On the Clebsch-Gordan problem in prime characteristic

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MARCH 2018

Abstract

We study the modules $\nabla(r) \otimes \Delta(s)$ for the group $SL_2(k)$, where k is an algebraically closed field of characteristic $p > 0$. We are primarily concerned with the decomposition of such modules into indecomposable summands, and construct a method which allows one to give the decomposition for any such module in any characteristic. We also develop a variety of other techniques to give the decomposition in particular cases. Furthermore, we give a complete account of exactly when such a module is a tilting module for all primes p, and give a number of results in a more general setting.

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Dedication and Acknowledgements

First and foremost, I'm particularly grateful to my supervisors Michael Bate and Stephen Donkin, for their constant support and encouragement. Indeed, I am grateful to the whole Department of Mathematics for making my time in York engaging and enjoyable. Thanks guys.

I would also like to acknowledge financial support generously given by EPSRC, and further financial support from the Department of Mathematics at the University of York, without which this work would not have happened.

Author's Declaration

I declare that this thesis is a presentation of original work and I am the sole author, except where indicated otherwise. This work has not previously been presented for an award at this, or any other, University. All sources are acknowledged as References.

CHAPTER 1

INTRODUCTION

In this thesis we investigate the modules $\nabla(r) \otimes \Delta(s)$ for the algebraic group $SL_2(k)$, where k is an algebraically closed field of prime characteristic, and $r, s \in \mathbb{N}$. Our primary goal is to solve a classic representation theory problem; to decompose each tensor product into a direct sum of indecomposable submodules.

Over a field of characteristic zero, the corresponding problem is well known and given by the celebrated Clebsch-Gordan equation,

$$V(r) \otimes V(s) = \bigoplus_{i=0}^{s} V(r+s-2i).$$

The elegance of this solution however, belies the complexity found over a field of prime characteristic. We will see very quickly though, that thanks to a result in [27], the Clebsch-Gordan coefficients (the integers $r+s-2i$) still have a crucial role to play.

In Chapter 2, we give all the necessary preliminary definitions and results for the later chapters. The subjects of algebraic groups and representation theory are both deep and wide ranging. In most cases we will give as general a result as possible, and refer the reader to the essential literature for further reading. In some cases it will be convenient to just give the results for $G = SL_2(k)$, in order to avoid the unnecessary complexity of the general result (in fact, this can already be seen in our use of natural numbers r and s as parameters). Since we will always be working over an algebraically closed field, we will not use the language of group schemes, however many results will be cited from Jantzen's book "Representations of Algebraic Groups" [21], which is given in this framework.

For an arbitrary reductive algebraic group G, very little is known about the modules $\nabla(\lambda)$ or $\Delta(\mu)$ for dominant weights λ and μ, let alone the modules $\nabla(\lambda)\otimes\Delta(\mu)$. For the case where $G = SL_2(k)$ however, one can use the realisation of $\nabla(r)$ as the r^{th} symmetric power of the natural module E to give a vector space basis to work with. Once we have our hands on such a basis, we can start to consider generating elements and begin to construct explicit maps. This is the approach taken in chapters 3 and 4, where we describe the endomorphism algebra of $\nabla(r)\otimes\Delta(s)$ in a number of cases.

Recent work by Stephen Doty and Anne Henke gives the result to an analogous problem; to decompose the tensor product of two simple modules for $SL_2(k)$. In their paper "Decomposition of tensor products of modular irreducibles for SL_2" [13], Doty and Henke give the tensor product $L(r)\otimes L(s)$ as a direct sum of twisted tensor products of tilting modules, using the Steinberg tensor product formula. This work relies on the fact that for SL_2, the indecomposable tilting modules are known and can be computed inductively (see [9]). In Chapter 5, we also utilise the powerful theory of tilting modules, and describe exactly when the module $\nabla(r)\otimes\Delta(s)$ is a tilting module. In general (and even just for SL_3), the indecomposable tilting modules are unknown. However, we hope that the methods developed in this chapter will generalise to give at least a partial result in some further cases, which may illuminate the general theory.

In 2012, as part of his Masters thesis [4], Mikaël Cavallin investigated the tensor product of two induced modules for SL_2, again over an algebraically closed field of positive characteristic. Using the duality between polynomial GL-modules and modules for the Schur algebra, Cavallin was able to decompose $\nabla(r)\otimes\nabla(s)$ as a direct sum of injective polynomial GL_2-modules.

The two cases described above are both variants of the problem we study in this thesis, and in fact, using that the simple module $L(ap^n-1)$ is equal to both $\nabla(ap^n-1)$ and $\Delta(ap^n-1)$, there is some overlap between all of these problems. In the final chapter, we give a general method to decompose any module $\nabla(r)\otimes\Delta(s)$ for a given characteristic p. More specifically, the method gives the good or Weyl filtration of each indecomposable summand. The ability to do this relies wholly on the known characters of the tilting modules for SL_2 and a result in [27] which gives sufficient conditions for $\nabla(r)\otimes\Delta(s)$ to be a tilting module. We then abstract the requirements necessary to do this, by introducing the notion of a Clebsch-Gordan module.

Chapter 2

Preliminaries

In this chapter we will introduce all the objects of study, and give the underlying theory upon which this thesis is based. None of the work presented here is new, and references are given where appropriate. For brevity, we will work in the context of affine algebraic groups, rather than the more general framework of group schemes. Whilst the theory of group schemes is immensely powerful, since we will only be considering algebraically closed fields, it's not necessary here. Some results, however, will be referenced in the context of group schemes.

In the main body of this thesis, we will be working primarily with the group $SL_2(k)$, for which many simplifications can be made. In this chapter though, we attempt to give a more general introduction to the theory we use, whilst giving explicitly those results for $SL_2(k)$ which will be essential.

2.1 Algebraic Groups

First we will introduce the main object of the theory, the algebraic group G. A first treatment of algebraic groups would not be complete without first doing some algebraic geometry, so we begin with varieties.

Since in this thesis it's sufficient to consider affine varieties, we will introduce only these here. For a more general treatment of varieties the reader is referred to [19, Chapter I], or the classic text by Robin Hartshorne [17].

2.1.1 Affine Varieties

This section follows lecture notes supplied by Stephen Donkin [11]. The intrinsic notion of an affine variety given, and the results that follow, can be found in the first chapter of [30]. We also take some results from the first chapter of [19].

First we must fix an algebraically closed field k. By affine n-space, denoted $\mathbb{A}^n$, we mean the set of all n-tuples, where each entry is an element of the field k. The first objects we will discuss are affine varieties. Intuitively, these can be thought of as subsets of affine n-space which consist of the solutions to some set of polynomial equations in n independent variables. For example, in $\mathbb{A}^2$ we have the line given by the equation $y = x$, and the circle given by the equation $x^2 + y^2 = 1$; these are both affine varieties. With this in mind, we will give a slightly more abstract definition, which has the advantage of being coordinate free.

For any set V, denote by $\mathrm{Map}(V,k)$ the set of all maps from V to k. We can regard this as an algebra via pointwise operations. For each $x \in V$, denote the evaluation map at x by $\varepsilon_x : \mathrm{Map}(V,k) \to k$, given by $\varepsilon_x(f) = f(x)$ for all $f \in \mathrm{Map}(V,k)$. Note that we will often restrict ε_x to some subalgebra of $\mathrm{Map}(V,k)$ without changing notation.

Definition 2.1.1. An *affine variety over k* is a pair (V,A) where V is a set and A is a finitely generated k-subalgebra of $\mathrm{Map}(V,k)$ such that the map

$$\begin{aligned} V &\longrightarrow \mathrm{Hom}_{k\text{-alg}}(A,k) \\ x &\longmapsto \varepsilon_x \end{aligned}$$

is a bijection.

Usually, if the pair (V,A) is an affine variety, we will simply say that V is an affine variety and denote A by $k[V]$. We will call $k[V]$ the coordinate algebra of V.

For example, take $V = \mathbb{A}^n$ and $A = k[T_1,\ldots,T_n]$, the polynomial algebra in n independent variables. The set A is made a subalgebra of $\mathrm{Map}(V,k)$ by defining $T_i(x) = x_i$, where x is the n-tuple $(x_1,\ldots,x_n) \in \mathbb{A}^n$. It's certainly true that A is finitely generated, so it remains to check that the map $x \mapsto \varepsilon_x$ is bijective. Injectivity is clear; if $\varepsilon_x = \varepsilon_y$ then for each $i = 1,\ldots,n$ we have that $x_i = \varepsilon_x(T_i) = \varepsilon_y(T_i) = y_i$, so $x = y$. To show surjectivity, suppose that $\theta \in \mathrm{Hom}_{k\text{-alg}}(A,k)$. Let $x \in V$ be given by $x_i = \theta(T_i)$, then since A is generated by the T_i, it's clear that $\theta = \varepsilon_x$. Hence the pair $(\mathbb{A}^n, k[T_1,\ldots,T_n])$ is an affine variety.

Next we will explore the rich structure of affine varieties by introducing the Zariski topology on them. Throughout, we will denote the complement of a subset $W \subset V$ by W^c, and the closure by $\overline{W}$. Let (V,A) be an affine variety, and let S be a subset of A. Define

$$\mathscr{V}(S) = \{ x \in V \ : \ f(x) = 0 \ \forall f \in S \}.$$

The sets $\mathscr{V}(S)$, for arbitrary $S \subseteq A$, are the closed sets of V. One can quickly show that, given some collection S_i of subsets of A, with $i \in I$, we have that

$$\mathscr{V}(\bigcup_{i \in I} S_i) = \bigcap_{i \in I} \mathscr{V}(S_i),$$

hence the intersection of closed sets is also closed. Given two subsets of A, say S_1 and S_2 we define their product by

$$S_1 S_2 := \{ s_1 s_2 \ : \ s_1 \in S_1, s_2 \in S_2 \} \subset A.$$

We have then, that $\mathscr{V}(S_1) \cup \mathscr{V}(S_2) = \mathscr{V}(S_1 S_2)$, so that finite unions of closed sets are closed. Finally we have that $\mathscr{V}(\{0\}) = V$, where 0 denotes the 0 map, and for any other constant $a \in k$, we have that $\mathscr{V}(\{a\}) = \emptyset$. Hence the sets $\mathscr{V}(S)$ define a topology on V, which is intrinsically linked to its structure as an affine variety. Indeed, we have the following.

Lemma 2.1.2. *Let V be an affine variety, and suppose that $W \subset V$ is a closed subset. Then W is an affine variety, and $k[W]$ is given by*

$$k[W] = \{ f|_W \ : \ f \in k[V] \}.$$

Proof. Let W be given by $\mathscr{V}(S)$ for some $S \subset k[V]$. Denote by B the set $\{ f|_W \ : \ f \in k[V] \}$, and let $\pi : k[V] \to B$ be the restriction map. Since V is an affine variety and $B = \pi(k[V])$ we have that B is finitely generated subalgebra of $\mathrm{Map}(W, k)$.

Let $\varepsilon'_x : B \to k$ denote evaluation at $x \in W$, and $\varepsilon_x : k[V] \to k$ evaluation at $x \in V$. Then $\varepsilon'_x \circ \pi = \varepsilon_x$, so if $\varepsilon'_x = \varepsilon'_y$ for some elements $x, y \in W$ then $\varepsilon_x = \varepsilon_y$ and thus $x = y$.

Next suppose that $\theta \in \mathrm{Hom}_{k-\mathrm{alg}}(B, k)$ so that $\theta \circ \pi \in \mathrm{Hom}_{k-\mathrm{alg}}(k[V], k)$ and thus $\theta \circ \pi = \varepsilon_x$ for some $x \in V$. Let $f \in S$, then $\varepsilon_x(f) = \theta \circ \pi(f) = 0$ since $f|_W = 0$, so in fact $x \in W$. Now let $g \in B$ so that $g = \pi(f)$ for some $f \in k[V]$. Then

$$\theta(g) = \theta \circ \pi(f) = \varepsilon_x(f) = f(x) = g(x) = \varepsilon'_x(g).$$

Hence we have that $\theta = \varepsilon'_x$ for some $x \in W$, so (W, B) is an affine variety. □

This lemma shows us that the examples given at the beginning of this section are indeed affine varieties.

Definition 2.1.3. Let V and W be affine varieties. A map $\phi : V \to W$ is called a *morphism of affine varieties* if for every $f \in k[W]$ we have that the composition $f \circ \phi$ is an element of $k[V]$. In this case the map $k[W] \to k[V]$ given by $f \mapsto f \circ \phi$ is called the comorphism of ϕ, and denoted ϕ^*.

One can show that morphisms of varieties are continuous with respect to the Zariski topology. Furthermore, it's clear that an inverse morphism exists precisely when the comorphism is an isomorphism of k-algebras. In this case we say that the morphism is an isomorphism of varieties. Notice that if a morphism of varieties is bijective then it is not necessarily an isomorphism of varieties.

For our first example of a morphism, let $\operatorname{Char} k = p > 0$. We define the Frobenius morphism by

$$\begin{aligned} F : \mathbb{A}^n &\longrightarrow \mathbb{A}^n \\ (x_1, \dots, x_n) &\longmapsto (x_1^p, \dots, x_n^p). \end{aligned}$$

Since k is algebraically closed, it's clear that this is a bijective morphism of varieties, but the inverse map is not a morphism.

Lemma 2.1.4. *Let $\phi : V \to W$ be a morphism of affine varieties, and suppose that $\phi^* : k[W] \to k[V]$ is surjective. Then the image of ϕ is closed in W and the restriction of ϕ to its image is an isomorphism.*

Proof. See [30, Lemma 1.5] for a proof. □

Now suppose that V is an affine variety, with $k[V]$ generated by the elements $f_1, \dots, f_n$. Define the map ϕ by

$$\begin{aligned} \phi : \ V &\longrightarrow \mathbb{A}^n, \\ x &\longmapsto (f_1(x), \dots, f_n(x)). \end{aligned}$$

Note that we have the composition $T_i \circ \phi = f_i$, so $T_i \circ \phi \in k[V]$. Since $k[\mathbb{A}^n]$ is generated by the T_i, we have that ϕ is a morphism of affine varieties. Furthermore, since $k[V]$ is generated by the $f_i = \phi^*(T_i)$ we have that the comorphism ϕ^* is surjective, so we can apply Lemma 2.1.4 to get that V is isomorphic to a closed set in $\mathbb{A}^n$. Hence we may think of affine varieties as the closed sets of affine n-space.

We would like to have the notion of a product of affine varieties, agreeing with the categorical notion of a product. We can consider the cartesian product of two affine varieties V and W as a set, but we must give some thought as to what the coordinate algebra $k[V \times W]$ should be. We observe that we have a k-algebra homomorphism, say $\Phi : k[V] \otimes k[W] \longrightarrow \operatorname{Map}(V \times W, k)$, given by

$$\Phi(f \otimes g)(v, w) = f(v)g(w),$$

for all $v \in V$ and $w \in W$. Furthermore, this map is injective (this can be seen by considering a k-basis for either $k[V]$ or $k[W]$), so in fact we may identify $k[V] \otimes k[W]$ with a subalgebra of $\operatorname{Map}(V \times W, k)$.

Lemma 2.1.5. *Let* $(V,k[V])$ *and* $(W,k[W])$ *be affine varieties, then the pair* $(V\times W,k[V]\otimes k[W])$ *is also an affine variety.*

Proof. It's clear that the algebra $k[V]\otimes k[W]$ is finitely generated, so we only need to show that the map $(v,w)\mapsto \varepsilon_{(v,w)}$ is bijective, for all $v\in V$, $w\in W$. First suppose that $\varepsilon_{(v_1,w_1)}=\varepsilon_{(v_2,w_2)}$ for some pairs $(v_1,w_1),(v_2,w_2)\in V\times W$. Then for all $f\in k[V]$ we have that $\varepsilon_{(v_1,w_1)}(f\otimes 1)=f(v_1)=f(v_2)=\varepsilon_{(v_2,w_2)}(f\otimes 1)$, so that $v_1=v_2$. Similarly we have that $w_1=w_2$ so $(v_1,w_1)=(v_2,w_2)$.

To show surjectivity suppose that $\theta: k[V]\otimes k[W]\to k$ is a k-algebra homomorphism. Restricting θ to $k[V]$ and $k[W]$ gives us two more algebra homomorphisms, i.e.

$$\begin{aligned}\theta_V : k[V] &\longrightarrow k,\\ f &\longmapsto \theta(f\otimes 1),\end{aligned}$$

and

$$\begin{aligned}\theta_W : k[W] &\longrightarrow k,\\ g &\longmapsto \theta(1\otimes g).\end{aligned}$$

Since both V and W are affine varieties we get that $\theta_V=\varepsilon_v$ for some $v\in V$ and $\theta_W=\varepsilon_w$ for some $w\in W$. Then it's clear that

$$\theta(f\otimes g)=\theta(f\otimes 1)\theta(1\otimes g)=\theta_V(f)\theta_W(g)=\varepsilon_v(f)\varepsilon_w(g)=\varepsilon_{(v,w)}(f\otimes g),$$

for all $f\in k[V]$, $g\in k[W]$. Since $k[V]\otimes k[W]$ is spanned by the pure tensors $f\otimes g$, we have proved the result. $\square$

This definition of the product of affine varieties is indeed the categorical product in the category of affine varieties (see [19, Proposition 2.4] for more on the categorical notion of products applied to varieties).

Next let V be an affine variety and consider the set

$$V_f=\{\, x\in V \ :\ f(x)\neq 0\,\},$$

where $f\in k[V]$. The sets V_f are called the principal open sets and form a basis of open sets for V under the Zariski topology [19, Section 1.5]. It's clear that the complement V_f^c is given by $\mathcal{V}(f)$, so V_f is indeed an open set.

Lemma 2.1.6. *Let V be an affine variety. For every $f\in k[V]$, the principal open set V_f is an affine variety, with coordinate algebra given by the localisation of $k[V]$ at f. Explicitly,*

$$k[V_f]=k[V]_f=\left\{\ \frac{g}{f^r}\ :\ g\in k[V],\ r\in\mathbb{N}\cup\{0\}\ \right\}.$$

Proof. See [19, Section 1.5] for an idea of the proof. □

Note that we can identify $k[V]_f$ as a subalgebra of $\mathrm{Map}(V_f, k)$ by defining

$$\frac{g}{f^r}(x) = \frac{g(x)}{f^r(x)},$$

since for all $x \in V_f$ we have that $f(x) \neq 0$ by definition.

We have now given the geometry of our affine varieties using the algebra, so a natural question is to ask whether we can give the algebra from the geometry. It turns out that we can, with some minor adjustments.

Let V be an affine variety, and consider a subset $W \subset V$. Define $\mathscr{I}(W)$ by

$$\mathscr{I}(W) = \{\, f \in k[V] \;:\; f(w) = 0 \;\forall w \in W \,\}.$$

Note in particular, that $\mathscr{I}(W)$ is an ideal of $k[V]$. We have that if W_1 and W_2 are two subsets of V, then $\mathscr{I}(W_1 \cup W_2) = \mathscr{I}(W_1) \cap \mathscr{I}(W_2)$, and if $W_1 \subset W_2$ then $\mathscr{I}(W_2) \subset \mathscr{I}(W_1)$. Furthermore we have that $\overline{W} = \mathscr{V}(\mathscr{I}(W))$, and for an ideal $I \subset k[V]$ we have that $I \subset \mathscr{I}(\mathscr{V}(I))$. Whilst this relationship is not quite one to one, we do have the following well known theorem. First recall the definition of the radical of an ideal I of a commutative ring R, denoted $\sqrt{I}$.

$$\sqrt{I} = \{\, r \in R \;:\; r^n \in I \text{ for some } n \in \mathbb{N} \,\}.$$

Theorem 2.1.7 (Hilbert's Nullstellensatz)**.** *Let V be an affine variety and I an ideal of $k[V]$. Then $\mathscr{I}(\mathscr{V}(I)) = \sqrt{I}$.*

Proof. See [19, Theorem 1.1] for an elementary proof. □

Note that if W is a closed set of an affine variety V, with $\mathscr{I}(W) = I$ for some ideal I of $k[V]$, then we have, by Lemma 2.1.2, that

$$k[W] = \{\, f|_W \;:\; f \in k[V] \,\}.$$

Since for all $f \in I$ and $v \in W$ we have that $f(v) = 0$, we may identify $k[W]$ with the quotient,

$$k[W] = k[V] \,/\, I.$$

Recall that a topological space is called irreducible if it cannot be written as union of two proper closed subsets, or equivalently if every non-empty open set is dense (i.e. if $W \subset V$ is open then $\overline{W} = V$). We remark that affine n-space is irreducible under the Zariski topology.

Lemma 2.1.8. *Let V be an affine variety, and suppose that the subset W is closed. Then W is irreducible if and only if the ideal $\mathscr{I}(W) \subset k[V]$ is a prime ideal.*

Proof. First suppose that W is irreducible, and let $f = gh \in I = \mathscr{I}(W)$. Then $W \subset \mathscr{V}(\{f\}) = \mathscr{V}(\{g\}) \cup \mathscr{V}(\{h\})$. But then $W = (W \cap \mathscr{V}(\{g\})) \cup (W \cap \mathscr{V}(\{h\}))$, a union of two closed sets. Hence we have that either $W \subset \mathscr{V}(\{g\})$, in which case $g \in I$, or $W \subset \mathscr{V}(\{h\})$, where $h \in I$, and so I is prime.

Conversely, let I be a prime ideal, and suppose that $W = W_1 \cup W_2$. Then $I = \mathscr{I}(W_1 \cup W_2) = \mathscr{I}(W_1) \cap \mathscr{I}(W_2)$, so we must have that either $\mathscr{I}(W_1) \subset \mathscr{I}(W_2)$ or $\mathscr{I}(W_2) \subset \mathscr{I}(W_1)$, and hence $W_2 \subset W_1$ or $W_1 \subset W_2$. □

Definition 2.1.9. A topological space V is called *Noetherian* if it satisfies the descending chain condition for closed subsets. That is, if for any sequence

$$W_1 \supseteq W_2 \supseteq W_3 \supseteq \dots$$

of closed subsets $W_i \subset V$, there exists an integer n such that for all $m > n$ we have $W_m = W_n$.

As an example of a Noetherian topological space, we have that $\mathbb{A}^n$ is Noetherian under the Zariski topology. This is clear since if the sets W_i form a descending chain in $\mathbb{A}^n$, we obtain an ascending chain of ideals $\mathscr{I}(W_i)$ in $k[T_1,\dots,T_n]$. Since this is a Noetherian ring, such a chain of ideals terminates. But $\mathscr{V}(\mathscr{I}(W_i)) = W_i$ for each i (since W_i is closed), so the chain $\{W_i\}$ must also terminate. This example shows that every affine variety is a Noetherian space.

Lemma 2.1.10. *Every non-empty, closed subset W of a Noetherian space V can be expressed as a finite union of closed irreducible subsets, i.e.*

$$W = W_1 \cup W_2 \cup \dots \cup W_m,$$

for some $m \in \mathbb{N}$. If $W_i \nsubseteq W_j$ for all i and j, then the W_i are uniquely determined, and called the irreducible components of W.

Proof. This can be found in [17, Proposition 1.5]. □

The following definition plays an important role in the next section.

Definition 2.1.11. A variety V is called *complete* if for all varieties W, the projection map

$$\pi_2 : V \times W \longrightarrow W,$$

sends closed sets to closed sets.

For example, all projective varieties are complete. A number of important properties of complete varieties are given in [19, Proposition 6.1].

2.1.2 Affine Algebraic Groups

In this section we will give the first definitions and results for affine algebraic groups. This will largely follow [19], but one may also consult the classic textbooks by Borel [3], and Springer [28].

Let G be a group. Then the group axioms give us maps μ and ι, where

$$\begin{aligned} \mu\colon G\times G &\longrightarrow G \\ (g,h) &\longmapsto gh \end{aligned}$$

is given by group multiplication, and

$$\begin{aligned} \iota\colon G &\longrightarrow G \\ g &\longmapsto g^{-1} \end{aligned}$$

is given by inversion.

Definition 2.1.12. Let G be a group and k an algebraically closed field. We say that G is an *affine algebraic group over* k if it is also an affine variety over k, and the maps μ and ι are morphisms of varieties.

The first examples of algebraic groups are $\mathbb{G}_a$ and $\mathbb{G}_m$. The group $\mathbb{G}_a$ is the additive group k, given the structure of variety as $\mathbb{A}^1$, so that its coordinate algebra is $k[T]$. The group $\mathbb{G}_m$ is the multiplicative group k^*, given the structure of variety as the principal open set $(\mathbb{A}^1)_T$, hence its coordinate algebra is given by $k[T]_T$, which is isomorphic to the algebra of Laurent polynomials $k[T,\frac{1}{T}]$. One can check that the multiplication and inversion maps are indeed morphisms of varieties. Furthermore, both of these groups are irreducible as varieties [19, Section 7.1].

Next we remark that any subgroup of an algebraic group that is also a closed subset is also an algebraic group. The direct product of any two algebraic groups is also an algebraic group, with variety structure that given in Lemma 2.1.5.

Definition 2.1.13. Let G and H be algebraic groups over the field k. A map $\phi: G\to H$ is a *morphism of algebraic groups* if it is both a group homomorphism and a morphism of varieties.

As in Definition 2.1.3, an isomorphism of algebraic groups is a morphism of algebraic groups whose inverse exists and is also a morphism of algebraic groups. Note that a group isomorphism may fail to be an isomorphism of algebraic groups. We will see an example of this shortly.

As a second example of an algebraic group, we remark that, for all $n\in\mathbb{N}$, the group $GL_n(k)$ is an affine algebraic group, given the structure of an affine variety as the principal open set

$(\mathbb{A}^{n^2})_{\det}$. Note in particular that the map $\det : \mathbb{A}^{n^2} \to k$, taking a matrix to its determinant, is an element of $k[T_{ij} : 1 \le i, j \le n]$, so that we have $k[GL_n(k)] = k[T_{ij} : 1 \le i, j \le n]_{\det}$. Furthermore, one can check that matrix multiplication and inversion are morphisms of varieties. This example plays a central role in algebraic group theory, since all affine algebraic groups are isomorphic to a closed subgroup of $GL_n(k)$, for some $n \in \mathbb{N}$ [19, Theorem 8.6].

Notice that if we restrict the map det to $GL_n(k)$ we obtain a morphism of algebraic groups,

$$\begin{aligned} \det : GL_n(k) &\longrightarrow \mathbb{G}_m, \\ M &\longmapsto \det(M), \end{aligned}$$

with comorphism given on the generators of $k[\mathbb{G}_m] = k[T, 1/T]$ as $\det^*(T) = \det$ and $\det^*(1/T) = 1/\det$, both of which are elements of $k[GL_n(k)]$.

For the main example in this thesis, consider the group $SL_2(k)$. It's clear already that $SL_2(k)$ is a closed subgroup of $GL_2(k)$, so it's an algebraic group in its own right. However, we will show that it's an affine algebraic group explicitly. We may think of the elements of $SL_2(k)$ as the set of points in $\mathbb{A}^4$ that satisfy the equation $x_{11}x_{22} - x_{12}x_{21} = 1$, so that $SL_2(k)$ is an affine variety with coordinate algebra given by

$$k[SL_2(k)] = k[T_{11}, T_{12}, T_{21}, T_{22}] \,/\langle T_{11}T_{22} - T_{12}T_{21} - 1\rangle.$$

Furthermore, the maps μ and ι are morphisms of varieties, since

$$\mu^*(T_{ij}) = T_{i1} \otimes T_{1j} + T_{i2} \otimes T_{2j} \in k[SL_2(k)] \otimes k[SL_2(k)],$$

and for ι we have

$$\begin{aligned} \iota^*(T_{11}) &= T_{22}, \qquad & \iota^*(T_{12}) &= -T_{12}, \\ \iota^*(T_{21}) &= -T_{21}, \qquad & \iota^*(T_{22}) &= T_{11}, \end{aligned}$$

all of which are in $k[SL_2(k)]$, so that $SL_2(k)$ is an affine algebraic group.

Recall from Section 2.1.1, that when the characteristic of k is p we have the Frobenius morphism $F : \mathbb{A}^n \to \mathbb{A}^n$. Identifying an affine algebraic group G as a closed subgroup of $GL_n(k)$ for some $n \in \mathbb{N}$, which itself we can think of as being embedded in $\mathbb{A}^{n^2}$, we can restrict the Frobenius morphism to

$$F : G \longrightarrow G.$$

This map is also a group homomorphism, essentially because we have $(x + y)^p = x^p + y^p$ for any $x, y \in k$. As such we have that the Frobenius map is a morphism of algebraic groups. We can extend this definition to the Frobenius morphism on a group scheme, where its kernel gives us a normal subgroup functor, denoted $\mathbf{G_1}$. For the k-algebra k, this allows us to define the first Frobenius kernel G_1 of the group G (see [21, Chapter I.9] for further details). We use this construction in Chapter 5.

Next we give an important class of morphisms of an algebraic group.

Definition 2.1.14. A morphism of algebraic groups $\chi : G \longrightarrow \mathbb{G}_m$ is called a *character* of G. Pointwise multiplication of characters (i.e. $(\chi\psi)(x) = \chi(x)\psi(x)$) makes the set of characters an Abelian group, denoted $X(G)$. The operation in this group is often denoted with a $+$.

Note that the character group for G may be trivial (e.g. in the case that $G = (G,G)$, since then if $g = xyx^{-1}y^{-1}$ then $\chi(g) = 1$). However, for a certain class of subgroup, the character group will be indispensable.

Dual to characters, we have the notion of cocharacters.

Definition 2.1.15. A morphism of algebraic groups $\lambda : \mathbb{G}_m \longrightarrow G$ is called a *cocharacter* of G. As before, the set of cocharacters forms an Abelian group, denoted $Y(G)$.

A cocharacter of G is sometimes called a one parameter multiplicative subgroup, or 1-psg. Notice that we may compose a character with a cocharacter to obtain a morphism $\mathbb{G}_m \longrightarrow \mathbb{G}_m$. Since the group of such morphisms is isomorphic to $\mathbb{Z}$, we obtain a map

$$\begin{aligned} \langle\,,\rangle : X(G) \times Y(G) &\longrightarrow \mathbb{Z}, \\ (\chi,\lambda) &\longmapsto \langle \chi, \lambda\rangle. \end{aligned}$$

As noticed in Lemma 2.1.10, an affine variety is a finite union of irreducible closed subsets, known as the irreducible components. In the case that G is an affine algebraic group, we can say a little more.

Lemma 2.1.16 ([19, Proposition 7.3])**.** *Let G be an affine algebraic group, then G can be written as a finite and disjoint union of irreducible subsets say,*

$$G = G_1 \sqcup G_2 \sqcup \ldots \sqcup G_n.$$

Denote by G° the unique component containing the identity element 1_G. Then G° is a closed normal subgroup of finite index in G, whose cosets are the other irreducible components. Furthermore, each closed subgroup of finite index in G contains G°. □

We say that an affine algebraic group G is connected if $G = G^\circ$. For example, the group $SL_n(k)$ is a connected algebraic group [19, Section 7.5].

In this thesis we will be interested in algebraic groups acting on a particular set. We will study this in the context of kG-modules (to be defined later), but first we give a formal definition for a group action.

Definition 2.1.17. We say a group G *acts* on a set X if there exists a map $\phi : G \times X \longrightarrow X$, where $\phi(g,x)$ is usually denoted $g \cdot x$, subject to the following conditions:

1. $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x \quad \forall\, g_1, g_2 \in G,\ x \in X,$

2. $1_g \cdot x = x \quad \forall\, x \in X.$

For example, the group G acts on itself by right multiplication ($g \cdot h = hg$), and left inverse multiplication ($g \cdot h = g^{-1}h$). In fact, these maps are morphisms of the variety G, whose comorphisms are particularly useful. Define the actions of G on $k[G]$ by

$$(\lambda_g f)(h) = f(g^{-1}h),$$
$$(\rho_g f)(h) = f(hg)$$

where $f \in k[G]$, and $g, h \in G$. These maps are often called left and right translation of functions.

Next we will give some important examples of subgroups of $GL_n(k)$. First we denote the group of all diagonal matrices in $GL_n(k)$ as $D_n(k)$. Notice that this subgroup is isomorphic to $\mathbb{G}_m \times \cdots \times \mathbb{G}_m$, the direct product of $\mathbb{G}_m$ with itself n times. Denote by $T_n(k)$ the group of upper triangular matrices, containing $D_n(k)$, and finally by $U_n(k)$ we denote the group of upper unipotent matrices, that is, those upper triangular matrices with 1's along the diagonal. These subgroups play a fundamental role in the theory of algebraic groups.

Definition 2.1.18. A *torus* T of an algebraic group G is a closed subgroup isomorphic to $D_n(k)$ for some $n \in \mathbb{N}$. We say that a closed subgroup of G is diagonalizable if it is isomorphic to a closed subgroup of $D_n(k)$.

Note that the character group $X(D_n(k))$ is isomorphic to the free Abelian group of rank n, $\mathbb{Z}^n$, where a basis is given by the characters $\chi_i(\mathrm{diag}(t_1, \ldots, t_n)) = t_i$. Later in this subsection, we will be interested in maximal tori (i.e. those that are not properly contained in any other). For the following definition, we will use [19, Theorem 8.6] to identify the algebraic group G as a subgroup of $GL_n(k)$, for some $n \in \mathbb{N}$.

Definition 2.1.19. A subgroup of an algebraic group G is called *unipotent* if all of its elements are unipotent; that is, they are the sum of the identity and a nilpotent element, or equivalently, their only eigenvalue is 1.

For example, the group $U_n(k)$ is a unipotent subgroup of the algebraic group $GL_n(k)$, and in fact, it can be shown that all unipotent subgroups are conjugate to a subgroup of $U_n(k)$, for some $n \in \mathbb{N}$ [19, Corollary 17.5].

Recall the derived series of a group G, given inductively by $\mathscr{D}^0(G) = G$ and $\mathscr{D}^{i+1}(G) = (\mathscr{D}^i(G), \mathscr{D}^i(G))$, where the product indicates the group commutator. We say that G is solvable if its derived series terminates. The notion of solvability is particularly well suited to algebraic groups, since each derived group $\mathscr{D}^i(G)$ is a closed normal subgroup of G, and is connected if G is [19, Section 17.3]. For example, a torus T of an algebraic group G is solvable. Since it's abelian, its first derived subgroup $\mathscr{D}^1(T)$ is equal to $\{1\}$.

Akin to the derived series, we have the descending central series. This series is also given inductively, by $\mathscr{C}^0(G) = G$ and $\mathscr{C}^{i+1}(G) = (G, \mathscr{C}^i(G))$. Again, each subgroup $\mathscr{C}^i(G)$ is closed, and connected if G is. We will call an algebraic group G nilpotent if $\mathscr{C}^n(G)$ is the trivial group for some $n \in \mathbb{N}$.

The radical of an algebraic group G, is defined to be the unique maximal, connected, normal, solvable subgroup of G, and will be denoted $R(G)$. The unipotent radical of G is the unique maximal, connected, normal unipotent subgroup of G, and will be denoted $R_u(G)$. The unipotent radical consists of the unipotent elements of $R(G)$.

Definition 2.1.20. An algebraic group G is called *semisimple* if its radical $R(G)$ is trivial, and it is called *reductive* if its unipotent radical $R_u(G)$ is trivial.

Note that since both $R(G)$ and $R_u(G)$ are normal subgroups, for any algebraic group G we can construct a reductive or semisimple algebraic group by quotienting by the appropriate radical. It's clear that a semisimple group is reductive. Our group of primary interest, $SL_2(k)$, is semisimple (see Section 2.1.5).

Next we give one of the central theorems of the theory of algebraic groups, which allows us to describe their structure in great detail.

Theorem 2.1.21 (Borel's Fixed Point Theorem)**.** *Let G be a connected, solvable algebraic group, and X a non-empty, complete variety on which G acts. Then G has a fixed point in X.*

Proof. See [19, Theorem 21.2] for a proof. □

Definition 2.1.22. A *Borel subgroup B* of an algebraic group G is a maximal, closed, connected solvable subgroup.

By maximal, we mean one that is not properly contained in any other. The study of connected, solvable algebraic groups can, in some senses, be reduced to the study of Borel subgroups. In particular we have that the maximal tori and maximal connected unipotent subgroups of G are those of the Borel subgroups of G [19, Corollary 21.3A], and the following theorem:

Theorem 2.1.23. *The Borel subgroups B of G are all conjugate, and their union is the whole group G. Furthermore, the maximal tori T of G are also all conjugate, and their union gives all the semisimple elements of G.*

Proof. This is a mixture of [19, Theorem 21.3, Corollary 21.3A, Theorem 22.2], where the proofs can be found. □

Since each maximal torus of G is conjugate to every other, their dimensions are equal. We call this dimension, the rank of G. The study of Borel subgroups leads to many remarkable properties of algebraic groups, and a much fuller account can be found in [19, Chapter IIX]. Here, however, we give one more result before moving on to the theory of root systems, which ultimately allows us to classify the simple algebraic groups, and plays a fundamental role in the representation theory.

Definition 2.1.24. A closed subgroup P of G is called *parabolic* if the quotient variety G/P is projective.

It can be shown that P is parabolic if and only if P contains a Borel subgroup B [19, Corollary 21.3B]. In particular, Borel subgroups are parabolic. Parabolic subgroups have the nice property that they can be decomposed in the following way.

Theorem 2.1.25 ([19, Theorem 30.2])**.** *A parabolic subgroup P of G can be decomposed as a semi-direct product $P = LU$, where $U = R_u(P)$ and L is reductive. The subgroup L is called a Levi factor, and any two Levi factors are conjugate by an element in U.* □

2.1.3 The Lie Algebra

For an algebraic group G, we may consider the set derivations of $k[G]$, that is, linear maps $\delta : k[G] \longrightarrow k[G]$ that satisfy the Leibniz condition $\delta(fg) = \delta(f)g + f\delta(g)$. Such a space is naturally a Lie algebra by defining the bracket to be $[\delta,\gamma] = \delta\gamma - \gamma\delta$. We will define the Lie algebra of the algebraic group G to be the space of left-invariant derivations of $k[G]$, i.e. those for which $\delta\lambda_g = \lambda_g\delta$ for all $g \in G$. It turns out that such a space can be identified with the tangent space of G at the identity element (where the tangent space of an affine variety is defined in [19, Section I.5]). As a finite dimensional vector space, the tangent space is also an affine variety. More details on the Lie algebra of an algebraic group can be found in [19, Chapter III]. It will suffice for our purposes to say that such an object exists.

2.1.4 Root Systems

We now make a brief digression to take a look at abstract root systems, where we will, for the most part, follow [18, Chapter III]. Everything that appears in this subsection takes place inside a Euclidean space E; that is a finite dimensional $\mathbb{R}$-vector space with a positive definite, symmetric bilinear form, denoted $(\,,\,)$. Such a form allows us to talk about angles between vectors (and indeed, to perform Euclidean geometry), which allows us to draw root systems, in 2 dimensions at least.

For any $\alpha \in E$, define by P_α the orthogonal hyperplane to α,

$$P_\alpha = \{\, \beta \in E \;:\; (\beta, \alpha) = 0 \,\}.$$

We can then define the reflection in this hyperplane, denoted σ_α. For any $\beta \in E$ this is given by

$$\sigma_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha.$$

The coefficient $\frac{2(\beta,\alpha)}{(\alpha,\alpha)}$ is often abbreviated to $\langle \beta, \alpha \rangle$, and the dual element $\frac{2\alpha}{(\alpha,\alpha)}$ to $\check{\alpha}$. Note that the product $\langle\,,\,\rangle$ is linear in only the first position.

Definition 2.1.26. A subset Φ of a Euclidean space E is called a *root system* in E if it satisfies the following.

1. The subset Φ is finite, spans E, and does not contain the vector 0.

2. For every element $\alpha \in \Phi$, the elements $\pm\alpha$ are the only multiples of α in Φ.

3. For every element $\alpha \in \Phi$, the reflection σ_α leaves Φ invariant.

4. For all $\alpha, \beta \in \Phi$ the quantity $\langle \beta, \alpha \rangle$ is an integer.

The elements $\alpha \in E$ are called the roots of Φ. Denote by $\mathscr{W}$ the subgroup of $GL(E)$ generated by all the reflections σ_α for $\alpha \in \Phi$. Since each reflection leaves Φ invariant, we may associate $\mathscr{W}$ with a subgroup of the symmetric group on Φ. In particular, we have that $\mathscr{W}$ is a finite group. We call $\mathscr{W}$ the Weyl group of Φ.

Definition 2.1.27. A subset $\Delta \subset \Phi$ is called a *base* of Φ if the following conditions hold.

1. Δ is a basis of E as an $\mathbb{R}$-vector space.

2. For each $\beta \in \Phi$ we can write $\beta = \sum_{\alpha\in\Delta} k_\alpha \alpha$, for some $k_\alpha \in \mathbb{Z}$ where either all the k_α are non-negative or non-positive.

If Δ is a base of Φ, then the elements of Δ are called the simple roots. The cardinality of such a base, which is equal to $\dim E$, is called the rank of the root system Φ.

The second condition above allows us to describe roots as being either positive or negative, where e.g. β is positive if $k_\alpha \geq 0$ for all the k_α in the expression for β. In this case we write $\beta > 0$. This allows us to define a partial ordering on the root system by saying that $\lambda < \mu$ if and only if $\mu - \lambda$ is a positive root.

The collection of positive roots is denoted Φ^+, and the collection of negative roots Φ^-. Note that we have $\Phi^- = -\Phi^+$ and $\Phi = \Phi^+ \sqcup \Phi^-$. Such a description, of course, depends on the base chosen, should it be possible to choose one at all. It is a theorem however, that a base always exists ([18, Theorem 10.1]), and that, up to transformation by elements in $\mathscr{W}$, all the possible bases are the same ([18, Theorem 10.3]).

One can show that the Weyl group $\mathscr{W}$ is generated by those σ_α for $\alpha \in \Delta$, for some chosen base Δ [18, Theorem 10.3]. In fact, it's clear that $\mathscr{W}$ is a finite Coxeter group, and as such contains a longest element, which we will denote ω_0.

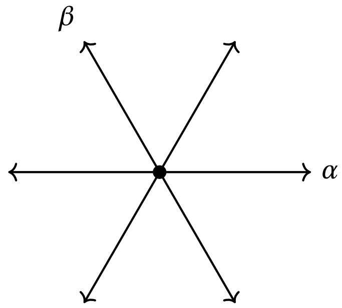

Figure 2.1: The root system $\mathbf{A_2}$

For example, the figure above shows the root system $\mathbf{A_2}$. A base is given by the roots $\{\alpha, \beta\}$, which gives us $\Phi^+ = \{\alpha, \beta, \alpha+\beta\}$. We have that $\alpha < \alpha+\beta$ and $\beta < \alpha+\beta$.

Definition 2.1.28. A root system Φ is called *irreducible* if it cannot be partitioned into the disjoint union of two proper subsets such that each root in one set is orthogonal to each root in the other.

For a root system Φ, we define the set of weights $\Lambda \subset E$ to be the set of $\lambda \in E$ such that $\langle \lambda, \alpha \rangle \in \mathbb{Z}$ for all $\alpha \in \Phi$. It's clear that Λ is a subgroup of E that contains Φ. For a fixed base Δ of Φ, we will say that a weight λ is dominant if the integer $\langle \lambda, \alpha \rangle$ is non-negative, for each $\alpha \in \Delta$. The set of dominant weights is denoted Λ^+.

Suppose now that $\Delta = \{\alpha_1,\dots,\alpha_l\}$, so that the set $\{\check{\alpha}_1,\dots,\check{\alpha}_l\}$ is also a basis of E. Let λ_i be the dual to $\check{\alpha}_i$ with respect to the form on E, i.e. such that

$$(\lambda_i,\check{\alpha}_j) = \frac{2(\lambda_i,\alpha_j)}{(\alpha_j,\alpha_j)} = \langle\lambda_i,\alpha_j\rangle = \delta_{ij}.$$

Such λ_i are clearly dominant, and are called the fundamental dominant weights, with respect to Δ. One can show that, for any $\lambda \in \Lambda$ we have that $\lambda = \sum_{i=0}^{l} m_i\lambda_i$ for some $m_i \in \mathbb{Z}$, and so in fact, Λ is a lattice in E. Furthermore, we have that $\lambda \in \Lambda^+$ if and only if $m_i \geq 0$ for all the m_i in the expression for λ.

Note that the partial order on Φ can be extended to Λ by $\mu < \lambda$ if and only if $\lambda - \mu$ is a dominant weight.

Definition 2.1.29. We will call a subset Π of Λ^+ *saturated of highest weight* λ if we have that Π consists only of the weight λ and all dominant weights $\mu \in \Lambda^+$ with the property that $\mu < \lambda$.

Note that this definition differs slightly from that given in [18]. Here we only want to consider dominant weights when talking about saturated subsets. Next we define the weight ρ, the half sum of positive roots, given as

$$\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha.$$

One can easily show that ρ is a dominant weight, and is also equal to the sum of the fundamental dominant weights ([18, Lemma 13.3A]).

We now return to the theory of algebraic groups to show that, for a semisimple algebraic group G, one can construct a root system. First, denote by $\mathfrak{g}$ the Lie algebra of G, and recall that, for $x \in G$, the map $\mathrm{Ad}x : \mathfrak{g} \longrightarrow \mathfrak{g}$ is an automorphism of $\mathfrak{g}$, and that the map $\mathrm{Ad} : G \longrightarrow \mathrm{Aut}(\mathfrak{g})$ which takes x to $\mathrm{Ad}x$ is a morphism of algebraic groups. As such, the image of a maximal torus T of G is diagonalizable in $GL(\mathfrak{g})$. Thus we may write the Lie algebra $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{g}^T \oplus \bigoplus_{\alpha\in\Phi(G,T)} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}^T$ is the space of fixed points of $\mathfrak{g}$ under T, and $\Phi(G,T)$ is the subset of $X(T)$ consisting of those α for which the space

$$\mathfrak{g}_\alpha := \{\, v \in \mathfrak{g} \;:\; \mathrm{Ad}x(v) = \alpha(x)v \;\forall\; x \in T \,\}$$

is non-zero. Next denote by $W(G,T)$ the quotient group given by $N_G(T)/C_G(T)$, and call this the Weyl group of G relative to T. Since we have $N_G(T)^\circ = C_G(T)^\circ$ [19, Corollary 16.3] and $C_G(T)$ is connected [19, Theorem 22.3], this group is a finite group. The Weyl group acts on the character $\chi \in X(T)$ in the following way; if $\sigma \in W(G,T)$ is represented by $n \in N_G(T)$, then we have $\sigma\chi(t) = \chi(n^{-1}tn)$. Similarly, for a cocharacter $\lambda \in Y(T)$ we have the action given by $\sigma\lambda(x) = n\lambda(x)n^{-1}$. Under such actions we have that $\langle \sigma\chi, \sigma\lambda \rangle = \langle \chi, \lambda \rangle$.

With somewhat more work, we have the following theorem

Theorem 2.1.30. *[19, Theorem 27.1] Let G be semisimple, and define $E = \mathbb{R} \otimes_{\mathbb{Z}} X(T)$ for a maximal torus T of G. Then $\Phi(G,T)$ is an abstract root system in E, whose rank is* rank$G$*, and whose Weyl group $\mathscr{W}$ is isomorphic to $W(G,T)$.* □

From the above statement, it would seem that the root system of G as given depends on the choice of maximal torus T. Of course, such a choice is largely irrelevant, as the resulting structures are all isomorphic (for example, the Weyl group of G relative to T is isomorphic for all maximal tori T). The inner product on E is given by the pairing $\langle , \rangle$ on $X(T) \times Y(T)$.

We note that the group of characters $X(T)$, under the above correspondence, is a subgroup of the weight lattice Λ [19, Section 31.1]. This allows us to make the following definition.

Definition 2.1.31. Let G be a semisimple group as above. We call the quotient group $\Lambda/X(T)$ the *fundamental group of G*. If this is trivial, then we say that G is *simply connected*.

As above, all choices of a maximal torus T are consistent. After a little bit more work, one finds that the simply connected algebraic groups correspond one to one with the irreducible root systems, and that the irreducible root systems may be classified (using Dynkin diagrams, see [18, Section 11]). From here, one is able to obtain a classification of simple algebraic groups (or to simple Lie algebras). The reader may consult [19, Chapter XI] for more details (or [18, Chapter IV.14] for the Lie algebra case).

Denote by $X^+(T)$ the intersection of $X(T)$ and the positive weights Λ^+. As an element of the weight lattice Λ, for each character χ we may write χ as sum of fundamental weights, say

$$\chi = \sum_{i=1}^{l} m_i \lambda_i,$$

with $m_i \in \mathbb{Z}$. When working over a field of positive characteristic p, we will write

$$X_r(T) = \left\{ \chi \in X^+(T) \,:\, m_i < p^r \text{ for } i = 1, \ldots, l \right\}.$$

The set of weights in $X_1(T)$ are often called the p-restricted weights, and play an important role in the representation theory of algebraic groups.

2.1.5 The Group SL_2

As shown in Section 2.1.2, the group $SL_2(k)$, where $k = \bar{k}$ is an algebraically closed field, is an algebraic group. In this thesis we will be primarily concerned with rational representations of $SL_2(k)$, so in this section we exhibit the particulars of the group.

As mentioned in the previous section, all maximal tori and Borel subgroups are conjugate. As one would hope, a particular choice of such groups does not affect the theory (up to isomorphisms). We pick the following subgroups explicitly, for ease of calculation. First let's choose the maximal torus T of $SL_2(k)$, given by

$$T = \left\{ h_t := \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mid t \in k^* \right\},$$

and the Borel subgroup B, containing T, given by

$$B = \left\{ \begin{pmatrix} t & x \\ 0 & t^{-1} \end{pmatrix} \mid t \in k^*, x \in k \right\}.$$

We have that the character group $X(T)$, that is, the group of algebraic group homomorphisms from T to $\mathbb{G}_m$, is given by the elements

$$\begin{aligned} \psi_r : \ T &\longrightarrow \mathbb{G}_m, \\ h_t &\longmapsto t^r, \end{aligned}$$

for all $r \in \mathbb{Z}$. Hence we may (and henceforth, will) associate $X(T)$ with the abelian group $\mathbb{Z}$. In particular we will use additive notation when describing this group.

As mentioned in Section 2.1.2, the group $SL_n(k)$ is a connected algebraic group. For the case of $SL_2(k)$, we have that the group is generated by the unipotent subgroups

$$U_{12} := \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in k \right\},$$

and

$$U_{21} := \left\{ \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} : y \in k \right\}.$$

One can then use [19, Proposition 7.5] to show that $SL_2(k)$ is connected. Furthermore, we have that since $SL_2(k)$ is equal to its own derived subgroup, it is neither solvable nor nilpotent, and in fact, since $SL_2(k)$ is simple [19, 27.5], its radical is trivial, and it is thus semisimple.

Recall that the Lie algebra $\mathfrak{sl}_2(k)$ has a basis given by

$$f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and that the morphism $\mathrm{Ad} : SL_2(k) \longrightarrow GL(\mathfrak{sl}_2(k))$ is simply given by conjugation by $x \in SL_2(k)$ [19, Proposition 10.3]. As such we can compute the action of T via Ad on $\mathfrak{sl}_2(k)$ directly.

$$\begin{aligned} \mathrm{Ad} h_t(f) &= \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ t^{-2} & 0 \end{pmatrix} \\ &= t^{-2} f. \end{aligned}$$

Similarly we have that $\mathrm{Ad} h_t(e) = t^2 e$ and that $\mathrm{Ad} h_t(h) = h$. Hence if $\alpha \in X(T)$ is the character $t \longmapsto t^2$ (which corresponds to the integer 2, as above), then we may write

$$\mathfrak{g} = \mathfrak{g}^T \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha},$$

where $\mathfrak{g}^T$ is the k-span of h, $\mathfrak{g}_\alpha$ is the k-span of e, and $\mathfrak{g}_{-\alpha}$ is the k-span of f. As such the root system $\Phi(G,T)$ is given by the set $\{\alpha, -\alpha\}$. This root system is called $\mathbf{A_1}$, and has rank 1.

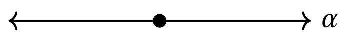

Figure 2.2: The root system $\mathbf{A_1}$

A base Δ is given by the element α, and the Weyl group $\mathscr{W}$ is generated by the reflection σ_α, and is thus the group consisting of two elements, $\{1, \sigma\}$, where in this case, the longest element ω_0 is given by σ.

Since this root system has rank one, we find that the dual basis $\{\lambda\}$ has to satisfy only the equation

$$(\lambda, \check{\alpha}) = \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} = 1,$$

and so is given by $\lambda = \alpha/2$. Hence we see that the weight lattice Λ is given by the integer multiples of $\alpha/2$. Associating α with the integer 2, we have that $\lambda = 1$, and the weights are given by the set $\mathbb{Z}$, which coincides exactly with $X(T)$, so $SL_2(k)$ is simply connected. Furthermore, it's clear that the partial ordering on Λ is given by the regular ordering of integers, so the dominant weights Λ^+ are given by the non-negative integers. The saturated subset of highest weight $r \in \mathbb{N}$ is given by the set of integers $\{0, 1, \ldots, r\}$.

2.2 Representation Theory

In this section we will begin to look at the representation theory of algebraic groups, particularly over an algebraically closed field of positive characteristic. For a comprehensive account of this subject area, the reader should consult [21]. Mostly we will work within the framework of modules for the group algebra kG, for the moment however, we will give some general definitions for modules of a k-algebra A. All modules will be left modules, unless otherwise stated.

Definition 2.2.1. An A-module V is called *simple* if V has no submodules except 0 and V. If a module can be written as a direct sum of simple modules, it is called *semisimple*.

Simple modules play a central role in representation theory, particularly for finite groups over a field of characteristic 0, where Maschke's theorem says that all modules are semisimple. Next we give the classical Schur's lemma.

Lemma 2.2.2 (Schur's Lemma)**.** *Let V be a finite dimensional, simple A-module. Then* $End_A(V) = k1_V$.

Proof. Let ρ be an endomorphism of V, with eigenvalue $\lambda \in k$ (such an eigenvalue exists, since k is algebraically closed). Then the endomorphism $\rho - \lambda 1_V$ is singular, so the image $(\rho - \lambda 1_V)V$ is a submodule properly contained in V. Since V is simple, it must be 0, so $\rho - \lambda 1_V = 0$, thus $\rho = \lambda 1_V$. □

Definition 2.2.3. For an A-module V, we define the *socle* of V, denoted $\operatorname{soc} V$, to be the sum of all the simple submodules of V. Similarly, we define the *radical* of V, denoted $\operatorname{rad} V$, to be the smallest submodule of V with semisimple quotient. We call this quotient the *head* of V.

For more details on the socle and radical, the reader may consult [1, Chapter I].

Now we introduce the group algebra kG, which is the k-algebra consisting of sums of the form $\sum_{g \in G} \lambda_g g$, where all but finitely many of the $\lambda_g \in k$ are non-zero. Multiplication is given by that in the group G and extending linearly. It is well known that every kG-module affords a representation

$$\rho : G \longrightarrow GL_n(k),$$

for some $n \in \mathbb{N}$, and vice-versa.

Definition 2.2.4. We call a kG-module *rational* if it is the union of finite dimensional submodules such that for each submodule W the map $G \to GL(W)$ is a morphism of algebraic groups.

From here on in, when we refer to a kG-module, we will mean a rational kG-module, unless otherwise stated.

We will usually be interested in finite dimensional rational modules, in which case we can talk about the corresponding representation $\rho : G \longrightarrow GL_n(k)$ for some $n \in \mathbb{N}$. In this case, the definition is equivalent to saying that for each i, j we have that the coefficient function $\rho_{ij} : G \longrightarrow k$ is in the coordinate algebra $k[G]$.

Recall that every finite dimensional rational kG-module V has a composition series (not necessarily unique) [21, I.2.14(5)], that is, a series of submodules

$$0 = V_0 < V_1 < \ldots < V_{n-1} < V_n = V,$$

where each quotient V_i/V_{i-1} is isomorphic to a simple module L. In such a case, L is called a composition factor of V, and the number of composition factors isomorphic to L is called the multiplicity of L in V, denoted $[V : L]$.

2.2.1 Canonical Constructions

In this subsection, we give some canonical ways of constructing rational kG-modules.

Given any two rational kG-modules, say V_1 and V_2, we can construct other rational kG-modules as follows. First we will introduce the direct sum, notated $V_1 \oplus V_2$. As a vector space, this is equal to the direct sum of the vector spaces V_1 and V_2, so every element $v \in V_1 \oplus V_2$ can be written as $v = v_1 + v_2$ for $v_1 \in V_1$ and $v_2 \in V_2$, and the intersection of the two subspaces V_1 and V_2 is 0. A basis is given by the union of a basis for V_1 and a basis for V_2. The action of G is simply given by $g \cdot v = g \cdot v_1 + g \cdot v_2$.

Next we introduce the tensor product of two modules $V_1 \otimes V_2$. Recall that if the set $\{v_1, ..., v_n\}$ is a basis of V_1, and $\{w_1, ..., w_m\}$ is a basis of V_2, then a basis of $V_1 \otimes V_2$ is given by the ordered pairs

$$\{\, v_i \otimes w_j \;:\; i = 1, ..., n,\; j = 1, ..., m \,\}.$$

We can turn this into a kG-module by defining the group action $g \cdot (v \otimes w) = (g \cdot v) \otimes (g \cdot w)$ and extending linearly for all $g \in G$, $v \in V_1$ and $w \in V_2$. As a rational kG-module, the tensor product $V_1 \otimes V_2$ has the same universal property as the vector space, but with linear maps replaced with module homomorphisms.

Let V be a rational kG-module. As a k-vector space, we can consider the dual vector space, consisting of linear maps from V to k,

$$V^* = \{\alpha : V \longrightarrow k \;:\; \alpha \text{ is linear }\}.$$

Recall that if V is finite dimensional, with basis $v_1,\dots,v_n$, then we can give V^* the dual basis $\alpha_1,\dots,\alpha_n$, where $\alpha_i(v_j)=\delta_{ij}$. The space V^* can be made into a rational kG-module by simply defining the G action by

$$[g\cdot\alpha](v)=\alpha(g^{-1}\cdot v).$$

One can check that all the required properties for a group action hold.

For each finite dimensional submodule W of a rational kG-module V, denote by ρ_W the map $G\to GL(W)$. We will denote by V^F the Frobenius twist of V. As an Abelian group, this is equal to V, but for each finite dimensional submodule W we have the map $G\to GL(W^F)$ is given by the composition $F\circ\rho_W$, where F is the Frobenius morphism as defined in Section 2.1.2. It's clear then, that V^F is also a rational kG-module. (See [21, I.9.10] for more details on the Frobenius twist.)

2.2.2 Projective and Injective Modules

For the following definitions, we let A be a k-algebra.

Definition 2.2.5. An A-module P is called *projective* if for any A-modules V and W with surjective morphism $\theta:V\longrightarrow W$, and a map $\phi:P\longrightarrow W$, there exists another map $\psi:P\longrightarrow V$ such that

$$\theta\circ\psi=\phi.$$

This property can be summarised with the following commutative diagram.

$$\begin{tikzcd} & P \arrow[dl, "\exists\psi"'] \arrow[d, "\forall\phi"] & \\ V \arrow[r, "\theta"] & W \arrow[r] & 0 \end{tikzcd}$$

Definition 2.2.6. An A-module I is called *injective* if for any A-modules V and W with injective morphism $\theta:V\longrightarrow W$, and a map $\phi:V\longrightarrow I$, there exists another map $\psi:W\longrightarrow I$ such that

$$\psi\circ\theta=\phi.$$

This property can be summarised with the following commutative diagram.

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r, "\theta"] \arrow[d, "\forall\phi"'] & W \arrow[dl, "\exists\psi"] \\ & I & \end{tikzcd}$$

By an injective resolution of a module W we mean an exact sequence

$$0 \longrightarrow W \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \dots$$

where each I^n is injective.

For any two kG-modules V and W, we will denote the set of module homomorphisms between them as $\mathrm{Hom}_G(V,W)$ (recall that a linear map $\phi : V \longrightarrow W$ is a kG-module homomorphism if for all $x \in kG$ and $v \in V$ we have $\phi(xv) = x\phi(v)$). Notice that, under pointwise addition, the set $\mathrm{Hom}_G(V,W)$ is an Abelian group.

Now suppose that W' is another kG-module, and let $\theta : W \longrightarrow W'$ be a module homomorphism. Then for any $\phi \in \mathrm{Hom}_G(V,W)$, we can obtain an element of $\mathrm{Hom}_G(V,W')$ by composing with θ. After checking the appropriate conditions, we find that we have a functor $\mathrm{Hom}_G(V,\cdot)$ from the category of kG-modules to the category of Abelian groups. Furthermore, this functor is left exact [21, I.4.2].

Since the Abelian category of rational kG-modules has enough injectives (i.e. each rational kG-module can be embedded in an injective module) [21, I.4.2], for any rational kG-module W we may construct an injective resolution, denoted I, as

$$0 \longrightarrow W \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \dots$$

Next we apply the left exact functor $\mathrm{Hom}_G(V,\cdot)$, for some fixed kG-module V, to obtain the complex

$$0 \longrightarrow \mathrm{Hom}_G(V,I^0) \longrightarrow \mathrm{Hom}_G(V,I^1) \longrightarrow \mathrm{Hom}_G(V,I^2) \longrightarrow \dots$$

which we denote $\mathrm{Hom}_G(V,I)$. Then we can define, for any $n \in \mathbb{N}$ the n^{th} right derived functor of $\mathrm{Hom}_G(V,W)$, denoted $R^n\mathrm{Hom}_G(V,W)$, as the n^{th} homology group $H^n(\mathrm{Hom}_G(V,I))$ of the above complex. This is abbreviated to $\mathrm{Ext}_G^n(V,W)$. Note that this definition would appear to depend on the injective resolution chosen for W, however, for any two injective resolutions of W there is a homomorphism between them, and any two such homomorphisms are homotopic (see [23, Section XX.6] for more details). In this manner, we can define the functor $\mathrm{Ext}_G^n(V,\cdot)$.

The groups $\mathrm{Ext}_G^n(V,W)$ can be identified as equivalence classes of exact sequences of kG-modules. In particular, for $n = 1$ we get the equivalence classes of short exact sequences of the form [21, II.2.12]

$$0 \longrightarrow W \longrightarrow M \longrightarrow V \longrightarrow 0.$$

Lemma 2.2.7. *For each $n \in \mathbb{N}$ and kG-modules V, W and W', we have natural isomorphisms*

$$\mathrm{Ext}^n_G(V, W \oplus W') \cong \mathrm{Ext}^n_G(V, W) \oplus \mathrm{Ext}^n_G(V, W').$$

Proof. This follows from the identity $\mathrm{Hom}_G(V, W \oplus W') \cong \mathrm{Hom}_G(V, W) \oplus \mathrm{Hom}_G(V, W')$, which is clear. □

This result allows us to define an inner product on rational kG-modules V and W by

$$(V, W) = \sum_{i \geq 0} (-1)^i \dim \mathrm{Ext}^i_G(V, W).$$

2.2.3 Weights

Let T be a maximal torus of G, so that the character group $X(T)$ is a subgroup of the weight lattice (as in the previous section). For any kG-module V and any $\lambda \in X(T)$ we have the λ weight space of V, given by

$$V^\lambda = \{\, v \in V : \ t.v = \lambda(t)v \text{ for all } t \in T \,\}.$$

Thus as a T-module we may decompose V into a sum of weight spaces

$$V = \bigoplus_{\lambda \in X(T)} V^\lambda,$$

where if V is finite dimensional, then only finitely many of the V^λ are non-zero. We say that λ is a weight of V if the weight space V^λ is non-zero, and we say that $v \in V^\lambda$ is a weight vector of weight λ. Note that this generalises the decomposition of $\mathfrak{g}$ given in Section 2.1.4, and we may think of the roots as being the weights of the adjoint representation Ad.

Next we introduce the character of a module, as an element of the ring $\mathbb{Z}X(T)$. In order to distinguish the two commutative operations (recall that $(X(T), +)$ is an Abelian group), we give $\mathbb{Z}X(T)$ the $\mathbb{Z}$-basis of formal exponentials x^λ for $\lambda \in X(T)$. We then give multiplication by $x^\lambda x^\mu = x^{\lambda+\mu}$. Note that the action of the Weyl group W on $X(T)$ gives us a $\mathbb{Z}$-linear action of W on $\mathbb{Z}X(T)$.

For the case of SL_2, we have only one non-identity element in W, and the action of this element is given by the mapping $x^r \longmapsto x^{-r}$.

Definition 2.2.8. Let V be a rational kG-module. The *character* of V is given by

$$\mathrm{Ch}\, V = \sum_{\lambda \in X(T)} (\dim V^\lambda)\, x^\lambda.$$

We have the following results on the character.

Lemma 2.2.9. *Let V_1, V_2 and W be rational kG-modules, such that we have a short exact sequence*

$$0 \longrightarrow V_1 \longrightarrow W \longrightarrow V_2 \longrightarrow 0.$$

Then $\operatorname{Ch} W = \operatorname{Ch} V_1 + \operatorname{Ch} V_2$.

Lemma 2.2.10. *For any two rational kG-modules V_1 and V_2 we have that* $\operatorname{Ch} V_1 \otimes V_2 = (\operatorname{Ch} V_1)(\operatorname{Ch} V_2)$.

Proof. These can be shown directly. □

From this one can easily show that the character gives an isomorphism between the Grothendieck ring of finite dimensional kG-modules and the ring of W invariants $\mathbb{Z}X(T)^W$.

For a reductive algebraic group, the simple modules can be characterized by highest weight.

Theorem 2.2.11. *For each dominant weight $\lambda \in X(T)^+$, there exists, up to isomorphism, a unique simple module of highest weight λ, denoted $L(\lambda)$.*

Proof. See [21, Proposition II.2.4a], where the simple module $L(\lambda)$ is described as the socle of the induced module $H^0(\lambda)$ (defined in Section 2.2.5 as $\nabla(\lambda)$). □

Now suppose that the subset $\pi \subset X(T)^+$ is a saturated set of weights. We may consider the subcategory $\mathscr{C}(\pi)$ of kG-modules M whose composition factors have the form $L(\mu)$ with $\mu \in \pi$. Such a subcategory is closed under taking quotients, extensions and sums. In particular, a kG-module M belongs to $\mathscr{C}(\pi)$ if and only if each dominant weight of M belongs to π [21, II.A.2(1)]. Denote by $O_\pi(M)$ the sum of all submodules of M that belong to $\mathscr{C}(\pi)$, i.e.

$$O_\pi(M) = \sum_{\substack{N \leq M \\ N \in \mathscr{C}(\pi)}} N.$$

Clearly $O_\pi(M) \in \mathscr{C}(\pi)$, and furthermore, it is the unique, largest submodule of M with this property. From this, for any saturated set of weights π, we obtain a functor

$$O_\pi : \{\text{rational } kG\text{-modules}\} \longrightarrow \mathscr{C}(\pi),$$

called the truncated functor associated to π. It's clear that for simple modules $L(\lambda)$ we have that $O_\pi(L(\lambda)) = L(\lambda)$ if $\lambda \in \pi$, and 0 otherwise. We have also that the functor O_π is left exact [21, II.A.1(2)].

2.2.4 Induction and Restriction

As in the case for finite groups, we may restrict representations of the algebraic group G to its subgroups, and induce representations of a subgroup H to representations of G.

First, let H be a closed subgroup of the algebraic group G, and suppose that V is a rational kG-module. The module V restricted to H, denoted $V|_H$ is a kH-module, isomorphic to V as a vector space, where H acts as it does on V.

For a rational module V we denote by $\mathrm{Map}(G,V)$ the set of maps $f : G \to V$ such that the image $f(G)$ lies in a finite dimensional subspace W of V, and the induced map $f : G \to W$ is a morphism of varieties. We can give $\mathrm{Map}(G,V)$ the structure of a rational kG-module by defining $xf : G \to V$ as $xf(y) = f(yx)$, for all $f \in \mathrm{Map}(G,V)$ and $x, y \in G$.

Next, for a closed subgroup H, we define the subset

$$\mathrm{Map}_H(G,V) = \left\{ f \in \mathrm{Map}(G,V) \ : \ f(hx) = hf(x) \text{ for all } h \in H, x \in G \right\}.$$

This is in fact a submodule of $\mathrm{Map}(G,V)$ called the induced module. From now on we will write this module as $\mathrm{Ind}_H^G V$.

Suppose now that we have two rational kH-modules, say W_1 and W_2, and let ϕ be a kH-module homomorphism between them. Define the map $\mathrm{Ind}_H^G \phi : \mathrm{Ind}_H^G W_1 \longrightarrow \mathrm{Ind}_H^G W_2$ by $\mathrm{Ind}_H^G \phi(f) = \phi \circ f$. It's clear that $\mathrm{Ind}_H^G \phi(f)$ commutes with the action of G, so we have that Ind_H^G is a functor from rational kH-modules to rational kG-modules. Furthermore, one can show that this functor is left exact (see [21, Proposition I.3.3]).

Just as we did in the construction of the functor $\mathrm{Ext}_G(V,\cdot)$, for a rational kH-module W we may construct an injective resolution

$$0 \longrightarrow W \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \dots$$

Applying the functor Ind_H^G to this resolution we obtain a complex

$$0 \longrightarrow \mathrm{Ind}_H^G I^0 \longrightarrow \mathrm{Ind}_H^G I^1 \longrightarrow \mathrm{Ind}_H^G I^2 \longrightarrow \dots$$

We may then consider the n^{th} homology of this complex, which we denote $R^n\mathrm{Ind}_H^G V$. As before, this definition does not depend on the injective resolution chosen. In this way, we define the functor $R^n\mathrm{Ind}_H^G$.

Theorem 2.2.12 (The Tensor Identity)**.** *Let H be a closed subgroup of G, V a kG-module and W a kH-module. For all $n \geq 0$ we have that*

$$R^n \mathrm{Ind}_H^G(V \otimes W) = V \otimes R^n \mathrm{Ind}_H^G(W).$$

Proof. For a proof (in the more general framework of group schemes), see [21, Proposition I.3.6]. □

2.2.5 The Induced and Weyl Modules

Let G be an arbitrary reductive algebraic group over an algebraically closed field of positive characteristic p. Choose a Borel subgroup B containing the maximal torus T so that we have the Levi decomposition $B = TU$, where U is the unipotent radical $R_u(B)$ [19, Corollary 26.2C]. Let $\lambda \in X(T)$ be a character and define the one dimensional kB-module k_λ by the action which is given trivially by U, and given by the character λ for T. i.e. if $v \in k_\lambda$ then

$$g.v = \begin{cases} v & : \ g \in U, \\ \lambda(g)v & : \ g \in T. \end{cases}$$

Definition 2.2.13. For $\lambda \in X(T)$ we define the *induced module* $\nabla(\lambda)$ by

$$\nabla(\lambda) := \mathrm{Ind}_B^G k_\lambda.$$

Let ω_0 be the longest element of the Weyl group W. We define the Weyl module by

$$\Delta(\lambda) := \nabla(-\omega_0\lambda)^*,$$

where the star indicates the usual dual module, as defined in Section 2.2.1.

The module $\Delta(\lambda)$ has the universal property that, for any other rational kG-module W generated by a highest weight vector w_+ of weight λ, there exists a unique kG-module homomorphism $\phi : \Delta(\lambda) \longrightarrow W$ such that $\phi(m_+) = w_+$, where $m_+ \in \Delta(\lambda)$ is a highest weight vector of weight λ. In fact, $\Delta(\lambda)$ is generated by a B-stable line, of weight λ, and so any other such kG-module is a homomorphic image of $\Delta(\lambda)$ [21, Lemma II.2.13b].

Theorem 2.2.14. *For all $\lambda \in X(T)^+$ we have that the socle of $\nabla(\lambda)$ is equal to $L(\lambda)$, the simple module of highest weight λ. Furthermore the head of $\Delta(\lambda)$ is also equal to $L(\lambda)$. In particular both $\nabla(\lambda)$ and $\Delta(\lambda)$ are indecomposable.*

Proof. A proof of this theorem can be found from [21, Corollary II.2.3] and [21, Proposition II.2.4b]. □

Now suppose that $\pi \subset X(T)^+$ is a saturated set of dominant weights, and recall the definition of the functor O_π as given in Section 2.2.3. We have

Lemma 2.2.15 ([21, II.A.2(2)]).

$$O_\pi(\nabla(\lambda)) = \begin{cases} \nabla(\lambda) & : \lambda \in \pi, \\ 0 & : \lambda \notin \pi. \end{cases}$$

Definition 2.2.16. Let $\lambda \in X(T)$, and define

$$A(\lambda) = \sum_{w \in W} \operatorname{sgn}(w) x^{w\lambda} \in \mathbb{Z}X(T).$$

It follows that $A(\lambda + \rho)$ is divisible by $A(\rho)$ in $\mathbb{Z}X(T)$, so we define

$$\chi(\lambda) = A(\lambda + \rho)/A(\rho) \in \mathbb{Z}X(T)^W,$$

where $\mathbb{Z}X(T)^W$ denotes the elements of $\mathbb{Z}X(T)$ that are invariant under the action of W. For dominant λ, the element $\chi(\lambda)$ will be of fundamental importance throughout this thesis, as it is the character of both the induced and Weyl module of highest weight λ. We prove this explicitly for $SL_2(k)$ in Section 2.2.7. We have the following result of Brauer, as given in [8, 2.2.3].

Theorem 2.2.17 (Brauer's Character Formula)**.** *For $\lambda \in X(T)$, and $\psi = \sum a_\mu x^\mu \in \mathbb{Z}X(T)^W$ we have*

$$\chi(\lambda)\psi = \sum_\mu a_\mu \chi(\lambda + \mu).$$

□

2.2.6 Steinberg's Tensor Product Theorem

Here we present the well known Steinberg Tensor Product theorem, which gives us a description of each simple module of highest weight λ in terms of the simple modules whose highest weight is p-restricted. Originally proved by Steinberg in [29], a much shorter proof was given by Cline, Parshall and Scott later in [6].

Theorem 2.2.18 (Steinberg's Tensor Product Theorem)**.** *Let G be a reductive algebraic group, and $\lambda \in X(T)$ such that $\lambda = \sum_{i=0}^n p^i \lambda_i$ for some $\lambda_i \in X_1(T)$. Then we have*

$$L(\lambda) = \bigotimes_{i=0}^n L(\lambda_i)^{F^i}.$$

2.2.7 Representations of SL_2

In this subsection we describe explicitly some representations for the group $SL_2(k)$. These are the objects that will be studied in this thesis.

Let E be the natural 2-dimensional module for $SL_2(k)$ (isomorphic to k^2 as a vector space). We will give E the basis $\{x_1, x_2\}$ so that the action of $SL_2(k)$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x_1 = ax_1 + cx_2, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} x_2 = bx_1 + dx_2,$$

and E is evidently rational. With this notation, for any $r \in \mathbb{N}$, the r^{th} symmetric power of E, denoted $S^r E$, has basis given by $\{ x_1^i x_2^j \ : \ i + j = r \}$. This vector space becomes a $kSL_2(k)$-module with the action of $g \in SL_2(k)$ given by $g(x_1^i x_2^j) = (gx_1)^i (gx_2)^j$, with usual polynomial multiplication, so that $S^r E$ is a rational $kSL_2(k)$-module. In particular, in $S^r E$ we have that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (x_1^i x_2^j) = (ax_1 + cx_2)^i (bx_1 + dx_2)^j = (\sum_{k=0}^{i} \binom{i}{k} a^k c^{i-k} x_1^k x_2^{i-k})(\sum_{l=0}^{j} \binom{j}{l} b^l d^{j-l} x_1^l x_2^{j-l}),$$

where in the second equality we use the binomial expansion.

For $t \in k^*$, denote by h_t the diagonal matrix $\operatorname{diag}(t, t^{-1})$ in $SL_2(k)$. Then we have that

$$h_t(x_1^i x_2^j) = (t^i x_1^i)(t^{-j} x_2^j) = t^{i-j}(x_1^i x_2^j),$$

where $i + j = r$. Hence the weights of $S^r E$ are the integers $r - 2j$ for $j = 0, 1, \ldots, r$, and each has a one dimensional weight space, i.e.

$$(S^r E)^{r-2j} = k\text{-span}\{ x_1^{r-j} x_2^j \}.$$

It's easy to show that over a field of characteristic 0, the module $S^r E$ is a simple module with highest weight r. Hence the symmetric powers of E give us a complete set of simple modules. If the field k has positive characteristic however, this is not the case, as we will now demonstrate.

Let $\operatorname{char} k = p > 0$ and consider the module $S^p E$. As above we can write down the action of $SL_2(k)$ on each basis vector. In particular we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (x_1^p) = \sum_{i=0}^{p} \binom{p}{i} a^i c^{p-i} x_1^i x_2^{p-i} = a^p x_1^p + c^p x_2^p,$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (x_2^p) = \sum_{i=0}^{p} \binom{p}{i} b^i d^{p-i} x_1^i x_2^{p-i} = b^p x_1^p + d^p x_2^p,$$

where, using Lucas' Theorem (Theorem A.3.1), we have that for all $i = 1, \ldots, p-1$ the binomial coefficient $\binom{p}{i}$ in k is equal to 0. Hence we see that the subspace given by $k\text{-span}\{x_1^p, x_2^p\}$ is in fact a submodule, so that $S^p E$ is not simple.

For $G = SL_2(k)$ and $r \in \mathbb{N}$ we have the well known equality $\nabla(r) = S^rE$ (see, for example [21, II.2.16]). Furthermore, since the root system of $SL_2(k)$ is given by $\{\alpha, -\alpha\}$, we have that $\Delta(r) = \nabla(r)^*$.

It's clear then, that $\dim \nabla(r) = r+1$, and, using the weight space calculations above, we can quickly write down the character of $\nabla(r)$ as

$$\mathrm{Ch}\,\nabla(r) = \sum_{i=0}^{r} x^{r-2i},$$

where we are using the association of $X(T)$ with $\mathbb{Z}$.

Lemma 2.2.19. *For $r \in X(T)^+$ we have the equality $\chi(r) = \mathrm{Ch}\,\nabla(r)$.*

Proof. First we note that under the association of $X(T)$ with $\mathbb{Z}$ we have that $\rho = 1$. Hence, by definition we have that $\chi(r) = A(r+1)/A(1)$. Since the Weyl group of SL_2 consists of two elements, we have

$$A(1) = x^1 - x^{-1}$$

and

$$A(r+1) = x^{r+1} - x^{-r-1} = (x^1 - x^{-1})(x^r + x^{r-2} + \cdots + x^{-r}).$$

Hence we have that $\chi(r) = \mathrm{Ch}\,\nabla(r)$. □

From the action on the dual space, defined in Section 2.2.1, we see that the weight of a dual vector is the negative of that of the vector. i.e., if $v \in V^\lambda$ has dual vector α, and $t \in T$, then

$$[t \cdot \alpha](v) = \alpha(t^{-1} \cdot v) = \alpha(\lambda(t^{-1})v) = \alpha(\lambda^{-1}(t)v) = \lambda^{-1}(t)\alpha(v).$$

From this, and the symmetry of $\chi(r)$, we have that $\mathrm{Ch}\,\Delta(r) = \mathrm{Ch}\,\nabla(r) = \chi(r)$.

In many cases we will find that we have, for a module M, a short exact sequence of the form

$$0 \longrightarrow \nabla(m_1) \longrightarrow M \longrightarrow \nabla(m_2) \longrightarrow 0,$$

for some m_1 and $m_2 \in \mathbb{N}$. When this sequence is not known to be split, we will often write

$$M = \frac{\nabla(m_2)}{\nabla(m_1)}.$$

The first extension groups are given in [14] (in their dual form for Weyl modules), and then all others are given in [27]. We will use the facts that, in such a sequence as above we must have that $m_2 > m_1$ and $m_1 \equiv m_2 (\mathrm{mod} 2)$ [14, Remark 3.1], and that the extension groups are either k or 0 [14, Proposition 3.3, Proposition 3.4], so up to scalars, there exists at most one non-split extension.

2.2.8 Tilting Modules

In the later chapters of this thesis we will use the theory of tilting modules for algebraic groups, developed by Donkin, to give results about the modules $\nabla(r) \otimes \Delta(s)$. As we will shortly see, the ideas of tilting modules are closely related to the modules we study, so it is natural that we investigate the link between them. Furthermore, the classification of tilting modules into modules of highest weight allows us to quickly give results for $G = SL_2(k)$; for example, we can decompose a tilting module into indecomposable summands using only the character.

For the following definitions, G may be an arbitrary reductive algebraic group, over an algebraically closed field of positive characteristic.

Definition 2.2.20. Let M be a kG-module. We say that M has a *good filtration* (or ∇-*filtration*) if there exists a sequence of submodules

$$0 = M_0 < M_1 < \ldots < M_{n-1} < M_n = M$$

such that for each $i = 1, \ldots, n$ the quotient module M_i/M_{i-1} is isomorphic to $\nabla(\lambda)$ for some $\lambda \in X(T)^+$. For such modules we write $M \in \mathscr{F}(\nabla)$. For a given λ, the number of times $\nabla(\lambda)$ appears in such a filtration will be denoted $(M : \nabla(\lambda))$, and is called the multiplicity of λ in M. This number is also the coefficient of $\chi(\lambda)$ in $\mathrm{Ch}\, M$, and so is independent of the filtration chosen.

Definition 2.2.21. Let N be a kG-module. We say that N has a *Weyl filtration* (or Δ-*filtration*) if there exists a sequence of submodules

$$0 = N_0 < N_1 < \ldots < N_{n-1} < N_n = N$$

such that for each $i = 1, \ldots, n$ the quotient module N_i/N_{i-1} is isomorphic to $\Delta(\lambda)$ for some $\lambda \in X(T)^+$. For such modules we write $N \in \mathscr{F}(\Delta)$. For a given λ, the number of times $\Delta(\lambda)$ appears in such a filtration will be denoted $(N : \Delta(\lambda))$, and is called the multiplicity of λ in M. As before, this number is the coefficient of $\chi(\lambda)$ in $\mathrm{Ch}\, N$, and so is independent of the filtration chosen.

Notice that if the module M has a good filtration, then the dual module M^* has a Weyl filtration, and vice-versa.

Lemma 2.2.22 ([21, Proposition II.4.16b])**.** *A kG-module V has a good filtration if and only if we have* $\mathrm{Ext}^1_G(\Delta(\lambda), V) = 0$ *for all* $\lambda \in X(T)^+$. □

Suppose that such a V can be decomposed into a direct sum $V = V_1 \oplus V_2$. We obtain then, using Lemma 2.2.7, that

$$\mathrm{Ext}^1_G(\Delta(\lambda), V) = \mathrm{Ext}^1_G(\Delta(\lambda), V_1) \oplus \mathrm{Ext}^1_G(\Delta(\lambda), V_2),$$

for all $\lambda \in X(T)^+$. From the above lemma, we obtain the following corollary.

Corollary 2.2.23. *If a kG-module V has a good filtration, then each summand of V has a good filtration.* $\square$

Next we give a useful character result for rational kG-modules $M \in \mathscr{F}(\nabla)$ and $N \in \mathscr{F}(\Delta)$, where $\operatorname{Ch} M = \sum a_\lambda \chi(\lambda)$ and $\operatorname{Ch} N = \sum b_\lambda \chi(\lambda)$ for integers a_λ and b_λ. Note that for such modules we have that, since $(\nabla(\lambda), \Delta(\mu)) = 1$ if $\lambda = \mu$, and 0 otherwise [21, Proposition II.4.13], the inner product (M,N) is equal to $\sum a_\lambda b_\lambda$.

Now, it follows from [21, Proposition II.4.16] that

$$\dim \operatorname{Hom}_G(M,N) = \sum_{\lambda \in \Lambda} (M : \nabla(\lambda))(N : \Delta(\lambda)).$$

This quantity is equal to $\sum a_\lambda b_\lambda$, so we have the following lemma.

Lemma 2.2.24. *For rational kG-modules M and N as above, we have that* $\dim \operatorname{Hom}_G(M,N) = (M,N)$. $\square$

Following [9], we make the following definition.

Definition 2.2.25. A kG-module M is a *tilting module* if M belongs to both $\mathscr{F}(\nabla)$ and $\mathscr{F}(\Delta)$.

Using the observation following Definition 2.2.21, we may also say that M is tilting if both M and M^* belong to either $\mathscr{F}(\nabla)$ or $\mathscr{F}(\Delta)$. Next we give the main theorem on tilting modules.

Theorem 2.2.26 ([9, Theorem 1.1])**.** *For each $\lambda \in X(T)^+$ there exists a unique indecomposable tilting module $T(\lambda)$, of highest weight λ. Furthermore, the weight λ occurs with multiplicity 1 in $T(\lambda)$.* $\square$

Theorem 2.2.27. *Every short exact sequence of tilting modules is split.*

Proof. We have that for any $\lambda, \mu \in X(T)^+$ [21, Proposition II.4.13],

$$\operatorname{Ext}^1_G(\nabla(\lambda), \Delta(\mu)) = 0.$$

It follows then, that if $M \in \mathscr{F}(\nabla)$ and $N \in \mathscr{F}(\Delta)$ we have $\operatorname{Ext}^1_G(M,N) = 0$. Hence, for any tilting modules T_1 and T_2 we have $\operatorname{Ext}^1_G(T_1, T_2) = 0$. $\square$

Every tilting module can be written uniquely, up to isomorphism and ordering, as a sum of these indecomposable tilting modules, so the above theorems give a complete classification of tilting modules (much like the case for semisimple modules). In general, it's not known what these indecomposable tilting modules look like. However, for SL_2, we have the following theorem, due to Donkin [9] (although we quote the closed formula given by Erdmann and Henke).

Theorem 2.2.28 ([15, Lemma 5]). *Let $u \in \mathbb{N} \cup \{0\}$. Then u can be written uniquely in the form $u = \sum_{i=0}^{m} u_i p^i$ where $p-1 \le u_i \le 2p-2$ for all $i < m$ and $0 \le u_m \le p-1$. Then*

$$T(u) = \bigotimes_{i=0}^{m} T(u_i)^{F^i}.$$

This result allows us to compute the character of the indecomposable tilting modules for SL_2 in a given characteristic. Then, for any tilting module for which we know the character, we may decompose it into indecomposable summands. This is because such a module must be a direct sum of the indecomposable tilting modules (in view of Theorem 2.2.27), and these are classified by highest weight. This procedure will be essential to the main result of this thesis.

2.2.9 Blocks

In this subsection we briefly introduce the idea of a block, following [21, Section II.7]. For a reductive algebraic group G, we can define an equivalence relation on the set of simple modules by saying that two simple modules L and L' are equivalent whenever $\mathrm{Ext}^1_G(L,L') \neq 0$. The equivalence classes given by this relation are called the *blocks* of G. Using the bijective mapping $\lambda \longmapsto L(\lambda)$ for $\lambda \in X(T)^+$, we may regard the blocks as a partition of $X(T)^+$.

Now by [21, Remark II.7.1.2], we have that for any indecomposable kG-module M, there exists a unique block $\mathscr{B}$ containing all the composition factors of M. We say M belongs to $\mathscr{B}$. For $\lambda, \mu \in X(T)^+$, consider the indecomposable modules $\nabla(\lambda)$ and $\nabla(\mu)$. It's clear that if λ and μ are in different blocks, then $\mathrm{Ext}^1_G(\nabla(\lambda), \nabla(\mu)) = 0$; if not, then there exists a non-split sequence

$$0 \longrightarrow \nabla(\mu) \longrightarrow M \longrightarrow \nabla(\lambda) \longrightarrow 0$$

for some indecomposable module M. Then M belongs to some block $\mathscr{B}$, but then λ and μ also belong to $\mathscr{B}$, a contradiction.

For a semisimple algebraic group, a full description of the blocks can be found in [7, Theorem 5.8]. In the case of $SL_2(k)$ we obtain the following.

Lemma 2.2.29. *For $a \in \mathbb{N}$, the block containing a is given by the set*

$$\mathscr{B}(a) = \left\{ w \cdot a + 2np^{r(a)+1} \ : \ n \in \mathbb{Z}, w \in \mathscr{W} \right\} \cap \mathbb{Z}_{\geq 0},$$

where the $\cdot$ action of $\mathscr{W}$ is given by $1 \cdot a = a$ and $\sigma \cdot a = -a-2$; and $r(a)$ is defined to be the non-negative integer satisfying $a+1 \in p^{r(a)}\mathbb{Z} \setminus p^{r(a)+1}\mathbb{Z}$. In particular, we have

$$\mathscr{B}(a) \subset \left\{ w \cdot a + 2np \ : \ n \in \mathbb{Z}, w \in \mathscr{W} \right\}.$$

□

In some cases we can use this result to show that a short exact sequence must be split.

2.3 The Universal Enveloping Algebra

Before discussing the universal enveloping algebra of a Lie algebra, we give some elementary results for abstract Lie algebras in order to establish some notation, and refer the reader to [18, Chapter I] for all the basic definitions that we don't give here.

In analogy to the case for algebraic groups, we can define the derived series for a Lie algebra $\mathfrak{g}$, given inductively as $\mathfrak{g}^{(0)} = \mathfrak{g}$ and $\mathfrak{g}^{(i)} = [\mathfrak{g}^{(i-1)}, \mathfrak{g}^{(i-1)}]$. We will say that $\mathfrak{g}$ is solvable if the derived series terminates. Furthermore, each Lie algebra contains a maximal solvable ideal [18, Proposition 3.1c], called the radical of $\mathfrak{g}$ and denoted $\operatorname{rad}\mathfrak{g}$. A Lie algebra $\mathfrak{g}$ is called semisimple if its radical, $\operatorname{rad}\mathfrak{g}$ is 0.

For any Lie algebra $\mathfrak{g}$ we can define the Killing form, given as $\kappa(x,y) = \mathrm{Tr}(\mathrm{ad}_x \mathrm{ad}_y)$. This is a symmetric, bilinear and associative (with respect to the Lie bracket) form on $\mathfrak{g}$. When working over a field of characteristic 0, the Killing form is non-degenerate if and only if $\mathfrak{g}$ is a semisimple Lie algebra [18, Theorem 5.1].

For the remainder of this subsection, we will assume that $\mathfrak{g}$ is a semisimple Lie algebra over a field of characteristic 0. Firstly, we have the root space decomposition given by

$$\mathfrak{g} = \mathfrak{h} \oplus \Big(\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \Big),$$

where $\mathfrak{h}$ is a maximal toral subalgebra, Φ is the root system (as in the algebraic group case) and $\mathfrak{g}_\alpha$ is the one-dimensional subspace given by

$$\mathfrak{g}_\alpha = \big\{ x \in \mathfrak{g} \ : \ [h,x] = \alpha(h)x \quad \forall h \in \mathfrak{h} \ \big\}.$$

We will often denote by x_α a basis vector for $\mathfrak{g}_\alpha$. In the case of $\mathfrak{sl}_2(\mathbb{C})$, since the root system is given by the set $\Phi = \{\alpha, -\alpha\}$, we have that $e = x_\alpha$, $f = x_{-\alpha}$ and a maximal toral subalgebra $\mathfrak{h}$ is given by the span of h.

A semisimple Lie algebra $\mathfrak{g}$ is built up from copies of $\mathfrak{sl}_2$ (this is the subject of [18, Chapter II]). In particular we have that for $x_\alpha \in \mathfrak{g}_\alpha$, there exists $y_\alpha \in \mathfrak{g}_{-\alpha}$ such that the elements $x_\alpha, y_\alpha, h_\alpha = [x_\alpha, y_\alpha]$ span a three dimensional subalgebra of $\mathfrak{g}$, isomorphic to $\mathfrak{sl}_2$ [18, Proposition 8.3].

Note that we have reused the symbol Φ here; this is no coincidence, as the root system corresponding to this decomposition is identical to that in Section 2.1.4. This is because the morphism

$$\begin{aligned} \mathrm{ad} : \mathfrak{g} &\longrightarrow \mathfrak{gl}(\mathfrak{g}) \\ x &\longmapsto \mathrm{ad}_x, \end{aligned}$$

where $\mathrm{ad}_x(y) = [x, y]$, is the differential of the morphism $\mathrm{Ad} : G \longrightarrow GL(\mathfrak{g})$ [19, Theorem 10.4].

2.3.1 The Tensor Algebra

Let $\mathfrak{g}$ be a Lie algebra over the field k. For any $n \in \mathbb{N}$ we can construct the vector space of n-fold tensor products, denoted by $\mathfrak{g}^{\otimes n}$. This is the vector space consisting of all sums of tensors of the form $X_1 \otimes X_2 \otimes \ldots \otimes X_n$, where each $X_i \in \mathfrak{g}$. If the set $\{X_1, \ldots, X_r\}$ is a basis of $\mathfrak{g}$, then a basis of $\mathfrak{g}^{\otimes n}$ is given by

$$\{\, X_{i_1} \otimes \ldots \otimes X_{i_n} \;:\; i_j = 1, \ldots, r \;\text{ for each } j \,\},$$

and so the dimension of $\mathfrak{g}^{\otimes n}$ is r^n. By convention we consider $\mathfrak{g}^{\otimes 0}$ to be the ground field k.

Next we construct an infinite dimensional vector space, given as the direct sum of all the vector spaces of n-fold tensor products

$$\mathscr{T}(\mathfrak{g}) = \bigoplus_{n \in \mathbb{N} \cup \{0\}} \mathfrak{g}^{\otimes n}.$$

We can make this into a k-algebra by defining the multiplication

$$(Y_1 \otimes \ldots \otimes Y_n) \cdot (Z_1 \otimes \ldots \otimes Z_m) = Y_1 \otimes \ldots \otimes Y_n \otimes Z_1 \otimes \ldots \otimes Z_m \in \mathfrak{g}^{\otimes (n+m)}.$$

This algebra is called the tensor algebra of $\mathfrak{g}$. Often we will write $\mathscr{T}^n(\mathfrak{g})$ for $\mathfrak{g}^{\otimes n}$, particularly when we want to emphasize that we are thinking about this vector space as a subspace of the tensor algebra.

Definition 2.3.1. The *Universal Enveloping Algebra* of a Lie algebra $\mathfrak{g}$, denoted $U(\mathfrak{g})$, is the quotient of the tensor algebra $\mathscr{T}(\mathfrak{g})$ by the ideal generated by the elements $X \otimes Y - Y \otimes X - [X, Y]$, for all $X, Y \in \mathfrak{g}$, i.e.

$$U(\mathfrak{g}) = \mathscr{T}(\mathfrak{g}) \,/\, \langle X \otimes Y - Y \otimes X - [X, Y] \rangle.$$

We could, in fact, introduce the universal enveloping algebra abstractly by describing the universal property it satisfies. For an arbitrary Lie algebra $\mathfrak{g}$ we can define the universal enveloping algebra as the unique pair $(U(\mathfrak{g}),\ i)$, where $U(\mathfrak{g})$ is an associative k-algebra with identity, and i is a linear map $i : \mathfrak{g} \to U(\mathfrak{g})$ that satisfies

$$i([x, y]) = i(x)i(y) - i(y)i(x),$$

for all $x, y \in \mathfrak{g}$, such that the following condition holds: For any associative k-algebra A with identity and a linear map $j : \mathfrak{g} \to A$ satisfying the above relation, there exists a unique k-algebra homomorphism $\phi : U(\mathfrak{g}) \to A$ such that $\phi \circ i = j$. This can be summarised by the following commutative diagram.

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "i"] \arrow[dr, "\forall j"'] & U(\mathfrak{g}) \arrow[d, "\exists!\ \phi"] \\ & A \end{tikzcd}$$

Of course, such a statement must be proved, and a proof can be found in [18, Section 17.2]. This universal property allows us to make a bijective map between modules for the Lie algebra $\mathfrak{g}$ and modules for $U(\mathfrak{g})$. For a $\mathfrak{g}$-module V, letting $A = \mathrm{End}_{\mathfrak{g}}(V)$ in the diagram, we have that each $\mathfrak{g}$-module can be extended to a $U(\mathfrak{g})$-module, and of course, each $U(\mathfrak{g})$-module can be restricted to a $\mathfrak{g}$-module. In fact, more is true, and, for a semisimple, simply connected algebraic group G over a field of characteristic 0, we have an equivalence of categories between finite dimensional $U(\mathfrak{g})$ modules and finite dimensional, rational kG-modules. This equivalence is given naturally by considering the differential $d\phi$, of the representation

$$\phi : G \longrightarrow GL(V),$$

for a kG-module V.

Next we will give an essential theorem on the universal enveloping algebra. First let's define the map

$$\begin{aligned} \pi : \mathscr{T}(\mathfrak{g}) &\longrightarrow U(\mathfrak{g}), \\ y_1 \otimes \ldots \otimes y_n &\longmapsto y_1 \ldots y_n \end{aligned}$$

as the canonical quotient map from $\mathscr{T}(\mathfrak{g})$ to $U(\mathfrak{g})$.

Theorem 2.3.2 (Poincaré-Birkhoff-Witt Theorem)**.** *Let the set $\{x_1, x_2, \ldots\}$ be a countable ordered basis of the Lie algebra $\mathfrak{g}$. Then the elements 1_k and $x_{i(1)} \ldots x_{i(m)} = \pi(x_{i(1)} \otimes \ldots \otimes x_{i(m)})$ for all $m \in \mathbb{N}$ such that $i(1) \leq i(2) \leq \ldots \leq i(m)$, form a basis of $U(\mathfrak{g})$.*

Proof. See [18, Section 17.4] for a proof. □

This theorem (often abbreviated to the PBW theorem) allows us to quickly give a basis for $U(\mathfrak{g})$, in the case that $\mathfrak{g}$ has countable dimension. Of course, we are only interested in a particular finite dimensional case, and for $\mathfrak{sl}_2(k)$ we can give this basis as the set of elements of the form

$$f^i h^n e^j,$$

where $i,j,n \in \mathbb{N}\cup\{0\}$, and by e^i we mean the image of i tensor products $e\otimes\cdots\otimes e$ under π.

Finally, we will look at a construction which will be used later in Chapter 4. For a Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, with the notation as above, we can define

$$U_n(\mathfrak{g}) := \mathbb{C}\text{-span}\,\{X_{i_1}\dots X_{i_m}\ :\ m\le n,\ X_{i_j}\in\mathfrak{g}\ \},$$

i.e., all those elements of $U(\mathfrak{g})$ with degree less than or equal to n. It's clear that $U_r(\mathfrak{g})U_s(\mathfrak{g})\subset U_{r+s}(\mathfrak{g})$. As such, we find that $U(\mathfrak{g})$ has a filtration given by

$$\mathbb{C} = U_0(\mathfrak{g})\subset U_1(\mathfrak{g})\subset U_2(\mathfrak{g})\subset\dots$$

From this we can form the graded ring, given as

$$\operatorname{gr}(U(\mathfrak{g})) := \bigoplus_{r=0}^{\infty} U_r(\mathfrak{g})/U_{r-1}(\mathfrak{g}),$$

where multiplication is defined in the obvious way. This ring is generated by the subset $\bar{\mathfrak{g}} = \{X+U_0(\mathfrak{g})\ :\ X\in\mathfrak{g}\ \}$. Using this, we see that $\operatorname{gr}(U(\mathfrak{g}))$ is in fact, commutative, as follows.

$$\begin{aligned}(X+U_0(\mathfrak{g})(Y+U_0(\mathfrak{g})) &= XY+U_1(\mathfrak{g})\\ &= YX+[X,Y]+U_1(\mathfrak{g})\\ &= YX+U_1(\mathfrak{g})\\ &= (Y+U_0(\mathfrak{g}))(X+U_0(\mathfrak{g}).\end{aligned}$$

As such, we may identify $\operatorname{gr}(U(\mathfrak{g}))$ with the symmetric algebra on $\bar{\mathfrak{g}}$, denoted $S(\bar{\mathfrak{g}})$.

2.3.2 Kostant $\mathbb{Z}$-forms

We begin this subsection with a defintion.

Definition 2.3.3. Let R be a subring of a field $\mathbb{F}$, such that R is a principal ideal domain, and $\mathbb{F}$ is the field of fractions of R. An *R-form* of an $\mathbb{F}$ vector space V is the R-span of an $\mathbb{F}$-basis of V.

In what follows, we will exclusively have $\mathbb{F}=\mathbb{Q}$ and $R=\mathbb{Z}$.

In analogy with Chevalley algebras, we can construct the hyperalgebra for a field of prime characteristic. The remainder of this section largely follows that in [18].

For Lie algebras, the idea is to construct a lattice inside the Lie algebra, denoted $\mathfrak{g}_{\mathbb{Z}}$, by finding a suitable $\mathbb{Z}$-basis, called a Chevalley basis, which has the property that the structure constants are integral, so the lattice is itself a Lie algebra. Tensoring over $\mathbb{Z}$ with the field $\mathbb{F}_p$ of p elements, $\mathfrak{g}_{\mathbb{F}_p} := \mathfrak{g}_{\mathbb{Z}}\otimes_{\mathbb{Z}}\mathbb{F}_p$, one obtains a Lie algebra over $\mathbb{F}_p$. Then, for any field extension k of $\mathbb{F}_p$

we can construct a Lie algebra over k by defining $\mathfrak{g}_k = \mathfrak{g}_{\mathbb{F}_p} \otimes_{\mathbb{F}_p} k$. The Lie algebra $\mathfrak{g}_k$ is called a Chevalley algebra.

Let $\mathfrak{g}$ be a semisimple Lie algebra with root system Φ, fix the ordering $\Phi^+ = \{\alpha_1, \dots \alpha_m\}$, and denote $h_i = h_{\alpha_i} = [x_{\alpha_i}, x_{\alpha_{-i}}]$.

Definition 2.3.4. A *Chevalley basis* of $\mathfrak{g}$ is a basis $\{x_\alpha \ : \ \alpha \in \Phi\} \cup \{h_i \ : \ 1 \le i \le l\}$ such that the following two conditions hold for all α and $\beta \in \Phi$:

1. $[x_\alpha, x_{-\alpha}] = h_\alpha$,

2. If $\alpha, \beta, \alpha+\beta \in \Phi$ with $[x_\alpha, x_\beta] = c_{\alpha\beta} x_{\alpha+\beta}$, then $c_{\alpha\beta} = -c_{-\alpha-\beta}$.

Theorem 2.3.5 (Chevalley)**.** *Let the set $\{x_\alpha \ : \ \alpha \in \Phi\} \cup \{h_i \ : \ 1 \le i \le l\}$ be a Chevalley basis of $\mathfrak{g}$. Then the structure constants $c_{\alpha\beta}$ are integers.* □

In fact, Chevalley's theorem tells us even more about the structure constants. For more details and a proof, see [18, Theorem 25.2].

Next we introduce some notation. For any commutative, associative $\mathbb{F}$-algebra A with 1 (where $\mathbb{F}$ is a field of characteristic 0), we can define the binomial element

$$\binom{h}{n} := \frac{h(h-1)\dots(h-n+1)}{n!} \in A,$$

where $h \in A$ and $n \in \mathbb{Z}$. We interpret $\binom{h}{0}$ to be 1, and for all negative n we have that $\binom{h}{n} = 0$.

Denote by A the m-tuple of integers given by $(a_1, a_2, \dots, a_m)$, and similarly for B and C. Define the following elements of the universal enveloping algebra $U(\mathfrak{g})$.

$$\begin{aligned} f_A &= \frac{x_{-\alpha_1}^{a_1}}{a_1!} \dots \frac{x_{-\alpha_m}^{a_m}}{a_m!}, \\ h_B &= \binom{h_1}{b_1} \dots \binom{h_l}{b_l}, \\ e_C &= \frac{x_{\alpha_1}^{c_1}}{c_1!} \dots \frac{x_{\alpha_m}^{c_m}}{c_m!}. \end{aligned}$$

Notice, in particular, that from the PBW theorem we have that the elements of the form $f_A h_B e_C$ form a basis of $U(\mathfrak{g})$.

The following theorem, due to Kostant, describes what we will call the Kostant $\mathbb{Z}$-form, to be denoted $U(\mathfrak{g})_{\mathbb{Z}}$, or just $U_{\mathbb{Z}}$ when the Lie algebra is clear.

Theorem 2.3.6 (Kostant)**.** *The subring of $U(\mathfrak{g})$ generated by all $x_\alpha^t/t!$ for $\alpha \in \Phi$ and $t \in \mathbb{N}$, is equal to the $\mathbb{Z}$-span of all elements of the form $f_A h_B e_C$.*

Proof. A proof of this theorem (and a much more detailed exposition) can be found in [18, Section 26]. □

For the Lie algebra $\mathfrak{sl}_2$, the Kostant $\mathbb{Z}$-form can be given as the $\mathbb{Z}$-span of the elements

$$\frac{f^i}{i!}\binom{h}{n}\frac{e^j}{j!},$$

for $i,j,n \in \mathbb{N}\cup\{0\}$.

2.3.3 The Hyperalgebra

We begin this subsection by giving the definition of the hyperalgebra, sometimes known also as the algebra of distributions (see [21, Section I.1.12] for further details).

Definition 2.3.7. For a field k and Lie algebra $\mathfrak{g}$, the *hyperalgebra*, denoted $U(\mathfrak{g})_k$ (or just U_k when the Lie algebra is clear) is given by the k-algebra

$$U_k = k \otimes_{\mathbb{Z}} U_{\mathbb{Z}}.$$

In the case $\mathfrak{g} = \mathfrak{sl}_2$, we will use the following notation

$$\begin{aligned} f_i &:= 1_k \otimes \frac{f^i}{i!}, \\ h_n &:= 1_k \otimes \binom{h}{n}, \\ e_j &:= 1_k \otimes \frac{e^j}{j!}. \end{aligned}$$

It follows then, from Theorem 2.3.6, that a k-basis for $U(\mathfrak{sl}_2)_k$ is given by elements of the form

$$f_i h_n e_j,$$

for $i,j,n \in \mathbb{N}\cup\{0\}$.

Our motivation for defining the hyperalgebra comes from Verma's conjecture, proved by Cline, Parshall and Scott in [5].

Theorem 2.3.8 (Verma's Conjecture)**.** *For a field k of prime characteristic, and a semisimple, simply connected algebraic group G with corresponding Lie algebra $\mathfrak{g}$, there is an equivalence of categories between finite dimensional, rational kG-modules and finite dimensional $U(\mathfrak{g})_k$-modules.*

Recall from Section 2.2.5 that the Weyl module $\Delta(s)$ has the universal property that it is generated by an element m_+ of weight s. When considered as a module for the hyperalgebra then, we can obtain a k-basis for $\Delta(s)$ by considering the elements

$$\{ f_i m_+ \ : \ i = 0, 1, \ldots, s \},$$

where $f_i \in U(\mathfrak{g})_k$ as above for each i. This follows by considering the character of $\Delta(s)$, and the fact that $f_i m_+$ has weight $s - 2i$.

CHAPTER 3

The Endomorphism Algebra

In this chapter we begin to develop the theory of the modules $\nabla(r) \otimes \Delta(s)$ for $SL_2(k)$ by looking at their endomorphism algebras. The results we obtain here will be essential for later chapters. At the end of this chapter we will give some explicit results for when $p = 2$.

3.1 Characteristic 0

First we will examine the case where the characteristic of the field is 0. It will be sufficient, for the most part, to use the field $\mathbb{Q}$, so we let $G = SL_2(\mathbb{Q})$. As mentioned in the previous chapter, we have that $\nabla(r) = \Delta(r) = V(r)$ is the unique simple module of highest weight r. In this case, the decomposition of the tensor product is given by the well known Clebsch-Gordan formula

$$V(r) \otimes V(s) = \bigoplus_{i=0}^{s} V(r+s-2i),$$

where we are assuming $r \geq s$. Our aim is to be able to describe the endomorphism algebra $\mathrm{End}_G(V(r) \otimes V(s))$ in such a way that allows us to describe the endomorphism algebra $\mathrm{End}_{U_k}(\nabla(r) \otimes \Delta(s))$ in positive characteristic.

Before doing this however, we show that the modules $V(r) \otimes V(s)$ are generated by a single element. This will greatly simplify our description of the endomorphism algebra in the next part. It turns out that this result holds not just for SL_2, but for an arbitrary reductive algebraic group, over an algebraically closed field of characteristic 0. The proof given here was supplied by Stephen Donkin.

Lemma 3.1.1. *Let G be a reductive algebraic group over K, an algebraically closed field of characteristic 0. For λ and $\mu \in X^+$ denote the simple modules $M = V(\lambda)$, $N = V(\mu)$, and let*

$m_+ \in M^\lambda$ be a highest weight vector, and $n_- \in N^{\omega_0\mu}$ a lowest weight vector. Then the module $M \otimes N$ is generated by $m_+ \otimes n_-$ as a KG-module.

Proof. To prove this statement, we will use the equivalence of categories outlined in Section 2.3 and consider M and N as modules for the universal enveloping algebra $U(\mathfrak{g})$. Denote by Z the module $U(\mathfrak{g})(m_+ \otimes n_-)$.

Let the positive roots be given by $\Phi^+ = \{\alpha_1, \ldots, \alpha_N\}$, and $A \in \mathbb{Z}^N$ be $(a_1, \ldots, a_N)$. Following the notation of Section 2.3.2 we have that

$$f_A(m_+ \otimes n_-) = f_A m_+ \otimes n_- \in Z.$$

If we denote by M' the set $\{m \in M \ : \ m \otimes n_- \in Z\}$, then since $M = U(\mathfrak{g})m_+$ (because M is simple), and m_+ has highest weight, it's clear that $M' = M$.

Now suppose that $M \otimes N \neq Z$, and let ν be a minimal weight of N such that $M \otimes N^\nu \not\subset Z$. Next, consider the weight $\nu - \omega_0\mu$ and write

$$S = \left\{B = (b_1, \ldots, b_N) \in \mathbb{Z}^N \ : \ \sum_{i=0}^{N} b_i \alpha_i = \nu - \omega_0\mu \right\}.$$

It's clear then that the weight space N^ν is equal to

$$N^\nu = \bigoplus_{B \in S} k e_B n_-,$$

since N is a simple module. However, we have that, for any $m \in M$, $e_B(m \otimes n_-) = m \otimes e_B n_- + v$ where

$$v \in \bigoplus_{\tau < \nu} M \otimes N^\tau$$

is a sum of elements of lower weight. It's clear that $e_B(m \otimes n_-) \in Z$, and since ν was chosen minimal, we have that $v \in Z$. Hence we must have that $m \otimes e_B n_- \in Z$. But $m \otimes e_B n_- \in M \otimes N^\nu$, so we obtain a contradiction and no such ν exists. Thus we have that $Z = M \otimes N$. □

3.1.1 The Endomorphism Algebra

Using the Clebsch-Gordan formula above, and since $V(r)$ is a simple module so that $\mathrm{End}_G(V(r)) \cong \mathbb{Q}$ [21, Proposition II.2.8], we can quickly give that

$$\begin{aligned} \mathrm{End}_G(V(r) \otimes V(s)) &= \bigoplus_{i=0}^{s} \mathrm{End}_G(V(r+s-2i)) \\ &\cong \bigoplus_{i=0}^{s} \mathbb{Q}. \end{aligned}$$

Hence the dimension of the endomorphism algebra of $V(r)\otimes V(s)$ is $s+1$. Furthermore we can immediately see that this algebra is commutative. Next we will exhibit a basis for the algebra $\mathrm{End}_G(V(r)\otimes V(s))$.

As a module in characteristic 0, we may consider $V(r)$ as a module for the universal enveloping algebra $U(\mathfrak{sl}_2(\mathbb{Q}))$ (or just $U_{\mathbb{Q}}$ for short), as discussed in Section 2.3. For the remainder of this section we will do this, without changing our notation for $V(r)$. Recall that by the PBW theorem, a basis of $U_{\mathbb{Q}}$ is given by the elements

$$\frac{e^i}{i!}\frac{h^k}{k!}\frac{f^j}{j!},$$

where $i,j,k\in\mathbb{N}_0$. For brevity we will write $e^i/i! = e^{(i)}$ and $f^j/j! = f^{(j)}$, and simply e for $e^{(1)}$, f for $f^{(1)}$. Pick non-zero $m_+\in V(r)$ of highest weight r, so that $hm_+ = rm_+$ and $e^{(i)}m_+ = 0$ for all $i\in\mathbb{N}$. Similarly pick a non-zero lowest weight vector $n_-\in V(s)$, so that $hn_- = -sn_-$ and $f^{(i)}n_- = 0$ for all $i\in\mathbb{N}$. Then, using Lemma 3.1.1 above, we have that $V(r)\otimes V(s)$ is generated by $m_+\otimes n_-$, which has weight $r-s$.

Now, given any endomorphism $\phi\in\mathrm{End}_{U_{\mathbb{Q}}}(V(r)\otimes V(s))$, it follows that ϕ is completely determined by the image of $m_+\otimes n_-$. Furthermore, since ϕ is an endomorphism, it commutes with the action of h, so the image of $m_+\otimes n_-$ must have the same weight as $m_+\otimes n_-$. Since this weight is $r-s$, we must have

$$\phi(m_+\otimes n_-)\in (V(r)\otimes V(s))^{r-s},$$

where, assuming $r\geq s$, we have

$$(V(r)\otimes V(s))^{r-s} = \mathbb{Q}\text{-span}\{m_+\otimes n_-, fm_+\otimes en_-,\dots, f^{(s)}m_+\otimes e^{(s)}n_-\}.$$

Hence we obtain a linear isomorphism

$$\begin{aligned}\mathrm{End}_{U_{\mathbb{Q}}}(V(r)\otimes V(s)) &\longrightarrow (V(r)\otimes V(s))^{r-s}\\ \phi &\longmapsto \phi(m+\otimes n_-).\end{aligned}$$

We can then give $\mathrm{End}_{U_{\mathbb{Q}}}(V(r)\otimes V(s))$ the basis $\{\phi_0,\phi_1,\dots,\phi_s\}$ where $\phi_i(m_+\otimes n_-) = f^{(i)}m_+\otimes e^{(i)}n_-$, and ϕ_0 is the identity endomorphism. We summarise the results of this section in the following lemma.

Lemma 3.1.2. *Let $G = SL_2(\mathbb{Q})$ and $r\geq s$. Let m_+ be a highest weight vector in $V(r)$ and n_- a lowest weight vector in $V(s)$. Then we have the following:*

1. *$V(r)\otimes V(s)$ is generated by $m_+\otimes n_-$ as a $U_{\mathbb{Q}}$-module.*

2. *The endomorphism algebra $\mathrm{End}_{U_{\mathbb{Q}}}(V(r)\otimes V(s))$ is commutative and has basis $\{\phi_0,\phi_1,\dots,\phi_s\}$, where the ϕ_i are as above.*

□

3.1.2 Actions of the Universal Enveloping Algebra

We will now give some explicit results on the action of the universal enveloping algebra, which will make our calculations in the later sections significantly easier. Some general results on the relations between elements of the universal enveloping algebra can be found in [18, 26.2], however we will be mostly interested in the actions of $e^{(i)}$ and $f^{(j)}$, particularly on highest weight vectors.

First we give the action of $f^{(j)}$ and $e^{(i)}$ on arbitrary tensor products. Let M and N be $U(\mathfrak{g})$-modules with $m \in M$ and $n \in N$. Since the Lie algebra acts as derivations on tensor products (i.e. $f(m \otimes n) = fm \otimes n + m \otimes fn$), we have the following

$$f^{(k)}(m \otimes n) = \sum_{i+j=k} f^{(i)}m \otimes f^{(j)}n,$$
$$e^{(k)}(m \otimes n) = \sum_{i+j=k} e^{(i)}m \otimes f^{(j)}n.$$

Next, we recall the action on the symmetric algebra of E, denoted $S(E)$, which we will consider as the algebra of polynomials over $\mathbb{Q}$ in the two variables x_1 and x_2. We have

$$f^{(j)}(x_1^a x_2^b) = \binom{a}{j} x_1^{a-j} x_2^{b+j}$$

and the weight of $f^{(j)}(x_1^a x_2^b)$ is $a - b - 2j$. Similarly we have

$$e^{(i)}(x_1^a x_2^b) = \binom{b}{i} x_1^{a+i} x_2^{b-i}$$

and the weight of $e^{(i)}(x_1^a x_2^b)$ is $a - b + 2i$. In particular we can think of the action of $f^{(1)}$ on $S(E)$ as $x_2 \partial_{x_1}$ and of the action of $e^{(1)}$ as $x_1 \partial_{x_2}$. Furthermore it's clear that for each i and $r \in \mathbb{N}$ we have that both $e^{(i)}$ and $f^{(i)}$ preserve the $\mathbb{Z}$-form $S^r E_{\mathbb{Z}}$, given by the $\mathbb{Z}$-span of the set $\{x_1^a x_2^b \,:\, a + b = r\}$.

Lemma 3.1.3. *Let V be a module for $U_{\mathbb{Q}}$ with highest weight m, and let v_+ be a highest weight vector. Then for all $i, j \in \mathbb{N}$ such that $j \geq i$,*

$$e^{(i)} f^{(j)} v_+ = \binom{m-(j-i)}{i} f^{(j-i)} v_+,$$

with weight $m - j + i$. If $i > j$ then we have

$$e^{(i)} f^{(j)} v_+ = 0.$$

Proof. The second statement is clear, since v_+ has highest weight m, and if $i > j$ then the weight of $e^{(i)}f^{(j)}v_+$ is greater than m. The first statement is proved by two inductions. First, we will show, by induction on j that $ef^{(j)}v_+ = (m-j+1)f^{(j-1)}v_+$. Afterwards, we will prove the result by induction on i.

First then, let's take $j = 1$ and consider efv_+. Since we have $ef - fe = h$ by the commutator relations in $\mathfrak{sl}_2$, we may write $efv_+ = (fe+h)v_+$. Furthermore, v_+ is a highest weight vector, so $ev_+ = 0$. Thus we have

$$efv_+ = hv_+ = mv_+,$$

giving us the base case. For the inductive step, let's assume that $ef^{(j)}v_+ = (m-j+1)f^{(j-1)}v_+$. Then we have

$$\begin{aligned} ef^{(j+1)}v_+ &= \frac{ef}{j+1}f^{(j)}v_+ \\ &= \frac{fe+h}{j+1}f^{(j)}v_+ \\ &= \frac{f}{j+1}ef^{(j)}v_+ + \frac{h}{j+1}f^{(j)}v_+ \\ &= (m-j+1)\frac{f}{j+1}f^{(j-1)}v_+ + \frac{m-2j}{j+1}f^{(j)}v_+, \end{aligned}$$

where for the first expression we have used the induction hypothesis, and in the second that the vector $f^{(j)}v_+$ has weight $m-2j$, and h has the effect of multiplying by the weight. Next, using that $ff^{(j-1)} = jf^{(j)}$ we obtain

$$\begin{aligned} ef^{(j+1)}v_+ &= \frac{j(m-j+1)}{j+1}f^{(j)}v_+ + \frac{m-2j}{j+1}f^{(j)}v_+ \\ &= \frac{j(m-j+1)+m-2j}{j+1}f^{(j)}v_+ \\ &= \frac{(j+1)(m-j)}{j+1}f^{(j)}v_+ \\ &= (m-j)f^{(j)}v_+, \end{aligned}$$

so the first result holds.

Next, we show the result by induction on i. Note that the previous induction gives us the base case $i = 1$. Before tackling the inductive step, we first note that we can rewrite the binomial coefficient so that the result becomes

$$e^{(i)}f^{(j)}v_+ = \frac{1}{i!}\Big(\prod_{i=1}^{i}(m-j+i)\Big)f^{(j-i)}v_+.$$

Using this as our induction hypothesis, we calculate

$$\begin{aligned}
e^{(i+1)}f^{(j)}v_+ &= \frac{e}{i+1}(e^{(i)}f^{(j)}v_+) \\
&= \frac{e}{i+1}\frac{1}{i!}\Big(\prod_{i=1}^{i}(m-j+i)\Big)f^{(j-i)}v_+ \\
&= \frac{1}{i+1!}\Big(\prod_{i=1}^{i}(m-j+i)\Big)ef^{(j-i)}v_+ \\
&= \frac{1}{i+1!}\Big(\prod_{i=1}^{i}(m-j+i)\Big)(m-(j-i)+1)f^{(j-i-1)}v_+ \\
&= \frac{1}{i+1!}\Big(\prod_{i=1}^{i}(m-j+i)\Big)(m-j+(i+1))f^{(j-i-1)}v_+ \\
&= \frac{1}{i+1!}\Big(\prod_{i=1}^{i+1}(m-j+i)\Big)f^{(j-i-1)}v_+
\end{aligned}$$

□

3.2 Moving to Positive Characteristic

In this section we will give a central result which allows us to begin to understand the endomorphism algebra of $\nabla(r)\otimes\Delta(s)$ over k, a field of positive characteristic p. First we will introduce some notation.

Over a field of characteristic 0, when considering elements of $V(r)\otimes V(s)$, we will use the natural basis of $\nabla(r)$ for $V(r)$ (as given in Section 2.2.5), and the basis $\{f^{(i)}m_+ \;:\; i=0,\dots,s\}$ of $\Delta(s)$ for $V(s)$, for a highest weight vector $m_+\in\Delta(s)$. This will correspond with the bases in the positive characteristic case. We define the following $\mathbb{Z}$-forms inside $V(r)$ and $V(s)$ respectively,

$$\nabla(r)_{\mathbb{Z}} = \mathbb{Z}\text{-span}\left\{x_1^i x_2^j \;:\; i+j=r\right\},$$

and

$$\Delta(s)_{\mathbb{Z}} = \mathbb{Z}\text{-span}\left\{f^{(i)}m_+ \;:\; i=0,\dots,s\right\},$$

where $m_+\in V(s)$ is a highest weight vector of weight s.

The result we prove is that we can construct all endomorphisms in positive characteristic from those in characteristic 0 that preserve the $\mathbb{Z}$-form $\nabla(r)_{\mathbb{Z}}\otimes_{\mathbb{Z}}\Delta(s)_{\mathbb{Z}}$. First however, we will give the dimension of the endomorphism algebra of $\nabla(r)\otimes\Delta(s)$. This result uses the ideas of polynomial $GL_n(k)$-modules, as introduced in Section A.1, as well as ideas introduced in Section A.2.

Lemma 3.2.1. *Let $r, s \in \mathbb{N}$ with $r \geq s$. Then the dimension of the algebra $End_{SL_2(k)}(\nabla(r) \otimes \Delta(s))$ is equal to $s+1$.*

Proof. First we note that

$$\begin{aligned}
\mathrm{End}_{SL_2(k)}(\nabla(r) \otimes \Delta(s)) &= \mathrm{Hom}_{SL_2(k)}(\nabla(r) \otimes \Delta(s), \nabla(r) \otimes \Delta(s)) \\
&= \mathrm{Hom}_{SL_2(k)}(\nabla(r), \nabla(r) \otimes \Delta(s) \otimes \Delta(s)^*) \\
&= \mathrm{Hom}_{SL_2(k)}(\nabla(r) \otimes \Delta(s)^*, \nabla(r) \otimes \Delta(s)^*) \\
&= \mathrm{Hom}_{SL_2(k)}(\nabla(r) \otimes \nabla(s), \nabla(r) \otimes \nabla(s)).
\end{aligned}$$

Now consider the $GL_2(k)$-module S^rE. When we restrict this module to $SL_2(k)$, we obtain $\nabla(r)$. Hence, for the purpose of finding the dimension, we may instead consider

$$\mathrm{Hom}_{GL_2(k)}(S^rE \otimes S^sE, S^rE \otimes S^sE),$$

using Lemma A.1.4, since $S^rE \otimes S^sE$ is a homogeneous polynomial $GL_2(k)$ module. As in Section A.2, we write $S^{(r,s)}E$ for $S^rE \otimes S^sE$.

Now, for any polynomial $GL_2(k)$-module X and weight α we have $\mathrm{Hom}_{GL_2(k)}(X^\circ, S^\alpha E) \cong X^\alpha$ (Lemma A.2.3), where X° is the contravariant dual (as described in Section A.2). Since the weight spaces of X and X° have the same dimension, we obtain that

$$\mathrm{Hom}_{GL_2(k)}(S^{(r,s)}E, S^{(r,s)}E) \cong (S^{(r,s)}E)^{(r,s)}.$$

We can calculate the (r,s) weight space of $S^{(r,s)}E$ as follows. We have that

$$\begin{aligned}
\mathrm{diag}(t_1,t_2) x_1^{r-i} x_2^i \otimes y_1^{s-j} y_2^j &= t_1^{r-i} t_2^i x_1^{r-i} x_2^i \otimes t_1^{s-j} t_2^j y_1^{s-j} y_2^j \\
&= t_1^{r+s-(i+j)} t_2^{i+j} x_1^{r-i} x_2^i \otimes y_1^{s-j} y_2^j,
\end{aligned}$$

for any $i = 0, \ldots, r$ and $j = 0, \ldots, s$. Hence the basis vector $x_1^{r-i} x_2^i \otimes y_1^{s-j} y_2^j$ is in $S^{(r,s)}E^{(r,s)}$ if and only if $i + j = s$. Since $r \geq s$ we have exactly $s+1$ ways to do this, taking $j = 0, \ldots, s$ and $i = s - j$. We obtain that

$$\dim \mathrm{Hom}_{GL_2(k)}(S^{(r,s)}E, S^{(r,s)}E) = s+1,$$

and hence

$$\dim \mathrm{End}_{SL_2(k)}(\nabla(r) \otimes \Delta(s)) = s+1.$$

$\square$

In particular, we notice that this dimension is equal to that in the characteristic 0 case. In the case $r < s$ we have that $\nabla(r) \otimes \Delta(s) = (\nabla(s) \otimes \Delta(r))^*$, and since these modules are of finite dimension, we can use the above result to get that $\dim(\mathrm{End}_{SL_2(k)}(\nabla(r) \otimes \Delta(s))) = r+1$.

Next we will give a more general result, which enables us to construct endomorphisms in prime characteristic from those in characteristic 0. We define $U_{\mathbb{F}_p} = \mathbb{F}_p \otimes_{\mathbb{Z}} U_{\mathbb{Z}}$.

Lemma 3.2.2. *Let M and N be two $U_{\mathbb{Z}}$ modules that are both finitely generated and torsion free over $\mathbb{Z}$. Then the map*

$$\mathbb{F}_p \otimes \mathrm{Hom}_{U_{\mathbb{Z}}}(M,N) \longrightarrow \mathrm{Hom}_{U_{\mathbb{F}_p}}(M_{\mathbb{F}_p}, N_{\mathbb{F}_p})$$

is injective.

Proof. Consider the short exact sequence of $\mathbb{Z}$-modules

$$0 \longrightarrow \mathrm{Hom}_{U_{\mathbb{Z}}}(M,N) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(M,N) \longrightarrow Q \longrightarrow 0,$$

where we consider Q as the quotient module in the usual way. Since both M and N are finitely generated and torsion free, it's clear that the first two modules are also torsion free. We will show that the module Q is also torsion free. Consider the element $q = \theta + \mathrm{Hom}_{U_{\mathbb{Z}}}(M,N) \in Q$, and suppose that for some non-zero $r \in \mathbb{Z}$ we have $rq = 0$ (i.e. that $r\theta \in \mathrm{Hom}_{U_{\mathbb{Z}}}(M,N)$). Then for any $u \in U_{\mathbb{Z}}$ and $m \in M$ we have

$$(r\theta)(um) = ur\theta(m),$$

or in other words

$$r(\theta(um) - u\theta(m)) = 0.$$

Now N is torsion free, so we must have $\theta(um) - u\theta(m) = 0$, and hence $\theta \in \mathrm{Hom}_{U_{\mathbb{Z}}}(M,N)$ and so $q = 0$.

Since Q is torsion free, and hence flat, we may tensor this sequence with the field $\mathbb{F}_p$ whilst maintaining exactness, to obtain the exact sequence

$$0 \longrightarrow \mathbb{F}_p \otimes \mathrm{Hom}_{U_{\mathbb{Z}}}(M,N) \longrightarrow \mathbb{F}_p \otimes \mathrm{Hom}_{\mathbb{Z}}(M,N) \longrightarrow \mathbb{F}_p \otimes Q \longrightarrow 0.$$

Hence we see that $\mathbb{F}_p \otimes \mathrm{Hom}_{U_{\mathbb{Z}}}(M,N)$ embeds into $\mathbb{F}_p \otimes \mathrm{Hom}_{\mathbb{Z}}(M,N)$. Next we use the fact that we have an isomorphism

$$\mathbb{F}_p \otimes \mathrm{Hom}_{\mathbb{Z}}(M,N) \simeq \mathrm{Hom}_{\mathbb{F}_p}(M_{\mathbb{F}_p}, N_{\mathbb{F}_p}),$$

so that we obtain the embedding

$$\mathbb{F}_p \otimes \mathrm{Hom}_{U_{\mathbb{Z}}}(M,N) \longrightarrow \mathrm{Hom}_{\mathbb{F}_p}(M_{\mathbb{F}_p}, N_{\mathbb{F}_p}).$$

Now since the image, under the above map, of each ϕ is in fact a $U_{\mathbb{F}_p}$-module homomorphism, and we can identify $\mathrm{Hom}_{U_{\mathbb{F}_p}}(M_{\mathbb{F}_p}, N_{\mathbb{F}_p})$ as a subalgebra of $\mathrm{Hom}_{\mathbb{F}_p}(M_{\mathbb{F}_p}, N_{\mathbb{F}_p})$, we have the embedding

$$\mathbb{F}_p \otimes \mathrm{Hom}_{U_{\mathbb{Z}}}(M,N) \longrightarrow \mathrm{Hom}_{U_{\mathbb{F}_p}}(M_{\mathbb{F}_p}, N_{\mathbb{F}_p}).$$

□

Now let's consider the case $M = N = \nabla(r)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Delta(s)_{\mathbb{Z}}$. It's clear that $\mathrm{Hom}_{U_{\mathbb{Z}}}(M,N)$ embeds into $\mathrm{Hom}_{U_{\mathbb{Q}}}(M,N)$, and since the latter is commutative, so is the former. Furthermore we have that $\mathrm{Hom}_{U_{\mathbb{Z}}}(M,N)$ is a $\mathbb{Z}$-form of $\mathrm{Hom}_{U_{\mathbb{Q}}}(M_{\mathbb{Q}}, N_{\mathbb{Q}})$, so that the $\mathbb{Z}$-rank of $\mathrm{Hom}_{U_{\mathbb{Z}}}(M,N)$ is equal to the $\mathbb{Q}$-dimension of $\mathrm{Hom}_{U_{\mathbb{Q}}}(M_{\mathbb{Q}}, N_{\mathbb{Q}})$, which, as discussed above, is $s+1$. Hence the image of $\mathbb{F}_p \otimes \mathrm{Hom}_{U_{\mathbb{Z}}}(M,N)$ in $\mathrm{Hom}_{U_{\mathbb{F}_p}}(M_{\mathbb{F}_p}, N_{\mathbb{F}_p})$ has dimension $s+1$. However, we know already, from Lemma 3.2.1, that $\dim \mathrm{Hom}_{U_{\mathbb{F}_p}}(M_{\mathbb{F}_p}, N_{\mathbb{F}_p}) = s+1$, so this is in fact an isomorphism. We summarise the result in a lemma.

Lemma 3.2.3. *The map*

$$\mathbb{F}_p \otimes_{\mathbb{Z}} \mathrm{End}_{U_{\mathbb{Z}}}(\nabla(r)_{\mathbb{Z}} \otimes \Delta(s)_{\mathbb{Z}}) \longrightarrow \mathrm{End}_{U_{\mathbb{F}_p}}(\nabla(r) \otimes \Delta(s))$$

is an isomorphism. In particular, $\mathrm{End}_{U_{\mathbb{F}_p}}(\nabla(r) \otimes \Delta(s))$ *is commutative.* □

We can construct endomorphisms of $\nabla(r) \otimes \Delta(s)$ by extending k-linearly endomorphisms in $\mathrm{End}_{U_{\mathbb{F}_p}}(\nabla(r) \otimes \Delta(s))$, where we are viewing $U_{\mathbb{F}_p}$ as a subalgebra of U_k, so we may naturally consider $\nabla(r)$ and $\Delta(s)$ as $U_{\mathbb{F}_p}$-modules. The lemma shows us that we may construct elements of $\mathrm{End}_{U_{\mathbb{F}_p}}(\nabla(r) \otimes \Delta(s))$ by considering endomorphisms of $V(r) \otimes V(s)$ that preserve the $\mathbb{Z}$-form $\nabla(r)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Delta(s)_{\mathbb{Z}}$. We will call this process 'base change '.

The module $\nabla(r) \otimes \Delta(s)$ is, in general, not generated by the single element $x_2^r \otimes m_+$, so when looking for endomorphisms that preserve the $\mathbb{Z}$-form we must consider the image under the endomorphism on a generating set of $\nabla(r) \otimes \Delta(s)$. For this it is certainly sufficient to take the set

$$\{\, x_1^i x_2^j \otimes m_+ \;:\; i+j=r \,\}.$$

Lemma 3.2.4. *The set* $S = \{\, x_1^i x_2^j \otimes m_+ \;:\; i+j=r \,\}$ *generates* $\nabla(r) \otimes \Delta(s)$ *as a* U_k*-module.*

Proof. The proof of this is identical to the second part of the proof of Lemma 3.1.1, with $\nabla(r) = M$, $\Delta(s) = N$ and letting n_- be a non-zero lowest weight element of $\Delta(s)$. The statement of the lemma allows us to assume that we already have $\nabla(r) \otimes n_- \subset Z$, where Z is the submodule generated by S. Note that we are utilising the fact that whilst $\Delta(s)$ is not necessarily simple, it is nonetheless generated as a U_k-module by the lowest weight element n_-. □

Now suppose that $\psi \in \mathrm{End}_{SL_2(\mathbb{Q})}(V(r)\otimes V(s))$ is an endomorphism preserving the $\mathbb{Z}$-form $\nabla(r)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Delta(s)_{\mathbb{Z}}$. We will denote by $\overline{\psi}$ the endomorphism,

$$\overline{\psi} = 1_k \otimes_{\mathbb{Z}} \psi|_{\mathbb{Z}},$$

in the algebra $\mathrm{End}_{U_k}(\nabla(r)\otimes\Delta(s))$, where $\psi|_{\mathbb{Z}}$ denotes the restriction of ψ to the $\mathbb{Z}$-form $\nabla(r)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Delta(s)_{\mathbb{Z}}$. Since we know that the dimensions of the endomorphism algebras in both characteristic 0 and p are equal, we would like to find a basis of $\mathrm{End}_{SL_2(\mathbb{Q})}(V(r)\otimes V(s))$ where each element preserves the $\mathbb{Z}$-form $\nabla(r)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Delta(s)_{\mathbb{Z}}$. Unfortunately, the basis given by the ϕ_i in the previous section does not, in general, do this, so we will need to take linear combinations of the elements ϕ_i.

Supposing we have found such a basis of $\mathrm{End}_{SL_2(\mathbb{Q})}(V(r)\otimes V(s))$, one further problem arises. This is that the set of endomorphisms given by the process of base change may no longer be linearly independent. In fact, certain endomorphisms may be 0 after the base change process, so we must pick our basis of $\mathrm{End}_{SL_2(\mathbb{Q})}(V(r)\otimes V(s))$ with this in mind.

3.3 Filtrations

The following theorem, known as Kempf's vanishing theorem, allows us to give an essential structure to the modules $\nabla(r)\otimes\Delta(s)$. Despite the name, the statement for a field of characteristic 0 had been known for some time. In 1976 Kempf's proof for a field of prime characteristic was published in [22].

Theorem 3.3.1 (Kempf's Vanishing Theorem)**.** *Let G be a reductive algebraic group, B a Borel subgroup. Then for all $\lambda \in X^+$ and $i > 0$ we have $R^i\mathrm{Ind}_B^G(k_\lambda) = 0$.* □

The next result and its derivation are taken from [27, Lemma 3.3], and are central to the results of this thesis. First, pick the maximal torus T and Borel subgroup B of $SL_2(k)$ as in Section 2.1.5, and associate $X(T)$ with $\mathbb{Z}$. Note that the module $\Delta(s)$, for $s \in \mathbb{N}$ has a B-module composition series with factors given by $k_{-s}, k_{-s+2}, \ldots, k_{s-2}, k_s$; this follows by looking at the character $\chi(s)$, and noticing that under the action of B the weight of a weight vector in $\Delta(s)$ cannot go up. Next suppose that $r \geq s-1$, and consider $\nabla(r)\otimes\Delta(s)$. Then

$$\nabla(r)\otimes\Delta(s) = (\mathrm{Ind}_B^G k_r)\otimes\Delta(s) = \mathrm{Ind}_B^G(k_r \otimes \Delta(s)),$$

where the second equality is by the tensor identity (Theorem 2.2.12). Now $k_r \otimes \Delta(s)$ has a B-module composition of $k_{r-s}, k_{r-s+2}, \ldots, k_{r+s}$, so we obtain for each a short exact sequence, given inductively by

$$0 \longrightarrow k_{r-s} \longrightarrow k_r \otimes \Delta(s) \longrightarrow M_0 \longrightarrow 0,$$

and

$$0 \longrightarrow k_{r-s+2i} \longrightarrow M_{i-1} \longrightarrow M_i \longrightarrow 0,$$

for $i = 1,\dots,s$. Applying the functor of induction to these sequences, and noting that all the terms with the derived functor vanish thanks to Kempf's vanishing theorem, we obtain the short exact sequences

$$0 \longrightarrow \nabla(r-s) \longrightarrow \nabla(r) \otimes \Delta(s) \longrightarrow N_0 \longrightarrow 0,$$

and

$$0 \longrightarrow \nabla(r-s+2i) \longrightarrow N_{i-1} \longrightarrow N_i \longrightarrow 0,$$

for $i = 1,\dots,s$. Hence we have shown that the module $\nabla(r) \otimes \Delta(s)$ has a good filtration, with sections given by the Clebsch-Gordan coefficients. This is in direct analogy with the characteristic 0 case. Furthermore, by taking the dual we obtain that if $r \le s$, the module $\nabla(r) \otimes \Delta(s)$ has a Weyl filtration, with the same coefficients. In summary,

Theorem 3.3.2 ([27, Lemma 3.3])**.** *Let $G = SL_2(k)$, for an algebraically closed field k of characteristic p, and suppose $r \ge s-1$ for $r,s \in \mathbb{N}$. Then the module $\nabla(r) \otimes \Delta(s)$ has a good filtration with sections $\nabla(r+s), \nabla(r+s-2),\dots,\nabla(r-s)$; and the module $\nabla(s) \otimes \Delta(r)$ has a Weyl filtration with sections $\Delta(r+s), \Delta(r+s-2),\dots,\Delta(r-s)$.* □

As an immediate corollary we notice that if $|r-s| \le 1$ then the module $\nabla(r) \otimes \Delta(s)$ is a tilting module. We will investigate this further in Chapter 5.

It's clear from Theorem 3.3.2 that the module $\nabla(r) \otimes \Delta(s)$ can have at most $s+1$ indecomposable summands, since every summand must also have a good (or, in the case $r < s$, Weyl) filtration (note that this agrees with Lemma 3.2.1). We remark now that our primary goal in this thesis, is to describe $\nabla(r) \otimes \Delta(s)$ in terms of either the good filtrations or Weyl filtrations of its indecomposable summands. In the next section we give some examples of this in characteristic 2.

3.4 Finding Endomorphisms

In Section 3.1.1 we gave the dimension of the endomorphism algebra of $\nabla(r) \otimes \Delta(s)$, which turned out to be independent of the characteristic of the field. Furthermore, we showed that in all cases this algebra is also commutative. In this section, we will use these results to help us understand some particular cases when the characteristic of the field k is 2. The idea will be to write the identity endomorphism as a sum of centrally primitive idempotents. Each such idempotent must correspond to a projection onto an indecomposable summand, so there is a one to one correspondence between the primitive idempotents in $\mathrm{End}_G(\nabla(r) \otimes \Delta(s))$ and the indecomposable summands. We will briefly outline these ideas now, remarking that, by definition, there is a one to one correspondence between the idempotents of an endomorphism algebra and the projections in that algebra. For more details on idempotents, see [12, Section 41].

For the following, k may be any field. Let M be a finite dimensional module for a k-algebra A, and write the identity element in $\mathrm{End}_A(M)$ as the sum of pairwise orthogonal primitive idempotents

$$1_M = e_1 + e_2 + \cdots + e_n.$$

Then we have

$$1_M(M) = M = e_1(M) + e_2(M) + \cdots + e_n(M).$$

Furthermore, such a sum is direct since if $m = e_i(m_i)$ and $m = e_j(m_j)$ then

$$m = e_i^2(m_i) = e_i e_j(m_j) = 0,$$

since the e_i are orthogonal. Writing $M_i = e_i(M)$, we have the decomposition

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_n.$$

Suppose that for some i the summand M_i can be written as a direct sum $M_{i_1} \oplus M_{i_2}$ of submodules. Then we can write $e_i = e_{i_1} + e_{i_2}$, where e_{i_1} is the restriction of e_i to M_{i_1}, and is thus also an idempotent, and similarly for e_{i_2}. However, such idempotents are orthogonal, contradicting the fact that e_i is primitive. Hence the M_i are indecomposable.

Now consider $\mathrm{End}_A(M_i)$. Since M_i is indecomposable, we have that $\mathrm{End}_A(M_i)$ is a local ring ([23, Proposition X.7.4]) so that $\dim \mathrm{End}_A(M_i) = 1 + \dim \mathrm{Rad}\big(\mathrm{End}_A(M_i)\big)$, where the radical $\mathrm{Rad}\big(\mathrm{End}_A(M_i)\big)$ is equal to the set of the nilpotent elements of $\mathrm{End}_A(M_i)$. Since

$$\mathrm{End}_A(M) = \bigoplus_{i,j=0}^{n} \mathrm{Hom}_A(M_i, M_j),$$

we have that the number of centrally primitive idempotents of $\mathrm{End}_A(M)$ is equal to the dimension of $\mathrm{End}_A(M)$ minus the span of all of its nilpotent elements.

3.4.1 $r = 2$

We will look at the modules $\nabla(2) \otimes \Delta(s)$, for arbitrary s, when k is an algebraically closed field of characteristic 2. By Theorem 3.3.2 we have that $\nabla(2) \otimes \Delta(s)$ has a Weyl filtration with sections $\Delta(s-2)$, $\Delta(s)$, and $\Delta(s+2)$. By Section 2.2.7, we have that there are the following possibilities for the decomposition of $\nabla(2) \otimes \Delta(s)$, for $s \geq 2$. These are given by

$$\Delta(s-2)\oplus\Delta(s)\oplus\Delta(s+2),\quad \frac{\Delta(s-2)}{\Delta(s)}\oplus\Delta(s+2),\quad \frac{\Delta(s-2)}{\Delta(s+2)}\oplus\Delta(s),\quad \Delta(s-2)\oplus\frac{\Delta(s)}{\Delta(s+2)},$$

and the possibility that $\nabla(2)\otimes\Delta(s)$ is indecomposable, where we are using the notation

$$\frac{M}{N}$$

to denote a non-split extension between modules M and N. We note that the final possibility can only happen when $s-2, s$ and $s+2$ are in the same block. For $p=2$, this can only happen when s is even, where the non-negative even numbers form a block.

To determine which of these decompositions occurs, we first note that in $V(2)\otimes V(s)$ we have that $e(x_2^2\otimes m_+)=2x_1x_2\otimes m_+$. Hence in $\nabla(2)\otimes\Delta(s)$ we have that $e(x_2^2\otimes m_+)=0$, so $x_1x_2\otimes m_+$ is not in the module generated by $x_2^2\otimes m_+$. However, we do have that $e_2(x_2^2\otimes m_+)=x_1^2\otimes m_+$, so by Lemma 3.2.4 we have that $\nabla(2)\otimes\Delta(s)$ is generated by the set

$$\{\, x_2^2\otimes m_+,\ x_1x_2\otimes m_+\,\}.$$

When constructing endomorphisms then, we will need to consider the images of these two elements in characteristic 0.

Now, by Section 3.2, we obtain from the characteristic 0 case that the algebra $\mathrm{End}_{U_k}(\nabla(2)\otimes\Delta(s))$ is commutative, and has dimension 3. Furthermore, in characteristic 0 we have that $\mathrm{End}_{U_{\mathbb{Q}}}(V(2)\otimes V(s))$ has basis $\{\phi_0,\phi_1,\phi_2\}$, where ϕ_i is given by

$$\phi_i(x_2^2\otimes m_+)=e^{(i)}(x_2^2)\otimes f^{(i)}(m_+).$$

We give in Table 3.1 the images of the endomorphisms over $\mathbb{Q}$, where in order to compute $\phi_i(x_1x_2\otimes m_+)$ we use that $x_1x_2\otimes m_+=\frac{e}{2}(x_2^2\otimes m_+)$ so that $\phi_i(x_1x_2\otimes m_+)=\frac{e}{2}\phi_i(x_2^2\otimes m_+)$.

$\mathbb{Q}$	$x_2^2\otimes m_+$	$x_1x_2\otimes m_+$
ϕ_0	$x_2^2\otimes m_+$	$x_1x_2\otimes m_+$
ϕ_1	$2x_1x_2\otimes fm_+$	$x_1^2\otimes fm_+ + sx_1x_2\otimes m_+$
ϕ_2	$x_1^2\otimes f^{(2)}m_+$	$\frac{(s-1)}{2}x_1^2\otimes fm_+$

Table 3.1: The endomorphisms ϕ_i on $V(2)\otimes V(s)$

From this table it's clear that if s is even, then the endomorphism ϕ_2 does not preserve the $\mathbb{Z}$-form $\nabla(2)_{\mathbb{Z}}\otimes_{\mathbb{Z}}\Delta(s)_{\mathbb{Z}}$. Furthermore, by simply clearing the denominators and considering $2\phi_2$ we find that after base change we have $\overline{2\phi_2}=\overline{\phi_1}$. We will, for the minute, ignore this case and consider the case when s is odd.

3.4.2 The Odd Case

Let $s \in \mathbb{N}$ be odd, say $s = 2u+1$ for some $u \in \mathbb{N}$. Here we have not only that each ϕ_i preserves the $\mathbb{Z}$-form $\nabla(2)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Delta(s)_{\mathbb{Z}}$, but the endomorphisms remain linearly independent when taken modulo 2. Table 3.2 shows this, where we have used $\bar{u}$ to denote the image of $u \in \mathbb{Z}$ under the map $\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \hookrightarrow k$.

k	$x_2^2 \otimes m_+$	$x_1x_2 \otimes m_+$
$\overline{\phi_0}$	$x_2^2 \otimes m_+$	$x_1x_2 \otimes m_+$
$\overline{\phi_1}$	0	$x_1^2 \otimes fm_+ + x_1x_2 \otimes m_+$
$\overline{\phi_2}$	$x_1^2 \otimes f_2m_+$	$\bar{u}x_1^2 \otimes fm_+$

Table 3.2: The endomorphisms $\overline{\phi_i}$ on $\nabla(2) \otimes \Delta(s)$

Next we would like to discern the number of linearly independent idempotents of $\mathrm{End}_{U_k}(\nabla(2) \otimes \Delta(s))$, since this will correspond to the number of indecomposable summands. To do this, we first, rather optimistically, compute $\overline{\phi_1}^2$. It's clear that $\overline{\phi_1}^2(x_2^2 \otimes m_+) = 0 = \overline{\phi_1}(x_2^2 \otimes m_+)$, so we calculate $\overline{\phi_1}^2(x_1x_2 \otimes m_+)$. Using that $x_1^2 \otimes fm_+ = e_2f(x_2^2 \otimes m_+)$ we obtain

$$\begin{aligned}
\overline{\phi_1}^2(x_1x_2 \otimes m_+) &= \overline{\phi_1}(x_1^2 \otimes fm_+) + \overline{\phi_1}(x_1x_2 \otimes m_+) \\
&= \overline{\phi_1}(e_2f(x_2^2 \otimes m_+) + \overline{\phi_1}(x_1x_2 \otimes m_+) \\
&= e_2f\overline{\phi_1}(x_2^2 \otimes m_+) + \overline{\phi_1}(x_1x_2 \otimes m_+) \\
&= e_2f(0) + \overline{\phi_1}(x_1x_2 \otimes m_+) \\
&= \overline{\phi_1}(x_1x_2 \otimes m_+).
\end{aligned}$$

Hence we see that $\overline{\phi_1}^2 = \overline{\phi_1}$, so $\overline{\phi_1}$ is an idempotent. Now we turn to $\overline{\phi_2}$, where will have to consider both cases for $\bar{u}$.

Once again, we want to compute $\overline{\phi_2}^2$ of our generating elements. To do this it will be necessary to write the image under $\overline{\phi_2}$ in terms of the generating elements. First we notice that

$$x_1^2 \otimes f_2m_+ = e_2(x_2^2 \otimes f_2m_+) + \bar{u}x_2^2 \otimes m_+,$$

so that we may write

$$x_1^2 \otimes f_2m_+ = e_2f_2(x_2^2 \otimes m_+) + \bar{u}x_2^2 \otimes m_+.$$

Here we have used that $e_2f_2m_+ = \bar{u}m_+$, since by Lemma 3.1.3 we have

$$e^{(2)}f^{(2)}m_+ = \binom{s}{2}m_+,$$

and we may compute the binomial coefficient as

$$\binom{s}{2} = \frac{(2u+1)!}{2!(2u-1)!} = \frac{(2u+1)(2u)}{2} = u(2u+1) \equiv u \,(\mathrm{mod}\ 2),$$

so that in k we have $\binom{s}{2} = \bar{u}$. This allows us to calculate $\overline{\phi_2}^2(x_2^2 \otimes m_+)$ as $(\bar{u} + e_2f_2)(x_1^2 \otimes f_2m_+)$ since

$$\overline{\phi_2}^2(x_2^2 \otimes m_+) = \overline{\phi_2}((e_2f_2 + \bar{u})(x_2^2 \otimes m_+)) = (\bar{u} + e_2f_2)\overline{\phi_2}(x_2^2 \otimes m_+) = (\bar{u} + e_2f_2)(x_1^2 \otimes f_2m_+).$$

Following this calculation through, using both Lemma 3.1.3 and that $f_2f_2 = 6f_4$, we obtain

$$\overline{\phi_2}^2(x_2^2 \otimes m_+) = (\bar{u}+1)(x_1^2 \otimes f_2m_+) + \bar{u}(x_2^2 \otimes m_+). \tag{3.1}$$

Supposing that $u \equiv 0\,(\mathrm{mod}\ 2)$ so that $\bar{u} = 0_k$, we have that $\overline{\phi_2}^2(x_2^2 \otimes m_+) = x_1^2 \otimes f_2m_+ = \overline{\phi_2}(x_2^2 \otimes m_+)$. Furthermore, we have that $\overline{\phi_2}(x_1x_2 \otimes m_+) = 0$ so that $\overline{\phi_2}^2 = \overline{\phi_2}$. This gives us our first result.

Lemma 3.4.1. *Let k be an algebraically closed field of characteristic 2, and $G = SL_2(k)$. Let $s = 4v+1$ for some $v \in \mathbb{N}$. Then the module $\nabla(2) \otimes \Delta(s)$ is a direct sum of Weyl modules, with highest weights given by the Clebsch-Gordan coefficients. Explicitly,*

$$\nabla(2) \otimes \Delta(s) = \Delta(s+2) \oplus \Delta(s) \oplus \Delta(s-2).$$

Proof. The calculations above show that the basis of $\mathrm{End}_{U_k}(\nabla(2) \otimes \Delta(s))$ given by $\bar{\phi}_0, \bar{\phi}_1$ and $\bar{\phi}_2$ consists only of idempotents. Hence we can write the identity endomorphism as a sum of three linearly independent idempotents, each corresponding to an indecomposable summand of $\nabla(2) \otimes \Delta(s)$. Since each summand must have a Weyl filtration, using Theorem 3.3.2 we obtain that the only possibility is the one given. □

Now we continue our previous calculations. It remains to compute $\overline{\phi_2}^2(x_1x_2 \otimes m_+) = \bar{u}\overline{\phi_2}(x_1^2 \otimes fm_+)$. Noticing that $x_1^2 \otimes fm_+ = fe(x_1x_2 \otimes m_+)$ we may write

$$\begin{aligned}
\overline{\phi_2}^2(x_1x_2 \otimes m_+) &= \bar{u}fe\overline{\phi_2}(x_1x_2 \otimes m_+)\\
&= \bar{u}^2fe(x_1^2 \otimes fm_+)\\
&= \bar{u}^2f(x_1^2 \otimes ef(m_+))\\
&= \bar{u}^2\bar{s}f(x_1^2 \otimes m_+)\\
&= \bar{u}^2\bar{s}x_1^2 \otimes fm_+.
\end{aligned}$$

k	$x_2^2 \otimes m_+$	$x_1x_2 \otimes m_+$
$\overline{\phi_0}$	$x_2^2 \otimes m_+$	$x_1x_2 \otimes m_+$
$\overline{\phi_1}$	0	$x_1^2 \otimes fm_+ + x_1x_2 \otimes m_+$
$\overline{\phi_1}^2$	0	$x_1^2 \otimes fm_+ + x_1x_2 \otimes m_+$
$\overline{\phi_2}$	$x_1^2 \otimes f_2m_+$	$\overline{u}x_1^2 \otimes fm_+$
$\overline{\phi_2}^2$	$(\overline{u}+1)(x_1^2 \otimes f_2m_+) + \overline{u}(x_2^2 \otimes m_+)$	$\overline{u}x_1^2 \otimes fm_+$

Table 3.3: The endomorphisms $\overline{\phi_i}$ and their squares on $\nabla(2) \otimes \Delta(2u+1)$.

We summarise all the previous calculations in Table 3.3.

Now in the case that $u \equiv 1 (\text{mod } 2)$ we can simplify these expressions to

$$\overline{\phi_2}^2(x_1x_2 \otimes m_+) = x_1^2 \otimes fm_+,$$

and

$$\overline{\phi_2}^2(x_2^2 \otimes m_+) = x_2^2 \otimes m_+.$$

Hence we see that $\overline{\phi_2}^2 = \overline{\phi_0} + \overline{\phi_1}$. Using this we notice that the endomorphism $\overline{\phi_2} + \overline{\phi_1} + \overline{\phi_0}$ is nilpotent:

$$(\overline{\phi_2}^2 + \overline{\phi_1} + \overline{\phi_0})^2 = \overline{\phi_2}^2 + \overline{\phi_1}^2 + \overline{\phi_0}^2 = \overline{\phi_1} + \overline{\phi_0} + \overline{\phi_1} + \overline{\phi_0} = 0.$$

We have found then, that the dimension of the span of nilpotent elements is equal to 1, so the number of summands of $\nabla(2) \otimes \Delta(s)$ when $s = 2u+1$ and $u \equiv 1 (\text{mod } 2)$ is equal to two. Next we will examine the Weyl filtrations of these summands.

There are three possibilities for the filtrations of our indecomposable summands. These are

$$\frac{\Delta(s-2)}{\Delta(s)} \oplus \Delta(s+2), \quad \frac{\Delta(s-2)}{\Delta(s+2)} \oplus \Delta(s), \quad \Delta(s-2) \oplus \frac{\Delta(s)}{\Delta(s+2)}.$$

We remark here that we may rule out the first possibility by showing that s and $s-2$ are in different blocks as follows. First we note that by Lemma 2.2.29 we have that

$$\mathscr{B}(s) \subset \{s + 4n : n \in \mathbb{Z}\} \cup \{-s-2+4n : n \in \mathbb{Z}\}.$$

Now, if $s-2 \in \mathscr{B}(s)$, then we must have that $s-2 = -s-2+4n$ for some $n \in \mathbb{Z}$. However, this implies that $2s = 4n$, or, in other words, $2 \mid s$, contradicting the fact that $s = 2u+1$.

Similarly, we can show that s and $s+2$ are in different blocks, and so we are left with only the second possibility.

In the spirit of this section, however, we will also determine which of these filtrations is correct by determining which summand splits. First we notice that $\overline{\phi_0} = 1 = (1-\overline{\phi_1}) + \overline{\phi_1}$, and we will write $\overline{\psi} = 1 - \overline{\phi_1}$.

k	$x_2^2 \otimes m_+$	$x_1x_2 \otimes m_+$
$\overline{\phi_1}$	0	$x_1^2 \otimes fm_+ + x_1x_2 \otimes m_+$
$\overline{\psi}$	$x_2^2 \otimes m_+$	$x_1^2 \otimes fm_+$

Table 3.4: The endomorphisms $\overline{\phi_1}$ and $\overline{\psi}$ on $\nabla(2) \otimes \Delta(s)$

We will denote by X_1 the image of $\overline{\psi}$ and by X_2 the image of $\overline{\phi_1}$ so that

$$\nabla(2) \otimes \Delta(s) = X_1 \oplus X_2.$$

Our plan is to consider the highest weight vectors and the dimensions of each individual weight space. The summand consisting of the split Weyl module will have a one dimensional space for each weight. First we consider the highest weight vector $x_1^2 \otimes m_+ = e_2(x_2^2 \otimes m_+)$. We have that

$$\begin{aligned} \overline{\phi_1}(x_1^2 \otimes m_+) &= e_2\overline{\phi_1}(x_2^2 \otimes m_+) = 0, \\ \overline{\psi}(x_1^2 \otimes m_+) &= e_2\overline{\psi}(x_2^2 \otimes m_+) = x_1^2 \otimes m_+. \end{aligned}$$

Next we look at $x_1x_2 \otimes m_+ \in \nabla(2) \otimes \Delta(s)^s$. We can read directly from the table that

$$\begin{aligned} \overline{\phi_1}(x_1x_2 \otimes m_+) &= x_1^2 \otimes fm_+ + x_1x_2 \otimes m_+, \\ \overline{\psi}(x_1x_2 \otimes m_+) &= x_1^2 \otimes fm_+. \end{aligned}$$

These calculations show us that X_1 has highest weight $s+2$ and X_2 has highest weight s. It remains then, to place $\Delta(s-2)$, which we will do by looking at the $s-2$ weight space of each summand. Consider first, the image of $x_2^2 \otimes m_+$, which again we can read straight from the table.

$$\begin{aligned} \overline{\phi_1}(x_2^2 \otimes m_+) &= 0, \\ \overline{\psi}(x_2^2 \otimes m_+) &= x_2^2 \otimes m_+. \end{aligned}$$

Next, writing $x_1x_2 \otimes fm_+ = f(x_1x_2 \otimes m_+) + x_2^2 \otimes m_+$ we have that

$$\begin{aligned}
\overline{\phi_1}(x_1x_2 \otimes fm_+) &= f\overline{\phi_1}(x_1x_2 \otimes m_+) + \overline{\phi_1}(x_2^2 \otimes m_+) \\
&= f(x_1^2 \otimes fm_+ + x_1x_2 \otimes m_+) \\
&= x_2^2 \otimes m_+ + x_1x_2 \otimes fm_+, \\
\overline{\psi}(x_1x_2 \otimes fm_+) &= f\overline{\psi}(x_1x_2 \otimes m_+) + \overline{\psi}(x_2^2 \otimes m_+) \\
&= f(x_1^2 \otimes fm_+) + x_2^2 \otimes m_+ \\
&= x_2^2 \otimes m_+.
\end{aligned}$$

Finally, we calculate the image of $x_1^2 \otimes f_2m_+ = f_2e_2(x_2^2 \otimes m_+) + x_2^2 \otimes m_+$.

$$\begin{aligned}
\overline{\phi_1}(x_1^2 \otimes f_2m_+) &= f_2e_2\overline{\phi_1}(x_2^2 \otimes m_+) + \overline{\phi_1}(x_2^2 \otimes m_+) \\
&= 0, \\
\overline{\psi}(x_1^2 \otimes f_2m_+) &= f_2e_2\overline{\psi}(x_2^2 \otimes m_+) + \overline{\psi}(x_2^2 \otimes m_+) \\
&= f_2e_2(x_2^2 \otimes m_+) + x_2^2 \otimes m_+ \\
&= x_1^2 \otimes f_2m_+.
\end{aligned}$$

Hence we have that $\dim(X_1^{s-2}) = 2$ so that X_1 is a non-split extension of $\Delta(s+2)$ and $\Delta(s-2)$. We obtain the following result.

Lemma 3.4.2. *Let k be an algebraically closed field of characteristic 2, and $G = SL_2(k)$. Let $s = 2u+1$ where $u \equiv 1 (\mathrm{mod}\ 2)$. Then the decomposition of $\nabla(2) \otimes \Delta(s)$ into indecomposables is given by*

$$\frac{\Delta(s-2)}{\Delta(s+2)} \oplus \Delta(s),$$

where the first summand is a non-split extension. □

3.4.3 The Even Case

We look now at the case where s is even, let's say $s = 2u$ for some $u \in \mathbb{N}$. Recall that Table 3.1 showed us that the endomorphism ϕ_2 did not preserve the $\mathbb{Z}$-form, and by simply clearing the denominators we ended up with linearly dependent endomorphisms after base change. Instead, we should consider the endomorphism $\psi = \phi_2 + \frac{1}{2}\phi_1$. This gives us the following table, where the endomorphisms are over $\mathbb{Q}$.

After the process of base change, we end up with the following linearly independent endomorphisms on $\nabla(2) \otimes \Delta(2u)$. Once again we can see that these depend on whether u is odd or even.

As before, we would like to determine the idempotent and nilpotent elements of this algebra. First we will compute $\overline{\phi_1}^2$. Note that

$$x_1^2 \otimes fm_+ = f(x_1^2 \otimes m_+) = fe_2(x_2^2 \otimes m_+),$$

$\mathbb{Q}$	$x_2^2 \otimes m_+$	$x_1x_2 \otimes m_+$
ϕ_0	$x_2^2 \otimes m_+$	$x_1x_2 \otimes m_+$
ϕ_1	$2x_1x_2 \otimes fm_+$	$x_1^2 \otimes fm_+ + sx_1x_2 \otimes m_+$
ψ	$x_1^2 \otimes f^{(2)}m_+ + x_1x_2 \otimes fm_+$	$u(x_1^2 \otimes fm_+ + x_1x_2 \otimes m_+)$

Table 3.5: The endomorphisms ϕ_0, ϕ_1 and ψ on $V(2) \otimes V(s)$

k	$x_2^2 \otimes m_+$	$x_1x_2 \otimes m_+$
$\overline{\phi_0}$	$x_2^2 \otimes m_+$	$x_1x_2 \otimes m_+$
$\overline{\phi_1}$	0	$x_1^2 \otimes fm_+$
$\overline{\psi}$	$x_1^2 \otimes f_2m_+ + x_1x_2 \otimes fm_+$	$\overline{u}(x_1^2 \otimes fm_+ + x_1x_2 \otimes m_+)$

Table 3.6: The endomorphisms $\overline{\phi_0}$, $\overline{\phi_1}$ and $\overline{\psi}$ on $\nabla(2) \otimes \Delta(s)$

so that $\overline{\phi_1}^2(x_1x_2 \otimes m_+) = fe_2\overline{\phi_1}(x_2^2 \otimes m_+) = 0$, and hence $\overline{\phi_1}^2 = 0$ and $\overline{\phi_1}$ is nilpotent.

Next we compute $\overline{\psi}^2$. First we give the elements in Table 3.6 in terms of the generators $x_2^2 \otimes m_+$ and $x_1x_2 \otimes m_+$. We have that

$$\begin{aligned} x_1^2 \otimes f_2m_+ &= f_2e_2(x_2^2 \otimes m_+) + x_2^2 \otimes m_+, \\ x_1x_2 \otimes fm_+ &= f(x_1x_2 \otimes m_+) + x_2^2 \otimes m_+, \\ x_1^2 \otimes fm_+ &= fe_2(x_2^2 \otimes m_+). \end{aligned}$$

This allows us to write

$$\overline{\psi}(x_2^2 \otimes m_+) = f_2e_2(x_2^2 \otimes m_+) + x_2^2 \otimes m_+ + f(x_1x_2 \otimes m_+) + x_2^2 \otimes m_+ = f_2e_2(x_2^2 \otimes m_+) + f(x_1x_2 \otimes m_+),$$

hence we have

$$\begin{aligned} \psi^2(x_2^2 \otimes m_+) &= f_2e_2\overline{\psi}(x_2^2 \otimes m_+) + f\overline{\psi}(x_1x_2 \otimes m_+) \\ &= f_2e_2(x_1^2 \otimes f_2m_+ + x_1x_2 \otimes fm_+) + \overline{u}f(x_1^2 \otimes fm_+ + x_1x_2 \otimes m_+) \\ &= f_2(x_1^2 \otimes e_2f_2m_+ + x_1^2 \otimes efm_+) + \overline{u}(x_2 \otimes m_+ + x_1x_2 \otimes fm_+). \end{aligned}$$

Now by Lemma 3.1.3 we have that $efm_+ = \bar{s}m_+ = 0$ since 2 divides s, and that

$$\begin{aligned} e^{(2)}f^{(2)}m_+ &= \binom{s}{2}m_+ \\ &= \frac{2u!}{2!(2u-2)!}m_+ \\ &= \frac{u(2u-1)!}{(2u-2)!}m_+ \\ &= u(2u-1)m_+ \end{aligned}$$

hence in k we have that $e_2f_2m_+ = \bar{u}m_+$. Continuing we have

$$\begin{aligned} \psi^2(x_2^2 \otimes m_+) &= f_2(\bar{u}x_1^2 \otimes m_+) + \bar{u}(x_2 \otimes m_+ + x_1x_2 \otimes fm_+) \\ &= \bar{u}(x_2^2 \otimes m_+ + x_1^2 \otimes f_2m_+ + x_2^2 \otimes m_+ + x_1x_2 \otimes fm_+) \\ &= \bar{u}(x_1^2 \otimes f_2m_+ + x_1x_2 \otimes fm_+) \\ &= \bar{u}\overline{\psi}(x_2^2 \otimes m_+). \end{aligned}$$

In the case that u is even we immediately obtain that $\overline{\psi}^2 = 0$. Since $\overline{\phi_1}$ is also nilpotent, we have that the identity is the only idempotent and the module $\nabla(2) \otimes \Delta(s)$ is indecomposable. Next we turn to $\overline{\psi}^2(x_1x_2 \otimes m_+)$. We have

$$\begin{aligned} \overline{\psi}^2(x_1x_2 \otimes m_+) &= \bar{u}\overline{\psi}(x_1^2 \otimes fm_+ + x_1x_2 \otimes m_+) \\ &= \bar{u}(fe_2\overline{\psi}(x_2^2 \otimes m_+) + \overline{\psi}(x_1x_2 \otimes m_+)) \\ &= \bar{u}(fe_2(x_1^2 \otimes f_2m_+ + x_1x_2 \otimes fm_+) + \bar{u}(x_1^2 \otimes fm_+ + x_1x_2 \otimes m_+) \\ &= \bar{u}(f(\bar{u}x_1^2 \otimes fm_+) + \bar{u}(x_1^2 \otimes fm_+ + x_1x_2 \otimes m_+) \\ &= \bar{u}(x_1^2 \otimes fm_+ + x_1^2 \otimes fm_+ + x_1x_2 \otimes m_+) \\ &= \bar{u}x_1x_2 \otimes m_+, \end{aligned}$$

where we have used that, since k has characteristic 2, $\bar{u}^2 = \bar{u}$. Using Table 3.6 we can see that $\overline{\psi}^2 = \bar{u}(\overline{\psi} + \overline{\phi_1})$. In the case that u is odd we have

$$\overline{\psi}^4 = \overline{\psi}^2 + \overline{\psi}\overline{\phi_1} + \overline{\phi_1}\overline{\psi} + \overline{\phi_1}^2 = \overline{\psi}^2 + \overline{\phi_1}^2 = \overline{\psi}^2,$$

so that $\overline{\psi}^2$ is idempotent. Hence we can give $\mathrm{End}_{U_k}(\nabla(2) \otimes \Delta(s))$ the basis $\{\overline{\phi_0},\ \overline{\phi_1},\ \overline{\psi} + \overline{\phi_1}\}$, and write $\overline{\phi_0} = \overline{\psi} + \overline{\phi_1} + (\overline{\phi_0} + \overline{\psi} + \overline{\phi_1})$.

It remains to determine the Weyl filtration in the case that u is odd. The primitive idempotents are given in Table 3.8.

By looking at the image of the highest weight vector $x_1^2 \otimes m_+ = e(x_1x_2 \otimes m_+)$ we can see that $\Delta(s+2)$, the module generated by $x_1^2 \otimes m_+$, belongs to the summand corresponding to $\overline{\psi} + \overline{\phi_1}$. Next, since $f(x_1^2 \otimes m_+) = x_1^2 \otimes fm_+$ and the weight space $\nabla(2) \otimes \Delta(s)^s$ has dimension 2, the element

k	$x_2^2 \otimes m_+$	$x_1x_2 \otimes m_+$
$\overline{\phi_0}$	$x_2^2 \otimes m_+$	$x_1x_2 \otimes m_+$
$\overline{\phi_1}$	0	$x_1^2 \otimes fm_+$
$\overline{\psi} + \overline{\phi_1}$	$x_1^2 \otimes f_2m_+ + x_1x_2 \otimes fm_+$	$(\overline{u}+1)x_1^2 \otimes fm_+ + \overline{u}x_1x_2 \otimes m_+$

Table 3.7: The endomorphisms $\overline{\phi_0}$, $\overline{\phi_1}$ and $\overline{\psi} + \overline{\phi_1}$ on $\nabla(2) \otimes \Delta(s)$

k	$x_2^2 \otimes m_+$	$x_1x_2 \otimes m_+$
$\overline{\psi} + \overline{\phi_1}$	$x_1^2 \otimes f_2m_+ + x_1x_2 \otimes fm_+$	$x_1x_2 \otimes m_+$
$\overline{\phi_0} + \overline{\psi} + \overline{\phi_1}$	$x_1^2 \otimes f_2m_+ + x_1x_2 \otimes fm_+ + x_2^2 \otimes m_+$	0

Table 3.8: The primitive idempotents when u is odd.

$x_1x_2 \otimes m_+$ generates $\Delta(s)$. Since this also belongs to the summand corresponding to $\overline{\psi} + \overline{\phi_1}$ we can write the following decomposition.

$$\nabla(2) \otimes \Delta(s) = \frac{\Delta(s)}{\Delta(s+2)} \oplus \Delta(s-2).$$

We summarise the results obtained for $\nabla(2) \otimes \Delta(s)$.

Theorem 3.4.3. *Let k be an algebraically closed field of characteristic 2, and $G = SL_2(k)$. For $s \in \mathbb{N}$, the module $\nabla(2) \otimes \Delta(s)$ decomposes into indecomposable summands in the following ways:*

$$\begin{aligned} s \equiv 0 \pmod 4 &\quad : \quad \textit{indecomposable}, \\ s \equiv 1 \pmod 4 &\quad : \quad \Delta(s+2) \oplus \Delta(s) \oplus \Delta(s-2), \\ s \equiv 2 \pmod 4 &\quad : \quad \frac{\Delta(s)}{\Delta(s+2)} \oplus \Delta(s-2), \\ s \equiv 3 \pmod 4 &\quad : \quad \frac{\Delta(s-2)}{\Delta(s+2)} \oplus \Delta(s). \end{aligned}$$

□

Chapter 4

Casimir Operator

In this chapter we will use a universal Casimir operator to define an endomorphism from which we can make some useful results. As in the previous chapter, we will first work over a field of characteristic 0. This time however, we will work over the p-adic numbers, $\mathbb{Q}_p$ (see Section A.4 for a brief introduction to p-adic numbers). The result we prove is as follows:

Theorem 4.0.1. *Let k be an algebraically closed field of characteristic p, and $G = SL_2(k)$. Let $r \geq s$, and denote by λ_i the Clebsch-Gordan coefficients in ascending order, so that $\lambda_i = r - s + 2i$. Let $z_i = \lambda_i^2 + 2\lambda_i$ and let*

$$Z_i = \prod_{\substack{0 \leq m \leq s \\ m \neq i}} (z_i - z_m)$$

for $i = 0, \ldots, s$. If p does not divide $|Z_i|$ for $i = 0, \ldots, s$ then

$$\nabla(r) \otimes \Delta(s) = \bigoplus_{i=0}^{s} \nabla(\lambda_i).$$

This theorem immediately shows us that for any given r and s there are only a handful of characteristics (finitely many) for which the module $\nabla(r) \otimes \Delta(s)$ is not a direct sum of induced modules, which can be seen as a direct analogy to the characteristic 0 case. We also find that there are more primes for which the module is a direct sum of induced modules, than just those that are 'sufficiently large', where we have $\nabla(r) = L(r)$.

We note that since

$$\lambda_i^2 = r^2 + s^2 - 2rs + 4ir - 4is + 4i^2,$$

we have that

$$\begin{aligned}z_i - z_j &= \lambda_i^2 - \lambda_j^2 + 2(\lambda_i - \lambda_j)\\ &= 4r(i-j) + 4s(i-j) + 4i^2 - 4j^2 + 2(2(i-j))\\ &= 4((r+s+1+i+j)(i-j).\end{aligned}$$

It's clear then, that 2 divides all the Z_i, so for the case $p = 2$, the result doesn't tell us anything. However, in proving the theorem, to which most of this chapter is devoted, we will still find some useful results for all primes.

4.1 Characteristic 0

In examining the algebra $\mathrm{End}_G(\nabla(r) \otimes \Delta(s))$, we will initially work in characteristic 0, for which it will be sufficient to take the field $\mathbb{Q}$. Our aim will be to find the centrally primitive idempotents that will correspond to the known decomposition in characteristic 0. Then, since there is a one to one correspondence between the idempotents in characteristic p and those over the p-adic integers in characteristic 0 (see [12, Theorem 44.3]), if we can say which idempotents preserve the lattice in $\nabla(r) \otimes \Delta(s)$ with p-adic integer coefficients, we will be able to say how many indecomposable summands the decomposition in characteristic p has.

As in the previous chapter, we will denote by $V(t)$ the simple module of highest weight t over a field of characteristic 0. We recall also that we may consider any $\mathbb{Q}G$-module as a module for the universal enveloping algebra $U_{\mathbb{Q}}$, and that when $r \geq s$ we have $\dim_{\mathbb{Q}}\mathrm{End}_{U_{\mathbb{Q}}}(V(r) \otimes V(s)) = s+1$.

4.1.1 The Casimir Operator

In this subsection we will derive the universal Casimir operator of the universal enveloping algebra $U(\mathfrak{sl}_2(\mathbb{C}))$, with respect to the Killing form κ. A detailed description of this topic can be found in [18, Section 22.1], but here we exhibit a quick way to derive it for $\mathfrak{sl}_2(\mathbb{C})$.

We know that, for a semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, the linear map induced from the Killing form, given as

$$\begin{aligned}\bar{\kappa} &: S^2(\mathfrak{g}) \longrightarrow \mathbb{C}\\ \bar{\kappa} &: XY \longmapsto \kappa(X,Y)\end{aligned}$$

is G-equivariant (where G acts via the adjoint action), which makes it a G-invariant in $S^2(\mathfrak{g})^*$. Using the isomorphisms of G-modules

$$S^2(\mathfrak{g})^* \simeq S^2(\mathfrak{g}^*) \simeq S^2(\mathfrak{g}) \simeq U_2(\mathfrak{g})/U_1(\mathfrak{g}),$$

we obtain a G-invariant in $U_2(\mathfrak{g})$ (using the notation defined in Section 2.3). This invariant is the universal Casimir operator. Such an element must be invariant under the action of a maximal torus in G, so must have weight 0. Furthermore, since the differential of Ad is ad, we have that our universal Casimir operator commutes with all elements of $\mathfrak{g}$, and thus lies in the centre $Z(U(\mathfrak{g}))$.

For the case $\mathfrak{sl}_2(\mathbb{C})$, using the PBW theorem we can write this element in the form

$$c = \alpha fe + \gamma h^2 + \delta h,$$

for some $\alpha, \gamma, \delta \in k$. Now using the fact that $c \in Z(U(\mathfrak{sl}_2(\mathbb{C})))$ we may find the coefficients as follows.

We have

$$fc = \alpha f^2 e + \gamma f h^2 + \delta f h,$$

and that this must equal cf, given by

$$\begin{aligned} cf &= \alpha fef + \gamma h^2 f + \delta hf \\ &= \alpha(fh + f^2 e) + \gamma(hfh - 2hf) + \delta(fh - 2f) \\ &= \alpha(fh + f^2 e) + \gamma(fh^2 - 4fh + 4f) + \delta(fh - 2f), \end{aligned}$$

where we have made repeated use of the commutator relations to describe cf in terms of the PBW basis. Equating these two expressions we see that we must have

$$\alpha fh + 4\gamma(f - fh) - 2\delta f = 0.$$

This tells us that we must have $\alpha = 4\gamma$ and $\delta = 2\gamma$. Up to scalars, this is enough information to define c, so we pick $\gamma = 1$ and obtain

$$c = 4fe + h^2 + 2h.$$

As a quick check, we calculate ec as

$$\begin{aligned} ec &= 4efe + eh^2 + 2eh \\ &= 4he + 4fe^2 - 2eh + heh - 4e + 2he \\ &= 4he + 4fe^2 + 4e - 2he - 2he + h^2 e - 4e + 2he \\ &= 2he + 4fe^2 + h^2 e, \end{aligned}$$

which does indeed equal ce. As a final remark, we note that, using the identity $[e,f]=h$, we can rewrite this as

$$c = h^2 + 2ef + 2fe.$$

Whilst this does not express c in our chosen PBW basis, it does exhibit it as a quadratic element of $U(\mathfrak{sl}_2(\mathbb{C}))$.

Since $c \in Z(U_{\mathbb{Q}})$, given any $U_{\mathbb{Q}}$-module M, we can consider multiplication by c as an endomorphism of M. We will define the endomorphism $\gamma_M : M \to M$ by $\gamma_M(m) = cm$ for all $m \in M$. When it's clear which module we are acting on we will denote this endomorphism by γ.

Let's now consider the action of $\gamma_{V(t)}$ on the simple module $V(t)$ of highest weight t, with highest weight vector denoted m_+. Since $V(t)$ is a simple module, Schur's lemma (Lemma 2.2.2) says that γ acts as a constant, so it's sufficient to look only at the effect of γ on the vector m_+. We obtain

$$\gamma(m_+) = 4fem_+ + (h^2+2h)m_+ = (h^2+2h)m_+,$$

where we are using that since m_+ has highest weight, we have $em_+ = 0$. Thus $\gamma(m_+) = (t^2 + 2t)m_+$, so γ acts as multiplication by the constant $t^2+2t \in \mathbb{Z}$ on $V(t)$.

Using the Clebsch Gordan decomposition of $V(r)\otimes V(s)$ given in Section 3.1, it's clear that the matrix of $\gamma_{V(r)\otimes V(s)}$ is a block diagonal matrix with each block given by the diagonal matrix of size $\dim V(r+s-2i) = r+s-2i+1$, and with all diagonal entries $(r+s-2i)^2 + 2(r+s-2i) =: z_i$. Furthermore we have that the minimal polynomial of $\gamma_{V(r)\otimes V(s)}$ is given by

$$m_\gamma(T) = \prod_{i=0}^{s}(T - z_i)$$

In particular, the ring $\mathbb{Q}[\gamma]$ has dimension $s+1$ over $\mathbb{Q}$. Since γ is an element of the algebra $\mathrm{End}_{U_{\mathbb{Q}}}(V(r)\otimes V(s))$, it's clear that $\mathbb{Q}[\gamma] \subseteq \mathrm{End}_{U_{\mathbb{Q}}}(V(r)\otimes V(s))$, but as the dimensions are the same they are in fact equal.

4.1.2 Idempotents

Given a commutative ring R and an R-module $M = M_1 \oplus \cdots \oplus M_n$, we may write $1 \in \mathrm{End}_R(M)$ uniquely as

$$1 = \varepsilon_1 + \cdots + \varepsilon_n$$

where each ε_i is a primitive idempotent and M_i are all indecomposable (as in the beginning of Section 3.4). In the case we are looking at, since we know $V(r)\otimes V(s) = \bigoplus_{i=0}^{s} V(\lambda_i)$, we would like to write the $s+1$ centrally primitive idempotents in terms of the basis $\{1, \gamma, \dots \gamma^s\}$

of $\mathrm{End}_{U_{\mathbb{Q}}}(V(r) \otimes V(s))$.

Now, for a moment, let A be a commutative, semisimple k algebra, where k is an algebraically closed field. As described above, we may write $1_A = \varepsilon_1 + \cdots + \varepsilon_n$. Consider $X(A)$, the set of all k algebra homomorphisms $A \to k$. For $\theta \in X(A)$ we have $1_k = \theta(\varepsilon_1) + \cdots + \theta(\varepsilon_n)$. Since

$$\theta(\varepsilon_i)^2 = \theta(\varepsilon_i^2) = \theta(\varepsilon_i)$$

and k is a field (containing only the idempotents 0_k and 1_k), we must have $\theta(\varepsilon_i) = 1_k$ for some unique i and $\theta(\varepsilon_j) = 0_k$ for all $j \neq i$. Using this, we can easily describe $X(\mathbb{Q}[\gamma])$, since if $\theta \in X(\mathbb{Q}[\gamma])$ we must have

$$0 = \theta(m_\gamma(\gamma)) = m_\gamma(\theta(\gamma)),$$

so $\theta(\gamma) = z_i$ for some $i = 0, \ldots, s$. Since the action of θ on γ completely describes θ we obtain a basis of $X(\mathbb{Q}[\gamma])$ consisting of the $s+1$ elements θ_i, where $\theta_i(\gamma) = z_i$ for $i = 0, \ldots, s$. Hence we have $\dim_{\mathbb{Q}} X(\mathbb{Q}[\gamma]) = s+1$.

Next, continuing the previous notation, given a basis $\{b_1, \ldots, b_n\}$ of A we can write each centrally primitive idempotent ε_j of A as

$$\varepsilon_j = \sum_{l=1}^{n} a_{jl} b_l$$

for some fixed j and some $a_{jl} \in k$. Given a basis $\{\theta_1, \ldots, \theta_n\}$ of $X(A)$ such that $\theta_i(\varepsilon_j) = \delta_{ij}$, applying θ_i, we obtain

$$\delta_{ij} = \theta_i(\varepsilon_j) = \sum_{l=1}^{n} a_{jl} \theta_i(b_l).$$

This can be written as the matrix equation $1_n = av$ where $v_{ij} = \theta_i(b_j)$ and the matrix a consists of the coefficients a_{jl}.

Reverting to our particular case, we would like to find the coefficients a_{jl} so that we can describe each idempotent ε_j as a polynomial in γ. We have the basis b_i of $\mathbb{Q}[\gamma]$ given by $\{1, \gamma, \ldots, \gamma^s\}$ and $\theta_i(\gamma^j) = \theta_i(\gamma)^j = z_i^j$ where j runs from 0 to s. This means that v is given by the $(s+1) \times (s+1)$ matrix

$$\begin{pmatrix} 1 & z_0 & z_0^2 & \ldots & z_0^s \\ 1 & z_1 & z_1^2 & \ldots & z_1^s \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 1 & z_s & z_s^2 & \ldots & z_s^s \end{pmatrix}.$$

Since this is a Vandermonde matrix, its inverse w is known and given as [25, Section 4]

$$w_{ij} = \begin{cases} (-1)^{s+1-i} \left(\dfrac{\sum\limits_{\substack{0 \le m_1 < \ldots < m_{s+1-i} \le s \\ m_1, \ldots, m_{s+1-i} \neq j}} z_{m_1} \ldots z_{m_{s+1-i}}}{\prod\limits_{\substack{0 \le m \le s \\ m \neq j}} \left(z_j - z_m\right)} \right) & : \ 1 \le i < s+1 \\ \\ \dfrac{1}{\prod\limits_{\substack{0 \le m \le s \\ m \neq j}} \left(z_j - z_m\right)} & : \ i = s+1 \end{cases}$$

and thus $\varepsilon_j = \sum_{i=1}^{s+1} w_{ij}\gamma^{i-1}$. Defining

$$Z_j := \prod_{\substack{0 \le m \le s \\ m \neq j}} (z_j - z_m),$$

we can take out the factor of $\frac{1}{Z_j} = w_{s+1,j}$ and we are left with

$$\varepsilon_j = \frac{1}{Z_j}\left(\sum_{i=1}^{s+1} a'_i \gamma^{i-1}\right)$$

where

$$a'_i = w_{ij} Z_j = \begin{cases} -1^{s+1-i} \sum\limits_{\substack{0 \le m_1 < \ldots < m_{s+1-i} \le s \\ m_1, \ldots, m_{s+1-i} \neq j}} z_{m_1} \ldots z_{m_{s+1-i}} & : \ 1 \le i < s+1 \\ 1 & : \ i = s+1 \end{cases}.$$

This gives us a complete set of centrally primitive idempotents in $\mathrm{End}_{U_{\mathbb{Q}}}(V(r) \otimes V(s))$ expressed as polynomials in γ. Since the z_i are integers we have that Z_j is an integer and it's clear from the above formula that a'_i are integers for $i = 1, \ldots, s+1$.

4.2 Moving to Prime Characteristic

As in Section 3.2, we would like to move from the well understood characteristic 0 case, to the characteristic p case. The idea here is exactly the same as in the previous chapter, except, since we are this time working over $\mathbb{Q}_p$, we will want to look for those endomorphisms that preserve the lattice $\nabla(r)_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \Delta(s)_{\mathbb{Z}_p}$, where $\mathbb{Z}_p$ is the valuation ring on $\mathbb{Q}_p$ with respect to the p-adic norm (as defined in Section A.4). Barring this slight change, all of the theory developed in the previous chapter goes through here too. In fact it will be sufficient to work inside $\mathbb{Q}$, and to just look at the valuation ring on $\mathbb{Q}$, given as

$$\mathbb{Z}_{(p)} := \mathbb{Q} \cap \mathbb{Z}_p = \left\{ \frac{r}{s} \mid r, s \in \mathbb{Z}, \ \gcd(r,s) = 1, \ p \nmid s \right\}.$$

In contrast to the previous chapter however, rather than finding endomorphisms that preserve $\nabla(r)_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \Delta(s)_{\mathbb{Z}_p}$ for a particular prime p, our aim this time will be to give sufficient conditions for p so that each idempotent obtained in the previous section preserves the lattice $\nabla(r)_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \Delta(s)_{\mathbb{Z}_p}$.

First, we give a helpful lemma.

Lemma 4.2.1. *The endomorphism γ preserves the lattice $\nabla(r)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Delta(s)_{\mathbb{Z}}$.*

As in Section 3.2, we have $\nabla(r)_{\mathbb{Z}} = \mathbb{Z}\text{-span}\left\{x_1^i x_2^j \ : \ i+j=r\right\}$, and $\Delta(s)_{\mathbb{Z}} = \mathbb{Z}\text{-span}\left\{f^{(i)}m_+ \ : \ i=0,\dots,s\right\}$.

Proof. γ is given by the action of $4fe+h^2+2h$. Since multiplication by h brings out the weight of a weight vector (all of which are integers), it's clear that acting with h^2+2h preserves $\nabla(r)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Delta(s)_{\mathbb{Z}}$, so it remains to check that acting with $4fe$ does too. We can check this explicitly by looking at the basis element $x_1^{r-i}x_2^i \otimes f^{(j)}m_+ \in V(r) \otimes V(s)$.

$$\begin{aligned}
4fe(x_1^{r-i}x_2^i \otimes f^{(j)}m_+) &= 4f(e(x_1^{r-i}x_2^i) \otimes f^{(j)}m_+ + x_1^{r-i}x_2^i \otimes ef^{(j)}m_+) \\
&= 4f(ix_1^{r-i+1}x_2^{i-1} \otimes f^{(j)}m_+ + (s-j+1)x_1^{r-i}x_2^i \otimes f^{(j-1)}m_+) \\
&= 4\Big[i\left[(r-i+1)x_1^{r-i}x_2^i \otimes f^{(j)}m_+ + (j+1)x_1^{r-i+1}x_2^{i-1} \otimes f^{(j+1)}m_+\right] \\
&\quad + (s-j+1)\left[(r-i)x_1^{r-i-1}x_2^{i+1} \otimes f^{(j-1)} + jx_1^{r-i}x_2^i \otimes f^{(j)}m_+\right]\Big]
\end{aligned}$$

Each coefficient is in $\mathbb{Z}$, proving the result. □

Since γ preserves the lattice $\nabla(r)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Delta(s)_{\mathbb{Z}}$, it's clear that any polynomial in γ with integer coefficients does too. Furthermore, as the above result shows that applying γ only produces integer coefficients, γ also preserves $\nabla(r)_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \Delta(s)_{\mathbb{Z}_p}$. When acting on $\nabla(r)_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \Delta(s)_{\mathbb{Z}_p}$ with the idempotents ε_j from the previous section, we can see that after expanding all terms out, the only new denominators will be Z_j. So if p does not divide Z_j for each j, then each ε_j will preserve $\nabla(r)_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \Delta(s)_{\mathbb{Z}_p}$ and give a centrally primitive idempotent in $\mathrm{End}_{U_k}(\nabla(r) \otimes \Delta(s))$ (as in [12, Theorem 44.3]). Since all such idempotents are gotten this way, the module $\nabla(r) \otimes \Delta(s)$, over the field k, can be decomposed into $s+1$ indecomposable summands. Since every summand must have a good filtration (Corollary 2.2.23), and we know that the good filtration of the whole module has exactly $s+1$ sections, the decomposition must be

$$\nabla(r) \otimes \Delta(s) = \bigoplus_{i=0}^{s} \nabla(\lambda_i).$$

This is the proof of Theorem 4.0.1.

As an aside, it's clear that the condition $p \nmid Z_j$ for $j = 0,\dots,s$, is equivalent to the condition that $p \nmid Z := \det V$ where

$$\det V = \prod_{0 \le i < j \le s} (z_j - z_i,)$$

since the factors of the product of the Z_j's and Z are identical. Furthermore, as mentioned at the start of this chapter, it's clear that the prime 2 always divides Z, so the theorem does not tell us about the decomposition in this case. In fact, in the case $p = 2$ we can find a counter example to the converse statement, which we do in the next section.

4.3 Further Applications

In proving the theorem above, we found an explicit form for each centrally primitive idempotent in characteristic 0. One might notice that instead of giving the explicit formula for the inverse of the Vandermonde matrix V we might very well have appealed to Cramers rule for inverting a matrix and said that we can take a factor of $\det V^{-1}$ out and all the matrix entries would lie in $\mathbb{Z}$. Then we would have reached same conclusion using the equivalent condition on p mentioned above (and arguably got there quicker). There are however, some uses in writing explicitly the centrally primitive idempotents, of which we now exhibit one.

Lemma 4.3.1. *Let k be an algebraically closed field of characteristic $p > 0$, and let $r \in \mathbb{N}$. Then*

$$\nabla(r) \otimes \Delta(1) = \begin{cases} \nabla(r+1) \oplus \nabla(r-1) & : \quad p \not| (r+1) \\ \\ \dfrac{\nabla(r+1)}{\nabla(r-1)} & : \quad p \mid (r+1). \end{cases}$$

In particular we notice that when $p = 2$ there exist modules for which the decomposition is a direct sum of induced modules. We should point out at this stage, that this result can be gotten by already known methods, one of which we outline in the next chapter.

Proof. Firstly we notice that the module $\nabla(r) \otimes \Delta(1)$ has a good filtration with only two sections, so the decompositions given are the only possible ones. In order to decide when each decomposition occurs, we look at when either of the two centrally primitive idempotents preserve the lattice $\nabla(r)_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \Delta(1)_{\mathbb{Z}_p}$, and give a non-trivial idempotent in $\mathrm{End}_{U_k}(\Delta(r) \otimes \nabla(1))$. In the case that either idempotent (and thus both) does, there are two summands. In the case that neither does, there is one summand.

Using the same notation as in Theorem 4.0.1, we have $\lambda_0 = r - 1$ and $\lambda_1 = r + 1$ giving $z_0 = r^2 - 1$ and $z_1 = r^2 + 4r + 3$. The Vandermonde matrix V is given by

$$V = \begin{pmatrix} 1 & z_0 \\ 1 & z_1 \end{pmatrix}$$

with determinant $z_1 - z_0 = 4(r+1)$. Its inverse matrix is then given by

$$V^{-1} = \frac{1}{z_1 - z_0} \begin{pmatrix} z_1 & -z_0 \\ -1 & 1 \end{pmatrix}$$

so that

$$\varepsilon_1 = \frac{1}{z_1 - z_0}(z_1 - \gamma), \quad \varepsilon_2 = \frac{1}{z_1 - z_0}(-z_0 + \gamma).$$

From here we can use the theorem to get the result for when $p \neq 2$. In order to complete the result, we must check explicitly whether either idempotent preserves $V(r)_{\mathbb{Z}_2} \otimes_{\mathbb{Z}_2} V(1)_{\mathbb{Z}_2}$. Fortunately, the module $V(r) \otimes V(1)$ is generated by the single vector $x_1^r \otimes fm_+$ (Lemma 3.1.1), which has weight $r - 1 = \lambda_0$, so it is sufficient to check for this vector only. We calculate

$$\begin{aligned}
\gamma(x_1^r \otimes fm_+) &= (4fe + h^2 + 2h)(x_1^r \otimes fm_+)\\
&= 4fe(x_1^r \otimes fm_+) + (\lambda_0^2 + 2\lambda_0)(x_1^r \otimes fm_+)\\
&= 4fe(x_1^r \otimes fm_+) + z_0 x_1^r \otimes fm_+\\
&= 4f(x_1^r \otimes m_+) + z_0 x_1^r \otimes fm_+\\
&= 4r x_1^{r-1} x_2 \otimes m_+ + (4 + z_0) x_1^r \otimes fm_+.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
\varepsilon_1(x_1^r \otimes fm_+) &= \frac{1}{(z_1 - z_0)}(z_1 x_1^r \otimes fm_+ - \gamma(x_1^r \otimes fm_+))\\
&= \frac{1}{(z_1 - z_0)}(z_1 x_1^r \otimes fm_+ - 4r x_1^{r-1} x_2 \otimes m_+ - (4 + z_0) x_1^r \otimes fm_+)\\
&= \frac{1}{(z_1 - z_0)}((z_1 - z_0 - 4) x_1^r \otimes fm_+ - 4r x_1^{r-1} x_2 \otimes m_+)\\
&= \frac{4r}{4(r+1)}(x_1^r \otimes fm_+ - x_1^{r-1} x_2 \otimes m_+)\\
&= \frac{r}{r+1}(x_1^r \otimes fm_+ - x_1^{r-1} x_2 \otimes m_+),
\end{aligned}$$

where we have used that $z_1 - z_0 = 4(r+1)$. Similarly, we obtain for ε_2,

$$\begin{aligned}
\varepsilon_2(x_1^r \otimes fm_+) &= \frac{1}{(z_1 - z_0)}(-z_0 x_1^r \otimes fm_+ + \gamma(x_1^r \otimes fm_+))\\
&= \frac{1}{(z_1 - z_0)}(-z_0 x_1^r \otimes fm_+ + 4r x_1^{r-1} x_2 \otimes m_+ + (4 + z_0) x_1^r \otimes fm_+)\\
&= \frac{1}{(z_1 - z_0)}(4 x_1^r \otimes fm_+ + 4r x_1^{r-1} x_2 \otimes m_+)\\
&= \frac{1}{r+1}(x_1^r \otimes fm_+ + r x_1^{r-1} x_2 \otimes m_+),
\end{aligned}$$

so it's clear, by looking at the denominators, that ε_1 and ε_2 only preserve $\nabla(r)_{\mathbb{Z}_2} \otimes_{\mathbb{Z}_2} \Delta(1)_{\mathbb{Z}_2}$ when 2 does not divide $r+1$. □

In particular, for $p = 2$ we have exhibited a case where $\nabla(r) \otimes \Delta(s)$ is a direct sum of induced modules, but 2 still divides the quantity Z, disproving the converse statement of Theorem 4.0.1. Finally, we remark that it would be interesting to discern whether this is the only prime where the converse fails, and whether we would be able to have the converse statement by simply dividing γ by some scalar.

CHAPTER 5

TILTING MODULES

In this chapter, we move away from the ideas of the previous two chapters, and investigate which of the modules $\nabla(r) \otimes \Delta(s)$ are tilting. Whilst this question is of interest in its own right, we note that if we know such a module is a tilting module, then we can quickly decompose it into a direct sum of indecomposable modules using the character. Some examples of this are given at the end of the chapter. Throughout, unless otherwise stated, k will be an algebraically closed field of characteristic p, and G will be the group $SL_2(k)$.

5.1 Preliminary Results

First, we will give several useful results which will be used in the next few sections. In particular we will make extensive use of the following well known result, which is a rewording of Lemma 4.3.1. As promised, we outline an alternative proof here.

Theorem 5.1.1. *There exists a short exact sequence given by*

$$0 \longrightarrow \nabla(r-1) \longrightarrow \nabla(r) \otimes E \longrightarrow \nabla(r+1) \longrightarrow 0,$$

and this is split if and only if p does not divide $r+1$.

Proof. That the sequence exists is clear by considering the ∇-filtration of $\nabla(r) \otimes E = \nabla(r) \otimes \Delta(1)$. If p does not divide $r+1$, the result follows by considering the blocks for $SL_2(k)$, as follows. We have, using Lemma 2.2.29, that

$$\mathscr{B}(r+1) \subset \{r+1+2pn \ : \ n \in \mathbb{Z}\} \cup \{-r-3+2pn \ : \ n \in \mathbb{Z}\}.$$

Now, if $r-1 \in \mathscr{B}(r+1)$, we must have that $r-1 = -r-3+2pn$ for some $n \in \mathbb{Z}$, so that $2r+2 = 2pn$ and $r+1 = pn$, a contradiction.

On the other hand, if p does divide $r+1$, then the module $E \otimes \nabla(r)$ is projective as a G_1-module, while neither $\nabla(r-1)$ nor $\nabla(r+1)$ are, so the sequence cannot be split. □

The next result extends Theorem 3.3.2.

Lemma 5.1.2. *If $r,s \in \{np-1,\ np,\ np+1,\dots,\ (n+1)p-1\}$ for some fixed $n \in \mathbb{N}$, then $\nabla(r)\otimes\Delta(s)$ is tilting.*

Proof. Suppose, for a contradiction, that we have $\nabla(r)\otimes\Delta(s)$ is not tilting for some $r,s \in \{np-1, np, np+1,\dots, np+p-1\}$, choosing r and s so that $r+s$ is minimal. If $r \notin \{np-1, np\}$ then by Theorem 5.1.1 we have

$$\nabla(r-1)\otimes E\otimes\Delta(s) = \nabla(r)\otimes\Delta(s)\oplus\nabla(r-2)\otimes\Delta(s).$$

Since r and s were chosen so that $r+s$ was minimal, we have that $\nabla(r-1)\otimes\Delta(s)$ is tilting, so that each summand on the right hand side is tilting, giving a contradiction. We may suppose then that $r \in \{np-1, np\}$, and similarly, that $s \in \{np-1, np\}$. But then we have that $|r-s| \le 1$, so by Theorem 3.3.2 we have $\nabla(r)\otimes\Delta(s)$ is tilting, contradicting our initial assumption. □

This result shows us that there are more tilting modules of the form $\nabla(r)\otimes\Delta(s)$ than those given in Theorem 3.3.2 for every characteristic p. Before we delve further into this investigation, we prove a couple of useful lemmas concerning tilting modules.

Lemma 5.1.3. *Let G be a semisimple, simply connected algebraic group, over an algebraically closed field k, of prime characteristic, and let T_1 and T_2 be tilting modules where T_1 is projective as a G_1-module. Then the tensor product $T_1 \otimes T_2^F$ is also a tilting module.*

Proof. First, since each tilting module has a unique decomposition (up to isomorphism) into indecomposable tilting modules, it's sufficient to prove the lemma in the case that T_1 is indecomposable. Now, let ρ be the half sum of all positive roots, and take T_1 to be the Steinberg module $\nabla((p-1)\rho) = \mathrm{St}$. In this case the result holds by [9, Proposition 2.1].

Next let $\lambda \in X^+$ be such that $T(\lambda)$ is projective as a G_1-module, so that $(\lambda, \check{\alpha}) \ge p-1$ for all simple roots α [9, Proposition 2.4]. Then, since $(\lambda-(p-1)\rho, \check{\alpha}) \ge 0$ for all simple roots α, we may write $\lambda = (p-1)\rho+\mu$ for some $\mu \in X^+$. It follows that the tilting module $\mathrm{St}\otimes T(\mu)$ has highest weight λ, and so $T(\lambda)$ is a summand of this module. Then $T(\lambda)\otimes T_2^F$ is a summand of the tilting module $(\mathrm{St}\otimes T_2^F)\otimes T(\mu)$, and is thus tilting itself. □

We will use this lemma throughout the chapter in the case $G = SL_2(k)$, and in conjunction with the facts that $\nabla(p-1) = \Delta(p-1)$ is a projective G_1-module [21, Proposition II.10.1], and that the tensor product of a projective G_1-module with another G_1-module is again projective.

Lemma 5.1.4. *Let $G = SL_2(k)$, let V be a tilting module, and define the module W by $H^0(G_1, V) = W^F$. Then W is a tilting module.*

Proof. First we remark that, since the action of G_1 on the kG-module $H^0(G_1, V)$ is trivial, there does indeed exist a kG-module W such that $H^0(G_1, V) = W^F$ (see [20, Section 2.1]).

As in Lemma 5.1.3, it suffices to prove this for $V = T(m)$ for some $m \in \mathbb{N}$. We can split this into three separate cases, the first of which deals with $0 \le m \le p-1$. For such m we have $T(m) = L(m)$ and so

$$H^0(G_1, T(m)) = \begin{cases} L(0) & : m = 0 \\ \\ 0 & : 1 \le m \le p-1. \end{cases}$$

Next we consider the case $m = p-1+t$ for $1 \le t \le p-1$. Here $T(m)$, considered as a G_1-module, is the injective envelope of $L(p-1-t)$ [9, Example 2.2.1]. In particular $L(p-1-t)$ is the socle of $T(p-1+t)$ so if $H^0(G_1, T(p-1+t)) \neq 0$ then $H^0(G_1, L(p-1-t)) \neq 0$. Considering the case $t = p-1$ separately we get

$$H^0(G_1, T(m)) = \begin{cases} 0 & : 1 \le t \le p-2. \\ \\ L(0) & : t = p-1. \end{cases}$$

For the remaining cases we will use induction by writing $m = p-1+t+pn$ for some $n \in \mathbb{N}$ and $0 \le t \le p-1$ so that we can write $T(m) = T(p-1+t) \otimes T(n)^F$. Taking the G_1 fixed points we get $H^0(G_1, T(m)) = H^0(G_1, T(p-1+t)) \otimes T(n)^F$ which by the previous case gives us

$$H^0(G_1, T(m)) = \begin{cases} 0 & : 0 \le t \le p-2 \\ \\ T(n)^F & : t = p-1. \end{cases}$$

so that

$$W = \begin{cases} 0 & : 0 \le t \le p-2 \\ \\ T(n) & : t = p-1. \end{cases}$$

and is thus tilting. $\square$

5.1.1 Main Theorem

Before stating the main theorem of this chapter, we will introduce some notation. Let $r \in \mathbb{N}$ and p a prime. We can write the base p expansion of r as

$$\sum_{i=0}^{n} r_i p^i,$$

where each $r_i \in \{0,\dots,p-1\}$, $r_n \neq 0$ and for all $j > n$ we have $r_j = 0$. We will say that r has p-length n (or just length n if the prime is clear), and write

$$\mathrm{len}_p(r) = n.$$

We define $\mathrm{len}_p(0) = -1$. Now given any pair $(r,s) \in \mathbb{N}^2$ with $r \neq s$, we can write

$$r = \sum_{i=0}^{n} r_i p^i, \quad s = \sum_{i=0}^{n} s_i p^i$$

where $n = \max(\mathrm{len}_p(r), \mathrm{len}_p(s))$ so that at least one of r_n and s_n is non zero. Now let m be the largest integer such that $r_m \neq s_m$ and let

$$\hat{r} = \sum_{i=0}^{m} r_i p^i, \quad \hat{s} = \sum_{i=0}^{m} s_i p^i$$

so that if $r > s$ we have $r_m > s_m$ and $\hat{r} > \hat{s}$. Using this notation we may write

$$r = \hat{r} + \sum_{i=m+1}^{n} r_i p^i, \quad s = \hat{s} + \sum_{i=m+1}^{n} s_i p^i = \hat{s} + \sum_{i=m+1}^{n} r_i p^i.$$

Notice in particular that $r - \hat{r} = s - \hat{s}$ and denote this quantity by $\varepsilon_p(r,s)$ so that p^{m+1} divides $\varepsilon_p(r,s)$. We will call the pair $(\hat{r},\hat{s})$ the primitive of (r,s), and say that (r,s) is a primitive pair if $(r,s) = (\hat{r},\hat{s})$. In the case $r = s$ we define $\hat{r} = \hat{s} = 0$, and we have $\varepsilon_p(r,s) = r = s$.

Theorem 5.1.5. *Let $G = SL_2(k)$, where k is an algebraically closed field of characteristic p, and for $r,s \in \mathbb{N}$ let the pair $(\hat{r},\hat{s})$ be the primitive of (r,s). The module $\nabla(r) \otimes \Delta(s)$ is a tilting module if and only if one of the following holds*

1. *$\hat{r} = ap^n + p^n - 1$ for some $a \in \{0,\dots,p-2\}$, $n \in \mathbb{N}$, and $\hat{s} < p^{n+1}$,*
2. *$\hat{s} = bp^n + p^n - 1$ for some $b \in \{0,\dots,p-2\}$, $n \in \mathbb{N}$, and $\hat{r} < p^{n+1}$.*

We briefly note that in order to give uniqueness to the expressions of $\hat{r}$ and $\hat{s}$ we only allow a and b up to $p-2$. (If, for example, $a = p-1$ we would have $\hat{r} = (p-1)p^n + p^n - 1 = p^{n+1} - 1$.)

Figure 5.1 illustrates which of the modules $\nabla(r) \otimes \Delta(s)$ are tilting for $r,s \leq 31$ and $p = 2$.

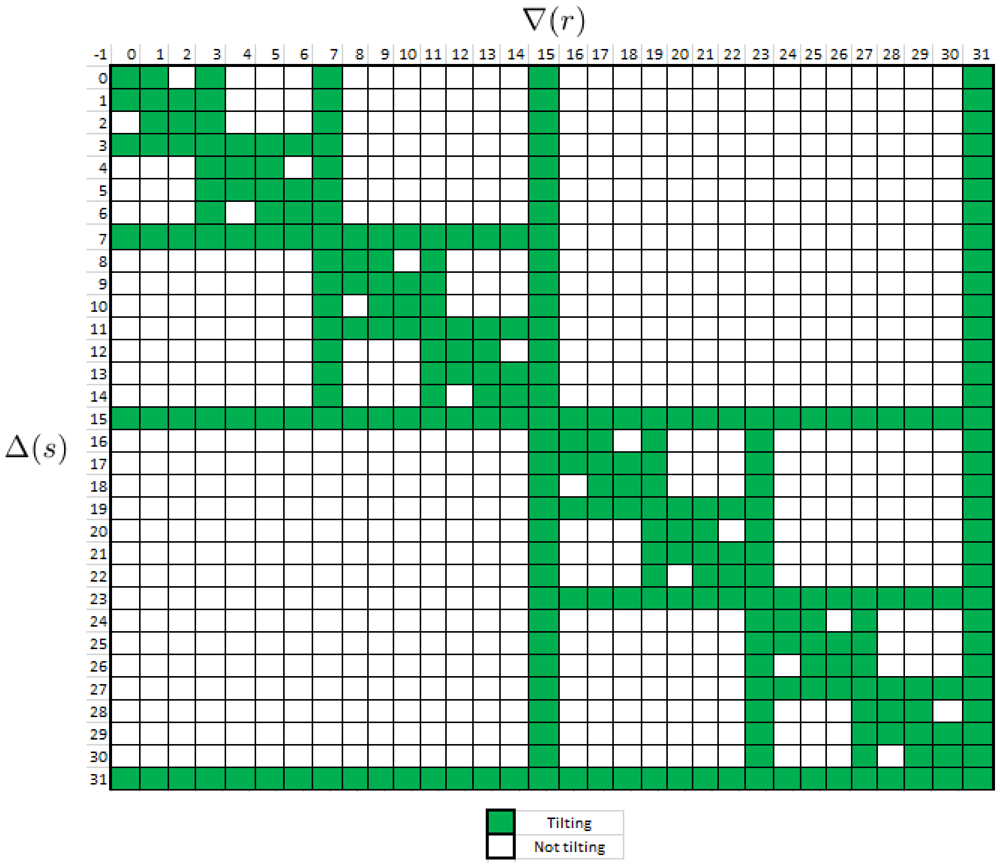

Figure 5.1: The modules $\nabla(r) \otimes \Delta(s)$ when $\mathrm{char}(k) = 2$.

5.2 Lemmas

In order to prove Theorem 5.1.5, we gather some elementary results on the modules $\nabla(r) \otimes \Delta(s)$. First we make an important observation.

Remark 5.2.1. Since the dual of a tilting module is also a tilting module, and we have the relation $(\nabla(r) \otimes \Delta(s))^* = \nabla(s) \otimes \Delta(r)$, it's clear that $\nabla(s) \otimes \Delta(r)$ is tilting if and only if $\nabla(r) \otimes \Delta(s)$ is tilting. Hence, for many of the results in this section, it's only necessary to prove the result for $r \geq s$.

Lemma 5.2.2. *Let $t, u \in \mathbb{N}$. The module $\nabla(p-1+pt) \otimes \Delta(p-1+pu)$ is tilting if and only if the module $\nabla(t) \otimes \Delta(u)$ is tilting.*

Proof. First recall the identities $\nabla(p-1+pt) = \nabla(p-1) \otimes \nabla(t)^F$ and $\Delta(p-1+pu) = \Delta(p-1) \otimes \Delta(u)^F$, found in [21, Proposition II.3.19]. Using these we may rewrite $\nabla(p-1+pt) \otimes \Delta(p-1+pu)$ as $\nabla(p-1) \otimes \Delta(p-1) \otimes (\nabla(t) \otimes \Delta(u))^F$.

Using Lemma 5.1.4 we easily obtain the forward implication. The reverse implication is also clear since $\nabla(p-1)\otimes\Delta(p-1)$ is tilting and projective as a G_1-module, so we can apply Lemma 5.1.3 □

Lemma 5.2.3. *Let $r = r_0 + pt$, $s = p-1+pu$ for some $0 \le r_0 \le p-2$ and $t,u \in \mathbb{N}$. Then $\nabla(r)\otimes\Delta(s)$ is tilting if and only if both $\nabla(t)\otimes\Delta(u)$ and $\nabla(t-1)\otimes\Delta(u)$ are tilting.*

Proof. First we assume that both $\nabla(t)\otimes\Delta(u)$ and $\nabla(t-1)\otimes\Delta(u)$ are tilting, and show that $\nabla(r)\otimes\Delta(s)$ is tilting. We will use the identity $\Delta(s) = \Delta(p-1)\otimes\Delta(u)^F$ as above, and the short exact sequence

$$0 \longrightarrow \nabla(r_0)\otimes\nabla(t)^F \longrightarrow \nabla(r) \longrightarrow \nabla(p-2-r_0)\otimes\nabla(t-1)^F \longrightarrow 0, \tag{5.1}$$

which can be found in [20, Satz 3.8, Bemerkung 2], in its dual form for Weyl modules. Tensoring sequence (5.1) with $\Delta(s)$ gives the following short exact sequence

$$\begin{aligned} 0 \longrightarrow \nabla(r_0)\otimes\Delta(p-1)\otimes(\nabla(t)\otimes\Delta(u))^F \longrightarrow \nabla(r)\otimes\Delta(s) \\ \longrightarrow \nabla(p-2-r_0)\otimes\Delta(p-1)\otimes(\nabla(t-1)\otimes\Delta(u))^F \longrightarrow 0. \end{aligned} \tag{5.2}$$

Now, for $0 \le r_0 \le p-2$, both $\nabla(r_0)\otimes\Delta(p-1)$ and $\nabla(p-2-r_0)\otimes\Delta(p-1)$ are tilting and projective as G_1-modules, so if both $\nabla(t)\otimes\Delta(u)$ and $\nabla(t-1)\otimes\Delta(u)$ are also tilting then by Lemma 5.1.3 both the second and fourth terms in sequence (5.2) are tilting. Hence we have that $\nabla(r)\otimes\Delta(s)$ is an extension of tilting modules. The only such extensions are split (e.g. by [21, Proposition II.4.16]), so we obtain $\nabla(r)\otimes\Delta(s)$ as a direct sum of two tilting modules, and hence is tilting itself.

For the converse statement, if $\nabla(r)\otimes\Delta(s) = \nabla(r_0+pt)\otimes\Delta(s)$ is tilting for some $r_0 \in \{0,1,\dots,p-2\}$, then each module $\nabla(v+pt)\otimes\Delta(s)$ for $v \in \{-1,0,1,\dots,p-1\}$ is tilting, so in particular the modules $\nabla(p-1+p(t-1))\otimes\Delta(s)$ and $\nabla(p-1+pt)\otimes\Delta(s)$ are tilting. This follows by induction, taking $v = r_0$ for the base case and using Theorem 5.1.1 for the induction step we obtain

$$\nabla(v+pt)\otimes E\otimes\Delta(s) = \big(\nabla(v+1+pt)\otimes\Delta(s)\big)\oplus\big(\nabla(v-1+pt)\otimes\Delta(s)\big),$$

so that both $\nabla(v+1+pt)\otimes\Delta(s)$ and $\nabla(v-1+pt)\otimes\Delta(s)$ are tilting. The result now follows from Lemma 5.2.2 applied to $\nabla(p-1+p(t-1))\otimes\Delta(s)$ and $\nabla(p-1+pt)\otimes\Delta(s)$. □

Remark 5.2.4. Note that by duality we obtain the corresponding result for when $r = p-1+pt$ and $s = s_0+pu$ for some $0 \le s_0 \le p-2$, and $t,u \in \mathbb{N}$. In this case we have that $\nabla(r)\otimes\Delta(s)$ is tilting if and only if both $\nabla(t)\otimes\Delta(u)$ and $\nabla(t)\otimes\Delta(u-1)$ are tilting.

It remains to determine which of the modules $\nabla(r)\otimes\Delta(s)$ are tilting when neither r nor s is congruent to $p-1$ modulo p. It turns out that this only occurs in the cases given in Lemma 5.1.2.

Lemma 5.2.5. *Let G be a semisimple, simply connected algebraic group over k, and let T be a G-module that is projective as a G_1-module. Then $\chi((p-1)\rho)$ divides* $\operatorname{Ch} T$.

Proof. This follows immediately from [10, 1.2(2)], since T must also be a projective B_1 module. □

We now revert to the case $G = SL_2(k)$ and obtain the following corollary.

Corollary 5.2.6. *For all $r \geq p-1$, the character of the Steinberg module $\nabla(p-1)$ divides that of the indecomposable tilting module $T(r)$ of highest weight r.*

Proof. By [9, Proposition 2.4] we have that for all $r \geq p-1$, the module $T(r)$ is a projective G_1-module. □

Recall that $\operatorname{Ch}\nabla(r) = \chi(r)$ (Lemma 2.2.19), and that this character is an element of $\mathbb{Z}[x, x^{-1}]$. We have that

$$\begin{aligned}\chi(r) &= x^r + x^{r-2} + \cdots + x^0 + \cdots + x^{-r}\\ &= \frac{1}{x^r}(x^{2r} + x^{2r-2} + \cdots + 1)\\ &= \frac{1}{x^r}\Big(\frac{x^{2r+2}-1}{x^2-1}\Big),\end{aligned}$$

so the roots of this equation are the $(2r+2)^{\text{th}}$ roots of unity, except ± 1. If $\chi(p-1)$ divides $\chi(r)$ then, we must have that the $2p^{\text{th}}$ roots of unity are also $(2r+2)^{\text{th}}$ roots of unity, which would imply that p divides $r+1$, i.e. that r is congruent to $p-1$ modulo p.

Hence we have shown that if both r and s are not congruent to $p-1$ modulo p, the character $\chi(p-1)$ does not divide $\operatorname{Ch}(\nabla(r)\otimes\Delta(s)) = \chi(r)\chi(s)$. Now suppose that $\nabla(r)\otimes\Delta(s)$ is tilting, and that $|r-s| > p-1$. By considering its good filtration (given in Theorem 3.3.2), we see that the decomposition of $\nabla(r)\otimes\Delta(s)$ into indecomposable tilting modules cannot contain any $T(j)$ for $j = 0,\ldots,p-1$. By Corollary 5.2.6 its character is divisible by $\chi(p-1)$ but the above calculation contradicts this. In summary:

Lemma 5.2.7. *For r and s both not congruent to $p-1$ modulo p, and $|r-s| > p-1$, the module $\nabla(r)\otimes\Delta(s)$ is not tilting.* □

There are now only a few more cases which we have not considered, which we deal with in the following lemma.

Lemma 5.2.8. *Let $r = r_0 + pt$ and $s = s_0 + pu$ with $r_0, s_0 \in \{0,1,\ldots,p-2\}$. Then $\nabla(r)\otimes\Delta(s)$ is tilting if and only if $t = u$.*

Proof. Assume that $\nabla(r) \otimes \Delta(s)$ is tilting, and suppose for a contradiction that $t \neq u$ with r and s chosen so that $r + s$ is minimal. Since $s \not\equiv p - 1 (\operatorname{mod} p)$ we have, by Theorem 5.1.1

$$\nabla(r) \otimes E \otimes \Delta(s) = \nabla(r) \otimes \Delta(s-1) \oplus \nabla(r) \otimes \Delta(s+1)$$

is tilting, and so the module $\nabla(r) \otimes \Delta(s-1)$ is tilting. Now, if $s_0 \neq 0$ then $s - 1 = s_0 - 1 + pu$ with $s_0 - 1 \geq 0$. Since r and s were chosen so that $r + s$ was minimal we must have that $t = u$, contradicting our initial assumption.

Similarly, if $r_0 \neq 0$, we obtain a contradiction. Now if $s_0 = r_0 = 0$, then since $t \neq u$ we must have $|r - s| \geq p$, so by Lemma 5.2.7 we obtain a contradiction.

For the converse, we assume $t = u$, so that we have $r, s \in \{np, np+1, \ldots, np+p-1\}$ for some $n \in \mathbb{N}$. Then by Lemma 5.1.2 the module $\nabla(r) \otimes \Delta(s)$ is tilting. □

Remark 5.2.9. Note that Lemma 5.2.8 shows us that if $\nabla(r) \otimes \Delta(s)$ is a tilting module, then we must have either at least one of r and s congruent to $p-1$ modulo p, or both r and s lie in the set $\{np, np+1, \ldots, np+p-2\}$ for some $n \in \mathbb{N}$.

We are now in a position where, given any r and s we could determine whether $\nabla(r) \otimes \Delta(s)$ is tilting by using the previous lemmas and induction. Figure 5.2 illustrates this for $p = 3$.

5.3 Proof of Theorem 5.2

In this section we prove Theorem 5.1.5 in two steps. The first is to show that for a primitive pair $(\hat{r}, \hat{s})$, we have that $\nabla(\hat{r}) \otimes \Delta(\hat{s})$ is a tilting module if and only if $\hat{r}$ and $\hat{s}$ are as described in the statement of the theorem. The second step is to show that for any pair (r, s) with primitive pair $(\hat{r}, \hat{s})$, we have that $\nabla(r) \otimes \Delta(s)$ is tilting if and only if $\nabla(\hat{r}) \otimes \Delta(\hat{s})$ is tilting. By the duality argument in Remark 5.2.1, we may assume that $r \geq s$ throughout.

Proposition 5.3.1. *Let (r, s) be a primitive pair. Then the module $\nabla(r) \otimes \Delta(s)$ is tilting if and only if*

$$r = p^n - 1 + ap^n, \;\; s < p^{n+1},$$

or

$$s = p^n - 1 + bp^n, \;\; r < p^{n+1},$$

for some $n \in \mathbb{N}$ and $a, b \in \{0, \ldots, p-2\}$.

Proof. ($\Rightarrow$) We assume that for a primitive pair (r, s), we have that $\nabla(r) \otimes \Delta(s)$ is tilting, and proceed by induction on $\operatorname{len}_p(r) = N$. For $N = 0$ we have that $r \leq p-1$ and so $r = ap^N + p^N - 1$ for

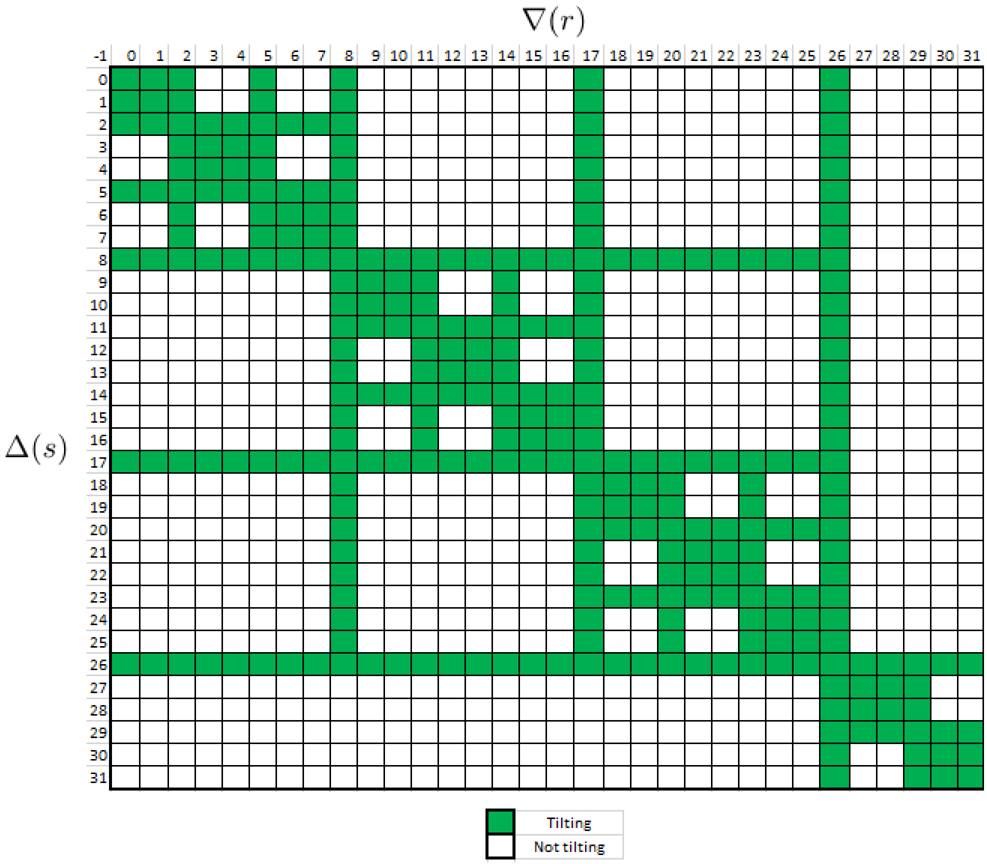

Figure 5.2: The modules $\nabla(r)\otimes\Delta(s)$ when $\mathrm{char}(k)=3$.

$a=0,\dots,p-2$, or in the case $r=p-1$ we have $r=p^{N+1}-1$. In each case we have that r is of the desired form, and $s<r<p^{N+1}$.

Next let's write $r=r_0+pt$ and $s=s_0+pu$, where $\mathrm{len}_p(t)=\mathrm{len}_p(r)-1$, and $\mathrm{len}_p(u)=\mathrm{len}_p(s)-1$. Since $\nabla(r)\otimes\Delta(s)$ is tilting, by Remark 5.2.9 we must have that either r_0 or s_0 is equal to $p-1$, or r and s both lie in the set $\{np, np+1,\dots,np+p-2\}$. However, since we are assuming that the pair (r,s) is primitive, we cannot have the second case. Hence either $r_0=p-1$ so that $r=p-1+pt$, or $s_0=p-1$ so that $s=p-1+pu$. Let's assume $r_0=p-1$, and note that the case $s_0=p-1$ is proved similarly.

Now we have two further cases to consider, the first is that $s_0=p-1$, and the second that $s_0\neq p-1$.

i.) Let's suppose that $s_0=p-1$, then by Lemma 5.2.2 we have that $\nabla(t)\otimes\Delta(u)$ is tilting. By induction we must have that t and u are of the form given in the statement of the theorem. If $t = p^{N-1} - 1 + ap^{N-1}$ for some $a \in \{0,\ldots,p-2\}$, and $u \leq p^N - 1$, then

$$r = p(p^{N-1} - 1 + ap^{N-1}) + p - 1 = p^N - 1 + ap^N,$$

and $s \leq p^N - p + s_0$, which is strictly less than p^{N+1} since $s_0 < p$. If we have that $u = p^{N-1} - 1 + ap^{N-1}$ and $t < p^N - 1$, then we obtain, in a similar manner, $s = p^N - 1 + ap^N$ and $r < p^{N+1}$.

ii.) For the second case, we suppose that $s_0 \neq p-1$, so that by Lemma 5.2.3 we have that $\nabla(t) \otimes \Delta(u)$ and $\nabla(t) \otimes \Delta(u-1)$ are tilting. By induction we have that the pairs (t,u) and $(t,u-1)$ are both of the form in the theorem. Since we cannot have that both u and $u-1$ are of the form $p^{N-1} - 1 + ap^{N-1}$, we must have that t is of this form and $u \leq p^N - 1$, so we complete the proof as above.

($\Leftarrow$) Now we prove the converse statement, that is, for a primitive pair (r,s), if $r = p^n - 1 + ap^n$ for some $a \in \{0,\ldots,p-2\}$, $n \in \mathbb{N}$, and $s < p^{n+1}$, then $\nabla(r) \otimes \Delta(s)$ is tilting. Once again, we use induction on n, with the case $n = 0$ being clear. For the inductive step, we have that if $r = p^n - 1 + ap^n = p - 1 + pt$ and $s = s_0 + pu < p^{n+1}$ for some t and u, then $t = p^{n-1} - 1 + ap^{n-1}$ and $u < p^n$. By induction the modules $\nabla(t) \otimes \Delta(u)$ and $\nabla(t) \otimes \Delta(u-1)$ are tilting, so by Lemma 5.2.3 (or Lemma 5.2.2 if $s_0 = p-1$) we have that $\nabla(r) \otimes \Delta(s)$ is tilting too. The case $s = p^n - 1 + bp^n$ and $r < p^{n+1}$ is obtained similarly. □

Proposition 5.3.2. *Let $(\hat{r},\hat{s})$ be the primitive of (r,s). Then $\nabla(r) \otimes \Delta(s)$ is tilting if and only if $\nabla(\hat{r}) \otimes \Delta(\hat{s})$ is tilting.*

Proof. Following Remark 5.2.9, we will first look at the case where at least one of r and s (and hence $\hat{r}$ and $\hat{s}$) is congruent to $p-1$. Note that if $r = s$, then we have $(\hat{r},\hat{s}) = (0,0)$, so in this case the result holds. Let's suppose then, that $r = p - 1 + pt$ and $s = s_0 + pu$, so that we have $\hat{r} = p - 1 + p\hat{t}$ and $\hat{s} = s_0 + p\hat{u}$. We remark that the other case, when $s = p - 1 + pu$ and $r = r_0 + pt$, is obtained in an identical manner.

As before, there are two cases to consider: $s_0 = p-1$ and $s_0 \neq p-1$. In both cases we will proceed by induction on $\mathrm{len}_p(r)$. Let's first consider the case $s_0 = p-1$, where, when $\mathrm{len}_p(r) = 0$, we have that $r = s = p-1$ which we have already covered. Now by Lemma 5.2.2 we have $\nabla(r) \otimes \Delta(s)$ is tilting if and only if $\nabla(t) \otimes \Delta(u)$ is tilting. By induction then we have that this is tilting if and only if $\nabla(\hat{t}) \otimes \Delta(\hat{u})$ is tilting. Applying Lemma 5.2.2 again we find that $\nabla(\hat{t}) \otimes \Delta(\hat{u})$ is tilting if and only if $\nabla(\hat{r}) \otimes \Delta(\hat{s})$ is tilting.

Next, we consider the case $s_0 \neq p-1$, where we may assume $r > s$. Again, the base case is easily obtained since this time the pair $(p-1, s_0)$ is primitive. For the inductive step, we will consider separately the cases $u \not\equiv 0 \mod p$ and $u \equiv 0 \mod p$. If $u \not\equiv 0 \mod p$ then, since $t > u$ it's clear that the pair $(\hat{t}, \widehat{u-1})$ is equal to the pair $(\hat{t}, \hat{u}-1)$. We then have that $\nabla(r) \otimes \Delta(s)$ is tilting if and only if $\nabla(t) \otimes \Delta(u)$ and $\nabla(t) \otimes \Delta(u-1)$ are tilting by Lemma 5.2.3. By induction, these are tilting if and only if both $\nabla(\hat{t}) \otimes \Delta(\hat{u})$ and $\nabla(\hat{t}) \otimes \Delta(\widehat{u-1})$ are tilting. Now $\nabla(\hat{t}) \otimes \Delta(\widehat{u-1}) = \nabla(\hat{t}) \otimes \Delta(\hat{u}-1)$, so we apply Lemma 5.2.3 again to obtain that these are tilting if and only if $\nabla(\hat{r}) \otimes \Delta(\hat{s})$ is tilting.

For the case $u \equiv 0 \mod p$, we treat each direction separately. If $\nabla(r) \otimes \Delta(s)$ is tilting, then by Lemma 5.2.3 we have that $\nabla(t) \otimes \Delta(u)$ is tilting, and by induction we have that $\nabla(\hat{t}) \otimes \Delta(\hat{u})$ is tilting. Now $\hat{u} \equiv 0 \mod p$, so by Theorem 5.1.1 we obtain

$$\nabla(\hat{t}) \otimes E \otimes \Delta(\hat{u}) = \nabla(\hat{t}) \otimes \Delta(\hat{u}+1) \oplus \nabla(\hat{t}) \otimes \Delta(\hat{u}-1).$$

The module on the left hand side is a tilting module, so $\nabla(\hat{t}) \otimes \Delta(\hat{u}-1)$ is also a tilting module. We apply Lemma 5.2.3 again to obtain that $\nabla(\hat{r}) \otimes \Delta(\hat{s})$ is tilting. For the reverse direction we have that if $\nabla(\hat{r}) \otimes \Delta(\hat{s})$ is tilting, then $\nabla(\hat{t}) \otimes \Delta(\hat{u})$ is tilting, so by induction $\nabla(t) \otimes \Delta(u)$ is also tilting. Now, as above, since $u \equiv 0 \mod p$ we have that $\nabla(t) \otimes \Delta(u-1)$ is also tilting, so we apply Lemma 5.2.3 to obtain that $\nabla(r) \otimes \Delta(s)$ is tilting.

What remains is to prove the result when both r and s lie in the set $\{np, np+1, \ldots, (n+1)p-2\}$ for some $n \in \mathbb{N}$. From Lemma 5.1.2 we know already that for such r and s the module $\nabla(r) \otimes \Delta(s)$ is tilting, so it's sufficient to show that $\nabla(\hat{r}) \otimes \Delta(\hat{s})$ is tilting. However, it's clear that in this case $\hat{r}$ and $\hat{s}$ lie in the set $\{0, \ldots, p-2\}$, and so $\nabla(\hat{r}) \otimes \Delta(\hat{s})$ is tilting. □

5.4 Example Decompositions

In this section we give some examples of decompositions of tilting modules, using the character. In order to do this, we will exploit the fact that every tilting module is uniquely determined by its character, and for the case $G = SL_2$, we can calculate the character of each indecomposable tilting module using Theorem 2.2.28 and Brauer's character formula (Theorem 2.2.17). The following characters, for the case $p = 2$, were calculated in this way.

$$\begin{aligned}
&\operatorname{Ch} T(0) = \chi(0) && \operatorname{Ch} T(9) = \chi(9) + \chi(5)\\
&\operatorname{Ch} T(1) = \chi(1) && \operatorname{Ch} T(10) = \chi(10) + \chi(8) + \chi(6) + \chi(4)\\
&\operatorname{Ch} T(2) = \chi(2) + \chi(0) && \operatorname{Ch} T(11) = \chi(11) + \chi(3)\\
&\operatorname{Ch} T(3) = \chi(3) && \operatorname{Ch} T(12) = \chi(12) + \chi(10) + \chi(4) + \chi(2)\\
&\operatorname{Ch} T(4) = \chi(4) + \chi(2) && \operatorname{Ch} T(13) = \chi(13) + \chi(9) + \chi(5) + \chi(1)\\
&\operatorname{Ch} T(5) = \chi(5) + \chi(1) && \operatorname{Ch} T(14) = \chi(14) + \chi(12) + \chi(10) + \chi(8) + \chi(6) + \chi(4) + \chi(2) + \chi(0)\\
&\operatorname{Ch} T(6) = \chi(6) + \chi(4) + \chi(2) + \chi(0) && \operatorname{Ch} T(15) = \chi(15)\\
&\operatorname{Ch} T(7) = \chi(7) && \operatorname{Ch} T(16) = \chi(16) + \chi(14)\\
&\operatorname{Ch} T(8) = \chi(8) + \chi(6) && \operatorname{Ch} T(17) = \chi(17) + \chi(13).
\end{aligned}$$

First, take $\nabla(15) \otimes \Delta(2)$, with character $\chi(17) + \chi(15) + \chi(13)$. Since such a tilting module must contain $T(17)$, we have that

$$\operatorname{Ch} \nabla(15) \otimes \Delta(2) - \operatorname{Ch} T(17) = \chi(15) = \operatorname{Ch} T(15),$$

hence we obtain $\nabla(15) \otimes \Delta(2) = T(17) \oplus T(15)$.

Next we look at $\nabla(7) \otimes \Delta(7)$, with character equal to that of $T(14)$, so we have $\nabla(7) \otimes \Delta(7) = T(14)$.

Consider now $\nabla(5) \otimes \Delta(4)$, with character $\chi(9) + \chi(7) + \chi(5) + \chi(3) + \chi(1)$. Taking away $\operatorname{Ch} T(9)$ we are left with $\chi(7) + \chi(3) + \chi(1)$. Next we take away $\operatorname{Ch} T(7)$ to obtain $\chi(3) + \chi(1)$, which is equal to the character of $T(3) \oplus T(1)$. Hence we have

$$\nabla(5) \otimes \Delta(4) = T(9) \oplus T(7) \oplus T(3) \oplus T(1).$$

Finally let's consider $\nabla(9) \otimes \Delta(8)$. The character is given by

$$\operatorname{Ch} \nabla(9) \otimes \Delta(8) = \sum_{i=0}^{8} \chi(17 - 2i).$$

Taking away the character of the tilting module of highest weight each time, as before, we obtain

$$\nabla(9) \otimes \Delta(8) = T(17) \oplus T(15) \oplus T(11) \oplus T(9) \oplus T(7) \oplus T(1).$$

We will see in the next chapter that the ability to decompose these modules like this is central to the final result.

CHAPTER 6

Final Results

In the final chapter, we give some further results derived from those of the previous chapter. First, we take another look at the pair $(\hat{r},\hat{s})$, when $\nabla(r)\otimes\Delta(s)$ is a tilting module. Next, we give an explicit surjective module homomorphism from $\nabla(ap^n-1)\otimes\Delta(s)$ to $\nabla(ap^n)\otimes\Delta(s-1)$ for some fixed $s \le ap^n-1$. Inspired by this, we then construct a more general surjective module homomorphism from $\nabla(r)\otimes\Delta(s)$ to $\nabla(r-1)\otimes\Delta(s+1)$. After some further investigation, we find that this is enough to allow us to calculate the good filtration of the indecomposable summands of arbitrary $\nabla(r)\otimes\Delta(s)$, where $r \ge s$. Finally, we formalise these results, and give some corollaries and examples.

6.1 Primitive Pairs

Firstly, using the notation from the previous chapter, we give the following lemma for all primes p.

Lemma 6.1.1. *Suppose that $\nabla(r)\otimes\Delta(s)$ is a tilting module, where $r = pt+p-1$ and $s = up+v$ for some $t,u\in\mathbb{N}$ and $v\in\{0,\dots,p-1\}$. Then we have the following decomposition for $\nabla(r)\otimes\Delta(s)$.*

$$\nabla(r)\otimes\Delta(s) = \nabla(p-1)\otimes\Delta(p-2-v)\otimes(\nabla(t)\otimes\Delta(u-1))^F \oplus \nabla(p-1)\otimes\Delta(v)\otimes(\nabla(t)\otimes\Delta(u))^F.$$

Proof. The proof follows that in Lemma 5.2.3, whereby we obtain the following short exact sequence.

$$0 \longrightarrow \nabla(p-1)\otimes\Delta(p-2-v)\otimes(\nabla(t)\otimes\Delta(u-1))^F \longrightarrow \nabla(r)\otimes\Delta(s) \longrightarrow \nabla(p-1)\otimes\Delta(v)\otimes(\nabla(t)\otimes\Delta(u))^F \longrightarrow 0.$$

By Lemma 5.2.3 we have that, since $\nabla(r)\otimes\Delta(s)$ is tilting, both $\nabla(t)\otimes\Delta(u)$ and $\nabla(t)\otimes\Delta(u-1)$ are tilting, so, using Lemma 5.1.3, we find that the second and fourth modules in the sequence are tilting. Hence the sequence must be split (Theorem 2.2.27), and $\nabla(r)\otimes\Delta(s)$ decomposes as the given direct sum. $\square$

Note that, since in the above case $\nabla(r)\otimes\Delta(s)$ is a tilting module, we can decompose it into its indecomposable summands. Such a decomposition would be at least as fine as the one just given, but we will make particular use of this one in the next lemma.

Recall that we say the pair (r,s) is a tilting pair if the module $\nabla(r)\otimes\Delta(s)$ is a tilting module. By theorem 5.1.5 we have that each primitive tilting pair (r,s) has the form given in the above lemma, so it holds for all primitive tilting pairs. Next we give the main result of this section.

Lemma 6.1.2. *Let (r,s) be a tilting pair. Then the module $\nabla(\hat{r})\otimes\Delta(\hat{s})$ is a submodule of $\nabla(r)\otimes\Delta(s)$.*

Proof. First we remark that when the pair (r,s) is a primitive tilting pair, we have that $(\hat{r},\hat{s}) = (r,s)$, so the result is trivial. To prove the non-trivial case, we first note that, as before, we can assume without loss of generality that $r \geq s$ by duality. Furthermore, it's sufficient to show that if the pair (r,s) is a primitive tilting pair, then we have $\nabla(r)\otimes\Delta(s)$ is a submodule of $\nabla(r+a_np^n)\otimes\Delta(s+a_np^n)$ for any $a_n \in \{0,1,\ldots,p-1\}$ and $n > \mathrm{len}_p(r)$. Then we can apply this result to every power of p appearing in $\epsilon_p(r,s)$. We will prove this by induction on $\mathrm{len}_p(r)$.

For the base case let us consider $\nabla(0)\otimes\Delta(0)\simeq k$, where $\mathrm{len}_p(0) = -1$. Since both $\nabla(0)\otimes\Delta(0)$ and $\nabla(a_np^n)\otimes\Delta(a_np^n)$ are tilting modules, the dimension of $\mathrm{Hom}_G(\nabla(0)\otimes\Delta(0),\nabla(a_np^n)\otimes\Delta(a_np^n))$ is equal to the inner product $(\nabla(0)\otimes\Delta(0),\nabla(a_np^n)\otimes\Delta(a_np^n))$ (Lemma 2.2.24). The character of $\nabla(0)\otimes\Delta(0)$ is $\chi(0)$, so clearly this inner product is 1. Hence there exists a non-zero homomorphism from $\nabla(0)\otimes\Delta(0)$ to $\nabla(a_np^n)\otimes\Delta(a_np^n)$. The kernel of such a homomorphism is a submodule of k, but since the homomorphism is non-zero we must have that it is injective, so we can embed $\nabla(0)\otimes\Delta(0)$ as a submodule of $\nabla(a_np^n)\otimes\Delta(a_np^n)$.

For the inductive step, let (r,s) be a primitive tilting pair, so that, by Proposition 5.3.1, we can assume $r = pt+p-1$ for some $t\in\mathbb{N}$ and $\mathrm{len}_p(r)\leq n$ (the case $s = pu+p-1$ is obtained likewise). Write $s = pu+v$ for some $u\in\mathbb{N}$ and $v\in\{0,\ldots,p-1\}$, and let

$$r' = r+ap^{n+1} = p(t+ap^n)+p-1, \quad s' = s+ap^{n+1} = p(u+ap^n)+v,$$

then we have (by Lemma 6.1.1) that

$$\begin{aligned}\nabla(r')\otimes\Delta(s') = \nabla(p-1)\otimes&\Delta(p-2-v)\otimes(\nabla(t+ap^n)\otimes\Delta(u+ap^n-1))^F\\ &\oplus\nabla(p-1)\otimes\Delta(v)\otimes(\nabla(t+ap^n)\otimes\Delta(u+ap^n))^F.\end{aligned}$$

By induction we have that $\nabla(t)\otimes\Delta(u-1)$ is a submodule of $\nabla(t+ap^n)\otimes\Delta(u+ap^n-1)$ and $\nabla(t)\otimes\Delta(u)$ is a submodule of $\nabla(t+ap^n)\otimes\Delta(u+ap^n)$. Hence we obtain injections

$$\nabla(p-1)\otimes\Delta(p-2-v)\otimes(\nabla(t)\otimes\Delta(u-1))^F \hookrightarrow \nabla(p-1)\otimes\Delta(p-2-v)\otimes(\nabla(t+ap^n)\otimes\Delta(u+ap^n))^F$$
$$\nabla(p-1)\otimes\Delta(v)\otimes(\nabla(t)\otimes\Delta(u))^F \hookrightarrow \nabla(p-1)\otimes\Delta(v)\otimes(\nabla(t+ap^n)\otimes\Delta(u+ap^n))^F.$$

But by Lemma 6.1.1 we have that

$$\begin{aligned}\nabla(r)\otimes\Delta(s) = &\nabla(p-1)\otimes\Delta(p-2-v)\otimes(\nabla(t)\otimes\Delta(u-1))^F\oplus\\ &\nabla(p-1)\otimes\Delta(v)\otimes(\nabla(t)\otimes\Delta(u))^F,\end{aligned}$$

and so $\nabla(r)\otimes\Delta(s)\hookrightarrow\nabla(r')\otimes\Delta(s')$. □

It follows that as a submodule of $\nabla(r)\otimes\Delta(s)$, the module $\nabla(\hat{r})\otimes\Delta(\hat{s})$ is equal to $O_\pi(\nabla(r)\otimes\Delta(s))$, where π is the saturated set of weights with highest weight $\hat{r}+\hat{s}$.

6.2 Short Exact Sequences

In this section we will exhibit a particular surjective module homomorphism between $\nabla(ap^n-1)\otimes\Delta(s)$ and $\nabla(ap^n)\otimes\Delta(s-1)$, where we assume that $s\le ap^n-1$, and $a\in\{1,\dots,p-1\}$. As in previous chapters, we consider kG-modules as modules for the hyperalgebra U_k.

Lemma 6.2.1. *The map $\phi:\nabla(ap^n-1)\otimes\Delta(s)\longrightarrow\nabla(ap^n)\otimes\Delta(s-1)$ taking the element $x_2^{ap^n-1}\otimes m_+$ to $x_1x_2^{ap^n-1}\otimes l_+$ defines a surjective module homomorphism, where m_+ is a highest weight vector of $\Delta(s)$, and l_+ is a highest weight vector of $\Delta(s-1)$.*

Proof. There are a number of statements in the above lemma that need proving, not least of which that ϕ exists and actually gives a module homomorphism. To this end, we recall Lemma 3.2.4 and remark that the module $\nabla(ap^n-1)\otimes\Delta(s)$ is in fact generated as a module by $x_2^{ap^n-1}\otimes m_+$ (this follows from the fact that $\nabla(ap^n-1)=\Delta(ap^n-1)=L(ap^n-1)$). Hence, in order to give a homomorphism explicitly, it's sufficient to just give the image of this element.

Next we show that the map given above exists. Since $\nabla(ap^n-1)\otimes\Delta(s)$ is generated by the element $x_2^{ap^n-1}\otimes m_+$ of weight $s+1-ap^n$, if we can show that

$$\dim\mathrm{Hom}_{U_k}(\nabla(ap^n-1)\otimes\Delta(s),\nabla(ap^n)\otimes\Delta(s-1))=\dim(\nabla(ap^n)\otimes\Delta(s-1))^{s+1-ap^n},$$

then we will have a linear isomorphism between the left and right sides. Hence we can give a basis of $\mathrm{Hom}_{U_k}(\nabla(ap^n-1)\otimes\Delta(s),\nabla(ap^n)\otimes\Delta(s-1))$ by the elements ϕ_i, where $\phi_i(x_2^{ap^n-1}\otimes m_+)=v_i$, and the elements v_i give a basis for $(\nabla(ap^n)\otimes\Delta(s-1))^{s+1-ap^n}$ (as in Section 3.1).

To this end, we notice that a basis for $(\nabla(ap^n)\otimes\Delta(s-1))^{s+1-ap^n}$ is given by the set

$$\left\{\, x_1^{1+i}x_2^{ap^n-1-i}\otimes f_i(l_+)\ :\ i=0,\dots,s-1\,\right\},$$

so that its dimension is s. Furthermore this basis contains the element $x_1x_2^{ap^n-1}\otimes l_+$. Next we calculate the dimension of $\mathrm{Hom}_{U_k}(\nabla(ap^n-1)\otimes\Delta(s),\nabla(ap^n)\otimes\Delta(s-1))$ by using that

$$\mathrm{Hom}_{U_k}(\nabla(ap^n-1)\otimes\Delta(s),\nabla(ap^n)\otimes\Delta(s-1))=\mathrm{Hom}_{U_k}(\nabla(ap^n-1)\otimes\Delta(ap^n),\nabla(s)\otimes\Delta(s-1)).$$

On the right hand side, we have that both modules are tilting modules by Theorem 3.3.2, so the dimension of the space of homomorphisms between them is given by their inner product (Lemma 2.2.24). Now

$$\mathrm{Ch}\,\nabla(ap^n-1)\otimes\Delta(ap^n)=\sum_{i=0}^{ap^n-1}\chi(2ap^n-1-2i)$$

and

$$\mathrm{Ch}\,\nabla(s)\otimes\Delta(s-1)=\sum_{i=0}^{s-1}\chi(2s-1-2i),$$

by the Clebsch-Gordan formula. Since both $2s-1$ and $2ap^n-1$ are odd and $s\le ap^n-1$, it's clear that the inner product is s. We have shown now that our homomorphisms exists and is well defined, so it remains to prove surjectivity. To do this, we will show that the image of ϕ contains a generating set of $\nabla(ap^n)\otimes\Delta(s-1)$.

By Lemma 3.2.4, we have that the set

$$S=\left\{\, x_1^{ap^n-i}x_2^{i}\otimes l_+\ :\ i=0,1,\dots,ap^n\,\right\}$$

certainly generates $\nabla(ap^n)\otimes\Delta(s-1)$. Since $\phi(x_2^{ap^n-1}\otimes m_+)=x_1x_2^{ap^n-1}\otimes l_+\in S$, we have that $e_i(x_1x_2^{ap^n-1}\otimes l_+)\in\mathrm{Im}\phi$ for each i from 0 to ap^n-1. Next, since l_+ is a highest weight vector of $\Delta(s-1)$, we have that

$$e_i(x_1x_2^{ap^n-1}\otimes l_+)=e_i(x_1x_2^{ap^n-1})\otimes l_+=\binom{ap^n-1}{i}x_1^{1+i}x_2^{ap^n-1-i}\otimes l_+,$$

for $i=1,\dots ap^n-1$. By Lucas' Theorem (Theorem A.3.1) we have that each binomial coefficient is non zero, so each element $x_1^ix_2^{ap^n-i}\otimes l_+$ lies in $\mathrm{Im}\phi$ for $i=1,\dots,ap^n$. Finally, we have that $f_{ap^n}(x_1^{ap^n}\otimes l_+)\in\mathrm{Im}\phi$, and

$$f_{ap^n}(x_1^{ap^n}\otimes l_+)=\sum_{j=0}^{ap^n}f_j(x_1^{ap^n})\otimes f_{ap^n-j}l_+=\sum_{j=0}^{ap^n}\binom{ap^n}{j}x_1^{ap^n-j}x_2^{j}\otimes f_{ap^n-j}l_+.$$

Once again we apply Lucas' theorem, but this time we get that $\binom{ap^n}{j}=0$ for all j except ap^n and 0. However, when $i=0$ we have that $f_{ap^n}l_+=0$ since $l_+\in\Delta(s-1)$ and $s-1<ap^n$. Hence we obtain that $f_{ap^n}(x_1^{ap^n}\otimes l_+)=x_2^{ap^n}\otimes l_+$. Now we have that $S\subset\mathrm{Im}(\phi)$, so ϕ is surjective. $\square$

The proposition shows us that we have a short exact sequence of the form

$$0 \longrightarrow K \longrightarrow \nabla(ap^n - 1) \otimes \Delta(s) \longrightarrow \nabla(ap^n) \otimes \Delta(s-1) \longrightarrow 0,$$

for some module K. By looking at the character of each module in the sequence we deduce that the character of K is given by

$$\begin{aligned}\operatorname{Ch} K &= \operatorname{Ch} \nabla(ap^n - 1) \otimes \Delta(s) - \operatorname{Ch} \nabla(ap^n) \otimes \Delta(s-1) \\ &= \sum_{i=0}^{s} \chi(ap^n - 1 - s + 2i) - \sum_{i=0}^{s-1} \chi(ap^n + 1 - s + 2i) \\ &= \chi(ap^n - 1 - s).\end{aligned}$$

By taking the left exact functor O_π (see Section 2.2.3), where π is the saturated set of weights with highest weight $ap^n - 1 - s$, we must have that

$$K = O_\pi(K) = O_\pi(\nabla(ap^n - 1) \otimes \Delta(s)) = \nabla(ap^n - 1 - s).$$

We can summarise with the following corollary.

Corollary 6.2.2. *Let $s \leq ap^n - 1$ for $n \in \mathbb{N}$ and $a \in \{1, \ldots, p-1\}$. The module $\nabla(ap^n) \otimes \Delta(s-1)$ is given by the quotient of $\nabla(ap^n - 1) \otimes \Delta(s)$ with the bottom induced module, $\nabla(ap^n - 1 - s)$, from its good filtration.* □

Next we are able to generalise the above result, by exhibiting a surjective module homomorphism

$$\psi : \nabla(r) \otimes \Delta(s) \longrightarrow \nabla(r+1) \otimes \Delta(s-1),$$

for $r, s \in \mathbb{N}$ with $r \geq s$. Furthermore, this homomorphism will be independent of the characteristic of k.

First recall that we have the module homomorphism given by multiplication,

$$\begin{aligned}\phi_r : S^r E \otimes E &\longrightarrow S^{r+1} E, \\ x_1^a x_2^b \otimes x_1 &\longmapsto x_1^{a+1} x_2^b, \\ x_1^a x_2^b \otimes x_2 &\longmapsto x_1^a x_2^{b+1},\end{aligned}$$

for all a and $b \in \mathbb{N}$ such that $a + b = r$. Since $S^r E = \nabla(r)$ and $\nabla(r)^* = \Delta(r)$ we can consider the dual homomorphism

$$\phi_r^* : \Delta(r+1) \longrightarrow E \otimes \Delta(r).$$

If we let $m_+ \in \Delta(r+1)$ be a highest weight vector, and l_+ a highest weight vector in $\Delta(r)$, then we must have that $\phi_r^*(m_+) = x_1 \otimes l_+$. From here we can compute the image of each basis element of $\Delta(r+1)$ by

$$\begin{aligned}
\phi_r^*(f_i m_+) &= f_i \phi_r^*(m_+) \\
&= f_i(x_1 \otimes l_+) \\
&= \sum_{j=0}^{i} f_j(x_1) \otimes f_{i-j} l_+ \\
&= x_1 \otimes f_i l_+ + x_2 \otimes f_{i-1} l_+,
\end{aligned}$$

for all $i = 1, \dots, r$. For $i = r+1$ we have that $\phi_r^*(f_{r+1} m_+) = x_2 \otimes f_r l_+$ since $l_+ \in \Delta(r)$, so $f_{r+1} l_+ = 0$.

Having defined these maps we may now give ψ as the composition

$$\psi = (\phi_r \otimes \mathrm{id}_{\Delta(s-1)}) \circ (\mathrm{id}_{\nabla(r)} \otimes \phi_{s-1}^*).$$

This is more easily visualised by the following commutative diagram.

$$\begin{tikzcd}
\nabla(r) \otimes \Delta(s) \arrow[rr, "\psi"] \arrow[dr, "\mathrm{id}_{\nabla(r)} \otimes \phi_{s-1}^*"'] & & \nabla(r+1) \otimes \Delta(s-1) \\
& \nabla(r) \otimes E \otimes \Delta(s-1) \arrow[ur, "\phi_r \otimes \mathrm{id}_{\Delta(s-1)}"'] &
\end{tikzcd}$$

Next we would like to show that the map ψ is surjective. Using the explicit maps above, we can calculate the image of ψ on the following basis.

$$\begin{aligned}
\psi(x_1^a x_2^b \otimes m_+) &= (\phi_r \otimes \mathrm{id}_{\Delta(s-1)})(x_1^a x_2^b \otimes x_1 \otimes l_+) \\
&= x_1^{a+1} x_2^b \otimes l_+, \qquad (6.1)
\end{aligned}$$

$$\begin{aligned}
\psi(x_1^a x_2^b \otimes f_i m_+) &= (\phi_r \otimes \mathrm{id}_{\Delta(s-1)})(x_1^a x_2^b \otimes x_1 \otimes f_i l_+ + x_1^a x_2^b \otimes x_2 \otimes f_{i-1} l_+) \\
&= x_1^{a+1} x_2^b \otimes f_i l_+ + x_1^a x_2^{b+1} \otimes f_{i-1} l_+, \qquad (6.2)
\end{aligned}$$

for $i = 1, \dots, s-1$. For $i = s$ we have

$$\begin{aligned}
\psi(x_1^a x_2^b \otimes f_s m_+) &= (\phi_r \otimes \mathrm{id}_{\Delta(s-1)})(x_1^a x_2^b \otimes x_2 \otimes f_{s-1} l_+) \\
&= x_1^a x_2^{b+1} \otimes f_{s-1} l_+. \qquad (6.3)
\end{aligned}$$

In particular, we notice from the first line that $x_1^{a+1} x_2^b \otimes l_+ \in \mathrm{Im}\psi$, for all a and b such that $a+b=r$. We claim also that $x_2^{r+1} \otimes l_+ \in \mathrm{Im}\psi$. Consider the image

$$\psi\Big(\sum_{i=1}^{s-1} (-1)^{i-1} x_1^{i-1} x_2^{r-i+1} \otimes f_i m_+\Big) = \sum_{i=1}^{s-1} (-1)^{i-1} \psi(x_1^{i-1} x_2^{r-i+1} \otimes f_i m_+),$$

which by eq. (6.2) is equal to

$$\sum_{i=1}^{s-1}(-1)^{i-1}\left(x_1^i x_2^{r-i+1}\otimes f_i l_+ + x_1^{i-1}x_2^{r-i+2}\otimes f_{i-1}l_+\right).$$

Now note that this sum is telescopic, so we are left only with the terms $x_2^{r+1}\otimes l_+ + (-1)^{s-2}x_1^{s-1}x_2^{r-s+2}\otimes f_{s-1}l_+$. But by eq. (6.3) we have

$$\psi(x_1^{s-1}x_2^{r-s+1}\otimes f_s m_+) = x_1^{s-1}x_2^{r-s+2}\otimes f_{s-1}l_+,$$

so we obtain that $x_2^{r+1}\otimes l_+ \in \mathrm{Im}\psi$. Hence we have that $\nabla(r+1)\otimes l_+ \subset \mathrm{Im}\psi$, and since this subset generates the module $\nabla(r+1)\otimes\Delta(s-1)$, we have that ψ is surjective.

Hence this map induces a short exact sequence

$$0 \longrightarrow K \longrightarrow \nabla(r)\otimes\Delta(s) \longrightarrow \nabla(r+1)\otimes\Delta(s-1) \longrightarrow 0,$$

where K can be seen as the kernel of ψ, a submodule of $\nabla(r)\otimes\Delta(s)$. Furthermore we can calculate the character of K as $\chi(r-s)$. Now, consider the saturated subset π of $X(T)$ consisting of those weights less than or equal to $r-s$. As before, applying the left exact functor O_π to the short exact sequence above gives us

$$0 \longrightarrow O_\pi(K) \longrightarrow O_\pi(\nabla(r)\otimes\Delta(s)) \longrightarrow O_\pi(\nabla(r+1)\otimes\Delta(s-1)) \longrightarrow \ldots$$

However, since the lowest Clebsch-Gordan coefficient of $\nabla(r+1)\otimes\Delta(s-1)$ is $r-s+2$ we have that $O_\pi(\nabla(r+1)\otimes\Delta(s-1)) = 0$. Furthermore, we have that $K = O_\pi(K)$ since its character is $\chi(r-s)$, so we obtain that

$$K = O_\pi(\nabla(r)\otimes\Delta(s)) = \nabla(r-s).$$

Theorem 6.2.3. *There exists a short exact sequence*

$$0 \longrightarrow \nabla(r-s) \longrightarrow \nabla(r)\otimes\Delta(s) \longrightarrow \nabla(r+1)\otimes\Delta(s-1) \longrightarrow 0,$$

for all $r,s\in\mathbb{N}$ where $r\geq s$. □

We remark that for any $r,s\in\mathbb{N}$ we can find some a and b in $\mathbb{N}$ such that $a+b=r+s, a\geq b$ and $|a-b|\leq 1$. By Theorem 3.3.2 we have that the module $\nabla(a)\otimes\Delta(b)$ is tilting, and so using the character we can decompose it into its indecomposable tilting summands for a particular prime p, as in Section 5.4. Since each Clebsch-Gordan coefficient $a+b-2i$ only occurs once in the good filtration of $\nabla(a)\otimes\Delta(b)$, there exists exactly one indecomposable summand for which $\nabla(a-b)$ occurs as a submodule. Let's say

$$\nabla(a)\otimes\Delta(b) = T(c_1)\oplus T(c_2)\oplus\ldots\oplus T(c_n),$$

and we may as well assume that $\nabla(a-b)$ appears in the good filtration of $T(c_1)$. Then by Theorem 6.2.3 we have that

$$\nabla(a+1)\otimes\Delta(b-1)=\frac{T(c_1)}{\nabla(a-b)}\oplus T(c_2)\oplus\ldots\oplus T(c_n).$$

There is however, one caveat. This is that an indecomposable tilting module, like $T(c_1)$, may not remain indecomposable under the quotient of the lowest submodule of its good filtration. It turns out however, that it does remain indecomposable, as we will shortly see. Not only this, but further quotients remain indecomposable. Assuming this, we can repeat the above procedure, taking the quotient of the lowest weight induced submodule. Each time this gives us the decomposition for $\nabla(a+i)\otimes\Delta(b-i)$ for $i=1,2,\ldots,b$. Hence, in principle, we may obtain the decomposition, in any characteristic, for the module $\nabla(r)\otimes\Delta(s)$, which was the original aim of this thesis.

I am thankful to Stephen Donkin for supplying the proof of the following proposition, which shows us that the indecomposable tilting modules remain indecomposable under taking multiple quotients.

Proposition 6.2.4. *For $G=SL_2(k)$, every indecomposable tilting module has simple socle and simple head.*

Proof. First we remark that since the indecomposable tilting modules $T(r)$ are all self dual, it's sufficient to prove only that they have simple socle. For $r\le p-1$ we have that $T(r)=L(r)$ and so the result is true. Next we consider r in the range $p\le r\le 2p-2$, so that we can write $r=p-1+t$ for some $t\le p-1$. In this case we have that the G_1-socle of $T(r)$ is $L(p-1-t)$ (as in the proof of Lemma 5.1.4), so it follows that its G-socle is simple too.

Next we consider the case $r>2p-2$, where we will write $r=s+pt$ with $p-1\le s\le 2p-2$, and $s=p-1+u$ with $0\le u\le p-1$. Then we have $T(r)=T(s)\otimes T(t)^F$, so that the G_1-socle of $T(r)$ is $L(p-1-u)\otimes T(t)^F$. Suppose then, that for some $v\ge 0$ we have $\mathrm{Hom}_G(L(v),T(r))\ne 0$, then we must also have that $\mathrm{Hom}_{G_1}(L(v),T(r))=\mathrm{Hom}_{G_1}(L(v),L(p-1-u)\otimes T(t)^F)\ne 0$, so that $v=p-1-u+pw$ for some $w\in\mathbb{N}$.

We have then, that

$$\begin{aligned}\mathrm{Hom}_G(L(v),T(r))&=\mathrm{Hom}_G(L(p-1-u)\otimes L(w)^F,L(p-1-u)\otimes T(t)^F)\\&=H^0(G,L(p-1-u)^*\otimes L(p-1-u)\otimes(L(w)^*\otimes T(t))^F)\\&=H^0(G/G_1,H^0(G_1,L(p-1-u)^*\otimes L(p-1-u))\otimes(L(w)^*\otimes T(t))^F)\\&=H^0(G/G_1,(L(w)^*\otimes T(t))^F)\\&=\mathrm{Hom}_G(L(w),T(t)),\end{aligned}$$

where we are using that for any module V we have $H^0(G_1, V^F) = V^F$. By induction we have that this is equal to k if $L(w)$ is the G-socle of $T(t)$, and 0 otherwise. Hence we have that

$$\mathrm{Soc}_G T(r) = L(u) \otimes (\mathrm{Soc}_G T(t))^F,$$

which by Steinberg's tensor product theorem (Theorem 2.2.18), is simple. □

6.3 Clebsch Gordan Modules

In this section, we formalise the ideas of the previous section, and introduce some notation. Let $T(r)$ be the indecomposable tilting module of highest weight r, and suppose it has the good filtration given by

$$0 = V_0 < V_1 < V_2 < \ldots < V_{t-1} < V_t = T(r),$$

where each quotient V_i/V_{i-1} is isomorphic to $\nabla(r_i)$ for some $r_i \in \mathbb{N}$. Define $T(r)_i = T(r)/V_{i-1}$ for $i = 1, \ldots, t$, and $T(r)_0 = T(r)$.

Lemma 6.3.1. *For all $r, s \geq 0$ we have that $(T(r) : \nabla(s))$ is either 1 or 0.*

Proof. We may write r as $r = 2m+1$ or $r = 2m$ for some $m \in \mathbb{N}$. Then $T(r)$ is a direct summand of the module $\nabla(m+1) \otimes \Delta(m)$ or $\nabla(m) \otimes \Delta(m)$. In either case, the Clebsch-Gordan formula for the character shows us that $(T(r) : \nabla(s)) \leq 1$. □

Definition 6.3.2. We will call a G-module V a *Clebsch-Gordan module* (or *CG-module* for short) if the following two properties hold:

1. V is a direct sum of modules of the form $T(r)_i$.
2. The composition factor $(V : \nabla(m))$ is either 1 or 0 for all $m \in \mathbb{N}$.

Lemma 6.3.3. *If V is a Clebsch-Gordan module and m is the minimal element of the set*

$$\{ i \ : \ (V : \nabla(i) \neq 0 \},$$

then V contains a unique submodule V_0 isomorphic to $\nabla(m)$, and the quotient module V/V_0 is also a Clebsch-Gordan module.

Proof. Let π denote the saturated set of weights corresponding to the set $\{i \in \mathbb{N} \cup \{0\} \ : \ i \leq m \}$. Since m was minimal we have that $V_0 = O_\pi(V) \cong \nabla(m)$. Since V is a Clebsch-Gordan module, it has decomposition

$$V = V_1 \oplus \ldots \oplus V_t,$$

where each $V_i \cong T(r_i)_{q_i}$ for some r_i and $q_i \in \mathbb{N} \cup \{0\}$. Hence we must have that

$$V_0 = O_\pi(V_1) \oplus \ldots \oplus O_\pi(V_t) \cong \nabla(m).$$

Since $\nabla(m)$ is indecomposable, we must have that $V_0 = O_\pi(V_u)$ for some $u \le t$. Therefore the quotient can be written

$$V/V_0 = V_u/V_0 \oplus \bigoplus_{i \neq u} V_i.$$

Finally we note that $V_u/V_0 = T(r_u)_{q_u+1}$, so we have that

$$V/V_0 = T(r_u)_{q_u+1} \oplus \bigoplus_{i \neq u} T(r_i)_{q_i}.$$

In particular we have that for each $j \neq m$ the composition factor $(V/V_0 : \nabla(j))$ is equal to $(V : \nabla(j))$, and

$$(V/V_0 : \nabla(m)) = (V : \nabla(m)) - 1 = 0.$$

Hence the quotient V/V_0 is a Clebsch-Gordan module. $\square$

Corollary 6.3.4. *For $r \ge s$ we have that the module $\nabla(r) \otimes \Delta(s)$ is a Clebsch-Gordan module.*

Proof. First we consider the case $|r-s| \le 1$, for which we have that $\nabla(r) \otimes \Delta(s)$ is tilting, and thus a Clebsch-Gordan module. For all other such r and s we have the short exact sequence from Theorem 6.2.3

$$0 \longrightarrow \nabla(r-s-2) \longrightarrow \nabla(r-1) \otimes \Delta(s+1) \longrightarrow \nabla(r) \otimes \Delta(s) \longrightarrow 0.$$

Using Lemma 6.3.3 and induction (with the case $|r-s| \le 1$ as the base case), we obtain that $\nabla(r) \otimes \Delta(s)$ is a Clebsch-Gordan module. $\square$

As a Clebsch-Gordan module, the indecomposable summands of $\nabla(r) \otimes \Delta(s)$ are all of the form $T(t)_i$, and these can be found from the known decomposition of the unique tilting module $\nabla(a) \otimes \Delta(b)$ where $a + b = r + s$ and $|a-b| \le 1$, by taking successive quotients. In particular we have that for all $r \in \mathbb{N}$, the induced module $\nabla(r)$ is a Clebsch-Gordan module.

We can in fact, take such successive quotients in one go, and we summarise the above procedure in the following result.

Corollary 6.3.5. *Let $r,s,t,u \in \mathbb{N}$ such that $r \ge s$, $t \ge u$, $r+s = t+u$ and $r-s < t-u$. Let π be the saturated set of weights consisting of those less than $t-u$. Then we have the following short exact sequence*

$$0 \longrightarrow O_\pi(\nabla(r) \otimes \Delta(s)) \longrightarrow \nabla(r) \otimes \Delta(s) \longrightarrow \nabla(t) \otimes \Delta(u) \longrightarrow 0. \quad \square$$

Recall that Lemma 6.1.2 told us that if $\nabla(r)\otimes\Delta(s)$ was a tilting module, then $\nabla(\hat{r})\otimes\Delta(\hat{s})$ was a submodule. As mentioned above, it's clear that if π is the saturated set of weights consisting of those less than or equal to $\hat{r}+\hat{s}$, then we have the equality

$$\nabla(\hat{r})\otimes\Delta(\hat{s}) = O_\pi(\nabla(r)\otimes\Delta(s)).$$

In light of the above result, we obtain the following corollary.

Corollary 6.3.6. *Let $\nabla(r)\otimes\Delta(s)$ be a tilting module with $r \geq s$. Then there exists the following short exact sequence.*

$$0 \longrightarrow \nabla(\hat{r})\otimes\Delta(\hat{s}) \longrightarrow \nabla(r)\otimes\Delta(s) \longrightarrow \nabla(\hat{r}+\hat{s}+1+\varepsilon_p(r,s))\otimes\Delta(\varepsilon_p(r,s)-1) \longrightarrow 0,$$

where the notation follows that in Section 5.1.1.

Proof. All that remains to be proven are the parameters for the third module in the sequence, since we know that this module is of the form $\nabla(t)\otimes\Delta(u)$ for some t and u in $\mathbb{N}$, by Corollary 6.3.5. By considering the character we obtain the following two equations

$$\begin{aligned} t+u &= r+s, \\ t-u &= \hat{r}+\hat{s}+2, \end{aligned}$$

the first of which we may rewrite as

$$t+u = \hat{r}+\hat{s}+2\varepsilon_p(r,s).$$

By solving these equations we obtain that $t = \hat{r}+\hat{s}+1+\varepsilon_p(r,s)$ and $u = \varepsilon_p(r,s)-1$. □

6.3.1 Example Decompositions

Following Section 5.4, we give some example decompositions when the prime p is equal to 2.

First let's consider $\nabla(16)\otimes\Delta(1)$. We have the short exact sequence

$$0 \longrightarrow \nabla(13) \longrightarrow \nabla(15)\otimes\Delta(2) \longrightarrow \nabla(16)\otimes\Delta(1) \longrightarrow 0.$$

Using the known decomposition of $\nabla(15)\otimes\Delta(2)$ from Section 5.4, we have that

$$\nabla(16)\otimes\Delta(1) = T(17)/\nabla(13)\oplus T(15) = \nabla(17)\oplus\nabla(15),$$

which agrees with Lemma 4.3.1.

Next we consider $\nabla(7)\otimes\Delta(7)$. We noticed in Section 5.4 that this module was indecomposable. It follows then that when $r+s=14$ we have that $\nabla(r)\otimes\Delta(s)$ is indecomposable. For example,

$$\nabla(8) \otimes \Delta(6) = T(14)/\nabla(0).$$

Finally, from the known decomposition of $\nabla(9) \otimes \Delta(8) = T(17) \oplus T(15) \oplus T(11) \oplus T(9) \oplus T(7) \oplus T(1)$, we give the following

$$\begin{aligned}
\nabla(10) \otimes \Delta(7) &= T(17) \oplus T(15) \oplus T(11) \oplus T(9) \oplus T(7),\\
\nabla(11) \otimes \Delta(6) &= T(17) \oplus T(15) \oplus \big(T(11)/\nabla(3)\big) \oplus T(9) \oplus T(7),\\
\nabla(12) \otimes \Delta(5) &= T(17) \oplus T(15) \oplus \big(T(11)/\nabla(3)\big) \oplus \big(T(9)/\nabla(5)\big) \oplus T(7),\\
\nabla(13) \otimes \Delta(4) &= T(17) \oplus T(15) \oplus \big(T(11)/\nabla(3)\big) \oplus \big(T(9)/\nabla(5)\big),\\
\nabla(14) \otimes \Delta(3) &= T(17) \oplus T(15) \oplus \big(T(11)/\nabla(3)\big),\\
\nabla(15) \otimes \Delta(2) &= T(17) \oplus T(15),\\
\nabla(16) \otimes \Delta(1) &= \big(T(17)/\nabla(13)\big) \oplus T(15)\\
\nabla(17) &= T(17)/\nabla(13).
\end{aligned}$$

APPENDIX

A

APPENDIX

A.1 Polynomial $GL_n(k)$-Modules

In Lemma 3.2.1 we use the notion of polynomial $GL_n(k)$-modules. In this section, we will briefly introduce them, and the main results we use concerning them. For further details the reader may consult [16]. Throughout this section, k is an infinite field of arbitrary characteristic, and $G = GL_n(k)$.

First recall that for each $i, j = 1, \dots, n$, we have the function $T_{ij} \in \mathrm{Map}(G,k)$, given by $T_{ij}(g) = g_{ij}$, the ij-entry of the matrix g. These functions generate a k-subalgebra of $\mathrm{Map}(G,k)$, which we denote $A_k(n)$. We will often refer to elements of $A_k(n)$ as *polynomial functions on G*.

Let V be a kG-module with corresponding representation ρ. Fix a basis $\{v_1, \dots, v_m\}$ for V, then we have maps $\rho_{ij} : G \longrightarrow k$ given by

$$\rho(g)v_j = \sum_{i=1}^{m} \rho_{ij}(g)v_i,$$

for all $g \in G$. These maps are called the *coefficient functions* of ρ, and their k-span is called the *coefficient space*, a subspace of $\mathrm{Map}(G,k)$ often denoted $\mathrm{cf}(V)$. It is easy to show that the coefficient space is independent of the basis chosen.

Definition A.1.1. A $GL_n(k)$-module V, with corresponding representation ρ, is called a *polynomial module* if for every i, j we have that the coefficient function ρ_{ij} is a polynomial function on G, i.e. $\rho_{ij} \in A_k(n)$, or equivalently, that $\mathrm{cf}(V) \subset A_k(n)$.

It's clear from the remark following Definition 2.2.4, that every polynomial module is also a rational module.

Recall that a polynomial is called *homogeneous of degree r* if it is expressible as a sum of monomials, each of degree r. The finite-dimensional space of homogeneous polynomials of degree r inside $A_k(n)$ is denoted $A_k(n,r)$. This allows us to give the grading,

$$A_k(n) = \bigoplus_{r\geq 0} A_k(n,r).$$

Definition A.1.2. Let V be a polynomial $GL_n(k)$-module. If $\mathrm{cf}(V) \subset A_k(n,r)$ then V is called a *homogeneous polynomial $GL_n(k)$-module of degree r.*

The central result concerning homogeneous polynomial $GL_n(k)$-modules is the following.

Theorem A.1.3 ([16, Theorem 2.2c]). *Every polynomial $GL_n(k)$-module V has a decomposition given by*

$$V = \bigoplus_{r\geq 0} V_r,$$

where for each r we have that V_r is a submodule of V such that V_r is a homogeneous polynomial $GL_n(k)$-module of degree r. □

We give the following result, used in Lemma 3.2.1.

Lemma A.1.4. *Let V and W be two homogeneous polynomial $GL_n(k)$-modules of some fixed degree $d \in \mathbb{N}$. Then we have an isomorphism of vector spaces*

$$\mathrm{Hom}_{GL_n(k)}(V,W) \cong \mathrm{Hom}_{SL_n(k)}(V,W).$$

Proof. First, it's clear that we may restrict a $GL_n(k)$ homomorphism to an $SL_n(k)$ homomorphism, and that restriction is linear. It remains to show then, that in this case restriction is also bijective.

Let Z be the center of $GL_n(k)$, given by

$$Z = \left\{ \mathrm{diag}(t,\ldots,t) \;:\; t \in k^* \right\},$$

and recall that we may write $GL_n(k)$ as the product $ZSL_n(k)$. Write $g \in GL_n(k)$ as zg_0 for $z \in Z$ and $g_0 \in SL_n(k)$. We may write z as $t\mathbb{1}_n$, so that z acts on V and W by multiplication by t^d. Hence we have, for any $\phi \in \mathrm{Hom}_{SL_n(k)}(V,W)$ and $v \in V$,

$$\phi(gv) = \phi(zg_0v) = \phi(t^d g_0 v) = t^d \phi(g_0 v) = t^d g_0 \phi(v) = zg_0\phi(v) = g\phi(v),$$

so ϕ is a $GL_n(k)$ homomorphism. □

A.2 Results on $SL_n(k)$-Modules

For the reader's convenience, we give the following results and proofs taken directly from [4]. These are used in Section 3.2.

Lemma A.2.1 ([4, Lemma 5.0.3])**.** *Let V be an indecomposable $GL_n(k)$-module, and W a subspace of V. Then W is a $kGL_n(k)$-submodule of V if and only if it is a $kSL_n(k)$-submodule of V.*

Proof. Let Z denote the center of $GL_n(k)$ and let $z \in Z$ be fixed. As a vector space, we can write V as

$$V = \bigoplus_{\lambda \in k} V^{\lambda},$$

where $V^{\lambda} = \{v \in V \ : \ zv = \lambda v\}$. Now let $\lambda \in k$, $v \in V^{\lambda}$ and $g \in GL_n(k)$. Then we have $z(gv) = (zg)v = (gz)v = \lambda gv$, and so in fact, the above decomposition is a decomposition as $kGL_n(k)$-modules. Since V is indecomposable, we must have that $V = V^{\lambda}$ for some $\lambda \in k$, hence z acts by scalar multiplication. Finally, using the fact that $GL_n(k) = Z \cdot SL_n(k)$ we get the result. □

Lemma A.2.2 ([4, Lemma 5.3.8])**.** *Let A be a finite dimensional algebra over k, V an A-module, and $e = e^2 \in A$ with $e \neq 0$. Then the k-vector spaces $\mathrm{Hom}_A(Ae, V)$ and eV are isomorphic.*

Proof. Consider the linear map $\phi : \mathrm{Hom}_A(Ae, V) \longrightarrow eV$ sending θ to $\theta(e)$, which is well defined since $\theta(e) = \theta(e)^2 = e\theta(e) \in eV$. It's clear that $\phi(\theta) = 0$ if and only if $\theta(Ae) = 0$ so ϕ is injective. Furthermore, for every $ev \in eV$ the map $\theta : se \longmapsto sev$ is a well defined morphism of A-modules and so ϕ is surjective. □

For the final lemma we will introduce some notation, and refer the reader to [16] and [26] for further details. Let $G = GL_n(k)$ and recall the definition of $A_k(n,r)$ from Section A.1. We denote the Schur algebra by $S_k(n,r)$; this is the linear dual of $A_k(n,r)$, with algebra structure given by the natural bialgebra structure on $A_k(n)$. It's well known that the category of homogeneous polynomial $GL_n(k)$-modules of degree r is equivalent to the category of $S_k(n,r)$ modules [26, Theorem 2.2.7].

Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda(n,r)$ be an n-composition (i.e. a sequence of n non-negative numbers summing to r). We will denote by $S^{\lambda}E$ the polynomial kG-module given by $S_1^{\lambda}E \otimes \cdots \otimes S_n^{\lambda}E$, where S^rE is the usual r^{th} symmetric power of the natural kG-module E. We have the following grading for the kG-module $A_k(n,r)$,

$$A_k(n,r) = \bigoplus_{\lambda \in \Lambda(n,r)} S^{\lambda}E.$$

It's known that for an n-composition $\alpha \in \Lambda(n,r)$, there exists an idempotent $\xi_\alpha \in S_k(n,r)$ such that $V^\alpha \cong \xi_\alpha V$ [26, Theorem 2.2.10].

For a polynomial $GL_n(k)$-module V, we denote by V° the contravariant dual of V. As a vector space this is equal to the linear dual, but the action is given as

$$g \cdot \alpha(v) = \alpha(g^{\mathrm{t}} v),$$

for all $g \in GL_n(k)$, $\alpha \in V^\circ$ and $v \in V$. The symbol g^{t} denotes the transpose of the matrix g. In particular then, since the transpose of an element in a torus T is equal to itself, the weights of V° are equal to those of V. Furthermore, it's clear that if $v \in V^\lambda$ and v has dual element α, then $\alpha \in (V^\circ)^\lambda$. The reader may consult [16, Section 2.7] for more on the contravariant dual.

Lemma A.2.3 ([4, Lemma 5.3.9])**.** *For an $S_k(n,r)$-module V, the k-vector spaces $\mathrm{Hom}_{S_k(n,r)}(V^\circ, S^\alpha E)$ and V^α are isomorphic.*

Proof. By Lemma A.2.2 with $A = S_k(n,r)$ and $e = \xi_\alpha$, the k-vector spaces $\mathrm{Hom}_{S_k(n,r)}(S_k(n,r)\xi_\alpha, V)$ and V^α are isomorphic. Then using the fact that, for any finite-dimensional k-algebra A we have

$$\mathrm{Hom}_A(V,W) \cong \mathrm{Hom}_A(W^\circ, V^\circ),$$

for A-modules V and W, we obtain that

$$\mathrm{Hom}_{S_k(n,r)}(V^\circ, A_k(n,r)\xi_\alpha) \cong V^\alpha.$$

Finally, we have that $A_k(n,r)\xi_\alpha \cong S^\alpha E$, giving the result. □

A.3 Lucas' Theorem

In many cases in this thesis it will be extremely useful to give a binomial coefficient modulo p in terms of the parameters. Fortunately, Lucas' theorem tells us exactly how to do that.

Theorem A.3.1. *Let $r, s \in \mathbb{N}$ with p-adic expansions given by*

$$r = \sum_{i=0}^{n} r_i p^i, \qquad s = \sum_{i=0}^{n} s_i p^i.$$

Then we have

$$\binom{r}{s} \equiv \prod_{i=0}^{n} \binom{r_i}{s_i} \pmod{p}.$$

The proof can be found in [24].

A.4 p-adic Numbers

In Chapter 4 we worked over the field of p-adic numbers $\mathbb{Q}_p$, for which we give a brief introduction here. We will follow loosely [2], to which we refer the reader for a more in-depth account.

Throughout we will fix a field k, and a prime number p. First we give a definition.

Definition A.4.1. A *valuation of rank* 1 of a field k is a mapping $|\cdot| : k \longrightarrow \mathbb{R}$ that satisfies the following properties for all a and $b \in k$:

1. $|a| \geq 0$ and $|a| = 0$ if and only if $a = 0$.

2. $|ab| = |a||b|$.

3. $|a+b| \leq |a| + |b|$.

If in addition we have that $|a+b| \leq \max(|a|,|b|)$ then $|\cdot|$ is called a non-archimedean valuation.

It's clear that taking the absolute value of a rational number gives a valuation of $\mathbb{Q}$. We are interested however, in the following alternative. First, fix a real number $c \in (0,1)$ ($c = 1/p$ is a common choice). Next, for any $x \in \mathbb{Q}$, we can factorize x as

$$x = p^{\alpha}\frac{a}{b}$$

for some $\alpha, a, b \in \mathbb{Z}$ such that $p \nmid a$ and $p \nmid b$. Now define

$$\begin{aligned} |\cdot|_p : \mathbb{Q} &\longrightarrow \mathbb{R} \\ x &\longmapsto c^{\alpha}. \end{aligned}$$

One can then show, that this is a non-archimedean valuation (see [2, Section I.1]). For a non-archimedean valuation $|\cdot|$, the subset $V \subset k$ given by

$$V = \{\, a \in k \;:\; |a| \leq 1 \,\},$$

forms a ring, called the valuation ring. Furthermore, its subset P consisting of those a such that $|a| < 1$ is the unique maximal ideal of V, and is prime. Hence the quotient V/P is a field, called the associated residue class field [2, Theorem I.2.3].

A field k with valuation $|\cdot|$ is called complete if every Cauchy sequence in k has a limit in k. Through a process detailed in [2, Section II.1], one can uniquely complete a field k with respect to a valuation on that field. We will denote the resulting field as $\hat{k}$. Furthermore, the characteristic of $\hat{k}$ is equal to that of k. For example, the real numbers $\mathbb{R}$ is the completion of $\mathbb{Q}$ with respect to the usual absolute value. We remark that for a non-archimedean valuation $|\cdot|$ we have $|k| = |\hat{k}|$ (where $|k|$ is the image of k in $\mathbb{R}$ under $|\cdot|$), and that for any Cauchy sequence $\{a_n\}$ with limit a there exists an $n \in \mathbb{N}$ such that $|a_n| = |a|$ [2, Theorem II.1.4].

Definition A.4.2. The *field of p-adic numbers*, denoted $\mathbb{Q}_p$, is the completion of $\mathbb{Q}$ with respect to the valuation $|\cdot|_p$.

Notice that the image of $\mathbb{Q}_p$ under the valuation $|\cdot|_p$ is given by the set

$$\left\{ |p|_p^n \;:\; n \in \mathbb{Z} \right\}.$$

For a p-adic number $a \in \mathbb{Q}_p$, we can write a as

$$a = \sum_{i=n}^{\infty} a_i p^i,$$

for some $a_i \in \mathbb{Z}$ and n is such that $|a|_p = |p|_p^n$. Note that this expression is not unique, however, there exists a unique expression with the a_i in the range $0, 1, \cdots, p-1$, called the canonical expansion. Those elements for which there exists an expansion with $n \geq 0$ will be called p-adic integers, the set of which will be denoted $\mathbb{Z}_p$. Notice that we have $\mathbb{Z}_p = \hat{V}_p$, the valuation ring on $\mathbb{Q}_p$ with respect to $|\cdot|_p$. This is given by the completion of the set

$$\mathbb{Z}_{(p)} = \left\{ p^{\alpha} \frac{a}{b} \in \mathbb{Q} \;:\; \alpha \geq 0 \right\}.$$

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