Unstable Cohomology Operations: Computational Aspects of Plethories

William Mycroft
School of Mathematics and Statistics
University of Sheffield

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Abstract

Generalised cohomology theories are a broad class of powerful invariants in algebraic topology. Unstable cohomology operations are a useful piece of structure associated to a such a theory and as a result the collection of these operations is of interest. Traditionally, these operations have been studied through the medium of Hopf rings. However, a Hopf ring does not readily admit algebraic structure corresponding to composition of operations.

Stacey and Whitehouse showed that the unstable cohomology operations naturally admit the structure of an esoteric algebraic gadget termed a plethory. This plethory contains all the information of the Hopf ring together with additional structure corresponding to the composition of operations.

In this thesis, I shall introduce the algebraic theory of plethories and extend with results which will aid computations. I will then illustrate, in a direct fashion, how the unstable cohomology operations admit the structure of a plethory and discuss the implications in this context. Finally, I shall perform some computations of the plethory of unstable cohomology operations for some familiar cohomology theories.

Contents

Nomenclature

Algebra

$\mathcal{V}$ variety of algebras.

$\mathcal{V}^*$ graded variety of algebras.

D linear dual.

$\langle -, - \rangle$ functional evaluation.

$\widehat{-}$ completion.

$\widehat{\otimes}$ completed tensor product.

$\circledast$ Sweedler product.

I augmentation ideal.

J coaugmentation quotient.

P primitives.

P_k k-primitives.

$\mathcal{A}$ super primitives.

$\langle - \rangle$ free module on a set.

$P_n[x]$ truncated polynomial algebra.

T tensor algebra.

S symmetric algebra.

S_k free functor on k-primitives.

Ψ free functor on super primitives.

$T_\odot$ tensor algebra w.r.t. $\odot$.

F Frobenius.

V Verschiebung.

$\#$ crossed product.

Hker Hopf kernel.

Hcoker Hopf cokernel.

Plethories

Δ^+ co-addition.

ε^+ co-zero.

σ co-additive inverse.

$\Delta^\times$ co-multiplication.

$\varepsilon^\times$ co-unit.

γ co-linear structure.

β alternative co-linear structure.

$\circ$ composition.

u composition unit map.

ι, ι_n composition unit element.

$\mathcal{I}$ initial plethory.

$\odot$ composition product.

$\widehat{\odot}$ completed composition product.

ω S^1-action.

Ω looping.

Hopf algebras and Hopf rings

$*$ $*$-multiplication.

$\circ$ $\circ$-multiplication.

ψ comultiplication.

ε counit.

$[\lambda]$ image of the right unit.

ι, ι_n augmentation.

r_* induced map.

$FHR(-)$ free Hopf ring.

λ-rings

λ^n n-th λ-operation.

ψ^n n-th Adams operation.

$P_n, P_{n,m}$ universal polynomials.

Λ universal λ-ring.

Ω representing plethory for Λ.

Topology

$h(-)$ ungraded cohomology theory.

$\underline{H}$ representing space of $h(-)$.

$E^*(-)$ graded cohomology theory.

E Ω-spectrum for E^*.

$\underline{E}_n$ n-th representing space of $E^*(-)$.

$\widehat{E}^*(-)$ completion of $E^*(-)$.

$\mathcal{E}^*(-)$ pro-finite $E^*(-)$.

$H^n(-;R)$ singular cohomology.

$K(G,n)$ Eilenberg-MacLane space.

$\mathcal{A}_p$ mod p Steenrod algebra.

$\mathcal{A}_p^*$ mod p dual Steenrod algebra.

P^n Steenrod power.

Sq^n Steenrod square.

β Bockstein.

$\mathrm{Vect}(-)$ isomorphism classes of complex vector bundles.

ξ_n canonical n-dimensional bundle.

$K(-)$ ungraded complex K-theory.

$K^*(-)$ graded complex K-theory.

U infinite unitary group.

BU classifying space of U.

$K(n)^*(-)$ n-th Morava K-theory.

T one point space.

Ω loop space functor.

Categories

$\mathbf{Ab}$ abelian groups.

$\mathbf{Alg}_k$ commutative unital k-algebras.

$\mathbf{Alg}_k^{\neg\mathrm{com}}$ associative unital k-algebras.

$\mathbf{FAlg}_k$ filtered k-algebras.

$\mathbf{CAlg}_k$ complete Hausdorff k-algebras.

$\mathbf{Bialg}_k$ bicommutative biunital k-bialgebras.

$\mathbf{Bialg}_k^{\neg\mathrm{com}}$ associative, cocommutative, biunital k-bialgebras.

FBialg$_k$ filtered k-bialgebras.

Biring$_{k,k'}$ k-k'-birings.

FBiring$_{k,k'}$ filtered k-k'-birings.

CBiring$_{k,k'}$ complete Hausdorff k-k'-birings.

Coalg$_k$ cocommutative counital coalgebras.

Ho Quillen's homotopy category.

Hopf$_k$ Hopf algebras over k.

HopfRing$_{k,k'}$ $k[k']$-Hopf rings.

Mod$_k$ k-modules.

$_k$**Mod**$_{k'}$ k-k'-bimodules.

FMod$_k$ filtered k-modules.

CMod$_k$ complete Hausdorff k-modules.

Monoid associative unital monoids.

Plethory$_k$ k-plethories.

CPlethory$_k$ complete Hausdorff k-plethories.

ΩPlethory$_k$ k-plethories with looping.

Ring commutative unital rings.

CRing complete Hausdorff rings.

SemiRing commutative unital semi-rings.

Set sets.

Top topological spaces.

TwBialg$_k$ twisted k-bialgebras.

Pro-$\mathcal{C}$ pro-objects in $\mathcal{C}$.

Introduction

The raison d'être of algebraic topology is to find computable algebraic invariants that classify topological spaces up to homeomorphism: deformations under which we are allowed to stretch and bend our object, but crucially not pinch or tear. A common 'joke' states that a topologist is a mathematician who cannot tell a donut from a coffee mug. The problem of finding such invariants has proved very difficult and we can only compute reasonable algebraic invariants in very special cases. However, great progress has been made towards the easier problem of classifying spaces up to homotopy equivalence: deformations under which we are additionally allowed to pinch solid regions down to a point.

Rather powerful homotopy invariants of a based space X are the *homotopy groups* denoted $\pi_n(X)$ and defined for all integers $n \geq 0$. These have the property that if $f\colon X \to Y$ is a continuous map of spaces then we have an induced map $f_*\colon \pi_n(X) \to \pi_n(Y)$ in a way which respects composition. Thus if X is homotopy equivalent to Y then we have induced isomorphisms $\pi_n(X) \cong \pi_n(Y)$ for all n. If we exclude somewhat pathological examples the converse statement is true by the Whitehead theorem and thus the homotopy groups provide a method to prove a continuous map is a homotopy equivalence. Unfortunately, the homotopy groups are extremely difficult to compute and even for simple spaces such as n-dimensional spheres some of the homotopy groups are still unknown. The computation of the homotopy groups of spheres is in some sense the holy grail of algebraic topology.

The *homology groups* of a space X are much more computable homotopy invariants denoted by $H_n(X)$ and defined for all integers $n \geq 0$. Just as for the homotopy groups maps of spaces induce maps of homology groups in a manner which is compatible with the composition. There are many powerful results which make such groups amenable to calculation and given a combinatorial expression of X we have efficient algorithms for computing the homology groups. Moreover, the homology groups are sufficient to answer many basic questions in algebraic topology. As an example, consider the problem of finding a retraction of the solid n-dimensional ball D^n onto its boundary S^{n-1}. That is to find a continuous map $r\colon D^n \to S^{n-1}$ such that $S^{n-1} \hookrightarrow D^n \xrightarrow{r} S^{n-1}$ is the identity on S^{n-1}. It is straightforward to compute that $H^{n-1}(D^n) \cong 0$ and $H^{n-1}(S^{n-1}) \cong \mathbb{Z}$. Thus if such a retraction were to exist then the identity map on the integers must factorise via zero, an impossibility.

Despite many uses, homology is not a complete invariant; there exist non-homotopy equivalent spaces which have isomorphic homology groups. Consider the torus $S^1 \times S^1$ and the wedge of spheres $S^2 \vee S^1 \vee S^1$. Computing homology we have the following result.

$$H_n(S^1 \times S^1) \cong H_n(S^2 \vee S^1 \vee S^1) \cong \begin{cases} \mathbb{Z} & n = 0, 2 \\ \mathbb{Z} \times \mathbb{Z} & n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

However, the spaces $S^1 \times S^1$ and $S^2 \vee S^1 \vee S^1$ are not homotopy equivalent as we would intuitively expect.

The *singular cohomology groups* $H^n(X)$ are defined in a categorically dual sense to the homology groups and in particular nice cases are just the linear duals of the homology groups. Not only are they also easily computable, they admit *cup products* or multiplications $H^n(X) \times H^m(X) \to H^{n+m}(X)$ making the collection of cohomology groups $H^*(X)$ into a graded (commutative, unital) ring. Moreover, these maps are natural with respect to X and thus continuous maps $f\colon X \to Y$ induce maps of graded rings $f^*\colon H^*(Y) \to H^*(X)$. Since a map of graded rings is at least a map of groups in each dimension, we expect this extra structure to strengthen our invariant. Computing the cohomology rings for the torus and our wedge of spheres we have $H^1(S^1 \times S^1) \cong H^1(S^2 \vee S^1 \vee S^1) \cong \mathbb{Z} \times \mathbb{Z}$. Letting α and β denote generators for $\mathbb{Z} \times \mathbb{Z}$ then in $H^*(S^2 \vee S^1 \vee S^1)$ we have $\alpha\beta = 0$, but in $H^*(S^1 \times S^1)$ we have $\alpha\beta = \gamma$, a generator for $H^2(S^1 \times S^1) = \mathbb{Z}$. Hence these two spaces are not homotopy equivalent. This is the first example we meet of our general motivation: the more algebraic structure we can pack into an invariant in a natural fashion, the stronger it becomes.

Abstracting some key properties of singular cohomology into a categorical setting we obtain the definition of a *generalised cohomology theory*. It turns out that many other important invariants in algebraic topology naturally fit into this framework. We work with “Quillen’s homotopy category” **Ho** which will exclude the pathological examples alluded to earlier and can be taken to consist of spaces X which are weakly homotopy equivalent to CW complexes and homotopy classes of continuous maps.

Definition 1. A *(generalised) cohomology theory* is a sequence of abelian group valued contravariant functors $E^n(-)\colon \mathbf{Ho} \to \mathbf{Ab}$ for each $n \in \mathbb{Z}$ which satisfy properties known as the *Eilenberg-Steenrod axioms.* We say $E^*(-)$ is *multiplicative* if it naturally takes values in (commutative, unital) E^*-algebras where $E^* = E^*(pt)$ is known as the *coefficient ring* of $E^*(-)$.

In addition to singular cohomology, familiar examples of multiplicative cohomology theories include real and complex topological K-theory, the Morava K-theories, complex cobordism, and many others. We shall henceforth assume all our cohomology theories are multiplicative. Continuing our theme of studying additional natural structure of our invariants leads to the notion of cohomology operations, a particular ubiquitous form of structure associated to a cohomology theory.

Definition 2. Let $E^*(-)$ be a cohomology theory. An *(unstable) cohomology operation* is a natural transformation $r\colon U \circ E^n(-) \to U \circ E^m(-)$ for some $n, m \in \mathbb{Z}$ where U denotes the forgetful functor $\mathbf{Ab} \to \mathbf{Set}$. Explicitly, for each space X we have a set map $r_X\colon E^n(X) \to E^m(X)$ such that if $f\colon X \to Y$ is a map of spaces, then the following diagram commutes.

$$\begin{array}{ccc} E^n(Y) & \xrightarrow{r_Y} & E^m(Y) \\ \big\downarrow{\scriptstyle f^*} & & \big\downarrow{\scriptstyle f^*} \\ E^n(X) & \xrightarrow{r_X} & E^m(X) \end{array}$$

For brevity, we shall denote a cohomology operation $r\colon E^n(-) \to E^m(-)$ and leave it implicit that we are viewing $E^n(-)$ and $E^m(-)$ as set-valued functors.

Familiar examples of cohomology operations are the Steenrod squares $Sq^n\colon H^*(-;\mathbb{F}_2) \to H^{*+n}(-;\mathbb{F}_2)$ for singular cohomology with coefficients in $\mathbb{F}_2$, the field with two elements. With this extra structure we can distinguish some non-homotopy equivalent spaces which have isomorphic cohomology rings. For example, consider the reduced suspension of the real projective plane $\Sigma\mathbb{R}P^2$ and the wedge of spheres $S^2 \vee S^3$. The singular $\mathbb{F}_2$-cohomology rings of these spaces are both 1-dimensional vector spaces in cohomological dimensions $0, 2$ and 3 and trivial in other dimensions. Moreover, both cohomology rings have no non-trivial products. However, if we let α, β denote the canonical basis element in dimensions 2 and 3 respectively, then we have $Sq^1(\alpha) = \beta$ in $H^*(\Sigma\mathbb{R}P^2;\mathbb{F}_2)$ but $Sq^1(\alpha) = 0$ in $H^*(S^2 \vee S^3;\mathbb{F}_2)$ and thus by naturality of the operations we see that these spaces cannot be homotopy equivalent.

The Steenrod squares have a very special property, they commute with the suspension isomorphism in the sense that $\Sigma Sq^n(x) = Sq^n(\Sigma x)$ for all $x \in H^*(X)$. Such operations are known as stable cohomology operations and in singular cohomology it turns out that all unstable cohomology operations are generated by stable operations in a suitable context. However, in general we have truly unstable operations such as the λ-operations in topological K-theory which do not arise naturally from stable operations.

As we have seen, the unstable cohomology operations are a useful piece of structure on our cohomology theories and utilising these increases the strength of our invariants. As a consequence, it seems prudent to study the collection of all unstable cohomology operations. By a famous theorem of E.H. Brown, generalised cohomology theories can be represented by Ω-spectra and this leads to a concise way of discussing cohomology operations.

Definition 3. An Ω-spectrum E is a sequence of based spaces $\underline{E}_n$ for $n \in \mathbb{Z}$ together with homotopy equivalences $\Omega \underline{E}_{n+1} \simeq \underline{E}_n$ where Ω denotes the loop space functor. We say E is a ring spectrum if it is equipped with a unital multiplication. In this case, we have induced maps on the representing spaces $\underline{E}_n \times \underline{E}_m \to \underline{E}_{n+m}$.

Theorem 4 (Brown's representability theorem)**.** Any graded cohomology theory $E^*(-)$ is represented in **Ho** by an Ω-spectrum E in the sense that we have universal elements $\iota_n \in E^n(\underline{E}_n, o) \subseteq E^n(\underline{E}_n)$ and isomorphisms $\mathbf{Ho}(X, \underline{E}_n) \cong E^*(X)$ given by $f \mapsto f^*\iota_n$ for all n. Moreover if $E^*(-)$ is multiplicative then E is a ring spectrum.

Brown's representability theorem together with Yoneda's lemma allows us to view a cohomology operation in the following equivalent ways, and we shall frequently pass between these.

1. As a cohomology operation $r\colon E^n(-) \to E^m(-)$.
2. As a cohomology class $r\iota_n \in E^m(\underline{E}_n)$.
3. As a representing map $r_U\colon \underline{E}_n \to \underline{E}_m$ in **Ho**.

As a consequence, understanding the collection of the cohomology operations is tantamount to understanding the bigraded object $E^*(\underline{E}_\bullet)$. This object is very highly structured. Not only is each $E^*(\underline{E}_n)$ naturally a graded E^*-algebra, but the representing spaces $\underline{E}_n$ of our cohomology theory are equipped with corresponding maps $\underline{E}_n \times \underline{E}_n \to \underline{E}_n$ and $\underline{E}_n \times \underline{E}_m \to \underline{E}_{n+m}$.

In nice cases additional structure on a space X induces additional structure on the cohomology $E^*(X)$. For example, the diagonal map $\Delta\colon X \to X \times X$ induces a map

$E^*(X \times X) \to E^*(X)$. In a multiplicative cohomology theory, we have an external product map $E^*(X) \otimes_{E^*} E^*(X) \to E^*(X \times X)$. The cup product multiplication on $E^*(X)$ is given by the following composition.

$$E^*(X) \otimes_{E^*} E^*(X) \to E^*(X \times X) \to E^*(X)$$

Now suppose X is an H-space with multiplication $\mu\colon X \times X \to X$. For example $X = \underline{E}_n \simeq \Omega \underline{E}_{n+1}$ with μ given by composition of loops. Now looking at the map on cohomology induced by μ we have the following maps.

$$E^*(X) \xrightarrow{\mu^*} E^*(X \times X) \leftarrow E^*(X) \otimes_{E^*} E^*(X)$$

Hence we see that we can only equip $E^*(X)$ with additional internal structure from μ in this way when the external product map is an isomorphism. In this case, the external product map is known as a *Künneth isomorphism* but sadly this very rarely holds.

To cope with this lack of Künneth isomorphisms, the historical approach to studying cohomology operations is to instead study the homology of the representing spaces $E_*(\underline{E}_\bullet)$ and leads to the notion of homology cooperations. The advantage of this approach is that homology groups are in general significantly smaller than cohomology groups and thus we much more frequently have a Künneth isomorphism in homology. Moreover, in sufficiently nice cases, the collection of cohomology operations is the linear dual of the set of homology cooperations and thus these objects encode precisely the same information.

It has been known since the work of Ravenel and Wilson [40] in 1976 that the homology cooperations naturally admit the structure of a *Hopf ring*. The theory of Hopf rings is very well developed [54] and is amenable to computations. In particular, Hopf rings can be neatly expressed in terms of generators and relations and the Hopf ring of homology cooperations has been calculated for many important cohomology theories. A summary of these results can be found in [15].

A fundamental piece of structure on the set of cohomology operations is composition. Given two operations $r\colon E^k(-) \to E^m(-)$ and $s\colon E^n(-) \to E^k(-)$, composition yields an operation $r \circ s\colon E^n(-) \to E^m(-)$. More generally, the cohomology operations act on the cohomology of a space X in the sense that if $x \in E^k(X)$ then $r(x) \in E^m(X)$ and we can realise the composition via the relation $(r \circ s)\iota_n = r(s\iota_n) \in E^m(\underline{E}_n)$. However, a Hopf ring contains no information which is naturally dual to the notion of composition and thus we cannot see arguably the most important piece of the structure.

A 1995 paper of Boardman, Johnson and Wilson [15] attempts to rectify this issue. In this work, they attempt to understand the action $E^m(\underline{E}_n) \times E^n(X) \to E^m(X)$ by studying the adjoint form $E^*(X) \to \mathcal{C}(E^*(\underline{E}_\bullet), E^*(X))$ where $\mathcal{C}$ is a suitable category of E^*-algebras. Under this point of view, the functor $\mathcal{C}(E^*(\underline{E}_\bullet), -)$ forms a comonad on $\mathcal{C}$ and cohomology algebras together with the action of the operations can be realised as coalgebras for this comonad. In some sense this completes the study of cohomology operations as it gives a compact categorical representation that encodes all the structure we have. Unfortunately, this definition is somewhat opaque and does not lend itself to computations. To combat this, Boardman et al. unpack this comonadic information by equipping the homology cooperations with the structure of an *enriched Hopf ring*: the augmentation of a Hopf ring by additional structure encoding the composition. Sadly, this additional structure is not internal to the homology cooperations and in particular we cannot express the entire enriched Hopf ring neatly in the language of generators and relations.

The more direct approach of studying the actions $E^m(\underline{E}_n) \times E^n(X) \to E^m(X)$ was abandoned by Boardman et al. due to the perceived lack of a monoidal structure which would suitably encode the non-linearity of this action. However there is a somewhat esoteric algebraic gadget which encodes precisely this information.

Abstractly, the collection of cohomology operations is an object which acts non-linearly on the cohomology algebras $E^*(X)$. In the same way that k-algebras are precisely the structure which act linearly on k-modules, we can ask what structure acts non-linearly on k-algebras. It turns out that in the ungraded setting this answer has been known in the universal algebra world since Tall and Wraith's work [50] in 1970. In 2005, Borger and Wieland [17] gave a more detailed treatment of such objects and coined the term *plethories*. A 2009 paper of Stacey and Whitehouse [43] generalised the notion of a plethory to the graded, topologised setting and show that this object naturally encodes the structure of the cohomology operations. Moreover, the action of the cohomology operations on a cohomology algebra can be encoded by a map $E^*(\underline{E}_\bullet) \odot E^*(X) \to E^*(X)$ where $\odot$ is a non-linear analogue of the tensor product.

There are technical difficulties that arise when constructing plethories in a topological setting. Stacey and Whitehouse take an aggressive approach to this problem, completing all the cohomology algebras. While this allows the theory to work smoothly, in general taking the completion of a cohomology algebra destroys information; we are unable to detect cohomology classes known as *phantoms*. In [11], Bauer takes a more subtle approach, working over categories of pro-algebras. Following this strategy to its natural conclusion leads to the notion of a *formal plethory* which encodes all the information of a plethory but can also detect phantoms. However, the extra complexity of formal plethories makes computations difficult. Moreover, in many cases of interest we can prove that no phantoms exist and so these two approaches encode the same information.

In the first chapter of this thesis, I give a detailed exposition of the theory of plethories in an abstract algebraic setting, building on the work of Tall and Wraith [50] and Borger and Wieland [17]. The main aim of my work is to be able to perform calculations with plethories and with this in mind I develop the computational theory of plethories. In particular, Proposition 1.2.16 gives an explicit formulation of the structure maps on the composition product and in Section 1.3.4 I detail the duality between enriched Hopf rings and plethories. In Section 1.4, I extend the definitions and basic results to the graded context, which has applications to the study of cohomology operations.

The second chapter extends the plethystic framework to a topological context and introduces the technical machinery necessary for formulating cohomology operations in this context. Theorem 2.1.59 gives a direct proof of an abstract result of Stacey and Whitehouse [43] which makes precise how the cohomology operations admit the structure of a plethory which acts on cohomology algebras. We then turn our attention to topological properties such as the suspension isomorphisms and complex orientation and detail the ramifications these properties have for our plethories. In particular, Theorem 2.2.16 illustrates how the looping of operations behaves with respect to the plethystic structure and Theorem 2.3.30 details the effects of complex orientation on the plethory of cohomology operations.

In the final chapter I compute examples of the plethory of cohomology operations for singular cohomology with coefficients over $\mathbb{Q}$ (Theorem 3.1.4), $\mathbb{F}_p$ (Theorem 3.1.25 and Theorem 3.1.55) and for topological complex K-theory (Theorem 3.2.34). In the process we illustrate the applications of the computational framework developed in the first chapter. In Section 3.3, I give partial results for the computation of the plethory of cohomology operations for the Morava K-theories and detail potentially fruitful lines of attack for future calculations.

Chapter 1

Abstract plethories

Plethories are fairly esoteric algebraic objects, but arise in a very natural context. Fix a commutative ring k. In the same way we realise commutative k-algebras as precisely the structure that acts linearly on k-modules, we can realise k-plethories as precisely the structure that acts non-linearly on commutative k-algebras. These objects were first studied by Tall and Wraith [50] in the case $k = \mathbb{Z}$ and in the case of general k by Borger and Wieland [17].

To best understand the structure of a plethory, we begin by studying representable endofunctors on the category of commutative k-algebras. This category admits a non-symmetric monoidal structure, corresponding to the composition of endofunctors and plethories are monoids in this category. I provide a detailed exposition of this construction, expanding on the work of [50] and [17] and introduce new results which will ease computations such as Proposition 1.2.16.

In the next section, we study the theory of plethories. I elaborate on many constructions of both [50] and [17] and introduce many new results with a focus on being able to perform computations. For example, in Section 1.3.4 I develop a purely algebraic description of an *enriched Hopf ring* introduced in [15] to study homology cooperations and detail the duality between such objects and plethories.

Finally, I detail the extension of plethories to the graded setting where we will be working for our main applications. However, much of this grading is superficial and it is more illuminating to study the theory in the ungraded case before highlighting the differences that arise in the graded context. It is my hope this chapter will provide a good introduction to the subject for those interested in both the graded and ungraded settings.

Throughout this thesis, all rings and algebras will be assumed to be commutative and unital and all coalgebras cocommutative and counital unless otherwise specified. When this will not cause confusion, unadorned tensor products will be assumed to be taken over the obvious ground ring.

1.1 Algebraic objects in categories

Consider the elementary definition of an abelian group: a set G with a specified unit element $e \in G$ such that for all elements $x, y \in G$ there exists a product $xy \in G$ and an inverse $x^{-1} \in G$ satisfying for all $x, y, z \in G$,

1. $xy = yx$,
2. $(xy)z = x(yz)$,
3. $xx^{-1} = x$,
4. $ex = x$.

We can express this categorically by saying G is a set G equipped with set maps $\mu\colon G \times G \to G$, $\omega\colon \{*\} \to G$ and $\nu\colon G \to G$, such that the following diagrams of sets commute.

$$\begin{tikzcd}
G \times G \arrow[r, "\tau"] \arrow[dr, "\mu"'] & G \times G \arrow[d, "\mu"] \\
& G
\end{tikzcd}
\qquad
\begin{tikzcd}
G \times G \times G \arrow[r, "\mu \times 1"] \arrow[d, "1 \times \mu"] & G \times G \arrow[d, "\mu"] \\
G \times G \arrow[r, "\mu"] & G
\end{tikzcd}$$

$$\begin{tikzcd}
G \arrow[r, "\Delta"] \arrow[d, "\omega \circ t"] & G \times G \arrow[d, "1 \times \nu"] \\
G & G \times G \arrow[l, "\mu"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\{*\} \times G \arrow[r, "\omega \times 1"] \arrow[dr, "\cong"'] & G \times G \arrow[d, "\mu"] \\
& G
\end{tikzcd}
\tag{1.1}$$

where $\Delta\colon G \to G \times G$ is the diagonal map $x \mapsto (x, x)$, $t\colon G \to \{*\}$ the unique map $x \mapsto *$, and $\tau\colon G \times G \to G \times G$ the twist map $(x, y) \mapsto (y, x)$.

If G and H are abelian groups then a group morphism is a map $f\colon G \to H$ such that $f(xy) = f(x)f(y)$, $f(x^{-1}) = f(x)^{-1}$, and $f(e) = e$ for all $x, y \in G$. The latter two properties are a consequence of the first, but this will not always be true in a more general context. Equivalently a group morphism is a map such that the following diagrams of sets commute.

$$\begin{tikzcd}
G \times G \arrow[r, "f \times f"] \arrow[d, "\mu"] & H \times H \arrow[d, "\mu"] \\
G \arrow[r, "f"] & H
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, "f"] \arrow[d, "\nu"] & H \arrow[d, "\nu"] \\
G \arrow[r, "f"] & H
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[d, "\omega"] \arrow[dr, "\omega"] & \\
G \arrow[r, "f"] & H
\end{tikzcd}
\tag{1.2}$$

We can now generalise the definition of an abelian group from the category of sets to arbitrary categories with finite products, and this methodology will work for a wide class of algebraic objects, not just abelian groups.

1.1.1 Group objects

Throughout the next 3 sections suppose $\mathcal{C}$ is a category with finite products $\times$ and a terminal object T.

Definition 1.1.1. We define an *abelian group object* in $\mathcal{C}$ to be an object $G \in \mathcal{C}$ together with the following morphisms.

$$\mu\colon G \times G \to G$$
$$\omega\colon T \to G$$
$$\nu\colon G \to G.$$

satisfying the commutative diagrams expressing the associativity, commutativity, inverse and unit axioms (1.1).

A morphism of abelian group objects in $\mathcal{C}$ is a morphism $f\colon G_1 \to G_2$ which commutes with the structure maps (1.2).

Example 1.1.2. The category of abelian groups is the category of abelian group objects in **Set**. The terminal object in **Set** is the one point set $\{*\}$ and the product is the Cartesian product. We typically write $\mu(x, y) = x + y$, $\omega(*) = 0$ and $\nu(x) = -x$.

Example 1.1.3. In $\mathbf{Coalg}_k$, the category of cocommutative counital coalgebras over a ring k, the terminal object is k and the product is the tensor product $\otimes_k$. The category $\mathbf{Hopf}_k$ of bicommutative Hopf algebras over k is the category of abelian group objects in $\mathbf{Coalg}_k$. If H is an abelian group object in $\mathbf{Coalg}_k$ then $\mu\colon H \otimes_k H \to H$ is the multiplication, $\eta\colon k \to H$ the unit and $\nu\colon H \to H$ the antipode.

Example 1.1.4. The category of abelian topological groups is the category of abelian group objects in **Top**, the category of topological spaces and continuous maps.

1.1.2 Ring and algebra objects

Definition 1.1.5. A *ring object* in $\mathcal{C}$ is an abelian group object R together with two additional morphisms

$$\phi\colon R \times R \to R$$
$$\eta\colon T \to R$$

satisfying relations expressing associativity, unit and the two distributive laws. A morphism of ring objects is a morphism of group objects which commutes with ϕ and η.

Following the notation of Boardman [14], we now introduce the analogue of a k-linear structure in our category $\mathcal{C}$.

Definition 1.1.6. A *k-module object* in $\mathcal{C}$ is an abelian group object M together with a group object morphism

$$\xi\lambda\colon M \to M$$

for each $\lambda \in k$ such that $\xi\colon k \to \mathcal{C}(M, M), \lambda \mapsto \xi\lambda$ is a map of rings. The ring structure on $\mathcal{C}(M, M)$ has addition given by

$$f + g\colon M \xrightarrow{\Delta} M \times M \xrightarrow{f \times g} M \xrightarrow{\mu} M$$

and multiplication given by composition. A morphism of k-module objects is a morphism of abelian group objects which commutes with each $\xi\lambda$.

Definition 1.1.7. A *k-algebra object* in $\mathcal{C}$ is a ring object A which is also a k-module object such that the two structures are related by the following commutative diagram.

$$\begin{tikzcd}
 & A \times A \arrow[dl, "\xi\lambda \times 1"'] \arrow[d, "\phi"] \arrow[dr, "1 \times \xi\lambda"] & \\
A \times A \arrow[dr, "\phi"'] & A \arrow[d, "\xi\lambda"] & A \times A \arrow[dl, "\phi"] \\
 & A &
\end{tikzcd} \tag{1.3}$$

A morphism of k-algebra objects is a morphism of ring objects which is also a morphism of k-module objects.

It will often be useful to consider the composition $\eta_R\lambda = \xi\lambda \circ \eta$ which yields a map $\eta_R\colon k \to \mathcal{C}(T, A)$ given by $\lambda \mapsto \eta_R\lambda$. The following commutative diagram shows we can recover ξ from η_R as the composition $\xi\lambda = A \cong T \times A \xrightarrow{\eta_R\lambda \times 1} A \times A \xrightarrow{\phi} A$.

$$\begin{tikzcd}
T \times A \arrow[r, "\eta \times 1"] \arrow[dr, "\cong"'] & A \times A \arrow[r, "\xi\lambda \times 1"] \arrow[d, "\phi"] & A \times A \arrow[d, "\phi"] \\
 & A \arrow[r, "\xi\lambda"] & A
\end{tikzcd} \tag{1.4}$$

Proposition 1.1.8. *The map $\eta_R\colon k \to \mathcal{C}(T, A)$ defined by $\lambda \mapsto \xi\lambda \circ \eta$ is a map of rings, where the ring structure on $\mathcal{C}(T, A)$ is induced by the ring object structure on A.*

Proof: Since $(\eta \times \eta) \circ \Delta = \Delta \circ \eta$, we have $\eta_R(\lambda + \lambda') = \eta_R\lambda + \eta_R\lambda'$ in $\mathcal{C}(T, A)$. The statement $\eta_R(\lambda\lambda') = (\eta_R\lambda)(\eta_R\lambda')$ in $\mathcal{C}(T, A)$ is equivalent to commutativity of the outer rectangle of the following diagram.

$$\begin{tikzcd}
T \arrow[r, "\eta"] \arrow[d, "\Delta"] & A \arrow[r, "\xi\lambda"] & A \arrow[r, "\xi\lambda'"] & A \\
T \times T \arrow[r, "\eta\times\eta"] & A \times A \arrow[u, "\phi"] \arrow[r, "\xi\lambda\times 1"] & A \times A \arrow[u, "\phi"] \arrow[r, "1\times\xi\lambda'"] & A \times A \arrow[u, "\phi"]
\end{tikzcd}$$

The first inner square commutes by the unital property, and the latter two commute by (1.3). Therefore the outer rectangle commutes. Finally, by definition, $\eta_R 1 = \eta$ which is the unit in $\mathcal{C}(T, A)$. Hence η_R is a ring map. □

We reinterpret what it means to be a morphism of k-algebra objects with respect to this alternative structure map η_R.

Proposition 1.1.9. *A morphism $f\colon A \to A'$ of ring objects is a morphism of k-algebra objects if and only if the following diagram of rings commutes.*

$$\begin{tikzcd}
k \arrow[r, "\eta_R"] \arrow[dr, "\eta_R"'] & \mathcal{C}(T, A) \arrow[d, "\mathcal{C}(T,f)"] \\
& \mathcal{C}(T, A')
\end{tikzcd} \tag{1.5}$$

Proof: Suppose f is a morphism of k-algebra objects. We have $f\circ\xi\lambda\circ\eta = \xi\lambda\circ f\circ\eta = \xi\lambda\circ\eta$ for all $\lambda \in k$. Hence $f \circ \eta_R\lambda = \eta_R\lambda$ and thus (1.5) commutes. Conversely, if (1.5) commutes then using (1.4) we see $f \circ \xi\lambda = \xi\lambda$ for all λ. □

1.1.3 General algebraic objects

All these constructions naturally fit into an abstract framework known as *general algebra* or *universal algebra*. Using this theory we can prove useful statements regarding our various types of algebraic objects all at once. Following [14] we will be slightly lax with the technical definitions, for a rigorous treatment close to our point of view see [43] or for a broad overview of the subject refer to [12].

It is worth mentioning that there are two main approaches to these constructions. We shall follow the *variety of algebras* method, although there is an equivalent approach using *Lawvere theories* as introduced in [33] with an expository treatment in [28].

Definition 1.1.10. A *variety of algebras* $\mathcal{V}$ is a collection of *operations* Ω, with an *arity map* $n\colon \Omega \to \mathbb{N}$ and a set of *identities* J expressing relations between the structure maps.

A $\mathcal{V}$*-algebra structure* on an object $A \in \mathcal{C}$ is a collection of morphisms $A^{\times n(\omega)} \to A$ for each $\omega \in \Omega$ which satisfy the relations expressed by J.

A morphism of $\mathcal{V}$-algebra objects $f: A \to B$ is a morphism in $\mathcal{C}$ such that the following diagram commutes for all operations ω.

$$\begin{tikzcd} A^{\times n(\omega)} \arrow[r, "f^{\times n(\omega)}"] \arrow[d, "\omega_A"] & B^{\times n(\omega)} \arrow[d, "\omega_B"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

We denote the category of $\mathcal{V}$-algebra objects in $\mathcal{C}$ and $\mathcal{V}$-algebra morphisms by $\mathcal{V}\mathcal{C}$.

Example 1.1.11. The variety of abelian groups has 3 operations $\mu, \eta, \sigma \in \Omega$ with $n(\mu) = 2, n(\eta) = 0, n(\sigma) = 1$. The identities in J are the axioms encoding the associativity, commutativity, inverse and unit properties (1.1).

Lemma 1.1.12 ([14, Lemma 7.6])**.** *Let A be a $\mathcal{V}$-algebra object in $\mathcal{C}$. If $F: \mathcal{C} \to \mathcal{D}$ preserves products then F induces to a functor $\mathcal{V}F: \mathcal{V}\mathcal{C} \to \mathcal{V}\mathcal{D}$.*

Corollary 1.1.13 ([14, Lemma 7.7])**.** *Let A be an object in $\mathcal{C}$. There is a bijection between $\mathcal{V}$-algebra object structures on A and lifts of the contravariant functor $\mathcal{C}(-, A): \mathcal{C} \to$ **Set** represented by A to a functor $\mathcal{C} \to \mathcal{V}$**Set**.*

Explicitly, the correspondence works is given as follows. Given an n-ary operation $\omega: A^{\times n} \to A$ the corresponding operation $\mathcal{C}(B, A)^{\times n} \to \mathcal{C}(B, A)$ sends $(f_1, \ldots, f_n)$ to the composition

$$B \xrightarrow{\Delta} B^{\times n} \xrightarrow{f^{\times n}} A^{\times n} \xrightarrow{\omega} A.$$

Conversely, given an n-ary operation $\omega: \mathcal{C}(B, A)^{\times n} \to \mathcal{C}(B, A)$, the corresponding operation $A^{\times n} \to A$ is given by $\omega(\pi_1, \ldots, \pi_n) \in \mathcal{C}(A^{\times n}, A)$ where $\pi_i \in \mathcal{C}(A^{\times n}, A)$ is the canonical projection onto the i-th factor.

1.1.4 Coalgebraic objects

There is a dual notion of coalgebraic objects in arbitrary categories. We construct these simply by 'turning around the arrows', replacing products with coproducts and exchanging terminal objects for initial objects.

In this section suppose $\mathcal{C}$ is a category with finite coproducts $\otimes$ and an initial object I.

Definition 1.1.14. Let $\mathcal{V} = (\Omega, n, J)$ be a variety of algebras. A co-$\mathcal{V}$-algebra object in $\mathcal{C}$ is an object A together with co-operations

$$\alpha\colon A \to A^{\otimes n(w)}$$

satisfying the categorical duals of the relations expressed in J.

A morphism of co-$\mathcal{V}$-algebra objects is a morphism such that categorical duals of the diagrams for a morphism of $\mathcal{V}$-algebra objects commute for all co-operations α.

We denote the category of co-$\mathcal{V}$-algebra objects in $\mathcal{C}$ and co-$\mathcal{V}$-algebra morphisms by $\mathcal{V}^c\mathcal{C}$.

This produces definitions of co-(abelian group), co-ring and co-k-algebra objects in $\mathcal{C}$.

Example 1.1.15. The category $\mathbf{Hopf}_k$ of (bicommutative) Hopf algebras over k is the category of co-(abelian group) objects in the category of (commutative) k-algebras $\mathbf{Alg}_k$.

We give the dual results to Lemma 1.1.12 and Corollary 1.1.13.

Lemma 1.1.16. *Let A be a co-$\mathcal{V}$-object in $\mathcal{C}$. If $F\colon \mathcal{C} \to \mathcal{D}$ preserves coproducts then F induces a functor $\mathcal{V}^c\mathcal{C} \to \mathcal{V}^c\mathcal{D}$.*

Corollary 1.1.17. *Let A be an object of $\mathcal{C}$. There is a bijection between co-$\mathcal{V}$-algebra object structures on A and lifts of the covariant functor $\mathcal{C}(A, -)\colon \mathcal{C} \to \mathbf{Set}$ represented by A to a functor $\mathcal{C} \to \mathcal{V}\mathbf{Set}$.*

1.2 Birings, plethories and P-algebras

A k-algebra is precisely the structure which acts linearly on k-modules. By this statement, we mean that for a finite k-module M, $\mathbf{Mod}_k(M, M)$ naturally has the structure of a non-commutative k-algebra where addition is pointwise and multiplication is given by composition. An action of a k-algebra A on M is a map of k-algebras $\phi\colon A \to \mathbf{Mod}_k(M, M)$. If N is another k-module with an action of A, then a k-module map $f\colon M \to N$ respects the action if the following diagram commutes for all $a \in A$.

$$\begin{tikzcd} M \arrow[r, "\phi_a"] \arrow[d, "f"] & M \arrow[d, "f"] \\ N \arrow[r, "\phi_a"] & N \end{tikzcd}$$

We denote the category of k-modules equipped with an action of A by $\mathbf{Mod}_A$. We can identify A with $\mathrm{Nat}(U, U)$, the set of natural transformations from the forgetful functor $U\colon \mathbf{Mod}_A \to \mathbf{Mod}_k$ to itself: we have an isomorphism of k-modules $\mathbf{Mod}_A(A, M) \cong M$ for all k-modules M and so by the Yoneda embedding $\mathrm{Nat}(U, U) \cong \mathbf{Mod}_A(A, A) \cong A$.

The above discussion relied on us having prior knowledge of the structure of a k-algebra. Recall that a k-k'-bimodule M naturally represents a functor $\mathbf{Mod}_k(M, -)\colon \mathbf{Mod}_k \to \mathbf{Mod}_{k'}$. Hence, given a k'-k''-bimodule N it is natural to consider the composition

$$\mathbf{Mod}_k \xrightarrow{\mathbf{Mod}_k(M,-)} \mathbf{Mod}_{k'} \xrightarrow{\mathbf{Mod}_{k'}(N,-)} \mathbf{Mod}_{k''}.$$

This functor is represented by the k-k''-bimodule $M \otimes_{k'} N$. Now $\otimes_k$ makes the category of k-k-bimodules monoidal and k-algebras are precisely monoids in this category. Moreover, if A is a k-algebra then $A \otimes_k -$ forms a monad on $\mathbf{Mod}_k$ and A-modules are precisely Eilenberg-Moore algebras for this monad. In other words, an A-module is a k-module equipped with an associative unital k-module map $A \otimes M \to M$.

We wish to generalise this to the setting of non-linear actions on commutative k-algebras. We consider representable functors $\mathbf{Alg}_k(B, -)\colon \mathbf{Alg}_k \to \mathbf{Alg}_{k'}$. It turns out that the composition of such a functor with another functor $\mathbf{Alg}_{k'}(B', -)\colon \mathbf{Alg}_{k'} \to \mathbf{Alg}_{k''}$ is representable by a non-linear analogue of the tensor product, denoted $B \odot_{k'} B'$. The functor $\odot_k$ makes the category of representable functors $\mathbf{Alg}_k \to \mathbf{Alg}_k$ monoidal and a k-plethory is a monoid in this category. If P is a k-plethory, then $P \odot_k -$ forms a monad on $\mathbf{Alg}_k$ and we denote the category of Eilenberg-Moore algebras for this monad by $\mathbf{Alg}_P$.

By an operation on a P-algebra, we mean a natural transformation from $U\colon \mathbf{Alg}_P \to \mathbf{Set}$ to itself. It turns out this functor is represented by P and so by the Yoneda embedding we have natural isomorphisms $\mathrm{Nat}(U, U) \cong \mathbf{Alg}_P(P, P) \cong P$. Therefore viewing $r \in P$ as a natural transformation $U \to U$, for each P-algebra A we have a map $P \to \mathbf{Set}(A, A)$ given by $r \mapsto r_A$. Hence we have realised k-plethories as precisely the structure which acts non-linearly on k-algebras.

Plethories were first introduced in 1970 by Tall and Wraith [50] under the name of a biring triple. They were given a more modern treatment in 2005 by Borger and Wieland [17]. In this section I will give a more detailed exposition of the basic definitions and properties of plethories. Along the way I will introduce some new results with a focus on being able to perform computations.

1.2.1 Birings

As discussed, we would like to study representable functors $\mathbf{Alg}_k \to \mathbf{Alg}_{k'}$ for commutative rings k and k'. By Corollary 1.1.17 this is equivalent to studying the co-k'-algebra objects in the category of k-algebras.

Definition 1.2.1. Define the category of *k-k'-birings* $\mathbf{Biring}_{k,k'}$ to be the category of co-k'-algebra objects in $\mathbf{Alg}_k$. Explicitly, a k-k'-biring is a k-algebra (B, ϕ, η) together with the following k-algebra morphisms.

$$\Delta^+ \colon B \to B \otimes B \qquad \text{(co-addition)}$$

$$\varepsilon^+ \colon B \to k \qquad \text{(co-zero)}$$

$$\sigma \colon B \to B \qquad \text{(co-additive inverse)}$$

$$\Delta^\times \colon B \to B \otimes B \qquad \text{(co-multiplication)}$$

$$\varepsilon^\times \colon B \to k \qquad \text{(co-unit)}$$

and for each $\lambda \in k'$,

$$\gamma\lambda \colon B \to B \qquad \text{(co-}k'\text{-linear structure)}$$

satisfying the relations for a co-k-algebra object, i.e. $(B, \phi, \eta, \Delta^+, \varepsilon^+, \sigma)$ is a bicommutative Hopf algebra, $\Delta^\times$ is coassociative, cocommutative with counit $\varepsilon^\times$, Δ^+ co-distributes over $\Delta^\times$ and the $\gamma\lambda$ respect the co-ring object structure suitably. For completeness, we list all these relations as commutative diagrams in Appendix A.

By the dual result of Proposition 1.1.8 we can encode the co-k'-linear structure as a map of rings

$$\beta \colon k' \to \mathbf{Alg}_k(B, k)$$

where $\beta\lambda = \varepsilon^\times \circ \gamma\lambda$. The ring structure on $\mathbf{Alg}_k(B, k)$ is induced by the co-ring object structure on B. We shall use this structure map more frequently than γ.

A map of k-k'-birings is a map of k-algebras $f \colon B \to B'$ respecting the additional structure in the sense that the diagrams in Appendix A commute.

Notation 1.2.2. We shall make frequent use of Sweedler notation by writing

$$\Delta^+(b) = \sum_{(b)} b_{(1)} \otimes b_{(2)},$$

$$\Delta^\times(b) = \sum_{(b)} b_{[1]} \otimes b_{[2]}.$$

When this will not cause confusion (for example when working with linear maps) we shall omit the summation symbol. We shall write $\Delta^+_{(n)}$ for the iterated co-addition $B \to B^{\otimes n}$ and similarly $\Delta^\times_{(n)}$ for the iterated co-multiplication.

The structure maps of the co-k'-algebra object may appear somewhat unnatural. The following example illustrates how these can arise by considering non-linear maps between two commutative rings.

Example 1.2.3. Let k and k' be commutative rings, and suppose k' is finite. Consider $B = \mathbf{Set}(k', k)$. Since k is a k-algebra object in $\mathbf{Set}$, B is naturally a k-algebra by Corollary 1.1.13. Now k' is finite and so we have an isomorphism $\mathbf{Set}(k' \times k', k) \cong \mathbf{Set}(k', k) \otimes \mathbf{Set}(k', k)$. Hence by Lemma 1.1.16, B is naturally a k-k'-biring. The structure maps are given as by

$$\begin{aligned}
\Delta^+ \colon \mathbf{Set}(k', k) &\xrightarrow{\mathbf{Set}(+,k)} \mathbf{Set}(k' \times k', k) \cong \mathbf{Set}(k', k) \otimes \mathbf{Set}(k', k) \\
\varepsilon^+ \colon \mathbf{Set}(k', k) &\xrightarrow{\mathbf{Set}(0,k)} \mathbf{Set}(\{*\}, k) \cong k \\
\sigma \colon \mathbf{Set}(k', k) &\xrightarrow{\mathbf{Set}(-,k)} \mathbf{Set}(k', k) \\
\Delta^\times \colon \mathbf{Set}(k', k) &\xrightarrow{\mathbf{Set}(\times,k)} \mathbf{Set}(k' \times k', k) \cong \mathbf{Set}(k', k) \otimes \mathbf{Set}(k', k) \\
\varepsilon^\times \colon \mathbf{Set}(k', k) &\xrightarrow{\mathbf{Set}(1,k)} \mathbf{Set}(\{*\}, k) \cong k \\
\beta\lambda \colon \mathbf{Set}(k', k) &\xrightarrow{\mathbf{Set}(\lambda\cdot,k)} \mathbf{Set}(k', k) \xrightarrow{\varepsilon^\times} k
\end{aligned}$$

and the relations between these maps are satisfied as a result of the relations between the structure maps on k'. The finiteness assumption is necessary so that $\mathbf{Set}(-, k)$ preserves coproducts. We are able to relax this condition later when we introduce topological plethories.

The view of birings as representable functors gives us more examples of birings, and we can use Corollary 1.1.17 to compute the corresponding structure maps.

Example 1.2.4. The initial k-k'-biring is k with the identity as each structure map. This represents the zero functor sending each k-algebra to the k'-algebra with one element.

Example 1.2.5. Consider the identity functor on $\mathbf{Alg}_k$. This is represented by the

k-k-biring, $\mathcal{I}$ with underlying k-algebra, $k[\iota]$. The structure maps are given by

$$\begin{aligned}
\Delta^+(\iota) &= 1 \otimes \iota + \iota \otimes 1 \\
\varepsilon^+(\iota) &= 0 \\
\sigma(\iota) &= -\iota \\
\Delta^\times(\iota) &= \iota \otimes \iota \\
\varepsilon^\times(\iota) &= 1 \\
\beta\lambda(\iota) &= \lambda.
\end{aligned}$$

Example 1.2.6. Consider the functor $A \mapsto A[[t]]$, taking a k-algebra to the k-algebra of power series over a single variable. This is represented by the k-algebra $B = k[x_0, x_1, \dots]$. The isomorphism $\theta\colon \mathbf{Alg}_k(k[x_0, x_1, \dots], A) \cong A[[t]]$ is $f \mapsto \sum f(x_i)t^i$ and the inverse is $\theta^{-1}(\sum a_i t^i)(x_n) = a_n$. Now the coaddition on B is the map $i_1 + i_2 \in \mathbf{Alg}_k(B, B \otimes B)$ which we can compute by noting

$$\begin{aligned}
(i_1 + i_2)(x_n) &= \theta^{-1}(\theta(i_1) + \theta(i_2))(x_n) \\
&= \theta^{-1}\left(\sum (x_i \otimes 1)t^i + \sum (1 \otimes x_n)t^i\right)(x_n) \\
&= \theta^{-1}\left(\sum (x_i \otimes 1 + 1 \otimes x_i)t^i\right)(x_n) \\
&= 1 \otimes x_n + x_n \otimes 1.
\end{aligned}$$

Arguing similarly for the other structure maps we have

$$\begin{aligned}
\Delta^+(x_n) &= 1 \otimes x_n + x_n \otimes 1 \\
\varepsilon^+(x_n) &= 0 \\
\sigma(x_n) &= -x_n \\
\Delta^\times(x_n) &= \sum_{i+j=n} x_i \otimes x_j \\
\varepsilon^\times(x_n) &= \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \\
\beta\lambda(x_n) &= \lambda.
\end{aligned}$$

Example 1.2.7. The k-algebra $k[t, t^{-1}]$ does not admit the structure of a biring since $\mathbf{Alg}_k(k[t, t^{-1}], A) = A^\times$, the set of invertible elements of A, and this does not naturally admit a ring structure. For example, there is no choice for the zero element.

Definition 1.2.8. Let B and B' be k-k'-birings. We define their tensor product to be the biring with underlying k-algebra $B \otimes B'$ and the structure maps defined analogously to the structure maps in the tensor product of Hopf algebras. This is the coproduct in $\mathbf{Biring}_{k,k'}$

1.2.2 The composition product

Let B be a k-k'-biring and A a k'-algebra. Consider the composition of the functors represented by these objects

$$\mathbf{Alg}_k \xrightarrow{\mathbf{Alg}_k(B,-)} \mathbf{Alg}_{k'} \xrightarrow{\mathbf{Alg}_{k'}(A,-)} \mathbf{Set}.$$

We now define the non-linear analogue of the tensor product which will represent this composition.

Definition 1.2.9. Define the *composition product* $\odot_{k'} \colon \mathbf{Biring}_{k,k'} \times \mathbf{Alg}_{k'} \to \mathbf{Alg}_k$ on objects by setting $B \odot_{k'} A$ to be the free k-algebra on the generators

$$\{b \odot a : b \in B, a \in A\}$$

quotiented by the ideal generated by relations

$$\begin{aligned}
(b_1 + b_2) \odot a &= b_1 \odot a + b_2 \odot a \\
(b_1 b_2) \odot a &= (b_1 \odot a)(b_2 \odot a) \\
\lambda \odot a &= \lambda \\
b \odot (a_1 + a_2) &= \sum_{(b)} (b_{(1)} \odot a_1)(b_{(2)} \odot a_2) \\
b \odot (a_1 a_2) &= \sum_{(b)} (b_{[1]} \odot a_1)(b_{[2]} \odot a_2) \\
b \odot \lambda &= \beta\lambda(b).
\end{aligned}$$

For morphisms $(f, g) \colon (B_1, A_1) \to (B_2, A_2)$ we define $f \odot g$ to be the algebra map

$$\begin{aligned}
B_1 \odot A_1 &\to B_2 \odot A_2 \\
(b \odot a) &\mapsto f(b) \odot g(a).
\end{aligned}$$

In the linear setting, evaluation naturally gives a map of k-modules $\mathbf{Mod}_k(M, N) \otimes M \to N$. Similarly, in the non-linear setting evaluation gives a map of k-algebras $\mathbf{Set}(k', k) \odot k' \to k$, $r \odot \lambda \mapsto r(\lambda)$ where $\mathbf{Set}(k', k)$ is the k-k'-biring of Example 1.2.3.

Recall the tensor-hom adjunction: for a k-k'-bimodule M the functor $M \otimes_{k'} - \colon \mathbf{Mod}_{k'} \to \mathbf{Mod}_k$ is left adjoint to $\mathbf{Mod}_k(M, -) \colon \mathbf{Mod}_k \to \mathbf{Mod}_{k'}$. We have the following non-linear analogue.

Proposition 1.2.10. *Let B be a k-k'-biring. The functor $B \odot -\colon \mathbf{Alg}_{k'} \to \mathbf{Alg}_k$ is left adjoint to the (lift of the) functor represented by B, $\mathbf{Alg}_k(B,-)\colon \mathbf{Alg}_k \to \mathbf{Alg}_{k'}$. Moreover, for a k-algebra A and k'-algebra A' the hom-set bijection is given by*

$$\begin{aligned} \mathbf{Alg}_k(B \odot A', A) &\cong \mathbf{Alg}_{k'}(A', \mathbf{Alg}_k(B, A)) \\ f &\mapsto \big[a' \mapsto \big[b \mapsto f(b \odot a')\big]\big] \\ \big[b \odot a' \mapsto g(a')(b)\big] &\leftarrow g. \end{aligned}$$

Proof: It is straightforward to check the given maps are well-defined, mutually inverse and are natural in A and B. $\square$

Corollary 1.2.11. *For a k-k'-biring B and a k'-algebra A', the functor $\mathbf{Alg}_k \xrightarrow{\mathbf{Alg}_k(B,-)} \mathbf{Alg}_{k'} \xrightarrow{\mathbf{Alg}_{k'}(A',-)} \mathbf{Set}$ is represented by $B \odot A'$.*

Proof: For a k-algebra A, we have $\mathbf{Alg}_k(B \odot A', A) \cong \mathbf{Alg}_{k'}(A', \mathbf{Alg}_k(B, A))$. $\square$

This adjunction allows us to easily compute some examples of the composition product.

Example 1.2.12. Let B be a k-k'-algebra. For all k-algebras A, we have

$$\mathbf{Alg}_k(B \odot k'[x], A) \cong \mathbf{Alg}_{k'}(k'[x], \mathbf{Alg}_k(B, A)) \cong \mathbf{Alg}_k(B, A)$$

Thus $B \odot k'[x] \cong B$.

Since $B \odot -$ is a left adjoint, it preserves colimits. It will prove useful for calculations to compute some of these isomorphisms explicitly.

Proposition 1.2.13. *For a k-k'-biring, B, and k'-algebras A, A' we have the following isomorphisms of k-algebras.*

$$\begin{aligned} B \odot (A \otimes A') &\cong (B \odot A) \otimes (B \odot A') \\ B \odot k' &\cong k \end{aligned}$$

These isomorphisms are determined on the algebra generators by

$$\begin{aligned} b \odot (a \otimes a') &\mapsto (b_{[1]} \odot a) \otimes (b_{[2]} \odot a') \\ (b \odot (a \otimes 1))(b' \odot (1 \otimes a')) &\leftarrow (b \odot a) \otimes (b' \odot a') \\ b \odot \lambda' &\mapsto \beta\lambda'(b) \\ \lambda &\leftarrow \lambda. \end{aligned}$$

Proof: From the standard proof of left adjoints preserving coproducts, we have the following isomorphisms.

$$\begin{aligned}\mathbf{Alg}_k(B \odot (A \otimes A'), X) &\cong \mathbf{Alg}_k(A \otimes A', \mathbf{Alg}_k(B, X)) \\ &\cong \mathbf{Alg}_k(A, \mathbf{Alg}_k(B, X)) \otimes \mathbf{Alg}_k(A', \mathbf{Alg}_k(B, X)) \\ &\cong \mathbf{Alg}_k(B \odot A, X)) \otimes \mathbf{Alg}_k(B \odot A', X) \\ &\cong \mathbf{Alg}_k((B \odot A) \otimes (B \odot A'), X).\end{aligned}$$

To obtain the isomorphism $(B \odot A) \otimes (B \odot A') \to B \odot (A \otimes A')$, we set $X = B \odot (A \otimes A')$ and track the image of the identity map through this composition. The appearance of $\Delta^{\times}$ in the reverse direction comes from the ring structure on $\mathbf{Alg}_k(B, X)$. The initial object case is similar. $\square$

Given a k-k'-bimodule M and a k'-module N, the tensor product $M \otimes N$ is a priori a k-module. However if N is not just a k'-module but a k'-k''-bimodule, then $M \otimes N$ is a k-k''-bimodule. Similarly, we have additional structure on $B \odot B'$ when B' is not only a k'-algebra, but a k'-k''-biring.

Proposition 1.2.14. *The composition product* $- \odot -\colon \mathbf{Biring}_{k,k'} \times \mathbf{Alg}_{k'} \to \mathbf{Alg}_k$ *lifts to a functor* $\mathbf{Biring}_{k,k'} \times \mathbf{Biring}_{k',k''} \to \mathbf{Biring}_{k,k''}$.

Proof: Let B be a k-k'-biring and B' be a k'-k''-biring. Using our adjunction we see

$$\mathbf{Alg}_k(B \odot B', -) \cong \mathbf{Alg}_{k'}(B', \mathbf{Alg}_k(B, -)) \cong \mathbf{Alg}_{k'}(B', -) \circ \mathbf{Alg}_k(B, -).$$

Hence $\mathbf{Alg}_k(B \odot B', -)$ lifts to a functor $\mathbf{Alg}_k \to \mathbf{Alg}_{k''}$ and so $B \odot B'$ admits the structure of a k-k''-biring. $\square$

Corollary 1.2.15. *Let B be a k-k'-biring, B' a k'-k''-biring and A a k-algebra. The hom-set bijection* $\mathbf{Alg}_k(B \odot B', A) \cong \mathbf{Alg}_{k'}(B', \mathbf{Alg}_k(B, A))$ *is an isomorphism of k''-algebras.*

Proof: The k''-algebra structure on $\mathbf{Alg}_k(B \odot B', A)$ is induced by the co-k''-algebra object structure on $B \odot B'$ which is in turn induced by the co-k''-algebra object structure on B'. $\square$

This abstract approach is very succinct but obfuscates the structure of the biring $B \odot B'$ which will be useful for performing calculations. In the following result we explicitly detail the structure maps on $B \odot B'$.

Proposition 1.2.16. *Let B be a k-k'-biring and B' a k'-k''-biring The structure maps on the k-k''-biring $B \odot B'$ are given by the following compositions.*

$$\begin{aligned}
\Delta^+ &\colon B \odot B' \xrightarrow{1 \odot \Delta^+} B \odot (B' \otimes B') \cong (B \odot B') \otimes (B \odot B') \\
\varepsilon^+ &\colon B \odot B' \xrightarrow{1 \odot \varepsilon^+} B \odot k' \cong k \\
\sigma &\colon B \odot B' \xrightarrow{1 \odot \sigma} B \odot B' \\
\Delta^\times &\colon B \odot B' \xrightarrow{1 \odot \Delta^\times} B \odot (B' \otimes B') \cong (B \odot B') \otimes (B \odot B') \\
\varepsilon^\times &\colon B \odot B' \xrightarrow{1 \odot \varepsilon^\times} B \odot k' \cong k \\
\beta\lambda &\colon B \odot B' \xrightarrow{1 \odot \beta\lambda} B \odot k' \cong k
\end{aligned}$$

Proof: Let i_1, i_2 be the canonical inclusions $B \odot B' \to (B \odot B') \otimes (B \odot B')$. The coaddition on $B \odot B'$ is given by $i_1 + i_2$ in the k''-algebra $\mathbf{Alg}_k(B \odot B', B \odot B' \otimes B \odot B')$. Now we have an isomorphism of k''-algebras

$$\theta \colon \mathbf{Alg}_k(B \odot B', (B \odot B') \otimes (B \odot B')) \xrightarrow{\cong} \mathbf{Alg}_{k'}(B', \mathbf{Alg}_k(B, B \odot B' \otimes B \odot B'))$$

and so $i_1 + i_2 = \theta^{-1}(\theta(i_1 + i_2)) = \theta^{-1}(\theta(i_1) + \theta(i_2))$. Here $\theta(i_1)$ and $\theta(i_2)$ are the maps

$$\begin{aligned}
\theta(i_1) &\colon b' \mapsto [b \mapsto b \odot b' \otimes 1_{B \odot B'}] \\
\theta(i_2) &\colon b' \mapsto [b \mapsto 1_{B \odot B'} \otimes b \odot b'].
\end{aligned}$$

Now addition in $\mathbf{Alg}_k(B, B \odot B' \otimes B \odot B')$ is induced by the coaddition on B' and so $\theta(i_1) + \theta(i_2)$ is given by the composition

$$\begin{aligned}
B' &\xrightarrow{\Delta^+_{B'}} B' \otimes B' \\
&\xrightarrow{\theta(i_1) \otimes \theta(i_2)} \mathbf{Alg}_k(B, B \odot B' \otimes B \odot B') \otimes \mathbf{Alg}_k(B, B \odot B' \otimes B \odot B') \\
&\xrightarrow{\phi} \mathbf{Alg}_k(B, B \odot B' \otimes B \odot B')
\end{aligned}$$

where ϕ is the multiplication on $\mathbf{Alg}_k(B, B \odot B' \otimes B \odot B')$. Hence,

$$(\theta(i_1) + \theta(i_2))(b') = \sum_{(b')} \theta(i_1)(b'_{(1)}) \theta(i_2)(b'_{(2)}).$$

Now since the multiplication in $\mathbf{Alg}_k(B, B \odot B' \otimes B \odot B')$ is induced by the comultiplication on B, we have

$$\theta(i_1)(b'_{(1)}) \theta(i_2)(b'_{(2)})(b) = \sum_{(b)} (b_{[1]} \odot b'_{(1)} \otimes b_{[2]} \odot b'_{(2)}).$$

However, we have a sum of these maps across the coaddition of b'. Suppose $\Delta^+(b') = \sum_{i=1}^{m} b'^i_{(1)} \otimes b'^i_{(2)}$, and $\Delta^+_{(m)}(b) = \sum_{j=1}^{n} b^j_{(1)} \otimes b^j_{(2)} \otimes \cdots \otimes b^j_{(m)}$. By the definition of addition in $\mathbf{Alg}_{k'}(B', \mathbf{Alg}_k(B, B \odot B' \otimes B \odot B'))$ we have

$$(\theta(i_1) + \theta(i_2))(b')(b) = \sum_{j=1}^{n} \prod_{i=1}^{m} \sum_{(b^j_{(i)})} (b^j_{(i)})_{[1]} \odot b'^i_{(1)} \otimes (b^j_{(i)})_{[2]} \odot b'^i_{(2)} = \Delta^+(b \odot b').$$

It is straightforward to check this is the same as the composition

$$\Delta^+ \colon B \odot B' \xrightarrow{1 \odot \Delta^+} B \odot (B' \otimes B') \cong B \odot B' \otimes B \odot B'.$$

The other structure maps are similar. □

As is the case with bimodules, specialising to the case $k = k' = k''$ yields a monoidal structure on our category. Recall the definition of $\mathcal{I}$ from Example 1.2.5.

Proposition 1.2.17. *The composition product* $- \odot - \colon \mathbf{Biring}_{k,k} \times \mathbf{Biring}_{k,k} \to \mathbf{Biring}_{k,k}$ *is associative with unit* $\mathcal{I}$*. Hence,* $(\mathbf{Biring}_{k,k}, \odot, \mathcal{I})$ *is a monoidal category.*

Proof: The functor represented by $B \odot B'$ is the functor represented by B composed with the functor represented by B'. Hence the associativity of $\odot$ follows from associativity of composition. The unit condition follows from the fact that $\mathcal{I}$ represents the identity functor $\mathbf{Alg}_k \to \mathbf{Alg}_k$. □

1.2.3 Plethories

Just as k-algebras, monoids in the category of k-k-bimodules, are precisely the structures that act linearly on k-modules; monoids in the category of k-k-birings will provide precisely the structure that acts non-linearly on k-algebras.

Definition 1.2.18. We define the category of *k-plethories* $\mathbf{Plethory}_k$ to be the category of monoids in $\mathbf{Biring}_{k,k}$. Explicitly, a k-plethory is a k-k-biring P together with two additional biring morphisms

$$\circ \colon P \odot P \to P \qquad \text{(composition)}$$

$$u \colon \mathcal{I} \to P \qquad \text{(unit)}$$

where $\mathcal{I}$ is the biring of Example 1.2.5 and these morphisms satisfy the following relations.

$$\begin{tikzcd} P \odot P \odot P \arrow[r, "\circ \odot 1"] \arrow[d, "1 \odot \circ"'] & P \odot P \arrow[d, "\circ"] \\ P \odot P \arrow[r, "\circ"] & P \end{tikzcd} \qquad \begin{tikzcd} \mathcal{I} \odot P \arrow[r, "u \odot 1"] \arrow[dr, "\cong"'] & P \odot P \arrow[d, "\circ"] & P \odot \mathcal{I} \arrow[l, "1 \odot u"'] \arrow[dl, "\cong"] \\ & P & \end{tikzcd} \tag{1.6}$$

We will write $r \circ s$ for $\circ(r \odot s)$ and ι for the image $u(\iota)$. Hence, composition is associative and satisfies $\iota \circ r = r = r \circ \iota$ for all $r \in P$.

A morphism of k-plethories $f\colon P \to Q$ is a k-k-biring map such that $f(r \circ s) = f(r) \circ f(s)$ and $f(\iota) = \iota$.

The following example illustrates how composition can arise in a natural way.

Example 1.2.19. If k is finite then the k-k-biring $\mathbf{Set}(k,k)$ of Example 1.2.3 is naturally a plethory with composition given by composition of maps.

Example 1.2.20. The unit object $\mathcal{I}$ in $\mathbf{Biring}_{k,k}$ naturally admits the structure of a k-plethory where the unit $u\colon \mathcal{I} \to \mathcal{I}$ is the identity and the composition $\circ$ is the natural isomorphism $\mathcal{I} \odot \mathcal{I} \cong \mathcal{I}$ given by composition of polynomials. By the definitions, $\mathcal{I}$ is the initial k-plethory.

Example 1.2.21. Analogous to the tensor algebra over a k-module we can construct a functor $F\colon \mathbf{Biring}_{k,k} \to \mathbf{Plethory}_k$ by 'freely adding composition'. For a k-k-biring B, define

$$T_\odot(B) = \bigotimes_{n\geq 0} B^{\odot n}.$$

The maps

$$B^{\odot r} \odot B^{\odot s} \to B^{\odot r+s}$$
$$(b_1 \odot \ldots \odot b_r) \odot (c_1 \odot \ldots \odot c_s) \mapsto b_1 \odot \ldots \odot b_r \odot c_1 \odot \ldots \odot c_s$$

induce an associative map

$$\circ\colon T_\odot(B) \odot T_\odot(B) = \bigotimes_{n\geq 0} B^{\odot n} \odot \bigotimes_{n\geq 0} B^{\odot n} \cong \bigotimes_{r,s\geq 0} B^{\odot r} \odot B^{\odot s} \to \bigotimes_{n\geq 0} B^{\odot n} = T_\odot(B).$$

Moreover, we have the unit map

$$u\colon \mathcal{I} = B^{\odot 0} \to P$$

For a map of k-k-birings $f\colon B_1 \to B_2$, we have a map $T_\odot(f)\colon T_\odot(B_1) \to T_\odot(B_2)$ induced by

$$B_1^{\odot n} \xrightarrow{f^{\odot n}} B_2^{\odot n}.$$

We can justify the analogy to the tensor algebra by noting that $T_\odot$ is left adjoint to the forgetful functor $U\colon \mathbf{Plethory}_k \to \mathbf{Biring}_{k,k}$.

The following result details the structure of a plethory at the level of the functor represented by the plethory.

Proposition 1.2.22. *Let P be a k-k-biring. There is a bijection between k-plethory structures on P and comonad structures on $\mathbf{Alg}_k(P,-)\colon \mathbf{Alg}_k \to \mathbf{Alg}_k$.*

Proof: Suppose P is a k-plethory and let A be a k-algebra. We have natural maps $\mathbf{Alg}_k(P,A) \to \mathbf{Alg}_k(P \odot P, A) \cong \mathbf{Alg}_k(P, \mathbf{Alg}_k(P,A))$ induced by the composition and $\mathbf{Alg}_k(P,A) \to \mathbf{Alg}_k(\mathcal{I},A) \cong A$ induced by the unit. Moreover, since P is a monoid these equip $\mathbf{Alg}_k(P,-)$ with the structure on a comonad. The converse direction is similar. $\square$

1.2.4 P-algebras

Just as we realised k-modules equipped with an action of a k-algebra A as Eilenberg-Moore algebras for the monad $A \otimes_k -$, we will realise k-algebras equipped with an action of a plethory as Eilenberg-Moore algebras for a suitable monad. Throughout this section we suppose P is a k-plethory.

Proposition 1.2.23. *$P \odot_k -$ is a monad on $\mathbf{Alg}_k$ with the natural transformations given by*

$$P \odot P \odot A \xrightarrow{\circ \odot 1} P \odot A$$

$$A \cong I \odot A \xrightarrow{u} P \odot A$$

Proof: The axioms are satisfied by the definition of P as a monoid in $\mathbf{Biring}_{k,k}$. $\square$

Definition 1.2.24. We define the category of *P-algebras* $\mathbf{Alg}_P$ to be the Eilenberg-Moore category of algebras for the monad $P \odot_k -$. Explicitly, a P-algebra is a k-algebra A together with a k-algebra map

$$\circ\colon P \odot A \to A$$

satisfying the usual axioms, namely for all $r, s \in P$, $a \in A$,

$$(r \circ s) \circ a = r \circ (s \circ a)$$

$$\iota \circ a = a.$$

A morphism of P-algebras $f\colon A_1 \to A_2$ is a map of k-algebras which commutes with $\circ$ in that for all $r \in P$, $a \in A$,

$$f(r \circ a) = r \circ f(a).$$

Unpacking the statement that $\circ$ is an algebra map gives the following formulae. For all $r, s \in P$, $\lambda \in k$, $a, a' \in A$,

$$\begin{aligned}
(r+s) \circ a &= (r \circ a) + (s \circ a) \\
(rs) \circ a &= (r \circ a)(s \circ a) \\
\lambda \circ a &= \lambda \\
r \circ (a + a') &= (r_{(1)} \circ a)(r_{(2)} \circ a') \\
r \circ (aa') &= (r_{[1]} \circ a)(r_{[2]} \circ a') \\
r \circ \lambda &= \beta\lambda(r).
\end{aligned}$$

Let P be a k-plethory. It is straightforward to check that k is the initial P-algebra. The P-action is given by the isomorphism $P \odot k \cong k$. Similarly, if A and A' are P-algebras then the tensor product $A \otimes A'$ of k-algebras is the coproduct in $\mathbf{Alg}_P$. The P-action is given by the composition

$$P \odot (A \otimes A') \cong P \odot A \otimes P \odot A' \to A \otimes A'.$$

Proposition 1.2.25. *Let A be a P-algebra. The structure maps $\eta \colon k \to A$ and $\mu \colon A \otimes A \to A$ are maps of P-algebras.*

Proof: Let $r \in P, \lambda \in k$. We have $r \circ \eta(\lambda) = \eta(\beta\lambda(r)) = \eta(r \circ \lambda)$. For $a, a' \in A$, $r \circ (aa') = (r_{[1]} \circ a)(r_{[2]} \circ a') = \mu(r \circ (a \otimes a'))$. $\square$

Several constructions from the linear world have analogues in this setting. Every k-module has a natural action of k. Given a k-module M we can construct the free A-module $A \otimes M$, and A is the universal A-module in that $\mathbf{Mod}_A(A, N) \cong N$ for all A-modules N.

Analogously, every k-algebra admits a unique $\mathcal{I}$-algebra structure. Given a k-algebra A the free P-algebra is the k-algebra $P \odot A$ with action

$$P \odot P \odot A \xrightarrow{\circ \odot 1} P \odot A.$$

This construction gives a functor $\mathbf{Alg}_k \to \mathbf{Alg}_P$ which is left adjoint to the forgetful functor $U \colon \mathbf{Alg}_P \to \mathbf{Alg}_k$. Any k-plethory P is naturally a P-algebra with the action $P \odot P \to P$ given by composition.

Proposition 1.2.26. *Given a P-algebra A and $a \in A$, there exists a unique P-algebra map $f \colon P \to A$ such that $f(\iota) = a$. Hence a P-algebra map $f \colon P \to A$ is uniquely determined by $f(\iota)$ and we have an isomorphism of P-algebras $\mathbf{Alg}_P(P, A) \cong A$.*

Proof: Define $f(r) = r \circ a$. This is trivially a P-algebra map with $f(\iota) = a$. If g is another map with this property then $f(r) - g(r) = f(r \circ \iota) - g(r \circ \iota) = r \circ f(\iota) - r \circ g(\iota) = 0$. $\square$

In many algebraic settings we are often interested in operations on some category of k-algebras, by which we mean natural transformations from the forgetful functor down to **Set** to itself. The study of such operations naturally fits into the framework of plethories.

Definition 1.2.27. An operation on P-algebras is a natural transformation $r\colon U \to U$ where U is the forgetful functor $U\colon \mathbf{Alg}_P \to \mathbf{Set}$. Hence an operation r on P-algebras is a collection of set maps $r_A\colon A \to A$ for all A in $\mathbf{Alg}_P$ such that if $f\colon A \to A'$ is a map of P-algebras, the following diagram commutes.

$$\begin{array}{ccc} A & \xrightarrow{r_A} & A \\ \big\downarrow f & & \big\downarrow f \\ A' & \xrightarrow{r_{A'}} & A' \end{array}$$

We denote the set of operations on P-algebras by $\mathrm{Op}(\mathbf{Alg}_P)$. The pointwise operations naturally make $\mathrm{Op}(\mathbf{Alg}_P)$ a k-algebra.

Proposition 1.2.28. *$\mathrm{Op}(\mathbf{Alg}_P)$ is naturally isomorphic to P as a k-algebra, and composition in $\mathrm{Op}(\mathbf{Alg}_P)$ corresponds to composition in P.*

Proof: Since the forgetful functor $U\colon \mathbf{Alg}_P \to \mathbf{Set}$ is represented by P, by the Yoneda embedding $\mathrm{Op}(\mathbf{Alg}_P) \cong \mathbf{Alg}_P(P, P) \cong P$. $\square$

The following result gives a conceptual way of understanding the co-k-algebra object structure maps in terms of how the operations respect the k-algebra structure on P-algebras.

Theorem 1.2.29. *Let P be a k-plethory. Then for $r \in P$,*

$$\begin{aligned}
\Delta^+(r) = \sum_{(r)} r_{(1)} \otimes r_{(2)} &\iff r \circ (x + y) = \sum_{(r)} (r_{(1)} \circ x)(r_{(2)} \circ y) \\
\varepsilon^+(r) = \mu &\iff r \circ 0 = \mu \\
\sigma(r) = s &\iff r \circ (-x) = s \circ x \\
\Delta^\times(r) = \sum_{(r)} r_{[1]} \otimes r_{[2]} &\iff r \circ (xy) = \sum_{(r)} (r_{[1]} \circ x)(r_{[2]} \circ y) \\
\varepsilon^\times(r) = \mu &\iff r \circ 1 = \mu \\
\beta\lambda(r) = \mu &\iff r \circ \lambda = \mu
\end{aligned}$$

for all P-algebras A and all $x, y \in A$.

Proof: ($\Longrightarrow$) This direction is immediate by the definition of a P-algebra.

($\Longleftarrow$) This statement holds for all P-algebras and in particular for P. Hence $r \circ (x+y) = \sum (r_{(1)} \circ x)(r_{(2)} \circ y)$ for all $x, y \in P$ but $r \circ (x+y) = \mu(\Delta^+(r) \circ (x \otimes y))$ and so $\Delta^+(r) = r_{(1)} \otimes r_{(2)}$. The remaining maps are similar. $\square$

1.2.5 A detailed example: λ-rings

In this section we will expand on a result of Tall and Wraith [50] and show how the theory of λ-rings fits neatly into the framework of plethystic algebra. A λ-ring is a commutative ring together with additional operations and these objects arise naturally in many areas of mathematics including K-theory and representation theory. We recall the basic definitions and results necessary for illustrating how the theory of λ-rings fits into a plethystic framework. For a detailed exposition of λ-rings, refer to [58].

Definition 1.2.30. A *λ-ring* is a ring R together with *λ-operations*: $\lambda^n \colon R \to R$ for $n \geq 0$ such that for all $x, y \in R$ we have the following properties.

1. $\lambda^0(x) = 1$,
2. $\lambda^1(x) = x$,
3. $\lambda^n(1) = 0$ for $n \geq 2$,
4. $\lambda^n(x+y) = \sum_{i+j=n} \lambda^i(x)\lambda^j(y)$,
5. $\lambda^n(xy) = P_n(\lambda^1(x), \ldots, \lambda^n(x); \lambda^1(y), \ldots, \lambda^n(y))$,
6. $\lambda^n(\lambda^m(x)) = P_{n,m}(\lambda^1(x), \ldots, \lambda^{nm}(x))$.

The polynomials $P_n \in \mathbb{Z}[x_1, \ldots x_n; y_1, \ldots y_n]$ and $P_{n,m} \in \mathbb{Z}[x_1, \ldots, x_{nm}]$ are the *universal polynomials* which can be described as follows. If $\sigma_1, \ldots, \sigma_n$ are the elementary symmetric polynomials in $x_1, \ldots, x_n$ and $\tau_1, \ldots, \tau_n$ are the elementary symmetric polynomials in $y_1, \ldots, y_n$ then P_n is the unique polynomial such that $P_n(\sigma_1, \ldots, \sigma_n; \tau_1, \ldots, \tau_n)$ is the coefficient of t^n in

$$\prod_{i,j}(1 + x_i y_j t).$$

If $\sigma_1, \ldots, \sigma_{nm}$ are the elementary symmetric polynomials in $x_1, \ldots, x_{nm}$ then $P_{n,m}$ is the unique polynomial such that $P_{n,m}(\sigma_1, \ldots, \sigma_{nm})$ is the coefficient of t^n in

$$\prod_{1 \leq i_1 < \cdots < i_m \leq nm} (1 + x_{i_1} \ldots x_{i_m} t).$$

We shall write $\lambda_t(x) = \sum \lambda^n(x)t^n$. A ring map $f\colon R \to S$ is a map of λ-rings if $\lambda^n f(x) = f\lambda^n(x)$ for all $x \in R$ and all $n \geq 0$. We denote the category of λ-rings by $\mathbf{Ring}_\lambda$.

Example 1.2.31. The ring of integers $\mathbb{Z}$ is naturally a λ-ring with operations given by

$$\lambda^n(k) = \binom{k}{n}.$$

Example 1.2.32. If X is a compact Hausdorff space then $K(X)$, the complex K-theory of X, naturally has the structure of a λ-ring. The λ-operations induced by the exterior power operations on vector bundles and we explain this in more detail in Section 3.2. The classical reference is [10].

Definition 1.2.33. Let R be a ring. Define the *universal λ-ring* on R to be the set of formal power series in R with constant term 1. That is to say,

$$\Lambda(R) = 1 + tR[[t]].$$

The addition, multiplication and λ-operations on $\Lambda(R)$ are defined by the following formulae. For $f = 1 + \sum_n a_n t^n, g = 1 + \sum_n b_n t^n$, we have

$$\begin{aligned} f + g &= (1 + \sum_n a_n t^n)(1 + \sum_n b_n t^n) \\ fg &= 1 + \sum_n P_n(a_1, \dots, a_n; b_1, \dots, b_n)t^n \\ \lambda^m(f) &= 1 + \sum_n P_{n,m}(a_1, \dots, a_{nm})t^n. \end{aligned}$$

Given a map of rings $f\colon R \to S$ we define a map of λ-rings $\Lambda(f)\colon \Lambda(R) \to \Lambda(S)$ by

$$\Lambda(f)\left(1 + \sum a_n t^n\right) = 1 + \sum f(a_n)t^n.$$

In particular, we can view Λ as an endofunctor on $\mathbf{Ring}$.

Proposition 1.2.34. *The functor $\Lambda\colon \mathbf{Ring} \to \mathbf{Ring}$ is represented by the ring of symmetric functions $\Omega = \mathbb{Z}[s_1, s_2, \dots]$. Explicitly, we have natural maps $\theta\colon \mathbf{Ring}(\Omega, R) \to \Lambda(R)$ and $\theta^{-1}\colon \Lambda(R) \to \mathbf{Ring}(\Omega, R)$ given by*

$$\begin{aligned} \theta(\phi) &= 1 + \sum \phi(s_n)t^n, \\ \theta^{-1}(1 + \sum a_n t^n)(s_m) &= a_m. \end{aligned}$$

Proof: It is straightforward to check the given maps are inverses. □

Hence by Corollary 1.1.17, the ring Ω admits the structure of a $\mathbb{Z}$-$\mathbb{Z}$-biring.

Corollary 1.2.35. *Let B denote the $\mathbb{Z}$-$\mathbb{Z}$-biring which is isomorphic to $\mathbb{Z}[s_1, s_2, \dots]$ as a ring and with coalgebraic structure given by the following formulae, where we write $s_0 = 1$.*

$$\begin{aligned}
\Delta^+(s_n) &= \sum_{i+j=n} s_i \otimes s_j \\
\varepsilon^+(s_n) &= 0 \\
\Delta^\times(s_n) &= \sum_{i+j=n} P_n(s_1 \otimes 1, \dots, s_n \otimes 1; 1 \otimes s_1, \dots, 1 \otimes s_n) \\
\varepsilon^\times(s_n) &= \begin{cases} 1 & n = 1 \\ 0 & \textit{otherwise.} \end{cases}
\end{aligned}$$

The antipode is defined inductively by

$$\begin{aligned}
\sigma(s_1) &= -s_1 \\
\sigma(s_n) &= -s_n - \sum_{i=1}^{n-1} \sigma(s_i) s_{n-i}.
\end{aligned}$$

We have an isomorphism of $\mathbb{Z}$-$\mathbb{Z}$-birings

$$\Omega \cong B.$$

Proof: Just as in Example 1.2.6, we compute the structure maps via the isomorphism $\theta \colon \mathbf{Ring}(\Omega, R) \to \Lambda(R)$. For the coaddition, note

$$\begin{aligned}
\theta(i_1) + \theta(i_2) &= (1 + \sum_i (s_i \otimes 1) t^i)(1 + \sum_j (1 \otimes s_j) t^j) \\
&= \sum_n \left(\sum_{i+j=n} s_i \otimes s_j \right) t^n.
\end{aligned}$$

Hence $\Delta^+(s_n) = (i_1 + i_2)(s_n) = \sum_{i+j=n} s_i \otimes s_j$. The remaining results are similar. □

However, we have additional structure on the representable functor Λ and this will induce the structure on a $\mathbb{Z}$-plethory on the representing ring Ω.

Proposition 1.2.36 ([58, Theorem 2.25])**.** *We have natural transfomations $\psi \colon \Lambda(R) \to \Lambda^2(R)$ and $\varepsilon \colon \Lambda(R) \to R$ given by*

$$\begin{aligned}
\psi(1 + \sum a_n t^n) &= 1 + \sum \left(1 + \sum P_{n,m}(a_1, \dots, a_{nm}) t^n\right) t^m \\
\varepsilon(1 + \sum a_n t^n) &= a_1.
\end{aligned}$$

Moreover, the triple $(\Lambda, \psi, \varepsilon)$ forms a comonad on **Ring***.*

Applying Proposition 1.2.22, we see that Ω admits the structure on a $\mathbb{Z}$-plethory.

Corollary 1.2.37. *We have an isomorphism of $\mathbb{Z}$-plethories*

$$\Omega \cong \mathbb{Z}[s_1, s_2, \dots]$$

where the $\mathbb{Z}$-$\mathbb{Z}$-biring structure is detailed in Corollary 1.2.35 and the composition is specified by the following formulae.

$$s_m \circ s_n = P_{m,n}(s_1, \dots, s_{mn})$$
$$\iota = s_1$$

Proof: We have natural maps

$$\mathbf{Ring}(\Omega, R) \cong \Lambda(R) \to \Lambda(R)^2 \cong \mathbf{Ring}(\Omega, \mathbf{Ring}(\Omega, R)) \cong \mathbf{Ring}(\Omega \odot \Omega, R)$$

and the composition $\Omega \odot \Omega \to \Omega$ is the image of the identity map $\Omega \to \Omega$ under this composition. Now the image of the identity in $\Lambda(\Omega)^2$ is the power series $1 + \sum (1 + \sum P_{n,m}(s_1, \dots, s_{nm}) t^n)\, t^m$ which corresponds to the map f with $f(s_m)(s_n) = P_{n,m}(s_1, \dots, s_{nm})$. Hence $s_m \circ s_n = P_{m,n}(s_1, \dots, s_{mn})$. The unit for the composition u is the image of the identity $\Omega \to \Omega$ under the sequence of natural maps

$$\mathbf{Ring}(\Omega, R) \cong \Lambda(R) \to R \cong \mathbf{Ring}(\mathcal{I}, R).$$

Hence $u(\iota) = s_1$. □

We can now realise λ-rings as algebras over our plethory Ω.

Corollary 1.2.38. *Any λ-ring A is naturally an Ω-ring and vice-versa.*

Proof: Given any λ-ring A we make A an Ω-algebra by defining $s_n \circ a = \lambda^n(a)$. Conversely, if A is an Ω-ring the operations $\lambda^n(a) = s_n \circ r$ make A into a λ-ring. □

By an operation on λ-rings we mean a natural transformation from the forgetful functor $U\colon \mathbf{Ring}_\lambda \to \mathbf{Set}$ as in [58, Definition 1.43]. Denote the collection of such operations by $\mathrm{Op}(\mathbf{Ring}_\lambda)$. We can now prove the verification principle for λ-rings which states that every operation on λ-rings is a polynomial in the λ-operations.

Corollary 1.2.39. *We have an isomorphism of rings* $\mathrm{Op}(\mathbf{Ring}_\lambda) \cong \Omega$. *Moreover, composition of operations corresponds to composition in Ω.*

Proof: Apply Proposition 1.2.28. □

1.3 Plethystic theory

We have seen plethories are powerful algebraic objects capable of encoding all the algebraic structure of the collection of operations on rings. Unfortunately and unavoidably, these objects are often extremely large and as a result can be unwieldy. The algebraic theory of plethories was first studied in detail by Borger and Wieland [17]. In this section we will study the structure of plethories and morphisms between them in an effort to isolate important structure of a plethory in a simpler context.

We have natural forgetful functors from the categories of birings and plethories to the more familiar categories of Hopf algebras, algebras and modules. All of these categories are extremely well studied and there is a very rich theory underpinning such objects. A major theme of this section will be extending results from these categories to the categories of birings and plethories.

The linear dual of a biring is an algebraic object known as a Hopf ring and will be much more familiar to topologists. However the Hopf ring contains no structure which is dual to the composition in a plethory. Extending an idea of Boardman and Johnson [15] we provide additional algebraic structure on a Hopf ring which dualises to the composition in a plethory.

The theory of Hopf algebras [36] [47] teaches us there is a lot of power in linearisation methods. These convert a complicated non-linear object, such as a Hopf algebra, and produce a simpler linear object, such as a module, in a way which hopefully preserves interesting information from the original object. Two particular useful and tractable examples of linearisation are the functor of primitives and the functor of indecomposables. These both map Hopf algebras to modules. Using a forgetful functor to the category of Hopf algebras, we are able to apply these functors to our categories of birings and plethories. However it seems natural that some of the extra structure on our objects may transfer to the image of our linearisation functors. In this section, we will detail how this works, and also discuss natural generalisations of these functors exclusive to the world of birings and plethories.

Throughout this section, we introduce several functors to and from the categories of plethories and related categories. It may prove helpful to refer to appendix Appendix B where these functors are illustrated diagrammatically.

1.3.1 Sub-birings and sub-plethories

Given an algebraic object A, a sub-object of A' is an algebraic object of the same type with an injective map $A' \to A$. Many properties of the algebraic object may not depend on the entire structure, but just that of a small sub-object and many problems are easier to solve by working with a specific sub-object.

When working with structures involving tensor products of modules there is a technical point we must be careful with. As illustrated by the following example, if N is a sub-module of M then in general $N \otimes N$ will not be a sub-module of $M \otimes M$.

Example 1.3.1. Let M be a k-module with sub-module $N \overset{i}{\hookrightarrow} M$. For an arbitrary k-module X the map $N \otimes X \xrightarrow{i\otimes 1} M \otimes X$ is not necessarily injective. For example, consider $k = M = \mathbb{Z}, N = n\mathbb{Z} \subseteq M, X = \mathbb{Z}/n\mathbb{Z}$ then the image of $N \otimes X$ under $i \otimes 1$ is zero. This is because in general $- \otimes X$ is not left exact.

Definition 1.3.2. Let M be a k-module with sub-module $N \overset{i}{\hookrightarrow} M$. For an arbitrary k-module X, we say P is an *X-pure* sub-module if $N \otimes X \xrightarrow{i\otimes 1} M \otimes X$ is injective. We say P is a *pure* sub-module if it is X-pure for all X.

This definition is useful for giving a simple classification of sub-coalgebras. Let C be a k-coalgebra and sub-module $C' \subseteq C$. A priori, $\psi_C(C') \subseteq C \otimes C$. However if C' is a k-coalgebra then we have a comultiplication $\psi_{C'} \colon C' \subseteq C' \otimes C'$ and counit $\varepsilon_{C'} \colon C' \to k$ and we can wonder under what conditions C' is a sub-coalgebra of C. The inclusion $i \colon C' \to C$ is a map of k-colagebras if the following diagrams commute.

$$\begin{tikzcd} C' \arrow[r, "i"] \arrow[d, "\psi_{C'}"] & C \arrow[d, "\psi_C"] \\ C' \otimes C' \arrow[r, "i\otimes i"] & C \otimes C \end{tikzcd} \qquad \begin{tikzcd} C' \arrow[r, "i"] \arrow[dr, "\varepsilon_{C'}"'] & C \arrow[d, "\varepsilon_C"] \\ & k \end{tikzcd}$$

The first of these diagrams requires $(i \otimes i)\psi_{C'}$ to be the restriction of ψ_C to C'. If C' is C-pure, then $C \otimes C'$ and $C' \otimes C$ are sub-modules of $C \otimes C$ and hence so is $C' \otimes C' = (C \otimes C') \cap (C' \otimes C)$ with inclusion given by $i \otimes i$. Thus, we require $\psi_{C'}$ to be the restriction of ψ_C to C'. The second diagram simply states that $\varepsilon_{C'}$ is the restriction of ε_C to C'. Hence, if C' is an arbitrary C-pure sub-module of C then the restriction of the structure maps on C make C' a sub-coalgebra if and only if $\psi(C') \subseteq C \otimes C$.

Proposition 1.3.3. *If M is a flat k-module then every sub-module of M is pure. In particular, if k is a field then all sub-modules are pure.*

Proof: This is the definition of a flat k-module. Moreover, over a field every k-module is flat. $\square$

Proposition 1.3.4. *Suppose B is a k-k'-biring. Let B' be a sub-algebra of B which is B-pure as a sub-module. The k-algebra B' together with the restrictions of the biring structure maps to B' is a sub-biring if and only if the following conditions hold.*

1. $\Delta^{+}(B') \subseteq B' \otimes B'$.

2. $\Delta^{\times}(B') \subseteq B' \otimes B'$.

Proof: If $i\colon B' \to B$ is an inclusion of k-k'-birings, the conditions hold immediately via the definition of a k-k'-biring map. Conversely, if conditions (1) and (2) hold then since the structure maps of B' are the restrictions of the structure maps on B it is immediate that the inclusion of algebras $i\colon B' \to B$ satisfies the commutative diagrams (see Appendix A) necessary to be a map of k-k'-birings. $\square$

Finding explicit sub-birings can often by quite difficult due to the vast amount of structure on birings. It can be easier to determine sub-birings by considering pointwise epi natural transformations between their representing functors. By the Yoneda lemma such a natural transformation will induce an inclusion of birings.

Example 1.3.5. Recall the k-k-biring B of Example 1.2.6 which represents the power series functor. The truncation of power series gives a pointwise epi natural transformation between the functor represented by B and the functor sending a k-algebra A to the truncated polynomial algebra $A[t]/(t^n)$. This functor is represented by the sub-biring of B given by $k[x_0, x_1, \ldots, x_n]$.

Just as $- \otimes X$ is not left exact for an arbitrary module X, nor is $- \odot X$ for a biring X. As an example, consider the initial $\mathbb{Z}$-$\frac{\mathbb{Z}}{n\mathbb{Z}}$-biring $\mathbb{Z}$ and the initial $\frac{\mathbb{Z}}{n\mathbb{Z}}$-$k''$-biring $\mathbb{Z}/n\mathbb{Z}$. The inclusion $i\colon n\mathbb{Z} \to \mathbb{Z}$ is a biring map, but the image of $n\mathbb{Z} \odot \mathbb{Z}/n\mathbb{Z}$ under $i \otimes 1$ is zero. However, as in the case of algebras, this does not cause us problems: given a k-algebra A and a sub-module $A' \overset{i}{\hookrightarrow} A$ we can define a multiplication on A' via the composite $A' \otimes A' \xrightarrow{i \otimes i} A \otimes A \xrightarrow{\mu} A$ without any need for purity conditions.

Proposition 1.3.6. *Let P be a k-plethory and suppose P' is a sub-biring of P. The sub-biring P' together with composition given by $P' \odot P' \xrightarrow{i \odot i} P \odot P \xrightarrow{\circ} P$ is a sub-plethory if and only if the following two conditions hold.*

1. $P' \circ P' \subseteq P'$.

2. $\iota \in P'$.

Proof: This is immediate from the definitions. $\square$

1.3.2 Augmentations

Recall for a Hopf algebra H, we define the *augmentation ideal* $IH = \ker \varepsilon$ and *coaugmentation quotient* $JH = \operatorname{coker} \eta$. These are very useful constructions that appear in various results and it turns out that they have natural generalisations to the setting of birings and plethories.

Definition 1.3.7. Let B be a k-k'-biring, we define the *augmentation ideal* of B to be $IB = \ker \varepsilon^+$.

By construction, IB is an algebra ideal of B. The following results hold in the setting of Hopf algebras and thus generalise to birings and plethories by applying the forgetful functor $\mathbf{Biring}_{k,k'} \to \mathbf{Hopf}_k$.

Proposition 1.3.8. *Let B be a k-k'-biring. For $x \in IB$, $\Delta^+(x)$ is in the image of the map $IB \otimes B \oplus B \otimes IB \to B \otimes B$.*

Proof: By definition of the co-unit, we have $x = \varepsilon^+(x_{(1)})x_{(2)}$ and so $\varepsilon^+(x_{(1)})\varepsilon^+(x_{(2)}) = \varepsilon^+(x) = 0$. The result follows. $\square$

We can use a similar argument to deduce the behaviour of the co-multiplication on the augmentation ideal.

Proposition 1.3.9. *Let B be a k-k'-biring. For $x \in IB$, $\Delta^\times(x)$ is in the image of the map $IB \otimes IB \to B \otimes B$.*

Proof: The image of $IB \otimes B$ in $B \otimes B$ is the kernel of the map $\varepsilon^+ \otimes 1$. By the dual of the relation expressing $x0 = 0$ in an algebra, we have $(\varepsilon^+ \otimes 1)(x_{[1]} \otimes x_{[2]}) = \varepsilon^+(x_{[1]}) \otimes x_{[2]} = \varepsilon^+(x) = 0$. Thus $\Delta^\times(x)$ is in the image of $IB \otimes B \to B \otimes B$. Similarly we show $\Delta^\times(x)$ is in the image of $B \otimes IB \to B \otimes B$ and the result follows.

The following statement gives a characterisation of elements of the augmentation ideal of a k-plethory P in terms of their actions on P-algebras and will be particularly relevant when we reach the topological setting.

Proposition 1.3.10. *Let P be a k-plethory. We have $r \in IP$ if and only if $r(0) = 0$ in all P-algebras.*

Proof: This is an application of Theorem 1.2.29. $\square$

We now study how the composition in a k-plethory P respects the augmentation ideal.

Proposition 1.3.11. *Let P be a k-plethory. The following statements are true.*

1. *If $r, s \in IP$ then $r \circ s \in IP$*
2. *If $s \in IP$ and $r \circ s \in IP$ then $r \in IP$.*

Proof: (1) If $r, s \in IP$ then for all P-algebras, $(r \circ s)(0) = r(s(0)) = r(0) = 0$ and so $r \circ s \in IP$.

(2) If $s, r \circ s \in IP$ then for all P-algebras, $0 = r(s(0)) = r(0)$ and so $r \in IP$. $\square$

We remark that since we are dealing with non-linear actions, if $r \in IP$ and $r \circ s \in IP$ then we do not necessarily have $s \in IP$. Indeed, we have $0 = r(s(0)) = \beta[s(0)](r)$ for all P-algebras.

1.3.3 Ideals

Just as in the world of algebras and Hopf algebras, it is useful to study *ideals* of birings and plethories. By an ideal of a biring (or plethory), we mean precisely the structure which upon quotienting the biring (resp. plethory) by we obtain another biring (resp. plethory). Being able to quotient out structure is extremely useful as it will allow us to exclude irrelevant information and work in a much simpler setting.

Definition 1.3.12. We define a *biring ideal* of B to be the kernel of a biring map $f\colon B \to B'$.

The following result provides a useful characterisation of biring ideals.

Proposition 1.3.13. *Let B be a k-k'-biring, $\mathcal{J}$ a sub-module of B and $\pi\colon B \to B/\mathcal{J}$ the canonical projection. The following statements are equivalent.*

1. *$\mathcal{J}$ is a biring ideal of B.*
2. *$B/\mathcal{J}$ is a k-k'-biring and $\pi\colon B \to B/\mathcal{J}$ is a biring homomorphism.*
3. *The following conditions hold.*
   (a) *$\mathcal{J}$ is an algebra ideal of B.*
   (b) *$\Delta^+(\mathcal{J}), \Delta^\times(\mathcal{J}) \subseteq \ker(\pi \otimes \pi)$,*

(c) $\varepsilon^+(\mathcal{J}) = \varepsilon^\times(\mathcal{J}) = 0$,

(d) $\sigma(\mathcal{J}) \subseteq \mathcal{J}$,

(e) For all $\lambda \in k'$, $\beta\lambda(\mathcal{J}) = 0$.

If $\mathcal{J}$ is B-pure, then condition (a) is equivalent to the following statement.

(a′) $\Delta^+(\mathcal{J}), \Delta^\times(\mathcal{J}) \subseteq B \otimes \mathcal{J} + \mathcal{J} \otimes B$.

Proof: (1) $\implies$ (3): Suppose $\mathcal{J} = \ker(f\colon B \to B')$ is a biring ideal. Replacing B' by $\mathrm{Im}(f)$ if necessary we can assume f is surjective.

We have the following commutative diagram of k-algebras.

$$\begin{tikzcd}
 & & B/\mathcal{J} \arrow[d, "\bar{f}"] \\
\mathcal{J} \arrow[r] & B \arrow[ur, "\pi"] \arrow[r, "f"] \arrow[d, "\Delta^+"] & B' \arrow[d, "\Delta^+"] \\
 & B \otimes B \arrow[r, "f\otimes f"] \arrow[dr, "\pi\otimes\pi"] & B' \otimes B' \\
 & & B/\mathcal{J} \otimes B/\mathcal{J} \arrow[u, "\bar{f}\otimes\bar{f}"']
\end{tikzcd}$$

where $\bar{f}$ is the k-algebra isomorphism $B/\mathcal{J} \to B'$.

Now $\Delta^+(\mathcal{J}) \subseteq \ker(f \otimes f)$. However since $\bar{f} \otimes \bar{f}$ is an isomorphism, we have $\ker(\pi \otimes \pi) = \ker(f \otimes f)$.

The remaining conditions are similar.

(3) $\implies$ (2): The conditions in (3) ensure the obvious structure maps for $B/\mathcal{J}$ are well defined. For example, Δ^+ induces a map of k-algebras

$$\frac{B}{\mathcal{J}} \to \frac{B \otimes B}{(B \otimes \mathcal{J} + \mathcal{J} \otimes B)} \cong \frac{B}{\mathcal{J}} \otimes \frac{B}{\mathcal{J}}.$$

(2) $\implies$ (1): Simply note $\mathcal{J} = \ker \pi$. $\square$

Example 1.3.14. Recall the k-k-biring $\mathcal{I}$ of Example 1.2.5 which represents the identity functor on $\mathbf{Alg}_k$. Let $B = \mathcal{I} \otimes \mathcal{I} \cong k[x, y]$. Then $\mathcal{J} = (x - y)$ is a biring ideal of B and $B/\mathcal{J} \cong \mathcal{I}$.

We now turn our attention to ideals of plethories.

Definition 1.3.15. We define a *plethystic ideal* of P to be the kernel of a map of plethories $f\colon P \to Q$.

Once again, we have the following useful characterisation of plethystic ideals.

Proposition 1.3.16. *Let P be a plethory, $\mathcal{J}$ a submodule of P and $\pi\colon P \to P/\mathcal{J}$ the canonical projection. The following statements are equivalent.*

1. *$\mathcal{J}$ is a plethystic ideal of P.*
2. *$P/\mathcal{J}$ is a plethory and $\pi\colon P \to P/\mathcal{J}$ is a map of plethories.*
3. *The following conditions hold.*
   (a) *$\mathcal{J}$ is a biring ideal of P.*
   (b) *$IP \circ \mathcal{J} \circ P \subseteq \mathcal{J}$ where IP is the augmentation ideal of P.*

Remark 1.3.17. If in condition 3 (b) we had the more natural looking $P \circ \mathcal{J} \circ P \subseteq \mathcal{J}$ then we would have $1 \circ x = 1 \in \mathcal{J}$ and so $\mathcal{J} = P$ and all ideals would be trivial.

Proof: (1) $\implies$ (3): Suppose $\mathcal{J} = \ker(f\colon P \to Q)$. Since f is a map of birings, $\mathcal{J}$ is a biring ideal. For $r \in IP, s \in P, x \in \mathcal{J}$, we have

$$\begin{aligned} f(r \circ x \circ s) &= f(r) \circ f(x) \circ f(s) \\ &= f(r) \circ 0 \circ f(s) \\ &= \varepsilon^+(f(r)) \\ &= \varepsilon^+(r) \\ &= 0 \end{aligned}$$

and so we see $r \circ x \circ s \in \mathcal{J}$.

(3) $\implies$ (2): By Proposition 1.3.13, $P/\mathcal{J}$ is a biring. We define the composition $\circ\colon P/\mathcal{J} \odot P/\mathcal{J} \to P/\mathcal{J}$ by

$$(r + \mathcal{J}) \circ (s + \mathcal{J}) = (r \circ s) + \mathcal{J}.$$

We check this map is well defined. Firstly, if $r \in \mathcal{J}$ then $r \circ s \in \mathcal{J}$. Now suppose $s + \mathcal{J} = s' + \mathcal{J}$, say $s' = s + j$ for some $j \in \mathcal{J}$. By the counit property, for any $r \in P$, $(1_P \otimes \varepsilon^+)\Delta^+(r) = r \otimes 1$ and so $\Delta^+(r) - r \otimes 1 \in \ker(1_P \otimes \varepsilon^+) = P \otimes IP$. We have

$$\begin{aligned} r \circ s' - r \circ s &= r \circ (s + j) - r \circ s \\ &= \mu(\Delta^+(r) \circ (s \otimes j)) - \mu((r \otimes 1) \circ (s \otimes j)) \\ &= \mu[(\Delta^+(r) - r \otimes 1) \circ (s \otimes j)] \\ &\in \mu[(P \otimes IP) \circ (P \otimes \mathcal{J})] \subseteq \mu(P \otimes \mathcal{J}) \subseteq \mathcal{J}. \end{aligned}$$

Hence, $r \circ s' + \mathcal{J} = r \circ s + \mathcal{J}$ thus our map is well defined. This composition clearly makes π a map of plethories.

(2) $\implies$ (1): Simply note $\mathcal{J} = \ker \pi$. □

Just as with finding explicit sub-birings, it can be difficult to find explicit ideals. Once again, it is often easier to find pointwise mono natural transformations between the representing functors and appealing to the Yoneda lemma.

Example 1.3.18. Recall the k-k-birings $\mathcal{I}$ of Example 1.2.5 which represents the identity functor and B of Example 1.2.6 which represents the power series functor. We have a pointwise mono natural transformation from the identity functor to the power series functor given by inclusion of a k-algebra A into the constant terms of $A[[t]]$. This induces a surjective k-k-biring map $B \to \mathcal{I}$ and we can easily see the kernel is $(x_1, x_2, \dots)$, a biring ideal of B.

1.3.4 Duality

The dual of a projective finitely generated (as a k-module) Hopf algebra is again a Hopf algebra [36]. Often the dual Hopf algebra, which still encodes the same information, is a nicer object to work with. An example of this is the Steenrod algebra over $\mathbb{F}_2$, the collection of stable cohomology operations for ordinary cohomology with coefficients in $\mathbb{F}_2$. As an algebra, this is generated by elements Sq^n which are subject to the rather complicated Adem relations,

$$Sq^n \circ Sq^m = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{m-n-1}{n-2i} Sq^{n+m-i} \circ Sq^i.$$

However, as shown by Milnor [35], the dual Hopf algebra is just a polynomial algebra and so has a much simpler algebraic structure. The price we pay is that the comultiplicative structure becomes more complicated in this setting.

In this section, we study the linear dual of a biring which is an object known as a Hopf ring and then extend this object with additional structure which is dual to the composition in a plethory.

Hopf rings have their origins in algebraic topology and are very well studied e.g. [27] [52] [53] [31]. An exposition of the subject can be found in [54] and we shall just recall the basic definitions. Knowledge of these objects will allow us to compute plethories of cohomology operations algebraically without having to delve into the topology, as the information we need has already been computed for the language of Hopf rings.

The terminology is somewhat misleading: a (bicommutative) Hopf algebra is an (abelian) group object in the category of cocommutative k-coalgebras and a Hopf ring is a k-algebra object in this category. It has been suggested [27] that the terms *coalgebraic group* and *coalgebraic k-algebra* would be more appropriate names for Hopf algebras and Hopf rings respectively. Unfortunately, history is on the side of the original names and we shall stick to those to avoid confusion.

Definition 1.3.19. We define the category of *$k[k']$-Hopf rings* to be the category of k'-algebra objects in $\mathbf{Coalg}_k$. Explicitly, a $k[k']$-Hopf ring is a k-coalgebra (H, ψ, ε) together with coalgebra maps.

$$\begin{aligned}
*\colon H \otimes H &\to H \\
\chi\colon H &\to H \\
\omega\colon k &\to H \\
\circ\colon H \otimes H &\to H \\
\eta\colon k &\to H
\end{aligned}$$

and a map of rings

$$\eta_R\colon k' \to \mathbf{Coalg}_k(k, H)$$

satisfying the usual relations for a k'-algebra object Definition 1.1.7. It is customary to write $[0] = \omega(1)$, $[1] = \eta(1)$ and $[\lambda] = \eta_R\lambda(1)$.

There is clearly some sort of duality between Hopf rings and birings. A $k[k']$-Hopf ring is a k'-algebra object in the category of k-coalgebras, and a k-k'-biring is a co-k-algebra object in the category of k-algebras. For a $k[k']$-Hopf ring or k-k'-biring X we define the linear dual of X to be $DX = \mathbf{Mod}_k(X, k)$. We shall say a Hopf ring (or biring or plethory) is projective and of finite type if it is projective and finitely generated as a k-module.

Proposition 1.3.20. *If $f\colon B \to B'$ is a map of projective k-k'-birings of finite type then DB and DB' are $k[k']$-Hopf rings and $Df\colon DB' \to DB$ is a map of $k[k']$-Hopf rings. Conversely, if $g\colon H \to H'$ is a map of projective $k[k']$-Hopf rings of finite type then $g\colon DH' \to DH$ is a map of k-k'-birings.*

Proof: We know $Df\colon DB' \to DB$ is a map of Hopf algebras. Moreover, we have induced maps

$$\begin{aligned}
\circ\colon DB \otimes DB \cong D(B \otimes B) &\xrightarrow{D\Delta^\times} DB \\
\eta_\circ\colon k \cong Dk &\xrightarrow{D\varepsilon^\times} DB \\
\eta_R\lambda\colon k \cong Dk &\xrightarrow{D\beta\lambda} DB.
\end{aligned}$$

Since the relations the structure maps of a Hopf ring satisfy are the categorical duals of the relations that the structure maps of B satisfy, we see DB is a Hopf ring. Similarly, DB' is a Hopf ring. Moreover the conditions for Df to be a map of Hopf rings are just the duals of the conditions for f to be a map of birings. The reverse direction is similar. □

However the structure of a Hopf ring contains no information which is naturally dual to composition. To rectify this situation, Boardman et al. [15] introduce the notion of an enriched Hopf ring in the context of homology operations. We abstract this definition to a purely algebraic context which makes things more intuitive. We require of Hopf rings to be projective and finitely generated as modules so that the linear dual is naturally a biring.

Definition 1.3.21. An *enriched $k[k]$-Hopf ring* is a projective $k[k]$-Hopf ring H of finite type together with a k-module map

$$\iota\colon H \to k \qquad \text{(augmentation)}$$

and for each x in the dual biring DH, a map of coalgebras

$$x_*\colon H \to H \qquad \text{(induced map)}$$

satisfying the following relations. For $r, s \in DH$, $x, y \in H$ and $\lambda \in k$,

$$\iota(x * y) = \varepsilon(x)\iota(y) + \iota(x)\varepsilon(y) \tag{1}$$

$$\iota(x \circ_H y) = \iota(x)\iota(y) \tag{2}$$

$$\iota([\lambda]) = \lambda \tag{3}$$

$$r_*(x * y) = \sum_{(x)} \sum_{(y)} \mathop{\Huge *}_{i=1}^{n} r^i_{(1)*}x_{(i)} \circ r^i_{(2)*}y_{(i)} \tag{4}$$

$$r_*(x \circ y) = \sum_{(x)} \sum_{(y)} \mathop{\Huge *}_{i=1}^{m} r^i_{[1]*}x_{(i)} \circ r^i_{[2]*}y_{(i)} \tag{5}$$

$$r_*([\lambda]) = [r([\lambda])] \tag{6}$$

$$(r_* \circ s)_* = r_* \circ s_* \tag{7}$$

$$r_* \circ \iota = r \tag{8}$$

$$\iota_* = 1_H \tag{9}$$

and subject to

$$(-)_* \colon DH \to \mathbf{Coalg}_k(H, H) \tag{10}$$

being a map of k-algebras. The k-algebra on $\mathbf{Coalg}_k(H, H)$ is induced by the k-algebra object structure on H by Corollary 1.1.13.

A map of enriched Hopf rings is a Hopf ring map $f \colon H \to H'$ such that the following diagrams commute for all $x \in DH'$.

$$\begin{tikzcd} H \arrow[r, "\iota"] \arrow[d, "f"'] & k \\ H' \arrow[ur, "\iota"'] & \end{tikzcd} \qquad \begin{tikzcd} H \arrow[r, "(Df)(x)_*"] \arrow[d, "f"] & H \arrow[d, "f"] \\ H' \arrow[r, "x_*"] & H' \end{tikzcd}$$

We can see how the additional structure encodes the dual of composition. The unit for composition is given by $\iota \in DH$ and for $x, y \in DH$, we define the composition $x \circ y$ by

$$x \circ y \colon H \xrightarrow{y_*} H \xrightarrow{x} k.$$

The following example is the motivation for the definition of the enrichment of a Hopf ring due to [15].

Example 1.3.22. Let $h(-)$ be an ungraded multiplicative cohomology theory (see Appendix D) with representing space $\underline{H}$ and denote the corresponding homology theory by $k(-)$. For example, take $h(-) = K(-)$ to be complex K-theory as detailed in Section 3.2

and then $\underline{H} = \mathbb{Z} \times BU$. Suppose additionally $k(\underline{H})$ is free as an $h(T)$-module so that we have duality $h(\underline{H}) \cong Dk(\underline{H})$. In this case, the homology cooperations $k(\underline{H})$ admits the structure of an enriched $h(T)[h(T)]$-Hopf ring. The augmentation is given by the composition

$$k(\underline{H}) \xrightarrow{\pi} Qk(\underline{H}) \xrightarrow{Q(\varepsilon)} h(T)$$

where $Q\colon \mathbf{Alg}^+_{h(T)} \to \mathbf{Mod}_{h(T)}$ denotes the functor of indecomposables from the category of augmented $h(T)$-algebras. The induced map for $x \in DH \cong h(\underline{H}) \cong \mathbf{Ho}(\underline{H}, \underline{H})$ is given by $x_* = k(x)\colon k(\underline{H}) \to k(\underline{H})$.

The following result details the duality between enriched Hopf rings and plethories. As we shall see in Chapter 3 this result is extremely useful for computations.

Proposition 1.3.23. *If $f\colon H \to H'$ is a map of enriched $k[k]$-Hopf rings then $Df\colon DH' \to DH$ is a map of k-plethories. Conversely, if $g\colon P \to P'$ is a map of projective k-plethories of finite type then $Dg\colon DP' \to DP$ is a map of enriched $k[k]$-Hopf rings.*

Proof: Let H be an enriched $k[k]$-Hopf ring. By Proposition 1.3.20, DH is a k-k-biring. Define the unit for composition $u\colon \mathcal{I} \to DH$ by $u(\iota) = \iota$. The conditions necessary for this to be a map of k-k-birings are precisely [(1), (2), (3)].

For example, for u to be a map of k-k-birings we require the following diagram to commute.

$$\begin{tikzcd}
\mathcal{I} \arrow[r, "u"] \arrow[d, "\Delta^\times"] & DH \arrow[d, "\Delta^\times"] \\
\mathcal{I} \otimes \mathcal{I} \arrow[r, "u \otimes u"] & DH \otimes DH \cong D(H \otimes H)
\end{tikzcd}$$

The statement $\Delta^\times(\iota) = \iota \otimes \iota$ in $DH \otimes DH \cong D(H \otimes H)$ is equivalent to commutativity of the following diagram which precisely expresses the condition $\iota(x \circ y) = \iota(x)\iota(y)$ for all $x, y \in H$.

$$\begin{tikzcd}
H \otimes H \arrow[r, "\circ"] \arrow[d, "\iota \otimes \iota"] & H \arrow[d, "\iota"] \\
k \otimes k \arrow[r, "\cong"] & k.
\end{tikzcd}$$

We define the composition $\circ\colon DH \odot DH \to DH$ for $x, y \in DH$ by

$$x \circ y\colon H \xrightarrow{y_*} H \xrightarrow{x} k.$$

For this map to be well defined, we need to show the relations in $DH \odot DH$ hold. The relation $(x_1 + x_2) \circ y = x_1 \circ y + x_2 \circ y$ holds since addition of maps in DH is pointwise.

The relations $(x_1x_2) \circ y = (x_1 \circ y)(x_2 \circ y)$ and $\lambda \circ y = \lambda$ hold precisely because y_* is a map of coalgebras. For example, the relation $(x_1x_2) \circ y = (x_1 \circ y)(x_2 \circ y)$ equivalent to commutativity of the following diagram.

$$\begin{tikzcd} H \arrow[r, "y_*"] \arrow[d, "\psi"] & H \arrow[r, "x_1x_2"] \arrow[d, "\psi"] & k \\ H \otimes H \arrow[r, "y_* \otimes y_*"] & H \otimes H \arrow[r, "x_1 \otimes x_2"] & k \otimes k \arrow[u, "\cong"'] \end{tikzcd}$$

The right hand square commutes by the definition of multiplication in DH and so our relation is equivalent to y_* respecting the comultiplication. The other relation is equivalent to y_* respecting the counit, since $\lambda \in DH$ is the map $\lambda\varepsilon \colon H \to k$.

The remaining three relations, $x \circ (y_1 + y_2) = (x_{(1)} \circ y_1)(x_{(2)} \circ y_2)$, $x \circ (y_1 + y_2) = (x_{[1]} \circ y_1)(x_{[2]} \circ y_2)$, and $x \circ \lambda = \beta\lambda(x)$ are satisfied as a consequence of condition (10). For example, the relation $x \circ (y_1 + y_2) = (x_{(1)} \circ y_1)(x_{(2)} \circ y_2)$ is equivalent to commutativity of the following diagram.

$$\begin{tikzcd} H \arrow[r, "(y_1+y_2)_*"] \arrow[d, "\psi"] & H \arrow[r, "x"] & k \\ H \otimes H \arrow[r, "y_{1*} \otimes y_{2*}"] & H \otimes H \arrow[r, "x_{(1)} \otimes x_{(2)}"] \arrow[u, "*"'] & k \otimes k. \arrow[u, "\cong"'] \end{tikzcd}$$

The right hand square commutes since $*$-multiplication induces Δ^+ on H^*, so we require the left hand square to commute. However, the composition in this square is simply addition in $\mathbf{Coalg}(H, H)$. Hence our composition $DH \odot DH \to DH$ is well defined.

The conditions (4), (5) and (6) precisely ensure that the composition is a map of k-k-birings. For example, consider the relation that composition respects Δ^+ in that we have an equality

$$\Delta^+(x \circ y) = \sum_{(x)} \prod_{i=1}^{n} \sum_{(x_{(i)})} x_{(i)[1]} \circ y^i_{(1)} \otimes x_{(i)[2]} \circ y^i_{(2)} \tag{1.7}$$

in $DH \otimes DH \cong D(H \otimes H)$ where $\Delta^+(y) = \sum_{i=1}^{n} y^i_{(1)} \otimes y^i_{(2)}$. The multiplication of $f_1, \ldots, f_n \in D(H \otimes H)$ is given by the composition

$$H \otimes H \xrightarrow{\psi^{(n)}_{H\otimes H}} (H \otimes H)^{\otimes n} \xrightarrow{\otimes_i f_i} k^{\otimes n} \cong k,$$

where $\psi^{(n)}_{H\otimes H}$ is the n-th iterated coproduct on $H \otimes H$. Thus the equality (1.7) is is equivalent to commutativity of the following diagram.

$$\begin{tikzcd}
H \otimes H \arrow[r, "\psi^{(n)}_{H\otimes H}"] \arrow[dd, "*"] & (H \otimes H)^{\otimes n} \arrow[r, "\bigotimes_i y^i_{(1)*} \otimes y^i_{(2)*}"] & (H \otimes H)^{\otimes n} \arrow[r, "\bigotimes_i x_{(i)[1]} \otimes x_{(i)[2]}"] \arrow[d, "\circ^{\otimes n}"] & (k \otimes k)^{\otimes n} \arrow[dd, "\cong"] \\
 & & H^{\otimes n} \arrow[d, "*"] & \\
H \arrow[rr, "y_*"] & & H \arrow[r, "x"] & k.
\end{tikzcd}$$

The right hand square commutes since $\bigotimes_i x_{(i)[1]} \otimes x_{(i)[2]} = (\Delta^{\times})^{\otimes n} \circ (\Delta^{+})^{(n)}(x)$, the co-multiplication $\Delta^{\times}$ on DH is induced by the $\circ$-product in H and the co-addition Δ^{+} on DH is induced by the $*$-product in H. Hence the equality (1.7) is equivalent to

$$y_*(a * b) = \sum_{(a)} \sum_{(b)} \mathop{\Huge *}_{i=1}^{n} y^i_{(1)*} a_{(i)} \circ y^i_{(2)*} b_{(i)}.$$

Finally, we need to show that $(DH, \circ, u)$ is a monoid in the category of birings. Associativity of the composition follows from (7) and the left and right unit relations follow from (8) and (9) respectively.

Conversely, for a k-plethory P we will show DP is an enriched $k[k]$-Hopf ring. By Proposition 1.3.20, DP is a $k[k]$-Hopf ring. We define the augmentation $DP \to k$ to be the image of $\iota \in P$ under the canonical isomorphism $P \cong DDP$. For each $x \in DDP \cong P$ we define the induced map $x_* \colon DP \to DP$ to be the dual of the map $-\circ x \colon P \to P$. These maps satisfy the defining properties of an enriched Hopf ring by the defining properties of a plethory. $\square$

1.3.5 Primitives

A useful linearisation functor on Hopf algebras is the functor of primitives. While the module of primitives is not a complete invariant of a Hopf algebra it contains a lot of useful information. A famous theorem of Milnor and Moore [36] states that under certain technical conditions that we can completely recover a Hopf algebra from its primitive elements. To avoid a clash of notation, in this section and later we will often use the alternative symbol Π to denote a plethory.

Definition 1.3.24. Let H be a Hopf algebra. We define the *k-module of primitives PH* by the exact sequence

$$0 \to PH \to JH \xrightarrow{\psi} JH \otimes JH$$

where $JH = \operatorname{coker} \eta$. Explicitly,

$$PH = \{x \in H \mid \psi x = 1 \otimes x + x \otimes 1\}.$$

Notice that if x is primitive then by the co-unit and antipode conditions we have $\varepsilon(x) = 0, \sigma(x) = -x$. Since a map of Hopf algebras necessarily sends primitive elements to primitive elements, we have a functor $P\colon \mathbf{Hopf}_k \to \mathbf{Mod}_k$ given as above on objects and by restriction on morphisms.

We have forgetful functors from our categories $\mathbf{Biring}_{k,k'}$ and $\mathbf{Plethory}_k$ to $\mathbf{Hopf}_k$ where the comultiplication on the Hopf algebra is the coaddition map on the biring. Hence if B is a biring (or plethory) an element is primitive if $\Delta^+(x) = 1 \otimes x + x \otimes 1$.

If Π is a k-plethory, then by Theorem 1.2.29 the primitive elements in Π are precisely the maps which are additive on all Π-algebras. That is to say, for all $A \in \mathbf{Alg}_\Pi$ we have $r \in P\Pi$ if and only if $r(x+y) = r(x) + r(y)$ for all $x, y \in A$.

However, birings and plethories have much more structure than a Hopf algebra and it is natural to ask what effect this has on the structure of the primitives.

Proposition 1.3.25. *Let B be a k-k'-biring. If $x \in PB$ then $\gamma\lambda(x) \in PB$ and $\Delta^\times x$ is in the image of the map $PB \otimes PB \to B \otimes B$.*

Proof: Let $x \in PB$. Since $\gamma\lambda\colon B \to B$ is a map of Hopf algebras, we see $\gamma\lambda(x) \in PB$. For the comultiplication, we note that the image of the map $PB \otimes B \to B \otimes B$ is the kernel of the map $f \otimes 1\colon B \otimes B \to B \otimes B \otimes B$ where $f(x) = \Delta^+(x) - 1 \otimes x - x \otimes 1$. By codistributivity (see Appendix A), we have $\Delta^+(x_{[1]}) \otimes x_{[2]} = (1 \otimes x_{[1]} + x_{[1]} \otimes 1) \otimes x_{[2]}$. Thus,

$$\begin{aligned}(f \otimes 1)x_{[1]} \otimes x_{[2]} &= (\Delta^+(x) - 1 \otimes x - x \otimes 1) \otimes x_{[2]} \\ &= \Delta^+(x_{[1]}) \otimes x_{[2]} - (1 \otimes x_{[1]} + x_{[1]} \otimes 1) \otimes x_{[2]} \\ &= 0.\end{aligned}$$

Hence $x_{[1]} \otimes x_{[2]}$ is in the image of $PB \otimes B \to B \otimes B$. Similarly, $x_{[1]} \otimes x_{[2]}$ is in the image of $B \otimes PB \to B \otimes B$ and the result follows. □

We now define some algebraic gadgets which will encode this structure. Since on our primitives the coaddition, co-zero and co-additive inverse are uniquely determined, we only need to retain the comultiplication, counit and co-linear structure.

Definition 1.3.26. We define a k-k'-coalgebra to be a k-k'-bimodule which is also a k-coalgebra. Explicitly, this is a k-coalgebra (C, ψ, ε) together with a right k'-action,

$$\varphi\colon C \otimes_{\mathbb{Z}} k' \to C$$

such that the following diagram commutes.

$$\begin{tikzcd}
& C \otimes k' \arrow[dl, "(1\otimes\tau)\circ(\psi\otimes 1)"'] \arrow[d, "\varphi"] \arrow[dr, "\psi\otimes 1"] & \\
C \otimes k' \otimes C \arrow[dr, "\varphi\otimes 1"'] & B \arrow[d, "\psi"] & C \otimes C \otimes k' \arrow[dl, "1\otimes\varphi"] \\
& C \otimes C &
\end{tikzcd}$$

A map of k-k'-coalgebras is a k-k'-bimodule map which is also a k-coalgebra map. We denote the category of k-k'-coalgebras and maps by $\mathbf{Coalg}_{k,k'}$.

Proposition 1.3.27. *The functor of primitives $P\colon \mathbf{Hopf}_k \to \mathbf{Mod}_k$ lifts to a functor*

$$P\colon \mathbf{Biring}^{pure}_{k,k'} \to \mathbf{Coalg}_{k,k'}$$

where $\mathbf{Biring}^{pure}_{k,k'}$ denotes the full sub-category of $\mathbf{Biring}_{k,k'}$ consisting of k-k'-birings B such that the sub-module PB is B-pure.

Proof: Let B be a k-k'-biring. Since PB is B-pure, we have $PB \otimes PB \subseteq B \otimes B$. Hence the content of Proposition 1.3.25 tells us that PB is a k-k'-coalgebra with $\psi = \Delta^{\times}$, $\varepsilon = \varepsilon^{\times}$ and right action given by $x \cdot \lambda = \gamma\lambda(x)$. If $f\colon B \to B'$ is a map of k-k'-birings, then clearly $Pf\colon PB \to PB'$ is a map of k-k'-coalgebras. $\square$

Example 1.3.28. For the k-k-biring $\mathcal{I}$ of Example 1.2.5, $P\mathcal{I} = \langle\iota\rangle$, the free k-module on a single generator. The structure maps are given by $\psi(\iota) = \iota \otimes \iota, \varepsilon(\iota) = 1, \iota \cdot \lambda = \lambda\iota$.

We now ponder the question of composition of primitive elements. Let Π be a k-plethory. If $r, s \in \Pi$ are primitive then for any Π-algebra A and any $x, y \in A$, we have $(r\circ s)(a+b) = r(s(a+b)) = r(s(a)+s(b)) = (r\circ s)(a) + (r\circ s)(b)$. Therefore by Theorem 1.2.29, $r \circ s$ is primitive. However there exist compositions of non-additive maps which are additive. For example consider the maps $x \mapsto x+1$ and $x \mapsto x-1$ in $\mathbf{Set}(k,k)$. Hence at the level of birings, we should expect a map $PB \otimes PB' \to P(B \odot B')$ for a k-k'-biring B and a k'-k''-birings B' which is not an isomorphism in general.

Proposition 1.3.29. *We define the tensor product of a k-k'-coalgebra C and a k'-k''-coalgebra C' to be the k-k''-bimodule*

$$C \otimes_{k'} C'$$

with coalgebraic structure given by

$$\psi\colon C \otimes C' \xrightarrow{\psi\otimes\psi} C \otimes C \otimes C' \otimes C' \xrightarrow{1\otimes\tau\otimes 1} C \otimes C' \otimes C \otimes C',$$
$$\varepsilon\colon C \otimes C' \xrightarrow{1\otimes\varepsilon} C \otimes k' \to C \xrightarrow{\varepsilon} k.$$

Proposition 1.3.30. *The category* $(\mathbf{Coalg}_{k,k}, \otimes_k, P\mathcal{I})$ *is monoidal.*

Proof: Associativity of $\otimes$ follows from coassociativity of ψ. Also, $C \otimes P\mathcal{I} \cong C \cong P\mathcal{I} \otimes C$ as k-k-bimodules and it is straightforward to check the maps $C \otimes P\mathcal{I}, c \otimes \iota \mapsto c$ and $P\mathcal{I} \otimes C, \iota \otimes c \mapsto c$ are coalgebra maps. $\square$

It turns out that monoids in $\mathbf{Coalg}_{k,k}$ have an equivalent formulation which may be more familiar. This definition is due to Borger and Wieland [17] and assuming flatness is the same as a $\times_k$-bialgebra in the sense of Sweedler [48].

Definition 1.3.31. Let A, B be non-commutative k-algebras. Form $A \otimes_k B$ with respect to the left action of k on both A and B (so $\lambda a \otimes b = a \otimes \lambda b$). Define the *Sweedler product* $A \circledast B$ to be the sub-module where the k-action given by right multiplication on A is equal to the k-action given by right multiplication on B. That is

$$A \circledast B = \left\{ \sum a \otimes b \in A \otimes B \mid \sum a \cdot \lambda \otimes b = \sum a \otimes b \cdot \lambda \text{ for all } \lambda \in k \right\}.$$

In fact, $A \circledast B$ is a sub-algebra of $A \otimes B$.

Definition 1.3.32. A *twisted k-bialgebra* is a not necessarily commutative k-algebra B equipped with a k-algebra map $\psi \colon B \to B \circledast B$ and a k-module map $\varepsilon \colon B \to k$ such that the following two conditions hold.

1. The composite $B \xrightarrow{\psi} B \circledast B \hookrightarrow B \otimes B$ is coassociative with counit ε.
2. $\varepsilon(1) = 1$ and $\varepsilon(ab) = \varepsilon(a\eta\varepsilon(b))$ for all $a, b \in B$.

A map of twisted k-bialgebras is a map of k-algebras which commutes with ε and ψ. We denote the category of twisted k-bialgebras by $\mathbf{TwBialg}_k$. Notice that if B is a twisted k-bialgebra and k is in the center of B in that $\lambda b = b\lambda$ for all $\lambda \in k$ and all $b \in B$ then B is just a (not necessarily commutative) k-bialgebra.

We now give an alternative definition of a twisted bialgebra which is better in keeping with the rest of our general framework.

Proposition 1.3.33. *The category of twisted k-bialgebras* $\mathbf{TwBialg}_k$ *is the category of monoids in* $\mathbf{Coalg}_{k,k}$.

Proof: Let $(B, \circ, u)$ be a monoid in $\mathbf{Coalg}_{k,k}$. By definition we have $\psi(a \circ b) = \psi(a) \circ \psi(b)$, $\varepsilon(\iota) = \iota$ and $\varepsilon(a \circ b) = \varepsilon(a \cdot \varepsilon(b))$. It remains to show ψ factors through the Sweedler product. Since $\circ$ is a map of k-k-bimodules, we have $a \otimes b \circ \lambda = a \circ \lambda \otimes b$. $\square$

Proposition 1.3.34. *The functor of primitives* $P\colon \mathbf{Biring}^{pure}_{k,k} \to \mathbf{Coalg}_{k,k}$ *is lax monoidal and hence lifts to a functor*

$$P\colon \mathbf{Plethory}^{pure}_k \to \mathbf{TwBialg}_k$$

where $\mathbf{Plethory}^{pure}_k$ *denotes the full subcategory of* $\mathbf{Plethory}_k$ *consisting of* k*-plethories* Π *such that the sub-module* $P\Pi$ *is* P*-pure.*

Proof: For a k-k'-biring B and k'-k''-biring B', consider the map $PB \otimes PB' \to P(B \odot B'), x \otimes y \mapsto x \odot y$. This is well defined since $\psi(x \odot y) = x_{[1]} \odot 1 \otimes x_{[2]} \odot y + x_{[1]} \odot y \otimes x_{[2]} \odot 1 = \varepsilon^{\times}(x_{[1]}) \otimes x_{[2]} \odot y + x_{[1]} \odot y \otimes \varepsilon^{\times}(x_{[2]}) = 1 \otimes x \odot y + x \odot y \otimes 1$ by the counit property. In addition it is a map of k-k''-coalgebras by the definitions. Moreover, $P\mathcal{I} \cong \langle e \rangle$ and these maps satisfy the necessary conditions for a monoidal functor. $\square$

As mentioned the primitive elements are precisely the elements which act additively on P-algebras. We can understand this action independently of the broader context by defining an additive analogue of $\odot$.

Definition 1.3.35. We define the *additive composition product* $\boxdot\colon \mathbf{Coalg}_{k,k} \times \mathbf{Alg}_k \to \mathbf{Alg}_k$ on objects by defining $C \boxdot A$ to be the free k-algebra on

$$\{c \boxdot a : c \in C, a \in A\}$$

and taking the quotient by the ideal generated by the following relations.

$$(c_1 + c_2) \boxdot a = c_1 \boxdot a + c_2 \boxdot a$$
$$(\lambda c) \boxdot a = \lambda(c \boxdot a)$$
$$c \boxdot (a_1 + a_2) = c \boxdot a_1 + c \boxdot a_2$$
$$c \boxdot (a_1 a_2) = (c_{(1)} \boxdot a_1)(c_{(2)} \boxdot a_2)$$
$$c \boxdot \lambda = (c \cdot \lambda) \boxdot 1$$

For morphisms $(f, g)\colon (C_1, M_1) \to (C_2, M_2)$ we define $f \boxdot g\colon C_1 \boxdot M_1 \to C_2 \odot M_2$ to be the k-algebra map given by $c \boxdot m \mapsto f(c) \boxdot g(m)$.

Proposition 1.3.36. *If* B *is a twisted* k*-bialgebra then* $B \boxdot -$ *is a monad on* $\mathbf{Alg}_k$*.*

Proof: Define $B \boxdot (B \boxdot A) \to B \boxdot A$ by $b \boxdot b' \boxdot a \mapsto (bb') \boxdot a$ and $P\mathcal{I} \boxdot A \to A$ by $\iota \boxdot a \mapsto a$. These maps are natural and the required compatibility conditions are satisfied since they are precisely those which make B a monoid in $\mathbf{Coalg}_{k,k}$. $\square$

Definition 1.3.37. Let B be a twisted k-bialgebra. Define the category of B*-algebras* to be the category of Eilenberg-Moore algebras for the monad $B \boxdot -$ by $\mathbf{Alg}_B$.

The following result shows that this definition encodes the action of the primitive elements on a Π-algebra.

Corollary 1.3.38. *Let Π be a Π-pure k-plethory so $P\Pi$ is a twisted k-bialgebra. If A is an Π-algebra then A naturally admits the structure of a $P\Pi$-algebra such that the following diagram commutes.*

$$\begin{tikzcd} \Pi \odot A \arrow[r, "\circ"] & A \\ P\Pi \boxdot A \arrow[u] \arrow[ur, "\circ"'] & \end{tikzcd}$$

where $P\Pi \boxdot A \to \Pi \odot A$ is the k-algebra map given by $r \boxdot a \mapsto r \odot a$.

Proof: First note $P\Pi \boxdot A \to \Pi \odot A$ is well defined since the relations in $P\Pi \boxdot A$ are satisfied in $\Pi \odot A$ by definition. Now we simply give A the structure of a $P\Pi$-algebra via the composition $P\Pi \boxdot A \to \Pi \odot A \xrightarrow{\circ} A$. □

For a twisted bialgebra B, let $U\colon \mathbf{Alg}_B \to \mathbf{Ab}$ denote the forgetful functor. We define an additive operation on B-algebras to be a natural transformation $U \to U$. In other words, for any B-algebra, A, we have a map $\mathrm{Nat}(U, U) \to \mathbf{Ab}(A, A)$. We can show that U is represented by B. Hence, by the Yoneda lemma we have $\mathrm{Nat}(U, U) \cong B$. Therefore, just as we realised k-plethories as precisely the structure that act non-linearly on k-algebras, we have realised k-twisted bialgebras as precisely the structure which acts additively on k-algebras.

1.3.6 k-Primitives

An element of a k-plethory Π is primitive if it acts additively on all Π-algebras. It is natural to ask about elements which act as k-linear maps on all Π-algebras.

Definition 1.3.39. Let B be a k-k-biring. We say $x \in B$ is *k-primitive* if it is primitive and $\beta\lambda(x) = \lambda\varepsilon(x)$. We denote the collection of k-primitive elements by $P_k B$. Since k-k-biring maps take k-primitive elements to k-primitive elements we have a functor $P_k\colon \mathbf{Biring}_{k,k} \to \mathbf{Set}$ given as above on objects and by restriction on morphisms.

If Π is a plethory, then by Theorem 1.2.29 the k-primitive elements in Π are precisely the maps which are k-linear on all Π-algebras. That is to say, for all $A \in \mathbf{Alg}_\Pi$ we have $r \in P_k\Pi$ if and only if $r(ax + by) = ar(x) + br(y)$ for all $x, y \in A$ and all $a, b \in k$.

We now wonder what structure we have on the collection of k-primitive elements. Since a primitive element is the same as a $\mathbb{Z}$-primitive element we expect this to be a very similar structure to the primitive elements.

Proposition 1.3.40. *Let B be a k-k-biring. If $x \in P_k B$ then $\Delta^{\times} x$ is in the image of the map $P_k B \otimes P_k B \to B \otimes B$.*

Proof: As in Proposition 1.3.25. $\square$

Since the co-additive and co-k-linear structure is uniquely determined on the k-primitives, the co-multiplicative structure is all we need to retain.

Corollary 1.3.41. *The functor of k-primitives lifts to a functor*

$$P_k \colon \mathbf{Biring}_{k,k}^{pure} \to \mathbf{Coalg}_k.$$

Proof: Let B be a k-k-biring. Since $P_k B$ is $P_k B$-pure, we have $P_k B \otimes P_k B \subseteq B \otimes B$. Hence $P_k B$ is a k-coalgebra with $\Delta = \Delta^{\times}$, $\varepsilon = \varepsilon^{\times}$. Moreover, if $f \colon B \to B'$ is a map of birings, then clearly $P_k f \colon P_k B \to P_k B'$ is a map of k-coalgebras. $\square$

Now since k-primitive elements are primitive, it is natural that the functor of k-primitives will factor via the functor of primitives.

Definition 1.3.42. Define the *action equaliser* $E \colon \mathbf{Coalg}_{k,k} \to \mathbf{Coalg}_k$ on objects to be the sub-k-module where the left and right k-actions agree and on morphisms by restriction. Explicitly, we have

$$E(C) = \{c \in C \mid \lambda c = c \cdot \lambda \text{ for all } \lambda \in k\}.$$

Theorem 1.3.43. *The functor of k-primitives P_k factors as the composition*

$$\mathbf{Biring}_{k,k} \xrightarrow{P} \mathbf{Coalg}_{k,k} \xrightarrow{E} \mathbf{Coalg}_k.$$

Proof: This is immediate from the definitions of the functors. $\square$

Now if $r, s \in \Pi$ are k-primitive then for any Π-ring A, $x, y \in A$ and any $a, b \in k$ we have $(r \circ s)(ax + by) = r(s(ax + by)) = r(as(x) + bs(y)) = a(r \circ s)(x) + b(r \circ s)(y)$. Therefore by Theorem 1.2.29, $r \circ s$ is k-primitive. As in the primitive case however there exist compositions of non-linear maps which are linear.

Proposition 1.3.44. *The functor of k-primitives $P_k \colon \mathbf{Biring}_{k,k}^{pure} \to \mathbf{Coalg}_k$ is lax monoidal and lifts to a functor*

$$P_k \colon \mathbf{Plethory}_k^{pure} \to \mathbf{Bialg}_k \subseteq \mathbf{TwBialg}_k.$$

Proof: By definition E is clearly lax monoidal and thus the result follows from Proposition 1.3.34 and Theorem 1.3.43. $\square$

Exactly as in the case for the primitives we can realise k-twisted bialgebras as precisely the structure which acts k-linearly on k-algebras.

1.3.7 Super primitives

Just as we have considered elements which act additively or k-linearly on all P-algebras, we can consider elements which act as k-algebra maps on all P-algebras. This is a very strong condition and in general there are a lot fewer such elements. Moreover, the collection of such elements carries much less structure. Nevertheless this can be a simple but useful invariant of plethories.

Definition 1.3.45. Let B be a biring, we say $x \in B$ is *super primitive* if

$$\begin{aligned}
\Delta^{+}(x) &= 1 \otimes x + x \otimes 1 \\
\varepsilon^{+}(x) &= 0 \\
\sigma(x) &= -x \\
\Delta^{\times}(x) &= x \otimes x \\
\varepsilon^{\times}(x) &= 1 \\
\beta\lambda(x) &= \lambda.
\end{aligned}$$

We denote the collection of super primitive elements by $\mathcal{A}(P)$. Since biring maps necessarily map super primitive elements to super primitive elements, we have a functor

$$\mathcal{A} \colon \mathbf{Biring}_{k,k} \to \mathbf{Set}.$$

Super primitive elements are sometimes refered to as 'ring-like' in the case $k = \mathbb{Z}$ to indicate they generalise the 'group-like' elements of a Hopf algebra. The analogous name for general k would presumably be 'k-algebra-like'. The following result is immediate from the definitions.

Proposition 1.3.46. *Let P be a k-plethory. For $r \in P$, the following statements are equivalent.*

1. *$r \in P$ is super primitive.*
2. *If A is a P-algebra, then the map $A \to A$, $a \mapsto r \circ a$ is a k-algebra map.*

3. The map $\mathcal{I} \to P$, $\iota \mapsto r$ is a map of birings.

As a consequence of (3), we have $\mathcal{A}(P) \cong$ $\mathbf{Biring}(\mathcal{I}, P)$.

Once again, the theory of λ-rings gives us an example of some super primitive elements. See Section 3.2 for more detail.

Example 1.3.47. Let A be a λ-ring. Recall the Adams operations $\psi^i \colon A \to A$ are defined by

$$-t\frac{d}{dt}\log \lambda_{-t}(x) = \sum_{i=1}^{\infty} \psi^i(x)t^i.$$

It is known that the Adams operations are ring maps. Hence the corresponding element in the plethory Π is super primitive. For this reason, super primitive elements are sometimes called Adams operations and this illuminates the choice of notation for the functor $\mathcal{A}$.

Since the composition of two algebra maps is again an algebra map, the composition on our biring induces an associative unital binary operation on the set of super primitive elements given by $(x, y) \mapsto x \circ y$. Indeed we have natural maps $\mathcal{A}(P) \times \mathcal{A}(P) \to \mathcal{A}(P \odot P)$ and $\mathcal{A}\mathcal{I} \cong \{e\}$. As in the context of primitive elements and linear maps, there exist compositions of non-algebra maps which are algebra maps and so $\mathcal{A}(P) \times \mathcal{A}(P) \to \mathcal{A}(P \odot P)$ is not necessarily an isomorphism. The content of this is the following result.

Corollary 1.3.48. *The functor of super primitives is lax monoidal and hence lifts to a functor*

$$\mathcal{A} \colon \mathbf{Plethory}_k \to \mathbf{Monoid}.$$

Proof: The map $\mathcal{A}(B) \times \mathcal{A}(B') \to \mathcal{A}(B \odot B')$ given by $(x, y) \to x \odot y$ is a natural map of sets which along with the natural isomorphism $\mathcal{A}\mathcal{I} \cong \{e\}$ satisfies the necessary conditions for a monoidal functor. $\square$

Since all super primitive elements are k-primitive, we expect $\mathcal{A}$ to factor via P_k just as P_k factored via the functor of primitives P.

Definition 1.3.49. Define the *group-like* functor $G \colon \mathbf{Coalg}_k \to \mathbf{Set}$ on objects to be the set of group-like elements and on morphisms by restriction. Explicitly,

$$G(C) = \{c \in C \,|\, \psi(x) = x \otimes x\}.$$

Proposition 1.3.50. *The functor of super primitives $\mathcal{A}$ factors as the composition*

$$\mathbf{Biring}_{k,k} \xrightarrow{P_k} \mathbf{Coalg}_k \xrightarrow{G} \mathbf{Set}$$

and this factorisation lifts to

$$\mathbf{Plethory}_k \xrightarrow{P_k} \mathbf{TwBialg}_k \xrightarrow{G} \mathbf{Monoid}.$$

Proof: The first statement is immediate from the definitions after we note that for a k-coalgebra, if $\psi(x) = x \otimes x$ then $\varepsilon(x) = 1$. For the second statement, we note that G is clearly lax monoidal and together with Proposition 1.3.44 we see all our functors lift. $\square$

We give a method of constructing simple plethories which generalises the construction of an algebra over a monoid, due to Tall and Wraith [50].

Definition 1.3.51. We define a functor $\Psi\colon \mathbf{Monoid} \to \mathbf{Plethory}_k$. For a monoid G, the underlying k-k-biring is

$$\Psi(G) = \bigotimes_{g \in G} \mathcal{I}$$

where the biring structure is induced by the biring structure on $\mathcal{I}$. Explicitly, if we write ψ_g for the generator in the g-th copy of $\mathcal{I}$ then we have $\Delta^+(\psi_g) = 1 \otimes \psi_g + \psi_g \otimes 1$, $\Delta^\times(\psi_g) = \psi_g \otimes \psi_g$. We define the composition by $\psi_g \circ \psi_h = \psi_{gh}$ and hence $e_{\Psi(G)} = \psi_{e_G}$.

Example 1.3.52. $\Psi(\{\iota\}) \cong \mathcal{I}$.

Proposition 1.3.53. *The functor of super primitives $\mathcal{A}$:* $\mathbf{Plethory} \to \mathbf{Monoid}$ *is right adjoint to* $\Psi\colon \mathbf{Monoid} \to \mathbf{Plethory}$.

Proof: [50, Theorem 3.3] deals with the case when $k = \mathbb{Z}$ and the proof passes to the case of general k without difficulty. $\square$

1.3.8 Linear plethories

Similarly to how we constructed the left adjoint to $\mathcal{A}$ by forcing all elements to be super primitive, we can construct left adjoints to the functor of primitives P and the functor of k-linear primitives P_k by forcing our elements to be additive (resp. k-linear). The construction for the k-linear case is due to Borger and Wieland [17].

Definition 1.3.54. Define a functor $S\colon \mathbf{Coalg}_{k,k'} \to \mathbf{Biring}_{k,k'}$ as follows. For a co-commutative coalgebra C, let $S(C)$ denote the symmetric algebra on the k-module C.

The structure maps are induced by the k-module maps from C given by

$$\begin{aligned}\Delta^{+}&\colon c\mapsto 1\otimes c+c\otimes 1\in S(C)\otimes S(C)\\ \varepsilon^{+}&\colon c\mapsto 0\in k\\ \sigma&\colon c\mapsto -c\in S(C)\\ \Delta^{\times}&\colon c\mapsto (i\otimes i)\psi(c)\in S(C)\otimes S(C)\\ \varepsilon^{\times}&\colon c\mapsto \varepsilon(c)\in k\\ \beta\lambda&\colon c\mapsto \varepsilon(c\cdot\lambda)\in k\end{aligned}$$

where $i\colon C\to S(C)$ denotes the canonical inclusion.

Define $S_k\colon \mathbf{Coalg}_k\to\mathbf{Biring}_{k,k}$ to be the composition

$$\mathbf{Coalg}_k\to\mathbf{Coalg}_{k,k}\xrightarrow{S}\mathbf{Biring}_{k,k}$$

where $\mathbf{Coalg}_k\to\mathbf{Coalg}_{k,k}$ is the functor which equips a coalgebra with the right k-action $c\cdot\lambda=\lambda c$. In other words, the co-linear structure on $S_k(C)$ is induced by $\beta\lambda\colon c\mapsto \lambda\varepsilon(c)\in k$.

We can lift these functions to monoids in the respective categories and obtain a method of constructing plethories. Let $\mathbf{Bialg}_k^{\neg\mathrm{com}}$ denote the category of cocommutative but not neccesarily commutative k-bialgebras i.e. the category of monoids in $\mathbf{Coalg}_k$

Proposition 1.3.55. *The functors S and S_k are strict monoidal and hence lift to functors*

$$\begin{aligned}S&\colon \mathbf{TwBialg}_k\to\mathbf{Plethory}_k\\ S_k&\colon \mathbf{Bialg}_k^{\neg com}\to\mathbf{Plethory}_k.\end{aligned}$$

Proof: For a k-k'-coalgebras B, a k'-k''-coalgebra B', and any k-algebra X we have

$$\begin{aligned}\mathbf{Alg}_k(S(B)\odot S(B'),X)&\cong\mathbf{Alg}_k(S(B),\mathbf{Alg}_k(S(B'),X))\\ &\cong\mathbf{Mod}_k(B,\mathbf{Mod}_k(B',X))\\ &\cong\mathbf{Mod}_k(B\otimes B',X)\\ &\cong\mathbf{Alg}_k(S(B\otimes B'),X).\end{aligned}$$

Hence $S(B\otimes B')\cong S(B)\odot S(B')$ as k-algebras under the map $b\otimes b'\mapsto b\odot b'$. It is straightforward to check this is a map of k-k''-birings. Also clearly $S(k)\cong k[\iota]=\mathcal{I}$. Hence S is strict monoidal. Moreover, the functor $\mathbf{Coalg}_k\to\mathbf{Coalg}_{k,k}$ is strict monoidal and hence so is S_k. $\square$

Theorem 1.3.56. *The functors S and S_k are left adjoint to the functors P and P_k respectively. These adjunctions lift to monoids in the appropriate categories. Diagrammatically, we have the following.*

$$S\colon \mathbf{Coalg}_{k,k} \leftrightarrows \mathbf{Biring}_{k,k}\colon P$$
$$S_k\colon \mathbf{Coalg}_{k} \leftrightarrows \mathbf{Biring}_{k,k}\colon P_k$$
$$S\colon \mathbf{TwBialg}_{k} \leftrightarrows \mathbf{Plethory}_{k}\colon P$$
$$S_k\colon \mathbf{Bialg}_{k}^{\neg com} \leftrightarrows \mathbf{Plethory}_{k}\colon P_k$$

Proof: Let $f\colon S(C) \to B$ be a map of k-k-birings. Define a map $g\colon C \to PB$ by $g(c) = f(c)$. By definition $c \in S(C)$ is primitive and hence so is $g(c) = f(c)$. Moreover, since f is a k-k-biring map, g is a k-k-coalgebra map. Conversely, if $g\colon C \to PB$ is a k-k-coalgebra map, then define $f\colon S(C) \to B$ to be the k-algebra map induced by the k-module map $C \to PB \hookrightarrow B$. We can check this is a map of k-k-birings and these constructions are mutally inverse. Therefore $\mathbf{Biring}_{k,k}(S(C), B) \cong \mathbf{Coalg}_{k,k}(C, PB)$ and we have shown our first adjunction. For the second we note that $\mathbf{Coalg}_k \to \mathbf{Coalg}_{k,k}$ is left adjoint to the forgetful functor $U\colon \mathbf{Coalg}_{k,k} \to \mathbf{Coalg}_k$ and so since $P_k = P \circ U$ we see S_k is left adjoint to P_k. Finally, since S, S_k, P and P_k are lax monoidal both of these adjunctions lift to monoids in the respective categories. □

Plethories generated in this way are entirely trivial in the sense that the action of such a plethory on an algebra is entirely determined by the action of the generating twisted bialgebra or cocommutative bialgebra. We give a name to such plethories.

Definition 1.3.57. We say a k-plethory is linear if $P \cong S_k(B)$ for some cocommutative k-bialgebra B.

We have the following classification theorem of Carlson. A topological generalisation of this result will give us a succinct expression for the plethory of cohomology operations for singular cohomology with rational coefficients.

Theorem 1.3.58 ([20, Theorem 1.1])**.** *Let k be a field of characteristic zero. Then any k-plethory is linear.*

1.3.9 Indecomposables

Another extremely useful functor on Hopf algebras is the functor of indecomposables. An element is indecomposable if it cannot be expressed as a non-trivial product. Hence the indecomposable elements are in some sense a minimal generating set for the Hopf algebra.

Definition 1.3.59. Let H be a Hopf algebra over k. We define the *k-module of indecomposables* of H by the exact sequence

$$IH \otimes IH \xrightarrow{\psi} IH \to QH \to 0$$

where $IH = \ker \varepsilon$ denotes the augmentation ideal. Explicitly,

$$QH = IH/(IH)^2.$$

Since a map of Hopf algebras $f\colon H \to H'$ necessarily has $f(IH^2) \subseteq (IH')^2$, we have a functor

$$Q\colon \mathbf{Hopf}_k \to \mathbf{Mod}_k.$$

Once again, we can ask what the effect the additional structure of a biring or plethory has on the structure of the indecomposable elements.

Proposition 1.3.60 ([17, Proposition 8.2])**.** *The functor of indecomposables $Q\colon \mathbf{Hopf}_k \to \mathbf{Mod}_k$ lifts to a functor*

$$Q\colon \mathbf{Biring}_{k,k'} \to {}_k\mathbf{Mod}_{k'}$$

Given a k-k'-biring B, the right k'-module structure on QB is given by

$$b \cdot \lambda = \gamma\lambda(b).$$

The comultiplication on B is a ring map and by Proposition 1.3.9 we have $\Delta^{\times}(IB) \subseteq IB \otimes IB$. Hence we have a well defined map

$$QB \xrightarrow{\Delta^{\times}} \frac{IB \otimes IB}{(IB)^2 \otimes (IB)^2} \to \frac{IB \otimes IB}{IB \otimes (IB)^2} \cong IB \otimes QB$$

where the second map is induced by the inclusion. Although the second map has non-trivial kernel, since $\Delta^{\times}$ is cocommutative the composition has trivial kernel. It turns out the image of this map lands in $PB \otimes QB$ and this makes QB into a comodule over PB.

Proposition 1.3.61 ([17, Proposition 11.1])**.** *There exists a unique map $\nu\colon QB \to IB \otimes QB$ such that the following diagram commutes*

$$\begin{tikzcd}
IB \arrow[r, "\Delta^{\times}"] \arrow[d, "q"] & IB \otimes IB \arrow[d, "1\otimes q"] \\
QB \arrow[r, "\nu"] & IB \otimes QB
\end{tikzcd}$$

Moreover, the image of ν is contained within $PB \otimes QB \subseteq IB \otimes QB$.

The composition induces a monoidal structure on the module of indecomposables. We denote the category of not necessarily commutative k-algebras by $\mathbf{Alg}_k^{\neg\mathrm{com}}$.

Proposition 1.3.62 ([17, Proposition 8.3])**.** *The functor of indecomposables $Q\colon \mathbf{Biring}_{k,k} \to {}_k\mathbf{Mod}_k$ is strong monoidal. In particular, Q lifts to a functor*

$$\mathbf{Plethory}_k \to \mathbf{Alg}_k^{\neg com}$$

where the algebra structure is induced by the composition on the plethory.

The collection of primitives of a k-plethory Π is a twisted k-bialgebra, in particular it is a k-algebra and so the Sweedler product $P\Pi \circledast Q\Pi$ is well defined.

Proposition 1.3.63. *For a k-plethory Π, the co-action map factors as*

$$\nu\colon Q\Pi \to P\Pi \circledast Q\Pi \subseteq P\Pi \otimes Q\Pi.$$

Proof: See [17, Prop 11.1].

1.3.10 The Frobenius and Verschiebung maps

Continuing our theme of extending results from the world of Hopf algebras to plethories and birings, we now focus on the situation where we are working with finite dimensional objects over fields of characteristic p. For such a Hopf algebra we have two very useful endomorphisms on Hopf algebras known as the Frobenius and the Verschiebung. Moreover the category of such Hopf algebras forms an abelian category and so $\mathbf{Hopf}(H, H')$ is an abelian group. Throughout this section, we fix a prime p.

Definition 1.3.64. Let H be a Hopf algebra over $\mathbb{F}_p$, then the *Frobenius* map $F\colon H \to H$ is the Hopf algebra map defined by $F(x) = x^p$. Suppose H is finite dimensional so DH is also a Hopf algebra and we have a canonical isomorphism $H \cong DDH$. We define the *Verschiebung* to be the Hopf algebra map given by

$$V\colon H \cong DDH \xrightarrow{DF_{DH}} DDH \cong H.$$

It is well known that the Verschiebung and Frobenius satisfy the relation $FV = VF = [p]$ where $[p]$ is p-times the identity map in the abelian group $\mathbf{Hopf}(H, H)$.

A more explicit definition of the Verschiebung can be found in [19] or [1]: for a k-module M, consider the action of the symmetric group Σ_p on $M^{\otimes p}$. Let $\mathcal{S}^p M \overset{i}{\hookrightarrow} M^{\otimes p}$ be the fixed sub-module under this action and $M^{\otimes p} \twoheadrightarrow \operatorname{Sym}^p M$ the quotient module by this action. We define a k-linear map $f\colon M \to \mathcal{S}^p M$ by $x \mapsto [x \otimes \cdots \otimes x]$ and [1, Theorem 2.5.6] shows there exists a unique map of k-modules $v\colon \mathcal{S}^p M \to M$ such that $f \circ v = ji$. The following result shows that for a Hopf ring H if $\psi^{(p)}(x) = \sum_i x^i_{(1)} \otimes \cdots \otimes x^i_{(p)}$ then $V(x)$ is the sum of the terms x_i with $x_i = x^i_{(1)} = \cdots = x^i_{(p)}$.

Proposition 1.3.65 ([1, Theorem 2.5.6]). *Let H be a Hopf algebra over $\mathbb{F}_p$. Since H is bicommutative, the p-fold multiplication descends to a map $\mu\colon \operatorname{Sym}^p H \to H$ and the p-fold comultiplication factors via a map $\psi^{(p)}\colon H \to \mathcal{S}^p H$. The Frobenius and Verschiebung maps are given by the compositions*

$$F\colon H \xrightarrow{f} \operatorname{Sym}^p H \xrightarrow{\mu} H$$
$$V\colon H \xrightarrow{\psi^{(p)}} \mathcal{S}^p H \xrightarrow{v} H.$$

The Frobenius and Verschiebung are maps of Hopf algebras. We now study how these maps respect the additional structure on a biring.

Proposition 1.3.66. *For a $\mathbb{F}_p$-$\mathbb{F}_p$-biring B, the Frobenius and Verschiebung satisfy the following conditions.*

1. $\Delta^\times \circ F = (F \otimes F) \circ \Delta^\times$.

2. *The following diagram commutes.*

$$\begin{tikzcd}
B \arrow[r, "V"] \arrow[d, "\Delta^\times"] & B \arrow[r, "\Delta^\times"] & B \otimes B \arrow[d, "1\otimes F"] \\
B \otimes B \arrow[rr, "V\otimes 1"] & & B \otimes B
\end{tikzcd}$$

Proof: For a Hopf ring H we have $V(x \circ y) = V(x) \circ V(y)$ and Frobenius reciprocity $V(x \circ F(y)) = V(x) \circ y$ (see Appendix E). Our result follows via duality. $\square$

The definition of the Verschiebung in terms of the p-fold iterated coproduct Δ^+ motivates an interesting construction. In a biring we have not one but two coproducts, and we can use the comultiplication $\Delta^\times$ to define a second Verschiebung.

Definition 1.3.67. For a $\mathbb{F}_p$-$\mathbb{F}_p$-biring B, we define the *multiplicative Verschiebung* $V^\times\colon B \to B$ to be the composition

$$V\colon B \xrightarrow{\Delta^\times_{(p)}} \mathcal{S}^p B \xrightarrow{v} B.$$

It is clear from the construction that this the multiplicative Verschiebung on a k-k'-biring is a k-module map, but its behaviour with respect to the Hopf algebraic structure on B seems hard to deduce. We now study how the Frobenius and Verschiebung maps interact with the composition in a plethory.

Proposition 1.3.68. *Let B and B' be $\mathbb{F}_p$-$\mathbb{F}_p$-birings. If $x \in B, y \in B'$ then writing $\Delta^+_{(p)}(y) = \sum_{i=1}^N y^i_{(1)} \otimes \cdots \otimes y^i_{(p)}$ and $\Delta^+_{(N)}(x) = \sum_{j=1}^M x^j_{(1)} \otimes \cdots \otimes x^j_{(N)}$ we have the following statements.*

1. *$F(x \circ y) = F(x) \circ y$.*
2. *$V(x \circ y) = \sum_{j=1}^M \sum_{i=1}^N V^\times(x^j_{(i)}) \circ V(y)$.*
3. *$V^\times(x \circ y) = \sum_{j=1}^M \sum_{i=1}^N V^\times(x^j_{(i)}) \circ V^\times(y)$.*

Proof: The first result is immediate. For the second, identifying $(B \odot B')^{\otimes p}$ with $B \odot ((B')^{\otimes p})$, we have

$$\begin{aligned}
\Delta^{+(p)}(x \odot y) &= x \odot (\sum_{i=1}^N y^i_{(1)} \otimes \cdots \otimes y^i_{(p)}) \\
&= \sum_{j=1}^M \sum_{i=1}^N x^j_{(i)} \odot (y^i_{(1)} \otimes \cdots \otimes y^i_{(p)}) \\
&= \sum_{j=1}^M \sum_{i=1}^N \sum_{(x^j_{(i)})} x^j_{(i)[1]} \odot y^i_{(1)} \otimes \cdots \otimes x^j_{(i)[p]} \odot y^i_{(p)} \in \mathcal{S}^p(B \odot B').
\end{aligned}$$

Applying the map $v \colon \mathcal{S}^p(B \odot B') \to B \odot B'$ and noting that $x \odot y = x' \odot y'$ if and only if $x = x'$ and $y = y'$ we have $V(x \odot y) = \sum_{j=1}^M \sum_{i=1}^N V^\times(x^j_{(i)}) \odot V(y)$. Now since $\circ$ is a map of birings, it commutes with V. The final result is proved in the same way. $\square$

When we are working over $\mathbb{F}_p$, the collection of primitive elements admits extra structure. Let H be a Hopf algebra over $\mathbb{F}_p$. We have seen that the module of primitives PH does not naturally admit a multiplication. However, for $x \in PH$, $\psi(F(x)) = F(\psi(x)) = F(1 \otimes x + x \otimes 1) = 1 \otimes F(x) + F(x) \otimes 1$ and so we see that the Frobenius on H induces a Frobenius on PH. Hence, PH admits the structure of an $\mathbb{F}_p[F]$-module. For an $\mathbb{F}_p$-plethory, the primitives (and k-primitives) naturally admit the structure of an $\mathbb{F}_p$-twisted bialgebra (resp. cocommutative $\mathbb{F}_p$-bialgebra), which is also an $\mathbb{F}_p[F]$-module.

Recall we can form the symmetric $\mathbb{F}_p$-plethory $S_{\mathbb{F}_p}(B)$ over a cocommutative $\mathbb{F}_p$-bialgebra B by freely including a multiplication. If B is naturally an $\mathbb{F}_p[F]$-module, we must take care to do so in a manner which respects the action of the Frobenius. This produces the following definition, detailed in [20].

Definition 1.3.69. Let B be a cocommutative $\mathbb{F}_p$-bialgebra which is also an $\mathbb{F}_p[F]$-module. Define the $\mathbb{F}_p$-plethory $S^{[p]}B$ to be the quotient of the $\mathbb{F}_p$-plethory $S_{\mathbb{F}_p}(B)$ by the ideal generated by the relations $F(x) = x^p$ for all $x \in B$. Now a map of cocommutative $\mathbb{F}_p$-bialgebras $B \to B'$ which respects the action of the Frobenius induces a map $S^{[p]}(B) \to S^{[p]}(B')$ and thus we have a functor

$$S^{[p]} \colon \mathbf{Bialg}_{\mathbb{F}_p}^{\neg\mathrm{com}} \to \mathbf{Plethory}_{\mathbb{F}_p}.$$

Carlson proves the following analogue of Theorem 1.3.58. We should remark that these constructions and following theorem generalise to the setting of plethories over any perfect field of characteristic p.

Theorem 1.3.70 ([20, Theorem 5.3])**.** *If Π is an $\mathbb{F}_p$-plethory with trivial Verschiebung ($V = 0$ on Π) then we have an isomorphism of $\mathbb{F}_p$-plethories $S^{[p]}(P\Pi) \cong \Pi$.*

1.4 Graded plethories

While for our main applications we will be working in a graded setting, this grading is mostly superficial and much of the theory from the ungraded case carries over to the graded setting without difficulty.

In this section, we briefly recap the definitions necessary for the theory of graded plethories and highlight any details specific to the graded case.

1.4.1 Graded algebraic objects

Fix some grading set Z, which we will also view as a discrete category. Later we will require this to be a monoid and we will typically take $Z = \mathbb{N}, \mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$ under addition. Consider the functor category $\mathcal{C}^Z$. We can view an object of this category as a collection of *components*, objects $C_n \in \mathcal{C}$ for $n \in Z$. A morphism between two such objects is a collection of morphisms $f \colon C_n \to C'_n$ for each $n \in Z$. Moreover $\mathcal{C}^Z$ inherits any colimits and limits which exist in $\mathcal{C}$.

Definition 1.4.1. A *variety* of graded algebras $\mathcal{V}^*$ is an ungraded variety (Ω, n, J) (see Definition 1.1.10) together with an *input map* $i \colon \Omega \to \coprod_{m \geq 0} Z^m$ and *output map* $o \colon \Omega \to Z$ which specify the components the operations act on. The 'arity' function $n \colon \Omega \to \mathbb{N}$ is determined by i via the composition

$$\Omega \xrightarrow{i} \coprod_{m \geq 0} Z^m \to \coprod_{m \geq 0} \{*\} \cong \mathbb{N}.$$

A $\mathcal{V}^*$-*algebra structure* on an object $A \in \mathcal{C}^Z$ is a collection of $\mathcal{C}$-morphisms $A_{i(\omega)_1} \times \cdots \times A_{i(\omega)_{n(\omega)}} \to A_{o(\omega)}$ for each $\omega \in \Omega$ which satisfy the relations expressed by J.

A morphism of $\mathcal{V}^*$-algebra objects $f\colon A \to B$ is a morphism such that the following diagram commutes for all operations ω.

$$\begin{array}{ccc} A^{\times n(\omega)} & \xrightarrow{f^{\times n(\omega)}} & B^{\times n(\omega)} \\ \downarrow{\scriptstyle \omega_A} & & \downarrow{\scriptstyle \omega_B} \\ A & \xrightarrow{\quad f \quad} & B \end{array}$$

We denote the subcategory of $\mathcal{V}^*$-algebra objects and morphisms in $\mathcal{C}$ by $\mathcal{V}^*\mathcal{C}$.

Example 1.4.2. If $(Z, +, 0)$ is a monoid then we can define the variety of graded monoids. This has an operation η with $i(\eta) = \emptyset, o(\eta) = 0$ and for each $i, j \in Z$ operations $\mu_{i,j}$ with $i(\mu_{i,j}) = (i, j), o(\mu_{i,j}) = i + j$. The identities in J are the axioms encoding the associativity and unit properties.

For $C \in \mathcal{C}^Z$, the covariant functor represented by C is given by $\mathcal{C}^Z(C, -)\colon \mathcal{C}^Z \to \mathbf{Set}$. By our usual adjunction, we will instead view this as a functor $\mathcal{C} \to \mathbf{Set}^Z$. We have the graded analogues of Lemma 1.1.12 and Corollary 1.1.13. The correspondences work in exactly the same way as the ungraded setting.

Lemma 1.4.3. *Let A be a $\mathcal{V}^*$-object in $\mathcal{C}$ and $F\colon \mathcal{C} \to \mathcal{D}$ be a product preserving functor. Then F restricts to a functor*

$$\mathcal{V}^*\mathcal{C} \to \mathcal{V}^*\mathcal{D}.$$

Corollary 1.4.4. *There is a bijection between $\mathcal{V}^*$-algebra object structures on A and lifts of the contravariant functor represented by A, $\mathcal{C}(-, A)\colon \mathcal{C} \to \mathbf{Set}^Z$, to a functor $\mathcal{C} \to \mathcal{V}^*$.*

Exactly as in the ungraded case, if $\mathcal{C}$ is a category with enough finite coproducts, we can define co-$\mathcal{V}^*$-algebra objects in $\mathcal{C}$ and the same dual results hold.

1.4.2 Graded plethories

Fix Z-graded commutative rings k, k' where $(Z, +, 0)$ is some monoid. We abuse our notation and when k is graded we shall write $\mathbf{Alg}_k$, $\mathbf{Coalg}_k$, etc. for the categories of graded k-algebras, k-coalgebras etc. Given a graded object X, we shall denote the *degree* of $x \in X$ by $|x|$.

We construct our algebraic objects as before except now we work with graded algebraic objects in categories of graded algebraic objects such as $\mathbf{Alg}_k$. Consequently we have two gradings; one arising from the graded objects in the category and another arising from the construction of the graded algebraic objects in that category.

Definition 1.4.5. A *bigraded k-k'-biring* is a (graded) co-k-algebra object in the category of (graded) k-algebras $\mathbf{Alg}_k$. Explicitly, this is a collection of graded k-algebras $B_\bullet = (B_n)_{n\in Z}$ together with graded k-algebra maps for each $n \in Z$

$$\begin{aligned}
\Delta^+ \colon B_n &\to B_n \otimes B_n \\
\varepsilon^+ \colon B_n &\to k \\
\sigma \colon B_n &\to B_n \\
\Delta^\times \colon B_n &\to \prod_{i+j=n} B_i \otimes B_j \\
\varepsilon^\times \colon B_0 &\to k
\end{aligned}$$

and a map of graded rings

$$\beta \colon k' \to \mathbf{Alg}_k(B_\bullet, k)$$

satisfying the relations for a co-k-algebra object. Given $x \in B$ we have $x \in B_n$ for some $n \in Z$. We define the $\bullet$-*degree* by $\deg_\bullet(x) = n$ and the $*$-*degree* to be $|x| \in Z$, the degree of x in the graded k-algebra B_n. We will sometimes write $\deg_*(x) = |x|$. For each $n \in Z$ and $i, j \in Z$ with $i + j = n$, we write $\Delta^\times_{i,j}$ for the composition $B_n \prod_{i+j=n} B_i \otimes B_j \xrightarrow{\pi} B_i \times B_j$ where π is the canonical projection. Moreover, we shall sometimes abuse notation and consider $\varepsilon^\times$ as a map $B_\bullet \to k$ which is zero on $\bullet$-components B_n with $n \neq 0$.

Example 1.4.6. When $Z = \mathbb{Z}$, the identity functor $\mathbf{Alg}_k \to \mathbf{Alg}_k$ is represented by the k-k-biring $\mathcal{I}$ with components

$$\mathcal{I}_n = \begin{cases} k[\iota_n] & n \text{ even} \\ \Lambda_k[\iota_n] & n \text{ odd} \end{cases}$$

where $|\iota_n| = n$. The structure maps are given by

$$\begin{aligned}
\Delta^+(\iota_n) &= 1 \otimes \iota_n + \iota_n \otimes 1 \\
\varepsilon^+(\iota_n) &= 0 \\
\sigma(\iota_n) &= -\iota_n \\
\Delta^\times(\iota_n) &= \sum_{r+s=n} \iota_r \otimes \iota_s \\
\varepsilon^\times(\iota_n) &= \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \\
\beta\lambda(\iota_n) &= \begin{cases} \lambda & |\lambda| = n \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Equivalently, the elements ι_n are the graded analogue of super primitive.

Example 1.4.7. Suppose $k = \bigoplus_{n\in Z} k_n, k' = \bigoplus_{m\in Z} k'_m$ where k_n, k_m are homogeneous. Consider the bigraded set $B = \mathbf{Set}(k', k)$ with grading $B^m_n = \mathbf{Set}(k'_n, k_m)$. The pointwise operations from k' make each B^*_n a graded k-algebra and if k' is of finite type then the k-algebra structure on k' makes B into graded k-k'-biring just as in the ungraded case.

The previous example illuminates what effect the grading on the plethory will have in terms of operations on k-algebras. An element $f \in \mathbf{Set}(k, k')$ with $\deg_\bullet(f) = n, \deg_*(f) = m$ is a set map $\mathbf{Set}(k'_n, k_m)$. Hence f is undefined on elements of k' not of degree n. The definitions turn out to be cleanest if we enforce that $f(x) = 0$ for all x not of degree n.

Definition 1.4.8. For a graded k-k'-biring B and graded k'-algebra A, we define the *graded composition product* to be the ungraded composition product $B \odot A$ quotiented by the ideal generated by the relations $b \odot a = 0$ whenever $\deg_\bullet(b) \neq |a|$.

We define a grading on $B \odot A$ by setting $|b \odot a| = \deg_*(b) = |b|$ and noting that this is compatible with the relations. Hence we have a functor

$$- \odot -\colon \mathbf{Biring}_{k,k'} \times \mathbf{Alg}_{k'} \to \mathbf{Alg}_k.$$

Exactly as in the ungraded case, the graded composition product lifts to a functor $\mathbf{Biring}_{k,k'} \times \mathbf{Biring}_{k',k''} \to \mathbf{Biring}_{k,k''}$ and this makes $(\mathbf{Biring}_{k,k}, \odot, \mathcal{I})$ a monoidal category.

Definition 1.4.9. We define the category of *graded k-plethories* $\mathbf{Plethory}_k$ to be the category of monoids in $\mathbf{Biring}_{k,k}$. Hence, a graded k-plethory is a graded k-k-biring P together with two additional graded biring morphisms

$$\circ\colon P \odot P \to P$$
$$u\colon \mathcal{I} \to P$$

satisfying the usual axioms for a monoid.

Example 1.4.10. The graded k-k-biring $\mathbf{Set}(k, k)$ of Example 1.4.7 naturally has the structure of a graded k-plethory. For $f, g \in \mathbf{Set}(k, k)$, say $f\colon k_s \to k_m$ and $g\colon k_n \to k_r$, the composition $f \circ g\colon k_n \to k_m$ is given by

$$f \circ g = \begin{cases} f \circ g & r = s \\ 0 & \text{otherwise.} \end{cases}$$

Exactly as in the ungraded case, if P is a graded k-plethory then the functor $P \odot -$ forms a monad on $\mathbf{Alg}_k$.

Definition 1.4.11. Let P be a graded k-plethory. We define the category of P*-algebras* $\mathbf{Alg}_P$ to be the Eilenberg-Moore category of algebras for the monad $P \odot -$. Explicitly, a P-algebra is a graded k-algebra A together with a graded algebra map

$$\circ \colon P \odot A \to A$$

satisfying the usual axioms.

Example 1.4.12. The graded ring k is naturally a P-algebra for the graded k-plethory $\mathbf{Set}(k, k)$. For $f \in \mathbf{Set}(k, k)$ we have $\phi(x) = 0$ for all $x \in k$ with $|x| \neq \deg_\bullet(\phi)$.

Definition 1.4.13. For $n \in Z$, define the functor $U^n \colon \mathbf{Alg}_P \to \mathbf{Set}$ to be the composition

$$\mathbf{Alg}_P \xrightarrow{U} \mathbf{Set}^Z \xrightarrow{-(n)} \mathbf{Set}$$

where U is the forgetful functor and $-(n)$ is evaluation at n. We define an *operation* of type (n, m) on P-algebras to be a natural transformation $r \colon U^n \to U^m$.

We denote the set of operations of all types on P-algebras by $\mathrm{Op}(\mathbf{Alg}_P)$. This is naturally a bigraded set by defining for $r \colon U^n \to U^m$, $\deg_\bullet(r) = n, \deg_*(r) = m$ and the point wise operations naturally make $\mathrm{Op}(\mathbf{Alg}_P)_n = \mathrm{Nat}(U^n, U^*)$ a graded k-algebra for each fixed n.

Exactly as in the ungraded case, we have an isomorphism of bigraded k-algebras $\mathrm{Op}(\mathbf{Alg}_P) \cong P$ and we have realised graded k-plethories as precisely the structure that acts on graded k-algebras.

Chapter 2

Plethories in topology

In this chapter I apply plethystic theory to study the unstable operations for a multiplicative cohomology theory satisfying suitable technical conditions. A priori, the unstable cohomology operations do not admit the structure of a plethory, but with a little technical machinery we can show they admit the structure of a topologised generalisation of a plethory. Moreover, in a suitable context, the cohomology algebra of a space is naturally a topological generalisation of an algebra over plethory. This is a consequence of an abstract result of Stacey and Whitehouse [43]. In Section 2.1, I motivate and give a new direct proof of this theorem.

As previously discussed, the approach chosen by Stacey and Whitehouse takes completions of the cohomology algebras and is therefore unable to detect phantom classes. The more general but more complex approach of formal plethories due to Bauer [11] gives a framework which can detect phantom classes. The definition works heavily with abstract categorical and algebro-geometric structures. I outline the details of this theory and reformulate it in more familiar language to highlight the similarities and differences to our framework.

In the remainder of this chapter, I discuss common properties of cohomology theories and analyse the implications of these properties for the plethory of unstable cohomology operations which has not been done before in a plethystic setting. All cohomology theories come equipped with suspension isomorphisms, and the effect of these isomorphisms on the plethory of operations is the focus of Section 2.2. This leads to the new definition of a *plethory with looping*, an abstract object which encodes this additional structure.

Many familiar cohomology theories come equipped with extra structure known as a complex orientation. The effect of this on the Hopf ring of homology cooperations has been well studied as in [40], [31] and [15]. In Section 2.3 I recall these results and derive the natural analogues of these in the dual world of birings before studying the effects on the monoidal plethystic structure. This leads to the new definition of a *complex oriented plethory* which encodes all additional structure induced by the complex orientation at the level of birings but perhaps only some of the structure at the plethystic level.

2.1 The plethory of unstable cohomology operations

We now turn to our main application of plethystic theory - studying the collection of unstable cohomology operations. We closely follow Boardman's definition [14] of cohomology theories, with details in Appendix D.

Definition 2.1.1. Let $E^*(-)\colon \mathbf{Ho} \to \mathbf{Ab}^{\mathbb{Z}}$, $F^*(-)\colon \mathbf{Ho} \to \mathbf{Ab}^{\mathbb{Z}}$ be graded cohomology theories. An unstable cohomology operation of type (E, F, n, m) is a natural transformation $r\colon U \circ E^n(-) \to U \circ F^m(-)$ where U is the forgetful functor $\mathbf{Ab} \to \mathbf{Set}$. Explicitly, for each space X we have a set map $r_X\colon E^n(X) \to F^m(X)$ such that if $f\colon X \to Y$ is a representative for a homotopy class of maps then the following diagram commutes.

$$\begin{array}{ccc} E^n(Y) & \xrightarrow{r_Y} & F^m(Y) \\ \big\downarrow{\scriptstyle f^*} & & \big\downarrow{\scriptstyle f^*} \\ E^n(X) & \xrightarrow{r_X} & F^m(X) \end{array}$$

For brevity, we shall often say an operation $r\colon E^n(-) \to F^m(-)$ and leave it implicit that we are viewing $E^n(-)$ and $F^m(-)$ as set-valued functors.

We wish to study the structure of the collection of unstable cohomology operations. To have a more concrete form for this, we can apply the Yoneda lemma and Brown's representability theorem (Theorem 4) to obtain the following useful identifications.

Proposition 2.1.2. *Let $E^*(-), F^*(-)$ be graded cohomology theories with representing spaces $\underline{E}_n, \underline{F}_n$ respectively. We have natural identifications between the following 3 sets.*

1. *The collection of unstable cohomology operations $E^n(-) \to F^m(-)$.*

2. *The homotopy classes of maps from $\underline{E}_n$ to $\underline{F}_m$.*

3. *The cohomology group $F^m(\underline{E}_n)$.*

Proof: By the Yoneda embedding, we have $\mathrm{Nat}(E^n(-), F^m(-)) \cong \mathbf{Ho}(\underline{E}_n, \underline{F}_m)$. However by Brown's representability theorem this is naturally isomorphic to $F^m(\underline{E}_n)$. $\square$

Hence studying the collection of unstable cohomology operations is equivalent to studying the bigraded collection of cohomology groups $F^m(\underline{E}_n)$ for $n, m \in \mathbb{Z}$. We shall denote this object by $F^*(\underline{E}_\bullet)$.

From this point on, we shall assume all our cohomology theories are multiplicative and hence naturally take values in algebras. We can ask what structure we have on $F^*(\underline{E}_\bullet)$. For all spaces X, $F^*(X)$ is an F^*-algebra, in particular $F^*(\underline{E}_n)$ is an F^*-algebra for all $n \in \mathbb{Z}$. However, our spaces $\underline{E}_n$ are not just any spaces. They are very highly structured representing spaces for the Ω-spectrum E.

If E is an Ω-spectrum with representing spaces $\underline{E}_n$, we have the following structure. Since $E^*(-)\colon \mathbf{Ho} \to \mathbf{Alg}^*_E$ is representable, by Corollary 1.1.13 we have an E^*-algebra object structure on $\underline{E}_\bullet$. We can realise the structure maps as follows.

- For each $n \in \mathbb{Z}$, the structure of a group object in $\mathbf{Ho}$ on $\underline{E}_n \simeq \Omega \underline{E}_{n+1}$, induced by the loop space structure.
- For each $n, m \in \mathbb{Z}$, we have multiplication maps $\phi\colon \underline{E}_n \times \underline{E}_m \to \underline{E}_n \wedge \underline{E}_m \to \underline{E}_{n+m}$ and a unit map $T \to \underline{E}_0$ where T denotes the one point space.
- For each $\lambda \in E^*$ of degree $\lambda = h$, we have the λ-action map $\underline{E}_n \cong T \times \underline{E}_n \xrightarrow{\lambda \times 1} \underline{E}_h \times \underline{E}_n \xrightarrow{\phi} \underline{E}_{n+h}$.

Thus, if our cohomology theory $F^*(-)\colon \mathbf{Ho} \to \mathbf{Alg}_{E^*}$ mapped products to coproducts then by Corollary 1.1.17, $F^*(\underline{E}_\bullet)$ would have the structure of a co-F^*-algebra object in $\mathbf{Alg}_{E^*}$; an E^*-F^*-biring. Sadly this is not the case. Fortunately, with a little bit of extra technical machinery we can derive a setting where under suitable conditions our cohomology theory will map products to coproducts.

Additionally, if $E^*(-) = F^*(-)$, then we can naturally compose operations of compatible bidegrees. This will make $E^*(\underline{E}_\bullet)$ a monoid in our suitable category of E^*-E^*-birings and have the structure of a generalisation of an E^*-plethory.

2.1.1 Filtrations and topologies

A multiplicative cohomology theory $E^*(-)$ comes equipped with external cross product maps in both homology and cohomology.

$$E^*(X) \otimes E^*(Y) \to E^*(X \times Y),$$
$$E_*(X) \otimes E_*(Y) \to E_*(X \times Y).$$

Given some structure on our spaces X we can often use these pairings to yield internal structure on the homology or cohomology groups. A simple piece of structure that all spaces X have is the diagonal map $\Delta\colon X \to X \times X$. The composition

$$E^*(X) \otimes E^*(X) \to E^*(X \times X) \xrightarrow{\Delta^*} E^*(X)$$

gives the familiar cup-product multiplication on $E^*(X)$ and makes the cohomology a graded E^*-algebra. In homology, the story is less straightforward and highlights some of the issues we will have to circumnavigate. We have maps

$$E_*(X) \xrightarrow{\Delta_*} E_*(X \times X) \leftarrow E_*(X) \otimes E_*(X).$$

Hence we can only obtain a comultiplication on $E_*(X)$ in this way if the cross product map is an isomorphism, known as a Künneth isomorphism. Fortunately, this is frequently the case.

Proposition 2.1.3 ([49, Theorem 13.75])**.** *If $E_*(X)$ or $E_*(Y)$ is a flat E^*-module, then the cross product map $E_*(X) \otimes E_*(Y) \to E_*(X \times Y)$ is an isomorphism of E^*-modules.*

As an immediate corollary we see that if $E_*(X)$ is a flat E^*-module, then $E_*(X)$ is a E^*-coalgebra. Now suppose X is an H-space and thus comes equipped with a unital multiplication $\mu\colon X \times X \to X$. On homology this induces a map

$$E_*(X) \otimes E_*(X) \to E_*(X \times X) \xrightarrow{\mu_*} E_*(X)$$

and gives $E_*(X)$ the structure of a Hopf algebra. However, the situation in cohomology is more complicated. We have maps

$$E^*(X) \xrightarrow{\mu^*} E^*(X \times X) \leftarrow E^*(X) \otimes E^*(X).$$

Once again, we will only obtain a multiplication on $E^*(X)$ in this manner if the cross product map is an isomorphism. Unfortunately, unlike in homology, this is rarely the case.

Consider the infinite complex projective space $\mathbb{C}P^\infty$. If $E^*(-)$ is a complex oriented cohomology theory (see Section 2.3) then $E^*(\mathbb{C}P^\infty) = E^*[[x]]$ and $E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) = E^*[[x_1, x_2]]$. The cross product map $E^*(\mathbb{C}P^\infty) \otimes E^*(\mathbb{C}P^\infty) \to E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$ is determined by $x \otimes 1 \mapsto x_1$, $1 \otimes x \mapsto x_2$. Since elements of the tensor product are finite linear combinations of pure tensors, this map is far from surjective. For example, the infinite series $\sum_{n\geq 0} x_1^n x_2^n$ does not lie in the image of the cross product map. However, taking increasing finite series $\sum_{n=0}^{N} x_1^n x_2^n$ we can get 'arbitrarily close' to this element.

The above example suggests it will be much more likely that we have a Künneth isomorphism in cohomology if we can topologise our cohomology algebras in a suitable manner. Since $E^*[[x_1]] \otimes E^*[[x_2]]$ is in some sense dense in $E^*[[x_1, x_2]]$, we will be able to realise the infinite series as an element of a suitable completion of the tensor product, denoted $E^*[[x_1]]\widehat{\otimes}E^*[[x_2]]$.

2.1.2 Filtered modules

As discussed, to have any hope of a Künneth isomorphism in cohomology, we need to equip our cohomology algebras with a topology. A sufficiently general way of equipping modules and algebras with a topology is via a filtration. In this section, we give a brief introduction to the subject of filtered modules, closely tailored to our purposes. For a complete treatment, see [38, Section 9]. We shall assume k to be a graded (commutative, unital) ring and work in the graded setting, although many results are identical in both the graded and ungraded contexts.

Definition 2.1.4. Let $\mathcal{O}$ be a partially ordered set. A *filtered k-module* is a (graded) k-module M together with submodules $F^\alpha M \subseteq M$ for each $\alpha \in \mathcal{O}$ satisfying the following conditions.

1. For all $\alpha \leq \beta$, $F^\beta M \subseteq F^\alpha M$.
2. For all $\alpha, \beta \in A$ there exists $\gamma \in A$ with $F^\gamma M \subseteq F^\alpha M \cap F^\beta M$.

A filtered k-module M is naturally a topological space with basis given by all translates of the filtration components. Explicitly, $\{x + F^\alpha M \mid x \in M, \alpha \in \mathcal{O}\}$ is a basis for the topology on M. We write $\mathbf{FMod}_k$ for the category of filtered k-modules and continuous k-module maps.

Example 2.1.5. Consider the k-module $k[x]$ with filtration (x^n) for $n \in \mathbb{N}$. Under this topology, we have $\lim_{n\to\infty} x^n = 0$.

Every k-module M can be given a discrete filtration $0 \subseteq M$ which induces the discrete topology. Hence we have an inclusion of categories $\mathbf{Mod}_k \hookrightarrow \mathbf{FMod}_k$ which is left adjoint to the forgetful functor $\mathbf{FMod}_k \to \mathbf{Mod}_k$.

Continuity of a map of filtered k-modules is a purely topological property. However, it is easier to work with if we restate it in terms of the filtration.

Proposition 2.1.6 ([38, Section 9, Proposition 8]). *Let M and N be filtered k-modules with indexing sets $\mathcal{O}$ and $\mathcal{O}'$ respectively. A map $f\colon M \to N$ of filtered k-modules is continuous if and only if for all $\beta \in \mathcal{O}'$ there exists $\alpha \in \mathcal{O}$ such that $f(F^\alpha M) \subseteq F^\beta N$.*

The fact that the filtration is by sub-modules forces the k-module structure maps to be continuous, and as a consequence we have the following result.

Proposition 2.1.7 ([38, Section 9, Propositions 5, 6 and 7]). *A filtered k-module M is naturally a k-module object in* **Top**.

Topological groups and hence k-module objects in **Top** are examples of uniform spaces. A uniform space is a topological space with additional structure that allows us to define uniform properties such as completeness, uniform continuity and uniform convergence. A reference for the general theory of uniform spaces is [18, Chapter 2]. However, discussing such spaces in full generality will take us away from our applications and so we will restrict ourself to just introducing the theory that is directly relevant.

Definition 2.1.8. A sequence of elements (x_n) in a filtered k-module M is *Cauchy* if for all $\alpha \in \mathcal{O}$ there exists some $N \geq 0$ such that $x_n - x_m \in F^\alpha M$ whenever $n, m \geq N$. The filtered k-module M is *complete* if every every Cauchy sequence is convergent. We write $\mathbf{CMod}_k \subseteq \mathbf{FMod}_k$ for the subcategory of complete Hausdorff filtered k-modules and continuous maps and note that the inclusion $\mathbf{Mod}_k \hookrightarrow \mathbf{FMod}_k$ factors as $\mathbf{Mod}_k \to \mathbf{CMod}_k \subseteq \mathbf{FMod}_k$.

Example 2.1.9. Recall the filtered k-module $k[x]$ of example Example 2.1.5. This is not complete: the sequence of partial sums $x_n = \sum_{k=0}^{n} x^k$ is Cauchy yet not convergent.

As is understood from a first course in real analysis, in general homeomorphisms do not preserve uniform properties. For example, the complete space $\mathbb{R}$ is homeomorphic to the open interval $(0, 1)$ which is not complete. However, the uniform structure on a filtered k-module is such that every continuous map is uniformly continuous. To prove this general statement would require us to delve into the theory of uniform spaces, so we just give a useful corollary of this fact.

Proposition 2.1.10. *Let $f\colon M \to N$ be a homeomorphism of filtered k-modules. If M is complete, then so is N.*

Proof: Let $\mathcal{O}$ and $\mathcal{O}'$ denote the indexing sets for the filtrations on M and N respectively. Write $g = f^{-1}\colon N \to M$ and let (y_n) be a Cauchy sequence in N. Defining $x_n = g(y_n)$, we claim this is a Cauchy sequence in M. For $\alpha \in \mathcal{O}$, by Proposition 2.1.6 there exists $\beta \in \mathcal{O}'$ such that $g(F^\beta N) \subseteq F^\alpha M$. Since (y_n) is Cauchy, there exists N such that for all $n, m \geq N$, $y_n - y_m \in F^\beta N$. Thus for all $n, m \geq N$, $x_n - x_m = g(y_n - y_m) \in g(F^\beta N) \subseteq F^\alpha M$ and thus (x_n) is Cauchy. Since M is complete, (x_n) converges to some $x \in M$. We claim (y_n) converges to $f(x)$. Let $\beta \in \mathcal{O}'$, by Proposition 2.1.6 there exists $\alpha \in \mathcal{O}$ such that $f(F^\alpha M) \subseteq F^\beta N$. Now since (x_n) converges to x, there exists N such that for all $n \geq N$, $x_n - x \in F^\alpha M$. Thus for $n \geq N$, $y_n - f(x) = f(x_n - x) \in f(F^\alpha M) \subseteq F^\beta M$ and hence (y_n) converges to y. $\square$

Any uniform space admits an up to isomorphism unique Hausdorff completion and in the case of filtered k-modules we have an explicit construction.

Proposition 2.1.11 ([14, Section 6])**.** *Let M be a filtered k-module and consider the canonical map $M \to \varprojlim_\alpha M/F^\alpha M$ where $\alpha \in \mathcal{O}$.*

1. *M is Hausdorff if and only if $M \to \varprojlim_\alpha M/F^\alpha M$ is injective.*
2. *M is complete if and only if $M \to \varprojlim_\alpha M/F^\alpha M$ is surjective.*

Definition 2.1.12. Define the *Hausdorff completion* of a filtered k-module M by

$$\widehat{M} = \varprojlim_\alpha M/F^\alpha M.$$

We write $c\colon M \to \widehat{M}$ for the canonical *completion map*.

Example 2.1.13. Recall the filtered k-module $k[x]$ of Example 2.1.5. The Hausdorff completion of $k[x]$ is

$$\widehat{k[x]} = \varprojlim_n k[x]/(x^n) = k[[x]].$$

The Hausdorff completion has many useful properties (beyond being Hasudorff complete) which justify the name.

Proposition 2.1.14 ([38, Section 9, Theorem 5])**.** *Let M be a filtered k-module with filtration indexed by $\mathcal{O}$. The completion $\widehat{M}$ has a canonical filtration with the same indexing set under which it is complete Hausdorff. Moreover, the following results hold.*

1. *The completion map $c\colon M \to \widehat{M}$ is continuous.*
2. $\ker c = \bigcap_\alpha F^\alpha M$.
3. *The image of the completion map $c(M)$ is dense in $\widehat{M}$.*

4. *For all $\alpha \in \mathcal{O}$, $\widehat{M}/F^\alpha \widehat{M} \cong M/F^\alpha M$.*

5. *If $f\colon M \to N$ is a map of filtered k-modules, there exists a unique map of filtered k-modules $\widehat{f}\colon \widehat{M} \to \widehat{N}$ such that the following diagram commutes.*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "c"] & N \arrow[d, "c"] \\ \widehat{M} \arrow[r, "\widehat{f}"] & \widehat{N} \end{tikzcd}$$

Corollary 2.1.15. *The completion is a functor $\widehat{-}\colon \mathbf{FMod}_k \to \mathbf{CMod}_k$ which is left adjoint to the inclusion $\mathbf{CMod}_k \subseteq \mathbf{FMod}_k$.*

Proof: Definition 2.1.12 defined our completion on objects and Proposition 2.1.14 defines it on morphisms. [38, Section 9, Theorem 6] asserts that we have a functor and the adjunction is a consequence of the uniqueness of the completion. $\square$

2.1.3 The filtered tensor product

An essential construction for working with modules and more complex objects is the tensor product. In this section we generalise this construction to the filtered setting.

Definition 2.1.16. Let M, N be filtered k-modules with indexing sets $\mathcal{O}$, $\mathcal{O}'$ respectively. For $(\alpha, \beta) \in \mathcal{O} \times \mathcal{O}'$, define

$$F^{\alpha,\beta}(M \otimes N) = \ker(M \otimes N \xrightarrow{q \otimes q} M/F^\alpha M \otimes N/F^\beta N).$$

The collection $F^{\alpha,\beta}(M \otimes N)$ gives a filtration on $M \otimes N$. If each $F^\alpha M$ is N-pure (see Definition 1.3.2) and each $F^\beta N$ is M-pure then we have

$$F^{\alpha,\beta}(M \otimes N) = F^\alpha M \otimes N \oplus M \otimes F^\beta N.$$

The following result shows that we have a functor $-\otimes -\colon \mathbf{FMod}_k \times \mathbf{FMod}_k \to \mathbf{FMod}_k$.

Proposition 2.1.17 ([14, Section 6])**.** *Let $f\colon M \to M'$, $g\colon N \to N'$ be maps of filtered k-modules. The map $f \otimes g\colon M \otimes N \to M' \otimes N'$ is continuous.*

Even if M and N are complete Hausdorff, the filtration topology on $M \otimes N$ is rarely complete. For example, recall the complete k-module $k[[x]]$ of Example 2.1.13. The sequence $\sum_{k=1}^n x^k \otimes x^k \in k[[x]] \otimes k[[x]]$ is Cauchy but not convergent.

Definition 2.1.18. We define the *completed tensor product* $-\widehat{\otimes}-\colon \mathbf{FMod}_k \times \mathbf{FMod}_k \to \mathbf{CMod}_k$ to be the composition

$$\mathbf{FMod}_k \times \mathbf{FMod}_k \xrightarrow{-\otimes-} \mathbf{FMod}_k \xrightarrow{\widehat{-}} \mathbf{CMod}_k.$$

Example 2.1.19. Recall the filtered k-module $k[x]$ of Example 2.1.5. We have

$$k[x]\widehat{\otimes}k[y] = \varprojlim_{n,m} \frac{k[x] \otimes k[y]}{(x^n) \otimes k[y] \oplus k[x] \otimes (y^m)} \cong \varprojlim_{n,m} \frac{k[x]}{(x^n)} \otimes \frac{k[y]}{(y^m)} \cong k[[x,y]].$$

The following result shows that completing either M or N before forming the completed tensor product has no effect on $M\widehat{\otimes}N$.

Proposition 2.1.20. *For filtered k-modules M and N, we have $\widehat{M}\widehat{\otimes}\widehat{N} \cong M\widehat{\otimes}N$.*

Proof: Let M and N be filtered k-modules with filtrations $F^\alpha M$ and $F^\beta N$ respectively. By Proposition 2.1.14,

$$\widehat{M}\widehat{\otimes}N = \varprojlim_{\alpha,\beta} \frac{\widehat{M}}{F^\alpha \widehat{M}} \otimes \frac{N}{F^\beta N} \cong \varprojlim_{\alpha,\beta} \frac{M}{F^\alpha M} \otimes \frac{N}{F^\beta N} = M\widehat{\otimes}N.$$

□

Proposition 2.1.21 ([14, Section 6])**.** *The categories $(\mathbf{FMod}_k, \otimes, k)$ and $(\mathbf{CMod}_k, \widehat{\otimes}, k)$ are symmetric monoidal, where k is given the discrete filtration.*

2.1.4 Filtered algebras

We would like to generalise our notion of filtered modules to more structured objects such as algebras, birings and plethories. As we will demonstrate, the categorical constructions for obtaining these objects from modules naturally respect the filtration.

Definition 2.1.22. We define the category of *filtered k-algebras* $\mathbf{FAlg}_k$ to be the category of monoids in $\mathbf{FMod}_k$. Similarly we define the category of *complete Hausdorff k-algebras* $\mathbf{CAlg}_k$ to be the category of monoids in $\mathbf{CMod}_k$.

Example 2.1.23. The k-module $k[x]$ of Example 2.1.5 together with the usual multiplication is a filtered k-algebra. Moreover, the completion $k[[x]]$ of Example 2.1.13 is a complete Hausdorff k-algebra.

Proposition 2.1.24. *The completion functor $\widehat{-}\colon \mathbf{FMod}_k \to \mathbf{CMod}_k$ lifts to a functor $\mathbf{FAlg}_k \to \mathbf{CAlg}_k$.*

Proof: Let A be a filtered k-algebra. A priori, $\widehat{A}$ is a completed k-module. However, the multiplication $A \otimes A \to A$ and unit $k \to A$ maps are continuous so we take their completion them to obtain filtered k-module maps $\widehat{A}\widehat{\otimes}\widehat{A} \to \widehat{A}$ and $k \to \widehat{A}$. By Proposition 2.1.14, these maps satisfy the axioms for a monoid in $\mathbf{CMod}_k$. Additionally, given a map of filtered k-algebras, taking the completion gives a map of complete Hausdorff k-algebras. $\square$

Unlike for k-modules, the category $\mathbf{CAlg}_k$ is not a priori a subcategory of $\mathbf{FAlg}_k$ since the multiplication in $\mathbf{CAlg}_k$ is a map $A\widehat{\otimes}A \to A$ rather than a map $A \otimes A \to A$. However, if A is a complete Hausdorff k-algebra, we can define a multiplication $A \otimes A \xrightarrow{c} A\widehat{\otimes}A \to A$ and view A as a filtered k-algebra. Since $A \otimes A$ is dense in $A\widehat{\otimes}A$, this gives an inclusion of categories $\mathbf{CAlg}_k \subseteq \mathbf{FAlg}_k$ which is right adjoint to the completion $\mathbf{FAlg}_k \to \mathbf{CAlg}_k$. Just as for k-modules, we have an inclusion of categories $\mathbf{Alg}_k \to \mathbf{CAlg}_k \subseteq \mathbf{FAlg}_k$ giving a k-algebra the discrete filtration and this inclusion is left adjoint to the forgetful functor $\mathbf{FAlg}_k \to \mathbf{Alg}_k$.

Those familiar with filtered rings and algebras may be more familiar with definitions which require a filtration by ideals, however this property is a consequence of our more abstract definition which is easier to generalise.

Proposition 2.1.25. *If A is a filtered k-algebra then A is isomorphic to a filtered k-algebra which is filtered by ideals.*

Proof: Let $\mathcal{O}$ be the indexing set for the filtration on A. Define A' to be the k-algebra A filtered by the ideals generated by $F^\alpha A$, that is $F^\alpha A' = (F^\alpha A)$ for $\alpha \in \mathcal{O}$. We claim the identity map $i\colon A \to A'$ is an isomorphism of filtered k'-algebras. Since μ is continuous, for each $\alpha \in \mathcal{O}$ there exists $\beta, \gamma \in \mathcal{O}$ such that $\mu(F^{\beta,\gamma}(A \otimes A)) \subseteq F^\alpha A$. However, $F^\beta A' = (F^\beta A), F^\gamma A' = (F^\gamma A) \subseteq \mu(F^{\beta,\gamma}(A \otimes A))$ and thus i is continuous. Conversely, for each $\alpha \in \mathcal{O}$, $F^\alpha A' = (F^\alpha A)$ so clearly $F^\alpha A \subseteq F^\alpha A'$. This shows i is an isomorphism of filtered k'-algebras. $\square$

Proposition 2.1.26 ([14, Section 6])**.** *In the category $\mathbf{FAlg}_k$, the initial object is the discrete k-algebra k and the coproduct is $\otimes$. In $\mathbf{CAlg}_k$ the initial object is again k and the coproduct is $\widehat{\otimes}$.*

2.1.5 Topological filtrations

For our applications we are not considering arbitrary modules and algebras, but cohomology algebras of topological spaces. The underlying topology induces a filtration on the cohomology algebra.

Definition 2.1.27. Let $E^*(-)$ be a (multiplicative) cohomology theory and X be a CW-complex. We define the *profinite filtration* of $E^*(X)$ to consist of the ideals

$$F^\alpha E^*(X) = \ker\left(E^*(X) \xrightarrow{i^*} E^*(X_\alpha)\right)$$

where $i\colon X_\alpha \to X$ is the inclusion and X_α runs through all finite subcomplexes of X. The resulting topology is called the profinite topology on $E^*(X)$.

The following results show our cohomology functor $E^*(-)\colon \mathbf{Ho} \to \mathbf{Alg}_{E^*}$ lifts to a functor $E^*(-)\colon \mathbf{Ho} \to \mathbf{FAlg}_{E^*}$.

Proposition 2.1.28. *If $f\colon X \to Y$ is a map of CW-complexes then $f^*\colon E^*(Y) \to E^*(X)$ is continuous with respect to the profinite topology.*

Proof: Let X_α be a finite subcomplex of X. Since f is cellular, $f(X_\alpha)$ is finite and hence $f(X_\alpha) \subseteq Y_\beta$ for some finite subcomplex $Y_\beta \subseteq Y$. Now if $y \in f^*(F^\beta E^*(Y))$, the composition $Y_\beta \to Y \xrightarrow{y} \underline{E}_n$ is null homotopic. Since

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "y"] & \underline{E}_n \\
X_\alpha \arrow[u] \arrow[r, "f|_{X_\alpha}"] & Y_\beta \arrow[u] &
\end{tikzcd}$$

commutes, we see that $f^*(y) \in F^\alpha E^*(X)$ and hence $f^*(F^\beta E^*(Y)) \subseteq F^\alpha E^*(X)$. □

The following result combined with Proposition 2.1.11 shows that the profinite topology is always complete. However, it is not necessarily Hausdorff.

Proposition 2.1.29 ([3, Theorem 1.8])**.** *The completion map $E^*(X) \to \widehat{E^*(X)} = \varprojlim_\alpha E^*(X)/F^\alpha E^*(X)$ is surjective.*

As we saw in Proposition 2.1.14, the completion map has kernel $\bigcap_\alpha F^\alpha E^*(X)$. The elements of this kernel are known as phantom classes and are those cohomology classes which are zero when restricted to any finite subcomplex of X.

Definition 2.1.30. A map of spaces $f\colon X \to Y$ is *phantom* if the restriction of f to any finite subcomplex of X is null-homotopic. We say a cohomology class $x \in E^n(X)$ is *phantom* if the representing map $X \to \underline{E}_n$ is phantom and call a cohomology operation *phantom* if the corresponding class is phantom.

Example 2.1.31. In [5], Adams and Walker construct a map $f\colon \Sigma\mathbb{C}P^\infty \to \bigvee_{\mathbb{N}} S^4$ which is null-homotopic when restricted to any finite subcomplex of $\Sigma\mathbb{C}P^\infty$.

As often occurs in topology, lack of Hausdorff-ness leads to many pathological examples. To avoid such issues we can take the Hausdorff completion of our cohomology functor.

Definition 2.1.32. Define the *Hausdorff completion* of $E^*(-)$, denoted $\widehat{E}^*(-)$ to be the following composition.

$$\mathbf{Ho} \xrightarrow{E^*(-)} \mathbf{FAlg}_{E^*} \xrightarrow{\widehat{-}} \mathbf{CAlg}_{E^*}.$$

Sadly, the Hausdorff completion of a cohomology theory contains strictly less information than the original theory. Indeed, we are taking the quotient by the ideal generated by the phantom classes. Fortunately, there are various results which prohibit the existence of phantom classes and thus in many cases $E^*(-) = \widehat{E}^*(-)$.

Theorem 2.1.33 ([14, Theorem 4.14]). *If $E_*(X)$ is a free E^*-module then $E^*(X)$ is complete Hausdorff.*

Since over a graded field all E^*-modules are free, this result ensures the absence of phantom classes in the ordinary cohomology theories with field coefficients, $H^*(-;\mathbb{F}_p), H^*(-;\mathbb{Q})$ as well as the Morava K-theories $K(n)$. We remark that for complex K-theory this is not the case, and there exist spaces X with phantom cohomology classes.

Armed with the Hausdorff completion of our cohomology theory, we can now precisely state the appropriate Künneth isomorphism. We need the following lemma.

Lemma 2.1.34 ([14, p. 603]). *The cross product map $\times\colon E^*(X) \otimes E^*(Y) \to E^*(X \times Y)$ is continuous with respect to the profinite topology.*

Completing the cross product yields a map $E^*(X)\widehat{\otimes}E^*(Y) \to \widehat{E}^*(X \times Y)$. A sufficient condition for this to be an isomorphism is that $E_*(X)$ and $E_*(Y)$ are free E^*-modules. In that case, $E_*(X \times Y) \cong E_*(X) \otimes E_*(Y)$, we see that $E_*(X \times Y)$ is also a free E^*-module and thus $E^*(X \times Y) \cong \widehat{E}^*(X \times Y)$.

Theorem 2.1.35 ([14, Theorem 4.19]). *If $E_*(X)$ and $E_*(Y)$ are free E^*-modules then we have an isomorphism of complete Hausdorff E^*-algebras*

$$E^*(X)\widehat{\otimes}E^*(Y) \xrightarrow{\cong} E^*(X \times Y).$$

It will prove convenient to have a name for spaces X whose filtered cohomology rings are complete Hausdorff and on which we have a Künneth isomorphism in cohomology.

Definition 2.1.36. We define the *E-Künneth homotopy category* $\mathbf{Ho}_E^\wedge$ to be the full subcategory of $\mathbf{Ho}$ consisting of spaces X such that $E_*(X)$ is a free E^*-module.

We can now rephrase Theorem 2.1.33 and Theorem 2.1.35 as the following statement.

Corollary 2.1.37. *If $E^*(-)\colon \mathbf{Ho} \to \mathbf{FAlg}_{E^*}$ is a cohomology theory then the restriction of this functor to $\mathbf{Ho}_E^\wedge$ lifts as $E^*(-)\colon \mathbf{Ho}_E^\wedge \to \mathbf{CAlg}_{E^*}$. Moreover, this restriction maps products to coproducts and terminal objects to initial objects.*

Now by Lemma 1.1.12, the restriction of $E^*(-)$ to $\mathbf{Ho}_E^\wedge$ maps algebraic objects in $\mathbf{Ho}_{E^*}^\wedge$ to coalgebraic objects in $\mathbf{CAlg}_{E^*}$.

Example 2.1.38. Consider the infinite complex projective space $\mathbb{C}P^\infty$. For a complex oriented cohomology theory (see Section 2.3), we have $E^*(\mathbb{C}P^\infty) \cong E^*[[x]]$. However, $\mathbb{C}P^\infty$ has additional structure: it is an H-space with multiplication $\mu\colon \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$. Thus, we have maps

$$E^*(\mathbb{C}P^\infty) \xrightarrow{\mu^*} E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \xleftarrow{\times} E^*(\mathbb{C}P^\infty) \otimes E^*(\mathbb{C}P^\infty).$$

As we have seen, the cross product is not an isomorphism. In fact $E^*(\mathbb{C}P^\infty) \otimes E^*(\mathbb{C}P^\infty) \cong E^*[[x_1]] \otimes E^*[[x_2]] \subsetneq E^*[[x_1, x_2]] \cong E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$. However, we can equip $E^*(\mathbb{C}P^\infty)$ with a 'completed comultiplication' via the composition

$$E^*(\mathbb{C}P^\infty) \xrightarrow{\mu^*} E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong E^*(\mathbb{C}P^\infty) \widehat{\otimes} E^*(\mathbb{C}P^\infty).$$

2.1.6 Filtered birings

We can now generalise our definition of a biring to a suitable topological setting. This will allows us to prove one of our major results: the collection of cohomology operations between two theories is a topologised version of a biring.

Definition 2.1.39. We define the category of *filtered k-k'-birings* $\mathbf{FBiring}_{k,k'}$ to be the category of co-k'-algebra objects in $\mathbf{FAlg}_k$. Similarly, we define the category of *complete Hausdorff k-k'-birings* $\mathbf{CBiring}_{k,k'}$ to be the category of co-k'-algebra objects in $\mathbf{CAlg}_k$.

Proposition 2.1.40. *The completion functor $\widehat{-}\colon \mathbf{FAlg}_k \to \mathbf{CAlg}_k$ lifts to a functor $\mathbf{FBiring}_{k,k'} \to \mathbf{CBiring}_{k,k'}$.*

Proof: As in Proposition 2.1.24. □

Unlike with k-algebras, we have no way of viewing the category $\mathbf{CBiring}_{k,k'}$ as a subcategory of $\mathbf{FBiring}_{k,k'}$. The comultiplications in $\mathbf{CBiring}_{k,k'}$ are maps $B \to B\widehat{\otimes}B$ and without losing information we have no way to convert these into maps $B \to B \otimes B$. However, just as for k-algebras we have inclusions of categories $\mathbf{Biring}_{k,k'} \hookrightarrow \mathbf{CBiring}_{k,k'}$

and $\mathbf{Biring}_{k,k'} \hookrightarrow \mathbf{FBiring}_{k,k'}$ giving a k-k'-biring the discrete filtration and the latter of these is left adjoint to the forgetful functor $\mathbf{FBiring}_{k,k'} \to \mathbf{Biring}_{k,k'}$. We have no forgetful functor $\mathbf{CBiring}_{k,k'} \to \mathbf{Biring}_{k,k'}$.

As we are interested in topological applications, we work primarily with complete Hausdorff birings. Up until now it has been illuminating to understand the complete Hausdorff setting as a special case of the filtered objects, but this approach will no longer work.

Consequently, much of our plethystic theory from Section 1.3 does not, a priori, make sense in the setting of complete Hausdorff birings, and we will have similar problems when we define complete Hausdorff plethories. Fortunately, this theory generalises without difficulty in the obvious way. For example, we can make the following definition which naturally generalises Proposition 1.3.13.

Definition 2.1.41. An *ideal* of a complete Hausdorff k-k'-biring B is a sub-(filtered module) I satisfying the following conditions.

1. I is an algebra ideal of B.
2. If $\pi\colon B \to B/I$ denotes the canonical quotient, $\Delta^+(I), \Delta^\times(I) \subseteq \ker(\pi \widehat{\otimes} \pi) \subseteq B \widehat{\otimes} B$.
3. $\varepsilon^+(I) = \varepsilon^\times(I) = 0$.
4. $\sigma(I) \subseteq I$.
5. For all $\lambda \in k'$, $\beta\lambda(I) = 0$.

If I is B-pure with respect to $\widehat{\otimes}$ then condition (2) is equivalent to the statement $\Delta^+(I), \Delta^\times(I) \subseteq B \widehat{\otimes} I \oplus I \widehat{\otimes} B$.

As in the case of filtered algebras, the categorical definition of filtered birings imposes strong conditions on the structure of the filtration ideals.

Proposition 2.1.42. *If B is a complete Hausdorff k-k'-biring then B is isomorphic to a complete Hausdorff k-k'-biring which is filtered by ideals of B.*

Proof: Let $\mathcal{O}$ be the indexing set for the filtration on B. Define B' to be the complete Hausdorff k-k'-biring B with filtration $F^\alpha B' = ((F^\alpha B))$ for $\alpha \in \mathcal{O}$ where $((F^\alpha B))$ denotes the smallest biring ideal of B containing $F^\alpha B$. Let $i\colon B \to B'$ be the identity map. As in Proposition 2.1.25, i is open. We now show i is continuous. Since Δ^+ and $\Delta^\times$ are continuous, for each $\alpha \in \mathcal{O}$, there exists $\beta, \gamma \in \mathcal{O}$ such that $\Delta^+(F^\beta B) \subseteq$

$\ker(\pi_\alpha \widehat{\otimes} \pi_\beta)$ and $\Delta^\times(F^\gamma B) \subseteq \ker(\pi_\alpha \widehat{\otimes} \pi_\beta)$ where π_α, π_β denote the canonical projections $B \to B/F^\alpha B$, $B \to B/F^\beta B$ respectively. Now by definition, there exists $\delta \in \mathcal{O}$ with $F^\delta M \subseteq F^\beta B \cap F^\gamma B$ which is a biring ideal. Since $((F^\alpha B))$ is the smallest biring ideal containing $F^\alpha B$, we have $((F^\alpha B)) \subseteq F^\delta B$. This shows i is continuous and is thus an isomorphism of complete Hausdorff k-k'-birings. $\square$

We can now give an elementary proof of a result of Stacey and Whitehouse [43, Corollary 5.4]. The proof found there is simply an application of a much more general abstract theorem.

Theorem 2.1.43. *Let $E^*(-), F^*(-)$ be (multiplicative) cohomology theories such that $E_*(\underline{F}_n)$ is a free E^*-module for each $n \in \mathbb{Z}$. Then $E^*(\underline{F}_\bullet)$ is a complete Hausdorff E^*-F^*-biring.*

Proof: The graded object of representing spaces $n \mapsto \underline{F}_n$ of the spectrum F naturally admits the structure of a graded F^*-algebra object in $\mathbf{Ho}_E^\wedge$. Now by Theorem 2.1.35, the Hausdorff complete cohomology functor $E^*(-)\colon \mathbf{Ho}_E^\wedge \to \mathbf{CAlg}_{E^*}$ takes products to coproducts and hence by Lemma 1.1.12 $E^*(\underline{F}_\bullet)$ is a co-F^*-algebra object in $\mathbf{CAlg}_{E^*}$. $\square$

As we saw in the non-topological case, it is illuminating to view birings through the lens of algebraic geometry and consider instead the functors they represent. The algebraic geometry in the topologised case is somewhat more complicated, and we follow [46] as our main reference and a few key results detailed in Appendix C. The topologised analogue of our discrete result is as follows.

Proposition 2.1.44. *Let B be a complete Hausdorff k-algebra. There is a bijection between complete Hausdorff k-k'-biring structures on B and lifts of* $\mathrm{Spf}_k(B)\colon \mathbf{Alg}_k \to$ $\mathbf{Set}$ *to a functor* $\mathbf{Alg}_k \to \mathbf{Alg}_{k'}$

Proof: By Theorem C.0.29 there is a bijection between complete Hausdorff k-k'-biring structures and k'-algebra objects in the category of solid formal schemes. However by Proposition C.0.30 the latter is in bijection with lifts of $\mathrm{Spf}_k(B)$. $\square$

Hence if B is a filtered k-k'-biring then for any k-algebra A, $\mathrm{Spf}_k(B)(A)$ is a k'-algebra. If A also carries a filtration, the filtration on A induces a filtration on $\mathrm{Spf}_k(B)(A)$. Moreover, if A is complete Hausdorff so is $\mathrm{Spf}_k(B)(A)$.

Proposition 2.1.45. *Let B be a filtered k-k'-biring. The functor* $\mathrm{Spf}_k(B)\colon \mathbf{Alg}_k \to$ $\mathbf{Alg}_{k'}$ *lifts to a functor* $\mathbf{FAlg}_k \to \mathbf{FAlg}_{k'}$ *which restricts to a functor* $\mathbf{CAlg}_k \to \mathbf{CAlg}_{k'}$.

Proof: Let $\mathcal{O}$ denote the indexing set for the filtration on B. Suppose A is a filtered k-algebra. The quotient maps $\pi_\alpha\colon A \to A/F^\alpha A$ induce k-algebra maps $\mathbf{FAlg}_k(B/F^\beta B, A) \to \mathbf{FAlg}_k(B/F^\beta B, A/F^\alpha A)$ for all $\beta \in B$ and thus k-algebra maps
$\pi_\alpha^*\colon \operatorname{Spf}_k(B)(A) \to \operatorname{Spf}_k(B)(A/F^\alpha A)$. We filter $\operatorname{Spf}_k(B)(A)$ by the ideals $F^\alpha \operatorname{Spf}_k(B)(A) = \ker \pi_\alpha^*$. Suppose further A is complete Hausdorff. We have

$$\varprojlim_\alpha \frac{\operatorname{Spf}_k(B)(A)}{F^\alpha \operatorname{Spf}_k(B)(A)} \cong \varprojlim_\alpha \operatorname{Spf}_k(B)(A/F^\alpha A) \cong \operatorname{Spf}_k(B)(\varprojlim_\alpha A/F^\alpha A) \cong \operatorname{Spf}_k(B)(A)$$

and hence $\operatorname{Spf}_k(B)(A)$ is complete. □

Example 2.1.46. If B is a discrete k-algebra, then $\operatorname{Spf}_k(B) = \operatorname{Spec}_k(B)$ and so Proposition 2.1.44 is a generalisation of our discrete correspondence between k-k'-birings and lifts of $\operatorname{Spec}_k(B)$.

Example 2.1.47. Let k', k be commutative rings. We saw in Example 1.2.3 that if k' is finite then $\mathbf{Set}(k', k)$ is a discrete k-k'-biring. For infinite k', $\mathbf{Set}(k', k)$ is a complete Hausdorff k-k'-biring. We filter $\mathbf{Set}(k', k)$ by the ideals $F^a \ker(\mathbf{Set}(k', k) \to \mathbf{Set}(k'_a, k))$ where k'_a ranges over all finite subsets of k'. Now $\operatorname{Spf}_k(\mathbf{Set}(k', k))(A) = \varinjlim_a \mathbf{Alg}_k(\mathbf{Set}(k'_a, k), A)$. Each of the terms in this colimit is a k'_a-algebra, and thus $\operatorname{Spf}(\mathbf{Set}(k', k))$ lifts to a functor $\mathbf{Alg}_k \to \mathbf{Alg}_{k'}$.

Example 2.1.48. Suppose k has characteristic p. Let $B = \mathbf{Set}(k, k) \otimes k[[e]]$ as a k-algebra and filter by the ideals $F^n B = \mathbf{Set}(k, k) \otimes (e^n)$ so $B/F^n B = \mathbf{Set}(k, k) \otimes k[e]/(e^n)$. Now $B/F^n B$ represents the functor $A \mapsto \operatorname{Nil}_n(A)^\wedge$. Hence $\operatorname{Spf}(B)(A) = \varinjlim_n \operatorname{Nil}_n(A)^\wedge = \operatorname{Nil}(A)^\wedge$. Therefore $\operatorname{Spf}(B)$ lifts to a functor $\mathbf{Alg}_k \to \mathbf{Alg}_k$ and thus B admits the structure of a complete Hausdorff k-k-biring. We note that $F^n B$ is only a biring ideal when n is a power of p. However, applying the method in the proof of Proposition 2.1.42, we see that B is isomorphic to B filtered only by $F^{p^n} B$.

2.1.7 The filtered composition product

In the discrete case, we saw that the category of birings was monoidal under the composition product. Just as we filtered the tensor product to generalise to filtered modules, we can filter the composition product to generalise to filtered birings and filtered algebras.

Definition 2.1.49. Let B be a filtered k-k'-biring and A a filtered k'-algebra with filtrations index by $\mathcal{O}, \mathcal{O}'$ respectively. We filter the composition product of Definition 1.2.9 by the ideals

$$F^{\beta,\alpha}(B \odot A) = \ker(B \odot A \xrightarrow{\pi_\beta \odot \pi_\alpha} B/F^\beta B \odot A/F^\alpha A)$$

for all $\beta \in \mathcal{O}, \alpha \in \mathcal{O}'$ where π_α and π_β denote the canonical projections $A \to A/F^\alpha A$ and $B \to B/F^\beta B$ respectively.

However, for our applications we are interested in complete Hausdorff k-k'-birings. Since we have no forgetful functor $\mathbf{CBiring}_{k,k'} \to \mathbf{Biring}_{k,k'}$ or inclusion $\mathbf{CBiring}_{k,k'} \to \mathbf{FBiring}_{k,k'}$, the composition product of a complete Hausdorff biring with a filtered algebra is not well defined. More explicitly, the relation $b \odot (a_1 + a_2) = \sum_{(b)}(b_{(1)} \odot a_1)(b_{(2)} \odot a_2)$ produces a possibly infinite sum, suggesting we need a completed version of the composition product. To motivate the definition, we remark that it is natural to have an analogous property to Proposition 2.1.20 i.e. completing the biring before taking the composition product should yield the same result as completing the filtered composition product.

Definition 2.1.50. Let B be a complete Hausdorff k-k'-biring and A a filtered k'-algebra. We define the *complete Hausdorff composition product* to be

$$B \widehat{\odot} A = \varprojlim_{\alpha,\beta} \frac{B}{F^\beta B} \odot \frac{A}{F^\alpha A}.$$

We give $B \widehat{\odot} A$ a filtration arising from the canonical projection maps under which is it clear that $B \widehat{\odot} A$ is complete Hausdorff.

$$F^{\beta,\alpha}(B \widehat{\odot} A) = \ker(B \widehat{\odot} A \to B/F^\beta B \odot A/F^\alpha A).$$

If $f\colon B \to B'$ is a map of complete Hausdorff k-k'-birings and $g\colon A \to A'$ is a map of filtered k'-algebras, we define $f \widehat{\odot} g$ to be induced by the maps $B \widehat{\odot} A \to B'/F^\beta B' \odot A'/F^\alpha A'$, $b \odot a \mapsto b + F^\beta B' \odot g(b) + F^\alpha A'$.

The following results show that the filtered composition product and complete Hausdorff composition product give functors $- \odot -\colon \mathbf{FBiring}_{k,k'} \times \mathbf{FAlg}_{k'} \to \mathbf{FAlg}_k$ and $- \widehat{\odot} -\colon \mathbf{CBiring}_{k,k'} \times \mathbf{FAlg}_{k'} \to \mathbf{CAlg}_k$.

Proposition 2.1.51. *If $f\colon B \to B'$ is a map of filtered k-k'-birings and $g\colon A \to A'$ is a map of filtered k'-algebras then the map $f \odot g\colon B \odot A \to B' \odot A'$ is continuous. Similarly, if f is a map of Hausdorff complete k-k'-birings and g a map of filtered k'-algebras then $f \widehat{\odot} g$ is continuous.*

Proof: Let $F^{\beta',\alpha'}(B' \odot A')$ be a filtration ideal of $B' \odot A'$. Since f and g are continuous, there exist α and β such that $f(F^\beta B) \subseteq F^{\beta'} B'$ and $g(F^\alpha A) \subseteq F^{\alpha'} A'$. Hence f and g induce maps $\bar{f}\colon B/F^\beta B \to B'/F^{\beta'} B'$ and $\bar{g}\colon A/F^\beta A \to A'/F^{\beta'} A'$ respectively. The following diagram commutes.

$$\begin{array}{ccccc} F^{\beta,\alpha}(B\odot A) & \hookrightarrow & B\odot A & \xrightarrow{\pi_\beta\odot\pi_\alpha} & \frac{B}{F^\beta B}\odot\frac{A}{F^\alpha A} \\ & & \downarrow{\scriptstyle f\odot g} & & \downarrow{\scriptstyle \bar{f}\odot\bar{g}} \\ & & B'\odot A' & \xrightarrow{\pi_{\beta'}\odot\pi_{\alpha'}} & \frac{B'}{F^{\beta'}B'}\odot\frac{A'}{F^{\alpha'}A'} \end{array}$$

Since the top row is zero, we see that $(\pi_{\beta'}\odot\pi_{\alpha'})(f\odot g)F^{\beta,\alpha}(B\odot A)=0$ and thus $(f\odot g)(F^{\beta,\alpha}(B\odot A))\subseteq F^{\beta',\alpha'}(B'\odot A')$. Now by Proposition 2.1.6, $f\odot g$ is continuous. An identical argument shows the same result for $\widehat{\odot}$. $\square$

We can now prove our motivating property for the definition of the complete Hausdorff composition product; an analogue of Proposition 2.1.20.

Proposition 2.1.52. *The following diagram commutes.*

$$\begin{array}{ccc} \mathbf{FBiring}_{k,k'}\times\mathbf{FAlg}_{k'} & \xrightarrow{\widehat{-}\times 1} & \mathbf{CBiring}_{k,k'}\times\mathbf{FAlg}_{k'} \\ \downarrow{\scriptstyle\odot} & & \downarrow{\scriptstyle\widehat{\odot}} \\ \mathbf{FAlg}_k & \xrightarrow{\widehat{-}} & \mathbf{CAlg}_k \end{array}$$

Proof: Let B be a filtered k-k'-biring and A a filtered k'-algebra. We have

$$\widehat{B\odot A}=\varprojlim_{\beta,\alpha}\frac{B}{F^\beta B}\odot\frac{A}{F^\alpha A}\cong\varprojlim_{\beta,\alpha}\frac{\widehat{B}}{F^\beta\widehat{B}}\odot\frac{A}{F^\alpha A}=\widehat{B}\widehat{\odot}A.$$

$\square$

As we saw in the discrete case for a k-k'-biring B, it is often easier to work with the adjoint functor $\mathrm{Spec}_k(B)$ rather than the composition product $B\odot -$. Unfortunately this result does not generalise to the topological setting. The obvious result would be an adjunction between $B\widehat{\odot}-$ and $\mathrm{Spf}_k(B)$. However, such a result only holds when restricting to categories of cocompact objects in $\mathbf{Alg}_k$: those k-algebras A such that there exist isomorphisms $\mathbf{Alg}_k(\varinjlim_\alpha A'_\alpha, A)\cong\varprojlim_\alpha\mathbf{Alg}_k(A'_\alpha,A)$ for all inverse systems A'_α. This condition turns out to be far too restrictive to be of any use.

However, the compact objects in $\mathbf{Alg}_k$: those k-algebras A such that there exist isomorphisms $\varinjlim_\alpha\mathbf{Alg}_k(A,A'_\alpha)\cong\mathbf{Alg}_k(A,\varinjlim_\alpha A'_\alpha)$ for all direct systems A'_α are much more ubiquitous, they are the finitely presented k-algebras. Using this fact, we can show that in certain contexts, the topological analogue of Corollary 1.2.11 holds. That is to say the composition product represents the composition of the represented functors.

Corollary 2.1.53. *Let B be a complete Hausdorff k-k'-biring and A a filtered k'-algebra with filtration indexed by $\mathcal{O}$. If $A/F^\alpha A$ is a finitely presented k'-algebra for all $\alpha\in\mathcal{O}$ then the composition*

$$\mathbf{Alg}_k\xrightarrow{\mathrm{Spf}_k(B)}\mathbf{Alg}_{k'}\xrightarrow{\mathrm{Spf}_{k'}(A)}\mathbf{Set}$$

is given by $\mathrm{Spf}_k(B\widehat{\odot}A)$.

Proof: For a k-algebra X,

$$\begin{aligned}
\mathrm{Spf}_k(B\widehat{\odot}A)(X) &= \varinjlim_{\beta,\alpha} \mathbf{Alg}_k(B/F^\beta B \odot A/F^\alpha A, X) \\
&= \varinjlim_{\alpha,\beta} \mathbf{Alg}_{k'}(A/F^\alpha A, \mathbf{Alg}_k(B/F^\beta B, X)). \\
&\cong \varinjlim_{\alpha} \mathbf{Alg}_{k'}(A/F^\alpha A, \varinjlim_{\beta} \mathbf{Alg}_k(B/F^\beta B, X)) \\
&= \mathrm{Spf}_{k'}(A)(\mathrm{Spf}_k(B)(X)).
\end{aligned}$$

□

Our condition that the quotients by the filtration ideals must be finitely presentably is always satisfied in our applications. Indeed, if X is a topological space and $E^*(X)$ is given the pro-finite filtration then $E^*(X)/F^\alpha E^*(X) \cong E^*(X_\alpha)$. Since X_α is a finite subcomplex of X, $E^*(X_\alpha)$ is of finite type and thus finitely presented. From this point on, we shall assume all our filtrations have quotients which are finitely presented as algebras.

Corollary 2.1.54. *Let B be a complete Hausdorff k-k'-biring. The functor $B\widehat{\odot}-\colon \mathbf{Alg}_{k'} \to \mathbf{Alg}_k$ lifts to a functor $\mathbf{CBiring}_{k',k''} \to \mathbf{CBiring}_{k,k''}$.*

Proof: Let B' be a filtered k'-k''-biring. Since $\mathrm{Spf}_k(B\widehat{\odot}B') = \mathrm{Spf}_{k'}(B') \circ \mathrm{Spf}_k(B)$ we see $\mathrm{Spf}_k(B\widehat{\odot}B')$ lifts to a functor $\mathbf{Alg}_k \to \mathbf{Alg}_{k''}$ and hence $B\widehat{\odot}B'$ is a complete Hausdorff k-k''-biring.

Proposition 2.1.55. *The category $(\mathbf{CBiring}_{k,k}, \widehat{\odot}, \mathcal{I})$ is monoidal.*

Proof: Just as in Proposition 1.2.17, this follows since $B\widehat{\odot}B'$ represents the composition $\mathrm{Spf}(B') \circ \mathrm{Spf}(B)$. □

2.1.8 Filtered plethories

Definition 2.1.56. We define the category of *complete Hausdorff k-plethories* $\mathbf{CPlethory}_k$ to be the category of monoids in $\mathbf{CBiring}_{k,k}$.

As for birings, we have an inclusion $\mathbf{Plethory}_k \hookrightarrow \mathbf{CPlethory}_k$, giving a plethory the discrete filtration. As we have seen before, the categorical construction forces additional structure on the filtration components.

Proposition 2.1.57. *If P is a filtered k-plethory then P is isomorphic to a k-plethory filtered by plethystic ideals.*

Proof: As in Proposition 2.1.25. $\square$

Example 2.1.58. Let k be a commutative ring. We saw in Example 1.2.19 that if k is finite then $\mathbf{Set}(k, k)$ is a discrete k-plethory. If k is infinite then from Example 2.1.47, $\mathbf{Set}(k, k)$ is a complete Hausdorff k-k-biring. Composition of maps naturally gives $\mathbf{Set}(k, k)$ the structure of a complete Hausdorff k-plethory.

We can now give a direct proof of a major result of Stacey and Whitehouse [43, Corollary 5.4]. The original proof is an application of a very abstract result.

Theorem 2.1.59. *Let $E^*(-)$ be a (multiplicative) cohomology theory. If $E_*(\underline{E}_n)$ is a free E^*-module for each $n \in \mathbb{Z}$ then $E^*(\underline{E}_\bullet)$ is a complete Hausdorff E^*-plethory.*

Proof: By Theorem 2.1.43, $E^*(\underline{E}_\bullet)$ is a complete Hausdorff E^*-E^*-biring. We define the composition $\circ\colon E^*(\underline{E}_\bullet)\widehat{\odot}E^*(\underline{E}_\bullet) \to E^*(\underline{E}_\bullet)$ by $r \circ s = s^*(r)$ and the unit $u\colon \mathcal{I} \to E^*(\underline{E}_\bullet)$ by $u(\iota_n) = \iota_n \in E^*(\underline{E}_n)$, the universal class. These maps make $E^*(\underline{E}_\bullet)$ a complete Hausdorff E^*-plethory by construction. $\square$

We also have the analogous result in the ungraded context which is proved in exactly the same way.

Theorem 2.1.60. *Let $h(-)$ be an ungraded (multiplicative) cohomology theory with representing space $\underline{H}$ and associated homology theory $k(-)$. If $k(\underline{H})$ is a free $h(T)$-module then $h(\underline{H})$ is a complete Hausdorff $h(T)$-plethory.*

Now we are in a situation to generalise Proposition 1.3.23 to the setting where we are not necessarily of finite type. This details the duality between homology and cohomology (Theorem D.0.36) in an abstract context. The analogous result also holds between Hopf rings and birings.

Theorem 2.1.61. *Let H be a enriched $k[k']$-Hopf ring which is free as a k-module. The linear dual $DH = \mathbf{Mod}_k(H, k)$ equipped with the dual-finite topology (see Definition D.0.35) naturally admits the structure of a complete Hausdorff k-plethory.*

Proof: The finite type conditions in Proposition 1.3.20 are to ensure that we have isomorphisms $D(H \otimes H) \cong DH \otimes DH$. In general we only have a map $DH \otimes DH \to D(H \otimes H)$ which is not surjective in general. However, if we take the completion, we have an isomorphism $DH\widehat{\otimes}DH \cong D(H \otimes H)$. The remainder of the proof is analogous to the proof in the finite type situation. $\square$

2.1.9 Filtered P-algebras

We wish to understand the objects which a complete Hausdorff k-plethory acts on. As in the discrete case, we realise these as the Eilenberg-MacLane algebras for a suitable monad.

Proposition 2.1.62. *Let P be a complete Hausdorff k-plethory. The functor $P\widehat{\odot}-\colon \mathbf{CAlg}_k \to \mathbf{CAlg}_k$ forms a monad on $\mathbf{CAlg}_k$.*

Proof: This is immediate since P is a monoid in the category of representable functors $\mathbf{CAlg}_k \to \mathbf{CAlg}_k$ where the monoid structure corresponds to composition of representable functors. $\square$

Definition 2.1.63. Let P be a complete Hausdorff k-plethory. We define the category of *complete Hausdorff P-algebras* $\mathbf{CAlg}_P$ to be category of Eilenberg-MacLane algebras for the monad $P\widehat{\odot}-\colon \mathbf{CAlg}_k \to \mathbf{CAlg}_k$.

If P is a discrete plethory and A a P-algebra then we can view A as a complete Hausdorff $i(P)$-algebra where $i\colon \mathbf{Plethory}_k \to \mathbf{CPlethory}_k$ is the inclusion. This yields an inclusion of categories $\mathbf{Alg}_P \to \mathbf{CAlg}_{i(P)}$.

Proposition 2.1.64. *If A is a complete Hausdorff P-algebra then A is isomorphic to a complete Hausdorff P-algebra with filtration by P-ideals.*

Proof: As in Proposition 2.1.25. $\square$

Example 2.1.65. If $P = \mathbf{Set}(k,k)$ is the complete Hausdorff P-algebra of Example 2.1.58 then the discrete k-algebra k is a complete Hausdorff P-algebra.

We can now give a direct proof of how the completed cohomology of spaces naturally forms an algebra over our plethory of cohomology operations.

Theorem 2.1.66. *Let $E^*(-)$ be a (multiplicative) cohomology theory and suppose $E_*(\underline{E}_n)$ is a free E^*-module for each n. For any space X, the completed cohomology $\widehat{E}^*(X)$ is a complete Hausdorff $E^*(\underline{E}_\bullet)$-algebra.*

Proof: An element $x \in \widehat{E}^*(X)$ is determined by the projections $i_\alpha^*(x) \in E^*(X_\alpha)$ for each finite subcomplex $X_\alpha \xhookrightarrow{i_\alpha} X$. Given an element $x_\alpha \in E^n(X_\alpha)$ viewed as a map $X_\alpha \to \underline{E}_n$, we define $r(x_\alpha) = x_\alpha^* r$. These elements are compatible and so define an element $r(x) \in \widehat{E}^*(X)$. We now define our action $E^*(\underline{E}_\bullet)\widehat{\odot}\widehat{E}^*(X) \to \widehat{E}^*(X)$ by $r \circ x = r(x)$. It is straightforward to show this map satisfies the required properties. $\square$

Suppose $E^*(E_\bullet)$ is a complete Hausdorff k-plethory. Recall that $\Delta^\times$ encodes the action of operations on products in an $E^*(E_\bullet)$-algebra. That is, on the internal product in $\widehat{E}^*(X)$ for any space X. However, since we can recover the external product from the internal product using natural maps we see that the $\Delta^\times$ also encodes the action of operations on the external products. Explicitly, if $r \in E^*(E_\bullet)$, $x \in \widehat{E}^*(X)$ and $y \in \widehat{E}^*(Y)$, we have $r(x \times y) = r_{[1]}(x) \times r_{[2]}(y)$ in $\widehat{E}^*(X \times Y)$ for any spaces X and Y.

2.1.10 Formal plethories

Our theory of complete Hausdorff plethories and associated objects is sufficient to perform calculations for many cohomology theories as we will see in Chapter 3. Unfortunately, it lacks full generality: due to only working with complete objects, we are unable to see phantom operations and phantom classes. Moreover, we require the homology of our representing spaces to be free E^*-modules.

In [11], Bauer manages to circumnavigate some of these issues by working in suitable categories of pro-objects: systems of objects from which we can recover the phantom classes as well as the completed cohomology. In this section we quickly introduce some of his important results which we have translated into a similar framework to our complete Hausdorff objects to allow for an easier comparison with our theory. For a full treatment of this approach from an algebro-geometric perspective, refer to [11].

Definition 2.1.67. For an arbitrary category $\mathcal{C}$ we denote the category of *pro-objects* in $\mathcal{C}$ by $\mathbf{Pro}(\mathcal{C})$. The objects of $\mathbf{Pro}(\mathcal{C})$ are functors $\mathcal{A} \to \mathcal{C}$ (i.e. $\mathcal{A}$-shaped diagrams in $\mathcal{C}$) where $\mathcal{A}$ is some small cofiltered category. If $F\colon \mathcal{A} \to \mathcal{C}$, $G\colon \mathcal{B} \to \mathcal{C}$ are two such objects, the morphisms are given by

$$\mathbf{Pro}(\mathcal{C})(F, G) = \varprojlim_{\beta \in \mathcal{B}} \varinjlim_{\alpha \in \mathcal{A}} \mathcal{C}(F(\alpha), G(\beta)).$$

We can think of a pro-object as a generalisation of the filtration of a filtered object. A pro-object contains all the information of the filtration, but has no need for a 'total' object which it filters. This philosophy motivates the following definition.

Definition 2.1.68. For a cohomology theory $E^*(-)$, define the *profinite cohomology*, $\mathcal{E}^*(-)\colon \mathbf{Top} \to \mathbf{Pro}(\mathbf{Alg}_{E^*})$ on objects by

$$\begin{aligned} \mathcal{E}^*(X)\colon \mathcal{A} &\to \mathbf{Alg}_{E^*} \\ \alpha &\mapsto E^*(X_\alpha) \end{aligned}$$

where the objects of $\mathcal{A}$ are the finite subcomplexes $X_\alpha \subseteq X$ and the morphisms are the inclusions of subcomplexes. A map $f\colon X \to Y$ of CW-complexes necessarily maps X_α to some finite subcomplex $Y_\beta \subseteq Y$ and thus we have maps $E^*(Y_\beta) \to E^*(X_\alpha)$. These maps are suitable compatible and hence f induces a map $\mathcal{E}^*(f) \in \mathbf{Pro}(\mathbf{Alg}_{E^*})(\mathcal{E}^*(Y), \mathcal{E}^*(X))$.

Many properties of a category $\mathcal{C}$ pass naturally to the category of pro-objects $\mathbf{Pro}(\mathcal{C})$. We list without proof some relevant properties of $\mathbf{Pro}(\mathbf{Mod}_k)$ and $\mathbf{Pro}(\mathbf{Alg}_k)$.

Proposition 2.1.69.

1. $\mathbf{Pro}(\mathbf{Mod}_k)$ *is an abelian category which admits a monoidal structure with unit object* k *and product of* $M\colon \mathcal{A} \to \mathbf{Mod}_k$ *and* $N\colon \mathcal{B} \to \mathbf{Mod}_k$ *given by*

$$\begin{aligned} \mathcal{A} \times \mathcal{B} &\to \mathbf{Mod}_k \\ (\alpha, \beta) &\mapsto M(\alpha) \otimes N(\beta). \end{aligned}$$

2. $\mathbf{Pro}(\mathbf{Alg}_k)$ *is the category of monoids in* $\mathbf{Pro}(\mathbf{Mod}_k)$*. This has coproduct and initial object given by* $\otimes$ *and* k *respectively.*

The Eilenberg-Steenrod axioms for a cohomology theory make sense in any abelian category and the following result shows we have the pro-analogue of a cohomology theory.

Theorem 2.1.70. *The functor* $\mathcal{E}^*(-)\colon \mathbf{Top} \to \mathbf{Pro}(\mathbf{Alg}_{E^*})$ *is homotopy invariant and satisfies the Eilenberg-Steenrod axioms on* $\mathbf{Pro}(\mathbf{Mod}_{E^*})$*.*

Recall the Milnor exact sequence:

$$0 \to \varprojlim\nolimits_a^1 \mathcal{E}^{*-1}(X) \to E^*(X) \to \varprojlim\nolimits_a \mathcal{E}^*(X) \to 0.$$

We see that from the profinite cohomology, we can recover both the complete Hausdorff cohomology $\widehat{E}^*(X) = \varprojlim_a E^*(X)$ and the phantom classes $\ker(E^*(X) \to \widehat{E}^*(X)) = \varprojlim_a^1 \mathcal{E}^*(X)$. Unfortunately this exact sequence is not split in general, and the cohomology $E^*(X)$ may be a non-trivial extension. Nevertheless, we have captured more information with the profinite cohomology than we could with just the complete Hausdorff cohomology.

The other major advantage of the profinite cohomology, is that we have a Künneth isomorphism under much weaker conditions.

Theorem 2.1.71 ([11, Corollary 4.21])**.** *Suppose* $\mathcal{E}_*(X)$ *is pro-flat. We have an isomorphism of pro-*E^**-algebras*

$$\mathcal{E}^*(X) \otimes \mathcal{E}^*(Y) \xrightarrow{\cong} \mathcal{E}^*(X \times Y).$$

Just as in the complete Hasudorff case, the existence of a Künneth isomorphism allows us to prove that the cohomology operations admit the structure of a biring.

Definition 2.1.72. The category of *formal k-k'-birings* is the category of co-k'-algebra objects $\mathbf{Pro}(\mathbf{Alg}_k)$. This is equivalent to $\mathbf{Pro}(\mathbf{Biring}_{k,k'})$.

The following result of Bauer illustrates how cohomology operations naturally fit into this setting of pro-objects. He requires one additional extra condition: the coefficient ring E^* must be a *Prüfer domain* and so in particular sub-modules of flat modules are flat.

Theorem 2.1.73 ([11, Corollary 4.22])**.** *Suppose $E^*(-), F^*(-)$ are (multiplicative) cohomology theories such that E^* is a Prüfer domain. If $\mathcal{E}_*(\underline{F}_n)$ is pro-flat for each n then $\mathcal{E}_*(\underline{F}_n)$ is a formal E^*-F^*-biring.*

Just like with the tensor product, since a pro-k-k'-biring is a system of k-k'-birings, the composition product has a straightforward extension to the world of pro-objects.

Definition 2.1.74. We define the *formal composition product* of a formal k-k'-biring $B\colon \mathcal{B} \to \mathbf{Biring}_{k,k'}$ and a pro-k'-algebra $A\colon \mathcal{A} \to \mathbf{Alg}_{k'}$ to be the following pro-k-algebra.

$$\begin{aligned} &\mathcal{B} \times \mathcal{A} \to \mathbf{Alg}_k \\ &(\beta, \alpha) \mapsto B(\beta) \odot A(\alpha). \end{aligned}$$

The formal composition product yields a functor $-\odot-\colon \mathbf{Pro}(\mathbf{Biring}_{k,k'}) \times \mathbf{Pro}(\mathbf{Alg}_{k'}) \to \mathbf{Pro}(\mathbf{Alg}_k)$ which lifts to a functor $\mathbf{Pro}(\mathbf{Biring}_{k,k'}) \times \mathbf{Pro}(\mathbf{Biring}_{k',k''}) \to \mathbf{Pro}(\mathbf{Biring}_{k,k''})$. Together with the unit object $\mathcal{I}$, the formal composition product makes $\mathbf{Pro}(\mathbf{Biring}_{k,k})$ a monoidal category.

Definition 2.1.75. The category of *formal k-plethories* is the category of monoids in $\mathbf{Pro}(\mathbf{Biring}_{k,k})$,

Theorem 2.1.76 ([11, Theorem 8.21])**.** *Suppose $E^*(-)$ is a (multiplicative) cohomology theory such that E^* is a Prüfer domain. If $\mathcal{E}_*(\underline{E}_n)$ is pro-flat for each n then $\mathcal{E}^*(\underline{E}_\bullet)$ is a formal k-plethory.*

Just as in the complete Hausdorff setting, for a formal k-plethory the functor $P \odot -$ forms a monad on $\mathbf{Pro}(\mathbf{Alg}_k)$ and provides a natural definition for algebras over a formal plethory.

Definition 2.1.77. Let P be a formal k-plethory. We define the category of *formal P-algebras* to the category of Eilenberg-Moore algebras for the monad $P \odot - \colon \mathbf{Pro}(\mathbf{Alg}_k) \to \mathbf{Pro}(\mathbf{Alg}_k)$.

Theorem 2.1.78 ([11, Theorem 8.21])**.** *Suppose $E^*(-)$ is a (multiplicative) cohomology theory such that E^* is a Prüfer domain. If $\mathcal{E}_*(\underline{E}_n)$ is pro-flat for each n then for any space X, the profinite cohomology $\mathcal{E}^*(X)$ is a formal $\mathcal{E}^*(\underline{E}_\bullet)$-algebra.*

The benefits of this approach are clear: it provides a framework for understanding cohomology operations under weaker assumptions and contains more information. However, many theories with interesting phantom classes are rather complicated and with current techniques it may prove too difficult to compute the associated pro-objects. Nevertheless, with advances in understanding of such theories this pro-algebraic approach may allow us to obtain useful insights not seen in the complete Hausdorff approach.

2.1.11 Based and primitive operations

In this section we study two particular types of operations which arise naturally in the topological context and have a clean expression in the language of plethories. The first of these are additive operations: those which act as group homomorphisms.

Proposition 2.1.79 ([15, Proposition 2.7])**.** *Let $r \colon E^n(-) \to E^m(-)$ be a cohomology operation. The following conditions are equivalent.*

1. *The natural transformation $r \colon E^n(-) \to E^m(-)$ is a natural transformation of abelian group valued functors.*
2. *The operation acts as a group homomorphism i.e. $r(x + y) = r(x) + r(y)$ for all spaces X and all $x, y \in E^n(X)$.*
3. *As a cohomology class $r \in PE^*(\underline{E}_n) \subseteq E^*(\underline{E}_n)$ where $PE^*(\underline{E}_n)$ denotes the module of primitives Definition 1.3.24.*
4. *The representing map $r_U \colon \underline{E}_n \to \underline{E}_m$ is a map of group objects in* $\mathbf{Ho}$.

Definition 2.1.80. An *additive cohomology operation* of type (n, m) is an operation $r \colon E^n(-) \to E^m(-)$ satisfying any of the conditions in Proposition 2.1.79.

Applying our plethystic theory of primitive elements Section 1.3.5 yields a slick formulation of the additive operations in an object which encodes all the algebraic structure.

Corollary 2.1.81. *Let $E^*(-)$ be a (multiplicative) cohomology theory such that $E_*(\underline{E}_n)$ is a free E^*-module for each $n \in \mathbb{Z}$. The set of additive operations is the twisted E^*-bialgebra $PE^*(\underline{E}_\bullet)$.*

Proof: By definition, the additive operations are simply the primitive elements in $E^*(\underline{E}_\bullet)$. Since $E^*(\underline{E}_\bullet))$ is a E^*-plethory by Theorem 2.1.59, the result follows by Proposition 1.3.34. $\square$

We remark that an analogous result holds for the additive operations $E^n(-) \to F^m(-)$ between two multiplicative cohomology theories.

As is well known, both the relative and reduced cohomology groups are extremely useful for performing calculations. As a result, it is of interest to know when operations restrict to relative and reduced cohomology groups.

Proposition 2.1.82. *Let $r\colon E^n(-) \to E^m(-)$ be an operation. The following conditions are equivalent.*

1. *$r(0) = 0$ in $E^*(T) = E^*$ where T is the one point space.*
2. *As a cohomology class, $r \in E^n(\underline{E}_m, o) \subseteq E^n(\underline{E}_m)$.*
3. *For any space X with subspace A, $r_X\colon E^n(X) \to E^m(X)$ restricts to a map $E^n(X, A) \to E^m(X, A)$.*
4. *The representing map $r\colon \underline{E}_n \to \underline{E}_m$ is homotopy equivalent to a map which preserves the base point.*

Proof: [15, Lemma 2.3] yields the result for the case $A = o$, the base point. For general A, we simply recall that $E^*(X, A) = E^*(X/A, o)$. $\square$

Definition 2.1.83. An operation $r\colon E^n(-) \to E^m(-)$ satisfying any of the equivalent properties in Proposition 2.1.82 is called *based*.

The based operations, despite being inherently topological arise naturally in our plethystic setting. Note that even when $E^*(\underline{E}_\bullet)$ is not a plethory, for example $E_*(\underline{E}_n)$ is not a free E^*-module, it is still an augmented algebra and thus the augmentation ideal $IE^*(\underline{E}_\bullet)$ (see Definition 1.3.7) is well defined.

Proposition 2.1.84. *Let $E^*(-)$ be a cohomology theory. We have $IE^*(\underline{E}_\bullet) = E^*(\underline{E}_\bullet, o)$.*

Proof: Since $\varepsilon^+ \colon E^*(\underline{E}_n) \to E^*$ is the map induced on cohomology by the inclusion of the base point $T \to \underline{E}_n$, we have $IE^*(\underline{E}_\bullet) = E^*(\underline{E}_\bullet, o)$. $\square$

Corollary 2.1.85. *An operation $r \in E^*(\underline{E}_\bullet)$ is based if and only if $r \in IE^*(\underline{E}_\bullet)$.*

Proof: This is immediate from Proposition 2.1.82 and Proposition 2.1.84. $\square$

Once again, an analogous result holds for the additive operations $E^n(-) \to F^m(-)$ between two multiplicative cohomology theories.

2.2 The suspension isomorphism

A graded cohomology theory is a collection of functors $E^n(-) \colon \mathbf{Ho} \to \mathbf{Ab}$ satisfying the Eilenberg-Steenrod axioms. As a consequence, for every space X we have the *suspension isomorphisms.* These are isomorphisms of abelian groups $\Sigma \colon E^n(X) \to E^{n+1}(S^1 \times X, o \times X)$ or equivalently $\Sigma \colon E^n(X, o) \cong E^{n+1}(\Sigma X, o)$ on reduced cohomology for all $n \in \mathbb{Z}$ where $\Sigma X = S^1 \wedge X$ denotes the reduced suspension. If we represent $E^*(-)$ by an Ω-spectrum E then the suspension isomorphisms are equivalent to the homotopy equivalences $\underline{E}_n \simeq \Omega \underline{E}_{n+1}$ on the level of representing spaces. This suspension isomorphism is an extra piece of structure on the algebras over our plethory of unstable cohomology operations. Since plethories are *precisely* the structure which acts on algebras, we will need extra structure on our plethory if we are to encode this additional information.

In a multiplicative cohomology theory, the suspension isomorphism has a simple expression. Recall that $E^*(S^1, o)$ is the free E^*-module on the canonical generator $u_1 = \Sigma 1_{E^*} \in E^1(S^1, o)$ and in the E^*-algebra $E^*(S^1)$ we have $u_1^2 = 0$.

Proposition 2.2.1 ([14, Equation 3.24])**.** *Let $E^*(-)$ be a multiplicative cohomology theory. For $x \in E^*(X)$, the* suspension *of x is given by $\Sigma x = u_1 \times x \in E^{*+1}(S^1 \times X, o \times X)$, where $u_1 \in E^1(S^1, o)$ is the canonical generator.*

Throughout the remainder of this section, we shall assume that $E^*(-)$ is a multiplicative cohomology theory and study operations $E^*(-) \to E^*(-)$. Much of the theory will hold in the study of operations $E^*(-) \to F^*(-)$ between two different multiplicative cohomology theories. However, as we will see the implications of the suspension isomorphisms are inherently tied to *actions* on spaces and we can only abstract this to our plethystic framework in the setting $E^*(-) = F^*(-)$. It should be possible to also tackle the more general setting if we were to devise a suitable notion of algebras over a biring which would encode the action of the F^*-E^*-biring $F^*(\underline{E}_\bullet)$ between completed cohomology rings $\widehat{E}^*(X)$ and $\widehat{F}^*(X)$ but we have not done this.

An immediate consequence of the suspension isomorphism is the construction of looping of based operations.

Definition 2.2.2. Let $r\colon E^n(-) \to E^m(-)$ be a based operation, we define the *looping of* r denoted $\Omega r\colon E^{n-1}(-) \to E^{m-1}(-)$ by the following commutative diagram.

$$\begin{tikzcd} E^{n-1}(X) \arrow[r, "\Sigma"] \arrow[d, "\Omega r"] & E^n(S^1 \times X, o \times X) \arrow[d, "r"] \\ E^{m-1}(X) \arrow[r, "\Sigma"] & E^m(S^1 \times X, o \times X) \end{tikzcd}$$

Thus, the looping of an operation encodes how a based operation acts on the suspension of a cohomology class via the relation $\Sigma(\Omega r)(x) = r(\Sigma x)$. We can only loop based operations as we require that they restrict to an operation on relative cohomology. The looping of a based operation is again based, so we can iteratively loop operations any finite number of times.

Proposition 2.2.3 ([15, Proposition 2.12])**.** *The map* $\Omega\colon F^*(\underline{E}_n, o) \to E^*(\underline{E}_{n-1}, o)$ *is a degree* -1 *map of* E^**-modules which is continuous with respect to the profinite topology.*

Following the general mantra of our plethystic theory, it will be interesting to understand how looped operations respect the algebraic structure on general cohomology rings. It is well known that the looping of any operation is additive and that the looping of a product is trivial.

Proposition 2.2.4 ([15, Corollary 2.18])**.** *The map* $\Omega\colon E^*(\underline{E}_n, o) \to E^*(\underline{E}_{n-1}, o)$ *factors as the following composition.*

$$E^*(\underline{E}_n, o) \xrightarrow{\pi} QE^*(\underline{E}_n) \to PE^*(\underline{E}_{n-1}) \subseteq E^*(\underline{E}_{n-1}, o)$$

where $QE^*(\underline{E}_n)$ *denotes the module of indecomposables (Definition 1.3.59),* $PE^*(\underline{E}_{n-1})$ *the module of primitives (Definition 1.3.24) and* $\pi\colon E^*(\underline{E}_n, o) = IE^*(rE_n) \to QE^*(\underline{E}_n)$ *the canonical projection.*

It is trivial to show that looping respects composition of operations.

Proposition 2.2.5. *For based operations* $r, s \in E^*(\underline{E}_\bullet, o)$*, we have* $\Omega(r \circ s) = \Omega(r) \circ \Omega(s)$*.*

Proof: This is immediate from the definition of looping. $\square$

Before computing the action of a looped operation on the multiplicative structure we first make some useful observations. In a multiplicative cohomology theory, we have $r(\Sigma x) = r(u_1 \times x) = r_{[1]}(u_1) \times r_{[2]}(x)$ and thus understanding the action of operations on suspensions is equivalent to understanding the action of the operations on $u_1 \in E^*(S^1, o)$.

Definition 2.2.6. We define the u_1-*evaluation map* $\omega\colon E^*(\underline{E}_1, o) \to E^*$ to be the composition

$$E^*(\underline{E}_1, o) \xrightarrow{\Omega} E^*(\underline{E}_0, o) \xrightarrow{\varepsilon^\times} E^*.$$

We will also write $\omega\colon E^*(\underline{E}_\bullet, o) \to E^*$ for the map which is identically zero on the other $\bullet$-components.

The following result motivates this definition.

Lemma 2.2.7. *For a based operation* $r \in E^*(\underline{E}_\bullet, o)$*, we have* $r(u_1) = \omega(r)u_1 \in E^*(S^1, o)$.

Proof: We have $r(u_1) = r(\Sigma 1_{E^*}) = \Sigma(\Omega r)(1_{E^*}) = \Sigma(\varepsilon^\times(\Omega(r))) = \varepsilon^\times(\Omega(r))u_1 = \omega(r)u_1$. $\square$

Proposition 2.2.8. *The* u_1*-evaluation map* $\omega\colon E^*(\underline{E}_\bullet) \to E^*$ *is an* E^**-module map which is zero on products and continuous with respect to the profinite topology.*

Proof: By the previous lemma, ω is the composition $E^*(\underline{E}_1, o) \xrightarrow{u_1^*} E^*(S^1, o) \cong E^*$ and thus an E^*-module map. Moreover since $u_1^2 = 0$, all products in $E^*(S^1, o)$ are trivial. Finally, since both Ω and $\varepsilon^\times$ are continuous, so is ω. $\square$

Corollary 2.2.9. *For a based operation* $r \in E^*(\underline{E}_\bullet, o)$*, we have* $r(\Sigma x) = \omega(r_{[1]})r_{[2]}(x)$.

Proof: We have $r(\Sigma x) = r(u_1 \times x) = r_{[1]}(u_1) \times r_{[2]}(x) = \omega(r_{[1]})r_{[2]}(x)$. $\square$

Now to compute the action of a looped operation on the multiplicative structure, we first recall how the suspension isomorphism respects products. In a multiplicative cohomology theory, $E^*(X)$ is canonically an $E^*(X)$-module. Moreover, the projection $\pi_2\colon S^1 \times X \to X$ induces a map $\pi_2^*\colon E^*(X) \to E^*(S^1 \times X)$. Since $E^*(S^1 \times X, o \times X)$ is an ideal of $E^*(S^1 \times X)$, π_2^* induces a $E^*(X)$-module structure on $E^*(S^1 \times X, o \times X)$ and this makes $\Sigma\colon E^*(X) \to E^*(S^1 \times X, o \times X)$ a degree 1 isomorphism of $E^*(X)$-modules. Explicitly, we have $\Sigma(xy) = (-1)^{|x|}(\pi_2^* x)\Sigma y$ for $x, y \in E^*(X)$ and as a special case, $\Sigma(\lambda x) = (-1)^{|\lambda|}\lambda(\Sigma x)$ for $\lambda \in E^*, x \in E^*(X)$. This result also holds in completed cohomology algebras $\widehat{E}^*(X)$.

Proposition 2.2.10. *Suppose $E_*(\underline{E}_n)$ is a free E^*-module for all $n \in \mathbb{Z}$. Let $r \in E^*(\underline{E}_\bullet, o)$ be a based operation and X a space. Using the usual notation $(\Delta^+, \varepsilon^+, \sigma, \Delta^\times, \varepsilon^\times)$ for the E^*-E^*-biring structure on $E^*(\underline{E}_n)$, for all $x, y \in \widehat{E}^*(X)$ and all $\lambda \in E^*$ the following statements are true.*

1. $\Omega r(xy) = (-1)^{\deg_*(r_{[1]})}(\sigma^{|x|} r_{[1]})(x)(\Omega r_{[2]})(y)$.
2. $\Omega r(1) = (-1)^{|\omega(r)|}\omega(r)$.
3. $\Omega r(\lambda) = \beta[(-1)^{|\lambda|}\lambda](r_{[1]})\omega(r_{[2]})$.

Proof: For (1), we have

$$\begin{aligned}
\Sigma(\Omega r)(xy) &= r(\Sigma(xy)) \\
&= r((-1)^{|x|}(\pi_2^* x)\Sigma y) \\
&= r_{[1]}(\pi_2^*((-1)^{|x|}x)) r_{[2]}(\Sigma y) \\
&= \pi_2^*((\sigma^{|x|} r_{[1]})(x))\Sigma(\Omega r_{[2]})(y) \\
&= \Sigma\left[(-1)^{\deg_*(r_{[1]})}(\sigma^{|x|} r_{[1]})(x)(\Omega r_{[2]})(y)\right].
\end{aligned}$$

Since Σ is an isomorphism our result follows. For (2), we note $\Sigma(\Omega r)(1) = r(u_1) = \omega(r)u_1 = \omega(r)\Sigma 1 = \Sigma(-1)^{|\omega(r)|}\omega(r)$ and our result follows. For (3), we have

$$\begin{aligned}
\Sigma(\Omega r)(\lambda) &= r(\Sigma\lambda) \\
&= r((-1)^{|\lambda|}\lambda\Sigma 1) \\
&= \beta((-1)^{|\lambda|}\lambda)(r_{[1]}) r_{[2]}(u_1) \\
&= \beta((-1)^{|\lambda|}\lambda)(r_{[1]})\omega(r_{[2]})u_1 \\
&= \Sigma\beta((-1)^{|\lambda|}\lambda)(r_{[1]})\omega(r_{[2]}).
\end{aligned}$$

The result follows. □

We remark that the signs, which also lead to to appearance of the antipode are mostly superfluous, and in a more abstract context get absorbed into the statement that Ω is a bidegree $(-1,-1)$ map of modules. We can easily compute how the u_1-evaluation map respects composition.

Corollary 2.2.11. *If $E_*(\underline{E}_n)$ is a free E^*-module for all $n \in \mathbb{Z}$ then for based operations $r, s \in E^*(\underline{E}_\bullet, o)$ we have*

$$\omega(r \circ s) = \beta[\omega(s)](r_{[1]})\omega(r_{[2]})$$
$$\omega(\iota_n) = \begin{cases} 1 & n = 1 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof: The statement about the unit for composition is immediate. For the composition, we have

$$\begin{aligned}(r \circ s)(u_1) &= r(\omega(s)u_1) \\ &= \beta[\omega(s)](r_{[1]})r_{[2]}(u_1) \\ &= \beta[\omega(s)](r_{[1]})\omega(r_{[2]})u_1.\end{aligned}$$

□

In the plethory of unstable cohomology operations, we can recover the looping from the much simpler u_1-evaluation map.

Proposition 2.2.12. *If $E_*(\underline{E}_n)$ is a free E^*-module for all n, then the following diagram commutes.*

$$\begin{tikzcd} E^*(\underline{E}_n) \arrow[r, "\Delta^{\times}_{1,n-1}"] \arrow[d, "\Omega"] & E^*(\underline{E}_1)\widehat{\otimes}E^*(\underline{E}_{n-1}) \arrow[d, "\omega\otimes 1"] \\ E^*(\underline{E}_{n-1}) & E^*\widehat{\otimes}E^*(\underline{E}_{n-1}) \arrow[l, "\cong"'] \end{tikzcd}$$

Proof: For $r \in E^*(\underline{E}_\bullet)$, we have $\Sigma(\Omega r)(x) = r(\Sigma x) = r(u_1 \times x) = r_{[1]}(u_1) \times r_{[2]}(x) = \omega(r_{[1]})u_1 r_{[2]}(x) = \Sigma\omega(r_{[1]})r_{[2]}(x)$. The result follows. □

2.2.1 Plethories with looping

We abstract the previous observations to a purely algebraic topological context leading to the notion of a *plethory with looping*. This will allow us to be cleaner and more flexible when working with the objects, as well as infer that we have most likely extracted all the useful information from the suspension isomorphism.

Definition 2.2.13. Let Π be a complete Hausdorff k-plethory. We say Π is a *k-plethory with looping* if it is equipped with a *S^1-action map*: a continuous map of k-modules $\omega\colon I\Pi \to k$ from the augmentation ideal to the ground ring satisfying the following properties.

1. If $\deg_\bullet(r) \neq 1$ then $\omega(r) = 0$.
2. For all $r, s \in I\Pi$, $\omega(rs) = 0$.
3. The composition $I\Pi \xrightarrow{\Delta^\times} I\Pi\widehat{\otimes}I\Pi \xrightarrow{\omega\widehat{\otimes}1} k\widehat{\otimes}I\Pi \cong I\Pi$ has image contained in $P\Pi$.
4. For all $r, s \in \Pi$, $\omega(r \circ s) = \beta[\omega(s)](r_{[1]})\omega(r_{[2]})$.

5. $\omega(\iota_1) = 1$.

A map $f\colon \Pi \to \Pi'$ is a map of k-plethories with looping if $\omega f(r) = f\omega(r)$ for all $r \in \Pi$. We denote the category of k-plethories with looping by $\mathbf{\Omega Plethory}_k$.

If Π is a k-plethory with looping, we define the *looping map* to be the continuous bidegree $(-1,-1)$ map of k-modules $\Omega\colon I\Pi \to I\Pi$ given by the composition

$$I\Pi \xrightarrow{\Delta^\times} I\Pi\widehat{\otimes}I\Pi \xrightarrow{\omega\widehat{\otimes}1} k\widehat{\otimes}I\Pi \cong I\Pi.$$

Proposition 2.2.14. *Let Π be a k-plethory with looping. The looping map $\Omega\colon I\Pi \to I\Pi$ satisfies the following properties.*

1. *Ω factors as $I\Pi \xrightarrow{\pi} Q\Pi \to P\Pi \subseteq I\Pi$.*

2. *For $r \in I\Pi$, $\Delta^\times\Omega r = r_{[1]} \otimes \Omega r_{[2]}$.*

3. *For $r, s \in I\Pi$, $\Omega(r \circ s) = \Omega r \circ \Omega s$.*

4. *For all $n \in \mathbb{Z}$, $\Omega(\iota_n) = \iota_{n-1}$.*

Proof: (1) Let $x, y \in I\Pi$. Since $\Delta^\times$ is a ring map, $\Delta^\times(xy) \in (I\Pi)^2\widehat{\otimes}(I\Pi)^2$. Since $\omega[(I\Pi)^2] = 0$ we have $\Omega[(I\Pi)^2] = 0$ and so Ω factors via $Q\Lambda$. By construction, the image of Ω is primitive.

(2) For $r \in IH$ and $x, y \in A$ in some Π-algebra A, we have

$$\begin{aligned}
\Omega r(xy) &= \omega(r_{[1]})r_{[2]}(xy) \\
&= \omega(r_{[1]})r_{[2]}(x)r_{[3]}(y) \\
&= r_{[1]}(x)(\omega(r_{[2]})r_{[3]})(y) \\
&= r_{[1]}(x)\Omega r_{[2]}(y).
\end{aligned}$$

The result follows by Theorem 1.2.29.

(3) Let $r, s \in I\Pi$ and x in some Π-algebra. In sumless Sweedler notation, we have

$$\begin{aligned}
(\Omega r \circ \Omega s)(x) &= \Omega r[\omega(s_{[1]})s_{[2]}(x)] \\
&= r_{[1]}(\omega(s_{[1]})) \cdot \omega(r_{[2]}) \cdot (r_{[3]} \circ s_{[2]})(x) \\
&= \beta[\omega(s_{[1]})](r_{[1]}) \cdot \omega(r_{[2]}) \cdot (r_{[3]} \circ s_{[2]})(x).
\end{aligned}$$

To compute $\Omega(r \circ s)$, recall

$$\Delta^\times(r \circ s) = \sum_{(r)} \prod_{i=1}^{n} \sum_{(r_{(i)})} r_{(i)[1]} \circ s^i_{[1]} \otimes r_{(i)[2]} \circ s^i_{[2]}$$

where $\Delta^{\times}(s) = \sum\limits_{i=1}^{n} s^i_{[1]} \otimes s^i_{[2]}$. Since $\omega((I\Pi)^2) = 0$, we have

$$\begin{aligned}(\omega \otimes 1)\Delta^{\times}(r \circ s) &= \sum_{(r)} \sum_{(s)} \omega(r \circ s_{[1]}) \otimes r_{[2]} \circ s_{[2]} \\ &= \sum_{(r)} \sum_{(s)} \beta[\omega(s_{[1]})](r_{[1]}) \cdot \omega(r_{[2]}) \otimes r_{[3]} \circ s_{[2]}\end{aligned}$$

The result follows.

(4) This is immediate since $\Delta^{\times}(\iota_n) = \sum_{r+s=n} \iota_r \otimes \iota_s$. □

Proposition 2.2.15. *Suppose Π is a k-plethory equipped with a continuous bidegree $(-1,-1)$ map of k-modules $\Omega\colon I\Pi \to I\Pi$ satisfying the conditions in Proposition 2.2.14 then the composition $I\Pi \xrightarrow{\Omega} I\Pi \xrightarrow{\varepsilon^{\times}} k$ equips $I\Pi$ with the structure of a complete Hausdorff k-plethory with looping.*

Proof: This is the content of the discussion in Section 2.2.

Theorem 2.2.16. *Suppose $E^*(-)$ is a (multiplicative) cohomology theory. If $E_*(\underline{E}_n)$ is a free E^*-module for each $n \in \mathbb{Z}$ then $E^*(\underline{E}_\bullet)$ is an E^*-plethory with looping.*

Proof: Recall that the E^*-module $E^*(S^1, o)$ has a canonical generator denoted u_1. This represents a map $S^1 \to \underline{E}_1$. The discussion in Section 2.2 showed that the map induced on cohomology $u_1^*\colon E^*(\underline{E}_1, o) \to E_*(S^1, o) \cong E^*$ satisfies the required properties. □

To be confident we have encapsulated all the important implications the suspension isomorphisms have on the plethory of cohomology operations we can compare our results to the well-studied implications of suspension on the Hopf ring of homology cooperations.

Consider the canonical generator $u_1 \in E^*(S^1, o)$, this represents a map $S^1 \to \underline{E}_1$ which induces a map $E_*(S^1, o) \to E^*(\underline{E}_1, o)$. The image of u_1 under this map is typically denoted $e \in E^1(\underline{E}_1, o)$. In the Hopf ring, we can consider the degree 1 map $e \circ -\colon E_*(\underline{E}_n) \to E_*(\underline{E}_{n+1})$ and in cases where we have duality $E^*(\underline{E}_n) \cong DE_*(\underline{E}_n)$, the dual of $\circ$-multiplication by e is looping of operations. Moreover, this information is all produced by the map induced on homology by u_1. In our plethystic setting, the information from the suspension isomorphisms is produced by the map induced by u_1 on cohomology. It should also be possible to formulate and prove this entirely abstractly if we devise a suitable notion of a Hopf ring equipped with the suspension element e.

2.2.2 Stable operations

Of particular prominence in topology are the stable operations: collections of operations $r_n\colon E^n(-) \to E^{n+h}(-)$ for $n \in \mathbb{Z}$ which commute with suspension in the sense that $r_n(\Sigma x) = \Sigma r_{n-1}(x)$ for all spaces X and $x \in E^*(X)$. Equivalently, we have $\Omega r_n = r_{n-1}$. A stable operation induces a map of spectra $E \to E$ which is by definition an element of our stable cohomology $E^*(E, o)$. Stable operations have interesting relationships with both the additive and the unstable operations and now we have abstracted our notion of looping to an algebraic context we can discuss this in detail.

Definition 2.2.17. Let P be a k-plethory with looping. We say a sequence $r \in \prod_{n\in Z} P_n$ is *stable* if $\Omega r_{n+1} = r_n$ for all n. Denote the set of all stable elements in $\prod_{n\in Z} P_n$ by $\mathrm{Stab}(P)$. Note by the looping condition, we have $\deg_*(r_n) - n$ is constant for all n and this gives a grading on $\mathrm{Stab}(P)$. Given a map $f\colon P \to P'$ of k-k'-birings with looping, we can define $\mathrm{Stab}(f)\colon \mathrm{Stab}(P) \to \mathrm{Stab}(P')$ by $\mathrm{Stab}(f)(r) = (f(r_n))_{n\in\mathbb{Z}}$. Hence we have a functor $\mathrm{Stab}(-)\colon \mathbf{\Omega Plethory}_k \to \mathbf{Mod}_k$.

The following result justifies this definition.

Theorem 2.2.18. *Suppose $E^*(-)$ is a (multiplicative) cohomology theory. If $E_*(\underline{E}_n)$ is a free E^*-module for each $n \in \mathbb{Z}$ then we have an isomorphism of k-modules $\mathrm{Stab}(E^*(\underline{E}_\bullet)) \cong E^*(E, o)$.*

Proof: Let $r\colon E \to E$ be a degree h self map of the representing Ω-spectrum for $E^*(-)$. Define $r_n\colon \underline{E}_n \to \underline{E}_{n+h}$ to be the component of r on the n-th representing space. Since r is a map of spectra we have $\Omega r_{n+1} = r_n$. Thus, viewing the r_n as elements of $E^{n+h}(\underline{E}_n)$, we have $(r_n)_{n\in\mathbb{Z}} \in \mathrm{Stab}(E^*(\underline{E}_\bullet))$. Conversely, if $(r_n)_{n\in\mathbb{Z}} \in \mathrm{Stab}(E^*(\underline{E}_\bullet))$ then we can define a map of Ω-spectrum $E \to E$ on each representing space by the r_n. These constructions are clearly inverses. $\square$

Another common used topological property which has a nice algebraic consequence is the stabilisation map of spectra. Thus induces a map $\sigma_n\colon E^*(E, o) \to E^*(\underline{E}_n)$ sending a stable operation to its restriction to the n-th degree cohomology.

Definition 2.2.19. For a k-plethory with looping Π, define the *stabilisation map* $\sigma_n\colon \mathrm{Stab}(\Pi) \to \Pi_n$ to be the canonical projection.

Proposition 2.2.20. *Let Π be a k-plethory with looping. The stabilisation map σ_n factors as $\mathrm{Stab}(\Pi) \to P\Pi_n \subseteq \Pi_n$.*

Proof: Since $\sigma_n(r) = r_n = \Omega r_{n+1}$, by Proposition 2.2.14 we have $\sigma_n(r) \in P\Pi_n$.

As usual, we are interested in the structure on the stable operations. Let Π be a k-plethory with looping. Since $\Omega(rs) = 0$ for all $rs \in \Pi$, it is clear that the multiplication in Π does not induce structure on $\mathrm{Stab}(\Pi)$ in any obvious fashion. Nevertheless, as is well known from topology, the composition of two stable operations is stable.

Theorem 2.2.21. *The functor* $\mathrm{Stab}(-)$ *lifts to a functor* $\mathbf{\Omega Plethory}_k \to \mathbf{Alg}_k^{\neg com}$. *If* Π *is a k-plethory with looping, the multiplication in* $\mathrm{Stab}(\Pi)$ *is induced by the composition in* Π.

Proof: Let Π be a k-plethory with looping. Define $\mu\colon \mathrm{Stab}(\Pi) \otimes \mathrm{Stab}(\Pi) \to \mathrm{Stab}(\Pi)$ as follows. For $r = (r_n), s = (s_n) \in \mathrm{Stab}(\Pi)$, we define $rs = \mu(r \otimes s) \in \mathrm{Stab}(\Pi)$ to be the unique element which has $\pi_n(rs) = r_{n+|s|} \circ s_n$ where π_n denotes the canonical projection $\mathrm{Stab}(\Pi) \to \Pi_n$. This is well defined since each r_n is primitive and

$$\begin{aligned}\Omega(r_{n+1+|s|} \circ s_{n+1}) &= \Omega r_{n+1+|s|} \circ \Omega s_{n+1} \\ &= r_{n+|s|} \circ s_n.\end{aligned}$$

Together with $1 \in \mathrm{Stab}(\Pi)$, defined to be the unique element with $\pi_n 1 = \iota_n$ this forms a not necessarily commutative algebra. □

While the action of a stable operation is a group homomorphism, we have not encoded how such operations respect the remaining algebraic structure, namely how they act on products and constants. It is well known [39] that the stable operations of a multiplicative cohomology theory form a Hopf algebroid. The extra structure provided by a Hopf algebroid precisely allows us to encode our action of operations on products and constants and it should be straightforward to show that $\mathrm{Stab}(-)$ lifts to take values in the category of Hopf algebroids.

2.3 Complex orientation

All cohomology theories come equipped with the suspension isomorphisms, allowing us to compute the cohomology of spheres. Many cohomology theories additionally come with structure known as a complex orientation, which allows us to compute the cohomology of complex projective spaces. The implications of a complex orientation have been well studied, refer to [4, Part II] and [32, Section 4.3 and 4.4] for a complete reference. We shall introduce only the definitions and results necessary for our purposes. Recall that for a cohomology theory $E^*(-)$, $E^*(S^2, o)$ is the free E^*-module on a canonical generator, typically denoted $u_2 \in E^2(S^2, o)$.

Definition 2.3.1. A *complex orientation* on a cohomology theory $E^*(-)$ is an element $x \in E^*(\mathbb{C}P^\infty, o)$ such that i^*x generates $E^*(S^2, o) \cong \langle u_2 \rangle$ where $i\colon S^2 \simeq \mathbb{C}P^1 \subseteq \mathbb{C}P^\infty$ is the canonical inclusion.

Note we do not require $i^*x = u_2$ just that $i^*x = \gamma u_2$ where γ is a unit in E^* and thus x is not necessarily in degree 2. This extra flexibility can result in cleaner formulae.

Example 2.3.2. Let $H^*(-;\mathbb{Z})$ denote singular cohomology with coefficients in $\mathbb{Z}$ as defined in Section 3.1. It is well known that $H^*(\mathbb{C}P^n;\mathbb{Z}) = \mathbb{Z}[x]/(x^{n+1})$. Hence $H^*(\mathbb{C}P^\infty;\mathbb{Z}) = \varprojlim_n H^*(\mathbb{C}P^n;\mathbb{Z}) = \mathbb{Z}[[\alpha]]$ and $x \in \widetilde{H}^2(\mathbb{C}P^\infty;\mathbb{Z})$ serves as a complex orientation.

Example 2.3.3. Let $K^*(-)$ denote complex topological K-theory as defined in Section 3.2. The canonical complex orientation is $[\xi] - 1 \in KU^0(\mathbb{C}P^\infty)$, where ξ is the universal line bundle over $\mathbb{C}P^\infty$.

The presence of a complex orientation leads to many well known and extremely useful properties. The most immediate consequence allows us to compute the cohomology algebras for complex projective spaces.

Proposition 2.3.4 ([32, Proposition 4.3.2])**.** *Let $E^*(-)$ be a (multiplicative) cohomology theory with complex orientation $x \in E^*(\mathbb{C}P^\infty, o)$. The following conditions hold.*

1. $E^*(\mathbb{C}P^n) \cong E^*[x]/(x^n)$.
2. $E^*(\mathbb{C}P^\infty) \cong E^*[[x]]$.
3. $E^*(\mathbb{C}P^n \times \mathbb{C}P^m) \cong E^*[x_1, x_2]/(x_1^n, x_2^m)$.
4. $E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong E^*[[x_1, x_2]]$

where x_1, x_2 are the images of x under the maps induced by the projections $\pi_1, \pi_2\colon \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$.

We remark that as a corollary of the above result, the degree of the complex orientation x must be even if the characteristic of E^* is not equal to 2: for if $|x|$ is odd then $x^2 = 0$.

Corollary 2.3.5 ([4, Part II. Lemma 2.14 and Corollary 2.18])**.** *Let $E^*(-)$ be a (multiplicative) cohomology theory with complex orientation $x \in E^n(\mathbb{C}P^\infty, o)$. The homology coalgebra $E_*(\mathbb{C}P^\infty)$ is the free E^*-module on generators β_i for $i \geq 0$, where $\beta_i \in E_{ni}(\mathbb{C}P^\infty)$ is dual to x^i. The coalgebra structure is given by*

$$\psi(\beta_k) = \sum_{i+j=k} \beta_i \otimes \beta_j.$$

Another very useful property of the complex orientation is the formal group law for the cohomology theory. Recall that $\mathbb{C}P^\infty$ is an H-space: it comes equipped with a continuous multiplication $\mu\colon \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$. There are two illuminating ways to realise this multiplication. The first identifies $\mathbb{CP}^\infty$ with $\mathbb{C}[[x]]/\sim$ where $f(x) \sim g(x)$ if and only if $f(x) = zg(x)$ for some $z \in \mathbb{C}$. Under this identification, μ is induced by multiplication of power series. An alternative realisation of μ is to recall that $\mathbb{CP}^\infty$ is a model for $BU(1)$, the classifying space of line bundles. If ξ denotes the universal line bundle over $\mathbb{C}P^\infty$ and $\pi_i\colon \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ the natural projections ($i = 1, 2$) then $\pi_1^*\xi \otimes \pi_2^*\xi$ is a line bundle over $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$ and thus classified by a map $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ which is the H-space multiplication on $\mathbb{C}P^\infty$.

Now $\mu\colon \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ induces a map on cohomology $E^*[[x]] \cong E^*(\mathbb{C}P^\infty) \xrightarrow{\mu^*} E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong E^*[[x_1, x_2]]$ which is determined by the image of x. This map produces an algebraic gadget known as a formal group law.

Definition 2.3.6. Let $F(x, y) = \sum_{i,j\geq 0} a_{ij}x^iy^j \in k[[x, y]]$ be a power series. We say F is a *formal group law* over k if the following conditions are true.

1. $F(x, 0) = x$ and $F(0, y) = y$. In other words, $a_{00} = 0, a_{10} = a_{01} = 1$.
2. $F(x, y) = F(y, x)$, that is $a_{ij} = a_{ji}$ for all i, j.
3. $F(x, F(y, z)) = F(F(x, y), z) \in k[[x, y, z]]$.

We will also write $x +_F y$ for $F(x, y)$. Under this notation, conditions (1), (2) and (3) say $+_F$ is a unital, commutative and associative binary operation. We shall write $[n]_F(x)$ for the iterated formal group law sum of n indeterminates: $x +_F \cdots +_F x \in k[[x]]$.

Proposition 2.3.7 ([32, Proposition 4.4.3])**.** *Let $\mu\colon \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ denote the H-space multiplication on $\mathbb{CP}^\infty$. If $E^*(-)$ is a cohomology theory with complex orientation $x \in E^*(\mathbb{C}P^\infty, o)$ then the power series*

$$F(x_1, x_2) = \mu^*(x) \in E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong E^*[[x_1, x_2]]$$

is a formal group law over E^*.

Definition 2.3.8. If $E^*(-)$ is a (multiplicative) cohomology theory with complex orientation $x \in E^*(\mathbb{C}P^\infty, o)$ then the power series of Proposition 2.3.7 is called the *formal group law* for $E^*(-)$. Notice that the formal group law depends on the choice of complex orientation for $E^*(-)$.

Example 2.3.9. The formal group law for ordinary cohomology $H^*(-;\mathbb{Z})$ together with the standard complex orientation (Example 2.3.2) is given by $F(x, y) = x + y$. This is the additive formal group law.

Example 2.3.10. The multiplicative formal group law $F(x, y) = x+y+xy$ is the formal group law for complex K-theory $K^*(-)$ together with the standard complex orientation (Example 2.3.3).

We have seen complex oriented cohomology theories give rise to formal group laws. The Landweber exact functor theorem provides a partial converse; given a formal group law F, it provides sufficient conditions for constructing a cohomology theory with formal group law F. This provides a method of constructing complex K-theory $K^*(-)$ via the multiplicative formal group law.

Finally, we have seen that the H-space structure μ on $\mathbb{C}P^\infty$ induces a map on cohomology $E^*(\mathbb{C}P^\infty) \to E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$ which is encoded by the formal group law. Since $E_*(\mathbb{C}P^\infty)$ is a free E^*-module, we have a Künneth isomorphism $E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong E^*(\mathbb{C}P^\infty)\widehat{\otimes}E^*(\mathbb{C}P^\infty)$ and we can thus give $E^*(\mathbb{C}P^\infty)$ the structure of a completed Hopf algebra. Similarly we can consider the map induced by μ on homology and give $E_*(\mathbb{C}P^\infty)$ the structure of a Hopf algebra.

Proposition 2.3.11 ([40, Lemma 3.3 and Theorem 3.4])**.** *Let $E^*(-)$ be a cohomology theory with complex orientation $x \in E^*(\mathbb{C}P^\infty, o)$ and formal group law F. The cohomology ring $E^*(\mathbb{C}P^\infty) \cong E^*[[x]]$ has the structure of a completed Hopf algebra with comultiplication $E^*(\mathbb{C}P^\infty) \to E^*(\mathbb{C}P^\infty)\widehat{\otimes}E^*(\mathbb{C}P^\infty)$ given by*

$$\psi(x) = F(x \otimes 1, 1 \otimes x).$$

Moreover, the homology coalgebra $E_*(\mathbb{C}P^\infty) \cong \langle \beta_0, \beta_1, \dots \rangle$ has the structure of a Hopf algebra with multiplication given by

$$\beta(s)\beta(t) = \beta(s +_F t)$$

in the ring in $E_*(\mathbb{C}P^\infty)[[s, t]]$ where $\beta(t) = \sum_i \beta_i t^i$.

2.3.1 Complex orientation and Hopf rings

The impact of complex orientation on the Hopf ring of homology coooperations has been well studied, see [40]. We will recall the important results, before abstracting to an algebraic setting. This abstraction will prove useful when we study the dual world of plethories and allow us to show that the dual of the effect of a complex orientation on the homology cooperations appears in our language of plethories.

Suppose $E^*(-), G^*(-)$ are multiplicative cohomology theories and recall that a complex orientation $x^G \in G^n(\mathbb{C}P^\infty, o)$ represents a map $\mathbb{C}P^\infty \to \underline{G}_n$ and thus induces a map $x^G_*\colon E_*(\mathbb{C}P^\infty) \to E_*(\underline{G}_n)$. If $E^*(-)$ is also complex oriented then we can compute the homology of $\mathbb{C}P^\infty$. Applying the map x^G_* to elements of $E_*(\mathbb{C}P^\infty)$ defines elements of $E_*(\underline{G}_n)$.

Definition 2.3.12. Let $E^*(-), G^*(-)$ be (multiplicative) cohomology theories with complex orientations $x^E \in E^m(\mathbb{CP}^\infty, o), x^G \in G^n(\mathbb{C}P^\infty, o)$. Define the elements $b_i \in E_{mi}(\underline{G}_n)$ by $b_i = x^G_*(\beta_i)$ where $\beta_i \in E_{mi}(\mathbb{C}P^\infty)$ is dual to $(x^E)^i$.

For notational ease, we define the formal power series $b(t) = \sum_i b_i t^i \in E_*(\underline{G}_n)[[t]]$.

This construction yields elements of our Hopf ring, but we have no information yet about the properties of these elements. Fortunately, these b_i have been well-studied, for example see [15, Section 15].

Proposition 2.3.13 ([15, Proposition 15.3])**.** *The elements $b_i \in E_{mi}(\underline{G}_n)$ satisfy the following properties.*

1. $b_0 = 1_0$.
2. $\psi b_k = \sum_{i+j=k} b_i \otimes b_j$ *or equivalently,* $\psi b(t) = b(t) \otimes b(t)$.
3. $\varepsilon b_k = 0$ *for* $k > 0$ *and* $\varepsilon b_0 = 1$ *or equivalently,* $\varepsilon b(t) = 1$.
4. $\chi b(s) = b(s)^{*-1}$ *where we can expand the right hand side by writing* $b(s) = 1 + \bar{b}(s)$.

The above proposition contains no statement about the interaction between the b_i, the $*$-multiplication and the $\circ$-multiplication. This information is encoded by the formal group law.

Definition 2.3.14. Let H be a k-k'-Hopf ring and F a formal group law over k. We shall write

$$x +_{[F]} y = \mathop{\huge *}_{i,j\geq 0} [a_{ij}] \circ x^{\circ i} \circ y^{\circ j}.$$

Since $*$-multiplication and $\circ$-multiplication come from the addition and multiplication on the k'-algebra object defining a Hopf ring, we can view this as a Hopf ring theoretic version of the power series $x +_F y$.

Theorem 2.3.15 ([40, Theorem 3.8])**.** *Let $E^*(-), G^*(-)$ be (multiplicative) cohomology theories with complex orientations $x^E \in E^m(\mathbb{C}P^\infty, o)$, $x^G \in G^n(\mathbb{C}P^\infty, o)$ and formal groups F_E, F_G respectively. In $E_*(\underline{G}_n)[[s,t]]$,*

$$b(s +_F t) = b(s) +_{[G]} b(t).$$

Example 2.3.16. Suppose $F_E(x,y) = F_G(x,y) = x+y$ then we have $b(s+t) = b(s)*b(t)$, and comparing coefficients we see, $b_i * b_j = \binom{i+j}{i} b_{i+j}$. Hence, if our map $x^G_*\colon E_*(\mathbb{C}P^\infty) \to E_*(\underline{G}_n)$ is an isomorphism, then $E_*(\underline{G}_n)$ is a divided power series algebra on a single generator. For example, if $E^*(-) = G^*(-) = H^*(-;\mathbb{Z})$ with the complex orientation in Example 2.3.2 then $x_*\colon H_*(\mathbb{C}P^\infty;\mathbb{Z}) \to H_*(K(\mathbb{Z},2);\mathbb{Z})$ where $K(\mathbb{Z},2)$ is the representing space for $H^2(-;\mathbb{Z})$. In this case, x_* is an isomorphism.

Example 2.3.17. For the multiplicative formal group law, $F_E(x,y) = F_G(x,y) = x + y + xy$, the answer is somewhat more complicated. For example, when $E^*(-)$ and $G^*(-)$ are complex K-theory $K^*(-)$ together with the complex orientation in Example 2.3.3. We have $b(s+t+st) = b(s) * b(t) * b(s) \circ b(t)$. If we consider the coefficient of $s^i t^j$ in

$$\begin{aligned} b(s+t+st) &= \sum_n b_n (s+t+st)^n \\ &= \sum_n \sum_{\alpha+\beta+\gamma=n} \binom{n}{\alpha,\beta,\gamma} b_n s^{\alpha+\gamma} t^{\beta+\gamma} \end{aligned}$$

then we see contributing terms will have $\alpha+\gamma = i, \beta+\gamma = j$ and $n \leq i+j \leq 2n$. Since $0 \leq \gamma \leq n$, the coefficient of $s^i t^j$ is

$$\sum_{n=\lceil \frac{i+j}{2} \rceil}^{i+j} \sum_{\gamma=0}^{n} \binom{n}{i-\gamma, j-\gamma, \gamma} b_n.$$

For the right hand side, we note

$$b(s) * b(t) * b(s) \circ b(t) = \sum_{\alpha,\beta,\gamma,\delta} b_\alpha * b_\beta * (b_\gamma \circ b_\delta) s^{\alpha+\gamma} t^{\beta+\delta}.$$

Hence the coefficient of $s^i t^j$ is

$$\sum_{\alpha+\gamma=i, \beta+\delta=j} b_\alpha * b_\beta * (b_\gamma \circ b_\delta) = \sum_{\alpha=0}^{i} \sum_{\beta=0}^{j} b_\alpha * b_\beta * (b_{i-\alpha} \circ b_{j-\beta}).$$

Using $b_0 = 1_0$, we have

$$\begin{aligned} \sum_{n=\lceil \frac{i+j}{2} \rceil}^{i+j} \sum_{\gamma=0}^{n} \binom{n}{i-\gamma, j-\gamma, \gamma} b_n =& b_i \circ b_j + \sum_{\alpha=1}^{i-1} b_\alpha * (b_{i-\alpha} \circ b_j) + \sum_{\beta=1}^{j-1} b_\beta * (b_i \circ b_{j-\beta}) \\ & + b_i * b_j + \sum_{\alpha=1}^{i-1} \sum_{\beta=1}^{j-1} b_\alpha * b_\beta * (b_{i-\alpha} \circ b_{j-\beta}) \end{aligned}$$

since $b_0 \circ b_k = 0$ unless $k = 0$ when $b_0 \circ b_0 = 1_0$. This gives an inductive formula to compute $b_i \circ b_j$ for any $i, j > 0$.

We now abstract this information to a purely algebraic setting. To do so, we first recall the construction of a free Hopf ring, originally due to Ravenel and Wilson [40].

Definition 2.3.18. Let $\mathbf{Coalg}_k^+$ denote the category of augmented k-coalgebras: k-coalgebras C with a k-module map $\eta\colon k \to C$ such that $\varepsilon\eta = 1_k$. We write $1 = \eta(1)$. Define the *free Hopf ring* functor $FHR\colon (\mathbf{Coalg}_k^+)^{\mathbb{Z}} \to \mathbf{HopfRing}_{k[k']}$ as follows. If $C_\bullet \in (\mathbf{Coalg}_k^+)^{\mathbb{Z}}$, identify $1 \in C_n$ with $[0_n]$, take all possible $*$ and $\circ$-products of $C_\bullet$ with itself and the elements $[\lambda]$ for $\lambda \in k'$ and then take the quotient by the defining relations of a Hopf ring, see [40, Lemma 1.12].

Similiarly, we can define the free $k[k']$-Hopf ring on a $\mathbb{Z}$-graded collection of Hopf algebras over k by taking the free Hopf ring on the underlying $\mathbb{Z}$-graded collection of augmented k-coalgebras and then taking the quotient by the multiplicative relations in the Hopf algebras. For example, this is used in [52].

The free Hopf ring satisfies the following universal property.

Proposition 2.3.19 ([40])**.** *There is a canonical map of augmented k-coalgebras $i\colon C_\bullet \to FHR(C_\bullet)$ such that for any k-k'-Hopf ring H and map of augmented k-coalgebras $f\colon C_\bullet \to H$, there exists a unique map of k-k'-Hopf rings $\widetilde{f}\colon FHR(C_\bullet) \to H$ such that the following diagram commutes.*

$$\begin{tikzcd} C_\bullet \arrow[r, "i"] \arrow[dr, "f"'] & FHR(C_\bullet) \arrow[d, "\widetilde{f}"] \\ & H \end{tikzcd}$$

The following augmented coalgebra will crop up frequently, so for brevity we give it a name.

Definition 2.3.20. Define the augmented k-coalgebra $C(m)$ to be the free k-module on generators β_i for $i \geq 0$ with $|\beta_i| = mi$. The coalgebra structure is given by $\psi(\beta_k) = \sum_{i+j=k} \beta_i \otimes \beta_j$ and the augmentation by $\beta_0 = 1$. As before, we shall write $\beta(t) = \sum_i \beta_i t^i \in C(m)[[t]]$.

Definition 2.3.21. Fix integers n, m and define an object of $(\mathbf{Coalg}_k^+)^{\mathbb{Z}}$ by

$$C_\bullet = l \mapsto \begin{cases} C(m) & l = n \\ \langle 1 \rangle & \text{otherwise.} \end{cases}.$$

For formal group laws F_1 over k and F_2 over k', let I_{F_1,F_2} be the Hopf ring ideal of $FHR(C_\bullet)$ generated by the coefficients of the power series

$$\beta(s +_{F_1} t) - [\beta(s) +_{[F_2]} \beta(t)] \in FHR(C_\bullet)[[s,t]].$$

Define the *bidegree* (n,m) *complex orientating* $k[k']$*-Hopf ring with respect to* F_1 *and* F_2 to be

$$H^{n,m}_{F_1,F_2} = \frac{FHR(C_\bullet)}{I_{F_1,F_2}}$$

Definition 2.3.22. Let F_1, F_2 be formal group laws over k and k' respectively. We say a k-k'-Hopf ring H is *bidegree* (n,m) *complex oriented with respect to formal group laws* F_1 *and* F_2 if there exists a map of $k[k']$-Hopf rings $H^{n,m}_{F_1,F_2} \to H$.

As $H^{n,m}_{F_1,F_2}$ is trivial outside of $\bullet$-degree n, we have a simpler sufficient condition for a Hopf ring to be complex oriented.

Proposition 2.3.23. *Let* F_1, F_2 *be formal group laws over* k *and* k' *respectively. A* $k[k']$*-Hopf ring* H *is bidegree* (n,m) *complex oriented with respect to formal group laws* F_1 *and* F_2 *if and only if there exists a map of augmented* k*-coalgebras*

$$f\colon C(m) \to H_n$$

such that $f(\beta(s +_{F_1} t)) = f(\beta(s)) +_{[F_2]} f(\beta(t))$ *in* $H_n[[s,t]]$.

Proof: Let $\phi\colon H^{n,m}_{F_1,F_2} \to H$ be a map of $k[k']$-Hopf rings. Let $C_\bullet$ be the augmented k-coalgebras of Definition 2.3.21 and consider the $k[k']$-Hopf ring map

$$FHR(C_\bullet) \xrightarrow{\pi} H^{n,m}_{F_1,F_2} \xrightarrow{\phi} H.$$

Restricting to the n-th $\bullet$-component gives a map $f\colon C(m) \to H_n$ of augmented k-coalgebras. Moreover, since f factors through $H^{n,m}_{F_1,F_2}$ we must have $f(\beta(s +_{F_1} t)) = f(\beta(s) +_{[F_2]} \beta(t))$. For the converse, suppose we have a map $f\colon C(m) \to H_n$ satisfying our hypothesis. This extends to a map $f_\bullet\colon C_\bullet \to H$ in the obvious way. Hence we have a map of $k[k']$-Hopf rings

$$\phi\colon FHR(C_\bullet) \to H.$$

Now,

$$\phi(\beta(s +_{F_1} t) - [\beta(s) +_{F_2} \beta(t)]) = f(\beta(s +_{F_1} t)) - f(\beta(s)) +_{F_2} f(\beta(t)) = 0.$$

Hence ϕ factors as $FHR(C_\bullet) \to H^{n,m}_{F_1,F_2} \to H$. □

The following result shows our definitions correctly encode the impact of a complex orientation on the Hopf ring of homology cooperations.

Corollary 2.3.24. *Let $E^*(-), G^*(-)$ be cohomology theories with complex orientations $x^E \in E^m(\mathbb{CP}^\infty, o), x^G \in G^n(\mathbb{C}P^\infty, o)$ and formal group laws F_E, F_G respectively. The $E^*[G^*]$-Hopf ring $E_*(\underline{G}_\bullet)$ is bidegree (n, m) complex oriented with respect to F_E, F_G.*

Proof: Apply Proposition 2.3.23 to $x^G_*\colon E_*(\mathbb{C}P^\infty) \to E^*(\underline{G}_n)$. The fact this map satisfies the hypothesis is the content of [40, Theorem 3.8]. $\square$

2.3.2 The impact on cohomology operations

Let P be a k-plethory and suppose we have a map of P-rings $f\colon k[[x]] \to k[[y, z]]$ given by $f(x) = F(y, z)$. By naturality, we require $r(f(x)) = r(F(y, z)) = f(r(x))$ for all $r \in P$. Now $F(y, z) \in k[[y, z]]$ and thus we can use the structure maps of our plethory to expand $r(F(y, z)) = r(\sum a_{ij} y^i z^j)$ in terms of actions of operations on y and z. This produces a 'comultiplication' $r \mapsto r_{\{1\}} \otimes r_{\{2\}}$ such that $r(y +_F z) = r_{\{1\}}(y) r_{\{2\}}(z)$.

Definition 2.3.25. Let $F(y, z) = \sum a_{ij} y^i z^j$ be a finite formal group law (so only N coefficients a_{ij} are non-zero), $n \in \mathbb{Z}$, and B a complete Hausdorff k-k'-biring. Define the *F-comultiplication* $\Delta^F\colon B_n \to B_n \widehat{\otimes} B_n$ to be the following composition.

$$B_n \xrightarrow{\Delta^+_{(N)}} \widehat{\bigotimes}_{i,j\geq 0} B_n \xrightarrow{\widehat{\otimes}_{i,j} \Delta^\times_{(1+i+j)}} \widehat{\bigotimes}_{i,j\geq 0} \left(B_{(i+j-1)n} \widehat{\otimes} B_n^{\widehat{\otimes} i} \widehat{\otimes} B_n^{\widehat{\otimes} j}\right) \xrightarrow{\widehat{\otimes}_{i,j}\left(\beta a_{i,j} \widehat{\otimes} 1^{\widehat{\otimes} i} \widehat{\otimes} 1^{\widehat{\otimes} j}\right)}$$
$$\xrightarrow{\cdots} \widehat{\bigotimes}_{i,j\geq 0} \left(k \widehat{\otimes} B_n^{\widehat{\otimes} i} \widehat{\otimes} B_n^{\widehat{\otimes} j}\right) \cong \widehat{\bigotimes}_{i,j\geq 0} \left(B_n^{\widehat{\otimes} i} \widehat{\otimes} B_n^{\widehat{\otimes} j}\right) \xrightarrow{\tau} \widehat{\bigotimes}_{i\geq 0} B_n^{\widehat{\otimes} i} \widehat{\otimes} \widehat{\bigotimes}_{j\geq 0} B_n^{\widehat{\otimes} j} \xrightarrow{\mu \widehat{\otimes} \mu} B_n \widehat{\otimes} B_n.$$

Here τ is the twist map which sends all copies of $B_n^{\otimes i}$ to the left and all copies of $B_n^{\otimes j}$ to the right. In sumless Sweedler notation, we shall write $\Delta^F(r) = r_{\{1\}} \otimes r_{\{2\}}$.

The above definition should pass to the non-finite case by considering finite truncations of an infinite group law and the passing to the limit.

Example 2.3.26. If F be the additive formal group law, then $\Delta^F = \Delta^+$.

Example 2.3.27. If F is the multiplicative formal group law and $n = 0$ then $\Delta^F(r) = r_{(1)} r_{(3)[1]} \otimes r_{(2)} r_{(3)[2]}$.

Proposition 2.3.28. *Let $k[[y, z]]$ be a P-algebra. For all $r \in P$, we have $r(F(y, z)) = r_{\{1\}}(y) r_{\{2\}}(z)$.*

Proof: Writing $\Delta^+(r) = \bigotimes_{i,j} r^{(i,j)}$, we have

$$\begin{aligned} r(F(y,z)) &= r(\sum_{i,j} a_{ij} y^i z^j) \\ &= \prod_{i,j} r^{(i,j)}(a_{ij} y^i z^j) \\ &= \prod_{i,j} r^{(i,j)}_{[1]}(a_{ij}) r^{(i,j)}_{[2]}(y) \dots r^{(i,j)}_{[1+i]}(y) r^{(i,j)}_{[2+i]}(z) \dots r^{(i,j)}_{[1+i+j]}(z) \\ &= \prod_{i,j} (\beta a_{ij})(r^{(i,j)}_{[1]})(r^{(i,j)}_{[2]} \dots r^{(i,j)}_{[1+i]})(y)(r^{(i,j)}_{[2+i]} \dots r^{(i,j)}_{[1+i+j]})(z). \end{aligned}$$

Comparing to our definition, we see this is precisely $r_{\{1\}}(y) r_{\{2\}}(z)$. □

Thus for the multiplicative formal group law, we see that knowing any two of $\Delta^+, \Delta^\times$ and Δ^F is sufficient to determine the third.

Definition 2.3.29. We say a k-k'-biring B is *bidegree* (n, m) *complex oriented with respect to formal group laws* F_1, F_2 if it is equipped with a map of k-bialgebras

$$(B_n, \mu, \Delta^{F_2}) \to k[[x]]$$

where $|x| = m$ and the coalgebra structure on $k[[x]]$ is determined by $\psi(x) = F_1(x \otimes 1, 1 \otimes x)$.

Theorem 2.3.30. *Suppose $E^*(-), G^*(-)$ have orientations $x_E \in E^n(\mathbb{C}P^\infty)$ and $x_G \in G^m(\mathbb{C}P^\infty)$ respectively with formal group laws F_E and F_G. If $E_*(\underline{G}_n)$ is a free E^*-module for each n then the complete Hausdorff E^*-G^*-biring $E^*(\underline{G}_\bullet)$ is bidegree (n, m) complex oriented with respect to F_E, F_G.*

Proof: The complex orientation for $G^*(-)$ induces a map of E^*-algebras $x_G^* \colon E^*(\underline{G}_m) \to E^*(\mathbb{C}P^\infty)$. Now $E^*(\mathbb{C}P^\infty) \cong E^*[[x_E]]$ and has a coalgebra structure induced by μ and given by $\psi(x_E) = F_E(x_E \otimes 1, 1 \otimes x_E)$. Placing the coalgebra structure Δ^{F_G} on $E^*(\underline{G}_m)$ it remains to show x_G^* is map of coalgebras. For $r \in E^*(\underline{G}_m)$, we have $(x_G^* \otimes x_G^*)\Delta^{F_G} r = r_{\{1\}}(x_G) \otimes r_{\{2\}}(x_G)$. Also, $\psi x_G^* r = \psi r(x_G)$ and since ψ is induced by the H-space structure on $\mathbb{C}P^\infty$ it commutes with r by naturality. Hence $\psi x_G^* r = r(1 \otimes x_G + F_E x_G \otimes 1)) = r_{\{1\}}(x_G) \otimes r_{\{2\}}(x_G)$ by Proposition 2.3.28. Hence see x_G^* is a bialgebra map. □

With these abstract algebraic definitions, we can prove that our definition of a complex orientated biring encapsulates the duals of all the well-studied implications of the complex orientation on the Hopf ring of homology cooperations.

Theorem 2.3.31. *Let H be a $k[k']$-Hopf ring which is free as a k-module and bidegree (n, m) complex orientated with respect to F_1, F_2. The dual complete Hausdorff k-k'-biring DH is bidegree (n, m) complex oriented with respect to F_1, F_2.*

Proof: We have seen from Theorem 2.1.61 that DH is a complete Hausdorff k-k'-biring. Let $f\colon C(m) \to H_n$ be the complex orienting map for H from Proposition 2.3.23. This induces a map of algebras $Df\colon DH_n \to DC(m) \cong k[[x]]$ where $|x| = m$. Moreover, we have $f(\beta(s +_{F_1} t)) = f(\beta(s)) +_{[F_2]} f(\beta(t))$ in $H_n[[s,t]]$. Since $\circ$-multiplication is dual to $\Delta^\times$ and $*$-multiplication is dual to Δ^+, we see equipping $DC(m) \cong k[[x]]$ with the comultiplication $\psi(x) = F_1(x \otimes 1, 1 \otimes x)$ and DH_n with Δ^{F_2} that Df is a map of coalgebras. $\square$

Up to this point we have only discussed birings, we now turn to the world of plethories. On a cohomology theory, complex orientation allows us to compute the cohomology of the complex projective spaces and thus these cohomology algebras have a natural action by the operations on our theory.

Definition 2.3.32. We define a *degree n complex oriented k-plethory with respect to a formal group law F* to be a complete Hausdorff k-plethory P which is a bidegree (n, n) complex oriented k-k-biring with respect to F, F and the k-algebra $k[[x]]$ with $|x| = n$ is a P-algebra.

We have some immediate consequences of this definition.

Proposition 2.3.33. *Suppose P is a degree n complex oriented k-plethory with respect to a formal group law F. The $\bullet$-degree n component of the unital composition map $u\colon \mathcal{I}_n \to P_n$ is injective.*

Proof: We have $\mathcal{I}_n = k[\iota_n]$. Suppose $u(\iota_n)^m = 0$ for some m. In the P-algebra $k[[x]]$, we have $0 = u(\iota_n)^m \circ x = (u(\iota_n) \circ x)^m = x^m \neq 0$ a contradiction. $\square$

Example 2.3.34. Let $k = \mathbb{F}_p$ and consider the algebra ideal $\mathcal{J}$ of $\mathcal{I}$ given by $\mathcal{J}_n = (\iota_n^p)$. This is a plethystic ideal and the quotient has $\bullet$-components $\mathbb{F}_p[\iota_n]/(\iota_n^p)$. By Proposition 2.3.33 we see the $\mathbb{F}_p$ plethory $\mathcal{I}/\mathcal{J}$ does not admit any complex orientations.

While we seem to have discovered all the consequences of a complex orientation on the biring of cohomology operations by comparing our answer to what is known about Hopf rings, it seems likely that there should be additional consequences of a complex orientation on the *plethory* of cohomology operations.

Chapter 3

Computations

After setting up a suitable plethystic framework we now turn our attention to our main application: computing the plethory of cohomology operations for some known cohomology theories.

We have two main approaches for performing these computations. If the cohomology operations have been well studied and there are many useful results, we are able to argue directly and compute the plethory of unstable cohomology operations just from this information.

For more esoteric cohomology theories this may not be the case. However, in many cases the enriched Hopf ring has been computed, usually via complex spectral sequence calculations. In this case, we can leverage our duality results to compute the plethory. We recall some useful computational results about Hopf rings in Appendix E.

In this chapter I give a complete description of the plethory of cohomology operations for singular cohomology with coefficients in $\mathbb{Q}$, $\mathbb{F}_2$ and $\mathbb{F}_p$. For each of these examples, I give two ways to obtain the description. One method uses well-known results about the cohomology operations and directly computes the plethory. The alternative method utilises our duality results from Section 1.3.4 and computes the plethory from a description of the enriched Hopf ring. Moreover, I use plethystic theory to make precise the heuristic that in singular cohomology, the stable operations freely generate the unstable operations.

I give a complete description of the plethory of cohomology operations for complex K-theory. As a result, we are able to show that in a plethystic context, the λ-operations together with looping generate all the unstable operations for complex K-theory.

I conclude the chapter by giving a partial description of the plethory of cohomology operations for the Morava K-theories. After pushing the existing plethystic theory as hard as possible to obtain these partial results, I offer a potential direction for calculating the remaining information in the plethory via a Hopf theoretic filtration and provide some conjectural results.

3.1 Singular cohomology

The first cohomology theory encountered in many courses in algebraic topology is known as singular cohomology. It has the advantage of being geometrically intuitive via the medium of simplical complexes and is amenable for doing many computations. For a detailed introduction to singular cohomology refer to [24] or [34].

Definition 3.1.1. Let R be some commutative ring. Define the *singular n-chains* $C_n(X;R)$ of a topological space to be the free R-module on the *singular n-simplices*: continuous maps $\sigma_n\colon \Delta^n \to X$ where Δ^n denotes the standard n-simplex.

The boundary map $\partial\colon C_{n+1}(X;R) \to C_n(X;R)$ makes $C_\bullet(X;R)$ into a chain complex. We define the chain complex of *singular co-chains* to be the dual chain complex $C^\bullet(X;R) = DC_\bullet(X;R)$ and define the *singular cohomology* $H^*(X;R)$ to be the homology of this chain complex.

The cup product equips $H^*(X;R)$ with the structure of an R-algebra. As these constructions are natural in X and homotopy invariant we have a contravariant functor $H^*(-;R)\colon \mathbf{Ho} \to \mathbf{Alg}_R$. We denote the corresponding reduced cohomology theory by $\widetilde{H}^*(-;R)$.

The Eilenberg-Steenrod axioms were devised by studying the abstract properties of the singular cohomology functor. As such singular cohomology is tautologically a cohomology theory. The representing spaces of singular cohomology are well studied and are known as the Eilenberg-MacLane spaces.

Definition 3.1.2. A space X is an *Eilenberg-MacLane space* (or a $K(G,n)$) for a group G and non-negative integer n if it satisfies the following.

$$\pi_k(K(G,n)) = \begin{cases} G & k = n \\ 0 & \text{o.w.} \end{cases}$$

Provided that if $n > 1$ then G is abelian, there exists a unique up to weak homotopy equivalence CW-complex which is a $K(G,n)$. We shall abuse notation and denote such a

CW-complex by $K(G,n)$. As an useful example, we remark that $K(G,0) = G$ considered as a discrete space.

If R is a commutative ring, then the Eilenberg-MacLane spaces $K(R,n)$ form the Ω-spectrum for singular cohomology $H^*(-;R)$. Explicitly, we have homotopy equivalences $\Omega K(R,n+1) \simeq K(R,n)$ and natural isomorphisms $H^*(X;R) \cong \mathbf{Ho}(X, K(R,n))$.

Now suppose k is a field. Since the coefficient ring, the cohomology of the one-point space, $H^*(T;k) = k$ is a field, Theorem 2.2.16 asserts that the unstable cohomology operations for singular cohomology $H^*(K(k,\bullet);k)$ admits the structure of a k-plethory with looping.

3.1.1 Singular cohomology with rational coefficients

A widely used coefficient ring for singular cohomology is the field of rationals, $\mathbb{Q}$. The cohomology theory $H^*(-;\mathbb{Q})$ is simple enough to be amenable to calculations but complex enough to retain important information about the topological spaces. It is known that in this case we have no interesting cohomology operations but nonetheless it gives us a suitable example for demonstrating our plethystic framework.

A direct approach

It is well known that in positive degrees the cohomology operations $H^*(-;\mathbb{Q}) \to H^*(-;\mathbb{Q})$ are multiplicatively generated by the identity operations.

Theorem 3.1.3. *For $n \geq 1$ we have isomorphisms of $\mathbb{Q}$-algebras*

$$H^*(K(\mathbb{Q},n);\mathbb{Q}) \cong \begin{cases} \Lambda[\iota_n] & n > 0 \textit{ odd} \\ \mathbb{Q}[\iota_n] & n > 0 \textit{ even} \end{cases}$$

where $\iota_n \in \widetilde{H}^n(K(\mathbb{Q},n);\mathbb{Q})$ is the universal class and represents the identity map $K(\mathbb{Q},n) \to K(\mathbb{Q},n)$.

This information is sufficient to compute the plethory of unstable cohomology operations.

Theorem 3.1.4. *Let P be the complete Hausdorff $\mathbb{Q}$-plethory with looping with $\bullet$-components*

$$P_n = \begin{cases} \mathbf{Set}(\mathbb{Q},\mathbb{Q}) & n = 0 \\ \Lambda[\iota_n] & n > 0 \textit{ odd} \\ \mathbb{Q}[\iota_n] & n > 0 \textit{ even} \end{cases}$$

where $|\iota_n| = n$ *and* $\mathbf{Set}(\mathbb{Q},\mathbb{Q})$ *is concentrated in degree* 0. *The plethystic structure on* $\mathbf{Set}(\mathbb{Q},\mathbb{Q})$ *is given by Example 2.1.58 and* ι_0 *is the identity map in* $\mathbf{Set}(\mathbb{Q},\mathbb{Q})$. *We have an isomorphism of* $\mathbb{Q}$*-plethories with looping* $H^*(K(\mathbb{Q},n);\mathbb{Q}) \cong P_\bullet$.

We remark that this information completely determines the plethystic structure since the unit map for the composition $\mathcal{I} \to H^*(K(\mathbb{Q},n);\mathbb{Q})$ is a map of $\mathbb{Q}$-plethories with looping, i.e. the operations $\iota_n \in H^*(K(\mathbb{Q},n);\mathbb{Q})$ are super primitive, the units for composition, and satisfy $\Omega\iota_n = \iota_{n-1}$.

Proof: For $n > 0$, Theorem 3.1.3 gives us an expression of $H^*(K(\mathbb{Q},n);\mathbb{Q})$ as an algebra. The plethystic structure and looping is immediate since ι_n represents the identity map. For $n = 0$, $K(\mathbb{Q},0) = \mathbb{Q}$, a disjoint union of points, and thus $H^*(K(\mathbb{Q},n);\mathbb{Q})$ is concentrated in degree 0. Now, $H^0(\mathbb{Q};\mathbb{Q}) \cong \mathbf{Ho}(\mathbb{Q},\mathbb{Q}) = \mathbf{Set}(\mathbb{Q},\mathbb{Q})$ and this has the correct biring structure by definition. $\square$

Recall Theorem 1.3.58 which states that every plethory over a field of characteristic 0 is linear. The following result shows that a topological generalisation of this result holds for $H^*(K(\mathbb{Q};\bullet))$. Recall the functor $S_k\colon \mathbf{Bialg}_k \to \mathbf{Plethory}_k$ of Definition 1.3.54. A topology on a k-bialgebra B induces a filtration on $S_k(B)$ given by the images $S_k(U) \to S_k(B)$ where $U \subseteq B$ ranges over the open sets. Hence composing with the completion, we can define a functor $\widehat{S}_k$ from the category of topological bialgebras to $\mathbf{CPlethory}_k$.

Theorem 3.1.5. *Let* $B_\bullet$ *be the bigraded* $\mathbb{Q}$*-bialgebra such that* B_n *is the free* $\mathbb{Q}$ *module on a single group-like generator in degree* n *for all* $n > 0$ *and the multiplication is induced by multiplication in* $\mathbb{Q}$. *For* $n > 0$, *equip* B_n *with the discrete topology and equip* B_0 *with the pro-finite topology.*

We have an isomorphism of complete Hausdorff $\mathbb{Q}$*-plethories* $\widehat{S}_k(B_\bullet) = H^*(K(\mathbb{Q},\bullet);\mathbb{Q})$.

Proof: Let ι_n denote the group-like generator for B_n. We have isomorphisms of $\mathbb{Q}$-algebras $S_k(B_n) \cong \mathbb{Q}[\iota_n]$ for n even and $S_k(B_n) \cong \Lambda[\iota_n]$ for n odd. For $n > 0$, this has the discrete topology and thus $\widehat{S}_k(B_n) = S_k(B_n)$. For $n = 0$, $S_k(\mathbb{Q}) = \mathbb{Q}[x]$ has open sets consisting of polynomials which agree on some finite subset $A \subseteq \mathbb{Q}$. Now for any set map $\phi\colon \mathbb{Q} \to \mathbb{Q}$ and finite subset $A \subset \mathbb{Q}$ we can find a polynomial that agrees with ϕ on A. Hence $\widehat{S}_k(B_0) \cong \mathbf{Set}(\mathbb{Q},\mathbb{Q})$. These algebra maps form a map of $\mathbb{Q}$-plethories by the definition of S_k. $\square$

Via the enriched Hopf Ring

The enriched Hopf ring of homology cooperations for singular cohomology with rational coefficients has been computed and to illustrate our machinery we re-derive Theorem 3.1.4 from this starting point.

Theorem 3.1.6 ([15, Theorem 17.5])**.** *The $\mathbb{Q}[\mathbb{Q}]$-Hopf ring $H_*(K(\mathbb{Q},\bullet);\mathbb{Q})$ is the free $\mathbb{Q}[\mathbb{Q}]$-Hopf ring on the primitive generator $e \in H_1(K(\mathbb{Q},1);\mathbb{Q})$ subject to the relations $e \circ [\lambda] = \lambda e$. The enrichment is determined by the induced maps $r_* e = [\langle r, 1_1\rangle] * [\langle r, e\rangle] \circ e$ for all $r \in DH_*(K(\mathbb{Q},\bullet);\mathbb{Q})$ and the augmentation determined by $\iota_1(e) = 1$.*

It is useful to convert such global descriptions into a local form, a description of $H^*(K(\mathbb{Q},n);\mathbb{Q})$ for each n.

Proposition 3.1.7. *We have isomorphisms of $\mathbb{Q}$-algebras*

$$H_*(K(\mathbb{Q},n);\mathbb{Q}) \cong \begin{cases} \mathbb{Q}[\mathbb{Q}] & n = 0 \\ \Lambda[e^{\circ n}] & n > 0 \text{ odd} \\ \mathbb{Q}[e^{\circ n}] & n > 0 \text{ even.} \end{cases}$$

Proof: This is immediate from the definition of a free Hopf ring. □

We can now give the following alternative proof of Theorem 3.1.4. For $n = 0$, $H_*(K(\mathbb{Q},0);\mathbb{Q})$ is the group Hopf algebra $\mathbb{Q}[\mathbb{Q}]$. Thus $H^*(K(\mathbb{Q},0);\mathbb{Q}) \cong D\mathbb{Q}[\mathbb{Q}] = \mathbf{Set}(\mathbb{Q},\mathbb{Q})$. For $n > 1$, $H_*(K(\mathbb{Q},n);\mathbb{Q})$ is the free $\mathbb{Q}$-algebra on a single primitive generator $e^{\circ n}$. Over $\mathbb{Q}$ these Hopf algebras are self-dual and hence $H^*(K(\mathbb{Q},n);\mathbb{Q})$ is the free $\mathbb{Q}$-algebra on a single primitive generator x_n which is dual to $e^{\circ n}$. Since $e^{\circ n} \circ e^{\circ m} = e^{\circ n+m}$ we see that $\Delta^{\times} x_n = \sum_{i+j=n} x_i \otimes x_j$. This shows we have an isomorphism of complete Hausdorff $\mathbb{Q}$-$\mathbb{Q}$-birings.

To unpack the composition, we first compute the augmentation ι_n as a functional $H_*(K(\mathbb{Q},n);\mathbb{Q}) \to \mathbb{Q}$. For $n > 0$ we take the basis of $H_*(K(\mathbb{Q},n);\mathbb{Q})$ given by powers of $e^{\circ n}$. Since $\varepsilon(e) = 0$, by Definition 1.3.21 we have $\langle \iota_n, (e^{\circ n})^{*k}\rangle = 1$ if $k = 1$ and 0 otherwise. Thus we see that $\iota_n = x_n$ for $n > 0$. For the case $n = 0$, by Definition 1.3.21 we see that ι_0 is the identity map in $\mathbf{Set}(\mathbb{Q},\mathbb{Q})$.

It remains to understand the composition in $D\mathbb{Q}[\mathbb{Q}]$. We take the usual basis of $\mathbb{Q}[\mathbb{Q}]$ consisting of $[\lambda]$ for $\lambda \in \mathbb{Q}$. Now for $r, s \in D\mathbb{Q}[\mathbb{Q}]$, we have $\langle r \circ s, [\lambda]\rangle = \langle r, s_*([\lambda])\rangle = \langle r, [s(\lambda)]\rangle$. This corresponds to the composition in $\mathbf{Set}(\mathbb{Q},\mathbb{Q})$. □

3.1.2 Singular cohomology with mod 2 coefficients

Another common choice of coefficient ring for singular cohomology is $\mathbb{F}_2$, the field with two elements. This is in some sense the simplest choice of coefficient ring where we have non-trivial cohomology operations and is thus a natural choice for our next plethystic computation.

Classical results of Steenrod [44] show that there exist stable operations on singular cohomology with coefficients in $\mathbb{F}_2$ which in some sense generalise the unstable operations given by squaring. These stable operations were given a convenient axiomatic characterisation in [45].

Theorem 3.1.8. *There exist stable operations for $n > 0$, called the* Steenrod squares*:*

$$Sq^n\colon H^*(X;\mathbb{F}_2) \to H^{*+n}(X;\mathbb{F}_2)$$

which can be uniquely characterised by the following axioms:

1. *Sq^n is a additive homomorphism $H^m(X,\mathbb{F}_2) \to H^{m+n}(X,\mathbb{F}_2)$ which is natural in X.*
2. *For $x \in H^n(X;\mathbb{F}_2)$, $Sq^n x = x^2$.*
3. *If $x \in H^*(X;\mathbb{F}_2)$ has $|x| < n$ then $Sq^n x = 0$.*
4. *We have the following* Cartan formula*. For all $x, y \in H^*(X,\mathbb{F}_2)$,*

$$Sq^n(xy) = \sum_{i+j=n} (Sq^i x)(Sq^j y). \tag{3.1}$$

Proof: [45, Chapter VII, Definition 6.1] provides a definition of the Steenrod squares and [45, Chapter VII, Theorem 6.7] shows they satisfy the desired axioms. A proof of uniqueness is given in [45, Chapter VIII, Theorem 3.10]. □

Definition 3.1.9. We define the mod 2 *Steenrod algebra* $\mathcal{A}_2$ to be the Hopf algebra of stable cohomology operations for singular cohomology over the field with two elements $\mathbb{F}_2$.

It is proved in [42] that the Steenrod squares generate the Steenrod algebra. However, the Steenrod algebra is far from free on these generators. The relations satisfied by the Steenrod squares are known as the Adem relations, conjectured in [55] before being proved by Adem in [6].

Theorem 3.1.10 ([6, Theorem 1.1]). *For all* $0 < n < 2m$*, we have the* Adem relations

$$Sq^n \circ Sq^m = \sum_{i=0}^{[n/2]} \binom{m-n-1}{n-2i} Sq^{n+m-i} \circ Sq^i \tag{3.2}$$

where the binomial coefficients are interpreted mod 2.

In 1953, Serre proved the following result describing the structure of $\mathcal{A}_2$.

Theorem 3.1.11 ([42]). *The Steenrod algebra* $\mathcal{A}_2$ *is the free graded algebra generated by* Sq^n $(n > 0)$ *in degree* n *quotiented by the ideal generated by the Adem relations* (3.2)*.*

The Hopf algebra structure is given by

$$\psi(Sq^n) = \sum_{i+j=n} Sq^i \otimes Sq^j,$$

$$\varepsilon(Sq^n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

where $Sq^0 = 1$*.*

To work with the Steenrod algebra, it will be useful to fix a choice of basis. There are numerous well-studied choices of basis for $\mathcal{A}_2$ and an in-depth discussion of these including change of basis formulae can be found in [37]. For our purposes we shall only need two of these choices of basis.

The Serre-Cartan basis When computing the structure of the mod 2 Steenrod algebra in [42], Serre found an explicit basis.

Definition 3.1.12. For any finite sequence $I = (i_1, \dots, i_n)$, we define $Sq^I = Sq^{i_1} \dots Sq^{i_n}$. We say a sequence I is *admissible* if for each t, $i_t \geq 2i_{t+1}$.

We define the *excess* of a sequence I by $e(I) = \sum i_t - 2i_{t+1}$. This quantity measures to what extent I exceeds the minimal requirement to be admissible.

Proposition 3.1.13 ([42]). *The set* $\{Sq^I \mid I \text{ is admissible}\}$ *is a basis for* $\mathcal{A}_2$*.*

Given two elements of the Serre-Cartan basis we do not have a closed form for their product, we must iteratively apply the Adem relations until we end up with a sum of Steenrod squares indexed by admissible sequences.

The Milnor basis While the multiplicative structure of the mod 2 Steenrod algebra is somewhat complex, the comultiplicative structure is straightforward. As a consequence, the dual Hopf algebra has a simple expression as an algebra due to Milnor.

Proposition 3.1.14 ([35])**.** *The linear dual of the Steenrod algebra, denoted $\mathcal{A}_2^*$, is the polynomial algebra $\mathbb{F}_2[\xi_1, \xi_2, \dots]$, where $|\xi_n| = 2^n - 1$.*

Definition 3.1.15. The dual basis to the monomial basis for $\mathcal{A}_2^*$ is known as the *Milnor basis*. We write $Sq(I)$ for the basis element dual to $\xi_1^{i_1} \dots \xi_n^{i_n}$.

We are able to express the product of Milnor basis elements by closed (albeit complex) formulae.

Definition 3.1.16. Consider a matrix of non-negative integers with all but finitely many entries zero.

$$X = \begin{bmatrix} * & x_{01} & x_{02} & \cdots \\ x_{10} & x_{11} & x_{12} & \cdots \\ x_{20} & x_{21} & x_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{3.3}$$

The first entry is never used and can be taken to be 0. Define sequences $I(X) = (i_1, \dots)$, $J(X) = (j_1, \dots)$, $T(X) = (t_1, \dots)$ and $b(X) \in \mathbb{Z}$ as follows.

$$\begin{aligned} i_r &= \sum_{s \geq 0} 2^s x_{rs} && \text{(weighted row sum)} \\ j_s &= \sum_{r \geq 0} x_{rs} && \text{(column sum)} \\ t_n &= \sum_{r+s=n} x_{rs} && \text{(diagonal sum)} \\ b(X) &= \frac{\prod_n t_n!}{\prod_{r,s} x_{rs}!} \end{aligned}$$

Note that $b(X) \in \mathbb{Z}$ as it is the product of multinomial coefficients.

Proposition 3.1.17 ([35, Theorem 4b])**.** *For sequences $I = (i_1, \dots)$ and $J = (j_1, \dots)$*

$$Sq(I)Sq(J) = \sum_X b(X) Sq(T(X))$$

where X ranges over all matrices of the form (3.3) *such that $I(X) = I$ and $J(X) = J$ and we take $b(X)$ modulo 2.*

A direct approach

We have sufficient knowledge of the cohomology operations for singular cohomology with $\mathbb{F}_2$ coefficients that we can directly understand the plethory of cohomology operations with a little work. In fact when proving the structure of the Steenrod algebra, Serre computes the mod 2 cohomology of the Eilenberg-MacLane spaces $K(\mathbb{F}_2, k)$ for each k.

Theorem 3.1.18 ([42, Théorème 2])**.** *For each $k \geq 0$, we have isomorphisms of $\mathbb{F}_2$-algebras*

$$H^*(K(\mathbb{F}_2, k); \mathbb{F}_2) \cong \mathbb{F}_2[\{Sq^I \iota_k : I \text{ is admissible and } e(I) < k\}] \tag{3.4}$$

where $\iota_k \in \widetilde{H}^k(K(\mathbb{F}_2, k); \mathbb{F}_2)$ is the universal class which represents the identity $K(\mathbb{F}_2, k) \to K(\mathbb{F}_2, k)$.

The condition $e(I) < k$ in the above theorem comes from the following result.

Lemma 3.1.19. *Let I be an admissible sequence.*

1. *If $e(I) > k$ then $Sq^I \iota_k = 0$.*
2. *If $e(I) = k$ then $Sq^I \iota_k = (Sq^J \iota_k)^{2^j}$ where j is the smallest integer such that $e(i_j, i_{j+1}, \dots) < k$ and $J = (i_j, i_{j+1}, \dots)$ with $e(J) < k$.*

Proof: [25, Lemma 1.33] provides a slightly weaker statement, but actually proves this result. □

By Theorem 1.2.29, the coalgebraic structure of our biring is equivalent to the action of the operations on the ring structure of (completed) cohomology algebras. As we saw in Proposition 2.2.20 the image of a stable operation under the stabilisation map $r \mapsto r\iota_k$ is primitive and thus all our generators for the $\mathbb{F}_2$-algebras in (3.4) are primitive. The action of the operations on products is encoded in the Cartan formula (3.1).

Lemma 3.1.20. *Fix a space X and any sequence I. For all $x, y \in H^*(X)$,*

$$Sq^I(xy) = \sum_{I' + I'' = I} Sq^{I'}(x) Sq^{I''}(y)$$

where the sum of two sequences I' and I'' is defined coordinate-wise after suitably appending zeros if necessary.

Proof: This is immediate from the Cartan formula and induction. □

Corollary 3.1.21. *In the $\mathbb{F}_2$-plethory $H^*(K(\mathbb{F}_2,\bullet);\mathbb{F}_2)$, we have*

$$\Delta^\times(Sq^I\iota_k) = \sum_{\substack{I'+I''=I\\k_1+k_2=k}} Sq^{I'}\iota_{k_1}\otimes Sq^{I''}\iota_{k_2},$$

$$\varepsilon^\times(Sq^I\iota_k) = \begin{cases}1 & I=0, k=0\\ 0 & \textit{otherwise.}\end{cases}$$

Proof: Apply Theorem 1.2.29 to Lemma 3.1.20. □

While this formula completely describes the comultiplicative structure of our plethory it is not in a closed form: we must repeatedly apply the Adem relations (3.2) to express the result in terms of our generators for $H^*(K(\mathbb{F}_2,k);\mathbb{F}_2)$ (see (3.4)). Noting that stably $Sq^I Sq^J = Sq^{i_1}\dots Sq^{i_n}Sq^{j_1}\dots Sq^{j_m}$ we see we will have similar issues when discussing the composition.

We can leverage our plethystic framework to provide a neater expression for the plethory of unstable cohomology operations. The intuition is that all our generators $Sq^I\iota_k$ in (3.4) are composites of 'atomic' elements $Sq^n\iota_k$ with $0<n<k$. However we note the elements $Sq^n\iota_n = \iota_n^2$ and $Sq^0\iota_k = \iota_k$ are non-zero.

Definition 3.1.22. Define the bigraded $\mathbb{F}_2$-$\mathbb{F}_2$-coalgebra $C_\bullet$ to consist of the $\mathbb{F}_2$-vector spaces

$$C_k = \langle Sq^n\iota_k \,|\, 0\le n\le k\rangle$$

where $\deg_*(Sq^n\iota_k) = k+n$. The comultiplicative structure is given by

$$\psi(Sq^n\iota_k) = \sum_{\substack{i+j=n\\k_1+k_2=k\\i<k_1, j<k_2}} Sq^i\iota_{k_1}\otimes Sq^j\iota_{k_2}$$

$$\varepsilon(Sq^n\iota_k) = \begin{cases}1 & \text{if } k=0 \text{ and } n=0,k\\ 0 & \text{otherwise.}\end{cases}$$

If we freely allow composition of the basis elements of $C_\bullet$ then we are clearly going to obtain all generators of $H^*(K(\mathbb{F}_2,\bullet);\mathbb{F}_2)$ as in (3.4). If we subject this composition to the unstable analogue of the Adem relations, then it turns out we obtain all the primitive elements of $H^*(K(\mathbb{F}_2,\bullet);\mathbb{F}_2)$. We remark that since $\mathbb{F}_2$ is in the centre of $PH^*(K(\mathbb{F}_2,k);\mathbb{F}_2)$ we have no twisting and $PH^*(K(\mathbb{F}_2,k);\mathbb{F}_2)$ is a cocommutative bialgebra under the composition and the comultiplication. Moreover, we are working over $\mathbb{F}_2$ and thus $PH^*(K(\mathbb{F}_2,k);\mathbb{F}_2)$ has the structure of an $\mathbb{F}_2[F]$ module where F denotes the Frobenius (see Definition 1.3.64).

Proposition 3.1.23. *As an $\mathbb{F}_2[F]$-module,*

$$PH^*(K(\mathbb{F}_2,k);\mathbb{F}_2) \cong \langle Sq^I\iota_k \,|\, I \text{ is admissible and } e(I) \le k\rangle.$$

The action of the Frobenius is given by $F(Sq^I\iota_k) = Sq^{I'}\iota_k$ where $I' = (k+|I|, i_1, \dots)$.

Proof: Given an arbitrary monomial m in the terms $Sq^I\iota_k$ where I is admissible and $e(I) < k$, we can see this is only primitive if and only if $m = Sq^I\iota_k$ or $m = F^n Sq^I\iota_k$ for some n. The result follows from Lemma 3.1.19. $\square$

The bialgebra structure on $PH^*(K(\mathbb{F}_2,k);\mathbb{F}_2)$ is induced by the Cartan formula (3.1) and the composition in the Steenrod algebra $\mathcal{A}_2$. Recall that the tensor algebra construction produces a functor $T\colon \mathbf{Coalg}_{\mathbb{F}_2,\mathbb{F}_2} \to \mathbf{TwBialg}_{\mathbb{F}_2}$.

Proposition 3.1.24. *Let $C_\bullet$ denote the $\mathbb{F}_2$-$\mathbb{F}_2$-coalgebra of Definition 3.1.22 and let $\mathscr{A}$ denote the two-sided ideal of the cocommutative bialgebra $T(C_\bullet)$ generated by the elements*

$$Sq^n\iota_{m+k}Sq^m\iota_k - \sum_{i=0}^{[\frac{n}{2}]}\binom{m-n-1}{n-2i}Sq^{n+m+i}\iota_{k+i}Sq^i\iota_k \text{ and } Sq^0\iota_k = 1_k$$

for $0 < n < 2m$ and for all $k \ge m$. Then $\mathscr{A}$ is a bialgebra ideal and we have a canonical isomorphism of bialgebras

$$\frac{T(C_\bullet)}{\mathscr{A}} \xrightarrow{\cong} PH^*(K(\mathbb{F}_2,\bullet);\mathbb{F}_2). \tag{3.5}$$

Moreover, if we equip $T(C_\bullet)$ with an $\mathbb{F}_2[F]$-module structure given by

$$F(Sq^{n_1}\iota_{k_1}\dots Sq^{n_t}\iota_{k_t}) = Sq^{k_1+n_1}\iota_{k_1+n_1}Sq^{n_1}\iota_{k_1}\dots Sq^{n_t}\iota_{k_t}$$

then this induces an $\mathbb{F}_2[F]$-module structure on $T(C_\bullet)/\mathscr{A}$ and makes (3.5) an isomorphism of $\mathbb{F}_2[F]$-modules.

Proof: The isomorphism of algebras is just an unstable version of the algebraic result in Proposition 3.1.13. By definition, the canonical map commutes with the comultiplication on the generators $Sq^n\iota_k$ and thus since $\Delta^\times$ is a map of algebras, our isomorphism is of bialgebras. Finally, by distributivity $F(\mathscr{A}) \subseteq \mathscr{A}$ and by construction the canonical map $T(C_\bullet)/\mathscr{A} \to PH^*(K(\mathbb{F}_2,\bullet);\mathbb{F}_2)$ is a map of $\mathbb{F}_2[F]$-modules. $\square$

Now we have an expression for the primitive elements we can invoke Theorem 1.3.70 and make precise the heuristic that the additive operations freely generate the unstable operations in $H^*(K(\mathbb{F}_2,\bullet);\mathbb{F}_2)$. Recall the functor $S^{[2]}\colon \mathbf{Bialg}_{\mathbb{F}_2}^{\neg\mathrm{com}} \to \mathbf{Plethory}_{\mathbb{F}_2}$ of Definition 1.3.69.

Theorem 3.1.25. *We have isomorphisms of $\mathbb{F}_2$-plethories*

$$H^*(K(\mathbb{F}_2,\bullet);\mathbb{F}_2) \cong S^{[2]}\left(\frac{T(C_\bullet)}{\mathscr{A}}\right) \cong \frac{S^{[2]}T(C_\bullet)}{(\mathscr{A})}$$

where $C_\bullet$ is the $\mathbb{F}_2$-$\mathbb{F}_2$-coalgebra of Definition 3.1.22, $\mathscr{A}$ is the ideal of Proposition 3.1.24 and $(\mathscr{A})$ is the algebra ideal of $S^{[2]}T(C_\bullet)$ generated by the elements of $\mathscr{A}$.

Proof: Since $H^*(K(\mathbb{F}_2,\bullet);\mathbb{F}_2)$ is primitively generated, the Verschiebung is zero. Hence by Theorem 1.3.70 we immediately have the first isomorphism. The second isomorphism is immediate from properties of the symmetric algebra. $\square$

Via the enriched Hopf Ring

The Hopf ring of homology cooperations $H_*(K(\mathbb{F}_2,\bullet);\mathbb{F}_2)$ is well understood and has a relatively simple form. We can leverage our knowledge of this together with our duality results to compute an alternative expression for the plethory $H^*(K(\mathbb{F}_2,\bullet);\mathbb{F}_2)$. We first recall standard results about the Hopf algebra $H_*(K(\mathbb{F}_2,1);\mathbb{F}_2)$.

The space $\mathbb{R}P^\infty$ is a $K(\mathbb{F}_2,1)$. Moreover, $\mathbb{R}P^\infty$ admits an H-space structure which can be realised analogously to the H-space structure on $\mathbb{C}P^\infty$ (see Section 2.3) and this models the loop space structure on $K(\mathbb{F}_2,1)$.

The homology $H_*(\mathbb{R}P^\infty;\mathbb{F}_2)$ is the free $\mathbb{F}_2$-vector space on the basis $c_0, c_1, \dots$ where $|c_i| = i$. The coalgebra structure is given by $\psi(c_n) = \sum_{i+j=n} c_i \otimes c_j$ and $\varepsilon(c_n) = 1$ if $n = 0$ and 0 otherwise. The algebra structure induced by the H-space structure is given by $c_i c_j = \binom{i+j}{i} c_{i+j}$. The following result shows that $H_*(\mathbb{R}P^\infty;\mathbb{F}_2)$ is generated just by the *accelerated elements* $c_{(i)} = c_{2^i} \in H_{2^i}(\mathbb{R}P^\infty;\mathbb{F}_2)$.

Proposition 3.1.26. *We have an isomorphism of $\mathbb{F}_2$-algebras*

$$H_*(K(\mathbb{F}_2,1);\mathbb{F}_2) \cong \frac{\mathbb{F}_2[c_{(0)}, c_{(1)}, \dots]}{(c_{(0)}^2, c_{(1)}^2, \dots)}$$

where $|c_{(i)}| = 2^i$. Moreover, if we give $\mathbb{F}_2[c_{(0)}, c_{(1)}, \dots]/(c_{(i)}^2)$ the coalgebra structure induced by $H_*(K(\mathbb{F}_2,1);\mathbb{F}_2)$ then $Vc_{(i)} = c_{(i-1)}$ for $i \geq 0$ and $Vc_{(0)} = 0$.

Proof: Define a map of algebras $\phi\colon \mathbb{F}_2[c_{(0)}, c_{(1)}, \dots] \to H_*(K(\mathbb{F}_2,1);\mathbb{F}_2)$ by $\phi(c_{(i)}) = c_{(i)} = c_{2^i}$. Now $\phi(c_{(i)}^2) = c_{2^i}^2 = \binom{2^{i+1}}{2^i} c_{2^{i+1}} = 0$ since $\binom{2^{i+1}}{2^i} \equiv 0 \mod 2$. Thus ϕ factors via a map $\mathbb{F}_2[c_{(0)}, c_{(1)}, \dots]/(c_{(i)}^2) \to H_*(K(\mathbb{F}_2,1);\mathbb{F}_2)$. To see this is an isomorphism we note that every integer $n \geq 0$ has a *unique* binary expansion $n = \sum_i n_i 2^i$ where $n_i \in \{0,1\}$ and $\phi(c_{(0)}^{n_0} c_{(1)}^{n_1} \dots) = c_{(0)}^{n_0} c_{(1)}^{n_1} \cdots = c_n$. $\square$

This result allows an extremely succinct expression of the Hopf ring of homology cooperations.

Theorem 3.1.27 ([54, Page 10])**.** *The Hopf ring $H_*(K(\mathbb{F}_2, \bullet); \mathbb{F}_2)$ is the free $\mathbb{F}_2[\mathbb{F}_2]$-Hopf ring on $H_*(K(\mathbb{F}_2, 1); \mathbb{F}_2)$.*

The advantage of this expression over the direct approach (Section 3.1.1) is that we will no longer have to deal with admissible sequences: all $\circ$-products of the elements $c_{(i)}$ are algebra generators.

Definition 3.1.28. We define the set of *multi-indices* to be the union

$$\bigcup_{n \geq 0} \mathbb{N}^{\times n}$$

where the inclusion maps are given by $(i_0, \dots, i_{n-1}) \mapsto (i_0, \dots, i_{n-1}, 0)$. Hence a multi-index is a sequence of non-negative integers $I = (i_0, i_1, \dots)$ with all but finitely many i_n zero. Since each $\mathbb{N}^{\times n}$ is an abelian monoid under the point-wise operations, so is the union.

We shall write 0 for the image of the map $\mathbb{N}^{\times 0} \to \bigcup_n \mathbb{N}^{\times n}$ and Δ_n for the the image of $(0, \dots, 0, 1) \in \mathbb{N}^{\times n+1}$ under the canonical inclusion to $\bigcup_n \mathbb{N}^{\times n}$.

Given a multi-index I, define the *length* by $|I| = \sum_n i_n$ and for $p \in \mathbb{Z}$ the *p-weighted length* by $|I|_p = \sum_n p^n i_n$.

For a multi-index I, define $s(I) = (0, i_0, i_1, \dots)$. If $i_0 = 0$ define $s^{-1}(I) = (i_1, i_2, \dots)$.

For any multi-index $I = (i_0, \dots, i_n)$, we define $c^{\circ I} = c_{(0)}^{\circ i_0} \circ \cdots \circ c_{(n)}^{\circ i_n} \in H_{|I|_2}(K(\mathbb{F}_2, |I|); \mathbb{F}_2)$. We make the convention that if $I = 0$ then $c^{\circ I} = [1] - [0]$. Over $\mathbb{F}_2$ we have $[1] - [0] = [1] + [0]$ but this convention will generalise to the case over fields of characteristic other than 2. Since $[0] \circ c_{(i)} = \varepsilon c_{(i)} = 0$, on the augmentation ideal the element $[1] - [0]$ retains the property of being the $\circ$-unit but has two major advantages over $[1]$: it squares to zero allowing a familiar expression of the algebra structure and the dual of $[1] - [0]$ is the identity operation on cohomology.

To compute the dual of the Hopf ring of homology cooperations, it proves useful to unpack the information into a 'local' form: a description of the Hopf algebras in each spacial degree.

Corollary 3.1.29. *We have isomorphisms of $\mathbb{F}_2$-algebras for all n*

$$H_*(K(\mathbb{F}_2, n); \mathbb{F}_2) = \bigotimes_I \Lambda_{\mathbb{F}_2}[c^{\circ I}]$$

where I ranges over all multi-indices with $|I| = n$.

Proof: The Hopf algebra $H_*(K(\mathbb{F}_2, n); \mathbb{F}_2)$ consists of $*$-products of $\circ$-products of our generators. Hence $H_*(K(\mathbb{F}_2, n); \mathbb{F}_2)$ is generated by $c^{\circ I}$ for all I with $|I| = n$. The relations follow from Frobenius reciprocity (see Proposition E.0.42). $\square$

We can recover the comultiplicative structure on $H_*(K(\mathbb{F}_2, n); \mathbb{F}_2)$ from Proposition 3.1.26 since the comultiplication ψ respects $\circ$-multiplication. Unfortunately we have no closed form for the comultiplication on our generators $c^{\circ I}$ since we must repeatedly rewrite elements in terms of the accelerated elements and apply the distributive laws of a Hopf ring. Nevertheless, we have a nice closed form for the Verschiebung on our generators. Since V respects $\circ$-multiplication, we have $Vc^{\circ I} = 0$ for $i_0 \neq 0$ and $Vc^{\circ I} = c^{s^{-1}I}$ for $i_0 = 0$.

Consider the monomial basis for $H_*(K(\mathbb{F}_2, n); \mathbb{F}_2)$ i.e. the basis consisting of elements of the form $c^{\circ I_1} * \cdots * c^{\circ I_N}$ with $|I_t| = n$. Let $x_I \in H^*(K(\mathbb{F}_2, |I|); \mathbb{F}_2) \cong DH_*(K(\mathbb{F}_2, |I|); \mathbb{F}_2)$ denote the element of the dual basis which is dual to $c^{\circ I}$. The other elements of the dual basis will remain anonymous.

Theorem 3.1.30. *Let $B_\bullet$ be the $\mathbb{F}_2$-$\mathbb{F}_2$-biring with $\bullet$-components given by*

$$B_n = \begin{cases} \mathbf{Set}(\mathbb{F}_2, \mathbb{F}_2) & n = 0 \\ \bigotimes\limits_{\substack{|I|=n \\ i_0>0}} \mathbb{F}_2[x_I] & n > 0 \end{cases} \tag{3.6}$$

where $|x_I| = |I|_2$, $\mathbf{Set}(\mathbb{F}_2, \mathbb{F}_2)$ has the biring structure of Example 1.2.3 and the remaining biring structure is specified by the following formulae.

$$\begin{aligned} \Delta^+(x_I) &= 1 \otimes x_I + x_I \otimes 1 \\ \sigma(x_I) &= x_I \\ \varepsilon^+(x_I) &= 0 \\ \Delta^\times(x_I) &= \sum_{I'+I''=I} x_{I'} \otimes x_{I''} \\ \varepsilon^\times(x_I) &= \begin{cases} 1 & I = 0 \\ 0 & \textit{otherwise.} \end{cases} \end{aligned}$$

We have an isomorphism of $\mathbb{F}_2$-$\mathbb{F}_2$-birings $H^*(K(\mathbb{F}_2, n); \mathbb{F}_2) \cong B_\bullet$.

Proof: For $n > 0$, $\pi_0 K(\mathbb{F}_2, n) = 0$ and thus $H_*(K(\mathbb{F}_2, n); \mathbb{F}_2)$ is a connected Hopf algebra. By Corollary 3.1.29 we see the Frobenius is trivial. Hence, by Proposition E.0.37, $PH_*(K(\mathbb{F}_2, n); \mathbb{F}_2) \subseteq QH_*(K(\mathbb{F}_2, n); \mathbb{F}_2) = \bigoplus_I \langle c^{\circ I} \rangle$. Since $c_{(0)}$ is primitive, by Proposition E.0.38, $c^{\circ I}$ is primitive whenever $i_0 > 0$. However, if $i_0 = 0$ then $V(c^{\circ I}) = c^{\circ s^{-1} I} \neq 0$. Hence $PH_*(K(\mathbb{F}_2, n); \mathbb{F}_2) = \bigoplus_{i_0 > 0} \langle c^{\circ I} \rangle$.

Now by Proposition E.0.39, $QH^*(K(\mathbb{F}_2, n); \mathbb{F}_2) = DPH_*(K(\mathbb{F}_2, n); \mathbb{F}_2) = \bigoplus_{i_0>0} \langle x_I \rangle$. Now, $\langle F^k x_I, c^{\circ s^k I} \rangle = \langle x_I, V^k c^{\circ s^k I} \rangle = \langle x_I, c^{\circ I} \rangle = 1$. Thus $F^k x_I \neq 0$ for any k. By Borel's structure theorem (Theorem E.0.40), $H^*(K(\mathbb{F}_2, n); \mathbb{F}_2)$ is isomorphic as an $\mathbb{F}_2$-algebra to a tensor product of monogenic Hopf algebras. Since we have no nilpotent elements, these must all be polynomial algebras and so the only possible relations between our generators would produce redundant generators. Hence as $\mathbb{F}_2$-algebras, $H^*(K(\mathbb{F}_2, n); \mathbb{F}_2) \cong \bigotimes_{i_0>0} \mathbb{F}_2[x_I]$. Since $PH^*(K(\mathbb{F}_2, n); \mathbb{F}_2) = DQH_*(K(\mathbb{F}_2, n); \mathbb{F}_2)$, we see that each x_I is primitive.

For the case when $n = 0$, $H_*(K(\mathbb{F}_2, 0); \mathbb{F}_2)$ is the Hopf algebra $\Lambda[y]$ where $y = [1] - [0]$. This has basis $[0] = 1, y$ and $\psi(y) = [1] \otimes [1] - [0] \otimes [0] = (y+1) \otimes (y+1) - 1 \otimes 1 = y \otimes y + 1 \otimes y + y \otimes 1$. Hence $H^*(K(\mathbb{F}_2, 0); \mathbb{F}_2) \cong \mathbb{F}_2[x_0]/(x_0^2 - x_0)$ where x_0 is dual to y. However, this is isomorphic to $\mathbf{Set}(\mathbb{F}_2, \mathbb{F}_2)$ via the isomorphism sending x_0 to the identity map and 1 to the constant map 1.

For the comultiplicative structure, we remark that $x_I \in PH^*(K(\mathbb{F}_2, n); \mathbb{F}_2) \cong DQH_*(K(\mathbb{F}_2, n); \mathbb{F}_2)$ and thus $\langle \Delta^\times x_I, u \otimes v \rangle = \langle x_I, u \circ v \rangle = 0$ unless $u \circ v \in QH_*(K(\mathbb{F}_2, n); \mathbb{F}_2)$. By Proposition E.0.41, we see $u \circ v \in QH_*(K(\mathbb{F}_2, n); \mathbb{F}_2)$ implies $u, v \in QH_*(K(\mathbb{F}_2, n); \mathbb{F}_2)$. By linearity it suffices to determine $\langle \Delta^\times x_I, c^{\circ I'} \otimes c^{\circ I''} \rangle = \langle x_I, c^{\circ I'} \circ c^{\circ I''} \rangle = \langle x_I, c^{\circ I' + I''} \rangle$. The result for $\Delta^\times$ follows. Finally, we have $\langle \varepsilon^\times x_I, 1 \rangle = \langle x_I, [1] \rangle = 1$ if and only if $I = 0$. □

To compute the composition, we must understand the enrichment of the Hopf ring which is most cleanly expressed formally. Hence, we let $c(t) = \sum_i c_i t^i \in H_*(K(\mathbb{F}_2, 1); \mathbb{F}_2)[[t]]$.

Theorem 3.1.31 ([15, Proposition 17.7])**.** *For all $r \in DH_*(K(\mathbb{F}_2, \bullet); \mathbb{F}_2) \cong H^*(K(\mathbb{F}_2, \bullet); \mathbb{F}_2)$, $r_* c_k$, the induced map applied to c_k, is the coefficient of t^k in the formal identity in $H_*(K(\mathbb{F}_2, \bullet); \mathbb{F}_2)[[t]]$*

$$r_* c(t) = \mathop{\ast}_{j \geq 0} c(t)^{\circ j} \circ [\langle r, c_j \rangle].$$

The augmentation is determined by

$$\iota_1(c_{(i)}) = \begin{cases} 1 & i = 0 \\ 0 & \textit{otherwise} \end{cases}$$

together with Definition 1.3.21.

It is straightforward to deduce the unit for the composition and this illuminates our choice of convention for the empty $\circ$-product $c^{\circ 0} = [1] - [0]$.

Corollary 3.1.32. *The canonical map $\mathcal{I} \to H^*(K(\mathbb{F}_2, \bullet))$ is given by $e_n \mapsto x_{n\Delta_0}$.*

Proof: For $n > 0$, since $\iota_n(x \circ y) = \sum_{r+s=n} \iota_r(x)\iota_s(y)$, we have $\iota_n(c^{\circ I}) = 1$ if and only if $I = n\Delta_0$. For the case $n = 0$, we note that $H_*(K(\mathbb{F}_2, 0); \mathbb{F}_2) = \mathbb{F}_2[\mathbb{F}_2]$. We take the basis $[0], [1] - [0]$ for $\mathbb{F}_2[\mathbb{F}_2]$ and x_0 is dual to $[1] - [0]$. Thus $\langle x_0, [0]\rangle = 0$ and $\langle x_0, [1]\rangle = \langle x_0, [1] - [0]\rangle + \langle x_0, [0]\rangle = 1$. Therefore $x_0 = \iota_0$. $\square$

To understand the composition in our plethory, we must compute $x_I \circ x_J$ for all our generators in (3.6). By Definition 1.3.21, this is equivalent to computing $\langle x_J, (x_I)_* u\rangle$ for all $u \in H_*(K(\mathbb{F}_2, \bullet); \mathbb{F}_2)$. It is straightforward to compute the induced maps on the generators of our Hopf ring.

Corollary 3.1.33. *For all multi-indices I, we have*

$$x_{I*}c_{(k)} = \begin{cases} c_{(k-i)}^{\circ 2^i} & I = \Delta_i, k \geq i \\ 0 & \textit{otherwise.} \end{cases}$$

Proof: For degree reasons, $x_{I*}c_{(k)} = 0$ unless $|I| = 1$ and so $I = \Delta_i$ for some i. Since $x_{\Delta_i} \in PH^*(K(\mathbb{F}_2, 1); \mathbb{F}_2) \cong DQH_*(K(\mathbb{F}_2, 1); \mathbb{F}_2)$, $\langle x_{\Delta i}, c_j\rangle = 0$ unless $c_j \in QH_*(K(\mathbb{F}_2, 1); \mathbb{F}_2) = \langle c_{(0)}, c_{(1)}, \ldots\rangle$. Since x_{Δ_i} is dual to $c_{(i)}$,

$$\langle x_{\Delta_i}, c_j\rangle = \begin{cases} 1 & j = 2^i \\ 0 & \text{otherwise.} \end{cases}$$

Hence from Theorem 3.1.31, $x_{\Delta_i *}c_k$ is the coefficient of t^k in $c(t)^{\circ 2^i}$. Since we are working over $\mathbb{F}_2$, the map $z \mapsto z^{\circ 2}$ is linear and hence $c(t)^{\circ 2^i} = \sum_j c_j^{\circ 2^i} t^{j2^i}$. Thus,

$$x_{\Delta_i *}c_k = \begin{cases} c_{\frac{k}{2^i}}^{\circ 2^i} & 2^i | k \\ 0 & \text{otherwise.} \end{cases}$$

$\square$

To compute the induced maps on general elements of our Hopf ring, we can apply the formulae in Definition 1.3.21. Unfortunately these expressions get unworkable very quickly. Fortunately there are nice combinatorial results which encode the formulae we need. These formulae are almost identical to Definition 3.1.16 encoding the multiplication of the stable Milnor basis elements, but contain a small extra bit of information able to keep the track of the unstable $\bullet$-degrees.

Definition 3.1.34. Consider a matrix of non-negative integers with all but finitely many entries zero.

$$X = \begin{bmatrix} x_{00} & x_{01} & x_{02} & \cdots \\ x_{10} & x_{11} & x_{12} & \cdots \\ x_{20} & x_{21} & x_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{3.7}$$

Define multi-indices $I(X) = (i_0, i_1, \dots)$, $J(X) = (j_0, j_1, \dots)$, $T(X) = (t_0, t_1, \dots)$ and a coefficient $b(X) \in \mathbb{Z}$ by

$$\begin{aligned} i_r &= \sum_{s\geq 0} 2^s x_{rs} && \text{(weighted row sum)} \\ j_s &= \sum_{r\geq 0} x_{rs} && \text{(column sum)} \\ t_n &= \sum_{r+s=n} x_{rs} && \text{(diagonal sum)} \\ b(X) &= \frac{\prod_n t_n!}{\prod_{rs} x_{rs}!} \end{aligned}$$

Example 3.1.35. Consider the following matrix.

$$X = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We have

$$\begin{aligned} I(X) &= (17, 38, 59) \\ J(X) &= (12, 15, 18) \\ T(X) &= (1, 6, 15, 14, 9) \\ b(X) &\equiv 0 \mod 2. \end{aligned}$$

Notation 3.1.36. For a multi-index $I = (i_0, \dots, i_n)$ we have $c^{\circ I} = c_{(0)}^{\circ i_0} \circ \dots c_{(n)}^{\circ i_n}$. A different way of discussing such $\circ$-products proves useful: for $1 \leq t \leq |I|$, define $\tilde{i}_t = \min\{k \,|\, t \leq \sum_{j=0}^{k} i_j\}$. Conversely, any collection of non-negative integers $\tilde{i}_1, \dots, \tilde{i}_N$ defines a multi-index $I = (i_0, i_1, \dots)$ by defining $i_n = \#\{t \,|\, \tilde{i}_t = n\}$.

For example, if $I = (1, 2, 0, 0, 3)$ then $c^{\circ I} = c_{(0)} \circ c_{(1)}^{\circ 2} \circ c_{(4)}^{\circ 3} = c_{(0)} \circ c_{(1)} \circ c_{(1)} \circ c_{(4)} \circ c_{(4)} \circ c_{(4)}$ and thus $\tilde{i}_1 = 0$, $\tilde{i}_2 = \tilde{i}_3 = 1$ and $\tilde{i}_4 = \tilde{i}_5 = \tilde{i}_6 = 4$.

Theorem 3.1.37. *For all multi-indices J, K with $|J| = |K|$, we have*

$$(x_J)_* c^{\circ K} = \sum_X b(X) c^{\circ I(X)}$$

where X ranges over all matrices of the form (3.7) *such that $J(X) = J$ and $T(X) = K$ and we take $b(X)$ modulo 2.*

Proof: If $c^{\circ K} = c_{(\tilde{k}_1)} \circ \dots \circ c_{(\tilde{k}_N)}$ then since x_J is primitive, by Definition 1.3.21 we have

$$x_{J*}(c^{\circ K}) = \sum_{J_1+\dots+J_N=J} x_{J_1*} c_{(\tilde{k}_1)} \circ \dots \circ x_{J_N*} c_{(\tilde{k}_N)}.$$

For degree reasons $x_{J*}c_{(i)} = 0$ unless $|J| = 1$ and hence any non-zero summand must have $J_t = \Delta_{\tilde{j}_t}$ for some $\tilde{j}_t$. By Corollary 3.1.33 we also need $\tilde{j}_t \leq \tilde{k}_t$ for this summand to be non-zero. For such a non-zero summand we have

$$x_{J_1*}c_{(\tilde{k}_1)} \circ \cdots \circ x_{J_N*}c_{(\tilde{k}_N)} = c^{\circ 2^{\tilde{j}_1}}_{(\tilde{k}_1 - \tilde{j}_1)} \circ \cdots \circ c^{\circ 2^{\tilde{j}_N}}_{(\tilde{k}_n - \tilde{j}_n)}.$$

We define $I = (i_0, i_1, \dots)$ to be the multi-index such that the above expression is $c^{\circ I}$. Explicitly, $i_n = \sum_{t \in \mathcal{T}_n} 2^{\tilde{j}_t}$ where $\mathcal{T}_n = \{t \,|\, \tilde{k}_t - \tilde{j}_t = n\}$. We claim $I = I(X)$ for some matrix X with $J(X) = J$ and $T(X) = K$. Define $X = (x_{rs})$ by

$$x_{rs} = \#\{t \colon \tilde{k}_t = r + s, \tilde{j}_t = s\}.$$

In words, we are starting with a matrix of zeros and for each t incrementing the entry in the $\tilde{k}_t$-th diagonal and $\tilde{j}_t$-th column by 1. For example, if $J = (1, 2)$ and $K = (1, 1, 1)$ then a non-zero summand could have $\tilde{j}_1 = 0, \tilde{j}_2 = \tilde{j}_3 = 1$, $\tilde{k}_1 = 0, \tilde{k}_2 = 1, \tilde{k}_3 = 2$ and would produce the matrix

$$X = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Notice $\sum_r x_{rs} = \#\left\{t \colon \tilde{j}_t = s\right\} = j_s$ and $\sum_{r+s=n} x_{rs} = \#\left\{t \colon \tilde{k}_t = n\right\} = k_n$ so $J(X) = J$ and $T(X) = K$. Moreover,

$$\begin{aligned} I(X)_r &= \sum_s 2^s x_{rs} \\ &= \sum_s 2^s \# \left\{t \colon \tilde{k}_t - \tilde{j}_t = r, \tilde{j}_t = s\right\} \\ &= \sum_{t \in \mathcal{T}_r} 2^{\tilde{j}_t} = i_r. \end{aligned}$$

Hence each summand is $c^{\circ I(X)}$ for some matrix X with $J(X) = J, T(X) = K$.

Conversely, let X be a matrix with $J(X) = J$ and $T(X) = K$. Suppose $I(X) = (i_0, i_1, \dots)$. Write $X = \sum_{t=1}^N E_{r_t s_t}$ where E_{rs} is the single-entry matrix with a one in row r and column s, and zeros elsewhere and (r_t, s_t) are such that $r_t + s_t = \tilde{k}_t$. In words, we start with the matrix X and pick a non-zero row and column index in the $\tilde{k}_t$-th diagonal. We then decrement that entry and repeat until we end up with the zero matrix. As an example, suppose $J = (1, 2)$, $K = (1, 1, 1)$ and

$$X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Since $\tilde{k}_1 = 0, \tilde{k}_2 = 1, \tilde{k}_3 = 2$, we have the decomposition $X = E_{00} + E_{01} + E_{11}$. Set $\tilde{j}_t = s_t$. Since $j_s = \sum_r x_{rs}$, we have $J = \Delta_{\tilde{j}_1} + \cdots + \Delta_{\tilde{j}_N}$. Now,

$$\begin{aligned} i_r &= \sum_s 2^s x_{rs} \\ &= \sum_s 2^s \# \{t \colon r_t = r, s_t = s\} \\ &= \sum_s 2^s \# \left\{t \colon r_t = \tilde{k}_t - \tilde{j}_t, \tilde{j}_t = s\right\} \\ &= \sum_{t \in \mathcal{T}_r} 2^{\tilde{j}_t} \end{aligned}$$

Hence $c^{\circ I(X)} = c^{\circ 2^{\tilde{j}_1}}_{(\tilde{k}_1 - \tilde{j}_1)} \circ \cdots \circ c^{\circ 2^{\tilde{j}_N}}_{(\tilde{k}_n - \tilde{j}_n)}$, and we see each matrix with $J(X) = J$ and $T(X) = K$ gives rise to a summand of $x_{J*}(c^{\circ K})$.

However, we made choices when defining our $\tilde{j}_t$'s. In choosing the components from the n-th diagonal, we have $(\sum_{r+s=n} x_{rs})! = t_n!$ choices. However, if picking two components from the same row and column this produces the same summand regardless of the order. So, in total, we have $\prod t_n! / \prod x_{rs}! = b(X)$ choices.

Therefore,

$$x_{J*} c^{\circ K} = \sum_{\substack{X \\ J(X)=J \\ T(X)=K}} b(X) c^{\circ I(X)}.$$

□

Theorem 3.1.38. *We have an isomorphism of $\mathbb{F}_2$-plethories*

$$H^*(K(\mathbb{F}_2, n); \mathbb{F}_2) \cong \begin{cases} \mathbf{Set}(\mathbb{F}_2, \mathbb{F}_2) & n = 0 \\ \bigotimes\limits_{\substack{|I|=n \\ i_0>0}} \mathbb{F}_2[x_I] & n > 0 \end{cases}$$

where the biring structure is given in Theorem 3.1.30 and the plethystic structure is specified by the following formulae.

$$x_I \circ x_J = \sum_X b(X) x_{T(X)}$$

$$\iota_n = x_{n\Delta_0}.$$

The sum is taken over matrices X as in (3.7) with $I(X) = I, J(X) = J$ and we take $b(X)$ modulo 2.

Proof: The composition $x_I \circ x_J$ is primitive and thus $\langle x_I \circ x_J, u\rangle = 0$ for all indecomposable $u \in H_*(K(\mathbb{F}_2;n);\mathbb{F}_2)$. Moreover from Theorem 3.1.37 we have,

$$\begin{aligned}\langle x_I \circ x_J, c^{\circ K}\rangle &= \langle x_I, x_{J*}c^{\circ K}\rangle \\ &= \langle x_I, \sum_{\substack{X \\ J(X)=J \\ T(X)=K}} b(X)c^{\circ I(X)}\rangle \\ &= \sum_{\substack{X \\ I(X)=I \\ J(X)=J \\ T(X)=K}} b(X).\end{aligned}$$

Hence, $x_I \circ x_J = \sum_X b(X)x_{T(X)}$. □

Example 3.1.39. Suppose $I = (0,1,2)$, $J = (1,1)$ then the only matrix X (up to adding zero rows and columns) with $I(X) = I, J(X) = J$ is

$$X = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

We have $T(X) = (0,1,0,1)$ and $b(X) = 1$. Therefore, $x_I \circ x_J = x_{(0,1,0,1)}$.

Example 3.1.40. Suppose $I = (4,2)$, $J = (2,2)$ then the matrices X_i with $I(X_i) = I, J(X_i) = J$ are

$$X_1 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } X_2 = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}.$$

We have $T(X_1) = (2,1,1)$, $b(X_1) \equiv 1 \mod 2$, $T(X_2) = (0,4,0)$ and $b(X_2) \equiv 0 \mod 2$. Therefore, $x_I \circ x_J = x_{(2,1,1)}$.

This formula for composition is very similar to the product formula for the Milnor basis. This is because we have $x_I = Sq(i_1, i_2, \dots)\iota_{|I|}$ in terms of the Milnor basis. The extra term in our matrix (3.7) simply allows us to keep track of degrees.

3.1.3 Singular cohomology with coefficients in $\mathbb{F}_p$

While a powerful invariant of homotopy classes of spaces, singular cohomology with coefficients in $\mathbb{F}_2$ is not complete, even when we account for the action of the plethory of unstable cohomology operations. For certain spaces we can obtain greater insight by studying singular cohomology over other coefficient rings. In this section we study singular cohomology with $\mathbb{F}_p$ coefficients for an odd prime p. Much of the theory over $\mathbb{F}_p$ is analogous to the $\mathbb{F}_2$ setting and so we be will somewhat brief with our results, referring to the $\mathbb{F}_2$ setting where possible. The Steenrod squares have a natural generalisation, which is a stable refinement of the p-th power map. Throughout this section, we fix an odd prime p.

Theorem 3.1.41 ([45]). *There exist stable operations for $n > 0$, called the* Steenrod powers*:*

$$P^n \colon H^*(X; \mathbb{F}_2) \to H^{*+2n(p-1)}(X; \mathbb{F}_2)$$

which can be uniquely characterised by the following axioms:

1. *P^n is a additive homomorphism $H^m(X, \mathbb{F}_2) \to H^{m+2n(p-1)}(X, \mathbb{F}_2)$ which is natural in X.*
2. *$P^n x = x^p$ whenever $|x| = n$.*
3. *If $|x| < n$ then $P^n x = 0$.*
4. *We have the* Cartan formula*,*

$$P^n(xy) = \sum_{i+j=n} (P^i x)(P^j y). \tag{3.8}$$

Definition 3.1.42. We define the mod p *Steenrod algebra* $\mathcal{A}_p$ to be the Hopf algebra of stable cohomology operations for singular cohomology over the field $\mathbb{F}_p$.

Unlike in the mod 2 situation, the Steenrod powers do not generate the Steenrod algebra. We need one additional operation, which arises from a more general context.

Definition 3.1.43. Suppose we have a short exact sequence of abelian groups,

$$0 \to A \to B \to C \to 0.$$

If we take the tensor product with $C^\bullet(X)$, the singular cochains in X, then the snake lemma yields group homomorphisms $H^n(X; C) \to H^{n+1}(X; A)$ for all n. These maps are natural in X and are known as the *Bockstein homomorphisms.*

Definition 3.1.44. We define the *Bockstein operation* $\beta \colon H^*(-; \mathbb{F}_p) \to H^{*+1}(-; \mathbb{F}_p)$ to be the Bockstein homomorphism arising from the following short exact sequence of abelian groups.

$$0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0$$

Proposition 3.1.45 ([45]). *The Bockstein operation $\beta \colon H^*(-; \mathbb{F}_p) \to H^{*+1}(-; \mathbb{F}_p)$ is stable.*

The Adem relations in $\mathcal{A}_p$ take the following form.

Theorem 3.1.46 ([6])**.** *In $\mathcal{A}_p$ we have, for $n < pm$,*

$$\begin{aligned} \beta^2 &= 0 \\ P^n P^m &= \sum_i (-1)^{n+i} \binom{(p-1)(m-i)-1}{n-pi} P^{n+m-i} P^i \end{aligned}$$

and for $n \leq pm$,

$$\begin{aligned} P^n \beta P^m =& \sum_i (-1)^{n+i} \binom{(p-1)(m-i)}{n-pi} \beta P^{n+m-i} P^i \\ &+ \sum_i (-1)^{n+i+1} \binom{(p-1)(m-i)-1}{n-pi-1} P^{n+m-i} \beta P^i \end{aligned}$$

where the signed binomial coefficients are taken modulo p.

The following result of Cartan shows that the Steenrod powers together the Bockstein operation generate the mod p Steenrod algebra $\mathcal{A}_p$.

Theorem 3.1.47 ([22])**.** *The Steenrod algebra $\mathcal{A}_p$ is the free graded algebra generated by P^n $(n > 0)$ in degree n and β in degree 1 subject to the Adem relations (Theorem 3.1.46).*

Writing $P^0 = 1$, the Hopf algebra structure is given by

$$\begin{aligned} \psi(P^n) &= \sum_{i+j=n} P^i \otimes P^j, \\ \psi(\beta) &= 1 \otimes \beta + \beta \otimes 1, \\ \varepsilon(P^n) &= \varepsilon(\beta) = 0. \end{aligned}$$

The Serre-Cartan and Milnor bases both have generalisations to the mod p case.

The Serre-Cartan basis In the mod p setting, the Serre-Cartan basis generalises the notion of an admissible sequence to incorporate the Bockstein operation.

Definition 3.1.48. For any finite sequence of non-negative integers $I = (\varepsilon_0, i_1, \varepsilon_1, \ldots, i_n, \varepsilon_n)$ with $\varepsilon_t \in \{0, 1\}$, we define $P^I = \beta^{\varepsilon_0} P^{i_1} \beta^{\varepsilon_1} \ldots P^{i_n} \beta^{\varepsilon_n}$. We say a sequence I is *admissible* if for each t, $i_t \geq p i_{t+1} + \varepsilon_t$.

We define the *excess* of a sequence I by $e(I) = \varepsilon_0 + \sum i_t - p i_{t+1} - \varepsilon_t$. This quantity measures to what extent I exceeds the minimal requirement to be admissible.

Proposition 3.1.49 ([22])**.** *The set $\{P^I \mid I \text{ is admissible}\}$ is a basis for $\mathcal{A}_p$.*

Given two elements of the Serre-Cartan basis we do not have a closed form for their product, we must iteratively apply the Adem relations until we end up with a sum of operations indexed by admissible sequences.

The Milnor basis Just as in the mod 2 case, while the multiplicative structure of $\mathcal{A}_p$ is somewhat complex, the comultiplicative structure is straightforward. Again, it turns out that the dual Hopf algebra $\mathcal{A}_p^*$ has a simple expression as an algebra.

Proposition 3.1.50 ([35, Theorem 2])**.** *The linear dual of the Steenrod Algebra $\mathcal{A}_p$ is the tensor product of a polynomial algebra with an exterior algebra,*

$$\mathcal{A}_p^* = \mathbb{F}_p[\xi_1, \dots] \otimes \Lambda[\tau_0, \tau_1, \dots]$$

where $|\xi_n| = 2p^k - 2$ and $|\tau_n| = 2p^k - 1$.

Definition 3.1.51. The dual basis to the monomial basis for $\mathcal{A}_p^*$ is known as the *Milnor basis.* We write $P(I, E)$ for the basis element dual to $\xi_1^{i_1} \dots \xi_n^{i_n} \otimes \tau_0^{\varepsilon_0} \dots \tau_n^{\varepsilon_n}$. If we let $Q_k = P(0, \Delta_k)$ denote the element dual to τ_k, and $P(I) = P(I, 0)$ denote the element dual to $\xi_1^{i_1} \dots \xi_n^{i_n}$ then up to sign we have $P(I, E) = Q_0^{\varepsilon_0} Q_1^{\varepsilon_1} \dots P(I)$.

We are able to express the product of Milnor basis elements by closed (albeit complex) formulae.

Proposition 3.1.52 ([35, Theorem 4a, 4b])**.** *The multiplication of elements of the Milnor basis is determined by the following formulae.*

$$\begin{aligned} Q_j Q_k &= -Q_k Q_j \\ P(I)Q_k - Q_k P(I) &= Q_{k+1} P(I - p^k \Delta_1) + Q_{k+2} P(I - p^k \Delta_2) + \dots \\ P(I)P(J) &= \sum_X b(X) P(T(X)) \end{aligned}$$

where X ranges over all matrices of the form (3.3) *such that $I(X) = I$ and $J(X) = J$ and we take $b(X)$ modulo p. It is understood we mean $P(I') = 0$ whenever $i'_t < 0$ for any t.*

A direct approach

Just as in the mod 2 case, we have sufficient knowledge to directly compute the plethory of cohomology operations for singular cohomology with $\mathbb{F}_p$ coefficients. The mod p cohomology of the Eilenberg-MacLane spaces $K(\mathbb{F}_p, k)$ was computed by Cartan while studying the Steenrod algebra.

Theorem 3.1.53 ([21])**.** *$H^*(K(\mathbb{F}_p, k); \mathbb{F}_p)$ is the polynomial ring*

$$\mathbb{F}_p[\{P^I \iota_k \mid I \text{ is admissible and } e(I) < k \text{ or } e(I) = k \text{ and } I = (1, j, J)\}]. \tag{3.9}$$

Although the Hopf algebras $H^*(K(\mathbb{F}_p, k); \mathbb{F}_p)$ are primitively generated, we cannot easily write closed formulae for the comultiplication or composition in our plethory since as in the mod 2 setting we have to invoke the Adem relations (Theorem 3.1.46) to express the formulae in terms of the generators (3.9). Nevertheless, in a plethystic context we can express all our elements in terms of the atomic elements $\beta\iota_k$ and $P^n\iota_k$.

Definition 3.1.54. Define the bigraded $\mathbb{F}_p$-$\mathbb{F}_p$-coalgebra $C_\bullet$ to be the collection of $\mathbb{F}_p$-vector spaces C_k with bases

$$\{P^n\iota_k \,|\, 0 \leq n \leq k\} \cup \{\beta\iota_k \,|\, k \geq 0\}$$

where $\deg_*(P^n\iota_k) = k + 2n(p-1)$ and $\deg_*(\beta\iota_k) = k+1$. The comultiplicative structure is given by

$$\begin{aligned}
\psi(P^n\iota_k) &= \sum_{\substack{i+j=n \\ k_1+k_2=k \\ i<k_1, j<k_2}} P^i\iota_{k_1} \otimes P^j\iota_{k_2} \\
\psi(\beta\iota_k) &= 1 \otimes \beta\iota_k + \beta\iota_k \otimes 1 \\
\varepsilon(Sq^n\iota_k) &= \begin{cases} 1 & \text{if } k = 0 \text{ and } n = 0, k \\ 0 & \text{otherwise} \end{cases} \\
\varepsilon(\beta\iota_k) &= 0.
\end{aligned}$$

Following the same reasoning as in the mod 2 case, we can find the following succinct expression of our plethory by freely adding composition via the tensor algebra construction T, quotienting by the Adem relations and then freely adding multiplication via the symmetric algebra construction $S^{[p]}$.

Theorem 3.1.55. *Let $\mathscr{A}$ denote the two-sided ideal of the cocommutative bialgebra $T(C_\bullet)$ generated by the following elements.*

1. *For $k \geq 0$, $\beta\iota_{k+1}\beta\iota_k$,*
2. *For $n < pm$ and $k \geq 0$,*

$$P^n\iota_{k+2m(p-1)}P^m\iota_k - \sum_i (-1)^{n+i}\binom{(p-1)(m-i)-1}{n-pi} P^{n+m-i}\iota_{k+2i(p-1)}P^i\iota_k,$$

3. *For $n \leq pm$ and $k \geq 0$,*

$$\begin{aligned}
&P^n\iota_{k+2m(p-1)+1}\beta\iota_{k+2m(p-1)}P^m\iota_k \\
&\quad - \sum_i (-1)^{n+i}\binom{(p-1)(m-i)}{n-pi}\beta\iota_{k+2(m+n)(p-1)}P^{n+m-i}\iota_{k+2i(p-1)}P^i\iota_k \\
&\quad + \sum_i (-1)^{n+i+1}\binom{(p-1)(m-i)-1}{n-pi-1}P^{n+m-i}\iota_{k+2i(p-1)+1}\beta\iota_{k+2i(p-1)}P^i\iota_k,
\end{aligned}$$

4. For $k \geq 0$, $P^0\iota_k - 1_k$.

The ideal $\mathscr{A}$ is a bialgebra ideal of $T(C_\bullet)$. If we equip $T(C_\bullet)$ with an $\mathbb{F}_p[F]$-module structure encoding the action of the Frobenius given by

$$F(P^{n_1}\iota_{k_1}\dots) = P^{k_1+n_1}\iota_{k_1+n_1}P^{n_1}\iota_{k_1}\dots$$
$$F(\beta\iota_{k_1}\dots) = P^{k_1+1}\iota_{k_1+1}\beta\iota_{k_1}\dots$$

then this induces an $\mathbb{F}_p[F]$-module structure on $T(C_\bullet)/\mathscr{A}$ and we have isomorphisms of $\mathbb{F}_p$-plethories

$$H^*(K(\mathbb{F}_p,\bullet);\mathbb{F}_p) \cong S^{[p]}\left(\frac{T(C_\bullet)}{\mathscr{A}}\right) \cong \frac{S^{[p]}T(C_\bullet)}{(\mathscr{A})}.$$

Proof: This is analogous to the proof of Theorem 3.1.25. □

Via the enriched Hopf ring

Just as in the mod 2 case, the Hopf ring of homology cooperations for singular cohomology with coefficients in $\mathbb{F}_p$ is well understood and has a relatively simple form. Once again, we can leverage our theory of duality to compute the plethory of cohomology operations via the enriched Hopf ring.

The infinite dimensional lens space $L(\infty,p)$ is a model for $K(\mathbb{F}_p,1)$. Moreover, $L(\infty,p)$ admits an H-space structure which models the loop-space structure on $K(\mathbb{F}_p,1)$. The homology $H_*(L(\infty,p);\mathbb{F}_p)$ is the free $\mathbb{F}_2$-vector space on the elements $a_i \in H_{2i}(L(\infty,p);\mathbb{F}_p)$ and $c_i \in H_{2i+1}(L(\infty,p);\mathbb{F}_p)$ for $i \geq 0$. The coalgebra structure is given by $\psi(a_n) = \sum_{i+j=n} a_i \otimes a_j$, $\psi(c_n) = \sum_{i+j=n} a_i \otimes c_j + c_i \otimes a_j$, $\varepsilon(a_n) = 1$ if $n = 0$ and 0 otherwise and $\varepsilon(c_n) = 0$. The algebra structure induced by the H-space structure is given by $a_i a_j = \binom{i+j}{i} a_{i+j}$ and $c_i = ea_i$ where $e = c_0$ is the suspension element. The following result shows that $H_*(\mathbb{R}P^\infty;\mathbb{F}_2)$ is generated by $e \in H_1(L(\infty,p);\mathbb{F}_p)$ together with the *accelerated elements* $a_{(i)} = a_{p^i} \in H_{2p^i}(L(\infty,p);\mathbb{F}_p)$.

Proposition 3.1.56. *We have an isomorphism of $\mathbb{F}_p$-algebras*

$$H_*(K(\mathbb{F}_p,1);\mathbb{F}_p) \cong \frac{\mathbb{F}_p[e,a_{(0)},a_{(1)},\dots]}{(a_{(0)}^p,a_{(1)}^p,\dots)}$$

where $|e| = 1$ and $|a_{(i)}| = 2p^i$. Moreover, in the Hopf algebra $H_*(K(\mathbb{F}_p,1);\mathbb{F}_p)$ we have $Va_{(i)} = a_{(i-1)}$ for $i \geq 0$ and $Va_{(0)} = Ve = 0$.

Proof: Just as in Proposition 3.1.26. $\square$

Wilson showed that these elements together with the elements $b_{(i)}$ induced by the complex orientation generate the Hopf ring of homology cooperations.

Theorem 3.1.57 ([52, Theorem 8.6]). *The Hopf ring $H_*(K(\mathbb{F}_p, \bullet); \mathbb{F}_p)$ is the free $\mathbb{F}_p[\mathbb{F}_p]$-Hopf ring on $H_*(K(\mathbb{F}_p, 1); \mathbb{F}_p)$ and $H_*(\mathbb{C}P^\infty; \mathbb{F}_p) \subseteq H_*(K(\mathbb{F}_p, 2); \mathbb{F}_p)$ subject to the relations $a_{(i)} \circ a_{(j)} = -a_{(j)} \circ a_{(i)}$ and $e \circ e = -b_{(0)}$.*

To discuss elements of our Hopf ring would be very difficult without some multi-index notation. From this point on, any multi-index denoted by I (including I', I'', etc.) will be a multi-index $I = (i_0, i_1, \dots)$ with $i_k \in \{0, 1\}$. We shall denote an arbitrary multi-index with any non-negative entries by J.

For such a pair of multi-indices $I = (i_0, \dots, i_n)$, $J = (j_0, \dots, j_m)$ define

$$\begin{aligned} a^I b^J &= a_{(0)}^{\circ i_0} \circ \cdots \circ a_{(n)}^{\circ i_n} \circ b_{(0)}^{\circ j_0} \circ \cdots \circ b_{(m)}^{\circ j_m} \in H_{2|I|_p + 2|J|_p}(K(\mathbb{F}_p, |I| + 2|J|); \mathbb{F}_p) \\ ea^I b^J &= e \circ a^I b^J \in H_{1+2|I|_p+2|J|_p}(K(\mathbb{F}_p, 1 + |I| + 2|J|); \mathbb{F}_p). \end{aligned}$$

Once again we make the convention that $a^0 b^0 = [1] - [0]$ (and so $ea^0 b^0 = e$), which has the same advantages as in the mod 2 situation.

Unpacking the information contained within our Hopf ring to a local form yields the following result. We introduce the notation $P_1[x]$ to denote the $\mathbb{F}_p$-algebra $\mathbb{F}_p[x]/(x^p)$ and $P[x]$ to denote $\mathbb{F}_p[x]$.

Corollary 3.1.58. *For all n, we have isomorphisms of $\mathbb{F}_p$-algebras*

$$H_*(K(\mathbb{F}_p, n); \mathbb{F}_p) = \bigotimes_{I,J} \Lambda[ea^I b^J] \otimes \bigotimes_{I,J} P_1[a^I b^J]$$

where in the first sum I, J ranges over all multi-indices with $1 + |I| + 2|J| = n$ and in the second we require $|I| + 2|J| = n$. Moreover, in the Hopf algebra $H_*(K(\mathbb{F}_p, n); \mathbb{F}_p)$, we have $Vea^I b^J = 0$ and

$$Va^I b^J = \begin{cases} a^{s^{-1}I} b^{s^{-1}J} & i_0 = j_0 = 0 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof: As in Corollary 3.1.29. $\square$

Remark 3.1.59. From this point onwards we shall neglect to specify the conditions on lengths of the multi-indices required to ensure the $\bullet$-degrees remain compatible and leave context to determine the requirements on these.

Just as in the mod 2 case, it is difficult to find a closed formula for the comultiplication. Luckily, we can use some Hopf algebraic theory to sidestep this obstacle.

Consider the monomial basis for $H_*(K(\mathbb{F}_p, n); \mathbb{F}_p)$ i.e. the basis consisting of elements of the form

$$ea^{I_1}b^{J_1} * \cdots * ea^{I_N}b^{J_N} * (a^{I'_1}b^{J'_1})^{*k_1} * \cdots * (a^{I'_M}b^{J'_M})^{*k_M}$$

for multi-indices I_t, J_t, I'_t, J'_t and $0 \leq k_t < p$. Let $x_{I,J}, y_{I,J} \in H^*(K(\mathbb{F}_p, n); \mathbb{F}_p) \cong DH^*(K(\mathbb{F}_p, n); \mathbb{F}_p)$ denote the elements of the dual basis which are dual to $a^I b^J$ and $ea^I b^J$ respectively. Let $x_{0,0}^n$ denote the element dual to $(a^{\circ 0}b^{\circ 0})^{*n} = ([1] - [0])^{*n}$ (so $x_{0,0}^1 = x_{0,0}$). The other elements of the dual basis will remain anonymous.

We first tackle the sub-plethory of operations of bidegree $(0, 0)$.

Proposition 3.1.60. *We have an isomorphism of ungraded $\mathbb{F}_p$-plethories*

$$H^0(K(\mathbb{F}_p, 0); \mathbb{F}_p) \cong \mathbf{Set}(\mathbb{F}_p, \mathbb{F}_p).$$

Moreover, under this isomorphism the element $x_{0,0}$ corresponds to the identity map $\mathbb{F}_p \to \mathbb{F}_p$.

Proof: Since $H_0(K(\mathbb{F}_p, 0); \mathbb{F}_p) \cong \mathbb{F}_p[\mathbb{F}_p]$ we have $H^0(K(\mathbb{F}_p, 0); \mathbb{F}_p) \cong \mathbf{Set}(\mathbb{F}_p, \mathbb{F}_p)$. By definition, the functional $x_{0,0} \in DH_0(K(\mathbb{F}_p, 0); \mathbb{F}_p)$ satisfies

$$\langle x_{0,0}, ([1] - [0])^n \rangle = \begin{cases} 1 & n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

We claim $\langle x_{0,0}, [n] \rangle = n$ for all n. The cases $n = 0$ and $n = 1$ follow since $([1] - [0])^0 = 1 = [0]$. We have $([1] - [0])^n = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} [i]$ and so

$$\begin{aligned} 0 &= \langle x_{0,0}, ([1] - [0])^n \rangle \\ &= \langle x_{0,0}, \sum_{i=0}^{n-1} (-1)^{n-i} \binom{n}{i} [i] \rangle + \langle x_{0,0}, [n] \rangle \\ &= \sum_{i=0}^{n-1} (-1)^{n-i} \binom{n}{i} i + \langle x_{0,0}, [n] \rangle \\ &= \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} i + \langle x_{0,0}, [n] \rangle - n. \end{aligned}$$

Now $n(x-1)^{n-1} = \frac{\mathrm{d}}{\mathrm{d}x}(x-1)^n = \frac{\mathrm{d}}{\mathrm{d}x} \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} x^i = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} i x^{i-1}$ and so setting $x = 1$ we see for $n > 1$ that $\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} i = 0$. Thus, $\langle x_{0,0}, [n] \rangle = n$. Our claim follows by induction. Since the isomorphism $D\mathbb{F}_p[\mathbb{F}_p] = \mathbf{Mod}_{\mathbb{F}_p}(\mathbb{F}_p[\mathbb{F}_p], \mathbb{F}_p) \cong \mathbf{Set}(\mathbb{F}_p, \mathbb{F}_p)$ is given by evaluation on the canonical basis elements of $\mathbb{F}_p[\mathbb{F}_p]$, our result follows. $\square$

Theorem 3.1.61. *We have an isomorphism of $\mathbb{F}_p$-$\mathbb{F}_p$-birings*

$$H^*(K(\mathbb{F}_p,n);\mathbb{F}_p) \cong \begin{cases} \mathbf{Set}(\mathbb{F}_p,\mathbb{F}_p) & n=0 \\ \bigotimes_{\substack{i_0>0 \\ or\ j_0>0}} P[x_{I,J}] \otimes \bigotimes_{I,J} \Lambda[y_{I,J}] & n>0 \end{cases}$$

where the $\mathbb{F}_p$-$\mathbb{F}_p$-biring structure is specified by the generators being primitive together with the following formulae.

$$\begin{aligned}
\Delta^\times(x_{I,J}) &= \sum_{\substack{I'+I''=I \\ J'+J''=J}} x_{I',J'} \otimes x_{I'',J''} + \sum_{\substack{I'+I''=I \\ J'+J''+\Delta_0=J}} y_{I',J'} \otimes y_{I'',J''} \\
\Delta^\times(y_{I,J}) &= \sum_{\substack{I'+I''=I \\ J'+J''=J}} x_{I',J'} \otimes y_{I'',J''} + y_{I',J'} \otimes x_{I'',J''} \\
\varepsilon^\times(x_{I,J}) &= \begin{cases} 1 & I=J=0 \\ 0 & otherwise \end{cases} \\
\varepsilon^\times(y_{I,J}) &= 0.
\end{aligned}$$

Proof: For $n=0$, since $K(\mathbb{F}_p,0)=\mathbb{F}_p$ is discrete, we have $H^*(K(\mathbb{F}_p,0);\mathbb{F}_p) = H^0(K(\mathbb{F}_p,0);\mathbb{F}_p)$ and the result follows from Proposition 3.1.60 . The remainder of the proof is identical to the proof of Theorem 3.1.30. □

To compute the composition, we must understand the enrichment of the Hopf ring. Once again this is most cleanly expressed formally. Let $a(t) = \sum_i a_i t^i \in H_*(K(\mathbb{F}_p,1);\mathbb{F}_p)[[t]]$ and $b(t) = \sum_i b_i t^i \in H_*(K(\mathbb{F}_p,2);\mathbb{F}_p)[[t]]$.

Theorem 3.1.62 ([15, Proposition 17.7])**.** *For all $r \in DH_*(K(\mathbb{F}_p,\bullet);\mathbb{F}_p) \cong H^*(K(\mathbb{F}_p,\bullet);\mathbb{F}_p)$, r_*a_k is the coefficient of t^k in the formal identity in $H_*(K(\mathbb{F}_p,\bullet);\mathbb{F}_p)[[t]]$*

$$r_*a(t) = \mathop{\ast}_i b(t)^{\circ i} \circ [\langle r, a_i\rangle] * \mathop{\ast}_j a(t) \circ b(t)^{\circ i} \circ [\langle r, c_i\rangle],$$

r_*b_k is the coefficient of t^k in the formal identity in $H_*(K(\mathbb{F}_p,\bullet);\mathbb{F}_p)[[t]]$

$$r_*b(t) = \mathop{\ast}_i b(t)^{\circ i} \circ [\langle r, b_i\rangle],$$

and $r_*e = [\langle r, 1_1\rangle] * [\langle r, e\rangle] \circ e$.

The augmentation is determined by $\iota_1(e)=1$ and $\iota_1(a_{(i)}) = \iota_2(b_{(j)}) = 0$ together with Definition 1.3.21.

As in the mod 2 setting, it is straightforward to compute the identity elements for composition.

Corollary 3.1.63. *The canonical map $\mathcal{I} \to H^*(K(\mathbb{F}_p, \bullet); \mathbb{F}_p)$ is given by*

$$\iota_n \mapsto \begin{cases} x_{0,m\Delta_0} & n = 2m \\ y_{0,m\Delta_0} & n = 2m+1. \end{cases}$$

Proof: As in Corollary 3.1.32. $\square$

Once again, we can easily compute the enrichment of our operations on the generators of the Hopf ring $H_*(K(\mathbb{F}_p, \bullet); \mathbb{F}_p)$.

Corollary 3.1.64. *For multi-indices I, J, the induced maps satisfy*

$$\begin{aligned}
x_{I,J*}a_{(k)} &= \begin{cases} b_{(k-i)}^{\circ p^i} & I = \Delta_i, J = 0, k \geq i \\ 0 & \textit{otherwise} \end{cases} \\
x_{I,J*}b_{(k)} &= \begin{cases} b_{(k-j)}^{\circ p^j} & I = 0, J = \Delta_j, k \geq j \\ 0 & \textit{otherwise} \end{cases} \\
x_{I,J*}e &= 0 \\
y_{I,J*}a_{(k)} &= \begin{cases} a_{(k)} & I = J = 0 \\ 0 & \textit{otherwise} \end{cases} \\
y_{I,J*}b_{(k)} &= 0 \\
y_{I,J*}e &= \begin{cases} e & I = J = 0 \\ 0 & \textit{otherwise.} \end{cases}
\end{aligned}$$

Proof: For degree reasons, $\langle x_{I,J}, a_{(k)} \rangle = 0$ unless $|I| + 2|J| = 1$ and hence $I = \Delta_i$ and $J = 0$ for some i. Since, $x_{\Delta_i,0} \in PH^*(K(\mathbb{F}_p, 1); \mathbb{F}_p) \cong DQH_*(K(\mathbb{F}_p, 1); \mathbb{F}_p)$, $\langle x_{\Delta i,0}, a_j \rangle = 0$ unless j is a power of p and $\langle x_{\Delta i,0}, c_j \rangle = 0$ unless $j = 0$. Since $x_{\Delta_i,0}$ is dual to $a_{(i)}$,

$$\langle x_{\Delta_i,0}, a_j \rangle = \begin{cases} 1 & j = p^i \\ 0 & \text{otherwise} \end{cases}$$

and $\langle x_{\Delta_i,0}, c_j \rangle = 0$. Hence from Theorem 3.1.62, $x_{\Delta_i,0*}a_k$ is the coefficient of t^k in $b(t)^{\circ p^i}$. Since we are working over $\mathbb{F}_p$, the map $z \mapsto z^{\circ p}$ is linear and hence $b(t)^{\circ p^i} = \sum_j b_j^{\circ p^i} t^{jp^i}$. Thus,

$$x_{\Delta_i*}a_k = \begin{cases} b_{\frac{k}{p^i}}^{\circ p^i} & p^i | k \\ 0 & \text{otherwise.} \end{cases}$$

The remaining results are similar. $\square$

By appealing to Definition 1.3.21, we can compute the induced maps on general elements of our Hopf ring. Once again, these formulae can get unwieldy very quickly. However, we have $x_{I,J} = P((i_1, i_2, \dots), J)\iota_{|I|+|J|}$ and $y_{I,J} = P((i_1, i_2, \dots), J)\iota_{|I|+|J|+1}$ in terms of the Milnor basis. Thus by utilising Proposition 3.1.52 we should be able to obtain neat combinatorial formulae as in the mod 2 case.

3.2 Complex K-theory

K-Theory is a classical and extremely well studied area of topology. It was founded to study vector bundles over topological spaces. With a little work, we can construct a ring from the collection of isomorphism classes of vector bundles over some fixed topological space. This construction gives rise to a functor which satisfies the Eilenberg-Steenrod axioms and is thus a cohomology theory. We give a brief recap of basic definitions and results. For a detailed expository treatment, refer to either [10] or [26].

We then turn our attention to operations of this theory. In the ungraded setting, there are many classical operations and these are detailed in [10]. In the graded setting, the Hopf ring of homology cooperations has been computed in [15] and [57] reformulates the framework of Boardman et al. in the more familiar language of filtered λ-rings. However, to the best of my knowledge, we are able to give a complete description of the cohomology operations in all degrees for the first time.

The most intuitive definition of K-theory arises from the study of vector bundles. Many of our constructions only work when our spaces are compact Hausdorff and so in the following we assume this is the case. For a compact Hausdorff space X, let $\mathrm{Vect}(X)$ denote the set of isomorphism classes of finite-dimensional complex vector bundles over X. We denote the isomorphism class of a vector bundle ξ by $[\xi]$. The fibre-wise direct sum and tensor product of vector bundles equips $\mathrm{Vect}(X)$ with the structure of a semi-ring (a ring without additive inverses) with the classes of the 0 and 1 dimensional trivial bundles $[\varepsilon^0]$ and $[\varepsilon^1]$ acting as the zero and unit elements respectively. Hence we shall write $n = [\varepsilon^n]$, the class of the n-dimensional trivial vector bundle.

A continuous map $f\colon X \to Y$ induces a map of semi-rings $f^*\colon \mathrm{Vect}(Y) \to \mathrm{Vect}(X)$ given by pulling a vector bundle back along f and f^* respects the semi-ring structure. Moreover, if f and g are homotopic then the pullback bundles are isomorphic.

Suppose X has base point o. Define a map of semi-rings $\varepsilon\colon \mathrm{Vect}(X) \to \mathbb{N}$ sending a class of vector bundles $[\xi]$ to the dimension of the fibre at the base point i.e. $\varepsilon[\xi] = \dim \pi^{-1}(o)$.

The lack of additive inverses in $\mathrm{Vect}(X)$ can make computations difficult. Fortunately there is a general construction of freely adjoining additive inverses due to Grothendieck.

Definition 3.2.1. We define the Grothendieck group of a monoid M to be the free abelian group $\mathcal{G}(M)$ consisting of formal differences $m - n$ for all $m, n \in M$ subject to the relations

$$m - n = m' - n' \iff m + n' = m' + n.$$

If $f\colon M \to N$ is a map of monoids then we define $\mathcal{G}(f)\colon \mathcal{G}(M) \to \mathcal{G}(N)$ by $\mathcal{G}(f)(m-n) = f(m) - f(n)$.

The Grothendieck group defines a functor $\mathcal{G}\colon \textbf{Monoid} \to \textbf{Ab}$. Moreover, if R is a commutative semi-ring then we can define a multiplication on $\mathcal{G}(R)$ by $(x-y)(x'-y') = xx' - xy' - x'y + yy'$. Thus $\mathcal{G}$ lifts to a functor $\textbf{SemiRing} \to \textbf{Ring}$. For a monoid (or semi-ring) M, we have a natural map of monoids (resp. semi-rings) $M \to \mathcal{G}(M)$ and we have the following universal property. If $f\colon M \to N$ is a map of monoids (or semi-rings) then $\mathcal{G}(f)$ is a map of groups (resp. rings).

Definition 3.2.2. We define the *ungraded K-theory* of a compact Hausdorff space X to be the ring $K(X) = \mathcal{G}(\mathrm{Vect}(X))$. A continuous map $f\colon X \to Y$ induces a ring map $\mathcal{G}(f^*)\colon K(Y) \to K(X)$, which we shall also denote by f^*.

If X is based, our dimension map $\varepsilon\colon \mathrm{Vect}(X) \to \mathbb{N}$ extends to a ring map $\varepsilon\colon K(X) \to \mathbb{Z}$. The kernel of this map defines the *reduced cohomology* $K(X, o)$. For a virtual bundle $x \in K(X)$, we refer to $\varepsilon(x) \in \mathbb{Z}$ as the *virtual dimension* of x. Hence the reduced cohomology consists of classes of virtual dimension 0.

A priori, elements of $K(X)$ consist of formal differences of vector bundles $[\xi]-[\gamma] \in K(X)$ which we call *virtual bundles*. The following result shows these have a more succinct representation.

Proposition 3.2.3 ([34, Chapter 24, Section 1])**.** *For compact Hausdorff spaces X, every virtual bundle $x \in K(X)$ can be written as $[\xi] - n$ for some vector bundle ξ and non-negative integer n.*

This result naturally leads us to the representing space for K-theory as follows. Suppose X is connected and let $[\xi] - n$ be a virtual bundle. Since X is connected, ξ has some constant dimension m. We can define a map $\phi\colon K(X) \to [X, \mathbb{Z} \times BU]$ by $\phi([\xi]-n)(x) \mapsto (m-n, f(x))$ where $f\colon X \to BU(m) \subseteq BU$ is the classifying map of ξ. This naturally extends to a map on all spaces since both $K(-)$ and $[-, \mathbb{Z} \times BU]$ map disjoint unions to Cartesian products. It turns out that ϕ is an isomorphism.

Proposition 3.2.4 ([34, Chapter 24, Section 1])**.** *For compact Hausdorff spaces X, we have a natural isomorphism $K(X) \cong [X, \mathbb{Z} \times BU]$.*

We have thus far restricted to the full-subcategory of **Ho** consisting of compact Hausdorff spaces. To extend our definition to the entire homotopy category we *define* our functor by the representing space $\mathbb{Z} \times BU$.

Definition 3.2.5. We define the *ungraded K-theory functor* $K(-)\colon \mathbf{Ho} \to \mathbf{Ring}$ to have representing space $\mathbb{Z} \times BU$. Explicitly, $K(X) = [X, \mathbb{Z} \times BU]$ for all CW-complexes X.

Theorem 3.2.6 ([10])**.** *The functor $K(-)\colon \mathbf{Ho} \to \mathbf{Ring}$ satisfies the Eilenberg-Steenrod axioms and is thus a multiplicative ungraded cohomology theory.*

3.2.1 Ungraded operations

Cohomology operations of the ungraded cohomology functor $K(-)$ are well studied. For a detailed reference see [10] or for a more abstract discussion see [58]. In this section we recall the basic definitions and properties of some well studied operations before computing the plethory of ungraded cohomology operations.

The λ-operations

Recall the construction of the exterior algebra: given a vector space V, we define the *exterior algebra* $\Lambda(V)$ to be the quotient of the tensor algebra $T(V)$ by the two-sided ideal I generated by the elements $v \otimes v$. We shall write $x_1 \wedge \cdots \wedge x_k \in \Lambda(V)$ for the image of $x_1 \otimes \cdots \otimes x_k \in T(V)$ under the quotient map. Let $\Lambda^k(V)$ denote the subspace spanned by elements of the form $v_1 \wedge \cdots \wedge v_k$. If V has dimension n then $\Lambda^k(V)$ has dimension $\binom{n}{k}$. This construction naturally extends to vector bundles over compact Hausdorff spaces.

Proposition 3.2.7 ([10, Chapter 3, §1])**.** *The exterior power bundles satisfy the following properties for all vector bundles ξ and ζ over a compact Hausdorff space X.*

1. $\Lambda^0\xi = \varepsilon^1$*, the trivial line bundle.*
2. $\Lambda^1\xi = \xi$.
3. $\Lambda^n\xi = 0$ *if* $n > \varepsilon[\xi]$.
4. $\Lambda^k f^*(\xi) = f^*(\Lambda^k\xi)$ *for* $f\colon Y \to X$.
5. $\Lambda^k(\xi \oplus \zeta) = \sum\limits_{i+j=k} \Lambda^i\xi \otimes \Lambda^i\zeta$.
6. *If* $\xi \cong \zeta$ *then* $\Lambda^k\xi \cong \Lambda^k\zeta$.

Properties (4) and (6) tell us we have natural transformations $\Lambda^k\colon \operatorname{Vect}(X) \to \operatorname{Vect}(X)$ which we can compose with the natural inclusion $\operatorname{Vect}(X) \to K(X)$. To extend these to operations $\lambda^i\colon K(X) \to K(X)$ via the Grothendieck construction we need a monoid homomorphism.

Define $\lambda_t\colon \mathrm{Vect}(X) \to K(X)[[t]]^\times$ by

$$\lambda_t[\xi] = \sum_{i\geq 0} [\Lambda^i \xi] t^i = 1 + [\xi]t + [\Lambda^2\xi]t^2 + \dots.$$

By property (5) we have $\lambda_t([\xi] \oplus [\zeta]) = \lambda_t[\xi]\lambda_t[\zeta]$ and so by the universal property of the Grothendieck group this extends to a group homomorphism $\lambda_t\colon K(X) \to K(X)[[t]]^\times$ by defining

$$\lambda_t([\xi] - [\zeta]) = \frac{\lambda_t(\xi)}{\lambda_t(\zeta)}.$$

Definition 3.2.8. Let X be a compact Hausdorff space. We define the λ-*operations* $\lambda^k\colon K(X) \to K(X)$ by setting $\lambda^k(x)$ to be the coefficient of t^n in the power series $\lambda_t(x)$.

By the naturality of the exterior power operations, the λ-operations are natural transformations $K(X) \to K(X)$. The properties of these λ-operations are well studied. Recall the definition of a λ-ring (Definition 1.2.30).

Proposition 3.2.9 ([58, Example 1.16])**.** *If X is any compact Hausdorff space then $K(X)$ is a λ-ring under the λ-operations and continuous maps induce λ-ring morphisms.*

Thus far we have only defined the λ-operations on compact Hausdorff spaces X. However, these operations naturally generalise to all CW-complexes.

Theorem 3.2.10. *The functor $K(-)\colon$ **Ho** $\to$ **Ring** lifts to a functor* **Ho** $\to$ $\mathbf{Ring}_\lambda$

Proof: By the Yoneda lemma, the λ-operations induce (homotopy classes of) maps $\mathbb{Z} \times BU \to \mathbb{Z} \times BU$. These maps induce a λ-ring structure on $K(X) = [X, \mathbb{Z} \times BU]$. $\square$

At this point it is natural to consider if any other constructions on vector spaces and vector bundles produce operations in K-theory. Another familiar construction on vector spaces is the symmetric algebra $S(V)$ formed by taking the quotient of the tensor algebra $T(V)$ by the two-sided ideal generated by the elements $v_1 \otimes v_2 - v_2 \otimes v_1$. We define S^kV to be the subspace spanned by classes of elements of the form $v_1 \otimes \cdots \otimes v_k$. This extends to give a natural transformation $\mathrm{Vect}(X) \to \mathrm{Vect}(X)$ and just as in the case of exterior powers, operations $s^k\colon K(X) \to K(X)$. However, it turns out that each s^k is expressible as a polynomial in $\lambda^1, \dots, \lambda^k$ and so this construction produces no additional operations.

It will be useful to understand how the λ-operations behave with respect to virtual dimension on based spaces. In particular, we show that each λ-operation is based.

Proposition 3.2.11. *Suppose X is a based space and $x \in K(X)$ has virtual dimension $\varepsilon(x) = d$. The virtual dimension of $\lambda^k x$ is*

$$\varepsilon(\lambda^k x) = \binom{d}{k} = \frac{d(d-1)\dots(d-(k-1))}{k!}.$$

Proof: Let X be a compact Hausdorff space and let $x = [\xi] - n \in K(X)$ have virtual dimension d. Since $\varepsilon[\xi] = d + n$,

$$\lambda_t(x) = \frac{\lambda_t[\xi]}{\lambda_t(n)} = \frac{\lambda_t[\xi]}{(1+t)^n}.$$

Since $(1+t)^{-n} = \sum \binom{-n}{i} t^i$, the coefficient of t^k in $\lambda_t(x)$ is

$$\lambda^k(x) = \sum_{i+j=k} \binom{-n}{i} \Lambda^j[\xi].$$

Therefore, $\varepsilon(\lambda^k x) = \sum \binom{-n}{i}\binom{n+d}{j}$. However, this is the coefficient of t^k in the power series $(1+t)^{n+d}(1+t)^{-n} = (1+t)^d$. Hence $\varepsilon(\lambda^k x)$ has virtual dimension $\binom{d}{k}$. The extension to general spaces is immediate since the virtual dimension only depends on a neighbourhood of the base point. □

Important insights can be obtained by understanding the representing maps of the λ-operations. Let $U(n) \to U\binom{n}{k}$ denote the exterior power representation of the unitary group for $0 \leq k \leq n$. Composing this with the inclusion yields a map $\Lambda^k \colon U(n) \to U$. If we apply the classifying space functor we get a map $\beta_n^k \colon BU(n) \to BU$ which represents the exterior power construction on n-dimensional vector bundles. Identifying BU with $\{0\} \times BU$, we can view β_n^k as an element of $K(BU(n), o)$.

Let X be a compact Hausdorff space and $x = [\xi] - n \in K(X, o)$. Since X is compact the representing map for x factors via $\mathbb{Z} \times BU(n)$ for some n. Hence

$$X \xrightarrow{x} \mathbb{Z} \times BU(n) \xrightarrow{\mathbb{Z} \times \beta_n^k} \mathbb{Z} \times BU$$

represents the virtual bundle $\Lambda^k[\xi] - \binom{n}{k} \in K(X, o)$.

Let $j \colon BU(n) \to BU(n+1)$ denote the canonical inclusion. Viewing β_n^k as an element of $K(BU(n), o)$ it is natural to study the element $j^* \beta_{n+1}^k \in K(BU(n), o)$. Since j classifies the construction $[\xi] \mapsto [\xi \oplus \varepsilon^1]$ and $\Lambda^k[\xi \oplus \varepsilon^1] = \Lambda^k[\xi] + \Lambda^{k-1}[\xi]$ we see $j^* \beta_{m+1}^k = \beta_m^k + \beta_m^{k-1}$ where we write $\beta_m^k = 0$ for $k > m$ or $k < 0$.

In the following we will require a result of Anderson [8, Theorem 1], which states that we have no phantom operations in complex K-theory, and thus we have an isomorphism $K(BU) \cong \varprojlim_n K(BU(n))$.

Proposition 3.2.12. *Let $i \in K(BU, o)$ be represented by the inclusion $BU \simeq \{0\} \times BU \subseteq \mathbb{Z} \times BU$ and let $\lambda_n^k = \sum_{i=0}^k \binom{-n}{i} \beta_n^{k-i} \in K(BU(n), o)$ where β_n^j is represented by $B\Lambda^j \colon BU(n) \to BU$. The following conditions are true.*

1. *If $j \colon BU(n) \to BU(n+1)$ is the canonical inclusion then $j^*\lambda_{n+1}^k = \lambda_n^k$.*
2. *The element $\lambda^k i \in K(BU, o) \cong \varprojlim_n K(BU(n), o)$ is the inverse limit of the $\lambda_n^k \in K(BU(n), o)$.*

Proof: (1) We have the following equalities.

$$\begin{aligned} j^*\lambda_{n+1}^k &= \sum_{i=0}^k \binom{-(n+1)}{i} i^*\beta_{n+1}^{k-i} \\ &= \sum_{i=0}^k \binom{-(n+1)}{i} \beta_n^{k-i} + \sum_{i=1}^k \binom{-(n+1)}{i-1} \beta_n^{k-i} \\ &= \beta_n^k + \sum_{i=1}^k \left[\binom{-(n+1)}{i} + \binom{-(n+1)}{i-1} \right] \beta_n^{k-i} \\ &= \sum_{i=0}^k \binom{-n}{i} \beta_n^{k-i} = \lambda_n^k. \end{aligned}$$

(2) Let X be a compact Hausdorff space and $x = [\xi] - n \in K(X, o)$. Since X is compact, the representing map for x factors via $\mathbb{Z} \times BU(n)$ for some n. Now the composition

$$X \xrightarrow{x} \mathbb{Z} \times BU(n) \xrightarrow{\mathbb{Z} \times \lambda_n^k} BU$$

represents the virtual bundle $\sum_{i=0}^k \binom{-n}{i} \left[\Lambda^{k-i}[\xi] - \binom{n}{k-i}\right] \in K(X, o)$. However since $\sum_{i=0}^k \binom{-n}{i}\binom{n}{k-i} = 0$ this is precisely $\sum_{i=0}^k \binom{-n}{i} \Lambda^{k-i}[\xi] = \lambda^k(x)$. The result follows. □

We have now understood a family of ungraded K-theory operations. To see how these realise all ungraded K-theory operations we must understand the K-theory of the representing space, $K(\mathbb{Z} \times BU)$. The following result is well known.

Theorem 3.2.13 ([51, Theorem 4.15])**.** *Let $i \in K(BU)$ be represented by the inclusion map $BU \simeq \{0\} \times BU \subseteq \mathbb{Z} \times BU$. We have an isomorphism of filtered rings*

$$K(BU) \cong \mathbb{Z}[[\lambda^1 i, \lambda^2 i, \dots]].$$

The filtration ideals on $\mathbb{Z}[[\lambda^1 i, \lambda^2 i, \dots]]$ are the kernels of the projections $\mathbb{Z}[[\lambda^1 i, \dots]] \to \mathbb{Z}[[\lambda^1 i, \dots, \lambda^n i]]$.

We remark that since $\varepsilon(i) = 0$, we have $\varepsilon(\lambda^k i) = 0$ for all k. Thus given a power series $x = f(\lambda^1 i, \lambda^2 i, \dots) \in K(BU)$ we have $\varepsilon(x) = f(0, 0, \dots)$ the constant term of f. Hence we can understand the elements of $K(BU)$ as self maps of BU together with an inclusion $BU \simeq \{d\} \times BU \subseteq \mathbb{Z} \times BU$ for some d. This geometrically realises the splitting $K(BU) \cong \mathbb{Z} \oplus K(BU, o)$.

Now since $\mathbb{Z} \times BU$ is a disjoint union $\coprod_{d \in \mathbb{Z}} \{d\} \times BU$ we have an isomorphism $K(\mathbb{Z} \times BU) \cong \prod_{d \in \mathbb{Z}} K(\{d\} \times BU)$ given by $x \mapsto (i_d^* x)_{d \in \mathbb{Z}}$ where $i_d \colon \{d\} \times BU \to \mathbb{Z} \times BU$ is the inclusion. Clearly we have isomorphisms $\theta \colon K(BU) \cong K(\{d\} \times BU)$. However, the image of i under this isomorphism is represented by the map $\{d\} \times BU \simeq BU \simeq \{0\} \times BU \subseteq \mathbb{Z} \times BU$. It is more natural to work with the element $i_d \in K(\{d\} \times BU)$ represented by the inclusion $\{d\} \times BU \to \mathbb{Z} \times BU$. By our discussion in the previous paragraph, $i_d = \theta(i) + d$. Thus, $\lambda^k i_d = \theta(\lambda^k i + d)$. However, from $\lambda^k i_d$ we can recover $\theta(\lambda^k i)$ by the relation $\theta(\lambda^k i) = \lambda^k(i_d - d) = \sum_{i+j=k} \lambda^i(i_d)\binom{-d}{j}$ and hence the $\lambda^k i_d$ are an algebraically independent generating set for $K(\{d\} \times BU)$.

Identifying $\lambda^k i_d$ with the sequence $(\dots, 0, \lambda^k i_d, 0, \dots)$ which is non-zero in the d-th entry, we can understand the elements $\lambda^k i_d \in K(\mathbb{Z} \times BU)$ as virtual dimension sensitive versions of the λ-operations. Explicitly, if X is a connected space and $x \in K(X)$ has virtual dimension d' then we have

$$\lambda^k i_d(x) = \begin{cases} \lambda^k(x) & \text{if } d = d' \\ 0 & \text{otherwise.} \end{cases}$$

Equivalently, we can note that $\lambda^k = \sum_{d \in \mathbb{Z}} \lambda^k i_d$.

Theorem 3.2.14. *We have an isomorphism of complete Hausdorff $\mathbb{Z}$-plethories*

$$K(\mathbb{Z} \times BU) \cong \prod_{d \in \mathbb{Z}} \mathbb{Z}[[\lambda^1 i_d, \lambda^2 i_d, \dots]]$$

where $i_d \in K(\{d\} \times BU)$ is represented by the inclusion $\{d\} \times BU \subseteq \mathbb{Z} \times BU$. Identifying $\lambda^k i_d$ with the sequence $(\dots, 0, \lambda^k i_d, 0, \dots)$, the plethystic structure is given by the *following formulae.*

$$\begin{aligned}
\Delta^+(\lambda^k i_d) &= \sum_{\substack{n+m=k \\ r+s=d}} \lambda^n i_r \otimes \lambda^m i_s \\
\varepsilon^+(\lambda^k i_d) &= 0 \\
\sigma(\lambda^k i_d) &= S_k(\lambda^1 i_{-d}, \ldots, \lambda^k i_{-d}) \\
\Delta^\times(\lambda^k i_d) &= \sum_{rs=d} P_k(\lambda^1 i_r \otimes 1, \ldots, \lambda^k i_r \otimes 1; 1 \otimes \lambda^1 i_s, \ldots, 1 \otimes \lambda^k i_s) \\
\varepsilon^\times(\lambda^k i_d) &= \begin{cases} 1 & \text{if } d = k = 1 \\ 0 & o.w. \end{cases} \\
\lambda^{k'} i_{d'} \circ \lambda^k i_d &= \begin{cases} P_{k',k}(\lambda^1 i_d, \ldots, \lambda^{kk'} i_d) & \text{if } d' = \binom{k}{d} \\ 0 & o.w. \end{cases} \\
\iota &= \sum_{d \in \mathbb{Z}} \lambda^1_d.
\end{aligned}$$

Here $P_k \in \mathbb{Z}[x_1, \ldots, x_k; y_1, \ldots, y_k]$ and $P_{k',k} \in \mathbb{Z}[x_1, \ldots, x_{kk'}]$ are the universal polynomials from Definition 1.2.30 and $S_k \in \mathbb{Z}[x_1, \ldots, x_k]$ is the antipode defined in Corollary 1.2.35.

Proof: To compute the plethystic structure, we appeal to Theorem 1.2.29. Let X be a connected space, and fix a choice of base point. Our results will then extend to all spaces by working on each connected component individually. Suppose $x, y \in K(X)$ have virtual dimension r, s respectively. Since $x + y$ has virtual dimension $r + s$, we have for $d = r + s$,

$$\begin{aligned}
\lambda^k i_d(x + y) &= \lambda^k(x + y) \\
&= \sum_{n+m=k} \lambda^n(x)\lambda^m(y) \\
&= \sum_{n+m=k} \lambda^n i_r(x)\lambda^m i_s(y)
\end{aligned}$$

and $\lambda^k_d(x + y) = 0$ for $d \neq r + s$. Our formula for Δ^+ follows and the result for $\Delta^\times$ is similar.

For ε^+, we note that $\lambda^k_d(0) = 0$ unless $d = 0$ since $0 = \varepsilon^0$ has virtual dimension 0. Moreover, we have $\lambda^k_0(0) = 0$ for $n \geq 0$. The result for $\varepsilon^\times$ is similar. The antipode follows from the relation $\lambda_t(-x) = \lambda_t(x)^{-1}$ and virtual dimension considerations.

For composition, if x has virtual dimension d then $\lambda^m(x)$ has virtual dimension $\binom{m}{d}$, and so we see $\lambda^n_{d'} \circ \lambda^m_d$ whenever $d' \neq \binom{m}{d}$. If $d' = \binom{m}{d}$, then:

$$\begin{aligned}(\lambda^{k'}_{d'} \circ \lambda^k_d)(x) &= (\lambda^{k'} \circ \lambda^k)(x) \\ &= P_{k',k}(\lambda^1(x), \ldots, \lambda^{kk'}(x)) \\ &= P_{k',k}(\lambda^1 i_d(x), \ldots, \lambda^{kk'} i_d(x)).\end{aligned}$$

Finally, since $\lambda^1 \in K(\mathbb{Z} \times BU)$ is the identity operation, we see that $\iota = \lambda^1 = \sum_{d \in \mathbb{Z}} \lambda^1 i_d$. □

The λ-operations generate a sub-plethory of $K(\mathbb{Z} \times BU)$ which has a familiar expression in terms of operations on λ-rings.

Corollary 3.2.15. *If Q denotes the sub-ring of $K(\mathbb{Z} \times BU)$ generated by the elements $\lambda^k = \sum_{d \in \mathbb{Z}} \lambda^k i_d$ then Q is a sub-plethory and we have an isomorphism of $\mathbb{Z}$-plethories $Q \cong \Omega$ where Ω is the plethory of Corollary 1.2.37.*

Proof: This is immediate from the definitions. □

An alternative computation gives an alternative expression for the plethory in terms of familiar structures. First recall that since $BU \simeq \{0\} \times BU$ represents $K(X, o)$, it admits an abelian group structure corresponding the direct product of vector bundles. Moreover, the abelian group object structure on $\mathbb{Z} \times BU$ is the product structure of the abelian group objects $\mathbb{Z}$ and BU.

Proposition 3.2.16. *The Hopf algebraic structure on $K(BU) = \mathbb{Z}[[\lambda^1 i, \lambda^2 i, \ldots]]$ is given by the following formulae.*

$$\begin{aligned}\psi(\lambda^k i) &= \sum_{n+m=k} \lambda^n i \otimes \lambda^m i \\ \varepsilon(\lambda^k i) &= 0 \\ \sigma(\lambda^k i) &= S_k(\lambda^1 i, \ldots, \lambda^k i)\end{aligned}$$

where $S_k \in \mathbb{Z}[x_1, \ldots, x_k]$ is the antipode defined in Corollary 1.2.35.

Proof: See [15, Theorem 5.7] or alternatively note that $BU \to \{0\} \times BU \subseteq \mathbb{Z} \times BU$ is a map of abelian group objects. □

Theorem 3.2.17. *The Künneth isomorphism*

$$K(\mathbb{Z} \times BU) \cong \mathbf{Set}(\mathbb{Z}, \mathbb{Z}) \widehat{\otimes} K(BU)$$

is an isomorphism of complete Hausdorff Hopf algebras.

Proof: Since the K-homology of $\mathbb{Z}$ and BU are free as abelian groups, we have an isomorphism of complete Hausdorff rings $K(\mathbb{Z} \times BU) \cong K(\mathbb{Z})\widehat{\otimes}K(BU)$. Now $K(\mathbb{Z}) = \mathbf{Ho}(\mathbb{Z}, \mathbb{Z} \times BU) \cong \mathbf{Set}(\mathbb{Z}, \mathbb{Z})$ since BU is connected. Since the abelian group object structure on $\mathbb{Z} \times BU$ is the product structure of the abelian groups $\mathbb{Z}$ and BU the Künneth isomorphism is an isomorphism of Hopf algebras. $\square$

Under the isomorphism $K(\mathbb{Z} \times BU) \cong \mathbf{Set}(\mathbb{Z}, \mathbb{Z})\widehat{\otimes}K(BU)$, the element $\lambda^k i_d$ corresponds to the element $\chi_d \otimes \lambda^k$ where χ_d denotes the indicator function given by

$$\chi_d(x) = \begin{cases} 1 & \text{if } x = d \\ 0 & \text{otherwise.} \end{cases}$$

This allows us to pull across the plethystic structure and realise $\mathbf{Set}(\mathbb{Z}, \mathbb{Z})\widehat{\otimes}K(BU)$ as a $\mathbb{Z}$-plethory.

The Adams operations

The λ-operations are neither additive nor multiplicative. Moreover, the composition of two λ-operations is rather complicated and as a result they can be somewhat unwieldy. Adams [2] found particular linear combinations of the λ-operations with pleasing properties: they act as ring homomorphisms and have straightforward expressions for composition. In a suitable context, these Adams operations can be viewed as a natural generalisation of the Steenrod powers to K-theory. The Adams operations can be defined for any λ-ring and we shall work primarily in this more abstract context.

Definition 3.2.18. Let R be a λ-ring. For $k \geq 0$, define the *Adams operations* $\psi^k \colon R \to R$ by $\psi^0(x) = 1$ and for $k > 0$,

$$\psi^k(x) = Q_k(\lambda^1(x), \ldots, \lambda^k(x))$$

where $Q_k \in \mathbb{Z}[x_1, \ldots, x_k]$ is the unique *Newton polynomial* such that if σ_i denotes the i-th elementary symmetric polynomial in $x_1, \ldots, x_k$ then

$$Q_k(\sigma_1, \ldots, \sigma_k) = x_1^k + \cdots + x_k^k.$$

Example 3.2.19. For a λ-ring R and all $x \in R$,

$$\begin{aligned} \psi^0(x) &= 1 \\ \psi^1(x) &= x \\ \psi^2(x) &= x^2 - 2\lambda^2(x) \\ \psi^3(x) &= x^3 - 3x\lambda^2(x) + 3\lambda^3(x). \end{aligned}$$

Theorem 3.2.20 ([58, Propositions 3.6 and 3.7])**.** *Let R be a λ-ring. For all $x, y \in R$ and all Adams operations ψ^k the following properties hold.*

1. $\psi^n(x+y) = \psi^n(x) + \psi^n(y)$.
2. $\psi^n(xy) = \psi^n(x)\psi^n(y)$.
3. $\psi^n(\psi^m(x)) = \psi^{n+m}(x)$.
4. *For a prime p, $\psi^p(x) = x^p + pz$ for some $z \in R$.*

The Adams operations have much nicer computational properties than the λ-operations. The following result gives conditions under which the Adams operations encode the same information as the λ-operations.

Proposition 3.2.21 ([58, Theorem 3.15])**.** *Suppose R is a torsion-free λ-ring. The λ-operations λ^k can be written as a homogenous polynomial with rational coefficients of degree k in $\psi^1, \ldots, \psi^k$ where ψ^j is given degree j.*

Unfortunately, for many spaces $K(X)$ and hence $\widehat{K}(X)$ is not $\mathbb{Z}$-torsion-free (e.g. $K(\mathbb{R}P^2) = \mathbb{Z} \oplus \mathbb{Z}/2$), and the Adams operations will generate a proper sub-plethory of the plethory of λ-operations. Nonetheless, this sub-plethory is often sufficient to prove many useful results (e.g. see [10, Theorem 3.2.3]). Recall the functor of super primitives $\mathcal{A}\colon \mathbf{Plethory}_{\mathbb{Z}} \to \mathbf{Monoid}$ of Definition 1.3.45 and the right adjoint $\Psi\colon \mathbf{Monoid} \to \mathbf{Plethory}_{\mathbb{Z}}$ of Definition 1.3.51.

Theorem 3.2.22. *We have an isomorphism of monoids $\mathcal{A}(K(\mathbb{Z} \times BU)) \cong \{\psi_k \,|\, k \geq 0\}$ where the monoidal structure is given by $\psi^k \psi^l = \psi^{k+l}$ for all k, l.*

Moreover, the free plethory $\Psi\mathcal{A}(K(\mathbb{Z}\times BU))$ is a sub-plethory of $K(\mathbb{Z}\times BU)$ and if $\widehat{K}(X)$ is $\mathbb{Z}$-torsion free then the $K(\mathbb{Z} \times BU)$-algebra structure of $\widehat{K}(X)$ is uniquely determined by the $\Psi\mathcal{A}(K(\mathbb{Z} \times BU))$-algebra structure.

Proof: By Proposition 1.3.46 the superprimitives are precisely the operations which act as ring homomorphisms and [30, Chapter IV, Theorem 7.13] asserts that these are precisely the Adams operations. Since there are no additive or multiplicative relations between the Adams operations, $\Psi\mathcal{A}(K(\mathbb{Z} \times BU))$ is a sub-plethory of $K(\mathbb{Z} \times BU)$. Lastly, by Proposition 3.2.21 if $\widehat{K}(X)$ is torsion free then the action of the λ-operations is uniquely determined by the action of the Adams operations. $\square$

3.2.2 Graded operations

Given an ungraded cohomology theory $h(-)$, we can extend to a cohomology theory $E^*(-)$ graded on the non-positive integers by defining $E^0(X) = h(X)$ and $E^{-n}(X, o) = h(\Sigma^n X, o)$ for all $n \geq 0$. Moreover the multiplicative structure can be transferred by the maps $E^{-n}(X, o) \otimes E^{-m}(X, o) = h(\Sigma^n X, o) \otimes h(\Sigma^m X, o) \to h(\Sigma^{n+m} X, o) = E^{-(n+m)}(X, o)$. In general there is no way to extend an ungraded cohomology theory to the positive integers, and the difficulty of the problem is best illustrated on the level of the representing spaces. When extending to non-positive degrees we take $\underline{E}_0 = \underline{H}$ and $\underline{E}_{-n} = \Omega^n \underline{H}$. To extend to the positive integers would require delooping the space $\underline{H}$ i.e. finding spaces $\underline{E}_n$ such that $\Omega^n \underline{E}_n \simeq \underline{H}$ and this is in general a very hard problem. Fortunately, we are able to deloop $\mathbb{Z} \times BU$ and this is a consequence of Bott's famous periodicity theorem.

Lemma 3.2.23 ([26, Corollary 2.3]). *We have an isomorphism of rings*

$$K(S^2) \cong \frac{\mathbb{Z}[\xi_1]}{([\xi_1] - 1)^2}$$

where ξ_1 denotes the canonical line bundle over $S^2 = \mathbb{C}P^1$.

Theorem 3.2.24 (Bott periodicity, [26, Theorem 2.2]). *For all compact Hausdorff spaces X, the external product map $K(X) \otimes K(S^2) \to K(X \times S^2)$ is an isomorphism of rings.*

As an immediate consequence of this result, we have isomorphisms $K(\Sigma^n X, o) \cong K(\Sigma^{n+2} X, o)$ given by taking the external product with $[\xi_1] - 1 \in K(S^2, o)$. Thus if we extend K-theory to the non-positive integers by defining $K^{-n}(X, o) = K(\Sigma^n X, o)$ we have periodicity $K^{-n}(X) \cong K^{-(n+2)}(X)$ given by multiplication by an element $u \in K^{-2}(T)$ corresponding to $[\xi_1] - 1 \in K(S^2, o)$ where T is the one point space. It is natural to extend this periodicity to define $K^n(X)$ for positive integer n.

Definition 3.2.25. We define the *graded K-theory* for compact Hausdorff spaces as follows. For non-positive degrees, we define $K^{-n}(X, o) = K(\Sigma^n X, o)$. For positive degrees, we define $K^n(X)$ to be the free $K(X)$-module on a single generator for n even and the free $K^{-1}(X)$-module on a single generator for n odd.

If we write u^{-1} for the generator of $K^2(T) \cong K^0(T) \cong \mathbb{Z}$, then this is compatible with the multiplicative structure in that $uu^{-1} = 1 \in K^0(T)$ and $(u^{-1})^n$ is the generator for $K^{2n}(T)$. Thus we can identify $K^{2n}(X)$ with $u^n K^0(X)$ and K^{2n-1} with $u^n K^{-1}(X)$ for all n.

Corollary 3.2.26. *The coefficient ring of graded K-theory is given by $K^* = K^*(T) = \mathbb{Z}[u, u^{-1}]$ where u is the canonical generator $[\xi_1] - 1 \in K(S^2, o) = K^{-2}(T)$.*

Proof: This is an immediate consequence of Definition 3.2.25 and Bott periodicity.

For many applications it simplifies matters to set $u = 1$ and view K-theory as $\mathbb{Z}/2$-graded.

It is straightforward to deduce the representing spaces of K-theory via Bott periodicity which on the level of representing spaces states we that we have a homotopy equivalence $\Omega^2(\mathbb{Z} \times BU) \simeq \mathbb{Z} \times BU$. Recalling that for any topological group G we have a homotopy equivalence $\Omega BG \simeq G$. Thus we have $\Omega(\mathbb{Z} \times BU) = \Omega(\{0\} \times BU) \simeq \Omega BU \simeq U$ and $\Omega U \simeq \mathbb{Z} \times BU$. This result illustrates how we extend our definition of graded K-theory to the entire homotopy category.

Definition 3.2.27. We define the *graded K-theory* $K^*(-)\colon \mathbf{Ho} \to \mathbf{Alg}_{K^*}$ to be represented by the 2-periodic Ω-spectrum with $\mathbb{Z} \times BU$ in even degrees and U in odd degrees.

Theorem 3.2.28 ([26, Section 2.2])**.** *The functor $K^*(-)\colon \mathbf{Ho} \to \mathbf{Alg}_{K^*}$ satisfies the Eilenberg-Steenrod axioms and is thus a cohomology theory.*

We have computed the plethory of unstable cohomology operations for the ungraded K-theory. We saw in Definition 3.2.25 that due to Bott periodicity we can extend K-theory to a graded cohomology theory. We now turn our attention to computing the plethory of operations for this graded theory. It is straightforward to compute the operations with even source degree.

Proposition 3.2.29. *Viewing $K(\mathbb{Z} \times BU)$ as concentrated in degree 0, we have an isomorphism of $\mathbb{Z}[u, u^{-1}]$-algebras $K^*(\mathbb{Z} \times BU) \cong K(\mathbb{Z} \times BU) \otimes_{\mathbb{Z}} \mathbb{Z}[u, u^{-1}]$ where $|u| = -2$.*

Proof: Since BU and hence $\mathbb{Z} \times BU$ has a cell structure consisting only of even-dimensional cells, the cellular cohomology is concentrated in even degrees and thus so is $K^*(\mathbb{Z} \times BU)$. By Bott perodicity we have isomorphisms of $\mathbb{Z}$-modules $K(\mathbb{Z} \times BU) = K^0(\mathbb{Z} \times BU) \cong K^{2n}(\mathbb{Z} \times BU)$ given by multiplication by u^{-n}. Hence we have an isomorphism of $\mathbb{Z}[u, u^{-1}]$-modules $K^*(\mathbb{Z} \times BU) \cong K(\mathbb{Z} \times BU) \otimes_{\mathbb{Z}} \mathbb{Z}[u, u^{-1}]$. By definition of the ring structure on $K^*(\mathbb{Z} \times BU)$ this is an isomorphism of algebras. □

We now turn our attention to operations of odd source degree. Since complex K-theory is represented in odd degrees by the infinite unitary group, we must compute the Hopf algebra $K^*(U)$. In addition, to be able to understand the whole plethystic structure we must understand how these operations relate to the operations of even source degree. The following result is well known.

Theorem 3.2.30 ([10, Theorem 2.7.17]). *Let $\Lambda^k \colon U(n) \to U\binom{n}{k} \subseteq U$ denote the exterior power representation and $\mu_n^k \in K^{-1}(U(n))$ denote the cohomology class represented by the map $U(n) \xrightarrow{\Lambda^k} U\binom{n}{k} \subseteq U$. The follow statements hold.*

1. *We have an isomorphism of K^*-algebras $K^*(U(n)) \cong \Lambda_{K^*}[\mu_n^1, \dots, \mu_n^n]$.*
2. *If $i \colon U(n-1) \to U(n)$ is the inclusion map then $i^*(\mu_n^k) = \mu_{n-1}^k + \mu_{n-1}^{k-1}$.*

We remark that the choice of degree for the elements $\mu_n^k \in K^*(U(n))$ is arbitrary and we could choose any odd degree. Our selection is motivated by a relation to the even degree operations: the looping of the λ-operations will be expressible in terms of the μ_n^k and we chose the λ-operations to be in spacial degree 0.

In Proposition 3.2.12 we expressed the λ-operations as linear combinations of elements represented by maps induced by the exterior power representation on the level of classifying spaces. Since we wish to relate our odd degree operations to the λ-operations it makes sense to choose the same linear combination.

Proposition 3.2.31. *Let $l_n^k = \sum_{i=0}^{k-1} \binom{-n}{i} \mu_n^{k-i} \in K^{-1}(U(n))$ for $k \leq n$. The following statements are true.*

1. *If $i \colon U(n-1) \to U(n)$ is the inclusion map then $i^*(l_n^k) = l_{n-1}^k$.*
2. *We have an isomorphisms of K^*-algebras $K^*(U(n)) \cong \Lambda_{K^*}[l_n^1, \dots, l_n^n]$.*
3. *We have an isomorphism of K^*-algebras $K^*(U) \cong \Lambda_{K^*}[l^1, l^2, \dots]$ where if $j \colon U(n) \to U$ denotes the inclusion then $j^* l^k = l_n^k$.*

Proof: (1)

$$\begin{aligned}
i^*(l_n^k) &= \sum_{i=0}^{k-1} \binom{-n}{i} i^*(\mu_n^{k-i}) \\
&= \sum_{i=0}^{k-1} \binom{-n}{i} \mu_{n-1}^{k-i} + \sum_{i=0}^{k-1} \binom{-n}{i} \mu_{n-1}^{k-i-1} \\
&= \sum_{i=0}^{k-1} \binom{-n}{i} \mu_{n-1}^{k-i} + \sum_{i=1}^{k} \binom{-n}{i-1} \mu_{n-1}^{k-i} \\
&= \mu_{n-1}^k + \sum_{i=1}^{k-1} (-1)^i \left[\binom{-n}{i} + \binom{-n}{i-1} \right] \mu_{n-1}^{k-i} \\
&= \mu_{n-1}^k + \sum_{i=1}^{k-1} \binom{-(n-1)}{i} \mu_{n-1}^{k-i} \\
&= l_{n-1}^k.
\end{aligned}$$

(2) We can write the μ_n^k as a linear combination of the l_n^k's.

(3) By [8][Theorem 1], $K^*(U) \cong \varprojlim K^*(U(n)) = \Lambda_{K^*}[l^1, l^2, \dots]$ where l^k are the sequences $(l_n^k)_{n>0}$. □

We can prove the result which motivated our definition for the odd degree operations l^k: they can be expressed as the looping of the λ-operations.

Corollary 3.2.32. *In $K^{-1}(\underline{K}_{-1})$, we have*

$$\Omega\lambda^k i_d = \begin{cases} l^k & \text{if } d = 0 \\ 0 & o.w. \end{cases}$$

Proof: We first show that $B\colon \mathbf{Ho}(U(n), U) \to \mathbf{Ho}(BU(n), BU)$ is a group homomorphism. Let $f, g\colon U(n) \to U$. The map $f + g$ is given by the composition

$$U(n) \xrightarrow{\Delta} U(n) \times U(n) \xrightarrow{f \times g} U \times U \xrightarrow{\oplus} U$$

where $\oplus$ denotes the H-space structure on U given by taking the direct sum of matrices. Now since B preserves products, $B(f+g)$ is given by the composition.

$$BU(n) \xrightarrow{\Delta} BU(n) \times BU(n) \xrightarrow{Bf \times Bg} BU \times BU \xrightarrow{\oplus} BU$$

where $\oplus$ is the H-space structure given by the classifying map of the direct sum of vector bundles. However, this is precisely $Bf + Bg$.

Hence, if we identify BU with $\{0\} \times BU$ then by definition we have $Bl_k^n\colon BU(n) \to BU$ is homotopy equivalent to the representing map of $\lambda_n^k i \in K(BU(n), o)$ where $i \in$

$K(BU(n), o)$ is represented by the canonical inclusion $i\colon BU(n) \simeq \{0\} \times BU(n) \subseteq \mathbb{Z} \times BU$. Moreover, B preserves direct limits and hence Bl^k is homotopy equivalent to the map represented by $\lambda^k \in K(BU, o)$.

Finally, since we have a natural isomorphism $\Omega B \cong 1$ we have a homotopy equivalence between l^k and the map represented by $\Omega\lambda^k i$. Moreover since $\Omega(\mathbb{Z}\times BU) = \Omega(\{0\}\times BU)$ we have $\lambda^k i_d = 0$ for $d \neq 0$. $\square$

This result is extremely useful, it illustrates that the λ-operations generate our entire plethory in a sense that allows looping. Since we are familiar with the properties of the λ-operations we now have sufficient information to be able to compute the plethory of unstable cohomology operations for K-theory.

Since our Ω-spectrum for K-theory is 2-periodic in a way which respects the ring spectrum structure, we are able to set $u = 1$ and compute the $\mathbb{Z}$-plethory of cohomology operations as graded over $\mathbb{Z}/2$ in both the source and target degrees.

Definition 3.2.33. Define the *linearisation* of the universal polynomials (see Definition 1.2.30) as follows. The polynomial $\widetilde{P}_n^L$ is the image of P_n under the natural map to indecomposables $\mathbb{Z}[x_1, \ldots, x_n; y_1, \ldots, y_n] \cong \mathbb{Z}[x_1, \ldots, x_n] \otimes \mathbb{Z}[y_1, \ldots, y_n] \to Q\mathbb{Z}[x_1, \ldots, x_n] \otimes \mathbb{Z}[y_1, \ldots, y_n]$, the polynomial $\widetilde{P}_n^R$ is the image of P_n under the under the natural map $\mathbb{Z}[x_1, \ldots, x_n; y_1, \ldots, y_n] \cong \mathbb{Z}[x_1, \ldots, x_n] \otimes \mathbb{Z}[y_1, \ldots, y_n] \to \mathbb{Z}[x_1, \ldots, x_n] \otimes Q\mathbb{Z}[y_1, \ldots, y_n]$ and $\widetilde{P}_{n,m}$ is the image of $P_{n,m}$ under the natural map $\mathbb{Z}[x_1, \ldots, x_{nm}] \to Q\mathbb{Z}[x_1, \ldots, x_{nm}]$.

Theorem 3.2.34. *As a $\mathbb{Z}/2$-$\mathbb{Z}/2$-bigraded $\mathbb{Z}$-plethory we have*

$$K^*(\underline{K}_n) \cong \begin{cases} \prod\limits_{d\in\mathbb{Z}} \mathbb{Z}[[\lambda^1 i_d, \lambda^2 i_d, \ldots]] & n = 0 \\ \Lambda_{\mathbb{Z}}[l^1, l^2, \ldots] & n = 1 \end{cases}$$

where $\deg_*(\lambda^k i_d) = 0$ *and* $\deg_*(l^k) = 1$. *The structure of a $\mathbb{Z}$-plethory with looping is determined by declaring that the l^k are primitive, the canonical map $K(\mathbb{Z} \times BU) \to$*

$K^*(\underline{K}_0)$ *is an isomorphism of Hopf algebras and the following formulae.*

$$\begin{aligned}
\Delta^\times(\lambda^k i_d) &= \sum_{rs=d} P_n(\lambda^1 i_r \otimes 1, \ldots, \lambda^k i_r \otimes 1; 1 \otimes \lambda^1 i_s, \ldots, 1 \otimes \lambda^k i_s) \\
\Delta^\times(l^k) &= \widetilde{P}^R_k(\lambda^1 i_0 \otimes 1, \ldots, \lambda^k i_0 \otimes 1; 1 \otimes l^1, \ldots, 1 \otimes l^k) \\
&\quad + \widetilde{P}^L_k(l^1 \otimes 1, \ldots, l^k \otimes 1; 1 \otimes \lambda^1 i_0, \ldots, 1 \otimes \lambda^k i_0) \\
\varepsilon^\times(\lambda^k i_d) &= \begin{cases} 1 & \textit{if } d = k = 1 \\ 0 & \textit{otherwise} \end{cases} \\
\lambda^{k'} i_{d'} \circ \lambda^k i_d &= \begin{cases} P_{k',k}(\lambda^1 i_d, \ldots, \lambda^{kk'} i_d) & \textit{if } d' = \binom{k}{d} \\ 0 & \textit{otherwise} \end{cases} \\
l^{k'} \circ l^k &= \widetilde{P}_{k',k}(l^1, \ldots, l^{kk'}) \\
\iota_0 &= \sum_{d \in \mathbb{Z}} \lambda^1_d \\
\iota_1 &= l^1 \\
\Omega(\lambda^k i_d) &= \begin{cases} l^k & \textit{if } d = 0 \\ 0 & \textit{o.w.} \end{cases} \\
\Omega l^k &= \sum_{d \in \mathbb{Z}} \widetilde{P}^L_k(-1, \ldots, (-1)^{k-1}; \lambda^1 i_d, \ldots, \lambda^k i_d).
\end{aligned}$$

Proof: We have already computed the Hopf algebra $K^*(\underline{K}_0)$. Moreover, since $l^k = \Omega\lambda^k i_0$, by Proposition 2.2.4 we see the l^k are primitive.

We appeal to Theorem 1.2.29 to compute the comultiplication. For $x, y \in K^0(X)$ we have already understood $\lambda^k i_d(xy)$ so it remains to understand the case where x and y are in odd dimension. Let $x, y \in K^{-1}(X)$ and consider $\Sigma x, \Sigma y \in K^0(\Sigma X_+, o)$. We have

$$\begin{aligned}
\lambda^k i_d(u^{-1}xy) &= \lambda^k i_d(\Sigma x \Sigma y) \\
&= \sum_{rs=d} P_k(\lambda^1 i_r(\Sigma x), \ldots, \lambda^k i_r(\Sigma x); \lambda^1 i_r(\Sigma y), \ldots, \lambda^k i_r(\Sigma y)).
\end{aligned}$$

However, in $K(\Sigma X_+)$ we have no non-trivial products and thus since P_k contains no linear terms this expression is zero.

To compute the comultiplication of the l^k we appeal to Proposition 2.2.10. For $|x| = 0$, $|y| = -1$ we have

$$l^k(xy) = \sum_{rs=d} (1 \otimes \Omega) P_k(\lambda^1 i_r \otimes 1, \ldots, \lambda^k i_r \otimes 1; 1 \otimes \lambda^1 i_s, \ldots, 1 \otimes \lambda^k i_s)(x \otimes y).$$

However, since Ω factors via the indecomposables, we have $(1 \otimes \Omega) P_n(\lambda^1 i_r \otimes 1, \ldots, \lambda^k i_r \otimes 1; 1 \otimes \lambda^1 i_s, \ldots, 1 \otimes \lambda^k i_s) = \widetilde{P}^R_k(\lambda^1 i_r \otimes 1, \ldots, \lambda^k i_r \otimes 1; 1 \otimes \Omega\lambda^1 i_s, \ldots, 1 \otimes \Omega\lambda^k i_s)$. Our result follows by cocommutativity.

We have already understood the composition $\lambda^{k'}i_{d'} \circ \lambda^k i_d$. For the composition $l^{k'} \circ l^k$, we have

$$\begin{aligned} l^{k'} \circ l^k &= \Omega(\lambda^{k'} i_0) \circ \Omega(\lambda^k i_0) \\ &= \Omega(\lambda^{k'} i_0 \circ \lambda^k i_0) \\ &= \Omega P_{k',k}(\lambda^1 i_0, \ldots, \lambda^{kk'} i_0). \end{aligned}$$

Now, Ω factors via the indecomposables and so this is precisely $\widetilde{P}_{k',k}(\Omega\lambda^1 i_0, \ldots, \Omega\lambda^{kk'} i_0) = \widetilde{P}_{k',k}(l^1, \ldots, l^{kk'})$.

It remains to compute $\Omega l^k = \Omega^2 \lambda^k i_0$. By Bott periodicity, we have

$$\begin{aligned} \Omega^2 \lambda^k i_0(x) &= \lambda^k i_0 \left[([\xi_1] - 1)x\right] \\ &= \sum_{d \in \mathbb{Z}} P_k(\lambda^1([\xi_1] - 1), \ldots, \lambda^k([\xi_1] - 1); \lambda^1 i_d(x), \ldots, \lambda^k i_d(x)). \end{aligned}$$

Now, $\lambda_t([\xi_1] - 1) = (1 + [\xi_1]t)/(1 + t)$ and so $\lambda^n([\xi_1] - 1) = (-1)^{n-1}([\xi_1] - 1)$. Since $([\xi_1] - 1)^2 = 0$, we have

$$\Omega l^k(x) = \Omega^2 \lambda^k i_0(x) = \sum_{d \in \mathbb{Z}} \widetilde{P}^L_k(-1, \ldots, (-1)^{k-1}; \lambda^1 i_d(x), \ldots, \lambda^k i_d(x)).$$

□

We remark that this plethory is not $\mathbb{Z}/2$-graded as a plethory with looping. For example, consider the Adams operation ψ^k viewed as an operation $K^0(-) \to K^0(-)$. If we let ψ^k_{-2} denote the same element but viewed as an operation $K^{-2}(-) \to K^{-2}(-)$, then we have $\Omega^2 \psi^k = k\psi^k_{-2}$.

Apparent from Theorem 3.2.34 is that if we were to make a suitable definition of a free plethory with looping, we should be able to express the plethory $K^*(\underline{K}_\bullet)$ as quotient of the free plethory with looping on the ungraded operations $K(\mathbb{Z} \times BU)$. I hope to formulate this precisely in future work.

3.3 The Morava K-theories

The Morava K-theories are powerful cohomology theories closed related to the theories of complex cobordism and Brown-Peterson cohomology. First introduced by Jack Morava in a series of unpublished works to obtain a better understanding of complex cobordism via the tools of algebraic geometry, they were introduced in the language of algebraic topology by a paper [29] of Johnson and Wilson in 1975. Since this paper, the deep relations between these theories and classical problems in algebraic topology have become apparent. Moreover, the Morava K-theories are reasonably computable. In this section we will briefly introduce the Morava K-theories before turning our attention to the study of their operations. For a more detailed introduction to the Morava K-theories including applications, refer to [56].

Theorem 3.3.1 ([56, Theorem 1.3])**.** *Let p be any prime. For all integers $n \geq 1$ there is a multiplicative, $2(p^n-1)$-periodic complex-oriented cohomology theory $K(n)^*(-)$ with coefficient ring $K(n)^* = \mathbb{F}_p[v_n, v_n^{-1}]$ where $|v_n| = -2(p^n-1)$ and the associated formal group law $F_n(x,y)$ satisfies the relation $[p]_{F_n}(x) = v_n x^{p^n}$. If p is odd, the product on $K(n)^*(-)$ is commutative, for $p=2$ it is non-commutative.*

Our theory of plethories is only applicable to *commutative* multiplicative cohomology theories so we shall restrict our attention to the Morava K-theories for p an odd prime.

3.3.1 Via the enriched Hopf ring

Wilson computed the Hopf ring for the Morava K-theories [53]. He gives his description locally, but we first give a global description of the Hopf ring. The test spaces needed for computing the Hopf ring are the complex projective space $\mathbb{C}P^\infty$ together with the $2p^n-1$ skeleton of the lens space $L^{2p^n-1} \subseteq L(\infty,p) = K(\mathbb{F}_p, 1)$.

Proposition 3.3.2 ([15, Theorem 17.16])**.** *We have an isomorphism of algebras*

$$K(n)_*(L^{2p^n-1}) \cong \frac{K(n)^*[e, a_{(0)}, \ldots, a_{(n-1)}]}{(a_{(0)}^p, a_{(1)}^p, \ldots)}$$

where $|e| = 1$ and $|a_{(i)}| = 2p^i$. The Hopf algebra structure is induced by the Hopf algebra structure on $H_*(L(\infty,p);\mathbb{F}_p)$ (Proposition 3.1.56).

Theorem 3.3.3 ([15, Theorem 17.19])**.** *The Hopf ring $K(n)_*\underline{K(n)}_\bullet$ is the free $K(n)^*[K(n)^*]$-Hopf ring on $K(n)_*(L^{2p^n-1}) \subseteq K(n)_*\underline{K(n)}_1$ and $K(n)_*(\mathbb{C}P^\infty) \subseteq K(n)_*\underline{K(n)}_2$ subject to the following relations.*

$$\begin{aligned}
e \circ e &= -b_{(0)} \\
a_{(i)} \circ a_{(j)} &= -a_{(j)} \circ a_{(i)} \\
b_{(i)}^{\circ p^n} &= v_n^{p^i} b_{(i)} \circ [v_n^{-1}] \\
e \circ b_{(0)}^{\circ p^n - 1} &= v_n e \circ [v_n^{-1}] \\
a_{(n-1)}^{*p} &= v_n a_{(0)} - a_{(0)} \circ b_{(0)}^{\circ p^n - 1} \circ [v_n].
\end{aligned}$$

Once again, we will need multi-index notation to discuss elements of our Hopf ring. From this point on, any multi-index denoted by I (including I', I'', etc.) will be a sequence of n integers $I = (i_0, \ldots, i_{n-1})$ with $i_k \in \{0, 1\}$ and any multi-index denoted by J (including J',J'', etc.) will be a finite length sequence of arbitrarily many integers $J = (j_0, \ldots, j_m)$ for some m with $0 \le j_k < p^n$.

For such a pair of multi-indices $I = (i_0, \ldots, i_{n-1})$, $J = (j_0, \ldots, j_m)$ define

$$\begin{aligned} a^I b^J &= a_{(0)}^{\circ i_0} \circ \cdots \circ a_{(n-1)}^{\circ i_{n-1}} \circ b_{(0)}^{\circ j_0} \circ \cdots \circ b_{(m)}^{\circ j_m} \in K(n)_{2|I|_p + 2|J|_p} \underline{K(n)}_{|I|+2|J|} \\ e a^I b^J &= e \circ a^I b^J \in K(n)_{1+2|I|_p+2|J|_p} \underline{K(n)}_{1+|I|+2|J|}. \end{aligned}$$

As in the case of ordinary cohomology we make the convention that $a^0 b^0 = [1] - [0]$ (and so $ea^0b^0 = e$).

To adequately manipulate these elements we require some additional functions on the set of multi-indices.

Definition 3.3.4. Define $\rho(I)$ to be the smallest k with $i_{n-k} = 0$. If $I = (1, \ldots, 1)$, then we make the convention that $\rho(I) = \infty$.

Define $t_0(I)$ to be the smallest k with $i_{n-k} = 1$. If $I = (0, \ldots, 0)$ then we make the convention that $t_0(I) = \infty$.

Define $l_1(I)$ to be the smallest k with $i_k = 0$. If $I = (1, \ldots, 1)$ then we make the convention that $l_1(I) = \infty$.

Define $l_1(J)$ to be the smallest k with $j_k < p^n - 1$. Note that since J is a finite sequence $l_1(J)$ is always finite.

Define $l_0(I)$ to be the smallest k with $i_k = 1$. If $I = (0, \ldots, 0)$ then we make the convention that $l_0(I) = \infty$.

Define $l_0(J)$ to be the smallest k with $j_k > 0$. If $J = (0, \ldots)$ then we make the convention that $l_0(J) = \infty$.

Define $c(I) = (i_{n-1}, i_0, \ldots, i_{n-2})$ and $c^{-1}(I) = (i_1, \ldots, i_{n-1}, i_0)$.

As in Definition 3.1.28 we define $s(J) = (0, j_0, j_1, \ldots)$. If $i_{n-1} = 0$ define $s(I) = (0, i_0, \ldots, i_{n-2})$. If $i_0 = 0$ define $s^{-1}(I) = (i_1, \ldots, i_{n-1}, 0)$ and if $j_0 = 0$ define $s^{-1}(J) = (j_1, j_2, \ldots)$.

We now unpack the global expression of our Hopf ring into the local form. Following [53], we exploit the periodicity of the spectrum $K(n)$, identify v_n with 1 and view the Hopf ring $K(n)_*(\underline{K(n)}_\bullet)$ as graded over $\mathbb{Z}/2(p^n - 1)$ in both degrees. As in the case of K-theory, this is sufficient to understand the plethory of cohomology operations and it is straightforward to convert our answer back into a $\mathbb{Z}$-$\mathbb{Z}$-bigraded object if we wish to study the action on cohomology algebras. Notice that under this identification the coefficient ring becomes $K(n)^* = \mathbb{F}_p$.

For readability, we shall write $\Lambda[x]$ for the exterior algebra over $\mathbb{F}_p$ on a single generator x and $P_k[x]$ for the truncated polynomial algebra $\mathbb{F}_p[x]/(x^{p^k})$ with the convention that $P_\infty[x] = \mathbb{F}_p[x]$.

Theorem 3.3.5 ([53, Theorem 1])**.** *We have an isomorphism of $\mathbb{F}_p$-algebras,*

$$K(n)_*(\underline{K(n)}_k) \cong \bigotimes_{j_0 < p^n - 1} \Lambda[a^I b^J e] \otimes \bigotimes_{\substack{i_0 = 0 \ or \\ j_0 < p^n - 1}} P_{\rho(I)}[a^I b^J]$$

where the first tensor product ranges over all multi-indices I, J with $|I| + 2|J| + 1 \equiv k \mod 2(p^n - 1)$ and the second tensor product ranges over all multi-indices with $|I| + 2|J| \equiv k \mod 2(p^n - 1)$.

Remark 3.3.6. From this point onwards we shall neglect to specify the conditions on lengths of the multi-indices required to ensure the $\bullet$-degrees remain compatible (modulo $2(p^n - 1)$) and leave context to determine the requirements on these.

The coalgebraic structure on the Hopf algebras $K(n)_*\underline{K(n)}_k$ in general produces rather complicated formulae. For example, if $I = \Delta_2$ and $J = 0$ we have

$$\psi(a^I b^J) = 1 \otimes a_{(2)} + a_{(2)} \otimes 1 + \sum_{\substack{i+j=p^2 \\ i,j \neq 0}} \frac{(p!)^{i_1}}{i_0! j_0! (i_1 p)!} (a_{(0)}^{*i_0} * a_{(1)}^{*i_1}) \otimes (a_{(0)}^{*j_0} * a_{(1)}^{*j_1})$$

where $i = i_0 + i_1 p$, $j = j_0 + j_1 p$ are the p-adic expansions of i and j respectively. In spite of this, we have a clean expression for the Verschiebung, which encodes a lot of important information about the comultiplication.

Proposition 3.3.7. *For multi-indices I, J we have*

$$V(a^I b^J) = \begin{cases} a^{s^{-1}I} b^{s^{-1}J} & i_0 = j_0 = 0 \\ 0 & \textit{otherwise} \end{cases}$$

and $V(ea^I b^J) = 0$.

Proof: Immediate from the fact that V respects $\circ$-multiplication. $\square$

The elements $a^I b^J \in K(n)_*\underline{K(n)}_\bullet$ for $i_0 = 1$ and $j_0 = p^n - 1$ are non-zero. The following result computes the Frobenius operator on $K(n)_*\underline{K(n)}_\bullet$ and consequently details how we can express these elements in terms of our generators. This is a generalisation of a result of Wilson [53, Proposition 1.2].

Proposition 3.3.8. *For multi-indices $I = (i_0, \dots, i_{n-1})$ and $J = (j_0, j_1, \dots)$ the following holds.*

1. *If $i_{n-1} = \dots = i_{n-m} = 1$ (i.e. $\rho(I) > m$) then*
$$(a^I b^J)^{*p^m} = \sum_{K \subseteq \{0,\dots,m-1\}} (-1)^{m(|I|+1)+|K|} a^{c^m I} b^{s^m J + (p^n-1)\Delta_K}.$$

2. *Let $m = \min(l_1(I), l_1(J))$. Then*
$$a^I b^J = (-1)^{m|I|}(a^{c^{-m}I} b^{s^{-m}J})^{*p^m} + \sum_{k=0}^{min(m,\rho(I))-1} (a^I b^{J-(p^n-1)\Delta_0})^{*p^k}.$$

Proof: (1) The case for $m = 1$ is [53, Proposition 1.2]. We now proceed by induction.

$$\begin{aligned}
(a^I b^J)^{*p^{m+1}} &= ((a^I b^J)^{*p^m})^{*p} \\
&= \sum_{K \subseteq \{0,\dots,m-1\}} (-1)^{m(|I|+1)+|K|} (a^{c^m I} b^{s^m J + (p^n-1)\Delta_K})^{*p} \\
&= \sum_{K \subseteq \{0,\dots,m-1\}} (-1)^{(m+1)|I|+m+|K|} \left[a^{c^{m+1} I} b^{s^{m+1} J + (p^n-1)(s\Delta_K + \Delta_0)} - v_n a^{c^{m+1} I} b^{s^{m+1} J + (p^n-1)s\Delta_K} \right] \\
&= \sum_{K \subseteq \{1,\dots,m\}} (-1)^{(m+1)|I|+m+|K|} \left[a^{c^{m+1} I} b^{s^{m+1} J + (p^n-1)(\Delta_K + \Delta_0)} - v_n a^{c^{m+1} I} b^{s^{m+1} J + (p^n-1)\Delta_K} \right] \\
&= \sum_{\substack{K \subseteq \{0,\dots,m\} \\ 0 \in K}} (-1)^{(m+1)|I|+m+1+|K|} a^{c^{m+1} I} b^{s^{m+1} J + (p^n-1)\Delta_K} \\
&\quad - \sum_{K \subseteq \{1,\dots,m\}} (-1)^{(m+1)|I|+m+|K|} a^{c^{m+1} I} b^{s^{m+1} J + (p^n-1)\Delta_K} \\
&= \sum_{K \subseteq \{0,\dots,m\}} (-1)^{(m+1)|I|+m+1+|K|} a^{c^{m+1} I} b^{s^{m+1} J + (p^n-1)\Delta_K}.
\end{aligned}$$

Hence our result follows.

(2) We prove the result in the case $\rho(I) = \infty$ by induction on m. If $m = 0$, the result holds trivially. Now suppose $m > 0$. Since $l_1(I), l_1(J) \geq m$, $i_0 = 1$ and $j_0 = p^n - 1$, so $J = ss^{-1}J + (p^n - 1)\Delta_0$. We now apply (1) to $c^{-1}I, s^{-1}J$ to obtain

$$(a^{c^{-1}I} b^{s^{-1}J})^{*p} = (-1)^{|I|}[a^I b^J - a^I b^{J-(p^n-1)\Delta_0}].$$

Rearranging yields

$$a^I b^J = (-1)^{|I|}(a^{c^{-1}I} b^{s^{-1}J})^{*p} + a^I b^{J-(p^n-1)\Delta_0}.$$

Now $\min(l_1(c^{-1}I), l_1(c^{-1}J)) = m - 1$, and so by the inductive hypothesis we have

$$a^{c^{-1}I} b^{s^{-1}J} = (-1)^{(m-1)|I|}(a^{c^{-m}I} b^{s^{-m}J})^{*p^{m-1}} + \sum_{k=0}^{m-2} (a^I b^{J-(p^n-1)\Delta_0})^{*p^k}.$$

Therefore, since the map $x \mapsto x^{*p}$ is linear,

$$\begin{aligned} a^I b^J &= (-1)^{|I|}\left[(-1)^{(m-1)|I|}(a^{c^{-m}I}b^{s^{-m}J})^{*p^{m-1}} + \sum_{k=0}^{m-2}(a^I b^{J-(p^n-1)\Delta_0})^{*p^k}\right]^{*p} + a^I b^{J-(p^n-1)\Delta_0} \\ &= (-1)^{m|I|}(a^{c^{-m}I}b^{s^{-m}J})^{*p^m} + \sum_{k=0}^{m-2}(a^I b^{J-(p^n-1)\Delta_0})^{*p^{k+1}} + \mu_*(v_n)a^I b^{J-(p^n-1)\Delta_0} \\ &= (-1)^{m|I|}(a^{c^{-m}I}b^{s^{-m}J})^{*p^m} + \sum_{k=0}^{m-1}(a^I b^{J-(p^n-1)\Delta_0})^{*p^k}. \end{aligned}$$

To prove the result for general $\rho(I)$ we note that for $k \geq \rho(I)$, $(a^I b^{J-(p^n-1)\Delta_0})^{*p^k} = 0$. $\square$

As in the case of singular cohomology, directly computing the dual Hopf algebra of $K(n)_*\underline{K(n)}_k$ is algebraically infeasible but we hope to find a generating set by first computing $PK(n)_*\underline{K(n)}_k$ and then applying Proposition E.0.39. To compute $PK(n)_*\underline{K(n)}_k$, we would like to appeal to Proposition E.0.37 but the algebraic structure on $K(n)_*\underline{K(n)}_k$ is not suitable for application of this result. An unpublished result of Tilman Bauer computes a Hopf algebra which is isomorphic to $K(n)_*\underline{K(n)}_k$ as an augmented coalgebra (and thus has the same primitives) but has algebraic structure satisfing the conditions of Proposition E.0.37.

Proposition 3.3.9. *Define the $\mathbb{F}_p$-vector space M_k to be*

$$M_k = \bigotimes_{j_0 < p^n - 1} \Lambda(e\alpha^I \beta^J) \otimes \bigotimes_{I,J} P_1(\alpha^I \beta^J)$$

where the first tensor product ranges over all multi-indices I, J with $|I| + 2|J| + 1 \equiv k \mod 2(p^n-1)$ and the second tensor product ranges over all multi-indices with $|I|+2|J| \equiv k \mod 2(p^n-1)$. We have isomorphisms of $\mathbb{F}_p$-vector spaces $M_k \to K(n)_*\underline{K(n)}_k$ defined on the monomial basis for M_k by

$$\begin{aligned} e\alpha^I\beta^J &\mapsto ea^I b^J \\ \alpha^I\beta^J &\mapsto a^I b^J \end{aligned}$$

and extending multiplicatively.

Proof: To show surjectivity we note that for any element of the monomial basis for $K(n)_*\underline{K(n)}_k$ we can replace any powers greater than p by using Proposition 3.3.8 (1). We show this map is injective by constructing a surjection $K(n)_*\underline{K(n)}_k \to M_k$. Since M_k and $K(n)_*\underline{K(n)}_k$ are of finite type, this will show that our map is an isomorphism. Proposition 3.3.8 (1) defines an obvious map of vector spaces $K(n)_*\underline{K(n)}_k \to M_k$. Moreover, Proposition 3.3.8 (2) gives an explicit element which this map will send to $\alpha^I\beta^J$ for all multi-indices I, J and so this map is surjective. $\square$

We *define* the coalgebra structure on M_k so that the map in Proposition 3.3.9 is an isomorphism of $K(n)^*$-coalgebras. Now consider the basis for $K(n)_*\underline{K(n)}_k$ which is the image of the monomial basis for M_k under the isomorphism $M_k \to K(n)_*\underline{K(n)}_k$. Explicitly it consists of elements of the form

$$ea^{I_1}b^{J_1}\dots ea^{I_m}b^{J_m}(a^{I'_1}b^{J'_1})^{*k_1}\dots(a^{I'_n}b^{J'_{m'}})^{*k_{m'}}$$

where $k_t < p$ for any multi-indices I_t, I'_t, J_t, J'_t provided that the first entry of each J_t is less than $p^n - 1$. In the dual basis for $K(n)^*\underline{K(n)}_k$, let $x_{I,J}$ denote the element dual to $a^I b^J$ and $y_{I,J}$ denote the element dual to $ea^I b^J$. The other elements will remain anonymous.

Theorem 3.3.10. *We have isomorphisms of $\mathbb{F}_p$-algebras*

$$K(n)^*\underline{K(n)}_k \cong \begin{cases} \mathbf{Set}(\mathbb{F}_p,\mathbb{F}_p)\otimes \bigotimes\limits_{\substack{I\neq I(0)\\ i_0=1 \text{ or}\\ j_0\geq 1}} P_{t_0(I)}[x_{I,J}] & k\equiv 0 \mod 2(p^n-1)\\ \bigotimes\limits_{j_0<p^n-1}\Lambda[y_{I,J}]\otimes \bigotimes\limits_{\substack{I\neq I(0)\\ i_0=1 \text{ or}\\ j_0\geq 1}} P_{t_0(I)}[x_{I,J}] & k\not\equiv 0 \mod 2(p^n-1). \end{cases}$$

Proof: For $k \not\equiv 0 \mod 2(p^n-1)$, we have $PK(n)_*\underline{K(n)}_k \cong PM_k$. By Proposition E.0.37, $PM_k \subseteq QM_k$. Since e, $a_{(0)}$, $b_{(0)}$ are primitive, ea^Ib^J is primitive for all I, J and a^Ib^J is primitive whenever $i_0 > 0$ or $j_0 > 0$. If $i_0 = j_0 = 0$ then $Va^Ib^J = a^{s^{-1}I}b^{s^{-1}J} \neq 0$ and so a^Ib^J is not primitive. Now $QK(n)^*\underline{K(n)}_k \cong DPK(n)^*\underline{K(n)}_k$ and so the elements $x_{I,J}$ with $i_0 > 0$ or $j_0 > 0$ together with the $y_{I,J}$ generate $K(n)^*\underline{K(n)}_k$.

For the truncations, if $t_0(I) < \infty$ then $\langle F^{t_0(I)}x_{I,J}, u\rangle = \langle x_{I,J}, V^{t_0(I)}u\rangle$. Now either $V^{t_0(I)}a^{I'}b^{J'} = 0$ or $V^{t_0(I)}a^{I'}b^{J'} = a^{I''}b^{J''}$ with $I'' = s^{-t_0(I)}I'$ and $J'' = s^{-t_0(I)}J'$. In particular $i''_{n-t_0(I)} = 0$ and so $I \neq I''$. Since V is an algebra map, we see $F^{t_0(I)}x_{I,J} = 0$. For $k < t_0(I)$, s^kI is well defined and we have $\langle F^kx_{I,J}, a^{s^kI}b^{s^kJ}\rangle = 1$ and thus $F^kx_{I,J} \neq 0$. Just as in Theorem 3.1.30 the result follows by Borel's theorem.

For $k \equiv 0 \mod 2(p^n-1)$, $\underline{K(n)}_k$ is not connected, but we can decompose $\underline{K(n)}_k = \mathbb{F}_p \times \underline{K(n)}'_k$ where $\underline{K(n)}'_k$ denotes the connected component of $\underline{K(n)}_k$ containing the base point. Now $QK(n)^*\underline{K(n)}'_k \cong DPK(n)_*\underline{K(n)}'_k \cong DPM_k$ and we can compute $K(n)^*\underline{K(n)}'_k$ as before.

Now via the Künneth isomorphism, we have $K(n)^*\underline{K(n)}_k \cong K(n)^*(\mathbb{F}_p)\otimes K(n)^*\underline{K(n)}'_k$. □

While the choice of basis for $K(n)_*\underline{K(n)}_k$ induced by M_k allowed us to easily compute $K(n)^*\underline{K(n)}_k$ as an algebra using some theoretical results, we have two major and related drawbacks. Firstly, it is not clear how the algebraic structure on $K(n)^*\underline{K(n)}_k$ respects our basis. By this we mean given some polynomial in the $x_{I,J}$ and $y_{I,J}$, we have no expression for this polynomial in terms of the dual basis. Secondly, to compute the comultiplication on the Hopf algebra $K(n)^*\underline{K(n)}_k$ we must understand the multiplication in $K(n)_*\underline{K(n)}_k$ on our basis elements. This itself is also non-trivial; we must appeal to Proposition 3.3.8 to express the product of basis elements as a linear combination of basis elements.

The first of these problems seems unavoidable, the algebraic structure on $K(n)^*\underline{K(n)}_k$ is inextricably tied to the coalgebraic structure on $K(n)_*\underline{K(n)}_k$. That said, while the coalgebraic structure on $K(n)_*\underline{K(n)}_k$ is fairly intractable, the Verschiebung has a simple expression given in Proposition 3.3.7. Hence, we should be able to understand how the Frobenius on $K(n)^*\underline{K(n)}_k$ respects our choice of basis. The following result gives the required formulae.

Proposition 3.3.11. *We have the following formulae in $K(n)^*\underline{K(n)}_k$.*

1. *If $i_0 = 1$ or $j_0 \geq 1$ then*

$$F^m x_{I,J} = \begin{cases} x_{s^m I, s^m J} & m < t_0(I) \\ 0 & \textit{otherwise.} \end{cases}$$

2. *If $i_0 = j_0 = 0$ and $m = \min(l_0(I), l_0(J)) < \infty$ then $x_{I,J} = F^m x_{s^{-m}I, s^{-m}J}$.*

Proof: For (1), by Theorem 3.3.10, $F^m x_{I,J} = 0$ for $m \geq t_0(I)$. Let $m < t_0(I)$ and thus s^{mI} is well defined. Now for any basis element u of $K(n)^*\underline{K(n)}_k$, we have $\langle F^m x_{I,J}, u\rangle = \langle x_{I,J}, V^m u\rangle$. Since V^m is a algebra map, this is zero unless $u = a^{s^m I} b^{s^m J}$.

(2) is an immediate consequence of (1) since if $I' = s^{-m}I, J' = s^{-m}J$ then $i'_0 = 1$ or $j'_0 \geq 1$ by definition. $\square$

For the other problem of computing the comultiplication on the Hopf algebra $K(n)^*\underline{K(n)}_k$, we can make some progress using theoretical results without explicitly having to understand the multiplication of our basis elements in $K(n)^*\underline{K(n)}_k$.

Proposition 3.3.12. *In the Hopf algebra $K(n)^*\underline{K(n)}_k$ we have the following formulae.*

1. *$\Delta^+(y_{I,J}) = 1 \otimes y_{I,J} + y_{I,J} \otimes 1$.*

2. *If $i_0 = 0$ or $j_0 < p^n - 1$ then $\Delta^+(x_{I,J}) = 1 \otimes x_{I,J} + x_{I,J} \otimes 1$.*

3. If $i_0 = 1$ and $j_0 = p^n - 1$

$$Vx_{I,J} = (-1)^{|I|} F^m x_{s^{-m}c^{-1}I, s^{-(m+1)}(J-(p^n-1)\Delta_0)}$$

where $m = \min(l_0(c^{-1}I), l_0(s^{-1}(J - (p^n - 1)\Delta_0))$.

Proof: Since all $a^I b^J e$ and $a^I b^J$ for $i_0 = 0$ or $j_0 < p^n - 1$ are indecomposable in $K(n)_*\underline{K(n)}_k$, by Proposition E.0.39 the corresponding dual elements $y_{I,J}$ and $x_{I,J}$ are primitive. For $i_0 = 1$ and $j_0 = p^n - 1$, let u be an element of our basis for $K(n)_*\underline{K(n)}_k$. Since $x_{I,J} \in QK(n)^*\underline{K(n)}_k$, we have $Vx_{I,J} \in QK(n)^*\underline{K(n)}_k \cong DPK(n)_*\underline{K(n)}_k$. Hence $\langle V(x_{I,J}), u\rangle = 0$ unless $u \in PK(n)_*\underline{K(n)}_k$. Now for $u = a^{I'}b^{J'} \in PK(n)_*\underline{K(n)}_k \cong PM_k$, we have $\langle V(x_{I,J}), a^{I'}b^{J'}\rangle = \langle x_{I,J}, Fa^{I'}b^{J'}\rangle$. By Proposition 3.3.8 this is zero if $i'_{n-1} = 0$ and if $i'_{n-1} = 1$, we have

$$\begin{aligned}\langle V(x_{I,J}), a^{I'}b^{J'}\rangle &= \langle x_{I,J}, (-1)^{|I|}a^{cI'}b^{sJ'+(p^n-1)\Delta_0} + (-1)^{|I|+1}v_n a^{cI'}b^{sJ'}\rangle \\ &= (-1)^{|I'|}\langle x_{I,J}, a^{cI'}b^{sJ'+(p^n-1)\Delta_0}\rangle + (-1)^{|I'|+1}v_n\langle x_{I,J}, a^{cI'}b^{sJ'}\rangle.\end{aligned}$$

Since $j_0 = p^n - 1$ this second term is always zero. The first term is zero unless $I = cI'$ and $J = sJ' + (p^n - 1)\Delta_0$ or equivalently $I' = c^{-1}I$ and $J' = s^{-1}(J - (p^n - 1)\Delta_0)$. Thus $Vx_{I,J} = (-1)^{|I|}x_{c^{-1}I, s^{-1}(J-(p^n-1))}$. Appealing to Proposition 3.3.11 to express this in terms of the generators for $K(n)^*\underline{K(n)}_k$ gives the desired result. $\square$

Of course, this does not give a complete expression of $K(n)^*\underline{K(n)}_k$ as a Hopf algebra; we have not computed the comultiplication on the non-primitive elements $x_{I,J}$ for $i_0 = 1$ and $j_0 = p^n - 1$.

3.3.2 The bialgebra of primitives

Our computational tricks for computing the plethystic structure of our cohomology operations from the enriched Hopf ring have relied heavily on having primitive generators and making use of Proposition E.0.39 to make the calculations manageable. Consequently, the presence of non-primitive generators in $K(n)^*\underline{K(n)}_k$ means we will be unable to compute the full plethystic structure by our usual methods. Nevertheless, a large proportion of our generators are primitive and we can hope to compute the plethystic structure on these generators. Alternatively, we should be able to compute the structure of the additive operations as a bialgebra.

Definition 3.3.13. For multi-indices I, I', J and J', define multi-indices $I \sqcup I'$, $J \bar{+} J'$ and $J \hat{+} J'$ and integers $\alpha(J, J')$ by the following formulae.

$$
\begin{aligned}
I \sqcup I' &= \begin{cases} I + I' & i_t + i'_t < 2 \text{ for all } t \\ 0 & \text{otherwise} \end{cases} \\
J \bar{+} J' &= (\overline{j_0 + j'_0}, \overline{j_1 + j'_1}, \dots) \\
J \hat{+} J' &= (\widehat{j_0 + j'_0}, \overline{j_1 + j'_1}, \dots)
\end{aligned}
$$

In the above, $\bar{k} = k - r(p^n - 1)$ where r is the least integer such that $k - r(p^n - 1) < p^n$ and $\widehat{k} = k - r'(p^n - 1)$ where r' is the least integer such that $k - r'(p^n - 1) < p^n - 1$.

The following lemma motivates these definitions. The sign conventions can be worked out by introducing a minus sign every time we interchange two elements of odd $\bullet$-degree.

Lemma 3.3.14. *In the Hopf ring $K(n)_* \underline{K(n)}_\bullet$ we have*

$$
\begin{aligned}
a^{I_1} b^{J_1} \circ a^{I_2} b^{J_2} &= \pm a^{I_1 \sqcup I_2} b^{J_1 \bar{+} J_2} \\
a^{I_1} b^{J_1} \circ e a^{I_2} b^{J_2} &= \pm e a^{I_1 \sqcup I_2} b^{J_1 \hat{+} J_2} \\
e a^{I_1} b^{J_1} \circ e a^{I_2} b^{J_2} &= \pm a^{I_1 \sqcup I_2} b^{\Delta_0 \bar{+} J_1 \bar{+} J_2}
\end{aligned}
$$

where $\Delta_0 \bar{+} J_1 \bar{+} J_2 = \Delta_0 \bar{+} (J_1 \bar{+} J_2)$.

Proof: This is immediate from the graded commutativity of $\circ$-multiplication together with the relations $e^{\circ 2} = -b_{(0)}$, $a_{(i)} \circ a_{(j)} = -a_{(j)} \circ a_{(i)}$, $b_{(k)}^{\circ p^n} = v_n^{p^k} b_{(k)}$ and $e \circ b_{(0)}^{\circ p^n - 1} = v_n e$. $\square$

Proposition 3.3.15. *We have an isomorphism of $\mathbb{F}_p$-coalgebras,*

$$
PK(n)^*(\underline{K(n)}_k) \cong \bigoplus_{j_0 < p^n - 1} \langle y_{I,J} \rangle \otimes \bigotimes_{\substack{i_0 = 0 \text{ or} \\ j_0 < p^n - 1}} \langle x_{I,J} \rangle.
$$

The coalgebraic structure is determined by the following formulae.

$$
\begin{aligned}
\Delta^\times(y_{I,J}) &= \sum_{\substack{I_1 \sqcup I_2 = I \\ J_1 \hat{+} J_2 = J}} \pm y_{I_1, J_1} \otimes x_{I_2, J_2} \pm v_n^{\alpha(J_1, J_2)} x_{I_1, J_1} \otimes y_{I_2, J_2} \\
\Delta^\times(x_{I,J}) &= \sum_{\substack{I_1 \sqcup I_2 = I \\ J_1 \bar{+} J_2 = J}} \pm x_{I_1, J_1} \otimes x_{I_2, J_2} + \sum_{\substack{I_1 \sqcup I_2 = I \\ \Delta_0 \bar{+} J_1 \bar{+} J_2 = J}} \pm y_{I_1, J_1} \otimes y_{I_2, J_2}
\end{aligned}
$$

Proof: The indecomposables of the cooperations $QK(n)_*(\underline{K(n)}_k)$ are spanned by the elements $ea^I b^J$ for $j_0 < p^n - 1$ and $a^I b^J$ for $i_0 = 0$ or $j_0 < p^n - 1$. The elements $ea^I b^J$ are dual to $y_{I,J}$ and for $i_0 = 1$ or $j_0 \geq 1$, the elements $a^I b^J$ are dual to $x_{I,J}$. By

Proposition 3.3.11, the elements $a^I b^J$ with $i_0 = j_0 = 0$ are dual to $x_{I,J}$ and are obtained by taking p-th powers of the $x_{I',J'}$ with $i' = 1$ or $j' \geq 1$. Hence since $PK(n)^*(\underline{K(n)}_k) \cong QK(n)_*(\underline{K(n)}_k)$, we have our isomorphism as $\mathbb{F}_p$-vector spaces.

The coalgebraic structure on $PK(n)^*(\underline{K(n)}_\bullet)$ is dual to the $\circ$-multiplication in $QK(n)_*(\underline{K(n)}_\bullet)$. Let $a^{I'}b^{J'}, a^{I''}b^{J''} \in QK(n)_*\underline{K(n)}_k$ and so $i'_0 = 0$ or $j'_0 < p^n - 1$ (resp. for i''_0, j''_0). Since $\langle \Delta^\times x_{I,J}, a^{I'}b^{J'} \otimes a^{I''}b^{J''} \rangle = \langle x_{I,J}, a^{I'}b^{J'} \circ a^{I''}b^{J''} \rangle$ the result is immediate from Lemma 3.3.14. □

The multiplication on the bialgebra $PK(n)^*\underline{K(n)}_\bullet$ is induced by the composition on $K(n)^*\underline{K(n)}_\bullet$. To understand this we must unpack the enrichment of the Hopf ring $K(n)_*\underline{K(n)}_\bullet$. This was computed by Boardman, Johnson and Wilson.

Theorem 3.3.16 ([15]). *For all r, r_*a_k is the coefficient of x^k in the formal identity*

$$r_*a(x) = \mathop{\huge *}_{i=0}^{p^n-1} b(x)^{\circ i} \circ [\langle r, a_i \rangle] * \mathop{\huge *}_{i=0}^{p^n-1} a(x) \circ b(x)^{\circ i} \circ [\langle r, c_i \rangle],$$

r_*b_k is the coefficient of x^k in the formal identity

$$r_*b(x) = [\langle r, 1_2 \rangle] * \mathop{\huge *}_{i=1}^{\infty} b(x)^{\circ i} \circ [\langle r, b_i \rangle],$$

and

$$r_*e = [\langle r, 1_1 \rangle] * [\langle r, e \rangle] \circ e.$$

The augmentation is determined by

$$\begin{aligned} \iota_1(e) &= 1 \\ \iota_1(a_{(k)}) &= 0 \\ \iota_2(b_{(k)}) &= \begin{cases} 1 & k = 0 \\ 0 & \textit{otherwise.} \end{cases} \end{aligned}$$

It is straightforward to compute the unit for composition for the augmentation.

Corollary 3.3.17. *The canonical map $\mathcal{I} \to K(n)^*\underline{K(n)}_\bullet$ is given by*

$$\iota_k \mapsto \begin{cases} x_{0,m\Delta_0} & k = 2m \\ y_{0,m\Delta_0} & k = 2m+1. \end{cases}$$

Proof: As in Corollary 3.1.32. □

Just as in the singular cohomology cases, we first understand the enrichment of our operations on the generators of our Hopf ring in the knowledge that we can appeal to Definition 1.3.21 to compute the enrichment of our operations on arbitrary elements.

Corollary 3.3.18. *In the enriched Hopf ring $K(n)_*\underline{K(n)}_\bullet$ we have the following formulae.*

$$x_{I,J*}a_{(k)} = \begin{cases} b_{(k-i)}^{\circ p^i} & I = \Delta_i, J = 0, k \geq i \\ 0 & \textit{otherwise} \end{cases}$$

$$x_{I,J*}b_{(k)} = \begin{cases} b_{(k-j)}^{\circ p^j} & I = 0, J = \Delta_j, k \geq j \\ 0 & \textit{otherwise} \end{cases}$$

$$x_{I,J*}e = 0$$

$$y_{I,J*}a_{(k)} = 0$$

$$y_{I,J*}b_{(k)} = 0$$

$$y_{I,J*}e = \begin{cases} e & I = J = 0 \\ 0 & \textit{otherwise} \end{cases}$$

Proof: Since $\deg_\bullet(x_{I,J}) = |I| + 2|J|$, for degree reasons we have $\langle x_{I,J}, a_i\rangle = \langle x_{I,J}, c_i\rangle = 0$, and thus $x_{I,J*}a_k = 0$ unless $I = \Delta_i, J = 0$. Since $x_{\Delta_i,0}$ is primitive, $\langle x_{\Delta_i,0}, u\rangle = 0$ if u is decomposable. Hence

$$\langle x_{\Delta_i,0}, a_k\rangle = \begin{cases} 1 & k = p^i \\ 0 & \text{otherwise} \end{cases}$$

$$\langle x_{\Delta_i,0}, c_k\rangle = 0.$$

Hence $x_{\Delta_i,0}a_k$ is the coefficient of x^k in $b(x)^{\circ p^i}$. Since we are working over a field of characteristic p, the map $z \mapsto z^{\circ p}$ is linear, and so

$$x_{\Delta_i,0*}(a_k) = \begin{cases} b_{\frac{k}{p^i}}^{\circ p^i} & p^i | k \\ 0 & \text{otherwise.} \end{cases}$$

For $x_{I,J*}b_{(k)}$, we notice for degree reasons we require either $I = 0, J = \Delta_j$ or $I = \Delta_i + \Delta_{i'}, J = 0$. In the former case, since x_{0,Δ_j} is primitive, we have $\langle x_{0,\Delta_j}, b_k\rangle = 1$ if and only if $k = p^j$ and so $x_{0,\Delta_j*}b_k$ is the coefficient of x^k in $b(x)^{\circ p^j}$. In the latter case, $\langle x_{I,J}, b_k\rangle = 0$ for all k and so $x_{I,J*}b_k = 0$. The result for $x_{I,J*}e$ is trivial.

Since $y_{I,J}$ is primitive, and dual to $ea^I b^J$, the remaining results are immediate. □

Just as the case for singular cohomology, using Definition 1.3.21 to compute the enrichment on arbitrary elements becomes complicated very quickly. Nonetheless, we are able to compute simple compositions by hand.

Proposition 3.3.19. *For multi-indices I, J with $i_0 = 0$ or $j_0 < p^n - 1$, then*

$$x_{I,J} \circ x_{\Delta_i,0} = \begin{cases} x_{\Delta_{m+i},0} & I = 0, J = p^i\Delta_m \textit{ and } m + i < n \\ 0 & \textit{otherwise.} \end{cases}$$

Proof: First note as $x_{\Delta_i,0} \in K(n)^{2p^i}\underline{K(n)}_1$, for degree reasons the composition $x_{I,J} \circ x_{\Delta_i,0}$ is zero unless $|I| + 2|J| = 2p^i$. Moreover, $\langle x_{I,J} \circ x_{\Delta_i,0}, -\rangle \colon K(n)^*\underline{K(n)}_1 \to K(n)^*\underline{K(n)}_1$. Since $x_{I,J} \circ x_{\Delta_i,0} \in PK(n)^*\underline{K(n)}_1 \cong DQK(n)^*\underline{K(n)}_1$ it suffices to determine $\langle x_{I,J} \circ x_{\Delta_i,0}, -\rangle$ on e and $a_{(k)}$.

We have $\langle x_{I,J} \circ x_{\Delta_i,0}, e\rangle = \langle x_{I,J}, x_{\Delta_i,0*}e\rangle = 0$ and

$$\begin{aligned}\langle x_{I,J} \circ x_{\Delta_i,0}, a_{(k)}\rangle &= \langle x_{I,J}, x_{\Delta_i,0*}a_{(k)}\rangle \\ &= \begin{cases} \langle x_{I,J}, b_{(k-i)}^{\circ p^i}\rangle & k \geq i \\ \langle x_{I,J}, 0\rangle & \text{otherwise.}\end{cases}\end{aligned}$$

Now $\langle x_{I,J}, b_{(k-i)}^{\circ p^i}\rangle$ is non-zero if and only if $I = 0$ and $J = p^i\Delta_{k-i}$. Our result follows. $\square$

The formulae for enrichment are identical to those for singular cohomology with coefficients in $\mathbb{F}_p$ (Corollary 3.1.64). As a result, we should be able to find a combinatorial expression for the composition which is somewhat related to the product formula for the Milnor basis elements (Proposition 3.1.52). Unfortunately, I have not been able to do this.

3.3.3 A useful filtration

The difficulties encountered in computing the plethory $K(n)^*\underline{K(n)}_\bullet$ stemmed from complexity of the coalgebraic structure on the Hopf ring $K(n)_*\underline{K(n)}_\bullet$. Nevertheless, we have a reasonable form for the Verschiebung and consequently can show that the non-primitive elements a^Ib^J for $i_0 = j_0 = 0$ are primitive modulo some iterations of the Verschiebung operator.

Lemma 3.3.20. *If I, J are multi-indices with $i_0 = j_0 = 0$ then in $K(n)_*\underline{K(n)}_\bullet$, we have $(V^m \otimes V^m)\psi(a^Ib^J) = 1 \otimes a^Ib^J + a^Ib^J \otimes 1$ where $m = \min(l_0(I), l_0(J))$.*

Proof: By the definition of m, we can write $I = s^{-m}I', J = s^{-m}J'$ where $i'_0 = 1$ or $j'_0 > 0$. Hence,

$$\begin{aligned}(V^m \otimes V^m)\psi(a^Ib^J) &= \psi(V^m a^Ib^J) \\ &= \psi(a^{I'}b^{J'}) \\ &= 1 \otimes a^{I'}b^{J'} + a^{I'}b^{J'} \otimes 1 \\ &= 1 \otimes V^m a^Ib^J + V^m a^Ib^J \otimes 1.\end{aligned}$$

$\square$

This suggests that a suitable filtration of $K(n)_*\underline{K(n)}_\bullet$ by kernels of the iterated Verschiebungs V^k might allow us to better understand the coalgebraic structure. We shall wish to apply homological algebraic techniques to reconstruct $K(n)_*\underline{K(n)}_\bullet$ from the associated graded object of the filtration and as such we should work in an abelian category.

Theorem 3.3.21 ([47]). *The category of bicommutative Hopf algebras over $\mathbb{F}_p$ is an abelian category. Given a Hopf algebra map $f\colon H \to H'$ the kernel and cokernels in this category, known as the* Hopf kernel *and* Hopf cokernel *respectively, are given by the following expressions.*

$$\begin{aligned}\operatorname{Hker} f &= \{x \in H | (1 \otimes f)\psi x = x \otimes 1\}\\ \operatorname{Hcoker} f &= \frac{H}{f(IH)H'}.\end{aligned}$$

Here, $f(IH)H'$ denotes the algebra ideal in H' generated by the image of the augmentation ideal IH under f.

Consider the following filtration of $K(n)_*\underline{K(n)}_k$ by sub-Hopf algebras.

$$\operatorname{Hker} V \subseteq \operatorname{Hker} V^2 \subseteq \cdots \subseteq \operatorname{Hker} V^k \subseteq \cdots \subseteq K(n)_*\underline{K(n)}_k$$

We can easily compute the algebraic structure of the sub-Hopf algebras $\operatorname{Hker} V^k$.

Lemma 3.3.22. *If $f\colon H \to H'$ is a map of Hopf rings then* $\operatorname{Hker} f$ *is an ideal for $\circ$-multiplication: for $x \in \operatorname{Hker} f$, $y \in H$ we have $x \circ y \in \operatorname{Hker} f$.*

Proof: Let $x \in \operatorname{Hker} f$ and $y \in H$. Writing $\psi(x) = x_{(1)} \otimes x_{(2)}$ and $\psi(y) = y_{(1)} \otimes y_{(2)}$ in sumless Sweedler notation we have

$$\begin{aligned}(1 \otimes f)\psi(x \circ y) &= (x_{(1)} \circ y_{(1)}) \otimes f(x_{(2)} \circ y_{(2)})\\ &= (x_{(1)} \otimes f(x_{(2)})) \circ (y_{(1)} \otimes f(y_{(2)}))\\ &= (x \otimes 1) \circ (y_{(1)} \otimes f(y_{(2)}))\\ &= (x \circ y_{(1)}) \otimes (1 \circ f(y_{(2)}))\\ &= x \circ (y_{(1)}\varepsilon f(y_{(2)})) \otimes 1\\ &= x \circ (y_{(1)}\varepsilon y_{(2)}) \otimes 1\\ &= x \circ y \otimes 1\end{aligned}$$

□

We provide the following conjectural result computing the sub-Hopf algebras of our filtration, and give a partial proof.

Conjecture 3.3.23. *We have the following isomorphisms of $\mathbb{F}_p$-algebras. For $k < n$,*

$$\operatorname{Hker} V^k \cong \bigotimes_{j_0 < p^n-1} \Lambda[ea^Ib^J] \otimes \bigotimes_{\substack{(i_0,j_0)\neq(1,p^n-1)\\ \exists t<k\colon (i_t,j_t)\neq(0,0)}} P_{\rho(I)}[a^Ib^J] \otimes \bigotimes_{\substack{(i_0,j_0)\neq(1,p^n-1)\\ i_{n-1}=1}} P_{\rho(I)-1}[(a^Ib^J)^{*p}],$$

and for $k \geq n$,

$$\operatorname{Hker} V^k \cong \bigotimes_{j_0 < p^n-1} \Lambda[ea^Ib^J] \otimes \bigotimes_{\substack{(i_0,j_0)\neq(1,p^n-1)\\ \exists t<k\colon (i_t,j_t)\neq(0,0)}} P_{\rho(I)}[a^Ib^J].$$

Partial Proof: If x is primitive, then $(1 \otimes V)\psi x = x \otimes 1$ and so $x \in \operatorname{Hker} V$. Hence $PK(n)_*\underline{K(n)}_k = PM_k$ and so $ea^Ib^J \in \operatorname{Hker} V \subseteq \operatorname{Hker} V^k$ for $j_0 < p^n - 1$ and $a^Ib^J \in \operatorname{Hker} V^k$ for all $i_0 > 0$ or $j_0 > 0$. For $i_0 = 1$ and $j_0 = p^n - 1$, we use Proposition 3.3.8 (2) to express a^Ib^J in terms of the generators of $K(n)_*\underline{K(n)}_k$ and see $(a^{I'}b^{J'})^{*p^m}$ with $(i'_0, j'_0) \neq (1, p^n - 1)$ is primitive whenever $m < \rho(I')$. Since $\operatorname{Hker} V \cong S^{[p]}PK(n)_*\underline{K(n)}_k$ this proves the result for $k = 1$.

Now for $k > 1$, notice that for $t < k$ we have

$$\psi a_{(t)} = 1 \otimes a_{(t)} + \sum_{i+j=p^t} a_i \otimes a_j + a_{(t)} \otimes 1$$

Since $V^k a_{(t)} = 0$ and for $j < p^t$ we can express a_j as a $*$-product of $a_{(0)}, \ldots a_{(t-1)}$ and thus $V^k a_j = 0$ we see that $a_{(t)} \in \operatorname{Hker} V^k$. Similarly, $b_{(t)} \in \operatorname{Hker} V^k$. Since $\operatorname{Hker} V^k$ is closed under $\circ$-multiplication by elements in $IK(n)_*\underline{K(n)}_k$ we see that $a^Ib^J \in \operatorname{Hker} V^k$ for I, J such that there exists $t < k$ with $i_t \neq 0$ or $j_t \neq 0$. Moreover, $\operatorname{Hker} V^k \subseteq K(n)_*(K(n)_\bullet)$ is a sub-Hopf algebra and thus contains the algebra generated by these elements. It remains to show that this is actually all of $\operatorname{Hker} V^k$. $\square$

Assuming this conjecture is true, we proceed with our argument. The inclusion maps $i_k\colon \operatorname{Hker} V^k \to \operatorname{Hker} V^{k+1}$ are monic in our category of Hopf algebras and therefore we can form short exact sequences.

$$\mathbb{F}_p \to \operatorname{Hker} V^k \to \operatorname{Hker} V^{k+1} \to \operatorname{Hcoker} i_k \to \mathbb{F}_p$$

In this context, the collection of Hopf algebras $\operatorname{Hcoker} i_k$ for $k \geq 0$ (where $\operatorname{Hcoker} i_0 = \operatorname{Hker} V$) forms the associated graded object of our filtration. These Hopf algebras turn out to be primitively generated.

Conjecture 3.3.24. *For $k > 0$, if $i_k\colon \operatorname{Hker} V^k \to \operatorname{Hker} V^{k+1}$ denotes the inclusion then we have isomorphisms of Hopf algebras,*

$$\operatorname{Hcoker} i_k \cong \bigotimes_{\substack{(i_0,j_0)\neq(1,p^n-1)\\ t<k \Rightarrow i_t=j_t=0\\ (i_k,j_k)\neq(0,0)}} P_1[a^Ib^J + \mathcal{I}]$$

where $\mathcal{I} = (i_k(I\,\mathrm{Hker}\,V^k))_{\mathrm{Hker}\,V^{k+1}}$ *and the generators are primitive.*

Proof: The augmentation ideal $I\,\mathrm{Hker}\,V^k$ is generated by the generators of $\mathrm{Hker}\,V^k$. Hence the ideal generated by the image of $I\,\mathrm{Hker}\,V^k$ in $\mathrm{Hker}\,V^{k+1}$ is

$$\mathcal{I} = (a^I b^J e, a^I b^J, (a^I b^J)^{*p^m})$$

Ao as algebras we have,

$$\mathrm{Hcoker}\, i_k = \frac{\mathrm{Hker}\,V^{k+1}}{\mathcal{I}} \cong \bigotimes_{\substack{(i_0,j_0)\neq(1,p^n-1)\\ t<k\Rightarrow i_t=j_t=0\\ (i_k,j_k)\neq(0,0)}} P_1[a^I b^J + \mathcal{I}].$$

It remains to show these generators are primitive. Let $a^I b^J + \mathcal{I}$ be such a generator. Suppose $i_k \neq 0$, we have $\psi(a^I b^J) = \psi(a_{(k)}) \circ \psi(a^{I-\Delta_k} b^J)$. Now $\psi(a_{(k)}) = 1 \otimes a_{(k)} + a_{(k)} \otimes 1 + \sum_{i+j=p^k} a_i \otimes a_j$ and $a_i \otimes a_j \in \mathcal{I} \otimes \mathcal{I}$. Hence by Lemma 3.3.22 we have $\psi(a^I b^J) = 1 \otimes a^I b^J + a^I b^J \otimes 1 + \mathcal{I} \otimes \mathcal{I}$ and so in $\mathrm{Hcoker}\, i_k \otimes \mathrm{Hcoker}\, i_k \cong \frac{\mathrm{Hker}\,V^{k+1} \otimes \mathrm{Hker}\,V^{k+1}}{\mathcal{I} \otimes \mathrm{Hker}\,V^{k+1} \oplus \mathrm{Hker}\,V^{k+1} \otimes \mathcal{I}}$,

$$\begin{aligned}\psi(a^I b^J + \mathcal{I}) &= \psi(a^I b^J) + \mathcal{I} \otimes \mathrm{Hker}\,V^{k+1} \oplus \mathrm{Hker}\,V^{k+1} \otimes \mathcal{I}\\ &= 1 \otimes a^I b^J + a^I b^J \otimes 1 + \mathcal{I} \otimes \mathrm{Hker}\,V^{k+1} \oplus \mathrm{Hker}\,V^{k+1} \otimes \mathcal{I}.\end{aligned}$$

□

Taking the $\mathbb{F}_p$-linear dual of our descending filtration of $K(n)_*\underline{K(n)}_k$ yields an ascending filtration of $DK(n)_*\underline{K(n)}_k \cong K(n)^*\underline{K(n)}_k$. Identifying $D\,\mathrm{Hker}\,V \cong \mathrm{Hcoker}\,DV \cong \mathrm{Hcoker}\,F_{K(n)^*\underline{K(n)}_k}$, this ascending filtration can be written as follows.

$$K(n)^*\underline{K(n)}_k \twoheadrightarrow \cdots \twoheadrightarrow \mathrm{Hcoker}\,F^2 \twoheadrightarrow \mathrm{Hcoker}\,F$$

Associated to this filtration are the short exact sequences

$$\mathbb{F}_p \to \mathrm{Hker}\,q_k \to \mathrm{Hcoker}\,F^{k+1} \twoheadrightarrow \mathrm{Hcoker}\,F^k \to \mathbb{F}_p$$

where $q_k\colon \mathrm{Hcoker}\,F^{k+1} \twoheadrightarrow \mathrm{Hcoker}\,F^k$ is the natural quotient map, the linear dual of i_k.

Due to the simple structure of the Hopf algebras in our filtration on homology, it is straightforward to compute the dual Hopf algebras in this filtration on cohomology.

We now turn our attention to the extension problem, and hence recovering $K(n)_*\underline{K(n)}_k$ or $K(n)^*\underline{K(n)}_k$ from the graded object associated to the filtration. As we have greater knowledge of the cooperations $K(n)_*\underline{K(n)}_k$, we will attempt to first solve the extension problem in homology and then take duals, converting this to a solution to the cohomological extension problem.

We need some theory on extensions of Hopf algebras. The trivial extension of two Hopf algebras H and H' is given by the tensor product $H \otimes H'$. However, we can construct non trivial extensions by introducing a twisting to the multiplication or the comultiplication. We follow work of Andruskiewitsch and Devoto [9] who studied extensions of Hopf algebras with very few conditions on (co-)associativity, (co-)commutativity and (co-)unitality building on [13]. Another useful reference is [7], and for a very brief introduction to this theory see Appendix F.

Conjecture 3.3.25. *Let i_k denote the inclusion $\operatorname{Hker} V^k \subseteq \operatorname{Hker} V^{k+1}$. First define*

$$\sigma_k^{I,J,I',J'}\colon P_1[a^I b^J + \mathcal{I}] \otimes P_1[a^{I'} b^{J'} + \mathcal{I}] \to P_{\rho(I)-1}[(a^I b^J)^{*p}]$$

$$\sigma_k^{I,J,I',J'}((a^I b^J + \mathcal{I})^{*i} \otimes (a^{I'} b^{J'} + \mathcal{I})^{*j}) = \begin{cases} (a^I b^J)^{*p} & \textit{if } (I,J) = (I',J') \textit{ and } i + j \geq p \\ 1 & \textit{if } i = j = 0 \\ 0 & \textit{otherwise.} \end{cases}$$

Now define $\sigma_k\colon \operatorname{Hcoker} i_k \otimes \operatorname{Hcoker} i_k \to \operatorname{Hker} V^k$ to be the composition

$$\bigotimes_{I,J} P_1[a^I b^J + \mathcal{I}] \otimes \bigotimes_{I',J'} P_1[a^{I'} b^{J'} + \mathcal{I}] \cong \bigotimes_{\substack{I,J \\ I',J'}} P_1[a^I b^J + \mathcal{I}] \otimes P_1[a^{I'} b^{J'} + \mathcal{I}]$$

$$\xrightarrow{\otimes \sigma_k^{I,J,I',J'}} \bigotimes_{I,J} P_{\rho(I)-1}[(a^I b^J)^{*p}] \subseteq \operatorname{Hker} V^k.$$

For $k < n$ we have an isomorphism of algebras

$$\operatorname{Hker} V^{k+1} \cong \operatorname{Hker} V^k \#_{\sigma_k} \operatorname{Hcoker} i_k$$

and for $k \geq n$ we have an isomorphism of algebras

$$\operatorname{Hker} V^{k+1} \cong \operatorname{Hker} V^k \otimes \operatorname{Hcoker} i_k.$$

Sketch proof: The isomorphism for $k \geq n$ is clear by inspection of both sides. For $k < n$, this is simple a generalisation of Example F.0.52 □

Of course we have not solved the extension problem as Hopf algebras. However, it feels like this should be possible if we use the correct co-2-cocycle to twist the comultiplication. Sadly, I have not been able to do this. We would then be able to study how the extra structure respects these extensions. We have seen that $\operatorname{Hker} V^k$ has nice behaviour with respect to $\circ$-multiplication (Lemma 3.3.22) and we could hope for similar results in the dual setting with the comultiplication $\Delta^\times$ and the composition.

Appendix A

Biring relations

The structure maps of a k-k' biring B satisfy the following relations.

Four relations expressing that Δ^+ is coassociative, cocommutative with counit ε^+ and antipode σ.

$$\begin{tikzcd}
B \arrow[r, "\Delta^+"] \arrow[d, "\Delta^+"] & B \otimes B \arrow[d, "\Delta^+ \otimes 1"] \\
B \otimes B \arrow[r, "1 \otimes \Delta^+"] & B \otimes B \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "\Delta^+"] \arrow[dr, "\Delta^+"'] & B \otimes B \arrow[d, "\tau"] \\
& B \otimes B
\end{tikzcd}$$

$$\begin{tikzcd}
B \arrow[r, "\varepsilon^+"] \arrow[dr, "\Delta^+"'] & k \arrow[r, "\eta"] & B \\
& B \otimes B \arrow[r, "\sigma \otimes 1"] & B \otimes B \arrow[u, "\mu"']
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "\Delta^+"] \arrow[dr, "\cong"'] & B \otimes B \arrow[d, "1 \otimes \varepsilon^+"] \\
& B \otimes k
\end{tikzcd}$$

Three relations expressing that $\Delta^\times$ is coassociative, cocommutative with counit $\varepsilon^\times$.

$$\begin{tikzcd}
B \arrow[r, "\Delta^\times"] \arrow[d, "\Delta^\times"] & B \otimes B \arrow[d, "1 \otimes \Delta^\times"] \\
B \otimes B \arrow[r, "\Delta^\times \otimes 1"] & B \otimes B \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
B \otimes B \arrow[r, "1 \otimes \varepsilon^\times"] & B \otimes k \\
B \arrow[u, "\Delta^\times"] \arrow[r, "c"'] & B \arrow[u, "\cong"']
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "\Delta^\times"] \arrow[dr, "\Delta^\times"'] & B \otimes B \arrow[d, "\tau"] \\
& B \otimes B
\end{tikzcd}$$

One relation expressing that comultiplication codistributes over coaddition.

$$\begin{tikzcd}
B \arrow[r, "\Delta^\times"] \arrow[d, "\Delta^+"] & B \otimes B \arrow[r, "1 \otimes \Delta^+"] & B \otimes B \otimes B \\
B \otimes B \arrow[r, "\Delta^\times \otimes \Delta^\times"] & B \otimes B \otimes B \otimes B \arrow[r, "1 \otimes \tau \otimes 1"] & B \otimes B \otimes B \otimes B \arrow[u, "\mu \otimes 1 \otimes 1"']
\end{tikzcd}$$

Three relations expressing that each $\gamma\lambda$ is a morphism of cogroup objects.

$$\begin{tikzcd}
B \arrow[r, "\gamma\lambda"] \arrow[d, "\Delta^+"] & B \arrow[d, "\Delta^+"] \\
B \otimes B \arrow[r, "\gamma\lambda \otimes \gamma\lambda"] & B \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "\gamma\lambda"] \arrow[dr, "\varepsilon^+"'] & B \arrow[d, "\varepsilon^+"] \\
& k
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "\gamma\lambda"] \arrow[d, "\sigma"] & B \arrow[d, "\sigma"] \\
B \arrow[r, "\gamma\lambda"] & B
\end{tikzcd}$$

Three relations expressing how $\gamma\lambda$ changes with respect to the ring structure on k'.

$$\begin{tikzcd} B \arrow[r, "\gamma(\lambda+\lambda')"] \arrow[d, "\Delta^+"] & B \\ B\otimes B \arrow[r, "\gamma\lambda\otimes\gamma\lambda'"] & B\otimes B \arrow[u, "\mu"'] \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "\gamma\lambda"] \arrow[dr, "\gamma(\lambda\lambda')"'] & B \arrow[d, "\gamma\lambda'"] \\ & B \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "\gamma 1"] \arrow[dr, "1"'] & B \arrow[d, equal] \\ & B \end{tikzcd}$$

One relation expressing how the colinear structure interacts with the comultiplication.

$$\begin{tikzcd} & B \arrow[dl, "\Delta^\times"'] \arrow[d, "\gamma\lambda"] \arrow[dr, "\Delta^\times"] & \\ B\otimes B \arrow[dr, "\gamma\lambda\otimes 1"'] & B \arrow[d, "\Delta^\times"] & B\otimes B \arrow[dl, "1\otimes\gamma\lambda"] \\ & B\otimes B & \end{tikzcd}$$

A map $f\colon B \to B'$ of k-algebras is a map of k-k'-birings if the following diagrams commute.

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "\Delta^+"] & B' \arrow[d, "\Delta^+"] \\ B\otimes B \arrow[r, "f\otimes f"] & B'\otimes B' \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "f"] \arrow[dr, "\varepsilon^+"'] & B' \arrow[d, "\varepsilon^+"] \\ & k \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "f"] \arrow[d, "\sigma"] & B' \arrow[d, "\sigma"] \\ B \arrow[r, "f"] & B' \end{tikzcd}$$

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "\Delta^\times"] & B' \arrow[d, "\Delta^\times"] \\ B\otimes B \arrow[r, "f\otimes f"] & B'\otimes B' \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "f"] \arrow[dr, "\varepsilon^\times"'] & B' \arrow[d, "\varepsilon^\times"] \\ & k \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "f"] \arrow[d, "\gamma\lambda"] & B' \arrow[d, "\gamma\lambda"] \\ B \arrow[r, "f"] & B' \end{tikzcd}$$

If we are using the equivalent co-k-linear structure $\beta\lambda = \varepsilon^\times \circ \gamma\lambda$, then we require the following diagram to commute.

$$\begin{tikzcd} k' \arrow[r, "\beta"] \arrow[dr, "\beta"'] & \mathbf{Alg}_k(B,k) \arrow[d, "\mathbf{Alg}_k(f,k)"] \\ & \mathbf{Alg}_k(B',k) \end{tikzcd}$$

Appendix B

Functor cartography

The following diagram details the domain and codomain relationships between some of the functors we defined in Section 1.3.

$$
\begin{tikzcd}
{}_k\mathbf{Mod}_k & \mathbf{Biring}_{k,k} \arrow[l, "Q"'] \arrow[r, bend left, "P"] \arrow[rr, bend left, "P_k"] \arrow[rrr, bend left, "\mathcal{A}"] \arrow[d, bend left, "T_\odot"] & \mathbf{Coalg}_{k,k} \arrow[l, bend left, "S"] \arrow[r, "E"] & \mathbf{Coalg}_k \arrow[ll, bend left, "S_k"] \arrow[r, "G"] & \mathbf{Set} \\
\mathbf{Alg}_k^{\neg\mathrm{com}} & \mathbf{Plethory}_k \arrow[l, "Q"'] \arrow[u, bend left, "U"] \arrow[r, bend left, "P"] \arrow[rr, bend left, "P_k"] \arrow[rrr, bend left, "\mathcal{A}"] & \mathbf{TwBialg}_k \arrow[l, bend left, "S"] \arrow[r, "E"] & \mathbf{Bialg}_k^{\neg\mathrm{com}} \arrow[ll, bend left, "S_k"] \arrow[r, "G"] & \mathbf{Monoid} \arrow[lll, bend left, "\Psi"]
\end{tikzcd}
$$

Appendix C

A primer on algebraic geometry

We recall a few basic results from algebraic geometry. For a more detailed reference see [23] or for a reference geared towards applications in algebraic topology, refer to [46].

Definition C.0.26. Let A be a k-algebra, we define the *spectrum* of A to be the functor $\mathrm{Spec}_k(A) = \mathbf{Alg}_k(A, -) \colon \mathbf{Alg}_k \to \mathbf{Set}$. Given a functor $X \colon \mathbf{Alg}_k \to \mathbf{Set}$, we define the *ring of functions* $\mathcal{O}_X = \mathrm{Nat}(X, \mathbb{A}^1)$ where $\mathbb{A}^1 = \mathrm{Spec}_k(k[x])$.

We define an *affine scheme* to be a representable functor $X \colon \mathbf{Alg}_k \to \mathbf{Set}$ and together with natural transformations this forms a category.

By the Yoneda lemma, we have $\mathcal{O}_{\mathrm{Spec}_k(A)} \cong A$ and thus we can recover A from $\mathrm{Spec}_k(A)$. This leads to the following result.

Proposition C.0.27 ([23])**.** *We have an anti-equivalence of categories from* $\mathbf{Alg}_k$ *to the category of affine schemes.*

We saw in Chapter 1, viewing k-algebras via their spectrum can be extremely enlightening. However, in the topologised setting, the story is a bit more complicated.

Definition C.0.28. Let A be a filtered k-algebra. We define the *formal spectrum* of A by

$$\mathrm{Spf}_k(A) = \varinjlim \mathbf{Alg}_k(A/F^aA, -) \colon \mathbf{Alg}_k \to \mathbf{Set}$$

. We define a *solid formal scheme* to be a functor $X \colon \mathbf{Alg}_k \to \mathbf{Set}$ which is of the form $\mathrm{Spf}_k(A)$ for some filtered k-algebra A. Together with natural transformations the solid formal schemes form a category.

Unlike in the discrete case, in general we are no longer able to recover A from $\mathrm{Spf}_k(A)$. Taking the direct limit is a bit too brutal and loses information. In turns out that we lose precisely the same information as when taking the completion and we have $\mathcal{O}_{\mathrm{Spf}_k(A)} \cong \widehat{A}$. Consequently, if $A = E^*(X)$ is a cohomology algebra, we are unable to see phantom classes in the corresponding solid formal scheme. However, if we are not interested in phantom classes then we have the following useful result.

Theorem C.0.29 ([23]). *The category of solid formal schemes is anti-equivalent to the category of completed Hausdorff k-algebras* $\mathbf{CAlg}_k$

Proposition C.0.30. *A $\mathcal{V}$-object in the category of solid formal schemes is the same as lifts of X to a functor* $\mathbf{Alg}_k \to \mathcal{V}$.

Proof: This is a consequence of Theorem C.0.29 and Corollary 1.1.17. □

Appendix D

Generalised cohomology theories

We follow Boardman's [14] definitions of generalised cohomology theories and here we just recall the main results and definitions.

Definition D.0.31. An *ungraded cohomology theory* is a contravariant functor $h\colon \mathbf{Ho} \to \mathbf{Ab}$ satisfying the following two *Eilenberg-Steenrod axioms.*

1. If $X = A \cup B$ where A, B are sub-complexes of a CW-complex X, and $y \in h(A)$ and $z \in h(B)$ agree on their restrictions to $A \cap B$ then there exists $x \in h(X)$ that lifts both y and z.
2. For any disjoint union $X = \coprod_i X_i$, the inclusions $X_i \to X$ induce an isomorphism $h(X) \cong \prod_i h(X_i)$.

Definition D.0.32. Given an ungraded cohomology theory, and a space X with base point o, we define the *reduced cohomology* by the split short exact sequence

$$0 \to h(X, o) \to h(X) \to h(o).$$

For a space X and a subspace A, we define the *relative cohomology* by the short exact sequence

$$0 \to h(X, A) \to h(X) \to h(A).$$

We remark that we can recover the absolute cohomology of a space X from the reduced homology by constructing the disjoint union of X with a new base point to form X^+. This yields an isomorphism $h(X^+, o) \cong h(X)$.

Definition D.0.33. A *graded cohomology theory* $E^*(-)$ is a collection of ungraded cohomology theories $E^n(-)$ for each $n \in \mathbb{Z}$, connected by natural *suspension isomorphisms*

$$\Sigma\colon E^n(X) \cong E^{n+1}(S^1 \times X, o \times X)$$

of abelian groups. On the level of reduced cohomology, we have isomorphisms $E^n(X, o) \cong E^{n+1}(\Sigma X, o)$ where $\Sigma X = S^1 \wedge X$ denotes the *reduced suspension* of X.

We have the following definition of a multiplicative graded cohomology theory, which has the obvious analogue in the ungraded setting.

Definition D.0.34. A graded cohomology theory $E^*(-)$ is *multiplicative* if it naturally takes values in commutative graded E^*-algebras where $E^* = E^*(T)$, the cohomology of the one point space is known as the *coefficient ring*. Equivalently, we have natural *cross product maps*

$$\times\colon E^*(X) \otimes_{E^*} E^*(Y) \to E^*(X \times Y).$$

Every cohomology theory has an associated homology theory $E_*(-)$, and in nice cases these are dual in the following sense.

Definition D.0.35. Let M be an k-module. We define the *dual-finite filtration* on $DM = \mathbf{Mod}_k(M, k)$ to consist of the sub-modules $F^L DM = \ker(DM \to DL)$ where L ranges through all finitely generated sub-modules of M. This topology induces the *dual-finite topology* on M.

Theorem D.0.36 ([14, Theorem 4.14])**.** *If $E_*(X)$ is a free E^*-module then we have an homeomorphism which is an isomorphism of abelian groups $E^*(X) \cong DE_*(X)$ between the profinite topology on $E^*(X)$ (see Definition 2.1.27) and the dual-finite topology on $DE_*(X)$.*

Appendix E

Hopf algebras and Hopf rings

In this section we state some useful results for performing computations involving Hopf algebras and Hopf rings. For a more detailed reference, refer to [36]. We assume that all our Hopf algebras are bicommutative and biunital.

Proposition E.0.37 ([36, Proposition 4.20])**.** *If H is a connected Hopf algebra over a field k of characteristic $p \neq 0$, then the natural morphism $PH \to QH$ is a monomorphism if and only if $FH_n = 0$ for $n > 0$ where $F\colon H \to H$ denotes the Frobenius map.*

Proposition E.0.38. *Let H be a Hopf ring with augmentation ideal IH. If $x \in PH$ is primitive and $y \in IH$ then $x \circ y \in PH$.*

Proof: We have the following sequence of equalities.

$$\begin{aligned}
\psi(x \circ y) &= \psi(x) \circ \psi(y) \\
&= (1 \otimes x + x \otimes 1) \circ (y_{(1)} \otimes y_{(2)}) \\
&= 1 \circ y_{(1)} \otimes x \circ y_{(2)} + x \circ y_{(1)} \otimes 1 \circ y_{(2)} \\
&= \varepsilon(y_{(1)}) \otimes x \circ y_{(2)} + x \circ y_{(1)} \otimes \varepsilon(y_{(2)}) \\
&= 1 \otimes x \circ (\varepsilon(y_{(1)})y_{(2)}) + x \circ (y_{(1)}\varepsilon(y_{(2)})) \otimes 1 \\
&= 1 \otimes x \circ y + x \circ y \otimes 1.
\end{aligned}$$

□

Proposition E.0.39 ([36, Proposition 3.10])**.** *Suppose H is a Hopf algebra over k. If $DH = \mathbf{Mod}_k(H, k)$, P denotes the functor of primitives and Q the functor of indecomposables then we have $PDH = DQH$. Moreover, if the underlying k-module of H is projective of finite type and*

$$IH \to IH \otimes IH \to QH \to 0$$

is split exact then $DPDH = QH$.

Theorem E.0.40 (Borel's theorem, [16])**.** *If H is a Hopf algebra over $\mathbb{F}_p$ then H is isomorphic as an algebra to the tensor product of monogenic Hopf algebras.*

Proposition E.0.41. *Let H be a Hopf ring and let $\pi\colon IH \to QH$ denote the canonical projection from the augmentation ideal to the indecomposables. For $x, y \in IH$, if $\pi(x \circ y) \neq 0$ then $\pi(x) \neq 0$ and $\pi(y) \neq 0$.*

Proof: Suppose $\pi(y) = 0$ so we can write $y = z * z'$ for $z, z' \in IH$. Now, $x \circ y = x \circ (z * z') = (x_{(1)} \circ z) * (x_{(2)} \circ z')$. However, $\varepsilon(x_{(1)} \circ z) = \varepsilon(x_{(1)})\varepsilon(z) = 0$ and similarly $\varepsilon(x_{(2)} \circ z) = 0$. Hence $\pi(x \circ y) = 0$ and the result follows. $\square$

Proposition E.0.42 ([41, Lemma 7.1])**.** *Let H be an $\mathbb{F}_p[\mathbb{F}_p]$-Hopf ring with Frobenius F and Verschiebung V. For all $x, y \in H$, we have $V(x \circ y) = V(x) \circ V(y)$ and the* Frobenius reciprocity *equation: $F(x \circ V(y)) = x \circ F(y)$.*

Appendix F

Extensions of Hopf algebras

We remark that the following is special case of much more general construction which relaxes conditions on (co-)associativity, (co-)commutativity and (co-)unitality. We introduce the minimal technical machinery necessary for our purposes while still obtaining some intuition. For a more detailed treatment refer to [9] and [7].

Definition F.0.43. Let H be a Hopf algebra over k, and A a k-algebra then we call a map $H \otimes A \to A, h \otimes a \mapsto h \cdot a$ an *action* if the following conditions hold.

1. $(gh) \cdot a = g \cdot (h \cdot a)$.
2. $1 \cdot a = a$.
3. $h \cdot (ab) = (h_{(1)} \cdot a)(h_{(2)} \cdot b)$.
4. $h \cdot 1 = \varepsilon(h)1$.

Example F.0.44. For any Hopf algebra H over k and k-algebra A, we have the *trivial action*

$$h \cdot a = \varepsilon(h)a.$$

Definition F.0.45. Given a Hopf algebra H over k and k-algebra A with action $H \otimes A \to A$, we define the *smash product* of A with H, $A\#H$ to be the vector space $A \otimes H$. Writing $a\#h$ for $a \otimes h$, the multiplication is given by

$$(a\#g)(b\#h) = a(g_{(1)} \cdot b)\#g_{(2)}h.$$

This makes $A\#H$ a (not necessarily commutative) algebra with unit $1\#1$.

Example F.0.46. For any Hopf algebra H over k and k-algebra A, if $H \otimes A \to A$ is the trivial action, then

$$A\#H = A \otimes H.$$

More generally, we can add a twisting to the multiplication in the smash product.

Definition F.0.47. For a Hopf algebra H over k and a k-algebra A with action $\cdot\colon H \otimes A \to A$, a 2-*cocycle* (relative to the action $\cdot$) is a map $\sigma\colon H \otimes H \to A$ satisfying: for $g, h, k \in H$,

1. $\sigma(1 \otimes h) = \sigma(h \otimes 1) = \varepsilon(h)1$.
2. $[g_{(1)} \cdot \sigma(h_{(1)} \otimes k_{(1)})]\sigma(g_{(2)} \otimes h_{(2)}k_{(2)}) = \sigma(g_{(1)} \otimes h_{(1)})\sigma(g_{(2)}h_{(2)} \otimes k)$.

Example F.0.48. For any Hopf algebra H over k and any k-algebra A, we have the *trivial* 2-*cocycle*, $H \otimes H \to A$ given by

$$\sigma(g \otimes h) = \varepsilon(g)\varepsilon(h)1.$$

Definition F.0.49. Given a Hopf algebra H over k, a k-algebra A together with an action $H \otimes A \to A$, and a 2-cocycle, σ, we define the *crossed product* $A\#_\sigma H$ to be the (not necessarily commutative) algebra to have underlying vector space $A \otimes H$ and multiplication given by

$$(a\#g)(b\#h) = a(g_{(1)} \cdot b)\sigma(g_{(2)}, h_{(1)})\#g_{(3)}h_{(2)}.$$

Example F.0.50. For a Hopf algebra H over k and a k-algebra A with action $\cdot\colon H \otimes A \to A$, if σ is the trivial 2-cocycle, then

$$A\#_\sigma H \cong A\#H.$$

Proposition F.0.51 ([7])**.** *Let A be a k-algebra, H a Hopf algebra over k and $\sigma\colon H \otimes H \to A$ a 2-cocycle. The crossed product $A\#_\sigma H$ is commutative if and only if $H \otimes A \to A$ is trivial and σ is symmetric i.e. $\sigma(g \otimes h) = \sigma(h \otimes g)$.*

As a consequence, we shall always restrict to the case where 2-cocycles are symmetric and the action is trivial. The following example illustrates how we can construct a non-trivial extension of a Hopf algebra and an algebra.

Example F.0.52. Let $A = \mathbb{F}_p[x]/(x^p)$, $H = \mathbb{F}_p[y]/(y^p)$ and suppose H acts on A by the trivial action. Let $\sigma\colon H \otimes H \to A$ be the 2-cocycle given by

$$\sigma(y^i \otimes y^j) = \begin{cases} x & \text{if } i + j \geq p \\ 1 & \text{if } i = j = 0 \\ 0 & \text{o.w.} \end{cases}$$

We claim that $(1\#y)^n = x^{\lfloor n/p \rfloor}\#y^{\bar{n}}$ where $\bar{n}$ denotes the reduction of n modulo p, and proceed by induction.

$$\begin{aligned}
(1\#y)^n &= (1\#y)(1\#y)^{n-1} \\
&= (1\#y)(x^{\lfloor n/p \rfloor}\#y^{\bar{n}}) \\
&= \sum_{i+j=\bar{n}} \binom{\bar{n}}{i} \left(x^{\lfloor n/p \rfloor}\sigma(1, y^i)\#y^{j+1} + x^{\lfloor n/p \rfloor}\sigma(y, y^i)\#y^j \right) \\
&= x^{\lfloor n/p \rfloor}\#y^{\bar{n}+1} + x^{\lfloor n/p \rfloor}\sigma(y, y^{\bar{n}})\#1 \\
&= \begin{cases} x^{\lfloor n/p \rfloor}\#y^{\bar{n}+1} & \text{if } \bar{n} \neq p-1 \\ x^{\lfloor n/p \rfloor+1}\#1 & \text{if } \bar{n} = p-1 \end{cases} \\
&= x^{\lfloor (n+1)/p \rfloor}\#y^{\overline{n+1}}
\end{aligned}$$

Hence $A\#_\sigma H$ is a vector space of dimension p^2 with an element of order p^2, and so

$$A\#_\sigma H = \frac{\mathbb{F}_p[1\#y]}{((1\#y)^{p^2})}.$$

There is a analogous construction which takes a Hopf algebra H over k and a k-coalgebra C, and produces a k-coalgebra which is isomorphic to $H \otimes C$ as a k-module, but incorporates a twisting defined by a *co-2-cocycle* $\tau \colon C \to H \otimes H$. Moreover, this construction can be combined with a twisting of the multiplication. Thus it is possible to produce a Hopf algebra over k which is a tensor product of two given Hopf algebras over k as k-module, but has twisting in both the multiplication and comultiplication. For details refer to [9, Theorem 2.20].

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