{(d_1d_2)d_3}$ is a diagram $d_1d_2$ stacked on top of the diagram $d_3$) which is produced from a contraction of each subpath of the path $P$ that lies wholly in $d_1d_2$.

Let $R$ be a path in the graph $\widehat{d_1(d_2d_3)}$ (where $\widehat{d_1(d_2d_3)}$ is a diagram $d_1$ stacked on top of the diagram $d_2d_3$) which is produced from a contraction of each subpath of the path $P$ that lies wholly in $d_2d_3$. Let $s_i \in \{0, 1\}$, $1 \leq i \leq h$ (resp. $\tilde{s_i} \in \{0, 1\}$, $1 \leq i \leq \tilde{h}$) be the number of the decoration on each edge which lies in the path $Q$ (resp. $R$) and in the diagram $d_1d_2$ (resp. $d_2d_3$), where these edges are in turn a contraction of the subpaths $p_i$ (resp. $\tilde{p_i}$).

Therefore, from the rule of the multiplication on $DPB(n)$, we have

$$r_i \equiv s_i \pmod 2 \qquad \text{and} \qquad \tilde{r_i} \equiv \tilde{s_i} \pmod 2.$$

Now let $A$ (resp. $B$) be the number of the decorated edges which lie in the path $Q$ (resp. $R$) and in the diagram $d_1d_2$ (resp. $d_2d_3$). So $A+z$ (resp. $x+B$) is the number of decorated edges in the path $Q$ (resp. $R$).

So we have

$$A = \sum_i s_i \equiv \sum_i r_i = x+y \pmod 2 \qquad \text{and} \qquad B = \sum_i \tilde{s_i} \equiv \sum_i \tilde{r_i} = y+z \pmod 2.$$

Therefore,

$$A + z \equiv (x + y) + z = x + (y + z) \equiv x + B \pmod 2.$$

This proves **(A)**.

**(B)** Let $i$ be an undecorated (resp. a decorated) isolated vertex in the top row of $(d_1d_2)d_3$ then either

(i) $i$ is an undecorated (resp. a decorated) isolated vertex in the top row of $d_1d_2$, **or**

(ii) there exists a path $P$ in $\widehat{(d_1d_2)d_3}$ that joins $i$ to an undecorated (resp. decorated) vertex in the middle row of $\widehat{(d_1d_2)d_3}$.

Assume (i) then there is either

(1) $i$ is an undecorated (resp. a decorated) isolated vertex in the top row of $d_1$ then (from **(II)**) $i$ is an undecorated (resp. a decorated) isolated vertex in the top row of $d_1(d_2d_3)$. **Or**

(2) there exists a path $Q$ in $\widehat{d_1d_2}$ that joins the vertex $i$ to an undecorated (resp. a decorated) vertex in the middle row of $\widehat{d_1d_2}$. From **(I)**, each edge in the path $Q$ which lies in $d_2$ is a contraction of itself in $\widehat{d_2d_3}$. Therefore $Q$ is a path is $\widehat{d_1(d_2d_3)}$. Consequently (by using **(II)**) $i$ in an undecorated (resp. a decorated) isolated vertex in the top row of $d_1(d_2d_3)$.

Assume (ii). So each edge in the path $P$ that lies in $d_1d_2$ is in turn a contraction of a path in $\widehat{d_1d_2}$. Then each such edge is lifted to a path in $\widehat{d_1d_2}$. This gives a path $\tilde{P}$ in the graph $\widehat{d_1d_2d_3}$. Now we contract each subpath of $\tilde{P}$ that lies wholly in $\widehat{d_2d_3}$ to an edge or isolated vertex in $d_2d_3$. This produces a path in $\widehat{d_1(d_2d_3)}$ that joins $i$ to an undecorated (resp. a decorated) vertex in the top row of $d_2$ which is a middle row of $\widehat{d_1(d_2d_3)}$. Therefore (from **(II)**) we have $i$ is an undecorated (resp. a decorated) isolated vertex in the top row of $d_1(d_2d_3)$.

**Similarly** By using the same process it can be shown that if $i$ is an undecorated (resp. a decorated) isolated vertex in the top row of $d_1(d_2d_3)$ then also in the top row of $(d_1d_2)d_3$.

This proves **(B)**.

**(C)** The proof is similar as **(B)**.

**(D)** There are the following cases:

**(i)** Let $P$ be a not closed path in the graph $\widehat{d_1d_2d_3}$ which does not connect to the top row of $d_1$ nor the bottom row of $d_3$.

Firstly, we will find the product $(d_1d_2)d_3$.

Contract each subpath of $P$ which lies wholly in $\widehat{d_1d_2}$. This produces a path $Q$ in $\widehat{(d_1d_2)d_3}$ which does not connect to the top row of $d_1d_2$ nor the bottom row of $d_3$, or (in the case when $P$ lies wholly in $\widehat{d_1d_2}$) possibly produces an undecorated isolated vertex in $\widehat{(d_1d_2)d_3}$ which does not lie on the top row of $d_1d_2$ nor the bottom row of $d_3$.

Therefore, in the result of the product $(d_1d_2)d_3$ the path $Q$ (or the isolated vertex) is removed and multiplied with a parameter $\delta'$.

Now we will find the product $d_1(d_2d_3)$.

Contract each subpath of $P$ which lies wholly in $\widehat{d_2d_3}$. This gives a path $R$ in $\widehat{d_1(d_2d_3)}$ which does not connect to the top row of $d_1$ nor the bottom row of $d_2d_3$, or (in the case when $P$ lies wholly in $\widehat{d_2d_3}$) possibly gives an undecorated isolated vertex in $\widehat{d_1(d_2d_3)}$ which does not lie on the top row of $d_1$ nor the bottom row of $d_2d_3$.

Therefore, in the result of the product $d_1(d_2d_3)$, the path $R$ or the isolated vertex is removed and multiplied with a parameter $\delta'$.

**Similarly** if $P$ is a path in $\widehat{d_1d_2d_3}$ with one of its endpoint (resp. both of its endpoints) is decorated with "$\square$", so in the result of $(d_1d_2)d_3$ and $d_1(d_2d_3)$ both of them will multiply with a parameter $\mu$ (resp. $\mu'$).

**(ii)** Let $P$ be a closed path in $\widehat{d_1d_2d_3}$ which does not connect to the top row of $d_1$ nor the bottom row of $d_3$ (by closed path we mean its endpoints coincide together. In other word, a closed path is a loop).

Let $x, y, z$ be the number of decorated edges which are in the path $P$ and lie in

$d_1, d_2, d_3$ respectively.

Let's first find the product $(d_1d_2)d_3$.

We contract each subpath of $P$ which lies wholly in $\widehat{d_1d_2}$. So we obtain a closed path $Q$ in $\widehat{(d_1d_2)d_3}$.

Let $A$ be the number of decorated edges in the path $Q$ which lies wholly in $d_1d_2$. Therefore, the number $A+z$ of decorated edges in $Q$ satisfies

$$A + z \equiv (x + y) + z \pmod 2.$$

(Note that the proof of this relation is as the proof in (**A**).)

If $A+z$ is even (resp. odd), then in the result of the multiplication of $(d_1d_2)d_3$ we remove a path $Q$, which is an undecorated (resp. decorated) loop, and multiply with a parameter $\delta$ (resp. $\delta_\circ$).

Now we will find a product $d_1(d_2d_3)$.

We contract each subpath of $P$ which lies wholly in $\widehat{d_2d_3}$. Then we obtain a closed path $R$ in $\widehat{d_1(d_2d_3)}$.

Let $B$ be the number of decorated edges in the path $R$ which lies wholly in $d_2d_3$. Therefore, the number $x+B$ of decorated edges in $R$ satisfies

$$x + B \equiv x + (y + z) \pmod 2.$$

(The proof of this relation is as the proof in (**A**).)

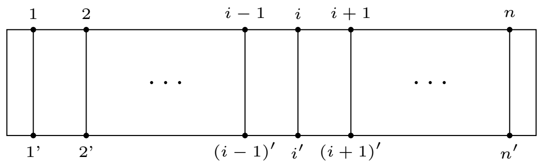
If $x+B$ is even (resp. odd), then in the result of the multiplication of $d_1(d_2d_3)$ we remove a path $R$, which is an undecorated (resp. decorated) loop, and multiply with a parameter $\delta$ (resp. $\delta_\circ$).

Since $A + z \equiv x + B \pmod 2$, so the paths $Q$ and $R$ are both an undecorated loop or both a decorated loop.

Therefore, in all cases the same multiplying parameters in both product $(d_1d_2)d_3$ and $d_1(d_2d_3)$ are obtained.

Hence, from **A**, **B**, **C** and **D**, we have $(d_1d_2)d_3 = d_1(d_2d_3)$. □

The identity element for multiplication on decorated partial Brauer diagrams $DPB(n)$ is the identity element of the undecorated partial Brauer diagrams, $\mathbb{I}$:

$\mathbb{I} =$ 


Now we will give a definition of a decorated partial Brauer algebra.

**Definition 3.3.8.** (**Decorated partial Brauer algebra**) Let $R$ be a commutative ring with identity, $\delta, \delta', \delta_\circ, \mu, \mu' \in R$ and $n$ a natural number. The *decorated partial Brauer algebra* denoted by $D\mathcal{P}\mathfrak{B}_n(\delta, \delta', \delta_\circ, \mu, \mu')$ is the free $R$-module with basis the decorated partial Brauer diagrams $DPB(n)$ and multiplication induced by the linear extension of the product on decorated partial Brauer diagrams defined in 3.3.

**Remark 3.3.9.** Note that $D\mathcal{P}\mathfrak{B}_n(\delta, \delta', \delta_\circ, \mu, \mu')$ is an associative and unital $R$-algebra by the previous results.

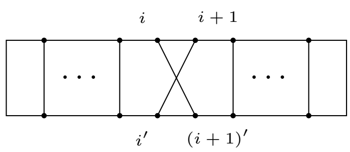
From Lemma 3.1.3 and equation 3.1 we have the following.

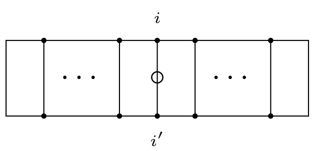
**Proposition 3.3.10.** *The dimension of* $D\mathcal{P}\mathfrak{B}_n(\delta, \delta_\circ, \delta', \mu, \mu')$ *is:*

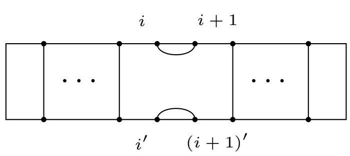
$$\dim(D\mathcal{P}\mathfrak{B}_n(\delta, \delta', \delta_\circ, \mu, \mu')) = \sum_{l=0}^{n} \binom{2n}{2l} 2^{2n-2l} 2^l (2l-1)!!$$

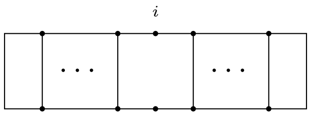
*where $l$ is the number of edges in the diagram.*

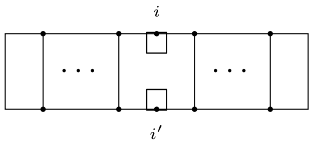
**Lemma 3.3.11.** *The decorated partial Brauer algebra, $D\mathcal{P}\mathfrak{B}_n$, is generated by the diagrams $s_i$, $e_i$, $1 \leq i \leq n-1$ and $g_i$, $p_i$, $q_i$, $1 \leq i \leq n$, where*

$s_i =$ 

,
$g_i =$ 


$e_i =$ 

$p_i =$ 

, $q_i =$ 

.

*Proof.* We first recall that the partial Brauer algebra, $\mathcal{P}\mathfrak{B}_n$, is generated by the diagrams $s_i$, $e_i$ for $1 \leq i \leq n-1$ and $p_i$ for $1 \leq i \leq n$ (section 2.4 in [8], proposition 20 in [15]).

Also, from the definition of decorated partial Brauer diagram, we observe that the decorated partial Brauer diagram is a partial Brauer diagram with potential decoration on any edge or any isolated vertex.

A proof goes by induction on the number of the decorations of any diagram $d \in DPB(n)$.

Let $d$ be any decorated partial Brauer diagram. If $d$ has no decoration then $d$ is a partial Brauer diagram and the result follows.
Now, suppose that $d$ has at least one decoration. We distinguish two cases:

**Case I:** Let $d$ has a decorated edge $\{i, j\}$ (say), $i, j \in \underline{n} \cup \underline{n'}$ are the endpoints of the edge $\{i, j\}$. So the diagram $d$ can be decomposed to a product $d = g_i b = b g_j$ that if $\{i, j\}$ is a decorated propagating line or $d = g_i b = g_j b$ (resp. $d = b g_i = b g_j$) if $\{i, j\}$ is a decorated arc in the top row (resp. bottom row) of $d$, where $b$ is the diagram $d$ with the decoration on the edge $\{i, j\}$ removed (i.e. $b$ is $d$ with strictly one less decoration).

**Case II:** Let $d$ has a decorated isolated vertex $i$ (say) in the top row (resp. bottom row) of $d$, $1 \leq i \leq n$. Then the diagram $d$ can be decomposed to a product $d = q_i c$ (resp. $d = c q_i$), where $c$ is a diagram $d$ with the decoration on the vertex $i$ removed (i.e. $c$ is $d$ with strictly one less decoration).

Consequently, from Cases **I** and **II** and by induction, any decorated partial Brauer diagram with any number of decorations is a product of the diagrams $g_i$, $q_i$, $p_i$, $1 \leq i \leq n$ and $s_i$, $e_i$, $1 \leq i \leq n-1$. $\square$

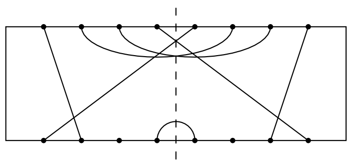
## 3.4 The Left-Right symmetric partial Brauer algebra

In this section we will define a subalgebra of the partial Brauer algebra $\mathcal{P}\mathfrak{B}_{2n}(\delta, \delta')$, called the Symmetric partial Brauer algebra, spanned by partial Brauer diagrams that are symmetric. Then we will demonstrate a correspondence between the set of decorated partial Brauer diagrams and the set of symmetric partial Brauer diagrams.

**Definition 3.4.1.** We say $D_1$ is a symmetric diagram if when $D_1$ is reflected about its central vertical axis the same diagram is obtained.

**Definition 3.4.2.** The set of *left-right symmetric partial Brauer diagrams*, denoted by $\mathcal{S}PB(2n)$, is the set of partial Brauer diagrams which are symmetric.

An example of a symmetric partial Brauer diagram is



.

**Lemma 3.4.3.** *The set $\mathcal{S}PB(2n)$ spans a subalgebra of $\mathcal{P}\mathfrak{B}_{2n}(\delta, \delta')$, denoted by $S\mathcal{P}\mathfrak{B}_{2n}(\delta, \delta')$.*

*Proof.* Firstly, note that the identity element of $PB(2n)$ is a symmetric diagram so it belongs to $\mathcal{S}PB(2n)$.

The multiplication of two diagrams $d_1$ and $d_2$ in $\mathcal{S}PB(2n)$ is, as in $PB(2n)$, given by concatenation. In this concatenation, by concatenating arcs from bottom row of $d_1$ with arcs from top row of $d_2$, some symmetric chains of lines form in the middle row of $d_1 d_2$. These symmetric chains may introduce the following:

1. Some of these chains may join pairs in the top row of $d_1$ or pairs in the bottom row of $d_2$ or vertices from top of $d_1$ with vertices from bottom of $d_2$, these new lines will be symmetric since they are introduced from symmetric chains.

2. Some of them may only join with the top of $d_1$ (resp. the bottom of $d_2$) which introduce symmetric isolated vertices in the top row of $d_1$ (resp. the bottom row of $d_2$).

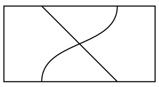
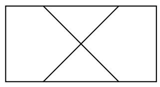
3. Some of these chains which do not connect to the top row of $d_1$ nor the bottom row of $d_2$ form symmetric closed loops or open strings in the middle row which are removed.

Also, from this concatenation, some symmetric isolated vertices may appear in the middle row which form from meeting two isolated vertices, these isolated vertices are also removed.
Therefore the diagram obtained after removing the middle row with connected components (closed loops, open strings or isolated vertices) consists of arcs and isolated vertices from the top row of $d_1$, arcs and isolated vertices from the bottom row of $d_2$ and new symmetric lines, isolated vertices which produced by the concatenation so this diagram is in $\mathcal{S}PB(2n)$ meaning that the set of $\mathcal{S}PB(2n)$ diagrams is closed under multiplication. □

**Definition 3.4.4.** Let $R$ be a commutative ring with identity, $\delta, \delta' \in R$, $n$ a natural number. *The left-right symmetric partial Brauer algebra* $S\mathcal{P}\mathfrak{B}_{2n}(\delta, \delta')$, is an associative unital subalgebra of the partial Brauer algebra with a basis consisting of symmetric partial Brauer diagrams.

In the following a process will be introduced to get a decorated partial Brauer diagram from a symmetric partial Brauer diagram and vice versa.

A decorated partial Brauer diagram can be obtained from a symmetric partial Brauer diagram as follows:
First draw the symmetric partial Brauer diagram so that there is a vertical axis of symmetry between the points $n$ and $n+1$. Draw it so no more than two lines are concurrent at any point. Also note that arcs do not just touch as this would violate condition $(L_5)$ in the Definition 3.2.2.

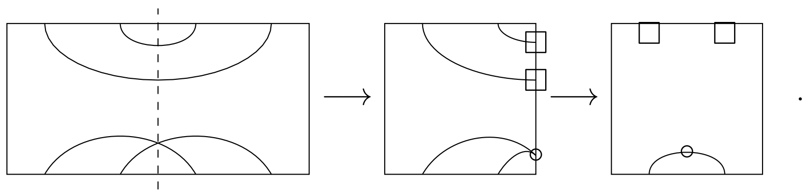
**Example 3.4.5.**  is a symmetric diagram which is redrawn  so that there is a same vertical axis of symmetry in the middle.

Now consider the left half of symmetric diagram after cutting along the axis of symmetry then

**($f_1$)** Lines crossing the axis which do not cross any other lines on the axis have a square "□" placed on the point on the axis. These lines are then contracted with their square.

**($f_2$)** For pairs of lines that intersect on the vertical axis of symmetric decorate this point with "$\circ$". This line can then be moved with the decoration "$\circ$" to form a decorated edge.

<u>Example</u>



This process defines a map $f : \mathcal{S}PB(2n) \longrightarrow DPB(n)$.

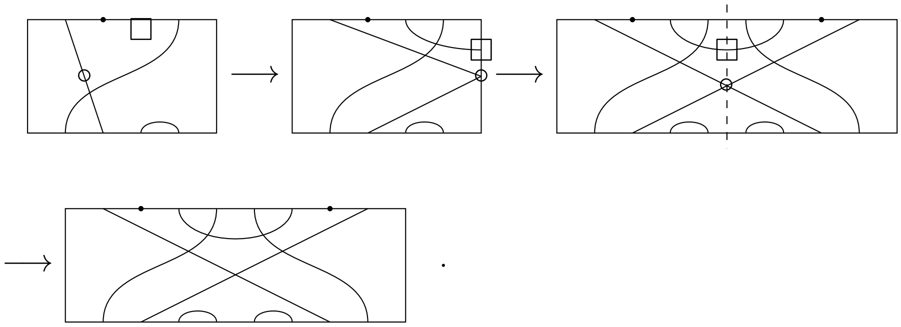
Also, a decorated partial Brauer diagram can be deformed to get a symmetric partial Brauer diagram by following steps:

**(1)** Deform the (both types of) decorations to touch the east wall of the rectangle.

**(2)** Take a reflection of the deformed diagram about the east wall.

**(3)** Remove the decoration from any line.

Example



This process defines a map $g : DPB(n) \longrightarrow \mathcal{S}PB(2n)$.

Clearly, from the definitions of $f$ and $g$ they are the inverse of each other as bijection. Therefore the following is obtained.

**Proposition 3.4.6.** *There is a bijection between the set of symmetric partial Brauer diagrams $\mathcal{S}PB(2n)$ and the set of decorated partial Brauer diagrams $DPB(n)$.*

# Chapter 4

# The decorated partial Brauer algebra is cellular

This chapter is devoted to establishing the cellularity of the decorated partial Brauer algebra. The main result in this chapter is Theorem 4.5.1 which shows that $D\mathcal{P}\mathfrak{B}_n$ is cellular. To prove this theorem, we apply Theorem 4.1.2 given by Green and Paget, which exhibits an algebra as an iterated inflation of existing cellular algebras which implies the cellularity of the algebra (Proposition 3.4 in [12]). For more details about iterated inflation we refer to [12], [14]. The second main result in this chapter is Theorem 4.7.1 which gives an indexing set of simple modules of the decorated partial Brauer algebra. Throughout this chapter, $K$ is a field and, unless otherwise stated, all tensors are over $K$.

## 4.1 Xi's Lemma

In [21] Xi's gave the following Lemma to provide a characterisation of iterated inflation of cellular algebras.

**Lemma 4.1.1.** *[21, Lemma 3.3]. Let $K$ be a field, $A$ a $K$-algebra with an involution $i$. Suppose there is a vector space decomposition*

$$A = \bigoplus_{j=1}^{m} V_j \otimes_K V_j \otimes_K B_j$$

*where $V_j$ is a vector space and $B_j$ is a cellular algebra with respect to an involution $\sigma_j$ and a cell chain $J_1^{(j)} \subset \cdots \subset J_{s_j}^{(j)} = B_j$ for each $j$. Define $J_t = \bigoplus_{j=1}^{t} V_j \otimes_K V_j \otimes_K B_j$. Assume that the restriction of $i$ on $V_j \otimes_K V_j \otimes_K B_j$ is given by $w \otimes v \otimes b \mapsto v \otimes w \otimes \sigma_i(b)$. If for each $j$ there is a bilinear form $\phi_j : V_j \otimes_K V_j \to B_j$ such that $\sigma_j(\phi_j(w, v)) = \phi_j(v, w)$ for all $w, v \in V_j$ and the multiplication of two elements in $V_j \otimes_K V_j \otimes_K B_j$ is governed by $\phi_j$ modulo $J_{j-1}$, that is, for $x, y, u, v \in V_j$ and $b, c \in B_j$, we have*

$$(x \otimes y \otimes b)(u \otimes v \otimes c) = x \otimes v \otimes b\phi_j(y, u)c$$

*modulo the ideal $J_{j-1}$, and if $V_j \otimes V_j \otimes J_l^{(j)} + J_{j-1}$ is an ideal in $A$ for all $l$ and $j$, then $A$ is a cellular algebra.*

Recently (in 2018) Green and Paget showed that this lemma is incorrect and they present the following replacement for Xi's lemma.

**Theorem 4.1.2.** *[7, Theorem 1] Let $A$ be a $K$-algebra, with an anti-involution $\sigma$. Suppose that we have, up to isomorphism of $K$-vector spaces, a $K$-vector space decomposition*

$$A \cong \bigoplus_{i \in I} V_i \otimes_K B_i \otimes_K V_i$$

*of $A$, where $I$ is a finite partially ordered set, each $V_i$ is a $K$-vector space, and each $B_i$ is a cellular algebra over $K$ with respect to an anti-involution $\sigma_i$ and cellular data $(\Lambda_i, M_i, C, \sigma_i)$. We shall henceforth consider $A$ to be identified with this direct sum of tensor products.*

*Suppose that for each $i \in I$, we have basis $\mathcal{V}_i$ for $V_i$ and a basis $\mathcal{B}_i$ for $B_i$ such that:*

1. *For each $i \in I$, we have for any $u, v \in \mathcal{V}_i$ and any $b \in \mathcal{B}_i$ that*

$$\sigma(u \otimes b \otimes v) = v \otimes \sigma_i(b) \otimes u.$$

2. *Let $\mathcal{A}$ be the basis of $A$ consisting of all elements $u \otimes b \otimes v$ for all $u, v \in \mathcal{V}_i$ and all $b \in \mathcal{B}_i$ as $i$ ranges over $I$. Then for any $i \in I$ we have maps $\phi_i : \mathcal{A} \times \mathcal{V}_i \to V_i$ and $\theta_i : \mathcal{A} \times \mathcal{V}_i \to B_i$ such that for any $u, v \in \mathcal{V}_i$ and any $b \in \mathcal{B}_i$, we have for any $a \in \mathcal{A}$ that*

$$a.(u \otimes b \otimes v) \equiv \phi_i(a, u) \otimes \theta_i(a, u) b \otimes v \pmod{J(< i)}$$

*where $J(< i) = \bigoplus_{l<i} V_l \otimes B_l \otimes V_l$.*

*Then $A$ is cellular with respect to $\sigma$ and the cellular data $(\Lambda, M, C, \sigma)$, where*

- *$\Lambda$ is the set $\{(i, \lambda) : i \in I \text{ and} \lambda \in \Lambda_i\}$ with the partial order defined by setting*

$$(i, \lambda) < (j, \mu) \text{ if } i < j \qquad \text{and} \qquad (i, \lambda) < (i, \mu) \text{ if } \lambda < \mu$$

  *(that is, lexicographic order);*

- *for $(i, \lambda) \in \Lambda$, $M(i, \lambda)$ is $\mathcal{V}_i \times M_i(\lambda)$;*

- *for $(i, \lambda) \in \Lambda$ and $(x, X), (y, Y) \in M(i, \lambda)$, let*

$$C^{(i,\lambda)}_{(x,X),(y,Y)} = x \otimes C^{\lambda}_{X,Y} \otimes y.$$

Green and Paget mention that we may use any bases of the cellular algebras $B_i$ to check the conditions of Theorem 4.1.2: we need not use the cellular bases of the $B_i$.

**Proposition 4.1.3.** *[7, Proposition 2] Let $A$ be an algebra satisfying the hypotheses of Theorem 4.1.2. Then the multiplication in each layer of $A$ is governed by a bilinear form as in Xi's lemma: for each $i \in I$ there is a unique $B_i$-valued $K$-bilinear form $\psi_i$ on $V_i$ such that for any $u, v, x, y \in V_i$ and $b, c \in B_i$, we have $\psi_i(y, u) = \sigma_i(\psi_i(u, y))$ and*

$$(x \otimes c \otimes y)(u \otimes b \otimes v) \equiv x \otimes c\psi_i(y, u) b \otimes v \pmod{J(< i)}.$$

**Proposition 4.1.4.** *[7, Proposition 3] Let $A$ be as in Theorem 4.1.2, let $(i, \lambda) \in \Lambda$, and let $\Delta^{\lambda}$ be the cell module of $B_i$ corresponding to $\lambda$. The cell module $\Delta^{(i,\lambda)}$ of $A$*

*may be obtained by equipping* $V_i \otimes \Delta^\lambda$ *with the action given, for* $a \in \mathcal{A}$, $x \in \mathcal{V}_i$ *and* $z \in \Delta^\lambda$, *by*

$$a(x \otimes z) = \phi_i(a, x) \otimes \theta_i(a, x)z.$$

## 4.2 The group algebra $K\widetilde{S_n}$

Let $\widetilde{S_n}$ be the set of decorated partial Brauer diagrams which only have propagating lines and do not have any isolated vertices. Then $\widetilde{S_n} \subset DPB(n)$ with multiplication induced from the multiplication in the decorated partial Brauer algebra and has the same identity element as the decorated partial Brauer which is the undecorated partial Brauer diagram with $n$ propagating lines. In the following we will show that $\widetilde{S_n}$ forms a group.

**Proposition 4.2.1.** *The set* $\widetilde{S_n}$ *is closed under the multiplication induced by* $D\mathcal{P}\mathfrak{B}_n(\delta, \delta_\circ, \delta', \mu, \mu')$, *and forms a group.*

*Proof.* Let $d_1, d_2 \in \widetilde{S_n}$. Firstly we want to show that $d_1 d_2 \in \widetilde{S_n}$.
Since $d_1$ and $d_2$ have no arcs nor isolated vertices in the top row nor the bottom row, there are no chains formed in the middle row of $d_1 d_2$, therefore it is not possible to have any arc or isolated vertex in the top row nor the bottom row of $d_1 d_2$. Furthermore there are no isolated component that can be produced in this product. So the resulting diagram $d_1 d_2$ has only propagating lines.

Since the identity element of $DPB(n)$ is an undecorated partial Brauer diagram with $n$ propagating lines then $\mathrm{id}_{DPB(n)} \in \widetilde{S_n}$.

Now let $d \in \widetilde{S_n}$ and $\hat{d}$ be the diagram obtained from $d$ by reflecting $d$ around its central horizontal axis, so $\hat{d} \in \widetilde{S_n}$ (since the reflecting does not change the number of propagating lines). Let $e$ be a decorated (resp. undecorated) propagating line joining $i$ to $j'$ in $d$, then it corresponds to a decorated (resp. undecorated) propagating line $e'$ which joins $j$ to $i'$ in $\hat{d}$. By concatenating $d, \hat{d}$ (resp. $\hat{d}, d$) we will get an undecorated propagating line $ee'$ (resp. $e'e$) joining $i$ to $i'$ in $d\hat{d}$ (resp. joining $j$ to $j'$ in $\hat{d}d$),

$i, j \in \{1, \ldots, n\}$, meaning that $d\hat{d} = \hat{d}d = \mathrm{id}_{\widetilde{S_n}}$. Thus $\hat{d}$ is the inverse element of $d$. Therefore $\widetilde{S_n}$ is a group. □

As a consequence of previous proposition, we have the following.

**Corollary 4.2.2.** *The group algebra* $K\widetilde{S_n}$ *is a subalgebra of* $D\mathcal{P}\mathfrak{B}_n(\delta, \delta_\circ, \delta', \mu, \mu')$*.*

Next we will show that $\widetilde{S_n}$ is isomorphic to the wreath product group $\mathbb{Z}_2 \wr S_n$.

**Proposition 4.2.3.** *There is an isomorphism between* $\widetilde{S_n}$ *and* $\mathbb{Z}_2 \wr S_n$*, where* $S_n$ *is the symmetric group and* $\mathbb{Z}_2 = \{0, 1\}$*.*

*Proof.* Let $\phi : \widetilde{S_n} \to \mathbb{Z}_2 \wr S_n$ be a map defined by $\phi(d) = (f, \pi),\ d \in \widetilde{S_n}$, where $\pi$ is an undecorated (underlying) permutation of $d$, $f \in \mathbb{Z}_2^n$ and $i^{th}$ entries of $f$ are
$$f(i) = \begin{cases} 0 & \text{if} \quad (i, i\pi) \quad \text{is undecorated,} \\ 1 & \text{if} \quad (i, i\pi) \quad \text{is decorated.} \end{cases}$$
Now let $\phi(d_1) = (f_1, \pi_1),\ \ \phi(d_2) = (f_2, \pi_2)$, we want to show that
$\phi(d_1 d_2) = \phi(d_1)\phi(d_2)$.
From Definition 2.3.1, we have $\phi(d_1)\phi(d_2) = (f_1, \pi_1).(f_2, \pi_2) = (f_1 + {}_{\pi_1}f_2, \pi_1\pi_2)$.
Let $\phi(d_1 d_2) = (g, \pi')$, where $\pi'$ is the product of the underlying permutations of $d_1$ and $d_2$. So $\pi' = \pi_1.\pi_2$, it remains to prove that $g = f_1 + {}_{\pi_1}f_2$. We have

$$g(i) = \begin{cases} 0 & \text{if} \quad (i, i\pi') \quad \text{is undecorated,} \\ 1 & \text{if} \quad (i, i\pi') \quad \text{is decorated.} \end{cases}$$

$$f_1(i) = \begin{cases} 0 & \text{if} \quad (i, i\pi_1) \quad \text{is undecorated,} \\ 1 & \text{if} \quad (i, i\pi_1) \quad \text{is decorated.} \end{cases}$$

and

$${}_{\pi_1}f_2 = f_2(i\pi_1) = \begin{cases} 0 & \text{if} \quad (i\pi_1, i\pi_1\pi_2) \quad \text{is undecorated,} \\ 1 & \text{if} \quad (i\pi_1, i\pi_1\pi_2) \quad \text{is decorated.} \end{cases}$$

Observe that if the propagating lines $(i, i\pi_1)$ and $(i\pi_1, i\pi_1\pi_2)$ are both decorated or undecorated so the propagating line $(i, i\pi_1\pi_2)$ will be undecorated, meaning that $f_1(i) + f_2(i\pi_1) = 0 = g(i)$ if $f_1(i) = f_2(i\pi_1)$.

If one of the propagating lines $(i, i\pi_1)$ and $(i\pi_1, i\pi_1\pi_2)$ is decorated and the other undecorated therefore the propagating line $(i, i\pi_1\pi_2)$ will be decorated, that means if $f_1(i) \neq f_2(i\pi_1)$ so $f_1(i) + f_2(i\pi_1) = 1 = g(i)$. Then

$$g(i) = (f_1 +_{\pi_1} f_2)(i) = f_1(i) + f_2(i\pi_1).$$

Therefore $\phi$ is homomorphism.

Now we want to show that $\phi$ is bijective.
Note that

$$\text{Ker } \phi = \{d \in \widetilde{S_n} \mid \phi(d) = \text{id}_{\mathbb{Z}_2 \wr S_n}\} = \{d \in \widetilde{S_n} \mid (f, \pi) = (\underline{0}, \text{id}_{S_n})\}.$$

Clearly $\phi(\text{id}_{\widetilde{S_n}}) = (\underline{0}, \text{id}_{S_n})$. If $\phi(d) = (\underline{0}, \text{id}_{S_n})$, $d \in \widetilde{S_n}$, then the underlying diagram of $d$ is the identity diagram of $S_n$ and the $\underline{0} = (0, 0, \ldots, 0)$ tells us that none of the lines are decorated. Therefore $d = \text{id}_{\widetilde{S_n}}$. Then Ker $\phi = \text{id}_{\widetilde{S_n}}$ and hence $\phi$ is injective.

Note that the set $\widetilde{S_n}$ is the set of symmetric group diagrams such that each propagating line can be decorated so we have:

$$|\widetilde{S_n}| = 2^n.|S_n| = 2^n.n! = |\mathbb{Z}_2 \wr S_n|.$$

Therefore $\phi$ is bijective and hence $\phi$ is an isomorphism. □

## 4.3 The $K$-vector space $V_l$

**Definition 4.3.1.** A *decorated partial Brauer half diagram* is a diagram with one row of $n$ vertices labelled $1, \ldots, n$ consisting of $k$ decorated or undecorated arcs, $l$ non-crossing undecorated propagating lines starting from points on this row towards points of infinity and the remaining $n - (2k + l)$ points are decorated or undecorated isolated vertices (vertices which are not connected to any edge), where $k \in \{0, \ldots, \lfloor \frac{n}{2} \rfloor\}$, $l \in \{0, \ldots, n\}$.

Let $h_l(DPB(n))$ denote the set of decorated partial Brauer lower half diagrams with $l$ non-crossing undecorated propagating lines and $V_l$ denote the $K$-vector space whose basis is $h_l(DPB(n))$.

**Lemma 4.3.2.**

$$|h_l(DPB(n))| = \sum_{k=0}^{\lfloor \frac{n-l}{2} \rfloor} \binom{n-2k}{l} \frac{n!}{(n-2k)!k!} 2^{n-(l+2k)}.$$

*Proof.* To draw $k$ arcs in $n$ vertices, firstly choose $2k$ vertices from $n$ to be the endpoints of the $k$ arcs. This gives $\binom{n}{2k}$ ways for a fixed $k$. Then choose two vertices of $2k$ to be an arc. I.e. to draw an arc, pick a vertex from $2k$ and join it with a randomly chosen vertex. For each choice we get two vertices less to choose from. So there are $(2k-1)$ choices for the first arc, $(2k-3)$ choices for the second and so on. Therefore the number of possibilities for drawing $k$ arcs between $n$ vertices is

$$\binom{n}{2k}(2k-1)(2k-3)\cdots 3.1 = \binom{n}{2k}(2k-1)!! = \frac{n!}{(2k)!(n-2k)!}\frac{(2k)!}{2^k k!} = \frac{n!}{2^k(n-2k)!k!}.$$

Now choose $l$ vertices from $n-2k$ to be propagating lines therefore, for fixed $l$, there are $\binom{n-2k}{l}\frac{n!}{2^k(n-2k)!k!}$ partial Brauer half diagrams with $k$ arcs and $l$ non-crossing propagating lines. The remaining $n-2k-l$ vertices represent isolated vertices. Since in the decorated partial Brauer half diagrams each arc and each isolated vertex can be decorated but not the propagating lines then there are

$$\binom{n-2k}{l}\frac{n!}{2^k(2k-1)!k!}2^k 2^{n-2k-l} = \binom{n-2k}{l}\frac{n!}{(2k-1)!k!}2^{n-2k-l}$$

decorated partial Brauer half diagrams with $l$ propagating lines. Take the sum over $k \in \{0, \ldots, \lfloor \frac{n-l}{2} \rfloor\}$ to get the all decorated partial Brauer half diagrams with fixed $l$ non-crossing undecorated propagating lines. □

## 4.4 An inflation of $K\widetilde{S}_l$ along $V_l$

We first recall the definition of an inflation.

**Definition 4.4.1.** [14, Definition 3.1]. Given a $K$-algebra $B$, a $K$-vector space $V$, and a bilinear form $\varphi : V \otimes V \to B$ with values in $B$, we define an associative algebra (possibly without unit) $A = A(B, V, \varphi)$ as follows: as a $K$-vector space, $A$ equals $V \otimes V \otimes B$. The multiplication is defined on basis elements as follows:

$$(a \otimes b \otimes x).(c \otimes d \otimes y) := a \otimes d \otimes x\varphi(b, c)y.$$

We need an additional property, namely an involution on $A$: assume there is an involution $\sigma$ on $B$. Assume moreover, that $\varphi$ satisfies $\sigma(\varphi(v, w)) = \varphi(w, v)$. Then we can define an involution $i$ on $A$ by putting $i(a \otimes b \otimes x) = b \otimes a \otimes \sigma(x)$.
This definition makes $A$ into an associative $K$-algebra (possibly without unit), and $i$ is an involutory anti-automorphism of $A$. We call $A$ an *inflation* of $B$ along $V$.

If $V$ has dimension one and the image of $\varphi$ contains the unit element of $B$, then $A$ clearly isomorphic to $B$. Otherwise, $A$ need not have a unit element, but it may contain idempotents.

The involution on $D\mathcal{P}\mathfrak{B}_n(\delta, \delta_\circ, \delta', \mu, \mu')$ is described in the following lemma.

**Lemma 4.4.2.** *The map* $i : D\mathcal{P}\mathfrak{B}_n(\delta, \delta', \delta_\circ, \mu, \mu') \to D\mathcal{P}\mathfrak{B}_n(\delta, \delta', \delta_\circ, \mu, \mu')$ *which sends the diagram* $d$ *to the diagram* $i(d)$ *which is the reflection of the diagram* $d$ *upside down, extended linearly to the whole algebra is an anti-involution.*

*Proof.* Clearly $i^2$ =id (see for example figure 1). It remains to show that $i(d_1d_2) = i(d_2)i(d_1)$ for all $d_1, d_2 \in DPB(n)$. However, this follows immediately from the way the product is defined in $D\mathcal{P}\mathfrak{B}_n(\delta, \delta', \delta_\circ, \mu, \mu')$ (see for example figure 2). □

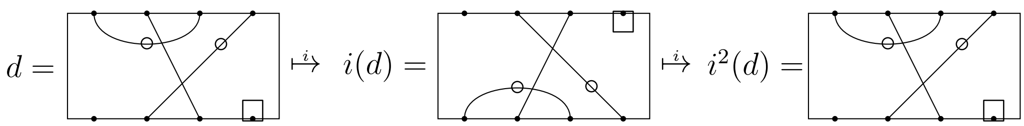



FIGURE 4.1: The involution map i.

**Definition 4.4.3.** For $d$ a decorated partial Brauer diagram let $\#(d)$ denote the number of propagating lines (decorated or undecorated).

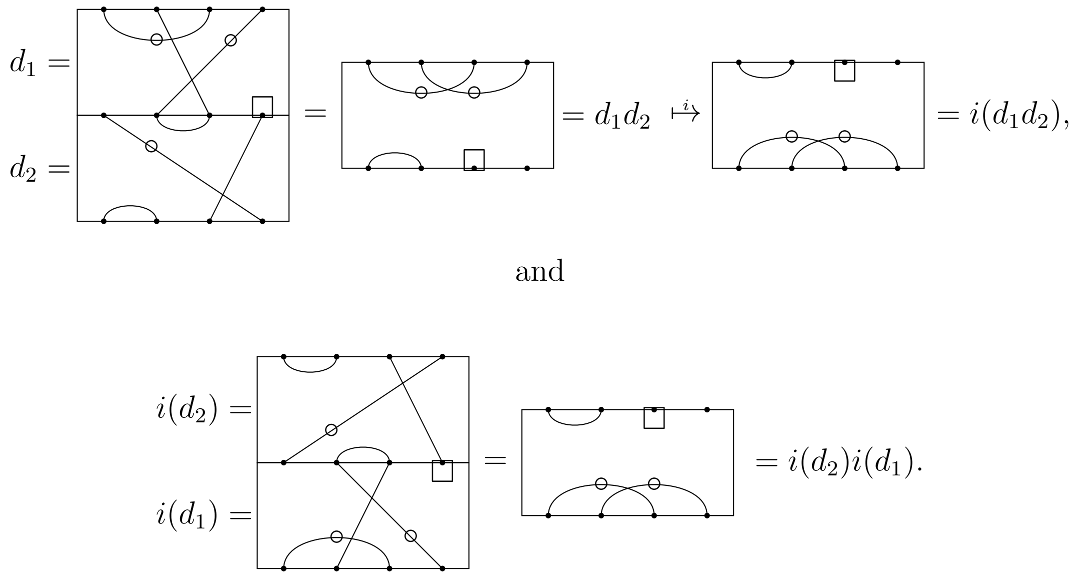



FIGURE 4.2: The map i is an anti-involution.

Note that the multiplication of decorated partial Brauer diagrams cannot increase the number of propagating lines, so we have the following fact:

**Lemma 4.4.4.** *For $d_1, d_2 \in DPB(n)$*

$$\#(d_1 d_2) \leq min\{\#(d_1), \#(d_2)\}.$$

Now let $J_l$ be a $K$-vector space spanned by all decorated partial Brauer diagrams with at most $l$ propagating lines, $l \in \{0, 1, \ldots, n\}$.

**Lemma 4.4.5.** *$J_l$ is a two-sided ideal in $D\mathcal{P}\mathfrak{B}_n(\delta, \delta', \delta_\circ, \mu, \mu')$.*

*Proof.* Let $d \in DPB(n)$ and $d' \in J_l$, we have $\#(d') \leq l$ so $\#(dd'), \#(d'd) \leq \min\{\#(d), \#(d')\} \leq l$. Therefore $dd'$ and $d'd \in J_l$ so $J_l$ is a two-sided ideal in $D\mathcal{P}\mathfrak{B}_n(\delta, \delta', \delta_\circ, \mu, \mu')$. □

Therefore we have a filtration of the decorated partial Brauer algebra by the two-sided ideals $J_l$:

$$0 \subset J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_{n-2} \subset J_{n-1} \subset J_n = D\mathcal{P}\mathfrak{B}_n(\delta, \delta', \delta_\circ, \mu, \mu')$$

and each quotient $J_l \diagup J_{l-1}$ is spanned by the decorated partial Brauer diagrams with $l$ propagating lines.

In the following lemmas we will show that the quotient $J_l \diagup J_{l-1}$ is isomorphic to the algebra $i(V_l) \otimes V_l \otimes K\widetilde{S}_l$ i.e. $J_l \diagup J_{l-1}$ is an inflation of $K\widetilde{S}_l$ along $V_l$.

**Lemma 4.4.6.** *For a fixed $l$, let $B_l$ denote the $K$-algebra $B_l = J_l \diagup J_{l-1}$. There is a bijective $K$-vector space homomorphism between $B_l$ and $i(V_l) \otimes V_l \otimes K\widetilde{S}_l$.*

*Proof.* Note that $B_l$ has basis of decorated partial Brauer diagrams with $l$ propagating lines and $i(V_l) \otimes V_l \otimes K\widetilde{S}_l$ has basis the set $\{i(x) \otimes y \otimes \pi \mid x, y \in h_l(DPB(n)),\ \pi \in \widetilde{S}_l\}$. Define a map $\psi : B_l \to i(V_l) \otimes V_l \otimes K\widetilde{S}_l$ as follows:
Let $d$ be a diagram in $B_l$ so $d$ consists of an element $i(x)$ (top half of $d$) where $x \in h_l(DPB(n))$, an element $y \in h_l(DPB(n))$ (bottom half of $d$) and an element $\pi_d = (f, \sigma_d) \in \widetilde{S}_l$, where $f \in \mathbb{Z}_2^n$ and $\sigma_d \in S_l$, ($S_l$ is symmetric group), which is defined as follows: renumber the top endpoints of the propagating lines in $d$ from left to right as $1, \ldots, l$ and their bottom endpoints as $1', \ldots, l'$ from left to right. Then put $\sigma_d(i) = j'$ if there is a propagating line in $d$ that joins $i$ with $j'$, meaning that $\{i, j'\} \in d$. The $i^{th}$ entry of $f$ is 0 if this propagating line is undecorated and 1 if it is decorated. This determines an element $\pi_d = (f, \sigma_d) \in \widetilde{S}_l$. Since $i(x), y$ and $\pi_d$ are uniquely determined by $d$ so $d$ gives a unique element $i(x) \otimes y \otimes \pi_d \in i(V_l) \otimes V_l \otimes K\widetilde{S}_l$. We therefore have a well-defined map $\psi$, which send a basis element of $B_l$ to basis element of $i(V_l) \otimes V_l \otimes K\widetilde{S}_l$, $\psi(d) = i(x) \otimes y \otimes \pi_d$ extended linearly to the whole algebra $B_l$.

Now we check that $\psi$ is a bijection.

Let $d = \Sigma_j \lambda_j d_j \in B_l$ where $d_j$ is a diagram (basis element) in $B_l$, $\lambda_j \in K$ such that $\psi(d) = 0$. So

$$\begin{aligned} 0 = \psi(d) &= \psi(\Sigma_j \lambda_j d_j) \\ &= \Sigma_j \lambda_j \psi(d_j) \qquad\qquad \text{(as } \psi \text{ is linear)} \\ &= \Sigma_j \lambda_j (i(x_j) \otimes y_j \otimes \pi_j). \end{aligned}$$

But the set $\{i(x) \otimes y \otimes \pi \mid x, y \in h_l(DPB(n)),\ \pi \in \widetilde{S}_l\}$ is a basis of $i(V_l) \otimes V_l \otimes K\widetilde{S}_l$, meaning that it is linearly independent. Therefore $\lambda_j = 0$ for all $j$. So $d = 0$ and then Ker $\psi = 0$. Hence $\psi$ is one-to-one.

We now show that $\psi$ is onto.
Let $u, v \in h_l(DPB(n))$ so each of them has $l$ non-crossing undecorated propagating lines. We produce a diagram $d$ by taking $i(u)$, $v$ and joining up the propagating lines to reproduce permutation $\sigma$ and decorate propagating lines according to the value of $f$. I.e. the diagram $d$ when we ignore all non-propagating lines and decorations produces the permutation $\sigma$ and the $j^{th}$ propagating line has a decoration if and only if $f(j) = 1$. It is clear that $\psi(d) = i(u) \otimes v \otimes (f, \sigma)$ so $\psi$ is onto.

Then the $K$-linear extension of the map $\psi$ to all of $B_l$ is a $K$-vector space isomorphism. □

In order to give a multiplication structure on $i(V_l) \otimes V_l \otimes K\widetilde{S}_l$, we need to define **the $K$-bilinear form $\varphi_l$**.

**Definition 4.4.7.** We will construct the bilinear form

$$\varphi_l : V_l \otimes i(V_l) \to K\widetilde{S}_l$$

(where $K\widetilde{S}_0$ is interpreted to be $K$) as follows:
Let $x \in h_l(DPB(n))$, $y \in i\big(h_l(DPB(n))\big)$ be half diagrams on $n$ vertices (labelled $1, \ldots, n$). Construct our $xy$ by identifying the labelled vertices of $x$ with the labelled vertices of $y$ so we will get a graph ($\Gamma$ say) which consists of:

**(i)** Decorated or undecorated isolated vertices which are not connected to any edge.

**(ii)** Decorated or undecorated paths where a path is a sequence of connected (decorated or undecorated) edges $a_1, a_2, \ldots, a_m$, these edges may be arcs or propagating lines.

There are five types of paths which may be formed in $\Gamma$:

1. A decorated or undecorated path which form a (decorated or undecorated) closed loop, (note that this path has no propagating lines).

2. A decorated or undecorated path which has no propagating lines (i.e. $a_1$ and $a_m$ are both arcs) and does not form a closed loop.

3. A decorated or undecorated path which has one propagating line (i.e. $a_1$ or $a_m$ is a propagating line.

4. A decorated or undecorated path that begins and ends with propagating lines both of them are in $x$ or in $y$ (i.e. $a_1, a_m \in x$ (or $a_1, a_m \in y$) are propagating lines).

5. A decorated or undecorated path which begins and ends with propagating lines one of them in $x$ and the other in $y$ (i.e. $a_1$ and $a_m$ are propagating lines, $a_1 \in x$, $a_m \in y$ (or $a_1 \in y, a_m \in x$)).

Now we will define $\varphi_l$ as follows:

**(I)** If the graph $\Gamma$ has any paths of type 3 or 4 then $\varphi_l(x, y) = 0$. (Note that in this case the number of propagating lines in $xy$ is less than $l$.)

**(II)** If the graph $\Gamma$ has only isolated vertices and paths of type 1 or 2 or 5, then

$$\varphi_l(x, y) = \delta^e \ (\delta')^o \ \delta_\circ^p \ \mu^q (\mu')^r \ \pi$$

where the scalars $\delta, \delta', \delta_\circ, \mu, \mu'$,$e, o, p, q, r$ are as defined in the multiplication of decorated partial Brauer diagrams, $\pi = (f, \sigma) \in \widetilde{S}_l$ is defined as follows:
Since we have basis elements $x$ and $y$, we can form the elements $z_1 := i(x) \otimes x \otimes \mathrm{id}$ and $z_2 := y \otimes i(y) \otimes \mathrm{id}$ in $i(V_l) \otimes V_l \otimes K\widetilde{S}_l$. The product $z_1 z_2$ is of the form $i(x) \otimes i(y) \otimes \varphi_l(x, y)$ and in $B_l$

$$\psi^{-1}(z_1)\psi^{-1}(z_2) = \delta^e \ (\delta')^o \ \delta_\circ^p \ \mu^q \ (\mu')^r \ d'$$

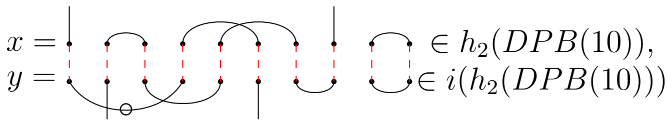
where $d' \in B_l$ consists of the top of $\psi^{-1}(z_1)$ (i.e. $i(x)$), the bottom of $\psi^{-1}(z_2)$ (i.e. $i(y)$) (note that as we only have paths of type 1,2 or 5 so $\#(\psi^{-1}(z_1)\psi^{-1}(z_2)) =$

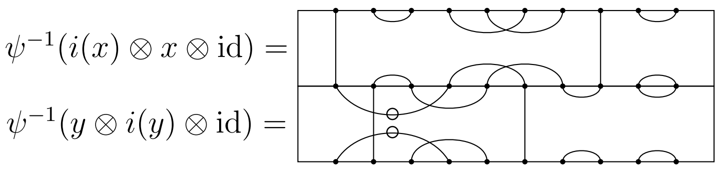
$\#d' = l$, consequently the top of $\psi^{-1}(z_1)$ and the bottom of $\psi^{1-}(z_2)$ won't be changed) and a permutation $\pi = (f, \sigma) \in \widetilde{S}_l$, where $\sigma = \begin{pmatrix} 1 & \cdots & l \\ 1\sigma & \cdots & l\sigma \end{pmatrix}$, $1 < \cdots < i < \cdots < l$ are the endpoints of propagating lines of the bottom of $\psi^{-1}(z_1)$ (i.e. $x$), $i\sigma' s$ are the endpoints of the propagating lines of the top of $\psi^{-1}(z_2)$ (i.e. $y$) and $f \in \mathbb{Z}_2^n$ has value 1 or 0 according to whether $(i, i\sigma)$ is decorated or not.

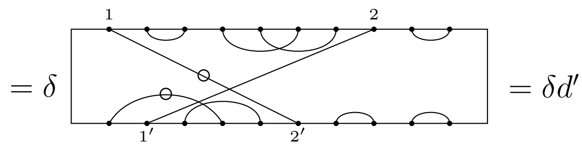
Then extend $\varphi_l$ linearly to the whole vector space $V_l \otimes i(V_l)$.

The following example illustrates the computing of $\varphi_l(x, y)$ in case $II$.

**Example 4.4.8.** Let




so $\pi = ((1, 0), (12))$ and then $\varphi_l(x, y) = \delta\pi$

**Lemma 4.4.9.** *For a fixed $l$, let $B_l$ denote the $K$-algebra $B_l = J_l / J_{l-1}$, then $B_l$ is isomorphic (as a $K$-algebra) to an inflation $i(V_l) \otimes V_l \otimes K\widetilde{S}_l$, where the multiplication in $i(V_l) \otimes V_l \otimes K\widetilde{S}_l$ is given by*

$$(a \otimes b \otimes x)(c \otimes d \otimes y) = a \otimes d \otimes x\varphi_l(b, c)y$$

*for $a, c \in i\big(h_l(DPB(n))\big)$, $b, d \in h_l(DPB(n))$ and $x, y \in \widetilde{S}_l$, which is the set of decorated partial Brauer diagrams having only $l$ propagating lines.*

*Proof.* From Lemma 4.4.6 we have seen that the map $\psi : B_l \to i(V_l) \otimes V_l \otimes K\widetilde{S}_l$ is a $K$-vector space isomorphism. To show that $\psi$ is a $K$-algebra isomorphism, it remains to show that

$$\psi(d_1 d_2) = \psi(d_1)\psi(d_2).$$

Let $d_1, d_2 \in B_l$, $\psi(d_1) = a\otimes b\otimes \pi_1$ and $\psi(d_2) = c\otimes d\otimes \pi_2$, where $a, c \in i\big(h_l(DPB(n))\big)$, $b, d \in h_l(DPB(n))$, $\pi_1 = (f_1, \sigma_1)$, $\pi_2 = (f_2, \sigma_2) \in \widetilde{S}_l$. So

$$\psi(d_1)\psi(d_2) = a \otimes d \otimes \pi_1 \varphi_l(b, c)\pi_2. \qquad (*)$$

We have the following cases:

Case 1: If $\#d_1d_2 < l$ meaning that $d_1d_2 \in J_{l-1}$ so $d_1d_2 = 0$ in $B_l$ and then $\psi(d_1d_2) = 0$. Also $\#d_1d_2 < l$ means there is a path of type 3 or 4 formed in the middle row of $d_1d_2$ so in this case we have $\varphi_l(b, c) = 0$. Therefore $\psi(d_1d_2) = 0 = \psi(d_1)\psi(d_2)$.

Case 2: If $\#d_1d_2 = l$, meaning that in the product $d_1d_2$ each vertex $i$, where $1 \le i \le l$ are the endpoints of propagating lines on the top half diagram of $d_1$ (i.e. in $a$), joins to $z\sigma_2$, $1 \le z\sigma_2 \le l$ are the endpoints of propagating lines on the bottom half diagram of $d_2$ (i.e. in $d$). Then, from the multiplication method of decorated partial Brauer diagrams, we have

$$d_1 d_2 = \delta^e\ (\delta')^o\ \delta_\circ^p\ \mu^q\ (\mu')^r\ d_3 \qquad (**)$$

where the scalars $\delta, \delta', \delta_\circ, \mu, \mu'$, $e, o, p, q, r$ are as defined in the multiplication of decorated partial Brauer diagrams which are formed from the paths in the middle row of $d_1d_2$, and $d_3 \in B_l$ consists of the top of $d_1$ (i.e. $a$), the bottom of $d_2$ (i.e. $d$) and $\pi = (g, \gamma) \in \widetilde{S}_l$, where $\gamma = \begin{pmatrix} 1 & \cdots & l \\ 1\gamma & \cdots & l\gamma \end{pmatrix} \in S_l$, $g \in \mathbb{Z}_2^n$ has value 1 or 0 according to $(i, i\gamma)$ is decorated or not, so

$$\psi(d_3) = a \otimes d \otimes \pi.$$

Note that since $i$ joins to $z\sigma_2$ in $d_1d_2$ that means $i\gamma = z\sigma_2$.

Also in this case since $i$ joins to $i\sigma_1$ in $d_1$ and $z\sigma_2$ joins to $z$ in $d_2$ so $i\sigma_1$ joins to $z$

in the middle row of $d_1d_2$ (i.e. in the graph $\Gamma$). As all propagating lines match up in the graph $\Gamma$, so no paths of type 3 or 4.

Put $\varphi_l(b,c) = \delta^e(\delta')^o\delta_\circ^p\mu^q(\mu')^r\ \pi'$.

Note that since $\varphi_l(b,c)$ is formed from the bottom row of $d_1$ and the top row of $d_2$ so the paths formed in it are the same paths that are formed in the middle row of $d_1d_2$ therefore $\varphi_l(b,c)$ has the same scalars as in $(**)$. Now we want to show that $\pi = \pi_1\pi'\pi_2$, where $\pi' = (f',\sigma') \in \widetilde{S}_l$, $\sigma' = \begin{pmatrix} 1\sigma_1 & \cdots & l\sigma_l \\ 1\sigma_1\sigma' & \cdots & l\sigma_1\sigma' \end{pmatrix} \in S_l$ and $f' \in \mathbb{Z}_2^n$ has value 1 or 0 according to whether $(i\sigma_1, i\sigma_1\sigma')$ is decorated or not. From $(*)$ we have

$$\psi(d_1)\psi(d_2) = \delta^e(\delta')^o\delta_\circ^p\mu^q(\mu')^r a \otimes d \otimes \pi_1\pi'\pi_2,$$

where

$$\begin{aligned}\pi_1\pi'\pi_2 &= (f_1,\sigma_1)(f',\sigma')(f_2,\sigma_2)\\ &= (f_1 + {}_{\sigma_1}f', \sigma_1\sigma')(f_2,\sigma_2)\\ &= (f_1 + {}_{\sigma_1}f' + {}_{\sigma_1\sigma'}f_2, \sigma_1\sigma'\sigma_2),\end{aligned}$$

and

$$(f_1 + {}_{\sigma_1}f' + {}_{\sigma_1\sigma'}f_2)(i) = f_1(i) + f'(i\sigma_1) + f_2(i\sigma_1\sigma'), \qquad 1 \le i \le l.$$

Note that vertex $i$ joins to $i\sigma_1$ in $d_1$ and $i\sigma_1$ joins to $z$ (where $1 \le z \le l$ are the endpoint of propagating lines in $c$) in the middle row of $d_1d_2$ but (by $\sigma'$) $i\sigma_1$ joins to $i\sigma_1\sigma'$, this implies that $z = i\sigma_1\sigma'$. Also, since in $d_2$ the vertex $z$ joins to $z\sigma_2 = i\sigma_1\sigma'\sigma_2$ implying that $i$ joins to $i\sigma_1\sigma'\sigma_2$ in $d_1d_2$, but (by $\gamma$) $i$ joins to $i\gamma$ therefore $\gamma = \sigma_1\sigma'\sigma_2$. Also we have the following:

- If the three lines $(i, i\sigma_1)$, $(i\sigma_1, i\sigma_1\sigma')$ and $(i\sigma_1\sigma', i\sigma_1\sigma'\sigma_2)$ are all undecorated or two of them are decorated and the third is undecorated then the line $(i, i\sigma_1\sigma'\sigma_2) = (i, i\gamma)$ will be undecorated so $f_1(i) + f'(i\sigma_1) + f_2(i\sigma_1\sigma') = 0 = g(i)$.

- If the three lines $(i, i\sigma_1)$, $(i\sigma_1, i\sigma_1\sigma')$ and $(i\sigma_1\sigma', i\sigma_1\sigma'\sigma_2)$ are all decorated or two of them are undecorated and the third is decorated then the line $(i, i\sigma_1\sigma'\sigma_2) =$

$(i, i\gamma)$ will be decorated so $f_1(i) + f'(i\sigma_1) + f_2(i\sigma_1\sigma') = 1 = g(i)$.

Then

$$f_1 + {}_{\sigma_1}f' + {}_{\sigma_1\sigma'}f_2 = g.$$

Therefore $\pi_1\pi'\pi_2 = \pi$. Hence in this case we also have $\psi(d_1d_2) = \psi(d_1)\psi(d_2)$. □

Let $DPB^l(n)$ denote the set of decorated partial Brauer diagrams with $l$ propagating lines. We have the following.

**Proposition 4.4.10.**

$$|DPB^l(n)| = \left( \sum_{k=0}^{\lfloor \frac{n-l}{2} \rfloor} \binom{n-2k}{l} \frac{n!}{(n-2k)!k!} 2^{n-(l+2k)} \right)^2 2^l l!.$$

*Proof.* Since (from Lemma 4.4.6) $B_l$ isomorphic to $i(V_l) \otimes V_l \otimes K\widetilde{S}_l$, then

$$\begin{aligned} |DPB^l(n)| &= |i\big(h_l(DPB(n))\big)|.|h_l(DPB(n))|.|\widetilde{S}_l| \\ &= |h_l(DPB(n))|^2.|\widetilde{S}_l| \qquad (\text{as } |i\big(h_l(DPB(n))\big)| = |h_l(DPB(n))|) \\ &= \left( \sum_{k=0}^{\lfloor \frac{n-l}{2} \rfloor} \binom{n-2k}{l} \frac{n!}{(n-2k)!k!} 2^{n-(l+2k)} \right)^2 .2^l l! \qquad (\text{using Lemma 4.3.2}). \end{aligned}$$

□

Note that the bilinear form $\varphi_l$ which defined in Definition 4.4.7 is not symmetric (i.e. $\varphi_l(x, y) \neq \varphi_l(y, x)$), however we have the following:

**Lemma 4.4.11.** *Let* $^{-}\colon K\widetilde{S}_l \to K\widetilde{S}_l$ *be the* $K$*-linear involution on* $K\widetilde{S}_l$ *defined via* $\bar{\pi} = \pi^{-1}$ *for all* $\pi \in \widetilde{S}_l$*. Then* $\overline{\varphi_l\big(x, y\big)} = \varphi_l\big(i(y), i(x)\big)$ *for all* $x \in h_l(DPB(n))$*,* $y \in i\big(h_l(DPB(n))\big)$*.*

*Proof.* From Definition 4.4.7 of $\varphi_l$, by reflecting the graph $\Gamma$ which is $xy$ upside down we will get the graph $i(\Gamma)$ which will be $i(xy) = i(y)i(x)$. Therefore the graph $i(\Gamma)$ has the same types of paths as in $\Gamma$ but replace $x, y$ by $i(y), i(x)$. So $\varphi_l(x, y)$ and $\varphi_l(i(y), i(x))$ have the same types of scalars. Also, if $\varphi_l(x, y) = 0$ then

$\varphi_l\big(i(y), i(x)\big) = 0.$

Now let $\varphi_l(x, y) \neq 0$ meaning that $\varphi_l(x, y) = \delta^e(\delta')^o\delta_\circ^p\mu^q(\mu')^r\ \pi$ where $\pi = (f, \sigma) \in \widetilde{S}_l$ and $\sigma = \begin{pmatrix} 1 & \cdots & l \\ 1\sigma & \cdots & l\sigma \end{pmatrix}$ takes vertices in $x$ (which are the endpoints of propagating lines in $x$) to vertices in $y$ (which are the endpoints of propagating lines in $y$).

Since turning diagrams upside down does not change scalars we have

$$\overline{\varphi_l(x, y)} = \delta^e(\delta')^o\delta_\circ^p\mu^q(\mu')^r\ \pi^{-1}$$

where $\pi^{-1} = ({}_{\sigma^{-1}}f, \sigma^{-1})$ , $\sigma^{-1} = \begin{pmatrix} 1\sigma & \cdots & l\sigma \\ 1 & \cdots & l \end{pmatrix}$ is the permutation formed in $i(\Gamma)$ by flipping $\sigma$ so $\sigma^{-1}$ takes vertices in $i(y)$ (which are the endpoints of propagating lines in $y$) to vertices in $i(x)$ (which are the endpoints of propagating lines in $x$). Since the map $f$ in $\pi$ has value 1 or 0 according to the line $(i, i\sigma)$ (which is from row 1 to row 2 in $\sigma$) is decorated or not and reflecting graphs upside down does not change the decoration of the lines so the map ${}_{\sigma^{-1}}f$ has value 1 or 0 according to the line $(i\sigma, i) = (j, j\sigma^{-1})$ in the permutation $\pi^{-1}$ (which corresponds to the line $(i, i\sigma)$ in the permutation $\pi$) is decorated or not.

Now, from the Definition 4.4.7 of $\varphi_l$ we have

$$\varphi_l(i(y), i(x)) = \delta^e(\delta')^o\delta_\circ^p\mu^q(\mu')^r\ \omega$$

where $\omega = (g, \gamma)$, $\gamma = \begin{pmatrix} 1 & \cdots & l \\ 1\gamma & \cdots & l\gamma \end{pmatrix}$, $1 < \cdots < j < \cdots < l$ are the endpoints of propagating lines in $i(y)$ (which are the endpoints of propagating lines in $y$), $1\gamma$, …, $j\gamma$, …, $l\gamma$ are the endpoints of propagating lines in $i(x)$ (which are the endpoints of propagating lines in $x$). Meaning that $\gamma$ takes vertices in $y$ to vertices in $x$. Therefore $\gamma = \sigma^{-1}$ and the map $g$ has value 1 or 0 according to whether the line $(j, j\gamma)$ is decorated or not. Since $j's$ are vertices in $y$ so $j = i\sigma$ for $i's$ vertices in $x$. Therefore $(j, j\gamma) = (i\sigma, i\sigma\sigma^{-1}) = (i\sigma, i)$ so $g = {}_{\sigma^{-1}}f$ and then $\omega = \pi^{-1}$. Hence $\varphi_l\big(i(y), i(x)\big) = \delta^e(\delta')^o\delta_\circ^p\mu^q(\mu')^r\pi^{-1} = \overline{\varphi_l(x, y)}$. □

The following Lemma describes the anti-involution on $i(V_l) \otimes V_l \otimes K\widetilde{S}_l$.

**Lemma 4.4.12.** *The map $\iota$ given by*

$$\iota(i(x)\otimes y\otimes \pi) = i(y)\otimes x\otimes \pi^{-1}$$

*where $x, y \in h_l(DPB(n))$ and $\pi \in \widetilde{S}_l$, is an anti-involution on $i(V_l)\otimes V_l \otimes K\widetilde{S}_l$.*

*Proof.* Let $d \in B_l$, $i(d) \in B_l$ be the reflection of $d$ through its horizontal axis. So the top (resp. bottom) of $d$ will be bottom (resp. top) of $i(d)$ and the element $\pi \in \widetilde{S}_l$ in $d$ will be $\pi^{-1} \in \widetilde{S}_l$ in $i(d)$.
By Lemma 4.4.9, we have $\psi(d) = i(x)\otimes y\otimes \pi$, $x, y \in h_l(DPB(n))$, $\pi \in \widetilde{S}_l$. Then

$$\psi(i(d)) = i(y)\otimes x\otimes \pi^{-1} = \iota\big(i(x)\otimes y\otimes \pi\big) = \iota(\psi(d)). \qquad (*)$$

Note that

$$\begin{aligned}\iota^2(i(x)\otimes y\otimes \pi) &= \iota^2(\psi(d)) = \iota\big(\iota(\psi(d))\big)\\ &= \iota\big(\psi(i(d))\big) = \psi(i^2(d)) = \psi(d) = i(x)\otimes y\otimes \pi.\end{aligned}$$

then $\iota^2 = \mathrm{id}$. Also

$$\begin{aligned}&\iota\Big(\big(i(x_1)\otimes y_1\otimes \pi_1\big).\big(i(x_2)\otimes y_2\otimes \pi_2\big)\Big)\\
&= \iota(\psi(d_1).\psi(d_2))\\
&= \iota\big(\psi(d_1d_2)\big) && \text{(as } \psi \text{ is an isomorphism)}\\
&= \psi\big(i(d_1d_2\big) && \text{(from (*))}\\
&= \psi\big(i(d_2).i(d_1)\big) && \text{(as } i \text{ is anti-involution)}\\
&= \psi\big(i(d_2)\big).\psi\big(i(d_1)\big) && \text{(as } \psi \text{ is an isomorphism)}\\
&= \iota\big(\psi(d_2)\big).\iota\big(\psi(d_1)\big) && \text{(from(*))}\\
&= \iota\big(i(x_2)\otimes y_2\otimes \pi_2\big).\iota\big(i(x_1)\otimes y_1\otimes \pi_1\big).\end{aligned}$$

Then $\iota$ is anti-involution on $i(V_l)\otimes V_l\otimes K\widetilde{S}_l$. □

**Remark 4.4.13.** Note that by Lemmas 4.4.6, 4.4.9 we can identify the set $DPB(n)$ with the set $\cup_{l=0}^{n} i\big(h_l(DPB(n))\big)\otimes h_l(DPB(n))\otimes \widetilde{S}_l$ via $\psi$ so, if no confusion can arise,

we will express for any element $d$ with $\#d = l$ in the basis of $D\mathcal{P}\mathfrak{B}_n(\delta, \delta', \delta_\circ, \mu, \mu')$ by its corresponding representation $x \otimes y \otimes \pi$ in the basis of $i(V_l) \otimes V_l \otimes K\widetilde{S}_l$. Note that with this identification we can write $J_l := \bigoplus_{k=0}^{l} i(V_k) \otimes V_k \otimes K\widetilde{S_k}$.
Also, with this identification we have the following
For $d_1,\ d_2 \in B_l$ so $d_1 = x_1 \otimes y_1 \otimes \pi_1$, $d_2 = x_2 \otimes y_2 \otimes \pi_2$ and, by Lemma 4.4.9, their multiplication is,

$$d_1 d_2 \equiv x_1 \otimes y_2 \otimes \pi_1 \varphi_l(y_1, x_2)\pi_2 \qquad \pmod{J_{l-1}}.$$

**Lemma 4.4.14.** *For each $0 \leq l \leq n$, there are maps $\phi_l : DPB(n) \times i(h_l(DPB(n))) \to i(V_l)$ and $\theta_l : DPB(n) \times i(h_l(DPB(n))) \to K\widetilde{S}_l$ such that for any $u, v \in h_l(DPB(n))$ and $\pi \in \widetilde{S}_l$, we have for any $d \in DPB(n)$ that*

$$d.(i(u) \otimes v \otimes \pi) \equiv \phi_l(d, i(u)) \otimes v \otimes \theta_l(d, i(u))\pi \pmod{J_{l-1}}$$

*where $J_{l-1} = \bigoplus_{k=0}^{l-1} i(V_k) \otimes V_k \otimes K\widetilde{S_k}$*

*Proof.* Let $d_1 = i(a) \otimes b \otimes \pi_1 \in DPB(n)$ with $\#d_1 = m$, $a, b \in h_m(DPB(n))$, $\pi_1 \in \widetilde{S_m}$ be any basis element of $D\mathcal{P}\mathfrak{B}_n(\delta, \delta_\circ, \delta', \mu, \mu')$ and $d_2 = i(u) \otimes v \otimes \pi_2$, $u, v \in h_l(DPB(n))$, $\pi_2 \in \widetilde{S}_l$. We want to show that

$$d_1 d_2 \equiv \phi_l(d_1, i(u)) \otimes v \otimes \theta_l(d_1, i(u))\pi_2 \qquad \pmod{J_{l-1}}$$

where $\phi_l(d_1, i(u)) \in i(V_l)$, $\theta_l(d_1, i(u)) \in K\widetilde{S}_l$ are independent of $\pi_2$.
We have the following cases:
*Case 1*: If $m \geq l$ then Lemma 4.4.4 implies that $\#d_1 d_2 \leq l$, and

- If $\#(d_1 d_2) < l$ then $\quad d_1 d_2 \equiv 0 \pmod{J_{l-1}}$.
- If $\#(d_1 d_2) = l = \#d_2$ then the bottom row of $d_1 d_2$ is "$v$" which is the bottom row of $d_2$. Consider the product of $d_1$ with $i(u)$, which is the top of $d_2$. This is formed by a series of concatenations: $i(a).\pi_1.b.i(u)$ and it will be of the form $\lambda x.\sigma$ where $\lambda \in K$, $x \in i\big(h_l(DPB(n))\big)$. The half diagram $x =$ top $(d_1.i(u))$

i.e. $x$ is the half diagram obtained by concatenating $d_1$ and $i(u)$, the scalar $\lambda$ arises from any isolated components that are removed from the product $d_1.i(u)$. The permutation $\sigma \in \widetilde{S}_l$ is the permutation induced from this concatenation which is independent of $\pi_2$. Then by concatenating this further with $\pi_2.v$ we will get $x.\sigma.\pi_2.v = x \otimes v \otimes \sigma\pi_2$. Therefore, from the above description we have $\phi_l(d_1, i(u)) = \lambda x \in i(V_l)$ and $\theta_l(d_1, i(u)) = \sigma \in \widetilde{S}_l$.
(See Example 4.4.16 which illustrates this product.)

Therefore in case, $m \geq l$, we have

$$\begin{aligned} d_1 d_2 &\equiv \lambda\ x \otimes v \otimes \sigma\pi_2 && \pmod{J_{l-1}} \\ &= \phi_l(d_1, i(u)) \otimes v \otimes \theta_l(d_1, i(u))\pi_2 && \pmod{J_{l-1}}. \end{aligned}$$

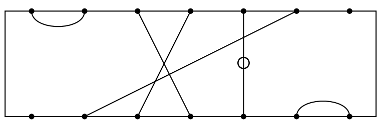
Note that in case $m = l$, then $x = i(a)$ and $\sigma = \pi_1\varphi_l(b, i(u))$.
*Case 2*: If $m < l$, then $\#(d_1 d_2) \leq m$ but $m < l$ means $d_1 d_2 \in J_m \subseteq J_{l-1}$ and then $d_1 d_2 \equiv 0 \pmod{J_{l-1}}$. □

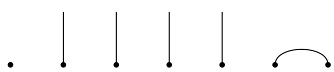
**Remark 4.4.15.** Note that, similarly, for $\phi_l : h_l(DPB(n)) \times DPB(n) \to V_l$ and $\theta_l : h_l(DPB(n)) \times DPB(n) \to K\widetilde{S}_l$ , $u, v \in h_l(DPB(n))$ and $\pi \in \widetilde{S}_l$, we can show that for any $d \in DPB(n)$

$$(i(u) \otimes v \otimes \pi).d \equiv i(u) \otimes \phi_l(v, d) \otimes \pi\theta_l(v, d) \qquad \pmod{J_{l-1}}$$

The following example illustrates the product $d.(i(u) \otimes v \otimes \pi)$ (in the previous lemma), where $d \in DPB(n)$, $u, v \in h_l(DPB(n))$ and $\pi \in \widetilde{S}_l$.

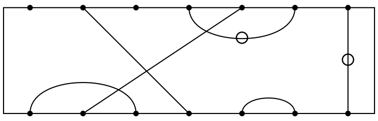
**Example 4.4.16.** Let

$d_1 =$ 

$=$  $\otimes$  $\otimes \Big((0,0,1,0),(134)\Big)$

$= a \otimes b \otimes \pi_1 \in DPB(7)$

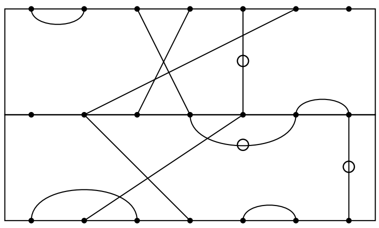
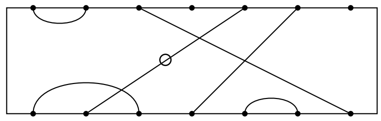
and

$d_2 =$ 

$=$  $\otimes \Big((0,0,1),(12)\Big)$

$= i(u) \otimes v \otimes \pi_2$

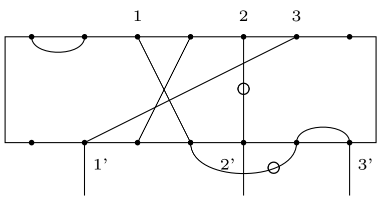
Then

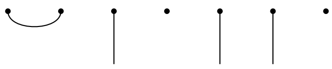
$d_1 d_2 =$  $= \delta'$ 

$= \delta'$  $\otimes \Big((0,1,0),(132)\Big)$

$= \delta' \ \ x \otimes v \otimes \pi'$

Also,

$x = d_1.i(u) =$ 

$= \delta'$  ,

$\sigma = \Big((1,1,0),(13)\Big)$ (which is the permutation induced from the product $d_1.i(u)$)
and
$\sigma.\pi_2 = \Big((1,1,0),(13)\Big).\Big((0,0,1),(12)\Big) = \Big((0,1,0),(132)\Big) = \pi'$
Then

$$d_1 d_2 = \delta' \ x \otimes v \otimes \sigma\pi_2$$
$$= \phi_3(d_1, i(u)) \otimes v \otimes \theta_3(d_1, i(u))\pi_2.$$

## 4.5 The main Theorem

**Theorem 4.5.1.** *Let $K$ be a field, $\delta, \delta_\circ, \delta', \mu, \mu' \in K$. Then the decorated partial Brauer algebra $D\mathcal{P}\mathfrak{B}_n(\delta, \delta_\circ, \delta', \mu, \mu')$ is a cellular algebra over $K$.*

*Proof.* By Lemma 4.4.6, the decorated partial Brauer algebra has a decomposition as a $K$-vector space

$$D\mathcal{P}\mathfrak{B}_n(\delta, \delta_\circ, \delta', \mu, \mu') = i(V_n) \otimes V_n \otimes K\widetilde{S}_n \oplus i(V_{n-1}) \otimes V_{n-1} \otimes K\widetilde{S}_{n-1} \\ \oplus \cdots \oplus i(V_1) \otimes V_1 \otimes K\widetilde{S}_1 \oplus i(V_0) \otimes V_0 \otimes K\widetilde{S}_0.$$

Note that $K\widetilde{S}_l$ is a cellular algebra with involution $\bar{\pi} = \pi^{-1}$ for all $\pi \in \tilde{S}_l$ (see Theorem 2.5.10). By Lemmas 4.4.12 and 4.4.14 , the above decomposition of $D\mathcal{P}\mathfrak{B}_n(\delta, \delta_\circ, \delta', \mu, \mu')$ satisfies the conditions in Theorem 4.1.2. Hence it is a cellular algebra. □

As a consequence of previous theorem and from Proposition 4.1.4 we have the following:

**Corollary 4.5.2.** *The cell modules of $D\mathcal{P}\mathfrak{B}_n(\delta, \delta_\circ, \delta', \mu, \mu')$ are*

$$\Delta_n(l, \lambda) = i(V_l) \otimes S^\lambda$$

*where $l \in \{0, 1, \ldots, n\}$, $\lambda$ is a bipartition of $l$ and $S^\lambda$ is a cell module of $K\widetilde{S}_l$ corresponding to $\lambda$.*

## 4.6 The $K$-bilinear form on cell modules

In this section we give the $K$-bilinear form on the set of cell modules $\Delta_n(l, \lambda)$. Firstly we describe the cellular basis for $D\mathcal{P}\mathfrak{B}_n(\delta, \delta_\circ, \delta', \mu, \mu')$ in the sense of [6]. Note that, from Lemma 4.4.6, the set

$$\{i(x) \otimes y \otimes \pi \ \mid \ x, y \in h_l(DPB(n)), \ \ \pi \in \widetilde{S}_l\}$$

forms a basis for the algebra $B_l \cong i(V_l) \otimes V_l \otimes K\widetilde{S}_l$. Therefore the $K$-algebra $D\mathcal{P}\mathfrak{B}_n(\delta, \delta_\circ, \delta', \mu, \mu')$ has a basis

$$\begin{aligned}&\coprod_{0 \le l \le n} \{i(x) \otimes y \otimes \pi \mid x, y \in h_l(DPB(n)), \quad \pi \in \widetilde{S}_l\} \\ &= \{i(x) \otimes y \otimes \pi \mid x, y \in h_l(DPB(n)), \quad \pi \in \widetilde{S}_l, \quad 0 \le l \le n\}.\end{aligned}$$

To describe the cellular basis of $D\mathcal{P}\mathfrak{B}_n(\delta, \delta_\circ, \delta', \mu, \mu')$, we need the following definitions.

**Definition 4.6.1.** For $n \ge 1$, define the poset $\Lambda(n)$ as follows.
Suppose $l \in \mathbb{N}$, $0 \le l \le n$. Let $P(l)$ be the set of bipartitions of $l$ ordered by dominance (Definition 2.4.11). Then, we have

$$\Lambda(n) = \{(l, \lambda) \mid 0 \le l \le n, \quad \lambda \in P(l)\}.$$

For $(l, \lambda), (l', \lambda') \in \Lambda(n)$ we define an ordered relation in $\Lambda(n)$ by

$$(l, \lambda) \le (l', \lambda') \text{ if } l < l' \text{ or } l = l' \text{ and } \lambda \trianglerighteq \lambda'$$

and we write $(l, \lambda) < (l', \lambda')$ if $(l, \lambda) \le (l', \lambda')$ and $(l, \lambda) \ne (l', \lambda')$.
It is clear that the order relation $\le$ in $\Lambda(n)$ is a partial order since the order $\lambda \trianglerighteq \lambda'$ is a partial order (Definition 2.4.11).

**Example 4.6.2.** Let $n = 2$. The set

$$\begin{aligned}\Lambda(n) = \Big\{&\Big(0, \big((0), (0)\big)\Big), \Big(1, \big((0), (1)\big)\Big), \Big(1, \big((1), (0)\big)\Big), \Big(2, \big((2), (0)\big)\Big), \\ &\Big(2, \big((0), (2)\big)\Big), \Big(2, \big((1, 1), (0)\big)\Big), \Big(2, \big((0), (1, 1)\big)\Big), \Big(2, \big((1), (1)\big)\Big)\Big\}.\end{aligned}$$

The order on $\Lambda(n)$ is as follows:

$$\begin{aligned}&\Big(0, \big((0), (0)\big)\Big) < \Big(1, \big((1), (0)\big)\Big) < \Big(1, \big((0), (1)\big)\Big) < \\ &\Big(2, \big((2), (0)\big)\Big) < \Big(2, \big((1, 1), (0)\big)\Big) < \Big(2, \big((1), (1)\big)\Big) < \Big(2, \big((0), (2)\big)\Big) < \Big(2, \big((0), (1, 1)\big)\Big).\end{aligned}$$

**Definition 4.6.3.** For $(l,\lambda)\in\Lambda(n)$, define

$$M(l,\lambda)=\{(x,\mathbf{s})\ \mid\ x\in h_l(DPB(n)),\ \mathbf{s}\in \mathrm{Std}\lambda,\ \lambda \text{ is a bipartition of } l\}.$$

Now we can use a cellular basis of $K\widetilde{S}_l$, which is $\{C^{\lambda}_{\mathbf{st}}\ \mid\ \mathbf{s},\mathbf{t}\in\mathrm{Std}\lambda,\ \lambda \text{ is a bipartition of } l\}$ (defined in Theorem 2.5.10), to obtain a cellular basis of $D\mathcal{P}\mathfrak{B}_n(\delta,\delta_\circ,\delta',\mu,\mu')$ as follows.

Let $(l,\lambda)\in\Lambda(n)$, $(x,\mathbf{s}),(y,\mathbf{t})\in M(l,\lambda)$. Define

$$C^{(l,\lambda)}_{(x,\mathbf{s})(y,\mathbf{t})}:=i(x)\otimes y\otimes C^{\lambda}_{\mathbf{st}}.$$

So by Lemma 4.4.6 the set

$$\mathfrak{M}=\{C^{(l,\lambda)}_{(x,\mathbf{s})(y,\mathbf{t})}\ \mid\ (x,\mathbf{s}),(y,\mathbf{t})\in M(l,\lambda),\ \lambda \text{ is a bipartition of } l,\ 0\le l\le n\}$$

forms a basis of $D\mathcal{P}\mathfrak{B}_n(\delta,\delta_\circ,\delta',\mu,\mu')$ (which proves condition $(C_1)$ of Definition 2.2.1).

Moreover, by Lemma 4.4.12, we have

$$\iota(C^{(l,\lambda)}_{(x,\mathbf{s})(y,\mathbf{t})})=i(y)\otimes x\otimes C^{\lambda}_{\mathbf{ts}}=C^{(l,\lambda)}_{(y,\mathbf{t})(x,\mathbf{s})}$$

(which is condition $(C_2)$ of Definition 2.2.1).

To prove $(C_3)$, it suffices to show that for any basis element $z$ of $D\mathcal{P}\mathfrak{B}_n(\delta,\delta_\circ,\delta',\mu,\mu')$, where $z=i(a)\otimes b\otimes\pi$, $a,b\in h_k(DPB(n))$, $\pi\in\widetilde{S_k}$ and $C^{(l,\lambda)}_{(x,\mathbf{s})(y,\mathbf{t})}\in\mathfrak{M}$, the product

$$zC^{(l,\lambda)}_{(x,\mathbf{s})(y,\mathbf{t})}\equiv\sum_{(x',\mathbf{s}')\in M(l,\lambda)} r_{(x',\mathbf{s}')}\ C^{(l,\lambda)}_{(x',\mathbf{s}')(y,\mathbf{t})}\qquad (\mathrm{mod}\ \breve{A}^{(l,\lambda)})\qquad (1)$$

where $(x',\mathbf{s}')$ depends only on $z$ and $(x,\mathbf{s})$, and

$$\breve{A}^{(l,\lambda)}=\mathrm{Span}\{C^{(k,\mu)}_{(x,\mathbf{s})(y,\mathbf{t})}\ \mid\ (x,\mathbf{s}),(y,\mathbf{t})\in M(k,\mu),\ (k,\mu)<(l,\lambda)\}$$

Notice in particular that $J_{l-1} \subseteq \breve{A}^{(l,\lambda)}$, where (from Lemma 4.4.6) $J_{l-1}$ has basis the set $\{C^{(k,\mu)}_{(x,\mathbf{s})(y,\mathbf{t})} \mid (x,\mathbf{s}),(y,\mathbf{t}) \in M(k,\mu),\ \mu \text{ is a bipartition of } k,\ k \le l-1\}$.

From Lemma 4.4.14 we have

$$\begin{aligned} zC^{(l,\lambda)}_{(x,\mathbf{s})(y,\mathbf{t})} &= (i(a)\otimes b\otimes \pi)(i(x)\otimes y\otimes C^{\lambda}_{\mathbf{st}}) \\ &\equiv \phi_l(z,i(x))\otimes y\otimes \theta_l(z,i(x))C^{\lambda}_{\mathbf{st}} \quad (\text{mod } \breve{A}^{(l,\lambda)}) \\ &\equiv \alpha i(x')\otimes y\otimes \sigma C^{\lambda}_{\mathbf{st}} \quad (\text{mod } \breve{A}^{(l,\lambda)}) \qquad (*) \end{aligned}$$

where $x' \in h_l(DPB(n))$ and $\sigma \in \widetilde{S}_l$ are independent of $C^{\lambda}_{\mathbf{st}}$, $\alpha \in K$.

Since $C^{\lambda}_{\mathbf{st}}$ is an element of a cellular basis of $K\widetilde{S}_l$, $\sigma \in \widetilde{S}_l$, then we have

$$\begin{aligned} \sigma C^{\lambda}_{\mathbf{st}} &\equiv \sum_{\mathbf{s}'\in Std\lambda} r_{\mathbf{s}'}\ C^{\lambda}_{\mathbf{s}'\mathbf{t}} \quad (\text{mod } \overline{A^{\lambda}}) \\ &= \sum_{\mathbf{s}'\in Std\lambda} r_{\mathbf{s}'}\ C^{\lambda}_{\mathbf{s}'\mathbf{t}} + \sum_{\mu\rhd\lambda} a_{\mathbf{pq}}C^{\mu}_{\mathbf{pq}} \qquad (**) \end{aligned}$$

where $\mathbf{s}'$ depends on $\sigma$ and $\mathbf{s}$, $r_{\mathbf{s}'} \in K$. By substituting $(**)$ in $(*)$ we get

$$\begin{aligned} zC^{(l,\lambda)}_{(x,\mathbf{s})(y,\mathbf{t})} &= \alpha i(x')\otimes y\otimes (\sum_{\mathbf{s}'\in Std\lambda} r_{\mathbf{s}'}C^{\lambda}_{\mathbf{s}'\mathbf{t}} + \sum_{\mu\rhd\lambda} a_{\mathbf{pq}}C^{\mu}_{\mathbf{pq}}) + \text{terms in } \breve{A}^{(l,\lambda)} \\ &= \alpha i(x')\otimes y\otimes \sum_{\mathbf{s}'\in Std\lambda} r_{\mathbf{s}'}\ C^{\lambda}_{\mathbf{s}'\mathbf{t}} + \alpha i(x')\otimes y\otimes \sum_{\mu\rhd\lambda} a_{\mathbf{pq}}C^{\mu}_{\mathbf{pq}} + \text{terms in } \breve{A}^{(l,\lambda)}. \end{aligned}$$

From the definition of $\breve{A}^{(l,\lambda)}$ we observe that the middle term in the above equation is in $\breve{A}^{(l,\lambda)}$, so we have

$$\begin{aligned} zC^{(l,\lambda)}_{(x,\mathbf{s})(y,\mathbf{t})} &= \alpha i(x')\otimes y\otimes \sum_{\mathbf{s}'\in Std\lambda} r_{\mathbf{s}'}\ C^{\lambda}_{\mathbf{s}'\mathbf{t}} + \text{terms in } \breve{A}^{(l,\lambda)} \\ &= \sum_{\mathbf{s}'\in Std\lambda} \alpha r_{\mathbf{s}'}(i(x')\otimes y\otimes C^{\lambda}_{\mathbf{s}'\mathbf{t}}) + \text{terms in } \breve{A}^{(l,\lambda)} \\ &\equiv \sum_{(x',\mathbf{s}')\in M(l,\lambda)} r_{(x',\mathbf{s}')}\ C^{(l,\lambda)}_{(x',\mathbf{s}')(y,\mathbf{t})} \quad (\text{mod } \breve{A}^{(l,\lambda)}). \end{aligned}$$

So the datum $(\Lambda(n), M, \mathfrak{M}, \iota)$ is a cell datum of $D\mathcal{P}\mathfrak{B}_n(\delta,\delta_\circ,\delta',\mu,\mu')$ and $(\mathfrak{M},\Lambda(n))$

is a cellular basis of $D\mathcal{P}\mathfrak{B}_n(\delta, \delta_\circ, \delta', \mu, \mu')$.

Note that by applying the anti-involution $\iota$ on (1) we get

$$C^{(l,\lambda)}_{(x,\mathbf{s})(y,\mathbf{t})}\iota(z) \equiv \sum_{(y',\mathbf{t}')\in M(l,\lambda)} r_{(y',\mathbf{t}')} C^{(l,\lambda)}_{(x,\mathbf{s})(y',\mathbf{t}')} \qquad (\text{mod } \check{A}^{(l,\lambda)}) \qquad (1')$$

From (1) and (1′) we deduce that the $K$-vector space

$$A^{(l,\lambda)} = \text{Span}\{C^{(k,\mu)}_{(x,s)(y,t)} = i(x) \otimes y \otimes C^{\mu}_{st} \mid (x,s),(y,t) \in M(k,\mu),\ (k,\mu) \leq (l,\lambda)\}$$

is an ideal of $D\mathcal{P}\mathfrak{B}_n(\delta, \delta_\circ, \delta', \mu, \mu')$. Also, since $\check{A}^{(l,\lambda)} = \Sigma_{(k,\mu)<(l,\lambda)} A^{(k,\mu)}$ then $\check{A}^{(l,\lambda)}$ is an ideal of $D\mathcal{P}\mathfrak{B}_n(\delta, \delta_\circ, \delta', \mu, \mu')$.

Since the cell (Specht) module $S^\lambda$ of $K\widetilde{S}_l$ (defined in Definition 2.5.11) has basis $\{C^{\lambda}_{\mathbf{t}} \mid \mathbf{t} \in \text{Std}\lambda\}$, we can say the set

$$\{i(x) \otimes C^{\lambda}_{\mathbf{t}} \mid x \in h_l(DPB(n)),\ C^{\lambda}_{\mathbf{t}} \text{ is an element from the basis of } S^\lambda\}$$

forms a basis for the cell module $\Delta_n(l,\lambda)$, with the action given, for $a \in DPB(n)$, by

$$a.(i(x) \otimes C^{\lambda}_{\mathbf{t}}) = \phi_l(a, i(x)) \otimes \theta_l(a, i(x)) C^{\lambda}_{\mathbf{t}}.$$

where $\phi_l(a, i(x)) \in i(V_l)$, $\theta_l(a, i(x)) \in K\widetilde{S}_l$.

**Proposition 4.6.4.**

$$\dim \Delta_n(l,\lambda) = \sum_{k=0}^{\lfloor \frac{n-l}{2} \rfloor} \binom{n-2k}{l} \frac{n!}{(n-2k)!k!}\ 2^{n-(l+2k)}. \dim S^\lambda.$$

*Proof.* Since the cell module $\Delta_n(l,\lambda)$ of $D\mathcal{P}\mathfrak{B}_n(\delta, \delta_\circ, \delta', \mu, \mu')$ has basis

$$\{C^{(l,\lambda)}_{(x,\mathbf{s})} = i(x) \otimes C^{\lambda}_{\mathbf{s}} \mid (x,\mathbf{s}) \in M(l,\lambda)\}.$$

Therefore,

$$
\begin{aligned}
\dim \Delta_n(l,\lambda) &= |M(l,\lambda)| \\
&= |h_l(DPB(n))|.|\mathrm{Std}(\lambda)| \\
&= |h_l(DPB(n))|.\dim S^{\lambda} \\
&= \sum_{k=0}^{\lfloor \frac{n-l}{2} \rfloor} \binom{n-2k}{l} \frac{n!}{(n-2k)!k!}\; 2^{n-(l+2k)}.\dim S^{\lambda} \quad \text{(From lemma 4.3.2).}
\end{aligned}
$$

□

We now give an example of a basis of a cell module.

**Example 4.6.5.** Let $n = 2$, $l = 1$.

Firstly we will find $S^{\lambda}$, the cell modules of $K\widetilde{S}_1$, where $\lambda$ is a bipartition of 1.

The bipartitions of $l = 1$ are:

$\lambda_1 = ((1),(0))$, $a = |\lambda^{(1)}| = 1$,

$\lambda_2 = ((0),(1))$, $a = |\lambda^{(1)}| = 0$, where $\lambda_1 \rhd \lambda_2$.

The standard $\lambda$-bitableaux are:

$\mathrm{Std}(\lambda_1)=\{\mathbf{t}_1 = (\boxed{1}, \emptyset)\}$, $\quad$ $\mathrm{Std}(\lambda_2)=\{\mathbf{t}_2 = (\emptyset, \boxed{1})\}$.

Now we will construct the elements $C_{\mathbf{st}} = d^*(\mathbf{s}) m_\lambda d(\mathbf{t})$, where

$m_\lambda = u_a^+ x_\lambda$, $u_a^+ = \prod_{i=1}^{a}(1 + s_{i,1}s_0 s_{1,i})$, $x_\lambda = \sum_{w \in S_\lambda} w$

so $u_0^+ = 1$, $u_1^+ = 1 + s_0$

So we have

$m_{\lambda_1} = u_1^+ x_{\lambda_1} = 1 + s_0$ $\quad$ and $\quad$ $m_{\lambda_2} = u_0^+ x_{\lambda_2} = 1$, and then

$C^{\lambda_1}_{\mathbf{t}_1\mathbf{t}_1} = 1 + s_0$ $\quad$ and $\quad$ $C^{\lambda_2}_{\mathbf{t}_2\mathbf{t}_2} = 1$, then

$A^{\lambda_1} = \mathrm{span}\{C^{\lambda_1}_{\mathbf{t}_1\mathbf{t}_1}\}$, $\overline{A^{\lambda_1}} = \emptyset$,

$A^{\lambda_2} = \mathrm{span}\{C^{\lambda_1}_{\mathbf{t}_1\mathbf{t}_1}, C^{\lambda_2}_{\mathbf{t}_2\mathbf{t}_2}\}$, $\overline{A^{\lambda_2}} = A^{\lambda_1}$.

Therefore, $S^{\lambda_1} = \mathrm{span}\{C^{\lambda_1}_{\mathbf{t}_1}\}$, where $C^{\lambda_1}_{\mathbf{t}_1} = (1 + s_0) + \overline{A^{\lambda_1}}$,

$S^{\lambda_2} = \mathrm{span}\{C^{\lambda_2}_{\mathbf{t}_2}\}$, $\quad$ where $C^{\lambda_2}_{\mathbf{t}_2} = 1 + \overline{A^{\lambda_2}}$.

Also, $h_1(DPB(2)) = \{x_1,\ x_2,\ x_3,\ x_4\}$, where

$$x_1 = \mid \;\; \bullet, \quad x_2 = \bullet \;\; \mid, \quad x_3 = \mid \;\; \square, \quad x_4 = \square \;\; \mid .$$

Therefore, $\Delta_2(1,\lambda_1) = \text{span}\{i(x_1)\otimes C^{\lambda_1}_{\mathbf{t}_1},\ i(x_2)\otimes C^{\lambda_1}_{\mathbf{t}_1},\ i(x_3)\otimes C^{\lambda_1}_{\mathbf{t}_1},\ i(x_4)\otimes C^{\lambda_1}_{\mathbf{t}_1}\}$

$\Delta_2(1,\lambda_2) = \text{span}\{i(x_1)\otimes C^{\lambda_2}_{\mathbf{t}_2},\ i(x_2)\otimes C^{\lambda_2}_{\mathbf{t}_2},\ i(x_3)\otimes C^{\lambda_2}_{\mathbf{t}_2},\ i(x_4)\otimes C^{\lambda_2}_{\mathbf{t}_2}\}$.

Now we will define a bilinear form, say $\Phi_{(l,\lambda)}$, on the cell modules $\Delta_n(l,\lambda)$ in the sense of Graham and Lehrer [6].

**Definition 4.6.6.** For $(l,\lambda)\in\Lambda(n)$, define $\Phi_{(l,\lambda)}:\Delta_n(l,\lambda)\times\Delta_n(l,\lambda)\to K$ by

$$C^{(l,\lambda)}_{(x,\mathbf{s})(y,\mathbf{t})}C^{(l,\lambda)}_{(x',\mathbf{s}')(y',\mathbf{t}')} \equiv \Phi_{(l,\lambda)}(y\otimes C^{\lambda}_{\mathbf{t}}, i(x')\otimes C^{\lambda}_{\mathbf{s}'})C^{(l,\lambda)}_{(x,\mathbf{s})(y',\mathbf{t}')} \pmod{\breve{A}^{(l,\lambda)}}$$

The following proposition describes how to get a bilinear form $\Phi_{(l,\lambda)}$ on the cell module $\Delta_n(l,\lambda)$ using a bilinear form $\phi_{(l,\lambda)}$ on the cell modules $S^{\lambda}$.

**Proposition 4.6.7.**

$$\Phi_{(l,\lambda)}(y\otimes C^{\lambda}_{t}, i(x')\otimes C^{\lambda}_{s'}) := \phi_{(l,\lambda)}\big(C^{\lambda}_{t}, \varphi_l(y,i(x'))\ C^{\lambda}_{s'}\big)$$

*where $(y\otimes C^{\lambda}_{t})$, $(i(x')\otimes C^{\lambda}_{s'})$ are in $\Delta_n(l,\lambda)$, $\phi_{(l,\lambda)}$ is the symmetric $K$-bilinear form on the cell module $S^{\lambda}$ of $K\tilde{S}_l$, where $\phi_{(l,\lambda)} = \langle\ ,\ \rangle$ as in Definition 2.5.11 and $\varphi_l$ is as defined in Definition 4.4.7.*

*Proof.* Let $C^{(l,\lambda)}_{(x,\mathbf{s})(y,\mathbf{t})} = i(x)\otimes y\otimes C^{\lambda}_{\mathbf{st}}$ and $C^{(l,\lambda)}_{(x',\mathbf{s}')(y',\mathbf{t}')} = i(x')\otimes y'\otimes C^{\lambda}_{\mathbf{s}'\mathbf{t}'}$ be elements from the cellular basis of $D\mathcal{P}\mathfrak{B}_n(\delta,\delta_\circ,\delta',\mu,\mu')$, then

$$C^{(l,\lambda)}_{(x,\mathbf{s})(y,\mathbf{t})}C^{(l,\lambda)}_{(x',\mathbf{s}')(y',\mathbf{t}')} \equiv i(x)\otimes y'\otimes\ C^{\lambda}_{\mathbf{st}}\big(\varphi_l(y,i(x'))\ C^{\lambda}_{\mathbf{s}'\mathbf{t}'}\big) \pmod{\breve{A}^{(l,\lambda)}}$$

From (the proof of) Proposition 2.9 in [16], we have

$$\langle C^{\lambda}_{\mathbf{s}}, aC^{\lambda}_{\mathbf{t}}\rangle C^{\lambda}_{\mathbf{ub}} \equiv C^{\lambda}_{\mathbf{us}}(aC^{\lambda}_{\mathbf{tb}}) \pmod{\breve{A}^{\lambda}}$$

where $C^{\lambda}_{\mathbf{s}}, C^{\lambda}_{\mathbf{t}} \in S^{\lambda}$, $a \in A$ ($A$ is an algebra).

Therefore, by using this property, we have got

$$\begin{aligned} C^{(l,\lambda)}_{(x,\mathbf{s})(y,\mathbf{t})} C^{(l,\lambda)}_{(x',\mathbf{s}')(y',\mathbf{t}')} &\equiv i(x) \otimes y' \phi_{(l,\lambda)}\big(C^{\lambda}_{\mathbf{t}}, \varphi_l(y, i(x'))C^{\lambda}_{\mathbf{s}'}\big)\ C^{\lambda}_{\mathbf{st}'} && \pmod{\breve{A}^{(l,\lambda)}} \\ &\equiv \phi_{(l,\lambda)}\big(C^{\lambda}_{\mathbf{t}}, \varphi_l(y, i(x'))C^{\lambda}_{\mathbf{s}'}\big)\ i(x) \otimes y' \otimes C^{\lambda}_{\mathbf{st}'} && \pmod{\breve{A}^{(l,\lambda)}} \\ &\equiv \phi_{(l,\lambda)}\big(C^{\lambda}_{\mathbf{t}}, \varphi_l(y, i(x'))C^{\lambda}_{\mathbf{s}'}\big)\ C^{(l,\lambda)}_{(x,s)(y',t')} && \pmod{\breve{A}^{(l,\lambda)}}. \end{aligned}$$

From definition 4.6.6 we have

$$C^{(l,\lambda)}_{(x,\mathbf{s})(y,\mathbf{t})} C^{(l,\lambda)}_{(x',\mathbf{s}')(y',\mathbf{t}')} \equiv \Phi_{(l,\lambda)}(y \otimes C^{\lambda}_{\mathbf{t}}, i(x') \otimes C^{\lambda}_{\mathbf{s}'}) C^{(l,\lambda)}_{(x,\mathbf{s})(y',\mathbf{t}')} \qquad \pmod{\breve{A}^{(l,\lambda)}}$$

Therefore we get the desired result. □

**Lemma 4.6.8.** *Suppose that at least one of the elements $\delta', \mu$ and $\mu'$ is a non-zero element in $K$. Then $\Phi_{(l,\lambda)} \neq 0$ if and only if the corresponding linear form $\phi_{(l,\lambda)}$ for cellular algebra $K\widetilde{S}_l$ is non-zero.*

*Proof.* From Definition 4.4.7 of the bilinear form $\varphi_l$ (in the case where the number of propagating lines in $xy = l$), we have

$$\varphi_l(x, y) = \delta^e\ (\delta')^o\ \delta^p_{\circ}\ \mu^q\ (\mu')^r\ \pi$$

where $\pi \in \widetilde{S}_l$ is as defined in Definition 4.4.7.

If $l = n$ then $\varphi_n(v_i, i(v_j)) = \text{id}$ for all $v_i, v_j \in h_n(DPB(n))$ and then, from Proposition 4.6.7, we have

$$\Phi_{(l,\lambda)}(v_i \otimes C^{\lambda}_{\mathbf{t}}), i(v_j) \otimes C^{\lambda}_{\mathbf{s}}) = \phi_{(l,\lambda)}(C^{\lambda}_{\mathbf{t}}, C^{\lambda}_{\mathbf{s}})$$

for $C^{\lambda}_{\mathbf{t}}, C^{\lambda}_{\mathbf{s}}$ basis elements of $S^{\lambda}$.

If $0 \leq l < n$ then there exist basis elements $v_i, v_j \in h_l(DPB(n))$ such that $v_i, v_j$ have $l$ propagating lines and $n - l$ decorated or undecorated isolated vertices (i.e. there are no arcs). In the case where the product of $v_i$ and $i(v_j)$ has $l$ propagating lines we have $\varphi_l(v_i, i(v_j)) = (\delta')^o\ \mu^q\ (\mu')^r$ id where $o$ is the number of undecorated isolated vertices meeting an undecorated isolated vertex, $q$ is the number of decorated isolated vertices meeting an undecorated isolated vertex and $r$ is the number of decorated isolated vertices meeting a decorated isolated vertex. (Note that we do not get a parameter $\delta$ or $\delta_\circ$ as there are no arcs so no loops can form, also the permutation is the identity as no arcs can swap the propagating lines.)

Now pick $v_1$ with $n-l$ undecorated isolated vertices, then $\varphi_l(v_1, i(v_1)) = (\delta')^{n-l}$ id and this is non-zero if and only if $\delta' \neq 0$. Similarly pick $v_2$ with $n-l$ decorated isolated vertices, then $\varphi_l(v_2, i(v_2)) = (\mu')^{n-l}$ id and this is non-zero if and only if $\mu' \neq 0$. Also, we have (in the case where the product of $v_1$ and $i(v_2)$ has $l$ propagating lines) $\varphi_l(v_1, i(v_2)) = \varphi_l(v_2, i(v_1)) = \mu^{n-l}$ id and this is non-zero if and only if $\mu \neq 0$. Thus overall $\varphi_l(v_i, i(v_j))$, $i, j \in \{1, 2\}$ is non-zero if and only if at least one of $\delta'$, $\mu$, $\mu'$ is non-zero.

Then, from proposition 4.6.7, we have for $l \geq 1$

$$\begin{aligned}
&\Phi_{(l,\lambda)}(v_i \otimes C^\lambda_{\mathbf{t}}, i(v_j) \otimes C^\lambda_{\mathbf{s}}) \\
&= \phi_{(l,\lambda)}(C^\lambda_{\mathbf{t}},\ \varphi_l(v_i, i(v_j))\ C^\lambda_{\mathbf{s}}) \\
&= \phi_{(l,\lambda)}(C^\lambda_{\mathbf{t}},\ \alpha^{n-l} \text{ id } C^\lambda_{\mathbf{s}}) \\
&= \alpha^{n-l}\ \phi_{(l,\lambda)}(C^\lambda_{\mathbf{t}}, C^\lambda_{\mathbf{s}})
\end{aligned}$$

where $C^\lambda_{\mathbf{t}}, C^\lambda_{\mathbf{s}}$ basis elements of the Specht module $S^\lambda$ of $K\widetilde{S}_l$ and

$$\alpha = \begin{cases} \delta', & \text{if } \ v_i = v_j = v_1 \\ \mu' & \text{if } \ v_i = v_j = v_2 \\ \mu, & \text{if } v_i = v_1 \text{ and } v_j = v_2 \text{ or vice versa.} \end{cases}$$

Since at least one of $\delta', \mu$ or $\mu'$ is non-zero this formula implies that $\Phi_{(l,\lambda)} \neq 0$ if its corresponding linear form $\phi_{(l,\lambda)}$ for cellular algebra $K\widetilde{S}_l$ is non-zero.
Conversely, if $\phi_{(l,\lambda)}(C^\lambda_{\mathbf{t}}, C^\lambda_{\mathbf{s}}) = 0$ for all $\mathbf{s}, \mathbf{t} \in \text{Std}(\lambda)$ then for all $\pi \in \widetilde{S}_l$, $\phi_{(l,\lambda)}(C^\lambda_{\mathbf{t}}, \pi C^\lambda_{\mathbf{s}}) = 0$. Then from Proposition 4.6.7 we have $\Phi_{(l,\lambda)} = 0$.

Note that, if $l = 0$ then $K\widetilde{S}_0$ is interpreted to be $K$ and then

$$i(V_0) \otimes V_0 \otimes K\widetilde{S}_0 \simeq i(V_0) \otimes V_0 \otimes K \simeq i(V_0) \otimes V_0$$

(since $i(V_0)$ and $V_0$ are $K$-modules). So, from the multiplication method of $i(V_l) \otimes V_l \otimes K\widetilde{S}_l$, for $l = 0$ and any elements $a$, $b$, $c$, $d \in h_0(DPB(n))$ we have

$$(i(a) \otimes b \otimes 1)(i(c) \otimes d \otimes 1) = \varphi_0(b, i(c))(i(a) \otimes d \otimes 1)$$

where $\varphi_0(b, i(c)) = \delta^e\ (\delta')^0\ \delta_\circ^p\ \mu^q\ (\mu')^r$ (note that there are no permutations as there are no propagating lines). Therefore, for $l = 0$, we have $\Phi_{(0,\lambda)} \neq 0$ when $\varphi_0 \neq 0$. Since at least one of the elements $\delta'$, $\mu$ or $\mu'$ is non-zero so $\varphi_0 \neq 0$, where $\varphi_0(v_1, i(v_1)) = (\delta')^n$, $\varphi_0(v_2, i(v_2)) = (\mu')^n$ and $\varphi_0(v_1, i(v_2)) = \mu^n$. Therefore, $\Phi_{(0,\lambda)} \neq 0$ when at least one of $\delta'$, $\mu$, $\mu'$ is non-zero. □

## 4.7 The indexing set of the simple modules for $D\mathcal{P}\mathfrak{B}_n$

In this section we give the indexing set of the simple modules when $K$ is a field of characteristic $p$, $p \neq 2$, using a result of Dipper and James for simple modules of $K(\mathbb{Z}_2 \wr S_n)$.

Recall that a partition $\lambda$ is $p$-restricted if $\lambda_i - \lambda_{i+1} < p$ ($p \neq 0$) for all $i$; if $p = 0$, then all partitions are $p$-restricted. The bipartition $\lambda = (\lambda_1, \lambda_2)$ is said to be $p$-restricted, $p \neq 2$, when both $\lambda_1$ and $\lambda_2$ are $p$-restricted.

**Theorem 4.7.1.** *Let $D\mathcal{P}\mathfrak{B}_n(\delta, \delta_\circ, \delta', \mu, \mu')$ be the decorated partial Brauer algebra over a field $K$ of characteristic $p$, $p \neq 2$ (possibly $p = 0$). If at least one of the elements $\delta', \mu$ or $\mu'$ is non-zero then the non-isomorphic simple modules are indexed by*

$$\{(l, \lambda)\ \mid\ 0 \leq l \leq n, \lambda \text{ is a } p\text{-restricted bipartition of } l\}.$$

*Proof.* Since from Theorem 4.5.1 $D\mathcal{P}\mathfrak{B}_n(\delta, \delta_\circ, \delta', \mu, \mu')$ is cellular and from Theorem (3.4) in [6] the simple $D\mathcal{P}\mathfrak{B}_n(\delta, \delta_\circ, \delta', \mu, \mu')$-modules are indexed by $\{(l, \lambda) \in \Lambda(n)\ \mid\ \Phi_{(l,\lambda)} \neq 0\}$. If $l \neq 0$ then it follows from Lemma 4.6.8, that $\Phi_{(l,\lambda)} \neq 0$ if and only if the corresponding linear form $\phi_{(l,\lambda)}$ for the cellular algebra

$K\widetilde{S}_l$ is non-zero. By using the result of Dipper and James (Theorem 5.3 in [3]) which states that $\phi_{(l,\lambda)} \neq 0$ if and only if $\lambda$ is a $p$-restricted bipartition of $l$, then we have $\Phi_{(l,\lambda)} \neq 0$ if and only if $\lambda$ is a $p$-restricted bipartition of $l$. If $l = 0$ then from the proof of Lemma 4.6.8, $\Phi_{(0,\lambda)} \neq 0$ when at least one of the elements $\delta', \mu$ or $\mu'$ is non-zero. This completes the proof of the Theorem. □

# Chapter 5

# Criteria for the decorated partial Brauer algebra to be quasi-hereditary

In this chapter we determine when the decorated partial Brauer algebra is quasi-hereditary.

## 5.1 Preparatory definitions

In this section we recall the definition of a quasi-hereditary algebra and some results which will be used to prove our main result.

**Definition 5.1.1.** [13] Let $K$ be a field and $A$ a $K$-algebra. An ideal $J$ in $A$ is called a *hereditary ideal* if $J$ is idempotent ($J^2 = J$), $J(\mathrm{rad}A)J = 0$ and $J$ is a projective left (or right) $A$-module. The algebra $A$ is called *quasi-hereditary* provided there is a finite chain $0 = J_1 \subset J_2 \subset \cdots \subset J_n = A$ of ideals in $A$ such that $J_j / J_{j-1}$ is a hereditary ideal in $A / J_{j-1}$ for all $j$. Such a chain is then called a heredity chain of the quasi-hereditary algebra $A$.

**Lemma 5.1.2.** *[13, Lemma 2.1] Let $A$ be a cellular algebra with involution $i$ and cell chain $0 = J_0 \subset J_1 \subset \cdots \subset J_n = A$. Then the following are equivalent.*

1. *The given cell chain of $A$ is a heredity chain (making $A$ into a quasi-hereditary algebra).*

2. *All $J_l$ satisfy $J_l^2 \nsubseteq J_{l-1}$.*

3. *$n$ equals the number of isomorphism classes of simple modules.*

The following gives the different kinds of cell ideals.

**Proposition 5.1.3.** *[11, Proposition 4.1] Let $A$ be a $K$-algebra ($K$ is any field) with an involution $i$ and $J$ a cell ideal. Then $J$ satisfies one of the following conditions:*

**(a)** *$J$ has square zero.*

**(b)** *There exists a primitive idempotent $e$ in $A$ such that $J$ is generated by $e$ as a two-sided ideal (i.e. $J = AeA$). In particular, $J^2 = J$. Moreover, $eAe$ equals $Ke \simeq K$, and multiplication in $A$ provides an isomorphism of $A$-bimodules $Ae \otimes_K eA \simeq J$. In other words $J$ is a heredity ideal.*

**Theorem 5.1.4.** *[3, Theorem 5.5] Let $K$ be a field of characteristic $p$, $p \neq 2$, then $K(\mathbb{Z}_2 \wr S_n)$ is semi-simple if and only if $p = 0$ or $p > n$.*

## 5.2 The main result

In the following we state when the decorated partial Brauer algebra over a field is quasi-hereditary.

**Theorem 5.2.1.** *Let $K$ be a field of characteristic $p$, $p \neq 2$, $\delta$, $\delta_\circ$, $\delta'$, $\mu$ and $\mu'$ are elements in $K$. Then the decorated partial Brauer algebra $D\mathcal{PB}_n(\delta, \delta_\circ, \delta', \mu, \mu')$ is quasi-hereditary if and only if*

**(i)** *at least one of the elements $\delta'$, $\mu$ or $\mu'$ is non-zero, and*

**(ii)** *$p$ is zero or strictly bigger than $n$.*

*Proof.* By Theorem 4.5.1, $D\mathcal{P}\mathfrak{B}_n(\delta,\delta_\circ,\delta',\mu,\mu') = \bigoplus_{j=0}^{n} i(V_j)\otimes V_j\otimes K\widetilde{S}_j$ is a cellular algebra with cell chain

$$W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_{\sum_{k=1}^{n} s_k}$$

where

$$W_j = \begin{cases} i(V_0)\otimes V_0\otimes K\widetilde{S}_0, & \text{if} \quad j=0. \\ \left(\bigoplus_{k=0}^{m-1} i(V_k)\otimes V_k\otimes K\widetilde{S_k}\right) \oplus i(V_m)\otimes V_m\otimes A^{\lambda^{(m)}_{j-\sum_{k=1}^{m-1}s_k}}, & \\ & \text{if} \quad \sum_{k=1}^{m-1} s_k < j \le \sum_{k=1}^{m} s_k. \end{cases}$$

and for all $1\le l\le n$, $1\le r\le s_l$, $\lambda_r^{(l)}$ is a bipartition of $l$ with $\lambda_1^{(l)} \rhd \lambda_2^{(l)} \rhd \cdots \rhd \lambda_{s_l}^{(l)}$. Also, $A^{\lambda_r^{(l)}} = \text{Span}\,\{C^{\mu}_{\mathbf{st}} \mid \mathbf{s},\mathbf{t}\in \text{Std}\ \mu,\ \mu \text{ is a bipartition of } l,\ \mu \unrhd \lambda_r^{(l)}\}$ is an ideal of $K\widetilde{S}_l$ and the chain

$$A^{\lambda_1^{(l)}} \subset A^{\lambda_2^{(l)}} \subset \cdots \subset A^{\lambda_{s_l}^{(l)}} = K\widetilde{S}_l$$

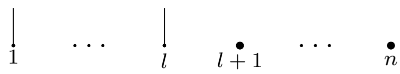
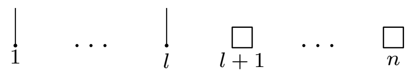
is a cell chain for the cellular algebra $K\widetilde{S}_l$.

To prove that the given cell chain of $D\mathcal{P}\mathfrak{B}_n(\delta,\delta_\circ,\delta',\mu,\mu')$ is a heredity chain (and $D\mathcal{P}\mathfrak{B}_n(\delta,\delta_\circ,\delta',\mu,\mu')$ is quasi-hereditary) we need to show that for all $0\le l\le n$ the square of $i(V_l)\otimes V_l\otimes B^{\lambda_r^{(l)}}$ (which is a cell ideal in the cell chain of $D\mathcal{P}\mathfrak{B}_n(\delta,\delta_\circ,\delta',\mu,\mu')$) is non-zero (Proposition 5.1.3), where $B^{\lambda_1^{(l)}} = A^{\lambda_1^{(l)}}$, $B^{\lambda_r^{(l)}} = A^{\lambda_r^{(l)}}\big/A^{\lambda_{r-1}^{(l)}}$, $1<r\le s_l$ is a subquotient in the cell chain of $K\widetilde{S}_l$. If the characteristic of $K$ ($p\neq 2$) is zero or strictly bigger than $n$ then by Theorem 5.1.4 $K\widetilde{S}_l$ is semi-simple for all $l\le n$. Let

$$\{\check{C}^{\lambda_r^{(l)}}_{\mathbf{uv}} = C^{\lambda_r^{(l)}}_{\mathbf{uv}} + A^{\lambda_{r-1}^{(l)}} \mid \mathbf{u},\mathbf{v}\in \text{Std}\ \lambda_r^{(l)},\ \lambda_r^{(l)} \text{ is a bipartion of } l\}$$

be a basis of $B^{\lambda_r^{(l)}}$. Then there are basis elements $\check{C}^{\lambda_r^{(l)}}_{\mathbf{uv}}$ and $\check{C}^{\lambda_r^{(l)}}_{\mathbf{u'v'}}$ such that their product is non-zero (since $K\widetilde{S}_l$ is semi-simple so (from Theorem (3.8) in [6]) the bilinear form $\phi_{(l,\lambda_r^{(l)})}\neq 0$). Since, for $l=n$, $i(V_n)\otimes V_n\otimes K\widetilde{S_n}\cong K\widetilde{S_n}$ the square of $i(V_n)\otimes V_n\otimes B^{\lambda_r^{(n)}}$, $1\le r\le s_n$ is not zero. Also, for all $0\le l<n$, there exists basis elements $v_1,v_2\in h_l(DPB(n))$ where $v_1$ has $l$ propagating lines and $n-l$ undecorated

isolated vertices, $v_2$ has $l$ propagating lines and $n-l$ decorated isolated vertices. For instance,

$v_1 =$  and $v_2 =$ 

Now consider the following products:

$$\begin{aligned} z_1 &:= (i(v_1)\otimes v_1 \otimes \check{C}^{\lambda_r^{(l)}}_{\mathbf{uv}})(i(v_1)\otimes v_1 \otimes \check{C}^{\lambda_r^{(l)}}_{\mathbf{u'v'}}) \\ &\equiv i(v_1)\otimes v_1 \otimes \check{C}^{\lambda_r^{(l)}}_{\mathbf{uv}}\,\varphi_l\big(v_1, i(v_1)\big)\check{C}^{\lambda_r^{(l)}}_{\mathbf{u'v'}} + \text{ lower terms} \\ &\equiv (\delta')^{n-l}\, i(v_1)\otimes v_1 \otimes \check{C}^{\lambda_r^{(l)}}_{\mathbf{uv}}\check{C}^{\lambda_r^{(l)}}_{\mathbf{u'v'}} + \text{ lower terms.} \end{aligned}$$

$$\begin{aligned} z_2 &:= (i(v_2)\otimes v_2 \otimes \check{C}^{\lambda_r^{(l)}}_{\mathbf{uv}})(i(v_2)\otimes v_2 \otimes \check{C}^{\lambda_r^{(l)}}_{\mathbf{u'v'}}) \\ &\equiv i(v_2)\otimes v_2 \otimes \check{C}^{\lambda_r^{(l)}}_{\mathbf{uv}}\,\varphi_l\big(v_2, i(v_2)\big)\check{C}^{\lambda_r^{(l)}}_{\mathbf{u'v'}} + \text{ lower terms} \\ &\equiv (\mu')^{n-l}\, i(v_2)\otimes v_2 \otimes \check{C}^{\lambda_r^{(l)}}_{\mathbf{uv}}\check{C}^{\lambda_r^{(l)}}_{\mathbf{u'v'}} + \text{ lower terms.} \end{aligned}$$

and

$$\begin{aligned} z_3 &:= (i(v_1)\otimes v_1 \otimes \check{C}^{\lambda_r^{(l)}}_{\mathbf{uv}})(i(v_2)\otimes v_2 \otimes \check{C}^{\lambda_r^{(l)}}_{\mathbf{u'v'}}) \\ &\equiv i(v_1)\otimes v_2 \otimes \check{C}^{\lambda_r^{(l)}}_{\mathbf{uv}}\,\varphi_l\big(v_1, i(v_2)\big)\check{C}^{\lambda_r^{(l)}}_{\mathbf{u'v'}} + \text{ lower terms} \\ &\equiv (\mu)^{n-l}\, i(v_1)\otimes v_2 \otimes \check{C}^{\lambda_r^{(l)}}_{\mathbf{uv}}\check{C}^{\lambda_r^{(l)}}_{\mathbf{u'v'}} + \text{ lower terms.} \end{aligned}$$

since at least one of the elements $\delta'$, $\mu$ or $\mu'$ is non-zero then at least one of the elements $z_1$,$z_2$ or $z_3$ is non-zero.

Conversely, for $J_{n-1} = \bigoplus_{k=0}^{n-1} i(V_k)\otimes V_k \otimes K\widetilde{S_k}$ which is an ideal in the cell chain of $D\mathcal{P}\mathfrak{B}_n(\delta,\delta_\circ,\delta',\mu,\mu')$, the quotient $D\mathcal{P}\mathfrak{B}_n(\delta,\delta_\circ,\delta',\mu,\mu')\big/J_{n-1} \simeq K\widetilde{S_n}$. Let

$$\Lambda = \{\lambda \mid \lambda \text{ is a bipartion of } n\}$$

and

$$\Lambda_\circ = \{\lambda \in \Lambda \mid \phi_\lambda \neq 0, \phi_\lambda \text{ is a bilinear form on cell modules of } K\widetilde{S_n}\}.$$

If $2 < p \leq n$ this means there exists $\mu \in \Lambda$ which is not $p$-restricted so $\phi_\mu = 0$ meaning that $\Lambda_\circ$ is strictly contained in $\Lambda$ but

$$\begin{aligned}|\Lambda| &= \text{the number of cell modules of } K\widetilde{S_n}\\ &= \text{the length of a cell chain of ideals of } K\widetilde{S_n}.\end{aligned}$$

So the length of a cell chain of $K\widetilde{S_n}$ and hence also that of $D\mathcal{P}\mathfrak{B}_n(\delta, \delta_\circ, \delta', \mu, \mu')$ is strictly bigger than the number of simple modules. Then by Lemma 5.1.2 the cell chain is not a hereditary chain and hence $D\mathcal{P}\mathfrak{B}_n(\delta, \delta_\circ, \delta', \mu, \mu')$ can not be quasi-hereditary.

Also, if $\delta' = \mu = \mu' = 0$, then the cell chain of $D\mathcal{P}\mathfrak{B}_n(\delta, \delta_\circ, \delta', \mu, \mu')$ contains a nilpotent ideal, $i(V_{n-1}) \otimes V_{n-1} \otimes K\widetilde{S_{n-1}}$, since any element $v_i \in h_{n-1}(DPB(n))$ has $n-1$ propagating lines and 1 decorated or undecorated isolated vertex then for any elements $v_i, v_j \in h_{n-1}(DPB(n))$ (in the case that the product of $v_i$ and $i(v_j)$ has $n-1$ propagating lines) we have

$$\varphi_{n-1}(v_i, i(v_j)) = \begin{cases} \delta' \text{ id}, & \text{if } v_i = v_j \text{ has one undecorated isolated vertex .} \\ \mu \text{ id}, & \text{if one of the elements } v_i, v_j \text{ has one decorated isolated vertex and the other has undecorated isolated vertex where these two isolated vertices meet together.} \\ \mu' \text{ id}, & \text{if } v_i = v_j \text{ has one decorated isolated vertex.} \end{cases}$$

So if $\delta' = \mu = \mu' = 0$ then $\varphi_{n-1}(v_i, i(v_j)) = 0$ for all $v_i, v_j \in h_{n-1}(DPB(n))$. Therefore the product of any basis elements of $i(V_{n-1}) \otimes V_{n-1} \otimes K\widetilde{S_{n-1}}$ is

$$\begin{aligned}&(i(v_1) \otimes v_2 \otimes C^\lambda_{\mathbf{st}})(i(v_3) \otimes v_4 \otimes C^\lambda_{\mathbf{s't'}})\\ &\equiv i(v_1) \otimes v_4 \otimes C^\lambda_{\mathbf{st}}\ \varphi_{n-1}\big(v_2, i(v_3)\big)\ C^\lambda_{\mathbf{s't'}} && \pmod{J_{n-2}}\\ &\equiv 0 && \pmod{J_{n-2}}.\end{aligned}$$

where $C^\lambda_{\mathbf{st}}, C^\lambda_{\mathbf{s't'}}$ are basis elements of $K\widetilde{S_{n-1}}$, $\lambda$ is a bipartition of $n-1$, then $(i(V_{n-1}) \otimes V_{n-1} \otimes K\widetilde{S_{n-1}})^2 = 0$ and hence $D\mathcal{P}\mathfrak{B}_n(\delta, \delta_\circ, \delta', \mu, \mu')$ can not be quasi-hereditary. $\square$

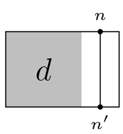
# Chapter 6

# Restriction rules for the cell modules

For $n \geq 1$, we can identify $D\mathcal{P}\mathfrak{B}_{n-1}(\delta, \delta_\circ, \delta', \mu, \mu')$ with a subalgebra of $D\mathcal{P}\mathfrak{B}_n(\delta, \delta_\circ, \delta', \mu, \mu')$ via an injective homomorphism

$$\mathfrak{i} : D\mathcal{P}\mathfrak{B}_{n-1}(\delta, \delta_\circ, \delta', \mu, \mu') \to D\mathcal{P}\mathfrak{B}_n(\delta, \delta_\circ, \delta', \mu, \mu')$$

which takes a diagram $d \in D\mathcal{P}\mathfrak{B}_{n-1}(\delta, \delta_\circ, \delta', \mu, \mu')$ to the diagram $\mathfrak{i}(d) \in D\mathcal{P}\mathfrak{B}_n(\delta, \delta_\circ, \delta', \mu, \mu')$ obtained by adding extra vertices $n$ and $n'$ to the right side of $d$ and joining them by an undecorated propagating line.

$\mathfrak{i}(d) =$ 


We can therefore consider the restriction of any $D\mathcal{P}\mathfrak{B}_n$-module to the subalgebra $D\mathcal{P}\mathfrak{B}_{n-1}$ to obtain a $D\mathcal{P}\mathfrak{B}_{n-1}$-module.

The aim of this chapter is to describe the restriction rule of the $D\mathcal{P}\mathfrak{B}_n(\delta, \delta_\circ, \delta', \mu, \mu')$ cell modules over $\mathbb{C}$ (Theorem 6.3.1). Throughout this chapter let $R = \mathbb{C}$.

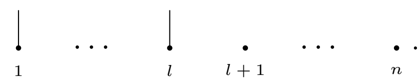
To prove this result we define $R$-submodules $\Delta_n^j(l, \lambda)$, $j = 1, 2, 3, 4$ of the cell module $\Delta_n(l, \lambda)$ (which is defined in Corollary 4.5.2). In section one we show that $\Delta_n^j(l, \lambda) \cong$

$\Delta_{n-1}(l,\lambda)$ for $j=1,2$ and $\Delta_n^3(l,\lambda) \cong \bigoplus_{\mu\to\lambda} \Delta_{n-1}(l-1,\mu)$. Section two is devoted to showing that $\frac{\Delta_n(l,\lambda)}{\bigoplus_{j=1}^{3} \Delta_n^j(l,\lambda)} \cong \bigoplus_{\lambda\to\nu} \Delta_{n-1}(l+1,\nu)$. In section three we describe the restriction rules for the cell modules.

Note that, throughout this chapter we will abbreviate the notation $D\mathcal{P}\mathfrak{B}_n(\delta,\delta',\delta_\circ,\mu,\mu')$ to $D\mathcal{P}\mathfrak{B}_n$ where the parameters are clear.

## 6.1 The modules $\Delta_n^1(l,\lambda)$, $\Delta_n^2(l,\lambda)$ and $\Delta_n^3(l,\lambda)$

**Definition 6.1.1.** For $0 \le l \le n$, let $DPB^l(n)$ denote the set of all decorated partial Brauer diagrams with exactly $l$ propagating lines.
Let $\overline{DPB^l(n)}$ denote the set of all decorated partial Brauer diagrams that have exactly $l$ propagating lines and the vertices $(l+1)'$, …, $n'$ in the bottom row are fixed undecorated isolated vertices, i.e. $\overline{DPB^l(n)}$ is the set of all decorated partial Brauer diagrams with $l$ propagating lines and fixed bottom $u_l$, where

$u_l =$ 


Denote by $\overline{B_n^l}$ the $\mathbb{C}$-space with basis $\overline{DPB^l(n)}$.

There is a left $D\mathcal{P}\mathfrak{B}_n$ action on $\overline{B_n^l}$ where, if $a$ is any decorated partial Brauer diagram, $d \in \overline{DPB^l(n)}$ then the product $ad$ is either a diagram with $l$ propagating lines and the bottom row of $ad$ is $u_l$ or zero if $\#(ad) < l$. So $\overline{B_n^l}$ is a left $D\mathcal{P}\mathfrak{B}_n$-module. Also this module is a right $\mathbb{C}\widetilde{S}_l$-module by the action permuting the vertices $\{1',\dots,l'\}$ and / or changing the decoration on the first $l$ propagating lines.

Note that $\overline{B_n^l}$ is a $\mathbb{C}$-subspace of $B_l \cong i(V_l) \otimes V_l \otimes \mathbb{C}\widetilde{S}_l$ (which is defined in Lemmas 4.4.6, 4.4.9), spanned by basis elements of $B_l$ with fixed bottom $u_l$. So, from Lemma 4.4.9 we have $\overline{B_n^l} \cong i(V_l) \otimes u_l \otimes \mathbb{C}\widetilde{S}_l$ which has a basis

$$\{v \otimes u_l \otimes \pi \mid v \in i\big(h_l(DPB(n))\big),\ \pi \in \widetilde{S}_l\}.$$

Also, since $\overline{B_n^l}$ is a right $\mathbb{C}\widetilde{S}_l$-module any basis element $d$ in $\overline{B_n^l}$ can be written as

$$d = v \otimes u_l \otimes \pi = (v \otimes u_l \otimes \mathrm{id})\pi,$$

where $v \in i\big(h_l(DPB(n))\big)$ and $\pi \in \widetilde{S}_l$.

Now for any element $v \in i\big(h_l(DPB(n))\big)$, let $F(v)$ denote a unique diagram in $\overline{DPB^l(n)}$ with top $v$ and $l$ propagating lines that are not decorated and do not cross each other. So $F(v)$ can be written as

$$F(v) = v \otimes u_l \otimes \mathrm{id}.$$

Recall that, for a given bipartition $\lambda$ of $l$, the cell module of $D\mathcal{P}\mathfrak{B}_n$ (defined in Corollary 4.5.2) is

$$\Delta_n(l, \lambda) = i(V_l) \otimes S^\lambda,$$

where $S^\lambda$ is a cell module of $\mathbb{C}\widetilde{S}_l$, and the set

$$\{v \otimes x \ \mid \ v \in i\big(h_l(DPB(n))\big),\ x \text{ is a basis element of } \ S^\lambda\}$$

is a basis of $\Delta_n(l, \lambda)$.

**Remark 6.1.2.** Let $\lambda$ be a bipartition of $l$, $u$ an arbitrary element of $h_l(DPB(n))$. define $i(V_l)\otimes u \otimes S^\lambda$ to be the $\mathbb{C}$-submodule of $A^{(l,\lambda)}/\check{A}^{(l,\lambda)}$ with basis $\{v\otimes u\otimes x+\check{A}^{(l,\lambda)} \mid v \in i(h_l(DPB(n)))$, $x$ is a basis element of $S^\lambda\}$. Then, by Lemma 4.4.14, $i(V_l) \otimes u \otimes S^\lambda$ is a left $D\mathcal{P}\mathfrak{B}_n$-module and the action of any basis element $a \in DPB(n)$ on a basis element of $i(V_l)\otimes u\otimes S^\lambda$ is independent of $u$, that is, $i(V_l)\otimes u\otimes S^\lambda \cong i(V_l)\otimes w\otimes S^\lambda$ for any $u, w \in h_l(DPB(n))$. Then (see [16], pg 17) the cell module

$$\Delta_n(l, \lambda) = i(V_l) \otimes S^\lambda \cong i(V_l) \otimes u \otimes S^\lambda$$

via the map $v \otimes x \mapsto v \otimes u \otimes x + \check{A}^{(l,\lambda)}$ where $u$ is a fixed non-zero element of $h_l(DPB(n))$, $x$ is a basis element of $S^\lambda$.

Now since $\mathbb{C}\widetilde{S}_l \otimes_{\mathbb{C}\widetilde{S}_l} S^\lambda \cong S^\lambda$, we have a vector space isomorphism,

$$\begin{aligned}\Delta_n(l,\lambda) &\cong i(V_l) \otimes_{\mathbb{C}} u_l \otimes_{\mathbb{C}} S^\lambda \\ &\cong i(V_l) \otimes_{\mathbb{C}} u_l \otimes_{\mathbb{C}} \mathbb{C}\widetilde{S}_l \otimes_{\mathbb{C}\widetilde{S}_l} S^\lambda \\ &\cong \overline{B_n^l} \otimes_{\mathbb{C}\widetilde{S}_l} S^\lambda.\end{aligned}$$

Clearly the set

$$\{F(v) \otimes x \mid v \in i\big(h_l(DPB(n))\big),\ x \text{ is a basis element of } S^\lambda\}$$

spans $\overline{B_n^l} \otimes_{\mathbb{C}\widetilde{S}_l} S^\lambda$. The correspondence $F(v) \otimes x \leftrightarrow v \otimes u_l \otimes x + \check{A}^{(l,\lambda)}$ together with the isomorphism tell us it is a basis of $\overline{B_n^l} \otimes_{\mathbb{C}\widetilde{S}_l} S^\lambda$.

The action of $D\mathcal{P}\mathfrak{B}_n$ on this module is as follows (implied in 4.5.2).
Let $a$ be a basis element of $D\mathcal{P}\mathfrak{B}_n$, $\#(a) = k$ and $F(v) \otimes x$ a basis element of $\overline{B_n^l} \otimes_{\mathbb{C}\widetilde{S}_l} S^\lambda$.
The action of $a$ on $F(v) \otimes x$ is $aF(v) \otimes x$ which is zero if $\#(aF(v)) < l$. Otherwise, $\#(aF(v)) = l$.
Since $a$ is a basis element, from Lemma 4.4.9, it can be written as $a = i(z_1) \otimes z_2 \otimes \pi$ where $z_1, z_2 \in h_k(DPB(n))$, $\pi \in \widetilde{S_k}$. From Lemma 4.4.14, we have

$$\begin{aligned}aF(v) = (i(z_1) \otimes z_2 \otimes \pi)(v \otimes u_l \otimes \mathrm{id}) &= c \otimes u_l \otimes \sigma\mathrm{id} \\ &= (c \otimes u_l \otimes \mathrm{id})\sigma = F(c)\sigma\end{aligned}$$

where $c = (av) \in i(h_l(DPB(n)))$ is the top half diagram induced from concatenation of $a$ with the top of $F(v)$, and $\sigma \in \widetilde{S}_l$ is the permutation induced from this concatenation. So,

$$a(F(v) \otimes x) = aF(v) \otimes x = F(c)\sigma \otimes x = F(c) \otimes \sigma x.$$

Now consider the following partition of the set $h_l(DPB(n))$ of half diagrams with $l$ propagating lines

$$h_l(DPB(n)) = \bigcup_{j=1}^{4} W_j,$$

where

- $W_1$ (resp. $W_2$) is a subset of $h_l(DPB(n))$ such that the vertex $n$ in each diagram $v \in W_1$ (resp. $W_2$) is an undecorated (resp. decorated) isolated vertex.
- $W_3$ is a subset of $h_l(DPB(n))$ such that the vertex $n$ in each $v \in W_3$ belongs to a propagating line.
- $W_4$ is a subset of $h_l(DPB(n))$ such that the vertex $n$ in each $v \in W_4$ belongs to a decorated or an undecorated arc.

Note that (from the above description) the sets $W_1$ and $W_2$ are not empty in the case $l \leq n-1$, $W_3$ is not empty in the case $1 \leq l \leq n$ and the set $W_4$ is not empty only in the case $l \leq n-2$.

For each $j = 1, 2, 3, 4$, let $\Delta_n^j(l, \lambda)$ be a $\mathbb{C}$-subspace of $\Delta_n(l, \lambda)$ with basis

$$\{F(v) \otimes x \mid v \in i(W_j),\ x \text{ is a basis element of } S^\lambda\}.$$

From the embedding $\mathfrak{i}$ of $D\mathcal{P}\mathfrak{B}_{n-1}$ into $D\mathcal{P}\mathfrak{B}_n$, we have the following.

**Lemma 6.1.3.** *For $j = 1, 2, 3$. Each $\Delta_n^j(l, \lambda)$ is a $D\mathcal{P}\mathfrak{B}_{n-1}$-module.*

*Proof.* Let $a$ be a basis element in $D\mathcal{P}\mathfrak{B}_{n-1}$, when embedded in $D\mathcal{P}\mathfrak{B}_n$, this diagram has an undecorated propagating line that joins the vertex $n$ in its top row to the vertex $n'$ in its bottom row and denoted by $\mathfrak{i}(a)$. Thus, for any basis element $F(v) \otimes x$ with $v \in i(W_j)$, $j = 1, 2, 3$, the vertex $n$ in $v$ has not been affected by the action of $a$ (which is the action of $\mathfrak{i}(a)$). Meaning that the action of $a$ fixes the vertex $n$ which means $(\mathfrak{i}(a)v) \in i(W_j)$, $j = 1, 2, 3$. Then the action of $\mathfrak{i}(a)$ on $F(v) \otimes x$ is $\mathfrak{i}(a)F(v) \otimes x$ which is zero if $\#(\mathfrak{i}(a)F(v)) < l$. Otherwise

$$\mathfrak{i}(a)F(v) = F(c)\sigma,$$

where $c = \mathfrak{i}(a)v \in i(W_j)$, the top of $\mathfrak{i}(a)F(v)$, and $\sigma \in \widetilde{S}_l$. Therefore we have,

$$\mathfrak{i}(a)(F(v) \otimes x) = \mathfrak{i}(a)F(v) \otimes x = F(c)\sigma \otimes x = F(c) \otimes \sigma x,$$

where $c = \mathfrak{i}(a)v \in i(W_j)$, $j = 1, 2, 3$, $\sigma x$ is an element of $S^\lambda$. □

It is clear that, from the description of $W_j, j = 1, 2, 3$, $\Delta_n^i(l, \lambda) \cap \Delta_n^j(l, \lambda) = 0$ for $i \neq j, i, j = 1, 2, 3$.
So we can write $\Delta_n^1(l, \lambda) + \Delta_n^2(l, \lambda) + \Delta_n^3(l, \lambda) = \bigoplus_{j=1}^3 \Delta_n^j(l, \lambda)$.
Put $M := \bigoplus_{j=1}^3 \Delta_n^j(l, \lambda)$. The quotient $\frac{\Delta_n(l,\lambda)}{M}$ is a $D\mathcal{P}\mathfrak{B}_{n-1}$-module, where $\frac{\Delta_n(l,\lambda)}{M}$ has basis

$$\{(F(v) \otimes x) + M \ \mid \ v \in i(W_4), \ x \text{ is a basis element in } \ S^\lambda\}.$$

Note that $\frac{\Delta_n(l,\lambda)}{M}$ can only be non-zero if $l \leq n - 2$.
In the following we will analyze each $\Delta_n^i(l, \lambda)$. Firstly, we recall the following definition.

**Definition 6.1.4.** (1) Let $\lambda$ be a partition of $n$, the elements $(i, j)$ of a Young diagram $[\lambda]$ are called nodes. A node $(i, \lambda_i)$ is called a *removable* node of $\lambda$ if $\lambda_i > \lambda_{i+1}$. A node $(i, \lambda_i + 1)$ of $[\lambda] \cup \{(i, \lambda_i + 1)\}$ is called an *addable* node of $\lambda$ if $i = 1$ or $i > 1$ and $\lambda_i < \lambda_{i-1}$.
The removable (resp. addable) nodes are the nodes which can be removed from (resp. added to) the Young diagram $[\lambda]$ to produce a Young diagram with $n-1$ (resp. $n+1$) nodes.

(2) Let $\lambda$ be a bipartition of $n$, the elements $(i, j, k)$ of Young diagram $[\lambda]$ are called nodes. Let $\gamma = (i, j, k)$, we say that:
$\gamma$ is a *removable* node if the element $\mu$ such that $[\mu] = [\lambda] - \{\gamma\}$ is still a bipartition (of rank $n - 1$, i.e. $|\mu| = n - 1$).
$\gamma$ is an *addable* node if the element $\mu$ such that $[\mu] = [\lambda] \cup \{\gamma\}$ is still a bipartition (of rank $n + 1$).

We use the notation $\mu \to \lambda$ to mean $\mu$ is obtained from $\lambda$ by removing a removable node (or, equivalently, $\lambda$ is obtained from $\mu$ by adding an addable node).

Let $S^\lambda \downarrow^{\mathbb{C}\widetilde{S_n}}_{\mathbb{C}\widetilde{S_{n-1}}}$ denote the restriction of $S^\lambda$ from $\mathbb{C}\widetilde{S_n}$ to $\mathbb{C}\widetilde{S_{n-1}}$ and $S^\lambda \uparrow^{\mathbb{C}\widetilde{S_{n+1}}}_{\mathbb{C}\widetilde{S_n}}$ denote the induced representation of $S^\lambda$ from $\mathbb{C}\widetilde{S_n}$ to $\mathbb{C}\widetilde{S_{n+1}}$. Since $\mathbb{C}\widetilde{S_n}$ is semi-simple, we have [18],[19]

$$S^\lambda \downarrow^{\mathbb{C}\widetilde{S_n}}_{\mathbb{C}\widetilde{S_{n-1}}} = \bigoplus_{\mu\to\lambda} S^\mu \quad \text{and} \quad S^\lambda \uparrow^{\mathbb{C}\widetilde{S_{n+1}}}_{\mathbb{C}\widetilde{S_n}} = \bigoplus_{\lambda\to\nu} S^\nu.$$

**Proposition 6.1.5.** *The module $\Delta_n^3(l,\lambda)$ is isomorphic to $\bigoplus_{\mu\to\lambda}\Delta_{n-1}(l-1,\mu)$ as a $D\mathcal{P}\mathfrak{B}_{n-1}$-module.*

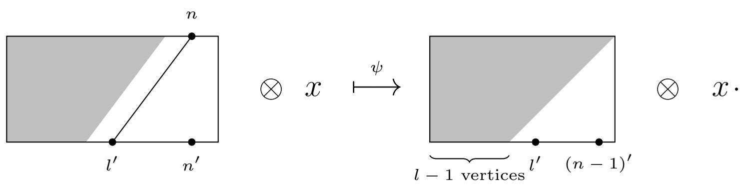
*Proof.* For any $v \in i(W_3)$, let $f_3(v)$ be a half diagram obtained from $v$ by removing the vertex $n$. Since in each $v \in i(W_3)$, the vertex $n$ has a propagating line, $f_3(v)$ has $(n-1)$ vertices and $(l-1)$ propagating lines so $f_3(v) \in i\big(h_{l-1}(DPB(n-1))\big)$. Note that, the map $f_3 : i(W_3) \longrightarrow i\big(h_{l-1}(DPB(n-1))\big)$ is a bijection between sets, since any half diagram $b \in i\big(h_{l-1}(DPB(n-1))\big)$ corresponds to a unique half diagram $b' \in i(W_3)$ that by adding a propagating line to the right of it. Write $f_3^{-1}(b) = b'$. By using $f_3$ we define the following $\mathbb{C}$-linear map

$$\psi : \Delta_n^3(l,\lambda) \longrightarrow \overline{B_{n-1}^{l-1}} \otimes_{\mathbb{C}\widetilde{S_{l-1}}} S^\lambda \downarrow^{\mathbb{C}\widetilde{S_n}}_{\mathbb{C}\widetilde{S_{n-1}}},$$

$$F(v)\otimes x \longmapsto F(f_3(v))\otimes x.$$

where $F(f_3(v)) = f_3(v)\otimes u_{l-1}\otimes \mathrm{id}$.

Note that the map $\psi$ takes a basis element $F(v)\otimes x$ of $\Delta_n^3(l,\lambda)$ to a basis element of $\overline{B_{n-1}^{l-1}} \otimes S^\lambda \downarrow^{\mathbb{C}\widetilde{S_n}}_{\mathbb{C}\widetilde{S_{n-1}}}$. It removes the vertices $n$, $n'$ together with the propagating line connecting $n$ with $l'$ and leaving everything else unchanged.

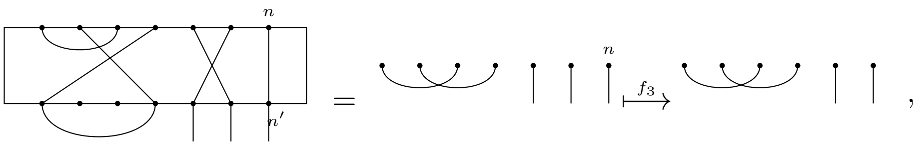


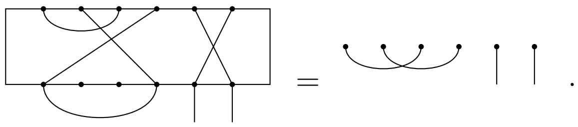

Since the map $f_3$ is a bijection, any basis element $F(b)\otimes x$ of $\overline{B_{n-1}^{l-1}} \otimes S_\lambda \downarrow$ ,where $F(b) = b\otimes u_{l-1}\otimes \mathrm{id}$, $b \in i\big(h_{l-1}(DPB(n-1))\big)$, has a unique $\psi$-preimage $F(f_3^{-1}(b))\otimes x$ where $f_3^{-1}(b)$ is obtained from $b$ by adding a vertex $n$ on the right-hand side of $b$ together with a propagating line on it. Therefore, $F(f_3^{-1}(b))\otimes x$ is obtained from $F(b)\otimes x$ by adding the vertices $n$, $n'$ and an undecorated propagating line that connects $n$ with $l'$ which is a basis element of $\Delta_n^3(l,\lambda)$. So $\psi$ is a bijection.

It remains to show that $\psi$ commutes with the action of $D\mathcal{P}\mathfrak{B}_{n-1}$.

Let $a$ be a basis element in $D\mathcal{P}\mathfrak{B}_{n-1}$. Since the action of a (which is $\mathfrak{i}(a)$) on $v \in i(W_3)$ fixes the vertex $n$ and the map $f_3$ only affects the vertex $n$ we have

$$af_3(v) = f_3(\mathfrak{i}(a)v)$$

$\mathfrak{i}(a)v =$ 


$af_3(v) =$ 

Now

$$\psi\big(a(F(v)\otimes x)\big) = \psi\big(aF(v)\otimes x\big) = \psi(\mathfrak{i}(a)F(v)\otimes x)$$
$$= \psi(F(c)\sigma\otimes x) = \psi(F(c)\otimes\sigma x) = F(f_3(c))\otimes\sigma x.$$

where $c = \mathfrak{i}(a)v = \operatorname{top}(\mathfrak{i}(a)F(v)) \in i(W_3)$ which is induced from concatenation $\mathfrak{i}(a)$ with $v$, since the action of $\mathfrak{i}(a)$ fixes $n$ so $c$ has a propagating line at $n$. The permutation $\sigma \in \widetilde{S_l}$, which is induced from the concatenation of $\mathfrak{i}(a)v$, permutes the vertices $\{1', \dots, (l-1)'\}$ and fixes $l'$. (Note that $\sigma$ can be identified with element of $\widetilde{S_{l-1}}$ since it permutes $l-1$ propagating lines.)

On the other hand,

$$\begin{aligned} a\big(\psi(F(v)\otimes x)\big) = a(F(f_3(v))\otimes x) &= aF(f_3(v))\otimes x \\ &= a(f_3(v)\otimes u_{l-1}\otimes \mathrm{id})\otimes x \\ &= (b\otimes u_{l-1}\otimes\sigma'\mathrm{id})\otimes x \\ &= (b\otimes u_{l-1}\otimes \mathrm{id})\otimes\sigma' x \\ &= F(b)\otimes\sigma' x \end{aligned}$$

where $b = af_3(v) = \text{top}(aF(f_3(v))) \in i\big(h_{l-1}(DPB(n-1))\big)$ which is induced from the concatenation of $af_3(v)$ and $\sigma' \in \widetilde{S_{l-1}}$ the permutation induced from the concatenation $af_3(v)$. But $af_3(v) = f_3(\mathfrak{i}(a)v) = f_3(c)$ then $F(b) = F(f_3(c))$ and $\sigma'$ is $\sigma$. Therefore,

$$\psi\big(a(F(v) \otimes x)\big) = a\big(\psi(F(v) \otimes x)\big).$$

Using the restriction rule for $\mathbb{C}\widetilde{S_l}$ to $\mathbb{C}\widetilde{S_{l-1}}$, we have

$$S^{\lambda} \downarrow^{\mathbb{C}\widetilde{S}_l}_{\mathbb{C}\widetilde{S}_{l-1}} \cong \bigoplus_{\mu \to \lambda} S^{\mu}$$

where the sum is over all bipartitions $\mu$ of $l-1$ that are obtained from $\lambda$ by removing one box. So,

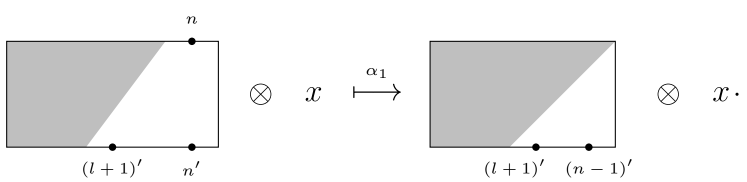
$$\Delta_n^3(l,\lambda) \cong \bigoplus_{\mu\to\lambda} \overline{B_{n-1}^{l-1}} \otimes_{\mathbb{C}\widetilde{S_{l-1}}} S^{\mu} \cong \bigoplus_{\mu\to\lambda} \Delta_{n-1}(l-1,\mu). \qquad \square$$

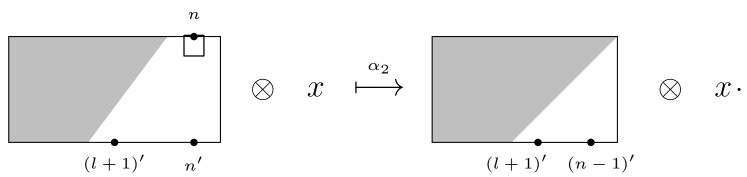
**Proposition 6.1.6.** *The module $\Delta_n^j(l,\lambda)$, $j = 1,2$ is isomorphic to $\Delta_{n-1}(l,\lambda)$ as an $DP\mathfrak{B}_{n-1}$-module.*

*Proof.* For $v \in i(W_1)$ (resp. $i(W_2)$), let $f_1(v)$ (resp. $f_2(v)$) be a half diagram obtained from $v$ by removing the vertex $n$ (resp. the vertex $n$ with its decoration). Note that $f_1(v),\ f_2(v) \in i\big(h_l(DPB(n-1))\big)$. The map $f_j$, $j = 1,2$ induces the following isomorphism of $DP\mathfrak{B}_{n-1}$-modules

$$\begin{aligned} \alpha_j : \Delta_n^j(l,\lambda) &\longrightarrow \overline{B_{n-1}^{l}} \otimes_{\mathbb{C}\widetilde{S}_l} S^{\lambda}, \\ F(v)\otimes x &\longmapsto F(f_j(v)) \otimes x, \qquad\qquad j = 1,2. \end{aligned}$$

These are illustrated below:






The proof that these are $D\mathcal{P}\mathfrak{B}_{n-1}$-module isomorphisms is similar to the proof of the previous Lemma 6.1.5. □

## 6.2 The quotient module $\frac{\Delta_n(l,\lambda)}{\oplus_{j=1}^{3}\Delta_n^j(l,\lambda)}$

This section is devoted to proving the following proposition:

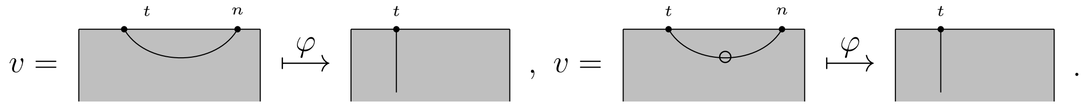
**Proposition 6.2.1.** *The $D\mathcal{P}\mathfrak{B}_{n-1}$-module $\frac{\Delta_n(l,\lambda)}{M}$ is isomorphic to*

$$\bigoplus_{\lambda\to\mu} \Delta_{n-1}(l+1,\mu).$$

The strategy of proof is as follows: Firstly, we define a map $\varphi$ from $i(W_4)$ to $i\big(h_{l+1}(DPB(n-1))\big)$. Then we define a map $f_4$ from $i(W_4)$ to $\overline{B_{n-1}^{l+1}}$ where $f_4(F(v)) = F(\varphi(v))\pi$, $v \in i(W_4)$, $\pi \in \widetilde{S_{l+1}}$. The map $f_4$ induces a linear map $\gamma$ from $\frac{\Delta_n(l,\lambda)}{M}$ to $\overline{B_{n-1}^{l+1}} \otimes_{\mathbb{C}\widetilde{S_{l+1}}} S^\lambda \uparrow$. We finally show that the map $\gamma$ is a $D\mathcal{P}\mathfrak{B}_{n-1}$-isomorphism in Lemmas 6.2.8 and 6.2.14.

We start with some definitions.

**Definition 6.2.2.** Let $\varphi : i(W_4) \longrightarrow i\big(h_{l+1}(DPB(n-1))\big)$ be a map defined as follows:
For $v \in i(W_4)$, let $\varphi(v)$ be the half diagram obtained from $v$ by removing the vertex $n$ together with its incident undecorated or decorated arc $\{t, n\}$ (say) and then adding a propagating line in the position of the vertex $t$. So the resulting half diagram, $\varphi(n)$, has $n-1$ vertices and $l+1$ propagating lines, which means that $\varphi(n) \in i\big(h_{l+1}(DPB(n-1))\big)$. This is illustrated below:

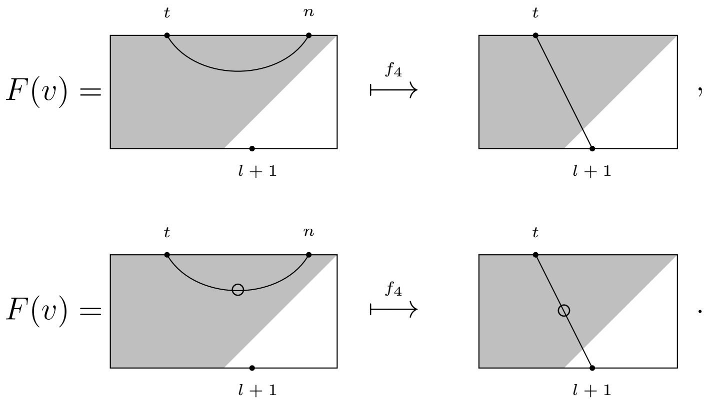



Recall, for $v \in i(W_4)$, $F(v) = v \otimes u_l \otimes \mathrm{id}$ is a diagram in $\overline{DPB^l(n)}$ with top $v$ and $l$ propagating lines that are not decorated and do not cross each other.

By using the map $\varphi$ we define the following:

**Definition 6.2.3.** For $v \in i(W_4)$, let $f_4(F(v))$ be the diagram obtained from $F(v)$ as follows: Firstly, remove the vertex $n$ together with its incident undecorated (resp. decorated) arc $\{t, n\}$ (say) and the bottom vertex $n'$. Next, connect the vertex $t$ in the top to the vertex $(l+1)'$ in the bottom by an undecorated (resp. a decorated) propagating line.

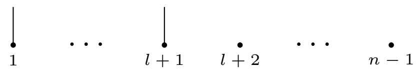
Note that the resulting diagram $f_4(F(v))$ has $n-1$ vertices in each row and $l+1$ propagating lines where the newly created propagating line $\{t, (l+1)'\}$ may be decorated, undecorated or may cross other propagating lines. This is illustrated below:




Therefore, $f_4(F(v))$ consists of top $\varphi(v)$, bottom $u_{l+1}$ and a permutation $\pi \in \widetilde{S_{l+1}}$ which gives any crossing or decoration for the new line. So $f_4(F(v))$ can be written as

$$f_4(F(v)) = \varphi(v) \otimes u_{l+1} \otimes \pi = (\varphi(v) \otimes u_{l+1} \otimes \mathrm{id})\pi = F(\varphi(v))\pi,$$

where $F(\varphi(v))$ is a diagram in $\overline{DPB^{l+1}(n-1)}$ with top $\varphi(v)$ and $l+1$ propagating lines that are not decorated and do not cross each other, and

$u_{l+1} =$ 

.

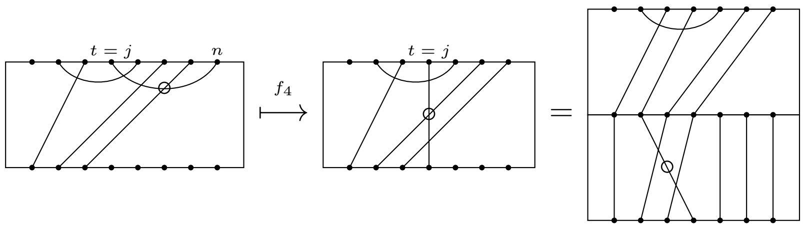
More formally, suppose that there are $j-1$ propagating lines to the left of $t$ in $f_4(F(v))$ so the newly created (decorated or undecorated) propagating line $\{t, (l+1)'\}$ will cross others and the remaining propagating lines are drawn so that they do not cross each other (note that the vertex $t$ is in the position of the $j^{th}$ propagating line so we put $t = j$). This means that the diagram

$$f_4(F(v)) = F(\varphi(v))\pi_{j,f}, \qquad 1 \leq j \leq l+1,$$

where $\pi_{j,f} = (f, \sigma_j)$, $\sigma_j = (j, l+1, l, \ldots, j+1)$ is a permutation in $S_{l+1}$ that maps $j$ to $(l+1)$ and then shifts the integers between $j+1$ and $l+1$ down by one, $f = (0, \ldots, 0, f(j), 0, \ldots, 0)$ where

$$f(j) = \begin{cases} 0, & \text{if the propagating line } \{j, (l+1)'\} \text{ is undecorated,} \\ 1, & \text{if the propagating line } \{j, (l+1)'\} \text{ is decorated.} \end{cases}$$

For example,

$F(v) =$ 

$= F(\varphi(v))\pi_{j,f}$ .

**Remark 6.2.4.** We view $S_l$ as a subgroup of $S_{l+1}$ via the embedding of $\{1, \ldots, l\} \subseteq \{1, \ldots, l, l+1\}$. This also induces an embedding of $\widetilde{S_l} \subseteq \widetilde{S_{l+1}}$ which is compatible with the natural embedding of $S_l \subseteq \widetilde{S_l}$ (via $\sigma \mapsto ((0, \ldots, 0), \sigma)$).

**Lemma 6.2.5.** *The set $\mathcal{T} = \{\pi_{j,f} \mid 1 \leq j \leq l+1,\ f(j) \in \{0,1\}\}$ forms a set of left coset representatives of $\widetilde{S_l}$ in $\widetilde{S_{l+1}}$, where $\pi_{j,f}$ is as in Definition 6.2.3.*

*Proof.* We first show that, for all $1 \leq j < k \leq l+1$, $\sigma_j S_l \neq \sigma_k S_l$.

For $1 \leq j < k \leq l+1$ we have,

$\sigma_j = (j, l+1, \ldots, k+1, k, k-1, \ldots, j+1)$ and $\sigma_k = (k, l+1, \ldots, k+1)$. So $\sigma_k^{-1} = (k, k+1, k+2, \ldots, l, l+1)$ and $\sigma_k^{-1}\sigma_j = (j, l+1, k-1, k-2, \ldots, j+1) \notin S_l$ because $\sigma_k^{-1}\sigma_j$ does not fix $l+1$. This implies that

$$\pi_{k,f}^{-1}\pi_{j,g} = ({}_{\sigma_k^{-1}}f, \sigma_k^{-1})(g, \sigma_j) = ({}_{\sigma_k^{-1}}f +_{\sigma_k^{-1}} g, \sigma_k^{-1}\sigma_j) \notin \widetilde{S}_l.$$

Then $\pi_{k,f}\widetilde{S}_l \neq \pi_{j,g}\widetilde{S}_l$ for all $1 \leq j < k \leq l+1$.

Now, note that since for $1 \leq j \leq l+1$, each $\pi_{j,f}$ has underlying permutation $\sigma_j$ with the propagating line $\{j, l+1\}$ either decorated or undecorated and $|\{\sigma_j = (j, l+1, \ldots, j+1),\ 1 \leq j \leq l+1\}| = l+1$, we have

$$|\mathcal{T}| = 2(l+1) = \frac{2^{l+1}(l+1)!}{2^l l!} = \frac{|\widetilde{S_{l+1}}|}{|\widetilde{S}_l|} = [\widetilde{S_{l+1}} : \widetilde{S}_l].$$

Hence the set $\{\pi_{j,f}\widetilde{S}_l \ \mid\ 1 \leq j \leq l+1,\ f(j) \in \{0,1\}\}$ is a set of left coset representations of $\widetilde{S}_l$ in $\widetilde{S_{l+1}}$. □

As a consequence of the previous lemma we have the following

**Corollary 6.2.6.** *Let $\pi_{j,f} \in \widetilde{S_{l+1}}$ be as in Definition 6.2.3, then*

$$\mathbb{C}\widetilde{S_{l+1}} = \oplus_{1 \leq j \leq l+1}\ \pi_{j,f}\ \mathbb{C}\widetilde{S}_l$$

*as a right $\mathbb{C}\widetilde{S}_l$-module.*

Then for a $\mathbb{C}\widetilde{S}_l$-module $S^\lambda$ we have

$$S^\lambda \uparrow_{\mathbb{C}\widetilde{S}_l}^{\mathbb{C}\widetilde{S_{l+1}}} = \mathbb{C}\widetilde{S_{l+1}} \otimes_{\mathbb{C}\widetilde{S}_l} S^\lambda = \oplus_{1 \leq j \leq l+1}(\pi_{j,f}\ \mathbb{C}\widetilde{S}_l \otimes_{\mathbb{C}\widetilde{S}_l} S^\lambda)$$

Since $\pi_{j,f}\mathbb{C}\widetilde{S}_l \otimes_{\mathbb{C}\widetilde{S}_l} S^\lambda \cong \pi_{j,f} \otimes_{\mathbb{C}\widetilde{S}_l} S^\lambda$ via $\mathbb{C}\widetilde{S}_l \otimes_{\mathbb{C}\widetilde{S}_l} S^\lambda \cong S^\lambda$, the set

$$\{\pi_{j,f} \otimes_{\mathbb{C}\widetilde{S}_l} x \ \mid\ \pi_{j,f} \in \mathcal{T},\ 1 \leq j \leq l+1,\ x \text{ is a basis element of } S^\lambda\}$$

is a basis of $S^\lambda \uparrow_{\mathbb{C}\widetilde{S_l}}^{\mathbb{C}\widetilde{S_{l+1}}}$.

Using the map $f_4$ we define a linear map from $\frac{\Delta_n(l,\lambda)}{M}$ to $\overline{B_{n-1}^{l+1}} \otimes_{\mathbb{C}\widetilde{S_{l+1}}} S^\lambda \uparrow$ (where $M = \bigoplus_{j=1}^{3} \Delta_n^j(l,\lambda))$ as follows:

**Definition 6.2.7.** Define

$$\gamma : \frac{\Delta_n(l,\lambda)}{M} \longrightarrow \overline{B_{n-1}^{l+1}} \otimes_{\mathbb{C}\widetilde{S_{l+1}}} S^\lambda \uparrow_{\mathbb{C}\widetilde{S_l}}^{\mathbb{C}\widetilde{S_{l+1}}},$$

$$F(v) \otimes x \longmapsto f_4(F(v))\pi_{j,f}^{-1} \otimes (\pi_{j,f} \otimes x).$$

where $v \in i(W_4)$ with arc $\{t,n\}$ and $j-1$ propagating lines on the left of $t$, the diagram $f_4(F(v)) = F(\varphi(v))\pi_{j,f}$ is as defined in Definition 6.2.3. Then extend $\gamma$ linearly to the whole $K$-module $\frac{\Delta_n(l,\lambda)}{M}$.

Note that $f_4(F(v))\pi_{j,f}^{-1} = F(\varphi(v)) = \varphi(v) \otimes u_{l+1} \otimes \mathrm{id}$ is a diagram in $\overline{B_{n-1}^{l+1}}$ with $l+1$ propagating lines that are not decorated and do not cross each other. Also, note that the set $\{F(b) \otimes (\pi_{j,f} \otimes x) \mid b \in i\big(h_{l+1}(DPB(n-1))\big),\ x$ is a basis element of $S^\lambda\}$ is a basis of $\overline{B_{n-1}^{l+1}} \otimes_{\mathbb{C}\widetilde{S_{l+1}}} S^\lambda \uparrow_{\mathbb{C}\widetilde{S_l}}^{\mathbb{C}\widetilde{S_{l+1}}}$.

In the following lemmas we show that $\gamma$ is a $D\mathcal{P}\mathfrak{B}_{n-1}$-isomorphism.

**Lemma 6.2.8.** *The map $\gamma$ is a bijection.*

*Proof.* We want to find the dimension of $\frac{\Delta_n(l,\lambda)}{M}$.
Note that since in any element of $W_4$ the vertex $n$ can be joined to any vertex from $1,\ldots,n-1$ by a decorated or undecorated arc, we obtain that

$$\begin{aligned}|W_4| &= 2(n-1)|h_l(DPB(n-2))| \\ &= 2(n-1).\sum_{k=0}^{\lfloor\frac{n-2-l}{2}\rfloor} \binom{n-2-2k}{l} \frac{(n-2)!}{(n-2-2k)!k!}\, 2^{n-2-(l+2k)} \text{ (by Lemma 4.3.2).}\end{aligned}$$

Therefore,

$$\begin{aligned}\dim \frac{\Delta_n(l,\lambda)}{M} &= |W_4|.\dim\ S^\lambda \\ &= 2(n-1).\sum_{k=0}^{\lfloor\frac{n-2-l}{2}\rfloor}\binom{n-2-2k}{l}\frac{(n-2)!}{(n-2-2k)!k!}2^{n-2-(l+2k)}.\dim\ S^\lambda.\end{aligned}$$

Now we find the dimension of $\overline{B_{n-1}^{l+1}}\otimes_{\mathbb{C}\widetilde{S_{l+1}}} S^\lambda\uparrow_{\mathbb{C}\widetilde{S}_l}^{\mathbb{C}\widetilde{S_{l+1}}}$. Since

$$\begin{aligned}\dim\ S^\lambda\uparrow_{\mathbb{C}\widetilde{S}_l}^{\mathbb{C}\widetilde{S_{l+1}}} &= \frac{|\widetilde{S_{l+1}}|}{|\widetilde{S}_l|}.\dim\ S^\lambda \\ &= \frac{2^{l+1}(l+1)!}{2^l l!}.\dim\ S^\lambda = 2(l+1).\dim\ S^\lambda.\end{aligned}$$

Then, by Lemma 4.3.2, we have

$$\begin{aligned}&\dim(\overline{B_{n-1}^{l+1}}\otimes_{\mathbb{C}\widetilde{S_{l+1}}} S^\lambda\uparrow_{\mathbb{C}\widetilde{S}_l}^{\mathbb{C}\widetilde{S_{l+1}}}) \\ &= |h_{l+1}(DPB(n-1))|.\dim S^\lambda\uparrow_{\mathbb{C}\widetilde{S}_l}^{\mathbb{C}\widetilde{S_{l+1}}} \\ &= \sum_{k=0}^{\lfloor\frac{n-1-(l+1)}{2}\rfloor}\binom{n-1-2k}{l+1}\frac{(n-1)!}{(n-1-2k)!k!}2^{n-1-(l+1+2k)}.2(l+1).\dim S^\lambda\end{aligned}$$

This equals

$$\begin{aligned}&2\sum_{k=0}^{\lfloor\frac{n-2-l}{2}\rfloor}\frac{(l+1)(n-1-2k)!}{(l+1)!(n-2-2k-l)!}\frac{(n-1)!}{(n-1-2k)!k!}2^{n-2-(l+2k)}.\dim S^\lambda \\ &= 2\sum_{k=0}^{\lfloor\frac{n-2-l}{2}\rfloor}(n-1-2k)\binom{n-2-2k}{l}\frac{(n-1)(n-2)!}{(n-1-2k)(n-2-2k)!k!}2^{n-2-(l+2k)}.\dim S^\lambda \\ &= 2(n-1)\sum_{k=0}^{\lfloor\frac{n-2-l}{2}\rfloor}\binom{n-2-2k}{l}\frac{(n-2)!}{(n-2-2k)!k!}2^{n-2-(l+2k)}.\dim S^\lambda.\end{aligned}$$

So $\dim\frac{\Delta_n(l,\lambda)}{M} = \dim(\overline{B_{n-1}^{l+1}}\otimes_{\mathbb{C}\widetilde{S_{l+1}}} S^\lambda\uparrow_{\mathbb{C}\widetilde{S}_l}^{\mathbb{C}\widetilde{S_{l+1}}})$.

Hence it suffices to check that $\gamma$ is onto.

Let $F(b)\otimes(\pi_{j,f}\otimes x)$ be a basis element in $\overline{B_{n-1}^{l+1}}\otimes S^\lambda\uparrow_{\mathbb{C}\widetilde{S}_l}^{\mathbb{C}\widetilde{S_{l+1}}}$, where $b\in i(h_{l+1}(DPB(n-1)))$ and $F(b)=b\otimes u_{l+1}\otimes \mathrm{id}$ a diagram in $\overline{B_{n-1}^{l+1}}$ with $l+1$ propagating lines that are not decorated and do not cross each other.

Choose $v \in i(W_4)$ such that $f_4(F(v)) = F(b)\pi_{j,f}$ (i.e. choose $v \in i(W_4)$ with arc $\{t, n\}$ and $j-1$ propagating lines to the left of $t$ and $\varphi(v) = b$). Thus

$$\begin{aligned}\gamma(F(v) \otimes x) &= f_4(F(v))\pi_{j,f}^{-1} \otimes (\pi_{j,f} \otimes x) \\ &= (F(b)\pi_{j,f})\pi_{j,f}^{-1} \otimes (\pi_{j,f} \otimes x) = F(b) \otimes (\pi_{j,f} \otimes x).\end{aligned}$$

So $\gamma$ is a bijection. □

It remains to show that $\gamma$ is a $D\mathcal{P}\mathfrak{B}_{n-1}$-homomorphism.

Using the definition of a linear map $\gamma$ we have the following:

**Lemma 6.2.9.** *Let $v \in i(W_4)$ with an arc $\{t, n\}$ and $j-1$ propagating lines on the left of $t$. For $\sigma \in \mathbb{C}\widetilde{S}_l$,*

$$\gamma(F(v) \otimes \sigma x) = f_4(F(v))\pi_{j,f}^{-1} \otimes (\pi_{j,f} \otimes \sigma x)$$

*where $x$ is a basis element of $S^\lambda$, $f_4$ and $\pi_{j,f}$ are as in definition 6.2.3.*

*Proof.* From Definition 2.5.11 of the Specht module $S^\lambda$ of $\mathbb{C}\widetilde{S}_l$, the set

$$\{C_t^\lambda \mid t \in \text{Std }(\lambda),\ \lambda \text{ is a bipartition of } l\}$$

is a basis of $S^\lambda$.

Since $x$ is a basis element of $S^\lambda$, put $x = C_t^\lambda$, $\sigma \in \mathbb{C}\widetilde{S}_l$, then from Definition 2.5.11, the action of $\sigma$ on $x$ is given by:

$$\sigma x = \sigma C_t^\lambda = \sum_{b \in \text{Std}(\lambda)} r_b C_b^\lambda.$$

Therefore,

$$\begin{aligned}F(v) \otimes \sigma x &= F(v) \otimes \sigma C_t^\lambda \\ &= F(v) \otimes \sum_{b \in \text{Std}(\lambda)} r_b C_b^\lambda = \sum_{b \in \text{Std}(\lambda)} r_b (F(v) \otimes C_b^\lambda).\end{aligned}$$

Then

$$\begin{aligned}
\gamma(F(v)\otimes\sigma x) &= \gamma\big(\sum_{b\in \mathrm{Std}(\lambda)} r_b(F(v)\otimes C_b^\lambda)\big) \\
&= \sum_{b\in \mathrm{Std}(\lambda)} r_b\gamma(F(v)\otimes C_b^\lambda) \qquad \text{(as } \gamma \text{ is linear)} \\
&= \sum_{b\in \mathrm{Std}(\lambda)} r_b\big(f_4(F(v))\pi_{j,f}^{-1}\otimes(\pi_{j,f}\otimes C_b^\lambda)\big) \\
&= f_4(F(v))\pi_{j,f}^{-1}\otimes(\pi_{j,f}\otimes \sum_{b\in \mathrm{Std}(\lambda)} r_b C_b^\lambda) \\
&= f_4(F(v))\pi_{j,f}^{-1}\otimes(\pi_{j,f}\otimes\sigma x). \qquad \square
\end{aligned}$$

The following lemmas show that the map $\gamma$ commutes with the generators $\{s_i, g_i, e_i, p_i, q_i\}$ of the $D\mathcal{P}\mathfrak{B}_{n-1}$.

**Lemma 6.2.10.**

$$\gamma\big(s_i(F(v)\otimes x)\big) = s_i\big(\gamma(F(v)\otimes x)\big)$$

*where $v\in i(W_4)$, with an arc $\{t,n\}$ and $j-1$ propagating lines on the left of $t$, $s_i$, $1\le i\le n-2$ are as in Lemma 3.3.11.*

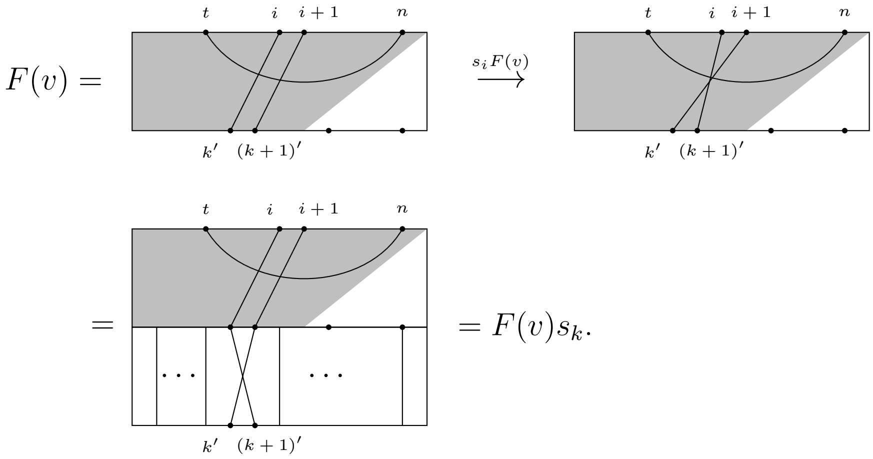
*Proof.* There are three cases to consider:

- Case $I$: Assume $t\neq i$, $i+1$. We distinguish the following two cases:

  1. Suppose $i$ and $i+1$ belong to propagating lines in $v$ then that also holds in $\varphi(v)$.
  
     Let $i$ be joined to $k'$ and $i+1$ joined to $(k+1)'$ by propagating lines in $F(v)$, $1\le k\le l-1$. Then the action of $s_i$ on $F(v)$ induces a permutation but does not change $v$. So we have

$$s_iF(v) = F(v)s_k.$$

This is illustrated below:




Then

$$
\begin{aligned}
\gamma\big(s_i(F(v)\otimes x)\big) &= \gamma(s_iF(v)\otimes x)\\
&= \gamma(F(v)s_k\otimes x)\\
&= \gamma(F(v)\otimes s_kx)\\
&= f_4(F(v))\pi_{j,f}^{-1}\otimes(\pi_{j,f}\otimes s_kx) \qquad \text{(from Lemma 6.2.9)}\\
&= F(\varphi(v))\otimes(\pi_{j,f}s_k\otimes x).
\end{aligned}
$$

Note that, if $i > t$ then $j \le k \le l-1$ and if $i+1 < t$ then $k+1 \le j-1$ (where $t$ is in the position of $j^{th}$ propagating line). Since $\pi_{j,f}$ shifted the integers between $j+1$ and $l+1$ down by one and fixed the lines on the left of $t$ we have

$$
\pi_{j,f}s_k = \begin{cases} s_{k+1}\pi_{j,f}, & \text{if } j \le k \le l-1;\\ s_k\pi_{j,f}, & \text{if } k+1 \le j-1. \end{cases}
$$

On the other hand, since $t$ is in the position of the $j^{th}$ propagating line in $\varphi(v)$, $F(\varphi(v))$ has $l+1$ propagating lines where the propagating lines to the right of $t$ have their bottom endpoints shifted up by one compared to $F(v)$. So we have the following:

If $i > t$ so in this case, the vertex $i$ in $F(\varphi(v))$ is joined to $(k+1)'$ and the vertex $i+1$ is joined to $(k+2)'$. Then we have

$$s_i F(\varphi(v)) = F(\varphi(v)) s_{k+1}.$$

Then

$$\begin{aligned} s_i\big(\gamma(F(v)\otimes x)\big) &= s_i\big(F(\varphi(v))\otimes(\pi_{j,f}\otimes x)\big) \\ &= F(\varphi(v))s_{k+1}\otimes(\pi_{j,f}\otimes x) \\ &= F(\varphi(v))\otimes(s_{k+1}\pi_{j,f}\otimes x) \\ &= F(\varphi(v))\otimes(\pi_{j,f}s_k\otimes x) = \gamma\big(s_i(F(v)\otimes x)\big). \end{aligned}$$

if $i+1<t$, then the vertex $i$ in $F(\varphi(v))$ is joined to $k'$ and $i+1$ is joined to $(k+1)'$. So

$$s_i F(\varphi(v)) = F(\varphi(v)) s_k$$

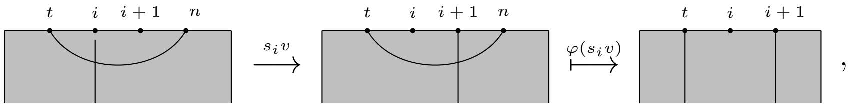
Therefore,

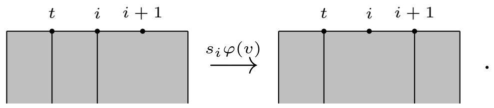
$$\begin{aligned} s_i\big(\gamma(F(v)\otimes x)\big) &= s_i\big(F(\varphi(v))\otimes(\pi_{j,f}\otimes x)\big) \\ &= F(\varphi(v))s_k\otimes(\pi_{j,f}\otimes x) \\ &= F(\varphi(v))\otimes(s_k\pi_{j,f}\otimes x) \\ &= F(\varphi(v))\otimes(\pi_{j,f}s_k\otimes x) = \gamma\big(s_i(F(v)\otimes x)\big). \end{aligned}$$

2. Suppose that one of the vertices $i$, $i+1$ or both of them does not belong to a propagating line in $v$ (i.e. one of the vertices $i, i+1$ belongs to a propagating line and the other is a decorated or undecorated isolated vertex (resp. the other belongs to an arc) or both of them are decorated or undecorated isolated vertex (resp. belong to an arc)) then that also holds in $\varphi(v)$. In this case the action of $s_i$ on $F(v)$ and also on $F(\varphi(v))$ does not introduce any permutation. Since $s_i$ does not affect the arc $\{t,n\}$ and the map $\varphi$ only affects the arc $\{t,n\}$ so we have

$$s_i\varphi(v) = \varphi(s_i v).$$

This is illustrated below:

$v =$ 


$\varphi(v) =$ 


(similarly for the other cases.)

Also, $s_iF(v) = F(c)$, where $c = s_iv \in i(W_4)$ and $s_iF(\varphi(v)) = F(s_i\varphi(v)) = F(\varphi(c))$. Then

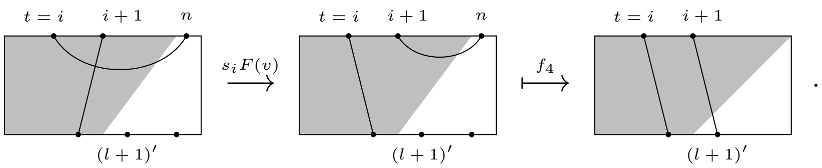
$$\begin{aligned}
\gamma\big(s_i(F(v)\otimes x)\big) &= \gamma(F(c)\otimes x)\\
&= F(\varphi(c))\otimes(\pi_{j,f}\otimes x)\\
&= s_iF(\varphi(v))\otimes(\pi_{j,f}\otimes x)\\
&= s_i\big(F(\varphi(v))\otimes(\pi_{j,f}\otimes x)\big) = s_i\big(\gamma(F(v)\otimes x)\big).
\end{aligned}$$

- Case II: Assume $t = i$, we have the following two cases:

1. If in $v$ the vertex $i+1$ belongs to a propagating line so it is also in $\varphi(v)$. In this case, the action of $s_i$ on $F(v)$ introduces a new arc $\{i+1, n\}$ and a propagating line in the position of $i$, and does not introduce any permutation. So

$$s_iF(v) = F(c), \tag{6.1}$$

where $c = s_iv \in i(W_4)$ is $v$ with a new arc $\{i+1, n\}$ and a propagating line in the position of $i$. Therefore in $f_4(F(c))$ the vertex $i+1$ is joined to $(l+1)'$.

$F(v) =$ 

Since there are $j-1$ propagating lines on the left of $t$ and in $f_4(F(c))$ the vertex $i = t$ belongs to a propagating line therefore there are $j$ propagating lines on the left of $i+1$ so (from the definition of $f_4$) we have

$$f_4(F(c)) = F(\varphi(c))\pi_{j+1,f}. \tag{6.2}$$

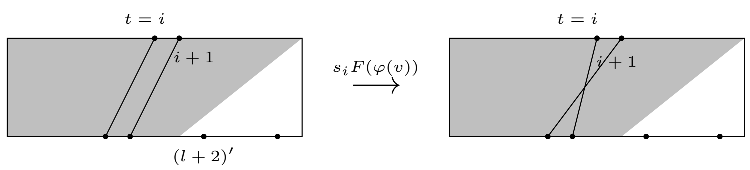
Then

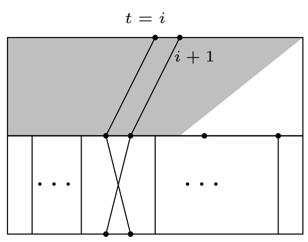
$$\begin{aligned}
\gamma\big(s_i(F(v)\otimes x)\big) &= \gamma(F(c)\otimes x) && \text{(from 6.1)}\\
&= f_4(F(c))\pi^{-1}_{j+1,f}\otimes(\pi_{j+1,f}\otimes x) && \text{(from 6.2)}\\
&= F(\varphi(c))\otimes(\pi_{j+1,f}\otimes x).
\end{aligned}$$

On the other hand, since $v$ has an arc $\{t, n\}$ then in $\varphi(v)$ there are propagating lines in the position of $t = i$ and $i+1$ (which are the positions of $j^{th}$ and $j^{th}+1$ propagating lines). So the action of $s_i$ on $F(\varphi(v))$ introduces a permutation $s_j$ and does not make any change in $\varphi(v)$ so we have,

$$s_iF(\varphi(v)) = F(\varphi(v))s_j. \tag{6.3}$$

This is illustrated bellow:

$F(\varphi(v)) =$ 


$=$ 

$= F(\varphi(v))s_j.$

Therefore,

$$\begin{aligned} s_i\big(\gamma(F(v)\otimes x)\big) &= s_i\big(F(\varphi(v))\otimes(\pi_{j,f}\otimes x)\big) \\ &= F(\varphi(v))s_j\otimes(\pi_{j,f}\otimes x) \qquad \text{(from 6.3)} \\ &= F(\varphi(v))\otimes(s_j\pi_{j,f}\otimes x), \end{aligned}$$

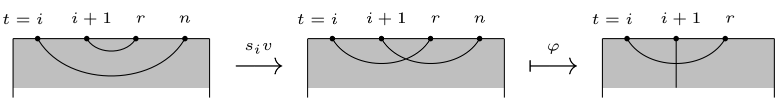
where $s_j\pi_{j,f} = \pi_{j+1,f}$, and since $c = s_iv$ is $v$ with a new arc $\{i+1,n\}$ and a propagating line in the position of $i$, then $\varphi(c)$ has propagating lines in the position of $i$ and $i+1$ which means $\varphi(c) = \varphi(v)$. Then

$$\gamma\big(s_i(F(v)\otimes x)\big) = s_i\big(\gamma(F(v)\otimes x)\big).$$

2. If in $v$ the vertex $i+1$ is joined by an arc to a vertex $r$ (say) (resp. is a decorated or an undecorated isolated vertex) so that also holds in $\varphi(v)$. The action of $s_i$ on $F(v)$ introduces new arcs $\{i+1,n\}$, $\{i,r\}$ (resp. a new arc $\{i+1,n\}$ and an isolated vertex in the position of $i$) and does not introduce any permutation. So

$$s_iF(v) = F(c).$$

where $c = s_iv \in i(W_4)$ is $v$ with new arcs $\{i+1,n\}$ and $\{i,r\}$ (resp. an isolated vertex in the position of $i$). So $\varphi(c)$ has a propagating line in the position of $i+1$.

$v =$  $= \varphi(c)$.


Since in $\varphi(v)$ the vertex $i+1$ is joined by an arc to a vertex $r$ (resp. isolated vertex) and the vertex $i$ belongs to a propagating line, the action of $s_i$ on $\varphi(v)$ is, $s_i\varphi(v) = \varphi(c)$ and then $s_iF(\varphi(v)) = F(\varphi(c))$.

$v =$  $= \varphi(c)$.

Therefore,

$$\begin{aligned}\gamma\big(s_i(F(v)\otimes x)\big) &= \gamma(F(c)\otimes x)\\ &= f_4(F(c))\pi_{j,f}^{-1}\otimes(\pi_{j,f}\otimes x)\\ &= F(\varphi(c))\otimes(\pi_{j,f}\otimes x)\\ &= s_iF(\varphi(v))\otimes(\pi_{j,f}\otimes x)\\ &= s_i\big(F(\varphi(v))\otimes(\pi_{j,f}\otimes x)\big)\\ &= s_i\big(\gamma(F(v)\otimes x)\big).\end{aligned}$$

- Case III: Assume $t = i+1$. The proof is similar to the case $t = i$. □

**Lemma 6.2.11.**

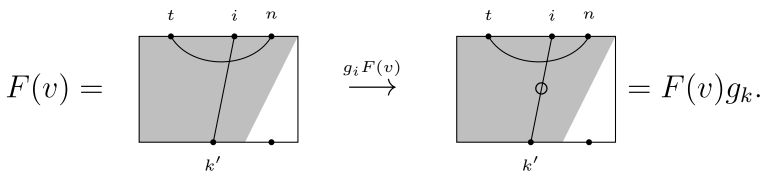
$$\gamma\big(g_i(F(v)\otimes x)\big) = g_i\big(\gamma(F(v)\otimes x)\big)$$

*where $v \in i(W_4)$, with an arc $\{t, n\}$ and $j-1$ propagating lines on the left of $t$, $g_i$, $1 \le i \le n-1$ are as in Lemma 3.3.11.*

*Proof.* Firstly, suppose that $i \neq t$. We have the following cases:

1. If in $v$ the vertex $i$ belongs to a propagating line then it is also in $\varphi(v)$. Suppose in $F(v)$, the vertex $i$ is joined to $k'$, $1 \le k \le l$, by an undecorated propagating line. Then the action of $g_i$ on $F(v)$ changes the decoration of the propagating line $\{i, k'\}$. So we have

$$g_iF(v) = F(v)g_k.$$



Then

$$\begin{aligned}
\gamma\big(g_i(F(v)\otimes x)\big) &= \gamma(F(v)g_k\otimes x)\\
&= \gamma(F(v)\otimes g_k x)\\
&= F(\varphi(v))\otimes(\pi_{j,f}\otimes g_k x) \qquad\qquad \text{(from Lemma 6.2.9)}\\
&= F(\varphi(v))\otimes(\pi_{j,f}g_k\otimes x).
\end{aligned}$$

Since there are $j-1$ propagating lines on the left of $t$, we have the following. If $i>t$ then $j\leq k\leq l$ and if $i<t$ then $k\leq j-1$ (where $t$ is in the position of $j^{th}$ propagating line). Since $\pi_{j,f}$ shifts the integers between $j+1$ and $l+1$ down by one and fixes those on the left of $t=j$ then we have,

$$\pi_{j,f}g_k = \begin{cases} g_k\pi_{j,f}, & \text{if } k\leq j-1;\\ g_{k+1}\pi_{j,f}, & \text{if } j\leq k\leq l.\end{cases}$$

On the other hand, since $F(\varphi(v))$ has $l+1$ propagating lines and the propagating lines on the right of $t$ have their bottom endpoints shifted up by one compared to $F(v)$, we have the following.
If $i>t$ then the vertex $i$ in $F(\varphi(v))$ joins to $(k+1)'$. So we have

$$g_iF(\varphi(v)) = F(\varphi(v))g_{k+1}.$$

Then

$$\begin{aligned}
g_i\big(\gamma(F(v)\otimes x)\big) &= g_i\big(F(\varphi(v))\otimes(\pi_{j,f}\otimes x)\big)\\
&= F(\varphi(v))g_{k+1}\otimes(\pi_{j,f}\otimes x)\\
&= F(\varphi(v))\otimes(g_{k+1}\pi_{j,f}\otimes x)\\
&= F(\varphi(v))\otimes(\pi_{j,f}g_k\otimes x) = \gamma\big(g_i(F(v)\otimes x)\big).
\end{aligned}$$

If $i<t$, meaning that the propagating line $\{i,k'\}$ is on the left of $t$, we have

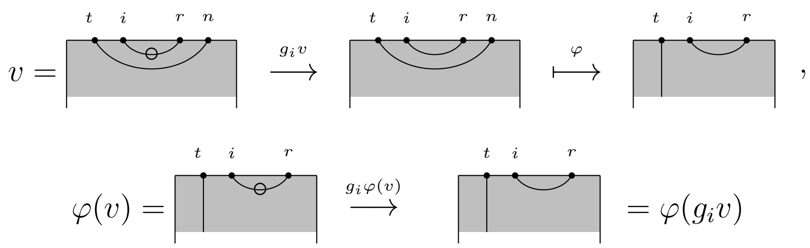
$$g_iF(\varphi(v)) = F(\varphi(v))g_k$$

and then

$$\begin{aligned} g_i\big(\gamma(F(v)\otimes x)\big) &= g_i\big(F(\varphi(v))\otimes(\pi_{j,f}\otimes x)\big)\\ &= F(\varphi(v))\otimes(g_k\pi_{j,f}\otimes x)\\ &= F(\varphi(v))\otimes(\pi_{j,f}g_k\otimes x) = \gamma\big(g_i(F(v)\otimes x)\big). \end{aligned}$$

2. If in $v$ the vertex $i$ is joined to a vertex $r$ by an undecorated (resp. a decorated) arc, then that also holds in $\varphi(v)$. The action of $g_i$ introduces a decorated (resp. undecorated) arc $\{i,r\}$ (i.e. the action of $g$ on $v$ (and also on $\varphi(v)$) only changes the decoration of the arc $\{i,r\}$). Let $g_iv = c \in i(W_4)$ so $\varphi(c)$ has a decorated (resp. undecorated) arc $\{i,r\}$ and a propagating line in the position of $t$.
Note that $\varphi(v)$ has an undecorated (resp. decorated) arc $\{i,r\}$ and a propagating line in the position of $t$, and the action of $g_i$ on $\varphi(v)$ only changes the decoration of the arc $\{i,r\}$. Therefore we get

$$g_i\varphi(v) = \varphi(g_iv) = \varphi(c).$$




Note that the action of $g_i$ on $v$ and also on $\varphi(v)$ does not introduce any permutation so we have,

$$g_iF(v) = F(g_iv) = F(c) \text{ and } g_iF(\varphi(v)) = F(g_i\varphi(v)) = F(\varphi(c)).$$

Then

$$\begin{aligned} \gamma\big(g_i(F(v))\otimes x)\big) &= \gamma(F(c)\otimes x)\\ &= F(\varphi(c))\otimes(\pi_{j,f}\otimes x)\\ &= g_iF(\varphi(v))\otimes(\pi_{j,f}\otimes x)\\ &= g_i\big(F(\varphi(v))\otimes(\pi_{j,f}\otimes x)\big) = g_i\big(\gamma(F(v)\otimes x)\big). \end{aligned}$$

3. If in $v$ the vertex $i$ is a decorated or an undecorated isolated vertex then this also true in $\varphi(v)$. In this case the action of $g_i$ on $v$ and also on $\varphi(v)$ does not make any change. So we have

$$\gamma\big(g_i(F(v) \otimes x)\big) = g_i\big(\gamma(F(v) \otimes x)\big).$$

Secondly, assume $t = i$.

**(a)** Suppose that the arc $\{t, n\}$ is an undecorated arc in $v$. So the action of $g_i$ on $v$ (and also on $F(v)$) gives a decorated arc $\{t, n\}$ and does not give any permutation so we have,

$$g_i F(v) = F(c), \tag{6.4}$$

where $c = g_i v \in i(W_4)$ is $v$ with a decorated arc $\{t, n\}$. This implies that

$$\varphi(c) = \varphi(v). \tag{6.5}$$

Now, since in $c$ the arc $\{t, n\}$ is decorated so in $f_4(F(c))$ the vertex $i = t$ is joined to the vertex $(l+1)'$ by a decorated propagating line. So, from the definition of $f_4$, we have
$f_4(F(c)) = F(\varphi(c))\tilde{\pi}_{j,f}$ where

$$\tilde{\pi}_{j,f} = ((0, \ldots, f(j), 0, \ldots, 0), \sigma_j) = ((0, \ldots, 0, 1, 0, \ldots, 0), \sigma_j)$$

while, $f_4(F(v)) = F(\varphi(v))\pi_{j,f}$, where $\pi_{j,f} = ((0, \ldots, 0), \sigma_j)$ since the arc $\{t, n\}$ is undecorated in $v$. That means $\tilde{\pi}_{j,f}$ is $\pi_{j,f}$ with a decorated line $\{j, (l+1)'\}$. This implies that

$$g_j \pi_{j,f} = \tilde{\pi}_{j,f}. \tag{6.6}$$

Also, note that in $F(\varphi(v))$ the vertex $t = i$ belongs to an undecorated propagating line. Therefore the action of $g_i$ introduces a decorated propagating line in the position of $i = t$ (which is a position of $j^{th}$ propagating line). So we have

$$g_i F(\varphi(v)) = F(\varphi(v)) g_j. \tag{6.7}$$

Then

$$\begin{aligned}
\gamma\big(g_i(F(v)\otimes x)\big) &= \gamma(g_iF(v)\otimes x) \\
&= \gamma(F(c)\otimes x) && \text{(from 6.4)} \\
&= f_4(F(c))(\tilde{\pi}_{j,f})^{-1}\otimes(\tilde{\pi}_{j,f}\otimes x) \\
&= F(\varphi(c))\otimes(\tilde{\pi}_{j,f}\otimes x) \\
&= F(\varphi(v))\otimes(g_j\pi_{j,f}\otimes x) && \text{(from 6.5, 6.6)} \\
&= F(\varphi(v))g_j\otimes(\pi_{j,f}\otimes x) \\
&= g_iF(\varphi(v))\otimes(\pi_{j,f}\otimes x) && \text{(from 6.7)} \\
&= g_i\big(F(\varphi(v))\otimes(\pi_{j,f}\otimes x\big) = g_i\big(\gamma(F(v)\otimes x)\big).
\end{aligned}$$

**(b)** If in $v$ the arc $\{t,n\}$ is decorated the proof is similar to the case $(a)$. □

**Lemma 6.2.12.**

$$\gamma\big(e_i(F(v)\otimes x)\big) = e_i\big(\gamma(F(v)\otimes x)\big)$$

*where $v\in i(W_4)$, with an arc $\{t,n\}$ and $j-1$ propagating lines on the left of $t$, $e_i$, $1\le i\le n-2$ are as in Lemma 3.3.11.*

*Proof.* Firstly, Assume $t\neq i,i+1$. We have the following cases:

1. If in $F(v)$ one of the vertices $i$ or $i+1$ belongs to a propagating line and the other is an undecorated or decorated isolated vertex or both of them belong to propagating lines, then that also holds in $F(\varphi(v))$. In this case the product $e_iF(v)$ has less than $l$ propagating (resp. the product $e_iF(\varphi(v)$ has less than $l+1$ propagating lines). Therefore, $e_iF(v)=0=e_iF(\varphi(v))$ and then

$$\gamma\big(e_i(F(v)\otimes x)\big)=0,$$

$$e_i\big(\gamma(F(v)\otimes x)\big)=e_iF(\varphi(v))\otimes(\pi_{j,f}\otimes x)=0.$$

2. If in $F(v)$ the vertices $i$, $i+1$ are joined together by a decorated or an undecorated arc or both of them are decorated or undecorated isolated vertices, then that also holds in $f_4(F(v))$. Then the products $e_iF(v)$ and $e_if_4(F(v))$ introduce

the same scalar $\lambda$ (say), where $\lambda$ is one of the parameters $\{\delta, \delta_\circ, \delta', \mu, \mu'\}$. So we have
$e_iF(v) = \lambda F(c)$ and $e_if_4(F(v)) = \lambda f_4(F(c))$ where $c = e_iv \in i(W_4)$. Then

$$\begin{aligned}
\gamma\big(e_i(F(v) \otimes x)\big) &= \gamma(e_iF(v) \otimes x) \\
&= \gamma(\lambda F(c) \otimes x) \\
&= \lambda\big(\gamma(F(c) \otimes x)\big) \qquad\qquad \text{(as } \gamma \text{ is linear)} \\
&= \lambda\big(f_4(F(c))\pi_{j,f}^{-1} \otimes (\pi_{j,f} \otimes x)\big) \\
&= e_if_4(F(v))\pi_{j,f}^{-1} \otimes (\pi_{j,f} \otimes x) \\
&= e_i\big(f_4(F(v))\pi_{j,f}^{-1} \otimes (\pi_{j,f} \otimes x)\big) = e_i\big(\gamma(F(v) \otimes x)\big).
\end{aligned}$$

3. If in $v$ the vertex $i$ is joined to a vertex $r$ (say) by an arc and the vertex $i+1$ belongs to a propagating line (resp. the vertex $i+1$ joins to a vertex $s$ (say) by an arc, resp. the vertex $i+1$ is an undecorated or decorated isolated vertex), then that also holds in $\varphi(v)$. Note that since the action of $e_i$ on $v$ and also on $\varphi(v)$ does not affect the vertex $t$ and the map $\varphi$ only affects $t$, we have

$$e_i\varphi(v) = \varphi(e_iv) = \varphi(c)$$

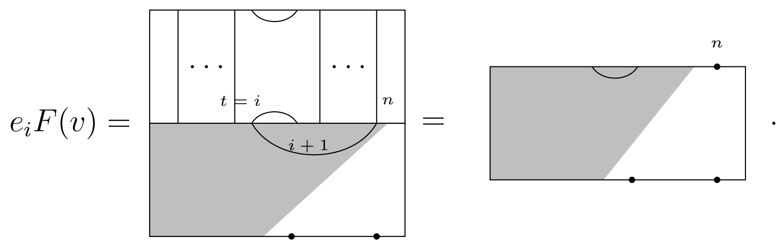
where $c = e_iv = \text{top}(e_iF(v))$.
Therefore $e_iF(v) = F(c)$ and $e_iF(\varphi(v)) = F(e_i\varphi(v)) = F(\varphi(e_iv)) = F(\varphi(c))$. Then

$$\begin{aligned}
\gamma\big(e_i(F(v) \otimes x)\big) &= \gamma(e_iF(v) \otimes x) \\
&= \gamma(F(c) \otimes x) \\
&= f_4(F(c))\pi_{j,f}^{-1} \otimes (\pi_{j,f} \otimes x) \\
&= F(\varphi(c)) \otimes (\pi_{j,f} \otimes x) \\
&= e_iF(\varphi(v)) \otimes (\pi_{j,f} \otimes x) \\
&= e_if_4(F(v))\pi_{j,f}^{-1} \otimes (\pi_{j,f} \otimes x) \\
&= e_i\big(f_4(F(v))\pi_{j,f}^{-1} \otimes (\pi_{j,f} \otimes x)\big) = e_i\big(\gamma(F(v) \otimes x)\big).
\end{aligned}$$

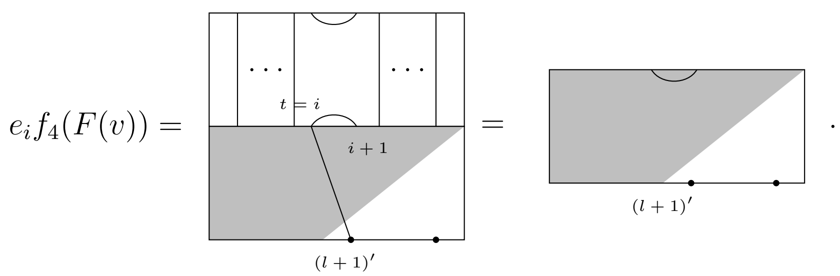
Note that, the dual of case (3) is similar to case (3).

Secondly, assume $t = i$ (the $t = i+1$ case is similar). We distinguish three cases:

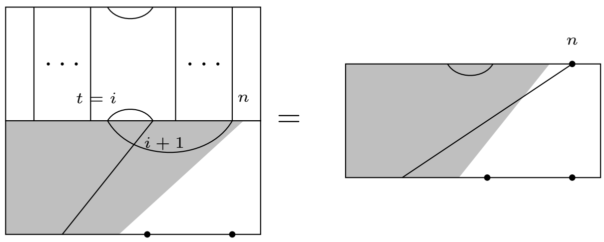
1. If in the diagram $F(v)$ the vertex $i+1$ is an undecorated (resp. a decorated) isolated vertex, then in the product $e_iF(v)$ we have an undecorated (resp. a decorated) isolated vertex in the position of $n$ in the top. This means that $\mathrm{top}(e_iF(v)) \in i(W_1)$ (resp. $i(W_2)$). So $e_iF(v) \otimes x = 0$ in $\frac{\Delta_n(l,\lambda)}{M}$ and then $\gamma\big(e_i(F(v) \otimes x)\big) = 0$.

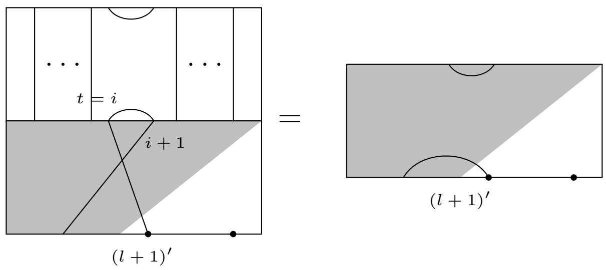
   On the other hand, in $f_4(F(v))$ the vertex $i$ is joined to the vertex $(l+1)'$ by a propagating line while the vertex $i+1$ is an undecorated (resp. a decorated) isolated vertex. Then in the product $e_if_4(F(v))$ we obtain an undecorated (resp. a decorated) isolated vertex in the position of $(l+1)'$ in the bottom. This means that $\#(e_if_4(F(v))) < l+1$, so $e_if_4(F(v))$ is zero in $\overline{B_{n-1}^{l+1}}$. Therefore, $e_i\gamma(F(v) \otimes x) = e_i\big(f_4(F(v))\pi_{j,f}^{-1} \otimes (\pi_{j,f} \otimes x)\big) = 0$.

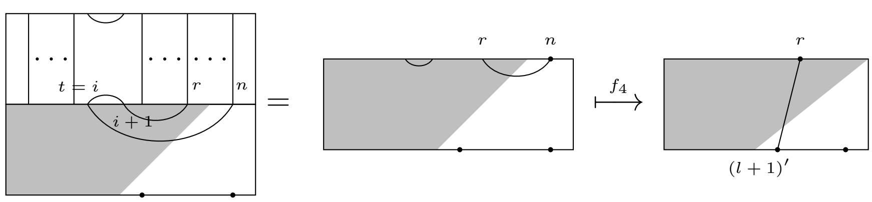
2. If in the diagram $F(v)$ the vertex $i+1$ belongs to a propagating line then we obtain in $e_iF(v)$ a propagating line that connects the vertex $n$ in the top of $e_iF(v)$ with a vertex in the bottom. This means that $\mathrm{top}(e_iF(v)) \in i(W_3)$. This implies that $e_iF(v) \otimes x = 0$ in the quotient $\frac{\Delta_n(l,\lambda)}{M}$ and then $\gamma(e_i(F(v) \otimes x)) = 0$.

$e_iF(v) =$  .


On the other hand, in $f_4(F(v))$ the vertices $i$ and $i+1$ are incident to the propagating lines. Therefore the product $e_i f_4(F(v))$ has less than $l+1$ propagating lines. Consequently $e_i f_4(F(v))$ is zero in $\overline{B_{n-1}^{l+1}}$. Then we have $e_i\gamma(F(v)\otimes x) = e_i\big(f_4(F(v))\pi_{j,f}^{-1}\otimes(\pi_{j,f}\otimes x)\big) = 0$.

$e_i f_4(F(v)) =$  .


3. If in the diagram $F(v)$ the vertex $t = i$ is connected to $n$ by an undecorated (resp. a decorated) arc and the vertex $i+1$ is connected to a vertex $r$ by an undecorated arc. Then in $e_iF(v)$ the vertex $n$ is connected to the vertex $r$ by an undecorated (resp. a decorated) arc. This implies that in $f_4(e_iF(v))$ the vertex $r$ is joined to the vertex $(l+1)'$ by an undecorated (resp. a decorated) propagating line.

$e_iF(v) =$  .

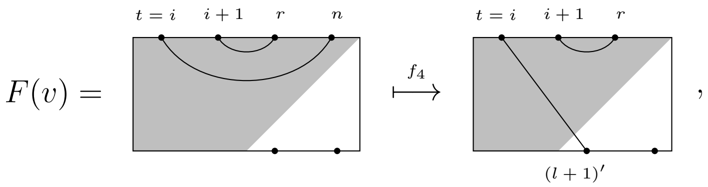
Suppose that in the diagram $e_iF(v)$ there are $w-1$ propagating lines on the left of $r$. So from the definition of $f_4$ we have

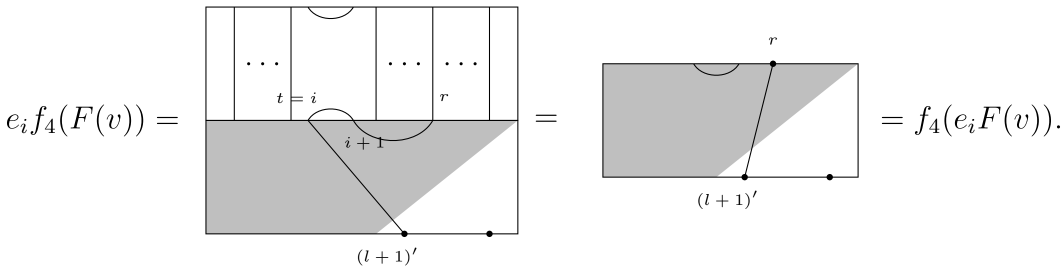
$$f_4(e_iF(v)) = F(\varphi(e_iF(v)))\pi_{w,g} \tag{6.8}$$

On the other hand, in the diagram $f_4(F(v))$ the vertex $i$ is joined to the vertex $(l+1)'$ by an undecorated (resp. decorated) propagating line and the vertex $i+1$ is joined to $r$ by an undecorated arc. Consequently, the vertex $r$ in $e_if_4(F(v))$ is joined to the vertex $(l+1)'$ by undecorated (resp. decorated) propagating line. Then we have

$$f_4(e_iF(v)) = e_if_4(F(v)). \tag{6.9}$$

This is illustrated below.




Similarly, if the vertex $i+1$ is connected to a vertex $r$ in $F(v)$ by a decorated arc then we have got $f_4(e_iF(v)) = e_if_4(F(v))$. Hence, we have

$$\begin{aligned}
\gamma\big(e_i(F(v)\otimes x)\big) &= \gamma(e_iF(v)\otimes x) \\
&= f_4(e_iF(v))\pi_{w,g}^{-1}\otimes(\pi_{w,g}\otimes x) && \text{(from 6.8)} \\
&= f_4(e_iF(v))\otimes(1\otimes x) \\
&= e_if_4(F(v))\otimes(1\otimes x) && \text{(from 6.9)} \\
&= e_i\big(f_4(F(v))\pi_{j,f}^{-1}\otimes(\pi_{j,f}\otimes x)\big) \\
&= e_i\big(\gamma(F(v)\otimes x)\big).
\end{aligned}$$

□

**Lemma 6.2.13.** **(a)** $\gamma\big(p_i(F(v)\otimes x)\big)=p_i\big(\gamma(F(v)\otimes x)\big)$,

**(b)** $\gamma\big(q_i(F(v)\otimes x)\big)=q_i\big(\gamma(F(v)\otimes x)\big)$

*where $v\in i(W_4)$, with an arc $\{t,n\}$ and $j-1$ propagating lines on the left of $t$, $p_i$, $q_i$, $1\le i\le n-1$ are as in Lemma 3.3.11.*

*Proof.* **(a)** Firstly, assume that $t\neq i$. We have the following cases:

1. If the vertex $i\neq t$ is an undecorated (resp. a decorated) isolated vertex in $F(v)$ then it is also in $f_4(F(v))$. In this case we have $p_iF(v)=\delta' F(v)$ (resp. $\mu F(v)$), also $p_if_4(F(v))=\delta' f_4(F(v))$ (resp. $\mu f_4(F(v))$). Consequently, we have

$$\begin{aligned}\gamma\big(p_i(F(v)\otimes x)\big)=\gamma(p_iF(v)\otimes x)&=\gamma(\lambda F(v)\otimes x)\\&=\lambda\big(\gamma(F(v)\otimes x)\big)\qquad\qquad\text{(as }\gamma\text{ is linear)}\\&=\lambda\big(f_4(F(v))\pi_{j,f}^{-1}\otimes(\pi_{j,f}\otimes x)\big)\\&=p_if_4(F(v))\pi_{j,f}^{-1}\otimes(\pi_{j,f}\otimes x)\\&=p_i\big(f_4(F(v))\pi_{j,f}^{-1}\otimes(\pi_{j,f}\otimes x)\big)\\&=p_i\big(\gamma(F(v)\otimes x)\big).\end{aligned}$$

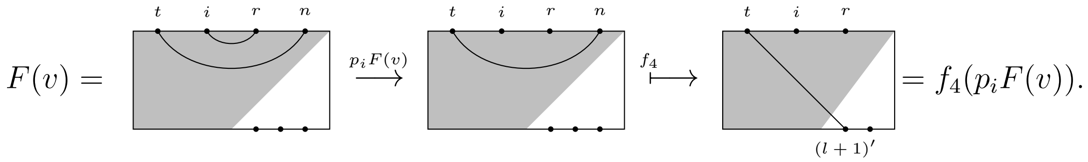
   where $\lambda=\delta'$ (resp. $\mu$).

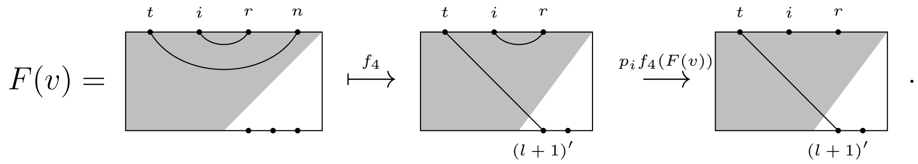
2. If the vertex $i$ is incident to a propagating line in $F(v)$ then that also holds in $f_4(F(v))$. Then we have $\#(p_iF(v))<l$, also $\#(p_if_4(F(v)))<l+1$. This implies that $p_iF(v)$ is zero in $\overline{B_n^l}$ and also $p_if_4(F(v))$ is zero in $\overline{B_{n-1}^{l+1}}$. Therefore, $\gamma\big(p_i(F(v)\otimes x)\big)=\gamma(p_iF(v)\otimes x)=0$ and
$p_i\big(\gamma(F(v)\otimes x)\big)=p_i\big(f_4(F(v))\pi_{j,f}^{-1}\otimes(\pi_j\otimes x)\big)=0$.

3. If the vertex $i$ is connected to a vertex $r$ (say) by a decorated or an undecorated arc in $F(v)$ then that also holds in $f_4(F(v))$. So in the product $p_iF(v)$ and also in $p_if_4(F(v))$ we have got an undecorated isolated vertex in the position of $i$ and $r$. Note that since the action of $p_i$ on $F(v)$ and also on $f_4(F(v))$ does not affect the vertex $t$ and $f_4$ only affects $t$, we have

$$f_4(p_iF(v))=p_if_4(F(v)).$$

This is illustrated below.




Let $p_iF(v) = F(c)$ where $c = p_iv = \text{top}(p_iF(v))$. Then we have

$$
\begin{aligned}
\gamma\big(p_i(F(v) \otimes x)\big) &= \gamma(F(c) \otimes x) \\
&= f_4(F(c)\pi_{j,f}^{-1} \otimes (\pi_{j,f} \otimes x) \\
&= f_4(p_iF(v))\pi_{j,f}^{-1} \otimes (\pi_{j,f} \otimes x) \\
&= p_if_4(F(v))\pi_{j,f}^{-1} \otimes (\pi_{j,f} \otimes x) \\
&= p_i(f_4(F(v))\pi_{j,f}^{-1} \otimes (\pi_{j,f} \otimes x)) \\
&= p_i\big(\gamma(F(v) \otimes x)\big).
\end{aligned}
$$

Secondly, suppose that $i = t$. Then in $p_iF(v)$ we have an undecorated isolated vertex in the position of $n$ in the top. This means that $\text{top}(p_iF(v)) \in i(W_1)$. This implies that $(p_iF(v) \otimes x)$ is zero in $\frac{\Delta_n(l,\lambda)}{M}$ and then $\gamma(p_i(F(v) \otimes x)) = 0$.

On the other hand, in $f_4(F(v))$ the vertex $i$ is joined to the vertex $(l+1)'$ by a propagating line. Consequently, in $p_if_4(F(v))$ we have got an undecorated isolated vertex in the position of $(l+1)'$ in the bottom. This means that $\#(p_if_4(F(v))) < l+1$ implying that $p_if_4(F(v))$ is zero in $\overline{B_{n-1}^{l+1}}$. Therefore,
$p_i\big(\gamma(F(v) \otimes x)\big) = p_i\big(f_4(F(v))\pi_{j,f}^{-1} \otimes (\pi_{j,f} \otimes x)\big) = 0.$
Hence $\gamma\big(p_i(F(v) \otimes x)\big) = p_i\big(\gamma(F(v) \otimes x)\big)$.

**(b)** The proof is similar to $(a)$. □

From the previous lemmas and Lemma 3.3.11 we have the following:

**Lemma 6.2.14.** *The map $\gamma$ is a $D\mathcal{P}\mathfrak{B}_{n-1}$-module homomorphism.*

Now we are in the position to prove the main result of this section.

*Proof of Proposition 6.2.1.* From Lemma 6.2.8 and Lemma 6.2.14 we have $\frac{\Delta_n(l,\lambda)}{M} \cong \overline{B_{n-1}^{l+1}} \otimes_{\mathbb{C}\widetilde{S_{l+1}}} S^\lambda \uparrow_{\mathbb{C}\widetilde{S_n}}^{\mathbb{C}\widetilde{S_{n+1}}}$. Since $S^\lambda \uparrow_{\mathbb{C}\widetilde{S_n}}^{\mathbb{C}\widetilde{S_{n+1}}} = \bigoplus_{\lambda\to\nu} S^\nu$, so

$$\frac{\Delta_n(l,\lambda)}{M} \cong \bigoplus_{\lambda\to\nu} \overline{B_{n-1}^{l+1}} \otimes_{\mathbb{C}\widetilde{S_{l+1}}} S^\nu = \bigoplus_{\lambda\to\nu} \Delta_{n-1}(l+1,\nu).$$

This completes the proof of the proposition. □

## 6.3 Main result

Now we are ready to state the main result of this chapter.

From Propositions 6.1.5, 6.1.6 and 6.2.1, we obtain the restriction rules for the cell modules.

**Theorem 6.3.1.** *Let $\lambda$ be a bipartition of $l$. Then we have the following:*

**(a)** *For $l = 0$, there is a short exact sequence of $D\mathcal{P}\mathfrak{B}_{n-1}$-modules as follows*

$$\begin{aligned} 0 \longrightarrow \Delta_{n-1}\big(0,(\emptyset,\emptyset)\big)^{\oplus 2} &\longrightarrow \Delta_n\big(0,(\emptyset,\emptyset)\big) \downarrow_{D\mathcal{P}\mathfrak{B}_{n-1}}^{D\mathcal{P}\mathfrak{B}_n} \\ &\longrightarrow \Delta_{n-1}\big(1,(\square,\emptyset)\big) \oplus \Delta_{n-1}\big(1,(\emptyset,\square)\big) \longrightarrow 0. \end{aligned}$$

**(b)** *For $n \geq 3$, $1 \leq l \leq n-2$, there is a short exact sequence of $D\mathcal{P}\mathfrak{B}_{n-1}$-modules as follows*

$$\begin{aligned} 0 \longrightarrow \Delta_{n-1}(l,\lambda)^{\oplus 2} \oplus \bigoplus_{\mu\to\lambda} \Delta_{n-1}(l-1,\mu) &\longrightarrow \Delta_n(l,\lambda) \downarrow_{D\mathcal{P}\mathfrak{B}_{n-1}}^{D\mathcal{P}\mathfrak{B}_n} \\ &\longrightarrow \bigoplus_{\lambda\to\nu} \Delta_{n-1}(l+1,\nu) \longrightarrow 0. \end{aligned}$$

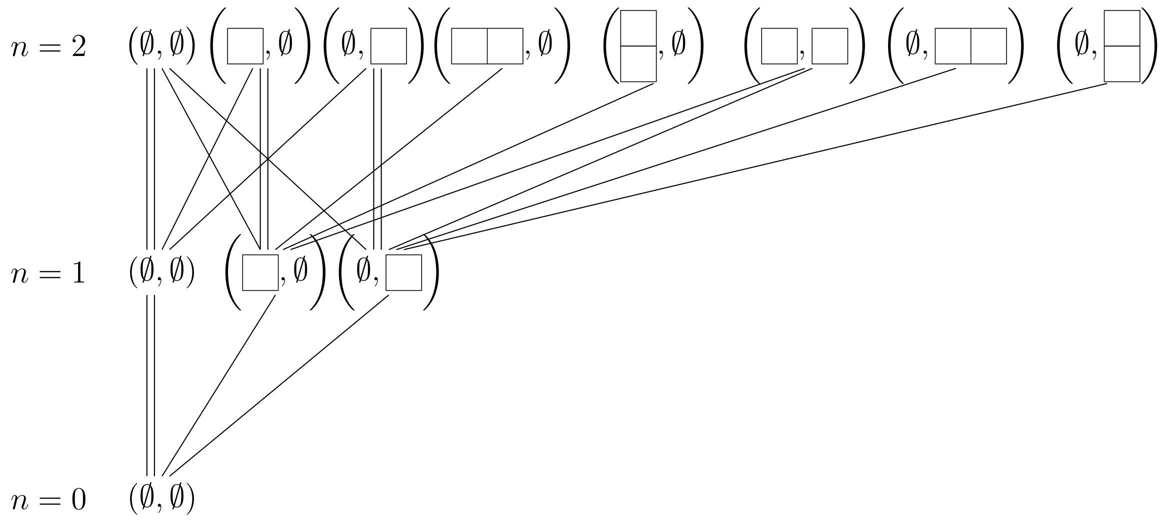


FIGURE 6.1: Bratelli diagram for the cell module $\Delta_n(l,\lambda)$, for $n \leq 2$.

**(c)** *For $l = n-1$, we have*

$$\Delta_n(n-1,\lambda)\downarrow^{D\mathcal{P}\mathfrak{B}_n}_{D\mathcal{P}\mathfrak{B}_{n-1}} \simeq \Delta_{n-1}(n-1,\lambda)^{\oplus 2} \oplus \bigoplus_{\mu\to\lambda} \Delta_{n-1}(n-1,\mu).$$

**(d)** *If $l = n$, the cell module of the $D\mathcal{P}\mathfrak{B}_n$ coincides with the Specht module of $\mathbb{C}\widetilde{S_n}$. Then we have*

$$\Delta_n(n,\lambda)\downarrow^{D\mathcal{P}\mathfrak{B}_n}_{D\mathcal{P}\mathfrak{B}_{n-1}} \simeq S^\lambda \downarrow^{\mathbb{C}\widetilde{S_n}}_{\mathbb{C}\widetilde{S_{n-1}}} = \bigoplus_{\mu\to\lambda} S^\mu.$$

In the following we represent the restriction rule for the cell module using a Bratteli diagram.

Let $\Lambda_n = \{\lambda \mid \lambda$ is a bipartition of $l$, $l$ is the number of the propagating lines of $\Delta_n(l,\lambda),\ 0 \leq l \leq n\}$.

Define the Bratteli diagram for the restriction rule for the cell module $\Delta_n(l,\lambda)$ to be a graph consisting of vertices in $n$-th level labelled by the bipartitions in $\Lambda_n$, $n \geq 0$, and edges between vertices in $(n-1)$-th level and $n$-th level i.e. between $\mu \in \Lambda_{n-1}$ and $\lambda \in \Lambda_n$, these edges are defined as follows: There are two edges between $\mu$ and $\lambda$ if $\lambda = \mu$ and one edge between $\mu$ and $\lambda$ if $\mu$ is obtained from $\lambda$ by removing or adding one box. (See Figure 6.1.)

## 6.4 Future work

This thesis makes only a start in understanding the representation theory of the decorated partial Brauer algebra. There are many future avenues of research that could be explored.

We would like to understand when the algebra is semisimple. We expect, like many of its diagram algebra cousins, that it is generically semisimple. Since the algebra is cellular, one way to prove this could be to show that the cell modules are generically simple, or equivalently, that the Gram determinant of each cell module is non-zero. This uses the work of Graham and Lehrer, [6], who showed that if $A$ is cellular $R$-algebra ($R$ a field) then $A$ is semisimple if and only if the non-zero cell modules of $A$ are simple modules.

Once we know it is generically semisimple, then the next question is what conditions on the parameters give a non-semisimple algebra. This may involve reparametrising in terms of quantum integers, as for other diagram algebras.

We then could begin to explore the non-generic representation theory of the algebra. Since the decorated partial Brauer algebra contains the symmetric group theory, this is a hard problem in general. But certainly we could expect to relate the representation theory of this algebra (and its decomposition numbers) to that of the Brauer algebra and hence the symmetric group.

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