Derived A-infinity Algebras:

Combinatorial models and obstruction theory

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Abstract

Let R be a commutative ring, and let A be a derived A_∞-algebra over R with structure maps m_{ij} for all $i \geqslant 0$, $j \geqslant 1$. In this thesis we construct a collection of based topological spaces V_{ij} which give rise to the notion of a DA_∞-space. The structure of these spaces gives new insight into the structure of a derived A_∞-algebra. We study the cell structure of these spaces via a combinatorial model using partitioned trees. We will prove that the singular chain complex on a DA_∞-space gives rise to a derived A_∞-algebra.

We go on to consider obstruction theories to the existence of the structure maps of a derived A_∞-algebra. The bigrading on A leads to choices of the order in which we develop the derived A_∞-structure. We give three different definitions of a "partial" derived A_∞-structure and in light of these definitions provide two different obstruction theories to extend a dA_{ij}^--structure to a dA_{ij} structure, plus an obstruction theory to extend a dA_{r-1}-structure to a dA_{r+1}-structure. In each case, the obstruction lies in a particular class of the Hochschild cohomology of the homology of A.

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Chapter 0

Introduction

An A_∞-algebra is a homotopy invariant version of an associative algebra and this notion has been extensively studied since its definition by James D. Stasheff [Sta63] in 1963. Keller [Kel01] provides a useful introduction to A_∞-algebra structures.

Stasheff [Sta63] defines the associahedra, denoted K_j for $j \geqslant 2$, a collection of convex polytopes of dimension $j-2$. It is well known that the k-cells of K_j are in bijection with bracketings of a word with j letters and $j-2-k$ sets of brackets and also planar trees with j leaves and $j-2-k$ internal edges. Perhaps less well known is a formula, $T(j+1,k) = \frac{1}{k+1}\binom{j-2}{k}\binom{j+k}{k}$, which counts the number of cells in K_j of dimension $j-2-k$. For completeness, and due to a lack of a proof in the literature, we prove this fact in Section 1.6 of this thesis.

An A_∞-space is an algebra over the operad of associahedra. The associahedra form a non-symmetric operad in the category of topological spaces, and an A_∞-space is an algebra over this operad. Stasheff [Sta63] shows that the singular chain complex of an A_∞-space admits the structure of an A_∞-algebra.

Livernet [Liv14] establishes an obstruction theory to A_∞-algebra structures on a differential $\mathbb{Z}$-graded R-module, A, equipped with a homotopy associative multiplication. She defines a "partial" A_∞-algebra structure, called an A_r-algebra, and for $r \geqslant 3$ shows that the obstruction to extend the underlying A_{r-1}-structure on A to an A_{r+1}-structure lies in a class of the Hochschild cohomology of the associative algebra $H(A)$.

In [Kad80], Kadeišvili studied an obstruction theory to the uniqueness of A_∞-algebra structures, also in terms of the Hochschild cohomology of $H(A)$. Furthermore, in the case of A_∞ ring spectra the obstructions to the existence of higher homotopies is studied by Robinson [Rob89].

Kadeišvili [Kad80] also classified all differential graded algebras over a field up to quasi-isomorphism. In order to generalise his results to work over a general commutative ring, Sagave [Sag10] introduced the notion of a derived A_∞-algebra. A derived A_∞-algebra is an $(\mathbb{N}, \mathbb{Z})$-bigraded R-module with R-linear maps m_{ij} of bidegree $(i, i+j-2)$, satisfying certain relations, for all $i \geqslant 0$, $j \geqslant 1$. Prior to this, Lapin [Lap02] introduced the related concept of a $D_\infty^{(s)}$-differential A_∞-algebra.

A derived A_∞-algebra has an underlying structure of a twisted chain complex (also known as a multicomplex). Twisted chain complexes were first introduced by Wall [Wal61] in his work on resolutions for extensions of groups. They can be considered as a generalisation of a double complex with one differential being a differential only up to homotopy and all higher coherences.

In [LRW13] an operadic description of a derived A_∞-algebra was developed. Derived A_∞-algebras are shown to be algebras over the operad dA_∞.

In this thesis we investigate combinatorial models and obstruction theories for derived A_∞-algebras. Throughout we will first consider the classical case of A_∞-algebras and the other special case of twisted chain complexes before introducing the more general theory for derived A_∞-algebras.

In Chapter 2 we will define a collection of based topological spaces V_{ij} for $i \geqslant 0$, $j \geqslant 1$, and show that these spaces form a non-symmetric $\mathbb{N}$-coloured operad, $\mathcal{V}$, in the category of based topological spaces. We give the definition of a DA_∞-space and show that this is a non-symmetric non-unital algebra over the operad $\mathcal{V}$. That is, a family of based topological spaces $X = \{X_n\}_{n\in\mathbb{N}}$ equipped with based maps

$$DA_{ij} : V_{ij} \wedge X_{p_1} \wedge \cdots \wedge X_{p_j} \to X_{p_1+\cdots+p_j+i}$$

satisfying some relations, for all $i \geqslant 0$, $j \geqslant 1$, and $(i,j) \neq (0,1)$.

In the same chapter we also define a collection of spaces T_i for $i \geqslant 1$ and a D_∞-space over these spaces to model the structure of a twisted chain complex. Given that $V_{i1} = T_i$, a DA_∞-space has an underlying structure of a D_∞-space when $j = 1$. There is also an operadic story here with the spaces T_i forming a non-symmetric $\mathbb{N}$-coloured operad, $\mathcal{T}$, and a D_∞-space being an algebra over this operad.

In Chapter 2 we will show that cells in the spaces V_{ij} are in bijection with partitioned trees with j leaves and i nodes (a vertex with exactly one child), and thus provide a counting argument for the number of cells in each dimension of V_{ij}. In particular, we show that the number of cells in dimension $(i + j - 2 - k)$ of V_{ij} is given by

$$\frac{1}{k+1}\binom{j+k}{k}\sum_{\alpha=0}^{k}(-1)^{k-\alpha}\binom{k+1}{k-\alpha}N_\alpha(i+j+\alpha-1, i+1)$$

where $N_\alpha(n,m) = \frac{\alpha+1}{n+1}\binom{n+1}{m+\alpha}\binom{n+1}{m-1}$. In this chapter we also show that the boundary of V_{ij} is homeomorphic to a wedge of spheres of dimension $i+j-3$.

The combinatorial structure of the spaces T_i is less complex since the cells of dimension $i - 1 - k$ in T_i are in bijection with partitions of i into $k + 1$ parts, and so there are clearly $\binom{i-1}{k}$ such cells. The space T_i is defined as a smash product of $(i - 1)$ copies of $I = [0, 1]$ with 0 taken as the basepoint, and thus the boundary of T_i is homeomorphic to a sphere of dimension $i-2$.

For Chapter 3, the main result is Theorem 3.3.1 in which we prove that taking the singular chain complex of a DA_∞-space results in a derived A_∞-

algebra. We see that we get a bigraded R-module with one grading from the chain complex and the other from the grading on the spaces. The structure maps m_{ij} result from the chain maps induced from the maps DA_{ij}, and the relations in the algebra result from the relations in spaces.

Finally, in Chapter 4 we study the obstructions to the existence of the structure maps of a twisted chain complex and a derived A_∞-algebra. We generalise the pre-Lie structure from [Liv14] to allow for an extra grading and define the Hochschild cohomology for a derived A_∞-algebra, as in [LRW13].

For the twisted chain complex case, we define a "partial" twisted chain complex structure in the obvious way, that is a stage r twisted chain complex has structure maps d_i for all $0 \leqslant i \leqslant r$ subject to the relations among these. Then in Theorem 4.4.3 we show that if A is a stage r twisted chain complex, then the obstruction to lift the underlying stage $(r-1)$-structure of A to a stage $(r+1)$-structure lies in

$$HH_{bicx}^{r+1,1,r-1}(H(A), H(A)) = H^{r+1}(\mathrm{Mor}(H(A), H(A))_*^{r-1}, [m_{11}, -]).$$

For the obstructions to the existence of the structure of a derived A_∞-algebra we define three different notions of a "partial" derived A_∞-structure. The different definitions come down to a choice of how to "build up" the structure. There is a choice to be made because a derived A_∞-algebra structure is bigraded.

The first definition is a DA_{ij}-structure in which we have all of the structure maps m_{pq} for $0 \leqslant p \leqslant i$ and $1 \leqslant q \leqslant j$. The second definition is a DA_{ij}^--structure which is a DA_{ij}-structure without the structure map m_{ij}. These definitions allow us to consider obstructions to lifting a DA_{ij}^--structure to a DA_{ij}-structure i.e. the obstructions to the existence of the structure map m_{ij}. In Theorem 4.5.3 we see that for A a vertical bicomplex such that $H(A)$ and $Z(A)$ are bigraded projective R-modules, if A is a dA_{ij}^--algebra with structure maps m_{pq},

- then the obstruction to extend the dA_{ij}^{-}-algebra structure to a dA_{ij}-algebra structure, by modifying the map $m_{(i-1)j}$, lies in

$$HH_{bicx}^{i,j,i+j-3}(H(A),H(A)) = H^{i}(\mathrm{Mor}(H(A)^{\otimes j},H(A))_{*}^{i+j-3},[m_{11},-]),$$

- and the obstruction to extend the dA_{ij}^{-}-algebra structure to a dA_{ij}-algebra structure, by modifying the map $m_{i(j-1)}$, lies in

$$HH_{dga}^{i,j,i+j-3}(H(A),H(A)) = H^{j}(\mathrm{Mor}(H(A)^{\otimes *},H(A))_{i}^{i+j-3},[m_{02},-]).$$

The third definition of a "partial" dA_∞-structure is a DA_r-structure with all structure maps m_{pq} such that $p \geqslant 0$, $q \geqslant 1$, and $p+q \leqslant r$. In Theorem 4.5.6 we show that if A is a dA_r-algebra, then the obstruction to lift the underlying dA_{r-1}-algebra structure on A to a dA_{r+1}-algebra structure lies in

$$HH_{bidga}^{r+1,r-2}(H(A),H(A)) = H^{r+1}(\prod_{n}\mathrm{Mor}(H(A)^{\otimes n},H(A))_{*-n}^{r-2},[m,-]).$$

We do not in this thesis consider the question of obstructions to the uniqueness of the structure of a twisted chain complex or a derived A_∞-algebra, however one could consider this for each of the cases above by following and generalising the approach of [Kad80].

Chapter 1

Background

1.1 Symmetric monoidal categories

In this section, we give the definition of a symmetric monoidal category and some key examples that will be used throughout the thesis. In particular, we define $\boldsymbol{CHau}$ the category of compactly generated Hausdorff spaces; $\boldsymbol{CHau}_*$ the category of pointed compactly generated Hausdorff spaces; $\boldsymbol{Mod}(\boldsymbol{R})$ the category of left R-modules over a commutative ring R; and $\boldsymbol{Chain}(\boldsymbol{R})$ the category of chain complexes of left R-modules over a commutative ring R.

Definition 1.1.1. A **monoidal category** is a tuple

$$(M, \otimes, I, \alpha, \lambda, \rho)$$

consisting of the following data.

1. M is a category.

2. The product $\otimes : M \times M \to M$ is a functor, called the **monoidal product** (or tensor product), where $M \times M$ is the product category.

3. I is an object in M, called the **$\otimes$-unit**.

4. α is a natural isomorphism

$$(X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$$

for all objects $X, Y, Z \in M$, called the **associativity isomorphism**.

5. λ and ρ are natural isomorphisms

$$\alpha : I \otimes X \to X \quad \text{and} \quad \rho : X \otimes I \to X$$

for all objects $X \in M$, called the **left unit** and the **right unit** respectively.

The data is required to satisfy the following two axioms.

Unit Axioms: The diagram

$$\begin{tikzcd}
(X \otimes I) \otimes Y \arrow[r, "\alpha"] \arrow[d, "\rho\otimes\mathrm{id}"'] & X \otimes (I \otimes Y) \arrow[d, "\mathrm{id}\otimes\lambda"] \\
X \otimes Y \arrow[r, "="] & X \otimes Y
\end{tikzcd}$$

is commutative for all objects $X, Y \in M$; and

$$\lambda = \rho : I \otimes I \xrightarrow{\cong} I.$$

Pentagon Axiom: The pentagon

$$\begin{tikzcd}
 & (W \otimes X) \otimes (Y \otimes Z) \arrow[dr, "\alpha"] & \\
((W \otimes X) \otimes Y) \otimes Z \arrow[ur, "\alpha"] \arrow[d, "\alpha\otimes\mathrm{id}"] & & W \otimes (X \otimes (Y \otimes Z)) \\
(W \otimes (X \otimes Y)) \otimes Z \arrow[rr, "\alpha"] & & W \otimes ((X \otimes Y) \otimes Z) \arrow[u, "\mathrm{id}\otimes\alpha"']
\end{tikzcd}$$

is commutative for all objects $W, X, Y, Z \in M$.

A strict monoidal category is a monoidal category in which the natural isomorphisms α, λ, and ρ are all identity maps. From this point onwards, we will drop α, λ, and ρ from the notation of a monoidal category.

Definition 1.1.2. A **symmetric monoidal category** is a pair (M, ξ) in which

1. $M = (M, \otimes, I)$ is a monoidal category;

2. ξ is a natural isomorphism

$$X \otimes Y \xrightarrow[\cong]{\xi_{X,Y}} Y \otimes X$$

   for objects $X, Y \in M$, called the **symmetry isomorphism**.

This data is required to satisfy the following three axioms.

Symmetry Axiom: The diagram

$$\begin{tikzcd} X \otimes Y \arrow[r, "\xi_{X,Y}"] \arrow[dr, "="] & Y \otimes X \arrow[d, "\xi_{Y,X}"] \\ & X \otimes Y \end{tikzcd}$$

is commutative for all objects $X, Y \in M$.

Compatibility with Units: The diagram

$$\begin{tikzcd} X \otimes I \arrow[r, "\xi_{X,Y}"] \arrow[d, "\rho"] & I \otimes X \arrow[d, "\lambda"] \\ X \arrow[r, "="] & X \end{tikzcd}$$

is commutative for all objects $X \in M$.

Hexagon Axiom: The following diagram is commutative for all objects $X, Y, Z \in M$:

$$
\begin{tikzcd}[column sep=small]
 & X \otimes (Z \otimes Y) \arrow[r, "\mathrm{id}\otimes\xi_{Z,Y}"] & X \otimes (Y \otimes Z) \arrow[dr, "\alpha^{-1}"] & \\
(X \otimes Z) \otimes Y \arrow[ur, "\alpha"] \arrow[dr, "\xi_{X\otimes Z,Y}"'] & & & (X \otimes Y) \otimes Z. \\
 & Y \otimes (X \otimes Z) \arrow[r, "\alpha^{-1}"] & (Y \otimes X) \otimes Z \arrow[ur, "\xi_{Y,X}\otimes\mathrm{id}"'] &
\end{tikzcd}
$$

We often drop ξ from the notation of a symmetric monoidal category. Throughout this thesis we will work in a few different symmetric monoidal categories. The key categories to consider are as follows.

1. $\boldsymbol{CHau}$**:** the category of compactly generated Hausdorff spaces with morphisms given by continuous maps, the product $\times$ as the monoidal product, and any one-point space as the $\otimes$-unit.

2. $\boldsymbol{CHau_*}$**:** the category of pointed compactly generated Hausdorff spaces, with morphisms given by continuous basepoint preserving maps, the smash product $\wedge$ as the monoidal product, and the two-point space as the $\otimes$-unit.

3. If R is a commutative ring, then the category $\boldsymbol{Mod}(\boldsymbol{R})$ of left R-modules with morphisms given by R-linear maps, tensor product $\otimes_R$, and R regarded as the left module over itself as the $\otimes$-unit.

4. If R is a commutative ring, then the category $\boldsymbol{Chain}(\boldsymbol{R})$ of chain complexes of left R-modules with morphisms given by chain maps, tensor product $X \otimes Y$ where

$$(X \otimes Y)_n = \bigoplus_{a+b=n} X_a \otimes Y_b$$

with differential $\partial_n : (X \otimes Y)_n \to (X \otimes Y)_{n-1}$ given by

$$\partial_n(x \otimes y) = \partial_a x \otimes y + (-1)^a x \otimes \partial_b y$$

where $x \in X_a$, $y \in Y_b$ and $a + b = n$. We have R concentrated in degree 0 as the $\otimes$-unit.

1.2 Graded modules and derived A_∞-Algebras

From this point onwards, we take R to be a commutative ring, and all tensor products are taken over R unless stated otherwise.

We consider a $\mathbb{Z}$-graded R-module, A, to be a collection of R-modules A^j for all $j \in \mathbb{Z}$ where A^j is said to be of degree j. A morphism of graded modules of degree v is a collection of morphisms of R-modules $A^j \to A^{j+v}$ for $j \in \mathbb{Z}$..

Definition 1.2.1. An ***A_∞*-algebra** over R is a $\mathbb{Z}$-graded R-module A, endowed with graded R-linear maps

$$m_n : A^{\otimes n} \to A, \quad n \geqslant 1$$

of degree $n - 2$ satisfying the following relation

$$\sum (-1)^{r+st} m_u (1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = 0$$

for each $n \geqslant 1$, where the sum runs over all decompositions $n = r + s + t$ and we put $u = r + 1 + t$.

Remark 1.2.2. Let $\mathcal{A}s$ be the associative operad in chain complexes. Then note that specifying an A_∞-algebra structure on a $\mathbb{Z}$-graded R-module, A, is equivalent to giving a square zero coderivation on the cofree coalgebra on A over the Kozul dual cooperad $\mathcal{A}s^{¡}$.

An $(\mathbb{N}, \mathbb{Z})$-bigraded R-module, A, is a collection of R-modules A_i^j for all $i \in \mathbb{N}$, $j \in \mathbb{Z}$ where A_i^j is said to be of bidegree (i, j). A morphism of bigraded modules of bidegree (u, v) is a collection of morphisms of R-modules

$A_i^j \to A_{i+u}^{j+v}$ for $i \in \mathbb{N}$, $j \in \mathbb{Z}$. The lower grading is called the horizontal degree and the upper grading the vertical degree.

Definition 1.2.3. A **twisted chain complex**, C, is an $(\mathbb{N}, \mathbb{Z})$-bigraded R-module, with maps $d_i : C \to C$ of bidegree $(i, i-1)$ for $i \geqslant 0$, satisfying

$$\sum_{i+p=u} (-1)^i d_i \circ d_p = 0 \tag{1.1}$$

for $u \geqslant 0$.

Definition 1.2.4. A **derived A_∞-algebra** (or dA_∞-algebra for short) is an $(\mathbb{N}, \mathbb{Z})$-bigraded R-module, A, with R-linear maps

$$m_{ij} : A^{\otimes j} \to A$$

of bidegree $(i, i+j-2)$ for each $i \geqslant 0$, $j \geqslant 1$, satisfying the equations

$$\sum_{\substack{u=i+p,\\ v=j+q-1,\\ j=1+r+t}} (-1)^{rq+t+pj} m_{ij}(1^{\otimes r} \otimes m_{pq} \otimes 1^{\otimes t}) = 0 \tag{1.2}$$

for all $u \geqslant 0$ and $v \geqslant 1$.

When derived A_∞-algebras were first defined by Sagave [Sag10], Sagave was thinking of these in terms of projective resolutions of the homology of a differential graded algebra. Sagave defined a derived A_∞-algebra as an $(\mathbb{N}, \mathbb{Z})$-bigraded R-module to avoid potential problems with taking total complexes. In this thesis we also use $(\mathbb{N}, \mathbb{Z})$ grading conventions but we note that some authors generalise to $(\mathbb{Z}, \mathbb{Z})$-bigraded R-modules.

It is also worth noting that in [Sta63] and [Sag10], A_∞-algebras and dA_∞-algebras are equipped with a unit condition that we do not include in our definition.

Remark 1.2.5. In [LRW13], the operad $d\mathcal{A}s$ (in vertical bicomplexes) is introduced, and it is shown that derived A_∞-algebras are $(d\mathcal{A}s)_\infty$-algebras. So specifying a derived A_∞-algebra strucure on an $(\mathbb{N}, \mathbb{Z})$-bigraded R-module, A, is equivalent to a square zero coderivation on the kozul dual operad $(d\mathcal{A}s)^{\text{i}}(A)$.

Recall that the Koszul sign rule applies to bigraded maps, that is

$$(f \otimes g)(x \otimes y) = (-1)^{pi+qj} f(x) \otimes g(y)$$

where g has bidegree (p, q) and x has bidegree (i, j). We will be applying this throughout, wherever necessary.

1.3 Operads

In this section we introduce the notion of an operad. Our main focus here is to define a non-symmetric operad by partial compositions and algebras over them. These definitions will be used in Chapter 2 to describe the structure of A_∞-spaces. This example comes from the work of Stasheff [Sta63] and was a motivating example in the definition of an operad. Here we present the story in the opposite order.

Let $\mathcal{C}$ be a symmetric monoidal category with monoidal product $\otimes$ and unit κ.

Definition 1.3.1 ([May97]). A **non-symmetric operad** $\mathcal{O}$ in $\mathcal{C}$ consists of objects $\mathcal{O}(j)$ for $j \geqslant 0$, a unit map $\eta : \kappa \to \mathcal{O}(1)$, and product maps

$$\gamma : \mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes \cdots \otimes \mathcal{O}(j_k) \to \mathcal{O}(j)$$

for $k \geqslant 1$, and $j_s \geqslant 0$, where $\sum j_s = j$. The γ are required to be associative and unital in the following senses.

1. The following associativity diagram commutes, where $\sum j_s = j$ and

$\sum i_t = i$; we set $g_s = j_1 + \cdots + j_s$, and $h_s = i_{g_{s-1}+1} + \cdots + i_{g_s}$ for $1 \leqslant s \leqslant k$:

$$\begin{tikzcd}
\mathcal{O}(k) \otimes (\bigotimes_{s=1}^{k} \mathcal{O}(j_s)) \otimes (\bigotimes_{r=1}^{j} \mathcal{O}(i_r)) \arrow[r, "(\gamma,\mathrm{id})"] \arrow[dd, "\text{shuffle}"] & \mathcal{O}(j) \otimes (\bigotimes_{r=1}^{j} \mathcal{O}(i_r)) \arrow[d, "\gamma"] \\
& \mathcal{O}(i) \\
\mathcal{O}(k) \otimes (\bigotimes_{s=1}^{k} (\mathcal{O}(j_s) \otimes (\bigotimes_{q=1}^{j_s} \mathcal{O}(i_{g_{s-1}+q})))) \arrow[r, "(\mathrm{id},\otimes_s \gamma)"] & \mathcal{O}(k) \otimes (\bigotimes_{s=1}^{k} \mathcal{O}(h_s)). \arrow[u, "\gamma"']
\end{tikzcd}$$

2. The following unit diagrams commute:

$$\begin{tikzcd}
\mathcal{O}(k) \otimes (\kappa)^k \arrow[r, "\cong"] \arrow[d, "(\mathrm{id},\eta^k)"] & \mathcal{O}(k) \\
\mathcal{O}(k) \otimes \mathcal{O}(1)^k \arrow[ur, "\gamma"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\kappa \otimes \mathcal{O}(j) \arrow[r, "\cong"] \arrow[d, "(\eta,\mathrm{id})"] & \mathcal{O}(j) \\
\mathcal{O}(1) \otimes \mathcal{O}(j). \arrow[ur, "\gamma"'] &
\end{tikzcd}$$

The following proposition gives an equivalent definition for a non-symmetric operad via partial compositions. This definition also appears in [Ger63] under the name “Pre-Lie system”. In Chapter 4 we will generalise this definition to a Pre-Lie system for trigraded modules over a commutative ring.

Proposition 1.3.2 ([LV12])**.** A non-symmetric operad $\mathcal{O}$ in $\mathcal{C}$ consists of objects $\mathcal{O}(j)$ for $j \geqslant 0$, a unit map $\eta : \kappa \to \mathcal{O}(1)$, and partial composition maps,

$$\circ_i : \mathcal{O}(m) \otimes \mathcal{O}(n) \to \mathcal{O}(m+n-1)$$

for all $1 \leqslant i \leqslant m$ satisfying the relations:

$$\lambda \circ_i (\mu \circ_j \nu) = (\lambda \circ_i \mu) \circ_{i+j-1} \nu \qquad \text{for} \quad 1 \leqslant i \leqslant l, 1 \leqslant j \leqslant m, \tag{1.3}$$

$$(\lambda \circ_i \mu) \circ_{k+m-1} \nu = (\lambda \circ_k \nu) \circ_i \mu \qquad \text{for} \quad 1 \leqslant i < k \leqslant l, \tag{1.4}$$

$$\kappa \circ_1 \lambda = \lambda = \lambda \circ_i \kappa. \tag{1.5}$$

for any $\lambda \in \mathcal{O}(l)$, $\mu \in \mathcal{O}(m)$ and $\nu \in \mathcal{O}(n)$.

We will now define an algebra over a non-symmetric operad.

Definition 1.3.3 ([May97]). Let $\mathcal{O}$ be a non-symmetric operad. An **$\mathcal{O}$-algebra** is an object A together with maps

$$\theta : \mathcal{O}(j) \otimes A^{\otimes j} \to A$$

for $j \geqslant 0$ that are associative and unital in the following senses.

1. The following associativity diagram commutes, where $j = \sum j_s$:

$$\begin{tikzcd}
\mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes \cdots \otimes \mathcal{O}(j_k) \otimes A^{\otimes j} \arrow[r, "(\gamma,\mathrm{id})"] \arrow[dd, "\text{shuffle}"] & \mathcal{O}(j) \otimes A^{\otimes j} \arrow[d, "\theta"] \\
 & A \\
\mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes A^{\otimes j_1} \otimes \cdots \otimes \mathcal{O}(j_k) \otimes A^{\otimes j_k} \arrow[r, "(\mathrm{id},\theta^k)"] & \mathcal{O}(k) \otimes A^{\otimes k}. \arrow[u, "\theta"']
\end{tikzcd}$$

2. The following unit diagram commutes:

$$\begin{tikzcd}
\kappa \otimes A \arrow[r, "\cong"] \arrow[d, "(\eta,\mathrm{id})"] & A \\
\mathcal{O}(1) \otimes A. \arrow[ur, "\theta"'] &
\end{tikzcd}$$

In what follows we will only be interested in non-unital algebras defined via partial compositions and so this equivalent formulation is given in the next proposition. I am not aware of a reference for this proposition in the case of classical operads, however it is a straightforward consequence of Proposition 1.3.2. In the next section we will give a more general result for coloured operads, of which this one is a direct consequence by restricting to the case with one colour.

Proposition 1.3.4. Let $\mathcal{O}$ be a non-symmetric operad. A non-unital **$\mathcal{O}$-algebra** is an object A together with maps

$$\theta : \mathcal{O}(j) \otimes A^{\otimes j} \to A$$

for $j \geqslant 0$ that are associative in the following sense.

For $i = 1, \ldots, m+n-1$,

$$\begin{tikzcd}
\mathcal{O}(m) \otimes \mathcal{O}(m) \otimes A^{\otimes m+n-1} \arrow[r, "(\circ_i, \mathrm{id}^{m+n-1})"] \arrow[dd, "\text{shuffle}"] & \mathcal{O}(m+n-1) \otimes A^{\otimes m+n-1} \arrow[d, "\theta"] \\
 & A \\
\mathcal{O}(m) \otimes A^{\otimes i-1} \otimes \mathcal{O}(n) \otimes A^{\otimes m+n-i} \arrow[r, "(\mathrm{id}^i, \theta, \mathrm{id}^{m-i})"] & \mathcal{O}(m) \otimes A^{\otimes m}. \arrow[u, "\theta"']
\end{tikzcd}$$

$\square$

1.4 Coloured operads

In this section we introduce the notion of a coloured operad. For reference throughout this section we refer to [Yau16] for a comprehensive introduction to coloured operads. We are specifically interested in a coloured operad without the symmetric group actions, which were first defined by Lambek as multicategories ([Lam69]).

Our main focus here is to define a non-symmetric coloured operad by partial compositions (or coloured pseudo-operads in [Yau16]) and algebras over them. These definitions will be used in Chapter 2 to describe the structure of D_∞-spaces and DA_∞-spaces.

We begin by defining a colour profile, the main purpose of which is to simplify the notation for the remainder of the section.

Definition 1.4.1 ([Yau16], 9.1)**.** Fix a non-empty set C, whose elements are called colours.

1. A ***C*-profile** is a finite sequence of elements of C, say, $\underline{c} = (c_1, \ldots, c_n)$.

2. Write $|\underline{c}|$ for the **length** of a C-profile as a finite sequence.

3. The **empty *C*-profile** is denoted by $\varnothing$.

4. The **set** of C-profiles is denoted by $\operatorname{Prof}(C)$.

5. Suppose $\underline{a} = (a_1, \ldots, a_m)$ and $\underline{b} = (b_1, \ldots, b_n) \in \operatorname{Prof}(C)$. Their **concatenation** is defined as the C-profile

$$(\underline{a}, \underline{b}) = (a_1, \ldots, a_m, b_1, \ldots, b_n).$$

We will now introduce the notion of a coloured operad in the monoidal category $(\mathcal{C}, \otimes, \kappa)$.

Definition 1.4.2 ([Yau16], 11.2)**.** Let C be a non-empty set. A **non-symmetric *C*-coloured operad** P in $\mathcal{C}$ consists of objects $P(\underline{c}; d)$ for each $\underline{c} \in \operatorname{Prof}(C)$ and $d \in C$. Plus, for each $c \in C$ a unit map $1_C : \kappa \to P(c; c)$, and product maps:

$$\gamma : P(\underline{c}; d) \otimes P(\underline{b_1}; c_1) \otimes \cdots \otimes P(\underline{b_n}; c_n) \to P(\underline{b}; d)$$

for each $\underline{c} = (c_1, \ldots, c_n) \in \operatorname{Prof}(C)$, and n other colour tuples $\underline{b_1}, \ldots, \underline{b_n} \in \operatorname{Prof}(C)$, with $\underline{b} = (\underline{b_1}, \ldots, \underline{b_n})$ their concatenation. The γ are required to be associative and unital in the following senses.

1. Suppose that

   - for each $1 \leqslant j \leqslant n$, $\underline{b}_j = (b_1^j, \ldots, b_{k_j}^j) \in \operatorname{Prof}(C)$ has length $k_j \geqslant 0$ such that at least one $k_j > 0$;
   - $\underline{a}_i^j \in \operatorname{Prof}(C)$ for each $1 \leqslant j \leqslant n$ and $1 \leqslant i \leqslant k_j$;

- for each $1 \leqslant j \leqslant n$,

$$\underline{a}_j = \begin{cases} (\underline{a}^j_1, \ldots, \underline{a}^j_{k_j}) & \text{if } k_j > 0, \\ \varnothing & \text{if } k_j = 0; \end{cases}$$

- $\underline{a} = (\underline{a}_1, \ldots, \underline{a}_n)$ is their concatenation.

Then the following associativity diagram commutes:

$$\begin{tikzcd}
P\binom{\underline{c}}{d} \otimes \left[\bigotimes_{j=1}^{n} P\binom{\underline{b}_j}{c_j}\right] \otimes \bigotimes_{j=1}^{n}\left[\bigotimes_{i=1}^{k_j} P\binom{\underline{a}^j_i}{b^j_i}\right] \arrow[r, "(\gamma, \mathrm{id})"] \arrow[dd, "\text{shuffle}"] & P\binom{\underline{b}}{d} \otimes \bigotimes_{j=1}^{n}\left[\bigotimes_{i=1}^{k_j} P\binom{\underline{a}^j_i}{b^j_i}\right] \arrow[d, "\gamma"] \\
& P\binom{\underline{a}}{d} \\
P\binom{\underline{c}}{d} \otimes \bigotimes_{j=1}^{n}\left[P\binom{\underline{b}_j}{c_j} \otimes \bigotimes_{i=1}^{k_j} P\binom{\underline{a}^j_i}{b^j_i}\right] \arrow[r, "(\mathrm{id}, \otimes_j \gamma)"] & P\binom{\underline{c}}{d} \otimes \bigotimes_{j=1}^{n} P\binom{\underline{a}_j}{c_j}, \arrow[u, "\gamma"']
\end{tikzcd}$$

where we have used the notation $P\binom{\underline{c}}{d} = P(\underline{c}; d)$ to make the diagram easier to read.

2. Suppose $d \in C$.

   - If $\underline{c} = (c_1, \ldots, c_n) \in \mathrm{Prof}(C)$ has length $n \geqslant 1$, then the right unit diagram

$$\begin{tikzcd}
P\binom{\underline{c}}{d} \otimes (\kappa)^n \arrow[r, "\cong"] \arrow[d, "(\mathrm{id}, \otimes 1_{c_j})"] & P\binom{\underline{c}}{d} \\
P\binom{\underline{c}}{d} \otimes \bigotimes_{j=1}^{n} P\binom{c_j}{c_j} \arrow[ur, "\gamma"'] &
\end{tikzcd}$$

     is commutative.

- If $\underline{b} \in \mathrm{Prof}(C)$ has length $|\underline{b}| \geqslant 0$, then the left unit diagram

$$\begin{tikzcd} \kappa \otimes P\binom{\underline{b}}{d} \arrow[r, "\cong"] \arrow[d, "(1_d,\mathrm{id})"'] & P\binom{\underline{b}}{d} \\ P\binom{d}{d} \otimes P\binom{\underline{b}}{d}. \arrow[ur, "\gamma"] & \end{tikzcd}$$

  is commutative.

The above is the definition of a non-symmetric coloured operad. In Proposition 1.4.5 we give an equivalent definition for non-symmetric coloured operads via partial compositions, but first we define partial composition of colour profiles.

Definition 1.4.3 ([Yau16], 16.1)**.** Let $\underline{c} = (c_1, \ldots, c_n)$ and $\underline{b} = (b_1, \ldots, b_m)$ be C-profiles. We define the **partial composition** of C-profiles by

$$\underline{c} \circ_i \underline{b} = (c_1, \ldots, c_{i-1}, b_1, \ldots, b_m, c_{i+1}, \ldots, c_n) \in \mathrm{Prof}(C)$$

for all $1 \leqslant i \leqslant n$.

Proposition 1.4.4 ([Yau16], 16.1)**.** For $\underline{c} = (c_1, \ldots, c_n)$ and $\underline{b} = (b_1, \ldots, b_m)$ in $\mathrm{Prof}(C)$ the partial composition of C-profiles satisfies the following associativity relations:

$$\underline{c} \circ_i (\underline{b} \circ_j \underline{a}) = (\underline{c} \circ_i \underline{b}) \circ_{i+j-1} \underline{a} \qquad \text{for } 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m \tag{1.6}$$

$$(\underline{c} \circ_i \underline{b}) \circ_{k+m-1} \underline{a} = (\underline{c} \circ_k \underline{a}) \circ_i \underline{b} \qquad \text{for } 1 \leqslant i < k \leqslant n. \tag{1.7}$$

□

Proposition 1.4.5 ([Yau16], 16.2 and 16.4)**.** A non-symmetric C-coloured operad P in $\mathcal{C}$ consists of objects $P(\underline{c}; d)$ for each $\underline{c} \in \mathrm{Prof}(C)$ and $d \in C$, together with partial composition maps:

$$\gamma_i : P(\underline{c}; d) \otimes P(\underline{b}; c_i) \to P(\underline{c} \circ_i \underline{b}; d)$$

for each $1 \leqslant i \leqslant n$ where $|\underline{c}| = n$. The γ are required to be associative in the following sense. For $\underline{c} = (c_1, \ldots, c_n), \underline{b} = (b_1, \ldots, b_m) \in \mathrm{Prof}(C)$ and $\underline{a} \in \mathrm{Prof}(C)$, the diagram

$$\begin{tikzcd}
P\binom{\underline{c}}{d} \otimes P\binom{\underline{b}}{c_i} \otimes P\binom{\underline{a}}{b_j} \arrow[r, "(\mathrm{id},\gamma_j)"] \arrow[dd, "(\gamma_i,\mathrm{id})"] & P\binom{\underline{c}}{d} \otimes P\binom{\underline{b}\circ_j\underline{a}}{c_i} \arrow[d, "\gamma_i"] \\
& P\binom{\underline{c}\circ_i(\underline{b}\circ_j\underline{a})}{d} \arrow[d, equal] \\
P\binom{\underline{c}\circ_i\underline{b}}{d} \otimes P\binom{\underline{a}}{b_j} \arrow[r, "\gamma_{i+j-1}"] & P\binom{(\underline{c}\circ_i\underline{b})\circ_{i+j-1}\underline{a}}{d}
\end{tikzcd} \tag{1.8}$$

commutes for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$; and the diagram

$$\begin{tikzcd}
P\binom{\underline{c}}{d} \otimes P\binom{\underline{b}}{c_i} \otimes P\binom{\underline{a}}{c_k} \arrow[r, "(\gamma_i,\mathrm{id})"] \arrow[d, "\mathrm{shuffle}"', "\cong"] & P\binom{\underline{c}\circ_i\underline{b}}{d} \otimes P\binom{\underline{a}}{c_k} \arrow[d, "\gamma_{k+m-1}"] \\
P\binom{\underline{c}}{d} \otimes P\binom{\underline{a}}{c_k} \otimes P\binom{\underline{b}}{c_i} \arrow[d, "(\gamma_k,\mathrm{id})"] & P\binom{(\underline{c}\circ_i\underline{b})\circ_{k+m-1}\underline{a}}{d} \arrow[d, equal] \\
P\binom{\underline{c}\circ_k\underline{a}}{d} \otimes P\binom{\underline{b}}{c_i} \arrow[r, "\gamma_i"] & P\binom{(\underline{c}\circ_k\underline{a})\circ_i\underline{b}}{d}
\end{tikzcd} \tag{1.9}$$

commutes for $1 \leqslant i < k \leqslant n$.

In what follows, we will present two equivalent definitions of an algebra over a non-symmetric coloured operad.

Definition 1.4.6 ([Yau16], 13.2)**.** Let P be a non-symmetric C-coloured operad in $\mathcal{C}$. A ***P*-algebra** is a family $X = \{X_c\}_{c \in C}$ of objects in $\mathcal{C}$, together with maps

$$\alpha : P(\underline{c}; d) \otimes X_{\underline{c}} \to X_d$$

where

$$X_{\underline{c}} = \begin{cases} X_{c_1} \otimes \cdots \otimes X_{c_n} & \text{if } |c| > 0, \\ \kappa & \text{if } |c| = 0, \end{cases}$$

that are associative and unital in the following senses.

1. For $d \in C$, $\underline{c} = (c_1, \ldots, c_n) \in \mathrm{Prof}(C)$ with length $n \geqslant 1$, $\underline{b}_j \in \mathrm{Prof}(C)$ for $1 \leqslant j \leqslant n$, and $\underline{b} = (\underline{b}_1, \ldots, \underline{b}_n)$; the following associativity diagram commutes:

$$\begin{tikzcd}
P\binom{\underline{c}}{d} \otimes \left[\bigotimes_{j=1}^{n} P\binom{\underline{b}_j}{c_j}\right] \otimes X_{\underline{b}} \arrow[r, "(\gamma,\mathrm{id})"] \arrow[d, "\mathrm{shuffle}"', "\cong"] & P\binom{\underline{b}}{d} \otimes X_{\underline{b}} \arrow[dd, "\alpha"] \\
P\binom{\underline{c}}{d} \otimes \bigotimes_{j=1}^{n} \left[P\binom{\underline{b}_j}{c_j} \otimes X_{\underline{b}_j}\right] \arrow[d, "(\mathrm{id},\otimes_j \alpha)"'] & \\
P\binom{\underline{c}}{d} \otimes X_{\underline{c}} \arrow[r, "\alpha"] & X_d.
\end{tikzcd}$$

2. For each colour $c \in C$, the following unit diagram commutes:

$$\begin{tikzcd}
\kappa \otimes X_c \arrow[r, "\cong"] \arrow[d, "(1_c,\mathrm{id})"'] & X_c \\
P\binom{c}{c} \otimes X_c. \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

Again, what we are really interested in is a non-unital algebra over a non-symmetric coloured operad, defined via partial compositions. The following proposition gives an equivalent definition for such an object.

Proposition 1.4.7 ([Yau16], 16.7)**.** Let P be a non-symmetric C-coloured operad. A non-unital P-algebra is a family $X = \{X_c\}_{c \in C}$ of objects in $\mathcal{C}$, together with maps

$$\alpha : P(\underline{c}; d) \otimes X_{\underline{c}} \to X_d$$

where

$$X_{\underline{c}} = \begin{cases} X_{c_1} \otimes \cdots \otimes X_{c_n} & \text{if } |c| > 0, \\ \kappa & \text{if } |c| = 0, \end{cases}$$

that are associative in the following sense.

For $d \in C$, $\underline{c} = (c_1, \ldots, c_n) \in \mathrm{Prof}(C)$ with length $n \geqslant 1$, and $\underline{b} \in \mathrm{Prof}(C)$; the following associativity diagrams commute:

$$\begin{tikzcd}
P\binom{c}{d} \otimes P\binom{\underline{b}}{c_j} \otimes X_{\underline{c}\circ_j \underline{b}} \arrow[r, "{(\gamma_j,\mathrm{id})}"] \arrow[d, "\mathrm{Shuffle}"', "\cong"] & P\binom{\underline{c}\circ_j \underline{b}}{d} \otimes X_{\underline{c}\circ_j \underline{b}} \arrow[dd, "\alpha"] \\
P\binom{c}{d} \otimes \bigotimes_{r=1}^{j-1} X_{c_r} \otimes \left[P\binom{\underline{b}}{c_j} \otimes X_{\underline{b}}\right] \otimes \bigotimes_{r=j+1}^{n} X_{c_r} \arrow[d, "{(\mathrm{id},\alpha,\mathrm{id})}"'] & \\
P\binom{c}{d} \otimes X_{\underline{c}} \arrow[r, "\alpha"] & X_d.
\end{tikzcd} \tag{1.10}$$

1.5 Trees

In this section we introduce background material on graphs and trees, leading to the definition of planar trees. Most of this is following the definitions of Yau [Yau16] with some small convention changes. For example, Yau defines rooted trees where we want our trees to have a root vertex but no root edge. Following on from this, we define a structure with a distinguished set of internal vertices which we refer to as a partitioned tree. The remainder of this section will be devoted to establishing properties of partitioned trees and a process for constructing them which will be used in Chapter 2 to consider a combinatorial description of the cell structure of the topological spaces V_{ij}.

We begin by giving Yau's definition of a graph and a directed graph. All graphs we consider in this thesis will be finite graphs.

Definition 1.5.1. A **graph** G is an ordered pair (V, E) of disjoint sets in which E is a subset of $V^{\underline{2}} = \{\{x, y\} \mid x, y \in V, x \neq y\}$.

1. An element in V is called an **abstract vertex**.

2. An element $e = \{x, y\} \in E$ is called an **edge** with abstract end-vertices $x \in V$ and $y \in V$.

3. We say that a graph is **finite** if both V and E are finite sets, and **non-empty** if both V and E are non-empty.

4. A **path** P in a graph G is an ordered list of abstract vertices

$$P = (x_0, x_1, \ldots, x_l)$$

for some $l \geqslant 1$ such that $e_i = \{x_{i-1}, x_i\} \in E$ for each $1 \leqslant i \leqslant l$. Call l the **length** of the path. We say that such a path is **from** x_0 **to** x_l, that each edge e_i is **in** P, and that P **contains** e_i.

5. A **trail** is a path $(x_0, \ldots, x_l)$ whose edges $e_i = \{x_{i-1}, x_i\}$ for $1 \leqslant i \leqslant l$ are all distinct.

6. A **cycle** is a path such that the abstract vertices x_j for $1 \leqslant j \leqslant l$ with $l \geqslant 3$ are all distinct and $x_0 = x_l$.

7. A **forest** is a graph with no cycles.

8. We say that a graph G is **connected** if for each pair of distinct abstract vertices $x, y \in V$, there exists a path P such that $x_0 = x$ and $x_l = y$.

Definition 1.5.2. A **directed graph** is a graph $G = (V, E)$ in which each edge is an ordered pair of abstract vertices.

1. Suppose $e = (x, y)$ is an edge in a directed graph, it will be depicted as follows.

$$x \longrightarrow y$$

   Call x and y its **initial vertex** and **terminal vertex** respectively. Call e an **outgoing** edge of x, and an **incoming** edge of y.

2. For an abstract vertex v in a directed graph, the **set of incoming edges** and **set of outgoing edges** are written as $\mathrm{in}(v)$ and $\mathrm{out}(v)$ respectively.

The next definition is based upon the definition of a directed (m, n)-graph in [Yau16] however we change conventions to allow an input (defined below) to have any number of outgoing edges. We also add the definitions of a "node" and a "child vertex" which will be useful language later in the section.

Definition 1.5.3. Suppose $m, n \geqslant 0$.

1. A **directed $(\boldsymbol{m}, \boldsymbol{n})$-graph** is a quadruple

$$G = (V, E, \mathrm{in}_G, \mathrm{out}_G)$$

   consisting of a directed graph (V, E) and disjoint subsets $\mathrm{in}_G = \{v | \, \mathrm{in}(v) = \varnothing\}$ and $\mathrm{out}_G = \{v | \, \mathrm{out}(v) = \varnothing\}$ where $|\mathrm{in}_G| = m$ and $|\mathrm{out}_G| = n$.

2. In such a directed (m, n)-graph G, we define the subset $V_{t_G} = \{v \in V | v \notin (\mathrm{in}_G \amalg \mathrm{out}_G)\}$.

3. An abstract vertex $v \in V$ in a directed (m, n)-graph G is called

   - an **input** if $v \in \mathrm{in}_G$;
   - an **output** if $v \in \mathrm{out}_G$;

- an **internal vertex** if $v \in V_{t_G}$;
- a **node** if $|\mathrm{out}(v)| = 1$;
- a **child vertex of x** for $x \in V$ if v is the terminal vertex of an element of $\mathrm{out}(x)$.

4. An edge $e = (x, y) \in E$ in a directed (m, n)-graph G is called

   - an **input edge** if $x \in \mathrm{in}_G$;
   - an **output edge** if $y \in \mathrm{out}_G$;
   - an **internal edge** if $x, y \in V_{t_G}$.

   An **external edge** is an edge that is an input edge, an output edge, or both.

5. The **set of child vertices of x** for $x \in V$ is denoted by $Ch(x)$.

6. The **set of internal edges** in G is denoted by Int_G.

Notice that removing the condition $|\mathrm{out}(v)| = 1$ for $v \in \mathrm{in}_G$ from Yau's definition removes the condition of having a root edge to the graph. The next definition allows us to put some extra structure on a graph by putting an ordering on the output vertices.

Definition 1.5.4. Suppose G is a directed (m, n)-graph for some $m, n \geqslant 0$. An **output labelling** of G is a bijection $\lambda : [n] \to \mathrm{out}_G$, where

$$[n] = \begin{cases} \{1, \ldots, n\} & \text{if} \quad n \geqslant 1, \\ \varnothing & \text{if} \quad n = 0. \end{cases}$$

In the next definition we change conventions from Yau's definition to have trees defined to "grow upwards" with one incoming edge and m-outgoing edges.

Definition 1.5.5. Suppose m is a positive integer. A ***m*-tree** T is a connected directed $(1, m)$-graph such that $|\text{in}(v)| = 1$ for each $v \in V_{t_T}$.

1. We call the single element of in_T the **root node** of T and denote this by rt_T.

2. An ***m*-corolla** is a m-tree T such that $V_{t_T} = \varnothing$.

A **tree** is an m-tree for some $m \geqslant 1$.

Remark 1.5.6. Notice that if T is an m-corolla then it is sufficient to specify $V = \{v_0, v_1, \ldots, v_m\}$ and $\text{in}_T = \{v_0\}$ since we must have $\text{out}_T = V \setminus \text{in}_T$ and $E = \{(v_0, v_1), (v_0, v_2), \ldots, (v_0, v_m)\}$.

Definition 1.5.7. A **planar tree** is a tree with an embedding into the strip $\mathbb{R} \times [0, 1]$ with the root sent to $\mathbb{R} \times \{0\}$ and the leaves sent to $\mathbb{R} \times \{1\}$, up to isotopies respecting these constraints. Such a structure induces an output labelling on the tree.

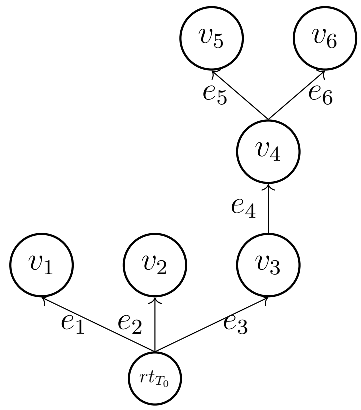
We now introduce the definition of a partitioned tree as a tree with a specified subset of vertices called the "cut set". From this point onwards, all trees that we consider will be planar trees.

Definition 1.5.8. Suppose $r, n \in \mathbb{N}$ such that $n \geqslant 1$. An ***r*-partitioned *n*-tree** is a planar n-tree $T = (V, E, \text{in}_T, \text{out}_T)$ with a specified subset $C \subseteq V_{t_T}$, such that $|C| = r$, which we call the **cut set**.

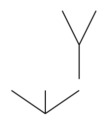
In particular, a 0-partitioned n-tree is just an n-tree.

For example, the tree

$T =$



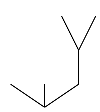
with cut set $C = \{v_3\}$, is a 1-partitioned 4-tree. We can represent this by

$T =$ 

in which the partition is represented by a gap and we drop the labels on the edges and vertices.

Definition 1.5.9. If T is an r-partitioned n-tree with cut set C, we recover the n-tree T by forgetting C. We call this the **closure of the partitioned tree $\boldsymbol{T}$**.

In the example above the closure of T would be the tree represented by



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Next we introduce the definition of an isomorphism of trees. Notice that by dropping the labelling of vertices and edges from our diagrams, we have in a sense already been considering isomorphism classes of trees.

Definition 1.5.10. Suppose $T_1 = (V_1, E_1, rt_{T_1}, \mathrm{out}_{T_1})$ and $T_2 = (V_2, E_2, rt_{T_2}, \mathrm{out}_{T_2})$ are two m-trees. An **isomorphism of trees**

$$\zeta : T_1 \to T_2$$

consists of two bijections

$$V_1 \xrightarrow{\zeta_V} V_2 \quad \text{and} \quad E_1 \xrightarrow{\zeta_E} E_2$$

that preserve edge orientations i.e.

$$e = (x, y) \in E_1 \quad \text{if and only if} \quad \zeta(e) = (\zeta(x), \zeta(y)) \in E_2$$

and the restrictions of ζ,

$$\mathrm{in}_{T_1} \xrightarrow{\zeta} \mathrm{in}_{T_2} \quad \text{and} \quad \mathrm{out}_{T_1} \xrightarrow{\zeta} \mathrm{out}_{T_2}$$

are bijections. Since all our trees are planar, the isomorphism is also required to respect the planar structre, and so must also preserve the output labelling.

Remark 1.5.11. It is important to note that there is at most one isomorphism between any two planar trees, or equivalently, all automorphisms are identities. Indeed, the output labelling condition means that any automorphism acts as the identity on leaves, and it must also commute with the parent map, and the claim follows easily from that. It is because of this that it is harmless to consider isomorphism classes of trees.

Next we introduce the grafting of two trees at a given vertex. The first definition gives grafting for a planar tree, then we describe how this can be extended to a grafting of partitioned planar trees.

Definition 1.5.12. Suppose $T_1 = (V_1, E_1, \mathrm{in}_{T_1}, \mathrm{out}_{T_1})$ is an n-tree, $v \in \mathrm{out}_{T_1}$, $T_2 = (V_2, E_2, \mathrm{in}_{T_2}, \mathrm{out}_{T_2})$ is an m-tree, and rt_{T_2} is the root of T_2, i.e. $\mathrm{in}_{T_2} = \{rt_{T_2}\}$, with V_1 and V_2 disjoint. Define the $(n + m - 1)$-tree

$$T = T_1 \circ_v T_2 = (V_T, E_T, \mathrm{in}_T, \mathrm{out}_T)$$

as the tree with

- $V_T = \frac{V_1 \amalg V_2}{(v \sim rt_{T_2})}$,

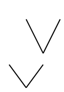
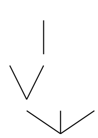
- $E_T = E_1 \amalg E_2$,
- $\mathrm{in}_T = \mathrm{in}_{T_1}$;
- and $\mathrm{out}_T = (\mathrm{out}_{T_1} \setminus \{v\}) \amalg \mathrm{out}_{T_2}$.

Call $T_1 \circ_v T_2$ the **grafting** of T_1 and T_2 via v. Such grafting has a clear compatibility with the planar structure of the trees. We can see that using the embeddings of T_1 and T_2 we have an embedding of $T_1 \circ_v T_2$ in $\mathbb{R} \times [0, 2]$ with leaves of T_1 at $\mathbb{R} \times \{1\}$ and leaves of T_2 at $\mathbb{R} \times \{2\}$. By performing an isotopy to horizontally scale T_2 to width δ where δ is less than the distance between v_{i-1} and v_{i+1} in $\mathbb{R} \times \{1\}$ we can then extend leaves in $\mathbb{R} \times \{1\}$ up to $\mathbb{R} \times \{2\}$. Finally we perform an isotopy to scale $T_1 \circ_v T_2$ from $\mathbb{R} \times [0, 2]$ to $\mathbb{R} \times [0, 1]$.

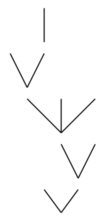
Definition 1.5.13. If $T_1 = (V_1, E_1, \mathrm{in}_{T_1}, \mathrm{out}_{T_1})$ is an r-partitioned tree with cut set C_1, and $T_2 = (V_2, E_2, \mathrm{in}_{T_2}, \mathrm{out}_{T_2})$ is an s-partitioned tree with cut set C_2, then we make the grafting $T_1 \circ_v T_2$ into a $(r + s + 1)$-partitioned tree by defining the cut set of $T_1 \circ_v T_2$ to be $C_1 \amalg C_2 \amalg \{v\}$. We call this a **partitioned grafting** and denote by $T_1 \wedge_v T_2$.

Remark 1.5.14. Notice that since T_1 is a planar n-tree we have a specified output labelling, $\lambda : [n] \to \mathrm{out}_{T_1}$, so $v \in \mathrm{out}_{T_1}$ has $\lambda(k) = v$ for some $k \in [n]$. As a result, we can denote $T_1 \wedge_v T_2$ by $T_1 \wedge_{\lambda(k)} T_2$, or $T_1 \wedge_k T_2$ for short.

Example 1.5.15. Let

$T_1 =$  , $T_2 =$  ,

where T_1 is a 1-partitioned 3-tree, and T_2 is a 2-partitioned 4-tree. Then the partitioned grafting

$T_1 \wedge_2 T_2 =$ 

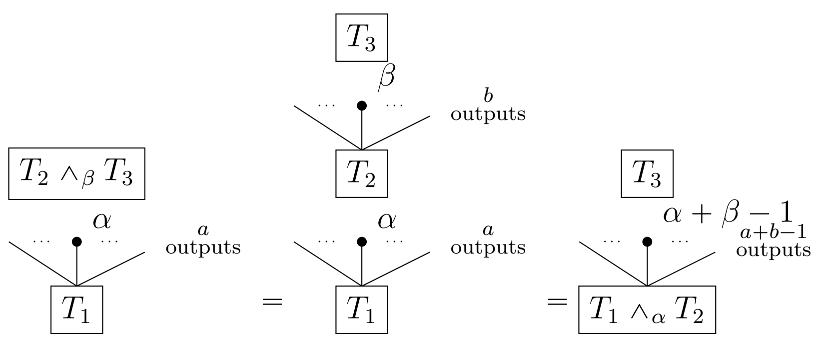
is a 4-partitioned 6-tree.

Lemma 1.5.16. Partitioned grafting is associative, i.e. if we have three planar partitioned trees T_1, T_2, T_3 where $|\mathrm{out}_{T_1}| = a$, and $|\mathrm{out}_{T_2}| = b$, then

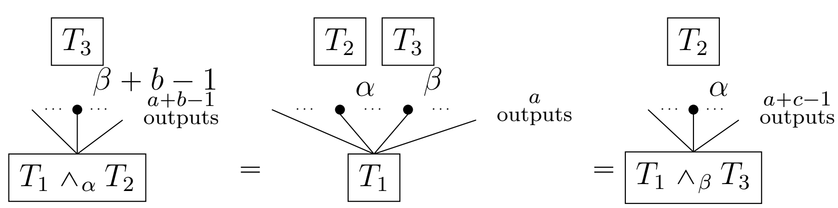
1. $T_1 \wedge_\alpha (T_2 \wedge_\beta T_3) = (T_1 \wedge_\alpha T_2) \wedge_{\alpha+\beta-1} T_3$, for $1 \leqslant \alpha \leqslant a$, $1 \leqslant \beta \leqslant b$, and

2. $(T_1 \wedge_\alpha T_2) \wedge_{\beta+b-1} T_3 = (T_1 \wedge_\beta T_3) \wedge_\alpha T_2$ for $1 \leqslant \alpha < \beta$.

Proof. The proof of this lemma follows directly from the definition of grafting. If we draw the structure of the trees on both sides, then relation 1 looks like:



and relation 2 looks like:



.

With planar structure on grafting defined as in Definition 1.5.12 it is clear that these associativity relations respect the planar structure up to isotopy. In particular the induced output labelling on $T_1 \wedge_\alpha (T_2 \wedge_\beta T_3)$ is equal to that on $(T_1 \wedge_\alpha T_2) \wedge_{\alpha+\beta-1} T_3$, and similarly for the second relation. $\square$

Remark 1.5.17. It is worth noting that the associativity relations for grafting of planar partitioned trees are exactly the associativity relations for partial compositions in a non-symmetric non-unital operad. So grafting is a kind of partial composition for planar partitioned trees.

The next definition gives two different ways to get a $(k+1)$-partitioned tree from a k-partitioned tree. In the following chapters, we will see that the splitting of a tree relates to the boundary component of the cell it represents in V_{ij}. The terminology D_∞-type and A_∞-type splitting is chosen to refer to the types of splittings in the trees that represent the cells of D_∞ and A_∞ spaces.

Definition 1.5.18. Let T be a k-partitioned m-tree with cut set C. A **splitting** of T at $\alpha \in V$ is a $(k+1)$-partitioned m-tree formed in one of the following ways.

1. If $\alpha \in V_{t_T}$ so $|\mathrm{out}(\alpha)| \geqslant 1$, and $\alpha \notin C$, we take T' to be T with the new cut set $C' = C \amalg \{\alpha\}$. We call this a ***D_∞*-type splitting**.

2. If $\alpha \in V \backslash \mathrm{out}_T$ and $|\mathrm{out}(\alpha)| = n_\alpha$ with $n_\alpha \geqslant 3$ then we choose J a proper non-empty interval in $Ch(\alpha)$ with $1 < |J| < n_\alpha$. Then T' is given by

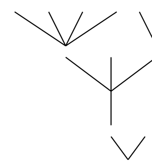
   - adding a new vertex β so that $V' = V \amalg \{\beta\}$;
   - adding edges (β, v) and removing edges (α, v) for all $v \in J$;
   - adding an edge (α, β);
   - adding β to the cut set so that $C' = C \amalg \{\beta\}$.

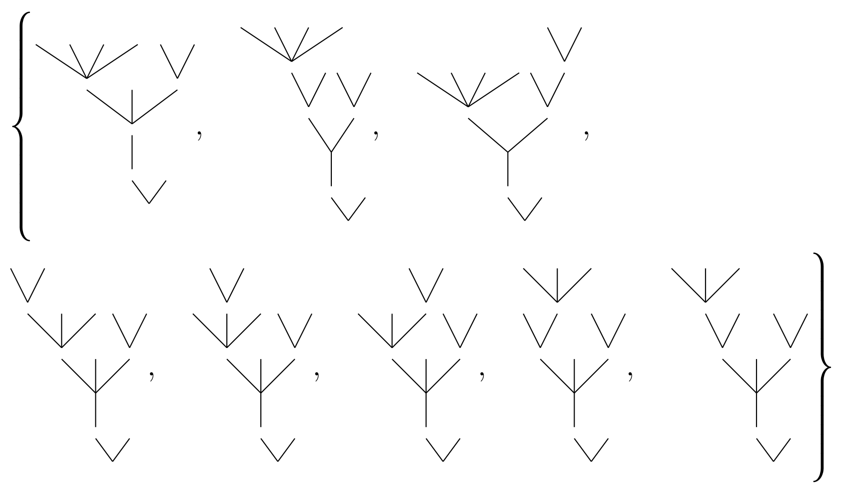
   We call this an ***A_∞*-type splitting**.

In both of the above cases, we refer to the node α as the **source of the splitting**.

Remark 1.5.19. If α has three or more children then there are $\frac{1}{2}(n_\alpha - 2)(n_\alpha + 1)$ A_∞-type splittings of T, where $n_\alpha = |\mathrm{out}(\alpha)|$. We prove this in Proposition 1.6.3 as the special case with $k = 1$.

Definition 1.5.20. Let t be a k-partitioned planar tree. We denote by $\mathbf{Sp}(\boldsymbol{t})$ the set of $(k+1)$-partitioned planar trees which are splittings of t.

Example 1.5.21. Consider the 3-partitioned tree $t =$  . Then the splittings of t are given by

$\mathrm{Sp}(t) =$ 

The first splitting listed is a D_∞-type splitting. The other two splittings on the top row are the two possible A_∞-type splittings of the 3-corolla in the middle of the tree. The five splittings on the bottom row are the five possible A_∞-type splittings of the 4-corolla at the top left of t.

Next we introduce decorated tree diagrams which are ordered sets of planar trees with some distinguished vertices. This will allow us to introduce a process for building planar partitioned trees. Recall that an ordered partition of $n \in \mathbb{N}$ is a collection of natural numbers n_i such that $\sum n_i = n$.

Definition 1.5.22. A ***n*-tree diagram of length $r+1$** is an ordered set $T = (t_0, \dots, t_r)$ of planar n_i-trees t_i for $0 \leqslant i \leqslant r$ with $n_0 + \cdots + n_r = n + r$.

Definition 1.5.23. A **decorated *n*-tree diagram of length $r+1$** is a n-tree diagram, $T = (t_0, \dots, t_r)$, with a specified subset $D \subseteq (\mathrm{out}_{t_0} \amalg \cdots \amalg \mathrm{out}_{t_r})$, of output vertices such that $|D| = r$.

1. We call D the **set of distinguished vertices**.

2. We can define the **set of distinguished vertices in t_i** by $D_i = D \cap \mathrm{out}_{t_i}$.

3. We define the **set of root vertices** by $Rt_T = \{rt_{t_0}, rt_{t_1}, \dots, rt_{t_r}\}$.

4. Since each $t_i \in T$ is a planar n_i-tree we have a specified output labelling, $\lambda_i : [n_i] \to \mathrm{out}_{t_i}$ for $0 \leqslant i \leqslant r$. This induces a labelling $\gamma : [r] \to D$ on D.

We can now introduce our **tree building process** which constructs a planar tree from a decorated tree partition diagram.

Definition 1.5.24. We define a process to construct a k-partitioned m-tree from a decorated m-tree diagram of length $k+1$. We will refer to this process as the **tree building process**.

We begin with a decorated m-tree diagram, $T = (t_0, \dots, t_k)$. Let $\gamma_i(T)$ denote the effect of gluing the root of t_{i+1} to the leftmost distinguished vertex of t_i (if one exists). Here t_{i+1} should be interpreted as t_0 if $i = k$ (i.e. if t_i is the last component). So

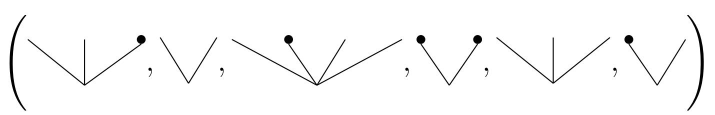
$$\gamma_i(T) = (t_0, \dots, t_{i-1}, (t_i \wedge_\alpha t_{i+1}), t_{i+2}, \dots, t_k)$$

where α is the leftmost distinguished vertex of t_i. We apply any valid sequence of γ's until we reach a diagram with a single component.

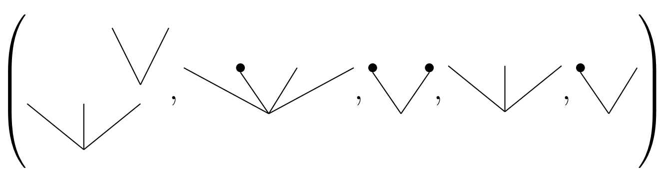
We will prove in Proposition 1.5.31 that the output of this process is well defined i.e. independant of the sequence of graftings. Notice that this would be clear if we did not allow the grafting of t_0 into t_k.

Remark 1.5.25. Notice that since each entry of the tree diagram is a 0-partitioned tree, the partitioned grafting taken in the tree building process assigns a cut set with k elements to the output tree in which the vertex at each grafting point is included in the cut set.

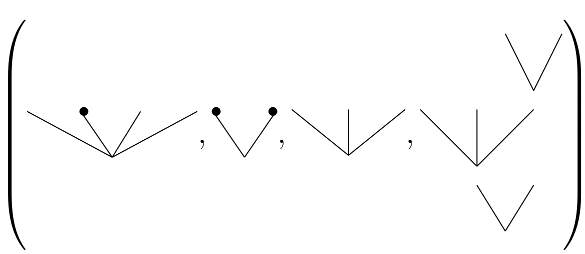
Example 1.5.26. Suppose we have the following decorated tree partition,

$(t_0, t_1, \ldots, t_5) =$ 

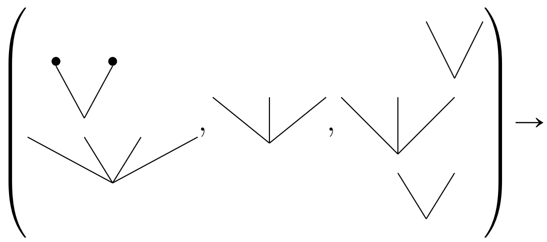
Then we could apply γ_1 to give,

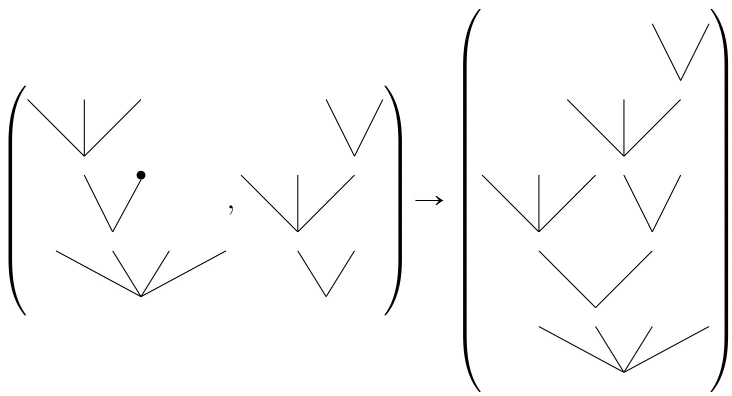


Now, we could apply γ_5 to give



The rest of the process could continue by applying γ_1's as follows:





Remark 1.5.27. Notice that the output of the tree building process is

$$T = (((((t_2 \wedge_2 t_3) \wedge_2 t_4) \wedge_5 t_5) \wedge_5 t_0) \wedge_7 t_1)$$

with cut set corresponding to the decorated vertices.

Definition 1.5.28. Suppose $T_1 = (V_1, E_1, rt_{T_1}, \mathrm{out}_{T_1})$ and $T_2 = (V_2, E_2, rt_{T_2}, \mathrm{out}_{T_2})$ are two k-partitioned m-trees, with cut sets C_1 and C_2 respectively. An **isomorphism of partitioned trees** is an isomorphism of trees that also preserves the cut set i.e. for $v \in C_1$, $\zeta_E(v) \in C_2$.

The following proposition highlights a special property of grafting of corollas. This result and the subsequent corollary will be useful in Section 1.6 when we think about applying the tree building process to decorated tree diagrams in which each entry is a corolla.

Proposition 1.5.29. Let C_i be an m_i-corolla for $i = 1, \ldots, 4$. We have $C_1 \circ_k C_2 \cong C_3 \circ_r C_4$ if and only if $C_1 \cong C_3$, $C_2 \cong C_4$, and $k = r$.

Proof. Suppose that $T \cong C_1 \circ_k C_2$. Then

- T has a unique internal vertex which we can call v;
- m_1 is one plus the number of leaves which are not children of v;
- m_2 is the number of children of v;
- k is one plus the number of leaves that are not children of v but that lie to the left of all children of v.

From these descriptions it is clear that m_1, m_2 and k are isomorphism invariants. □

Corollary 1.5.30. Two partitioned graftings of corollas are isomorphic if and only if their graftings are isomorphic, i.e. $C_1 \wedge_k C_2 \cong C_3 \wedge_r C_4$ if and only if $C_1 \circ_k C_2 \cong C_3 \circ_r C_4$.

Proof. If $C_1 \wedge_k C_2 \cong C_3 \wedge_r C_4$ then clearly $C_1 \circ_k C_2 \cong C_3 \circ_r C_4$. If $C_1 \circ_k C_2 \cong C_3 \circ_r C_4$ then by Proposition 1.5.29 we have $C_1 \cong C_3$, $C_2 \cong C_4$, and $k = r$. Hence the cut set for each tree would be $\{k\} = \{r\}$ so $C_1 \wedge_k C_2 \cong C_3 \wedge_r C_4$. □

From this point onwards we will be working with isomorphism classes of trees. For simplicity we just refer to these as trees. Our next proposition shows that two decorated m-tree diagrams of length $k + 1$ which are cyclic permutations of one another will produce the same partitioned tree. Possibly a little more surprising a result is Proposition 1.5.32 which shows that two decorated partition diagrams which are not cyclic permutations of one another cannot produce the same partitioned tree.

Proposition 1.5.31. A decorated m-tree diagram A has a unique output tree $\tau(A)$ from the tree building process of Definition 1.5.24, and two decorated m-tree diagrams of length $k + 1$ will produce the same k-partitioned

m-tree via the tree building process if they are cyclic permutations of one another.

Proof. Let us begin by considering a decorated m-tree diagram $A = (t_0, \ldots, t_k)$. We will show that if $i < j$ then $\gamma_{j-1}\gamma_i(A) = \gamma_i\gamma_j(A)$.

If $j > i + 1$ then clearly

$$\begin{aligned}\gamma_{j-1}\gamma_i(A) &= (t_0, \ldots, (t_i \wedge_\alpha t_{i+1}), t_{i+2}, \ldots, t_{j-1}, (t_j \wedge_\beta t_{j+1}), \ldots, t_k) \\ &= \gamma_i\gamma_j(A).\end{aligned}$$

If $j = i + 1$, then

$$\gamma_i\gamma_{i+1}(A) = (t_0, \ldots, t_{i-1}, t_i \wedge_\alpha (t_{i+1} \wedge_\beta t_{i+2}), \ldots, t_k)$$

and

$$\gamma_i\gamma_i(A) = (t_0, \ldots, t_{i-1}, (t_i \wedge_\alpha t_{i+1}) \wedge_{\alpha+\beta-1} t_{i+2}, \ldots, t_k).$$

So $\gamma_{j-1}\gamma_i(A) = \gamma_i\gamma_j(A)$ by Lemma 1.5.16. Now we proceed by induction on $|A|$.

If $|A| = 1$ then there are no steps to take. If $|A| = 2$ then there is only one possible choice of grafting and so $\tau(A)$ is unique.

Now for $k \geqslant 3$ assume at for $|A| \leqslant k$ there is a unique resulting tree $\tau(A)$. Then for $|B| = k + 1$, $|\gamma_i(B)| = k$ so has a unique output $\tau(\gamma_i(B))$ by the induction assumption, and $|\gamma_j(B)| = k$ so has a unique output $\tau(\gamma_j(B))$, where we take $i < j$ without loss of generality. Now since $\gamma_{j-1}\gamma_i(B) = \gamma_i\gamma_j(B)$, we must have $\tau(\gamma_i(B)) = \tau(\gamma_j(B))$ and hence $\tau(B)$ is unique.

Finally since the operation γ_i have an obvious compatibility with cyclic permutation, we see that cyclically permuting A does not affect the resulting tree $\tau(A)$. □

Proposition 1.5.32. Two decorated m-tree diagrams of length $k+1$ that are not cyclic permutations of one another will produce different k-partitioned

m-trees via the tree building process.

Proof. Suppose we have two decorated m-tree diagrams of length $k+1$ that result in the same partitioned tree but are not cyclic permutations of one another. Say we have $T = (t_0, \ldots, t_k)$, and $B = (b_0, \ldots, b_k)$. Since by Proposition 1.5.31 we know that cyclic permutation will not affect the resulting tree, we can perform a cyclic permutation so that the root of t'_0 is the root of the resulting tree $\tau(T)$, and the root of b'_0 is the root of the resulting tree $\tau(B)$.

Now we can repeatedly apply γ_1 to each diagram until we have a unique resulting tree (since we have performed a cyclic permutation in order to make this choice valid). Since the two partition diagrams are not equal, at some point $t_n \neq b_n$ for $1 \leqslant n \leqslant k$. However, since we are applying the same grafting to both diagrams, t_n and c_n will be grafted into the same position in their respective trees, and so the partitioned trees cannot be the same. □

Corollary 1.5.33. The tree building process produces a unique k-partitioned m-tree from each decorated m-tree diagram of length $k+1$ up to cyclic permutation, i.e. two decorated m-tree diagrams of length $k+1$ produce the same k-partitioned m-tree under the tree building process if and only if they are cyclic permutations of one another.

Proof. This follows directly from Proposition 1.5.31 and Proposition 1.5.32. □

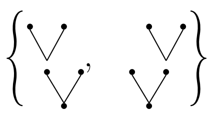
1.6 Counting sets of trees

In this section we will define all the necessary combinatorial structure for the rest of the thesis. The key points of this section are the definition of the set of trees $\mathcal{T}^k_{i,j}$ and Propositions 1.6.4 and 1.6.5 which provide arguments for counting the number of elements of $\mathcal{T}^k_{i,j}$. These results will be used in Chapter 2 for counting the number of cells in each dimension of V_{ij}.

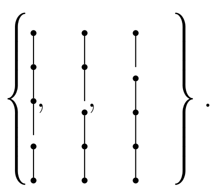
We specify planar trees by the number of outputs, the number of partitions, and the number of nodes.

Definition 1.6.1. For $i, k \geqslant 0$, $j \geqslant 1$, we denote by $\mathcal{T}^k_{i,j}$ the set of k-partitioned planar j-trees with i nodes, in which for any vertex v with $|\mathrm{out}(v)| \geqslant 2$, the children must be leaves, or nodes, or cut points.

So in Example 1.5.21, t is in $\mathcal{T}^3_{1,8}$. Some other small examples are:

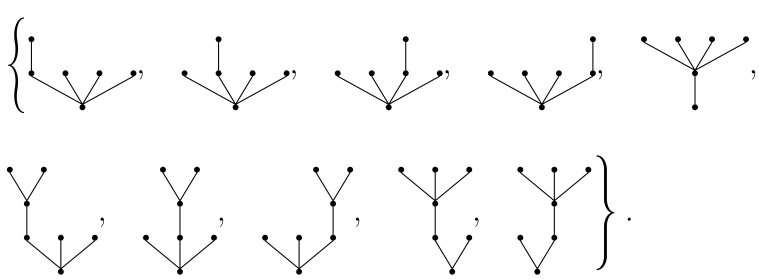
$\mathcal{T}^1_{0,3} =$ 

and,

$\mathcal{T}^1_{4,1} =$ 

Now we will consider a more complex example.

Example 1.6.2. We consider the elements of $\mathcal{T}^0_{1,4}$. The trees in this set must have 4 outputs, 1 node, and no partitions so for any node with more than one outgoing edge, its child vertices have a maximum of one outgoing edge. From these conditions we can see that the elements of $\mathcal{T}^0_{1,4}$ are:



Notice that the first five trees are the 4-corolla with a single edge affixed in all possible different positions, while the second five trees are the five possible A_∞-type splittings of the 4-corolla with a single edge between the two pieces.

The following three propositions provide counting arguments for the elements of $\mathcal{T}_{i,j}^k$. We first restrict to $i = 0$ and Proposition 1.6.3 provides a counting argument for this case, then Proposition 1.6.4 provides a counting argument for the special case of $k = 0$. We do not restrict to $j = 1$ because restricting to 1-trees reduces this case to counting ordered partitions of i elements. Proposition 1.6.5 uses the special cases to provide a counting argument for general $\mathcal{T}_{i,j}^k$.

Proposition 1.6.3. The number of trees in $\mathcal{T}_{0,n}^k$ is given by $T(n+1,k) = \frac{1}{k+1}\binom{n-2}{k}\binom{n+k}{k}$.

Proof. For the elements of $\mathcal{T}_{0,n}^k$, there are no nodes, so we must take every internal vertex as a cut point. Thus, we are just counting trees with n leaves, no nodes, and k internal vertices. So we construct and count all trees of this type.

We begin by taking an ordered partition of the $(n+k)$ edges into $(k+1)$ parts, in which each part is greater than or equal to two (because we are not allowed to have any nodes). To do this we take an ordered partition of $(n+k)-(k+1) = n-1$ into $(k+1)$ parts and then add one to each part. There are $\binom{n-2}{k}$ ordered partitions of this type. This gives us a n-tree diagram of length $k+1$ in which each element is a corolla with at least two edges.

In order to apply the tree building process we need to choose k of the $(n+k)$ outputs to be distinguished. There are $\binom{n+k}{k}$ ways of doing this. This gives us a decorated n-tree diagram of length $k+1$ to which we can apply the tree building process.

So far we have $\binom{n-2}{k}\binom{n+k}{k}$ different decorated partition diagrams. However, by Corollary 1.5.33 we know that decorated n-tree diagrams only produce a unique k-partitoned n-tree up to cyclic permutation. The cyclic group of order k acts freely on the set of decorated n-tree diagrams, because if a diagram has stabiliser of order m then m must divide both the number $k+1$ of roots and the number k of decorated points, so $m = 1$.

So for each decorated n-tree diagram of length $k+1$, we have counted the same k-partitioned n-tree $k+1$ times. Hence, the number of n-trees in $\mathcal{T}_{0,n}^k$ is given by $T(n+1,k) = \frac{1}{k+1}\binom{n-2}{k}\binom{n+k}{k}$. $\square$

Proposition 1.6.4. The number of trees in $\mathcal{T}_{i,j}^0$ is given by the Narayana number $N(i+j,j) = \frac{1}{i+j}\binom{i+j}{j}\binom{i+j}{j-1}$.

Proof. Suppose $T \in \mathcal{T}_{ij}^0$. We begin by adding a node at the root of T. Now let m be the number of non-nodal internal vertices. By assumption, beneath every non-nodal internal vertex we have at least one node. We can remove one such node in each case, leaving a tree with i internal vertices. This construction gives a bijection from $\mathcal{T}_{ij}^0$ to the full set of trees with j leaves and i internal vertices.

Now let us say that a Narayana path of type (n,j) is a sequence $u \in \{-1,1\}^{2n}$ such that $\sum_{i=1}^m u_i \geqslant 0$ for all m and $\sum_{i=1}^{2n} u_i = 0$ and there are j peaks (i.e. adjacent pairs of the form $(1,-1)$). Suppose that T is a tree with j leaves and i internal vertices. We can walk clockwise around the tree, starting on the left hand side of the root, recording a $+1$ for each upwards step and a -1 for each downward step. There are $i+j$ edges, and we walk up the left hand side of each one and down the right hand side, giving $2(i+j)$ steps in total. There is a peak for each leaf. Thus, we have a path in $\mathcal{N}(i+j,j)$. It is not hard to see that this gives a bijection between trees and Narayana paths.

We know from Petersen [Pet15] that Narayana paths are counted by the Narayana numbers. Hence there are $N(i+j,j)$ trees in $\mathcal{T}_{ij}^0$. $\square$

Proposition 1.6.5. Let

$$F_k(x,y) = \sum_{\substack{i\geqslant 0,\\ j\geqslant 1}} |\mathcal{T}_{ij}^k| x^i y^j$$

then

$$F_0(x,y) = \sum_{(i,j)\neq(0,1)} N(i+j,j)x^i y^j$$

and

$$F_k(x,y) = \frac{1}{(k+1)!}\frac{\partial^k}{\partial y^k}F_0^{k+1}.$$

Proof. The claim is equivalent to saying that number of trees in $\mathcal{T}_{i,j}^k$ is given by

$$\frac{1}{k+1}\binom{j+k}{k} \sum_{\substack{i=u_0+\cdots+u_k,\\ j+k=v_0+\cdots+v_k,\\ u_\alpha\geqslant 0, v_\alpha\geqslant 1 \text{ for all } \alpha,\\ (u_\alpha,v_\alpha)\neq(0,1) \text{ for any } \alpha}} \prod_{\alpha=0}^{k} N(u_\alpha+v_\alpha, v_\alpha).$$

The elements of $\mathcal{T}_{i,j}^k$ are k-partitioned j-trees with i nodes. Notice that a k-partitioned tree is the same as a grafting of $(k+1)$ 0-partitioned trees. So for any $t \in \mathcal{T}_{i,j}^k$, $t = t_{u_0,v_0} \wedge \cdots \wedge t_{u_k,v_k}$ where $t_{u_\alpha,v_\alpha} \in \mathcal{T}_{u_\alpha,v_\alpha}^0$ for $0 \leqslant \alpha \leqslant k$ and $u_0 + \cdots + u_k = i$, and $v_0 + \cdots + v_k = j+k$. We take $v_0 + \cdots + v_k = j+k$ because we want the whole tree to have j outputs, and each time we graft two trees one output becomes an internal edge.

For each set $\mathcal{T}_{u_\alpha,v_\alpha}^0$, there are $N(u_\alpha+v_\alpha, v_\alpha)$ trees (by Proposition 1.6.4). So we count the number of possible tree diagrams of the form $(t_{u_0,v_0},\ldots,t_{u_k,v_k})$. There are

$$\sum_{\substack{i=u_0+\cdots+u_k,\\ j+k=v_0+\cdots+v_k,\\ u_\alpha\geqslant 0, v_\alpha\geqslant 1,\\ (u_\alpha,v_\alpha)\neq(0,1)}} N(u_0+v_0,v_0)\cdots N(u_k+v_k,v_k)$$

such tree diagrams.

There are a total of $j+k$ outputs in the subtrees and we must choose k

of them to be distinguished so there are $\binom{j+k}{k}$ possible ways of doing this. Finally, by Corollary 1.5.33, we must divide by $k+1$ to account for the fact that cyclic permutation means we have counted each partitioned tree $k+1$ times.

Therefore, the number of trees in $\mathcal{T}_{i,j}^{k}$ is given by

$$\frac{1}{k+1}\binom{j+k}{k} \sum_{\substack{i=u_0+\cdots+u_k,\\ j+k=v_0+\cdots+v_k,\\ u_\alpha\geqslant 0, v_\alpha\geqslant 1 \text{ for all } \alpha,\\ (u_\alpha,v_\alpha)\neq(0,1) \text{ for any } \alpha}} \prod_{\alpha=0}^{k} N(u_\alpha+v_\alpha, v_\alpha).$$

□

We now aim to simplify this result using the identity given in the following Proposition.

Proposition 1.6.6 ([Def])**.** Let us define

$$G_k(s,t) = \sum_{\substack{n>0,\\ l\geqslant 0}} \frac{k+1}{n}\binom{n}{l}\binom{n}{l+k+1} s^n t^l,$$

Then $G_k(s,t) = G_0(s,t)^{k+1}$. □

Remark 1.6.7. If we let

$$A(n,k,l) = \sum_{\substack{i_0+\cdots+i_k=n,\\ j_0+\cdots+j_k=l,\\ i_t\geqslant 1, j_t\geqslant 0}} \prod_{t=0}^{k} N(i_t, j_t+1),$$

then the claim is equivalent to $A(n,k,l) = \frac{k+1}{n}\binom{n}{l}\binom{n}{l+k+1}$.

Proposition 1.6.8. Let

$$N_r(n,k) = \frac{r+1}{n}\binom{n}{k+r}\binom{n}{k-1}$$

be the generalised Narayana number. Then

$$\sum_{\substack{i=u_0+\cdots+u_k,\\ j+k=v_0+\cdots+v_k,\\ u_\alpha\geqslant 0, v_\alpha\geqslant 1,\\ (u_\alpha,v_\alpha)\neq(0,1)}} \prod_{\alpha=0}^{k} N(u_\alpha+v_\alpha, u_\alpha+1) = \sum_{\alpha=0}^{k}(-1)^{k-\alpha}\binom{k+1}{k-\alpha}N_\alpha(i+j+\alpha, i+1).$$

Proof. We will prove the proposition by induction on k. We begin by considering the initial cases $k=0$ and $k=1$. When $k=0$,

$$\begin{aligned}
\sum_{\substack{i=u_0,\\ j+k=v_0,\\ u_\alpha\geqslant 0, v_\alpha\geqslant 1,\\ (u_\alpha,v_\alpha)\neq(0,1)}} \prod_{\alpha=0}^{0} N(u_\alpha+v_\alpha, u_\alpha+1) &= N(i+j, i+1) \\
&= \frac{1}{i+j}\binom{i+j}{i+1}\binom{i+j}{i} \\
&= N_0(i+j, i+1).
\end{aligned}$$

When $k=1$, we consider the sum as the sum over all pairs in which we allow $(u_\alpha, v_\alpha) = (0,1)$ and use Proposition 1.6.6, then subtract the cases where one of the two pairs is $(0,1)$, i.e.

$$\begin{aligned}
&\sum_{\substack{i=u_0+u_1,\\ j+k=v_0+v_1,\\ u_\alpha\geqslant 0, v_\alpha\geqslant 1,\\ (u_\alpha,v_\alpha)\neq(0,1)}} \prod_{\alpha=0}^{1} N(u_\alpha+v_\alpha, u_\alpha+1) \\
&\qquad = \sum_{\substack{i=u_0+u_1,\\ j+k=v_0+v_1,\\ u_\alpha\geqslant 0, v_\alpha\geqslant 1}} \prod_{\alpha=0}^{1} N(u_\alpha+v_\alpha, u_\alpha+1) - 2N(i+j, i+1) \\
&\qquad = N_1(i+j+1, i+1) - 2N_0(i+j, i+1) \\
&\qquad = \sum_{\alpha=0}^{1}(-1)^{1-\alpha}\binom{2}{1-\alpha}N_\alpha(i+j+\alpha, i+1).
\end{aligned}$$

Now let

$$L(i,j,k) = \sum_{\substack{i=u_0+\cdots+u_k,\\ j+k=v_0+\cdots+v_k,\\ u_\alpha\geqslant 0, v_\alpha\geqslant 1,\\ (u_\alpha,v_\alpha)\neq(0,1)}} \prod_{\alpha=0}^{k} N(u_\alpha+v_\alpha, u_\alpha+1),$$

and

$$B(i,j,k) = \sum_{\alpha=0}^{k}(-1)^{k-\alpha}\binom{k+1}{k-\alpha}N_\alpha(i+j+\alpha, i+1),$$

and assume that the proposition holds for all $k \leqslant t$. Then for $k = t+1$ we have

$$\begin{aligned}
L(i,j,t+1) = & \sum_{\substack{i=u_0+\cdots+u_{t+1},\\ j+k=v_0+\cdots+v_{t+1},\\ u_\alpha\geqslant 0, v_\alpha\geqslant 1}} \prod_{\alpha=0}^{t+1} N(u_\alpha+v_\alpha, u_\alpha+1) - \\
& \binom{t+2}{1}\begin{bmatrix}\text{sum with}\\ \text{exactly one}\\ \text{pair}=(0,1)\end{bmatrix} - \cdots - \binom{t+2}{t+1}\begin{bmatrix}\text{sum with}\\ \text{exactly } t+1\\ \text{pairs}=(0,1)\end{bmatrix} \\
= & A(i+j+t+1, t+1, i) - \sum_{s=0}^{t}\binom{t+2}{t+1-s}B(i,j,s).
\end{aligned}$$

Now

$$\begin{aligned}
&\sum_{s=0}^{t}\binom{t+2}{t+1-s}B(i,j,s) \\
&= \sum_{s=0}^{t}\sum_{r=0}^{s}(-1)^{s-r}\binom{t+2}{s+1}\binom{s+1}{s-r}N_r(i+j+r, i+1) \\
&= \sum_{r=0}^{t}\sum_{s=r}^{t}(-1)^{s-r}\binom{t+2}{s+1}\binom{s+1}{s-r}N_r(i+j+r, i+1) \\
&= \sum_{r=0}^{t}\sum_{s=r}^{t}(-1)^{s-r}\binom{t+2}{r+1}\binom{t+1-r}{s-r}N_r(i+j+r, i+1)
\end{aligned}$$

$$= \sum_{r=0}^{t} \binom{t+2}{r+1} \left[\sum_{s=r}^{t} (-1)^{s-r} \binom{t+1-r}{s-r} \right] N_r(i+j+r, i+1)$$
$$= \sum_{r=0}^{t} \binom{t+2}{r+1} \left[\sum_{\beta=0}^{t-r} (-1)^{\beta} \binom{t+1-r}{\beta} \right] N_r(i+j+r, i+1)$$
$$= \sum_{r=0}^{t} (-1)^{t-r} \binom{t+2}{r+1} N_r(i+j+r, i+1)$$

where the final step uses the identity $\sum_{\beta=0}^{n} (-1)^{\beta} \binom{n}{\beta} = 0$. We know from Proposition 1.6.6 that $A(i+j+t+1, t+1, i) = N_{t+1}(i+j+t+1, i+1)$. So we have

$$\begin{aligned} L(i,j,t+1) &= N_{t+1}(i+j+t+1, i+1) \\ &\qquad - \sum_{r=0}^{t} (-1)^{t-r} \binom{t+2}{r+1} N_r(i+j+r, i+1) \\ &= \sum_{r=0}^{t+1} (-1)^{t+1-r} \binom{t+2}{r+1} N_r(i+j+r, i+1) \\ &= B(i,j,t+1) \end{aligned}$$

as required. □

Corollary 1.6.9. The number of trees in $\mathcal{T}_{i,j}^k$ is given by

$$\frac{1}{k+1} \binom{j+k}{k} \sum_{\alpha=0}^{k} (-1)^{k-\alpha} \binom{k+1}{k-\alpha} N_{\alpha}(i+j+\alpha, i+1) \tag{1.11}$$

where $N_r(n,k) = \frac{r+1}{n} \binom{n}{k+r} \binom{n}{k-1}$.

Proof. This follows directly from Proposition 1.6.5 and Proposition 1.6.8. □

Remark 1.6.10. To see that this restricts to the result we expect in the case

$i = 0$ we need to use the binomial identity

$$\sum_{r=0}^{k}(-1)^r \binom{k+1}{r}\binom{m+k-r}{m} = \binom{m-1}{k}. \tag{1.12}$$

From the book Concrete Mathematics ([GKP94], equation 5.25) we have the identity

$$\sum_{r \leqslant l}(-1)^r \binom{l-r}{m}\binom{s}{r-n} = (-1)^{l+m}\binom{s-m-1}{l-m-n}.$$

The identity we require is a special case of this with $n = 0$, $s = k+1$, and $l = m+k$.

So when $i = 0$ in equation (1.11) we have

$$\begin{aligned}
|\mathcal{T}_{0,j}^k| &= \frac{1}{k+1}\binom{j+k}{k}\sum_{r=0}^{k}(-1)^{k-r}\binom{k+1}{k-r}N_r(j+r,1) \\
&= \frac{1}{k+1}\binom{j+k}{k}\sum_{r=0}^{k}(-1)^{k-r}\binom{k+1}{k-r}\frac{r+1}{j+r}\binom{j+r}{r+1}\binom{j+r}{0} \\
&= \frac{1}{k+1}\binom{j+k}{k}\sum_{r=0}^{k}(-1)^{k-r}\binom{k+1}{k-r}\binom{j+r-1}{j-1} \\
&= \frac{1}{k+1}\binom{j+k}{k}\sum_{\alpha=0}^{k}(-1)^{\alpha}\binom{k+1}{\alpha}\binom{j-1+k-\alpha}{j-1} \\
&= \frac{1}{k+1}\binom{j+k}{k}\binom{j-2}{k} \quad \text{by (1.12).}
\end{aligned}$$

So this agrees with Proposition 1.6.3.

Furthermore, if we restrict to the case $j = 1$ we get

$$|\mathcal{T}_{i,1}^k| = \frac{1}{k+1}\binom{k+1}{k}\sum_{r=0}^{k}(-1)^{k-r}\binom{k+1}{k-r}N_r(i+r+1,i+1)$$

$$
\begin{aligned}
&= \sum_{r=0}^{k} (-1)^{k-r} \binom{k+1}{k-r} \frac{r+1}{i+r+1} \binom{i+r+1}{i+r+1} \binom{i+r+1}{i} \\
&= \sum_{r=0}^{k} (-1)^{k-r} \binom{k+1}{k-r} \binom{i+r}{i} \\
&= \sum_{\alpha=0}^{k} (-1)^{\alpha} \binom{k+1}{\alpha} \binom{i+k-\alpha}{i} \\
&= \binom{i-1}{k} \quad \text{by (1.12).}
\end{aligned}
$$

Chapter 2

Topological Models

The aim of this chapter is to introduce the notion of an A_∞-space, D_∞-space, and DA_∞-space. In Chapter 3 we will see that taking singular chains on these structures gives an A_∞-algebra, twisted chain complex, and derived A_∞-algebra respectively. The first case is classical and due to Stasheff [Sta63], while the other two are new constructions.

We introduce three collections of topological spaces and give examples of the construction for some low dimensional spaces in each. In this chapter, we will also discuss the non-symmetric coloured operad structure of these spaces.

2.1 A_∞-Structures

An A_∞-space is a topological space with a multiplication which is not strictly associative but is associative up to homotopy in a strong sense. So we have a topological space, X, and a multiplication map, $M_2 : X \times X \to X$. Then we want to consider a homotopy $M_3 : I \times X^3 \to X$ such that $M_3(0, x_1, x_2, x_3) = M_2(M_2(x_1, x_2), x_3)$ and $M_3(1, x_1, x_2, x_3) = M_2(x_1, M_2(x_2, x_3))$. An illustration of this homotopy is given by figure 2.1.

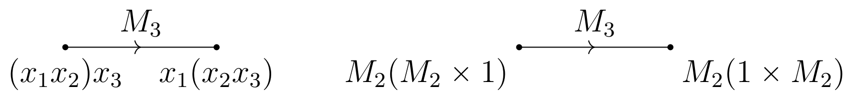


Figure 2.1: Stasheff polytope K_3, homotopy M_3

We want to continue to generalise this to a higher associativity condition for multiplication of four variables. There are five different ways to fully bracket four variables in a fixed order, and we already have some maps between them given by compositions of M_3 and M_2, as shown in figure 2.2. These maps give us the boundary of a pentagon and we call this pentagon K_4, and so we want to define a homotopy $M_4 : K_4 \times X^4 \to X$.

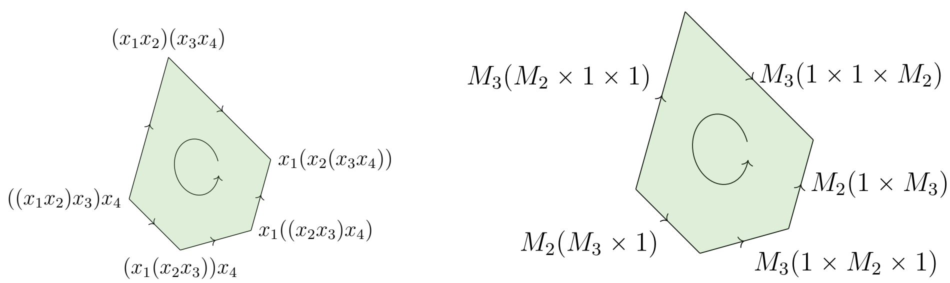


Figure 2.2: Stasheff polytope K_4

To consider generalising this idea to associativity for multiplication of more variables, we first need to define a collection of convex polytopes called the associahedra. The associahedron K_i is a convex polytope of dimension $(i-2)$, in which the vertices are in bijection with the number of ways of fully associating i items. In the brief discussion above, we have already seen K_3 and K_4.

We will now discuss Stasheff's original construction of the spaces and give a few examples. We also note that there are many different realisations of the associahedron, for example the realisation given by Loday and Vallette in Appendix C of [LV12] was used to create the images in this section.

Definition 2.1.1 ([Sta63])**.** [Stasheff's Construction] To construct the **as-**

sociahedron K_i, we consider inserting a set of parentheses into a word of length i, $x_1 ... x_i$. To each such insertion there corresponds a cell on the boundary of K_i. If the brackets enclose x_k through to x_{k+s-1} then this cell is taken to be the image of $K_r \times K_s$ under a homeomorphism which we call $\partial_k(r, s)$ where $r + s = i + 1$. Two cells intersect only on their boundaries according to two relations:

$$\partial_j(r, s+t-1)(1 \times \partial_k(s,t)) = \partial_{j+k-1}(r+s-1,t)(\partial_j(r,s) \times 1) \tag{2.1}$$

for $1 \leqslant j \leqslant r$ and $1 \leqslant k \leqslant s$, and

$$\partial_{j+s-1}(r+s-1,t)(\partial_k(r,s) \times 1) = \partial_k(r+t-1,s)(\partial_j(r,t) \times 1)(1 \times T) \tag{2.2}$$

for $1 \leqslant k < j \leqslant r$, where $T : K_s \times K_t \to K_t \times K_s$ permutes the factors. Starting with K_2 being a one point space, we obtain the boundary for each K_i by induction.

This is a cellular decomposition of the sphere S^{i-3}. Then K_i is the cone on its boundary, so as a space K_i is homeomorphic to D^{i-2} and we have a particular cellular decomposition of the boundary.

Example 2.1.2 (Construction of K_4)**.** We consider all possible ways of inserting a single pair of matching brackets into a four letter word, that is $(x_1x_2x_3)x_4$, $x_1(x_2x_3)x_4$, $x_1(x_2x_3x_4)$, $x_1x_2(x_3x_4)$, and $(x_1x_2)x_3x_4$. To each such insertion we have a cell on the boundary of K_4 under the maps $\partial_1(2,3)$, $\partial_2(3,2)$, $\partial_2(2,3)$, $\partial_3(3,2)$, and $\partial_1(3,2)$ respectively. The pieces on the boundary of K_4 are shown in Figure 2.3. Notice that each piece is a copy of $K_2 \times K_3$. The relations for these pieces are:

$$\partial_1(2,3)(1 \times \partial_1(2,2)) = \partial_1(3,2)(\partial_1(2,2) \times 1) \tag{2.3}$$

$$\partial_1(2,3)(1 \times \partial_2(2,2)) = \partial_2(3,2)(\partial_1(2,2) \times 1) \tag{2.4}$$

$$\partial_2(2,3)(1 \times \partial_1(2,2)) = \partial_2(3,2)(\partial_2(2,2) \times 1) \tag{2.5}$$

(a) $\partial_1(2,3)$

(b) $\partial_2(3,2)$

(c) $\partial_2(2,3)$

(d) $\partial_3(3,2)$

(e) $\partial_1(3,2)$

Figure 2.3: The pieces in K_4

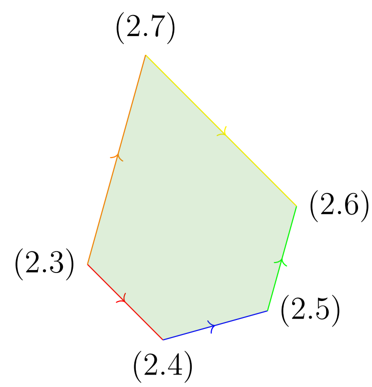


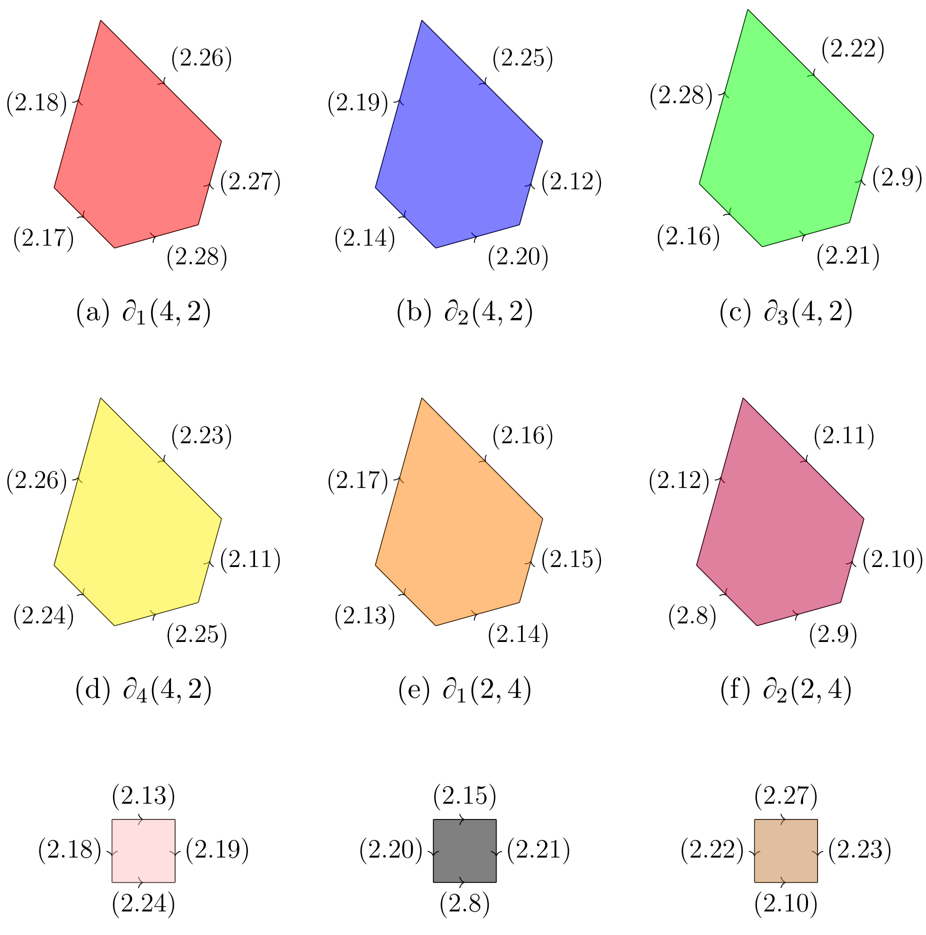
Figure 2.4: The space K_4

$$\partial_2(2,3)(1 \times \partial_2(2,2)) = \partial_3(3,2)(\partial_2(2,2) \times 1) \tag{2.6}$$

$$\partial_3(3,2)(\partial_1(2,2) \times 1) = \partial_1(3,2)(\partial_2(2,2) \times 1)(1 \times T). \tag{2.7}$$

We use these relations to construct the boundary of K_4 and then take the cone to get the space K_4 as shown in figure 2.4.

Example 2.1.3 (Construction of K_5)**.** We consider all possible pairs of numbers $r, s \geqslant 2$ such that $r + s = 6$. That is, $(2,4)$, $(3,3)$, and $(4,2)$. So, we have cells on the boundary as shown in Figure 2.5. Notice that the pieces (a), (b), (c), (d), (e), and (f) are homeomorphic to $K_2 \times K_4$, while the pieces (g), (h), and (i) are homeomorphic to $K_3 \times K_3$. The relations for these pieces



(a) $\partial_1(4,2)$ (b) $\partial_2(4,2)$ (c) $\partial_3(4,2)$

(d) $\partial_4(4,2)$ (e) $\partial_1(2,4)$ (f) $\partial_2(2,4)$

(g) $\partial_1(3,3)$ (h) $\partial_2(3,3)$ (i) $\partial_3(3,3)$

Figure 2.5: The pieces in K_5

are:

$$\partial_2(2,4)(1\times\partial_1(2,3)) = \partial_2(3,3)(\partial_2(2,2)\times 1) \tag{2.8}$$
$$\partial_2(2,4)(1\times\partial_2(3,2)) = \partial_3(4,2)(\partial_2(2,3)\times 1) \tag{2.9}$$
$$\partial_2(2,4)(1\times\partial_2(2,3)) = \partial_3(3,3)(\partial_2(2,2)\times 1) \tag{2.10}$$
$$\partial_2(2,4)(1\times\partial_3(3,2)) = \partial_4(4,2)(\partial_2(2,3)\times 1) \tag{2.11}$$
$$\partial_2(2,4)(1\times\partial_1(3,2)) = \partial_2(4,2)(\partial_2(2,3)\times 1) \tag{2.12}$$
$$\partial_1(2,4)(1\times\partial_1(2,3)) = \partial_1(3,3)(\partial_1(2,2)\times 1) \tag{2.13}$$
$$\partial_1(2,4)(1\times\partial_2(3,2)) = \partial_2(4,2)(\partial_1(2,3)\times 1) \tag{2.14}$$
$$\partial_1(2,4)(1\times\partial_2(2,3)) = \partial_2(3,3)(\partial_1(2,2)\times 1) \tag{2.15}$$
$$\partial_1(2,4)(1\times\partial_3(3,2)) = \partial_3(4,2)(\partial_1(2,3)\times 1) \tag{2.16}$$
$$\partial_1(2,4)(1\times\partial_1(3,2)) = \partial_1(4,2)(\partial_1(2,3)\times 1) \tag{2.17}$$
$$\partial_1(3,3)(1\times\partial_1(2,2)) = \partial_1(4,2)(\partial_1(3,2)\times 1) \tag{2.18}$$
$$\partial_1(3,3)(1\times\partial_2(2,2)) = \partial_2(4,2)(\partial_1(3,2)\times 1) \tag{2.19}$$
$$\partial_2(3,3)(1\times\partial_1(2,2)) = \partial_2(4,2)(\partial_2(3,2)\times 1) \tag{2.20}$$
$$\partial_2(3,3)(1\times\partial_2(2,2)) = \partial_3(4,2)(\partial_2(3,2)\times 1) \tag{2.21}$$
$$\partial_3(3,3)(1\times\partial_1(2,2)) = \partial_3(4,2)(\partial_3(3,2)\times 1) \tag{2.22}$$
$$\partial_3(3,3)(1\times\partial_2(2,2)) = \partial_4(4,2)(\partial_3(3,2)\times 1) \tag{2.23}$$
$$\partial_4(4,2)(\partial_1(2,3)\times 1) = \partial_1(3,3)(\partial_2(2,2)\times 1)(1\times T) \tag{2.24}$$
$$\partial_4(4,2)(\partial_2(3,2)\times 1) = \partial_2(4,2)(\partial_3(3,2)\times 1)(1\times T) \tag{2.25}$$
$$\partial_4(4,2)(\partial_1(3,2)\times 1) = \partial_1(4,2)(\partial_3(3,2)\times 1)(1\times T) \tag{2.26}$$
$$\partial_3(3,3)(\partial_1(2,2)\times 1) = \partial_1(4,2)(\partial_2(2,3)\times 1)(1\times T) \tag{2.27}$$
$$\partial_3(4,2)(\partial_1(3,2)\times 1) = \partial_1(4,2)(\partial_2(3,2)\times 1)(1\times T). \tag{2.28}$$

We use these relations to construct the boundary of K_5 and then take the cone to get the space K_5 as shown in figure 2.6.

The encoding of the cell structure of an associahedron by planar trees

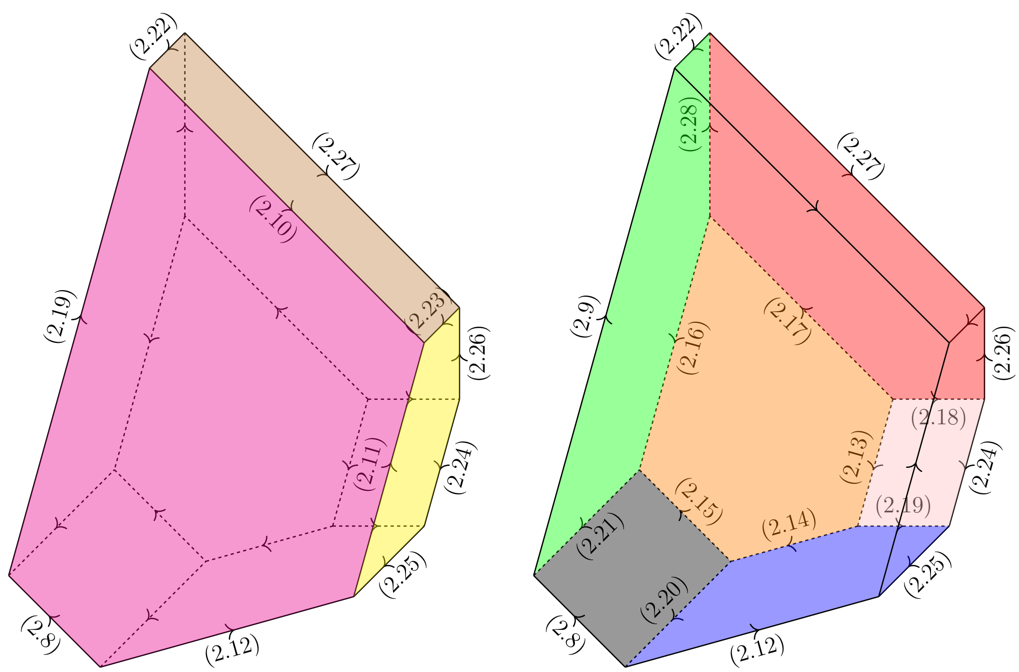


Figure 2.6: Stasheff Polytope K_5

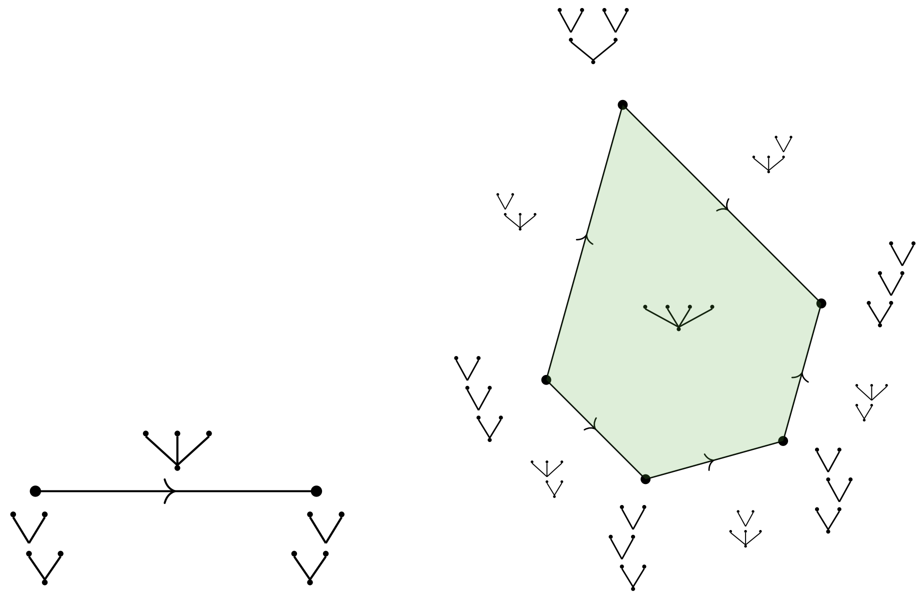


Figure 2.7: Stasheff Polytope K_3 and K_4 with planar tree labels

is well documented in the literature (see for example [LV12][Appendix C]). Several realisations of the associahedra as polytopes are known, but here we are only concerned with the structure as a finite cell complex. Figure 2.7 shows this representation for the polytopes K_3 and K_4 with splittings drawn with a small gap as in Section 1.5. We recall that collapsing and expanding an internal edge allows us to move from a cell to its boundary, we think of this as adding or removing a splitting.

However, a counting argument for the number of faces of each dimension is not well documented in the literature. This may well be known, but I have been unable to find a reference, and so I will present a proof of this using the counting arguments from Section 1.5.

Proposition 2.1.4 (Appendix C, [LV12])**.** The cells of dimension k in K_n are in bijection with the planar trees having n leaves and $n-1-k$ vertices. $\square$

Proposition 2.1.5. The number of cells in dimension $(n-2-k)$ in the associahedron K_n is given by $T(n+1,k) = \frac{1}{k+1}\binom{n-2}{k}\binom{n+k}{k}$.

Proof. The cells of dimension $(n-2-k)$ in K_n are in bijection with planar trees with n leaves, and $k+1$ vertices. It is easy to see that this is equivalent to trees with n leaves, k internal edges (i.e. k A_∞-splittings) and no nodes. So the cells of dimension $(n-2-k)$ in K_n are in bijection with elements of $\mathcal{T}_{0,n}^k$, and by Proposition 1.6.3 we know that there are $T(n+1,k)$ such trees. $\square$

The following two results are also well known in the work of Stasheff but will be useful in the final section of this chapter when we consider the structure of the spaces V_{ij}.

Proposition 2.1.6 (Proposition 3, [Sta63])**.** The space K_i is homeomorphic to $I^{i-2} \cong D^{i-2}$. $\square$

Proposition 2.1.7. The associahedra $\{K_n\}_{n\geqslant 2}$ form a non-symmetric non-unital operad, $\mathcal{K}$, in the category of topological spaces.

Proof. This follows directly from the definition of a non-symmetric operad via partial compositions in Proposition 1.3.2. The structure maps, $\partial_k(r,s):$ $K_r \times K_s \to K_{r+s-1}$, give the partial compositions, and relations 2.1 and 2.2 are equivalent to the relations for the partial compositions. More details can be found in [Sta97]. $\square$

In the next definition, we use the spaces K_i to define an A_∞-space. In [Sta63] Stasheff defines an A_∞-space with a unit condition for the multiplication which we omit here.

Definition 2.1.8. A space X admits an $\boldsymbol{A_\infty}$**-structure** if and only if there exist maps $M_i : K_i \times X^i \to X$ for $i \geqslant 2$ such that

$$\begin{aligned} &M_i(\partial_k(r,s)(\rho,\sigma), x_1, ..., x_i) = \\ &\qquad\qquad M_r(\rho, x_1, .., x_{k-1}, M_s(\sigma, x_k, ..., x_{k+s-1}), x_{k+s}, ..., x_i), \end{aligned} \tag{2.29}$$

for $\rho \in K_r$, $\sigma \in K_s$, $r+s=i+1$. The pair $(X,\{M_i\})$ is called an $\boldsymbol{A_\infty}$**-space**.

Proposition 2.1.9 ([Sta97])**.** An A_∞-space is an algebra over the operad $\mathcal{K} = \{K_n\}$ in the category of topological spaces.

Proof. This follows directly from Proposition 1.3.4. We see that with the structure maps, $M_i : K_i \times X^i \to X$ for $2 \leqslant i \leqslant n$, relation (2.29) is equivalent to satisfying the associativity diagram of Proposition 1.3.4. $\square$

Example 2.1.10. A natural example of an A_∞-space is the loop space, ΩX of a based topological space X, with basepoint $*$. We can take the composition of two loops a, b where:

$$\begin{aligned} & a : I \to X & & b : I \to X \\ s.t. \quad & a(0) = a(1) = * & & b(0) = b(1) = *. \end{aligned}$$

Then $a \circ b : I \times I \to X$ is given by the formula

$$a \circ b = \begin{cases} a(2i) & 0 \leqslant i \leqslant \frac{1}{2}, \\ b(2i-1) & \frac{1}{2} \leqslant i \leqslant 1. \end{cases}$$

We can easily see that when composition is defined in this way, it is not associative, i.e. $(a \circ b) \circ c \neq a \circ (b \circ c)$. However, we can define a homotopy between the two ways of associating:

$$\begin{aligned} & M_3 : (\Omega X)^3 \times I \to \Omega X \\ s.t. \quad & M_3(a,b,c,t) = \begin{cases} a((2-t)2i) & 0 \leqslant i \leqslant \frac{1+t}{4}, \\ b(4i-1-t) & \frac{1+t}{4} \leqslant i \leqslant \frac{2+t}{4}, \\ c((2i-1)+2t(i-1)) & \frac{2+t}{4} \leqslant i \leqslant 1. \end{cases} \end{aligned}$$

If we consider the multiplication map, $M_2 : \Omega X \times \Omega X \to \Omega X$, which takes two loops (a, b) to the composite $a \circ b$, then M_3 is a homotopy between $M_2(M_2 \times 1)$ and $M_2(1 \times M_2)$. Continuing in this manner for composition of loops naturally gives rise to maps $M_i : (\Omega X)^i \times K_i \to \Omega X$ which satisfy the conditions for an A_∞-space. This is because the lack of associativity in a loop space is a result of "how fast" we travel round each loop, and so we get higher homotopies by varying speeds.

2.2 D_∞-Structures

In this section we will see that we can define a D_∞-space, constructed to be a topological version of a twisted chain complex. The idea is that we want to capture the essence of the twisted chain complex structure in the category of topological spaces. We recall the first few relations for a twisted chain complex below.

$$
\begin{tikzcd}
C_0^{-1} \arrow[r, "d_1"] \arrow[d, "d_0"] & C_1^{-1} \arrow[r, "d_1"] \arrow[d] & C_2^{-1} \arrow[r, "d_1"] \arrow[d] & C_3^{-1} \arrow[r, "d_1"] \arrow[d] & C_4^{-1} \arrow[d, "d_0"] \\
C_0^{0} \arrow[r] \arrow[d, "d_0"] \arrow[urr, red, "d_2"] & C_1^{0} \arrow[r] \arrow[d] \arrow[urr, red, "d_2"] & C_2^{0} \arrow[r] \arrow[d] \arrow[urr, red, "d_2"] & C_3^{0} \arrow[r] \arrow[d] & C_4^{0} \arrow[d, "d_0"] \\
C_0^{1} \arrow[r] \arrow[d, "d_0"] \arrow[urr, red] \arrow[uurrr, blue] & C_1^{1} \arrow[r] \arrow[d] \arrow[urr, red] \arrow[uurrr, blue, "d_3"] & C_2^{1} \arrow[r] \arrow[d] \arrow[urr, red, "d_3"] & C_3^{1} \arrow[r] \arrow[d] & C_4^{1} \arrow[d, "d_0"] \\
C_0^{2} \arrow[r, "d_1"] \arrow[urr, red] \arrow[uurrr, blue] & C_1^{2} \arrow[r, "d_1"] \arrow[urr, red] \arrow[uurrr, blue] & C_2^{2} \arrow[r, "d_1"] \arrow[urr, red] & C_3^{2} \arrow[r, "d_1"] & C_4^{2}
\end{tikzcd}
$$

Figure 2.8: Some of the maps in a twisted chain complex

1. $d_0 \circ d_0 = 0$, i.e. d_0 is a differential,

2. $d_0 \circ d_1 - d_1 \circ d_0 = 0$, i.e. d_1 commutes with d_0,

3. $d_0 \circ d_2 + d_2 \circ d_0 = d_1 \circ d_1$, i.e. d_2 is a chain homotopy with respect to the differential d_0 between $d_1 \circ d_1$ and 0,

4. $d_0 \circ d_3 - d_3 \circ d_0 = d_1 \circ d_2 - d_2 \circ d_1$,

5. $d_0 \circ d_4 + d_4 \circ d_0 = d_1 \circ d_3 - d_2 \circ d_2 + d_3 \circ d_1$,

Figure 2.9: The space T_2 (left) and T_3 (right)

Notice that we have a differential d_0 and another map d_1 which is not a differential but is a differential up to chain homotopy. To model this situation in topological spaces, we will consider a family of based topological spaces $X = \{X_n\}_{n\in\mathbb{N}}$ and take D_1 to be a map $D_1 : X_n \to X_{n+1}$. We then want D_2 to be a homotopy between D_1^2 and the constant map at the basepoint, $D_2 : I_* \wedge X_n \to X_{n+2}$. This gives us the space which we will call T_2 as shown in figure 2.9. We then define a family of spaces T_i in order to fit with the relations. The space T_3 is shown in figure 2.9 and T_4 is shown in figure 2.11. We will now give a formal construction of the spaces T_i for $i \geqslant 1$.

Definition 2.2.1. We define the topological space T_i for $i \geqslant 1$ by $T_i = I^{\wedge(i-1)}$ where we take 0 as the basepoint in $I = [0, 1]$.

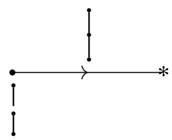
Remark 2.2.2. The boundary of T_i is homeomorphic to S^{i-2}. This follows directly from the definition of T_i. Notice that $\partial T_i \cong \partial D^{i-1} = S^{i-2}$.

Proposition 2.2.3. The $(i-1-k)$-cells of T_i are in bijection with the trees in $\mathcal{T}_{i,1}^k$, that is k-partitioned 1-trees with i nodes. There are $\binom{i-1}{k}$ such cells.

Proof. We think of T_i as

$$T_i = I_1 \wedge I_2 \wedge \cdots \wedge I_{i-1}$$

where each I_n is an interval in which we think of 0 as the basepoint, 1 as the



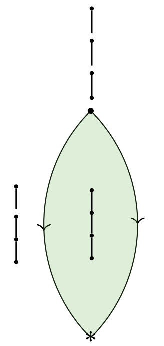


Figure 2.10: The space T_2 (left) and T_3 (right)

vertex v_n and the 1-cell as e_n. Then T_i has one $(i-1)$-cell given by

$$e_1 \wedge e_2 \wedge \cdots \wedge e_{i-1}.$$

An $(i-1-k)$-cell of T_i is given by choosing k of the edges $e_1, \ldots, e_{i-1}$ to be replaced by their respective vertices v_n. Recall from Remark 1.6.10 that there are $\binom{i-1}{k}$ trees in $\mathcal{T}_{i,1}^k$. We give a bijection to an element of $\mathcal{T}_{i,1}^k$ by letting each element of our smash product represent an internal vertex of a 1-tree with i edges. Then the k edges replaced by vertices in the smash product form the cut set of vertices in the tree. □

In the following proposition we show that the spaces T_i can be used to define an $\mathbb{N}$-coloured non-symmetric non-unital operad. We work in the symmetric monoidal category $CHau_*$ of based topological spaces with monoidal product given by the smash product and unit S^0.

Proposition 2.2.4. We can define an $\mathbb{N}$-coloured non-symmetric operad, $\mathcal{T}$, in which all operations have arity 1, in the category of based topological spaces, from the spaces T_i for $i \geqslant 1$ by

$$\mathcal{T}(p; p+i) = \begin{cases} T_i & \forall p \in \mathbb{N}, i \geqslant 1, \\ * & \text{otherwise,} \end{cases}$$

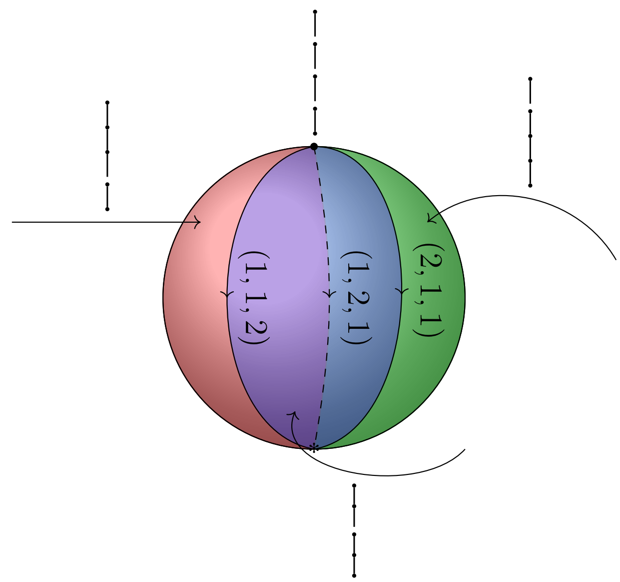


Figure 2.11: The space T_4

and $\mathcal{T}(q_1, \ldots, q_n; r) = *$ if $n \geqslant 2$.

Proof. We have natural face inclusion maps $\partial(r, s) : T_r \wedge T_s \to T_{r+s}$ given by

$$\rho \wedge \sigma \to \rho \wedge 1 \wedge \sigma$$

where $\rho \in T_r$ and $\sigma \in T_s$. Now consider the definition of a non-symmetric non-unital coloured operad via partial compositions as in Proposition 1.4.5. Then we have partial compositions

$$\gamma : \mathcal{T}(p + s; p + s + r) \wedge \mathcal{T}(p; p + s) \to \mathcal{T}(p; p + s + r)$$

for all $p \in \mathbb{N}$, given by $\partial(r, s) : T_r \wedge T_s \to T_{r+s}$. The partial composition is trivial otherwise by definition.

It is then easy to check that diagram (1.8) of Proposition 1.4.5 is exactly equivalent to

$$\partial(t, r+s)(1 \times \partial(r,s)) = \partial(t+r, s)(\partial(t,r) \times 1)$$

in all non-trivial cases, and this clearly holds by definition. Since this operad is in arity one, diagram (1.9) is always trivial in this case. □

We will now define a D_∞-structure over a family of based topological spaces. In Proposition 2.2.6 we show that this structure is equivalent to an algebra over the operad $\mathcal{T}$. In Chapter 3 we will show that taking singular chains on this structure results in a twisted chain complex.

Definition 2.2.5. A ***D*$_\infty$-structure** on a family of based spaces $X = \{X_n\}_{n \in \mathbb{N}}$ is a collection of based maps $D_i : T_i \wedge X_n \to X_{n+i}$ for $i \geqslant 1$ and any $n \in \mathbb{N}$ such that

$$D_i(\partial(r,s)(\rho, \sigma), x) = D_r(\rho, D_s(\sigma, x)) \tag{2.30}$$

for $\rho \in T_r$, $\sigma \in T_s$, $x \in X_n$, with $r + s = i$.

A ***D*$_\infty$-space** is a family of based spaces $X = \{X_n\}_{n \in \mathbb{N}}$ together with a D_∞-structure.

Proposition 2.2.6. A D_∞-space is an algebra over the $\mathbb{N}$-coloured operad $\mathcal{T}$ in the category of based topological spaces.

Proof. This follows immediately from Proposition 1.4.7. We have a family $X = \{X_n\}_{n \in \mathbb{N}}$ of objects in Top_*, together with maps

$$D_i : \mathcal{T}(n; n+i) \wedge X_n \to X_{n+i}$$

and relation (2.30) is exactly equivalent to diagram (1.10) of Proposition 1.4.7. □

Remark 2.2.7. I am not currently aware of any examples of D_∞-spaces, however one place in which they might arise is as semi-simplicial (up to homotopy) H-spaces.

2.3 DA_∞-Structures

In this final section of this chapter, we view the previous two cases as special cases of a more general structure, constructed to give a geometric model of a derived A_∞-algebra. We already know that a derived A_∞-algebra has structure maps

$$m_{ij} : A^{\otimes j} \to A$$

of bidegree $(i, i + j - 2)$ for each $i \geqslant 0$, $j \geqslant 1$. A derived A_∞-algebra has an underlying twisted chain complex structure given by the maps m_{i1}. Additionally, if we restrict the structure of a derived A_∞-algebra to the case $i = 0$ by considering the special case where $m_{ij} = 0$ if $i > 0$, we have the structure of an A_∞-algebra.

In what follows we will construct based topological spaces V_{ij} for $i \geqslant 0$, $j \geqslant 1$, and $(i,j) \neq (0,1)$. When $j = 1$, V_{ij} will be equal to T_i, and when $i = 0$, V_{ij} will be equal to $(K_i)_+ = K_i \amalg *$. These spaces will be used to define a DA_∞-space, which will give us both a multiplication which is associative up to homotopy and a map of based spaces which is a differential up to homotopy, as well as compatibility between these and all higher coherences.

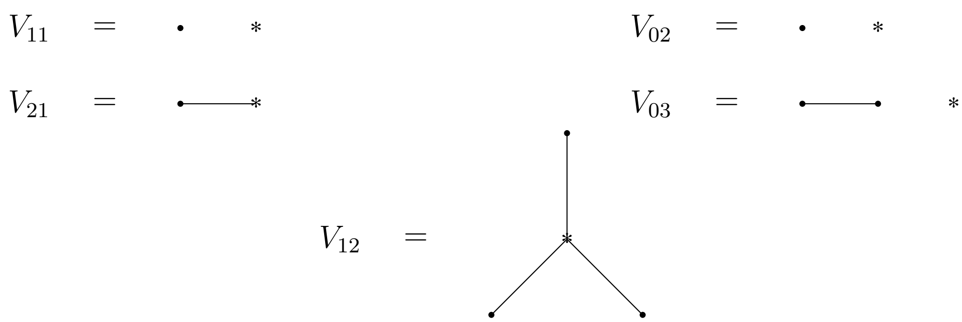


Figure 2.12: Initial spaces for construction of V_{ij}

Figure 2.12 shows the spaces V_{ij} for low values of i and j. The space V_{11} is going to be used to give a map $DA_{11} : V_{11} \wedge X_p \to X_{p+1}$ with a map $DA_{21} : V_{21} \wedge X_p \to X_{p+2}$ giving a homotopy between DA_{11}^2 and the constant map at the basepoint. The space V_{02} will be used to construct a map $DA_{02} : V_{02} \wedge X_p \wedge X_q \to X_{p+q}$ which is our multiplication map, and a map $DA_{03} : V_{03} \wedge X_p \wedge X_q \wedge X_r \to X_{p+q+r}$ gives homotopy associativity for DA_{02}.

Definition 2.3.1. We first begin by defining $(K_1)_+ = S^0$ so that taking a smash product with $(K_1)_+$ is equal to the identity. Then we define the collection of based spaces V_{ij} for $i \geqslant 0$, $j \geqslant 1$, and $(i,j) \neq (0,1)$ as:

$$V_{ij} := \bigvee_{t \in \mathcal{T}^0_{i,j}} T_{r+1} \wedge (K_{\mathrm{out}(v_0)})_+ \wedge \cdots \wedge (K_{\mathrm{out}(v_r)})_+$$

where $t \in \mathcal{T}^0_{i,j}$ has a root vertex v_0, and r internal vertices, labelled $v_1, \ldots, v_r$.

Remark 2.3.2. In this definition we are working with pointed spaces V_{ij}. It is clear that when $j = 1$ in this definition, there is just one tree in $\mathcal{T}^0_{i,1}$. This tree has $i - 1$ internal vertices, and each vertex has just one output. So $V_{i1} = T_i \wedge (K_1)_+ \wedge \cdots \wedge (K_1)_+ \cong T_i$. We also want the definition of V_{ij} when $i = 0$ to give $V_{0j} \cong (K_j)_+$. Clearly there is just one tree in $\mathcal{T}^0_{0,j}$ and that is the j-corolla. This tree has no internal vertices and so $V_{0j} = T_1 \wedge (K_j)_+ \cong (K_j)_+$.

Remark 2.3.3. It may also be possible to define the spaces V_{ij} in terms of the set of planar trees with a length function, however there was insufficient time to work out the details of this.

Proposition 2.3.4. There is a bijection between $\mathcal{T}^k_{i,j}$ and $(i+j-2-k)$-cells of V_{ij}.

Proof. Clearly by definition of V_{ij} there is a bijection between the top cells of V_{ij} and trees in $\mathcal{T}^0_{i,j}$. We know that the top cell of each $K_{\mathrm{out}(v)}$ is in dimension $\mathrm{out}(v)-2$ and the top cell of each T_{r+1} is in dimension r. So the top cells of V_{ij} lie in dimension

$$r + \sum_{s=0}^{r} (\mathrm{out}(v_s) - 2) = r + (i+j+r) - 2(r+1) = i+j-2.$$

For each $t \in \mathcal{T}^0_{i,j}$ where t has a root vertex, v_0 and r internal vertices, labelled $v_1, \ldots, v_r$, then t can be formed by a grafting of corollas $c_0, \ldots, c_r$ in which each c_n is an $\mathrm{out}(v_n)$-corolla. The internal vertices (i.e. the grafting points/ $v_1, \ldots, v_r$) correspond to the internal vertices in the tree of $T \in \mathcal{T}^0_{i,1}$.

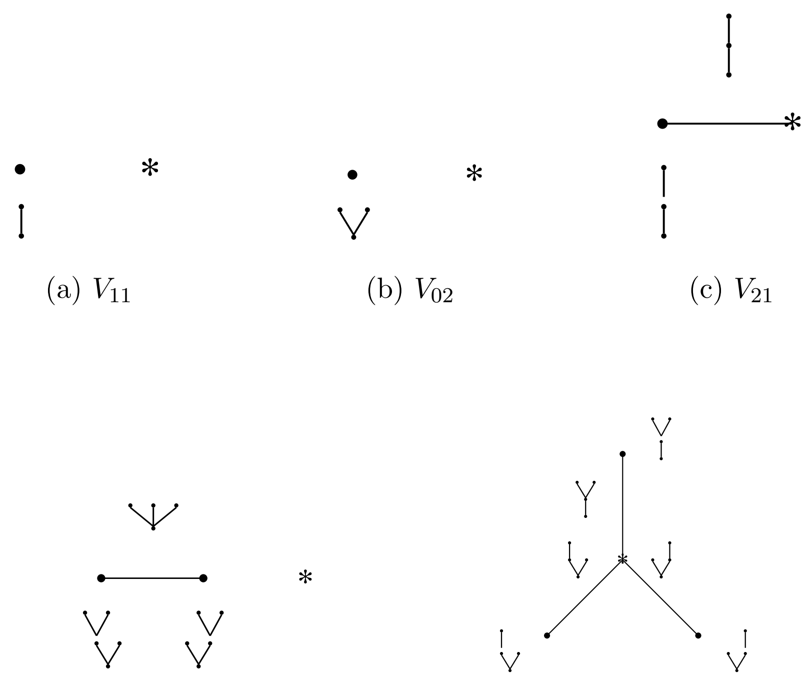
We know that cells in lower dimensions in V_{ij} are smash products of cells in lower dimensions of the T_{r+1} and the $K_{\mathrm{out}(v)}$. So by Proposition 2.1.4 and Proposition 2.2.3 the bijection extends to all $(i+j-2-k)$-cells of V_{ij} by A_∞-splitting of the components $(K_{\mathrm{out}(v_n)})_+$ and D_∞-splittings of T_{r+1}. □

Proposition 2.3.5. The number of cells in dimension $(i+j-2-k)$ of V_{ij} is given by

$$\frac{1}{k+1}\binom{j+k}{k} \sum_{\alpha=0}^{k} (-1)^{k-\alpha} \binom{k+1}{k-\alpha} N_\alpha(i+j+\alpha, i+1)$$

where $N_r(n,k) = \frac{r+1}{n}\binom{n}{k+r}\binom{n}{k-1}$.

Proof. This follows directly from Proposition 2.3.4 and the counting argument for the number of trees in $\mathcal{T}^k_{i,j}$ given in Corollary 1.6.9. □



(a) V_{11} (b) V_{02} (c) V_{21}

(d) V_{03} (e) V_{12}

Figure 2.13: Initial spaces for construction of V_{ij}

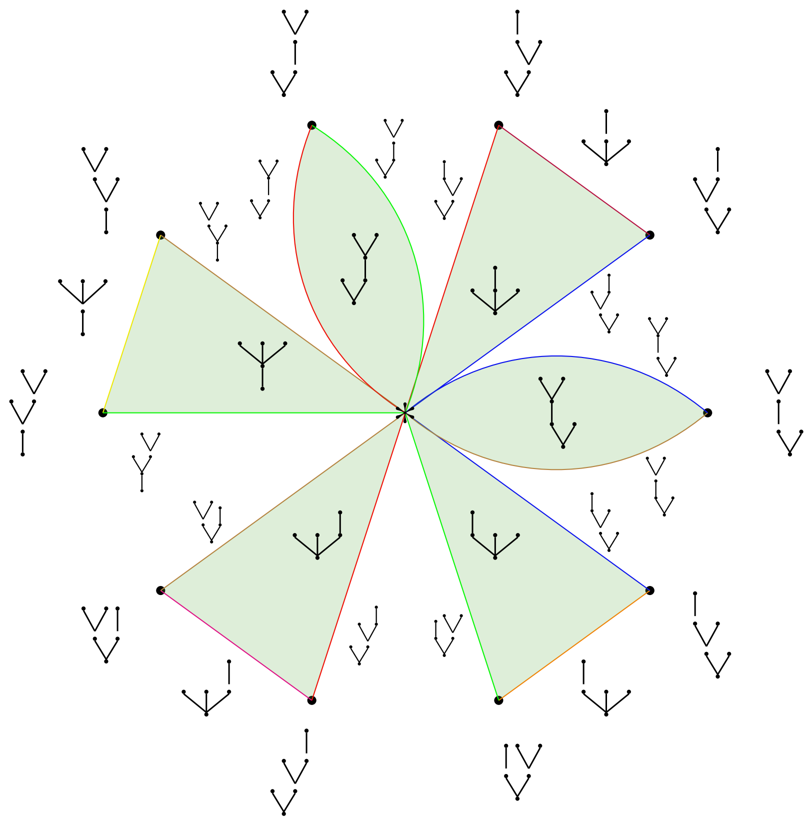


Figure 2.14: Combinatorial description of V_{13}

Proposition 2.3.6. The boundary of V_{ij} with $i > 0$ is homeomorphic to a wedge of $N(i+j, j)$ spheres of dimension $i + j - 3$.

Proof. By definition,

$$V_{ij} := \bigvee_{t \in \mathcal{T}^0_{i,j}} T_{r+1} \wedge (K_{\mathrm{out}(v_0)})_+ \wedge \cdots \wedge (K_{\mathrm{out}(v_r)})_+.$$

So,

$$\begin{aligned}
\partial V_{ij} &= \bigvee_{t \in \mathcal{T}^0_{i,j}} \partial(T_{r+1} \wedge (K_{\mathrm{out}(v_0)})_+ \wedge \cdots \wedge (K_{\mathrm{out}(v_r)})_+) \\
&\cong \bigvee_{N(i+j,j)} \partial(D^r \wedge (D^{\mathrm{out}(v_0)-2})_+ \wedge \cdots \wedge (D^{\mathrm{out}(v_r)-2})_+) \\
&= \bigvee_{N(i+j,j)} \partial(D^{r+(\mathrm{out}(v_1)+\cdots+\mathrm{out}(v_r))-2(r+1)}) \\
&= \bigvee_{N(i+j,j)} \partial(D^{r+(i+j+r)-2r-2}) \\
&= \bigvee_{N(i+j,j)} \partial(D^{i+j-2}) \\
&= \bigvee_{N(i+j,j)} S^{i+j-3}.
\end{aligned}$$

Hence, $\partial V_{ij} \cong \bigvee_{N(i+j,j)} S^{i+j-3}$. □

Remark 2.3.7. When $i = 0$, $V_{0j} \cong (K_j)_+$ and so $\partial V_{0j} \cong \partial(D^{j-2})_+ \cong (S^{j-3})_+$.

In the following proposition, we use the face maps $\partial_k(r,s) : K_r \times K_s \to K_{r+s-1}$ of the associahedra, and the face maps $\partial(r,s) : T_r \wedge T_s \to T_{r+s}$ of the spaces T_i to form face maps $\partial_k((u,v),(p,q)) : V_{uv} \wedge V_{pq} \to V_{u+p,v+q-1}$ on the spaces V_{ij}. These face maps will use the A_∞ maps when we have performed an A_∞-splitting, and the D_∞ maps when we have performed a D_∞-splitting.

Proposition 2.3.8. We have natural face maps $\partial_k((u,v),(p,q)) : V_{uv} \wedge V_{pq} \to V_{u+p,v+q-1}$ which satisfy the relations

$$\begin{aligned}
&\partial_k((u,v),(p+a,q+b-1))(1 \times \partial_r((p,q),(a,b))) = \\
&\quad \partial_{k+r-1}((u+p,v+q-1),(a,b))(\partial_k((u,v),(p,q)) \times 1)
\end{aligned} \tag{2.31}$$

for $1 \leqslant k \leqslant v$ and $1 \leqslant r \leqslant q$; and

$$\begin{aligned}\partial_{k+b-1}((p+a, q+b-1), (u,v))(\partial_r((p,q),(a,b)) \times 1) = \\ \partial_r((u+p, v+q-1), (a,b))(\partial_k((p,q),(u,v)) \times 1)(1 \times T)\end{aligned} \tag{2.32}$$

for $1 \leqslant r < k \leqslant q$ where $T : V_{ab} \wedge V_{uv} \to V_{uv} \wedge V_{ab}$ permutes the factors.

Proof. We first observe that we can consider a subcomplex of V_{ij} in the following way:

$$\begin{aligned}\bigvee_{\substack{T \in \mathcal{T}^0_{u+p,v+q-1}, \\ \text{s.t. } t_1 \wedge_k t_2 \in \mathrm{Sp}(T), \\ \text{with } t_1 \in \mathcal{T}^0_{u,v}, t_2 \in \mathcal{T}^0_{p,q}}} & T_{r+1} \wedge (K_{\mathrm{out}(v_0)})_+ \wedge \cdots \wedge (K_{\mathrm{out}(v_r)})_+ \\ & \subseteq \bigvee_{T \in \mathcal{T}^0_{u+p,v+q-1}} T_{r+1} \wedge (K_{\mathrm{out}(v_0)})_+ \wedge \cdots \wedge (K_{\mathrm{out}(v_r)})_+\end{aligned}$$

where $u+p = i$ and $v+q-1 = j$. Also notice that we can write $V_{uv} \wedge V_{pq}$ as

$$\begin{aligned}& (\bigvee_{t_1 \in \mathcal{T}^0_{u,v}} T_{\alpha+1} \wedge (K_{\mathrm{out}(v_0)})_+ \wedge \cdots \wedge (K_{\mathrm{out}(v_\alpha)})_+) \bigwedge \\ & \qquad (\bigvee_{t_2 \in \mathcal{T}^0_{p,q}} T_{\beta+1} \wedge (K_{\mathrm{out}(u_0)})_+ \wedge \cdots \wedge (K_{\mathrm{out}(u_\beta)})_+) \\ = & \bigvee_{\substack{t_1 \in \mathcal{T}^0_{u,v}, \\ t_2 \in \mathcal{T}^0_{p,q}}} T_{\alpha+1} \wedge T_{\beta+1} \wedge (K_{\mathrm{out}(v'_0)})_+ \wedge \cdots \wedge (K_{\mathrm{out}(v'_\alpha)})_+ \\ & \qquad \wedge (K_{\mathrm{out}(u_0)})_+ \wedge \cdots \wedge (K_{\mathrm{out}(u_\beta)})_+\end{aligned}$$

where $(K_{\mathrm{out}(v'_0)})_+, \ldots, (K_{\mathrm{out}(v'_\alpha)})_+$ are arranged such that the kth leaf of t_1 is a child of v'_α to enable us to specify where the grafting takes place. Now we can define a face map:

$$\partial_k((u,v),(p,q)) : V_{uv} \wedge V_{pq} \to V_{u+p,v+q-1}$$

by

$$\partial_k((u,v),(p,q))(t_\alpha,t_\beta,k_1,\dots,k_{\alpha+\beta+2})$$
$$= \begin{cases} \partial(\alpha+1,\beta+1)\wedge 1^{\alpha+\beta+2}(t_\alpha,t_\beta,k_1,\dots,k_{\alpha+\beta+2}) & \substack{\text{if } \mathrm{out}(v'_\alpha)=1 \\ \text{or } \mathrm{out}(u_0)=1,} \\ 1^{\alpha+2}\wedge\partial_k(\mathrm{out}(v'_\alpha),\mathrm{out}(u_0))\wedge 1^{\beta}(t_\alpha,t_\beta,k_1,\dots,k_{\alpha+\beta+2}) & \text{otherwise.} \end{cases}$$

The two required associativity relations hold due to the associtivity conditions satisfied by $\partial(\alpha+1,\beta+1)$ and $\partial_k(\mathrm{out}(v'_\alpha),\mathrm{out}(u_0))$, and the associativity of grafting of trees given in Lemma 1.5.16 □

The following proposition describes how the spaces V_{ij} can be used to define a $\mathbb{N}$-coloured non-symmetric non-unital operad, $\mathcal{V}$. This gives a nice description of the structure on the spaces V_{ij} and also a simple way to define a DA_∞-space as an algebra over $\mathcal{V}$, as we will prove in Proposition 2.3.12.

Proposition 2.3.9. We can define a $\mathbb{N}$-coloured non-symmetric non-unital operad, $\mathcal{V}$, in Top_*, from the spaces V_{ij} for $i \geqslant 0$, $j \geqslant 1$, $(i,j) \neq (0,1)$ by

$$\mathcal{V}(\underline{c};d) = \begin{cases} V_{ij} & \begin{array}{l}\text{for all } \underline{c}=(c_1,\dots,c_j)\in\mathrm{Prof}(\mathbb{N}) \\ \quad\text{such that } c_1+\cdots+c_j+i=d,\end{array} \\ \\ * & \text{otherwise.} \end{cases}$$

Proof. We have partial compositions

$$\gamma_k : \mathcal{V}(\underline{c};d)\wedge\mathcal{V}(\underline{b};c_k)\to\mathcal{V}(\underline{c}\circ_k\underline{b};d)$$

given by $\partial_k((u,v),(p,q)) : V_{uv}\wedge V_{pq}\to V_{u+p,v+q-1}$ if $|\underline{c}| = v$, $|\underline{b}| = q$, $d = c_1+\cdots+c_v+u$ and $c_k = b_1+\cdots+b_q+p$; and trivial otherwise.

It is then easy to see that in all non-trivial cases, diagrams (1.8) and (1.9) from Proposition 1.4.5 are exactly the associativity conditions given in Proposition 2.3.8. □

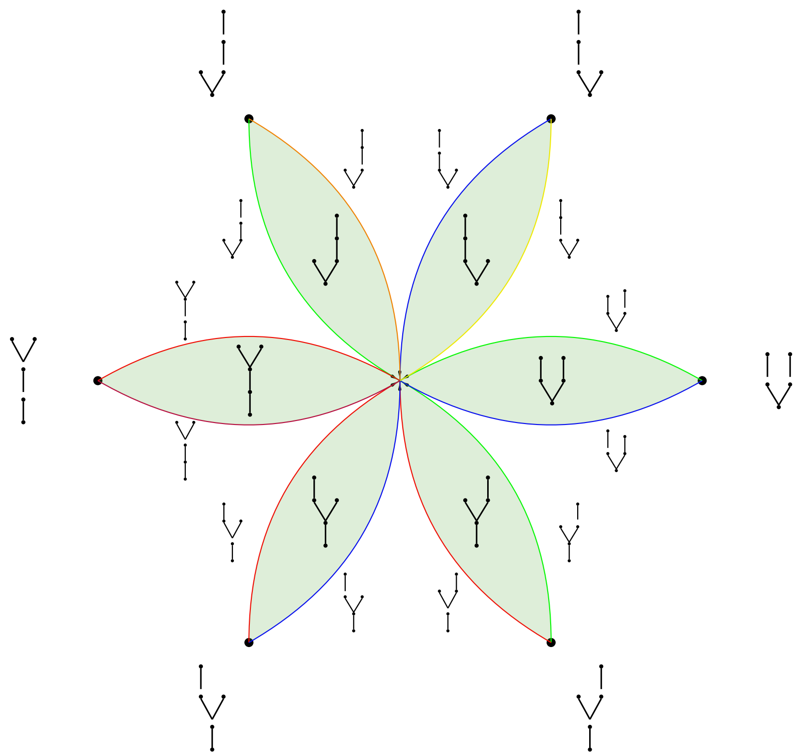


Figure 2.15: Combinatorial description of V_{22}

We will now give the definition of a DA_∞-space. In Proposition 2.3.12 we will see that a DA_∞-space is an algebra over $\mathcal{V}$.

Definition 2.3.10. A family of based spaces $X = \{X_n\}_{n\in\mathbb{N}}$ admits a ***DA***$_\infty$**-structure** if and only if there exist based maps

$$DA_{ij} : V_{ij} \wedge X_{p_1} \wedge \cdots \wedge X_{p_j} \to X_{p_1+\cdots+p_j+i}$$

such that

$$\begin{aligned} &DA_{ij}(\partial_k((u,v),(p,q))(\rho,\theta), x_1, \ldots, x_j) = \\ &\qquad DA_{uv}(\rho, x_1, \ldots, x_{k-1}, DA_{pq}(\theta, x_k, \ldots, x_{k+q-1}), x_{k+q}, \ldots, x_j) \end{aligned} \tag{2.33}$$

for $\rho \in V_{uv}$, $\theta \in V_{pq}$, with $u+p-i$, $v+q=j+1$, and $1 \leqslant k \leqslant v$; and $x_r \in X_{p_r}$ for $r=1,\dots,j$.

A family of based spaces $X=\{X_n\}_{n\in\mathbb{N}}$ with a DA_∞-structure is called a $\boldsymbol{DA_\infty}$**-space**.

Remark 2.3.11. Recall from Definition 1.2.4 that in a derived A_∞-algebra, we have relations

$$\sum_{u=i+p, v=j+q-1, j=1+r+t} (-1)^{rq+t+pj} m_{ij}(1^{\otimes r} \otimes m_{pq} \otimes 1^{\otimes t}) = 0$$

for all $u \geqslant 0$ and $v \geqslant 1$. It should now be possible to see the similarity between the right hand side of relation (2.33) and the relations for a derived A_∞-algebra. This indicates a connection between the maps DA_{ij} restricted to the boundary of V_{ij}, and the relations of a derived A_∞-algebra. In Chapter 3 we will prove that taking singular chains on a DA_∞-space gives rise to a derived A_∞-algebra.

Proposition 2.3.12. A DA_∞-space is an algebra over the $\mathbb{N}$-coloured operad $\mathcal{V}$ in the category of based topological spaces.

Proof. This follows immediately from Proposition 1.4.7. We have a family $X=\{X_n\}_{n\in\mathbb{N}}$ of objects in Top_*, together with maps

$$DA_{ij} : \mathcal{V}(\underline{c}; d) \wedge X_{\underline{c}} \to X_d$$

for $\underline{c}=(c_1,\dots,c_j) \in \mathrm{Prof}(\mathbb{N})$ and $d = c_1+\cdots+c_j+i \in \mathbb{N}$. It is then straightforward to check that relation (2.33) is exactly equivalent to diagram (1.10) of Proposition 1.4.7 in all non-trivial cases. □

Chapter 3

Passage to Algebra

In this section we see the relationship between the spaces studied so far and the algebras they are designed to model. In particular, we want to show that the singular chain complex on a D_∞-space is a twisted chain complex, and more generally that the singular chain complex on a DA_∞-space is a derived A_∞-algebra. The idea is not to show that all derived A_∞-algebras can be derived from the singular chains on some DA_∞-space but rather that any DA_∞-space provides an example of a derived A_∞-algebra via singular chains.

Before we look at each of our three usual cases, we first briefly recall the definition of the tensor product of chain complexes and the statement of the Eilenberg-Zilber theorem.

Definition 3.0.1. Let C and C' be chain complexes. We make the tensor product $C \otimes C'$ into a chain complex

$$(C \otimes C')_n = \bigoplus_{a+b=n} C_a \otimes C'_b$$

with differential $\partial_n : (C \otimes C')_n \to (C \otimes C')_{n-1}$ given by

$$\partial_n(x \otimes x') = \partial_a x \otimes x' + (-1)^a x \otimes \partial_b x'$$

where $x \in C_a$, $x' \in C'_b$ and $a + b = n$.

Theorem 3.0.2 (Eilenberg-Zilber Theorem [EZ53])**.** Let X and Y be topological spaces. Then there exist chain maps

$$\begin{aligned} F : C_*(X \times Y) &\to C_*(X) \otimes C_*(Y), \\ EZ : C_*(X) \otimes C_*(Y) &\to C_*(X \times Y), \end{aligned}$$

such that $F \circ EZ$ and $EZ \circ F$ are chain homotopic to the identity.

Definition 3.0.3. If we have two based topological spaces X, Y then we have a quotient map from $X \times Y$ to $X \wedge Y$ which we will denote by

$$\pi : X \times Y \to X \wedge Y.$$

3.1 A_∞-Spaces to A_∞-Algebras

For our classical case, Stasheff defined an A_∞-space specifically so that taking chains on an A_∞-space gives an A_∞-algebra. Recall from Definition 2.1.8 that a space X admits an ***A_∞*-structure** if and only if there exist maps $M_i : K_i \times X^i \to X$ for $i \geqslant 2$ such that

$$\begin{aligned} &M_i(\partial_k(r, s)(\rho, \sigma), x_1, ..., x_i) = \\ &M_r(\rho, x_1, .., x_{k-1}, M_s(\sigma, x_k, ..., x_{k+s-1}), x_{k+s}, ..., x_i), \\ &\qquad\qquad\qquad\qquad \text{for } \rho \in K_r, \sigma \in K_s, r + s = i + 1. \end{aligned} \tag{3.1}$$

The pair $(X, \{M_i\})$ is called an A_∞-space.

We can see that taking chains on this structure will give a graded R-module with chain maps induced from the maps M_i on spaces. Recall from Definition 1.2.1 that an A_∞-algebra over R is a $\mathbb{Z}$-graded R-module A, endowed with graded R-linear maps

$$m_n : A^{\otimes n} \to A, \quad n \geqslant 1$$

of degree $2 - n$ satisfying the following relation

$$\sum (-1)^{r+st} m_u(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = 0$$

for each $n \geqslant 1$, where the sum runs over all decompositions $n = r + s + t$ and we put $u = r + 1 + t$.

Theorem 3.1.1 ([Sta63])**.** If X admits an A_∞-structure $\{M_i\}$, then $C_*(X)$ admits the structure of an A_∞-algebra by defining $m_1 = \partial$ and for $i > 1$, $m_i(u_1 \otimes \cdots \otimes u_i) = M_{i\#}(\kappa_i \otimes u_1 \otimes \cdots \otimes u_i)$ where κ_i is a suitable generator of $C_*(K_i)$. $\square$

Remark 3.1.2. Stasheff does not give a proof of this result but from the statement we can see that the maps in the algebra should be those induced from the space after applying the Eilenberg-Zilber map so that the map $M_{i\#}$ goes from $(C_*(K_i) \otimes C_*(X)^{\otimes i})_n$ to $C_n(x)$. The generator κ_i should be the generator that represents the top cell of K_i in $C_{i-2}(K_i)$, then a choice of orientation on the space gives the sign conventions in the algebra.

Remark 3.1.3. [MSS02] Since the associahedra are regular cell complexes with the operad structure given by cellular inclusions $K_r \times K_s \to K_{r+s-1}$, their cellular chain complexes $C_*(K_n)$ form a non-symmetric chain operad which is precisely the non-symmetric operad $\mathcal{A}ss_\infty$ for A_∞-algebras.

3.2 D_∞-Spaces to D_∞-Algebras

In this section we consider the relationship between a D_∞-space and a twisted chain complex. Recall from Definition 2.2.5 that a family of based spaces $X = \{X_n\}_{n\in\mathbb{N}}$ admits a $\boldsymbol{D_\infty}$**-structure** if and only if there exist based maps

$D_i : T_i \wedge X_n \to X_{n+i}$ for $i \geqslant 1$ and any $n \in \mathbb{N}$ such that

$$D_i(\partial(r,s)(\rho,\sigma),x) = D_r(\rho, D_s(\sigma,x)) \tag{3.2}$$

for $\rho \in T_r$, $\sigma \in T_s$, $x \in X_n$, with $r+s=i$. A D_∞-space is a family of based spaces $X = \{X_n\}_{n\in\mathbb{N}}$ together with a D_∞-structure.

We can see that when we take singular chains on this structure we will get two gradings, one from the chain complex, and the other we inherit from the grading on the spaces. We will have a $(\mathbb{N},\mathbb{Z})$-bigraded R-module $C_*(X_*,R)$ with $C_n(X_p,R)$ in bidegree (p,n) and $C_n(X_p,R)=0$ if $n<0$.

Recall from Definition 1.2.3 that a twisted chain complex, C, is an $(\mathbb{N},\mathbb{Z})$-bigraded R-module, with maps $d_i : C \to C$ of bidegree $(i, i-1)$ for $i \geqslant 0$, satisfying

$$\sum_{i+p=u} (-1)^i d_i d_p = 0 \text{ for } u \geqslant 0. \tag{3.3}$$

In the following theorem we will show how to obtain such a structure with the maps d_i derived from the induced chain maps on D_i. Similarly, relation 3.3 is derived using the structure of the space T_i and relations 3.2.

Before stating the theorem, we briefly discuss the chain maps induced from the structure maps relating the spaces $T_i = I^{\wedge(i-1)}$. We consider a generator $\tau_i \in C_{i-1}(T_i)$ where we take τ_p to be the $(i-1)$'th power of the obvious chain $u_1 \in C_1(I)$ with respect to the Eilenberg-Zilber product. It should be clear that $d(u_1 \otimes \cdots \otimes u_{p-1}) = \sum_{t=1}^{p-1} (-1)^{t-1} u_1 \otimes \cdots \otimes \hat{u}_t \otimes \cdots \otimes u_{i-1}$. Hence we have

$$d_{T_i}(\tau_i) = \sum_{r+s=i} (-1)^{r-1} D(r,s)(\tau_r \otimes \tau_s). \tag{3.4}$$

where $D(r,s)$ is the induced chain map

$$D(r,s) : C_{r-1}(T_r) \otimes C_{s-1}(T_s) \to C_{r+s-2}(T_{r+s})$$

which sends $\tau_r \otimes \tau_s$ to $\tau_r \otimes 1 \otimes \tau_s$.

Theorem 3.2.1. Let $X = \{X_n\}_{n\in\mathbb{N}}$ be a D_∞-space. Then $C_*(X, R)$, the singular chain complex on X, is a twisted chain complex.

Proof. We take chains on each based space, X_p for $p \in \mathbb{N}$, to obtain a collection of graded R-modules $C_*(X_p, R)$ with differentials ∂_p, for all $p \in \mathbb{N}$. This results in a bigraded R-module $C_n(X_p)$ for $n \in \mathbb{Z}$ and $p \in \mathbb{N}$ with a map of bigraded modules d_0 of bidegree $(0, -1)$ given by $d_0(x) = \partial_p(x)$ for $x \in C_*(X_p, R)$.

Now since X is a D_∞-space, we consider the chain maps induced by the map D_i for any $i \geqslant 1$. We can see from Figure 3.1 that a sequence of maps and restrictions enables us to obtain from D_i a map $d_i : C_{n-i+1}(X_p) \to C_n(X_{p+i})$ for each $p \in \mathbb{N}$ and any $n \geqslant 0$, i.e. a map of bigraded modules of bidegree $(i, i-1)$.

Now, clearly D_i'' is a chain map, so we know that the following diagram commutes.

$$\begin{tikzcd}
\bigoplus\limits_{a+b=n} C_a(T_i) \otimes C_b(X_p) \arrow[r, "D_i''"] \arrow[d, "d_{T_i}\otimes 1 + 1 \otimes d_0"] & C_n(X_p) \arrow[d, "d_0"] \\
\bigoplus\limits_{r+s=n-1} C_r(T_i) \otimes C_s(X_p) \arrow[r, "D_i''"] & C_{n-1}(X_p)
\end{tikzcd}$$

This tells us that when we restrict to $a = i-1$ we have

$$\begin{aligned}
d_0\overline{D_i}(\tau_i \otimes -) &= \overline{D_i}(d_{T_i} \otimes 1)(\tau_i \otimes -) + \overline{D_i}(1 \otimes d_0)(\tau_i \otimes -) \\
&= \overline{D_i}(d_{T_i}(\tau_i) \otimes -) + (-1)^{i-1}\overline{D_i}(\tau_i \otimes d_0(-)) \\
&= \sum_{r+s=i} (-1)^{r-1}\overline{D_i}(D(r,s)(\tau_r \otimes \tau_s) \otimes -) + (-1)^{i-1}\overline{D_i}(\tau_i \otimes d_0(-)).
\end{aligned} \tag{3.5}$$

$$
\begin{tikzcd}
C_n(T_i \wedge X_p) \arrow[r, "D'_i"] & C_n(X_{p+i}) \\
C_n(T_i \times X_p) \arrow[u, "\pi_*"] & \\
(C_*(T_i) \otimes C_*(X_p))_n \arrow[u, "EZ"] & \\
\bigoplus_{a+b=n} C_a(T_i) \otimes C_b(X_p) \arrow[u, equal] \arrow[uuur, bend right=15, "D''_i"] & \\
C_{i-1}(T_i) \otimes C_{n-i+1}(X_p) \arrow[u, hook] \arrow[uuuur, bend right=15, "\overline{D_i}"] & \\
R\left\{\tau_i\right\} \otimes C_{n-i+1}(X_p) \arrow[u, hook, "\text{Inclusion}"] & \\
C_{n-i+1}(X_p) \arrow[u, "\text{Natural Isomorphism}"] \arrow[uuuuuur, bend right=25, "d_i"'] &
\end{tikzcd}
$$

Figure 3.1: Diagram showing the sequence of maps and restrictions to obtain d_i

for $\tau_i \in C_{i-1}(T_i)$. Notice that the last step of this equality comes from equation 3.4.

Now since X is a D_∞-space,

$$D_i(\partial(r,s)(\rho,\sigma),-) = D_r(\rho, D_s(\sigma,-))$$

for $\rho \in T_r$, $\sigma \in T_s$, with $r+s=i$. Thus by considering induced chain maps, we have

$$\overline{D_i}(D(r,s)(\tau_r \otimes \tau_s) \otimes -) = \overline{D_r}(\tau_r \otimes \overline{D_s}(\tau_s \otimes -)).$$

So

$$\begin{aligned}(d_0\overline{D_i}(\tau_i \otimes -) &= \sum_{r+s=i} (-1)^{r-1}\overline{D_i}(D(r,s)(\tau_r \otimes \tau_s) \otimes -) + (-1)^{i-1}\overline{D_i}(\tau_i \otimes d_0(-)) \\ &= \sum_{r+s=i} (-1)^{r-1}\overline{D_r}(\tau_r \otimes \overline{D_s}(\tau_s \otimes -)) + (-1)^{i-1}\overline{D_i}(\tau_i \otimes d_0(-)).\end{aligned} \tag{3.6}$$

Finally, from Figure 3.1 we see that

- $\overline{D_i}(\tau_i \otimes -) = d_i(-)$, and
- $\overline{D_r}(\tau_r \otimes \overline{D_s}(\tau_s \otimes -)) = \overline{D_r}(\tau_r \otimes d_s(-)) = d_r(d_s(-))$,

so we have

$$d_0 d_i + (-1)^i d_i d_0 = \sum_{r+s=i} (-1)^{r-1} d_r d_s$$

as required. □

Remark 3.2.2. Notice that since the spaces T_i form a non-symmetric $\mathbb{N}$-coloured operad $\mathcal{T}$, we have an induced $\mathbb{N}$-coloured non-symmetric operad in chain complexes $C_*(\mathcal{T})$ with structure maps given by the maps $D(r,s)$ and the obvious relations. It may then be possible to argue that there is a map of operads from $C_*(\mathcal{T})$ to the operad $\mathcal{D}_\infty$, and we know that algebras over the operad $\mathcal{D}_\infty$ are twisted chain complexes from [LRW13], however I have not had time to work out the details of this.

3.3 DA_∞-Spaces to dA_∞-Algebras

In this section we generalise the above argument to investigate the relationship between a DA_∞-space and a derived A_∞-algebra. Recall from Definition 2.3.10 that a DA_∞-space is a family of based spaces $X = \{X_n\}_{n\in\mathbb{N}}$ along with based maps

$$DA_{ij} : V_{ij} \wedge X_{p_1} \wedge \cdots \wedge X_{p_j} \to X_{p_1+\cdots+p_j+i}$$

such that

$$\begin{aligned} &DA_{ij}(\partial_k((u,v),(p,q))(\rho,\theta),x_1,\ldots,x_j) = \\ &\qquad DA_{uv}(\rho,x_1,\ldots,x_{k-1},DA_{pq}(\theta,x_k,\ldots,x_{k+q-1}),x_{k+q},\ldots,x_j) \end{aligned} \tag{3.7}$$

for $\rho \in V_{uv}$, $\theta \in V_{pq}$, with $u+p-i$, $v+q=j+1$, and $1 \leqslant k \leqslant v$; and $x_r \in X_{p_r}$ for $r = 1,\ldots,j$.

Again we can see that taking singular chains on this structure will give a bigraded R-module with one grading coming from the chain complex and the other coming from the grading on the spaces. Recall from Definition 1.2.4 that a derived A_∞-algebra is an $(\mathbb{N},\mathbb{Z})$-bigraded R-module, A, with R-linear maps

$$m_{ij} : A^{\otimes j} \to A$$

of bidegree $(i, i+j-2)$ for each $i \geqslant 0$, $j \geqslant 1$, satisfying the equations

$$\sum_{\substack{u=i+p,\\ v=j+q-1,\\ j=1+r+t}} (-1)^{rq+t+pj} m_{ij}(1^{\otimes r} \otimes m_{pq} \otimes 1^{\otimes t}) = 0 \tag{3.8}$$

for all $u \geqslant 0$ and $v \geqslant 1$. We will see in the proof of the following theorem that maps m_{ij} can be derived from the chain maps induced by DA_{ij} and that a relation of the form of equation 3.8 can be derived from the structure of the space V_{ij} and relation 3.7.

Before stating the theorem, we briefly discuss the chain maps induced from the structure maps relating the spaces V_{ij}. Notice that from the construction of V_{ij} in Definition 2.3.1 we have maps

$$\partial_k((p,q),(r,s)) : V_{pq} \wedge V_{rs} \to V_{ij}$$

where $p + r = i$, and $q + s = j + 1$. So we have induced chain maps

$$\partial'_k((p,q),(r,s)) : C_n(V_{pq} \wedge V_{rs}) \to C_n(V_{ij}).$$

We can take a sequence of maps and restrictions as shown in Figure 3.2.

Then if we consider this specifically in the case $n = i + j - 3$ we have

$$\begin{tikzcd} & & C_{i+j-2}(V_{ij}) \arrow[d, "d_{V_{ij}}"] \\ C_{p+q-2}(V_{pq}) \otimes C_{r+s-2}(V_{rs}) \arrow[r, hook, "I"] & \bigoplus_{a+b=i+j-3} C_a(V_{pq}) \otimes C_b(V_{rs}) \arrow[r, "\partial_k((p,q),(r,s))''"] & C_{i+j-3}(V_{ij}). \end{tikzcd}$$

$$
\begin{tikzcd}
C_n(V_{pq} \wedge V_{rs}) \arrow[r, "\partial_k((p,q),(r,s))'"] & C_n(V_{ij}) \\
C_n(V_{pq} \times V_{rs}) \arrow[u, "\pi_*"] & \\
(C_*(V_{pq}) \otimes C_*(V_{rs}))_n \arrow[u] & \\
\bigoplus_{a+b=n} C_a(V_{pq}) \otimes C_b(V_{rs}) \arrow[u, equal] \arrow[uuur, bend right, "\partial_k((p,q),(r,s))''"'] &
\end{tikzcd}
$$

Figure 3.2: Diagram showing the sequence of maps to obtain $\partial_k((p,q),(r,s))''$

Let us denote the composition $\partial_k((p,q),(r,s))''I$ by $D_k((p,q),(r,s))$, then we can choose a generator τ_{pq} in $C_{p+q-2}(V_{pq})$ for each $p+q \geqslant 2$ such that

$$d_{V_{ij}}(\tau_{ij}) = \sum_{\substack{p+r=i,\\ q+s=j+1,\\ 1\leqslant k\leqslant q}} (-1)^{(k-1)s+(j-k)+rq+(p+r-1)} D_k((p,q),(r,s))(\tau_{pq}\otimes\tau_{rs}). \quad (3.9)$$

The sign $(-1)^{(k-1)s+(j-k)+rq+(p+r-1)}$ is consistent with a choice of orientation on the cells of V_{ij}. Notice that since V_{ij} is a wedge of smash products of T_α's and K_β's, we take τ_{ij} to be a sum of products of the generators τ_α and κ_β with the induced map $D_k((p,q),(r,s))$ consistent with the map given in Proposition 2.3.8.

Theorem 3.3.1. Let the family of based spaces $X = \{X_n\}_{n\in\mathbb{N}}$ be a DA_∞-space. Then $C_*(X_p)$, the singular chain complex on X is a bigraded R-module with the structure of a derived A_∞-algebra.

Proof. We take chains on each based space, X_p for $p \in \mathbb{N}$, to obtain a collection of graded R-modules $C_*(X_p, R)$ with differentials ∂_p, for all $p \in \mathbb{N}$. This results in a bigraded R-module $C_*(X_*, R)$ with $C_n(X_p)$ of bidegree (p, n) for $n \in \mathbb{Z}$ and $p \in \mathbb{N}$ with a map of bigraded modules m_{01} of bidegree $(0, -1)$ given by $m_{01}(x) = \partial_p(x)$ for $x \in C_*(X_p)$. By convention we take $C_n(X_*) = 0$ for $n < 0$.

Now we consider the chain maps induced by the maps DA_{ij} for $i \geqslant 0$, $j \geqslant 1$. If we let $A_{p,n} := C_n(X_p)$ then we can see from figure 3.3 that a sequence of maps and restrictions enables us to obtain from DA_{ij} a map $m_{ij} : A^{\otimes j} \to A$ of bidegree $(i, i + j - 2)$.

$$
\begin{tikzcd}
C_n(V_{ij} \wedge X_{p_1} \wedge \cdots \wedge X_{p_j}) \arrow[r, "D'_{ij}"] & C_n(X_{p_1+\cdots+p_j+i}) \\
C_n(V_{ij} \times X_{p_1} \times \cdots \times X_{p_j}) \arrow[u, "\pi_*"'] & \\
(C_*(V_{ij}) \otimes C_*(X_{p_1}) \otimes \cdots \otimes C_*(X_{p_j}))_n \arrow[u, "EZ"'] & \\
\bigoplus_{a_0+a_1+\cdots+a_j=n} C_{a_0}(V_{ij}) \otimes C_{a_1}(X_{p_1}) \otimes \cdots \otimes C_{a_j}(X_{p_j}) \arrow[u, equal] \arrow[uuur, bend right, "D''_{ij}"'] & \\
\bigoplus_{a_1+\cdots+a_j=n+2-i-j} C_{i+j-2}(V_{ij}) \otimes C_{a_1}(X_{p_1}) \otimes \cdots \otimes C_{a_j}(X_{p_j}) \arrow[u, hook] \arrow[uuuur, bend right, "\overline{D_{ij}}"'] & \\
\bigoplus_{a_1+\cdots+a_j=n+2-i-j} R\{\tau_{ij}\} \otimes C_{a_1}(X_{p_1}) \otimes \cdots \otimes C_{a_j}(X_{p_j}) \arrow[u, hook, "\text{Inclusion}"'] & \\
\bigoplus_{a_1+\cdots+a_j=n+2-i-j} C_{a_1}(X_{p_1}) \otimes \cdots \otimes C_{a_j}(X_{p_j}) \arrow[u, "\text{Natural Isomorphism}"'] \arrow[uuuuuur, bend right, "m_{ij}"'] &
\end{tikzcd}
$$

Figure 3.3: Commutative diagram showing a sequence of maps and restrictions to obtain m_{ij}

Since D''_{ij} is a chain map we know that the following diagram commutes:

$$\begin{tikzcd} \bigoplus_{a_0+a_1+\cdots+a_j=n} C_{a_0}(V_{ij}) \otimes C_{a_1}(X_{p_1}) \otimes \cdots \otimes C_{a_j}(X_{p_j}) \arrow[r, "D''_{ij}"] \arrow[d, "d_{V_{ij}}\otimes 1^{\otimes j} + \sum_{t=1}^{j} 1^{\otimes t}\otimes m_{01}\otimes 1^{\otimes j-t}"] & C_n(X_{p_1+\cdots+p_j+i}) \arrow[d, "m_{01}"] \\ \bigoplus_{b_0+b_1+\cdots+b_j=n-1} C_{b_0}(V_{ij}) \otimes C_{b_1}(X_{p_1}) \otimes \cdots \otimes C_{b_j}(X_{p_j}) \arrow[r, "D''_{ij}"] & C_{n-1}(X_{p_1+\cdots+p_j+i}) \end{tikzcd}$$

This tells us that when we restrict to $a_0 = i + j - 2$ we have

$$\begin{aligned} m_{01}\overline{D_{ij}}(\tau_{ij} \otimes - \otimes \cdots \otimes -) &= \overline{D_{ij}}(d_{V_{ij}}(\tau_{ij} \otimes 1^{\otimes j})(- \otimes \cdots \otimes -) \\ &+ \sum_{t=1}^{j}(-1)^{i+j-2}\overline{D_{ij}}(\tau_{ij} \otimes 1^{\otimes t-1} \otimes m_{01} \otimes 1^{\otimes j-t})(- \otimes \cdots \otimes -), \quad (3.10) \end{aligned}$$

where $\tau_{ij} \in C_{i+j-2}(V_{ij})$. Using equation 3.9 and rearranging we get

$$\begin{aligned} & m_{01}\overline{D_{ij}}(\tau_{ij} \otimes - \otimes \cdots \otimes -) \\ &\qquad + \sum_{t=1}^{j}(-1)^{i+j-2}\overline{D_{ij}}(\tau_{ij} \otimes 1^{\otimes t-1} \otimes m_{01} \otimes 1^{\otimes j-t})(- \otimes \cdots \otimes -) \\ &= \overline{D_{ij}}(d_{V_{ij}}(\tau_{ij}) \otimes 1^{\otimes j})(- \otimes \cdots \otimes -) \\ &= \sum_{\substack{p+r=i,\\ q+s=j+1,\\ 1\leqslant k\leqslant q}} (-1)^{(k-1)s+(j-k)+rq+(p+r-1)} \\ &\qquad\qquad \overline{D_{ij}}(D_k((p,q),(r,s))(\tau_{pq} \otimes \tau_{rs}) \otimes - \otimes \cdots \otimes -). \quad (3.11) \end{aligned}$$

Now since X is a DA_∞-space,

$$\begin{aligned} & DA_{ij}(\partial_k((u,v),(p,q)) \wedge 1^{\wedge j})(\rho, \theta, -, \ldots, -) = \\ &\qquad DA_{uv}(1^{\wedge k} \wedge DA_{pq} \wedge 1^{\wedge j-k})(\rho, -, \ldots, -, \theta, -, \ldots, -) \end{aligned}$$

for $\rho \in V_{uv}$, $\theta \in V_{pq}$, with $u + p - i$, $v + q = j + 1$, and $1 \leqslant k \leqslant v$. Thus we have an induced equality of chain maps

$$
\begin{aligned}
&\overline{D_{ij}}(D_k((p,q),(r,s))(\tau_{pq}\otimes\tau_{rs})\otimes 1^{\otimes j})(-\otimes\cdots\otimes -)\\
&\quad = \overline{D_{pq}}(\tau_{pq}\otimes 1^{\otimes k-1}\otimes \overline{D_{rs}}(\tau_{rs}\otimes 1^{\otimes s})\otimes 1^{\otimes q-s-k})(-\otimes\cdots\otimes -).
\end{aligned} \tag{3.12}
$$

So

$$
\begin{aligned}
&m_{01}\overline{D_{ij}}(\tau_{ij}\otimes -\otimes\cdots\otimes -)+\\
&\qquad\qquad \sum_{t=1}^{j}(-1)^{i+j-2}\overline{D_{ij}}(\tau_{ij}\otimes 1^{\otimes t-1}\otimes m_{01}\otimes 1^{\otimes j-t})(-\otimes\cdots\otimes -)\\
&= \sum_{\substack{p+r=i,\\ q+s=j+1,\\ 1\leqslant k\leqslant q}} (-1)^{(k-1)s+(j-k)+rq+(p+r-1)}\\
&\qquad\qquad \overline{D_{ij}}(D_k((p,q),(r,s))(\tau_{pq}\otimes\tau_{rs})\otimes -\otimes\cdots\otimes -)\\
&= \sum_{\substack{p+r=i,\\ q+s=j+1,\\ 1\leqslant k\leqslant q}} (-1)^{(k-1)s+(j-k)+rq+(p+r-1)}\overline{D_{pq}}(\tau_{pq}\otimes 1^{\otimes k-1}\otimes\\
&\qquad\qquad \overline{D_{rs}}(\tau_{rs}\otimes 1^{\otimes s})\otimes 1^{\otimes q-s-k})(-\otimes\cdots\otimes -).
\end{aligned} \tag{3.13}
$$

Finally, from Figure 3.3 we see that

- $\overline{D_{ij}}(\tau_{ij}\otimes -\otimes\cdots\otimes -) = m_{ij}(-\otimes\cdots\otimes -)$, and
- $\overline{D_{pq}}(\tau_{pq}\otimes -,\cdots\otimes -\otimes\overline{D_{rs}}(\tau_{rs}\otimes -\otimes\cdots\otimes -)\otimes\cdots\otimes -)$
  $= m_{pq}(-\otimes\cdots\otimes -\otimes m_{rs}(-\otimes\cdots\otimes -)\otimes\cdots\otimes -)$.

So we have

$$
\begin{aligned}
&m_{01}m_{ij}+\sum_{t=1}^{j}(-1)^{i+j-1}m_{ij}(1^{\otimes t-1}\otimes m_{01}\otimes 1^{\otimes j-t}) =\\
&\qquad \sum_{\substack{p+r=i,\\ q+s=j+1\\ 1\leqslant k\leqslant q}} (-1)^{(k-1)s+(j-k)+rq+(i-1)}m_{pq}(1^{\otimes k-1}\otimes m_{rs}\otimes 1^{\otimes j-k})
\end{aligned} \tag{3.14}
$$

which we multiply throughout by $(-1)^{i-1}$ and rearrange to get

$$\sum_{\substack{i=p+r,\\ j+1=q+s,\\ q=1+k+t}} (-1)^{ks+t+rq} m_{pq}(1^{\otimes k} \otimes m_{rs} \otimes 1^{\otimes t}) = 0$$

as required.

□

Remark 3.3.2. Notice that since the spaces V_{ij} form a non-symmetric $\mathbb{N}$-coloured operad $\mathcal{V}$, we have an induced $\mathbb{N}$-coloured non-symmetric operad in chain complexes $C_*(\mathcal{V})$ with structure maps given by the maps $D_k((p,q),(r,s))$ and the obvious relations. It may then be possible to argue that there is a map of operads from $C_*(\mathcal{V})$ to the operad $(d\mathcal{A}s)_\infty$, and we know that algebras over the operad $(d\mathcal{A}s)_\infty$ are derived A_∞-algebras from [LRW13], however I have not had time to work out the details of this.

Chapter 4

Obstruction Theory

In this chapter we establish three different obstruction theories for the existence of dA_∞-algebra structures on an $(\mathbb{N}, \mathbb{Z})$-bigraded R-module A. These three theories arise from two fundamentally different approaches, the first by considering building the bigraded structure one piece at a time (and this is studied in two different ways), and the second using a total degree approach where the structure is added several maps at a time by arity and horizontal degree. In each case, we work in terms of the relevant Hochschild cohomology of $H(A)$.

We present separately the special case of obstructions to the existence of twisted chain complex structures on an $(\mathbb{N}, \mathbb{Z})$-bigraded R-module. For the special case of A_∞-algebra structures, this question has already been answered by Livernet [Liv14]. We follow the same lines of approach as Livernet in avoiding the common assumptions on the underlying R-module of having no 2-torsion and being $\mathbb{N}$-graded, directly applying her results and generalising where necessary.

Throughout this chapter we work over a commutative ring R, and consider an $(\mathbb{N}, \mathbb{Z})$-bigraded R-module A, where A is a collection of R-modules A_i^j for $i \in \mathbb{N}$, $j \in \mathbb{Z}$.

4.1 Homology of bigraded R-modules of morphisms

In this section, we present a generalisation of Chapter 3 of [Liv14] to vertical bicomplexes. That is, we establish an isomorphism

$$H(\mathrm{Mor}(C^{\otimes n}, C)) \to \mathrm{Mor}(H(C)^{\otimes n}, H(C))$$

where C is a vertical bicomplex. In order to do this, we place some unavoidable projectivity conditions on C, which will later become conditions needed for the obstruction theories we develop.

Definition 4.1.1. Let C and D be vertical bicomplexes, i.e. bigraded R-modules together with a vertical differential $d_C : C_i^j \to C_i^{j+1}$ of bidegree $(0,1)$. We denote by $\mathbf{Mor}(\boldsymbol{C}, \boldsymbol{D})$ the vertical bicomplex given by

$$\mathrm{Mor}(C, D)_u^v = \prod_{\alpha,\beta} \mathrm{Hom}_R(C_\alpha^\beta, D_{\alpha+u}^{\beta+v})$$

with vertical differential $\partial : \mathrm{Mor}(C, D)_u^v \to \mathrm{Mor}(C, D)_u^{v+1}$ given by $\partial f = d_D f - (-1)^v f d_C$ for $f \in \mathrm{Mor}(C, D)_u^v$.

- The bigraded module of **cycles** in C is $Z(C)$ where $Z_i^j(C) = \mathrm{Ker}(d_C : C_i^j \to C_i^{j+1})$.
- The bigraded module of **boundaries** in C is $B(C)$ where $B_i^j(C) = \mathrm{Im}(d_C : C_i^{j-1} \to C_i^j)$.
- The **homology** of C is the bigraded module $H(C)$ where $H_i^j(C) := H^j(C_i^*) = Z_i^j(C)/B_i^j(C)$.

The map $f \in \mathrm{Mor}(C, D)$ is a morphism of vertical bicomplexes if and only if $\partial f = 0$. In particular, $f(Z(C)) \subset Z(D)$ and $f(B(C)) \subset B(D)$. So, if $f \in \mathrm{Mor}(C, D)_u^v$ is such that $\partial f = 0$, then f defines a map $\overline{f} \in \mathrm{Mor}(H(C), H(D))_u^v$

as $\overline{f}([c]) = [f(c)]$. Moreover, if $f = \partial u$ for some $u \in \mathrm{Mor}(C, D)_u^{v-1}$, then $f(Z(C)) \subset B(D)$ and $\overline{f} = 0$. Thus there is a well defined map of bigraded modules

$$\begin{aligned}\mathcal{H}_{C,D} : H(\mathrm{Mor}(C, D)) &\to \mathrm{Mor}(H(C), H(D))\\ [f] &\mapsto \overline{f}.\end{aligned}$$

Definition 4.1.2. We say that a vertical bicomplex, C, satisfies assumption (A) if the following two sequences are split exact:

$$0 \to Z(C) \to C \xrightarrow{d_C} B(C) \to 0$$
$$0 \to B(C) \to Z(C) \to H(C) \to 0.$$

Proposition 4.1.3. Let C and D be vertical bicomplexes satisfying assumption (A).

1. Given $g \in \mathrm{Mor}(H(C), H(D))$, there exists $f \in \mathrm{Mor}(C, D)$ such that $\partial f = 0$ and $\overline{f} = g$.

2. For $f \in \mathrm{Mor}(C, D)$ satisfying $\partial f = 0$ and $\overline{f} = 0 \in \mathrm{Mor}(H(C), H(D))$ there exists $u \in \mathrm{Mor}(C, D)$ such that $\partial u = f$.

Consequently, the map $\mathcal{H}_{C,D} : H(\mathrm{Mor}(C, D)) \to \mathrm{Mor}(H(C), H(D))$ is an isomorphism of bigraded modules and the vertical bicomplex $\mathrm{Mor}(C, D)$ satisfies assumption (A).

Proof. This follows directly from Proposition 3.3 of [Liv14] in which the author proves the exact same statement for C, D dg-modules so in this proposition we are just including an extra grading. □

Corollary 4.1.4. Let C be a vertical bicomplex such that $Z(C)$ and $H(C)$ are projective bigraded modules. For every $n \geqslant 1$, there exists an isomorphism of bigraded modules

$$\varphi_n : H(\mathrm{Mor}(C^{\otimes n}, C)) \to \mathrm{Mor}(H(C)^{\otimes n}, H(C)).$$

Proof. Again Livernet [Liv14] proves this result for a dg-module C and so the proof of this corollary follows her lines of argument but with an extra grading. $\square$

Remark 4.1.5. We let $C_i^{j,k}(A,A) = \mathrm{Mor}(A^{\otimes j}, A)_i^k$. Then we have the isomorphism $\varphi_j : H(C_i^{j,*}(A,A)) \to C_i^{j,*}(H(A), H(A))$.

4.2 Lie structures and Hochschild cohomology

In this section we follow the sign conventions as in [LRW13] and present some of their main results around Hochschild cohomology. It is worth noting that a similar result can be found in [RW11], and differs from the one presented here by sign convention. The one stated here is the more general result and the case we will use later in the obstruction theory.

Definition 4.2.1. Given a vertical bicomplex A, the trigraded R-module $C_*^{*,*}(A,A)$ is defined by

$$C_k^{n,i}(A,A) = \mathrm{Mor}(A^{\otimes n}, A)_k^i.$$

Then we can define a graded R-module $CH^*(A,A)$ given by

$$CH^N(A,A) = \prod_{n \geqslant 1} \prod_{\substack{k,j \\ k+j+n=N}} C_k^{n,j}(A,A),$$

where the grading is the total grading, that is, an element in $C_k^{n,j}(A,A)$ has total degree $j+k+n$.

We describe a graded Lie structure on $CH^{*+1}(A,A)$.

Definition 4.2.2. Let C be a dg-R-module. A graded pre-Lie algebra structure on X is a graded R-linear map $\circ : X \otimes X \to X$ satisfying

$$\forall f, g, h \in X, \quad (f \circ g) \circ h - f \circ (g \circ h) = (-1)^{|g||h|}(f \circ h) \circ g - (-1)^{|g||h|} f \circ (h \circ g).$$

Definition 4.2.3. Let C be a dg-R-module. A graded Lie algebra structure on C is a bracket operation $[-,-] : C \otimes C \to C$ satisfying

$$[f, g] = -(-1)^{|f||g|}[g, f],$$
$$(-1)^{|f||h|}[f, [g, h]] + (-1)^{|g||f|}[g, [h, f]] + (-1)^{|h||g|}[h, [f, g]] = 0.$$

Proposition 4.2.4 ([LRW13])**.** The composition product,

$$f \circ g = \sum_{r=0}^{n-1} (-1)^{(n+1)(m+1)+r(m+1)+j(n+1)+k|g|} f(1^{\otimes r} \otimes g \otimes 1^{\otimes n-r-1}) \in C_{k+l}^{n+m-1,i+j}(A, A)$$

for $f \in C_k^{n,i}(A, A)$ and $g \in C_l^{m,j}(A, A)$ endows $CH^{*+1}(A, A)$ with the structure of a weight graded pre-Lie algebra, with weight given by $|f| = k+n+i-1$.

Corollary 4.2.5. The bracket

$$[f, g] = f \circ g - (-1)^{|f||g|} g \circ f \quad \text{for } f \in C_k^{n,i}(A, A) \text{ and } g \in C_l^{m,j}(A, A)$$

gives rise to a graded Lie algebra structure on $CH^{*+1}(A, A)$.

Proof. A graded pre-Lie algebra as stated above always gives rise to a graded Lie algebra with the given bracket operation. A proof of this general result can be found in Theorem 1 of [Ger63]. □

Remark 4.2.6. In fact we can actually go further than this and say that

$$f \circ g = \sum_{r=1}^{n} f \circ_r g$$

where $f \circ_r g = (-1)^{(n+1)(m+1)+(r+1)(m+1)+j(n+1)+k|g|} f(1^{\otimes r-1} \otimes g \otimes 1^{\otimes n-r})$ defines a weight graded pre-Lie system in the sense of the following definition.

Definition 4.2.7. Let $\mathcal{O}$ be an $(\mathbb{N}, \mathbb{N}, \mathbb{Z})$-trigraded R-module. A weight graded pre-Lie system on $\mathcal{O}$ is a sequence of maps, called composition maps,

$$\circ_u : \mathcal{O}_k^{n,i} \otimes \mathcal{O}_l^{m,j} \to \mathcal{O}_{k+l}^{n+m-1,i+j}, \quad \forall \quad 1 \leqslant u \leqslant n$$

satisfying the relations: for every $f \in \mathcal{O}_k^{n,i}$, $g \in \mathcal{O}_l^{m,j}$ and $h \in \mathcal{O}_a^{b,d}$,

$$\begin{aligned} f \circ_u (g \circ_v h) &= (f \circ_u g) \circ_{v+u-1} h, && \forall \quad 1 \leqslant u \leqslant n \text{ and } 1 \leqslant v \leqslant m, \\ (f \circ_u g) \circ_{v+m-1} h &= (-1)^{|g||h|}(f \circ_v h) \circ_u g, && \forall \quad 1 \leqslant u < v \leqslant n. \end{aligned}$$

Remark 4.2.8. Notice that this definition contains an extra $\mathbb{N}$ grading compared to the one presented in [Liv14], however that definition can be recovered from the one above by considering the horizontal grading to be zero throughout.

The following two results are generalisations of Lemma 2.10 and Proposition 2.13 from [Liv14]. We omit the proofs for these due to them being analogous to those in [Liv14].

Lemma 4.2.9. Let $(\mathcal{O}, \circ)$ be a weight graded pre-Lie system. Let $g \in \mathcal{O}$ be an element of odd weight. Then for all $f \in \mathcal{O}$, one has

$$(f \circ g) \circ g = f \circ (g \circ g) \quad \text{and} \tag{4.1}$$

$$[f, g \circ g] = -[g, [g, f]] = -[g \circ g, f]. \tag{4.2}$$

□

Proposition 4.2.10. Let A be a vertical bicomplex with vertical differential m_{01}. There is an induced differential ∂ on $C(A, A)$ which satisfies, for all $g \in C(A, A)$,

$$\partial f = [m_{01}, f]; \tag{4.3}$$

$$\partial(f \circ g) = \partial f \circ g + (-1)^{|f|} f \circ \partial g; \tag{4.4}$$

$$\partial[f, g] = [\partial f, g] + (-1)^{|f|}[f, \partial g]. \tag{4.5}$$

As a consequence, $CH(A, A)$ is a differential (weight) graded Lie algebra.

Proof. The differential m_{01} is considered as an element of $C_0^{1,1}(A, A)$, so has weight 1. Hence, for all $f \in C_k^{n,i}(A, A)$ we have

$$\begin{aligned}
\partial f &= m_{01} f - (-1)^i \sum_{r=1}^{n} f(1^{\otimes r-1} \otimes m_{01} \otimes 1^{\otimes n-r}) \\
&= m_{01} \circ f - (-1)^i (-1)^{n+1+k} f \circ m_{01} \\
&= m_{01} \circ f - (-1)^{|f|} f \circ m_{01} \\
&= [m_{01}, f].
\end{aligned}$$

The proof for 4.4 and 4.5 are easy calculations and can be found without the extra grading in [Liv14]. □

Recall from Definition 1.2.4 that a dA_∞-algebra is an $(\mathbb{N}, \mathbb{Z})$-bigraded R-module, A, with R-linear maps $m_{ij} \in C_i^{j,i+j-2}(A, A)$ for each $i \geqslant 0$, $j \geqslant 1$, satisfying the equations

$$\sum_{\substack{u=i+p,\\ v=j+q-1,\\ j=1+r+t}} (-1)^{rq+t+pj} m_{ij}(1^{\otimes r} \otimes m_{pq} \otimes 1^{\otimes t}) = 0$$

for all $u \geqslant 0$, $v \geqslant 1$. This system of equations is equivalent to

$$\sum_{\substack{u=i+p,\\ v=j+q-1}} m_{ij} \circ m_{pq} = 0 \tag{4.6}$$

for all $u \geqslant 0$ and $v \geqslant 1$.

Proposition 4.2.11. Let

$$\mathcal{O}_{ij} = \sum_{\substack{i=a+p,\\ j=b+q-1,\\ (a,b),(p,q)\neq(0,1)}} m_{ab} \circ m_{pq} \in C_i^{j,i+j-3}(A, A).$$

Then $\partial\mathcal{O}_{ij} = 0$.

Proof.

$$\begin{aligned}
\partial\mathcal{O}_{ij} &= \sum_{\substack{i=a+p,\\ j=b+q-1,\\ (a,b),(p,q)\neq(0,1)}} \partial(m_{ab} \circ m_{pq}) \\
&= \sum_{\substack{i=a+p,\\ j=b+q-1,\\ (a,b),(p,q)\neq(0,1)}} \partial m_{ab} \circ m_{pq} - m_{ab} \circ \partial m_{pq} \\
&= \sum_{\substack{i=c+e+g,\\ j=d+f+h=2,\\ (c,d),(e,f),(g,h)\neq(0,1)}} (m_{cd} \circ m_{ef}) \circ m_{gh} - m_{cd} \circ (m_{ef} \circ m_{gh}) \\
&= 0.
\end{aligned}$$

The individual summands vanish as a result of the Jacobi relation when $m_{cd} \neq m_{ef} \neq m_{gh}$; the pre-Lie relation when $m_{ef} \neq m_{gh}$ and ($m_{cd} = m_{ef}$ or $m_{cd} = m_{gh}$); and by Lemma 4.2.9 when $m_{ef} = m_{gh}$. $\square$

Lemma 4.2.12. Let A be an dA_∞-algebra, with structure maps m_{ij}. The maps

$$\partial = [m_{01}, -] : C_k^{n,i}(A, A) \to C_k^{n,i+1}(A, A),$$

$$d^\tau = [m_{11}, -] : C_k^{n,i}(A, A) \to C_{k+1}^{n,i}(A, A),$$

and

$$d^\mu = [m_{02}, -] : C_k^{n,i}(A, A) \to C_k^{n+1,i}(A, A)$$

satisfy

$$\begin{aligned}
\partial d^\tau &= -d^\tau \partial, \\
\partial d^\mu &= -d^\mu \partial.
\end{aligned}$$

Proof. The proof is the same for both parts of this lemma and so here we only present the proof of the first equality.

From the relations on a dA_∞-algebra we know that $\partial m_{11} = [m_{01}, m_{11}] = 0$. So for $f \in C_k^{n,i}(A, A)$

$$\begin{aligned}
\partial d^\tau f &= [m_{01}, [m_{11}, f]] \\
&= -(-1)^{|m_{01}||m_{11}|}(-1)^{|m_{01}||f|}[m_{11}, [f, m_{01}]] \\
&\qquad - (-1)^{|m_{01}||f|}(-1)^{|m_{11}||f|}[f, [m_{01}, m_{11}]] \\
&= -(-1)^{|f|}[m_{11}, [f, m_{01}]] \\
&= -(-1)^{|f|}(-1)^{|f|}[m_{11}, [m_{01}, f]] \\
&= -d^\tau \partial f.
\end{aligned}$$

□

The remainder of this section is devoted to describing the Hochschild cohomology of A via $CH(A, A)$. We also consider the special cases when A is a bidga, bicomplex or a dga.

Definition 4.2.13. A bidga is a dA_∞-algebra with $m_{ij} = 0$ for $i + j \geqslant 3$.

Remark 4.2.14. An equivalent definition is that a bidga is a monoid in the category of bicomplexes with vertical and horizontal differentials given by m_{01} and m_{11}, and associative multiplication given by m_{02}.

Definition 4.2.15 ([LRW13])**.** Let m be a formal sum $m = \sum m_{ij}$ and (A, m) be a dA_∞-algebra. Then the Hochschild cohomology of A is defined as

$$HH^*(A, A) := H^*(CH(A, A), [m, -]).$$

Remark 4.2.16 ([LRW13]). When A is a bidga with $m = m_{11} + m_{02}$, i.e. A is a bidga with trivial vertical differential, the external grading is preserved by both bracketing with m_{11} and m_{02}. Hence we can, as in [RW11, Section 3.1], consider bigraded Hochschild cohomology

$$HH^{s,r}_{bidga}(A, A) = H^s(\prod_n C^{n,r}_{*-n}(A, A), [m, -]).$$

Remark 4.2.17. In addition to the above, when A is a bicomplex with trivial vertical differential, the arity and vertical grading are preserved by bracketing with m_{11}. As a result we can consider a trigraded Hochschild cohomology

$$HH^{s,n,r}_{bicx}(A, A) := H^s(C^{n,r}_*(A, A), [m_{11}, -]).$$

When A is a graded algebra with an associative multiplication $m = m_2$, i.e. A is a dga with trivial vertical differential, the grading is preserved by bracketing with m_2. We think of A as a bigraded module concentrated in horizontal degree zero, with an associative multiplication $m = m_{02}$, and then we can consider bigraded Hochschild cohomology

$$HH^{0,n,r}_{dga}(A, A) = H^n(C^{*,r}_0(A, A), [m_{02}, -]).$$

4.3 Obstruction theory for A_∞-structures

Here we recall the main theorem of [Liv14]. It is worth noting that our conventions on notation and bidegree differ slightly from Livernet's and so here the result has been written to be consistent with our notation so far. We easily recover the Lie structure on $\mathrm{End}(A)$ defined in [Liv14] by setting $l = k = 0$ in Proposition 4.2.4. The results in Section 4.1 are precisely a bigraded generalisation of the results in Section 3 of [Liv14] and thus the original results are easily recovered by ignoring the horizontal grading in those presented above.

Definition 4.3.1. Let $r > 0$ be an integer. A graded R-module A is an A_r-algebra if there exists a collection of elements $m_i \in C_0^{i,i-2}(A,A)$ for $1 \leqslant i \leqslant r$ such that (in the notation of Proposition 4.2.4)

$$\sum_{i+j=n+1} m_i \circ m_j = 0$$

for all $1 \leqslant n \leqslant r$.

Remark 4.3.2. Given an A_r-algebra with $r \geqslant 3$, the graded R-module $H(A)$ is a graded associative algebra (i.e. a dga with trivial differential) with multiplication induced from m_2, so we can consider the Hochschild cohomology $HH_{dga}^{0,n,t}(H(A), H(A))$.

Theorem 4.3.3 ([Liv14], Theorem 4.8)**.** Let $r \geqslant 3$. Let A be a dg-module such that $H(A)$ and $Z(A)$ are graded projective R-modules. Assume A is an A_r-algebra, with structure maps $m_i \in C_0^{i,i-2}(A,A)$ for $1 \leqslant i \leqslant r$. The obstruction to lift the A_{r-1}-structure of A to an A_{r+1}-structure lies in $HH_{dga}^{0,r+1,r-2}(H(A), H(A))$.

This theorem tells us that $\mathcal{O}_{0,r+1}$ gives rise to an element

$$\overline{\mathcal{O}_{0,r+1}} \in C_0^{r+1,r-2}(H(A), H(A)),$$

and if the class of $\overline{\mathcal{O}_{0,r+1}}$ vanishes in $HH^{0,r+1,r-2}_{dga}(H(A),H(A))$ then there exist maps m_{r+1} and m'_r which extend the A_{r-1}-structure of A to an A_{r+1}-structure.

In fact the statement is stronger than this and the following Proposition shows that if an extension exists, then the class of $\overline{\mathcal{O}_{0,r+1}}$ vanishes and so we can say that an extension exists if and only if the class of $\overline{\mathcal{O}_{0,r+1}}$ vanishes in $HH^{0,r+1,r-2}_{dga}(H(A),H(A))$.

Proposition 4.3.4. Let $r \geqslant 3$. Let A be a dg-module such that $H(A)$ and $Z(A)$ are graded projective R-modules. Assume A is an A_r-algebra, with structure maps $m_i \in C_0^{i,i-2}(A,A)$ for $1 \leqslant i \leqslant r$. If an extension of the A_{r-1}-structure of A to an A_{r+1}-structure exists, then the class of $\overline{\mathcal{O}_{0,r+1}}$ vanishes in $HH^{0,r+1,r-2}_{dga}(H(A),H(A))$.

Proof. By assumption, we have relations

$$\partial m_n = -\sum_{\substack{i+j=n+1,\\ i,j>1}} m_i \circ m_j \quad \text{for} \quad n \leqslant r.$$

Notice that

$$\begin{aligned}\mathcal{O}_{0,r+1} &= \sum_{\substack{i+j=r+2,\\ i,j>1}} m_i \circ m_j \\ &= m_2 \circ m_r + m_r \circ m_2 + \sum_{\substack{i+j=r+2,\\ i,j>2}} m_i \circ m_j.\end{aligned}$$

Now if an extension of the A_{r-1}-structure to an A_{r+1}-structure exists, then we have $m_1, .., m_{r-1}$ as above and also two new elements, $m'_r \in C_0^{r,r-2}(A,A)$ and $m_{r+1} \in C_0^{r+1,r-1}(A,A)$ with relations

$$\partial m'_r = -\sum_{\substack{i+j=r+1,\\ i,j>1}} m_i \circ m_j \tag{4.7}$$

$$\partial m_{r+1} = -m_2 \circ m_r' - m_r' \circ m_2 - \sum_{\substack{i+j=r+2, \\ i,j>2}} m_i \circ m_j. \tag{4.8}$$

We see that $\partial m_r = \partial m_r'$ so $\partial(m_r - m_r') = 0$, and

$$\begin{aligned} \partial m_{r+1} + \mathcal{O}_{0,r+1} &= m_2 \circ (m_r - m_r') + (m_r - m_r') \circ m_2 \\ &= d^\mu(m_r - m_r'). \end{aligned}$$

We can check that

$$\partial(\partial m_{r+1} + \mathcal{O}_{0,r+1}) = \partial d^\mu(m_r - m_r') = -d^\mu \partial(m_r - m_r') = -d^\mu(0) = 0$$

and so we have a map $\overline{\partial m_{r+1} + \mathcal{O}_{0,r+1}}$. Now

$$\overline{\partial m_{r+1} + \mathcal{O}_{0,r+1}} = \overline{d^\mu(m_r - m_r')} = d^\mu(\overline{m_r - m_r'})$$

but also

$$\overline{\partial m_{r+1} + \mathcal{O}_{0,r+1}} = \overline{\partial m_{r+1}} + \overline{\mathcal{O}_{0,r+1}} = \overline{\mathcal{O}_{0,r+1}}$$

since $\partial m_{r+1} \in \operatorname{Im} \partial$ so $\overline{\partial m_{r+1}} = 0$.

Now we have shown that $\overline{\mathcal{O}_{0,r+1}} = d^\mu(\overline{m_r - m_r'})$, in particular $\overline{\mathcal{O}_{0,r+1}} \in \operatorname{Im} d^\mu$, and so $[\overline{\mathcal{O}_{0,r+1}}]$ vanishes in $HH_{dga}^{0,r+1,r-2}(H(A), H(A))$. $\square$

4.4 Obstruction theory for twisted chain complexes

In this section we consider twisted chain complexes, another special case of the obstruction theory which may be of independent interest to some readers. As above these results can be recovered from the more general results in the following section.

Here we are working with $(\mathbb{N}, \mathbb{Z})$-bigraded R-modules with all structure maps in arity one so we can use the isomorphism from Corollary 4.1.4 with $n = 1$ to get

$$\varphi : H(\mathrm{Mor}(A, A)) \to \mathrm{Mor}(H(A), H(A)).$$

We can also specialise the Lie structure from Section 4.2 to get $f \circ g = (-1)^{k|g|} fg$ for $f \in C_k^{1,i}(A, A)$ and $g \in C_l^{1,j}(A, A)$.

Definition 4.4.1. A stage r twisted chain complex, A, is an $(\mathbb{N}, \mathbb{Z})$-bigraded R-module with maps $d_i : A \to A$ of bidegree $(i, i-1)$ for $0 \leqslant i \leqslant r$, satisfying

$$\sum_{i+p=u} d_i \circ d_p = 0 \quad \text{for} \quad 0 \leqslant u \leqslant r.$$

Remark 4.4.2. If we have a stage r twisted chain complex, with $r \geqslant 2$, then the relation when $u = 1$ implies

$$d_0 d_1 - d_1 d_0 = 0 \quad \text{i.e.} \quad \partial d_1 = 0,$$

so $\overline{d_1} \in C_1^{1,0}(H(A), H(A))$ is well defined. Additionally, the relation when $u = 2$ implies

$$d_0 d_2 + d_2 d_0 = d_1 d_1 \quad \text{i.e.} \quad \partial d_2 = d_1 d_1,$$

Thus $\bar{d}_1 \bar{d}_1 = 0$ and $\bar{d}_1$ is a differential for $H(A)$ (the induced differential on $\mathrm{Mor}(H(A), H(A))$ is $[\overline{d_1}, -]$).

Hence, $H(A)$ is a bicomplex with trivial vertical differential and we can consider the Hochschild cohomology $HH_{bicx}^{s,1,t}(H(A), H(A))$.

Theorem 4.4.3. Let $r \geqslant 2$. Let A be an $(\mathbb{N}, \mathbb{Z})$-bigraded R-module with vertical differential $d_0 : A_s^t \to A_s^{t+1}$ such that $H(A, d_0)$ and $Z(A, d_0)$ are $(\mathbb{N}, \mathbb{Z})$-bigraded projective R-modules. Assume A is a stage r twisted chain complex. Then the obstruction to lift the stage $(r-1)$-structure of A to a stage $(r+1)$-structure lies in $HH^{r+1,1,r-1}_{bicx}(H(A), H(A))$.

Proof. By assumption we have

$$\sum_{i+p=u} d_i \circ d_p = 0 \quad \text{for} \quad 0 \leqslant u \leqslant r$$

or equivalently

$$\partial d_n = \sum_{\substack{i+p=u, \\ i,p>0}} -d_i \circ d_p \quad \text{for} \quad 0 \leqslant u \leqslant r.$$

We begin by defining

$$\mathcal{O}_{r+1} = \sum_{\substack{i+p=r+1, \\ i,p>0}} d_i \circ d_p.$$

Then,

$$\begin{aligned}
\partial \mathcal{O}_{r+1} &= \sum_{\substack{i+p=r+1, \\ i,p>0}} \partial(d_i \circ d_p) \\
&= \sum_{\substack{i+p=r+1, \\ i,p>0}} \partial(d_i) \circ d_p - d_i \circ \partial(d_p) \\
&= -\sum_{\substack{i+p=r+1, \\ i,p>0}} \sum_{\substack{s+t=i, \\ s,t>0}} (d_s \circ d_t) \circ d_p + \sum_{\substack{i+p=r+1, \\ i,p>0}} \sum_{\substack{u+v=p, \\ u,v>0}} d_i \circ (d_u \circ d_v) \\
&= \sum_{\substack{a+b+c=r+1, \\ a,b,c>0}} -(d_a \circ d_b) \circ d_c + d_a \circ (d_b \circ d_c) \\
&= \sum_{\substack{a+b+c=r+1, \\ a,b,c>0}} (-1)^{b+1} d_a d_b d_c + (-1)^b d_a d_b d_c \\
&= 0.
\end{aligned}$$

So, $\partial \mathcal{O}_{r+1} = 0$ and $\mathcal{O}_{r+1}$ gives rise to an element $\overline{\mathcal{O}_{r+1}} \in C^{1,r-1}_{r+1}(H(A), H(A))$. Now,

$$\begin{aligned}
\partial\left(\sum_{\substack{a+b=r+2,\\ a,b>1}} (-1)^a d_a d_b\right) &= \sum_{\substack{a+b=r+2,\\ a,b>1}} \partial(d_a \circ d_b) \\
&= \sum_{\substack{a+b=r+2,\\ a,b>1}} \partial(d_a) \circ d_b - d_a \circ \partial(d_b) \\
&= \sum_{\substack{s+t+v=r+2,\\ s,t>0,v>1}} -(d_s \circ d_t) \circ d_v + d_v \circ (d_s \circ d_t) \\
&= \sum_{\substack{s+t=r+1,\\ s,t>0}} (d_s \circ d_t) \circ d_1 - d_1 \circ (d_s \circ d_t) \\
&= \mathcal{O}_{r+1} \circ d_1 - d_1 \circ \mathcal{O}_{r+1} \\
&= -[d_1, \mathcal{O}_{r+1}].
\end{aligned}$$

So $\partial[d_1, \mathcal{O}_{r+1}] = 0$, and $[d_1, \mathcal{O}_{r+1}] \in \operatorname{Im} \partial$ so $\overline{[d_1, \mathcal{O}_{r+1}]} = 0$. It can easily be checked that $\overline{[d_1, \mathcal{O}_{r+1}]} = [\overline{d_1}, \overline{\mathcal{O}_{r+1}}]$.

If the class of $\overline{\mathcal{O}_{r+1}}$ vanishes in $HH^{r+1,1,r-1}_{bicx}(H(A), H(A))$ then there exists an element $u \in C^{1,r-1}_r(H(A), H(A))$ such that $[d_1, u] = \overline{\mathcal{O}_{r+1}}$. We apply the isomorphism φ to obtain an element $d'_r \in C^{1,r-1}_r(A, A)$ such that $\partial d'_r = 0$ and $\overline{d'_r} = u$. Now,

$$\overline{[d_1, d'_r]} = [\overline{d_1}, \overline{d'_r}] = [d_1, u] = \overline{\mathcal{O}_{r+1}}.$$

So $\overline{[d_1, d'_r] - \mathcal{O}_{r+1}} = 0 \in C^{1,r-1}_{r+1}(H(A), H(A))$ and thus there exists an element $d_{r+1} \in C^{1,r}_{r+1}(A, A)$ such that

$$\begin{aligned}
\partial d_{r+1} &= [d_1, d'_r] - \mathcal{O}_{r+1} \\
&= [d_1, d'_r - d_r] - \sum_{\substack{i+p=r+1,\\ i,p>1}} (-1)^i d_i d_p.
\end{aligned}$$

The collection $\{d_0, d_1, ..., d_{r-1}, d_r - d'_r, d_{r+1}\}$ form a stage $r+1$ twisted chain complex structure on A. Thus the class of $\overline{\mathcal{O}_{r+1}}$ is an obstruction and if $\left[\overline{\mathcal{O}_{r+1}}\right]$ vanishes in $HH^{r+1,1,r-1}_{bicx}(H(A), H(A))$ then we can extend the stage $(r-1)$-structure on A to a stage $(r+1)$ twisted chain complex structure on A. □

4.5 Obstruction theory for derived A_∞-structures

In this final section of the chapter we present the two main results, Theorem 4.5.3 and Theorem 4.5.6. Though the two different theorems present a choice of how to build the structure of a dA_∞-algebra, the proofs are largely similar and follow the same line of argument. We begin by defining the different notions of "partial" dA_∞-structure.

Definition 4.5.1. Let $i \geqslant 0$, $j \geqslant 1$ be integers. An $(\mathbb{N}, \mathbb{Z})$-bigraded R-module, A, is a dA^{-}_{ij}-algebra if there exist elements $m_{pq} \in C^{q,p+q-2}_p(A, A)$ for all $0 \leqslant p \leqslant i$, $1 \leqslant q \leqslant j$, with $(p,q) \neq (i,j)$, satisfying the equations

$$\sum_{\substack{u=c+p,\\ v=d+q-1}} m_{cd} \circ m_{pq} = 0$$

for all $0 \leqslant u \leqslant i$, $1 \leqslant v \leqslant j$ with $(u,v) \neq (i,j)$.

Definition 4.5.2. Let $i \geqslant 0$, $j \geqslant 1$ be integers. An $(\mathbb{N}, \mathbb{Z})$-bigraded R-module, A, is a dA_{ij}-algebra if there exist elements $m_{pq} \in C^{q,p+q-2}_p(A, A)$ for all $0 \leqslant p \leqslant i$, $1 \leqslant q \leqslant j$, satisfying the equations

$$\sum_{\substack{u=c+p,\\ v=d+q-1}} m_{cd} \circ m_{pq} = 0$$

for all $0 \leqslant u \leqslant i$, $1 \leqslant v \leqslant j$.

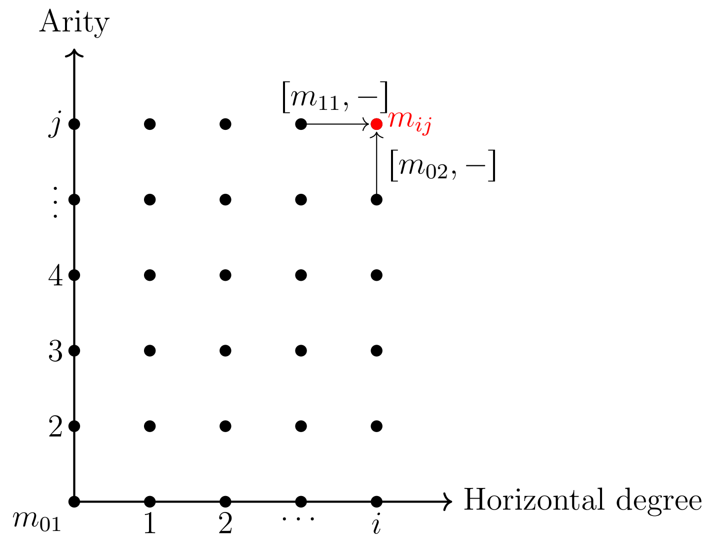


Figure 4.1: The maps in a dA_{ij}^{-}-algebra / dA_{ij}-algebra

In the following theorem we are going to consider obstructions to extending a dA_{ij}^{-}-algebra structure to a dA_{ij}-algebra structure.

Theorem 4.5.3. Let $i \geqslant 1$, $j \geqslant 2$ be integers such that $i + j > 3$.

Let A be a vertical bicomplex such that $H(A)$ and $Z(A)$ are bigraded projective R-modules. Assume A is a dA_{ij}^{-}-algebra with structure maps $m_{pq} \in C_p^{q,p+q-2}(A, A)$.

4.5.3.1 Then after modifying $m_{(i-1)j}$, the obstruction to extend the modified dA_{ij}^{-}-algebra structure to a dA_{ij}-algebra structure lies in

$$HH_{bicx}^{i,j,i+j-3}(H(A), H(A)).$$

4.5.3.2 Then after modifying $m_{i(j-1)}$, the obstruction to extend the modified

dA_{ij}^{-}-algebra structure to a dA_{ij}-algebra structure lies in

$$HH_{dga}^{i,j,i+j-3}(H(A), H(A)).$$

Proof. Let $\partial = [m_{01}, -]$, $d^{\tau} = [m_{11}, -]$, $d^{\mu} = [m_{02}, -]$.

Note that $|m_{pq}| = p+q+(p+q-2)-1 = 2p+2q-3$ is odd for all $p \geqslant 0$, $q \geqslant 1$.

By assumption we have

$$\sum_{\substack{u=a+p,\\ v=b+q-1}} m_{ab} \circ m_{pq} = 0$$

for all $0 \leqslant u \leqslant i$, $1 \leqslant v \leqslant j$ with $(u, v) \neq (i, j)$. Or equivalently,

$$\partial m_{uv} = - \sum_{\substack{u=a+p,\\ v=b+q-1,\\ (a,b),(p,q)\neq(0,1)}} m_{ab} \circ m_{pq}.$$

We have

$$\mathcal{O}_{ij} = \sum_{\substack{i=a+p,\\ j=b+q-1,\\ (a,b),(p,q)\neq(0,1)}} m_{ab} \circ m_{pq} \in C_i^{j,i+j-3}(A, A),$$

where $\partial \mathcal{O}_{ij} = 0$ by Proposition 4.2.11. So $\mathcal{O}_{ij}$ gives rise to an element $\overline{\mathcal{O}_{ij}} \in C_i^{j,i+j-3}(H(A), H(A))$. For 4.5.3.1 we notice that

$$\partial \left(\sum_{\substack{a+p=i+1,\\ b+q=j+1,\\ (a,b),(p,q)\neq(0,1),(1,1)}} m_{ab} \circ m_{pq} \right)$$

$$
\begin{aligned}
&= \sum_{\substack{a+p=i+1,\\ b+q=j+1,\\ (a,b),(p,q)\neq(0,1),(1,1)}} \partial m_{ab}\circ m_{pq} - m_{ab}\circ\partial m_{pq}\\
&= \sum_{\substack{a+p=i+1,\\ b+q=j+1,\\ (a,b),(p,q)\neq(0,1),(1,1)}} \partial m_{ab}\circ m_{pq} - m_{pq}\circ\partial m_{ab}\\
&= \sum_{\substack{c+e+p=i+1,\\ d+f+q=j+2,\\ (c,d),(e,f),(p,q)\neq(0,1),\\ (p,q)\neq(1,1)}} -(m_{cd}\circ m_{ef})\circ m_{pq} + m_{pq}\circ(m_{cd}\circ m_{ef})\\
&= \sum_{\substack{c+e=i,\\ d+f=j+1,\\ (c,d),(e,f)\neq(0,1)}} -(m_{cd}\circ m_{ef})\circ m_{11} + m_{11}\circ(m_{cd}\circ m_{ef})\\
&= \mathcal{O}_{ij}\circ m_{11} - m_{11}\circ\mathcal{O}_{ij}\\
&= -[m_{11},\mathcal{O}_{ij}]\\
&= -d^{\tau}\mathcal{O}_{ij}.
\end{aligned}
$$

As a consequence, $d^{\tau}(\overline{\mathcal{O}_{ij}}) = 0$ and $\overline{\mathcal{O}_{ij}}$ represents a class in

$$HH^{i,j,i+j-3}_{bicx}(H(A),H(A)) = H^{i}(C^{j,i+j-3}_{*}(H(A),H(A)),d^{\tau}).$$

If $[\overline{\mathcal{O}_{ij}}] = 0$ then there exists $u \in C^{j,i+j-3}_{i-1}(H(A),H(A))$ such that $d^{\tau}u = \overline{\mathcal{O}_{ij}}$. By Corollary 4.1.4 there exists $m'_{(i-1)j} \in C^{j,i+j-3}_{i-1}(A,A)$ such that $\partial m'_{(i-1)j} = 0$

and $\overline{m'_{(i-1)j}} = u$. So

$$\begin{aligned}
\overline{[m_{11}, m'_{(i-1)j}]} = \overline{d^\tau m'_{(i-1)j}} &= d^\tau u \\
&= \overline{\mathcal{O}_{ij}} \\
&= \overline{\sum_{\substack{i=a+p,\\ j=b+q-1,\\ (a,b),(p,q)\neq(0,1)}} m_{ab} \circ m_{pq}} \\
&= \overline{[m_{11}, m_{(i-1)j}] + \sum_{\substack{i=a+p,\\ j=b+q-1,\\ (a,b),(p,q)\neq(0,1),(1,1)}} m_{ab} \circ m_{pq}}.
\end{aligned}$$

Hence,

$$\overline{[m_{11}, m_{(i-1)j} - m'_{(i-1)j}] + \sum_{\substack{i=a+p,\\ j=b+q-1,\\ (a,b),(p,q)\neq(0,1),(1,1)}} m_{ab} \circ m_{pq}} = 0.$$

By Corollary 4.1.4, there exists $m_{ij} \in C_i^{j,i+j-2}(A, A)$ such that

$$\partial m_{ij} = [m_{11}, m_{(i-1)j} - m'_{(i-1)j}] + \sum_{\substack{i=a+p,\\ j=b+q-1,\\ (a,b),(p,q)\neq(0,1),(1,1)}} m_{ab} \circ m_{pq}.$$

As a consequence, the collection

$$\{m_{pq} | 0 \leqslant p \leqslant i, 1 \leqslant q \leqslant j, (p,q) \neq (i-1, j)\} \cup \{m_{(i-1)j} - m'_{(i-1)j}\}$$

gives A the structure of a dA_{ij}-algebra.

For 4.5.3.2 we notice that

$$\begin{aligned}
\partial\left(\sum_{\substack{a+p=i,\\ b+q=j+2,\\ (a,b),(p,q)\neq(0,1),(0,2)}} m_{ab}\circ m_{pq}\right) &= \sum_{\substack{a+p=i,\\ b+q=j+2,\\ (a,b),(p,q)\neq(0,1),(0,2)}} \partial m_{ab}\circ m_{pq} - m_{ab}\circ\partial m_{pq} \\
&= \sum_{\substack{a+p=i,\\ b+q=j+2,\\ (a,b),(p,q)\neq(0,1),(0,2)}} \partial m_{ab}\circ m_{pq} - m_{pq}\circ\partial m_{ab} \\
&= \sum_{\substack{c+e+p=i,\\ d+f+q=j+3,\\ (c,d),(e,f),(p,q)\neq(0,1),\\ (p,q)\neq(0,2)}} \begin{array}{c} -(m_{cd}\circ m_{ef})\circ m_{pq} \\ +m_{pq}\circ(m_{cd}\circ m_{ef}) \end{array} \\
&= \sum_{\substack{c+e=i,\\ d+f=j+1,\\ (c,d),(e,f)\neq(0,1)}} \begin{array}{c} -(m_{cd}\circ m_{ef})\circ m_{02} \\ +m_{02}\circ(m_{cd}\circ m_{ef}) \end{array} \\
&= \mathcal{O}_{ij}\circ m_{02} - m_{02}\circ\mathcal{O}_{ij} \\
&= -[m_{02},\mathcal{O}_{ij}] \\
&= -d^{\mu}\mathcal{O}_{ij}.
\end{aligned}$$

As a consequence, $d^{\mu}(\overline{\mathcal{O}_{ij}})=0$ and $\overline{\mathcal{O}_{ij}}$ represents a class in

$$HH^{i,j,i+j-3}_{dga}(H(A),H(A)) = H^j(C_i^{*,i+j-3}(H(A),H(A)),d^{\mu}).$$

If $[\overline{\mathcal{O}_{ij}}]=0$ then there exists $u\in C_i^{j-1,i+j-3}(H(A),H(A))$ such that $d^{\mu}u = \overline{\mathcal{O}_{ij}}$. By Corollary 4.1.4 there exists $m'_{i(j-1)}\in C_i^{j-1,i+j-3}(A,A)$ such that

$\partial m'_{i(j-1)} = 0$ and $\overline{m'_{i(j-1)}} = u$. So

$$\begin{aligned}\overline{[m_{02}, m'_{i(j-1)}]} = \overline{d^{\mu} m'_{i(j-1)}} &= d^{\mu} u \\ &= \overline{\mathcal{O}_{ij}} \\ &= \overline{\sum_{\substack{i=a+p,\\ j=b+q-1,\\ (a,b),(p,q)\neq(0,1)}} m_{ab} \circ m_{pq}} \\ &= \overline{[m_{02}, m_{i(j-1)}] + \sum_{\substack{i=a+p,\\ j=b+q-1,\\ (a,b),(p,q)\neq(0,1),(0,2)}} m_{ab} \circ m_{pq}}.\end{aligned}$$

Hence,

$$\overline{[m_{02}, m_{i(j-1)} - m'_{i(j-1)}] + \sum_{\substack{i=a+p,\\ j=b+q-1,\\ (a,b),(p,q)\neq(0,1),(0,2)}} m_{ab} \circ m_{pq}} = 0.$$

By Corollary 4.1.4, there exists $m_{ij} \in C_i^{j,i+j-2}(A, A)$ such that

$$\partial m_{ij} = [m_{02}, m_{i(j-1)} - m'_{i(j-1)}] + \sum_{\substack{i=a+p,\\ j=b+q-1,\\ (a,b),(p,q)\neq(0,1),(0,2)}} m_{ab} \circ m_{pq}.$$

As a consequence, the collection

$$\{m_{pq} | 0 \leqslant p \leqslant i, 1 \leqslant q \leqslant j, (p,q) \neq (i, j-1)\} \cup \{m_{i(j-1)} - m'_{i(j-1)}\}$$

gives A the structure of a dA_{ij}-algebra. $\square$

Instead of building up the structure maps m_{ij} one by one, we may consider taking collections of maps m_{ij} with $i + j = \alpha$ and look at the obstructions to building up the structure by adding a whole collection in one go.

Definition 4.5.4. Let $r \geqslant 1$ be an integer. An $(\mathbb{N}, \mathbb{Z})$-bigraded R-module,

A, is a dA_r-algebra if there exist collections of maps $M_\alpha = (m_{pq})_{\substack{p+q=\alpha,\\ p\geqslant 0, q\geqslant 1}} \in \prod_{p+q=\alpha} C_p^{q,p+q-2}(A,A)$ for all $\alpha \leqslant r$, satisfying the relations

$$\left(\sum_{\substack{u=c+p,\\ v=d+q-1,\\ u\geqslant 0, v\geqslant 1}} m_{cd} \circ m_{pq} \right)_{u+v=\beta} = 0$$

for all $\beta \leqslant r$. Equivalently,

$$\partial M_\beta = (\partial m_{uv})_{u+v=\beta} = \left(- \sum_{\substack{u=c+p,\\ v=d+q-1,\\ (c,d),(p,q)\neq(0,1)}} m_{cd} \circ m_{pq} \right)_{u+v=\beta}.$$

Remark 4.5.5. If A is a dA_1-algebra then A is a vertical bicomplex. The induced differential on $C(A,A)$ is $\partial = [m_{01}, -]$.

If A is a dA_2-algebra then we have $\partial m_{11} = 0$ and $\partial m_{02} = 0$, so there are induced elements $\overline{m_{11}} \in C_1^{1,0}(H(A),H(A))$ and $\overline{m_{02}} \in C_0^{2,0}(H(A),H(A))$.

If A is a dA_3-algebra then $\overline{m_{11}} \circ \overline{m_{11}} = 0$, so $H(A)$ is a bicomplex with trivial vertical differential. In addition, we have $\overline{m_{02}} \circ \overline{m_{02}} = 0$ and $\overline{m_{11}} \circ \overline{m_{02}} + \overline{m_{02}} \circ \overline{m_{11}} = 0$, so $\overline{m_{02}}$ is an associative multiplication on $H(A)$. Hence the bigraded module $H(A)$ is a bidga with trivial vertical differential.

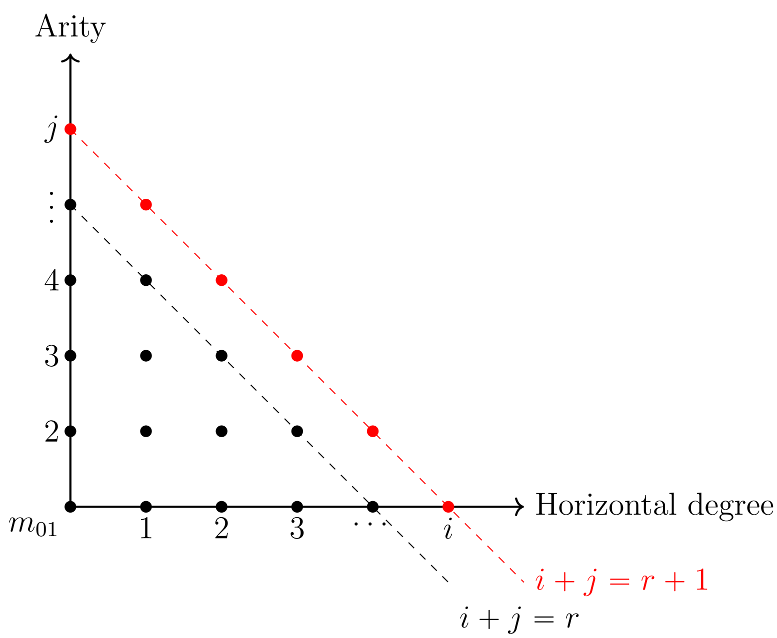


Figure 4.2: The maps in a dA_r-algebra/ dA_{r+1}-algebra

Theorem 4.5.6. Let $r > 3$ be an integer.

Let A be a vertical bicomplex such that $H(A)$ and $Z(A)$ are bigraded projective R-modules. Assume A is a dA_r-algebra with structure maps $M_\alpha = (m_{pq})_{\substack{p+q=\alpha,\\ p\geqslant 0, q\geqslant 1}} \in \prod_{\substack{p+q=\alpha,\\ p\geqslant 0, q\geqslant 1}} C_p^{q,p+q-2}(A,A)$. Then the obstruction to lift the underlying dA_{r-1}-algebra structure on A to a dA_{r+1}-algebra structure lies in

$$HH_{bidga}^{r+1,r-2}(H(A),H(A)).$$

Proof. Let $\partial = [m_{01}, -]$, $d^{\mathrm{Tot}} = [m_{11} + m_{02}, -]$.

Note that $|m_{pq}| = p + q + (p + q - 2) - 1 = 2p + 2q - 3$ is odd.

Let us define

$$\begin{aligned}\mathcal{O}_{r+1} &= (\mathcal{O}ij)_{i+j=r+1}\\ &= \left(\sum_{\substack{i=a+p,\\ j=b+q-1,\\ (a,b),(p,q)\neq(0,1)}} m_{ab}\circ m_{pq} \right)_{i+j=r+1} \in \prod_{i+j=r+1} C_i^{j,i+j-3}(A,A).\end{aligned}$$

Then

$$\partial\mathcal{O}_{r+1} = (\partial\mathcal{O}_{ij})_{i+j=r+1} = (0)_{i+j=r+1} = 0$$

by Proposition 4.2.11.

So $\mathcal{O}_{r+1}$ gives rise to a collection of elements

$$\overline{\mathcal{O}_{r+1}} \in \prod_{i+j=r+1} C_i^{j,i+j-3}(H(A),H(A)).$$

We notice that

$$\begin{aligned}\left(d^{\mathrm{Tot}}\mathcal{O}_{r+1}\right)_{uv} &= \Big([m_{11}+m_{02},(\mathcal{O}_{ij})_{i+j=r+1}]\Big)_{uv}\\ &= \Big([m_{11},(\mathcal{O}_{ij})_{i+j=r+1}] + [m_{02},(\mathcal{O}_{ij})_{i+j=r+1}]\Big)_{uv}\\ &= [m_{11},\mathcal{O}_{(u-1)v}] + [m_{02},\mathcal{O}_{u(v-1)}] \quad \text{with } u+v=r+2.\end{aligned}$$

So

$$\partial\left(\sum_{\substack{a+p=u,\\ b+q=v+1,\\ (a,b),(p,q)\neq(0,1),(1,1),(0,2)}} m_{ab}\circ m_{pq}\right)_{u+v=r+2}$$

$$= \left(\sum_{\substack{a+p=u,\\ b+q=v+1,\\ (a,b),(p,q)\neq(0,1),(1,1),(0,2)}} \partial m_{ab}\circ m_{pq} - m_{ab}\circ \partial m_{pq}\right)_{u+v=r+2}$$

$$= \left(\sum_{\substack{a+p=u,\\ b+q=v+1,\\ (a,b),(p,q)\neq(0,1),(1,1),(0,2)}} \partial m_{ab}\circ m_{pq} - m_{pq}\circ \partial m_{ab}\right)_{u+v=r+2}$$

$$= \left(\sum_{\substack{c+e+p=u,\\ d+f+q=v+2,\\ (c,d),(e,f),(p,q)\neq(0,1),\\ (p,q)\neq(1,1),(0,2)}} -(m_{cd}\circ m_{ef})\circ m_{pq} + m_{pq}\circ(m_{cd}\circ m_{ef})\right)_{u+v=r+2}$$

$$= \left(\sum_{\substack{c+e=u-1,\\ d+f=v+1,\\ (c,d),(e,f)\neq(0,1)}} -(m_{cd}\circ m_{ef})\circ m_{11} + m_{11}\circ(m_{cd}\circ m_{ef})\right)_{u+v=r+2}$$

$$+ \left(\sum_{\substack{c+e=u,\\ d+f=v,\\ (c,d),(e,f)\neq(0,1)}} -(m_{cd}\circ m_{ef})\circ m_{02} + m_{02}\circ(m_{cd}\circ m_{ef})\right)_{u+v=r+2}$$

$$= \left(\mathcal{O}_{(u-1)v}\circ m_{11} - m_{11}\circ\mathcal{O}_{(u-1)v} + \mathcal{O}_{u(v-1)}\circ m_{02} - m_{02}\circ\mathcal{O}_{u(v-1)}\right)_{u+v=r+2}$$

$$= \left(-[m_{11},\mathcal{O}_{(u-1)v}] - [m_{02},\mathcal{O}_{u(v-1)}]\right)_{u+v=r+2}$$

$$= -d^{\mathrm{Tot}}\mathcal{O}_{ij}.$$

As a consequence, $d^{\mathrm{Tot}}(\overline{\mathcal{O}_{r+1}}) = 0$ and $\overline{\mathcal{O}_{r+1}}$ represents a class in

$$HH^{r+1,r-2}_{bidga}(H(A), H(A)).$$

If $[\overline{\mathcal{O}_{r+1}}] = ([\mathcal{O}_{ij}])_{i+j=r+1} = 0$ then there exists

$$U = (u_{ij})_{i+j=r} \in \prod_{i+j=r} C_i^{j,i+j-2}(H(A), H(A))$$

such that $d^{\mathrm{Tot}} U = \overline{\mathcal{O}_{r+1}}$. By Corollary 4.1.4 there exists

$$M'_r = \left(m'_{ij}\right)_{i+j=r} \in \prod_{i+j=r} C_i^{j,i+j-2}(A, A)$$

such that $\partial M'_r = 0$ and $\overline{M'_r} = U$. So

$$\begin{aligned}
\overline{[m_{11}, m'_{(i-1)j}] + [m_{02}, m'_{i(j-1)}]} &= \left(\overline{d^{\mathrm{Tot}} M'_r}\right)_{ij} \\
&= (d^{\tau} U)_{ij} = \overline{\mathcal{O}_{ij}} \\
&= \overline{\sum_{\substack{i=a+p,\\ j=b+q-1,\\ (a,b),(p,q)\neq(0,1)}} m_{ab} \circ m_{pq}} \\
&= \overline{[m_{11}, m_{(i-1)j}] + [m_{02}, m_{i(j-1)}]} \\
&\quad \overline{+ \sum_{\substack{i=a+p,\\ j=b+q-1,\\ (a,b),(p,q)\neq(0,1),(1,1),(0,2)}} m_{ab} \circ m_{pq}}.
\end{aligned}$$

Hence,

$$\left(\overline{[m_{11}, m_{(i-1)j} - m'_{(i-1)j}] + [m_{02}, m_{i(j-1)} - m'_{i(j-1)}]} \right.$$
$$\left. \overline{+ \sum_{\substack{i=a+p,\\ j=b+q-1,\\ (a,b),(p,q)\neq(0,1),(1,1),(0,2)}} m_{ab} \circ m_{pq}} \right)_{i+j=r+1} = 0.$$

By Corollary 4.1.4, there exists $M_{r+1} = (m_{ij})_{i+j=r+1} \in \prod_{i+j=r+1} C_i^{j,i+j-2}(A, A)$ such that

$$\begin{aligned}\partial M_{r+1} &= (\partial m_{ij})_{i+j=r+1} \\ &= \left([m_{11}, m_{(i-1)j} - m'_{(i-1)j}] + [m_{02}, m_{i(j-1)} - m'_{i(j-1)}] \right. \\ &\qquad \left. + \sum_{\substack{i=a+p,\\ j=b+q-1,\\ (a,b),(p,q)\neq(0,1),(1,1),(0,2)}} m_{ab} \circ m_{pq} \right)_{i+j=r+1}.\end{aligned}$$

As a consequence, the collection $\{M_1, M_2, \cdots, M_{r-1}, M_r - M'_r, M_{r+1}\}$ is a dA_{r+1}-algebra structure on A extending the dA_{r-1}-algebra structure. $\square$

Appendices

Appendix A

Construction of V_{23}

In this appendix we give the details of the structure of the space V_{23} using Definition 2.3.1. This is an extra example which may be of interest to readers wanting to see a case of the construction of a space V_{ij} with $i + j = 5$. There are of course other examples we could consider, such as V_{32} or V_{14}.

There are 20 trees in the set $\mathcal{T}^0_{2,3}$. It is straightforward to check that 10 of these trees correspond to copies of $T_3 \wedge (K_3)_+$ in V_{23} and the other 10 correspond to copies of $T_4 \wedge (K_2)_+ \wedge (K_2)_+$. Hence we see that the space V_{23} is as shown in figure A.1.

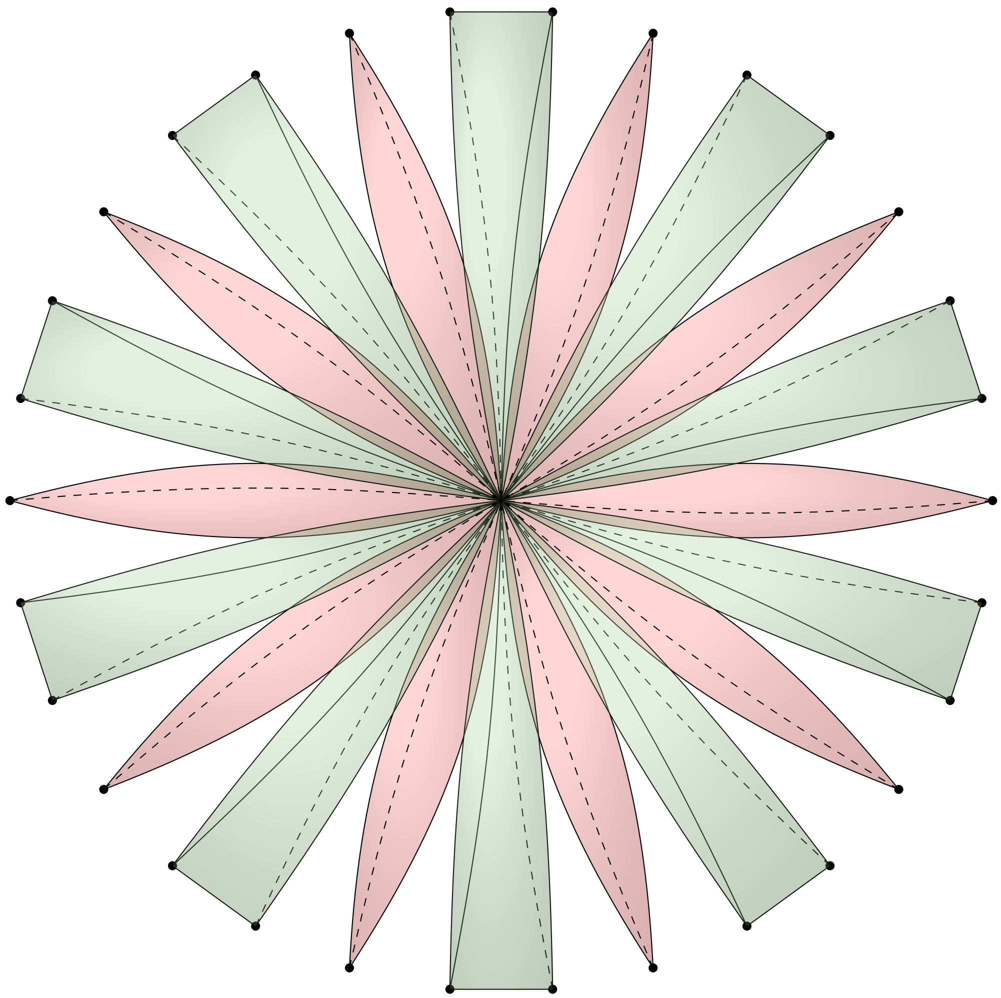


Figure A.1: The space V_{23}

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