Some Representation Theory Of The Dilute Blob Algebra

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"your dream is possible when you decide to be possible"

Eman Alluqmani

The candidate confirms that the work submitted is her own and that appropriate credit has been given where reference has been made to the work of others.

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Never, never forget my inspiration and educator in this life, my love "my dad" may God rest his soul. In the eyes of the world you are my father but in my eyes you are the whole world.

Abstract

The main objective of this thesis is to define a new class of multi-parameter algebras, called the dilute blob algebra $db_n(p, q, r, s)$, which is a generalization of the Motzkin algebra.

After we define basis diagrams of the dilute blob algebra, we give generators for the dilute blob algebra. A bijection between basis diagrams of the dilute blob algebra and basis diagrams of the left-right symmetric Motzkin algebra is also studied.

We prove that the dilute blob algebra is cellular in the sense of Graham and Lehrer and construct the left cell modules. We then compute the dimension of these cell modules and the dimension of a dilute blob algebra. We define an inner product on these cell modules. Then we prove that the cell modules are cyclic.

Moreover, we study the Gram matrix to determine when the cell module with $n-1$ propagating lines is simple. We also prove that the cell modules are generically simple over the complex field, thus the dilute blob algebra is generically semisimple over the complex field.

We give a necessary and sufficient condition for a dilute blob algebra to be quasi-hereditary. Explicit restriction rules for the cell modules are given and we find the Bratelli diagram for $n \leq 4$. We also study induction of the cell modules.

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Chapter 1

Introduction

The aim of this thesis is to introduce a new family of diagram algebras. When we say *diagram algebra* we mean an associative, unital algebra whose basis elements are described by diagrams and multiplication is described by diagram concatenation.

There are many known diagrams algebras such as the *partition algebra*, the *planar partition algebra*, the *left-right symmetric partition algebra*, the *partial Brauer algebra*, the *Brauer algebra*, the *Motzkin algebra*, the *Temperley-Lieb algebra*, the *left-right symmetric Temperley-Lieb algebra* and the *blob algebra*. We will give the definition of each one of these algebras and explain the relations between each of them and show how our new algebra fits in with these algebras.

Now let R be a commutative ring with identity and fix $p, q \in R$. All the following algebras are defined over R. We begin with the partition algebra $P_n(p, q)$ that appeared in Martin's work ([18], [19], [20]) and later in Jones's work [17]. Martin and Jones defined the partition algebra as a generalization of the Temperley-Lieb algebra to study the Potts model in statistical mechanics. The representation theory of the partition algebra is extensively studied. See for example, Martin ([19], [20]), Halverson and Ram [14] and Doran and Wales [9].

The partition algebra is a tower of finite dimensional algebras that has a basis consisting of diagrams. These diagrams have a rectangular frame consisting of two rows, the top row and the bottom row. We arrange n vertices labelled by $1, 2, \ldots, n$

in the top row and $1', 2', \ldots, n'$ in the bottom row. We then join vertices by edges, some vertices may not join with any other vertices. Here is an example of a basis diagram for the partition algebra $P_6(p, q)$:

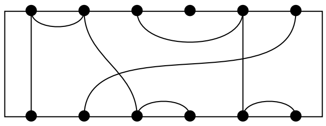


FIGURE 1.1: An example of a basis diagram for $P_6(p, q)$.

The dimension of $P_n(p, q)$ is the number of ways to partition a set of $2n$ elements which is known to be the Bell number B_{2n} (see, for example [3]).
The product of two basis diagrams x, y in the partition algebra is obtained by concatenating diagrams as follows: place x on the top of y and identify the vertices of the bottom row of x with the vertices on the top row of y. This forms a new diagram consisting of a top row, middle row and bottom row. Possibly in the middle row occurs some of loops, paths and isolated vertices that are not joined with the top row and the bottom row. We remove these components and multiply the resulting diagram with p^α and q^β where α is the number of loops and β is the number of paths or isolated vertices that removed from the middle row. An example is given as follows:

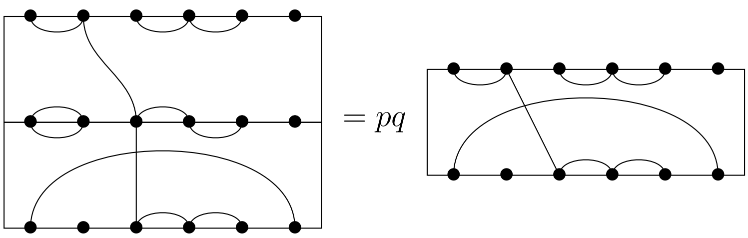


FIGURE 1.2: An example of the multiplication of two diagrams in $P_6(p, q)$.

The planar partition algebra $P_n^\square(p, q)$ is a subalgebra of the partition algebra $P_n(p, q)$. The planar partition algebra consists of basis diagrams from the partition algebra such that the edges do not cross in the diagrams.

The left-right symmetric partition algebra $SP_{2n}(p,q)$ is a subalgebra of the partition algebra $P_n(p,q)$. Consider the basis diagrams of the partition algebra that are left-right symmetric under reflection in the middle vertical axis. These diagrams are the basis of the left-right symmetric partition algebra $SP_{2n}(p,q)$. See an example as follows:

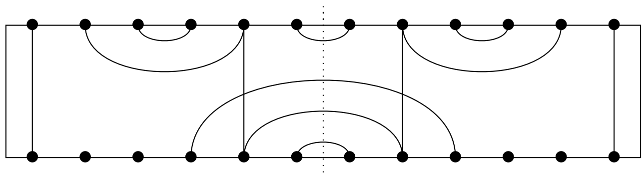


FIGURE 1.3: An example of a left-right symmetric partition diagram in $SP_{12}(p,q)$.

The partial Brauer algebra $PB_n(p,q)$ is a subalgebra of the partition algebra. Martin and Mazorchuk in [21] studied the representation theory of this algebra. The partial Brauer algebra consists of basis diagrams that are in the partition algebra such that these diagrams only allow an edge to connect any two distinct vertices and may be there is some vertices that are not connected with any other vertices. An example is given as follows:

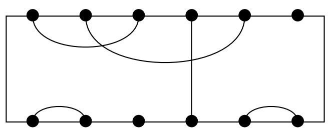


FIGURE 1.4: An example of a diagram in $PB_6(p,q)$.

The Brauer algebra $B_n(p)$ is a subalgebra of the partial Brauer algebra $PB_n(p,q)$ that has basis diagrams that are in $PB_n(p,q)$ such that every two distinct vertices are joined with an edge. See an example of a diagram in $B_n(p)$.

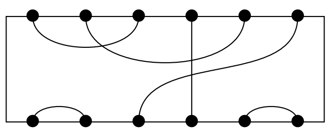


FIGURE 1.5: An example of a diagram in $B_6(p)$.

The Motzkin algebra is a subalgebra $M_n(p,q)$ (see [2]) of the planar partition algebra $P_n^{\square}(p,q)$ and the partial Brauer algebra $PB_n(p,q)$. The basis diagrams of the Motzkin algebra are the diagrams such that the edges in the diagram do not cross and every two distinct vertices might be joined with an edge. An example of a Motzkin diagram is given below:

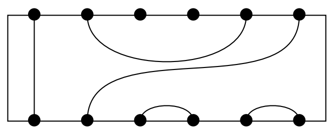


FIGURE 1.6: An example of a diagram in $M_6(p,q)$.

The left-right symmetric Motzkin algebra $SM_{2n}(p,q)$ is a subalgebra of the Motzkin algebra $M_n(p,q)$ and the left-right symmetric partition algebra $SP_{2n}(p,q)$ that consists of basis diagrams that are in the Motzkin algebra that are left-right symmetric under reflection in the middle vertical axis. The following is an example of the diagram in the left-right symmetric Motzkin algebra:

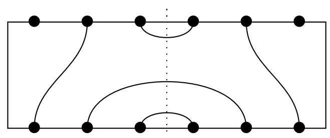


FIGURE 1.7: An example of a diagram in $SM_6(p,q)$.

The Temperley-Lieb algebra $TL_n(p)$ (see for example [26] and [29]) is a subalgebra of the Brauer algebra $B_n(p)$ and the Motzkin algebra $M_n(p,q)$. The basis diagrams of $TL_n(p)$ are the diagrams in the Motzkin algebra such that every two distinct vertices are joined with an edge and the edges in the diagram do not cross. An example of a diagram in Temperley-Lieb algebra given as follows:

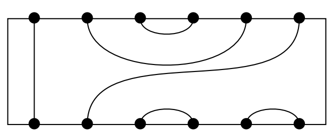


FIGURE 1.8: An example of a diagram in $TL_6(p)$.

The left-right symmetric Temperley-Lieb algebra $STL_{2n}(p)$ is a subalgebra of Temperley-Lieb algebra $TL_n(p)$ and left-right symmetric Motzkin algebra $SM_{2n}(p,q)$. The subset of the basis diagrams of Temperley-Lieb algebra that consists of left-right symmetric diagrams under reflection in the middle axis is the basis of $STL_{2n}(p)$.

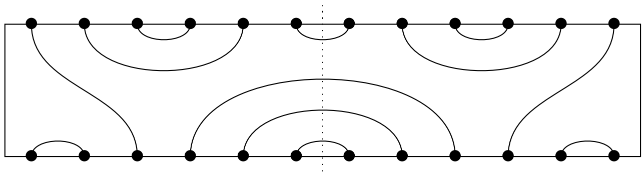


FIGURE 1.9: An example of a diagram in $STL_{12}(p)$.

For $p' \in R$, the blob algebra $b_n(p,p')$ (see for example [22] and [23]) is the generalization of the Temperley-Lieb algebra $TL_n(p)$ and when $p' = 1$ the blob algebra is isomorphic to the left-right symmetric Temperley-Lieb algebra as Green proved in [13]. The basis diagrams of $b_n(p,p')$ consists of the basis diagrams of the Temperley-Lieb algebra that are undecorated and decorated in various lines by (single) blobs, with the condition that no line to the left of the rightmost propagating line is decorated; and to the right of it only the outermost arcs may be decorated. An example of a basis diagram of blob algebra is given below:

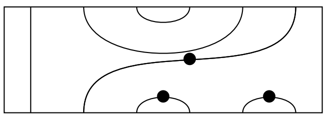
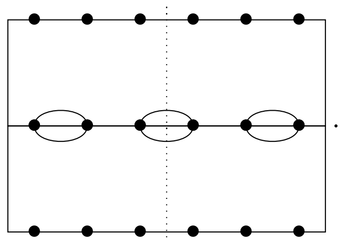
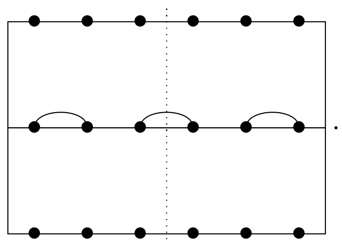


FIGURE 1.10: An example of a diagram in $b_6(p,p')$.

The multiplication on the diagrams of $b_n(p,p')$ is similar to the multiplication on the diagrams of $TL_n(p)$, except it is now possible to introduce loops decorated with a singe blob and lines with two blobs. We remove the loops in the middle row of the product diagram and multiply the resulting diagram by p^{α} and $p'^{\alpha'}$ where α is the number of undecorated loops and α' is the number of decorated loops and we replace any multiple blobs with single blob.

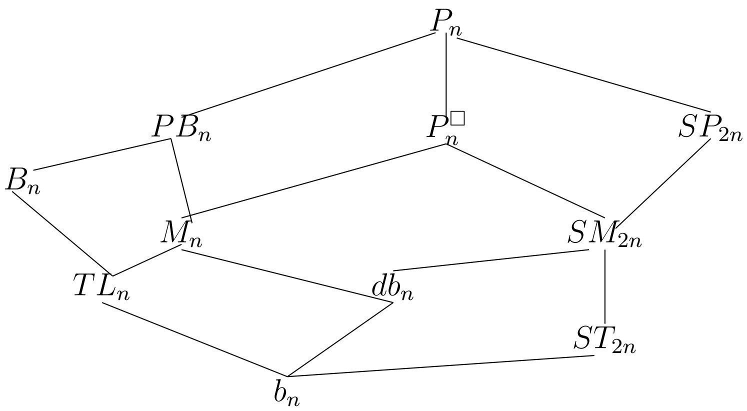
In this thesis we will introduce a new family of multi parameter algebras, the dilute blob algebra over a commutative ring R with identity and study some representation theory of the dilute blob algebra.
Our motivation to introduce the dilute blob algebra is that when we multiply two diagrams in left-right symmetric Motzkin algebra we can have for example the following product diagrams:





In the first figure we can distinguish between a loop on the axis of symmetry and ones that is not on the axis of symmetry. So we can have two parameters p and s where p for the loop that is not on the axis of symmetry and s for the loop that is on the axis of symmetry.
Similarly, in the second figure we can also distinguish between a path on the axis of symmetry and the ones that is not on the axis of symmetry. So we can have two parameters q and r where q for the path that is not on the axis of symmetry and r for the other one.
Therefore, we can define a deformation of the Motzkin algebra with four parameters $p, q, r, s \in R$ which is our algebra the dilute blob algebra denoted by $db_n(p, q, r, s)$.
Similarly, we can define a deformation of the Temperley-Lieb algebra with two parameters which is the blob algebra.

Now we can draw a graph that represents the relations between all the algebras that we have defined. We denote algebras without their parameters.



1.1 Structure of this thesis

Chapter 2 is devoted to the necessary background results from representation theory. In particular the theory of cellular algebras, quasi-hereditary algebras and reviewing of the Temperley-Lieb algebras, the left-right symmetric Temperley-Lieb algebras, the blob algebras and the Motzkin algebras and some results regarding to their representation theories.

In chapter 3 we introduce a unital associative algebra $db_n(p,q,r,s)$ which depends on the parameters p,q,r,s. The algebra db_n is defined over any commutative ring R with identity and has an R-basis of dilute blob diagrams. We show that the algebra $db_n(p,q,r,s)$ is generated by certain diagrams t_i, r_i, l_i for $1 \leqslant i \leqslant n-1$ and the diagrams v_n, w_n when $q \in R$ is invertible, where the subalgebra $M_n(p)$, generated by t_i, r_i and l_i for $1 \leqslant i \leqslant n-1$ is the Motzkin algebra with dimension the Motzkin number $\mathfrak{M}_{2n} = \sum_{k=0}^{n} \dfrac{1}{k+1}\binom{2n}{2k}\binom{2k}{k}$. We end this chapter by studying the bijection between dilute blob diagrams and left-right symmetric Motzkin diagrams which was in fact the first step we used to define the basis diagrams of the dilute blob algebra.

The main objective of the fourth chapter is to prove that the dilute blob algebra is cellular, construct the cell modules and find their dimension. By using the dimension of the cell module we find the dimension of the dilute blob algebra. We then define a bilinear form on the cell modules and we use it to prove that the cell modules are cyclic when the parameters p, q, r and s are invertible in R.

The aim of chapter 5 is to use the Gram matrix to prove that the cell module $\Delta_n(n-1)$ is simple if and only if $q \neq 0$ and $qs \neq r^2$. Also, we introduce some calculations when $n = 1, 2$ to investigate when the cell module is simple. In theorem 5.2.1 we prove that the cell modules are generically simple over the complex field.

Chapter 6 is devoted to a very important result about the representation theory of the dilute blob algebra which is theorem 6.2.5 that proves that the dilute blob algebra is quasi-heredity over a field F when $q \neq 0$.

In chapter 7 we study restriction, draw the Bratteli diagram for the restriction rule for $n \leq 4$ and find a spanning set for the induced module of the cell modules. We end this chapter by giving future directions for research.

Throughout this thesis we assume that R is a commutative ring with identity.

Chapter 2

Background

We devote this chapter to reviewing known results which we will require during this thesis. We start in section 2.1 to remind the reader of the definition of a class of finite dimensional algebras, namely cellular algebras and we briefly present basic results about their representation theory. Next we recall the definition of quasi-hereditary algebra which is closely related to a lot of cellular algebras. We then define the Temperley-Lieb algebra and present some needed results about its representation theory. Roughly speaking the Temperley-Lieb algebra is a finite dimensional algebra that possesses a basis which can be described by diagrams. Multiplication of the basis can be defined by concatenation of diagrams. We also briefly define two related important algebras for our study that are known as the left-right symmetric Temperley-Lieb algebra and the blob algebra. In the final section we define the Motzkin algebra which is a subalgebra of the dilute blob algebra, following this we define half Motzkin diagrams which are very useful in finding the dimension of the dilute blob algebra.

2.1 Cellular algebras

We devote this section to recall the definition of a *cellular algebra* which was originally defined by Graham and Lehrer [11] in terms of a cellular basis, and present some results of its representation theory.

Definition 2.1.1. [11, Definition 1.1] Let R be a commutative ring with identity. A *cellular algebra* over R is an associative (unital) algebra A, together with *cell datum* $(\Lambda, M, C, *)$ where

(C1) Λ is a partially ordered set(poset) and for each $\lambda \in \Lambda, M(\lambda)$ is a finite set such that $C : \coprod_{\lambda\in\Lambda} M(\lambda) \times M(\lambda) \to A$ is a injective map with image an R-basis of A.

(C2) If $\lambda \in \Lambda$ and $u, w \in M(\lambda)$, write $C(u, w) = C^{\lambda}_{uw} \in A$. Then $*$ is an R-linear anti-involution of A (An involution means $*^2 = *$) such that $(C^{\lambda}_{uw})^* = C^{\lambda}_{wu}$.

(C3) If $\lambda \in \Lambda$ and $u, w \in M(\lambda)$ then for any element $x \in A$ we have

$$xC^{\lambda}_{uw} \equiv \sum_{z\in M(\lambda)} r_x(z, u)C^{\lambda}_{zw} \mod A^{<\lambda} \tag{2.1}$$

where $r_x(z, u) \in R$ is independent of w and where $A^{<\lambda}$ is the R-submodule of A generated by $\{C^{\lambda''}_{vg} : \lambda'' < \lambda, v, g \in M(\lambda'')\}$.

The basis is called a *cellular basis.*

Definition 2.1.2. [11, Definition 2.1] Let A be a cellular algebra with cell datum $(\Lambda, M, *, C)$. For each $\lambda \in \Lambda$ define the (left)*cell* A-module C^{λ} to be the R-module with basis $\{C^{\lambda}_u \mid u \in M(\lambda)\}$ and A-action defined by

$$xC^{\lambda}_u = \sum_{z\in M(\lambda)} r_x(z, u)C^{\lambda}_z \quad (x \in A, u \in M(\lambda)) \tag{2.2}$$

where $r_x(z, u)$ is the element of R defined in definition 2.1.1 (C3).

Definition 2.1.3. [16, section 1.5] Let V be a finite dimensional vector space with an inner product $\langle , \rangle$. The *Gram matrix*, G, is defined with respect to a basis $v_1, v_2, \ldots, v_k$ of V by letting the $(i, j)^{th}$ entry of G be $\langle v_i, v_j\rangle$.

By (2.1) there is a unique bilinear form $\langle\ ,\ \rangle_\lambda : C^\lambda \times C^\lambda \to R$ such that for $u_1, w_1, u_2, w_2 \in M(\lambda)$, $\langle w_1, u_2\rangle_\lambda$ is given by

$$C^\lambda_{u_1w_1} C^\lambda_{u_2w_2} \equiv \langle w_1, u_2\rangle_\lambda C^\lambda_{u_1w_2} \mod A^{<\lambda}. \tag{2.3}$$

Regarding to $\langle\ ,\ \rangle_\lambda$ define the Gram matrix G_λ of C^λ with respect to the basis of cell module in definition 2.1.2.

Proposition 2.1.4. *[11, proposition 2.4],[25, proposition 2.9] Keep the notation above. Suppose that $\lambda \in \Lambda$ and let $x, y \in C^\lambda$. Then*

(i) The form $\langle\ ,\ \rangle_\lambda$ is symmetric, i.e. $\langle x, y\rangle_\lambda = \langle y, x\rangle_\lambda$.

(ii) $\langle a^*x, y\rangle_\lambda = \langle x, ay\rangle_\lambda$ for all $a \in A$.

Now assume that A is a cellular algebra over a field F.

Definition 2.1.5. [11, Definition 3.1], [25, 2.10] For $\lambda \in \Lambda$, define the radical of the bilinear form on C^λ:

$$\mathrm{rad}(\lambda) = \{x \in C^\lambda : \langle x, y\rangle_\lambda = 0 \quad \text{for all } y \in C^\lambda\}. \tag{2.4}$$

The following results are important for the representation theory of cellular algebras.

Proposition 2.1.6. *[11, Proposition 3.2] Let $\lambda \in \Lambda$. Then*

(i) $\mathrm{rad}(\lambda)$ *is an A-submodule of C^λ.*

(ii) If $\langle\ ,\ \rangle_\lambda \neq 0$, the quotient $C^\lambda/\mathrm{rad}(\lambda)$ is irreducible.

(iii) If $\langle\ ,\ \rangle_\lambda \neq 0$, $\mathrm{rad}(\lambda)$ *is the radical of C^λ (the radical of C^λ is the smallest submodule of C^λ with semisimple quotient).*

Theorem 2.1.7. *[25, Theorem 2.16] Let $\Lambda_0 = \{\lambda \in \Lambda : \langle\ ,\ \rangle_\lambda \neq 0\}$. Then*
The set $\{L(\lambda) := C^\lambda/\mathrm{rad}(\lambda) : \lambda \in \Lambda_0\}$ is a complete set of pairwise inequivalent irreducible A-modules.

Proposition 2.1.8. *[25, Proposition 2.12] Suppose $\lambda, \mu \in \Lambda$ such that $L(\lambda) \neq 0$. Let D be a proper submodule of C^λ, and suppose that $\theta : C^\mu \to C^\lambda/D$ is a non-zero A-module homomorphism, then $\lambda \geq \mu$.*

Theorem 2.1.9. *[25, Corollary 2.21], [11, Theorem 3.8] The following are equivalent.*

(i) The algebra A is semisimple.

(ii) The non-zero cell modules C^λ are irreducible and pairwise inequivalent (This means $C^\lambda = C^\lambda/\operatorname{rad}(\lambda)$ for all $\lambda \in \Lambda$).

(iii) $\operatorname{rad}(\lambda) = 0$ *for all $\lambda \in \Lambda$.*

(iv) $\det G_\lambda \neq 0$ *for each $\lambda \in \Lambda$.*

2.2 Quasi-hereditary algebras

In this section, we recall the definition of another class of finite dimensional algebras, the *quasi-hereditary* algebras introduced by Cline, Parshall and Scott, [27] and how they are related with cellular algebras.

Throughout this section F is a fixed field and A is a finite dimensional F-algebra. Let $\operatorname{rad}(A)$ be the Jacobson radical of A, $\operatorname{rad}(A)$ is the largest two-sided nilpotent ideal of A.

Definition 2.2.1. A two-sided ideal $\mathfrak{J}$ in the F-algebra is idempotent if $\mathfrak{J}^2 = \mathfrak{J}$.

Lemma 2.2.2. *[7, Lemma C1]*

(1) A two-sided ideal $\mathfrak{J}$ in algebra A is idempotent if and only if $\mathfrak{J} = AeA$, for some idempotent element $e \in A$.

(2) Given an idempotent two-sided ideal $\mathfrak{J} = AeA$, the algebra eAe is semisimple if and only if $\mathfrak{J} \cdot \operatorname{rad}(A) \cdot \mathfrak{J} = 0$.

Definition 2.2.3. [4, Section 3] A two-sided ideal $\mathfrak{J}$ of A is said to be a *heredity ideal* if the following three conditions hold:

(i) $\mathfrak{J}^2 = \mathfrak{J}$.

(ii) $\mathfrak{J}$ is projective as a left A-module.

(iii) $\mathfrak{J} \cdot \mathrm{rad}(A) \cdot \mathfrak{J} = 0$.

Lemma 2.2.4. *[7, Lemma C.4] Let $\mathfrak{J}$ be an idempotent two-sided ideal in A. Then $\mathfrak{J}$ is a heredity ideal if and only if the following two conditions hold:*

(i) The multiplication map $\eta : Ae \otimes_{eAe} eA \to AeA$ is bijective, where $\mathfrak{J} = AeA$, for some idempotent $e \in A$.

(ii) $\mathfrak{J} \cdot \mathrm{rad}(A) \cdot \mathfrak{J} = 0$.

Definition 2.2.5. [4, Section 3] A finite dimensional algebra A over the field F is called *quasi-hereditary* provided there is a sequence

$$0 = \mathfrak{J}_0 \subset \mathfrak{J}_1 \subset \cdots \subset \mathfrak{J}_t = A$$

of ideals in A such that $\mathfrak{J}_i/\mathfrak{J}_{i-1}$ is a heredity ideal in $A/\mathfrak{J}_{i-1}$ for $i = 1, 2, \ldots, t$. Such a sequence of ideals is called a *heredity chain* in A.

Proposition 2.2.6. *[25, corollary 2.23] Let A be a finite dimensional cellular algebra over a field F with a cell datum $(\Lambda, M, C, *)$ and assume $\Lambda = \Lambda_0$. Then A is a quasi-hereditary algebra.*

2.3 The Temperley-Lieb algebra $TL_n(p)$

The Temperley-Lieb algebras was originally introduced by Temperley and Lieb in [28] in the study of transfer matrices in lattice models.
In this section, we recall the diagrammatic definition of the Temperley-Lieb algebra $TL_n(p)$ where $p \in R$ and then define it by generators and relations. The dimension of $TL_n(p)$ and its cell modules are reviewed.

Definition 2.3.1. A *set partition* of a finite set X is a set of non-empty subsets $\{X_1, X_2, \ldots\}$ of X such that

$$\bigcup_i X_i = X \text{ and } X_i \cap X_j = \emptyset \text{ whenever } i \neq j.$$

The subset X_i in a set partition is called a *part.*
Let $\mathcal{P}_X$ denote the set of all set partitions of X.

For $n \in \mathbb{N}$, we define $\underline{n} = \{1, 2, \ldots, n\}$, $\underline{n}' = \{1', 2', \ldots, n'\}$ and $\underline{n}'' = \{1'', 2'', \ldots, n''\}$. We consider the following total order:

$$1 < 2 < \cdots < n \ , \ 1' < 2' < \cdots < n' \text{ and } 1'' < 2'' < \cdots < n''.$$

Let $\mathcal{P}_n$ be the set of all partitions of the set $\underline{n} \cup \underline{n}'$.

Definition 2.3.2. A *Temperley-Lieb set partition* of the set $\underline{n} \cup \underline{n}'$ is a set partition $T \in \mathcal{P}_n$ such that T satisfies the following properties:

(i) $|t_i| = 2$ for all $t_i \in T$.

(ii) If $t_i = \{a, c : a, c \in \underline{n}\}$ and $t_j = \{b, d : b, d \in \underline{n}\}$ are elements of T, then these elements should not satisfy $a < b < c < d$. Similarly, if $a, b, c, d \in \underline{n}'$.

(iii) If $t_i = \{a, d' : a \in \underline{n}, d' \in \underline{n}'\}$ and $t_j = \{b, c' : b \in \underline{n}, c' \in \underline{n}'\}$ are elements of T, then these elements should not satisfy either $a < b$ and $c' < d'$; or $b < a$ and $d' < c'$.

(iv) If $t_i = \{a, b : a, b \in \underline{n}\}$ and $t_j = \{c, d' : c \in \underline{n}, d' \in \underline{n}'\}$ are elements of T, then these elements should not satisfy $a < c < b$. Similarly, if $a, b, c \in \underline{n}', d' \in \underline{n}$.

We denote $\mathcal{T}_n$ for the subset of $\mathcal{P}_n$ that consists of all Temperley-Lieb set partitions.

We can represent each element $T \in \mathcal{T}_n$ by a diagram on the set $\underline{n} \cup \underline{n}'$, which are called *vertices*: fix a rectangular frame with n nodes on the top row of the rectangle representing the vertices of the set $\underline{n}$ (increasing from left to right) and with n

nodes on the bottom row of the rectangle representing the vertices of the set $\underline{n}'$ and connecting the two vertices that belong to the same part by a line bounded by the rectangular frame, and the vertices are the endpoints of the line, and no two lines crossing.

It is useful to think of the rectangle as being embedded in the plane as $[0, n+1] \times [0, 1]$. We put the lower left corner of the rectangle at the origin point $(0, 0)$ such that each vertex i (respectively, i') is located at the point $(i, 1)$ (respectively, $(i', 0)$).

The diagram representing a Temperley-Lieb set partition is not unique, since there are many different ways to draw the lines. Two such diagrams are regarded as the same diagram if they encode the same set partition.

We call a diagram representing $T \in \mathcal{T}_n$ a *Temperley-Lieb n-diagram.* Then denote by T_n the set of all Temperley-Lieb n-diagrams.

A line in a Temperley-Lieb n-diagram with one endpoint in the top row and one endpoint in the bottom row is called a *propagating line* and the one with both endpoints in the same row is called an *arc*. An example is given in Figure 2.1.

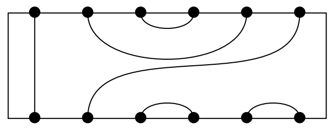


FIGURE 2.1: An example of a Temperley-Lieb 6-diagram.

We define a Temperley-Lieb algebra as in [11].

Definition 2.3.3. Fix $p \in R$, we define the *Temperley-Lieb algebra* $TL_n(p)$ to be the R-algebra with basis T_n. The multiplication $x \cdot y$ of $x, y \in T_n$ is obtained by concatenating x and y as follows. Place x above y and identify the bottom row vertices in x with the corresponding top row vertices in y. This forms a new Temperley-Lieb n-diagram (since x and y do not have crossing lines then concatenating x and y form a diagram with no crossing lines), possibly with number of closed loops in the middle row. We remove these loops and multiply the resulting diagram by p^α, where α is the number of closed loops removed from the middle row. Figure 2.2 gives an example of this multiplication.

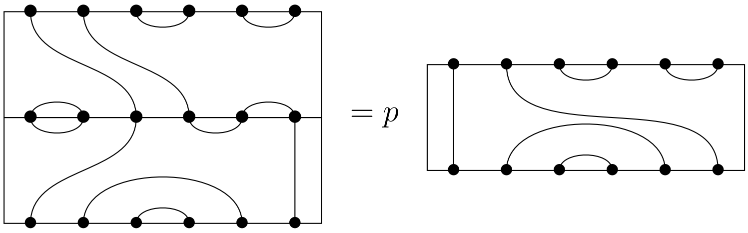
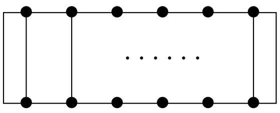
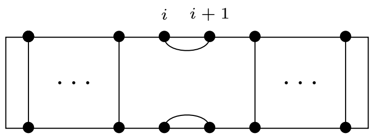


FIGURE 2.2: Multiplication of two diagrams in $TL_6(p)$.

Note that multiplication in algebra $TL_n(p)$ can not increase the number of propagating lines.

The algebra $TL_n(p)$ is generated by the set $\{\mathbb{I}_n, U_1, \ldots, U_{n-1}\}$ where

$\mathbb{I}_n =$  , $U_i =$ 

.

The diagram $\mathbb{I}_n$ is the identity element of $TL_n(p)$, and the elements U_i satisfy the following relations:

$$U_i^2 = pU_i \quad \text{for all } i; \tag{2.5}$$

$$U_iU_j = U_jU_i \quad \text{for} \quad |i-j| \geqslant 2; \tag{2.6}$$

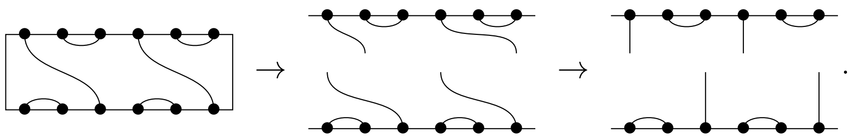
$$U_iU_jU_i = U_i \quad \text{for} \quad |i-j| = 1. \tag{2.7}$$

The dimension of the algebra $TL_n(p)$ is the n-th Catalan number

$$\mathcal{C}_n = \frac{(2n)!}{(n+1)!n!}. \tag{2.8}$$

see for example Theorem 2.4 in [26].

The Temperley-Lieb n-diagram can be cut in half horizontally in such a way that only propagating lines are cut (once). This produces a top half diagram and a bottom half diagram, the floating lines in the half diagrams are straightened out and called *defects*. For example,



In section 3 in [26], the R-module Z_n that is spanned by all top half diagrams that are resulted from cutting diagrams in T_n is naturally a left TL_n-module under the concatenation of diagrams with half diagrams. Now define a left TL_n-submodule $Z_{n,c} \subseteq Z_n$ that is spanned by top half diagrams that contain exactly c arcs.
Graham and Lehrer proved in [11, example 1.4] that the Temperley-Lieb algebra $TL_n(p)$ is cellular over any commutative ring R with identity, with involution $*$ acting by reflecting each diagram horizontally, and $\Lambda = \{0, 1, \ldots, [\frac{n}{2}]\}$ with reverse of the natural order. The cell modules for the algebra $TL_n(p)$ are denoted by $Z_{n,c}$.

The dimension of $Z_{n,c}$ is given from [26, equation 2.9] as follows:

$$\dim Z_{n,c} = \binom{n}{c} - \binom{n}{c-1}. \tag{2.9}$$

More details about Temperley -Lieb algebras and their representation theory can be found in [26] and [29].

2.4 The left-right symmetric Temperley-Lieb algebra $STL_{2n}(p)$ and the blob algebra $b_n(p, p')$

In this section we give a brief review about the left-right symmetric Temperley-Lieb subalgebra $STL_{2n}(p)$ of the Temperley-Lieb algebra $TL_{2n}(p)$ and the blob algebra

$b_n(p, p')$ where $p, p' \in R$.

2.4.1 The left-right symmetric Temperley-Lieb algebra $STL_{2n}(p)$

We define the left-right symmetric Temperley-Lieb algebra $STL_{2n}(p)$ as in [13] and recall its dimension.

Definition 2.4.1. A *left-right symmetric Temperley Lieb $2n$-diagram* is a Temperley-Lieb $2n$-diagram which is symmetric under reflection in the middle vertical axis.

An example is given in figure 2.3.

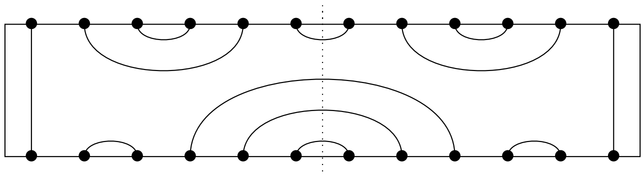


FIGURE 2.3: A left-right symmetric Temperley Lieb 12-diagram.

Lemma 2.4.2. *[13, Lemma 5.6] The left-right symmetric diagrams of TL_{2n} span a subalgebra of TL_{2n} of dimension $\binom{2n}{n}$.*

Definition 2.4.3. We call the algebra spanned by the left-right symmetric diagrams of TL_{2n} the *left-right symmetric Temperley-Lieb algebra* and denote it by $STL_{2n}(p)$. Let ST_{2n} be the set of all left-right symmetric Temperley-Lieb $2n$-diagrams in T_{2n}.

Definition 2.4.4. A *half symmetric Temperly-Lieb $2n$-diagram* is a diagram obtained by cutting horizontally a diagram in ST_{2n} such that each propagating line is cut once, and no other line is cut. This obtains a well defined pair of half diagrams which are the top half $2n$-diagram and the bottom half $2n$-diagram that have floating lines. The floating lines are straightened out and called *defects*.

For example,

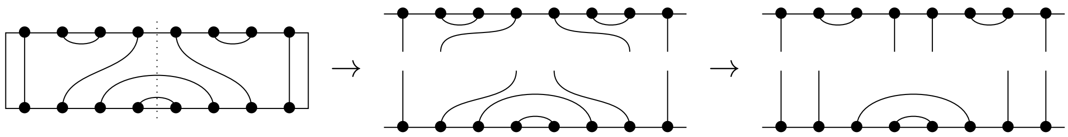


FIGURE 2.4: An example of splitting a left-right symmetric Temperley-Lieb 8-diagram into two. The dotted line here just indicates the axis of symmetry.

It is clear that the possible number of propagating lines in a diagram in ST_{2n} is $0, 2, \ldots, 2n$.

Definition 2.4.5. For $l \in \{0, 2, 4, \ldots, 2n\}$, let $ST_{2n}^{|\rangle}(l), ST_{2n}^{\langle|}(l)$ be the sets of top half $2n$-diagrams and bottom half $2n$-diagrams respectively with l defects constructed by cutting diagrams in ST_{2n} with l propagating lines as in definition 2.4.4. Let $ST_{2n}(l)$ be the subset of ST_{2n} that contains diagrams with exactly l propagating lines.

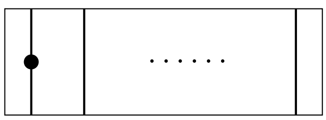
It is always possible to construct a unique diagram in $ST_{2n}(l)$ from two diagrams in $ST_{2n}^{|\rangle}(l)$ and $ST_{2n}^{\langle|}(l)$ by joining the propagating lines. So we have the following bijection:

$$ST_{2n}(l) \leftrightarrow ST_{2n}^{|\rangle}(l) \times ST_{2n}^{\langle|}(l) \quad \text{for all} \quad l \in \{0, 2, 4, \ldots, 2n\}. \tag{2.10}$$

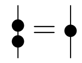
2.4.2 The blob algebra $b_n(p, p')$

In this subsection, we recall the definition of (left) blob algebra $b_n(p, p')$ where $p, p' \in R$ as an algebra given by a diagram basis and some useful results that will use in the next subsection.

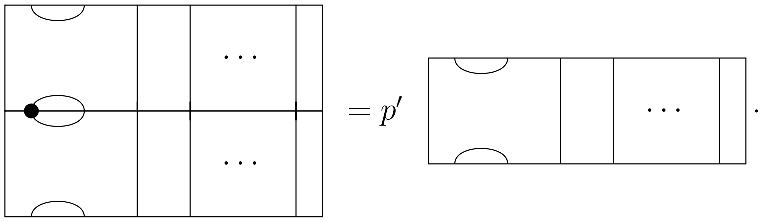
Definition 2.4.6. [22, Subsection 2.1] For $p, p' \in R$, define the (left)*blob algebra* $b_n(p, p')$ as the generalisation obtained by including an additional idempotent *blob generator* to the (diagram) generators of the Temperley-Lieb algebra, and additional

$e =$  .

relations given by



and



Definition 2.4.7. Let $B_n^{\bullet}$ be the basis set of the blob algebra consists of Temperley-Lieb diagrams in T_n and diagrams with decoration of various lines by (single) blobs, with the condition that no line to the right of the leftmost propagating line is decorated; and to the left of it only the outermost arcs may be decorated.

Definition 2.4.8. For $l \in \mathbb{N}$, let $B_n^{\bullet}(l)$ be the subset of of $B_n^{\bullet}$ that consists of blob diagrams with l propagating lines, one of which is decorated by blob; and let $B_n^{\bullet}(-l)$ be the subset of $B_n^{\bullet}$ with l propagating lines, all undecorated.

Remark 2.4.9. [22, Section 2] The propagating lines of a diagram in $B_n^{\bullet}$ is indexed by the set $\{n, n-2, n-4, \ldots, 2-n, -n\}$.

Definition 2.4.10. A *half blob diagram* is a diagram obtained by cutting a blob diagram from east to west in such a way that only propagating lines are cut (for definiteness a propagating line that is decorated with a blob is replaced with a propagating line that is decorated with two blobs and cut between the blobs). This produces a well defined pair of half diagrams which are the top half diagram and the bottom half diagram. The lines that result from cutting propagating lines are called *defects*.

See for example figure 2.5.

Definition 2.4.11. For $l \in \{n, n-2, n-4, \ldots, 2-n, -n\}$, let $B_n^{\bullet|\rangle}(l), B_n^{\bullet\langle|}(l)$ be the sets of top half blob diagrams and bottom half blob diagrams respectively with l defects obtained by cutting blob diagrams in $B_n^{\bullet}(l)$ that have l propagating lines.

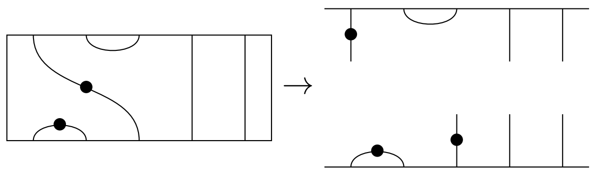


FIGURE 2.5: Splitting a diagram in $B_5^\bullet(3)$ into two halves.

Since any two half blob diagram in $B_n^{\bullet|\rangle}(l)$ and $B_n^{\bullet\langle|}(l)$ connect in a unique way, as in figure 2.5. Therefore, for $l \in \{n, n-2, n-4, \ldots, 2-n, -n\}$ we have a bijection:

$$B_n^\bullet(l) \leftrightarrow B_n^{\bullet|\rangle}(l) \times B_n^{\bullet\langle|}(l). \tag{2.11}$$

The blob algebra is a cellular algebra [12, Section 2] and its left (right) cell modules C_n^l are the R-free module with basis $B_n^{\bullet|\rangle}(l)$ ($B_n^{\bullet\langle|}(l)$) and they are indexed by $l \in \{n, n-2, n-4, \ldots, 2-n, -n\}$. The cell modules C_n^l and C_n^{-l} have the same dimension

$$\dim C_n^l = \binom{n}{\frac{n-l}{2}}. \tag{2.12}$$

For more details see [22, Section 2] and [10, subsection 2.1].

2.4.3 A bijection between $B_n^\bullet$ and ST_{2n}

As mentioned in [13, Section 5] there is a bijection between diagrams in ST_{2n} and diagrams in $B_n^\bullet$. For an example of this correspondence see figure 2.6.

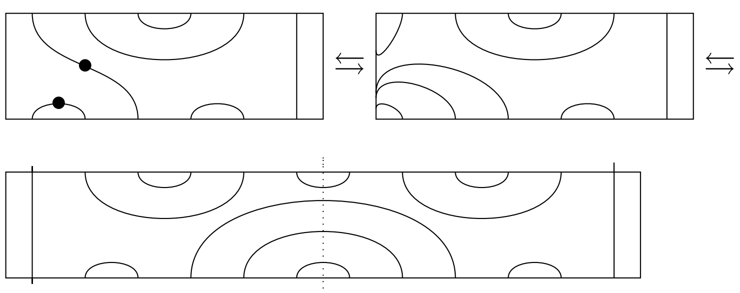


FIGURE 2.6: An example of a bijection between a blob 6-diagram for $b_6(p, p')$ and a left-right symmetric Temperley-Lieb 12-diagram for $STL_{12}(p)$.

Therefore, from equations (2.10) and (2.11) we obtain a bijection

$$ST_{2n}^{|\rangle}(2k) \times ST_{2n}^{\langle|}(2k) \leftrightarrow ST_{2n}(2k) \leftrightarrow B_n^{\bullet}(l) \leftrightarrow B_n^{\bullet|\rangle}(l) \times B_n^{\bullet\langle|}(l). \tag{2.13}$$

for $k \in \{0, 1, \ldots, n\}$ and $l \in \{n, n-2, n-4, \ldots, 2-n, -n\}$.

Our aim now is to find for any $l \in \{n, n-2, n-4, \ldots, 2-n, -n\}$ the corresponding value $k \in \{0, 1, \ldots, n\}$.

Lemma 2.4.12. *let $d \in B_n^{\bullet}(l)$ correspond to $x \in ST_{2n}(2k)$.*

(i) If n is an even number, then

$$\begin{cases} k = -l & \textit{if } l = 0, -2, \ldots, -n \\ k = l - 1 & \textit{if } l = 2, 4, \ldots, n. \end{cases}$$

(ii) If n is an odd number, then

$$\begin{cases} k = -l & \textit{if } l = -1, -3, \ldots, -n \\ k = l - 1 & \textit{if } l = 1, 3, \ldots, n. \end{cases}$$

Proof. If $l \in \{0, , -1, -2, \ldots, -n\}$, then the diagram d has l undecorated propagating lines, and therefore d corresponds to the diagram x with $k = l$ propagating lines. If $l \in \{1, 2, 3, \ldots, n\}$, then the diagram d has l propagating lines including one propagating line decorated with a blob, and therefore d corresponds to the diagram x with $k = l-1$ propagating lines. □

Corollary 2.4.13. *Let $|ST_{2n}^{|\rangle}(2k)|$ denote the number of top half diagrams in $ST_{2n}^{|\rangle}(2k)$.*

(i) If n is an even number, then

$$|ST_{2n}^{|\rangle}(2k)| = \begin{cases} \binom{n}{\frac{n-k}{2}} & \text{if } k = 0, 2, \ldots, n \\ \\ \binom{n}{\frac{n-(k+1)}{2}} & \text{if } k = 1, 3, \ldots, n-1. \end{cases}$$

(ii) If n is an odd number, then

$$|ST_{2n}^{|\rangle}(2k)| = \begin{cases} \binom{n}{\frac{n-k}{2}} & \text{if } k = 1, 3, \ldots, n \\ \\ \binom{n}{\frac{n-(k+1)}{2}} & \text{if } k = 0, 2, \ldots, n-1. \end{cases}$$

Proof. This follows immediately from equations (2.12), (2.13) and lemma 2.4.12. □

The next result shows how the algebra $STL_{2n}(p)$ is a special case of the blob algebra $b_n(p, p')$.

Lemma 2.4.14. *[13, Lemma 5.7] If p is invertible, the algebra of $STL_{2n}(p^2)$ is isomorphic to the blob algebra $b_n(p, 1)$.*

2.5 The Motzkin Algebra $M_n(p, q)$

In this section, we recall the definition of the Motzkin algebra using Motzkin diagrams. Also, we recall the dimension of this algebra and its generating set.

Definition 2.5.1. A *Motzkin set partition* of the set $\underline{n} \cup \underline{n}'$ is a set partition $M \in \mathcal{P}_n$ such that it satisfies (ii), (iii), (iv) of definition 2.3.2 and $|t_i| \leq 2$ for all $t_i \in M$. Denote by $\mathcal{M}_n$ the set of all Motzkin set partitions of the set $\underline{n} \cup \underline{n}'$.

We can draw a *Motzkin n-diagram* that represents an element of $\mathcal{M}_n$ in the same way as drawing a Temperley-Lieb diagram, and since the size of the parts of a Motzkin

set partition are at most two, then a Motzkin n-diagram may have vertices not connected with any other vertices. We call these vertices *isolated vertices*.
We call a line in a Motzkin n-diagram, d, with one endpoint in the top row and one endpoint in the bottom row a *propogating line* and a line with both endpoints in the same row an *arc*. An example is given in figure 2.7.

As for Temperley-Lieb diagrams, the diagram representing a Motzkin set partition

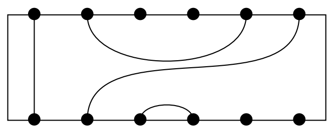


FIGURE 2.7: An example of a Motzkin 6-diagram.

is not unique. Two Motzkin n-diagrams are considered equal if they represent the same set partition.

We denote by MO_n the set of all Motzkin n-diagrams, and $MO_n(k)$ the set of all Motzkin n-diagrams with k propagating lines.

From [2, subsection 2.1], the number of the Motzkin n-diagrams is the same as the number of ways of drawing any number of non-intersecting chords among $2n$ points on a circle, which is known to be the Motzkin number $\mathfrak{M}_{2n}$. A Motzkin n-diagram with n lines is a Temperley-Lieb n-diagram and the number of Temperley-Lieb n-diagrams is equal to the n-th Catalan number as given in equation (2.8)

$$\mathfrak{M}_{2n} = \sum_{k=0}^{n} \binom{2n}{2k} \mathcal{C}_k = \sum_{k=0}^{n} \frac{1}{k+1} \binom{2n}{2k} \binom{2k}{k}. \tag{2.14}$$

Definition 2.5.2. A *half Motzkin diagram* is a diagram obtained by cutting horizontally a diagram in MO_n such that each propagating line is cut once, and no other line is cut. This gives a well defined pair of half diagrams which are the top half n-diagram and the bottom half n-diagram that have floating lines. The floating lines are straightened out and called *defects*.

For $k \in \{0, 1, \ldots, n\}$, let $MO_n^{|\rangle}(k)$, $MO_n^{\langle|}(k)$ be the sets of top half n-diagrams and bottom half n-diagrams respectively with k defects constructed by cutting diagrams in $MO_n(k)$.

Definition 2.5.3. [2, subsection 2.2] For fixed $p, q \in R$, the *Motzkin algebra* $M_n(p, q)$ is defined to be the set of linear combinations of Motzkin n-diagrams. The multiplication $x \cdot y$ of two Motzkin n-diagrams x and y is found by concatenating x and y in the following way: place x on top y and identify the vertices in the bottom row of x with the vertices in the top row of y. This new diagram may contain a number of closed loops, isolated vertices and paths in the middle row that are not connected to the top and bottom rows of the diagram. Then we remove these components and multiply the final result by $p^\alpha q^\beta$ where α is the number of closed loops and β is the number of isolated vertices and paths that arise in the middle row.

An example is given in figure 2.8.

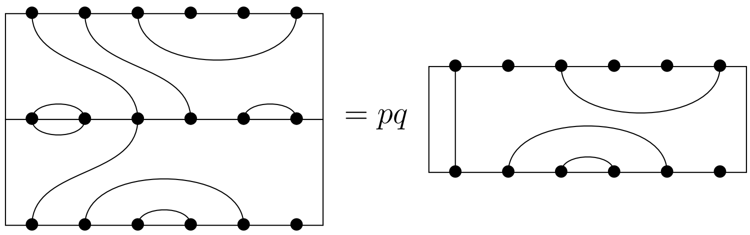


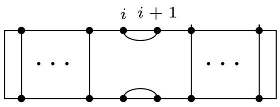
FIGURE 2.8: An example of multiplication of two diagrams in $M_6(p, q)$.

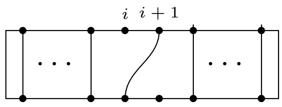
Belletete and Saint-Aubin in [1] define the dilute Temperley-Lieb algebra $dTL_n(\beta)$ where $\beta \in \mathbb{C}$ that has a basis coinciding with the basis of the Motzkin algebra $M_n(p, q)$, but the multiplication defined on the basis elements of both algebras are different. In [1, Corollarly 3.6] the authors calculate the number of the top half diagrams of dilute Temperley-Lieb n-diagrams with k defects which is the same the number of top half diagram in $MO_n^{|\rangle}(k)$. Thus, we have the size of $MO_n^{|\rangle}(k)$, denoted by $|MO_n^{|\rangle}(k)|$, is given by the formula,

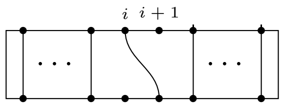
$$|MO_n^{|\rangle}(k)| = \sum_{c=0}^{\left[\frac{n-k}{2}\right]} \binom{n}{k+2c} \dim Z_{k+2c,c}. \tag{2.15}$$

where $Z_{n,c}$ is as in section 2.3.

Proposition 2.5.4. *[2, Proposition 2.13] The Motzkin algebra $M_n(p, 1)$ is generated by $\mathbb{I}_n$ and the diagrams t_i, r_i, l_i for $1 \leq i \leq n-1$ where*

$t_i =$ 

, $r_i =$ 

, $l_i =$ 

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Remark 2.5.5. The diagrams t_i, r_i and l_i for $1 \leqslant i \leqslant n-1$ satisfy the relations which are given in [15, Theorem 4.1].

Chapter 3

The dilute blob algebra $db_n(p, q, r, s)$

In this chapter we introduce a new class of finite dimensional algebras, the dilute blob algebra the main object of study in this thesis. This algebra is a generalisation of the Motzkin algebra. In the first section we define dilute blob set partitions that are represented by diagrams called dilute blob diagrams. These diagrams are the basis elements of a dilute blob algebra. We also introduce some concepts that will be required in the next chapters. In the second section we define a generating set of a dilute blob algebra when $q \in R$ is invertible. The final section is devoted to define the left-right symmetric Motzkin algebra which is a subalgebra of Motzkin algebra. Then we show the correspondence between dilute blob diagrams and left-right symmetric Motzkin diagrams.

Recall from definition 2.5.1 that the set $\mathcal{M}_n$ denotes the set of all Motzkin set partitions of the set $\underline{n} \cup \underline{n}'$.

3.1 Definition and structure

The purpose of this section is to define the dilute blob algebra. Firstly, we construct a dilute blob set partition by using a Motzkin set partition that is defined in definition 2.5.1. Then we explain how we represent a dilute blob set partition by a diagram drawn in the plane and we called it a dilute blob diagram. Secondly, in order to multiply two dilute blob diagrams we need to define a product diagram. Next, we define relations on a product diagram to obtain a dilute blob diagram. These procedures allow us to define the dilute blob algebra. We end this section by defining a left module of the dilute blob algebra.

Definition 3.1.1. Given $T \in \mathcal{M}_n$, then $A(T)$ is the subset of T that consists of the elements a_i that satisfy the following properties:

(i) $|a_i| = 1$.

(ii) If $a_i = \{a \colon a \in \underline{n}\}$, then there is no set $\{b, c \colon b, c \in \underline{n}\} \in T$ such that $b < a < c$, and there is no set $\{b, c' \colon b \in \underline{n}, c' \in \underline{n'}\} \in T$ such that $a < b$.

(iii) If $a_i = \{a' \colon a' \in \underline{n'}\}$, then there is no set $\{b', c' \colon b', c' \in \underline{n'}\}$ such that $b' < a' < c'$, and there is no set $\{b, c' \colon b \in \underline{n}, c' \in \underline{n'}\} \in T$ such that $a' < c'$.

Definition 3.1.2. A *dilute blob set partition* is an element of the set

$$\{(T, d) : T \in \mathcal{M}_n, d \in \mathscr{P}(A(T))\} \tag{3.1}$$

where $\mathscr{P}(A(T))$ is the power set of $A(T)$. We denote the set of all dilute blob set partitions by $\mathcal{DB}_n$.

Definition 3.1.3. We can rewrite the set in (3.1) to be the set

$$\{x : x \in T \setminus d\} \cup \{y_\Box : y \in T \cap d\}. \tag{3.2}$$

Example 3.1.4. 1. If $A = (\{\{1, 1'\}, \{2\}, \{2'\}\}, \{\{2'\}\}) \in \mathcal{DB}_2$, then $A = \{\{1, 1'\}, \{2\}, \{2'\}_\Box\}$.

2. If $B = (\{\{1, 1'\}, \{2, 3\}, \{2', 3'\}\}, \{\phi\}) \in \mathcal{DB}_3$, then $B = \{\{1, 1'\}, \{2, 3\}, \{2', 3'\}\}$.

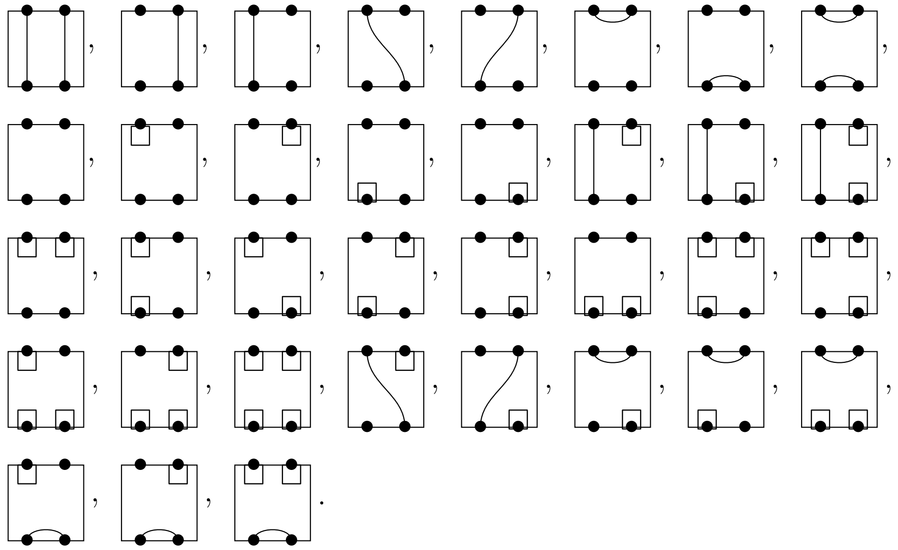
From definition 3.1.3, we have that an element of the set $\mathcal{DB}_n$ has elements from a Motzkin set partition T, and elements that belong to $T \cap d$. Therefore, we represent an element of $\mathcal{DB}_n$ as we have represented a Motzkin set partition T by a diagram as explained in section 2.5, and then add squares that decorate the vertices of the elements belonging to $T \cap d$. Such vertices are called *decorated vertices.*

Since $d \in \mathscr{P}(A(T))$ and from the properties of the elements of $A(T)$ in definition 3.1.1, we will have that decorated vertices should not be located either between the end points of an arc or in the top row (bottom row) on the left side of any propagating line.

Definition 3.1.5. We define a *dilute blob n-diagram* to be an n-diagram that represents an element belonging to $\mathcal{DB}_n$. We denote the set of all dilute blob n-diagrams by dB_n.

As with a Temperley-Lieb n-diagram and a Motzkin n-diagram, also here the dilute blob n-diagram that represents an element belonging to $\mathcal{DB}_n$ is not unique.
Two dilute blob n-diagrams are considered equal if they represent the same element in $\mathcal{DB}_n$.
A line in a dilute blob n-diagram with one endpoint in the top row and one endpoint in the bottom row is called a *propagating line* and one with both endpoints in the same row is called an *arc.* A vertex in a dilute blob n-diagram that is not connected with any other vertex is called an *isolated vertex.* An isolated vertex decorated with a single square is called a *decorated vertex.*
It is useful to keep in mind that every dilute blob n-diagram d encodes a dilute blob set partition. Hence, if $\{i, j'\} \in d$ for $i \in \underline{n}$ and $j' \in \underline{n}'$ then there is a propagating line between vertices i and j' in d, $\{i, j\} \in d$ for $i, j \in \underline{n}$ (respectively $\underline{n}'$) then there is an arc between vertices i and j in d, $\{i\} \in d$ for $i \in \underline{n}$ (respectively $\underline{n}'$) then i is an isolated vertex in d, and $\{i\}_\square$ for $i \in \underline{n}$ (respectively $\underline{n}'$) then i is a decorated vertex with a single square in d.

Example 3.1.6. The set dB_2 has 35 elements which are:



In order to be able to multiply dilute blob n-diagrams as for the Temperley-Lieb and Motzkin algebras, we define the notion of a *product n-diagram*.

Definition 3.1.7. A *product n-diagram*, $\Gamma(a, b)$, is two dilute blob n-diagrams a, b stacked on top each other and we visualise it as follows:
we draw a in a rectangular frame embedded in a plane with vertices 1 to n (increasing from left to right) on the top row and vertices $1'$ to n' (increasing from left to right) on the bottom row. We then relabel the vertices of b from i to i' on the top row and i' to i'' on the bottom row. We draw the rectangular frame for b below a with the top row of b identified with the bottom row of a. The edges of the two diagrams are drawn as usual, and so are the isolated vertices and decorated vertices.

See for example figure 3.1.

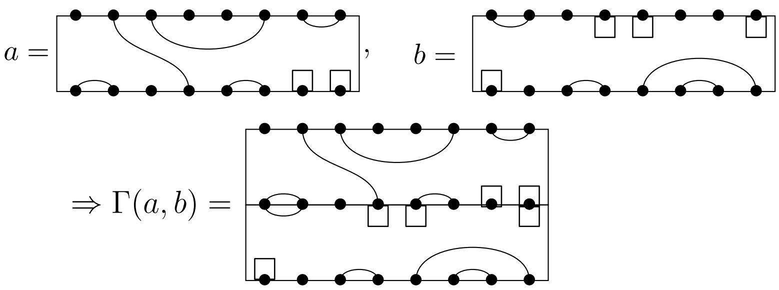


FIGURE 3.1: An example of a product 8-diagram $\Gamma(a, b)$.

Similarly, we can define the product n-diagram $\Gamma(a, b, \dots, c)$ for any finite number of dilute blob n-diagrams. See figure 3.2 for an example of $\Gamma(a, b, c)$

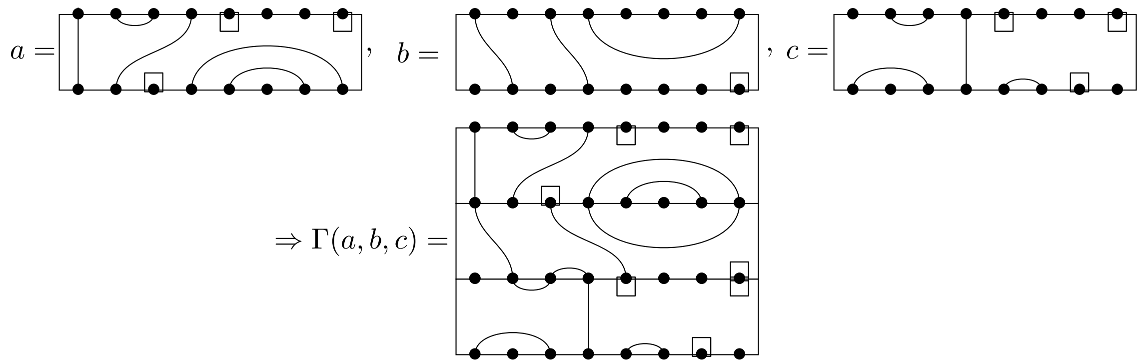


FIGURE 3.2: An example of the product diagram $\Gamma(a, b, c)$.

Definition 3.1.8. Given two diagrams $a, b \in dB_n$, we call the connected components in the middle row of the product diagram $\Gamma(a, b)$ that are not connected with the top and bottom rows of the diagram $\Gamma(a, b)$ *isolated components.*

Isolated components that can occur are:
loops, paths, isolated decorated or undecorated vertices, isolated vertices decorated with two squares and paths with endpoints decorated or not by single square as illustrated in figure 3.3.

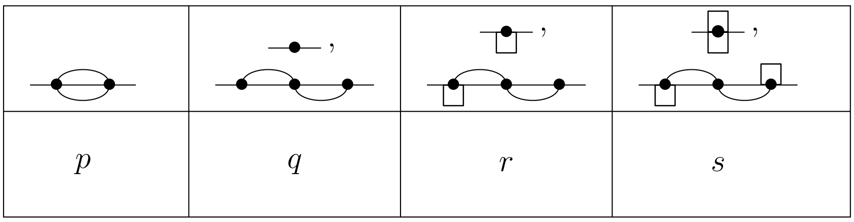


FIGURE 3.3: Table classifying all the isolated components with their corresponding parameters in db_n.

In the following any isolated vertex decorated or undecorated is considered a path of length zero.

Definition 3.1.9. Let p, q, r and s be elements in R, and $a, b \in dB_n$. We define a set of relations to obtain a dilute blob n-diagram from a product n-diagram $\Gamma(a,b)$ as follows:

(1) A product n-diagram $\Gamma(a,b)$ with closed loops is identified with $p^{\alpha(a,b)}$ times the same diagram with the closed loops omitted, where $\alpha(a,b)$ is the number of closed loops removed in the product n-diagram.

(2) A product n-diagram $\Gamma(a,b)$ with a path which lies entirely in the middle row is identified with $q^{\beta(a,b)}$ times the same diagram with these components omitted, where $\beta(a,b)$ is the number of such components.

(3) A product n-diagram $\Gamma(a,b)$ with a path with one of its endpoints decorated with a single square which lies entirely in the middle row is identified with $r^{\gamma(a,b)}$ times the same diagram with these components omitted, where $\gamma(a,b)$ is the number of such components.

(4) A product n-diagram $\Gamma(a,b)$ with a path with both its end vertices decorated with squares which lies entirely in the middle row is identified with $s^{\delta(a,b)}$ times the same diagram with these components omitted, where $\delta(a,b)$ is the number of such components.

(5) A product n-diagram $\Gamma(a,b)$ with a path connecting between two vertices (that are not decorated), one of them in the middle row and the other one either in the top row or in the bottom row is identified to be the same diagram with omitting this path and replacing it with its endpoint, that is either at the top row or bottom row.

(6) A product n-diagram $\Gamma(a,b)$ with a path connecting between two vertices, one of them in the middle row and decorated with a single square and the other one (that is not decorated with a single square) either in the top row or in the bottom row is identified with the same diagram omitting this path and replacing it with its endpoint decorated with a single square, either at the top row or bottom row.

Definition 3.1.10. Given a, b two diagrams in dB_n we form the product $ab \in RdB_n$ as follows:

(1) Form the product n-diagram $\Gamma(a, b)$ as in definition 3.1.7.

(2) Reduce the product n-diagram to a product n-diagram with no isolated components by applying relations (1) to (4) in definition 3.1.9.

(3) Apply relations (5) and (6) in definition 3.1.9 and replace any remaining paths by single edges.

(4) Remove the middle row and relabel each vertex in the bottom row i'' to i'.

We denote the resulting diagram by $a \circ b$. The product is

$$ab = p^{\alpha(a,b)} q^{\beta(a,b)} r^{\gamma(a,b)} s^{\delta(a,b)} a \circ b. \tag{3.3}$$

For example, for $n = 8$

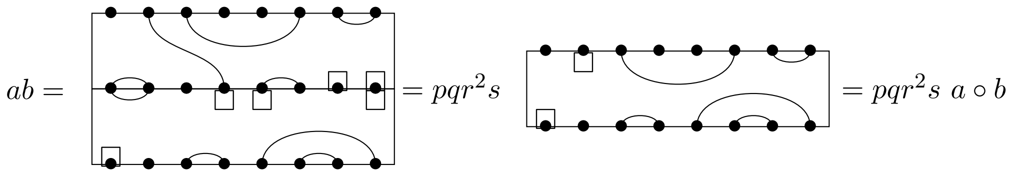


FIGURE 3.4: An example of the multiplication $ab \in dB_8$.

Lemma 3.1.11. *For any diagrams $a, b \in dB_n$, we have $a \circ b \in dB_n$.*

Proof. It is clear from the structure of the product n-diagram $\Gamma(a, b)$ that the diagram $a \circ b$ consists of non crossing edges since both a, b have non crossing edges. Any arcs, isolated vertices decorated or undecorated in the top row of a and in the bottom row of b are also in the top row and bottom row respectively of $a \circ b$ and by applying (3) in definition 3.1.10. Therefore, we can not have decorated vertices between the endpoints of any arcs or in the left side of any propagating lines in $a \circ b$. Hence, $a \circ b$ is a dilute blob n-diagram. $\square$

Since it does not matter what order we apply the relations, the product ab is well defined.

In order to prove the product ab (as in definition 3.1.10) satisfies the associative property, we need to prove that both product $(ab)c$ and $a(bc)$ have the same number of each type of the isolated components for any $a, b, c \in dB_n$. For example, we need to prove the number of closed loops in the middle row of $\Gamma(a, b)$ $(\alpha(a, b))$ adds to the number of closed loops in the middle row of $\Gamma(a \circ b, c)$ $(\alpha(a \circ b, c))$ equal to the number of closed loops in the middle row of $\Gamma(b, c)$ $(\alpha(b, c))$ adds to the number of closed loops in the middle row of $\Gamma(a, b \circ c)$ $(\alpha(a, b \circ c))$. For this purpose we state and prove the following proposition.

Proposition 3.1.12. *Let a, b and c be diagrams in dB_n. Then*

$$\alpha(a, b) + \alpha(a \circ b, c) = \alpha(a, b \circ c) + \alpha(b, c) \tag{3.4}$$

$$\beta(a, b) + \beta(a \circ b, c) = \beta(a, b \circ c) + \beta(b, c), \tag{3.5}$$

$$\gamma(a, b) + \gamma(a \circ b, c) = \gamma(a, b \circ c) + \gamma(b, c), \tag{3.6}$$

$$\delta(a, b) + \delta(a \circ b, c) = \delta(a, c \circ b) + \delta(b, c). \tag{3.7}$$

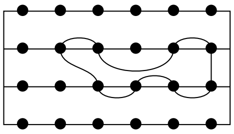
Proof. Firstly, to prove equation (3.4) consider the following:

(p1) If a loop appears in the middle row of the diagram $\Gamma(a, b)$, then definitely it will appear in the middle row of the diagram $\Gamma(a, b \circ c)$. Therefore, $\alpha(a, b) = \alpha(a, b \circ c)$.

(p2) If a loop appears in the middle row of the diagram $\Gamma(b, c)$, then it also will appear in middle row of the diagram $\Gamma(a \circ b, c)$. Therefore, $\alpha(b, c) = \alpha(a \circ b, c)$.

(p3) If there is a path in the middle row of the diagram $\Gamma(a, b)$ starting in the vertex i' and ending in the vertex j' in the middle row of the $\Gamma(a, b)$ such that $i' < j'$, there is a path in the middle row of the diagram $\Gamma(b, c)$ starting in the vertex

k' and ending in the vertex l' in the middle row of $\Gamma(b, c)$ such that $k' < l'$, and b has two propagating lines $\{i, k'\}$ and $\{j, l'\}$. This construction will result to appear a loop in the middle row of the diagrams $\Gamma(a \circ b, c)$ and $\Gamma(a, b \circ c)$. Therefore, $\alpha(a \circ b, c) = \alpha(a, b \circ c)$. As pictured in the example below

$\Gamma(a, b, c) =$ 

Hence, from p1-p3 we have proved equation (3.4).
Similarly, we can prove equations (3.5), (3.6) and (3.7) with appropriate changes. □

To satisfy that $(ab)c = a(bc)$ for any $a, b, c \in dB_n$, we need also to prove that $(a \circ b) \circ c = a \circ (b \circ c)$. So we have the following theorem.

Theorem 3.1.13. *The product in equation* (3.3) *is associative.*

Proof. Let a, b and c be diagrams in dB_n. Consider the products diagrams $\Gamma(a \circ b, c)$, $\Gamma(a, b \circ c)$ and $\Gamma(a, b, c)$. It is clear (ignoring isolated components) that the three product diagrams are the same. The following explains this explicitly:

(1) If there is a path starting from the top row of a and ending in the bottom row of c, then it will be a propagating line in $(a \circ b) \circ c$ and $a \circ (b \circ c)$ i.e following through lines is associative.

(2) Any arcs, isolated vertices and decorated vertices in the top row of a are also in the top row of both diagrams $(a \circ b) \circ c$ and $a \circ (b \circ c)$. Similarly any arcs, isolated vertices and decorated vertices in the bottom row of c are added to the bottom row of both diagrams $(a \circ b) \circ c$ and $a \circ (b \circ c)$.

(3) If there is a path starting from the propagating line $\{i, j'\} \in a$ and terminating in an isolated vertex (a decorated vertex) $\{k\} \in b$ ($\{k'\} \in a$). This would appear the isolated vertex (the decorated vertex) $\{i\}$ in the top row of both diagrams $(a \circ b) \circ c$ and $a \circ (b \circ c)$.

(4) If there is a path ending in the propagating line $\{i, j'\} \in c$ and terminating in an isolated vertex (a decorated vertex) $\{k'\} \in b$ ($\{k\} \in c$). This would appear the isolated vertex (decorated vertex) $\{j'\}$ in the bottom row of both $(a \circ b) \circ c$ and $a \circ (b \circ c)$.

(5) If there is a path starting from the propagating line $\{i, j'\} \in a$ and terminating in an isolated vertex (a decorated vertex) $\{k'\} \in$ b ($\{k\} \in c$). This would appear the isolated vertex (decorated vertex) $\{i\}$ in the top row of both diagrams $(a \circ b) \circ c$ and $a \circ (b \circ c)$.

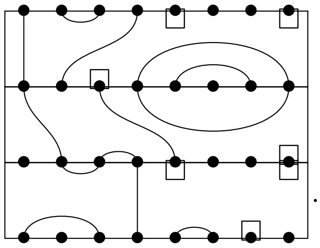
(6) If there is a path ending in the propagating line $\{i, j'\} \in c$ and terminating in an isolated vertex (a decorated vertex) $\{k\} \in b$ ($\{k'\} \in a$). This would appear the isolated vertex (decorated vertex) $\{j'\}$ in the bottom row of both diagrams $(a \circ b) \circ c$ and $a \circ (b \circ c)$.

(7) If there is a path starting from a propagating line $\{i, j'\} \in a$ and ending in another propagating line $\{k, l'\} \in a$. This would appear an arc $\{i, k\}$ in the top row of both diagrams $(a \circ b) \circ c$ and $a \circ (b \circ c)$.

(8) If there is a path starting with a propagating line $\{i, j'\} \in c$ and ending in another propagating line $\{k, l'\} \in c$. This would appear an arc $\{j', l'\}$ in the bottom row of both diagrams $(a \circ b) \circ c$ and $a \circ (b \circ c)$.

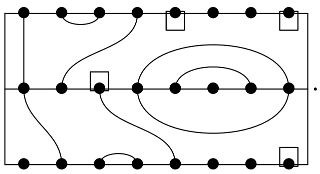
Hence, from 1-8 we have $(a \circ b) \circ c = a \circ (b \circ c)$, and from proposition 3.1.12 that tells us the parameters that appear in $(ab)c$ and $a(bc)$ are equal. Therefore, we have $(ab)c = a(bc)$ for all $a, b, c \in dB_n$. □

Example 3.1.14. Consider the following product of diagrams in dB_8.

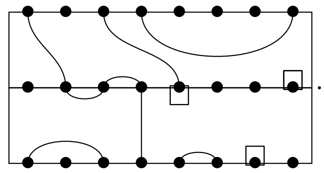


Regardless of the order of the multiplication, we have a path starting from $\{1, 1'\}$ in the first digram and ending at $\{4, 4'\}$ in the last diagram, which will result the propagating line $\{1, 4'\}$ in the multiplication diagram. A loop, an isolated vertex and a path appear in the middle row between the first diagram and the second diagram. The isolated vertex $\{4\}$ will appear in the multiplication diagram that resulted from terminating the propagating line $\{4, 2'\}$ in the isolated vertex $\{2\}$ in the top row of the second diagram. Isolated vertices and decorated vertex with two squares appear in the middle row between the second diagram and the third diagram. The propagating line $\{3, 5'\}$ in the second diagram terminates in two decorated vertices one is $\{3'\}_\square$ in the bottom row of the first diagram and the other one is $\{5\}_\square$ in the top row of the last diagram.

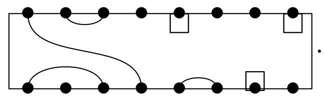
If we multiply the top two diagrams first we have



If we compose the last two diagrams first we have



In either case, the same number of each isolated components appears, all components of the top row and bottom row of the multiplication diagram appear, just in the opposite order. Consequently, regardless of the order of multiplication we get the same diagram. In this case, the multiplication diagram is



Now we can write the main result of our thesis.

Theorem 3.1.15. *Let R be a commutative ring with identity and fix $p, q, r, s \in R$. The* dilute blob algebra *$db_n(p, q, r, s)$ is an associative R-algebra and its identity element is the diagram $\mathbb{I}_n$ (as in figure 3.5) with basis dB_n and multiplication defined on the basis elements in dB_n and then extended bilinearly to all of $db_n(p, q, r, s)$.*

In what follows we will mostly omit the parameters p, q, r, s from our notation, writing simply db_n for $db_n(p, q, r, s)$.

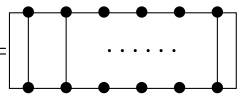
$\mathbb{I}_n =$ 

FIGURE 3.5: The identity element of the algebra $db_n(p, q, r, s)$.

Definition 3.1.16. The *rank* of a diagram $x \in dB_n$, denoted $\mathrm{rank}(x)$, is the number of propagating lines in the diagram x. We extend rank to scalar multiples of a diagram via $\mathrm{rank}(ax) = \mathrm{rank}(x)$ for $x \in dB_n, a \in R$.

Lemma 3.1.17. *Given $x_1, x_2 \in dB_n$. Then*

$$\operatorname{rank}(x_1x_2) \leq \min\left(\operatorname{rank}(x_1), \operatorname{rank}(x_2)\right). \tag{3.8}$$

Proof. The multiplication of two diagrams in dB_n, can not create any additional propagating lines, two propagating lines can become an arc, and a propagating line can contract either to an isolated vertex or to a decorated vertex with a single square.

$\square$

Corollary 3.1.18. *For an integer λ with $0 \leqslant \lambda \leqslant n$, let I_λ be the R-*span *of the diagrams in dB_n of rank less than or equal to λ. Then I_λ is a two-sided ideal in db_n, and we have the tower of ideals,*

$$0 \subset I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_n = db_n. \tag{3.9}$$

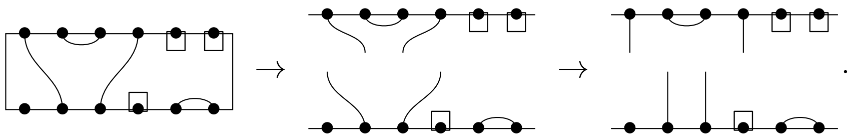
Proof. Given $x, x_\lambda \in dB_n$ such that $\operatorname{rank}(x_\lambda) \leqslant \lambda$. Thus $x_\lambda \in I_\lambda$, and by lemma 3.1.17 we have,

$$\operatorname{rank}(xx_\lambda) = \operatorname{rank}(x_\lambda x) \leqslant \lambda.$$

Therefore, $xx_\lambda \in I_\lambda$, and $x_\lambda x \in I_\lambda$, and therefore I_λ is a two sided ideal in db_n. $\square$

Definition 3.1.19. A *half dilute blob diagram* is a diagram obtained by cutting horizontally a dilute blob diagram in the middle in such a way that only propagating lines are cut (once). This produces a well defined pair of half diagrams, a top half diagram and a bottom half diagram that have floating lines. The floating lines are straightened out and called *defects*.

For example,



Definition 3.1.20. For $\lambda \in \{0,1,\ldots,n\}$ define $dB_n^{|\rangle}(\lambda), dB_n^{\langle|}(\lambda)$ to be the sets of top half diagrams and bottom half diagrams respectively with λ defects obtained by cutting dilute blob diagrams in $dB_n(\lambda)$ that have λ propagating lines.

We can form a unique dilute blob diagram by these two half diagrams that have the same number of defects by joining the defects in the unique way. So we have a bijection:

$$dB_n(\lambda) \leftrightarrow dB_n^{|\rangle}(\lambda) \times dB_n^{\langle|}(\lambda) \qquad \text{for all} \qquad \lambda \in \{0,1,\ldots,n\}. \tag{3.10}$$

Definition 3.1.21. Denote by $|d\rangle$ and $\langle d|$ the top half dilute blob diagram and bottom half dilute blob diagram respectively given from cutting a dilute blob n-diagram d.

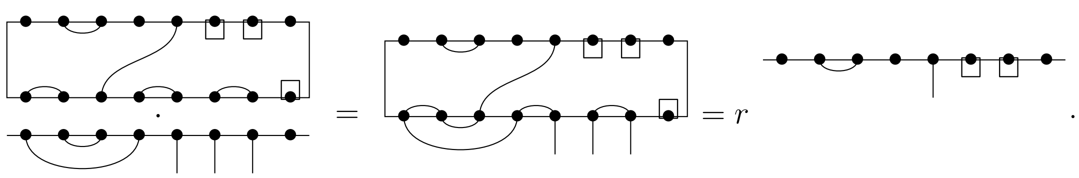
Note $d = |d\rangle\langle d|$, where defects are joined up.

Lemma 3.1.22. *Let V_n be the free R-module with basis $\cup_{\lambda=0}^{n} dB_n^{|\rangle}(\lambda)$. Then V_n is a left db_n-module with the action defined by concatenating a dilute blob diagram in dB_n with a half diagram, then proceeding as for multiplication in db_n. The result is a half diagram.*

Proof. Let $x, y \in db_n$ and v be a half diagram in V_n, then $(xy)v = x(yv)$ since we multiply as we would multiply diagrams in db_n. This proves that the action defined in V_n is associative.

It is clear that $\mathbb{I}_n v = v$ for all half diagrams $v \in V_n$. □

We give an example to demonstrate the action:



Similarly, we can have a right db_n-module, if we consider the free R-module with basis $\cup_{\lambda=0}^{n} dB_n^{\langle|}$.

Lemma 3.1.23. *For $0 \leqslant \lambda \leqslant n$ define the free R-submodule $W_n(\lambda)$ of V_n spanned by half diagrams having at most λ number of defects. Then $W_n(\lambda)$ is a db_n-submodule of V_n, and forms a filtration of V_n:*

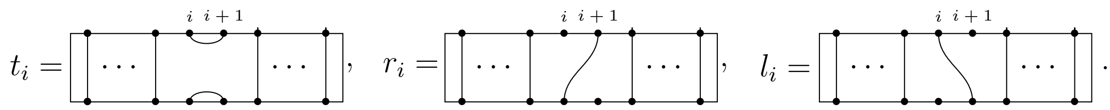
$$W_n(0) \subset W_n(1) \subset \cdots \subset W_n(n) = V_n. \tag{3.11}$$

Proof. As in lemma 3.1.17, the multiplication of two dilute blob diagrams in dB_n can not increase the number of propagating lines. Also, the action of a dilute blob diagram in dB_n on a half diagram can not increase the number of its defects. Therefore, if $x \in dB_n$ and $a \in W_n(\lambda)$, then xa can not have more than λ defects, so $xa \in W_n(\lambda)$. $\square$

3.2 A generating set for the algebra $db_n(p, q, r, s)$

In this section, we define the dilute blob algebra by generators. Assume that $q \in R$ is invertible.

For $1 \leq i \leq n-1$, consider the following diagrams in dB_n,



It is well known these diagrams generate the Motzkin algebra as mentioned in proposition 2.5.4. For $1 \leq i \leq n$, let

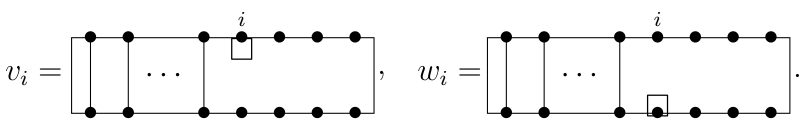


Diagram multiplication shows that

$$v_i = l_i v_{i+1} \quad \text{for} \quad 1 \leq i \leq n-1. \tag{3.12}$$

Similarly,

$$w_i = w_{i+1} r_i \quad \text{for} \quad 1 \leq i \leq n-1. \tag{3.13}$$

Definition 3.2.1. Let $H_n \subset db_n$ denote the subalgebra of db_n spanned by the identity element $\mathbb{I}_n$ and the diagrams v_i for $1 \leq i \leq n$. Analogously, let $\overline{H}_n$ denote the subalgebra of db_n spanned by the identity element $\mathbb{I}_n$ and the diagrams w_i for $1 \leq i \leq n$.

Proposition 3.2.2. *If $q \in R$ is invertible then the dilute blob algebra db_n is generated by the identity diagram $\mathbb{I}_n$ and the diagrams t_i, r_i, l_i for $1 \leq i \leq n-1$, and the diagrams v_n, w_n.*

Proof. We claim that every diagram $d \in db_n$ can be factored as

$$d = \xi m_1 v m_2 w m_3 \tag{3.14}$$

where $\xi \in R$, m_1, m_2 and m_3 are Motzkin diagrams, v is a diagram in H_n and w is a diagram in $\overline{H}_n$.

This factorization can be done as follows. The diagram m_1 is obtained from the diagram d as follows:

1. If d has a propagating line $\{i, j'\}$, then m_1 has a propagating line $\{i, i'\}$.
2. If d has an isolated vertex $\{i\}$, then m_1 has an isolated vertex $\{i\}$.
3. If d has an arc $\{i, j\}$, then m_1 has an arc $\{i, j\}$.
4. If d has a decorated vertex $\{i\}_\square$, then m_1 has a propagating line $\{i, i'\}$.
5. Any unused vertices in the bottom row of m_1 are isolated vertices.

The diagram v is obtained from the diagram d as follows:

1. If d has a decorated vertex $\{i\}_\square$, then v has a decorated vertex $\{i\}_\square$.

   Consider the least element $i \in \underline{n}$ such that the vertex $\{i\}_\square$ is decorated, and put propagating lines $\{a, a'\}$ to the left of the vertex $\{i\}_\square$. All other unused vertices are isolated.

2. If d has no decorated vertex in the top row, then v is the identity diagram.

The diagram m_2 is obtained from the diagram d as follows:

1. If d has a propagating line $\{i, j'\}$, then so does m_2. All other unused vertices are isolated.

2. If d has no propagating lines, then m_2 is the identity diagram.

The diagram w is obtained from the diagram d as follows:

1. If d has a decorated vertex $\{i'\}_\square$, then w has a decorated vertex $\{i'\}_\square$. Consider the least element $i' \in \underline{n'}$ such that the vertex $\{i'\}$ is decorated, and put propagating lines $\{b, b'\}$ to the left of the vertex $\{i'\}_\square$. All other unused vertices are isolated.

2. If d has no decorated vertices in the bottom row, then w is the identity diagram.

The diagram m_3 is obtained from the diagram d as follows:

1. If d has a propagating line $\{i, j'\}$, then m_3 has a propagating line $\{j, j'\}$.

2. If d has an isolated vertex $\{j'\}$, then m_3 has an isolated vertex $\{j'\}$.

3. If d has an arc $\{i', j'\}$, then m_3 has an arc $\{i', j'\}$.

4. If d has a decorated vertex $\{j'\}_\square$, then m_3 has a propagating line $\{j, j'\}$.

5. Any unused vertices in the bottom row of m_3 are isolated vertices.

The scalar $\xi = \dfrac{1}{q^\beta}$ where β is the number of isolated components which are isolated vertices and paths in the product $m_1 v m_2 w m_3$.

We claim that $d = \xi m_1 v m_2 w m_3$. Consider a propagating line $\{i, j'\}$ in the diagram d. Then if $\{a\}_\square \in d$ then $i < a$, and if $\{b'\}_\square \in d$ then $j' < b'$.

Now $\{i, j'\} \in m_1 v m_2 w m_3$ as:

1. $\{i,i'\} \in m_1$ by construction of m_1 (property 1),

2. $\{i,i'\} \in v$ as $i < a$ for any $\{a\}_\square \in v$ by construction of v (property 1).

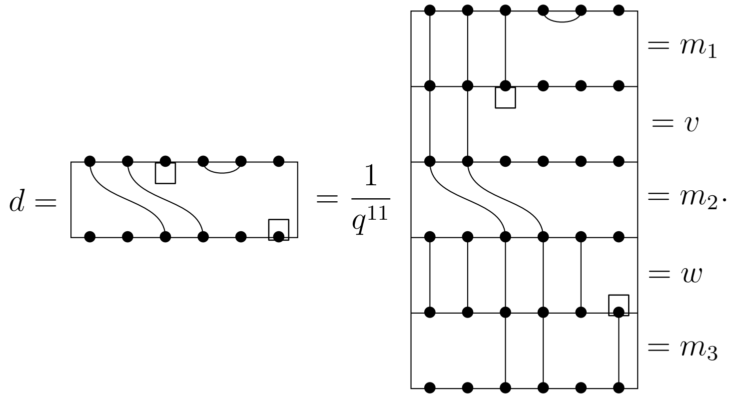
3. $\{i,j'\} \in m_2$ by construction of m_2 (property 1).

4. $\{j,j'\} \in w$ by construction of w as $j' < b'$ for any $\{b'\}_\square \in d$ (property 1).

5. $\{j,j'\} \in m_3$ by construction of m_3 (property 1).

Hence, $\{i,j'\} \in m_1vm_2wm_3$.
Consider a decorated vertex $\{i\}_\square \in d$. Then $\{i\}_\square \in m_1vm_2wm_3$ as $\{i,i'\} \in m_1$ by construction of m_1 (property 4), and $\{i\}_\square \in v$ by construction of v (property 1).
Consider a decorated vertex $\{j'\}_\square \in d$. Then $\{j'\}_\square \in m_1vm_2wm_3$ as $\{j'\}_\square \in w$ by construction of w (property 1) and $\{j,j'\} \in m_3$ by construction of m_3 (property 4).
Consider an isolated vertex $\{i\} \in d$. Then $\{i\} \in m_1vm_2wm_3$ as $\{i\} \in m_1$ by the construction of m_1 (property 2).
Similarly for $\{j'\} \in d$. Then $\{j'\} \in m_1vm_2wm_3$ as $\{j'\} \in m_3$ by construction of m_3 (property 2).
Consider an arc $\{i,j\} \in d$. Then $\{i,j\} \in m_1vm_2wm_3$ as $\{i,j\} \in m_1$ by construction of m_1 (property 3).
Similarly an arc $\{i',j'\} \in d$. Then $\{i',j'\} \in m_1vm_2wm_3$ as $\{i',j'\} \in m_3$ by construction of m_3 (property 3).
Therefore, $d = \dfrac{1}{q^\beta} m_1vm_2wm_3$ as $\xi = \dfrac{1}{q^\beta}$ where β is the number of isolated vertices and paths in the product $m_1vm_2wm_3$. $\square$

For example,



3.3 Bijection between left-right symmetric Motzkin diagrams and dilute blob diagrams

Recall the definition of the Motzkin algebra $M_n(p, q)$ from definition 2.5.3. In this section we define the left-right symmetric Motzkin algebra and then we prove it is a subalgebra of $M_{2n}(p, q)$. We end this section by making explicit the bijection between the basis diagrams of the left-right symmetric Motzkin algebra and the basis diagrams of the dilute blob algebra.

Definition 3.3.1. A *left-right symmetric Motzkin $2n$-diagram* is a Motzkin $2n$-diagram that is symmetric under reflection in the middle vertical axis. Let SMO_{2n} be the set of all left-right symmetric Motzkin $2n$-diagrams.

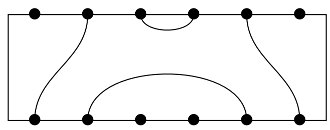


FIGURE 3.6: An example of a left-right symmtric Motzkin 6-diagram.

Definition 3.3.2. Fix $p, q \in R$, define $SM_{2n}(p, q)$ to be the free R-submodule of $M_{2n}(p, q)$ with a basis consisting of the basis elements of $M_{2n}(p, q)$ that are left-right symmetric diagrams.

Lemma 3.3.3. *The free R-submodule $SM_{2n}(p, q)$ is a subalgebra of $M_{2n}(p, q)$.*

Proof. Since the identity element $\mathbb{I}_{2n}$ of $M_{2n}(p,q)$ is a left-right symmetric diagram, therefore, $\mathbb{I}_{2n} \in SM_{2n}(p,q)$.
Also, the multiplication of any diagrams in $SM_{2n}(p,q)$ is the multiplication in $M_{2n}(p,q)$. However, concatenating two left-right symmetric diagrams also results in a left-right symmetric diagram. Therefore the multiplication is closed in $SM_{2n}(p,q)$. Hence, $SM_{2n}(p,q)$ is subalgebra of $M_{2n}(p,q)$. □

Proposition 3.3.4. *There is a bijection between left-right symmetric Motzkin $2n$-diagrams in SMO_{2n} and dilute blob n-diagrams in dB_n.*

Proof. Consider a dilute blob n-diagram d in dB_n. Replace each square on a decorated vertex with a line that connects such vertex with the east wall in such a way that the lines are not crossing. Now consider the diagram union its reflection in the east wall, which is a diagram for SMO_{2n}.
Conversely, consider a left-right symmetric Motzkin $2n$-diagram a. Split a vertically in the middle into two symmetrical parts. Consider the left part, then replace each line that is resulting from cutting an arc with a square decorating the endpoint vertex of the line. The resulting diagram is a dilute blob n-diagram. □

For example,

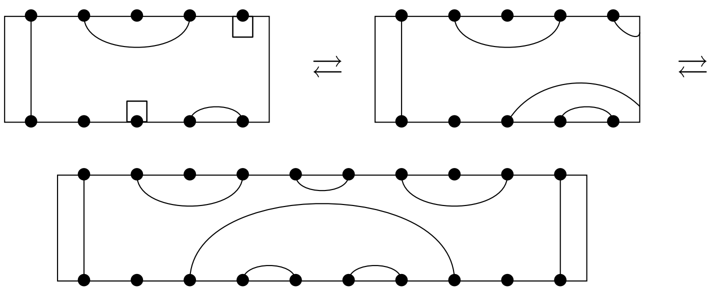


FIGURE 3.7: An example of a bijection between a diagram in dB_5 and a diagram in SMO_{10}.

Chapter 4

Cellularity Of $db_n(p, q, r, s)$

In this chapter, we answer some fundamental questions about the representation theory of db_n. In particular, in proposition 4.1.1 we show that the dilute blob algebra db_n over any unital commutative ring R is cellular in the sense of Graham and Lehrer [11], then we define the cell modules $\Delta_n(\lambda)$ of db_n, which have a basis $dB_n^{|\rangle}(\lambda)$. In proposition 4.2.3 we prove that there is a one-to-one correspondence between half diagrams in $dB_n^{|\rangle}(\lambda)$ and half diagrams in the set of top half Motzkin n-diagrams that has $\lambda, \lambda+1, \ldots, n$ defects. We use this bijection to find the dimension of the cell module $\Delta_n(\lambda)$. Then we introduce a bilinear form on cell modules and then we use the bilinear form to prove that the cell modules are cyclic.

4.1 The dilute blob algebra $db_n(p, q, r, s)$ is a cellular algebra

The aim of this section is showing how the dilute blob algebra db_n satisfies the axioms of cellular algebra in the sense of Graham and Lehrer [11].

Proposition 4.1.1. *The dilute blob algebra db_n is a cellular algebra.*

Proof. We prove that the dilute blob algebra db_n satisfies axioms C1-C3 of definition 2.1.1. The cell datum is $(\Lambda_{db_n}, dB_n^{|\rangle}(\lambda), C, *)$ where $\Lambda_{db_n} = \{0, 1, 2, \ldots, n\}$ with the usual (total) order, $<$, of integers.

We define $C : \sqcup_{0\leq\lambda\leq n} dB_n^{|\rangle}(\lambda) \times dB_n^{|\rangle}(\lambda) \to db_n$ to be a map that sends the pair $(u, w) \in dB_n^{|\rangle}(\lambda) \times dB_n^{|\rangle}(\lambda)$ with λ defects to the diagram $C^{\lambda}_{u\overline{w}} \in dB_n$ where $\overline{w}$ is the bottom mirror image of w.

A map $*$ is defined by

$$* : db_n \to db_n \quad \text{where} \qquad (C^{\lambda}_{u\overline{w}})^* = C^{\lambda}_{w\overline{u}}. \tag{4.1}$$

This map sends every diagram in dB_n to its reflection around a horizontal line. An example is given in figure 4.1 for the case $n = 6$. We then extend $*$ linearly to all of db_n

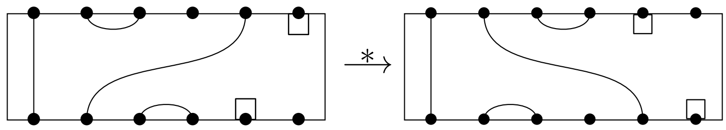


FIGURE 4.1: An example of the map $*$ that reflects the basis element of db_n around a horizontal line to get another basis element of db_n.

Since every diagram in dB_n can be cut into a unique top half diagram and a unique bottom half diagram, the map C is an injective map. Moreover, every basis element is in $\operatorname{im} C$, so (C1) is satisfied.

We now show that $*$ is an anti-involution. Let $C^{\lambda}_{u\overline{w}}, C^{\lambda'}_{u'\overline{w}'} \in dB_n$.

Clearly $(C^{\lambda}_{u\overline{w}} C^{\lambda'}_{u'\overline{w}'})^* = (C^{\lambda'}_{u'\overline{w}'})^* (C^{\lambda}_{u\overline{w}})^*$ as multiplying diagrams then flipping is the same as flipping then multiplying.

Also,

$$(C^{\lambda}_{u\overline{w}})^{*^2} = ((C^{\lambda}_{u\overline{w}})^*)^* = (C^{\lambda}_{w\overline{u}})^* = C^{\lambda}_{u\overline{w}}.$$

Therefore, $*$ is an anti-involution. Hence, we satisfy (C2).

Now from lemma 3.1.17 we have

$$\operatorname{rank}(C^{\lambda'}_{u'\overline{w}'} C^{\lambda}_{u\overline{w}}) \leqslant \min(\operatorname{rank}(C^{\lambda'}_{u'\overline{w}'}), \operatorname{rank}(C^{\lambda}_{u\overline{w}})) = \min(\lambda', \lambda).$$

Assume that $\operatorname{rank}(C^{\lambda'}_{u'\overline{w}'}C^{\lambda}_{u\overline{w}}) = l$. Then $l \leq \lambda$, and so we have the following two cases:

Case (1) If $l < \lambda$. Then

$$C^{\lambda'}_{u'\overline{w}'}C^{\lambda}_{u\overline{w}} \equiv 0 \mod db_n^{<\lambda}.$$

Case (2) If $l = \lambda$. Then we have this case when $\lambda \leqslant \lambda'$ and all λ propagating lines connect with λ' propagating lines from $C^{\lambda'}_{u'\overline{w}'}$. Then the multiplication does not change the bottom half of $C^{\lambda}_{u\overline{w}}$. Thus, we have

$$C^{\lambda'}_{u'\overline{w}'}C^{\lambda}_{u\overline{w}} = p^{\alpha}q^{\beta}r^{\gamma}s^{\delta}C^{\lambda}_{u''\overline{w}}$$

where $u'' \in dB_n^{|\rangle}(\lambda)$. Here $p^{\alpha}q^{\beta}r^{\gamma}s^{\delta}$ depends on $\overline{w}', u$ and does not depend on $\overline{w}$.

Therefore, we satisfy (C3).
Hence, the algebra db_n is a cellular algebra. □

4.2 Cell modules of the algebra $db_n(p, q, r, s)$

In this section, we define the cell modules of the dilute blob algebra db_n, find their dimensions and prove they are cyclic modules.

Definition 4.2.1. For each $\lambda \in \Lambda_{db_n}$, the (left) cell module of db_n corresponding to λ is

$$\Delta_n(\lambda) = \operatorname{span}_R\{C^{\lambda}_u : u \in dB_n^{|\rangle}(\lambda)\}$$

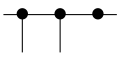
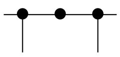
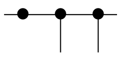
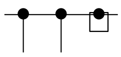
where the action of db_n is defined by

$$xC^{\lambda}_u = \sum_{z \in dB_n^{|\rangle}} r_x(z, u)C^{\lambda}_z \quad (x \in db_n, u \in dB_n^{|\rangle})$$

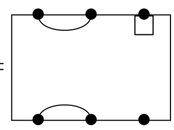
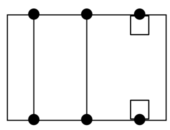
where $r_x(z, u)$ is the element of R defined in definition 2.1.1 (C3).

Since the elements of the basis of $\Delta_n(\lambda)$ are in one-to-one correspondence with elements of $dB_n^{|\rangle}(\lambda)$, we shall identify $C_u^\lambda \leftrightarrow u \in dB_n^{|\rangle}(\lambda)$. Then $dB_n^{|\rangle}(\lambda)$ forms a basis of the cell module $\Delta_n(\lambda), \lambda \in \Lambda_{db_n}$.

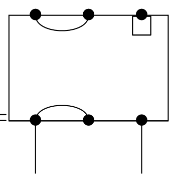
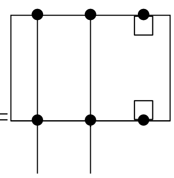
Example 4.2.2. The cell module $\Delta_3(2)$ for the algebra db_3 has basis consists of the following top half diagrams:

$u_1 =$  , $u_2 =$  , $u_3 =$  , $u_4 =$  .

If x, y two diagrams in dB_3 such that

$x =$  , $y =$  .

Then

$x \cdot u_2 =$  $= 0$, and $y \cdot u_1 =$  $= ru_4$.

Now, recall from definition 2.5.2 that $MO_n^{|\rangle}(k)$ is the set of top half Motzkin n-diagrams with k defects.

Proposition 4.2.3. *For all $\lambda \in \{0, 1, \ldots, n\}$, there is a bijection between the half diagrams in $dB_n^{|\rangle}(\lambda)$ and the half diagrams in $\bigcup_{i=\lambda}^n MO_n^{|\rangle}(i)$.*

Proof. Consider a half diagram $a \in dB_n^{|\rangle}(\lambda)$, if there are no decorated vertices in a, then $a \in MO_n^{|\rangle}(\lambda)$. Otherwise, replace every square decorating a vertex with a defect. We will have a half diagram in $\bigcup_{i=\lambda}^n MO_n^{|\rangle}(i)$.
Conversely, consider a half diagram $b \in MO_n^{|\rangle}(\lambda)$, then $b \in dB_n^{|\rangle}(\lambda)$. If $b \in MO_n^{|\rangle}(i)$ with $i \in \{\lambda+1, \lambda+2, \ldots, n\}$, then we will fix the first λ defects start from the left side and then replace the other defects with squares decorating the vertices. The resulting half diagrams are in $dB_n^{|\rangle}(\lambda)$.
It is clear that both procedures are the converse of each other. $\square$

For example,

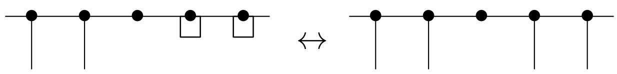


FIGURE 4.2: An example of the bijection between two elements of $dB_5^{|\rangle}(2)$ and $MO_5^{|\rangle}(4)$.

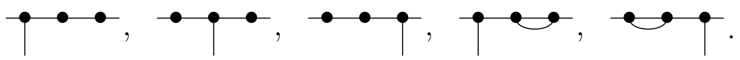
Recall from equation (2.15) $|MO_n^{|\rangle}(k)|$, so we have the following corollary.

Corollary 4.2.4. *The dimension of cell module $\Delta_n(\lambda)$ is*

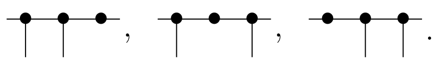
$$\dim_R \Delta_n(\lambda) = \sum_{i=\lambda}^{n} \left(\sum_{c=0}^{\left[\frac{n-i}{2}\right]} \binom{n}{i+2c} \left(\binom{i+2c}{c} - \binom{i+2c}{c-1} \right) \right). \tag{4.2}$$

Proof. It is an immediate consequence of proposition 4.2.3 and equation (2.15). □

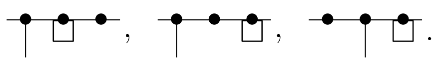
Example 4.2.5. Here is an example for the module $\Delta_3(1)$. The set $MO_3^{|\rangle}(1)$ has the following diagrams which are some of the basis diagrams of $\Delta_3(1)$



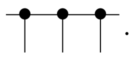
The set $MO_3^{|\rangle}(2)$ consists of the following diagrams:



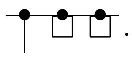
These diagrams correspond to some basis diagrams of $\Delta_3(1)$ respectively which are



Finally, the set $MO_3^{|\rangle}(3)$ consists of a unique diagram



This diagram corresponds to a basis diagram of $\Delta_3(1)$ which is



Therefore, $\dim_R \Delta_3(1) = 9$.

Proposition 4.2.6. *The dimension of the dilute blob algebra db_n is given by*

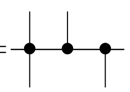
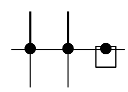
$$\dim_R db_n = \sum_{\lambda=0}^{n} \left(\dim_R \Delta_n(\lambda)\right)^2. \tag{4.3}$$

Proof. This is an immediate result from the algebra db_n being a cellular algebra with the map C as in the proof of proposition 4.1.1 and the basis set of the cell module $\Delta_n(\lambda)$ being the set $dB_n^{|\rangle}(\lambda)$. $\square$

On every cell module $\Delta_n(\lambda)$ there is a unique bilinear form as mentioned in section 2.1. We can simplify the definition of the bilinear form as follows:

Definition 4.2.7. We define the bilinear form $\langle\ ,\ \rangle_\lambda : \Delta_n(\lambda) \times \Delta_n(\lambda) \to R$ as follows. If $x, y \in dB_n^{|\rangle}(\lambda)$, then $\langle x, y\rangle_\lambda$ is defined to be the mirror image of x by a horizontal reflection, and gluing it on the top of y. If there is a defect in x that does not connect with a defect in y, then $\langle x, y\rangle_\lambda = 0$. Otherwise, $\langle x, y\rangle_\lambda = p^\alpha q^\beta r^\gamma s^\delta$. We then extend $\langle\ ,\ \rangle_\lambda$ bilinearly to all of $\Delta_n(\lambda)$.

Example 4.2.8. Consider u_1, u_2 and u_4 in example 4.2.2, we have

$\langle u_1, u_2\rangle_2 =$  $= 0, \quad \langle u_1, u_4\rangle_2 =$  $= r.$

Definition 4.2.9. For $\lambda = \{0, 1, \cdots, n\}$ define $Y_n(\lambda)$ to be the subset of $dB_n^{|\rangle}(\lambda)$ that has half diagrams having precisely λ defects and $n - \lambda$ isolated vertices.

Note that the R-submodule $\operatorname{span}_R Y_n(\lambda)$ is not a db_n-submodule.

Proposition 4.2.10. *For an invertible $q \in R$, $\Delta_n(\lambda)$ is cyclic with any non-zero element of $Y_n(\lambda)$ being a generator.*

Proof. Let y be a half diagram in $Y_n(\lambda)$, then $\langle y, y\rangle_\lambda = q^{n-\lambda} = \zeta$. For any element $b \in \Delta_n(\lambda)$, then $b = \sum_{i=1}^{m} a_i x_i$ for $a_i \in R$ and $x_i \in dB_n^{|\rangle}(\lambda)$. Consider the following equation:

$$\left(\sum_{i=1}^{m} a_i C^{\lambda}_{x_i \overline{y}}\right) y = \sum_{i=1}^{m} a_i \langle y, y\rangle_\lambda x_i = \zeta b.$$

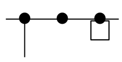
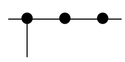
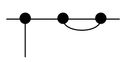
Therefore, $\dfrac{1}{\zeta}\sum_{i=1}^{m} a_i C^{\lambda}_{x_i\overline{y}} y = b$. Thus, y generates $\Delta_n(\lambda)$. $\square$

In general, we have

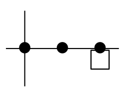
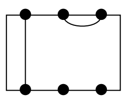
Proposition 4.2.11. *A half diagram a is a generator of $\Delta_n(\lambda)$ if there exists a half diagram $b \in \Delta_n(\lambda)$ such that $\langle a, b\rangle_\lambda$ is an invertible in R.*

Proof. Let b be a half diagram in $\Delta_n(\lambda)$ such that $\langle b, a\rangle_\lambda = \zeta$ is an invertible in R. For any half diagram $x \in \Delta_n(\lambda)$, both x and b have the same number of defects and therefore we may consider $C^{\lambda}_{x\bar{b}}$. Hence, $\dfrac{1}{\zeta} C^{\lambda}_{x\bar{b}} a = x$ and this proves that $(db_n)a = \Delta_n(\lambda)$. $\square$

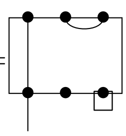
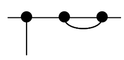
Example 4.2.12. Consider the cell module $\Delta_3(1)$.

Let $a =$  , $b =$  and $x =$  .

Then

$\langle b, a\rangle_1 =$  $= qr$, $C^1_{x\bar{b}} =$  .

Then

$\frac{1}{qr} C^1_{x\bar{b}} \cdot a =$  $= \frac{1}{qr}$  $= x$

Then, if $q, r \in R$ are invertible then a generates $\Delta_3(1)$.

Chapter 5

Generic semisimplicity of $db_n(p, q, r, s)$

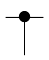
Theorem 2.1.9 gives us a technique to know when the cell modules of the dilute blob algebra are simple. The technique is calculating the Gram matrix $G_n(\lambda)$ for the bilinear form $\langle\ ,\ \rangle_\lambda$ for the corresponding cell module $\Delta_n(\lambda)$, and then we find when the determinant of $G_n(\lambda)$ ($\det(G_n(\lambda))$) is non-zero. Therefore, as stated in the theorem 2.1.9 the cell module $\Delta_n(\lambda)$ is simple if and only $\det(G_n(\lambda)) \neq 0$. In this chapter we study the Gram matrix $G_n(\lambda)$ of the cell module $\Delta_n(\lambda)$ for $\lambda = 0, \ldots, n$ when $n = 1, 2$ to identify when the module $\Delta_n(\lambda)$ is simple. We then show that for $n > 1$ the module $\Delta_n(n-1)$ is simple if and only $q \neq 0$ and $qs \neq r^2$. We end this chapter by proving that the module $\Delta_n(\lambda)$ is generically simple over the complex field which implies that the dilute blob algebra is generically semisimple over the complex field.

5.1 Gram matrix

Throughout this section, we assume that F is a field and $R = F$. Let $\Delta_n(\lambda)$ be the cell module with basis $dB_n^{|\rangle}(\lambda)$ as given in definition 3.1.20. Recall the definition of the bilinear form $\langle\ ,\ \rangle_\lambda$ from definition 4.2.7 and the definition of the radical of the bilinear form on cell module from definition 2.1.5. Let $G_n(\lambda)$ be the Gram matrix of the inner product $\langle\ ,\ \rangle_\lambda$ on the cell module $\Delta_n(\lambda)$ as given in section 2.1.

For $n \geq 1$ the module $\Delta_n(n)$ is spanned by the top half diagram $x \in dB_n^{|\rangle}(n)$ consisting of n defects, and then it has dimension 1. Then $\Delta_n(n)$ is simple for $n \geqslant 1$.

Now we discuss when the cell module is simple in the following examples:
Take $n = 1$. The algebra db_1 is 5-dimensional. The corresponding cell modules are

$\Delta_1(1) = \langle$  $\rangle$

$\Delta_1(0) = \langle a =$, b=  $\rangle$

and therefore the Gram matrix of $\Delta_1(0)$ with respect to the previous basis with the same order is

$$G_1(0) = \begin{pmatrix} q & r \\ r & s \end{pmatrix}$$

and $\det(G_1(0)) = qs - r^2$. Thus the module $\Delta_1(0)$ is simple if and only if $qs \neq r^2$. Let us find $\operatorname{rad}\Delta_1(0)$

$$\operatorname{rad}\Delta_1(0) = \{x \in \Delta_1(0) : \langle x, y\rangle_0 = 0 \quad \text{for all} \quad y \in \Delta_1(0)\}.$$

Let x and y be elements in $\Delta_1(0)$. Therefore, we can write x and y as linear combinations of the basis elements of $\Delta_1(0)$. Let $a_i, b_i \in F$ for $i = 1, 2$ and

$$x = a_1 a + a_2 b,$$
$$y = b_1 a + b_2 b.$$

Suppose that $\langle x, y\rangle_0 = 0$ for all $y \in \Delta_1(0)$. By using $G_1(0)$ we have the following equations:

$$qa_1b_1 + ra_1b_2 + ra_2b_1 + sa_2b_2 = 0.$$

This equation can be written as

$$(qa_1 + ra_2)b_1 + (ra_1 + sa_2)b_2 = 0.$$

The above equation is for all values of b_1 and b_2 . Thus, we can write

$$qa_1 + ra_2 = 0, \tag{5.1}$$

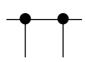
$$ra_1 + sa_2 = 0. \tag{5.2}$$

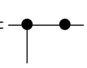
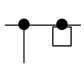
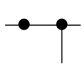
We get non-zero solutions to the above equations if $\det G_1(0) = 0$. This implies $qs - r^2 = 0$. If $qs \neq r^2$, then $x = 0$, which implies that $\operatorname{rad} \Delta_1(0) = \{0\}$.

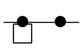
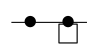
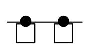
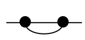
When $qs = r^2$ the equations (5.1) and (5.2) satisfy that $a_1 = -\frac{r}{q}a_2$. Therefore,

$$\operatorname{rad} \Delta_1(0) = F\langle -\tfrac{r}{q}a + b \rangle.$$

In the case $n = 2$, the algebra db_2 is 35-dimensional. We have three cell modules $\Delta_2(2)$, $\Delta_2(1)$ and $\Delta_2(0)$ as follows:

$\Delta_2(2) = \langle$  $\rangle$,

$\Delta_2(1) = \langle a =$  , b=  , $c =$  $\rangle$, and

$\Delta_2(0) = \langle a =$, b=  , $c =$  , $d =$  , $e =$  $\rangle$.

The Gram matrix of $\Delta_2(1)$ with respect to the previous basis with the same order is

$$G_2(1) = \begin{pmatrix} q & r & 0 \\ r & s & 0 \\ 0 & 0 & q \end{pmatrix}$$

and $\det G_2(1) = q(qs - r^2)$. Thus the module $\Delta_2(1)$ is simple unless $q(qs - r^2) = 0$. Let us find $\operatorname{rad} \Delta_2(1)$

$$\operatorname{rad} \Delta_2(1) = \{x \in \Delta_2(1) : \langle x, y\rangle_1 = 0 \quad \text{for all} \quad y \in \Delta_2(1)\}.$$

Let x and y be elements in $\Delta_2(1)$. Therefore, we can write x and y as linear combinations of the basis elements of $\Delta_2(1)$. Let $a_i, b_i \in F$ for $i = 1, 2, 3$ and

$$x = a_1 a + a_2 b + a_3 c,$$
$$y = b_1 a + b_2 b + b_3 c.$$

Suppose that $\langle x, y\rangle_1 = 0$ for all $y \in \Delta_2(1)$. By using $G_2(1)$ we have the following equations:

$$qa_1b_1 + ra_1b_2 + ra_2b_1 + sa_2b_2 + qa_3b_3 = 0.$$

This equation can be written as

$$(qa_1 + ra_2)b_1 + (ra_1 + sa_2)b_2 + qa_3b_3 = 0.$$

The above equation is for all values of b_1, b_2 and b_3. Thus, we can write

$$qa_1 + ra_2 = 0, \tag{5.3}$$
$$ra_1 + sa_2 = 0, \tag{5.4}$$
$$qa_3 = 0. \tag{5.5}$$

We get non-zero solutions to the above equations if $\det G_2(1) = 0$. This implies that $q(qs - r^2) = 0$. If $q \neq 0$ and $qs \neq r^2$, then $x = 0$, which implies $\operatorname{rad} \Delta_2(1) = \{0\}$.

When $q = 0$ and $qs \neq r^2$ the equations (5.3), (5.4) and (5.5) satisfy $a_1 = a_2 = 0$ and a_3 arbitrary. This implies that

$$\operatorname{rad} \Delta_2(1) = F\langle c\rangle,$$

which is a one dimensional space.
When $qs = r^2$ and $q \neq 0$ the equations (5.3), (5.4) and (5.5) satisfy $a_3 = 0$ and $a_1 = -\frac{r}{q}a_2$. Therefore,

$$\operatorname{rad} \Delta_2(1) = F\langle -\tfrac{r}{q}a + b\rangle.$$

Also, the Gram matrix of the module $\Delta_2(0)$ with respect to the previous basis with the same order is

$$G_2(0) = \begin{pmatrix} q^2 & qr & qr & r^2 & q \\ qr & qs & r^2 & rs & r \\ qr & r^2 & qs & rs & r \\ r^2 & rs & rs & s^2 & s \\ q & r & r & s & p \end{pmatrix}$$

and $\det G_2(0) = q^2(r^2 - qs)^4(p-2)$. Thus the module $\Delta_2(0)$ is simple unless $q^2(r^2 - qs)^4(p-2) = 0$.

From the last example we can recognise that it is not easy to calculate the determinant of $G_n(\lambda)$. Therefore, it is not easy to know in which value of the parameters p, q, r and s that make the cell module $\Delta_n(\lambda)$ simple for high rank n and $\lambda = 0, 1, \ldots, n-2$.

Proposition 5.1.1. *For $n > 1$ the cell module $\Delta_n(n-1)$ is simple if and only if $q \neq 0$ and $qs \neq r^2$.*

Proof. Order the basis elements of $\Delta_n(n-1)$ by letting the first diagram consists of defects from the vertex 1 to the vertex $n-1$ and an isolated vertex in the vertex n. The second diagram is similar to the first one with put in the vertex n a decorated vertex. Other diagrams will consist of $n-1$ defects and an isolated vertex, we do not need to order them. Therefore, we have

$$G_n(n-1) = \left(\begin{array}{c|c} G_1(0) & 0 \\ \hline 0 & q\mathbf{1}_{n-1} \end{array}\right)$$

where $\mathbf{1}_{n-1}$ the $n-1 \times n-1$ identity matrix. Hence the module $\Delta_n(n-1)$ is simple if and only if $qs \neq r^2$ and $q \neq 0$. □

5.2 Generic semisimplicity of the algebra $db_n(p, q, r, s)$ over the complex field

We will use the notion *generically* to mean that the property holds on an Zariski open dense subset of parameter space, as defined in [5, Section 1]. We will prove that the property holds for all but finite number of parameters.

Our strategy to prove that the cell module $\Delta_n(\lambda)$ is generically simple is proving that the leading term of the $\det(G_n(\lambda)$ is not identically zero which implies that $\det(G_n(\lambda)) \neq 0$.

Theorem 5.2.1. *For $\lambda \in \{0, 1, \ldots, n\}$, the module $\Delta_n(\lambda)$ is generically simple over the complex field.*

Proof. By computing $G_n(\lambda)$ for $\lambda \in \{0, 1, \ldots, n\}$, we have $\det(G_n(\lambda))$ is a polynomial P in the parameters considered as indeterminates. We need to show that P is not identically zero. Now for any order of the basis of the cell module $\Delta_n(\lambda)$, consider the diagonal entries of $G_n(\lambda)$ that are $\langle a, a\rangle_\lambda$ where a is a basis diagram of $\Delta_n(\lambda)$. If a is a diagram only consisting of $n-\lambda$ isolated vertices, then $\langle a, a\rangle_\lambda$ is $q^{n-\lambda}$ which is the maximal power of q that can occur in the Gram matrix and it occurs only once. Similarly, if a only consists of $n-\lambda$ decorated vertices, then we have $s^{n-\lambda}$ in the diagonal which is the maximal power of s that occurs in the Gram matrix and it occurs only once. If a only has isolated vertices and decorated vertices, then $q^\beta s^\delta$ where $\beta + \delta = n - \lambda$ which is maximal total degree of a monomial in q and s and occurs only once in that row or column.

Now, we consider how the loop occurs on the diagonal. Consider the following cases:

1. Suppose that $n-\lambda$ even , so the maximal number of loops is $\frac{n-\lambda}{2}$. To achieve $\frac{n-\lambda}{2}$ loops we need the $\frac{n-\lambda}{2}$ arcs to match up exactly. i.e. this occurs when we have a half diagram with $\frac{n-\lambda}{2}$ arcs and we take the inner product with itself. Thus the maximal power of p (namely $\frac{n-\lambda}{2}$) occurs on the diagonal, and only on the diagonal. Similarly, the maximal total degree for $p^\alpha q^\beta s^\delta$ occurs on the diagonal and only on the diagonal with $2\alpha+\beta+\delta = n-\lambda$.

2. Suppose that $n-\lambda$ odd. Then $\frac{n-\lambda-1}{2}$ is the maximal number of loops. We similarly get that $p^\alpha q^\beta s^\delta$ has maximal total degree on the diagonal with $2\alpha+\beta+\delta = n-\lambda$.

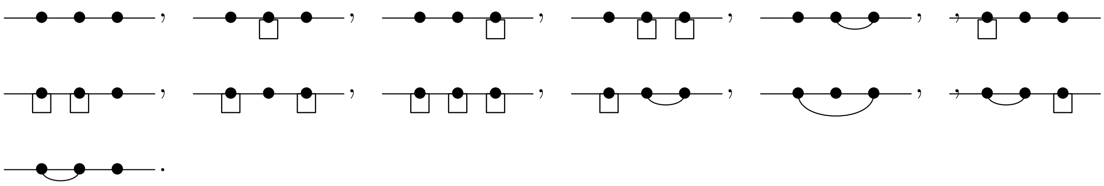
Therefore, every entry on the diagonal has maximal total degree for that row and uniquely for that row. The product of the diagonal entries gives the leading term of the polynomial P, this shows that P is not identically zero. So for a fixed n, there is a finite set of values for this polynomial p to be zero. Thus the set of parameters which give non-zero polynomial p has finite complement and hence is dense in parameter space. Therefore, $\det(G_n(\lambda))$ is generically non-zero. □

Corollary 5.2.2. *The dilute blob algebra db_n is generically semisimple over the complex field.*

Proof. It is an immediate consequence from theorem 5.2.1 and applying theorem 2.1.9. □

To see that the diagonal of $G_n(\lambda)$ consists of maximal entries as explained above see the following example:

Example 5.2.3. Consider $n = 3$ and the cell module $\Delta_3(0)$ is spanned by the diagrams:



The Gram matrix of the module $\Delta_3(0)$ with respect to the illustrated above basis with the same order is

$$G_3(0) = \begin{pmatrix}
q^3 & q^2r & q^2r & qr^2 & q^2 & q^2r & qr^2 & qr^2 & r^3 & qr & q^2 & qr & q^2 \\
q^2r & q^2s & qr^2 & qrs & qr & qr^2 & qrs & r^3 & r^2s & r^2 & qr & r^2 & qr \\
q^2r & qr^2 & q^2s & qrs & qr & qr^2 & r^3 & qrs & r^2s & r^2 & qr & qs & qr \\
qr^2 & qrs & qrs & qs^2 & qs & r^3 & r^2s & r^2s & rs^2 & rs & r^2 & rs & r^2 \\
q^2 & qr & qr & qs & qp & qr & r^2 & r^2 & rs & pr & q & r & q \\
q^2r & qr^2 & qr^2 & r^3 & qr & q^2s & qrs & qrs & r^2s & qs & qr & r^2 & qr \\
qr^2 & qrs & r^3 & r^2s & r^2 & qrs & qs^2 & r^2s & rs^2 & rs & r^2 & rs & qs \\
qr^2 & r^3 & qrs & r^2s & r^2 & qrs & r^2s & qs^2 & rs^2 & rs & qs & rs & r^2 \\
r^3 & r^2s & r^2s & rs^2 & rs & r^2s & rs^2 & rs^2 & s^3 & s^2 & rs & s^2 & rs \\
qr & r^2 & r^2 & rs & pr & qs & rs & rs & s^2 & ps & r & s & r \\
q^2 & qr & qr & r^2 & q & qr & r^2 & qs & rs & r & pq & r & q \\
qr & r^2 & qs & rs & r & r^2 & rs & rs & s^2 & s & r & ps & pr \\
q^2 & qr & qr & r^2 & q & qr & qs & r^2 & rs & r & q & pr & pq
\end{pmatrix}$$

If you look to the entries of the diagonal of $\det(G_3(0))$, you will recognise for example that 3 is the unique maximal power of q which is also maximal in the first row and first column.

Also, the entry q^2s has a total degree equal 3 which is uniquely and maximal total degree for q^2s in the second row and second column, in the third row and third column and in the sixth row and sixth column.

Chapter 6

On quasi-heredity for $db_n(p, q, r, s)$

In this chapter we prove that the dilute blob algebra db_n over a field F is quasi-hereditary when the parameter $q \in F$ is non-zero. Axiom (2) of the towers of recollement studied in [6] gives us some machinery to prove that the dilute blob algebra is quasi-hereditary. We will only recall the required axioms.

Let F be a algebraically closed field. For $n \geq 0$ let A_n be a family of finite dimensional F-algebras, with idempotent e_n in A_n. We recall the following axioms from [6, axiom (A1), axiom (A2)].

Axiom (A1). For each $n \geq 2$ we have an isomorphism

$$\phi_n \colon A_{n-2} \to e_n A_n e_n.$$

Set $e_{n,0} = 1$ in A_n, and for $1 \leq i \leq \frac{n}{2}$ define new idempotents in A_n by setting $e_{n,i} = \phi_n(e_{n-2,i-1})$. To these elements we associate corresponding quotients of A_n by setting $A_{n,i} = A_n/(A_n e_{n,i+1} A_n)$.

Axiom (A2).

(i) The algebra $A_n/A_n e_n A_n$ is semisimple.

(ii) For $n \geq 0$ and $0 \leq i \leq \frac{n}{2}$, setting $e = e_{n,i}$ and $A = A_{n,i}$, the surjective multiplication map $Ae \otimes_{eAe} eA \to AeA$ is a bijection.

By [8, Statement 7] or [24, Definition 3.3.1 and remarks following] the following Axiom is equivalent to Axiom (A2).

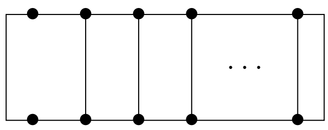
Axiom (A2'). For each $n \in \mathbb{N}$ the algebra A_n is quasi-hereditary, with heredity chain of the form

$$0 \subset \cdots \subset A_n e_{n,i} A_n \subset \cdots \subset A_n e_{n,0} A_n = A_n.$$

6.1 An idempotent subalgebra of the dilute blob algebra

In this section, we construct an idempotent e_n of the dilute blob algebra and use this to construct an isomorphism between the rank $n-1$ dilute blob algebra and the algebra $e_n db_n e_n$.

Fix a non-zero element $q \in F$ and define an idempotent in db_n

$$e_n = \frac{1}{q}$$

.

Proposition 6.1.1. *For $q \neq 0$ then for each $n > 1$ we have an algebra isomorphism*

$$\phi_n \colon e_n db_n e_n \to db_{n-1}. \tag{6.1}$$

Proof. Since the structure of the diagram e_n consists of isolated vertices at vertex 1 and vertex $1'$, and the rest of the diagram as the structure of $\mathbb{I}_{n-1}$, so for any diagram $D \in db_n$, the multiplication diagram $e_n D e_n$ has isolated vertices at vertex 1 and vertex $1'$, and the rest of the diagram is a diagram D' that is in db_{n-1}. For $n > 1$ define a map

$$\phi_n \colon e_n db_n e_n \to db_{n-1}$$

that sends a diagram $e_nDe_n \in e_ndb_ne_n$ to a diagram $D' \in db_{n-1}$ multiplied by q, obtained by removing the isolated vertices 1 and $1'$ in the diagram e_nDe_n, as illustrated in figure 6.1.

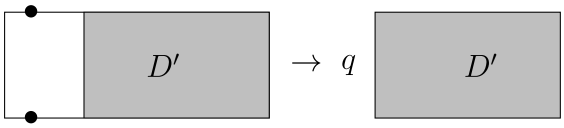


FIGURE 6.1: $\phi_n(e_nDe_n) \in db_{n-1}$ for $e_nDe_n \in e_ndb_ne_n$.

Then we can extend by linearity to any elements of $e_ndb_ne_n$.
Note that $\phi_n(e_n) = \mathbb{I}_{n-1}$ where e_n is the identity for $e_ndb_ne_n$.

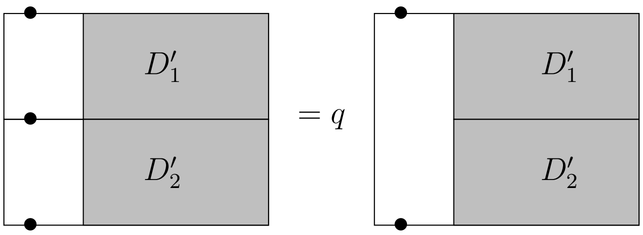


FIGURE 6.2: The diagram $(e_nD_1e_n)(e_nD_2e_n)$.

Now from figure 6.2, we have for any $D_1, D_2 \in dB_n$

$$\phi_n\left((e_nD_1e_n)(e_nD_2e_n)\right) = q^2D_1'D_2'$$

where $D_1', D_2' \in dB_{n-1}$ as shown in figure 6.2.
On the other hand, we know that

$$\phi_n(e_nD_1e_n) = qD_1' \qquad \text{and} \qquad \phi_n(e_nD_2e_n) = qD_2'.$$

Hence,

$$\phi_n(e_nD_1e_n)\phi_n(e_nD_2e_n) = q^2D_1'D_2'.$$

So, we have

$$\phi_n\left((e_nD_1e_n)(e_nD_2e_n)\right) = \phi_n(e_nD_1e_n)\phi_n(e_nD_1e_n).$$

Then the map ϕ_n is a homomorphism.

Conversely, for $n > 1$ define a map

$$\iota_n : db_{n-1} \to e_n db_n e_n$$

That sends a diagram $D' \in db_{n-1}$ to a diagram $\frac{1}{q}e_nDe_n \in e_ndb_ne_n$, obtained by adding an isolated vertex at the top row and bottom row at the left hand side of the diagram D' as illustrated in figure 6.3.

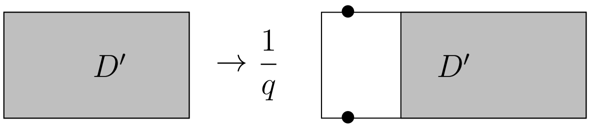


FIGURE 6.3: $\iota_n(D') \in e_ndb_ne_n$ for $D' \in db_{n-1}$.

We can extend by linearity to all of db_{n-1}.

Note that $\iota_n(\mathbb{I}_{n-1}) = e_n$.

By looking to the figure 6.2, we have

$$\iota_n(D_1'D_2') = \frac{1}{q}\frac{1}{q}(e_nD_1e_n)(e_nD_2e_n) = \frac{1}{q^2}(e_nD_1e_n)(e_nD_2e_n).$$

Also,

$$\iota_n(D_1') = \frac{1}{q}e_nD_1e_n, \text{ and } \iota_n(D_2') = \frac{1}{q}e_nD_2e_n.$$

Therefore,

$$\iota_n(D_1')\iota_n(D_2') = \frac{1}{q^2}e_nD_1e_ne_nD_2e_n.$$

So, we have

$$\iota_n(D_1'D_2') = \iota_n(D_1')\iota_n(D_2'),$$

hence the map ι_n is a homomorphism.

It is clear that the homomorphisms ϕ_n and ι_n are inverses of each other. Hence, they are isomorphisms. □

6.2 The dilute blob algebra $db_n(p, qr, s)$ is quasi-hereditary

Definition 6.2.1. Consider the homomorphism ι_n defined in the previous section, set $e_{n,0} = \mathbb{I}_n$ in db_n, and for $1 \leqslant i \leqslant n$ define new idempotents in db_n by setting

$$e_{n,i} = \iota_n(e_{n-1,i-1}).$$

To these elements we associate corresponding quotients of db_n by setting

$$db_{n,i} = db_n/(db_n e_{n,i+1} db_n).$$

Now recall from corollary 3.1.18 the ideal I_λ that is spanned by all dilute blob diagrams with λ or less propagating lines.

Lemma 6.2.2. *For $n \geqslant 1$ we have*

$$db_n e_{n,i} db_n = I_{n-i}. \tag{6.2}$$

Proof. It is clear that $db_n e_{n,i} db_n$ is a two sided ideal generated by $e_{n,i}$. Since multiplication in the dilute blob algebra cannot increase the propagating number as proved in lemma 3.1.17, and $e_{n,i}$ has $n - i$ propagating lines. Then $db_n e_{n,i} db_n$ consists of dilute blob diagrams with $n - i$ or less propagating lines.

Let x be a dilute blob diagram which has $n - i$ or less propagating lines. We will show that $x \in db_n e_{n,i} db_n$ as follows:

(i) Let x have $n - i$ propagating lines. We take the diagram D to be the diagram with top half the same as the top half of x and bottom half the same as the

bottom half of $e_{n,i}$, and the diagram D' to be diagrams with top half the same as the top half of $e_{n,i}$, and bottom half the same as the bottom half of x.
Then $x = \frac{1}{q^{2i}} De_{n,i}D' \in db_n e_{n,i} db_n$.

(ii) Let x have $0 \leq m < n-i$ propagating lines. We take D to be the diagram with top half the same as the top half of x and bottom half with m propagating lines, the first i vertices being isolated vertices and the next m vertices with propagating lines and the remaining $n-m-i$ vertices being isolated. We take D' to be the diagram with bottom half the same as the bottom half of x and top half is the mirror image of the bottom half of D.
Then $x = \frac{1}{q^{n-m+i}} De_{n,i}D' \in db_n e_{n,i} db_n$.

Therefore, $db_n e_{n,i} db_n$ is the ideal that is spanned by dilute blob diagrams with $n-i$ or less propagating lines, and hence $db_n e_{n,i} db_n = I_{n-i}$. □

Proposition 6.2.3. *For $n \geqslant 1$ the algebra $db_n/db_n e_n db_n$ is semisimple.*

Proof. By lemma 6.2.2 we have in the quotient $db_n/db_n e_n db_n$ that all dilute blob diagrams with $n-1$ or less propagating lines are identified with 0. Hence, the quotient algebra is spanned by the unique diagram containing only propagating lines which is the identity element $\mathbb{I}_n$. So we can write

$$db_n/db_n e_n db_n = \mathrm{span}_F\{\mathbb{I}_n\}.$$

Therefore, $db_n/db_n e_n db_n$ is a semisimple algebra. □

Proposition 6.2.4. *For $n \geqslant 0$ and $0 \leqslant i \leqslant n$, the surjective multiplication map $db_{n,i}e_{n,i} \otimes_{e_{n,i}db_{n,i}e_{n,i}} e_{n,i}db_{n,i} \to db_{n,i}e_{n,i}db_{n,i}$ is a bijection.*

Proof. The idempotent element $e_{n,i}$ has $n-i$ propagating lines for all $0 \leq i \leq n$. Therefore, for each $e_{n,i+1}$ by lemma 6.2.2 the ideal $db_n e_{n,i+1} db_n$ of db_n consists of all dilute blob diagrams with $n-(i+1)$ propagating lines or less. It follows immediately from this that in the quotient algebra $db_n/(db_n e_{n,i+1} db_n) = db_{n,i}$ all dilute blob diagrams with $n-(i+1)$ propagating lines or less are identified with 0. Thus, the quotient $db_{n,i}$ has a basis indexed by dilute blob diagrams with at least $n-i$ propagating lines. This implies that $e_{n,i}db_{n,i}e_{n,i}$ is the algebra that has basis consisting of dilute blob diagrams that have exactly $n-i$ propagating lines, where the first i vertices on the top row and bottom row on the left side of the basis diagrams are isolated vertices and the rest of the diagram has $n-i$ propagating lines. This means that the basis of algebra $e_{n,i}db_{n,i}e_{n,i}$ consists of one element which is $e_{n,i}$, so we can write

$$e_{n,i}db_{n,i}e_{n,i} = \mathrm{span}_F\{e_{n,i}\},$$

and therefore $e_{n,i}db_{n,i}e_{n,i}$ is a semisimple algebra.

Since $\dim_F e_{n,i}db_{n,i}e_{n,i} = 1$, then $e_{n,i}db_{n,i}e_{n,i} \cong F$.

For each $0 \leq i \leq n$, the subquotient $db_{n,i}e_{n,i}$ has a basis consisting of dilute blob diagrams with exactly $n-i$ propagating lines, where the first i vertices on the left side of the bottom row are isolated vertices. Thus $db_{n,i}e_{n,i}$ is the left cell module $\Delta_n(n-i)$.

Similarly, The subquotient $e_{n,i}db_{n,i}$ has a basis consisting of dilute blob diagrams with exactly $n-i$ propagating lines , where the first i vertices on the left side of the top row are isolated vertices. Thus $e_{n,i}db_{n,i} = *(\Delta_n(n-i))$ (right cell module) where $*$ as defined in equation (4.1), and therefore $\dim_F e_{n,i}db_{n,i} = \dim_F db_{n,i}e_{n,i}$.

Therefore, we have $\dim_F db_{n,i}e_{n,i} \otimes_{e_{n,i}db_{n,i}e_{n,i}} e_{n,i}db_{n,i} = (\dim_F \Delta_n(n-i))^2$.

The algebra $db_{n,i}e_{n,i}db_{n,i}$ has a basis consisting of dilute blob diagrams with exactly $n-i$ propagating lines. We know from equation (3.10) that every dilute blob diagram with $n-i$ propagating lines can construct uniquely from two half dilute blob diagrams that have $n-i$ defects. Therefore, $\dim_F db_{n,i}e_{n,i}db_{n,i} = (\dim_F \Delta_n(n-i)^2$.

Therefore, $\dim_F db_{n,i}e_{n,i} \otimes_{e_{n,i}db_{n,i}e_{n,i}} e_{n,i}db_{n,i} = \dim_F db_{n,i}e_{n,i}db_{n,i}$.

Define the multiplication map

$$\Psi : db_{n,i}e_{n,i} \otimes_{e_{n,i}db_{n,i}e_{n,i}} e_{n,i}db_{n,i} \to db_{n,i}e_{n,i}db_{n,i}$$
$$ae_{n,i} \otimes e_{n,i}a' \to ae_{n,i}a'$$

for any basis element $ae_{n,i}$ of $db_{n,i}e_{n,i}$ and any basis element $e_{n,i}a'$ of $e_{n,i}db_{n,i}$.
Since for any basis element $ae_{n,i}a' \in db_{n,i}e_{n,i}db_{n,i}$, the element $ae_{n,i} \otimes e_{n,i}a'$ is a basis element of $db_{n,i}e_{n,i} \otimes e_{n,i}db_{n,i}$, Ψ is surjective.
Since $\dim_F db_{n,i}e_{n,i} \otimes_{e_{n,i}db_{n,i}e_{n,i}} e_{n,i}db_{n,i} = \dim_F db_{n,i}e_{n,i}db_{n,i}$, then Ψ is injective. Thus Ψ is a bijection. □

Theorem 6.2.5. *For $p, q, r, s \in F$, $q \neq 0$ and $n > 0$, the dilute blob algebra $db_n(p, q, r, s)$ is quasi-hereditary, with heredity chain of the form*

$$0 \subset I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_n = db_n. \tag{6.3}$$

Proof. From propositions 6.2.3 and 6.2.4, we have that the dilute blob algebra db_n satisfies axiom (A2) from the tower of recollement in [6, (A2)], and this axiom is equivalent to the axiom (A2') [6, (A2')] that states that for each $n \geqslant 0$ the algebra db_n is quasi-hereditary, with heredity chain of the form:

$$0 \subset \cdots \subset db_n e_{n,i} db_n \subset \cdots db_n e_{n,0} db_n = db_n$$

from lemma 6.2.2 we have $db_n e_{n,i} db_n = I_{n-i}$. □

Chapter 7

Restriction and Induction for the $db_n(p, q, r, s)$ cell modules

This chapter is devoted to studying restriction and induction for the cell modules $\Delta_n(\lambda)$. The first step, for the study of the restriction, is to decide how the subalgebra db_n is embedded into db_{n+1}. The embedding that we shall use is defined by adding a propagating line at the front of the db_n-diagrams. This natural embedding gives a tower of algebras

$$db_1 \subset db_2 \subset db_3 \subset \cdots \tag{7.1}$$

The module $\Delta_n(\lambda)$ seen as a db_{n-1}-module will be called the restriction of $\Delta_n(\lambda)$ and denoted by $\Delta_n(\lambda) \downarrow$. We end the first section by drawing the Bratteli diagram for the restriction rule for $n \leq 4$.

After studying restriction, we then work towards induced modules of the cell modules $\Delta_n(\lambda)$ denoted by $\Delta_n(\lambda) \uparrow$ which is defined by the tensor product as follows:

$$\Delta_n(\lambda) \uparrow = db_{n+1} \otimes_{db_n} \Delta_n(\lambda). \tag{7.2}$$

These are db_{n+1}-modules in which the action given by $x(y \otimes_{db_n} d) = (xy) \otimes_{db_n} d$ for all $x, y \in db_{n+1}$ and $d \in \Delta_n(\lambda)$. The subscript db_n on the $\otimes$ means that we have $(xy) \otimes_{db_n} d = x \otimes_{db_n} yd$ for all $x \in db_{n+1}$, $y \in db_n$ and $d \in \Delta_n(\lambda)$, we use this to construct a generating set of the module $\Delta_n(\lambda) \uparrow$.

7.1 Restriction for the cell modules of $db_n(p, q, r, s)$

In this section we work towards proving proposition 7.1.2 which describes the restriction rules for the cell modules $\Delta_n(\lambda)$. We begin by defining an inclusion map from db_{n-1} into db_n.

Lemma 7.1.1. *For all $n \geq 2$, db_{n-1} can be identified as a subalgebra of db_n.*

Proof. Define an inclusion $i : db_{n-1} \hookrightarrow db_n$ obtained by adding a propagating line immediately to the front of the db_{n-1} diagram and then extend by linearity to all of db_{n-1}. It is clear that the inclusion i sends the identity diagram of db_{n-1} to the identity diagram of db_n, and for any two diagrams x, y in db_{n-1} we have

$$i(xy) = i(x)i(y)$$

as multiplying x in y then adding a propagating line in the front of the diagram xy is the same as adding a propagating line in the front of x and y then multiplying.
Therefore, the inclusion i is a homomorphism.
Also, since different diagrams in db_{n-1} have different images in db_n, then i is injective and therefore db_{n-1} is isomorphic to the subalgebra $\mathrm{im}(i)$ of db_n. □

Proposition 7.1.2. *Consider the above inclusion, denote the corresponding restriction of $\Delta_n(\lambda)$ to a db_{n-1}-module by $\Delta_n(\lambda) \downarrow$. Then, we have for all $n \geq 2$ the following short exact sequences or isomorphisms:*

(i) For $\lambda = \{1, 2, \ldots, n-2\}$

$$0 \to \Delta_{n-1}(\lambda) \oplus \Delta_{n-1}(\lambda - 1) \to \Delta_n(\lambda) \downarrow \to \Delta_{n-1}(\lambda + 1) \to 0. \tag{7.3}$$

(ii) For $\lambda = 0$

$$0 \to \Delta_{n-1}(0) \oplus \Delta_{n-1}(0) \to \Delta_n(0) \downarrow \to \Delta_{n-1}(1) \to 0. \tag{7.4}$$

(iii) For $\lambda = n-1$

$$\Delta_n(n-1) \downarrow \cong \Delta_{n-1}(n-1) \oplus \Delta_{n-1}(n-2). \tag{7.5}$$

(iv) For $\lambda = n$

$$\Delta_n(n) \downarrow \cong \Delta_{n-1}(n-1). \tag{7.6}$$

Proof. (i) Suppose that $1 \leq \lambda \leq n-2$. We will prove the first part of exact sequence (7.3). Define an inclusion $\theta : \Delta_{n-1}(\lambda) \oplus \Delta_{n-1}(\lambda-1) \hookrightarrow \Delta_n(\lambda) \downarrow$ as follows: for a half diagram $d_1 \in \Delta_{n-1}(\lambda)$ and a half diagram $d_2 \in \Delta_{n-1}(\lambda-1)$

$$\theta(d_1, d_2) = \overline{d_1} + \overline{d_2}$$

where $\overline{d_1}$ is the half diagram d_1 with an isolated vertex added to the front, and $\overline{d_2}$ is the half diagram d_2 with a defect added to the front. We then extend linearly to all of $\Delta_{n-1}(\lambda) \oplus \Delta_{n-1}(\lambda-1)$.

It is clear that $\theta(\Delta_{n-1}(\lambda)) \cap \theta(\Delta_{n-1}(\lambda-1)) = 0$ as no half diagram can have both an isolated vertex and a defect at the front. Firstly, to show that θ is a db_{n-1} homomorphism, let a be a diagram in db_{n-1}, d_1 be a half diagram in $\Delta_{n-1}(\lambda)$ and d_2 be a half diagram in $\Delta_{n-1}(\lambda-1)$. Then

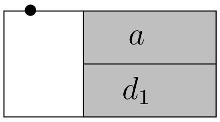
$$\theta(a(d_1, d_2)) = \theta(ad_1, ad_2) = \overline{ad_1} + \overline{ad_2}.$$

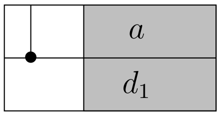
Also,

$$a\theta(d_1, d_2) = a(\overline{d_1} + \overline{d_2}) = i(a)(\overline{d_1} + \overline{d_2}) = i(a)\overline{d_1} + i(a)\overline{d_2}$$

where i is the inclusion as described in proof of lemma 7.1.1.

Now, by considering the diagrams $\overline{ad_1}$ and $i(a)\overline{d_1}$ we have

$\overline{ad_1} =$ 

, $i(a)\overline{d_1} =$ 

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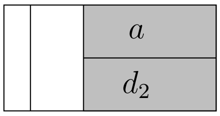
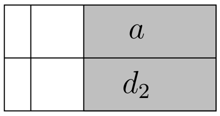
clearly, they are equal.

Also, compare $\overline{ad_2}$ and $i(a)\overline{d_2}$

$\overline{ad_2} =$ 

, $i(a)\overline{d_2} =$ 

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clearly, they are equal. Therefore,

$$\theta(a(d_1, d_2)) = a\theta(d_1, d_2)$$

and so θ is a homomorphism.

Secondly, θ is injective since distinct elements in $\Delta_{n-1}(\lambda) \oplus \Delta_{n-1}(\lambda-1)$ have distinct images in $\Delta_n(\lambda)\downarrow$.

Now, define a map $\Phi : \Delta_n(\lambda)\downarrow \to \Delta_{n-1}(\lambda+1)$ that sends a half diagram in $\Delta_n(\lambda)$ that has a defect or isolated vertex at the vertex 1 to zero in $\Delta_{n-1}(\lambda+1)$. Otherwise, it removes the arc and its endpoint at the vertex 1 and then puts at the other endpoint a defect. We then extend linearly to of all $\Delta_n(\lambda)\downarrow$.

To see that Φ is a homomorphism, suppose a is a diagram in db_{n-1} and x is a half diagram in $\Delta_n(\lambda)\downarrow$. We need to consider the following cases:

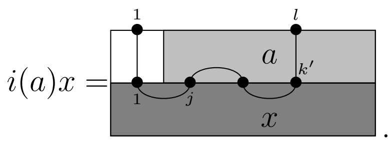
Case (1): x has a defect at vertex 1, then $a\Phi(x) = 0$ as $\Phi(x) = 0$. Also, $i(a)x$ has a defect at vertex 1, then $\Phi(i(a)x) = 0$.

Case (2): x has an isolated vertex at vertex 1, then $a\Phi(x) = 0$ as $\Phi(x) = 0$. Also, $i(a)x$ has an isolated vertex at vertex 1, then $\Phi(i(a)x) = 0$.

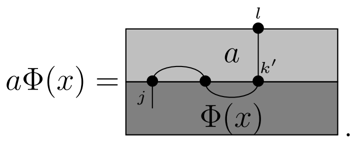
Case (3): x has an arc $\{1, j\}$, and therefore $\Phi(x)$ has a defect at the vertex j. To study this case consider the following possibilities:

(3a) If there is a propagating line $\{l, k'\} \in i(a)$ that connects with the propagating line $\{1, 1'\} \in i(a)$ via a path in $i(a)x$, then $i(a)x$ will have an arc $\{1, l\}$ and $\{1, j\}$

from x that will form part of this path. Therefore, $\Phi(i(a)x)$ will have a defect at the vertex l. As in the following figure:



On the other hand, $a\Phi(x)$ will have a defect at the vertex l since the defect at the vertex j in $\Phi(x)$ will link with the propagating line $\{l, k'\} \in a$ via a path in $a\Phi(x)$ essentially the same path in $i(a)x$. As in the following figure.



Hence, $\Phi(i(a)x) = a\Phi(x)$.

(3b) If there is an isolated vertex $\{c'\} \in i(a)$ or $\{h\} \in x$ connecting with the propagating line $\{1, 1'\} \in i(a)$ via a path in $i(a)x$, then $i(a)x$ will have an isolated vertex at the vertex 1, so $\Phi(i(a)x) = 0$.

On the other hand, $a\Phi(x)$ would then link the defect at the vertex j in $\Phi(x)$ with an isolated vertex via a path in $a\Phi(x)$ and therefore the number of the defects will be fewer than $\lambda + 1$. Hence $a\Phi(x) = 0$ in $\Delta_{n-1}(\lambda + 1)$.

(3c) If there is a decorated vertex $\{c'\}_\square \in i(a)$ or $\{h\}_\square \in x$ connecting with the propagating line $\{1, 1'\} \in i(a)$ via a path in $i(a)x$, then $i(a)x$ will have a decorated vertex at the vertex 1. So the number of defects in $i(a)x$ is zero and therefore $i(a)x = 0$ in $\Delta_n(\lambda)$. Hence, $\Phi(i(a)x) = 0$.

On the other hand, $a\Phi(x)$ will link the defect at the vertex j in $\Phi(x)$ with a decorated vertex via a path in $a\Phi(x)$. So the propagating lines in a can not connect with any defect in $\Phi(x)$. Hence, the number of defects in $a\Phi(x)$ is zero and therefore $a\Phi(x) = 0$ in $\Delta_{n-1}(\lambda + 1)$.

(3d) If there is a defect at the vertex $k > j$ in x connecting with the propagating line $\{1, 1'\} \in i(a)$ via a path in $i(a)x$, then $i(a)x$ will have a defect at the vertex 1, so $\Phi(i(a)x) = 0$.

On the other hand, $a\Phi(x)$ would then link two defects in $\Phi(x)$ via a path in $a\phi(x)$ and therefore the number of the defects will be fewer than $\lambda+1$. Hence $a\Phi(x)=0$ in $\Delta_{n-1}(\lambda+1)$.

Therefore, from the three cases Φ is a homomorphism.

To see that Φ is surjective, let y be a half diagram in $\Delta_{n-1}(\lambda+1)$, then y has at least one defect since $\lambda+1 \geq 1$. Then add on the front of y a vertex and close the first defect on the left hand side of y onto this new vertex. This is then a half diagram in $\Delta_n(\lambda)\downarrow$ whose image by Φ is y.

Clearly, $\operatorname{im}\theta \subset \ker\Phi$. To prove $\ker\Phi \subset \operatorname{im}\theta$, we know that Φ sends every half diagram in $\Delta_n(\lambda)$ that has a defect or an isolated vertex at vertex 1 to zero in $\Delta_{n-1}(\lambda+1)$, otherwise the image of the half diagram is a distinct half diagram in $\Delta_{n-1}(\lambda+1)$. Hence, if $v=\sum_{i=1}^{m}\zeta_i v_i \in \Delta_n(\lambda)\downarrow \setminus \operatorname{im}\theta$ where $\zeta_i \in R$ and $v_i \in \Delta_n(\lambda)\downarrow$. Then there is at least one $j \in \{1,2,\cdots,m\}$ such that v_j has an arc $\{1,k\}$, and therefore $\Phi(v_j)\neq 0$. Hence, $\Phi(v)\neq 0$, so $\ker\Phi \subset \operatorname{im}\theta$. Then, we have $\ker\Phi = \operatorname{im}\theta$.

(ii) Suppose that $\lambda=0$.

Define an inclusion $\theta_0 : \Delta_{n-1}(0)\oplus\Delta_{n-1}(0) \hookrightarrow \Delta_n(0)\downarrow$ as follows: for half diagrams $d_1, d_2 \in \Delta_{n-1}(0)$

$$\theta_0(d_1,d_2)=\overline{d_1}+\overline{d_2}$$

where $\overline{d_1}$ is the half diagram d_1 with an isolated vertex added to the front, and $\overline{d_2}$ is the half diagram d_2 with a decorated vertex added to the front. We then extend linearly to all of $\Delta_{n-1}(0)\oplus\Delta_{n-1}(0)$.

It is clear that $\theta_0(\Delta_{n-1}(0))\cap\theta_0(\Delta_{n-1}(0))=0$ as no half diagram can have both an isolated vertex and a decorated vertex at the front. To show that θ_0 is a db_{n-1} homomorphism, let a be a diagram in db_{n-1}, d_1, d_2 be half diagrams in $\Delta_{n-1}(0)$. Then

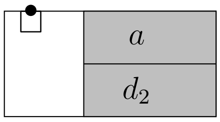
$$\theta_0(a(d_1,d_2))=\theta_0(ad_1,ad_2)=\overline{ad_1}+\overline{ad_2}.$$

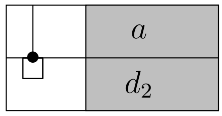
Also,

$$a\theta_0(d_1,d_2) = a(\overline{d_1}+\overline{d_2}) = i(a)(\overline{d_1}+\overline{d_2}) = i(a)\overline{d_1}+i(a)\overline{d_2}$$

where i is the inclusion as described in proof of lemma 7.1.1.

We have $\overline{ad_1} = i(a)\overline{d_1}$ as explained in (i) . Also, compare $\overline{ad_2}$ and $i(a)\overline{d_2}$

$\overline{ad_2} =$ 

, $i(a)\overline{d_2} =$ 

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clearly, they are equal. Therefore,

$$\theta_0(a(d_1,d_2)) = a\theta_0(d_1,d_2)$$

and so θ_0 is a homomorphism.

Moreover, θ_0 is injective since distinct elements in $\Delta_{n-1}(0) \oplus \Delta_{n-1}(0)$ have distinct images in $\Delta_n(0)\downarrow$.

Now, define a map $\Phi_0 : \Delta_n(0)\downarrow \to \Delta_{n-1}(1)$ that sends a half diagram in $\Delta_n(0)$ that has an isolated vertex or a decorated vertex at the vertex 1 to zero in $\Delta_{n-1}(1)$. Otherwise, it removes the arc and its endpoint at the vertex 1 and then puts at the other endpoint a defect. We then extend linearly to all of $\Delta_n(0)\downarrow$.

To see that Φ_0 is a db_{n-1}- homomorphism, suppose a is a diagram in db_{n-1} and x is a half diagram in $\Delta_n(0)\downarrow$. We need to consider the following cases:

Case (1): x has an isolated vertex (respectively decorated vertex) at vertex 1, then $a\Phi_0(x) = 0$ as $\Phi_0(x) = 0$. Also, $i(a)x$ has an isolated vertex (respectively decorated vertex) at vertex 1, then $\Phi_0(i(a)x) = 0$.

Case (2): x has an arc $\{1, j\}$, this case is similar to case (3) in (i) but without the need to do (c) as there are no defects in x.

Therefore, Φ_0 is a homomorphism.

To see that Φ_0 is surjective, let y be a half diagram in $\Delta_{n-1}(1)$, then y has one defect. Then add on the front of y a vertex and close the defect onto this new vertex. This is then a half diagram in $\Delta_n(0)\downarrow$ whose image by Φ_0 is y.

Clearly, $\operatorname{im}\theta_0 \subset \ker\Phi_0$. To prove $\ker\Phi_0 \subset \operatorname{im}\theta_0$, we know that Φ_0 sends every half diagram in $\Delta_n(0)$ that has an isolated vertex or a decorated vertex at vertex 1 to zero in $\Delta_{n-1}(1)$, otherwise the image of the half diagram is a distinct half diagram in $\Delta_{n-1}(1)$. Hence, if $v = \sum_{i=1}^{m} \zeta_i v_i \in \Delta_n(0)\downarrow \setminus \operatorname{im}\theta_0$ where $\zeta_i \in R$ and $v_i \in \Delta_n(0)\downarrow$. Then there is at least one $j \in \{1, 2, \cdots, m\}$ such that v_j has an arc $\{1, k\}$, and therefore $\Phi_0(v_j) \neq 0$. Hence, $\Phi_0(v) \neq 0$, so $\ker \Phi_0 \subset \operatorname{im}\theta_0$. Then, we have $\ker \Phi_0 = \operatorname{im}\theta_0$.

(iii) Suppose that $\lambda = n-1$. Define an inclusion $\theta_{n-1} : \Delta_{n-1}(n-1) \oplus \Delta_{n-1}(n-2) \hookrightarrow \Delta_n(n-1)\downarrow$ as follows: for a half diagram $d_1 \in \Delta_{n-1}(n-1)$ and a half diagram $d_2 \in \Delta_{n-1}(n-2)$

$$\theta_{n-1}(d_1, d_2) = \overline{d_1} + \overline{d_2}$$

where $\overline{d_1}$ is the half diagram d_1 with an isolated vertex added to the front, and $\overline{d_2}$ is the half diagram d_2 with a defect added to the front.

To proof that θ_{n-1} is an injective homomorphism is similar to the proof that θ is an injective homomorphism in (i).

Moreover, θ_{n-1} is surjective. Suppose y is a half diagram in $\Delta_n(n-1)\downarrow$ and therefore y can not have an arc, it has $n-1$ defects and the other vertex is an isolated vertex or a decorated vertex. The half diagram y would have a defect or isolated vertex at vertex 1, and if y has a decorated vertex then this must be at the vertex n. If y has an isolated vertex at the vertex 1, then the other vertices have defects. So by removing the isolated vertex we will have the unique half diagram that has $n-1$ defects in $\Delta_{n-1}(n-1)$ whose image under θ_{n-1} is y. If y has a defect at the vertex 1, then by removing the defect we will have a half diagram in $\Delta_{n-1}(n-2)$ whose image under θ_{n-1} is y.

Therefore θ_{n-1} is surjective and therefore is an isomorphism.

(iv) Suppose that $\lambda = n$. Define an inclusion $\theta_n : \Delta_{n-1}(n-1) \hookrightarrow \Delta_n(n)\downarrow$ that sends a unique half diagram in $\Delta_{n-1}(n-1)$ that has $n-1$ defects to a unique half diagram in $\Delta_n(n)\downarrow$ that has n defects. To show that θ_n is a db_{n-1}-homomorphism, let a be a diagram in db_{n-1} and d be the half diagram in $\Delta_{n-1}(n-1)$ that has $n-1$ defects. Then ad is zero unless a is the identity diagram of db_{n-1}, then $ad = d$. Therefore,

$\theta_n(ad)$ is equal to zero or $\theta_n(d)$.

On the other hand, if a is the identity diagram of db_{n-1} then $i(a)\theta_n(d) = \theta_n(d)$. Otherwise, $i(a)\theta_n(d) = 0$.

Hence, θ_n is a homomorphism.

Since both $\Delta_{n-1}(n-1)$ and $\Delta_n(n)\downarrow$ are one dimensional, so θ_n is an isomorphism. $\square$

Note that the exact sequences gives a relationship between the dimensions of the cell modules:

(i) For $1 \leq \lambda \leq n-2$, we have

$$\dim \Delta_n(\lambda) = \dim \Delta_{n-1}(\lambda) + \dim \Delta_{n-1}(\lambda - 1) + \dim \Delta_{n-1}(\lambda + 1). \tag{7.7}$$

(ii) For $\lambda = 0$, we have

$$\dim \Delta_n(0) = 2 \dim \Delta_{n-1}(0) + \dim \Delta_{n-1}(1). \tag{7.8}$$

(iii) For $\lambda = n-1$

$$\dim \Delta_n(n-1) = \dim \Delta_{n-1}(n-1) + \dim \Delta_{n-1}(n-2). \tag{7.9}$$

We can describe the restriction rule that is studied in proposition 7.1.2 by a *Bratteli diagram* which is a graph with vertices arranged into level $n \in \mathbb{N} \cup \{0\}$ such that each vertex on level n corresponds to the label of the cell module $\Delta_n(\lambda)$ which is the number of defects λ and each edge in this graph corresponds to a factor in the restriction of the module $\Delta_n(\lambda)$ to $n-1$.

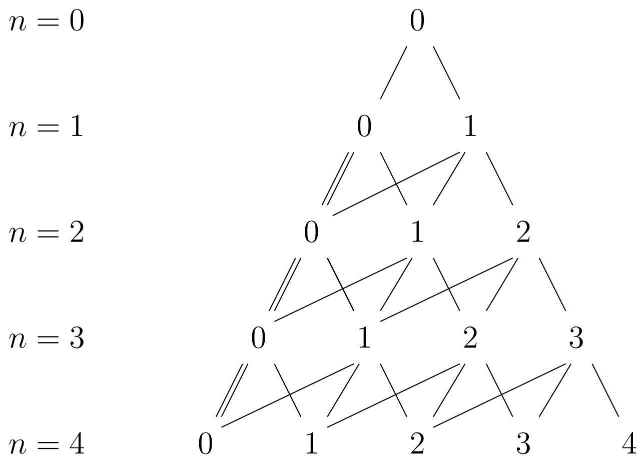


FIGURE 7.1: The Bratteli diagram for the restriction rule for $n \leq 4$.

7.2 Induction of the cell modules of $db_n(p, q, r, s)$

In this section we construct a generating set of the induced modules $\Delta_n(\lambda)\uparrow$. Assume that the parameters p, q, r, s are in invertible in R. Recall the inclusion i from proof of lemma 7.1.1.

Definition 7.2.1. The induced module of $\Delta_n(\lambda)$, denoted by $\Delta_n(\lambda)\uparrow$, is defined by the tensor product

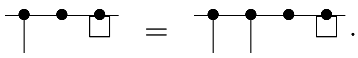
$$\Delta_n(\lambda)\uparrow = db_{n+1} \otimes_{db_n} \Delta_n(\lambda).$$

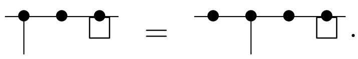
Firstly, we will find a finite generating set for $\Delta_n(\lambda)\uparrow$ where $\lambda \in \{0, 1, \ldots, n\}$. From proposition 4.2.11, we have that $\Delta_n(\lambda)$ is a cyclic module. Let z be a half diagram in $\Delta_n(\lambda)$ such that $\Delta_n(\lambda) = db_n z$, then

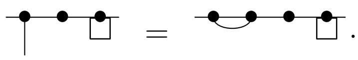
$$\Delta_n(\lambda)\uparrow = db_{n+1} \otimes_{db_n} (db_n z) = db_{n+1} \otimes_{db_n} z. \tag{7.10}$$

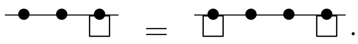
For $j \in \{-1, 0, 1\}$ and $\lambda \in \{0, 1, \ldots, n\}$, define three maps $\theta_j : \Delta_n(\lambda) \to \Delta_{n+1}(\lambda + i)$ such that for any half diagram $d \in \Delta_n(\lambda)$, $\theta_1(d)$ is the half diagram d with a defect added to the front, and $\theta_0(d)$ is the half diagram d with an isolated vertex added to the front. The last map θ_{-1} is the half diagram d with adds a vertex to the front of d

and closes the nearest defect to the new vertex into an arc, if there is no such defect in d then $\theta_{-1}(d) = 0$. Also, we can define $\Phi_0 : \Delta_n(0) \to \Delta_{n+1}(0)$ that sends a half diagram $d \in \Delta_n(0)$ to d with a decorated vertex added to the front of d. Here are some examples.

$\theta_1 :$ 

$\theta_0 :$ 

$\theta_{-1} :$ 

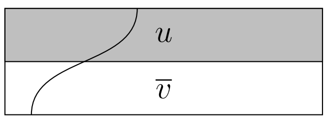
$\Phi_0 :$ 

Let $C^{\mu}_{u\overline{v}}$ be a diagram in dB_{n+1} where $\mu \in \Lambda_{n+1} = \{0, 1, \ldots, n+1\}$, $u \in dB^{|\rangle}_{n+1}(\mu)$ and $\overline{v}$ is bottom mirror image of $v \in dB^{|\rangle}_{n+1}(\mu)$. Then

1. If v has a defect at the vertex 1. Then we can write

$$C^{\mu}_{u\overline{v}} = \frac{1}{\zeta} C^{\mu}_{u\overline{v}} C^{\mu}_{v\overline{v}}$$

where $\zeta = \langle v, v \rangle_{\mu}$ (is the scalar of the product of the isolated components in the product $C^{\mu}_{u\overline{v}} C^{\mu}_{v\overline{v}}$), as pictured below:

$C^{\mu}_{u\overline{v}} =$ 

, $C^{\mu}_{v\overline{v}} =$  .

Let v' be the half diagram obtained from v by deleting the vertex 1 and the defect, so $v = \theta_1(v')$. Then

$$C^{\mu}_{v\overline{v}} = i(C^{\mu-1}_{v'\overline{v'}}).$$

Therefore,

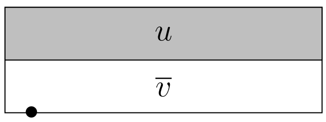
$$\begin{aligned}
C^{\mu}_{u\overline{v}} \otimes z &= \frac{1}{\zeta} C^{\mu}_{u\overline{v}} C^{\mu}_{v\overline{v}} \otimes z \\
&= \frac{1}{\zeta} C^{\mu}_{u\overline{v}} i(C^{\mu-1}_{v'\overline{v'}}) \otimes z \\
&= \frac{1}{\zeta} C^{\mu}_{u\overline{v}} \otimes C^{\mu-1}_{v'\overline{v'}} z.
\end{aligned}$$

This tensor is zero if $C^{\mu-1}_{v'\overline{v'}} z$ is zero. Since v has a defect at the vertex 1, then the element $C^{\mu}_{u\overline{v}} \otimes z$ can only be non-zero if $C^{\mu-1}_{v'\overline{v'}} z$ is non-zero where $v = \theta_1(v')$.

2. If v has an isolated vertex at the vertex 1. Then we can write

$$C^{\mu}_{u\overline{v}} = \frac{1}{\zeta} C^{\mu}_{u\overline{v}} C^{\mu}_{v\overline{v}}$$

where ζ is the scalar of the product of the isolated components in the product $C^{\mu}_{u\overline{v}} C^{\mu}_{v\overline{v}}$, as pictured below

$C^{\mu}_{u\overline{v}} =$ 

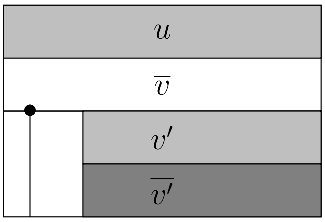
, $C^{\mu}_{v\overline{v}} =$  .

Since v contains an isolated vertex at vertex 1, then the isolated vertex is one of the isolated components in $C^{\mu}_{u\overline{v}} C^{\mu}_{v\overline{v}}$. Therefore, $\zeta = q\eta$ where η is the product of the other isolated components.

Let v' be the half diagram obtained from v by deleting the vertex 1 and the isolated vertex, so $v = \theta_0(v')$. Then

$$C^{\mu}_{u\overline{v}} = \frac{1}{\eta} C^{\mu}_{u\overline{v}} i(C^{\mu}_{v'\overline{v'}}).$$

As in the following figure



.

Therefore,

$$\begin{aligned} C^{\mu}_{u\overline{v}} \otimes z &= \frac{1}{\eta} C^{\mu}_{u\overline{v}} i(C^{\mu}_{v'\overline{v'}}) \otimes z \\ &= \frac{1}{\eta} C^{\mu}_{u\overline{v}} \otimes C^{\mu}_{v'\overline{v'}} z. \end{aligned}$$

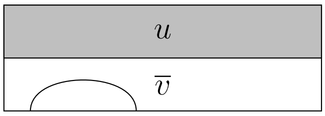
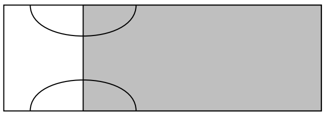
This tensor is zero if $C^{\mu}_{v'\overline{v'}}z$ is zero. Hence, when v has an isolated vertex at the vertex 1, then the element $C^{\mu}_{u\overline{v}} \otimes z$ can only be non-zero if $C^{\mu}_{v'\overline{v'}}z$ is non-zero where $v = \theta_0(v')$.

3. If v has a decorated vertex at the vertex 1 ($\mu = 0$), this case similar to case (2) with small differences which are $\zeta = s\eta$ and $v = \Phi_0(v')$.

4. If v has an arc that links the vertex 1 with the vertex k. Then we can write

$$C^{\mu}_{u\overline{v}} = \frac{1}{\zeta} C^{\mu}_{u\overline{v}} C^{\mu}_{v\overline{v}}$$

where ζ is the scalar of the product of the isolated components in the product $C^{\mu}_{u\overline{v}} C^{\mu}_{v\overline{v}}$. As in the following figure

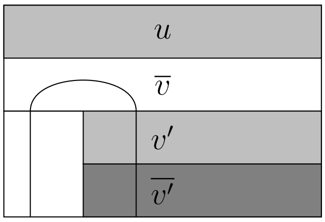
$C^{\mu}_{u\overline{v}} =$  , $C^{\mu}_{v\overline{v}} =$  .

Since v contains an arc ending at vertex 1, then the loop is one of the isolated components in $C^{\mu}_{u\overline{v}} C^{\mu}_{v\overline{v}}$. Therefore, $\zeta = p\eta$ where η is the product of the other isolated components.

Let v' be the half diagram obtained from v by deleting the vertex 1 and putting a defect at vertex k, so $v = \theta_{-1}(v')$. Then

$$C^{\mu}_{u\overline{v}} = \frac{1}{\eta} C^{\mu}_{u\overline{v}} i(C^{\mu+1}_{v'\overline{v'}}).$$

As in the following figure



.

Therefore,

$$\begin{aligned} C^{\mu}_{u\overline{v}} \otimes z &= \frac{1}{\eta} C^{\mu}_{u\overline{v}} i(C^{\mu+1}_{v'\overline{v'}}) \\ &= \frac{1}{\eta} C^{\mu}_{u\overline{v}} \otimes C^{\mu+1}_{v'\overline{v'}} z. \end{aligned}$$

This tensor is zero if $C^{\mu+1}_{v'\overline{v'}} z$ is zero. Hence, when v has an arc ending at vertex 1, then the element $C^{\mu}_{u\overline{v}} \otimes z$ can only be non-zero if $C^{\mu+1}_{v'\overline{v'}} z$ is non-zero where $v = \theta_{-1}(v')$.

Proposition 7.2.2. *Assume that p, q, r, s are invertible in R, $u \in dB^{|\rangle}_{n+1}(\mu)$ and suppose z is a half diagram that generates the cell db_n-module $\Delta_n(\lambda)$. Then for $\lambda \in \{0, 1, 2, \ldots, n\}$*

$$\Delta_n(\lambda) \uparrow = \operatorname{span} G \tag{7.11}$$

where G is the finite set

$$\begin{aligned} G =& \{C^{\mu}_{u\overline{\theta_1(v)}} \otimes z : v \in dB^{|\rangle}_n(\mu - 1) \text{ and } C^{\mu-1}_{v\overline{v}} z \neq 0\} \\ & \cup \{C^{\mu}_{u\overline{\theta_0(v)}} \otimes z : v \in dB^{|\rangle}_n(\mu) \text{ and } C^{\mu}_{v\overline{v}} z \neq 0\} \\ & \cup \{C^{0}_{u\overline{\Phi_0(v)}} \otimes z : v \in dB^{|\rangle}_n(0)\} \\ (\text{if } \lambda \neq 0)\ & \cup \{C^{\mu}_{u\overline{\theta_{-1}(v)}} \otimes z : v \in dB^{|\rangle}_n(\mu + 1) \text{ and } C^{\mu+1}_{v\overline{v}} z \neq 0\}. \end{aligned}$$

7.3 Future work

In this section we make some suggestions for future directions to continue research in this topic.

This thesis laid the foundation of the representation theory of the dilute blob algebra. There are a number of interesting open questions could possibly be answered.

One of the most interesting questions is determining the simple modules of the algebra. Since the dilute blob algebra is cellular, we investigate when the cell modules are simple. In theorem 5.2.1 we prove that the cell modules are generically simple. We still need to know what the value of the parameters that make the cell modules simple. We can do that by known techniques that apply for cellular algebras (over a field) by calculating the Gram matrix of the cell modules and then finding the values of the parameters that make the determinant of the Gram matrix non-zero. Therefore, we could work to find a formula to the Gram matrix and a formula of its determinant as in the Temperley-Lieb algebra and the Motzkin algebra.

We then could explore the non-generic representation theory of the algebra. We might expect the representation theory of this algebra to be similar to that for the blob algebra (which is turn shares some features with the Temperley-Lieb algebra).

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