The Morava Cohomology of Finite General Linear Groups

Samuel M A Hutchinson

A thesis submitted in partial fulfilment of the requirements for the degree of

Doctor of Philosophy

School of Mathematics and Statistics

May 2018

Supervisor: Prof. Neil Strickland

University of Sheffield

Abstract

In this thesis, for all finite heights n and odd primes p, we compute the Morava E-theory and Morava K-theory of general linear groups over finite fields $\mathbb{F}$ of order $q \equiv 1 \mod p$. We rephrase the problem in terms of $\mathcal{V}_*$, the graded groupoid of vector spaces over $\mathbb{F}$, and focus on the graded algebra and coalgebra structures induced from the direct sum functor. We use character theory to determine the ranks of $E^0 B\mathcal{V}_*$ and $K^0 B\mathcal{V}_*$, and use this information to reverse engineer the Atiyah-Hirzebruch spectral sequence for $K^* B\mathcal{V}_*$ and $K_* B\mathcal{V}_*$. We then use this result in two ways: we deduce that the algebra and coalgebra structures are free commutative and cofree cocommutative respectively, and we identify a lower bound for the nilpotence of the canonical top normalised Chern class in $K^0 B\mathcal{V}_{p^k}$. Following this we make use of algebro-geometric and Galois theoretic techniques to determine the indecomposables in Morava E-theory and K-theory, before using this calculation in conjunction with K-local duality and the nilpotence lower bound to determine the primitives of the coalgebra structure in Morava K-theory. Along the way, we show that $E^0 B\mathcal{V}_*$ and $K^0 B\mathcal{V}_*$ have structures similar to that of a graded Hopf ring, but with a modified version of the compatibility relation. We call such structures "graded faux Hopf rings".

Acknowledgements

Firstly, I would like to thank my supervisor, Professor Neil Strickland. I am very grateful to him for his help, and for sharing some of his vast knowledge and insights with me. I thank my office mates from J25 for offering a great source of entertainment. Thank you to my family for a lifetime of love, support and encouragement. Finally, I dedicate this thesis to Jenna. I don't know where I would be without your love and support. Thank you for listening to me when I have rambled on, and frankly, thank you for all the ways you have changed my life.

Contents

Introduction

0.1 Background

The fundamental idea of homotopy theory is to study spaces using homotopy invariants. Cohomology theories offer an important class of such invariants that naturally exist in the context of stable homotopy theory. Examples of cohomology theories include ordinary cohomology, K-theory and cobordism. For a large class of cohomology theories (referred to as "complex-oriented") there is a close relationship to the theory of 1-dimensional commutative formal group schemes (formal groups) from algebraic geometry.

By considering the classification of formal groups in characteristic p using their heights n, the chromatic theory of homotopy has enabled topologists to determine a lot about the global picture of the stable homotopy category. Fundamental to this are the Morava E and K-theories, which are periodic cohomology theories constructed to have particular associated formal groups relating to the prime p and height n from the aforementioned classification.

One of the main reasons that computations in K-theory and ordinary cohomology are tractable is that they both have geometric descriptions. This is often useful in determining the properties of a cohomology theory and in generally understanding the information captured by a cohomology theory.

In contrast, the Morava cohomologies are constructed purely algebraically (from their formal groups) and not a lot is understood about what they represent geometrically. We do know that for $n = 1$ Morava K and E-theory are essentially mod p and p-adic K-theory respectively, but to gain a better understanding in the higher height cases, we expect to have to do some computations. Characteristic classes of bundles offer a good starting point for understanding these cohomology theories, and these are best understood through the cohomology of the associated classifying spaces.

Attempts to extend the methods of chromatic homotopy to the setting of equivariant homotopy have been made, but there is still much that is not understood.

Calculations of E^*BG and K^*BG should offer insight into the coefficients of equivariant versions of the Morava cohomologies, which provides another motivation for studying the Morava cohomology of classifying spaces.

In 1980, Ravenel and Wilson computed the Morava K-theory of Eilenberg-Maclane spaces for all finite abelian groups ([33]), and various progress has been made on the case of non-abelian classifying spaces since. Some important computations include the Morava cohomology of wreath products of C_p ([19]); the counterexample to the conjecture that the Morava K-theory of all groups is concentrated in even degrees ([22]); and the Morava cohomology of symmetric groups ([34]), which is closely related to the construction of power operations in Morava E-theory ([3]).

There are two main calculational tools that we will use in this thesis that are specific to studying the Morava cohomology of finite groups. The first is generalised character theory as in [17], which is motivated by the behaviour of the $n = 1$ case, complex K-theory. By the Atiyah-Segal completion theorem, KU^0BG is equal to the completion of the representation ring $R(G)$ at its augmentation ideal. Using this one can obtain a "character theory" for K-theory from the ordinary character theory of representations. As the Morava E-theories offer higher height analogues of p-adic K-theory, this suggests that E^0BG should behave something like the completion of a "higher height representation ring" of G. This gives another reason to study E^*BG.

It turns out that one can generalise this complex K-theoretic version of character theory to all Morava E-theories. This provides a complete description of E^0BG after a flat extension of the coefficient ring, which is incredibly useful.

The second calculational tool is the K-local duality theory of finite groupoids developed in [36]. This shows that the K-localisation of BG has a natural Frobenius object structure. This puts considerable restrictions on the ring structures of E^0BG and K^0BG. In particular, this enables us to construct K-local transfers for all maps of finite groupoids.

As well as these tools, it has borne fruit to consider the associated affine formal scheme $\mathrm{spf}(E^0BG)$. From a functorial perspective, this often turns out to be a moduli scheme that classifies objects related to the formal group $\mathbb{G}$ in a manner that reflects how BG classifies principal G-bundles.

0.2 Outline of the Problem

In this thesis, we will study the Morava cohomology of the groups $GL_d(\mathbb{F})$ when $\mathbb{F}$ is a finite field of order $q \equiv 1 \mod p$, where p is the prime number associated with E and K. Work has already been done to study this, particularly by

Tanabe [40], and Marsh [24]. In [40], Tanabe constructs an Eilenberg-Moore spectral sequence from the cohomological Lang fibre square to relate the Morava K-theory of $BG(\mathbb{F})$ to the Morava K-theory of $BG(\bar{\mathbb{F}})$, where G is a Chevalley group and $\bar{\mathbb{F}}$ is an algebraic closure of $\mathbb{F}$. In particular, he shows that

$$K^*BGL_d(\mathbb{F}) = K^*BGL_d(\bar{\mathbb{F}})_\Gamma$$

where Γ is the dense subgroup of $Gal(\bar{\mathbb{F}}, \mathbb{F})$ generated by the Frobenius automorphism. This isomorphism was lifted to Morava E-theory in [24].

This offers a nice description, but there is more structure to be understood. In particular, the group $GL_d(\mathbb{F})$ is equivalent to the groupoid $\mathcal{V}_d$ of d-dimensional $\mathbb{F}$-vector spaces. If we consider the graded groupoid

$$\mathcal{V}_* = \bigoplus_{d \geq 0} \mathcal{V}_d$$

then the direct sum functor of vector spaces induces a monoid structure on the graded space $B\mathcal{V}_*$. By taking the corresponding restriction and transfer maps, we get commutative graded algebra and cocommutative graded coalgebra structures on $E^0B\mathcal{V}_*$ and $K^0B\mathcal{V}_*$. The main aim of this thesis is to understand what these structures look like.

In some sense, the symmetric groups may be morally thought of as general linear groups over a field of order one (which does not exist). Using this perspective, we might hope to gain some insight into the finite general linear case by looking at Strickland's computation in [34]. The symmetric group Σ_d is equivalent to the groupoid of sets of size d. The disjoint union offers structure analogous to the direct sum and in this situation, the induced commutative algebra and cocommutative coalgebra structures fit together to form a Hopf algebra. This enables Strickland to utilise a theorem of Borel and deduce that these structures are free (polynomial) and cofree respectively.

As expected, things behave similarly in the finite general linear case, so one might hope to emulate the symmetric groups argument. We find that all of the axioms of a Hopf algebra are satisfied, *except* the compatibility relation between the product and coproduct. In the symmetric groups case, this relation is proved by applying the Mackey property of transfers to a certain homotopy pullback square of groupoids. The argument that the square is a homotopy pullback square depends crucially on the fact that the set of subsets of a set forms a distributive lattice.

In the general linear case, the lattice of subspaces of a vector space is not distributive (except in trivial cases). This stops the analogous diagram from being a homotopy pullback square and prevents us from deducing the compatibility relation. In fact, by looking at the character theory of $\mathcal{V}_*$, one can show that this relation cannot possibly hold. This shows that an alternative path is required if we want to prove that the algebra and coalgebra structures are free.

We will also seek to understand the indecomposables and primitives of the algebra and coalgebra structures respectively. Once again, we draw inspiration from the symmetric groups case. There, the cup product induces an algebra structure on the indecomposables, and the primitives form a free module over the indecomposables of rank one. An essential part of the argument for identifying a generator for the primitives requires the determination of the nilpotence of a certain Euler class in $K^*B\Sigma_{p^m}$. This is done by considering the nilpotence of its image in K^*BW_m, where W_m is the Sylow p-subgroup of Σ_{p^m} - a wreath product of cyclic groups of order p. The indecomposables are then determined using algebro-geometric techniques in conjunction with rank calculations deduced from character theory.

In [24], Marsh determined the character theory of $\mathcal{V}_*$ and successfully employed a similar approach to determine the primitives of $E^0B\mathcal{V}_*$ for $* \leq p$. However, the Sylow p-subgroups of $GL_d(\mathbb{F})$ are more complicated than the symmetric groups case, and the characteristic classes involved seem to have more complicated relations. This makes it difficult to pin down exactly how the methods used in [24] can be generalised to determine the primitives for $* > p$.

0.3 Summary

The main results of this thesis can be summarised as follows.

Define maps of graded groupoids

$$\sigma : \mathcal{V}_*^2 \to \mathcal{V}_*$$

$$(U, W) \mapsto U \oplus W$$

$$\eta : * \to \mathcal{V}_*$$

$$* \mapsto 0$$

where $*$ is the trivial groupoid considered as a graded object concentrated in degree 0.

Let p denote the associated prime and n denote the chromatic level (height) of E and K. We assume that $\mathbb{F}$ is a finite field of order q such that $q \equiv 1 \mod p$, and write $\bar{\mathbb{F}}$ for a fixed choice of algebraic closure of $\mathbb{F}$. Define $v := v_p(q-1)$ to be the p-adic valuation of $q-1$.

We define positive integers

$$N_0 := p^{nv}$$

$$N_k := p^{n(k+v)-k} - p^{n(k+v-1)-k}$$

for $k > 0$.

Theorem 1. *$E^0 B\mathcal{V}_*$ and $K^0 B\mathcal{V}_*$ are polynomial algebras under the algebra structure $(\sigma_!, \eta_!)$ generated over E^0 and K^0 respectively by the set*

$$\{s_k^i \mid 0 \leq k, 0 \leq i < N_k\}$$

where $\sigma_!$ and $\eta_!$ are transfers of σ and η, and $s_k := c_{p^k}$ is the top normalised $\bar{\mathbb{F}}$-Chern class of the canonical $\bar{\mathbb{F}}$-linear representation of $\mathcal{V}_{p^k}$ (see 1.6.10).

This is proved in chapters 10 and 11.

Theorem 2. *The indecomposables of $E^0 B\mathcal{V}_*$ and $K^0 B\mathcal{V}_*$ under $\sigma_!$ are concentrated in pth power degrees. For $k \geq 0$, we have*

$$QE^0 B\mathcal{V}_{p^k} = E^0[s_k]/h_k(s_k)$$

$$QK^0 B\mathcal{V}_{p^k} = K^0[s_k]/s_k^{N_k}$$

where $h_k(s_k)$ is a Weierstrass polynomial of degree N_k.

This is proved in chapter 11.

Define

$$\bar{N}_k := \sum_{i=0}^{k} N_i.$$

Theorem 3. *The primitives of $E^0 B\mathcal{V}_*$ and $K^0 B\mathcal{V}_*$ under σ^* form free homogeneous modules of rank 1 over their respective indecomposables, such that*

$$PK^0 B\mathcal{V}_* = K^0 \otimes_{E^0} PE^0 B\mathcal{V}_*.$$

Moreover, for $k \geq 0$, $PK^0 B\mathcal{V}_{p^k}$ is generated over $QK^0 B\mathcal{V}_{p^k}$ by $s_k^{\bar{N}_{k-1}}$.

This is proved in chapter 12.

Along the way, we also show that $E^0 B\mathcal{V}_*$ and $K^0 B\mathcal{V}_*$ each have what we call a "graded faux Hopf ring" structure. This is similar to the notion of a graded Hopf ring, but with a modified version of the compatibility relation, which is controlled by a distinguished element w in the 4-fold tensor product. At one point, we suspected that this would enable us to apply an adapted version of the theorem of Borel on Hopf algebras, and deduce the polynomial algebra structure from theorem 1. However, attempts at this failed and it appears that the graded faux Hopf ring structure doesn't provide enough information for such an approach on its own.

We now give an outline of the thesis.

In chapter 1, we start by reviewing the basic theory of complex orientable cohomology, and the relation between formal group laws and complex cobordism. We then proceed to define what we mean by Morava E-theory and K-theory, and recall some basic facts about the Morava cohomology of spaces and spectra. In the second half of the chapter we look at some relevant computations of the Morava cohomology of groups. In particular, we review the case of cyclic groups, and Tanabe's work in [40] on the case of finite general linear groups.

In chapter 2, we review the homotopy theory of groupoids. In particular, we focus on the interaction between functors of groupoids and homotopy theory, and we recall the standard properties of stable transfers. In the last section, we introduce the notion of a "regular p-local equivalence" of groupoids, which gives a class of particularly well-behaved p-local equivalences and will be useful in chapters 7 and 8.

In chapter 3, we note that the groups $GL_d(\mathbb{F})$ are equivalent to the groupoids $\mathcal{V}_d$ of d-dimensional vector spaces over $\mathbb{F}$ and recast the discussion in terms of the graded groupoid $\mathcal{V}_*$. We study the structure on $\mathcal{V}_*$ and its stabilisation induced by the diagonal map δ, the direct sum map σ, and its transfer $\sigma^!$ to conclude that $\mathcal{V}_*$ behaves just like a graded Hopf cosemiring object, with the exception that the compatibility relation between σ and $\sigma^!$ fails to commute. After this, we translate these results to Morava cohomology and then study the cohomological effects of extending the base field $\mathbb{F}$.

In chapter 4, we review some basic theory in formal algebraic geometry and study how it can be applied to topology. We take a fairly simplistic perspective using complete Noetherian semi-local rings. We begin with some basic definitions and observations. We then take a detour to give a brief description of how Galois theory can be applied to integrally closed domains, which will be useful later on for determining the function rings of certain formal schemes. After this, we look at some formal schemes that arise from studying the Morava cohomology of groups, focusing mainly on examples relevant to the finite general linear case.

In chapter 5, we recall the basic statement of generalised character theory, and then identify the character theory of $\mathcal{V}_*$, essentially following the proof of Marsh in [24]. This turns out to be the semiring of $\mathbb{F}$-linear representations of a group Θ^*, which in particular is a commutative monoid on the set of irreducibles.

In chapter 6, we discuss the theory of K-local duality for finite groupoids developed in [37]. First of all, we review some standard theory about Frobenius algebras, and look at their relationship to Gorenstein local rings. We then recall the main statements in the theory of K-local duality for finite groupoids, and apply them to the case of $\mathcal{V}_*$, before providing a discussion of the properties of K-local transfers in a fashion analogous to the discussion of stable transfers in chapter 2. We conclude the chapter by recalling the interaction between K-local duality and character theory, and then applying it to the case of $\mathcal{V}_*$. By the end of this chapter, the statements of theorem 1 and theorem 2 seem reasonable.

In chapter 7, we study a different product and coproduct on the K-localisation of $\Sigma^\infty_+ B\mathcal{V}_*$, which we denote $L\mathcal{V}_*$. We consider the graded groupoid of short exact sequences of $\mathbb{F}$-vector spaces and define the Harish-Chandra product and coproduct on $L\mathcal{V}_*$ as an alternative to the direct sum product and coproduct. This has a construction reminiscent of parabolic induction. We show that this is actually just a twisted version of the direct sum product and coproduct structure, and in particular, provides a monoid and comonoid structure on $L\mathcal{V}_*$.

In chapter 8, we define graded groupoids of perfect and hollow square diagrams of $\mathbb{F}$. We show that these graded groupoids are p-locally equivalent to $\mathcal{V}_*^4$ and fit into a homotopy pullback square. By applying the Mackey property to this square, we deduce a p-local compatibility diagram between σ and $\sigma^!$ that is very similar to the ordinary compatibility diagram for a Hopf bimonoid, but with an extra twisting element. In the last two sections, we compute the twisting element modulo the maximal ideal, and in character theory, and deduce some of its properties.

In chapter 9, we amalgamate all of the structure so far identified on $E^0B\mathcal{V}_*$ and $K^0B\mathcal{V}_*$ and abstract it to give the definition of a "faux Hopf ring". This is similar to a Hopf ring, but with the compatibility relation between the addition algebra structure and diagonal coalgebra structure replaced with a modified compatibility relation to accommodate the work of chapter 8. We then proceed to construct some basic theory of faux Hopf rings and their "modules". In particular, we observe that K-local duality implies that $E^0B\mathcal{V}_*$ and $K^0B\mathcal{V}_*$ are self-dual as modules over themselves.

In chapter 10, we calculate the differentials in the Atiyah-Hirzebruch spectral sequences for $K_*B\mathcal{V}_*$ and $K^*B\mathcal{V}_*$, up to unit multiples. The E_2-pages are given by Quillen's results on the mod p (co)homology of $\mathcal{V}_*$, and to compute the differentials, we show that the homological spectral sequence is multiplicative with respect to the direct sum algebra structure, as well as making use of the cohomological multiplicativity with respect to the cup product. We conclude that the homological E^∞-page is a polynomial algebra with respect to the direct sum algebra structure, and then use this to deduce that the commutative algebra and cocommutative coalgebra structures of $K_0B\mathcal{V}_*$, $K^0B\mathcal{V}_*$, $E_0^\vee B\mathcal{V}_*$, and $E^0B\mathcal{V}_*$ are free and cofree respectively. We also use the spectral sequence to determine a lower bound for the nilpotence of $s_k = c_{p^k}$ in $K^0B\mathcal{V}_{p^k}$.

In chapter 11, we begin by using the results from the previous chapter to determine some restrictions on the form of the indecomposables and primitives. Then, motivated by the character theory of $B\mathcal{V}_*$ in chapter 5, we use the results from chapters 9 and 10, as well as the Galois theoretic and moduli-theoretic techniques from chapter 4 to determine the form of the indecomposables of $E^0B\mathcal{V}_*$ and $K^0B\mathcal{V}_*$. This also enables us to give a canonical set of generators for the algebra structures, and thus prove theorem 1 and theorem 2.

In chapter 12, we apply the theory from chapter 6 to show that the rings of indecomposables in each degree are Frobenius algebras. We use this result, faux Hopf ring theory and K-local duality to show that the primitives of $E^0B\mathcal{V}_*$ and $K^0B\mathcal{V}_*$ form free modules of rank 1 over the respective indecomposables. We then use Frobenius algebra theory to determine a criterion for a primitive element to be a generator for the primitives over the indecomposables in $K^0B\mathcal{V}_*$. Finally, we implement this criterion in conjunction with the lower bound on the nilpotence of s_k deduced in chapter 10 to inductively determine a generator in $K^0B\mathcal{V}_*$ and prove theorem 3.

In the appendix, we collect a variety of standard results from commutative algebra that we will find useful at various stages of this thesis.

Remark 0.3.1. Although we have done our best to avoid the problem, inevitably some symbols have been used to notate more than one object in this document. We hope that, given the context, the meaning is always clear.

Chapter 1

Morava Cohomology

In this chapter, we recall some basic definitions and properties of the cohomology theories and groups that we will be working with. We also review some computations that will be relevant to us, including the work of Tanabe on the case of Chevalley groups, which we apply in the general linear case.

In the first section, we recall the fundamental results of complex orientable cohomology theory as found in [1] and [31]. In particular, we recall the relation between formal group laws and complex cobordism, the Landweber exact functor theorem, and show that even-periodic cohomology theories are complex orientable.

In the second section, we define what we mean by Morava E-theory and Morava K-theory, and in the third section, we give some basic facts about the Morava cohomology of spaces and spectra, mainly following [18].

In the fourth section, we recall the standard calculations of E^*BA and K^*BA for A a finite cyclic group, using Gysin sequences. We also give a general statement about the ring structure of E^0BG and K^0BG for G a finite group.

In the fifth section, we make precise the fields and groups that we will be working with. In the last two sections, we review the calculations of the Morava K-theory of $BGL_d(\bar{\mathbb{F}})$ and $BGL_d(\mathbb{F})$ as studied in [40]. The computation over $\bar{\mathbb{F}}$ makes use of the ordinary homology as given in [14], and the computation over $\mathbb{F}$ makes use of an Eilenberg-Moore spectral sequence. We also use standard techniques, as in [24], to lift these results to Morava E-theory.

1.1 Complex Orientable Cohomology

In this section, we recall the fundamental results of complex orientable cohomology theory as found in [1] and [31].

In ordinary cohomology we have a notion of Chern classes for complex vector bundles. These carry important information about the twisting of vector bundles, and facilitate a lot of calculations in topology. The initial aim of complex oriented cohomology theory is to find different cohomology theories that also have a good theory of Chern classes. The theory of classifying spaces shows that Chern classes in ordinary cohomology are pullbacks of universal Chern classes constructed in the following calculations ([26, p. 199])

$$H^*(BU(d)) \cong \mathbb{Z}[c_1, \ldots, c_d] = \mathbb{Z}[x_1, \ldots, x_d]^{\Sigma_d} = H^*(BU(1)^d)^{\Sigma_d}$$

where $U(1)^d \subset U(d)$ is a maximal torus, c_i is the ith symmetric polynomial in $\{x_j\}$, and we have chosen a degree 2 generator x to give a presentation

$$H^* BU(1) = \mathbb{Z}[x].$$

It turns out that we only need a small assumption in order to translate these calculations to a generalised cohomology theory.

Recall that we have a canonical inclusion map

$$i : S^2 \simeq \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^\infty \simeq BS^1 \simeq BU(1).$$

Definition 1.1.1. A commutative multiplicative cohomology theory (or commutative ring spectrum) E is called *complex orientable* if the map

$$i^* : \tilde{E}^2 BS^1 \to \tilde{E}^2(S^2) \cong E^0$$

is surjective. We define a *complex orientation* to be an element x in the preimage of 1.

This definition is justified by the following result.

Theorem 1.1.2. *If E is a cohomology theory with a complex orientation x, then for $m \geq 1$, we have*

(1)
$$E^* \mathbb{C}P^n = E^*[x]/(x^n).$$

(2)
$$E^* \mathbb{C}P^\infty = E^* BU(1) = E^* BS^1 = E^*[\![x]\!].$$

(3)
$$E^*(BU(1)^m) = E^*B(S^1)^m = E^*[\![x_1, \dots, x_m]\!].$$

(4)
$$E^*BGL_m(\mathbb{C}) = E^*BU(m) = E^*[\![c_1, \dots, c_m]\!] = (E^*BU(1)^m)^{\Sigma_m}.$$

(5)
$$E^*BU = E^*[\![c_1, c_2, \dots]\!].$$

where c_i is the symmetric polynomial in $\{x_j\}$.

Proof. This can be found in [1, Part II, Lemma 2.5, Lemma 4.3] or [31, Lemma 4.1.4].

(1), (2), (3) follow from Atiyah-Hirzebruch spectral sequences that collapse at the E_2-page. (4) and (5) follow by computing the E-homology using an Atiyah-Hirzebruch spectral sequence (in particular, the comultiplication under the diagonal), and then dualising.

□

Examples: (1) K-theory is complex orientable ([1, Part II, Example 2.3]).

(2) Complex cobordism is naturally complex *oriented* via the canonical map ([1, Part II, 2.4] or [31, Example 4.1.4])

$$x_{MU} : BU(1) \simeq MU(1) \to MU.$$

If we have a map of commutative multiplicative cohomology theories/ring spectra $E_1 \xrightarrow{f} E_2$, then by naturality, if x_1 is a complex orientation on E_1, then $f_*(x_1)$ is a complex orientation on E_2. Thus we can make the following definition.

Definition 1.1.3. Define a functor

$$\mathrm{CxOr} : \mathrm{CRingSp} \to \mathrm{Sets}$$

that sends a commutative ring spectrum E to the set of complex orientations on E.

Theorem 1.1.4. *If x_{MU} is the canonical complex orientation of complex cobordism, then as a homotopy ring spectrum, MU represents the functor* CxOr*, i.e. we have a natural isomorphism*

$$\mathrm{HoRingSp}(MU, E) \xrightarrow{\sim} \mathrm{CxOr}(E)$$

$$f \mapsto f_* \circ x_{MU}$$

where the left hand side denotes the set of homotopy classes of homotopy ring spectra maps.

Proof. This can be found in [1, Part II, Lemma 4.6] or [31, Lemma 4.1.13]. □

Remark 1.1.5. In other words, the pair (MU, x_{MU}) is the universal complex oriented commutative ring spectrum.

Definition 1.1.6. For a ring R, a *formal group law* with coefficients in R is defined to be a formal power series

$$F(x_1, x_2) = x_1 +_F x_2 = \sum_{i,j \geq 0} a_{i,j} x_1^i x_2^j \in R[\![x_1, x_2]\!]$$

such that

$$F(x, 0) = F(0, x) = x$$

$$F(x_1, x_2) = F(x_2, x_1)$$

$$F(x_1, F(x_2, x_3)) = F(F(x_1, x_2), x_3).$$

If we have a map of commutative rings $R_1 \xrightarrow{f} R_2$, then it is easy to see that if

$$F(x_1, x_2) = \sum_{i,j \geq 0} a_{i,j} x_1^i x_2^j$$

is a formal group law over R_1, then

$$(f_* F)(x_1, x_2) := \sum_{i,j \geq 0} f(a_{i,j}) x_1^i x_2^j$$

is a formal group law over R_2. Thus we can make the following definition.

Definition 1.1.7. Define a functor

$$\mathrm{FGL} : \mathrm{Rings} \to \mathrm{Sets}$$

that sends a commutative ring R to the set of formal group laws over R.

Theorem 1.1.8. *There exists a ring L (the Lazard ring) that represents the functor* FGL*, i.e. there are natural isomorphisms*

$$\mathrm{Rings}(L, R) \xrightarrow{\sim} \mathrm{FGL}(R).$$

Moreover, L is a polynomial ring

$$L = \mathbb{Z}[t_1, t_2, \dots]$$

with a natural grading such that $\dim(t_i) = 2$.

Proof. This can be found in [1, Part II, Theorem 5.1, Theorem 7.1] or [31, Theorem A2.1.10].

□

As S^1 is a commutative topological group, BS^1 inherits an H-space structure. We write

$$\mu : BS^1 \times BS^1 \to BS^1$$

for the multiplication.

Lemma 1.1.9. *If x is a complex orientation on a cohomology theory E, then the image of x under the map*

$$\mu^* : E^* BS^1 \to E^*(BS^1 \times BS^1)$$

is a formal group law $x_1 +_F x_2$ over E^*.

Proof. This can be found in [1, Part II, Lemma 2.7] or [31, Lemma 4.1.4]. This follows by functoriality from the associativity, commutativity and unit of m.

□

If x is a complex orientation for a commutative ring spectrum E, then its associated formal group law induces a graded ring map

$$L \to E_*.$$

The universality of MU in 1.1.4 inspires the next theorem.

Theorem 1.1.10 (Quillen)**.** *The map classifying the formal group law of the complex orientation x_{MU}*

$$L \to MU_*$$

is an isomorphism of graded rings and thus the graded ring MU_* represents the functor FGL. In particular

$$MU_* = \mathbb{Z}[t_1, t_2, \dots]$$

where $deg(t_i) = 2i$.

Proof. This can be found in [1, Part II, Theorem 8.1, Theorem 8.2] or [31, Theorem 4.1.6].

□

Remark 1.1.11. (1) This relationship extends to include an identification of MU-cooperations MU_*MU with the set of strict isomorphisms between graded formal group laws. The corresponding affine scheme can be seen to act on the set of formal group laws [31, Theorem 4.1.11].

(2) There is extensive theory exploiting this relationship between complex cobordism and formal group laws to inform our understanding of global stable homotopy theory.

One question we can now ask is whether, given a graded ring map $MU_* \to R_*$, and thus a formal group law F over R_*, can we construct a (co)homology theory that has F as its associated formal group law. The obvious way to construct such a homology theory is to define

$$R_*(X) := MU_*(X) \otimes_{MU_*} R_*.$$

By checking, one finds that the only obstruction to this being a homology theory is the preservation of long exact sequences when tensoring. The next result is called the Landweber exact functor theorem, and it provides a criterion for sufficient flatness.

Definition 1.1.12. For a formal group law F and an integer m, we define the *m-series* to be the mth iteration

$$[m](x) := x +_F x +_F \cdots +_F x.$$

This is clearly divisible by x and we define the *reduced m-series* to be

$$\langle m \rangle(x) := [m](x)/x.$$

Definition 1.1.13. For a formal group law F and a prime p, we write v_i for the coefficient of x^{p^i} in the p-series $[p](x)$.

Theorem 1.1.14 (Landweber)**.** *Let M_* be a graded module over MU_* corresponding to a graded formal group law F. Then $MU_*(X) \otimes_{MU_*} M_*$ is a homology theory if and only if, for every prime number p, the sequence $(v_0 = p, v_1, v_2, \dots) \in MU_*$ is a regular sequence for M_*.*

Proof. This is proved in [23, Theorem 2.6].

□

Remark 1.1.15. The proof centres around the observation that an equivariant form of flatness suffices, where the action comes from the MU-cooperations.

Definition 1.1.16. We say that a ring spectrum E is *even-periodic* if E^* is concentrated in even degrees, and there exists an invertible element $u \in E^2$.

Lemma 1.1.17. *If E is an even-periodic ring spectrum with periodic element u, then E is complex orientable. Moreover, $E^1(BS^1) = 0$ and if y is a complex orientation, and $x := u^{-1}y$, then $E^0(BS^1) = E^0[\![x]\!]$.*

Proof. This is standard. We have an Atiyah-Hirzebruch spectral sequence

$$H^*(BS^1; E^*) \Rightarrow E^* BS^1.$$

This collapses at the E_2-page because it is concentrated in even bidegree. Thus, as the map

$$i^* : H^*(BS^1; E^*) \to H^*(S^2; E^*)$$

is surjective, so is the map

$$i^* : E^* BS^1 \to E^*(S^2)$$

by A.0.19 and thus also the map on reduced cohomology.

□

Definition 1.1.18. If E is an even-periodic ring spectrum, then we define a *normalised complex orientation* to be an element $x \in E^0(BS^1)$ such that $E^0 BS^1 = E^0[\![x]\!]$.

In particular, if u is a periodic element and y is a complex orientation, then $x := u^{-1}y$ is a *normalised complex orientation.*

Remark 1.1.19. If E is even-periodic and complex oriented and x is a normalised complex orientation, then we get a map

$$E^0 BS^1 \xrightarrow{\mu^*} E^0 BS^1 \hat{\otimes} E^0 BS^1$$

$$x \mapsto F_0(x_1, x_2) = x_1 +_{F_0} x_2$$

where $\hat{\otimes}$ denotes the completed tensor product with respect to the (x)-adic topology. Then F_0 is a formal group law over E^0. This is a normalised version of the formal group law F over E^*.

1.2 Morava K and E-Theory

In this section, we define what we mean by Morava E-theory and Morava K-theory.

From this point forward, p is an odd prime and n is a non-negative integer.

Remark 1.2.1. We omit the case $p = 2$ for convenience due to the technical difficulties relating to the lack of commutativity of K in that case.

Definition 1.2.2. For a formal group law F over a ring R, we say that the pair (R, F) is *admissible* of type p, n if R is a complete local Noetherian ring with characteristic p residue field, such that the reduction F mod $\mathfrak{m}_R$ has *height* n. In other words, the coefficients of the p-series

$$[p](x) = \sum_{i=1}^{\infty} u_i x^i$$

satisfy $p = u_1, \ldots, u_{p^n-1} \in \mathfrak{m}_R$ and $u_{p^n} \notin \mathfrak{m}_R$.

We say that an admissible pair has the *universality condition* if $p = u_1, u_p \ldots, u_{p^{n-1}}$ form a regular sequence in R, generating $\mathfrak{m}_R$.

Remark 1.2.3. In particular, if (R, F) is an admissible pair of type p, n with the universality condition, then R is a regular local ring such that $\dim(R) = n$.

Definition 1.2.4. For a ring R, we write $\sim$ for the equivalence relation on R given by: $a \sim b$ iff a is a unit multiple of b.

Remark 1.2.5. If (R, F) is an admissible pair, then $[p](x)$ has Weierstrass degree p^n (A.0.2) and so by Weierstrass preparation (A.0.3) there exists a unique polynomial $f(x) \in R[x]$ of degree p^n such that $[p](x) \sim f(x)$ in $R[\![x]\!]$.

Definition 1.2.6. If E is an even-periodic commutative ring spectrum, we say that E is *admissible* of type p, n if there exists a normalised complex orientation such that the corresponding formal group law (E^0, F_0) is admissible of type p, n, where F_0 is the formal group law from 1.1.19.

We say that an admissible spectrum E has the *universality condition* if (E^0, F_0) has the universality condition.

Remark 1.2.7. This definition is essentially equivalent to the definition of Morava E-theory in terms of Lubin-Tate universal deformations.

Theorem 1.2.8. *For every pair p, n, there exists an admissible spectrum E with the universality condition. Moreover, the multiplication of such spectra E can be lifted to an E_∞-structure in an essentially unique way.*

Proof. By Lubin-Tate theory (e.g. [37, Section 6]), admissible pairs (R, F) with the universality condition exist in the form of Lubin-Tate universal deformations. The formal group law defines a map

$$MU_* \to E_*.$$

Then to construct the homology theory, we define

$$E_*(X) := MU_*(X) \otimes_{MU_*} E_*.$$

By assumption, the sequence $(p, u_p, u_{p^2}, \dots,)$ is regular on E_*, and thus satisfies Landweber's criterion 1.1.14.

The existence of an E_∞-structure on E is proved in [15, Corollary 7.6].

□

Theorem 1.2.9. *For an admissible spectrum E with the universality condition, for any $\underline{i} := (i_0, \dots, i_{n-1}) \in \mathbb{N}^n$, there exists a ring spectrum*

$$E_{\underline{i}} := E/(u_1^{i_0}, u_p^{i_1} \dots, u_{p^{n-1}}^{i_{n-1}})$$

with quotient map $E \to E_{\underline{i}}$ and coefficient ring

$$E^*_{\underline{i}} = E^*/(u_1^{i_0}, u_p^{i_1} \dots, u_{p^{n-1}}^{i_{n-1}}).$$

Proof. The sequences $(u_1^{i_0}, u_p^{i_1} \dots, u_{p^{n-1}}^{i_{n-1}})$ are regular, so the proposed coefficient rings are regular quotients. By the theory of E-modules in [35], such rings are "strongly realisable" by ring spectra.

□

Definition 1.2.10. For an admissible spectrum E with the universality condition, we define the corresponding K-theory to be the spectrum

$$K := E_{(1,\dots,1)}$$

whose coefficient ring is the field $K^* = E^*/\mathfrak{m}_{E^*}$.

This is essentially a 2-periodic version of "Morava K-theory". We write F_0 for the corresponding formal group law.

Remark 1.2.11. The induced formal group law on K has a p-series of the form

$$[p](x) = \sum_{i=p^n}^{\infty} u_i x^i$$

with $u_{p^n} \neq 0$. It is straightforward to see that $[p](x)$ is a unit multiple of x^{p^n} in $K^*[\![x]\!]$. Notationally,

$$[p](x) \sim x^{p^n}.$$

Lemma 1.2.12. *If E is an admissible spectrum with the universality condition, then E is K-local.*

Proof. This follows from [18, Lemma 5.2]. $\square$

From now on, we fix a choice of p, n and an admissible spectrum E with the universality condition.

1.3 Morava Cohomology of Spaces and Spectra

In this section, we give some basic facts about the Morava cohomology of spaces and spectra, mainly following [18].

Definition 1.3.1. For a spectrum X, we write $L_K X$ for the Bousfield localisation of X with respect to K.

Theorem 1.3.2. *For X a space/spectrum*

(1) If X is a finite CW-complex, then E^*X and K^*X are finitely generated over E^* and K^* respectively.

(2) $E^*L_KX = E^*X$ and $K^*L_KX = K^*X$.

(3) E^*X is finitely generated over E^* iff K^*X is finite dimensional over K^*.

(4) If E^*X is free and finitely generated over E^*, then

$$K^*X = K^* \otimes_{E^*} E^*X = E^*X/(\mathfrak{m}_{E^*}E^*X).$$

(5) If E^*X is finitely generated over E^* and K^*X is concentrated in even degrees, then E^*X is free over E^* and concentrated in even degrees.

Proof. (1) At each dimension, we have cofibre sequences describing the attachment of the cells at that dimension. This then follows easily by induction on cell dimension via the corresponding long exact sequences.

(2) K is a K-module spectrum, and thus K-local by [1, Lemma 13.1] or [32, Lemma 7.1.5]. E is K-local by [18, Lemma 5.2]. The statements then follow easily from the definition of Bousfield localisation.

(3) [18, Proposition 2.4].

(4) and (5) are essentially the finitely generated version of [18, Proposition 2.5].

□

Remark 1.3.3. If K^*X is finite dimensional and concentrated in even degrees, then $K^*X = K^* \otimes_{K^0} K^0X$ and $E^*X = E^* \otimes_{E^0} E^0X$ so it suffices to study K^0X and E^0X.

Corollary 1.3.4. *If we have a map of spaces/spectra $X \xrightarrow{f} Y$, then K^*f is an isomorphism iff E^*f is an isomorphism.*

Proof. The cofibre sequence/distinguished triangle of f induces long exact sequences in cohomology for $R = E, K$.

$$\cdots \to R^*C(f) \to R^*X \xrightarrow{f^*} R^*Y \to R^{*+1}C(f) \to \cdots .$$

Thus f^* is an isomorphism iff $R^*C(f) = 0$, so the problem is reduced to proving that $K^*X = 0$ iff $E^*X = 0$, i.e. the case where f is an initial map.

If $E^*X = 0$, then in particular, it is finitely generated and free over E^*, so by 1.3.2, $K^*X = K^* \otimes_{E^*} E^*X = 0$. Conversely, if $K^*X = 0$, then by 1.3.2, E^*X is finitely generated, free, and concentrated in even degrees. Using 1.3.2 again, we have that

$$E^*X/\mathfrak{m}_{E^*} = K^*X = 0.$$

By Nakayama's lemma/A.0.9, it follows that $E^*X = 0$.

□

Lemma 1.3.5. *If X is a CW-complex with finitely many cells in each dimension, then*

$$E^0X = \lim E^0X/I_k = \lim E^0(X_k)$$

$$K^0X = \lim K^0X/I_k = \lim K^0(X_k)$$

where X_k *is the* k*-skeleton of* X*, and for* $R = E, K$

$$I_k := Ker(R^0 X \to R^0 X_{k-1}).$$

Proof. By [4, Theorem 4.3], we have a Milnor exact sequence

$$0 \to \lim{}^1 E^{-1}(X_k) \to E^0 X \to \lim E^0 X_k \to 0.$$

By assumption, X_k is a finite complex, and so by 1.3.2, $E^{-1} X_k$ is finitely generated over E^0 and thus linearly compact with respect to the $\mathfrak{m}_{E^0}$-adic topology. By [20, Théorème 7.1], inverse limits of filtered diagrams of linearly compact modules are exact, so $\lim^1 E^{-1}(X_k) = 0$ and we have an isomorphism

$$E^0 X \to \lim E^0 X_k.$$

We have maps

$$E^0 X \twoheadrightarrow E^0 X / I_k \rightarrowtail E^0 X_{k-1}$$

which fit together to form maps of filtered diagrams, and so the above isomorphism factors as

$$E^0 X \to \lim E^0 X / I_k \to \lim E^0 X_k.$$

As the composition is an isomorphism, the right hand map must be surjective. However, as inverse limits are left exact, the right hand map must also be injective, and thus an isomorphism. Thus, the left hand map must also be an isomorphism.

The argument for Morava K-theory is similar.

□

Recall that a *semi-local ring* is a ring with a finite number of maximal ideals.

Proposition 1.3.6. *If* X *is a connected CW-complex with finitely many cells in each dimension, then* $E^0 X$ *and* $K^0 X$ *are local rings, and maps of spaces induce continuous maps on* K*-theory and* E*-theory with respect to the* $\mathfrak{m}$*-adic topology, where* $\mathfrak{m}$ *is the maximal ideal.*

Moreover, if $E^0 X$ *is finitely generated over* E^0 *(or equivalently,* $K^0 X$ *is finite dimensional over* K^0*), then* $E^0 X$ *and* $K^0 X$ *are complete Noetherian semi-local rings.*

Proof. As $E^*(X \amalg Y) = E^*X \times E^*Y$ and $K^*(X \amalg Y) = K^*X \times K^*Y$, in order to prove that E^*X and K^*X are semi-local, it suffices to prove that for X connected, E^*X and K^*X are local rings.

Assuming that X is connected, let

$$i : * \to X$$

$$\pi : X \to *$$

be the inclusion of and projection to a point respectively, and let $\mathfrak{m}$ be the kernel of the map

$$\epsilon : E^*X \xrightarrow{i^*} E^* \to K^*.$$

As K^* is a field, $\mathfrak{m}$ is maximal. We also see that $Ker(i^*) = I_1$ (using the notation from 1.3.5). To see that E^*X is local, it suffices to show that if $\epsilon(x) \neq 0$, then x is invertible. Assuming $\epsilon(x) \neq 0$, then $u := i^*(x)$ is invertible in E^* because E^* is local with residue field K^*. If we let $t := 1 - \pi^*(u)^{-1}x$, then we see that $t \in I_1$.

By [39, Proposition 13.67], the skeletal filtration is multiplicative, i.e. $I_j I_k \subseteq I_{j+k}$. In particular, $I_1^k \subseteq I_k$ for all k, so $t^k \in I_k$. By 1.3.5

$$E^*X = \lim E^*X/I_k$$

so the series

$$\bar{t} := 1 + t + t^2 + \dots$$

converges in E^*X, and moreover we have

$$\bar{t}\pi^*(u)^{-1}x = 1$$

so x is invertible. K^*X is local similarly, but in that case $i^* = \epsilon$, which makes the argument easier. Maps are then continuous, because the maximal ideals are defined in terms of the terminal maps i.

□

Proposition 1.3.7. *If X is a CW-complex with finitely many cells in each dimension, such that K^*X is finite dimensional over K^*, then E^0X is a complete Noetherian semi-local ring of dimension n and K^0X is a complete Noetherian semi-local ring of dimension 0 (i.e. an Artinian ring).*

Proof. It suffices to prove this for X connected.

By 1.3.2, E^*X is finitely generated over E^*. As E^* and K^* are complete Noetherian local rings, E^*X and K^*X are also complete Noetherian local rings by 1.3.6 and A.0.12.

The maps $* \to X \to *$ necessarily compose to the identity up to homotopy, so in particular we have split injections

$$E^* \rightarrowtail E^*X$$

$$K^* \rightarrowtail K^*X.$$

The statements about dimension then follow from A.0.13, because the finite generation over E^* and K^* respectively makes these extensions integral.

□

Definition 1.3.8. We define the *covariant E-theory* of a spectrum X to be

$$E_*^\vee X := \pi_*(L_K(E \wedge X)).$$

Remark 1.3.9. $E_*^\vee X$ is the right covariant analogue of E^*X primarily from a duality point of view. However, it is *not* a homology theory because it does not send infinite wedges to coproducts.

Theorem 1.3.10. *For X a space/spectrum*

(1) $E_*^\vee X$ is finitely generated over E_* iff K_*X is finite dimensional over K_*.

(2) If $E_*^\vee X$ is free and finitely generated over E_*, then

$$K_*X = K_* \otimes_{E_*} E_*^\vee X = E_*^\vee X/(\mathfrak{m}_{E_*} E_*^\vee X).$$

(3) If $E_*^\vee X$ is finitely generated over E_* and K_*X is concentrated in even degrees, then $E_*^\vee X$ is free over E_* and concentrated in even degrees.

(4) If X is dualisable in the K-local category, then $K_*DX = K^{-*}X$, and $E_*^\vee DX = E^{-*}X$.

Proof. (1), (2), and (3) come from [18, Proposition 8.4].

(4) Working in the K-local homotopy category with behaviour as in [18, Theorem 7.1], as X is dualisable, $DX \wedge_K K \simeq F_K(X, K)$, so the duality adjunction gives isomorphisms

$$\begin{aligned} K^m X = [X, \Sigma^m K] &= [S^{-m} \wedge_K X, K] = [S^{-m}, F_K(X, K)] \\ &= [S^{-m}, DX \wedge_K K] = K_{-m} DX. \end{aligned}$$

Similarly, we have $DX \wedge_K E \simeq F_K(X, E)$, so the duality adjunction gives isomorphisms

$$\begin{aligned} E^m X = [X, \Sigma^m E] = [S^{-m} \wedge_K X, E] &= [S^{-m}, F_K(X, E)] \\ &= [S^{-m}, DX \wedge_K E] = E^\vee_{-m} DX. \end{aligned}$$

□

Remark 1.3.11. If K_*X is finite dimensional and concentrated in even degrees, then $K_*X = K_* \otimes_{K_0} K_0X$ and $E^\vee_* X = E_* \otimes_{E_0} E^\vee_0 X$.

Theorem 1.3.12. *For a K-local spectrum X, the following are equivalent:*

(1) X is K-locally dualisable.

(2) K_*X is finite dimensional over K_*.

(3) K^*X is finite dimensional over K^*.

(4) $E^\vee_* X$ is finitely generated over E_*.

(5) E^*X is finitely generated over E^*.

Proof. This is proved in [18, Theorem 8.6].

□

Lemma 1.3.13. *We have natural isomorphisms*

$$K^0X = \mathrm{Hom}_{K_0}(K_0X, K_0).$$

Moreover, if K_0X is finite dimensional over K_0, then the K-Kronecker pairing is perfect

$$K^0X \otimes_{K_0} K_0X \to K_0.$$

If $E^\vee_0 X$ is finitely generated and free over E_0, then the E-Kronecker pairing is perfect

$$E^0X \otimes_{E_0} E^\vee_0 X \to E_0.$$

Proof. The K-theory case is true by [18, Proposition 3.4]. When K_0X is finite dimensional, it necessarily has perfect duality.

For the E-theory case, we have a natural map

$$f : E^0X \to \mathrm{Hom}_{E_0}(E^\vee_0 X, E_0).$$

When $E^\vee_0 X$ is finitely generated and free, so is E^0X, and it follows by 1.3.10 and 1.3.2 that $K^0 \otimes_{E^0} f$ is the map

$$K^0X \to \mathrm{Hom}_{K_0}(K_0X, K_0)$$

which we know to be an isomorphism. By A.0.9, f must also be an isomorphism. As $E^\vee_0 X$ is finitely generated and free over E_0, it has perfect duality.

□

The following statement gives us Künneth isomorphisms.

Lemma 1.3.14. *Let X, Y be spectra such that K_*X and K_*Y are concentrated in even degrees, then we have the following:*

(1) We have an isomorphism

$$K_0(X \wedge Y) = K_0X \otimes_{K_0} K_0Y.$$

(2) If X and Y are K-locally dualisable, then we have an isomorphism

$$K^0(X \wedge Y) = K^0X \otimes_{K^0} K^0Y.$$

(3) If $E^\vee_* X$ and $E^\vee_* Y$ are free and finitely generated over E_*, then we have an isomorphism

$$E^\vee_0(X \wedge Y) = E^\vee_0 X \otimes_{E_0} E^\vee_0 Y.$$

(4) If E^*X and E^*Y are free and finitely generated over E^*, then we have an isomorphism

$$E^0(X \wedge Y) = E^0X \otimes_{E^0} E^0Y.$$

Proof. (1) As K_* is a field, K_*X is free (and thus flat) over K_* for all X, so we have Künneth isomorphisms in Morava K-theory by [39, Theorem 13.75]. In particular,

$$K_0(X \wedge Y) = K_0X \otimes_{K_0} K_0Y \oplus K_1X \otimes_{K_0} K_{-1}Y$$

but $K_1X = 0$, so the right hand term disappears.

(2) Apply (1) to DX using $K^0X = K_0DX$ as in 1.3.10.

(3) We have a natural map

$$f : E^\vee_0 X \otimes_{E_0} E^\vee_0 Y \to E^\vee_0(X \wedge Y).$$

By 1.3.10, $K_0 \otimes_{E_0} f$ is equal to the map

$$K_0 X \otimes_{K_0} K_0 Y \to K_0(X \wedge Y)$$

which is an isomorphism by (1). Thus, as X, Y and $X \wedge Y$ have finite-dimensional K-homology concentrated in even degrees, f is a map of free E_0-modules, and thus restricts to an isomorphism

$$E_0^\vee X \otimes_{E_0} E_0^\vee Y = E_0^\vee(X \wedge Y)$$

by A.0.9.

(4) By 1.3.12, X and Y are K-locally dualisable. We have a natural map

$$f : E^0 X \otimes_{E^0} E^0 Y \to E^0(X \wedge Y).$$

By 1.3.2, $K^0 \otimes_{E^0} f$ is equal to the map

$$K^0 X \otimes_{K^0} K^0 Y \to K^0(X \wedge Y)$$

which is an isomorphism by (2). Thus, as X, Y and $X \wedge Y$ have finite-dimensional K-cohomology concentrated in even degrees, f is a map of free E^0-modules, and thus restricts to an isomorphism

$$E^0 X \otimes_{E^0} E^0 Y = E^0(X \wedge Y)$$

by A.0.9.

$\square$

1.4 Morava Cohomology of Groups

In this section, we begin by recalling the standard caclulations of E^*BA and K^*BA for A a finite cyclic group, using Gysin sequences. We then recall a result of Ravenel and deduce its effect on the ring structures of E^0BG and K^0BG for G a finite group.

Lemma 1.4.1. *If A is a cyclic group of order m, and $i : A \rightarrowtail S^1$ is a generator of $A^* = \mathrm{Hom}(A, S^1)$, then we have Gysin short exact sequences*

$$0 \to E^*BS^1 \xrightarrow{\bullet[m](x)} E^*BS^1 \xrightarrow{i^*} E^*BA \to 0$$

$$0 \to K^*BS^1 \xrightarrow{\bullet[m](x)} K^*BS^1 \xrightarrow{i^*} K^*BA \to 0.$$

If we write x for the pullback of a complex orientation along i, then we get

$$E^*BA = E^*[\![x]\!]/[m](x)$$

$$K^*BA = K^*[\![x]\!]/[m](x).$$

Proof. [17, Lemma 5.7].

The map i gives us an exact sequence

$$0 \to A \to S^1 \xrightarrow{\bullet m} S^1 \to 0.$$

We can extend this to give a fibre sequence. In particular, this includes a fibration

$$S^1 \to BA \to BS^1.$$

This gives us a Gysin long exact sequence

$$\cdots \to E^{*-2}BS^1 \xrightarrow{\bullet [m](x)} E^*BS^1 \xrightarrow{i^*} E^*BA \to \cdots.$$

Recall from 1.1.2 that $E^*BS^1 = E^*[\![x]\!]$ for a complex orientation x. E^* is a regular local ring, and thus a domain by 1.2.3, so E^*BS^1 is also a domain. In particular, $[m](x)$ is not a zero-divisor, so the Gysin sequence splits and yields a short exact sequence

$$0 \to E^{*-2}[\![x]\!] \xrightarrow{\bullet [m](x)} E^*[\![x]\!] \to E^*BA \to 0$$

which gives us the result. The same argument works for K-theory as K^* is a field.

∎

Corollary 1.4.2. *If A is a cyclic group of order m such that $v_p(m) = d$, and $A(p^d)$ is the unique cyclic subgroup of order p^d, then the restriction maps*

$$E^*BA \to E^*BA(p^d)$$

$$K^*BA \to K^*BA(p^d)$$

are isomorphisms. Moreover, in E-theory, $[p^d](x)$ is a Weierstrass series of degree p^{dn}, so $E^*BA(p^d)$ is free over E^* of rank p^{dn} and we have

$$K^*BA(p^d) = K^*[x]/x^{p^{dn}}.$$

Proof. If we write $m = rp^d$ for r coprime to p, then in $E^*[\![x]\!]$ and $K^*[\![x]\!]$

$$[r](x) = rx + O(x^2)$$

so $[r](x) \sim x$. It follows that

$$[m](x) = [r]([p^d](x)) \sim [p^d](x)$$

and so the restriction maps are isomorphisms. By 1.2.5, $[p^d](x)$ is a Weierstrass series of degree p^{dn} so $E^*BA(p^d)$ is free over E^* by Weierstrass preparation. It also follows that in K-theory

$$[p^d](x) \sim x^{p^{dn}}$$

from which the last statement follows.

□

Theorem 1.4.3. *For G a finite group, E^*BG and K^*BG are finitely generated as modules over E^* and K^* respectively. Moreover, E^0BG is a complete local Noetherian ring of dimension n and K^0BG is a complete local Noetherian ring of dimension 0 (i.e. an Artinian ring).*

Proof. For the first part, the case for K-theory is proved in [30], but also follows from [17, Theorem B]. The case for E-theory follows by 1.3.2.

For the second part, BG can be given the structure of a connected CW-complex with finitely many cells in each dimension via the geometric realisation of the nerve, so E^*BG and K^*BG are complete local rings of the claimed dimensions by 1.3.7.

□

1.5 General Linear Groups over Finite Fields

In this section, we give definitions of the fields and groups that we will be working with.

From now on, we fix a field $\mathbb{F}$ of order q, where q is a prime power, with prime different from p. In particular, for ease of calculation, we will assume that $q \equiv 1 \mod p$. This is not a huge specialisation, because for any prime power q coprime to p, if a is the order of q in $\mathbb{Z}/p^\times$ then $q^a \equiv 1 \mod p$. We fix notation for the p-adic valuation of $q-1$

$$v := v_p(q-1).$$

We also fix an algebraic closure $\bar{\mathbb{F}}$ of $\mathbb{F}$. Recall that the Galois group $Gal(\bar{\mathbb{F}}, \mathbb{F})$ acts upon $\bar{\mathbb{F}}$ with fixed field $\mathbb{F}$. $Gal(\bar{\mathbb{F}}, \mathbb{F})$ is isomorphic to the profinite completion of the integers, $\hat{\mathbb{Z}}$, and the Frobenius automorphism

$$F_q : \bar{\mathbb{F}} \to \bar{\mathbb{F}}$$

$$a \mapsto a^q$$

generates a dense subgroup $\Gamma \leq Gal(\bar{\mathbb{F}}, \mathbb{F})$ isomorphic to $\mathbb{Z}$. Moreover, $\mathbb{F}$ is the set of fixed points of F_q and so $\mathbb{F} = \bar{\mathbb{F}}^{\Gamma}$. We define the subgroups and corresponding intermediate fields to be

$$\Gamma(m) := m\Gamma = \langle F_q^m \rangle$$

$$\mathbb{F}(m) := \bar{\mathbb{F}}^{\Gamma(m)}.$$

This makes $\mathbb{F}(m)$ into a field of order q^m, and in particular, $\mathbb{F} = \mathbb{F}(1)$. For the Galois group of the extension $\mathbb{F} \rightarrowtail \mathbb{F}(m)$, we write

$$\Gamma_m := Gal(\mathbb{F}(m), \mathbb{F}) = \Gamma/\Gamma(m) \cong \mathbb{Z}/m = \langle F_q \rangle.$$

Proposition 1.5.1. *The group $GL_1(\mathbb{F}(m))$ is cyclic of order $q^m - 1$. Moreover, its Sylow p-subgroup is cyclic of order $p^{v+v_p(m)}$.*

Proof. As $GL_1(\mathbb{F}(m)) = \mathbb{F}(m)^{\times}$, the first part is standard. The second part follows because by A.0.1 we have

$$v_p(q^m - 1) = v_p(q - 1) + v_p(m).$$

□

By the functoriality of GL_d, the action of Γ on $\bar{\mathbb{F}}$ induces an action on $GL_d(\bar{\mathbb{F}})$.

Proposition 1.5.2. *We have*

$$GL_d(\mathbb{F}) = GL_d(\bar{\mathbb{F}})^{\Gamma}$$

$$GL_d(\mathbb{F}(m)) = GL_d(\bar{\mathbb{F}})^{\Gamma(m)}.$$

Proof. If we think in terms of matrices, then Γ acts on the entries. The statement is then easy to see.

□

1.6 Morava Cohomology of $BGL_d(\bar{\mathbb{F}})$

In this section, we compute $E^*BGL_d(\bar{\mathbb{F}})$ and $K^*BGL_d(\bar{\mathbb{F}})$, where $\bar{\mathbb{F}}$ is the algebraic closure of $\mathbb{F}$, essentially as in [40, Corollary 2.12]. We deduce the Morava K-theory with an Atiyah-Hirzberuch spectral sequence from the calculation of the ordinary mod p homology given in [14]. We then use 1.3.4 to lift this result to E-theory as in [24].

Theorem 1.6.1. *For all d, there exists a set of maps*

$$BGL_d(\bar{\mathbb{F}}) \to BGL_d(\mathbb{C})$$

which are natural with respect to homomorphisms of group schemes, such that the induced maps on mod p homology and cohomology

$$H_*(BGL_d(\bar{\mathbb{F}}); \mathbb{Z}/p) \to H_*(BGL_d(\mathbb{C}); \mathbb{Z}/p)$$

$$H^*(BGL_d(\mathbb{C}); \mathbb{Z}/p) \to H^*(BGL_d(\bar{\mathbb{F}}); \mathbb{Z}/p)$$

are isomorphisms.

Proof. The first isomorphism follows immediately from [14, Theorem 1.4] applied to the connected, reductive $\mathbb{Z}$-group schemes GL_d. The second isomorphism follows from the duality between homology and cohomology over a field (via the universal coefficient theorem).

□

Remark 1.6.2. This set of natural maps depends upon various choices relating $\bar{\mathbb{F}}$ to $\mathbb{C}$. In particular, the map $BGL_1(\bar{\mathbb{F}}) \to BGL_1(\mathbb{C})$ is induced by a homomorphism

$$GL_1(\bar{\mathbb{F}}) \rightarrowtail GL_1(\mathbb{C}).$$

Theorem 1.6.3. *For all d, the induced maps*

$$K^*BGL_d(\mathbb{C}) \to K^*BGL_d(\bar{\mathbb{F}})$$

$$K_*BGL_d(\bar{\mathbb{F}}) \to K_*BGL_d(\mathbb{C})$$

are isomorphisms.

Proof. The map from 1.6.1 induces maps of Atiyah-Hirzebruch spectral sequences

$$E^r_{*,*}(BGL_d(\bar{\mathbb{F}})) \to E^r_{*,*}(BGL_d(\mathbb{C}))$$

$$E_r^{*,*}(BGL_d(\mathbb{C})) \to E_r^{*,*}(BGL_d(\bar{\mathbb{F}})).$$

By 1.6.1, the maps of E_2-pages are isomorphisms. It follows that the maps on E_r-pages must be isomorphisms (in fact the spectral sequences collapse at their E_2-pages), and so by strong convergence, the maps on K-homology and K-cohomology must also be isomorphisms.

□

Corollary 1.6.4. *For all d, the induced map*

$$E^*BGL_d(\mathbb{C}) \to E^*BGL_d(\bar{\mathbb{F}})$$

is an isomorphism.

Proof. This is proved in [24, Proposition 4.81]. It follows immediately from 1.6.3 and 1.3.4.

□

Remark 1.6.5. In particular, $E^*BGL(\bar{\mathbb{F}})$ and $K^*BGL(\bar{\mathbb{F}})$ are concentrated in even degrees.

These results prompt the following definition.

Definition 1.6.6. For R a multiplicative cohomology theory, we define an $\bar{\mathbb{F}}$-*orientation* to be a choice of element $x \in R^2BGL_1(\bar{\mathbb{F}})$ such that

$$R^*BGL_1(\bar{\mathbb{F}}) = R^*[\![x]\!].$$

We define a *normalised* $\bar{\mathbb{F}}$*-orientation* to be a choice of element $x \in R^0BGL_1(\bar{\mathbb{F}})$ such that

$$R^0BGL_1(\bar{\mathbb{F}}) = R^0[\![x]\!].$$

We say that R is $\bar{\mathbb{F}}$-*orientable* if an $\bar{\mathbb{F}}$-orientation exists. In the presence of a normalised $\bar{\mathbb{F}}$-orientation, the map on R^0 induced by the group multiplication of $GL_1(\bar{\mathbb{F}})$ yields a formal group law.

Remark 1.6.7. If we let $R = K, E$ and we choose a complex orientation, then the isomorphism in 1.6.3/1.6.4 induces an $\bar{\mathbb{F}}$-orientation on R and by remark 1.6.2, the formal group laws from the complex and $\bar{\mathbb{F}}$-orientations agree.

Definition 1.6.8. For all d, we fix a maximal torus $T_d \cong GL_1^d$ in the group scheme GL_d.

Remark 1.6.9. There is only one conjugacy class of maximal tori, so the induced map

$$BT_d(k) \to BGL_d(k)$$

is independent of the choice of maximal torus up to homotopy.

Theorem 1.6.10. *For k a field with characteristic p, and for all d, the inclusion of the maximal torus induces isomorphisms*

$$K^*BGL_d(\bar{\mathbb{F}}) \xrightarrow{\sim} K^*BT_d(\bar{\mathbb{F}})^{\Sigma_d}$$
$$E^*BGL_d(\bar{\mathbb{F}}) \xrightarrow{\sim} E^*BT_d(\bar{\mathbb{F}})^{\Sigma_d}.$$

If we choose a normalised $\bar{\mathbb{F}}$-orientation x, then we have

$$K^*BGL_d(\bar{\mathbb{F}}) = K^*[\![c_1, \ldots, c_d]\!]$$
$$E^*BGL_d(\bar{\mathbb{F}}) = E^*[\![c_1, \ldots, c_d]\!]$$

where c_i are the corresponding elementary symmetric polynomials in $\{x_i\}$.

Proof. This is proved in [24, Corollary 4.82].

There is an obvious Σ_d-action on T_d and the map

$$T_d \to GL_d$$

is Σ_d-invariant up to conjugacy. By naturality, we have a commutative diagram

$$\begin{tikzcd}
BGL_d(\bar{\mathbb{F}}) \arrow[r] & BGL_d(\mathbb{C}) \\
BT_d(\bar{\mathbb{F}}) \arrow[r] \arrow[u] & BT_d(\mathbb{C}) \arrow[u]
\end{tikzcd} .$$

For $R = K, E$, the horizontal maps on cohomology give isomorphisms, and by 1.1.2 the right hand vertical map gives

$$R^*BGL_d(\mathbb{C}) \xrightarrow{\sim} R^*BT_d(\mathbb{C})^{\Sigma_d}$$

so the same must be true over $\bar{\mathbb{F}}$. The last statements follow from the fundamental theorem of symmetric formal power series as for 1.1.2.

□

Remark 1.6.11. If we choose a normalised $\bar{\mathbb{F}}$-orientation, then this gives a theory of normalised Chern classes of $\bar{\mathbb{F}}$-vector bundles. (This is further discussed in [24, Section 4.3.7].)

Remark 1.6.12. An important conclusion from this section is that $GL_1(\bar{\mathbb{F}}) = \bar{\mathbb{F}}^\times$ behaves just like S^1 in Morava cohomology. In particular, when working with groups and groupoids related to $\mathbb{F}$ and $\bar{\mathbb{F}}$, it will be more natural to base statements around $\bar{\mathbb{F}}^\times$ instead of S^1.

1.7 Morava Cohomology of $BGL_d(\mathbb{F})$

In this section, we recall the descriptions of $E^*BGL_d(\mathbb{F})$ and $K^*BGL_d(\mathbb{F})$. Tanabe determines the Morava K-theory in [40] using an Eilenberg-Moore spectral sequence. We follow [24] and use a further result from [40], along with 1.3.4 as in the previous section to lift the K-theory result to E-theory.

The action of Γ on $GL_d(\bar{\mathbb{F}})$ necessarily induces actions on the K and E-cohomology of $BGL_d(\bar{\mathbb{F}})$, and so the restriction maps for the inclusion $GL_d(\mathbb{F}) \rightarrowtail GL_d(\bar{\mathbb{F}})$ factor through maps

$$E^*BGL_d(\bar{\mathbb{F}})_\Gamma \to E^*BGL_d(\mathbb{F})$$

$$K^*BGL_d(\bar{\mathbb{F}})_\Gamma \to K^*BGL_d(\mathbb{F})$$

where

$$R_G := R/(\{x - gx \mid x \in R, g \in G\})$$

is the *ring of coinvariants* of the action of G.

Remark 1.7.1. The induced map

$$F_q : GL_1(\bar{\mathbb{F}}) \to GL_1(\bar{\mathbb{F}})$$

$$a \mapsto a^q$$

is actually the qth-power automorphism of the group $GL_1(\bar{\mathbb{F}})$. If $R = K, E$, by 1.6.6, if we let x be a normalised $\bar{\mathbb{F}}$-orientation, then this implies that the corresponding map on cohomology is given by

$$F_q^* : R^*BGL_1(\bar{\mathbb{F}}) \to R^*BGL_1(\bar{\mathbb{F}})$$

$$x \mapsto [q](x).$$

As F_q induces natural ring maps on $E^*BGL_d(\bar{\mathbb{F}})$, for these specific spaces F_q^* behaves like an Adams operation ψ^q.

Theorem 1.7.2. *The map*

$$K^*BGL_d(\bar{\mathbb{F}})_\Gamma \to K^*BGL_d(\mathbb{F})$$

is an isomorphism.

Proof. This follows from [40, Theorem 2.7] applied to the connected, reductive $\mathbb{Z}$-group schemes GL_d. We sketch the argument used there. In [13, Theorem 12.2], Friedlander constructs a homotopy commutative diagram

$$\begin{array}{ccc} BGL_d(\mathbb{F}) & \xrightarrow{\quad D \quad} & BGL_d(\mathbb{C})^\wedge_p \\ \Big\downarrow D & & \Big\downarrow \Delta \\ BGL_d(\mathbb{C})^\wedge_p & \xrightarrow{id \times \phi} & (BGL_d(\mathbb{C}) \times BGL_d(\mathbb{C}))^\wedge_p \end{array} .$$

called the cohomological Lang fibre square. Here, $(-)^\wedge_p$ denotes p-completion, Δ is the diagonal, D is induced from the map in 1.6.1, ϕ is a characteristic 0 lifting of the Frobenius map constructed via [13, Proposition 8.8], and the induced map between the fibres of D and Δ is an isomorphism on mod p (co)homology.

Using the mod p (co)homology equivalence $BGL_d(\bar{\mathbb{F}}) \to BGL_d(\mathbb{C})$ from 1.6.1, we get a homotopy pullback square

$$\begin{array}{ccc} BGL_d(\mathbb{F})^\wedge_p & \longrightarrow & BGL_d(\bar{\mathbb{F}})^\wedge_p \\ \Big\downarrow & & \Big\downarrow \Delta \\ BGL_d(\bar{\mathbb{F}})^\wedge_p & \xrightarrow{id \times F_q} & (BGL_d(\bar{\mathbb{F}}) \times BGL_d(\bar{\mathbb{F}}))^\wedge_p \end{array} .$$

From this homotopy pullback square we get an Eilenberg-Moore spectral sequence

$$E_2 = \mathrm{Tor}_{K^*BGL_d(\bar{\mathbb{F}})^2}(K^*BGL_d(\bar{\mathbb{F}}), K^*BGL_d(\bar{\mathbb{F}})) \Rightarrow K^*BGL_d(\mathbb{F})$$

We find that this spectral sequence collapses to its edge homomorphism, which gives the required isomorphism.

□

Remark 1.7.3. In particular, $K^*BGL_d(\mathbb{F})$ is finite dimensional and concentrated in even degrees.

Lemma 1.7.4. *The sequence $c_1 - F_q^*(c_1), \ldots, c_d - F_q^*(c_d)$ is a regular sequence on $K^*BGL_d(\bar{\mathbb{F}})$.*

Proof. This follows immediately from [40, Proposition 4.6].

□

Corollary 1.7.5. *The map*

$$E^*BGL_d(\bar{\mathbb{F}})_\Gamma \to E^*BGL_d(\mathbb{F})$$

is an isomorphism.

Proof. We prove this similarly to the proof given in [24, Proposition 6.3].

By 1.3.2, we know that $E^*BGL_d(\mathbb{F})$ is finitely generated, free, concentrated in even dimensions, and

$$K^*BGL_d(\mathbb{F}) = K^* \otimes_{E^*} E^*BGL_d(\mathbb{F}).$$

By right exactness of the tensor product, we also have that

$$K^*BGL_d(\bar{\mathbb{F}})_\Gamma = K^* \otimes_{E^*} E^*BGL_d(\bar{\mathbb{F}})_\Gamma.$$

By 1.4.3, $K^*BGL_d(\bar{\mathbb{F}})_\Gamma = K^*BGL_d(\mathbb{F})$ is finitely generated, and hence so is $E^*BGL_d(\bar{\mathbb{F}})_\Gamma$ by lifting generators.

By A.0.9, it suffices to show that $(p, u_1, \ldots, u_{n-1})$ is a regular sequence on both $E^*BGL_d(\bar{\mathbb{F}})_\Gamma$ and $E^*BGL_d(\mathbb{F})$. By freeness and the definition of E, this is true for the latter.

The sequence $(p, u_1, \ldots, u_{n-1})$ is regular on $E^*BGL_d(\bar{\mathbb{F}})$ by freeness, with quotient $K^*BGL_d(\bar{\mathbb{F}})$, and by 1.7.4, the sequence $(c_1 - F_q^*(c_1), \ldots, c_d - F_q^*(c_d))$ is regular on $K^*BGL_d(\bar{\mathbb{F}})$. Combining these gives us a regular sequence

$$(p, u_1, \ldots, u_{n-1}, c_1 - F_q^*(c_1), \ldots, c_d - F_q^*(c_d))$$

on $E^*BGL_d(\bar{\mathbb{F}})$. As this ring is local Noetherian, we can reorder this (e.g. [25, p. 127]) to give a regular sequence

$$(c_1 - F_q^*(c_1), \ldots, c_d - F_q^*(c_d), p, u_1, \ldots, u_{n-1}).$$

This shows that $(p, u_1, \ldots, u_{n-1})$ is a regular sequence on

$$E^*BGL_d(\bar{\mathbb{F}})/(c_1 - F_q^*(c_1), \ldots, c_d - F_q^*(c_d)) = E^*BGL_d(\bar{\mathbb{F}})_\Gamma.$$

$\square$

Remark 1.7.6. In particular, by 1.4.3 and 1.3.2, this implies that $E^*BGL_d(\mathbb{F})$ is finitely generated and free over E^*, and concentrated in even degrees.

Chapter 2

Groupoids

In this chapter, we cover relevant material from the homotopy theory of groupoids.

In the first section, we recall the basic statements about the homotopy theory of groupoids, as made explicit in [36]. This includes the definition of the model category, as well as the construction of homotopy pullbacks.

In the second section, we consider how the basic properties of functors between groupoids interact with the homotopy theory. In particular, we look at fullness, faithfulness and essential surjectivity from a homotopy theoretic perspective. We conclude by describing the behaviour of homotopy fibres in certain situations.

In the third section, we review the basic properties of stable transfers, such as Frobenius reciprocity, most of which follow from the Mackey property. We phrase these results in such a way as to make best use of them in later chapters, and we prove them in ways that will immediately translate for K-local transfers in chapter 6.

In the final section, we introduce a class of well-behaved p-local equivalences of groupoids, which we call "regular p-local equivalences". These are preserved by composition and homotopy pullbacks. We will make use of these in later chapters.

2.1 Model Category of Groupoids

In this section, we recall some of the key results from the homotopy theory of groupoids as discussed in [36].

Definition 2.1.1. We write Gpds for the category of groupoids. Let G be a groupoid. For $a, b \in Obj(G)$ we write $G(a, b)$ for the set of homomorphisms from a to b (otherwise written as $\mathrm{Hom}_G(a, b)$). When $a = b$, we will sometimes write $G(a) := G(a, a)$.

We say that G is *hom-finite* if the set $G(a, b)$ is finite for all $a, b \in Obj(G)$.

We say that G is *finite* if G is hom-finite and moreover, the set of isomorphism classes of G is finite.

Definition 2.1.2. Let $f : H \to G$ be a functor of groupoids.

(1) f is a *weak equivalence* if it is an equivalence of the underlying categories.

(2) f is a *cofibration* if it is injective on objects.

(3) f is a *fibration* if for all $a \in H, b \in G$ and isomorphisms $g : f(a) \to b$, there exists a lift of g starting at a in H, $h : a \to a'$.

Theorem 2.1.3. *The definitions in 2.1.2 give* Gpds *the structure of a right proper, closed model category.*

Proof. [36, Theorem 6.7, Proposition 6.8].

□

Definition 2.1.4. For groupoids H, G, we write $[H, G]$ for the set of homotopy classes (i.e. natural isomorphism classes) of functors $H \to G$.

By [36, 6.5] Gpds is Cartesian closed, so we define $\mathcal{H}(H, G)$ to be the corresponding hom-groupoid, whose objects are functors $\alpha : H \to G$, i.e. the set $\mathrm{Hom}(H, G)$, and whose morphisms are natural isomorphisms of functors.

This has $\pi_0(\mathcal{H}(H, G)) = [H, G]$

Lemma 2.1.5. *For H, G finite groups we have*

$$[H, G] = \mathrm{Hom}(H, G)/conj_G$$

where $conj_G$ denotes the equivalence relation such that $a \sim b$ if there exists $g \in G$ such that $b = conj_g \circ a$.

Proof. Unwinding the definition, a natural isomorphism $a \Rightarrow b$ of two group homomorphisms $a, b : H \to G$ is given by an element of G

$$g : *_G \to *_G$$

such that $b(h) = a(h)^g$ for all $h \in H$, i.e. $b = conj_g \circ a$.

□

Definition 2.1.6. We write

$$B : \mathrm{HoGpds} \to \mathrm{HoSpaces}$$

for the functor that takes a groupoid to its classifying space.

Proposition 2.1.7. *The functor B preserves homotopy pullbacks.*

Proof. By [36, 6.2], there is a lift of B to a functor of categories

$$\mathrm{Gpds} \to \mathrm{Spaces}$$

that preserves coproducts, finite limits, and fibrations. In particular, any homotopy pullback in HoGpds can be lifted to a genuine pullback of fibrations in Gpds. Then the image of such a diagram must also be a pullback of fibrations, and thus must become a homotopy pullback in HoSpaces.

□

Lemma 2.1.8. *A functor of groups $H \to G$ is a fibration iff it is surjective.*

Proof. [36, Remark 6.5]. By definition, a functor of groups is a fibration iff for all morphisms g in G, there exists a lift h of g, i.e. the functor is surjective.

□

By 2.1.7, homotopy pullbacks in HoGpds map to homotopy pullbacks in HoSpaces. We can give a concrete description of a homotopy pullback of groupoids as follows.

Definition 2.1.9. Let $H \xrightarrow{h} G$ and $K \xrightarrow{k} G$ be functors. Define the *concrete homotopy pullback* of h and k to be the groupoid L with objects and morphisms

$$Obj(L) := \{(a, c, g) \mid (a, c) \in Obj(H) \times Obj(K), g \in \mathrm{Hom}_G(h(a), k(c))\}$$

$$\mathrm{Hom}((a, c, g), (a', c', g')) = \{(r, s) \in \mathrm{Hom}(a, a') \times \mathrm{Hom}(c, c') \mid g' \circ h(r) = k(s) \circ g\}.$$

We can define functors out of L in the obvious way

$$h' : L \to K$$

$$k' : L \to H$$

as well as a natural isomorphism $\phi : hk' \to kh'$.

Proposition 2.1.10. *For functors $H \xrightarrow{h} G$ and $K \xrightarrow{k} G$, the concrete homotopy pullback is a genuine homotopy pullback in the model category* Gpds.

Proof. [36, Definition 6.9, Remark 6.10].

□

Definition 2.1.11. For a fibration of groupoids $H \xrightarrow{f} G$ and an object $x \in Obj(G)$, define the *concrete homotopy fibre* of f at x, F_x to be the concrete homotopy pullback of f along the inclusion $* \xrightarrow{x} G$.

Theorem 2.1.12. *For G a finite groupoid, E^*BG and K^*BG are finitely generated as modules over E^* and K^* respectively. Moreover, E^0BG is a complete semi-local Noetherian ring of dimension n and K^0BG is a complete local Noetherian ring of dimension* 0 *(i.e. an Artinian ring).*

Proof. As a finite groupoid is equivalent to a finite disjoint union of finite groups, this follows immediately from 1.4.3.

□

2.2 Functors of Groupoids

In this section, we look at how the notions of fullness, faithfulness, and essential surjectivity interact with the homotopy theory of groupoids. We conclude with a description of homotopy fibres when a functor is faithful or full.

Definition 2.2.1. For a groupoid G, define the 0*th homotopy group* of G to be

$$\pi_0(G) := [*, G].$$

If $f : H \to G$ is a functor of groupoids, then we have an induced map

$$\pi_0(f) : \pi_0(H) \to \pi_0(G)$$

and π_0 defines a functor

$$\pi_0 : \mathrm{HoGpds} \to \mathrm{Sets}.$$

Remark 2.2.2. $\pi_0(G)$ is just the set of components of G and we have isomorphisms

$$\pi_0(G) = \pi_0(BG).$$

Lemma 2.2.3. *If G is a groupoid, $x, y \in Obj(G)$, and $g : x \to y$ is a morphism, then we have isomorphisms*

$$g \circ - : G(x, x) \to G(x, y)$$
$$- \circ g : G(y, y) \to G(x, y).$$

In particular, the map

$$conj_g : G(x, x) \to G(y, y)$$

is an isomorphism.

Proof. It is easy to see that $g \circ -$ has inverse $g^{-1} \circ -$, and $- \circ g$ has inverse $- \circ g^{-1}$. The map

$$conj_g : G(x, x) \to G(y, y)$$

is an isomorphism, with inverse $conj_{g^{-1}}$.

□

Lemma 2.2.4. *For a groupoid G, and representatives $x \in Obj(G)$ for each component $[x] \in \pi_0(G)$, we have an equivalence*

$$G \simeq \coprod_{[x] \in \pi_0(G)} G(x).$$

Proof. It suffices to prove that the inclusion

$$\coprod_{[x] \in \pi_0(G)} G(x) \to G$$

is an equivalence. It is straightforward to see that it is full, and faithful. As all objects in a component are isomorphic, it is also essentially surjective, and thus it must be an equivalence.

□

Definition 2.2.5. We say that a homotopy class of functors $f : H \to G$ is π_0-*injective/surjective* if $\pi_0(f)$ is injective/surjective respectively. We say that f is π_0-*bijective* or a π_0-*isomorphism* if f is π_0-injective and π_0-surjective.

Lemma 2.2.6. *A functor f is essentially surjective iff the homotopy class of f is π_0-surjective. If f is full, then it is π_0-injective.*

Proof. This follows straightforwardly from the definitions.

□

Lemma 2.2.7. *If $f, g : H \to G$ are functors of groupoids, such that $f \simeq g$, then f is faithful/full/essentially surjective iff g is faithful/full/essentially surjective respectively.*

Proof. By assumption we have a natural isomorphism

$$\eta : f \Rightarrow g.$$

So for objects x, y in H the composition

$$H(x,y) \xrightarrow{f} G(f(x), f(y)) \xrightarrow{post(\eta_y)} G(f(x), g(y)) \xrightarrow{pre(\eta_x^{-1})} G(g(x), g(y))$$

is equal to g on the set $H(x, y)$, where *post* and *pre* denote postcomposition and precomposition isomorphisms. It follows that f is faithful/full iff g is faithful/full.

If x is an object in H, and a is an object in G, then there is an isomorphism $a \to f(x)$ iff there is an isomorphism $a \to g(x)$, by postcomposing with η_x or its inverse. It follows that f is essentially surjective iff g is essentially surjective.

□

Lemma 2.2.8. *If H, G are groupoids and we have functors*

$$f : H \to G$$

$$g : G \to H$$

such that $gf \simeq \mathrm{id}_H$, then f is π_0-injective, and g is π_0-surjective.

Proof. As $gf \simeq \mathrm{id}$, by functoriality we have

$$\pi_0(g)\pi_0(f) = \mathrm{id}$$

so $\pi_0(f)$ is split injective and $\pi_0(g)$ is split surjective.

□

Lemma 2.2.9. *If $f : H \to G$ is a homotopy section (i.e. $gf \simeq \mathrm{id}_H$ for some g), then f is faithful.*

Proof. By assumption we have a natural isomorphism

$$\eta : gf \Rightarrow id.$$

So for objects x, y in H the composition

$$H(x,y) \xrightarrow{f} G(f(x), f(y)) \xrightarrow{g} H(gf(x), gf(y))$$
$$\xrightarrow{post(\eta_y)} H(gf(x), y) \xrightarrow{pre(\eta_x^{-1})} H(x,y)$$

is the identity, where *post* and *pre* denote postcomposition and precomposition isomorphisms. In particular, the first map is injective, so f is faithful.

$\square$

Lemma 2.2.10. *For a functor of groupoids $g : G \to H$, if g is a homotopy retraction (i.e. $gf \simeq \mathrm{id}_H$ for some f) and a π_0-injection, then g is full.*

Proof. Firstly, note that by 2.2.8, g is π_0-surjective, and so π_0-bijective, and so f is also π_0-bijective. Suppose a, b are objects in G. As f is π_0-surjective, there exist objects x, y and isomorphisms $r : f(x) \to a$, $s : f(y) \to b$. We can form a commutative diagram

$$\begin{tikzcd}
H(x,y) \arrow[r, "f"] & G(f(x), f(y)) \arrow[r, "g"] \arrow[d, "post(s)"'] & H(gf(x), gf(y)) \arrow[d, "post(g(s))"] \\
& G(f(x), b) \arrow[r, "g"] \arrow[d, "pre(r^{-1})"'] & H(gf(x), g(b)) \arrow[d, "pre(g(r^{-1}))"] \\
& G(a,b) \arrow[r, "g"] & H(g(a), g(b))
\end{tikzcd}.$$

The vertical maps are precomposition and postcomposition isomorphisms. By the argument from 2.2.9, the top map is an isomorphism. In particular the top g is surjective, so it follows that the bottom g is also surjective. Thus g is full.

$\square$

Lemma 2.2.11. *For a map of groupoids $H \xrightarrow{f} G$ and an object $x \in Obj(G)$, if we define*

$$\tilde{Z} := \{z \in Obj(H) \mid f(z) \simeq x\}$$

then we have the following:

(1) If f is full and $z \in \tilde{Z}$, then

$$F_x \simeq Ker(H(z) \to G(f(z))).$$

(2) If f is faithful and $Z \subseteq \tilde{Z}$ is a subset containing precisely one element for each conjugacy class, then the homotopy fibres are equivalent to discrete groupoids; specifically

$$F_x \simeq \amalg_{z \in Z} G(f(z))/H(z)$$

where $G(f(z))/H(z)$ is considered as a set.

Proof. Firstly, observe that an isomorphism $x \to x'$ gives a natural isomorphism between the maps

$$x : * \to G$$

$$x' : * \to G$$

which induces an equivalence of homotopy pullback diagrams, and in particular, an equivalence $F_x \simeq F_{x'}$. The statements above are evident when the component of x is outside of the image of f, because F_x is trivial in those situations, so it is sufficient to prove each case when x is in the image of f.

Next, observe that if y, y' are objects in H, and we have a map $r : y \to y'$ in H, then we get a commutative diagram

$$\begin{tikzcd} H(y) \arrow[r, "f"] \arrow[d, "conj_r"'] & G(f(y)) \arrow[d, "conj_{f(r)}"] \\ H(y') \arrow[r, "f"'] & G(f(y')) \end{tikzcd}$$

where *conj* denotes a conjugation isomorphism. In the respective cases, this induces isomorphisms of groups and sets respectively

$$Ker(H(y) \xrightarrow{f} G(f(y))) \to Ker(H(y') \xrightarrow{f} G(f(y')))$$

$$G(x)/H(y) \to G(x)/H(y').$$

Therefore, it is sufficient to prove each case when $f(y) = x$ for all $y \in Z$.

We may factorise f as

$$H \xrightarrow{f_0} H' \xrightarrow{f_1} G$$

where f_0 is an equivalence, and f_1 is a fibration. As f_0 is an equivalence, for all objects y in H, it induces isomorphisms

$$f_0 : H(y) \to H'(f_0(y)).$$

In particular, in the respective cases, for all objects y in H, this gives isomorphisms

$$Ker(H(y) \xrightarrow{f} G(f(y))) \to Ker(H(f_0(y)) \xrightarrow{f_1} G(f(y)))$$

$$G(x)/H(y) \to G(x)/H'(f_0(y)).$$

Therefore, it suffices to prove each case when f is a fibration. In this situation, it suffices to take the ordinary pullback, P. P has objects

$$Obj(P) = \{y \in Obj(H) \mid f(y) = x\}$$

and morphisms

$$P(y, y') = \{r \in H(y, y') \mid f(r) = \mathrm{id}_x\}.$$

When f is full, given $y, y' \in Obj(P)$, by fullness, we have an isomorphism $r : y \to y'$ in H such that $f(r) = \mathrm{id}_x$, so P is connected. It is easy to see that

$$P(z) = Ker(H(z) \to G(f(z)))$$

which gives part (1). When f is faithful, we see that $P(y, y')$ has either one or no elements, so P is discrete. For $z \in Z$, as f is a fibration, for each $g \in G(x)$, there exists a lift $r_g : z \to z_g$ in H such that $f(r_g) = g$. In particular, we must have $r_h = id_z$ for all $h \in H(z)$, so this defines a map

$$\amalg_{z \in Z} G(f(z))/H(z) \to P.$$

It suffices to show that this is essentially surjective. Given $y \in Obj(P)$, there exists $z \in Obj(P)$ and a map $r : y \to z$ in H. However, if $g = f(r)$, then $f(r_{g^{-1}} r) = id_x$, so $r_{g^{-1}} r : y \to z_g$ is a map in P.

$\square$

Remark 2.2.12. In particular, for a homomorphism of groups $H \xrightarrow{f} G$, the homotopy fibre is equivalent to the group $Ker(f)$ if f is surjective, and to the set G/H if f is injective.

2.3 Transfers of Groupoids

In this section, we will prove some fundamental statements about stable transfers of groupoids.

Definition 2.3.1. For a groupoid G, we write

$$SG := \Sigma_+^\infty BG$$

and we write

$$S := \Sigma^\infty S^0 = S1$$

for the sphere spectrum, where 1 is the terminal groupoid.

Remark 2.3.2. For groupoid functors f, we will abuse notation and write f when we really mean Bf or Sf. What we mean should always be clear from the context.

Definition 2.3.3. Define the *cohomotopy* of a spectrum X to be

$$\pi^0(X) := [X, S].$$

where $[-,-]$ denotes the group of homotopy classes of maps of spectra. As S is a commutative ring spectrum, when $X = \Sigma_+^\infty Y$ is the suspension of a space, this is a ring with unit

$$1 : X \to S$$

defined as the stabilisation of the terminal map

$$Y \to \{pt\}.$$

Remark 2.3.4. For a groupoid G, the stabilisation of the terminal map

$$\epsilon : G \to 1$$

coincides with the unit $1 \in \pi^0(SG)$.

Definition 2.3.5. For a multiplicative cohomology theory R, and a space X, we write

$$(- \bullet -) : R^*(X) \otimes R^*(X) \to R^*(X)$$

$$a \otimes b \mapsto a \bullet b$$

for the *cup product*. Often, this will just be written as the concatenation of elements.

Definition 2.3.6. For a space X, if $u \in \pi^0(\Sigma_+^\infty X)$, then we define

$$\mu_u : \Sigma_+^\infty X \xrightarrow{\delta} \Sigma_+^\infty X \wedge \Sigma_+^\infty X \xrightarrow{u \wedge id} \Sigma_+^\infty X.$$

Lemma 2.3.7. *For a multiplicative cohomology theory R, a space X, and $u \in \pi^0(\Sigma^\infty_+ X)$, we have*

$$\mu_u^* = (u \bullet -) : R^*(X) \to R^*(X)$$

where u is considered as an element of $R^0(X)$ via the map

$$\pi^0(\Sigma^\infty_+ X) = [\Sigma^\infty_+ X, S] \to [\Sigma^\infty_+ X, R] = R^0(X)$$

induced by the unit $S \to R$.

Proof. This essentially follows by writing out the definitions of μ_u and of cup product multiplication by u in $R^*(X)$.

□

Lemma 2.3.8. *For spaces X, Y and elements $u, v \in \pi^0(\Sigma^\infty_+ X)$ and $w \in \pi^0(\Sigma^\infty_+ Y)$, we have*

$$\mu_{u+v} = \mu_u + \mu_v \in [\Sigma^\infty_+ X, \Sigma^\infty_+ X]$$
$$\mu_u \wedge \mu_w = \mu_{u \otimes w} \in [\Sigma^\infty_+ (X \times Y), \Sigma^\infty_+ (X \times Y)]$$
$$\mu_{uv} = \mu_v \circ \mu_u \in [\Sigma^\infty_+ X, \Sigma^\infty_+ X]$$
$$\mu_1 = \mathrm{id} \in [\Sigma^\infty_+ X, \Sigma^\infty_+ X].$$

where $+$ denotes sum and $\circ$ denotes composition in the stable homotopy category.

Proof. For the first statement we have distributivity of $\wedge$ over $+$ (i.e. $\vee$)

$$(u + v) \wedge \mathrm{id} = (u \wedge \mathrm{id}) + (v \wedge \mathrm{id}).$$

The result follows by appending δ and using the additivity of the stable homotopy category.

The second statement follows straightforwardly from the fact that π^0 is a lax monoidal functor and thus has natural maps

$$\pi^0(\Sigma^\infty_+ X) \otimes \pi^0(\Sigma^\infty_+ Y) \to \pi^0(\Sigma^\infty_+ X \wedge \Sigma^\infty_+ Y) = \pi^0(\Sigma^\infty_+ (X \times Y))$$

$$u \otimes w \mapsto u \wedge w$$

and

$$(u \wedge w \wedge \mathrm{id}_X \wedge \mathrm{id}_Y)\delta_{X \times Y} = [(u \wedge \mathrm{id}_X) \wedge (w \wedge \mathrm{id}_Y)](\delta_X \wedge \delta_Y).$$

in $[\Sigma^\infty_+ (X \times Y), \Sigma^\infty_+ (X \times Y)]$.

The third statement follows using the associativity of δ and the fact that $uv = \delta^*(u \wedge v)$

$$\begin{aligned}(v \wedge \mathrm{id})\delta(u \wedge \mathrm{id})\delta &= (v \wedge \mathrm{id})(u \wedge \mathrm{id}^2)(\mathrm{id} \wedge \delta)\delta \\ &= (u \wedge v \wedge \mathrm{id})(\delta \wedge \mathrm{id})\delta \\ &= ((uv) \wedge \mathrm{id})\delta \in [\Sigma^\infty_+ X, \Sigma^\infty_+ X].\end{aligned}$$

The fourth statement follows immediately from the fact that $\epsilon : G \to 1$ is a counit for δ by 2.3.4.

□

Theorem 2.3.9. *For faithful functors of groupoids $f : H \to G$ with finite fibres, there exist stable transfer maps*

$$f^! : SG \to SH.$$

If f, g are faithful functors, then

(1) $(fg)^! = g^! f^!$

(2) $\mathrm{id}^! = \mathrm{id}$

(3) $(f \amalg g)^! = f^! \vee g^!$

(4) $(f \times g)^! = f^! \wedge g^!$

(5) The Mackey property holds, i.e. for a homotopy pullback diagram of groupoids

$$\begin{array}{ccc} A & \xrightarrow{k} & B \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow{g} & D \end{array}$$

we have $g^! f = hk^!$.

Proof. This follows by [36, Proposition 6.25, Section 7].

□

Remark 2.3.10. The Mackey property is just a reformulation of the double coset formula for transfers in terms of groupoids.

Remark 2.3.11. The remainder of the statements in this section are essentially direct consequences of the properties of transfers in 2.3.9.

Proposition 2.3.12. *For a map of groupoids $H \xrightarrow{f} G$ and an element $u \in \pi^0(SG)$, we have*

$$\mu_u \circ f = f \circ \mu_{f^*(u)} : SH \to SG.$$

Proof. This follows from the commutative diagram

$$\begin{tikzcd}
BH \arrow[r, "\delta"] \arrow[d, "f"] & BH \wedge BH \arrow[r, "f \wedge \mathrm{id}"] \arrow[d, "f \wedge f"] & BG \wedge BH \arrow[r, "u \wedge \mathrm{id}"] \arrow[d, "\mathrm{id} \wedge f"] & BH \arrow[d, "f"] \\
BG \arrow[r, "\delta"'] & BG \wedge BG \arrow[r, "\mathrm{id}"'] & BG \wedge BG \arrow[r, "u \wedge \mathrm{id}"'] & BG
\end{tikzcd}.$$

The top line is $\mu_{f^*(u)}$ and the bottom line is μ_u.

□

There are two useful forms of Frobenius reciprocity when working with groupoids.

Lemma 2.3.13. *For a faithful map of groupoids $H \xrightarrow{f} G$ with finite fibres, and an element $u \in \pi^0(SG)$, we have*

$$f^! \circ \mu_u = \mu_{f^*(u)} \circ f^! : SG \to SH.$$

Proof. Firstly, we note that the following square is a homotopy pullback

$$\begin{tikzcd}
H \arrow[r, "f"] \arrow[d, "{(f, \mathrm{id})}"'] & G \arrow[d, "\delta"] \\
H \times G \arrow[r, "\mathrm{id} \times f"'] & G \times G
\end{tikzcd}.$$

This is proved in [36, Proposition 8.5] (it follows because the fibres of f are equivalent to the fibres of $\mathrm{id} \times f$), so by the Mackey property 2.3.9, we get a stable equality

$$(\mathrm{id} \wedge f^!)\delta = (f, \mathrm{id})f^!.$$

Observe that we also have a commutative diagram

$$\begin{tikzcd}
H \arrow[r, "\delta"] \arrow[dr, "{(f, \mathrm{id})}"'] & H \times H \arrow[d, "f \times \mathrm{id}"] \\
& H \times G
\end{tikzcd}$$

which gives a stable equality

$$(f, \mathrm{id}) = (f \wedge \mathrm{id})\delta.$$

Now using these two equalities we get stable equalities

$$\begin{aligned} f^! \mu_u &= f^!(u \wedge \mathrm{id})\delta \\ &= (u \wedge \mathrm{id})(\mathrm{id} \wedge f^!)\delta \\ &= (u \wedge \mathrm{id})(f, \mathrm{id})f^! \\ &= (u \wedge \mathrm{id})(f \wedge \mathrm{id})\delta f^! \\ &= \mu_{f^*(u)} f^!. \end{aligned}$$

□

Lemma 2.3.14. *For a faithful functor of groupoids $H \xrightarrow{f} G$ with finite fibres, stably we have*

$$ff^! = \mu_u : SG \to SG$$

where $u := f_!(1) \in \pi^0(SG)$.

Proof. If ϵ is the terminal groupoid map from H, then it is easy to see that

$$(\epsilon \times \mathrm{id}) \circ (\mathrm{id}, f) = f : G \to G.$$

Using the stablisation of this fact, and the reflection of the Mackey square from 2.3.13, we get a commutative diagram

$$\begin{tikzcd} SG \arrow[r, "f^!"] \arrow[d, "\delta"'] & SH \arrow[d, "(\mathrm{id}, f)"'] \arrow[dr, "f"] & \\ SG \wedge SG \arrow[r, "f^! \wedge \mathrm{id}"'] & SH \wedge SG \arrow[r, "\epsilon \wedge \mathrm{id}"'] & SG \end{tikzcd}.$$

The statement follows from this.

□

Theorem 2.3.15. *For an injective map of groups $H \xrightarrow{f} G$ of finite index, on ordinary (co)homology, we have $f_!(1) = [G : H]$ and therefore*

$$f_* f^! : H_*(BG) \to H_*(BG)$$
$$f_! f^* : H^*(BG) \to H^*(BG)$$
$$x \mapsto [G : H]x.$$

Proof. This can be found in [42, Lemma 6.7.17].

□

Lemma 2.3.16. *Let $f : H \to G$ and $g : G \to H$ be groupoid maps such that f is faithful with finite fibres, $gf = \mathrm{id}$ and $u := f_!(1)$, then*

$$f^! = g\mu_u : SG \to SH.$$

Moreover, if f, g are inverse equivalences, then $f^! = g$, $g^! = f$, $f_!(1) = 1$, and $g_!(1) = 1$.

Proof. For the first statement we have

$$f^! = gff^! = g\mu_u : SG \to SH$$

by 2.3.14.

The second statement follows by functoriality of transfers, and the fact that $f^*(1) = 1$ and $g^*(1) = 1$.

□

Lemma 2.3.17. *For $R = E, K$, let $A' \xrightarrow{f} A$ be the inclusion of a cyclic group of order i into a cyclic group of order j, with $m := j/i$. Then we have*

$$f_!(1) = \langle m \rangle([i](x)) \in R^0 BA$$

where x is induced by a complex orientation as in 1.4.1.

Proof. This result is standard. Observe that we can form a pullback diagram of groups

$$\begin{tikzcd} A' \arrow[r, "f"] \arrow[d, "e"'] & A \arrow[d, "g"] \\ 1 \arrow[r, "h"] & A/A' \end{tikzcd}$$

As the right hand map is surjective, it is a fibration and thus the diagram is a homotopy pullback. By the Mackey property, we have

$$f_!(1) = f_!(e^*(1)) = g^*(h_!(1))$$

If we write

$$R^* B(A/A') = R^*[\![y]\!]/[m](y)]$$

$$R^*BA = R^*[\![x]\!]/[j](x)]$$

as in 1.4.1, and let a be a generator for A, then the multiplication by i map factors through g

$$\bullet i : A \xrightarrow{g} A/A' \rightarrowtail A$$

from which we see that $g^*(y) = [i](x)$. $h_!(1)$ is determined by the case where A' is the trivial group, with $i = 1$ and $j = m$. This is computed in [41, Lemma 2.1], which gives $h_!(1) = \langle m \rangle(y)$. Combining these observations gives the result for the general case.

□

2.4 p-Local Groupoids

In this section, we determine sufficient conditions for a functor of groupoids to be a p-local equivalence. This defines a class of p-local equivalences, which we call "regular". We then show that compositions and homotopy pullbacks of regular p-local equivalences are also regular p-local equivalences.

Note: All groupoids in this section are assumed finite.

Lemma 2.4.1. *In the poset of Bousfield classes*

$$\langle S^0_{(p)} \rangle \geq \langle K \rangle.$$

Thus $L_K L_{(p)} = L_K$. Moreover, when X is a connective spectrum

$$L_{(p)} X \simeq L_{H\mathbb{Z}_{(p)}} X.$$

Proof. By construction, K is an E-module and so in particular an $S^0_{(p)}$-module, so by [1, Lemma 13.1] it is already p-local. We must show that $S^0_{(p)}$-acyclics are also K-acyclics.

Suppose $S^0_{(p)} \wedge X \simeq 0$. Then as K is p-local and p-localisation is smashing, $K \simeq K \wedge S^0_{(p)}$, so

$$K \wedge X \simeq K \wedge S^0_{(p)} \wedge X \simeq 0.$$

The last statement follows by [5, Theorem 3.1].

□

Definition 2.4.2. For a groupoid G, we write

$$S_{(p)}G := (\Sigma^\infty_+ BG)_{(p)}$$

for the p-localisation of SG.

Recall the standard theorem from group homology.

Theorem 2.4.3. *For a finite group G and an abelian group A, if multiplication by $|G|$ acts invertibly on A, then*

$$H_*(G; A) = \begin{cases} A & \textit{if } * = 0 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. This follows from [42, Proposition 6.1.10].

□

Definition 2.4.4. We say that a groupoid G is *coprime to p* if for all objects $x \in G$, $|G(x, x)|$ is coprime to p.

Proposition 2.4.5. *If $f : H \to G$ is a π_0-bijective, full map of finite groupoids with fibres coprime to p, then f is a p-local equivalence, i.e. $S_{(p)}f$ is an equivalence. Moreover, it is a K-local equivalence.*

Proof. As every groupoid is equivalent to a disjoint union of groups, we can reduce to the case where $H \to G$ is a surjective map of groups such that the kernel K has order coprime to p. By 2.4.1, it suffices to prove that we have a p-local equivalence. We have a fibre sequence

$$BK \to BH \to BG$$

which gives us a Serre spectral sequence in p-local homology

$$E^2_{*,*} = H_*(BG; H_*(K)) \Rightarrow H_*(BH).$$

There is also an induced map to the trivial spectral sequence

$$E^2_{*,*} = H_*(BG) \Rightarrow H_*(BG).$$

But as K has order coprime to p, $H_*(K) = H_0(pt) = \mathbb{Z}_{(p)}$, so the spectral sequence collapses and the map on E^2-pages is an isomorphism, and thus the map on the targets is also an isomorphism.

It follows that we have a $H\mathbb{Z}_{(p)}$-local equivalence, and thus as everything is connective, this is a p-local equivalence by 2.4.1.

□

Definition 2.4.6. If $f : H \to G$ is a π_0-bijective, full map of finite groupoids with fibres coprime to p then we call it a *regular p-local equivalence.*

Remark 2.4.7. By 2.2.6, this is equivalent to f being essentially surjective and full with fibres coprime to p.

Proposition 2.4.8. *If $f : H \to G$ is a π_0-bijective, faithful functor of finite groupoids with fibres F_x such that $|\pi_0(F_x)| = [G(f(x)) : H(x)]$ is coprime to p, then*

$$ff^! : SG \to SG$$

is a p-local equivalence. In particular, $S_{(p)}f$ is split epi and $S_{(p)}(f^!)$ is split mono.

Proof. As every groupoid is equivalent to a disjoint union of groups, we can reduce to the case where $H \to G$ is an injective map of groups with index $[G : H]$ coprime to p. On homology, we get a composite

$$H_*(BG) \xrightarrow{f^!} H_*(BH) \xrightarrow{f_*} H_*(BG)$$

but by Frobenius reciprocity in ordinary homology 2.3.15, this is just multiplication by $[G : H]$, which is an isomorphism as $[G : H]$ is coprime to p.

□

Proposition 2.4.9. *If $A \xrightarrow{f} B \xrightarrow{g} C$ are regular p-local equivalences, then so is gf.*

Proof. As f and g are π_0-isomorphisms, gf is a π_0-isomorphism. By 2.2.10, gf is also full. It suffices to prove that for the case when A, B, C are groups, the kernel of gf is coprime to p. In this case, we have

$$|Ker(gf)| = |Ker(g)||Ker(f)|$$

which proves the statement.

□

Proposition 2.4.10. *Homotopy pullbacks preserve regular p-local equivalences of groupoids.*

Proof. Let f be a regular p-local equivalence. As we are in a model category, we can factor f as $f_1 f_0$ where f_0 is an equivalence and f_1 is a fibration. We can form a diagram

$$\begin{tikzcd}
A \arrow[r, "g''"] \arrow[d, "h_0"'] & B \arrow[d, "f_0"] \\
C \arrow[r, "g'"] \arrow[d, "h_1"'] & D \arrow[d, "f_1"] \\
E \arrow[r, "g"'] & F
\end{tikzcd}$$

where the bottom square is the ordinary pullback of f_1 along g, and the top square is the concrete homotopy pullback of f_0 along g'. As f_1 is a fibration, the bottom square is also a homotopy pullback square, so $h := h_1 h_0$ is a homotopy pullback of f along g. By 2.1.3, as f_0 is an equivalence, so is h_0.

f_1 is full, so as h_1 is the ordinary pullback, it is also full. As f_1 is π_0-surjective and a fibration, it is genuinely surjective on objects, and so h_1 is also surjective on objects and thus π_0-surjective. Moreover, as h_1 is full, it is also π_0-injective. It follows that h is full and π_0-bijective.

By definition, f has fibres coprime to p. As h is a homotopy pullback of f along g, f and h have equivalent fibres, so by 2.4.5, h is also a regular p-local equivalence.

$\square$

Chapter 3

Groupoids of Vector Spaces

The main point of this chapter is to replace $GL_d(\mathbb{F})$ with the groupoid of d-dimensional vector spaces over $\mathbb{F}$, which is equivalent. By considering the graded groupoid $\mathcal{V}_*$ we can identify lots of structure on the Morava cohomology of finite general linear groups.

In the first section, we recall the definition of a graded Hopf semiring object, along with its categorical dual. The rest of this section is devoted to identifying the structure on $\mathcal{V}_*$ coming from the diagonal map δ, the direct sum map σ, and its stable transfer $\sigma^!$. We conclude that these structures almost fit together to form a graded Hopf cosemiring. The only issue is that the compatibility relation between σ and $\sigma^!$ fails.

In the second section, we use Künneth isomorphisms to translate these results into Morava E-theory and K-theory, and we describe the behaviour of σ^*.

In the last section, we consider finite field extensions $\mathbb{F}(m)/\mathbb{F}$, and in particular, the induced maps coming from the forgetful functors $U_m : \mathcal{V}(m)_* \to \mathcal{V}_*$. We show that these maps preserve the σ^*-coalgebra structures, and we use representation theory to compute their effect on cohomology.

3.1 Direct Sum Product and Coproduct

In this section, we first show that $GL_d(k)$ is equivalent to the groupoid $\mathcal{V}_d$ of d-dimensional vector spaces over k. We then recall the definition of a graded Hopf (co)semiring object. Following 2.3.1, $S\mathcal{V}_*$ is the graded spectrum with $\Sigma^\infty_+ B\mathcal{V}_d$ in degree d. We show that the structure on $S\mathcal{V}_*$ coming from the diagonal map δ, the direct sum map σ, and its stable transfer $\sigma^!$ almost fits together to form a graded Hopf cosemiring structure. The only issue is that the compatibility relation between σ and $\sigma^!$ fails. At the end of this section, we remark about why this relation fails.

Definition 3.1.1. For a field k, define $\mathcal{V}(k)$ to be the groupoid of finite dimensional k-vector spaces and k-linear isomorphisms. For $d \geq 0$ also write $\mathcal{V}(k)_d$ for the full subgroupoid of d-dimensional k-vector spaces in $\mathcal{V}(k)$. We write $\mathcal{V}(k)_*$ for the graded groupoid with degree d part $\mathcal{V}(k)_d$ and we write $\mathcal{V}(k)_*^m$ for the m-fold graded Cartesian product of $\mathcal{V}(k)_*$.

In particular, we write $\mathcal{V}_* := \mathcal{V}(\mathbb{F})_*$, $\mathcal{V}(m)_* := \mathcal{V}(\mathbb{F}(m))_*$, and $\bar{\mathcal{V}}_* := \mathcal{V}(\bar{\mathbb{F}})_*$.

Remark 3.1.2. We will primarily be interested in the graded versions.

Definition 3.1.3. We say that a graded groupoid G_* is *connected* if $G_0 \simeq *$.

Lemma 3.1.4. *For a field k and for all $d \geq 0$ we have*

$$\mathcal{V}(k)_d \simeq GL_d(k).$$

Proof. Over a field k, for each non-negative integer d, it is a standard fact that there is only one isomorphism class of d-dimensional k-vector spaces. If we take the vector space k^d, then $\mathcal{V}_d$ is equivalent to the automorphism group $\mathcal{V}_d(k^d, k^d)$, which is equal to $GL_d(k)$ by definition.

□

Remark 3.1.5. In particular, $\mathcal{V}(k)_*$ is a connected graded groupoid.

Definition 3.1.6. In a symmetric monoidal category $\mathcal{C}$, let τ denote the natural twist isomorphism

$$\tau : A_1 \otimes A_2 \to A_2 \otimes A_1$$
$$(a_1, a_2) \mapsto (a_2, a_1).$$

We also let tw denote the twist isomorphism

$$\mathrm{tw} := \mathrm{id} \otimes \tau \otimes \mathrm{id} : A_1 \otimes A_2 \otimes A_3 \otimes A_4 \to A_1 \otimes A_3 \otimes A_2 \otimes A_4$$
$$(a_1, a_2, a_3, a_4) \mapsto (a_1, a_3, a_2, a_4).$$

Definition 3.1.7. In a symmetric monoidal category $\mathcal{C}$, a *graded bimonoid* A_* is a monoid in the category of graded comonoids in $\mathcal{C}$. Equivalently, it is a comonoid in the category of graded monoids in $\mathcal{C}$. Explicitly, it consists of a graded monoid structure (μ, ι) on A_*, and a graded comonoid structure (ψ, π) on A_*, such that the following compatibility relations hold

$$\psi\mu = \mu^{\otimes 2}\mathrm{tw}\psi^{\otimes 2}$$
$$\pi\mu = \pi \otimes \pi$$

$$\psi\iota = \iota \otimes \iota$$
$$\pi\iota = \mathrm{id}.$$

A graded bimonoid in Mod_R is called a *graded bialgebra.*

Definition 3.1.8. If A_* is a graded bimonoid, then an *antipode* χ for A is defined to be a graded endomorphism

$$\chi : A_* \to A_*$$

such that

$$\mu(\mathrm{id} \otimes \chi)\psi = \iota\pi = \mu(\chi \otimes \mathrm{id})\psi.$$

Definition 3.1.9. In a symmetric monoidal category $\mathcal{C}$, a *graded Hopf monoid* A_* is defined to be a graded bimonoid with an antipode χ.

A graded Hopf monoid in Mod_R is called a *graded Hopf algebra.*

Definition 3.1.10. In a symmetric monoidal category $\mathcal{C}$, a *graded Hopf semiring object* A_* is a graded semiring object in the category of comonoids in $\mathcal{C}$. Explicitly, it is a graded object in $\mathcal{C}$ with an "addition" commutative graded monoid structure (ρ, γ), a "multiplication" homogeneous commutative monoid structure (μ, ι), and a "diagonal" cocommutative graded comonoid structure (ψ, π), such that (μ, ι, ψ, π) is a bimonoid (in the ungraded sense), $(\rho, \gamma, \psi, \pi)$ is a graded bimonoid, and the following distributivity and annihilation relations hold:

$$\mu(\mathrm{id} \otimes \rho) = \rho(\mu \otimes \mu)\mathrm{tw}(\psi \otimes \mathrm{id}^{\otimes 2})$$
$$\mu(\gamma \otimes \mathrm{id}) = \gamma\pi$$

We say a graded Hopf semiring object A is *connected* if $A_0 = I$, where I is the monoidal unit in $\mathcal{C}$, and γ and π are the inclusion of and projection to I respectively. In other words, γ and π are isomorphisms that are inverse to each other.

Remark 3.1.11. When we say that (μ, ι) is *homogeneous*, we mean that for $d \geq 0$, on each A_d we have a monoid structure (μ_d, ι_d), and (μ, ι) is the sum of these monoid structures. In particular, it is not graded.

Dually, we have the following definition.

Definition 3.1.12. In a symmetric monoidal category $\mathcal{C}$, a *graded Hopf cosemiring object* A is a graded cosemiring object in the category of monoids in $\mathcal{C}$. Explicitly, it is a graded object in $\mathcal{C}$ with a "coaddition" cocommutative graded comonoid structure (θ, ϵ), a "comultiplication" homogeneous cocommutative comonoid structure (ψ, π), and a "codiagonal" commutative graded monoid structure (μ, ι), such that (μ, ι, ψ, π) is a bimonoid (in the ungraded sense), $(\mu, \iota, \theta, \epsilon)$ is a graded bimonoid, and the following codistributivity and coannihilation relations hold:

$$(\mathrm{id} \otimes \theta)\psi = (\mu \otimes \mathrm{id}^{\otimes 2})\mathrm{tw}(\psi \otimes \psi)\theta$$

$$(\epsilon \otimes \mathrm{id})\psi = \iota\epsilon$$

We say a graded Hopf cosemiring object A is *connected* if $A_0 = I$, where I is the monoidal unit in $\mathcal{C}$, and ι and ϵ are the inclusion of and projection to I respectively. In other words, ι and ϵ are isomorphisms that are inverse to each other.

Remark 3.1.13. When we say that (ψ, π) is *homogeneous*, we mean that (ψ, π) is a sum of comonoid structures on A_d for each $d \geq 0$, and in particular, is not graded.

Definition 3.1.14. For a field k, define functors

$$\sigma : \mathcal{V}(k)_*^2 \to \mathcal{V}(k)_*$$

$$(U, W) \mapsto U \oplus W$$

$$\eta : * \to \mathcal{V}(k)_*$$

$$* \mapsto 0$$

$$\delta : \mathcal{V}(k)_* \to \mathcal{V}(k)_*^2$$

$$V \mapsto (V, V)$$

$$\epsilon : \mathcal{V}(k)_* \to *$$

$$V \mapsto *.$$

Remark 3.1.15. If we consider $*$ as a graded object concentrated in degree 0, then σ and η are graded maps. However, δ and ϵ are homogeneous (and not graded).

In the rest of this section, we will show that these maps lead to a structure on $S\mathcal{V}_*$ that satisfies all of the axioms for a graded Hopf cosemiring object, except for the "coaddition-codiagonal" compatibility diagram.

Lemma 3.1.16. *The functors σ, η, δ are all faithful.*

Proof. If $f : V_1 \to V_1'$ and $g : V_2 \to V_2'$ are isomorphisms in $\mathcal{V}_*$, then f and g are determined by the isomorphism $f \oplus g$ on $V_1 \oplus V_2$ by restricting to the subspaces V_1 and V_2 respectively. This shows that σ is faithful.

η is trivially faithful, because $*$ only has one morphism.

If $f : V \to V'$ is an isomorphism in $\mathcal{V}_*$, then $\delta(f) = (f, f)$, so f is determined by projecting to one of the factors. This shows that δ is faithful.

$\square$

Remark 3.1.17. In particular, this result implies that we can form stable transfers of these maps by 2.3.9.

Lemma 3.1.18. *For all fields k, we have a commutative graded monoid structure (σ, η) on $\mathcal{V}(k)_*$ in $HoGpd$.*

Proof. We have

$$\sigma(\sigma \times \mathrm{id})(V_1, V_2, V_3) = (V_1 \oplus V_2) \oplus V_3$$
$$\sigma(\mathrm{id} \times \sigma)(V_1, V_2, V_3) = V_1 \oplus (V_2 \oplus V_3).$$

$\oplus$ is a product in $\mathcal{V}(k)$ so the universal property makes $\mathcal{V}(k)$ into a Cartesian monoidal category, and thus we have an associator natural isomorphism

$$(V_1 \oplus V_2) \oplus V_3 \to V_1 \oplus (V_2 \oplus V_3)$$

so σ is associative. We also have a twist isomorphism

$$V_1 \oplus V_2 \to V_2 \oplus V_1$$

which makes σ commutative. Lastly, we have a unitor isomorphism

$$0 \oplus V \to V$$

which makes η into a unit.

$\square$

Lemma 3.1.19. *For all fields k, we have a cocommutative homogeneous comonoid structure (δ, ϵ) on $\mathcal{V}(k)_*$ in $HoGpd$.*

Proof. To see coassociativity, we have

$$(\delta \times \mathrm{id})\delta(V) = (V, V, V) = (\mathrm{id} \times \delta)\delta(V).$$

To see cocommutativity, we have

$$\tau\delta(V) = (V, V) = \delta(V).$$

To see that ϵ is a counit, we have

$$(\epsilon \times \mathrm{id})\delta(V) = (*, V) \simeq V \simeq (V, *) = (\mathrm{id} \times \epsilon)\delta(V).$$

$\square$

Lemma 3.1.20. *We have a cocommutative graded comonoid structure* $(\sigma^!, \eta^!)$ *on* $S\mathcal{V}_*$.

Proof. By 2.3.9, $(-)^!$ is functorial and by 3.1.16, σ, η are faithful. Each graded piece of the m-fold graded Cartesian product $\mathcal{V}_*^m$ is a finite groupoid, so we can apply $(-)^!$. By taking the transfers of the monoid diagrams of (σ, η) in 3.1.18, we get coassociativity, counitality, and cocommutativity of $(\sigma^!, \eta^!)$.

$\square$

Lemma 3.1.21. *For all fields* k*, we have an (ungraded) bimonoid structure* $(\sigma, \eta, \delta, \epsilon)$ *on* $\mathcal{V}(k)_*$ *in HoGpd.*

Proof. By 3.1.18 and 3.1.19, it remains to prove the compatibility relations.

The first compatibility relation in 3.1.7 follows from the equalities

$$\delta\sigma(V_1, V_2) = (V_1 \oplus V_2, V_1 \oplus V_2)$$

$$\sigma^2 \mathrm{tw} \delta^2(V_1, V_2) = (V_1 \oplus V_2, V_1 \oplus V_2).$$

The second relation follows because $*$ is terminal in Gpds. The third relation follows from

$$\delta\eta(*) = (0, 0) = (\eta \times \eta)(*).$$

The fourth relation follows because $*$ is terminal in Gpds.

$\square$

Lemma 3.1.22. *We have*

$$\eta^!\sigma = \eta^! \wedge \eta^!$$

$$\sigma^!\eta = \eta \wedge \eta$$

$$\eta^!\eta = \mathrm{id}.$$

Moreover,

$$\eta : * \to \mathcal{V}_0$$

$$\eta^! : S\mathcal{V}_0 \to S$$

are equivalences.

Proof. Observe that we have a homotopy commutative diagram

$$\begin{tikzcd} \mathcal{V}_* \times \mathcal{V}_* \arrow[r, "\sigma"] & \mathcal{V}_* \\ * \arrow[u, "\eta^2"] \arrow[r, "\mathrm{id}"'] & * \arrow[u, "\eta"'] \end{tikzcd}.$$

In positive degrees, η is the zero map, and for degree 0, η and σ are essentially identity maps, so in particular, this square is a homotopy pullback square (in each degree). Applying the Mackey property (2.3.9) horizontally and vertically gives the first two relations. Similarly, we also have a pullback diagram

$$\begin{tikzcd} * \arrow[r, "\eta"] & \mathcal{V}_* \\ * \arrow[u] \arrow[r] & * \arrow[u, "\eta"'] \end{tikzcd}.$$

The third relation follows by applying the Mackey property to this. We see that η is an equivalence in degree 0 because $\mathcal{V}_0 \simeq \mathrm{Aut}(0) \simeq *$, and it thus follows by functoriality of the transfer that $\eta^!$ is also an equivalence.

□

Remark 3.1.23. $\eta^! \circ \eta = \mathrm{id}$ implies the connectivity of the algebra and coalgebra structures.

Theorem 3.1.24. *If we let $(\sigma^!, \eta^!)$ be the coaddition comonoid, and (δ, ϵ) be the comultiplication comonoid on $S\mathcal{V}_*$, then the maps*

$$(\sigma^!, \eta^!, \delta, \epsilon, \sigma, \eta)$$

*satisfy all of the relations for a Hopf cosemiring object from 3.1.12, **except** for the compatibility relation*

$$\sigma^! \sigma = \sigma^{\wedge 2} \mathrm{tw} (\sigma^!)^{\wedge 2}.$$

Proof. The amalgamation of 3.1.20, 3.1.21 and 3.1.22 proves all of the relations except for codistributivity and coannihilation.

If we consider the commutative square

$$\begin{tikzcd}
\mathcal{V}_*^3 \arrow[r, "\mathrm{id} \times \sigma"] & \mathcal{V}_*^2 \\
\mathcal{V}_*^4 \arrow[u, "\sigma \times \mathrm{id}"] & \\
\mathcal{V}_*^4 \arrow[u, "\mathrm{tw}"] & \\
\mathcal{V}_*^2 \arrow[u, "\delta^2"] \arrow[r, "\sigma"'] & \mathcal{V}_* \arrow[uuu, "\delta"']
\end{tikzcd}.$$

The fibre of $\mathrm{id} \times \sigma$ at (x, y) is $* \times F_y(\sigma) \simeq F_y(\sigma)$, where $F_y(\sigma)$ is the fibre of σ at y. For all $(x, y) \in \mathcal{V}_*^2$, the induced map between horizontal fibres is an equivalence, and so the diagram is a homotopy pullback.

By applying the Mackey property to this diagram, we have the codistributivity relation

$$(\mathrm{id} \wedge \sigma^!)\delta = (\sigma \wedge \mathrm{id}^{\wedge 2})\mathrm{tw}(\delta \wedge \delta)\sigma^!.$$

For the coannihilation relation, we have a commutative diagram

$$\begin{tikzcd}
\mathcal{V}_* \arrow[r, "\delta"] & \mathcal{V}_* \times \mathcal{V}_* \\
* \arrow[u, "\eta"] \arrow[r, "\eta"'] & \mathcal{V}_* \arrow[u, "\eta \times \mathrm{id}"']
\end{tikzcd}.$$

In positive degrees, η is the zero map, and for degree 0, η and δ are essentially identity maps, so in particular, this square is a homotopy pullback square. By applying the Mackey property to this diagram, we have the coannihilation relation

$$(\eta^! \wedge \mathrm{id})\delta = \eta\eta^!.$$

□

Remark 3.1.25. There is also a monoid structure coming from the tensor product of vector spaces. This combines with $\oplus$ to give a different Hopf semiring structure on $\mathcal{V}_*$, but we will not have a need for this structure.

Remark 3.1.26. If we were to try to prove the compatibility relation

$$\sigma^! \sigma = \sigma^{\wedge 2} \mathrm{tw} (\sigma^!)^{\wedge 2}$$

the obvious strategy would be to show that the diagram

$$\begin{tikzcd}
\mathcal{V}_*^4 \arrow[r, "\sigma^2"] \arrow[d, "\sigma^2 \mathrm{tw}"'] & \mathcal{V}_*^2 \arrow[d, "\sigma"] \\
\mathcal{V}_*^2 \arrow[r, "\sigma"'] & \mathcal{V}_*
\end{tikzcd}$$

is a homotopy pullback (in each degree). The concrete homotopy pullback $\mathcal{P}_*$ is the graded groupoid of 5-tuples

$$(U_1, U_2, W_1, W_2, f)$$

such that f is an isomorphism

$$f : U_1 \oplus U_2 \to W_1 \oplus W_2.$$

There is an obvious functor

$$\mathcal{V}_*^4 \to \mathcal{P}_*$$

$$(V_1, V_2, V_3, V_4) \mapsto (V_1 \oplus V_2, V_3 \oplus V_4, V_1 \oplus V_3, V_2 \oplus V_4, \mathrm{id}).$$

If the subspaces of a vector space formed a distributive lattice, then we could define an inverse in terms of the intersections between the U_i's and W_j's, and this map would be an equivalence. An analogous method works if we replace vector spaces with sets, but in the case of vector spaces, this diagram fails to be a pullback and the compatibility relation fails. We will see in chapter 8 that $\mathcal{V}_*$ satifies an adjusted compatibility relation instead.

Lemma 3.1.27. *The natural map*

$$T : \mathcal{V}_* \rightarrowtail \bar{\mathcal{V}}_*$$

$$V \mapsto \bar{\mathbb{F}} \otimes_{\mathbb{F}} V$$

is a map of graded (σ, η)-monoids and homogeneous (δ, ϵ)-comonoids.

Proof. The preservation of the monoid structure follows because the tensor product naturally distributes over the direct sum, so there is a natural isomorphism

$$\bar{\mathbb{F}} \otimes_{\mathbb{F}} (V_1 \oplus V_2) \cong (\bar{\mathbb{F}} \otimes_{\mathbb{F}} V_1) \oplus (\bar{\mathbb{F}} \otimes_{\mathbb{F}} V_2).$$

There is also an obvious natural isomorphism

$$\bar{\mathbb{F}} \otimes_{\mathbb{F}} 0_{\mathbb{F}} \cong 0_{\bar{\mathbb{F}}}.$$

The preservation of the comonoid structure follows because

$$\delta(T(V)) = (\bar{\mathbb{F}} \otimes_{\mathbb{F}} V, \bar{\mathbb{F}} \otimes_{\mathbb{F}} V) = (T \times T)(\delta(V))$$

$$\epsilon(T(V)) = * = \epsilon(V).$$

□

3.2 Cohomology of Groupoids of Vector Spaces

In this section, we translate the work of the previous section to show that the maps δ, σ, and $\sigma^!$ induce various structures on the Morava E-theory and Morava K-theory. Once we have Künneth isomorphisms, existence of these structures follows from the counterpart structures in the previous section by functoriality. At the end, we also describe the behaviour of σ^*.

Definition 3.2.1. For k a field, we define graded E_0-modules

$$E^0 B\mathcal{V}(k)_* = \bigoplus_{d \geq 0} E^0 B\mathcal{V}(k)_d$$

$$E_0^\vee B\mathcal{V}(k)_* = \bigoplus_{d \geq 0} E_0^\vee B\mathcal{V}(k)_d$$

and graded K_0-vector spaces

$$K^0 B\mathcal{V}(k)_* = \bigoplus_{d \geq 0} K^0 B\mathcal{V}(k)_d$$

$$K_0 B\mathcal{V}(k)_* = \bigoplus_{d \geq 0} K_0 B\mathcal{V}(k)_d.$$

By 2.1.12, when $k = \mathbb{F}$, these are of finite type.

Remark 3.2.2. We treat these as graded objects to take advantage of the finiteness properties in each degree when $k = \mathbb{F}$, and to avoid technical issues relating to completions and infinite products of E_0-modules.

Lemma 3.2.3. *We have graded Künneth isomorphisms* $E_0^\vee B\mathcal{V}_*^m = (E_0^\vee B\mathcal{V}_*)^{\otimes m}$, $E^0 B\mathcal{V}_*^m = (E^0 B\mathcal{V}_*)^{\otimes m}$, $K_0 B\mathcal{V}_*^m = (K_0 B\mathcal{V}_*)^{\otimes m}$, $K^0 B\mathcal{V}_*^m = (K^0 B\mathcal{V}_*)^{\otimes m}$.

Proof. This follows by 2.1.12 and 1.3.14.

□

Theorem 3.2.4. *On $E^0 B\mathcal{V}_*$ and $K^0 B\mathcal{V}_*$ (or with any cohomology theory with appropriate Künneth isomorphisms), if we let $(\sigma_!, \eta_!)$ be the addition monoid, and (δ^*, ϵ^*) be the multiplication monoid, then the maps*

$$(\sigma_!, \eta_!, \delta^*, \epsilon^*, \sigma^*, \eta^*)$$

satisfy all of the relations of a connected graded Hopf semiring, except for the compatibility relation

$$\sigma^* \sigma_! = (\sigma_!)^{\otimes 2} \mathrm{tw} (\sigma^*)^{\otimes 2}.$$

Proof. We have Künneth theorems by 3.2.3, so this follows immediately from 3.1.24 by functoriality.

□

Remark 3.2.5. Here, δ^* is just the cup product.

Dually, we have the following.

Theorem 3.2.6. *On $E_0^\vee B\mathcal{V}_*$ and $K_0 B\mathcal{V}_*$ (or with any homology theory with appropriate Künneth isomorphisms), if we let $(\sigma^!, \eta^!)$ be the coaddition comonoid, and (δ_*, ϵ_*) be the comultiplication comonoid, then the maps*

$$(\sigma^!, \eta^!, \delta_*, \epsilon_*, \sigma_*, \eta_*)$$

satisfy all of the relations of a graded Hopf cosemiring, except for the compatibility relation

$$\sigma^! \sigma_* = (\sigma_*)^{\otimes 2} \mathrm{tw} (\sigma^!)^{\otimes 2}.$$

Proof. We have Künneth theorems by 3.2.3, so this follows immediately from 3.1.24 by functoriality.

□

Lemma 3.2.7. *For $k = \mathbb{C}, \bar{\mathbb{F}}$, we have graded isomorphisms $E^0 B\mathcal{V}(k)^m_* = (E^0 B\mathcal{V}(k)_*)^{\hat{\otimes} m}$, $K^0 B\mathcal{V}(k)^m_* = (K^0 B\mathcal{V}(k)_*)^{\hat{\otimes} m}$, where $\hat{\otimes}$ denotes the graded completed tensor product with respect to the maximal ideals.*

Proof. The case for $k = \mathbb{C}$ is standard: let R be a complex-oriented cohomology theory, then by the Künneth theorem for ordinary cohomology, we get $H^* B\mathcal{V}(\mathbb{C})^m_* = H^* B\mathcal{V}(\mathbb{C})^{\otimes m}_*$. The corresponding Atiyah-Hirzebruch spectral sequence is in even bidegree, so it collapses and it is easy to see that we have

$$R^* B\mathcal{V}(\mathbb{C})^m_* = R^* B\mathcal{V}(\mathbb{C})^{\hat{\otimes} m}_*.$$

As $R^* B\mathcal{V}(\mathbb{C})$ is concentrated in even degrees, we get

$$R^0 B\mathcal{V}(\mathbb{C})^m_* = R^0 B\mathcal{V}(\mathbb{C})^{\hat{\otimes} m}_*.$$

From the map in 1.6.1, we can construct maps

$$B\bar{\mathcal{V}}^m_* \to B\mathcal{V}(\mathbb{C})^m_*.$$

The case for $k = \bar{\mathbb{F}}$ then follows from the complex case by essentially arguing in the same way as section 1.6 that the maps induced on K-theory and E-theory are isomorphisms.

□

Proposition 3.2.8. *For $k = \mathbb{C}, \bar{\mathbb{F}}$, and $R = E, K$, the maps (σ^*, η^*) make $R^0 B\mathcal{V}(k)_*$ into a graded comonoid in the category of graded complete Noetherian local algebras over R^0.*

Proof. The monoidal product in the category of graded complete Noetherian local algebras over a local ring R is the graded completed tensor product with respect to the maximal ideals. The statement then follows by 3.1.18 and 3.2.7.

□

Lemma 3.2.9. *For $k = \mathbb{C}, \bar{\mathbb{F}}$, $R = E, K$, and $s = u + t$, we have*

$$\sigma^* : R^0 B\mathcal{V}(k)_s \to R^0 B\mathcal{V}(k)_t \hat{\otimes}_{R^0} R^0 B\mathcal{V}(k)_u$$

$$c_m \mapsto \sum_{i+j=m} c_i \otimes c_j.$$

Proof. For $k = \mathbb{C}, \bar{\mathbb{F}}$, by the coassociativity of σ^*, we have a commutative diagram

$$\begin{tikzcd}
R^0 B\bar{\mathcal{V}}_s \arrow[rr, "\sigma^*"] \arrow[dr, "\sigma^*"'] & & R^0 B\bar{\mathcal{V}}_1^s \\
& R^0 B\bar{\mathcal{V}}_t \hat{\otimes} R^0 B\bar{\mathcal{V}}_u \arrow[ur, "(\sigma^*)^{\otimes 2}"'] &
\end{tikzcd}$$

where the top and bottom right maps are injective, and so the bottom left map is also injective. The image of c_m under the top map is the mth elementary symmetric polynomial in $x_1, \ldots, x_s$. One can check that this is also where $\sum_{i+j=m} c_i \otimes c_j$ maps to, and so the statement follows.

□

Remark 3.2.10. The maps from 1.6.4 yield isomorphisms of coalgebras

$$K^0 B\bar{\mathcal{V}}_* \cong K^0 B\mathcal{V}(\mathbb{C})_*$$

$$E^0 B\bar{\mathcal{V}}_* \cong E^0 B\mathcal{V}(\mathbb{C})_*.$$

Lemma 3.2.11. *For $R = E, K$, and $s = u + t$, we have*

$$\sigma^* : R^0 B\mathcal{V}_s \to R^0 B\mathcal{V}_t \otimes_{R^0} R^0 B\mathcal{V}_u$$

$$c_m \mapsto \sum_{i+j=m} c_i \otimes c_j.$$

Proof. For $k = \mathbb{F}$, by 3.1.27, 3.2.7, and 3.2.3, we have a commutative diagram

$$\begin{tikzcd}
R^0 B\mathcal{V}(k)_s \arrow[r, "\sigma^*"] \arrow[d, "i^*"'] & R^0 B\mathcal{V}(k)_t \hat{\otimes}_{R^0} R^0 B\mathcal{V}(k)_u \arrow[d, "(i \times i)^*"] \\
R^0 B\mathcal{V}(k)_s \arrow[r, "\sigma^*"'] & R^0 B\mathcal{V}(k)_t \otimes_{R^0} R^0 B\mathcal{V}(k)_u
\end{tikzcd}.$$

$c_m \in R^0 B\mathcal{V}_s$ is defined to be the image of $c_m \in R^0 B\bar{\mathcal{V}}_s$ under i^*, so the result follows by 3.2.9.

□

3.3 Field Extensions and Groupoids of Vector Spaces

In this section, we consider the finite field extensions $\mathbb{F}(m)/\mathbb{F}$. Base changing along these extensions induces maps on the associated groupoids of vector spaces. In particular, we consider the associated forgetful functors and their relationship to the structure on $\mathcal{V}_*$. Finally, we make use of representation theory to compute σ^* in Morava cohomology.

Definition 3.3.1. For $m \geq 1$ we define

$$U_m : \mathcal{V}(m)_* \to \mathcal{V}_*$$

to be the forgetful functor along the map $\mathbb{F} \rightarrowtail \mathbb{F}(m)$. We also define a functor

$$T_m : \mathcal{V}_* \to \mathcal{V}(m)_*$$
$$V \mapsto \mathbb{F}(m) \otimes_{\mathbb{F}} V.$$

Remark 3.3.2. There is no such forgetful functor

$$U : \bar{\mathcal{V}}_* \to \mathcal{V}_*$$

for the extension $\mathbb{F} \rightarrowtail \bar{\mathbb{F}}$ in this context because U would take finite dimensional vector spaces to infinite dimensional vector spaces. However, we already know that there is still a well-defined map

$$T : \mathcal{V}_* \rightarrowtail \bar{\mathcal{V}}_*.$$

Proposition 3.3.3. *For $m \geq 1$, if we grade $\mathcal{V}(m)_*$ by putting $\mathcal{V}(m)_d$ in degree md for all d, then the map*

$$U_m : \mathcal{V}(m)_* \rightarrowtail \mathcal{V}_*$$

is a map of graded monoids in HoGpds *with respect to the structures coming from (σ, η).*

Proof. For a pair of $\mathbb{F}(m)$-vector spaces (V_1, V_2), we have natural inclusion maps $V_i \rightarrowtail V_1 \oplus V_2$ for $i = 1, 2$. Applying U_m, we get maps $U_m(V_i) \rightarrowtail U_m(V_1 \oplus V_2)$ and this induces a natural map

$$U_m(V_1) \oplus U_m(V_2) \to U_m(V_1 \oplus V_2)$$

which is clearly a natural isomorphism. Clearly, we also have a natural isomorphism

$$0 \to U_m(0)$$

which implies that U_m is unital.

∎

Corollary 3.3.4. *For $m \geq 1$, if we grade $E^0 B\mathcal{V}(m)_*$ by putting $E^0 B\mathcal{V}(m)_d$ in degree md for all d, then the map*

$$U_m^* : E^0 B\mathcal{V}_* \to E^0 B\mathcal{V}(m)_*$$

is a map of graded coalgebras with respect to the structures coming from (σ^*, η^*).

Proof. As we have Künneth isomorphisms, this follows immediately from 3.3.3.

□

Definition 3.3.5. Write $V(m)_d$ for the canonical d-dimensional $\mathbb{F}(m)$-linear representation of $GL_d(\mathbb{F}(m))$ (or equivalently, $\mathcal{V}(m)_d$). This is induced by the identity map

$$\mathrm{id} : \mathcal{V}(m)_d \to \mathcal{V}(m)_d.$$

Write $U(m)_d$ for $V(m)_d$ treated as an $\mathbb{F}$-linear representation of $GL_d(\mathbb{F}(m))$. This is induced by the map of groupoids

$$U_m : \mathcal{V}(m)_d \to \mathcal{V}_{md}.$$

Lemma 3.3.6. *For extensions of fields $F/L/K$ such that L/K is finite and Galois with Galois group G, we have an isomorphism of rings*

$$F \otimes_K L \to \mathrm{Hom}_F(F[G], F) = \mathrm{Map}(G, F)$$

$$a \otimes b \mapsto (g \mapsto a.g(b))$$

where $\mathrm{Map}(G, F)$ *has pointwise addition and multiplication.*

Proof. This is standard. We prove it as in [24, Lemma 5.17], but with a little more detail. For each $g \in G$, we have a field automorphism, and so we can construct an extension

$$L \xrightarrow{g} L \to F.$$

Putting these together, we can form the product of these maps

$$\prod_{g \in G} g : L \to \prod_{g \in G} F \cong \mathrm{Map}(G, F) = \mathrm{Hom}_F(F[G], F).$$

If $i : L \rightarrowtail F$ is the given extension, we can also form a map

$$\prod_{g \in G} i : L \to \prod_{g \in G} F \cong \mathrm{Map}(G, F) = \mathrm{Hom}_L(L[G], L).$$

It is easy to see that K equalises these maps, so we get an induced injective map of rings

$$F \otimes_K L \to \mathrm{Map}(G, F) = \mathrm{Hom}_F(F[G], F)$$
$$a \otimes b \mapsto (g \mapsto a.g(b)).$$

As this is a map of finite dimensional vector spaces, it suffices to show that the F-linear dual is an isomorphism. This is the F-linear map

$$F[G] \to \mathrm{Hom}_K(L, F)$$
$$g \mapsto (g^* : L \to L).$$

This is injective, because g^* is injective, so by [2, Theorem 41.1], $\{g^*\}$ is linearly independent in $\mathrm{Hom}_K(L, F)$. By considering dimensions over F, the map must be an isomorphism.

$\square$

Proposition 3.3.7. *We have an isomorphism of $\bar{\mathbb{F}}$-linear representations of $\mathcal{V}(m)_d$*

$$\bar{\mathbb{F}} \otimes_{\mathbb{F}} U(m)_d = \bigoplus_{i=0}^{m-1} (F_q^i)_*(\bar{\mathbb{F}} \otimes_{\mathbb{F}(m)} V(m)_d).$$

where F_q is the Frobenius automorphism as in section 1.5.

Proof. If we apply 3.3.6 to the extensions $\bar{\mathbb{F}}/\mathbb{F}(m)/\mathbb{F}$ then we get an isomorphism of rings

$$\bar{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}(m) \to \mathrm{Map}(\Gamma_m, \bar{\mathbb{F}}).$$

In particular, this gives us an isomorphism of left $\bar{\mathbb{F}}$-modules and right $\mathbb{F}(m)$-modules

$$\bar{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}(m) \to \bigoplus_{i=0}^{m-1} (F_q^i)_*(\bar{\mathbb{F}})$$

where, if $\mathbb{F}(m) \xrightarrow{j} \bar{\mathbb{F}}$ is the canonical inclusion, then $(F_q^i)_*(\bar{\mathbb{F}}) = \bar{\mathbb{F}}$ has the right $\mathbb{F}(m)$-module structure given by

$$\bar{\mathbb{F}} \otimes \mathbb{F}(m) \xrightarrow{id \otimes F_q^i} \bar{\mathbb{F}} \otimes \mathbb{F}(m) \xrightarrow{id \otimes j} \bar{\mathbb{F}} \otimes \bar{\mathbb{F}} \xrightarrow{m} \bar{\mathbb{F}}.$$

Tensoring up with an $\mathbb{F}(m)$-vector space V will also give an isomorphism, and so in particular, we have an isomorphism of left $\bar{\mathbb{F}}$-modules and right $\mathbb{F}(m)$-modules

$$\bar{\mathbb{F}} \otimes_{\mathbb{F}} U(m)_d \to \bigoplus_{i=0}^{m-1} (F_q^i)_*(\bar{\mathbb{F}}) \otimes_{\mathbb{F}(m)} V(m)_d = \bigoplus_{i=0}^{m-1} (F_q^i)_*(\bar{\mathbb{F}} \otimes_{\mathbb{F}(m)} V(m)_d).$$

In other words, this is an $\bar{\mathbb{F}}$-linear isomorphism of $\mathcal{V}(m)_d$-representations.

□

Proposition 3.3.8. *We have*

$$U_m^* : E^0 B\mathcal{V}_{md} \to E^0 B\mathcal{V}(m)_d$$
$$c_i \mapsto \sigma_i(x_1, [q](x_1), \ldots, [q^{d-1}](x_1), x_2, [q](x_2), \ldots, [q^{d-1}](x_m)).$$

Proof. By 3.3.7, we have an isomorphism of $\bar{\mathbb{F}}$-linear representations of $\mathcal{V}(m)_d$

$$\bar{\mathbb{F}} \otimes_{\mathbb{F}} U(m)_d \to \bigoplus_{i=0}^{m-1} (F_q^i)_*(\bar{\mathbb{F}}) \otimes_{\mathbb{F}(m)} V(m)_d = \bigoplus_{i=0}^{m-1} (F_q^i)_*(\bar{\mathbb{F}} \otimes_{\mathbb{F}(m)} V(m)_d).$$

It follows that we have a homotopy commutative diagram of groupoids

$$\begin{tikzcd} \mathcal{V}(m)_d \arrow[r, "U"] \arrow[d, "((F_q^i)_*)_{i=0}^{m-1}"'] & \mathcal{V}_{md} \arrow[d, "T"] \\ \bar{\mathcal{V}}_d^m \arrow[r, "\sigma_m"'] & \bar{\mathcal{V}}_{md} \end{tikzcd}$$

where $(F_q^i)_*$ is induced by tensoring along the map

$$\mathbb{F}(m) \xrightarrow{F_q^i} \mathbb{F}(m) \xrightarrow{j} \bar{\mathbb{F}}.$$

Thus we get a commutative diagram

$$\begin{tikzcd} E^0 B\mathcal{V}(m)_d & E^0 B\mathcal{V}_{md} \arrow[l, "U^*"'] \\ E^0 B\bar{\mathcal{V}}_d^m \arrow[u, "\amalg_{i=0}^{m-1} (F_q^i)^*"] & E^0 B\bar{\mathcal{V}}_{md} \arrow[l, "\sigma_m^*"] \arrow[u, "T^*"'] \end{tikzcd}.$$

As T^* is surjective by 1.7.5, the description of U^* follows straightforwardly from the behaviour of F_q^* as explained in 1.7.1.

□

Remark 3.3.9. The induced map

$$U_m^* : E^0 B\mathcal{V}_{m*} \to E^0 B\mathcal{V}(m)_*$$

preserves a lot of the structure available, but we will not need to know this for our purposes.

Chapter 4

Formal Algebraic Geometry

The main aim of this chapter is to consider some relevant examples of formal schemes that arise from studying the Morava cohomology of groups. There is a large amount that can be said about the theory of formal schemes and the examples that appear in topology (see [37], [38]). As such, in this thesis we restrict ourselves to essential theory and work in a simplistic setting, using complete Noetherian semi-local rings for our foundations.

We look at $\mathrm{spf}(E^0BG)$ for groups G, because it turns out that the geometry often reflects the topology. In particular, classifying spaces are essentially moduli objects and the corresponding schemes often have good descriptions as moduli schemes related to the formal group $\mathrm{spf}(E^0BS^1)$.

In the first section, we make some basic definitions and highlight some fundamental results that we will need.

In the second section, we give a brief exposition of the standard application of Galois theory to integrally closed domains. If we have a smooth formal scheme with a group action, then this provides a useful method for determining the function ring of the quotient by that action.

In the remainder of the sections, we look at some specific formal schemes that arise from studying the Morava cohomology of relevant groups.

In the third section, we show that the groups S^1 and $GL_1(\bar{\mathbb{F}})$ yield one-dimensional commutative formal group schemes (i.e. formal groups) $\mathbb{G}$ and $\mathbb{H}$ respectively.

In the fourth section, we study divisors on formal curves as in [37]. In particular, for the curve $\mathbb{H}$, the formal scheme of divisors of degree d is naturally isomorphic to the formal scheme coming from $\bar{\mathcal{V}}_d$.

In the fifth section, we consider the "multiplication by m" maps on formal groups, and their kernels. In the case of $\mathbb{H}$, these are naturally related to the formal schemes coming from the groups $GL_1(\mathbb{F}(m))$.

In the last section, in the presence of a formal group, we consider formal schemes that morally represent "elements of order p^k". By considering transfers, we see that these formal schemes also manifest topologically in the case of $\mathbb{H}$.

4.1 Formal Schemes

In this section, we define the foundations that we will be using in this chapter.

We will take a functorial approach to affine algebraic geometry. A more advanced theory of formal schemes with a view to applications in algebraic topology has been developed in [38], but for simplicity, we restrict attention to what might otherwise be termed "local affine formal schemes".

Recall that a *semi-local ring* is a ring with a finite number of maximal ideals.

Definition 4.1.1. We write FRings_{loc} for the category of complete Noetherian semi-local rings and local homomorphisms.

We define a formal scheme to be a representable functor

$$X : \mathrm{FRings}_{loc}^{op} \to \mathrm{Sets}$$

and write FSchemes for the category of formal schemes and natural transformations. For a formal scheme X, we will also write $\mathrm{FSchemes}_X$ for the category of formal schemes over X

Definition 4.1.2. For a complete Noetherian semi-local ring R, we write $\mathrm{spf}(R)$ for the formal scheme represented by R.

Example 4.1.3. For $n \in \mathbb{N}$, define a functor

$$\hat{\mathbb{A}}^n : \mathrm{FRings}_{loc}^{op} \to \mathrm{Sets}$$
$$R \mapsto J(R)^n$$

where $J(R)$ is the Jacobson radical of R. This is canonically isomorphic to $\mathrm{spf}(\mathbb{Z}[\![x_1, \ldots, x_n]\!])$.

Example 4.1.4. For a finite groupoid G, we know by 2.1.12 that E^0BG is a complete Noetherian semi-local ring over E^0, and thus we get a formal scheme $\mathrm{spf}(E^0BG)$ over $\mathrm{spf}(E^0)$.

Note that we can also form an ordinary scheme $\mathrm{spec}(E^0BG)$, but by doing so we lose information.

Lemma 4.1.5. *We have an equivalence of categories*

$$\mathrm{spf} : \mathrm{FRings}_{loc}^{op} \xrightarrow{\sim} \mathrm{FSchemes}.$$

Proof. This essentially follows from the Yoneda lemma.

□

Definition 4.1.6. Given a formal scheme $X = \mathrm{spf}(R)$, we write $\mathcal{O}_X := R$ when we wish to refer to the representing object of X, so that $\mathrm{spf}(\mathcal{O}_X) = X$ and $\mathcal{O}_{\mathrm{spf}(R)} = R$.

We say that X is *connected* if $\mathcal{O}_X$ is local.

We define the *special fibre*, X_0, of $X = \mathrm{spf}(R)$ to be $\mathrm{spf}(R/J(R))$.

Example 4.1.7. The special fibre of $\mathrm{spf}(E^0)$ is $\mathrm{spf}(K^0)$.

Definition 4.1.8. For a formal scheme X, the *dimension* of X, $\dim(X)$ is defined to be the Krull dimension of $\mathcal{O}_X$.

Definition 4.1.9. A map of formal schemes $X \xrightarrow{f} Y$ is finitely generated/flat/very flat if $f^* : \mathcal{O}_Y \to \mathcal{O}_X$ makes $\mathcal{O}_X$ into a finitely generated/flat/free $\mathcal{O}_Y$-module.

Remark 4.1.10. If X, Y are connected, then flatness is equivalent to very flatness because of the complete local property.

Recall that a ring is *regular* if its localisation at each prime ideal is a regular local ring.

Definition 4.1.11. We say that a formal scheme X is *smooth* if $\mathcal{O}_X$ is a regular semi-local ring.

Definition 4.1.12. A map of formal schemes $X \xrightarrow{f} Y$ is a *closed inclusion* if $f^* : \mathcal{O}_Y \to \mathcal{O}_X$ is surjective. If f is the inclusion of a subscheme and also a closed inclusion, then we say that X is a *closed subscheme* of Y.

A map of formal schemes $X \xrightarrow{f} Y$ is an *epimorphism* if $f^* : \mathcal{O}_Y \to \mathcal{O}_X$ is injective.

Lemma 4.1.13. *If X, Y are formal schemes that are finite over Z, then $X \times_Z Y$ is also a formal scheme, and its ring of functions is given by*

$$\mathcal{O}_{X \times Y} = \mathcal{O}_X \hat{\otimes}_{\mathcal{O}_Z} \mathcal{O}_Y$$

where the right hand side refers to the completed tensor product.

Proof. The tensor product is the coproduct in the category of Noetherian local rings, and for a Noetherian local ring R, the completion map $R \to \hat{R}$ is the universal local map to a complete Noetherian local ring. Combining these two facts shows that the completed tensor product is the coproduct in the category of complete semilocal Noetherian R-algebras.

□

Lemma 4.1.14. *If X, Y are formal schemes over Z such that X is finite over Z, then the natural map*

$$\mathcal{O}_X \otimes_{\mathcal{O}_Z} \mathcal{O}_Y \to \mathcal{O}_X \hat{\otimes}_{\mathcal{O}_Z} \mathcal{O}_Y$$

is an isomorphism.

Proof. This can be rephrased as: if R is a complete semi-local Noetherian ring and S, T are complete semi-local Noetherian R-algebras such that S is finitely generated over R, then the natural map

$$S \otimes_R T \to S \hat{\otimes}_R T$$

is an isomorphism. It is easy to see that $S \otimes_R T$ is a finitely generated module over T, so by A.0.12, it must be a complete semi-local ring.

□

4.2 Galois Theory

In this section, we recall some important results about the Galois theory of integrally closed domains. This is essentially a direct application of the Galois theory of fields. Later on, these results will be useful for determining the rings of functions of certain formal schemes.

Definition 4.2.1. We will say that a finite free extension S/R is *Galois* if the $\mathrm{Aut}_R(S)$-fixed subring of S is equal to R, i.e.

$$R = S^{\mathrm{Aut}_R(S)}.$$

In this case, we define the *Galois group* of the extension to be

$$Gal(S/R) := \mathrm{Aut}_R(S).$$

Remark 4.2.2. This agrees with the usual definition of a Galois extension of fields.

Definition 4.2.3. If $f : X \twoheadrightarrow Y$ is an epimorphism of formal schemes, then we say that f is a *Galois covering* if $f^* : \mathcal{O}_Y \rightarrowtail \mathcal{O}_X$ is a Galois extension.

Definition 4.2.4. For a finite free extension S/R, define the *degree* to be

$$[S : R] := \mathrm{rk}_R(S).$$

Remark 4.2.5. This agrees with the usual definition for field extensions. It is also easy to see that the degree is multiplicative, i.e. for finite, free extensions S/R and T/S we have

$$[T : R] = [T : S].[S : R].$$

Proposition 4.2.6. *If S/R is a finite free extension of integrally closed domains, then S/R is Galois iff the extension $Frac(S)/Frac(R)$ of fraction fields is Galois.*

When this is true, we have

$$Gal(S/R) = Gal(Frac(S)/Frac(R))$$

$$[S : R] = [Frac(R) : Frac(S)] = |Gal(Frac(S)/Frac(R))| = |Gal(S/R)|.$$

Proof. This is standard and essentially proved in [37, Lemma 2.5, Lemma 2.6].

Write $K := Frac(R)$, $L := Frac(S)$, $H := \mathrm{Aut}_R(S)$, $G := \mathrm{Aut}_K(L)$. As S is free over R of degree $d := [S : R]$, by A.0.6 we have

$$L = S \otimes_R K \cong R^d \otimes_R K = K^d$$

so $[S : R] = [L : K]$. As $L = S \otimes_R K$, tensoring along $R \rightarrowtail K$ gives us an injection

$$H \rightarrowtail G$$

so $|H| \leq |G|$.

If we assume that S/R is Galois, then by A.0.7 we have

$$L^H = (S \otimes_R K)^H = S^H \otimes_R K = R \otimes_R K = K.$$

It follows that we also have $L^G = K$, and so L/K is Galois with $|G| = d$. Moreover, by the Fundamental theorem of Galois theory, we must have $H = G$. This proves one direction and the final statements.

If we assume that L/K is Galois, then $|G| = [L : K] = d$. Suppose $f \in G$. Then as S is finitely generated over R, it is integral over R, and so $f(S)$ is integral over $f(R) = R$ and thus also integral over S. As S is integrally closed, this implies that $f(S) = S$, and we see that $H = G$. We have

$$R \subseteq S^G \subseteq K$$

but as S is integral over R, S^G is also integral over R. As R is integrally closed, it follows that $S^G = R$ and thus S/R is Galois.

□

Remark 4.2.7. By A.0.14 and A.0.15, we can use Galois theory to study regular local rings/smooth schemes.

4.3 Formal Groups

In this section, we study formal group schemes, particularly over a local ring with characteristic p residue field. We show that the groups S^1 and $GL_1(\bar{\mathbb{F}})$ yield formal groups $\mathbb{G}$ and $\mathbb{H}$ respectively.

Definition 4.3.1. A *one-dimensional commutative formal group* $\mathbb{G}$ over a formal scheme X is a commutative group in the category $\mathrm{FSchemes}_X$ such that, as a formal scheme, $\mathbb{G}$ is abstractly isomorphic to $\hat{\mathbb{A}}^1 \times X$. In other words, $\mathcal{O}_{\mathbb{G}} \cong \mathcal{O}_X[\![x]\!]$.

Remark 4.3.2. From now on, when we refer to "formal groups", they will always be one-dimensional and commutative. In any other situation, we will refer to a group in formal schemes as a "formal group scheme".

Definition 4.3.3. For X a formal scheme and $\mathbb{G}$ a formal group over X, a *coordinate* for $\mathbb{G}$ is a choice of map

$$x : \mathbb{G} \to \hat{\mathbb{A}}^1$$

that induces an isomorphism

$$\mathbb{G} \xrightarrow{\sim} \hat{\mathbb{A}}^1 \times X.$$

In other words, it is a choice of $x \in \mathcal{O}_{\mathbb{G}}$ such that $\mathcal{O}_{\mathbb{G}} = \mathcal{O}_X[\![x]\!]$.

Remark 4.3.4. Equivalently, a coordinate is a generator for $\mathrm{Ker}(0^* : \mathcal{O}_{\mathbb{G}} \to \mathcal{O}_X)$, where 0 is the unit of the group $\mathbb{G}$.

Remark 4.3.5. If we choose a coordinate x, then by considering the multiplication map

$$\mathcal{O}_{\mathbb{G}} \to \mathcal{O}_{\mathbb{G}\times\mathbb{G}} = \mathcal{O}_{\mathbb{G}} \hat{\otimes}_{\mathcal{O}_X} \mathcal{O}_{\mathbb{G}}$$

$$x \mapsto x_1 +_F x_2$$

we get a *formal group law* $x_1 +_F x_2$.

Lemma 4.3.6. *Let X be a formal scheme such that the special fibre X_0 has characteristic p. If $f : \mathbb{H} \to \mathbb{G}$ is a non-zero homomorphism of formal groups over X, then f is finite and flat of degree p^h for some $h \geq 0$.*

Moreover, if x and y are coordinates on $\mathbb{H}$ and $\mathbb{G}$ respectively, then there exists $0 \neq u \in \mathcal{O}_{X_0}$ such that

$$f_0^*(y) \equiv ux^{p^h} \mod x^{p^h+1}.$$

In particular, $f^*(y)$ has Weierstrass degree p^h (A.0.2).

Proof. [37, Lemma 3.1, Proposition 3.2].

□

Definition 4.3.7. With the context of 4.3.6, we define h to be the *height* of the homomorphism f. We define the *height* of $\mathbb{G}$ to be the height of the "multiplication by p" endomorphism $p : \mathbb{G} \to \mathbb{G}$.

Lemma 4.3.8. *The formal scheme $\mathbb{G}_E := \mathrm{spf}(E^0BS^1)$ is a formal group over $\mathrm{spf}(E^0)$ and $\mathbb{G}_K := \mathrm{spf}(K^0BS^1)$ is a formal group over $\mathrm{spf}(K^0)$. Moreover, $\mathbb{G}_K$ is the pullback of $\mathbb{G}_E$ along the special fibre map $\mathrm{spf}(K^0) \to \mathrm{spf}(E^0)$.*

Proof. The group structure is induced functorially from the group structure of S^1. By 1.1.2, we see that both instances are formal groups. The last statement follows from the fact that

$$K^0BS^1 = E^0 \otimes_{K^0} E^0BS^1.$$

□

Remark 4.3.9. From this perspective, a *complex orientation* on E/K induces a coordinate on $\mathbb{G}_E/\mathbb{G}_K$, which yields a formal group law as discussed earlier.

Lemma 4.3.10. *The formal scheme $\mathbb{H}_E := \mathrm{spf}(E^0BGL_1(\bar{\mathbb{F}}))$ is a formal group over $\mathrm{spf}(E^0)$ and $\mathbb{H}_K := \mathrm{spf}(K^0BGL_1(\bar{\mathbb{F}}))$ is a formal group over $\mathrm{spf}(K^0)$. Moreover, $\mathbb{H}_K$ is the pullback of $\mathbb{H}_E$ along the special fibre map $\mathrm{spf}(K^0) \to \mathrm{spf}(E^0)$.*

Proof. The group structure is induced functorially from the group structure of $GL_1(\bar{\mathbb{F}})$. By 1.6.10, we see that both instances are formal groups. The last statement follows from the fact that

$$K^0 BGL_1(\bar{\mathbb{F}}) = E^0 \otimes_{K^0} E^0 BGL_1(\bar{\mathbb{F}}).$$

$\square$

Remark 4.3.11. From this perspective, an $\bar{\mathbb{F}}$-*orientation* on E/K induces a coordinate on $\mathbb{H}_E/\mathbb{H}_K$, which yields a formal group law as discussed earlier.

Remark 4.3.12. The action of Γ on $GL_1(\bar{\mathbb{F}})$ induces an action on $\mathbb{H}_R$. By 1.7.1, for a point $a \in \mathbb{H}_R$, the Frobenius automorphism acts as $F_q(a) = q.a$, i.e. multiplication by q.

Remark 4.3.13. When the context is clear, we will write $\mathbb{H}$ for $\mathbb{H}_E$.

Lemma 4.3.14. *$\mathbb{H}_E$ is non-canonically isomorphic to $\mathbb{G}_E$, and $\mathbb{H}_K$ is non-canonically isomorphic to $\mathbb{G}_K$.*

Proof. Recall that a choice of injection $\bar{\mathbb{F}}^\times \rightarrowtail S^1$ induces isomorphisms

$$E^0(BS^1) \cong E^0(BGL_1(\bar{\mathbb{F}}))$$
$$K^0(BS^1) \cong K^0(BGL_1(\bar{\mathbb{F}}))$$

by 1.6.4 and 1.6.3.

$\square$

Remark 4.3.15. We mentioned earlier that when talking about general linear groups over $\mathbb{F}$, it is natural to base everything around $GL_1(\bar{\mathbb{F}}) = \bar{\mathbb{F}}^\times$ instead of S^1.

As a result of this, when working with $\bar{\mathbb{F}}$-based objects, it will be more natural to make statements about the corresponding formal schemes in terms of $\mathbb{H}_E$ and $\mathbb{H}_K$ than in terms of $\mathbb{G}_E$ and $\mathbb{G}_K$.

4.4 Divisors on Curves

In this section, we study divisors on a formal curve. We show that in the case of $\mathbb{H}$, the formal schemes of divisors on $\mathbb{H}$ are related to the Morava cohomology $\bar{\mathcal{V}}_*$.

Definition 4.4.1. We say that a formal scheme C over X is a *formal curve* over X if C is isomorphic to $\hat{\mathbb{A}}^1 \times X$. In other words, $\mathcal{O}_C \cong \mathcal{O}_X[\![x]\!]$ for some $x \in \mathcal{O}_C$.

Example 4.4.2. By definition, the underlying scheme of a formal group is a formal curve.

Definition 4.4.3. For a formal curve C over a formal scheme X, we define a *divisor* to be a closed formal subscheme $D \subseteq C$ such that D is finite and flat over X. We define the *degree* of D to be the rank of $\mathcal{O}_D$ as an $\mathcal{O}_X$-module.

If Y is a formal scheme over X, we call a divisor D on the formal curve $C \times_X Y$ a *divisor over* Y.

Remark 4.4.4. Elsewhere, these divisors would be described as "effective", but these are the only divisors that will concern us so we stick with this terminology.

Definition 4.4.5. For a formal curve C over a formal scheme X, define a functor

$$\mathrm{Div}_m(C) : \mathrm{FSchemes}_X^{op} \to \mathrm{Sets}$$

$$Y \mapsto \{D \,|D \text{ is a divisor over } Y \text{ of degree } m\}.$$

Lemma 4.4.6. *For C a formal curve over a formal scheme X, $\mathrm{Div}_m(C)$ is a formal scheme. In particular, a choice of coordinate on C induces an isomorphism $\mathrm{Div}_m(C) \cong \hat{\mathbb{A}}^m_X$. In terms of function rings, if x is a coordinate on C, then we have an isomorphism*

$$\mathcal{O}_{\mathrm{Div}_m(C)} = \mathcal{O}_X[\![c_1, \ldots, c_m]\!].$$

Proof. We prove this as in [37, Section 4].

Define a functor

$$\mathrm{MPoly}_m(C) : \mathrm{FSchemes}_X^{op} \to \mathrm{Sets}$$

by sending a scheme Y over X to the set of monic polynomials $f(x)$ of degree m with coefficients in $\mathcal{O}_Y$ such that $f(x) \equiv x^m \bmod J_Y$, where J_Y is the Jacobian of $\mathcal{O}_Y$. If we fix a coordinate x on C, then for $f(x) \in \mathrm{MPoly}_m(C)$ it is easy to see that $\mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{O}_C/f(x)$ is finitely generated and free over $\mathcal{O}_Y$ with a basis $\{1, x, \ldots, x^{m-1}\}$. Thus we get a natural map

$$\mathrm{MPoly}_m(C) \to \mathrm{Div}_m(C)$$

$$f(x) \mapsto \mathrm{spf}(\mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{O}_C/f(x)).$$

Given a divisor D on C over Y, we have an isomorphism of $\mathcal{O}_Y$-modules $\mathcal{O}_D \cong \mathcal{O}_Y^m$, so we define $f_D(x)$ to be the characteristic polynomial of the "multiplication by x" map on $\mathcal{O}_D$. By Cayley-Hamilton, multiplication by $f_D(x)$ is zero on $\mathcal{O}_D$, and in particular $f_D(x).1 = 0$, so we have a surjection

$$\mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{O}_C/f_D(x) \to \mathcal{O}_D.$$

$f_D(x)$ is a monic polynomial of degree m, so $\mathcal{O}_C/f_D(x)$ has rank $\leq m$ over $\mathcal{O}_Y$, but as the above map is surjective, it must have rank $\geq m$, so the rank of $\mathcal{O}_C/f_D(x)$ must be equal to m and the above map must be an isomorphism. It follows that we must also have $f_D(x) \equiv x^m \bmod J_Y$.

This gives an natural map

$$\mathrm{Div}_m(C) \to \mathrm{MPoly}_m(C).$$

It is straightforward to see that this is inverse to the previous map. We can also construct a natural map

$$\mathrm{MPoly}_m(C) \to \hat{\mathbb{A}}^m$$
$$\sum_{i=0}^{m}(-1)^i c_i x^{m-i} \mapsto (c_1, c_2, \ldots, c_m)$$

which is clearly an isomorphism of functors. The result follows.

□

Definition 4.4.7. If x is a coordinate on C and D is a divisor on C, then we define the *equation* of D to be the characteristic polynomial $f_D(x)$ (as in 4.4.6).

Definition 4.4.8. If D_1 and D_2 are divisors on a curve C with a coordinate x, then we define their sum $D_1 + D_2$ to be the divisor defined by the equation $f_{D_1}(x)f_{D_2}(x)$. By applying A.0.3, we find that this is independent of our choice of coordinate.

Definition 4.4.9. Let C be a formal curve over a formal scheme X, and let x be a coordinate on C. If Y is a formal scheme over X, and $a \in C(Y)$, define $[a]$ to be the divisor over Y with equation $f_{[a]}(x) = x - a^*(x)$. If $S \subseteq C(Y)$ is a finite subset, define a divisor over Y

$$[S] := \sum_{a \in S} [a].$$

This has equation

$$f_S(x) = \prod_{a \in S} f_{[a]}(x) = \prod_{a \in S} (x - a^*(x)).$$

Lemma 4.4.10. *For C a formal curve over X, we have a natural isomorphism*

$$C^m/\Sigma_m \xrightarrow{\sim} \mathrm{Div}_m(C).$$

If we fix a coordinate, then this is given by

$$\mathcal{O}_X[\![c_1, \ldots, c_m]\!] \xrightarrow{\sim} \mathcal{O}_X[\![x_1, \ldots, x_m]\!]^{\Sigma_m}$$

$$c_i \mapsto \sigma_i(x_1, \dots, x_m)$$

where σ_i is the ith elementary symmetric polynomial.

Proof. We prove this as in [37, Section 4].

We can define a natural transformation

$$C^m \to \mathrm{Div}_m(C)$$

$$(a_1, \dots, a_m) \mapsto \sum_{i=1}^{m} [a_i].$$

If we fix a coordinate x on C, then the equation of $\sum_{i=1}^{m} [a_i]$ is

$$\prod_{i=1}^{m} (x - a_i^*(x))$$

so the map on rings of functions is given by

$$\mathcal{O}_X[\![c_1, \dots, c_m]\!] = \mathcal{O}_X[\![x_1, \dots, x_m]\!]$$

$$c_i \mapsto \sigma_i(x_1, \dots, x_m).$$

The result then follows by the fundamental theorem of symmetric power series.

□

Lemma 4.4.11. *For $R = E, K$, we have a natural isomorphism over* $\mathrm{spf}(R^0)$

$$\mathrm{spf}(R^0 B\bar{\mathcal{V}}_m) = \mathrm{Div}_m(\mathbb{H}_R).$$

Proof. Write $\mathbb{H}$ for $\mathbb{H}_R$. The map $\sigma : \bar{\mathcal{V}}_1^m \to \bar{\mathcal{V}}_m$ induces a map

$$\mathbb{H}^m = \mathrm{spf}(R^0(B\bar{\mathcal{V}}_1^m)) \to \mathrm{spf}(R^0 B\bar{\mathcal{V}}_m).$$

By the commutativity of σ, this is clearly Σ_m-invariant, and so factors through a map

$$\mathbb{H}^m/\Sigma_m \to \mathrm{spf}(E^0 B\bar{\mathcal{V}}_m).$$

The corresponding map on function rings is then an isomorphism by 1.6.10.

If we fix an $\bar{\mathbb{F}}$-orientation x for R, then this gives a coordinate for $\mathbb{H}$, and $\bar{\mathbb{F}}$-Chern classes are symmetric polynomials in the x_i.

□

Remark 4.4.12. In a similar way, we can show that for $R = E, K$ we have a natural isomorphism over $\mathrm{spf}(R^0)$

$$\mathrm{spf}(R^0 B\mathcal{V}(\mathbb{C})_m) = \mathrm{Div}_m(\mathbb{G}_R)$$

using 1.1.2.

Definition 4.4.13. For $R = E, K$, given an m-dimensional $\bar{\mathbb{F}}$-vector bundle V over a space X, we have a corresponding map $X \to B\bar{\mathcal{V}}_m$, and thus a map

$$\mathcal{O}_{\mathrm{Div}_m(\mathbb{H}_R)} = R^0 B\bar{\mathcal{V}}_m \to R^0 X.$$

If $R^0 X$ is a complete Noetherian semi-local R^0-algebra, then we get a divisor $D(V)$ of $\mathbb{H}_R$ over $\mathrm{spf}(R^0 X)$. If we fix a coordinate, then its equation is

$$f_D(x) = \sum_{i=0}^{m} (-1)^i c_i(V) x^{m-i}$$

where $c_i(V)$ is the ith Chern class of V.

Remark 4.4.14. The inverse of the map constructed in 4.4.11 is induced by the classifying map for the divisor $D(V_m)$, where V_m is the canonical $\bar{\mathbb{F}}$-linear representation of $\bar{\mathcal{V}}_m$.

Lemma 4.4.15. *For $R = E, K$ we have a natural isomorphism over* $\mathrm{spf}(R^0)$

$$\mathrm{spf}(R^0 B\mathcal{V}_m) = \mathrm{Div}_m(\mathbb{H}_R)^\Gamma.$$

Proof. This follows from 4.4.11, 1.7.2 and 1.7.5.

□

Definition 4.4.16. For a formal curve C over a formal scheme X, write $\mathrm{Div}_*(C)$ for the graded formal scheme with degree m part $\mathrm{Div}_m(C)$.

Proposition 4.4.17. *For $R = E, K$, the maps $(\mathrm{spf}(\sigma^*), \mathrm{spf}(\eta^*))$ make $\mathrm{spf}(R^0 B\bar{\mathcal{V}}_*) = \mathrm{Div}_*(\mathbb{H}_R)$ and $\mathrm{spf}(R^0 B\mathcal{V}_*) = \mathrm{Div}_*(\mathbb{H}_R)^\Gamma$ into graded monoids in the category of formal schemes.*

Proof. This follows immediately from 3.2.8, 3.2.4.

□

Remark 4.4.18. If we write

$$+ := \mathrm{spf}(\sigma^*) : \mathrm{Div}_*(\mathbb{H}_R) \times \mathrm{Div}_*(\mathbb{H}_R) \to \mathrm{Div}_*(\mathbb{H}_R)$$

and fix a coordinate x for $\mathbb{H}_R$, then by 3.2.9, we find that $f_{D+D'}(x) = f_D(x)f_{D'}(x)$. In particular, this implies that this map sends a pair of divisors to their sum. It is then unsurprising that the sum of two Γ-invariant divisors is Γ-invariant.

Remark 4.4.19. Similarly, there is a graded monoid structure on $\mathrm{spf}(R^0 B\mathcal{V}(\mathbb{C})_*) = \mathrm{Div}_*(\mathbb{G}_R)$ that behaves like addition of divisors, using 4.4.12.

Proposition 4.4.20. *For $R = E, K$, the map*

$$\mathrm{spf}(U_m^*) : \mathrm{Div}_*(\mathbb{H}_R)^{\Gamma(m)} \to \mathrm{Div}_{m*}(\mathbb{H}_R)^{\Gamma}$$

is a map of graded monoids in the category of formal schemes with respect to the structures from $(\mathrm{spf}(\sigma^*), \mathrm{spf}(\eta^*))$.

Proof. This follows immediately from 3.3.4.

□

Remark 4.4.21. By 3.3.8, this behaves as

$$\mathrm{Div}_*(\mathbb{H})^{\Gamma(m)} \to \mathrm{Div}_{m*}(\mathbb{H})^{\Gamma}$$

$$D \mapsto \sum_{\gamma \in \Gamma_m} \gamma(D) = \sum_{i=0}^{m-1} F_q^i(D).$$

4.5 Finite Subgroups

In this section, we consider the "multiplication by m" endomorphisms on formal groups. We show that the kernels of these maps are subgroup divisors. In the case of $\mathbb{H}$, these arise topologically from the groups $GL_1(\mathbb{F}(m))$.

Throughout this section, we assume that $\mathbb{G}$ is a formal group over a formal scheme X with characteristic p residue field.

Definition 4.5.1. For a formal group $\mathbb{G}$ over a formal scheme X, we define a *finite subgroup* of $\mathbb{G}$ to be a divisor that is a formal subgroup scheme over X (but not a formal group as defined earlier). In other words, it inherits the group axioms from $\mathbb{G}$.

Definition 4.5.2. For every positive integer m, we get an endomorphism of formal groups

$$\mathbb{G} \xrightarrow{m} \mathbb{G}$$

that we call *multiplication by m*. If x is a coordinate on a formal group $\mathbb{G}$ then we get a formal group law F as in 4.3.5. It follows that we have

$$m_{\mathbb{G}}^*(x) = [m](x)$$

where $[m](x)$ is the m-series from 1.1.12.

Definition 4.5.3. For $m > 0$, we define a functor

$$\mathbb{G}(m) : \mathrm{FSchemes}_X \to \mathrm{Sets}$$

$$Y \mapsto \{a \in \mathbb{G}(Y) \mid m.a = 0\}.$$

Remark 4.5.4. There is an obvious canonical natural transformation

$$\mathbb{G}(m) \to \mathbb{G}.$$

Lemma 4.5.5. *$\mathbb{G}(m)$ is a formal subgroup scheme of $\mathbb{G}$ over X. Moreover, if x is a coordinate on $\mathbb{G}$, then we have a natural description*

$$\mathcal{O}_{\mathbb{G}(m)} = \mathcal{O}_X[\![x]\!]/[m](x).$$

Proof. If we take an element $a \in \mathbb{G}(Y)$, then this corresponds to a map of complete Noetherian semi-local rings

$$a : \mathcal{O}_X[\![x]\!] \to \mathcal{O}_Y.$$

If we let $x_a := a(x)$, then $m.a = 0$ iff $[m](x_a) = 0$, which implies the structure of the ring of functions and shows that the canonical map

$$\mathbb{G}(m) \to \mathbb{G}$$

is a closed inclusion. By 4.3.6, $[m](x)$ has finite Weierstrass degree, so $\mathbb{G}(m)$ is finite and flat over $\mathcal{O}_X$ and thus a divisor on $\mathbb{G}$. To see that $\mathbb{G}(m)$ inherits the group structure, if $m.a = 0$ and $m.b = 0$, then $m(a+b) = ma + mb = 0$, so the multiplication map restricts. As $m.0 = 0$ the unit map also restricts to $\mathbb{G}(m)$.

∎

Remark 4.5.6. This implies that $\mathbb{G}(m)$ is the categorical kernel of the "multiplication by m" map

$$Ker(\mathbb{G} \xrightarrow{m} \mathbb{G}).$$

Lemma 4.5.7. *For $m > 0$ with $v_p(m) = d$, we have*

$$\mathbb{G}(p^d) = \mathbb{G}(m).$$

Proof. We have a natural inclusion

$$\mathbb{G}(p^d) \to \mathbb{G}(m).$$

If we fix a coordinate, then it suffices to show that $[m](x) \sim [p^d](x)$. To prove this, it suffices to show that $[m](x) \sim x$ when m is coprime to p. But in such cases, m is a unit in $\mathcal{O}_X$ and it is easy to inductively construct an inverse of $[m](x)$ in $\mathcal{O}_X[\![x]\!]$.

$\square$

Proposition 4.5.8. *For $m \geq 0$, $R = E, K$, $\mathrm{spf}(R^0 B\mathcal{V}(m)_1)$ is a formal group scheme over $\mathrm{spf}(R^0)$. Moreover, we have a natural isomorphism of formal group schemes over $\mathrm{spf}(R^0)$*

$$\mathrm{spf}(R^0 B\mathcal{V}(m)_1) = \mathbb{H}_R(q^m - 1) = \mathbb{H}(p^{v+v_p(m)}).$$

Proof. We have that $\mathcal{V}(m)_1 \simeq GL_1(\mathbb{F}(m))$. By standard field theory, $GL_1(\mathbb{F}(m)) = \mathbb{F}(m)^\times$ is a cyclic group of order $q^m - 1$. The inclusion $\mathbb{F}(m) \rightarrowtail \mathbb{F}$ induces a canonical generator of $\mathrm{Hom}(\mathbb{F}(m)^\times, \bar{\mathbb{F}}^\times)$. It thus follows by 1.4.1 and 4.5.5 that the natural map

$$\mathrm{spf}(R^0 B\mathcal{V}(m)_1) \to \mathbb{H}_R$$

induces a natural isomorphism

$$\mathrm{spf}(R^0 B\mathcal{V}(m)_1) = \mathbb{H}_R(q^m - 1).$$

By A.0.1, $v_p(q^m - 1) = v + v_p(m)$, which gives the second equality by 4.5.7.

$\square$

4.6 Level Structures and Orders of Elements

In the final section of this chapter, we look at the divisor that morally represents "elements of order p^k" on a formal group. By considering transfers, we give a topological description of these divisors in the case of $\mathbb{H}$.

Throughout this section, we assume that $\mathbb{G}$ is a formal group over a formal scheme X with characteristic p residue field.

Definition 4.6.1. For A a finite abelian group, $\mathbb{G}$ a formal group over a formal scheme X, Y a formal scheme over X, and a group homomorphism $A \xrightarrow{\phi} \mathbb{G}(Y)$, we let $A(p) := Ker(p : A \to A)$ and define a divisor

$$[\phi A(p)] := \sum_{a \in A(p)} [\phi(a)].$$

We say that ϕ is an *A-level structure over Y* if $[\phi A(p)] \leq \mathbb{G}(p)$ as divisors.

We define a functor

$$\mathrm{Level}(A, \mathbb{G}) : \mathrm{FSchemes}_X \to \mathrm{Sets}$$
$$Y \mapsto \{A\text{-level structures over } Y\}.$$

Remark 4.6.2. The condition that $[\phi A(p)] \leq \mathbb{G}(p)$ is equivalent to the condition that the associated divisor $[\phi(A)] \leq \mathbb{G} \times_X Y$ is a subgroup divisor.

Lemma 4.6.3. *For A a finite group,* $\mathrm{Level}(A, \mathbb{G})$ *is a formal scheme over X.*

Proof. [37, Proposition 7.2].

□

Definition 4.6.4. We write D_A for the ring of functions of $\mathrm{Level}(A, \mathbb{G})$.

Definition 4.6.5. Let $k \geq 0$, and Y a formal scheme over X. If $a \in \mathbb{G}(Y)$, and $p^k.a = 0$, then we define a divisor of $\mathbb{G}$

$$[p^{k-1}\langle a \rangle] := \sum_{i=0}^{p-1} [ip^{k-1}.a]$$

over Y. We say that a has *exact order p^k* if $p^k.a = 0$ and $[p^{k-1}\langle a \rangle] \leq \mathbb{G}(p)$ as divisors.

We define a functor

$$\mathrm{Ord}_{p^k}(\mathbb{G}) : \mathrm{FSchemes}_X \to \mathrm{Sets}$$
$$Y \mapsto \{a \in \mathbb{G}(Y) \mid a \text{ has exact order } p^k\}.$$

Remark 4.6.6. We have a canonical natural transformation

$$\mathrm{Ord}_{p^k}(\mathbb{G}) \to \mathbb{G}(p^k).$$

Lemma 4.6.7. *If A is a cyclic group of order p^k and a is a generator of A, then we have a natural isomorphism of functors*

$$\mathrm{Level}(A, \mathbb{G}) \to \mathrm{Ord}_{p^k}(\mathbb{G})$$
$$\phi \mapsto \phi(a).$$

In particular, $\mathrm{Ord}_{p^k}(\mathbb{G})$ *is a formal scheme over X.*

Proof. It is immediate that the map is a natural isomorphism by the definitions. It follows by 4.6.3 that $\mathrm{Ord}_{p^k}(\mathbb{G})$ is a formal scheme over X.

□

Definition 4.6.8. We write D_k for the ring of functions of $\mathrm{Ord}_{p^k}(\mathbb{G})$.

Proposition 4.6.9. *$\mathrm{Ord}_{p^k}(\mathbb{G})$ is finite and flat over X. If $\mathbb{G}$ is an admissible formal group with the universality condition, of height n, $\mathrm{Ord}_{p^k}(\mathbb{G})$ is smooth of dimension n.*

Proof. This follows from [37, Theorem 7.3] by 4.6.7.

□

Proposition 4.6.10. *If $\mathbb{G}$ is an admissible formal group with the universality condition, of height n, the map*

$$\mathrm{Ord}_{p^k}(\mathbb{G}) \to X$$

has degree

$$|\Theta(p^k) \setminus \Theta(p^{k-1})| = |\mathrm{Ord}_{p^k}(\Theta)| = p^{nk} - p^{n(k-1)}$$

where $\Theta = (\mathbb{Q}_p/\mathbb{Z}_p)^n$.

Proof. If we note that $|\mathrm{Mon}(\mathbb{Z}/p^k, \Theta)| = |\mathrm{Ord}_{p^k}(\Theta)|$, then this follows from [37, Section 8] by 4.6.7.

□

We can translate this result into algebra to give the following.

Corollary 4.6.11. *Let $k \geq 0$, $\mathbb{G}$ be an admissible formal group with the universality condition of height n, and x a coordinate on $\mathbb{G}$. Then the canonical map of schemes*

$$\mathrm{Ord}_{p^k}(\mathbb{G}) \to \mathbb{G}(p^k)$$

is a closed inclusion and induces an isomorphism

$$D_k = \frac{\mathcal{O}_X[\![x]\!]}{\langle p\rangle([p^{k-1}](x))} = \frac{\mathcal{O}_X[x]}{h_k(x)}$$

where $\langle p\rangle([p^{k-1}](x))$ has Weierstrass degree $p^{nk} - p^{n(k-1)}$, and $h_k(x) \sim \langle p\rangle([p^{k-1}](x))$ is the corresponding Weierstrass polynomial over E^0. Moreover, modulo $\mathfrak{m}_{\mathcal{O}_X}$, we have

$$\mathcal{O}_{X_0} \otimes_{\mathcal{O}_X} D_k = \mathcal{O}_{X_0}[x]/(x^{p^{nk}-p^{n(k-1)}}).$$

Proof. We use the techniques from [37, Proposition 4.6] and [37, Section 8].

If we think in terms of the base formal scheme $Y := \mathbb{G}(p^k)$ with $\mathcal{O}_Y = \mathcal{O}_X[\![x]\!]/[p^k](x)$, then observe that the divisors $\mathbb{G}(p)$ and $[p^{k-1}\langle a\rangle]$ are defined on $\mathbb{G}$ over Y with corresponding equations $g_k(t)$ and $f(t)$. There are unique elements b_i such that

$$f(t) \equiv \sum_{i=0}^{m} b_i t^i \mod g_k(t)$$

for some m, and it is clear that $[p^{k-1}\langle a\rangle] \leq \mathbb{G}(p)$ iff $b_i = 0$, so we see that $D_k = \mathcal{O}_Y/(b_0, \ldots, b_m)$. In particular, $\mathrm{Ord}_{p^k}(\mathbb{G})$ is a closed formal subscheme of $\mathbb{G}(p^k)$.

It is easy to see that the power series $[p^{k-1}](x)$ of Weierstrass degree $p^{n(k-1)}$ divides the power series $[p^k](x)$ of Weierstrass degee p^{nk} in $\mathcal{O}_X[\![x]\!]$ with quotient $\langle p\rangle([p^{k-1}](x))$, which must have Weierstrass degree $p^{nk} - p^{n(k-1)}$. If we let $g_j(x)$ be the Weierstrass polynomial corresponding to $[p^j](x)$ and $h_j(x)$ be the Weierstrass polynomial corresponding to $\langle p\rangle([p^{j-1}](x))$, then we find that $g_{k-1}(x)$ divides $g_k(x)$ with quotient $h_k(x)$.

If x_a is a root of $g_{k-1}(x)$ (equivalently $[p^{k-1}](x)$) in R, then $p^{k-1}.a = 0$ so $[p^{k-1}\langle a\rangle] = p.[0]$, which is not a subdivisor of $\mathbb{G}(p)$, because x^p does not divide $[p](x)$. Consequently, the canonical map

$$\mathcal{O}_{\mathbb{G}(p^k)} = \mathcal{O}_X[\![x]\!]/[p^k](x) \twoheadrightarrow D_k$$

factors through the quotient $\mathcal{O}_X[\![x]\!]/\langle p\rangle([p^{k-1}](x))$. This is free of rank $p^{nk} - p^{n(k-1)}$, so by 4.6.10, the map

$$\mathcal{O}_X[\![x]\!]/\langle p\rangle([p^{k-1}](x)) \twoheadrightarrow D_k$$

must be an isomorphism.

□

Definition 4.6.12. For $k \geq 0$, let $i : \mathcal{V}(p^{k-1})_1 \rightarrowtail \mathcal{V}(p^k)_1$ be the canonical inclusion formed by tensoring along $\mathbb{F}(p^{k-1}) \to \mathbb{F}(p^k)$. Then we define

$$I_{tr} := Im(i_! : E^0 B\mathcal{V}(p^{k-1})_1 \to E^0 B\mathcal{V}(p^k)_1).$$

Proposition 4.6.13. *For $R = E, K$, and $k \geq 0$, we have natural isomorphisms*

$$\mathrm{spf}(R^0 B\mathcal{V}(p^k)_1/I_{tr}) \cong \mathrm{Ord}_{p^{k+v}}(\mathbb{G}).$$

Equivalently, we have $E^0 B\mathcal{V}(p^k)_1/I_{tr} = D_{k+v}$.

Proof. By Frobenius reciprocity, we have that $I_{tr} = (i_!(1))$. If we let x be a coordinate for $\mathbb{G}$, and $m := (q^{p^k} - 1)/(q^{p^{k-1}} - 1)$, then by 2.3.17 we have that

$$i_!(1) = \langle m \rangle ([q^{p^{k-1}} - 1](x)).$$

By A.0.1 and the arguments in 4.5.7, we see that $[q^{p^j} - 1](x) \sim [p^{j+v}](x)$, and $[m](x) \sim [p](x)$. In particular, we find that

$$i_!(1) \sim \langle p \rangle ([p^{k+v-1}](x)).$$

Consequently, the natural map

$$R^0 B\bar{\mathcal{V}}_1 \to R^0 B\mathcal{V}(p^k)_1 / I_{tr}$$

is just the surjection

$$R^0[\![x]\!] \to \frac{\mathcal{O}_X[\![x]\!]}{\langle p \rangle ([p^{k-1}](x))}.$$

By 4.6.11, this gives a natural isomorphism to $\mathrm{Ord}_{p^{k+v}}(\mathbb{H}_R)$.

$\square$

Chapter 5

Generalised Character Theory

The ideas in generalised character theory are inspired by the combination of the Atiyah-Segal completion theorem, which identifies KU^0BG with a completion of the representation ring $R(G)$, and character theory, which identifies $\mathbb{C} \otimes_{\mathbb{Z}} R(G)$ with the character ring.

In the first section of this chapter, we recall the basic statement of generalised character theory from [17]. We then recall from [3] that we can define Adams operations in Morava E-theory analogous to those in complex K-theory.

In the second section, we identify the character theory of $\mathcal{V}_*$, essentially following the proof of Marsh in [24]. This turns out to have a natural description in terms of the semiring of $\mathbb{F}$-linear representations of a group Θ^*. In particular, this is a commutative monoid on the set of irreducibles.

5.1 Generalised Character Theory

In this section, we recall the main statement of generalised character theory from [17], and note the existence of Adams operations in Morava E-theory as proved in [3].

Definition 5.1.1. Write Θ for the p-divisible topological group $(\mathbb{Q}_p/\mathbb{Z}_p)^n$ (where n is the height of $\mathbb{G}_E$). We also write Θ^* for the dual $\mathrm{Hom}(\Theta, S^1) \cong \mathbb{Z}_p^n$.

Remark 5.1.2. Θ should be thought of as an approximation to $\mathbb{G}_E$.

Recall that for groupoids H, G, in 2.1.4 we defined $[H, G]$ to be the set of homotopy classes (i.e. natural isomorphism classes) of functors $H \to G$.

Theorem 5.1.3. *There exists a faithfully flat graded ring extension L^0 (also* *written L_0) of $p^{-1}E^0$ concentrated in even degrees, such that for all finite groups, G, there is a natural isomorphism of L^0-algebras*

$$L^0 \otimes_{E^0} E^0BG \xrightarrow{\sim} \mathrm{Map}([\Theta^*, G], L^0).$$

If E^0BG is free over E^0, then dually, we have natural isomorphisms of L_0-modules

$$L_0 \otimes_{E_0} E_0^\vee BG \xrightarrow{\sim} L_0\{[\Theta^*, G]\}.$$

Proof. The first statement can be found in [17, Theorem C]. The second statement follows from the first statement by dualising and applying 1.3.13.

□

We can generalise this to a statement in terms of finite groupoids.

Corollary 5.1.4. *For G a finite groupoid, there is a natural isomorphism of L^0-algebras*

$$L^0 \otimes_{E^0} E^0BG \xrightarrow{\sim} \mathrm{Map}([\Theta^*, G], L^0).$$

If E^0BG is free over E^0, then dually, we have natural isomorphisms of L_0-modules

$$L_0 \otimes_{E_0} E_0^\vee BG \xrightarrow{\sim} L_0\{[\Theta^*, G]\}.$$

Proof. As a finite groupoid is equivalent to a finite disjoint union of finite groups, this follows from 5.1.3.

□

Remark 5.1.5. The ring L^0 can be constructed as follows. If we define $\Theta_k^* := \Theta^*/p^k\Theta^*$, then we can form an inverse system

$$\cdots \to \Theta_k^* \to \Theta_{k-1}^* \to \ldots$$

where $\Theta_k^* \cong (\mathbb{Z}/p^k)^n$. From this, we get a direct system

$$\cdots \to E^0B\Theta_{k-1}^* \to E^0B\Theta_k^* \to \ldots$$

For each $k > 0$, and each homomorphism $f : \Theta_k^* \to S^1$, we get a map

$$f^* : E^0BS^1 \to E^0B\Theta_k^*$$

To construct L^0 from this, we take the colimit $\mathrm{colim} E^0B\Theta_k^*$ and then localise at the multiplicative set generated by the elements $f^*(x)$ for all non-zero homomorphisms f.

Definition 5.1.6. For a groupoid G, we define

$$L^0BG := L^0 \otimes_{E^0} E^0BG$$
$$L_0BG := L_0 \otimes_{E_0} E_0^\vee BG.$$

General power operations can be constructed using the E_∞-structure on E. By considering level structures on $\mathbb{G}_E$, one can construct additive power operations from subgroups of Θ (or $\mathbb{G}_E$). This is studied in [3].

Theorem 5.1.7. *For spaces X and for all $k \geq 0$, there exist natural transformations of rings*

$$\psi^{p^k} : E^0X \to E^0X$$

such that for x an Euler class

$$\psi^{p^k}(x) = [p^k](x).$$

Proof. This is proved in [3, Theorem 1, Theorem 2] and the following comments.

□

Definition 5.1.8. We refer to ψ^{p^k} as the *p^kth Adams operation.*

5.2 Character Theory of $\mathcal{V}_*$

In this section, we study the character theory of $\mathcal{V}_*$, as in [24].

Definition 5.2.1. If Θ^* has the p-adic topology, and we make $\bar{\mathcal{V}}_d$ and $GL_d(\bar{\mathbb{F}})$ into topological groupoids with the discrete topology on both objects and morphisms, then we define

$$\mathrm{Rep}(\Theta^*; \mathbb{F})_d := [\Theta^*, \mathcal{V}_d] \cong [\Theta^*, GL_d(\mathbb{F})]$$
$$\mathrm{Rep}(\Theta^*; \bar{\mathbb{F}})_d := [\Theta^*, \bar{\mathcal{V}}_d]_{cts} \cong [\Theta^*, GL_d(\bar{\mathbb{F}})]_{cts}$$

the sets of (continuous) $\mathbb{F}$ and $\bar{\mathbb{F}}$-linear representations of Θ^*. We write $\mathrm{Rep}(\Theta^*; \mathbb{F})_*$ and $\mathrm{Rep}(\Theta^*; \bar{\mathbb{F}})_*$ for the corresponding graded sets.

In particular, we define a group

$$\Phi := \mathrm{Hom}_{cts}(\Theta^*, GL_1(\bar{\mathbb{F}})) = \mathrm{Rep}(\Theta^*; \bar{\mathbb{F}})_1.$$

Lemma 5.2.2. *For $k = \mathbb{F}, \bar{\mathbb{F}}$, we have a commutative graded monoid structure (σ_*, η_*) on $\mathrm{Rep}(\Theta^*; k)_*$.*

Proof. This follows by 3.1.18.

□

Remark 5.2.3. This monoid structure is just the direct sum of representations

$$\sigma_*([U],[V]) = [U \oplus V]$$

$$\eta_*[*] = [0].$$

Moreover, we have the following statement.

Proposition 5.2.4. *On $L^0 B\mathcal{V}_*$, if we let $(\sigma_!, \eta_!)$ be the addition monoid, and (δ^*, ϵ^*) be the multiplication monoid, then the maps*

$$(\sigma_!, \eta_!, \delta^*, \epsilon^*, \sigma^*, \eta^*)$$

satisfy all of the relations of a connected graded Hopf semiring, except for the compatibility relation

$$\sigma_! \sigma^* = (\sigma^*)^{\otimes 2} \mathrm{tw}(\sigma_!)^{\otimes 2}.$$

Proof. This follows by tensoring the result from 3.2.4 along $E^0 \to L^0$.

□

Lemma 5.2.5. *If Θ^* has the p-adic topology, and G is a discrete groupoid (with discrete objects and morphisms), then any continuous functor*

$$\Theta^* \to G$$

factors through $\Theta^*/p^m\Theta^*$ for some m. Moreover, if G is a finite groupoid, then there exists m such that

$$[\Theta^*, G] = [\Theta^*/p^m\Theta^*, G].$$

Proof. Let $f : \Theta^* \to G$ be a continuous functor. As G is discrete, 0 is open in $G(f(*), f(*))$, so $Ker(f) = f^{-1}(0)$ is an open neighbourhood of 0 in Θ^*, and thus is a union of sets of the form

$$p^{m_1}\mathbb{Z}_p \times \cdots \times p^{m_n}\mathbb{Z}_p$$

and thus contains $p^m\Theta^*$ for some m.

For the second statement, if G is finite, then the kernel of a functor $f : \Theta^* \to G$ has finite index in Θ^*, and is thus open. The same argument as before shows that $Ker(f) \supseteq p^m\Theta^*$ for some m. Observe that as Θ^* is finitely generated, there can only be finitely many functors $\Theta^* \to G$, so we can take the maximum value of m across this set to give the result.

$\square$

Lemma 5.2.6. *Φ is non-canonically isomorphic to Θ.*

Proof. The Sylow p-subgroup of $GL_1(\bar{\mathbb{F}})$ is isomorphic to $\mathbb{Q}_p/\mathbb{Z}_p$, so by 5.2.5, we have

$$[\Theta^*, GL_1(\bar{\mathbb{F}})]_{cts} = [\Theta^*, \mathbb{Q}_p/\mathbb{Z}_p]_{cts} \cong \Theta.$$

$\square$

Remark 5.2.7. Φ should be thought of as an approximation to $\mathbb{H}_E$. From this perspective, this lemma is analogous to 4.3.14.

Remark 5.2.8. As mentioned in remark 1.6.12, it will be more natural to phrase statements in terms of $\bar{\mathbb{F}}^\times$ than in terms of S^1. This will include phrasing statements in terms of Φ instead of Θ.

Proposition 5.2.9. *We have isomorphisms of monoids*

$$\mathrm{Rep}(\Theta^*; \bar{\mathbb{F}})_* = \mathbb{N}[[\Theta^*, GL_1(\bar{\mathbb{F}})]_{cts}] = \mathbb{N}[\Phi]$$

where $\mathbb{N}[\Phi]$ is the free monoid on Φ.

Proof. Given a continuous $\bar{\mathbb{F}}$-linear Θ^*-representation V, by 5.2.5, it factors through a representation of $\Theta^*/p^m\Theta^*$ for some m. By Maschke's theorem, the representations of $\Theta^*/p^m\Theta^*$ are completely reducible, and as Θ^*/p^m is abelian, the irreducibles all have dimension 1.

It follows that all continuous $\bar{\mathbb{F}}$-linear representations of Θ^* have unique descriptions as sums of continuous 1-dimensional representations.

$\square$

Remark 5.2.10. (1) This identifies Φ as $\mathrm{Irr}(\Theta^*; \bar{\mathbb{F}})$, the set of irreducible $\bar{\mathbb{F}}$-linear representations of Θ^*. In particular, they are all 1-dimensional.

(2) We may alternatively write $\mathbb{N}[\Phi] \cong \amalg_{d=0}^{\infty} \Phi^d/\Sigma_d$, which better reflects the grading over the representations by dimension.

Observe that Γ acts on $\mathrm{Rep}(\Theta^*; \bar{\mathbb{F}})_*$, which translates to an action on $\mathbb{N}[\Phi]$ via the action on Φ.

Proposition 5.2.11. *We have an isomorphism of monoids*

$$\mathrm{Rep}(\Theta^*; \mathbb{F})_* = \mathrm{Rep}(\Theta^*; \bar{\mathbb{F}})_*^\Gamma = \mathbb{N}[\Phi]^\Gamma = \mathbb{N}[\Phi_\Gamma]$$

where Φ_Γ is the set of Γ-orbits on Φ, and for $\phi \in \Phi_\Gamma$, the corresponding representation is given by the direct sum over the orbit.

Proof. The symmetric monoidal functor

$$T : \mathcal{V} \rightarrowtail \bar{\mathcal{V}}$$

$$V \mapsto \bar{\mathbb{F}} \otimes_{\mathbb{F}} V$$

induces a map of monoids

$$\mathrm{Rep}(\Theta^*; \mathbb{F})_* \to \mathrm{Rep}(\Theta^*; \bar{\mathbb{F}})_*.$$

The action of Γ on $\bar{\mathcal{V}}$ induces an action on $\mathrm{Rep}(\Theta^*; \bar{\mathbb{F}})$, and it is easy to see that the above map lands in the Γ-fixed points. It is straightforward to see that by 5.2.9

$$\mathrm{Rep}(\Theta^*; \bar{\mathbb{F}})_*^\Gamma = \mathbb{N}[\Phi]^\Gamma = \mathbb{N}[\Phi_\Gamma].$$

We wish to show that the map

$$\mathrm{Rep}(\Theta^*; \mathbb{F})_* \to \mathrm{Rep}(\Theta^*; \bar{\mathbb{F}})_*^\Gamma$$

is an isomorphism. Firstly, observe that by A.0.18, the map

$$\mathrm{Rep}(\Theta^*; \mathbb{F})_* \to \mathrm{Rep}(\Theta^*; \mathbb{F}(m))_*$$

is injective for all $m \geq 1$. Thus, taking the colimit, we have injectivity of our map.

To show that the map is surjective, it suffices to show that Φ_Γ is in the image. If $(\phi, V) \in \Phi$ is an irreducible representation of Θ^*, then by 5.2.5, $\phi^{p^M} = 0$ for some M. By A.0.1

$$v_p(q^k - 1) = v + v_p(k)$$

so it follows that the Γ-orbit of ϕ is finite. If we let m denote the orbit size of ϕ, then $\phi^{q^m} = \phi$, so $V = \bar{\mathbb{F}} \otimes_{\mathbb{F}(m)} W$ for some $\mathbb{F}(m)$-linear Θ^*-representation W.

By 3.3.7, we have that

$$\bar{\mathbb{F}} \otimes_{\mathbb{F}} U(W) = \bigoplus_{i=0}^{m-1} (F_q^i)^*(V).$$

But the right hand side is the Γ-orbit sum of V, which is the corresponding element of Φ_Γ. This shows that Φ_Γ is in the image as required.

□

Remark 5.2.12. Under this identification, $\mathbb{F}$-linear irreducibles of Θ^* correspond to Γ-orbits in Φ, where each irreducible is given by the orbit-sum.

Definition 5.2.13. We define

$$\mathrm{Irr}(\Theta^*; \mathbb{F})_d := (\Phi_\Gamma)_d$$

the set of irreducible $\mathbb{F}$-linear representations of Θ^* of dimension d. Equivalently, this is the set of Γ-orbits in Φ of size d.

We write $\mathrm{Irr}(\Theta^*; \mathbb{F})_*$ for the graded set with degree d part equal to $\mathrm{Irr}(\Theta^*; \mathbb{F})_d$, the set of irreducible k-linear representations of G of dimension d.

Corollary 5.2.14. *We have an isomorphism of algebras over* L_0

$$L_0 B\mathcal{V}_* = L_0\{\mathrm{Rep}(\Theta^*; \mathbb{F})_*\} = L_0[\mathrm{Irr}(\Theta^*; \mathbb{F})_*] = L_0[\Phi_\Gamma]$$

and an isomorphism of coalgebras

$$L^0 B\mathcal{V}_* = \mathrm{Map}(\mathrm{Rep}(\Theta^*; \mathbb{F})_*, L^0) = \mathrm{Map}(\mathbb{N}[\Phi_\Gamma], L^0).$$

Proof. This follows from 5.2.11.

□

Remark 5.2.15. This implies that $L_0\{\mathrm{Irr}(\Theta^*; \mathbb{F})_*\} = L_0\{\Phi_\Gamma\}$ generates the graded algebra $L_0 B\mathcal{V}_*$ under (σ_*, η_*) and $\mathrm{Map}(\mathrm{Irr}(\Theta^*; \mathbb{F})_*, L^0) = \mathrm{Map}(\Phi_\Gamma, L^0)$ cogenerates the graded coalgebra $L^0 B\mathcal{V}_*$ under (σ^*, η^*).

Definition 5.2.16. Define

$$\mathrm{Ord}_{p^k}(\Phi) := \Phi(p^k) \setminus \Phi(p^{k-1}).$$

Definition 5.2.17. Recall that n is the height of E and K, and $v = v_p(q-1)$ is the p-adic valuation of $q-1$. Define

$$N_0 := p^{nv}$$
$$N_k := p^{n(k+v)-k} - p^{n(k+v-1)-k}$$

for $k > 0$.

Proposition 5.2.18. *We have that* $\mathrm{Irr}(\Theta^*; \mathbb{F})_*$ *is concentrated in pth power degrees, and in particular for all* $k \geq 0$

$$\mathrm{Irr}(\Theta^*; \mathbb{F})_{p^k} = \mathrm{Ord}_{p^{k+v}}(\Phi)_\Gamma = [\Phi(p^{k+v}) \setminus \Phi(p^{k+v-1})]_\Gamma$$

and $|\mathrm{Irr}(\Theta^*; \mathbb{F})_{p^k}| = N_k$.

Proof. If $\phi \in \Phi$, then the dimension of the representation corresponding to $\mathrm{Orb}_\Gamma(\phi)$ (the Γ-orbit of ϕ) is equal to $|\mathrm{Orb}_\Gamma(\phi)|$. By A.0.1, $(q^d - 1).\phi = 0$ iff $p^{v+v_p(d)}.\phi = 0$. Thus we have that $\mathrm{Irr}(\Theta^*; \mathbb{F})_d = 0$ unless $d = p^k$ in which case

$$\mathrm{Irr}(\Theta^*; \mathbb{F})_{p^k} = \mathrm{Ord}_{p^{k+v}}(\Phi)_\Gamma = [\Phi(p^{k+v}) \setminus \Phi(p^{k+v-1})]_\Gamma.$$

It is easy to see that $p^k N_k = |\mathrm{Ord}_{p^{k+v}}(\Phi)|$, but we have already seen that this set splits as a disjoint union of Γ-orbits of size p^k, so $N_k = |\mathrm{Ord}_{p^{k+v}}(\Phi)_\Gamma|$.

□

Corollary 5.2.19. *The Hilbert series for the graded algebra $L_0 B\mathcal{V}_*$ is*

$$\sum_{i \geq 0} \mathrm{rk}_{L_0}(L_0 B\mathcal{V}_i) t^i = \prod_{j \geq 0} (1 - t^{p^j})^{-N_j}.$$

Proof. The Hilbert series of a graded polynomial algebra with m_r generators in degree r for each r is given by

$$\prod_{r \geq 0} (1 - t^r)^{-m_r}$$

so this follows by 5.2.14 and 5.2.18.

□

Corollary 5.2.20. *The Hilbert series for the graded algebras $E_0^\vee B\mathcal{V}_*$ and $K_0 B\mathcal{V}_*$ are*

$$\sum_{i \geq 0} \mathrm{rk}_{E_0}(E_0^\vee B\mathcal{V}_i) t^i = \prod_{j \geq 0} (1 - t^{p^j})^{-N_j}$$

$$\sum_{i \geq 0} \dim_{K_0}(K_0 B\mathcal{V}_i) t^i = \prod_{j \geq 0} (1 - t^{p^j})^{-N_j}$$

respectively.

Proof. As $E_0^\vee B\mathcal{V}_*$ is free, $\mathrm{rk}_{E_0}(E_0^\vee B\mathcal{V}_i) = \mathrm{rk}_{L_0}(L_0 B\mathcal{V}_i)$. By 1.7.6 and 1.3.2, $\dim_{K_0}(K_0 B\mathcal{V}_i) = \mathrm{rk}_{E_0}(E_0^\vee B\mathcal{V}_i)$. Thus, both cases follow from 5.2.19.

□

Chapter 6

K-Local Duality of Groupoids

In this chapter, we recall the K-local duality theory for finite groupoids developed in [36]. The main statement is that the K-localisation of G has a natural Frobenius object structure.

In the first section, we look at some of the basic theory of Frobenius objects. We highlight the relationship between local Frobenius algebras and Gorenstein local rings, and then study how a Frobenius algebra structure interacts with ideals via annihilators.

In the second section, we recall the main statements in the theory of K-local duality for finite groupoids from [36], and apply them to the case of $\mathcal{V}_*$.

In the third section, we present a discussion of the properties of K-local transfers that is analogous to that of stable transfers in chapter 2.

In the fourth section, we recall the interaction between K-local duality and character theory as observed in [36]. In particular, we focus on the form of K-local transfers in character theory.

In the last section, we apply the results from the fourth section to the character theory of $\mathcal{V}_*$. In particular, we identify the form of $\sigma_!$ in character theory. By the end of this chapter, the statements of theorem 1 and theorem 2 seem reasonable.

6.1 Frobenius Algebras and Gorenstein Local Rings

In this section, we look at some basic theory relating to Frobenius objects, Frobenius algebras, and Gorenstein local rings.

Definition 6.1.1. If R is a local Noetherian ring such that $\dim(R) = n$, and M is a finitely generated R-module, then M is a *canonical module/dualising module* for R if

$$\mathrm{Ext}^m_R(R/\mathfrak{m}, M) \cong \begin{cases} R/\mathfrak{m} & \text{if } m = n \\ 0 & \text{otherwise} \end{cases}.$$

We say that R is a *Gorenstein local ring of dimension n* if R is a canonical module for itself.

Lemma 6.1.2. *If R is a regular local ring then it is also a Gorenstein local ring. If R is a Gorenstein local ring then it is also a Cohen-Macaulay local ring.*

Proof. Both statements follow from [12, Theorem 18.1 (5')].

□

Lemma 6.1.3. *Let R be a Noetherian local ring and $x_1, \dots, x_m$ a regular sequence on R. Then R is Gorenstein iff $R/(x_1, \dots, x_m)$ is Gorenstein.*

Proof. This follows inductively from [12, Proposition 21.10].

□

Definition 6.1.4. If $\mathcal{C}$ is a closed symmetric monoidal category with unit I, we define a *Frobenius object in $\mathcal{C}$* to be an object A along with maps

$$\mu : A \otimes A \to A$$

$$\psi : A \to A \otimes A$$

$$\eta : I \to A$$

$$\epsilon : A \to I$$

such that (A, μ, η) is a commutative monoid, (A, ψ, ϵ) is a cocommutative comonoid, and the Frobenius laws hold

$$(\mathrm{id} \otimes \mu) \circ (\psi \otimes \mathrm{id}) = \psi \circ \mu = (\mu \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \psi).$$

For a ring k, we define a *Frobenius algebra over k* to be a Frobenius object in Mod_k.

Further discussion of the theory of Frobenius objects can be found in [36, Section 3].

Definition 6.1.5. If $\mathcal{C}$ is a closed symmetric monoidal category with unit I, and A is an object in $\mathcal{C}$, we define an *inner product* to be a map

$$b : A \otimes A \to I$$

such that $b\tau = b$, and the adjoint map $b^{\#} : A \to DA$ is an isomorphism.

Definition 6.1.6. If $\mathcal{C}$ is a closed symmetric monoidal category with unit I, we define a *commutative semigroup in* $\mathcal{C}$ to be a pair (A, μ) such that A is an object in $\mathcal{C}$ and μ is a map

$$\mu : A \otimes A \to A$$

that is commutative and associative.

If (A, μ) is a commutative semigroup in $\mathcal{C}$, then we define a *Frobenius form* on (A, μ) to be a map

$$\epsilon : A \to I$$

such that the composition $\epsilon\mu$ is an inner product.

Remark 6.1.7. For any object $A \in \mathcal{C}$, there is a canonical map

$$A \to D^2 A.$$

This map induces a map

$$\mathrm{Hom}(A \otimes A, A) \to \mathrm{Hom}(A \otimes A, D^2 A) = \mathrm{Hom}(A \otimes A \otimes DA, I) = \mathrm{Hom}(A \otimes DA, DA).$$

If A is a monoidal object in $\mathcal{C}$, then there is a canonical element of the left hand hom-set, so there is also a canonical map

$$A \otimes DA \to DA.$$

One can show that this is an A-module structure.

Lemma 6.1.8. *Let $\mathcal{C}$ be a compact closed symmetric monoidal category, and (A, μ, η) be a commutative monoid in $\mathcal{C}$. Then Frobenius object structures extending (A, μ, η) are in bijective correspondence with isomorphisms of A-modules*

$$A \xrightarrow{\sim} DA.$$

Proof. This is standard. Write I for the dualising object. Given a Frobenius object structure extending (A, μ, η), by [36, Lemma 3.9] we have an inner product

$$b : A \otimes A \xrightarrow{\mu} A \xrightarrow{\epsilon} I$$

and thus an adjoint isomorphism

$$b^\# : A \xrightarrow{\sim} DA$$

which can be checked to be a map of A-modules. Given an A-module isomorphism

$$c^\# : A \xrightarrow{\sim} DA$$

we can use its adjoint and η to get a Frobenius form for μ

$$\epsilon : A \xrightarrow{\eta \otimes id} A \otimes A \xrightarrow{c} I.$$

By [36, Lemma 3.9, Proposition 3.10], this yields a Frobenius object structure extending (A, μ, η). One can check that these processes are inverse to each other.

□

Definition 6.1.9. If A is a Frobenius object, then we call the isomorphism of A-modules

$$\Theta : A \to \mathrm{Hom}(A, I)$$

the *self-duality isomorphism.*

Remark 6.1.10. If A is a Frobenius algebra over a ring k, then A acts on $\mathrm{Hom}_k(A, k)$ by $a.f : b \mapsto f(ab)$, and so the self-duality isomorphism

$$\Theta : A \to \mathrm{Hom}_k(A, k)$$

is determined by $\theta := \Theta(1)$.

Proposition 6.1.11. *Let R be a local ring and let A be a local R-algebra, such that R is Gorenstein, A is Cohen-Macaulay, and* $\dim(R) = \dim(A)$*. Then A is Gorenstein iff A has a Frobenius algebra structure over R.*

Proof. If we assume that A is Gorenstein, then by [12, Theorem 21.15], $\mathrm{Hom}_R(A, R)$ is a canonical module, but by [12, Corollary 21.14], all canonical modules are isomorphic so as A is Gorenstein, there exists an isomorphism of modules

$$A \cong \mathrm{Hom}_R(A, R)$$

giving a commutative Frobenius algebra structure to A. Conversely, if we assume that A has a commutative Frobenius algebra structure, then we have an isomorphism

$$A \cong \mathrm{Hom}_R(A, R).$$

By [12, Theorem 21.15], the right hand side is a canonical module. Hence so is the left hand side, and so A is Gorenstein.

$\square$

The most important R-modules relating to an R-algebra A are the ideals in A, and their quotients. When A is a Frobenius algebra, there are certain cases where we can give some nice descriptions of the duals of these modules.

Lemma 6.1.12. *If $I \lhd A$ is an ideal, then $ann_A(I)$ is isomorphic to the R-linear dual of A/I.*

Proof. Recall that the duality isomorphism is given by

$$A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$$

$$a \mapsto \theta(a.-)$$

where θ is the image of 1. Observe that we have a submodule

$$\mathrm{Hom}_R(A/I, R) \to \mathrm{Hom}_R(A, R)$$

and this is precisely

$$\{f : A \to R \mid f(I) = 0\}.$$

The duality isomorphism therefore restricts to a map

$$ann_A(I) \to \mathrm{Hom}_R(A/I, R).$$

This is injective because the self-duality isomorphism is injective. To show surjectivity, observe that any element $f \in \mathrm{Hom}_R(A/I, R)$ has a unique description as $\theta(a.-)$ for some $a \in A$ by the self-duality isomorphism. We must show that $a \in ann_A(I)$. By assumption $f(I) = \theta(a.I) = 0$. If we take $s \in I$ and $b \in A$ then

$$\theta((a.s).b) = b.\theta(a.s) = b.\theta(0) = 0.$$

Consequently, $\Theta(a.s) = \theta((a.s).-) = 0$ is the zero map, so $a.s = 0$ in A. As this applies for all $s \in I$, we find that $a \in ann_A(I)$.

$\square$

Lemma 6.1.13. *If A is a Frobenius algebra over R, then if the exact sequence of R-modules*

$$0 \to I \to A \to A/I \to 0$$

splits, then the exact sequence of R-modules

$$0 \to ann_A(I) \to A \to A/ann_A(I) \to 0$$

also splits. In this case, the second exact sequence is the dual of the first.

Proof. By the splitting lemma, splittings of the first exact sequence biject with sections of $A \to A/I$. Also by the splitting lemma, splittings of the second exact sequence biject with retractions of $ann_A(I) \to A$.

By the previous lemma, these maps are dual R-modules, and so section of the first give retractions of the second. If there is a splitting of the first exact sequence, then we get a dual split exact sequence

$$0 \to \mathrm{Hom}_R(A/I, R) \to \mathrm{Hom}_R(A, R) \to \mathrm{Hom}_R(I, R) \to 0.$$

The left hand non-trivial map is dual to the map $ann_A(I) \to A$, so the short exact sequence must be isomorphic to the exact sequence

$$0 \to ann_A(I) \to A \to A/ann_A(I) \to 0.$$

□

Remark 6.1.14. As a result, if I is a summand in A then $ann_A(I)$ is also a summand in A.

Corollary 6.1.15. *If R is Noetherian and A is a finitely generated Frobenius algebra over R such that $I \lhd A$ is a summand, then $ann_A(ann_A(I)) = I$. In particular, in the previous lemma, the first exact sequence is the dual of the second.*

Proof. This follows by applying 6.1.13 twice and using strong dualisablity of finitely generated R-modules.

□

Definition 6.1.16. For a local ring R with maximal ideal $\mathfrak{m}$, we define the *socle* of R to be the annihilator of the maximal ideal:

$$soc(R) := ann(\mathfrak{m}).$$

Lemma 6.1.17. *If R is a local Frobenius k-algebra that is finite dimensional over k, then $soc(R)$ is dual to $R/\mathfrak{m}$.*

Proof. This follows immediately from 6.1.12.

□

Remark 6.1.18. In all situations that we will encounter, $k \cong R/\mathfrak{m}$, making $soc(R)$ 1-dimensional.

6.2 K-Local Duality of Groupoids

In this section, we recall the main statements of K-local duality from [36, Section 8]. We then apply them to $E^0 B\mathcal{V}_*$ and $K^0 B\mathcal{V}_*$.

Note: In this section, we will be working K-locally. In particular, $\wedge$ refers to the K-local smash product.

Definition 6.2.1. For a groupoid G, we write

$$LG := L_K(\Sigma^\infty_+ BG)$$

and we write

$$L_K S := L_K \Sigma^\infty S^0 = L1$$

for the K-local sphere spectrum, where 1 is the terminal groupoid.

Remark 6.2.2. For a functor of groupoids f, we will abuse notation and write f when we really mean Bf or Sf. What we mean should always be clear from the context.

Definition 6.2.3. For a groupoid G define δ to be the diagonal map

$$\delta : G \to G \times G$$

and define ϵ to be the terminal map

$$\epsilon : G \to 1.$$

Theorem 6.2.4. *For a finite groupoid G, the map*

$$b_G : LG \wedge LG \xrightarrow{L(\delta^!)} LG \xrightarrow{L\epsilon} S$$

is an inner product on LG.

Proof. This is proved in [36, Proposition 8.3].

□

Definition 6.2.5. For a map of finite groupoids $f : H \to G$, we define the *K-local transfer*

$$(Lf)^! : LG \to LH$$

to be the adjoint of Lf under the natural inner products b_H and b_G of 6.2.4. This defines a functor

$$(L-)^! : \mathrm{Gpds}^{op} \to \mathcal{K}$$

$$G \mapsto LG$$

where $\mathcal{K}$ is the K-local homotopy category.

Theorem 6.2.6. *For G a finite groupoid, LG has a natural structure of a commutative Frobenius object with maps $((L\epsilon)^!, (L\delta)^!, L\delta, L\epsilon)$.*

Proof. This is proved in [36, Theorem 8.7].

□

Theorem 6.2.7. *If G is a finite groupoid and K^*BG is concentrated in even degrees, then K^0BG has a Frobenius algebra structure over K^0 and E^0BG has a Frobenius algebra structure over E^0, both given by the maps $((L\epsilon)^*, (L\delta)^*, (L\delta)_!, (L\epsilon)_!)$.*

Proof. By 1.3.2 we have $K^0BG = K^0LG$ and $E^0BG = E^0LG$, and by 1.3.14, we have Künneth isomorphisms, so by functoriality applied to 6.2.6, we get the result.

□

Remark 6.2.8. Here, the pair (δ^*, ϵ^*) induce the usual cup-product ring structure.

Corollary 6.2.9. *For a finite groupoid G such that K^*BG is concentrated in even degrees, K^0BG is a Gorenstein local ring and E^0BG is a Gorenstein local ring.*

Proof. By Hironaka's criterion ([28, Theorem 25.16]), E^0BG/K^0BG is Cohen-Macaulay iff it is free over E^0/K^0, so this follows by 6.1.11.

□

Lemma 6.2.10. *For G a finite groupoid such that K^*BG concentrated in even degrees, we have isomorphisms of K^0BG-modules and E^0BG-modules*

$$K^0BG \cong \mathrm{Hom}_{K^0}(K^0BG, K^0) = K_0BG$$

$$E^0BG \cong \mathrm{Hom}_{E^0}(E^0BG, E^0) = E_0^\vee BG$$

respectively, where K_0BG and $E_0^\vee BG$ are modules via the cap product.

Proof. The category of finitely generated projective modules over k forms a compact closed symmetric monoidal category by [36, Example 2.2]. As everything is finitely generated and free over E^0/K^0, by 6.2.7, 6.1.8 and 1.3.13, we have isomorphisms

$$K^0BG \cong \mathrm{Hom}_{K^0}(K^0BG, K^0) = K_0BG$$
$$E^0BG \cong \mathrm{Hom}_{E^0}(E^0BG, E^0) = E_0^\vee BG.$$

The left hand map is an isomorphism of K^0BG/E^0BG-modules as K^0BG/E^0BG is a Frobenius algebra. It is standard that under the Kronecker pairing, the cup product translates to the cap product action.

□

Lemma 6.2.11. *If $f : H \to G$ is a faithful functor of groupoids, then*

$$L(f^!) = (Lf)^!.$$

Proof. [36, Proposition 8.5].

□

Proposition 6.2.12. *The isomorphisms of 6.2.10*

$$E^0B\mathcal{V}_* \cong \mathrm{Hom}(E^0B\mathcal{V}_*, E^0) = E_0^\vee B\mathcal{V}_*$$

$$K^0B\mathcal{V}_* \cong \mathrm{Hom}_{K^0}(K^0B\mathcal{V}_*, K^0) = K_0B\mathcal{V}_*$$

are isomorphisms of the algebra structures $(\sigma_!, \eta_!)$, $((\sigma^*)^\vee, (\eta^*)^\vee)$, and (σ_*, η_*) respectively, and isomorphisms of the coalgebra structures (σ^*, η^*), $((\sigma_!)^\vee, (\eta_!)^\vee)$, and $(\sigma^!, \eta^!)$ respectively (see 3.2.4, 3.2.6).

Proof. If H, G are groupoids such that K^*BG and K^*BH are concentrated in even degrees, and we have a map $f : H \to G$, then under the isomorphisms from the proof of 6.2.10

$$E^0BG \cong \mathrm{Hom}_{E^0}(E^0BG, E^0) = E_0^\vee BG$$

$$K^0BG \cong \mathrm{Hom}_{K^0}(K^0BG, K^0) = K_0BG$$

$f^*/f_!$ on the left hand side corresponds to the dual of $f_!/f^*$ respectively in the middle, because K-locally, transfers are just adjoints of restrictions. Then the dual of $f_!/f^*$ in the middle corresponds to $f^!/f_*$ respectively on the right hand side, by the naturality of the Kronecker pairing. Using this, by the definition of the algebra and coalgebra structures, the statement follows.

□

6.3 K-Local Transfers of Groupoids

In this section, we look at the fundamental properties of K-local transfers. These are completely analogous to those of stable transfers in chapter 2. This is essentially because the Mackey property generalises for K-local transfers.

Note: In this section, we will be working K-locally. In particular, $\wedge$ refers to the K-local smash product and for a map of groupoids f, we will write $f^!$ for the K-local transfer.

Definition 6.3.1. Define the *K-local cohomotopy* of a spectrum X to be

$$\pi^0_K(X) := [X, L_K S] = [L_K X, L_K S].$$

where $[-,-]$ denotes the group of homotopy classes of maps of spectra. As $L_K S$ is a commutative ring spectrum, when $X = \Sigma^\infty_+ Y$ is the suspension of a space, this is a ring with unit

$$1 : L_K \Sigma^\infty_+ Y \to L_K S$$

defined as the K-localisation of the terminal map

$$Y \to \{pt\}.$$

Remark 6.3.2. For a groupoid G, the localisation of the terminal map

$$\epsilon : G \to 1$$

coincides with the unit $1 \in \pi^0_K(LG)$.

Recall that for a multiplicative cohomology theory R, and a space X, we write

$$(- \bullet -) : R^*(X) \otimes R^*(X) \to R^*(X)$$
$$a \otimes b \mapsto a \bullet b$$

for the *cup product.* Often, this is just written as the concatenation of elements.

Definition 6.3.3. For a space X, if $u \in \pi_K^0(\Sigma_+^\infty X)$, then we define

$$\mu_u : L_K\Sigma_+^\infty X \xrightarrow{\delta} L_K\Sigma_+^\infty X \wedge L_K\Sigma_+^\infty X \xrightarrow{u\wedge\mathrm{id}} L_K\Sigma_+^\infty X.$$

Lemma 6.3.4. *For a K-local multiplicative cohomology theory R, a space X, and $u \in \pi_K^0(\Sigma_+^\infty X)$, we have*

$$\mu_u^* = (u \bullet -) : R^0(X) \to R^0(X)$$

where u is considered as an element of $R^0(X)$ via the map

$$\pi_K^0(\Sigma_+^\infty X) = [\Sigma_+^\infty X, L_K S] \to [\Sigma_+^\infty X, R] = R^0(X)$$

induced by the unit $L_K S \to R$.

Proof. This essentially follows by writing out the definitions of μ_u and of cup product multiplication by u in $R^0(X)$.

□

Lemma 6.3.5. *For spaces X, Y and elements $u, v \in \pi_K^0(\Sigma_+^\infty X)$ and $w \in \pi_K^0(\Sigma_+^\infty Y)$, we have*

$$\mu_{u+v} = \mu_u + \mu_v \in [L_K\Sigma_+^\infty X, L_K\Sigma_+^\infty X]$$

$$\mu_u \wedge \mu_w = \mu_{u\otimes w} \in [L_K\Sigma_+^\infty (X\times Y), L_K\Sigma_+^\infty (X\times Y)]$$

$$\mu_{uv} = \mu_v \circ \mu_u \in [L_K\Sigma_+^\infty X, L_K\Sigma_+^\infty X]$$

$$\mu_1 = \mathrm{id} \in [L_K\Sigma_+^\infty X, L_K\Sigma_+^\infty X].$$

where $+$ denotes sum and $\circ$ denotes composition in the K-local homotopy category.

Proof. This is essentially the same proof as for 2.3.8.

□

Remark 6.3.6. We will abuse notation by writing $f^!$ instead of $(Lf)^!$ when it is clear that we are working K-locally. This is reasonable given 6.2.11.

Theorem 6.3.7. *For functors of finite groupoids $f, g : H \to G$, the K-local transfer maps satisfy the following*

(1) $(fg)^! = g^! f^!$

(2) $\mathrm{id}^! = \mathrm{id}$

(3) $(f \amalg g)^! = f^! \vee g^!$

(4) $(f \times g)^! = f^! \wedge g^!$

(5) The generalised Mackey property holds, i.e. for a homotopy pullback diagram of groupoids

$$\begin{tikzcd} A \arrow[r, "k"] \arrow[d, "h"'] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

we have $g^! f = h k^!$.

Proof. [36, Section 8].

□

Remark 6.3.8. The remainder of the statements in this section are essentially direct consequences of the properties of K-local transfers in 6.3.7.

Proposition 6.3.9. *For a map of finite groupoids* $H \xrightarrow{f} G$ *and an element* $u \in \pi^0_K(LG)$*, we have*

$$\mu_u \circ f = f \circ \mu_{f^*(u)} : LH \to LG.$$

Proof. This is essentially the same proof as for 2.3.12.

□

There are two useful forms of K-local Frobenius reciprocity when working with groupoids.

Lemma 6.3.10. *For a map of finite groupoids* $H \xrightarrow{f} G$ *and an element* $u \in \pi^0(\Sigma^\infty_+ BG)$ *or* $u \in \pi^0_K(LG)$*, we have*

$$f^! \circ \mu_u = \mu_{f^*(u)} \circ f^! : LG \to LH.$$

Note: This is the opposite way around to the restriction.

Proof. This is essentially the same proof as for 2.3.13.

□

Lemma 6.3.11. *For a map of finite groupoids $H \xrightarrow{f} G$, K-locally we have*

$$ff^! = \mu_u : LG \to LG$$

where $u := f_!(1) \in \pi_K^0(LG)$.

Proof. This is essentially the same proof as for 2.3.14.

□

Lemma 6.3.12. *For $u \in \pi_K^0(LG)$, $\mu_u : LG \to LG$ is self-adjoint.*

Proof. By the definition of the adjoint, it suffices to show that $b_G(\mu_u \wedge \mathrm{id}) = b_G(\mathrm{id} \wedge \mu_u)$ where b_G is the inner product from 6.2.4. Recall that $b_G := \epsilon\delta^!$. To see this, we have K-local equalities

$$\begin{aligned} b_G(\mu_u \wedge \mathrm{id}) &= \epsilon\delta^!(\mu_u \wedge \mathrm{id}) = \epsilon\delta^! \mu_{u\otimes 1} \\ &= \epsilon\mu_{\delta^*(u\otimes 1)}\delta^! = \epsilon\mu_u\delta^! \end{aligned}$$

where the second equality follows by 2.3.8, the third equality follows by 2.3.13, and the fourth equality follows because $\delta^*(u \otimes 1) = u \bullet 1 = u$. A reflected argument shows that we also have a K-local equality

$$b_G(\mathrm{id} \wedge \mu_u) = \epsilon\mu_u\delta^!$$

which shows that μ_u is self-adjoint.

□

Lemma 6.3.13. *Stably or K-locally, if f and g are functors of finite groupoids, such that $gf = \mathrm{id}$ then $f^! g^! = \mathrm{id}$.*

Proof. This follows from the functoriality of K-local transfers.

□

Lemma 6.3.14. *Let $f : H \to G$ and $g : G \to H$ be K-local maps of finite groupoids such that $gf = \mathrm{id}$ and $u := f_!(1)$, then*

$$f^! = g\mu_u : LG \to LH.$$

Moreover, if f, g are K-local inverses, and $v := g_!(1)$, then

$$u.g^*(v) = 1$$

$$v.f^*(u) = 1.$$

Proof. For the first statement the proof is essentially the same proof as for 2.3.16.

For the second statement, by 6.3.10, 6.3.11, and 6.3.13 we have K-local equalities

$$\begin{aligned}\mu_u \mu_{g^*(v)} &= f f^! \mu_{g^*(v)} = f \mu_{f^*(g^*(v))} f^! \\ &= f \mu_v f^! = f g g^! f^! \\ &= \mathrm{id}.\end{aligned}$$

The statement follows by considering the action on 1 in K-local cohomotopy. The last equality is similar.

□

6.4 Duality and Transfers in Character Theory

In this section, we see how the induced Frobenius algebra structure on $L^0BG = L^0 \otimes_{E^*} E^*BG$ behaves in character theory, as studied in [36].

Definition 6.4.1. Given a finite groupoid G, define $C(G) := \mathbb{Q}\{\pi_0(G)\}$ and define an inner product on this by

$$([a], [b]) := |G(a, b)|.$$

Remark 6.4.2. This is just a weighted version of the inner product coming from the standard basis for $C(G)$.

Definition 6.4.3. For a functor of finite groupoids $f : H \to G$, we define

$$f_* := \mathbb{Q}\{\pi_0 f\} : C(H) \to C(G)$$
$$[a] \mapsto [f(a)]$$

and we also define $f^!$ to be the adjoint of f_*

$$f^! : C(G) \to C(H)$$
$$[b] \mapsto \sum_{[a]\in(\pi_0 f)^{-1}[b]} \frac{|G(b,b)|}{|H(a,a)|}[a].$$

Write f^* for the dual of f_*, and write $f_!$ for the dual of $f^!$.

Recall that δ and ϵ refer to the diagonal and terminal maps on G respectively.

Proposition 6.4.4. *or G a finite groupoid, $C(G)$ has a natural structure of a Frobenius algebra with maps $(\epsilon^!, \delta^!, \delta_*, \epsilon_*)$.*

Dually, $C(G)^\vee := \mathrm{Hom}_{\mathbb{Q}}(C(G), \mathbb{Q})$ has a natural structure of a Frobenius algebra with maps $(\epsilon^*, \delta^*, \delta_!, \epsilon_!)$.

Proof. This is proved in [36, Section 10].

□

Theorem 6.4.5. *For finite groupoids G, the character theory map*

$$L^0 BG \xrightarrow{\sim} \mathrm{Map}([\Theta^*, G], L) = L^0 \otimes_{\mathbb{Q}} C(\mathcal{H}(\Theta^*, G))^\vee$$

is an isomorphism of Frobenius algebras over L^0. If E^*BG is free over E^*, then the dual map

$$L_0^\vee BG \xrightarrow{\sim} L_0\{[\Theta^*, G]\} = L_0 \otimes_{\mathbb{Q}} C(\mathcal{H}(\Theta^*, G))$$

is also an isomorphism of Frobenius algebras.

Proof. The first statement is proved in [36, Theorem 10.2]. The second statement follows by dualising.

□

6.5 Transfers in the Character Theory of $\mathcal{V}_*$

In this section, we study the interaction of K-local duality with the character theory of $\mathcal{V}_*$. In particular, it remains to identify the algebra structure $(\sigma_!, \eta_!)$.

Proposition 6.5.1. *$L^0 B\mathcal{V}_*$ is a free commutative graded algebra under $(\sigma_!, \eta_!)$. Dually, $L_0 B\mathcal{V}_*$ is a cofree cocommutative graded coalgebra under $(\sigma^!, \eta^!)$.*

Proof. This follows by dualising the results in 5.2.14, because the algebra structure $(\sigma_!, \eta_!)$ is adjoint to the coalgebra structure (σ^*, η^*).

□

Remark 6.5.2. This result offers good evidence for theorem 1.

The inner product on $C(\mathcal{H}(\Theta^*, \mathcal{V}_*))$ is determined by the following result.

Lemma 6.5.3. *If V is an $\mathbb{F}$-linear Θ^*-representation with irreducible decomposition*

$$V = \bigoplus_{\phi \in \Phi_\Gamma} V_\phi^{d_\phi}$$

then

$$([V],[V]) = |\mathrm{Aut}_{\Theta^*}(V)| = \sum_{\phi \in \Phi_\Gamma} |GL_{d_\phi}(\mathbb{F}(|\phi|))|$$

where $|\phi|$ is the size of the orbit represented by ϕ.

Proof. By Schur's lemma, it suffices to assume that $V = V_\phi^d$ for some $\phi \in \Phi_\Gamma$ with $|\phi| = m$. By the proof of 5.2.11, $V_\phi = (U_m)_*(W)$ for a 1-dimensional $\mathbb{F}(m)$-linear Θ^*-representation W, where U_m is the forgetful functor from section 3.3. The corresponding map

$$\Theta^* \to GL_1(\mathbb{F}(m))$$

has image A a cyclic group of order p^{v+m}. By Schur's lemma, $End_{\mathbb{F}(m)}(W) \cong \mathbb{F}(m)$. If we pick a generator a of A and an $\mathbb{F}(m)$-linear generator x for W, then this determines a surjective map of $\mathbb{F}$-linear representations of Θ^*

$$\mathbb{F}[A] \twoheadrightarrow V_\phi$$

$$a \mapsto x.$$

It is then easy to see that an $\mathbb{F}$-linear endomorphism of V_ϕ is determined by the image of x, so we must also have $End_{\mathbb{F}}(V_\phi) \cong \mathbb{F}(m)$. It follows that

$$\mathrm{Aut}_{\mathbb{F}}(V_\phi^d) \cong GL_d(\mathbb{F}(m)).$$

□

Corollary 6.5.4. *If W_1, W_2 are $\mathbb{F}$-linear Θ^*-representations with irreducible decompositions*

$$W_1 = \bigoplus_{\phi \in \Phi_\Gamma} V_\phi^{s_\phi}$$

$$W_2 = \bigoplus_{\phi \in \Phi_\Gamma} V_\phi^{t_\phi}$$

then we have a bijection

$$\mathrm{Hom}_{\Theta^*}(W_1, W_2) \cong \bigoplus_{\phi \in \Phi_\Gamma} \mathrm{Mat}_{s_\phi, t_\phi}(\mathbb{F}(|\phi|))$$

where $\mathrm{Mat}_{s,t}(k)$ is the set of $s \times t$ matrices with coefficients in k.

Proof. In the proof of 6.5.3, we showed that $End_{\Theta^*}(V_\phi) \cong \mathbb{F}(m)$. The statement then follows easily by Schur's lemma.

□

Proposition 6.5.5. *We have*

$$\sigma_![V] = \sum_{[U \oplus W] = [V]} \frac{|\mathrm{Aut}_{\Theta^*}(V)|}{|\mathrm{Aut}_{\Theta^*}(U)||\mathrm{Aut}_{\Theta^*}(W)|}([U] \otimes [W])$$

$$\eta_![V] = \begin{cases} 1 & \textit{if } [V] = [0] \\ 0 & \textit{otherwise} \end{cases}.$$

Proof. This follows straightforwardly from 6.4.3.

□

Corollary 6.5.6. *We have natural identifications*

$$PL_0 B\mathcal{V}_* = L_0\{\mathrm{Irr}(\Theta^*; \mathbb{F})_*\}$$

$$QL^0 B\mathcal{V}_* = \mathrm{Map}(\mathrm{Irr}(\Theta^*; \mathbb{F})_*, L^0).$$

Proof. By 6.5.5, it is easy to see that $[V] \in L_0 B\mathcal{V}_*$ is primitive iff V is irreducible, but also the term $[U] \otimes [W]$ has a non-zero coefficient in $\sigma_![V]$ if $[U \oplus W] = [V]$, so $x \in L_0 B\mathcal{V}_*$ is primitive iff $x \in L_0\{Irr(\Theta^*; \mathbb{F})_*\}$.

The second statement follows by dualising.

□

Remark 6.5.7. This result offers good evidence for theorem 2.

Chapter 7

Harish-Chandra (Co)monoid Structures

If we take the graded groupoid of short exact sequences of $\mathbb{F}$-vector spaces, we can define an alternative product and coproduct on the K-localisation $L\mathcal{V}_*$. This is reminiscent of parabolic induction, so we call these the Harish-Chandra product and coproduct.

In the first section, we make the definitions needed in this chapter. In the second section, we compare the Harish-Chandra product and coproduct with σ and $\sigma^!$ respectively. We show that the Harish-Chandra structures are actually just twisted versions of the direct sum product and coproduct structures.

In the third section, we identify the image of the twisting element $u \in \pi_K^0(L\mathcal{V}_*^2)$ in character theory, and in the fourth section, we deduce that u satisfies a 2-cocycle condition. This implies that the Harish-Chandra product and coproduct form monoid and comonoid structures on $L\mathcal{V}_*$.

7.1 Harish-Chandra Product and Coproduct

In this section, we define an alternative product and coproduct on $L\mathcal{V}_*$ reminiscent of Harish-Chandra induction.

Definition 7.1.1. Define $\mathcal{ES}_*$ to be the graded groupoid with degree d part given by the groupoid of short exact sequences of $\mathbb{F}$-vector spaces and isomorphisms of short exact sequences

$$U \rightarrowtail V \twoheadrightarrow W$$

such that $\dim(V) = \dim(U) + \dim(W) = d$. If we let $\mathcal{ES}_{i,j}$ be the groupoid of short exact sequences such that $\dim(U) = i$ and $\dim(W) = j$, then

$$\mathcal{ES}_d = \coprod_{i+j=d} \mathcal{ES}_{i,j}.$$

Define graded functors

$$\alpha : \mathcal{ES}_* \to \mathcal{V}_*^2$$
$$(U \rightarrowtail V \twoheadrightarrow W) \mapsto (U, W)$$
$$\beta : \mathcal{ES}_* \to \mathcal{V}_*$$
$$(U \rightarrowtail V \twoheadrightarrow W) \mapsto V.$$

Lemma 7.1.2. *α and β are fibrations, and β is faithful.*

Proof. To see that α is a fibration, given a short exact sequence $E = (U \xrightarrow{i} V \xrightarrow{\pi} W)$ in $\mathcal{ES}_*$, and a morphism

$$h = (u, w) : \alpha(E) = (U, W) \to (U', W')$$

we can define a short exact sequence

$$E' := (U' \xrightarrow{iu^{-1}} V \xrightarrow{w\pi} W').$$

Then we have a commutative diagram

$$\begin{tikzcd}
U \arrow[r, "i"] \arrow[d, "u"] & V \arrow[r, "\pi"] \arrow[d, "\mathrm{id}"] & W \arrow[d, "w"] \\
U' \arrow[r, "iu^{-1}"'] & V \arrow[r, "w\pi"'] & W'
\end{tikzcd}$$

so the morphism $g := (u, \mathrm{id}, w)$ lifts h, i.e. $\alpha(g) = h$, which proves that α is a fibration. To see that β is a fibration, given a short exact sequence $E = (U \xrightarrow{i} V \xrightarrow{\pi} W)$ in $\mathcal{ES}_*$, and a morphism

$$t : \alpha(E) = V \to V'$$

we can define a short exact sequence

$$E'' := (U \xrightarrow{ti} V' \xrightarrow{\pi t^{-1}} W).$$

Then we have a commutative diagram

$$\begin{tikzcd} U \arrow[r, "i"] \arrow[d, "\mathrm{id}"] & V \arrow[r, "\pi"] \arrow[d, "t"] & W \arrow[d, "\mathrm{id}"] \\ U \arrow[r, "ti"'] & V' \arrow[r, "\pi t^{-1}"'] & W \end{tikzcd}$$

so the morphism $s := (\mathrm{id}, t, \mathrm{id})$ lifts t, i.e. $\beta(s) = t$, which proves that β is a fibration. To see that β is faithful, observe that if we have short exact sequences

$$E = (U \xrightarrow{i} V \xrightarrow{\pi} W)$$

$$E' = (U' \xrightarrow{i'} V' \xrightarrow{\pi'} W')$$

and two maps

$$(a_1, b, c_1), (a_2, b, c_2) : E \to E'$$

with $\beta(a_1, b, c_1) = b = \beta(a_2, b, c_2)$, then

$$i'a_1 = bi = i'a_2$$

$$c_1\pi = \pi' b = c_2\pi$$

so as i' is a monomorphism, and π is an epimorphism, $a_1 = a_2$, and $c_1 = c_2$, which proves that β is faithful.

□

Remark 7.1.3. As β is faithful, we can form its transfer stably. However, α is not faithful so we don't have an ordinary transfer, but we can still consider its transfer K-locally.

Definition 7.1.4. Define the *Harish-Chandra graded product and graded coproduct* to be

$$\sigma' : L\mathcal{V}_* \wedge L\mathcal{V}_* \xrightarrow{\alpha^!} L\mathcal{ES}_* \xrightarrow{\beta} L\mathcal{V}_*$$

$$\psi' : L\mathcal{V}_* \xrightarrow{\beta^!} L\mathcal{ES}_* \xrightarrow{\alpha} L\mathcal{V}_* \wedge L\mathcal{V}_*.$$

Remark 7.1.5. When defining α we made a choice. We could equally have defined it to be what is now $\tau\alpha$. It is important to keep track of this choice, however, we will see later that the effect of σ' and ψ' on E-theory and K-theory is independent of this choice anyway.

7.2 Comparison of (Co)products

In this section, we compare the Harish-Chandra product and coproduct with the K-localisation of the direct sum product and coproduct. We show that they are actually very closely related.

Definition 7.2.1. Define a graded functor

$$\lambda : \mathcal{V}_*^2 \to \mathcal{ES}_*$$

$$(U, W) \mapsto (U \rightarrowtail U \oplus W \twoheadrightarrow W).$$

Lemma 7.2.2. *We have*

$$\sigma = \beta\lambda$$

$$\sigma^! = \lambda^! \beta^!.$$

Proof. For the first statement we have

$$\beta\lambda(U, W) = \beta(U \rightarrowtail U \oplus W \twoheadrightarrow W) = U \oplus W = \sigma(U, W).$$

The second statement follows from the functoriality of transfers 6.3.7.

□

In order to compare σ and σ', we need to compare λ and $\alpha^!$.

Definition 7.2.3. Given a short exact sequence

$$U_1 \xrightarrow{i} U_2 \xrightarrow{p} U_3$$

define

$$K(\underline{U}) := Ker(\mathcal{ES}_*(\underline{U}) \xrightarrow{\alpha} \mathcal{V}_*^2(U_1, U_3))$$

and define a map

$$f : \mathrm{Hom}(U_3, U_1) \to K(\underline{U})$$

$$m \mapsto (f(m)_1, f(m)_2, f(m)_3)$$

by $f(m)_i := \mathrm{id}$ for $i \neq 2$ and

$$f(m)_2 := \mathrm{id} + imp$$

This map is well-defined because $pi = 0$ and because $f_i = \mathrm{id}$ for $i \neq 2$.

Lemma 7.2.4. *The map*

$$f : \mathrm{Hom}(U_3, U_1) \to K(\underline{U})$$

is an isomorphism of groups.

Proof. Firstly, we show that we have a homomorphism. It is clear that $f(0) = \mathrm{id}$. As $pi = 0$, we have

$$\begin{aligned}(\mathrm{id} + im_1p) \circ (\mathrm{id} + im_2p) &= \mathrm{id} + im_1p + im_2p + im_1pim_2p \\ &= \mathrm{id} + im_1p + im_2p \\ &= \mathrm{id} + i(m_1 + m_2)p.\end{aligned}$$

To see that f is injective, suppose that $\mathrm{id}+im_1p = \mathrm{id}+im_2p$. Then $im_1p = im_2p$. As i is monic and p is epic, we see that $m_1 = m_2$.

To see that f is surjective, suppose $k \in K(\underline{U})$. Then we see that $k_1, k_3 = \mathrm{id}$. Define $n := k_2 - id$. Then we see that $ni = 0 = pn$, so by the identification of i and p as a kernel and cokernel respectively, n factors uniquely as

$$n : U_2 \xrightarrow{p} U_3 \xrightarrow{m} U_1 \xrightarrow{i} U_2.$$

From this it is clear that $k = f(m)$.

□

Proposition 7.2.5. *We have*

$$\alpha\lambda = \mathrm{id}.$$

Moreover, α is a regular p-local equivalence with p-local inverse λ.

Proof. Firstly, it is easy to see that

$$\alpha\lambda(U_1, U_3) = (U_1, U_3).$$

To see that α is a π_0-isomorphism, firstly, note α is π_0-surjective by the first statement and 2.2.8. To see π_0-injectivity, note that every short exact sequence of vector spaces has a splitting, and there is only one isomorphism class of split short exact sequences. It follows by 2.2.10 that α is full

Now given an object $\underline{U}$ in $\mathcal{ES}_*$, we have a short exact sequence

$$0 \to K(\underline{U}) \to \mathcal{ES}_*(\underline{U}) \xrightarrow{\alpha} \mathcal{V}_*^2(\alpha(\underline{U})) \to 0.$$

But by 7.2.4, $K(\underline{U}) = \mathrm{Hom}(U_3, U_1)$, which has order $q^{\dim(U_3)\dim(U_1)}$, which is congruent to 1 mod p and thus coprime to p. It follows by 2.2.11 that α is a regular p-local equivalence, and λ must be its p-local inverse.

□

Remark 7.2.6. Morally, this means that p-locally, exact sequences of $\mathbb{F}$-vector spaces behave like they have a unique splitting.

Proposition 7.2.7. *K-locally, we have*

$$\sigma' = \sigma\mu_{u^{-1}}$$

$$\psi' = \mu_{u^{-1}}\sigma^!$$

where $u := \lambda^*(\lambda_!(1)) = \alpha_!(1)^{-1} \in \pi^0_K(L\mathcal{V}^2_*)$.

Proof. Observe that by 7.2.5 and 6.3.14, $\lambda^*(\lambda_!(1)) = \alpha_!(1)^{-1}$ in $\pi^0_K(L\mathcal{V}^2_*)$.

For the first statement, we have

$$\begin{aligned}\sigma' &= \beta\alpha^! = \beta\lambda\alpha\alpha^! \\ &= \beta\lambda\mu_{u^{-1}} = \sigma\mu_{u^{-1}}\end{aligned}$$

where the second equality follows from 7.2.5, the third equality follows from 6.3.11 and the final equality follows from 7.2.2.

For the second statement, we have

$$\begin{aligned}\psi' &= \alpha\beta^! = \alpha\alpha^!\lambda^!\beta^! \\ &= \mu_{u^{-1}}\lambda^!\beta^! = \mu_{u^{-1}}\sigma^!.\end{aligned}$$

□

Definition 7.2.8. We will refer to $u := \alpha_!(1)^{-1} \in \pi^0_K(L\mathcal{V}^2_*)$ as the *Harish-Chandra twisting element.*

7.3 Identification of u in Generalised Character Theory

In this section, we study how the Harish-Chandra twisting element u behaves in character theory.

Observe that for a groupoid G, an object of $\mathcal{H}(\Theta^*, G)$ is just an object of G carrying a Θ^*-action, and an isomorphism in $\mathcal{H}(\Theta^*, G)$ is the same as a Θ^*-equivariant isomorphism in G. For $d \geq 0$, the groupoid $\mathcal{ES}_d$ is a groupoid of diagrams in the category of $\mathbb{F}$-vector spaces $\mathrm{Vect}(\mathbb{F})$, i.e. of the form $\mathcal{H}(D, \mathrm{Vect}(\mathbb{F}))$ for some diagram category D.

Definition 7.3.1. For categories $\mathcal{C}, \mathcal{D}$, we write $F(\mathcal{C}, \mathcal{D})$ for the category of functors from $\mathcal{C}$ to $\mathcal{D}$.

Lemma 7.3.2. *If $G = F(D, H)$ is the groupoid of diagrams of shape D in the category H, then*

$$\mathcal{H}(\Theta^*, G) \simeq \mathcal{H}(D, F(\Theta^*, H)).$$

Proof. For categories $\mathcal{C}, \mathcal{D}$, $\mathcal{H}(\mathcal{C}, \mathcal{D})$ is just the maximal subgroupoid of the functor category $F(\mathcal{C}, \mathcal{D})$. Cat is a Cartesian closed category, so we can construct equivalences

$$F(\Theta^*, F(D, H)) \simeq F(\Theta^* \times D, H) \simeq F(D, F(\Theta^*, H)).$$

The result follows by taking the maximal subgroupoids on either side.

□

Remark 7.3.3. In words, the groupoid of Θ^*-equivariant D-shaped diagrams in H is equivalent to the groupoid of D-shaped diagrams in the category of Θ^*-equivariant objects in H.

In the next lemma, we consider the maps

$$\mathcal{H}(\Theta^*, \alpha) : \mathcal{H}(\Theta^*, \mathcal{ES}_*) \to \mathcal{H}(\Theta^*, \mathcal{V}_*^2)$$
$$\mathcal{H}(\Theta^*, \lambda) : \mathcal{H}(\Theta^*, \mathcal{V}_*^2) \to \mathcal{H}(\Theta^*, \mathcal{ES}_*).$$

Proposition 7.3.4. *We have*

$$\mathcal{H}(\Theta^*, \alpha) \circ \mathcal{H}(\Theta^*, \lambda) = \mathrm{id}.$$

Moreover, $\mathcal{H}(\Theta^*, \alpha)$ is a regular p-local equivalence with p-local inverse $\mathcal{H}(\Theta^*, \lambda)$.

For an object $(U, W) \in \mathcal{H}(\Theta^*, \mathcal{V}_*^2)$ the fibre is equivalent to the group

$$\mathrm{Hom}_{\Theta^*}(W, U).$$

Proof. $\mathcal{H}(\Theta^*, \mathcal{V}_*)$ is a faithful, essentially wide subcategory of the category of $\mathbb{F}$-linear representations of Θ^*, and by 5.2.5 and Maschke's theorem, short exact sequences of $\mathbb{F}$-linear representations of Θ^* always split.

Using these facts, the proof is then essentially identical to the amalgamation of the proofs of 7.2.4 and 7.2.5.

□

Proposition 7.3.5. *Under the character theory isomorphism from 5.1.3, the Harish-Chandra twisting element from 7.2.8 is*

$$u : \mathrm{Rep}(\Theta^*; \mathbb{F})^2_* \to L^0$$

$$[U, W] \mapsto |\mathrm{Hom}_{\Theta^*}(W, U)|.$$

Proof. Recall that $u := \alpha_!(1)^{-1} \in E^0 B\mathcal{V}^2_*$. By 7.3.4, $\mathcal{H}(\Theta^*, \alpha)$ is a π_0-isomorphism and full. Write $[b] := [U, W] \in \mathrm{Rep}(\Theta^*; \mathbb{F})^2_*$. By 6.4.3, in character theory we have

$$\begin{aligned} \mathcal{H}(\Theta^*, \alpha)^! : C(\mathcal{H}(\Theta^*, \mathcal{V}^2_*)) &\to C(\mathcal{H}(\Theta^*, \mathcal{ES}_*)) \\ [b] &\mapsto \sum_{[a'] \in (\pi_0 \mathcal{H}(\Theta^*, \alpha))^{-1}[b]} |K_{a'}|^{-1}[a'] = |K_a|^{-1}[a] \end{aligned}$$

where $[a] = [U \rightarrowtail V \twoheadrightarrow W]$ is the unique isomorphism class of $\mathcal{H}(\Theta^*, \alpha)^{-1}[b]$ and if $b' := \mathcal{H}(\Theta^*, \alpha)(a')$ then $K_{a'}$ is defined to be the kernel of the surjective group homomorphism

$$\mathcal{H}(\Theta^*, \mathcal{ES}_*)(a') \to \mathcal{H}(\Theta^*, \mathcal{V}^2_*)(b').$$

By 6.4.5 we have $u^{-1} = \mathcal{H}(\Theta^*, \alpha)_!(1) \in L^0 \otimes_{\mathbb{Q}} C(\mathcal{H}(\Theta^*, \mathcal{V}^2_*))$ and so

$$u^{-1}([b]) = |K_a|^{-1}$$

By 7.3.4, K_a is isomorphic to $\mathrm{Hom}_{\Theta^*}(W, U)$, which gives the result.

□

Corollary 7.3.6. *The image of the Harish-Chandra twisting element u in $E^0 B\mathcal{V}^2_*$ and $K^0 B\mathcal{V}^2_*$ is symmetric, i.e.*

$$\tau^*(u) = u.$$

Proof. By 6.5.4, we see that

$$|\mathrm{Hom}_{\Theta^*}(U, W)| = |\mathrm{Hom}_{\Theta^*}(W, U)|$$

because

$$|\mathrm{Mat}_{s,t}(k)| = |\mathrm{Mat}_{t,s}(k)|.$$

□

Remark 7.3.7. It follows that on E-theory and K-theory, $(\sigma')^*$ and $(\psi')^*$ are cocommutative and commutative respectively.

7.4 (Co)monoidal Structures

In this section, we show that u^m satisfies a 2-cocycle condition in character theory for all $m \in \mathbb{Z}$. We will use this to prove that twists of σ and $\sigma^!$ by u^m give monoid and comonoid structures on $L\mathcal{V}_*$ respectively.

Lemma 7.4.1. *The Harish-Chandra twisting element u from 7.2.8 satisfies the 2-cocycle condition*

$$(u \otimes 1) \bullet (\sigma \wedge \mathrm{id})^*(u) = (1 \otimes u) \bullet (\mathrm{id} \wedge \sigma)^*(u)$$

in $E^*B\mathcal{V}_*^2$ and $K^*B\mathcal{V}_*^2$.

Proof. By 5.2.3, in character theory the left hand side's value on a triple of Θ^*-representations (W_1, W_2, W_3) is

$$\begin{aligned}|\mathrm{Hom}_{\Theta^*}(W_1, W_2)||\mathrm{Hom}_{\Theta^*}(W_1 \oplus W_2, W_3)|& \\ = |\mathrm{Hom}_{\Theta^*}(W_1, W_2)||\mathrm{Hom}_{\Theta^*}(W_1, W_3)||\mathrm{Hom}_{\Theta^*}(W_2, W_3)|.&\end{aligned}$$

The right hand side's value on (W_1, W_2, W_3) is

$$\begin{aligned}|\mathrm{Hom}_{\Theta^*}(W_1, W_2)||\mathrm{Hom}_{\Theta^*}(W_1, W_2 \oplus W_3)|& \\ = |\mathrm{Hom}_{\Theta^*}(W_1, W_2)||\mathrm{Hom}_{\Theta^*}(W_1, W_3)||\mathrm{Hom}_{\Theta^*}(W_2, W_3)|.&\end{aligned}$$

Thus the two sides are equal in character theory, so as $E^*B\mathcal{V}_*$ is free over E^*, the two sides are equal in E-theory, and thus also in K-theory.

□

Corollary 7.4.2. *For $m \in \mathbb{Z}$, the Harish-Chandra twisting element u^m satisfies the 2-cocycle condition*

$$(u^m \otimes 1) \bullet (\sigma \wedge \mathrm{id})^*(u^m) = (1 \otimes u^m) \bullet (\mathrm{id} \wedge \sigma)^*(u^m)$$

in $E^*B\mathcal{V}_*^2$ and $K^*B\mathcal{V}_*^2$.

Proof. $(\sigma \wedge \mathrm{id})^*$ and $(\mathrm{id} \wedge \sigma)^*$ are multiplicative, so this statement is just the mth power of 7.4.1.

□

Proposition 7.4.3. *For all $m \in \mathbb{Z}$, on $E^0B\mathcal{V}_*$ and $K^0B\mathcal{V}_*$, we have a commutative graded monoid structure $(\sigma_! \circ (u^m \bullet -), \eta_!)$ and a cocommutative graded comonoid structure $((u^m \bullet -) \circ \sigma^*, \eta^*)$ where u is the Harish-Chandra twisting element.*

Proof. We will prove this for u, but the same proof applies for u^m for all m. Commutativity and cocommutativity follow immediately from the commutativity and cocommutativity of $\sigma_!$ and σ^*, along with the symmetry of u.

For coassociativity of the comonoid, observe that by 6.3.5, 6.3.9, and the associativity of σ from 3.1.18, the bottom left, bottom right, and top right squares in the following diagram commute

$$\begin{tikzcd}
L\mathcal{V}_*^3 \arrow[r, "\mu_{1\otimes u}"] \arrow[d, "\mu_{u\otimes 1}"] & L\mathcal{V}_*^3 \arrow[r, "\mathrm{id}\wedge\sigma"] \arrow[d, "\mu_{(\mathrm{id}\wedge\sigma)^*(u)}"] & L\mathcal{V}_*^2 \arrow[d, "\mu_u"] \\
L\mathcal{V}_*^3 \arrow[r, "\mu_{(\sigma\wedge\mathrm{id})^*(u)}"] \arrow[d, "\sigma\wedge\mathrm{id}"] & L\mathcal{V}_*^3 \arrow[r, "\mathrm{id}\wedge\sigma"] \arrow[d, "\sigma\wedge\mathrm{id}"] & L\mathcal{V}_*^3 \arrow[d, "\sigma"] \\
L\mathcal{V}_*^2 \arrow[r, "\mu_u"'] & L\mathcal{V}_*^2 \arrow[r, "\sigma"'] & L\mathcal{V}_*
\end{tikzcd}.$$

By 7.4.1, u satisfies the 2-cocycle condition in cohomology, so the top left square (and thus whole diagram) commutes after applying E or K. By 1.3.2, $E^*L\mathcal{V}_* = E^*B\mathcal{V}_*$ and $K^*L\mathcal{V}_* = K^*B\mathcal{V}_*$. This shows that $(u \bullet -) \circ \sigma^*$ is coassociative.

To see that η^* is a counit, we must show that

$$(\eta^* \otimes \mathrm{id})(u \bullet -)\sigma^* = \mathrm{id}.$$

However, by the multiplicativity of η^*, we have

$$(\eta^* \otimes \mathrm{id})(u \bullet -) = [(\eta \otimes \mathrm{id})^*(u) \bullet -](\eta^* \otimes \mathrm{id}) = \eta^* \otimes \mathrm{id}$$

because in character theory

$$(\eta \otimes \mathrm{id})^*(u)(V) = |\mathrm{Hom}_{\Theta^*}(0, V)| = 1$$

so $(\eta^* \otimes \mathrm{id})(u) = 1$. It follows that η^* is a counit, because it is a counit for σ^*.

For associativity of the monoid, we have the following diagram, where the bottom left, bottom right, and top right squares commute by 6.3.10, and the coassociativity of $\sigma^!$ from 3.1.20

$$\begin{tikzcd}
L\mathcal{V}_*^3 & L\mathcal{V}_*^3 \arrow[l, "\mu_{1\otimes u}"'] & L\mathcal{V}_*^2 \arrow[l, "\mathrm{id}\wedge\sigma^!"'] \\
L\mathcal{V}_*^3 \arrow[u, "\mu_{u\otimes 1}"'] & L\mathcal{V}_*^3 \arrow[l, "\mu_{(\sigma\wedge\mathrm{id})^*(u)}"'] \arrow[u, "\mu_{(\mathrm{id}\wedge\sigma)^*(u)}"'] & L\mathcal{V}_*^3 \arrow[l, "\mathrm{id}\wedge\sigma^!"'] \arrow[u, "\mu_u"'] \\
L\mathcal{V}_*^2 \arrow[u, "\sigma^!\wedge\mathrm{id}"'] & L\mathcal{V}_*^2 \arrow[l, "\mu_u"] \arrow[u, "\sigma^!\wedge\mathrm{id}"'] & L\mathcal{V}_* \arrow[l, "\sigma^!"] \arrow[u, "\sigma^!"']
\end{tikzcd}.$$

By 7.4.1, u satisfies the 2-cocycle condition in cohomology, so the top left square (and thus whole diagram) commutes after applying E or K. This shows that $\sigma_! \circ (u \bullet -)$ is associative.

To see that $\eta_!$ is a unit, we must show that

$$\sigma_!(u \bullet -)(\eta_! \otimes \mathrm{id}) = \mathrm{id}.$$

However, by the Frobenius reciprocity, we have

$$(u \bullet -)(\eta_! \otimes \mathrm{id}) = (\eta_! \otimes \mathrm{id})[(\eta \otimes \mathrm{id})^*(u) \bullet -] = \eta_! \otimes \mathrm{id}$$

because as shown before, $(\eta^* \otimes \mathrm{id})(u) = 1$. It follows that $\eta_!$ is a unit, because it is a unit for $\sigma_!$.

□

Corollary 7.4.4. *On $E^0 B\mathcal{V}_*$ and $K^0 B\mathcal{V}_*$, we have a commutative graded monoid structure $((\psi')^*, \eta_!)$ and a cocommutative graded comonoid structure $((\sigma')^*, \eta^*)$.*

Proof. This follows immediately from 7.2.7 and 7.4.3.

□

Chapter 8

Compatibility Relations

The main aim of this chapter is to address the absence of a compatibility relation between σ and $\sigma^!$. As explained in 3.1.26, the underlying reason that we don't have a compatibility diagram is that a certain commutative diagram is not a pullback square. This happens because the set of subspaces of a vector space doesn't form a distributive lattice.

In this chapter, we study precisely how the bimonoid compatibility axiom fails for the structure $(\sigma, \sigma^!)$ on $S\mathcal{V}_*$. We will show that the failure of the compatibility relation is controlled by elements $u \in \pi^0(S\mathcal{V}_*^2)$ and $v \in \pi^0(S\mathcal{V}_*^4)$. Moreover, p-locally u and v become invertible and we can combine them to give a single element $w \in \pi^0(S_{(p)}\mathcal{V}_*^4)$ that controls the failure of the relation.

In section 1, we show that we can construct a homotopy pullback square similar to that desired, but with $\mathcal{ES}_*$ in place of $\mathcal{V}_*^2$, and a graded groupoid of perfect squares $\mathcal{PS}_*$ in place of $\mathcal{V}_*^4$. We then show that $\mathcal{V}_*^4$ is p-locally equivalent to $\mathcal{PS}_*$ and recall from the previous chapter that $\mathcal{V}_*^2$ and $\mathcal{ES}_*$ are p-locally equivalent.

In section 2, we apply the Mackey property to this homotopy pullback, to conclude that p-locally, we have what looks like a compatibility diagram, but with an extra twisting by an element $w \in \pi_0(S_{(p)}\mathcal{V}_*^4)$.

In the third section, we compute the twisting element w modulo the maximal ideal, and in the fourth section, we compute the twisting element in character theory. In particular, we use character theory to observe that w satisfies a 2-cocycle condition, much like u in the previous chapter.

8.1 Perfect and Hollow Squares

In this section, we define several types of diagrams of vector spaces using short exact sequences and show that the graded groupoids of these diagrams are equivalent to $\mathcal{V}_*^4$. The work is similar to that relating $\mathcal{ES}_*$ to $\mathcal{V}_*^2$ in section 7.2. We also construct a homotopy pullback diagram to replace the diagram from 3.1.26, which is not a homotopy pullback.

Definition 8.1.1. Define a *perfect square* to be a diagram in the category of $\mathbb{F}$-vector spaces, $\mathrm{Vect}(\mathbb{F})$, of the form

$$\begin{tikzcd}
U_1 \arrow[r, tail, "j_1"] \arrow[d, tail, "i_1"] & U_4 \arrow[r, two heads, "q_1"] \arrow[d, tail, "i_2"] & U_7 \arrow[d, tail, "i_3"] \\
U_2 \arrow[r, tail, "j_2"] \arrow[d, two heads, "p_1"] & U_5 \arrow[r, two heads, "q_2"] \arrow[d, two heads, "p_2"] & U_8 \arrow[d, two heads, "p_3"] \\
U_3 \arrow[r, tail, "j_3"] & U_6 \arrow[r, two heads, "q_3"] & U_9
\end{tikzcd}$$

such that all of the rows and columns are exact, the top left square is a pullback and the bottom right square is a pushout.

Define $\mathcal{PS}_*$ to be the graded groupoid whose degree d part is the groupoid whose objects are perfect squares with $\dim(U_5) = d$, and whose morphisms are isomorphisms of perfect squares (considered as diagrams in $\mathrm{Vect}(\mathbb{F})$). If we take a 4-tuple $\underline{d} = (d_1, d_3, d_7, d_9)$ and let $\mathcal{PS}_{\underline{d}}$ be the groupoid of short exact sequences such that $\dim(U_1) = d_1$, $\dim(U_3) = d_3$, $\dim(U_7) = d_7$ and $\dim(U_9) = d_9$, then

$$\mathcal{PS}_d = \coprod_{d_1+d_3+d_7+d_9=d} \mathcal{PS}_{\underline{d}}.$$

Definition 8.1.2. Define projection functors that give the central horizontal and vertical short exact sequences

$$\gamma : \mathcal{PS}_* \to \mathcal{ES}_*$$

$$\underline{U} \mapsto (U_2 \xrightarrow{i_2} U_5 \xrightarrow{p_2} U_8)$$

$$\delta : \mathcal{PS}_* \to \mathcal{ES}_*$$

$$\underline{U} \mapsto (U_4 \xrightarrow{j_2} U_5 \xrightarrow{q_2} U_6).$$

Also, define a functor

$$\tau : \mathcal{PS}_* \to \mathcal{PS}_*$$

that takes a perfect square to its reflection in the diagonal from the top left to the bottom right, i.e. $\tau(\underline{U})_i = U_{j_i}$ with $j_i = i$ for $i = 1, 5, 9$, and

$$j_2 = 4, j_4 = 2, j_3 = 7, j_7 = 3, j_6 = 8, j_8 = 6.$$

This is clearly involutive, i.e. $\tau^2 = \mathrm{id}$, and we have

$$\delta = \gamma\tau.$$

Proposition 8.1.3. *We have a homotopy pullback square*

$$\begin{tikzcd}
\mathcal{ES}_* \arrow[r, "\beta"] & \mathcal{V}_* \\
\mathcal{PS}_* \arrow[u, "\gamma"] \arrow[r, "\delta"'] & \mathcal{ES}_* \arrow[u, "\beta"']
\end{tikzcd}$$

where β is as defined in 7.1.1.

Proof. Firstly, observe that the square commutes on the nose:

$$\beta\gamma(\underline{U}) = \beta(U_2 \xrightarrow{i_2} U_5 \xrightarrow{p_2} U_8) = U_5$$

$$\beta\delta(\underline{U}) = \beta(U_4 \xrightarrow{j_2} U_5 \xrightarrow{q_2} U_6) = U_5.$$

Recall from 7.1.2 that β is a fibration, so the homotopy pullback is equivalent to the ordinary pullback. The ordinary pullback is the graded groupoid with degree d part given by the groupoid of diagrams in $\mathrm{Vect}(\mathbb{F})$ of the form

$$\begin{tikzcd}
 & U_4 \arrow[d, tail, "j_2"] & \\
U_2 \arrow[r, tail, "i_2"] & U_5 \arrow[r, two heads, "p_2"] \arrow[d, two heads, "q_2"] & U_8 \\
 & U_6 &
\end{tikzcd}$$

such that $\dim(U_5) = d$, and the row and column are exact. We refer to these diagrams as *exact crosses*, and write $\mathcal{EC}_*$ for the graded groupoid of exact crosses and isomorphisms of exact crosses. Consequently, we get a limit functor

$$\phi : \mathcal{PS}_* \to \mathcal{EC}_*$$

that takes a perfect square to its underlying central exact cross. We will show that this is an equivalence. To see this, define a functor

$$\psi : \mathcal{EC}_* \to \mathcal{PS}_*$$

that takes an exact cross $\underline{U}$ to the perfect square whose central exact cross is $\underline{U}$, and whose corners are as follows: U_1 is the pullback of i_2 and j_2; $U_3 := Im(p_2 j_2)$; $U_7 := Im(q_2 i_2)$; and U_9 is the pushout of p_2 and q_2.

The resulting diagram has exact rows and columns by the first isomorphism theorem. Moreover, it is functorial because pullback, pushout, image, and coimage constructions can all be made into functors from categories of diagrams to categories of diagrams.

It is clear that we have $\phi\psi = \text{id}$ on the nose, so ϕ is π_0-surjective by 2.2.8. It is also easy to see that given a perfect square, U_1 is a pullback of i_2 and j_2; $U_3 \cong Im(q_2 i_2)$; $U_7 \cong Im(p_2 j_2)$; and U_9 is a pushout of p_2 and q_2. By the uniqueness of these constructions, we see that ϕ is also π_0-injective, and by 2.2.10, thus also full.

To see that ϕ is faithful, observe that if we have a map of perfect squares

$$f : \underline{U} \to \underline{U'}$$

then the map $U_5 \xrightarrow{f_5} U_5'$ determines all the other components f_i. This information (i.e. f_5) is remembered by the map of exact crosses, so ϕ is faithful. Thus ϕ is an equivalence, which completes the proof.

□

Remark 8.1.4. We will show that this pullback diagram can be related to a diagram in terms of $\mathcal{V}_*$. This will enable us to apply the Mackey property to a diagram related to $\mathcal{V}_*$.

Definition 8.1.5. Define a *hollow square* to be a diagram in Vect($\mathbb{F}$) of the form

$$\begin{tikzcd}
U_1 \arrow[r, tail, "j_1"] \arrow[d, tail, "i_1"'] & U_4 \arrow[r, two heads, "q_1"] & U_7 \arrow[d, tail, "i_3"] \\
U_2 \arrow[d, two heads, "p_1"'] & & U_8 \arrow[d, two heads, "p_3"] \\
U_3 \arrow[r, tail, "j_3"'] & U_6 \arrow[r, two heads, "q_3"'] & U_9
\end{tikzcd}$$

such that all of the rows and columns are exact.

Define $\mathcal{HS}_*$ to be the graded groupoid whose degree d part is the groupoid whose objects are hollow squares such that

$$\begin{aligned} d &= \dim(U_2) + \dim(U_8) = \dim(U_4) + \dim(U_6) \\ &= \dim(U_1) + \dim(U_3) + \dim(U_7) + \dim(U_9) \end{aligned}$$

and whose morphisms are isomorphisms of hollow squares (considered as diagrams in $\mathrm{Vect}(\mathbb{F})$).

Definition 8.1.6. Define a functor

$$\rho_1 : \mathcal{PS}_* \to \mathcal{HS}_*$$

by sending a perfect square to its outer hollow square.

Define another functor by sending a hollow square to the corners of that hollow square

$$\begin{aligned} \rho_2 : \mathcal{HS}_* &\to \mathcal{V}_*^4 \\ \underline{U} &\mapsto (U_1, U_3, U_7, U_9). \end{aligned}$$

We also define their composite

$$\begin{aligned} \rho : \mathcal{PS}_* \xrightarrow{\rho_1} \mathcal{HS}_* &\xrightarrow{\rho_2} \mathcal{V}_*^4 \\ \underline{U} &\mapsto (U_1, U_3, U_7, U_9). \end{aligned}$$

Definition 8.1.7. Define another functor in the opposite direction

$$\pi : \mathcal{V}_*^4 \to \mathcal{PS}_*$$

that takes (W_1, W_2, W_3, W_4) to the following direct sum perfect square

$$\begin{tikzcd} W_1 \arrow[r, tail] \arrow[d, tail] & W_1 \oplus W_3 \arrow[r, two heads] \arrow[d, tail] & W_3 \arrow[d, tail] \\ W_1 \oplus W_2 \arrow[r, tail] \arrow[d, two heads] & W_1 \oplus \cdots \oplus W_4 \arrow[r, two heads] \arrow[d, two heads] & W_3 \oplus W_4 \arrow[d, two heads] \\ W_2 \arrow[r, tail] & W_2 \oplus W_4 \arrow[r, two heads] & W_4 \end{tikzcd}.$$

Remark 8.1.8. When defining ρ_2 and π we have made a choice. We could equally have defined ρ_2 with $U_1, \ldots, U_4$ reordered, and we could correspondingly reorder $W_1, \ldots, W_4$ before applying π. It is important to keep track of this choice, because our conclusions will depend upon it.

Definition 8.1.9. Given a perfect square $\underline{U}$, define

$$K_1(\underline{U}) := Ker(\mathcal{PS}_*(\underline{U}) \xrightarrow{\rho_1} \mathcal{HS}_*(\rho_1(\underline{U})))$$

and define a map

$$f : \mathrm{Hom}(U_9, U_1) \to K_1(\underline{U})$$

$$m_{91} \mapsto (f(m_{91})_a)$$

where $f_a := \mathrm{id}$ for $a \neq 5$ and

$$f(m_{91})_5 := \mathrm{id} + i_2 j_1 m_{91} q_3 p_2$$

This map is well-defined because $p_2 i_2 = 0$, $q_2 i_2 j_1 = q_2 j_2 i_1 = 0$, $q_3 p_2 j_2 = p_3 q_2 j_2 = 0$, and because $f(m_{91})_a = \mathrm{id}$ for $a \neq 5$.

Definition 8.1.10. Given a hollow square $\underline{U}$, define

$$K_2(\underline{U}) := Ker(\mathcal{HS}_*(\underline{U}) \xrightarrow{\rho_2} \mathcal{V}_*^4(\rho_2(\underline{U})))$$

and define a map

$$g : \mathrm{Hom}(U_3, U_1) \times \mathrm{Hom}(U_7, U_1) \times \mathrm{Hom}(U_9, U_3) \times \mathrm{Hom}(U_9, U_7) \to K_2(\underline{U})$$

$$m_{ij} \mapsto (g(m_{ij})_a)$$

where $g(m_{ij})_a := \mathrm{id}$ for $a = 1, 3, 7, 9$ and

$$g(m_{ij})_2 := \mathrm{id} + i_1 m_{31} p_1$$

$$g(m_{ij})_4 := \mathrm{id} + j_1 m_{71} q_1$$

$$g(m_{ij})_6 := \mathrm{id} + j_3 m_{93} q_3$$

$$g(m_{ij})_8 := \mathrm{id} + i_3 m_{97} p_3.$$

This map is well-defined because $p_a i_a = 0$, $q_a j_a = 0$, and because $g_a = \mathrm{id}$ for $a = 1, 3, 7, 9$.

Lemma 8.1.11. *For $\underline{U}$ a perfect/hollow square respectively, the maps*

$$f : \mathrm{Hom}(U_9, U_1) \to K_1(\underline{U})$$

$$g : \mathrm{Hom}(U_3, U_1) \times \mathrm{Hom}(U_7, U_1) \times \mathrm{Hom}(U_9, U_3) \times \mathrm{Hom}(U_9, U_7) \to K_2(\underline{U})$$

are isomorphisms of groups.

Proof. Firstly, we show that we have homomorphisms. It is clear that $f(0) = \mathrm{id}$ and $g(0) = \mathrm{id}$. For f, as $p_2 i_2 = 0$

$$(\mathrm{id} + i_2 j_1 m_{91} q_3 p_2) \circ (\mathrm{id} + i_2 j_1 m'_{91} q_3 p_2) = \mathrm{id} + i_2 j_1 (m_{91} + m'_{91}) q_3 p_2$$

similar to 7.2.4. For g, as $p_a i_a = 0$ and $q_a j_a = 0$, we have

$$g_2(m_{ij}) \circ g_2(m'_{ij}) = (\mathrm{id} + i_1 m_{31} p_1) \circ (\mathrm{id} + i_1 m'_{31} p_1) = \mathrm{id} + i_1 (m_{31} + m'_{31}) p_1$$

just as in 7.2.4, and similarly for g_4, g_6, g_8.

To see that f is injective, suppose that $\mathrm{id} + i_2 j_1 m_{91} q_3 p_2 = \mathrm{id} + i_2 j_1 m'_{91} q_3 p_2$. Then $i_2 j_1 m_{91} q_3 p_2 = i_2 j_1 m'_{91} q_3 p_2$. As $i_2 j_1$ is monic and $q_3 p_2$ is epic, we see that $m_{91} = m'_{91}$.

To see that g is injective, suppose that $\mathrm{id} + i_1 m_{31} p_1 = \mathrm{id} + i_1 m'_{31} p_1$. Then $i_1 m_{31} p_1 = i_1 m'_{31} p_1$. As i_1 is monic and p_1 is epic, we see that $m_{31} = m'_{31}$. Similar arguments work for g_4, g_6, g_8, so g is injective.

To see that f is surjective, suppose $k \in K(\underline{U})$. Then we see that $k_a = \mathrm{id}$ for $a \neq 5$. Define $n := k_5 - id$. Then we see that $n i_2 = 0$, $n j_2 = 0$, $p_2 n = 0$, and $q_2 n = 0$, so n factors through the cokernels of i_2 and j_2, which are (U_6, p_2) and (U_8, q_2) respectively, and it factors through the kernels of p_2 and q_2, which are (U_4, i_2) and (U_2, j_2) respectively. Thus it also factors through the pushout of p_2 and q_2, i.e. U_9, and through the pullback of i_2 and j_2, i.e. U_1.

We conclude that n is of the form $i_2 j_1 m q_3 p_2$ for some $m \in \mathrm{Hom}(U_9, U_1)$. From this it is clear that $k = f(m)$.

To see that g is surjective, suppose $k \in K(\underline{U})$. Then we see that $k_a = \mathrm{id}$ for $a = 1, 3, 7, 9$. Define $n_a := k_a - id$ for $a = 2, 4, 6, 8$. Then we see that $n_2 i_1 = 0$ and $p_1 n_2 = 0$, so by the identification of i_1 and p_1 as a kernel and cokernel respectively, n_2 factors uniquely as

$$n_2 : U_2 \xrightarrow{p_1} U_3 \xrightarrow{m_{31}} U_1 \xrightarrow{i_1} U_2.$$

Each n_a factors similarly for $a = 4, 6, 8$ through the respective ends of the corresponding short exact sequences. From this it is clear that $k = g(m_{ij})$.

□

Proposition 8.1.12. *We have*

$$\rho\pi = \mathrm{id}.$$

Moreover, ρ is a regular p-local equivalence with p-local inverse π.

Proof. Firstly, it is easy to see that

$$\rho\pi(\underline{U}) = \underline{U}.$$

We will show that both ρ_1 and ρ_2 are regular p-local equivalences, and it will follow by 2.4.9 that ρ is as well. We will show this for ρ_2 first.

It is straightforward to check that $\mathcal{HS}_*$ fits into an ordinary pullback square

$$\begin{tikzcd}
\mathcal{ES}^2_* \arrow[r, "\alpha^2"] & \mathcal{V}^4_* \\
\mathcal{HS}_* \arrow[u, "\zeta_1"] \arrow[r, "\zeta_2"'] & \mathcal{ES}^2_* \arrow[u, "\alpha^2"']
\end{tikzcd}$$

and that ρ_2 is the composition map

$$\mathcal{HS}_* \xrightarrow{\zeta_i} \mathcal{ES}^2_* \xrightarrow{\alpha^2} \mathcal{V}^4_*.$$

As α is a fibration by 7.1.2, so is α^2, and so this is a homotopy pullback square. By 2.4.10, we see that ζ_1 and ζ_2 are regular p-local equivalences, and by 2.4.10, ρ_2 is also a regular p-local equivalence.

Next, we show that ρ_1 is a regular p-local equivalence. By the first statement, 2.2.8 implies that ρ is π_0-surjective, so as ρ_2 is π_0-bijective, ρ_1 must be π_0-surjective as well.

Next, we show that $\pi_0\rho_1$ is injective. To see this, first note that as $\pi_0\rho_2$ is bijective, every hollow square is isomorphic to the outer square of a perfect square $\pi(\underline{W})$ of the form

$$\begin{tikzcd}
W_1 \arrow[r, tail] \arrow[d, tail] & W_1 \oplus W_3 \arrow[r, two heads] \arrow[d, tail] & W_3 \arrow[d, tail] \\
W_1 \oplus W_2 \arrow[r, tail] \arrow[d, two heads] & W_1 \oplus \cdots \oplus W_4 \arrow[r, two heads] \arrow[d, two heads] & W_3 \oplus W_4 \arrow[d, two heads] \\
W_2 \arrow[r, tail] & W_2 \oplus W_4 \arrow[r, two heads] & W_4
\end{tikzcd}.$$

Thus, it suffices to show that any perfect square $\underline{U}$ with the same outer square as $\pi(\underline{W})$ is isomorphic to $\pi(\underline{W})$. In order to do this, it suffices to construct an isomorphism

$$k : W_1 \oplus W_2 \oplus W_3 \oplus W_4 \to U_5$$

that commutes with the corresponding maps i_2, j_2, p_2, q_2 in $\pi(\underline{W})$ and $\underline{U}$. If we look at i_2, j_2 then for each perfect square we get a map from the pushout to the central element, and similarly we get a map to the pullback of p_2, q_2. Commutation with i_2, j_2, p_2, q_2 then follows if we have commutation with these pushouts and pullbacks, i.e. if we have a commutative diagram

$$\begin{tikzcd}
W_1 \oplus W_2 \oplus W_3 \arrow[r, tail, "i"] \arrow[d, tail, "i'"] & U_5 \arrow[d, two heads, "p"] \\
W_1 \oplus W_2 \oplus W_3 \oplus W_4 \arrow[ur, "k"] \arrow[r, two heads, "p'"] & W_2 \oplus W_3 \oplus W_4
\end{tikzcd}.$$

If we take a section s of p and let s_4 be the restriction to W_4, then ps_4 is just the inclusion of W_4 and if we define

$$k := i + s_4$$

then it is straightforward to see that this makes the diagram commute. By the five lemma, k is necessarily an isomorphism. This proves that ρ_1 is π_0-injective.

It follows that ρ_1 is a π_0-isomorphism, and so ρ is also a π_0-isomorphism. By 2.2.10, ρ is full, and so as ρ_2 is full, ρ_1 must be full.

By 2.2.11, in order to show that we have a p-local equivalence we need to compute the order of the kernel for each isomorphism class of perfect squares $\underline{U}$. By 8.1.11

$$|K_1(\underline{U})| = |\mathrm{Hom}(U_9, U_1)|.$$

It is straightforward to see that $|\mathrm{Hom}(V_1, V_2)|$ has order $q^{\dim(V_1)\dim(V_2)}$, which is congruent to $1 \mod p$ and thus coprime to p. It follows that ρ_1 is a regular p-local equivalence, and so ρ is a regular p-local equivalence, and π must be its p-local inverse.

□

8.2 Adjusted Compatibility Relations

In this section, we apply the Mackey property to the homotopy pullback square from 8.1.3. Using the various p-local equivalences that we have constructed, and Frobenius reciprocity, we determine a twisted p-local compatibility diagram.

Definition 8.2.1. Define a twist map

$$\tau : \mathcal{V}^4_* \to \mathcal{V}^4_*$$

$$(W_1, W_2, W_3, W_4) \mapsto (W_1, W_3, W_2, W_4).$$

Proposition 8.2.2. *We have a homotopy commutative diagram such that λ, π are p-local equivalences and the middle square is a homotopy pullback square*

$$\begin{tikzcd}
& \mathcal{V}^2_* \arrow[d, "\lambda"] \arrow[dr, "\sigma"] & & \\
& \mathcal{ES}_* \arrow[r, "\beta"] & \mathcal{V}_* & \\
& \mathcal{PS}_* \arrow[u, "\gamma"] \arrow[r, "\delta"] & \mathcal{ES}_* \arrow[u, "\beta"] & \mathcal{V}^2_* \arrow[l, "\lambda"'] \arrow[ul, "\sigma"'] \\
\mathcal{V}^4_* \arrow[uuur, "\sigma^2"] \arrow[ur, "\pi"] \arrow[urrr, "\sigma^2\tau"'] & & &
\end{tikzcd}.$$

Proof. We have already seen that the middle square commutes and is a homotopy pullback square in 8.1.3. The far top and right triangles commute by 7.2.2.

We must now show that the bottom and left triangles commute. We start with the left triangle.

π maps (W_1, W_2, W_3, W_4) to the direct sum perfect square. Then γ sends this to the central horizontal short exact sequence

$$W_1 \oplus W_2 \rightarrowtail W_1 \oplus W_2 \oplus W_3 \oplus W_4 \twoheadrightarrow W_3 \oplus W_4.$$

Going the other way, σ^2 sends (W_1, W_2, W_3, W_4) to $(W_1 \oplus W_2, W_3 \oplus W_4)$. Then λ sends this to the same short exact sequence. Essentially the same argument applies to morphisms.

For the bottom triangle, π maps (W_1, W_2, W_3, W_4) to the direct sum perfect square. Then δ sends this to the central vertical short exact sequence

$$W_1 \oplus W_3 \rightarrowtail W_1 \oplus W_2 \oplus W_3 \oplus W_4 \twoheadrightarrow W_2 \oplus W_4.$$

Going the other way, $\sigma^2\tau$ sends (W_1, W_2, W_3, W_4) to $(W_1 \oplus W_3, W_2 \oplus W_4)$. Then λ sends this to the short exact sequence.

$$W_1 \oplus W_3 \rightarrowtail W_1 \oplus W_3 \oplus W_2 \oplus W_4 \twoheadrightarrow W_2 \oplus W_4.$$

Applying the isomorphism tw to the middle term, it is easy to see that these two short exact sequences are naturally isomorphic. Essentially the same argument applies to morphisms.

$\square$

Proposition 8.2.3. *We have a stable commutative diagram*

$$\begin{tikzcd}
 & \mathcal{V}_*^2 & & \\
\mathcal{V}_*^2 \arrow[r, "\lambda"] & \mathcal{ES}_* \arrow[u, "\lambda^!"] & \mathcal{V}_* \arrow[l, "\beta^!"] \arrow[ul, "\sigma^!"'] & \\
\mathcal{V}_*^4 \arrow[u, "\sigma^2"] \arrow[r, "\pi"] & \mathcal{PS}_* \arrow[u, "\gamma"'] \arrow[d, "\pi^!"'] & \mathcal{ES}_* \arrow[l, "\delta^!"] \arrow[u, "\beta"] \arrow[d, "\lambda^!"] & \mathcal{V}_*^2 \arrow[l, "\lambda"] \arrow[ul, "\sigma"'] \\
 & \mathcal{V}_*^4 & \mathcal{V}_*^2 \arrow[l, "\tau(\sigma^!)^2"] &
\end{tikzcd}.$$

Proof. As β, λ, and π are faithful, we can certainly construct all of these maps stably.

The commutation of the left square and right triangle follows immediately from the corresponding parts of 8.2.2. The top triangle and bottom square commute by functoriality of the transfer 2.3.9 applied to the corresponding parts of 8.2.2. Care must be taken to note that $(\sigma^2)^! = (\sigma^!)^2$, and that τ is an involutive equivalence and thus

$$\tau^! = \tau^{-1} = \tau.$$

Finally, the middle square commutes as a result of the Mackey property 2.3.9.

$\square$

Corollary 8.2.4. *We have p-locally that*

$$\sigma^!\sigma = \mu_u\sigma^2\mu_v\tau(\sigma^!)^2\mu_u$$

where $u := \lambda^*\lambda_!(1)$ and $v := \pi_!(1)^{-1}$. Moreover, u and v are units p-locally.

Proof. This essentially involves piecing together the information from 8.2.3. From the diagram, we have that after p-completion

$$\sigma^!\sigma = \lambda^!\lambda\sigma^2\pi^{-1}(\pi^!)^{-1}\tau(\sigma^!)^2\lambda^!\lambda.$$

Now observe that by 2.3.14 and 2.3.12

$$\begin{aligned}\lambda^!\lambda &= \lambda^{-1}(\lambda\lambda^!)\lambda \\ &= \lambda^{-1}\mu_{\lambda_!(1)}\lambda \\ &= \lambda^{-1}\lambda\mu_{\lambda^*\lambda_!(1)}\end{aligned}$$

and observe that by 2.3.14 and 2.3.8

$$\pi^{-1}(\pi^!)^{-1} = (\pi^!\pi)^{-1} = \mu^{-1}_{\pi_!(1)} = \mu_{\pi_!(1)^{-1}}.$$

□

Remark 8.2.5. The definition of u given here agrees K-locally with the Harish-Chandra twisting element defined in 7.2.8.

Theorem 8.2.6. *We have p-locally that*

$$\sigma^!\sigma = \sigma^2\mu_w\tau(\sigma^!)^2$$

where $w := v.(\sigma^2)^*(u).(\sigma^2\tau)^*(u)$. Equivalently, we have a p-local commutative diagram

$$\begin{tikzcd}
S_{(p)}\mathcal{V}_*^2 & & S_{(p)}\mathcal{V}_* \arrow[ll, "\sigma^!"'] \\
S_{(p)}\mathcal{V}_*^4 \arrow[u, "\sigma^2"] & & \\
& S_{(p)}\mathcal{V}_*^4 \arrow[ul, "\mu_w"] & S_{(p)}\mathcal{V}_*^2 \arrow[l, "(\sigma^2)^!"] \arrow[uu, "\sigma"']
\end{tikzcd}.$$

Remark 8.2.7. w is not unique in having this property because of the choice we made when defining ρ and π.

Proof. Starting with the previous lemma, apply Frobenius reciprocity (2.3.13) to $(\sigma^!)^2 \circ \mu_u$ and apply 2.3.12 to $\tau \circ \mu_{(\sigma^2)^*(u)}$ and $\mu_u \circ \sigma^2$ to get

$$\sigma^!\sigma = \sigma^2\tau\mu_{(\sigma^2\tau)^*(u)}\mu_v\mu_{(\sigma^2)^*(u)}(\sigma^!)^2.$$

Composing the middle terms using 2.3.8 gives the result.

□

Proposition 8.2.8. *For a characteristic p field k, in $H^*(B\mathcal{V}^2_*; k)$ we have $u = 1$, and in $H^*(B\mathcal{V}^4_*; k)$ we have $v = w = 1$.*

Proof. Note first that this statement makes sense because the Eilenberg-Maclane spectrum Hk is a module spectrum over $S^0_{(p)}$ and thus p-local by [1, Lemma 13.1], so $H^*(X; k) = H^*(X_{(p)}; k)$.

By standard group cohomology, Frobenius reciprocity implies that for an injective map of groups, $f : H \to G$, $f_!(1) = [G : H]$. But the corresponding indices for λ and π are equal to the sizes of the kernels of α and ρ and we have seen in 7.2.5 and 8.1.12 that these are congruent to $1 \mod p$, so u and v are both equal to 1.

It then follows that $w = 1$ by its definition.

□

8.3 Identifying u, v, and w

In this section, we compute the elements u, v, and w modulo the maximal ideals.

Lemma 8.3.1. *In $K^*(B\mathcal{V}_i \times \mathcal{V}_j)$, we have*

$$u_{ij} \equiv 1 \mod \mathfrak{m}$$

where u_{ij} is the component of u in $K^*(B\mathcal{V}_i \times \mathcal{V}_j)$.

Proof. Recall that $u = \lambda^*\lambda_!(1)$. As λ^* is continuous, it suffices to prove that

$$\lambda_!(1) \equiv 1 \mod \mathfrak{m}$$

in the local ring $K^*B\mathcal{ES}_{i,j}$. We have a unique homotopy class of maps

$$x_{ij} : * \to \mathcal{ES}_{i,j}$$

$$* \mapsto \underline{U} := (U \rightarrowtail V \twoheadrightarrow W).$$

Then it suffices to prove that we have

$$x^*_{ij}(\lambda_!(1)) = 1$$

in Morava K-theory. Consider the homotopy pullback square

$$\begin{tikzcd}
\mathcal{V}_i \times \mathcal{V}_j \arrow[r, "\lambda"] & \mathcal{ES}_{i,j} \\
F_{i,j} \arrow[u] \arrow[r, "\phi"'] & * \arrow[u, "x_{ij}"']
\end{tikzcd}.$$

Then by the Mackey property, $x_{ij}^*(\lambda_!(1)) = \phi_!(1)$. Moreover, as λ is faithful, by 2.2.11, $F_{i,j}$ is discrete of order

$$\frac{|\mathcal{ES}_{i,j}(\underline{U})|}{|(\mathcal{V}_i \times \mathcal{V}_j)(U,W)|} = |Ker(\alpha : \mathcal{ES}_{i,j}(\underline{U}) \to \mathcal{V}_i(U) \times \mathcal{V}_j(W))|.$$

But then $\phi = \amalg_{a \in F_{i,j}} \mathrm{id}$, so by 2.3.9

$$\phi_!(1) = \sum_{a \in F_{i,j}} 1 = |F_{i,j}|.$$

But by 7.2.4, the kernel above is equal to

$$|\mathrm{Hom}(W,U)| = q^{\dim(W)\dim(U)} \equiv 1 \mod p.$$

$\square$

Lemma 8.3.2. *For each* 4*-tuple* $\underline{i} = (i_1, i_2, i_3, i_4)$*, in* $K^*(B\mathcal{V}_{i_1} \times \cdots \times \mathcal{V}_{i_4})$*, we have*

$$v_{\underline{i}} \equiv 1 \mod \mathfrak{m}$$

where $v_{\underline{i}}$ *is the component of* v *in* $K^*(B\mathcal{V}_{i_1} \times \cdots \times \mathcal{V}_{i_4})$*.*

Proof. It suffices to prove that

$$\pi_!(1) \equiv 1 \mod \mathfrak{m}$$

in the local ring $K^*(B\mathcal{V}_{i_1} \times \ldots \mathcal{V}_{i_4})$. We have a unique homotopy class of maps

$$x_{\underline{i}} : * \to \mathcal{V}_{i_1} \times \ldots \mathcal{V}_{i_4}$$

$$* \mapsto \underline{W}.$$

Then it suffices to prove that we have

$$x_{\underline{i}}^*(\pi_!(1)) = 1$$

in Morava K-theory. Consider the homotopy pullback square

$$\begin{tikzcd}
\mathcal{PS}_{\underline{i}} \arrow[r, "\pi"] & \mathcal{V}_{i_1} \times \dots \mathcal{V}_{i_4} \\
F_{\underline{i}} \arrow[u] \arrow[r, "\phi"'] & * \arrow[u, "x_{\underline{i}}"']
\end{tikzcd}.$$

Then by the Mackey property, $x_{\underline{i}}^*(\pi_!(1)) = \phi_!(1)$. Moreover, as π is faithful, by 2.2.11, $F_{\underline{i}}$ is discrete of order

$$\frac{|\mathcal{PS}_{\underline{i}}(\underline{W})|}{|\mathcal{V}_{i_1}(W_1) \times \dots \times \mathcal{V}_{i_4}(W_4)|} = |Ker(\rho : \mathcal{PS}_{\underline{i}}(\underline{U}) \to (\mathcal{V}_{i_1} \times \dots \times \mathcal{V}_{i_4})(U_1, U_3, U_7, U_4)|.$$

But then $\phi = \amalg_{a \in F_{\underline{i}}} \mathrm{id}$, so by 2.3.9

$$\phi_!(1) = \sum_{a \in F_{\underline{i}}} 1 = |F_{\underline{i}}|.$$

But by 8.1.11, the kernel above is equal to

$$\begin{aligned} &|\mathrm{Hom}(U_9, U_1)||\mathrm{Hom}(U_3, U_1)||\mathrm{Hom}(U_7, U_1)||\mathrm{Hom}(U_9, U_3)||\mathrm{Hom}(U_9, U_7)| \\ &\qquad \equiv 1 \mod p. \end{aligned}$$

□

Corollary 8.3.3. *In* $K^*(B\mathcal{V}_*^4)$

$$w \equiv 1 \mod \mathfrak{m}_4.$$

Proof. This follows from the continuity of σ^*, the definition of w in 8.2.6, and the previous two lemmas.

□

Remark 8.3.4. It is possible to give good descriptions of the homotopy pullback squares of λ with itself and π with itself, which enable one to give more refined descriptions of u and v^{-1}.

8.4 Identification of v and w in Generalised Character Theory

In this section, we compute v and w in character theory. We then use this computation to observe that w satisfies various relations, including a 2-cocycle condition.

In the next proposition, we consider the maps

$$\mathcal{H}(\Theta^*, \rho) : \mathcal{H}(\Theta^*, \mathcal{PS}_*) \to \mathcal{H}(\Theta^*, \mathcal{V}_*^4)$$

$$\mathcal{H}(\Theta^*, \pi) : \mathcal{H}(\Theta^*, \mathcal{V}_*^4) \to \mathcal{H}(\Theta^*, \mathcal{PS}_*).$$

For the next two results, write $[W_{\underline{i}}]$ for the element

$$[W_1, W_2, W_3, W_4] \in [\Theta^*, \mathcal{V}_*^4].$$

Proposition 8.4.1. *We have*

$$\mathcal{H}(\Theta^*, \rho) \circ \mathcal{H}(\Theta^*, \pi) = \mathrm{id}.$$

Moreover, $\mathcal{H}(\Theta^*, \rho)$ is a regular p-local equivalence with p-local inverse $\mathcal{H}(\Theta^*, \pi)$.

For an object $(W_{\underline{i}}) \in \mathcal{H}(\Theta^*, \mathcal{V}_*^4)$ the fibre has order

$$|\mathrm{Hom}_{\Theta^*}(W_2, W_1)||\mathrm{Hom}_{\Theta^*}(W_3, W_1)||\mathrm{Hom}_{\Theta^*}(W_4, W_2)|$$
$$|\mathrm{Hom}_{\Theta^*}(W_4, W_3)||\mathrm{Hom}_{\Theta^*}(W_4, W_1)|.$$

Proof. $\mathcal{H}(\Theta^*, \mathcal{V}_*)$ is a faithful, essentially wide subcategory of the category of $\mathbb{F}$-representations of Θ^*, and by 5.2.5 and Maschke's theorem, short exact sequences of $\mathbb{F}$-representations of Θ^* always split.

Using these facts, the proof is then essentially identical to the amalgamation of the proofs of 8.1.11 and 8.1.12.

□

Proposition 8.4.2. *We have*

$$v : \mathrm{Rep}(\Theta^*; \mathbb{F})^4 \to L$$

$$[W_{\underline{i}}] \mapsto (|\mathrm{Hom}_{\Theta^*}(W_2, W_1)||\mathrm{Hom}_{\Theta^*}(W_3, W_1)||\mathrm{Hom}_{\Theta^*}(W_4, W_2)|$$
$$|\mathrm{Hom}_{\Theta^*}(W_4, W_3)||\mathrm{Hom}_{\Theta^*}(W_4, W_1)|)^{-1}.$$

Proof. Recall that $v := \rho_!(1)$. By 8.4.1, $\mathcal{H}(\Theta^*, \rho)$ is a π_0-isomorphism and full. By 6.4.3, in character theory we have

$$v[W_{\underline{i}}] = \sum_{[a] \in (\pi_0 \rho^{-1})[W_{\underline{i}}]} |K_a|^{-1} = |K_a|^{-1}$$

where $[a]$ is the unique isomorphism class $[U_{\underline{j}}]$ of $\rho^{-1}[W_{\underline{i}}]$ and

$$K_a = Ker(\mathcal{PS}_*(U_{\underline{j}}) \to \mathcal{V}^2_*(W_{\underline{i}})).$$

By 8.4.1, this is isomorphic to

$$|\mathrm{Hom}_{\Theta^*}(W_2, W_1)||\mathrm{Hom}_{\Theta^*}(W_3, W_1)||\mathrm{Hom}_{\Theta^*}(W_4, W_2)| \\ |\mathrm{Hom}_{\Theta^*}(W_4, W_3)||\mathrm{Hom}_{\Theta^*}(W_4, W_1)|$$

which gives the result.

□

Proposition 8.4.3. *We have*

$$w : \mathrm{Rep}(\Theta^*; \mathbb{F})^4_* \to L$$
$$[W_{\underline{i}}] \mapsto |\mathrm{Hom}_{\Theta^*}(W_4, W_1)||\mathrm{Hom}_{\Theta^*}(W_2, W_3)|^2.$$

Proof. By 7.3.5 and 5.2.2 we have

$$(\sigma^2)^*(u) : \mathrm{Rep}(\Theta^*; \mathbb{F})^4_* \to L$$

$$[W_{\underline{i}}] \mapsto |\mathrm{Hom}_{\Theta^*}(W_1 \oplus W_2, W_3 \oplus W_4)|^{-1} \\ = (|\mathrm{Hom}_{\Theta^*}(W_1, W_3)||\mathrm{Hom}_{\Theta^*}(W_1, W_4)||\mathrm{Hom}_{\Theta^*}(W_2, W_3)||\mathrm{Hom}_{\Theta^*}(W_2, W_4)|)^{-1}$$

and

$$\tau(\sigma^2)^*(u) : \mathrm{Rep}(\Theta^*; \mathbb{F})^4_* \to L$$

$$[W_{\underline{i}}] \mapsto |\mathrm{Hom}_{\Theta^*}(W_1 \oplus W_3, W_2 \oplus W_4)|^{-1} \\ = (|\mathrm{Hom}_{\Theta^*}(W_1, W_2)||\mathrm{Hom}_{\Theta^*}(W_1, W_4)||\mathrm{Hom}_{\Theta^*}(W_3, W_2)||\mathrm{Hom}_{\Theta^*}(W_3, W_4)|)^{-1}.$$

By definition, $w = v.(\sigma^2)^*(u).(\sigma^2\tau)^*(u)$ and the multiplication in character theory is pointwise, so as $|\mathrm{Hom}_{\Theta^*}(V_1, V_2)| = |\mathrm{Hom}_{\Theta^*}(V_2, V_1)|$ for Θ^*-reps V_1, V_2 by 6.5.4, the result follows by simply multiplying these two expressions and the expression from 8.4.2.

□

Remark 8.4.4. By looking at the coefficients of terms, commutativity of 8.2.6 in character theory essentially becomes the statement that for U, W, U', W'

$$\sum_{\{W_i\}} |\mathrm{Hom}_{\Theta^*}(W_4, W_1)||\mathrm{Hom}_{\Theta^*}(W_2, W_3)|^2 \frac{|\mathrm{Aut}_{\Theta^*}(U) \times \mathrm{Aut}_{\Theta^*}(W)|}{|\prod_i \mathrm{Aut}_{\Theta^*}(W_i)|} \\ = \frac{|\mathrm{Aut}_{\Theta^*}(U' \oplus W')|}{|\mathrm{Aut}_{\Theta^*}(U') \times \mathrm{Aut}_{\Theta^*}(W')|}$$

where the sum runs over 4-tuples of isomorphism classes $(W_1, \dots, W_4)$ such that $W_1 \oplus W_3 \cong U$, $W_2 \oplus W_4 \cong W$, $W_1 \oplus W_2 \cong U'$, $W_3 \oplus W_4 \cong W'$. This is essentially a generalisation of the q-Vandermonde identity.

This suggests that it should be possible to prove the character theoretic version of 8.2.6 combinatorially.

Remark 8.4.5. Recall that in a symmetric monoidal category $\mathcal{C}$, tw denotes the twist isomorphism

$$\mathrm{tw} : \mathrm{id} \otimes \tau \otimes \mathrm{id} : A_1 \otimes A_2 \otimes A_3 \otimes A_4 \to A_1 \otimes A_3 \otimes A_2 \otimes A_4$$
$$(a_1, a_2, a_3, a_4) \mapsto (a_1, a_3, a_2, a_4).$$

Lemma 8.4.6. *The image of w in $E^0 B\mathcal{V}_*^4$ and $K^0 B\mathcal{V}_*^4$ satisfies the 2-cocycle condition, unit condition, and twist condition*

$$(w \otimes 1^{\otimes 2}) \bullet ((\sigma^2 \mathrm{tw}) \wedge \mathrm{id}^2)^*(w) = (1^{\otimes 2} \otimes w) \bullet (\mathrm{id}^2 \wedge (\sigma^2 \mathrm{tw}))^*(w)$$
$$(w \bullet -) \circ (\eta_!^{\otimes 2} \otimes \mathrm{id}^{\otimes 2}) = (\eta_!^{\otimes 2} \otimes \mathrm{id}^{\otimes 2})$$
$$w = \mathrm{tw}^*(w).$$

Proof. The K-theory case follows from the E-theory case by tensoring along $E^0 \to K^0$. The E-theory case will follow by character theory. The twist condition follows easily by 8.4.3. By Frobenius reciprocity

$$(w \bullet -) \circ (\eta_!^{\otimes 2} \otimes \mathrm{id}^{\otimes 2}) = (\eta_!^{\otimes 2} \otimes \mathrm{id}^{\otimes 2}) \circ (\eta^*(w) \bullet -).$$

But $\eta^*(w) = 1$ because $|\mathrm{Hom}_{\Theta^*}(0, W)| = 1$. This proves the unit condition. Finally, we prove the 2-cocycle condition. Let $[W_{\underline{i}}] = [W_1, \dots, W_6] \in \mathrm{Rep}(\Theta^*; \mathbb{F})_*^6$. The value of the left hand side of the 2-cocycle condition on $[W_{\underline{i}}]$ is

$$\begin{aligned}
&w(W_1, \dots, W_4) \bullet w(W_1 \oplus W_3, W_2 \oplus W_4, W_5, W_6) \\
&= |\mathrm{Hom}_{\Theta^*}(W_4, W_1)||\mathrm{Hom}_{\Theta^*}(W_2, W_3)|^2 \\
&\qquad |\mathrm{Hom}_{\Theta^*}(W_6, W_1 \oplus W_3)||\mathrm{Hom}_{\Theta^*}(W_2 \oplus W_4, W_5)|^2 \\
&= |\mathrm{Hom}_{\Theta^*}(W_1, W_4)||\mathrm{Hom}_{\Theta^*}(W_1, W_6)||\mathrm{Hom}_{\Theta^*}(W_3, W_6)| \\
&\qquad (|\mathrm{Hom}_{\Theta^*}(W_2, W_3)||\mathrm{Hom}_{\Theta^*}(W_2, W_5)||\mathrm{Hom}_{\Theta^*}(W_4, W_5)|)^2.
\end{aligned}$$

The value of the right hand side of the 2-cocycle condition on $[W_{\underline{i}}]$ is

$$\begin{aligned}
&w(W_3, \dots, W_6) \bullet w(W_1, W_2, W_3 \oplus W_5, W_4 \oplus W_6) \\
&= |\mathrm{Hom}_{\Theta^*}(W_6, W_3)||\mathrm{Hom}_{\Theta^*}(W_4, W_5)|^2 \\
&\qquad |\mathrm{Hom}_{\Theta^*}(W_4 \oplus W_6, W_1)||\mathrm{Hom}_{\Theta^*}(W_2, W_3 \oplus W_5)|^2 \\
&= |\mathrm{Hom}_{\Theta^*}(W_1, W_4)||\mathrm{Hom}_{\Theta^*}(W_1, W_6)||\mathrm{Hom}_{\Theta^*}(W_3, W_6)| \\
&\qquad (|\mathrm{Hom}_{\Theta^*}(W_2, W_3)||\mathrm{Hom}_{\Theta^*}(W_2, W_5)||\mathrm{Hom}_{\Theta^*}(W_4, W_5)|)^2.
\end{aligned}$$

These values are equal, so the relation holds in character theory.

$\square$

Chapter 9

Faux Hopf Rings and their Modules

The main aim of this chapter is to amalgamate all of the structure that we have identified on $E^0B\mathcal{V}_*$ and $K^0B\mathcal{V}_*$. As noted in 3.2.4, these objects satisfy almost all of the axioms of a graded Hopf ring, except for the compatibility relation. In the previous chapter, we showed that, p-locally we have a modified version of the compatibility relation involving a twisting element w.

In the first section of this chapter, we abstract the resulting structure to give the definition of a graded faux Hopf semiring, and of a graded faux Hopf ring. We first show that for a graded faux Hopf semiring A, the 2-cocycle condition on w yields adjusted algebra and coalgebra structures on $A^{\otimes 2}$. We then consider the existence of antipodes, and show that $E^0B\mathcal{V}_*$ and $K^0B\mathcal{V}_*$ are indeed graded faux Hopf rings. After this we define the indecomposables and primitives of a graded faux Hopf ring. We then show that the indecomposables naturally form an algebra over the base ring, and the primitives naturally form a module over this algebra.

In the second section, we develop the notion of a module over a graded faux Hopf ring. We observe that such a module can be dualised in an obvious sense, and show that $E^0B\mathcal{V}_*$ and $K^0B\mathcal{V}_*$ are self-dual as modules over themselves, via K-local duality. After this, we extend the definitions of indecomposables and primitives to modules over a graded faux Hopf ring. We then investigate the interaction between these definitions and the indecomposables of the graded faux Hopf ring.

Remark 9.0.1. Ideally, in this chapter we would spell out proofs with commutative diagrams to assert categorical invariance, however due to the required size and number of such diagrams this is impractical. Instead, we present elemental arguments that hopefully can be generalised without too much difficulty. It may help the reader to figure out the corresponding diagrams themselves.

9.1 Faux Hopf Semirings and Rings

In this section, we define faux Hopf semirings and faux Hopf rings, before studying the basic consequences of these definitions. In particular, we show that $E^0 B\mathcal{V}_*$ and $K^0 B\mathcal{V}_*$ are faux Hopf rings.

Definition 9.1.1. A *graded faux Hopf semiring* over a ring k is a graded k-module A with an "addition" commutative graded algebra structure $(*, i_0)$, a "multiplication" commutative homogeneous algebra structure $(\bullet, i_1)$, and a "diagonal" cocommutative graded coalgebra structure (Δ, π) satisfying all of the usual graded Hopf semiring axioms, except with a modified compatibility relationship between $*$ and Δ. Explicitly, $(\bullet, i_1, \Delta, \pi)$ is a bialgebra (in the ungraded sense), and the following distributivity and annihilation relations hold

$$\bullet \circ (\mathrm{id} \otimes *) = * \circ (\bullet \otimes \bullet) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\Delta \otimes \mathrm{id}^{\otimes 2}) : A^{\otimes 3} \to A$$

$$\bullet \circ (i_0 \otimes \mathrm{id}) = i_0 \circ \pi : A \to A.$$

We also have the usual bialgebra-type relations between $(*, i_0)$ and (Δ, π)

$$\pi \circ * = \pi \otimes \pi : A^{\otimes 2} \to k$$

$$\Delta \circ i_0 = i_0 \otimes i_0 : k \to A^{\otimes 2}$$

$$\pi \circ i_0 = \mathrm{id}_k : k \to k.$$

In place of the usual compatibility diagram for $(*, \Delta)$, there is a unit $w \in (A^{\otimes 2})^{\otimes 2}$, which must satisfy the 2-cocycle condition in $(A^{\otimes 2})^{\otimes 3}$, an addition unit condition, and a twist condition

$$(w \otimes 1) \bullet ([\mathrm{tw} \circ \Delta^{\otimes 2}] \otimes \mathrm{id}^{\otimes 2})(w) = (1 \otimes w) \bullet (\mathrm{id}^{\otimes 2} \otimes [\mathrm{tw} \circ \Delta^{\otimes 2}])(w) \in A^{\otimes 6}$$

$$(w \bullet -) \circ (i_0^{\otimes 2} \otimes \mathrm{id}^{\otimes 2}) = (i_0^{\otimes 2} \otimes \mathrm{id}^{\otimes 2}) : A^{\otimes 2} \to A^{\otimes 4}$$

$$w = \mathrm{tw}(w) \in A^{\otimes 4}$$

as well as a modified compatibility diagram

$$\begin{tikzcd}
A^{\otimes 2} \arrow[rr, "*"] \arrow[d, "\Delta^{\otimes 2}"'] & & A \arrow[dd, "\Delta"] \\
A^{\otimes 4} \arrow[dr, "w \bullet -"'] & & \\
& A^{\otimes 4} \arrow[r, "*^{\otimes 2} \circ \mathrm{tw}"'] & A^{\otimes 2}
\end{tikzcd}.$$

Remark 9.1.2. When we say that $(\bullet, i_1)$ is *homogeneous*, we mean that for $d \geq 0$, on each A_d we have an algebra structure $(\bullet_d, (i_1)_d)$, and $(\bullet, i_1)$ is the sum of these algebra structures. In particular, it is not graded.

Remark 9.1.3. In terms of elements $a, b, c \in A$, the distributivity and annihilation relations can be written respectively as

$$a \bullet (b * c) = *(\Delta(a) \bullet (b \otimes c)) \in A$$

$$[0] \bullet a = i_0 \circ \pi(a) \in A$$

where $[0] := i_0(1) \in A$.

Remark 9.1.4. Recall that a semiring is like a ring, except that elements need not have additive inverses. In particular, there is a multiplication and addition. These satisfy the usual distributivity relation of a ring, but they also satisfy an annihilation relation

$$0.a = 0,$$

which is otherwise lost without additive inverses. One can abstract this notion of a semiring to give a categorical definition, and in this way, a graded faux Hopf semiring is similar to (though not the same as) a semiring in the category of graded coalgebras.

Definition 9.1.5. If A is a graded faux Hopf semiring, then define maps

$$*' := *^{\otimes 2} \circ \mathrm{tw} \circ (w \bullet -) : A^{\otimes 4} \to A^{\otimes 2}$$

$$\Delta' := (w \bullet -) \circ \mathrm{tw} \circ \Delta^{\otimes 2} : A^{\otimes 2} \to A^{\otimes 4}.$$

Proposition 9.1.6. *If A is a graded faux Hopf semiring, then $(*', i_0^{\otimes 2})$ defines an algebra structure on $A^{\otimes 2}$, and $(\Delta', \pi^{\otimes 2})$ defines a coalgebra structure on $A^{\otimes 2}$.*

Proof. This is similar to 7.4.3. Firstly, we note that it is standard that if $(*, i_0)$ defines an algebra structure on A, then $(*_2 := *^{\otimes 2} \circ \mathrm{tw}, i_0^{\otimes 2})$ defines an algebra structure on $A^{\otimes 2}$. In these terms, we have

$$*' = *_2 \circ (w \bullet -) : (A^{\otimes 2})^{\otimes 2} \to A^{\otimes 2}.$$

To see that $*'$ is associative, for $x, y, z \in A^{\otimes 2}$, we have

$$\begin{aligned}(x *' y) *' z &= *_2(w \bullet [*_2(w \bullet (x \otimes y)) \otimes z]) \\ &= *_2(w \bullet [(*_2 \otimes \mathrm{id}^{\otimes 2})(w \otimes 1) \bullet (x \otimes y \otimes z)]) \\ &= *_2((*_2 \otimes \mathrm{id}^{\otimes 2})((w \otimes 1) \bullet (\Delta \otimes \mathrm{id})(w) \bullet (x \otimes y \otimes z))) \in A^{\otimes 2}.\end{aligned}$$

By reflecting this argument, we also have

$$x *' (y *' z) = *_2((*_2 \otimes \mathrm{id}^{\otimes 2})((1 \otimes w) \bullet (\mathrm{id} \otimes \Delta)(w) \bullet (x \otimes y \otimes z))) \in A^{\otimes 2}.$$

By the 2-cocycle condition, these two expressions are equal, so $*'$ is associative. To see that $i_0^{\otimes 2}$ is a unit, we must show that

$$*_2 \circ (w \bullet -) \circ (i_0^{\otimes 2} \otimes \mathrm{id}^{\otimes 2}) = \mathrm{id}^{\otimes 2} : A^{\otimes 2} \to A^{\otimes 2}.$$

However, by the addition unit condition

$$(w \bullet -) \circ (i_0^{\otimes 2} \otimes \mathrm{id}^{\otimes 2}) = (i_0^{\otimes 2} \otimes \mathrm{id}^{\otimes 2}) : A^{\otimes 2} \to A^{\otimes 2}$$

so as $i_0^{\otimes 2}$ is a unit for $*_2$, it is also a unit for $*'$.

The argument for $(\Delta', \pi^{\otimes 2})$ is essentially the dual of the argument for $(*', i_0^{\otimes 2})$.

□

Remark 9.1.7. In terms of elements $a, b \in A$, the modified compatibility relation can be written as

$$\Delta(a * b) = \Delta(a) *' \Delta(b) \in A^{\otimes 2}.$$

Definition 9.1.8. A *graded faux Hopf ring* is a graded faux Hopf semiring A with a graded *additive antipode* map

$$S : A \to A$$

that makes the following diagram commute

$$\begin{tikzcd} A^{\otimes 2} \arrow[rr, "S \otimes \mathrm{id}"] & & A^{\otimes 2} \arrow[d, "*"] \\ A \arrow[u, "\Delta"] \arrow[r, "\pi"'] & k \arrow[r, "i_0"'] & A \end{tikzcd}.$$

Remark 9.1.9. As Δ is cocommutative and $*$ is commutative, this diagram also commutes if we replace $S \otimes \mathrm{id}$ with $\mathrm{id} \otimes S$.

Remark 9.1.10. With these definitions, a graded Hopf ring is the same as a graded faux Hopf ring such that $w = [1]^{\otimes 4} \in A^{\otimes 4}$, where $[1] := i_1(1) \in A$.

Remark 9.1.11. Following on from 9.1.4, recall that a ring is a semiring with additive inverses. Analogously, a graded faux Hopf ring is a graded faux Hopf semiring with an additive antipode. From this perspective, the additive antipode corresponds to the additive inverse map.

Lemma 9.1.12. *If A is a faux Hopf semiring then it has at most one additive antipode. Moreover, if there is an additive antipode then it is given by*

$$([-1] \bullet -) : A \to A$$

where $[-1] := S([1])$.

Proof. Firstly, we prove that if an additive antipode exists, then it is unique. This follows with the same proof as for Hopf algebras. Suppose we have two additive antipodes $S_1, S_2 : A \to A$. We consider the map

$$*_3(S_1 \otimes \mathrm{id} \otimes S_2)\Delta_3 : A \to A$$

where

$$\begin{aligned} *_3 &= *(id \otimes *) : A^{\otimes 3} \to A \\ \Delta_3 &= (id \otimes \Delta)\Delta : A \to A^{\otimes 3} \end{aligned}$$

Then using the associativity of $*$, coassociativity of Δ, and the definition of an additive antipode, we have

$$\begin{aligned} *_3(S_1 \otimes \mathrm{id} \otimes S_2)\Delta_3 &= *(\mathrm{id} \otimes *)(S_1 \otimes \mathrm{id}^{\otimes 2})(\mathrm{id}^{\otimes 2} \otimes S_2)(\mathrm{id} \otimes \Delta)\Delta \\ &= *(S_1 \otimes \mathrm{id})(\mathrm{id} \otimes *)(\mathrm{id}^{\otimes 2} \otimes S_2)(\mathrm{id} \otimes \Delta)\Delta \\ &= *(S_1 \otimes (i_0\pi))\Delta \\ &= *(\mathrm{id} \otimes i_0)S_1(\mathrm{id} \otimes \pi)\Delta \\ &= S_1. \end{aligned}$$

Reflecting this argument shows that also

$$*_3(S_1 \otimes \mathrm{id} \otimes S_2)\Delta_3 = S_2$$

so $S_1 = S_2$.

For the second statement, by the distributivity and annihilation relations, we consider the map

$$* \circ (([-1] \bullet -) \otimes \mathrm{id}) \circ \Delta : A \to A$$

We have

$$\begin{aligned} * \circ (([-1] \bullet -) \otimes \mathrm{id}) \circ \Delta(x) &= * \circ ([-1] \otimes [1] \bullet^{\otimes 2} -) \circ \Delta(x) \\ &= x \bullet *([-1] \otimes [1]) \\ &= x \bullet (* \circ (S \otimes \mathrm{id}) \circ \Delta)([1]) \\ &= x \bullet (i_0 \circ \pi([1])) \\ &= x \bullet [0] \\ &= i_0\pi(x) \in A \end{aligned}$$

where we have used the fact that $\Delta([1]) = [1] \otimes [1]$.

$\square$

Definition 9.1.13. We say a faux Hopf semiring A over a ring k is *connected* if $A_0 = k$. This implies that i_0 and π are isomorphisms in degree 0.

We say a faux Hopf semiring A is *grouplike* if A_0 is an ungraded Hopf algebra under $(*, \Delta)$ with antipode S_0.

Remark: In particular, any connected faux Hopf semiring is grouplike.

Lemma 9.1.14. *If A is a grouplike faux Hopf semiring, then A has an additive antipode, and is thus a faux Hopf ring.*

Proof. This is the same proof as for Hopf algebras. We define the mth graded part of S inductively. The base case $m = 0$ exists via the grouplike assumption. Assuming S is defined for gradings less than m, we define a k-linear map

$$S_m : A_m \xrightarrow{(\mathrm{id} \otimes i_0) - \Delta} \sum_{i+j=m, i<m} A_i \otimes A_j \xrightarrow{\sum S_i \otimes \mathrm{id}} \sum_{i+j=m, i<m} A_i \otimes A_j \xrightarrow{*} A_m.$$

For $x \in A_m$, if we write $\Delta(x) = x \otimes 1 + 1 \otimes x + \sum x' \otimes x''$ then we get

$$S(x) = -x - \sum S(x') * x'' \in A_m.$$

We thus have

$$*(S \otimes \mathrm{id})\Delta(x) = S(x) + x + \sum S(x') * x'' = 0 = i_0 \pi(x) \in A$$

as required.

$\square$

Theorem 9.1.15. *$K^0 B\mathcal{V}_*$ is a connected graded faux Hopf ring over K^0, and $E^0 B\mathcal{V}_*$ is a connected graded faux Hopf ring over E^0. The addition algebra is given by $(\sigma_!, \eta_!)$; the multiplication algebra is given by (δ^*, ϵ^*); the diagonal coalgebra is given by (σ^*, η^*); w is given by the image of the element w in 8.2.6.*

Proof. It is straightforward to check that most of the axioms of 9.1.1 come from 3.2.4. The only ones that remain are those involving w.

We have Künneth isomorphisms by 3.2.3. By functoriality, 1.3.2 and 3.2.3, we can apply E or K to K-local diagrams to yield corresponding diagrams in terms of $E^0 B\mathcal{V}_*$ or $K^0 B\mathcal{V}_*$.

By 2.4.1, $L_K L_{(p)} = L_K$, so we can K-localise the p-local diagram in 8.2.6, and define w to be the image in $E^0 B\mathcal{V}_*$ or $K^0 B\mathcal{V}_*$ of the element w defined in 8.2.6. By 6.3.4, we see that $\mu_w^* = (w \bullet -)$, so this gives us the compatibility diagram.

By 8.4.6, w satisfies the 2-cocycle condition for σ^*, the addition unit condition for $\eta_!$, and the twist condition, so $E^0 B\mathcal{V}_*$ and $K^0 B\mathcal{V}_*$ are graded faux Hopf semirings. It is clear that $E^0 B\mathcal{V}_*$ and $K^0 B\mathcal{V}_*$ are connected, because $\mathcal{V}_*$ is connected by 3.1.5, so by 9.1.14, there exist unique antipodes making $E^0 B\mathcal{V}_*$ and $K^0 B\mathcal{V}_*$ into graded faux Hopf rings.

□

Corollary 9.1.16. *We have an isomorphism of faux Hopf rings over K^0*

$$K^0 \otimes_{E^0} E^0 B\mathcal{V}_* = K^0 B\mathcal{V}_*.$$

Proof. The map of ring spectra $E \to K$ induces a natural transformation of cohomology theories, and so we get a natural map

$$E^0 B\mathcal{V}_* \to K^0 B\mathcal{V}_*$$

of faux Hopf semirings over E^0. By 1.3.2, the induced map

$$K^0 \otimes_{E^0} E^0 B\mathcal{V}_* \to K^0 B\mathcal{V}_*$$

is an isomorphism of faux Hopf semirings. Both sides have an antipode, so by uniqueness, they must be the same antipode, making this into an isomorphism of faux Hopf rings.

□

Theorem 9.1.17. *For a characteristic p field k, $H^*(B\mathcal{V}_*; k)$ is a connected bigraded Hopf ring over k. The addition algebra is given by $(\sigma_!, \eta_!)$; the multiplication algebra is given by (δ^*, ϵ^*); the diagonal coalgebra is given by (σ^*, η^*).*

Proof. As k is a field, it is standard that we have Künneth isomorphisms. We can apply $H^*(-; k)$ to 3.1.24 to give all of the axioms of a bigraded Hopf semiring, except for the compatibility relation for $(\sigma_!, \sigma^*)$.

As k has characteristic p, Hk is an $S^0_{(p)}$-module spectrum and thus p-local by [1, Lemma 13.1], so we can apply $H^*(-; k)$ to the relation in 8.2.4 to get a relation in terms of $H^*(B\mathcal{V}_*; k)$. By 8.2.8, $u = 1$ and $v = 1$ in $H^*(B\mathcal{V}_*^4; k)$, and by 2.3.7, we see that $\mu_u^* = \mathrm{id}$ and $\mu_v^* = \mathrm{id}$, so the relation we get is just the ordinary compatibility relation for a bialgebra. This makes $H^*(B\mathcal{V}_*; k)$ into a bigraded Hopf semiring.

It is clear that $H^*(B\mathcal{V}_*; k)$ is connected, because $\mathcal{V}_*$ is connected. As a graded Hopf ring is a special case of a graded faux Hopf ring, by 9.1.14, there exists a unique antipode making $H^*(B\mathcal{V}_*; k)$ into a bigraded Hopf ring.

□

Remark 9.1.18. In the case of $H^*(B\mathcal{V}_*; k)$, as we have elements in odd cohomological degree, we need to modify our assumptions slightly to account for the graded commutativity of the cup product.

Definition 9.1.19. For a faux Hopf semiring A, we define

$$IA := Ker(\pi : A \to k).$$

We define the *indecomposables* of A to be

$$\begin{aligned} QA &:= Coker(IA \otimes IA \xrightarrow{*} IA) \\ &= IA/IA^{*2}. \end{aligned}$$

This is a graded quotient k-module.

Definition 9.1.20. For a faux Hopf semiring A, we define

$$JA := Coker(i_0 : k \to A).$$

We define the *primitives* of A to be

$$\begin{aligned} PA &:= Ker(JA \xrightarrow{\Delta} JA \otimes JA) \\ &\cong \{x \in A \mid \Delta(x) = x \otimes [0] + [0] \otimes x\}. \end{aligned}$$

where $[0] = i_0(1)$ as in 9.1.3. This is a graded sub-k-module.

Remark 9.1.21. It is also possible to define a different (ungraded) module of primitives by using the multiplication unit i_1, however, we will not consider that here.

Lemma 9.1.22. *For a connected faux Hopf ring A, $IA_0 = 0$, and for $m > 0$, $IA_m = A_m$. Moreover, IA is a $\bullet$-ideal in A.*

Proof. $\pi : A \to k$ is a graded map, so $\pi(A_m) = 0$ for $m > 0$. However, by connectedness, $\pi : A_0 \to k$ is an isomorphism, so $IA = \bigoplus_{m>0} A_m$. The statement now follows by the homogeneity of $\bullet$.

□

Proposition 9.1.23. *For a connected faux Hopf ring A, there is a unique commutative k-algebra structure on the degree m indecomposables QA_m for $m > 0$, such that the map*

$$A_m \to QA_m$$

is a map of k-algebras, where A_m has the $\bullet$-product. In other words, QA inherits a homogeneous k-algebra structure from $\bullet$.

Proof. As $A_m \to QA_m$ is surjective, if the homogeneous maps

$$\bullet : A_m \otimes A_m \to A_m$$
$$i_1 : k \to A_m$$

descend to give k-linear maps

$$\bullet : QA_m \otimes QA_m \to QA_m$$
$$i_1 : k \to QA_m$$

then QA_m must also inherit the associativity, commutativity and unit relations. We have

$$QA = IA/IA^{*2}$$
$$QA \otimes QA = \frac{IA \otimes IA}{IA \otimes IA^{*2} + IA^{*2} \otimes IA}$$

so to prove that $\bullet$ descends to a map on QA, it suffices to show that

$$IA \bullet (IA^{*2}) \subseteq IA^{*2}$$
$$(IA^{*2}) \bullet IA \subseteq IA^{*2}.$$

By distributivity as in 9.1.3, for $a, b, c \in IA$ we have

$$a \bullet (b * c) = *(\Delta(a) \bullet (b \otimes c)).$$

But as IA is a $\bullet$-ideal, $\Delta(a) \bullet (b \otimes c) \in IA \otimes IA$, so $a \bullet (b * c) \in *(IA \otimes IA) = IA^{*2}$. The second part is similar.

To define i_1, we just construct the composition

$$k \xrightarrow{i_1} A_m \to QA_m.$$

□

Remark 9.1.24. Here, we couldn't use the functoriality of Q to show that QA is a k-algebra, because the structure maps don't exist in any obvious category upon which we could apply $k \otimes_A -$.

9.2 Modules over a Faux Hopf Ring

In this section, we define modules over a faux Hopf ring and study the basic consequences of the definition. In particular, we show that K-local duality implies that $E^0 B\mathcal{V}_*$ and $K^0 B\mathcal{V}_*$ are self-dual as modules over themselves.

Definition 9.2.1. For a bialgebra $(A, \bullet, i_1, \Delta, \pi)$ over k, a *bialgebra module* M over A is a graded k-module with a coalgebra structure (Δ, π) and a homogeneous linear action

$$\bullet : A \otimes M \to M$$

such that we have associativity and unit relations

$$\bullet \circ (\bullet \otimes \mathrm{id}) = \bullet \circ (\mathrm{id} \otimes \bullet) : A^{\otimes 2} \otimes M \to M$$

$$\bullet \circ (i_1 \otimes \mathrm{id}) = \mathrm{id} : M \to M.$$

We must also have bialgebra-type relations between $\bullet$ and the coalgebra structure (Δ, π)

$$\Delta \circ \bullet = \bullet^{\otimes 2} \circ \mathrm{tw} \circ \Delta^{\otimes 2} : A \otimes M \to M^{\otimes 2}$$

$$\pi \circ \bullet = \pi \otimes \pi : A \otimes M \to k.$$

Remark 9.2.2. In terms of elements $a, b \in A$ and $x, y \in M$, the associativity, unit and compatibility relations can be written as

$$a \bullet (b \bullet x) = (a \bullet b) \bullet x \in M$$

$$[1] \bullet x = x \in M$$

$$\Delta(a \bullet x) = \Delta(a) \bullet \Delta(x) \in M^{\otimes 2}$$

$$\pi(a \bullet x) = \pi(a) \otimes \pi(x) \in k.$$

Definition 9.2.3. For A a faux Hopf semiring, an *A-module* is a graded k-module M along with an "addition" graded commutative algebra structure $(*, i_0)$, a "diagonal" graded cocommutative coalgebra structure (Δ, π), and a homogeneous linear action

$$\bullet : A \otimes M \to M$$

such that $(M, \bullet, \Delta, \pi)$ is a bialgebra module over A, and the following distributivity and annihilation relations hold

$$\bullet \circ (\mathrm{id} \otimes *) = * \circ (\bullet \otimes \bullet) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\Delta \otimes \mathrm{id}^{\otimes 2}) : A \otimes M^{\otimes 2} \to M$$

$$\bullet \circ (* \otimes \mathrm{id}) = * \circ (\bullet \otimes \bullet) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\mathrm{id}^{\otimes 2} \otimes \Delta) : A^{\otimes 2} \otimes M \to M$$

$$\bullet \circ (i_0 \otimes \mathrm{id}) = i_0 \circ \pi : M \to M$$

$$\bullet \circ (\mathrm{id} \otimes i_0) = i_0 \circ \pi : A \to M.$$

We also have bialgebra-type relations between $(*, i_0)$ and (Δ, π)

$$\pi \circ * = \pi \otimes \pi : M^{\otimes 2} \to k$$

$$\Delta \circ i_0 = i_0 \otimes i_0 : k \to M^{\otimes 2}$$

$$\pi \circ i_0 = \mathrm{id}_k : k \to k.$$

In place of the usual compatibility diagram for $(*, \Delta)$, M must also satisfy a modified compatibility diagram between $*$ and Δ

$$\begin{tikzcd}
M^{\otimes 2} \arrow[rr, "*"] \arrow[d, "\Delta^{\otimes 2}"'] & & M \arrow[dd, "\Delta"] \\
M^{\otimes 4} \arrow[dr, "w \bullet -"'] & & \\
& M^{\otimes 4} \arrow[r, "*^{\otimes 2} \circ \mathrm{tw}"'] & M^{\otimes 2}
\end{tikzcd}.$$

A consequence of this is that $*' := *^{\otimes 2} \circ \mathrm{tw} \circ (w \bullet -)$ defines a new algebra structure on $M^{\otimes 2}$, and $\Delta' := (w \bullet -) \circ \mathrm{tw} \circ \Delta^{\otimes 2}$ defines a new coalgebra structure on $M^{\otimes 2}$.

A map of A-modules must preserve all the structure.

Remark 9.2.4. In terms of elements $a, b \in A$ and $x, y \in M$, the distributivity, annihilation, and modified compatibility relations can be written respectively as

$$a \bullet (x * y) = *(\Delta(a) \bullet (x \otimes y)) \in M$$

$$(a * b) \bullet x = *((a \otimes b) \bullet \Delta(x)) \in M$$

$$[0] \bullet x = i_0 \circ \pi(x) \in M.$$

where $[0] := i_0(1) \in M$.

Remark 9.2.5. It is straightforward to check using the definitions that if A is a faux Hopf semiring, then A is naturally an A-module.

Remark 9.2.6. Recall that an ordinary module M over an ordinary semiring A is similar to a module over a ring, except that over a semiring, the elements of a module need not have additive inverses. In particular, there is a multiplicative action of A on M and an addition on M that makes it into a commutative monoid. These satisfy the normal distributivity relations of a module, but they also satisfy annihilation relations

$$0.m = 0 = a.0,$$

which are otherwise lost without additive inverses. One can abstract this notion of a module over a semiring to give a categorical definition. This is the analogy to keep in mind when thinking about modules over a graded faux Hopf semiring.

Proposition 9.2.7. *If A is a graded faux Hopf semiring, then $(*', i_0^{\otimes 2})$ defines an algebra structure on $M^{\otimes 2}$, and $(\Delta', \pi^{\otimes 2})$ defines a coalgebra structure on $M^{\otimes 2}$.*

Proof. This is essentially the same as the proof of 9.1.6.

□

Remark 9.2.8. In terms of elements $x, y \in M$, the modified compatibility relation can be written as

$$\Delta(x * y) = \Delta(x) *' \Delta(y) \in M^{\otimes 2}.$$

Definition 9.2.9. If A is a faux Hopf semiring, and M is an A-module, then an *additive antipode* for M is a map

$$S : M \to M$$

that makes the following diagram commute

$$\begin{tikzcd}
M^{\otimes 2} \arrow[rr, "S \otimes \mathrm{id}"] & & M^{\otimes 2} \arrow[d, "m"] \\
M \arrow[u, "\Delta"] \arrow[r, "\epsilon"'] & k \arrow[r, "\eta"'] & M
\end{tikzcd}$$

as well as its reflection with $\mathrm{id} \otimes S$ in place of $S \otimes \mathrm{id}$.

Remark 9.2.10. If we consider the A-module M as analogous to a module over a semiring, and thus in particular, a commutative monoid under addition, then the additive antipode for M corresponds to an additive inverse map.

Lemma 9.2.11. *If A is a faux Hopf ring and M is an A-module, then M has a unique additive antipode given by*

$$S := ([-1] \bullet -) : M \to M.$$

Proof. The proof is essentially identical to the amalgamation of the proofs of 9.1.12 and 9.1.14.

□

Definition 9.2.12. For a faux Hopf ring A over k, and an A-module M that is projective and of finite type over k, we can define an A-module structure on the k-linear dual

$$M^\vee = \mathrm{Hom}_k(M, k)$$

with addition algebra structure $(\Delta^\vee, \pi^\vee)$, diagonal coalgebra structure $(*^\vee, i_0^\vee)$, and homogeneous linear action

$$\bullet : A \otimes M^\vee \mapsto M^\vee$$

$$a \bullet f : x \mapsto f(a \bullet x).$$

By the symmetric nature of the axioms, one can easily check that this defines an A-module structure.

Proposition 9.2.13. *For the faux Hopf ring $A = K^0 B\mathcal{V}_*/E^0 B\mathcal{V}_*$, we have an A-module $K_0 B\mathcal{V}_*/E_0^\vee B\mathcal{V}_*$. The addition algebra is given by (σ_*, η_*); the action is given by the cap product; and the diagonal coalgebra is given by $(\sigma^!, \eta^!)$. Moreover, the isomorphisms*

$$K_0 B\mathcal{V}_* = \mathrm{Hom}_{K_0}(K^0 B\mathcal{V}_*, K_0)$$

$$E_0^\vee B\mathcal{V}_* = \mathrm{Hom}_{E_0}(E^0 B\mathcal{V}_*, E_0)$$

from 1.3.13 are isomorphisms of A-modules.

Proof. It suffices to prove that the isomorphisms preserve the structure. The A-module structures then follow from those of the duals. The isomorphisms perserve the structure because, by 6.2.12, they send the algebra structure (σ_*, η_*) to the algebra structure $((\sigma^*)^\vee, (\eta^*)^\vee)$, and the coalgebra structure $(\sigma^!, \eta^!)$ to the coalgebra structure $((\sigma_!)^\vee, (\eta_!)^\vee)$. Finally, it is standard that under this isomorphism, the cap product becomes the map

$$a \bullet f : x \mapsto f(a \bullet x)$$

as in 6.2.10. This completes the proof.

□

Definition 9.2.14. For a faux Hopf ring A and an A-module M that is projective and of finite type over k, we say that M is *self-dual* if there is an isomorphism of A-modules

$$M^\vee \cong M.$$

In particular, under this isomorphism the addition algebra structures $(\Delta^\vee, \pi^\vee)$ and $(*, i_0)$ correspond, the diagonal coalgebra structures $(*^\vee, i_0^\vee)$ and (Δ, π) correspond, and the homogeneous linear actions of A (i.e. $\bullet$) correspond.

Theorem 9.2.15. *For the faux Hopf ring $A = K^0 B\mathcal{V}_*/E^0 B\mathcal{V}_*$, the A-modules $M = K^0 B\mathcal{V}_*/E^0 B\mathcal{V}_*$ and $N = K_0 B\mathcal{V}_*/E_0^\vee B\mathcal{V}_*$ are self-dual.*

Proof. If we let $R = K^0/E^0$ respectively, by 6.2.12, we have an isomorphism

$$M \cong N = M^\vee.$$

By the argument from 6.2.10, this sends the cup product action to the cap product action. By 6.2.12, this isomorphism preserves the algebra structure and coalgebra structure. Thus it is an isomorphism of A-modules. This shows that M is self-dual. As N is isomorphic to M, it follows that N is also self-dual.

□

Definition 9.2.16. For a faux Hopf semiring A, and a module M over A, we define

$$IM := Ker(\pi : M \to k).$$

We define the *indecomposables* of M to be

$$\begin{aligned} QM &:= Coker(IM \otimes IM \xrightarrow{*} IM) \\ &= IM/IM^{*2}. \end{aligned}$$

This is a graded quotient k-module.

Definition 9.2.17. For a faux Hopf semiring A, and a module M over A, we define

$$JM := Coker(i_0 : k \to M).$$

We define the *primitives* of M to be

$$\begin{aligned} PM &:= Ker(JM \xrightarrow{\Delta} JM \otimes JM) \\ &\cong \{x \in M \mid \Delta(x) = x \otimes [0] + [0] \otimes x\}. \end{aligned}$$

where $[0] = i_0(1)$ as in 9.2.4. This is a graded sub-k-module.

Proposition 9.2.18. *For a connected faux Hopf ring A and an A-module M, the indecomposables QM form a homogeneous $\bullet$-module over the homogeneous k-algebra QA.*

Proof. As $M_m \to QM_m$ is surjective, by the associativity and unit conditions of the $\bullet$-action of A on M, we will have a homogeneous QA-module structure on QM if for all $m > 0$, $\bullet$ descends to a map

$$\bullet : QA_m \otimes QM_m \to QM_m.$$

In other words, we need to show that $\bullet$ descends to a map

$$\frac{IA \otimes IM}{IA^{*2} \otimes IM + IA \otimes IM^{*2}} \to \frac{IM}{IM^{*2}}.$$

For $a, b \in IA$, $x, y \in IM$, by distributivity we have

$$(a * b) \bullet x = *((a \otimes b) \bullet \Delta(x)) \in IM^{*2}$$

$$a \bullet (x * y) = *(\Delta(a) \bullet (x \otimes y)) \in IM^{*2}$$

which shows that the map does indeed descend, giving us the QA-module structure.

□

Proposition 9.2.19. *For a connected faux Hopf ring A and an A-module M, the primitives PM form a homogeneous $\bullet$-module over the ring QA.*

Proof. Observe that if $x \in A$ and $y \in PM$, then

$$\begin{aligned}\Delta(x \bullet y) &= \Delta(x) \bullet \Delta(y) \\ &= \Delta(x) \bullet (y \otimes [0] + [0] \otimes y) \\ &= (x \bullet y) \otimes [0] + [0] \otimes (x \bullet y) \in M \otimes M.\end{aligned}$$

So $x \bullet y$ is primitive and as $[1] \bullet y = y$, $\bullet$ restricts to a module structure

$$\bullet : A \otimes PM \to PM.$$

Now in order to prove that this descends to a QA-module action on PM, as $A \to QA$ is surjective, we need only prove that this map descends to a k-linear map

$$\bullet : QA \otimes PM \to PM.$$

To see this, let $w, x \in IA$ and $y \in PM$, then by distributivity we have

$$\begin{aligned}(w * x) \bullet y &= *((w \otimes x) \bullet \Delta(y)) \\ &= *((w \otimes x) \bullet (y \otimes [0] + [0] \otimes y)) \\ &= 0 \in M\end{aligned}$$

because $w \bullet [0] = 0 = x \bullet [0]$. It follows that the above map descends.

□

Proposition 9.2.20. *For a connected faux Hopf ring A and an A-module M that is projective and of finite type over k, we have a natural isomorphism of QA-modules*

$$P(M^\vee) = Q(M)^\vee.$$

Proof. We have exact sequences

$$0 \to IM \to M \xrightarrow{\pi} k \to 0$$
$$0 \to k \xrightarrow{\pi^\vee} M^\vee \to J(M^\vee) \to 0.$$

Dualising the first gives an exact sequence

$$0 \to k \xrightarrow{\pi^\vee} M^\vee \to (IM)^\vee \to 0$$

because $\mathrm{Ext}^1_k(k, k) = 0$, and thus induces an isomorphism of $\bullet$-modules

$$J(M^\vee) \to (IM)^\vee \tag{9.1}$$

As M is projective of finite type, the natural map

$$J(M^\vee) \otimes J(M^\vee) \to (IM \otimes IM)^\vee$$

is an isomorphism, so we have exact sequences

$$IM \otimes IM \xrightarrow{*} IM \to QM \to 0$$
$$0 \to P(M^\vee) \to JM^\vee \xrightarrow{*^\vee} J(M^\vee) \otimes J(M^\vee)$$

where the second is dual to the first. In particular, the $\bullet$-isomorphism 9.1 restricts to a $\bullet$-isomorphism

$$P(M^\vee) \to (QM)^\vee$$

as required.

□

Chapter 10

(Co)algebra Structure

The main conclusion of this chapter is that $E^0 B\mathcal{V}_*$ and $K^0 B\mathcal{V}_*$ are polynomial algebras. We prove this by computing the Atiyah-Hirzebruch spectral sequences for $K_* B\mathcal{V}_*$ and $K^* B\mathcal{V}_*$.

In the first section, we use 1.6.1 to determine the structure of the ordinary mod p homology and cohomology of $\bar{\mathcal{V}}_*$, particularly with respect to the algebra and coalgebra structures from σ_* and σ^* respectively. In the second section, we recall Quillen's results on the ordinary mod p homology and cohomology of $\mathcal{V}_*$.

In the third section, we determine the structure induced by σ and δ on the spectral sequences. In the fourth section, we compute the differentials on the Atiyah-Hirzebruch spectral sequences for $\mathcal{V}_*$. The results of Quillen provide the E_2-pages, and 5.2.20 provides the dimensions of the E_∞-pages. We then use this information, and the structure induced by σ and δ, to reverse engineer the behaviour of the differentials, up to units in K_0. In particular, we deduce that the homological E_∞-page is polynomial.

In the fifth section, we firstly use our description of the E^∞-page to obtain a lower bound on the nilpotence of the top Chern classes $s_k \in K^0 B\mathcal{V}_{p^k}$. We then use the polynomial structure on the E_∞-page to show that the algebra and coalgebra structures induced by σ and $\sigma^!$ on $K_0 B\mathcal{V}_*$, $K^0 B\mathcal{V}_*$, $E_0^\vee B\mathcal{V}_*$, and $E^0 B\mathcal{V}_*$ are free commutative and cofree cocommutative respectively.

Note: In this chapter, in the presence of a field k (in particular, K_*) we will write $P[\{x_i\}]$ for the polynomial algebra generated by the set $\{x_i\}$, and $E[\{a_i\}]$ for the exterior algebra generated by the set $\{a_i\}$.

10.1 Ordinary (Co)homology of $\bar{\mathcal{V}}_*$

In this section, we use 1.6.1 to describe the ordinary mod p homology and cohomology of $\bar{\mathcal{V}}_*$.

Throughout this section, k is a field of characteristic p.

Lemma 10.1.1. *We have*

$$H^*(B\bar{\mathcal{V}}_1; k) = P[x]$$

$$H_*(B\bar{\mathcal{V}}_1; k) = H^*(B\bar{\mathcal{V}}_1; k)^* = k\{b_0, b_1, \dots\}$$

where b_i is the dual of x^i (i.e. $(b_i, x^j) = \delta_{ij}$)

Proof. This follows from the standard computations of $H^*(BS^1; k)$ and $H_*(BS^1; k)$ by 1.6.1. If we choose a complex orientation, then this implicitly yields a choice of x.

□

Remark 10.1.2. Applying the Künneth theorem over the field k, we have

$$H^*(B\bar{\mathcal{V}}_1^m; k) = P[x_1, \dots, x_m].$$

Lemma 10.1.3. *For $m \geq 0$ we have bigraded isomorphisms*

$$H_*(B\bar{\mathcal{V}}_*^m; k) = H_*(B\bar{\mathcal{V}}_*; k)^{\otimes m}$$

$$H^*(B\bar{\mathcal{V}}_*^m; k) = H^*(B\bar{\mathcal{V}}_*; k)^{\otimes m}.$$

Proof. The statement for homology follows by the Künneth isomorphism over a field. $H^*(B\bar{\mathcal{V}}_m; k)$ is of finite type by 1.6.1, so the statement for cohomology follows by dualising the statement for homology.

□

Lemma 10.1.4. *For k a field with characteristic p, and for all d, the map $\sigma : \bar{\mathcal{V}}_1^d \to \bar{\mathcal{V}}_d$ induces isomorphisms*

$$H^*(B\bar{\mathcal{V}}_d; k) \xrightarrow{\sim} H^*(B\bar{\mathcal{V}}_1^d; k)^{\Sigma_d}$$

$$H_*(B\bar{\mathcal{V}}_1^d; k)_{\Sigma_d} \xrightarrow{\sim} H_*(B\bar{\mathcal{V}}_d; k).$$

Proof. There is an obvious Σ_d-action on $\mathcal{V}(k)_1^d$ and the map

$$\mathcal{V}(k)_1^d \to \mathcal{V}(k)_d$$

is Σ_d-invariant. By naturality, we have a commutative diagram

$$\begin{array}{ccc} B\bar{\mathcal{V}}_d & \longrightarrow & B\mathcal{V}(\mathbb{C})_d \\ \uparrow & & \uparrow \\ B\bar{\mathcal{V}}_1^d & \longrightarrow & B\mathcal{V}(\mathbb{C})_1^d \end{array}.$$

By 1.6.1, the horizontal maps on (co)homology give isomorphisms. By the standard result in [26, p. 199] and its dual, the right hand vertical map induces isomorphisms

$$H^*(B\mathcal{V}(\mathbb{C})_d; k) \xrightarrow{\sim} H^*(B\mathcal{V}(\mathbb{C})_1^d; k)^{\Sigma_d}$$
$$H_*(B\mathcal{V}(\mathbb{C})_1^d; k)_{\Sigma_d} \xrightarrow{\sim} H_*(B\mathcal{V}(\mathbb{C})_d; k)$$

so the same must be true over $\bar{\mathbb{F}}$.

□

Definition 10.1.5. If k is a ring, A is a k-algebra and $V \rightarrowtail A$ is an injective map of k-modules, then we say that V *generates* A if the adjoint map $T(V) \to A$ from the free tensor algebra is surjective.

Dually, if A is a k-coalgebra and $A \twoheadrightarrow V$ is a surjective map of k-modules, then we say that V *cogenerates* A if the adjoint map $A \to T(V)$ to the cofree tensor coalgebra is injective.

Theorem 10.1.6. *For a field k of characteristic p, the maps (σ_*, η_*) make $H_*(B\bar{\mathcal{V}}_*; k)$ into the free commutative graded algebra over k, generated by $H_*(B\bar{\mathcal{V}}_1; k)$.*

Dually, the maps (σ^*, η^*) make $H^*(B\bar{\mathcal{V}}_*; k)$ into the cofree cocommutative graded coalgebra over k, cogenerated by the projection to $H^*(B\bar{\mathcal{V}}_1; k)$.

Proof. By 10.1.3, we have Künneth isomorphisms, so by 3.1.18 we have algebra and coalgebra structures respectively.

By 10.1.4, the restriction map

$$H^*(B\bar{\mathcal{V}}_m; k) \to H^*(B\bar{\mathcal{V}}_1^m; k)^{\Sigma_m} = (H^*(B\bar{\mathcal{V}}_1; k)^{\otimes m})^{\Sigma_m}$$

is an isomorphism. Dually, we have an isomorphism

$$(H_*(B\bar{\mathcal{V}}_1; k)^{\otimes m})_{\Sigma_m} = H_*(B\bar{\mathcal{V}}_1^m; k)_{\Sigma_m} \to H_*(B\bar{\mathcal{V}}_m; k).$$

It follows that algebra $H_*(B\bar{\mathcal{V}}_*; k)$ and coalgebra $H^*(B\bar{\mathcal{V}}_*; k)$ are free commutative and cofree cocommutative respectively.

□

Corollary 10.1.7. *We have an isomorphism of graded algebras*

$$H_*(B\bar{\mathcal{V}}_*;k) = P[b_0, b_1, \dots]$$

where $H_*(B\bar{\mathcal{V}}_*;k)$ has the algebra structure from (σ_*, η_*).

Proof. This follows immediately from 10.1.6 and 10.1.1.

□

Definition 10.1.8. If a group G acts on an R-module M, and $x \in M$, define $Sum_G(x)$ to be the sum of elements in the G-orbit of x, $Orb_G(x)$.

Definition 10.1.9. In $H^*(B\bar{\mathcal{V}}_1^m;k)$, for $1 \le i \le m$, define

$$c_{i,m} := Sum_{\Sigma_m}(x_1 \dots x_i).$$

Corollary 10.1.10. *If we choose an $\bar{\mathbb{F}}$-orientation, for all $m \ge 0$, we have an isomorphism of algebras*

$$H^*(B\bar{\mathcal{V}}_m;k) = P[c_{1,m}, \dots, c_{m,m}]$$

where $H^*(B\bar{\mathcal{V}}_m;k)$ has the algebra structure from the cup product.

Proof. This follows from 10.1.4 by the fundamental theorem of symmetric polynomials.

□

10.2 Ordinary (Co)homology of $\mathcal{V}_*$

In this section, we recall Quillen's results on the ordinary mod p homology and cohomology of $\mathcal{V}_*$.

Lemma 10.2.1. *We have*

$$H^*(B\mathcal{V}_1;k) = P[x] \otimes E[a]$$

$$H_*(B\mathcal{V}_1;k) = k\{b_0, b_1, \dots\} \oplus k\{e_0, e_1, \dots\}$$

where $|a| = 2$, b_i is the dual of x^i (i.e. $(b_i, x^j) = \delta_{ij}$), and e_i is the dual of $x^i a$ (i.e. $(e_i, x^j a) = \delta_{ij}$).

Proof. If we let A be the cyclic quotient of $GL_1(\mathbb{F})$ of order p^v, then by 2.4.5, the quotient map induces (co)homological equivalences in characteristic p. The cohomological statement then follows by [7, Proposition 4.5.1], and as the cohomology is free of finite type, dualising gives the homological statement.

$\square$

Remark 10.2.2. Applying the Künneth theorem over the field k, we have

$$H^*(B\mathcal{V}_1^m; k) = P[x_1, \dots, x_m] \otimes_k E[a_1, \dots, a_m].$$

Lemma 10.2.3. *For a field k of characteristic p, for $m \geq 0$ we have graded isomorphisms*

$$H_*(B\mathcal{V}_*^m; k) = H_*(B\mathcal{V}_*; k)^{\otimes m}$$
$$H^*(B\mathcal{V}_*^m; k) = H^*(B\mathcal{V}_*; k)^{\otimes m}.$$

Proof. The statement for homology follows by the Künneth isomorphism over a field. The statement for cohomology follows by dualising the statement for homology.

$\square$

Theorem 10.2.4. *For a field k of characteristic p, the maps (σ_*, η_*) make $H_*(B\mathcal{V}_*; k)$ into the free graded-commutative graded algebra over k, generated by $H_*(B\mathcal{V}_1; k)$.*

Dually, the maps (σ^*, η^*) make $H^*(B\mathcal{V}_*; k)$ into the cofree graded-cocommutative coalgebra over k, cogenerated by the projection to $H^*(B\mathcal{V}_1; k)$.

Proof. We have Künneth isomorphisms, so by 3.1.18 we have graded algebra and coalgebra structures respectively. By [29, Theorem 3], $H_*(B\mathcal{V}_*; k)$ is free, graded-commutative over k, generated by $H_*(B\mathcal{V}_1; k)$. The second statement follows from this by dualising.

$\square$

Corollary 10.2.5. *We have an isomorphism of graded algebras*

$$H_*(B\mathcal{V}_*; k) = P[b_0, b_1, \dots] \otimes E[e_0, e_1, \dots]$$

where $H_*(B\mathcal{V}_*; k)$ has the algebra structure from (σ_*, η_*).

Proof. This follows immediately from 10.2.4 and 10.2.1, but it is really just [29, Theorem 3].

$\square$

Definition 10.2.6. In $H^*(B\mathcal{V}_1^m; k)$, for $1 \leq i \leq m$, define

$$c_{i,m} := Sum_{\Sigma_m}(x_1 \dots x_i)$$
$$v_{i,m} := Sum_{\Sigma_m}(x_1 \dots x_i a_{i+1}).$$

Corollary 10.2.7. *For all $m \geq 0$, we have an isomorphism of algebras*

$$H^*(B\mathcal{V}_m; k) = P[c_{1,m}, \dots, c_{m,m}] \otimes E[v_{1,m}, \dots, v_{m,m}]$$

where $H^*(B\mathcal{V}_m; k)$ has the algebra structure from the cup product.

Proof. This is proved in [29, Section 9].

□

Remark 10.2.8. Recall by 3.1.27 that we have a map of (σ, η) graded monoids

$$T : \mathcal{V}_* \to \bar{\mathcal{V}}_*.$$

Proposition 10.2.9. *We have a graded coalgebra and homogeneous algebra map with respect to the structures (σ_*, η_*) and (δ_*, ϵ_*) respectively*

$$T^* : H^*(B\bar{\mathcal{V}}_*; k) \to H^*(B\mathcal{V}_*; k)$$
$$c_{i,m} \mapsto c_{i,m}.$$

We have a graded algebra and homogeneous coalgebra map with respect to the structures (σ^*, η^*) and (δ^*, ϵ^*) respectively

$$T_* : H_*(B\mathcal{V}; k) \to H_*(B\bar{\mathcal{V}}; k)$$
$$b_i \mapsto b_i$$
$$e_i \mapsto 0.$$

Proof. Preservation of the algebra and coalgebra structures follows by functoriality from 3.1.27.

The description of T^* follows because $x \in H^*(B\mathcal{V}_1; k)$ is defined via T^* as the image of $x \in H^*(B\bar{\mathcal{V}}_1; k)$.

The description of i_* follows by considering the dual of the first map

$$T_* : \mathrm{Hom}_k(H^*(B\mathcal{V}_1; k), k) \to \mathrm{Hom}_k(H^*(B\bar{\mathcal{V}}_1; k), k)$$
$$\delta_{x^i} \mapsto \delta_{x^i}$$
$$\delta_{x^i a} \mapsto 0.$$

But under the Kronecker pairing isomorphism, we defined $b_i := \delta_{x^i}$, $e_i := \delta_{x^i a}$, so the result follows.

□

10.3 Atiyah-Hirzebruch Spectral Sequences

In the following section, we study structures induced by σ and δ on the Atiyah-Hirzebruch spectral sequences for $\mathcal{V}_*$ and $\bar{\mathcal{V}}_*$.

We have Atiyah-Hirzebruch spectral sequences

$$E^2_{*,*}(B\mathcal{V}_*) = H_*(B\mathcal{V}_*; K_*) \Rightarrow K_*B\mathcal{V}_*$$

$$E_2^{*,*}(B\mathcal{V}_*) = H^*(B\mathcal{V}_*; K^*) \Rightarrow K^*B\mathcal{V}_*$$

$$E^2_{*,*}(B\bar{\mathcal{V}}_*) = H_*(B\bar{\mathcal{V}}_*; K_*) \Rightarrow K_*B\bar{\mathcal{V}}_*$$

$$E_2^{*,*}(B\bar{\mathcal{V}}_*) = H^*(B\bar{\mathcal{V}}_*; K^*) \Rightarrow K^*B\bar{\mathcal{V}}_*$$

such that the rth differentials have tridegree

$$|d_r| = (-r, r-1, 0)$$

$$|d_r| = (r, 1-r, 0)$$

respectively, where the first degree is (co)homological, the second comes from K_*/K^*, and the third comes from the $\mathcal{V}_*/\bar{\mathcal{V}}_*$ grading. The generators in homology have degrees

$$|b_i| = (2i, 0, 1)$$

$$|e_i| = (2i+1, 0, 1)$$

$$|u| = (0, 2, 0).$$

The generators in cohomology have degrees

$$|c_{i,m}| = (2i, 0, m)$$

$$|v_{i,m}| = (2i+1, 0, m)$$

$$|u| = (0, -2, 0)$$

where we have fixed a unit $u \in K_2 = K^{-2}$.

Proposition 10.3.1. *For a CW-complex X, we have a natural isomorphism of Atiyah-Hirzebruch spectral sequences*

$$E_r^{*,*}(X) \to \mathrm{Hom}_{K_*}(E^r_{*,-*}(X), K_*)$$

such that the map on E_2-pages coincides with the duality isomorphism

$$H^*(X; K^*) \to \mathrm{Hom}_{K_*}(H_*(X; K_{-*}), K_*).$$

In other words, the map on rth pages is an isomorphism of chain complexes induced by taking the homology of the map on $(r-1)$th pages. Moreover, when $E^2_{*,*}(X)$ is of finite type, this induces a perfect duality between the homological and cohomological spectral sequences.

Proof. The existence of such a map follows from [21, Proposition 4.2.10] (it is stated in less generality, but the proof still works). By the universal coefficient theorem for a field, the map of E_2-pages is an isomorphism. By taking homology with respect to the differentials, this inductively implies that the map is an isomorphism of E_r-pages for all $r \geq 2$.

If $E^2_{*,*}(X)$ is of finite type, then inductively, $E^r_{*,*}(X)$ is of finite type for all $r \geq 2$, so the duality must be perfect.

□

Remark 10.3.2. In terms of pairings, when the spectral sequence is of finite type, we may say that the Kronecker pairing on the E_2-pages extends to a perfect pairing of spectral sequences.

Lemma 10.3.3. *For CW-complexes X, Y, we have natural isomorphisms of spectral sequences*

$$E^r_{*,*}(X) \otimes_{K_*} E^r_{*,*}(Y) \to E^r_{*,*}(X \times Y)$$
$$E_r^{*,*}(X) \otimes_{K^*} E_r^{*,*}(Y) \to E_r^{*,*}(X \times Y)$$

where the tensor product of two chain complexes is given the graded Leibniz differential

$$d(x \otimes y) = d(x) \otimes y + (-1)^{|x|} x \otimes d(y).$$

Moreover, the maps on E_2-pages coincide with the Künneth isomorphisms

$$H_*(X; K_*) \otimes_{K_*} H_*(Y; K_*) \to H_*(X \times Y; K_*)$$
$$H^*(X; K^*) \otimes_{K^*} H^*(Y; K^*) \to H^*(X \times Y; K^*)$$

respectively.

Proof. We use the theory and terminology from [9] and [10].

If we can construct the maps, then by the Künneth isomorphism for a field, the maps of E_2-pages are isomorphisms. By taking homology this inductively implies that the maps of E_r-pages are isomorphisms for all $r \geq 2$.

The cohomological case is proved in [10, Section 3]. We outline here the proof of the homological case, which is similar.

Write X_m for the m-skeleton of X, and similarly for Y. Define $(X \times Y)_m := \bigcup_{i+j=m}(X_i \times Y_j)$. We can naturally form sequences of cofibrations

$$\varnothing \rightarrowtail X_0 \rightarrowtail X_1 \rightarrowtail \dots$$

$$\varnothing \rightarrowtail Y_0 \rightarrowtail Y_1 \rightarrowtail \dots$$

$$\varnothing \rightarrowtail (X \times Y)_0 \rightarrowtail (X \times Y)_1 \rightarrowtail \dots$$

with colimits X, Y, and $X \times Y$ respectively.

If we let $V_m := K \wedge X_{-m}$, for all $m \in \mathbb{Z}$ (noting that X_{-m} is empty for $m > 0$), then we get induced rigid homotopy cofibre sequences

$$V_m \to V_{m-1} \to K \wedge (X_{-m}/X_{-m+1}).$$

By [9, C.6], these are also rigid homotopy fibre sequences. They fit together to give a tower of rigid homotopy fibre sequences, which induces an exact couple and homotopy spectral sequence with

$$E_1^{p,q}(X) = \pi_p(X_{-q}/X_{-q+1}) = C_{-q}(X; K_{p+q})$$

converging to $K_*(X)$. In particular, this is isomorphic to the homological Atiyah-Hirzebruch spectral sequence (after adjusting gradings).

Similarly, we can define $W_m := K \wedge Y_{-m}$, $Z_m := K \wedge K \wedge (X \times Y)_{-m}$ to get spectral sequences converging to K_*Y and $(K \wedge K)_*(X \times Y)$ respectively.

We have inclusion maps

$$(X)_i \times (Y)_j \to (X \times Y)_{i+j}$$

from which we can construct a pairing on towers

$$V \wedge W \to Z$$

as in [9, Section 5]. By [9, Theorem 6.1], this induces a pairing on spectral sequences. A cellular argument as in [10, Theorem 3.4] allows us to identify the pairing on E^1-pages in terms of the chain level Eilenberg-Zilber map, and thus identify the pairing on E^2-pages as the cross product. As K_* is concentrated in even dimensions, there is no issue with signs as in [10, Section 2]

□

Proposition 10.3.4. *The spectral sequences $E^r_{*,*}(B\mathcal{V}_*)$ and $E^r_{*,*}(B\bar{\mathcal{V}}_*)$ are comultiplicative with respect to (δ_*, ϵ_*), and multiplicative with respect to (σ_*, η_*).*

Dually, the spectral sequences $E_r^{*,*}(B\mathcal{V}_*)$ and $E_r^{*,*}(B\bar{\mathcal{V}}_*)$ are multiplicative with respect to (δ^*, ϵ^*), and comultiplicative with respect to (σ^*, η^*).

Proof. As we have Künneth isomorphisms of spectral sequences by 10.3.3, this follows by functoriality from 3.1.19 and 3.1.20.

□

Remark 10.3.5. The statement for (δ^*, ϵ^*) is just the standard statement that the cohomological spectral sequence is multiplicative with respect to the cup product.

Remark 10.3.6. (δ_*, ϵ_*) and (δ^*, ϵ^*) are bigraded maps (not $\mathcal{V}_*$-graded), whereas (σ_*, η_*) and (σ^*, η^*) are trigraded maps.

Remark 10.3.7. There is a lot more structure available, but this proposition summarises all we will need for the spectral sequence calculation.

Lemma 10.3.8. *The maps*

$$T_* : E^r_{*,*}(B\mathcal{V}_*) \to E^r_{*,*}(B\bar{\mathcal{V}}_*)$$

$$T^* : E_r^{*,*}(B\bar{\mathcal{V}}_*) \to E_r^{*,*}(B\mathcal{V}_*)$$

are trigraded. The first preserves the coalgebra structure (δ_*, ϵ_*) and algebra structure (σ_*, η_*). The second preserves the algebra structures (δ^*, ϵ^*), and coalgebra structure (σ^*, η^*).

Proof. By 3.1.27, T is a map of homogeneous (δ, ϵ)-comonoids and graded (σ, η)-monoids, so the result follows by functoriality of the spectral sequence construction.

□

10.4 Spectral Sequence Calculations for $\mathcal{V}_*$

In this section, we compute the differentials on the Atiyah-Hirzebruch spectral sequences for $\mathcal{V}_*$. In particular, we deduce that $E^\infty_{*,*}(B\mathcal{V}_*)$ is polynomial under (σ_*, η_*).

From now on in this chapter, we denote the multiplication from σ_* on $E^r_{*,*}(B\mathcal{V}_*)$ by $*$, and we drop notation for the multiplication from δ^* on $E_r^{*,*}(B\mathcal{V}_*)$ (i.e.

the cup product) and just denote the cup product of two elements by their concatenation.

Definition 10.4.1. Define

$$\bar{N}_k := \sum_{i=0}^{k} N_i.$$

Definition 10.4.2. For sequences of non-negative integers α and β of lengths $|\alpha| = r$ and $|\beta| = s$ with $m := r + s$, define elements of $H_*(B\mathcal{V}_m; K_*) = H_*(B\mathcal{V}_1^m; K_*)_{\Sigma_m}$

$$b_{\alpha,\beta} := b_{\alpha_1} * \cdots * b_{\alpha_r} * e_{\beta_1} * \cdots * e_{\beta_s}$$

and elements of $H^*(B\mathcal{V}_m; K^*) = H^*(B\mathcal{V}_1^m; K^*)^{\Sigma_m}$

$$x_{\alpha,\beta} := Sum_{\Sigma_m}(x_1^{\alpha_1} \dots x_r^{\alpha_r} x_{r+1}^{\beta_1} a_{r+1} \dots x_{r+s}^{\beta_s} a_{r+s}).$$

Definition 10.4.3. For $i, k \geq 0$, define sequences

$$\alpha_{i,k} := (\bar{N}_{k-1} + i, \dots, \bar{N}_{k-1} + i)$$

$$\beta_{i,k} := ()$$

$$\alpha'_{i,k} := (\bar{N}_{k-1} + i, \dots, \bar{N}_{k-1} + i, \bar{N}_{k-2} + i, \dots, \bar{N}_{k-2} + i, \dots, \bar{N}_0 + i)$$

$$\beta'_{i,k} := (i)$$

where $|\alpha_{i,k}| = p^k$, and for each $j \leq k-1$ there are $p^j(p-1)$ copies of $\bar{N}_j + i$ in $\alpha'_{i,k}$. Then for $i, k \geq 0$, define elements of $H_*(B\mathcal{V}_*; K_*)$

$$b_{i,k} := b_{\alpha_{i,k},\beta_{i,k}}$$

$$e_{i,k} := b_{\alpha'_{i,k},\beta'_{i,k}}$$

and elements of $H^*(B\mathcal{V}_*; K^*)$

$$s_{i,k} := x_{\alpha_{i,k},\beta_{i,k}}$$

$$a_{i,k} := x_{\alpha'_{i,k},\beta'_{i,k}}.$$

Remark 10.4.4. Write $a_k := a_{0,k}$, $s_k := c_{p^k,p^k}$. Then we have

$$s_{i,k} = s_k^{\bar{N}_{k-1}+i}$$

$$a_{i,k} = s_k^i a_k$$

$$b_{i,k} = b_{\bar{N}_{k-1}+i}^{*p^k}$$

$$e_{i,k} = (\prod_{j=0}^{k-1} b_{\bar{N}_j+i}^{*p^j(p-1)}) * e_i.$$

Remark 10.4.5. In particular, we have inductive descriptions

$$b_{i,k} = b^p_{N_{k-1}+i,k-1}$$

$$e_{i,k} = b^{p^{k-1}(p-1)}_{\bar{N}_{k-1}+i,k-1} e_{i,k-1}.$$

Lemma 10.4.6. *Under the Kronecker pairing, the basis*

$$\{b_{\alpha,\beta} \mid \alpha, \beta \text{ decreasing }, |\alpha| + |\beta| = m\}$$

for $H_*(B\mathcal{V}_m)$ is dual to the basis

$$\{x_{\alpha,\beta} \mid \alpha, \beta \text{ decreasing }, |\alpha| + |\beta| = m\}$$

for $H^*(B\mathcal{V}_m)$. In particular, under this identification, $b_{i,k}$ is dual to $s_{i,k} = s_k^{\bar{N}_{k-1}+i}$ and $e_{i,k}$ is dual to $a_{i,k} = s_k^i a_k$.

Proof. Assume that V is a vector space with a basis $\{f_i\}$, and the dual, $V^\vee$ has dual basis $\{\delta_i\}$. For $m \geq 0$, we can construct a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_k((V^{\otimes m})_{\Sigma_m}, k) \arrow[r] \arrow[d] & ((V^\vee)^{\otimes m})^{\Sigma_m} \arrow[d] \\
\mathrm{Hom}_k((V^{\otimes m}), k) \arrow[r] & (V^\vee)^{\otimes m}
\end{tikzcd}.$$

It is standard that over a field, the top and bottom maps are isomorphisms. If we let δ_i be the dual of f_i, then we see that the image of $\delta_{i_1} \otimes \cdots \otimes \delta_{i_m}$ in $\mathrm{Hom}_k((V^{\otimes m}), k)$ is

$$Sum_{\Sigma_m}(\delta_{i_1} \otimes \cdots \otimes \delta_{i_m}).$$

In particular, the dual of the basis for $(V^{\otimes m})_{\Sigma_m}$

$$\{f_{i_1} \otimes \cdots \otimes f_{i_m} \mid i_1 \geq \cdots \geq i_m\}$$

is given by

$$\{Sum_{\Sigma_m}(\delta_{i_1} \otimes \cdots \otimes \delta_{i_m}) \mid i_1 \geq \cdots \geq i_m\}.$$

By the universal coefficient theorem, we have a natural isomorphism $H^*(B\mathcal{V}_1; k) = \mathrm{Hom}_k(H_*(B\mathcal{V}_1; k), k)$. We also have Künneth isomorphisms. So applying this to the basis for $H_*(B\mathcal{V}_1^m; k)$

$$\{b_{\alpha,\beta} \mid |\alpha| + |\beta| = m\}$$

we have a basis for $H_*(B\mathcal{V}_m; k)$

$$\{b_{\alpha,\beta} \mid \alpha, \beta \text{ decreasing }, |\alpha| + |\beta| = m\}$$

with dual basis

$$\{x_{\alpha,\beta} \mid \alpha, \beta \text{ decreasing }, |\alpha| + |\beta| = m\}$$

for $H^*(B\mathcal{V}_m; k)$.

□

Lemma 10.4.7. *Let k be a ring, V be a k-module, A be a commutative k-algebra generated by V, and M be an A-module. If $A \xrightarrow{d} M$ is a derivation, then d is uniquely determined by the composition*

$$V \to A \xrightarrow{d} M.$$

Dually, let C be a cocommutative k-coalgebra cogenerated by V, and M be an A-comodule. If $M \xrightarrow{d} A$ is a coderivation, then d is uniquely determined by the composition

$$M \xrightarrow{d} A \to V.$$

Proof. For the first statement, the d is determined by the composition

$$T(V) \twoheadrightarrow A \xrightarrow{d} M.$$

As d is a derivation, we can describe this composition as

$$y_1 \otimes \cdots \otimes y_m \mapsto \sum_{i=1}^{m} y_1 \ldots y_{i-1} d(y_i) y_{i+1} \ldots y_m$$

which shows that the composition is determined by the restriction of d to V. The proof of the second statement is dual to that of the first statement.

□

Definition 10.4.8. When a derivation d is determined by the composition

$$f : V \to A \xrightarrow{d} M$$

we say that d is *generated* by f. Dually, when a coderivation is determined by the composition

$$f : M \xrightarrow{d} A \to V$$

we say that d is *cogenerated* by f.

Definition 10.4.9. Define

$$r_{-1} := 2$$

$$r_k := 2p^{n(k+v)}$$

for $k \geq 0$.

Remark 10.4.10. Recall that

$$N_0 := p^{nv}$$

$$N_k := p^{n(k+v)-k} - p^{n(k+v-1)-k}$$

for $k > 0$, and that we have inductive relations

$$\bar{N}_k := \sum_{i=0}^{k} N_i$$

$$\bar{N}_k = N_k + \bar{N}_{k-1}.$$

Definition 10.4.11. We write $a \sim_0 b$ to indicate that $a = u_0 b$ for some unit $u_0 \in K_0$.

Proposition 10.4.12. *The only non-trivial differentials in the spectral sequence*

$$E^2_{*,*}(B\mathcal{V}_*) = H_*(B\mathcal{V}_*; K_*) \Rightarrow K_* B\mathcal{V}_*$$

are $\delta^k := d^{r_k - 1}$ *for* $k \geq 0$.

For $r_{k-1} \leq r < r_k$*, as a* K_**-algebra under* σ_**,* $E^r_{*,*}(B\mathcal{V}_*)$ *is the free graded-commutative graded algebra generated by*

$$K_*\{b_{i,j} \mid j < k, 0 \leq i < N_j\} \oplus K_*\{b_{i,k} \mid 0 \leq i\} \oplus K_*\{e_{i,k} \mid 0 \leq i\}$$

i.e.

$$E^r_{*,*}(B\mathcal{V}_*) = P[\{b_{i,j} \mid j < k, 0 \leq i < N_j\} \amalg \{b_{i,k} \mid 0 \leq i\}] \otimes E[\{e_{i,k} \mid 0 \leq i\}]$$

and δ^k *is generated under the Leibniz rule for the* σ_**-algebra structure by*

$$\delta^k(b_{i,j}) \sim_0 \begin{cases} u^{p^{n(k+v)}-1} e_{i-N_k,k} & \text{if } j = k, i \geq N_k \\ 0 & \text{otherwise} \end{cases}$$

$$\delta^k(e_{i,j}) = 0.$$

Proposition 10.4.13. *The only non-trivial differentials in the spectral sequence*

$$E_2^{*,*}(B\mathcal{V}_*) = H^*(B\mathcal{V}_*; K^*) \Rightarrow K^*B\mathcal{V}_*$$

are $\delta_k := d_{r_k-1}$ *for* $k \geq 0$.

For $r_{k-1} \leq r < r_k$*, as a* K^**-coalgebra under* σ^**,* $E_r^{*,*}(B\mathcal{V}_*)$ *is the cofree graded-cocommutative graded coalgebra cogenerated by*

$$K^*\{s_j^{\bar{N}_{j-1}+i} \mid 0 \leq j, 0 \leq i < N_j\} \oplus K^*\{s_k^{\bar{N}_{k-1}+i} \mid 0 \leq i\} \oplus K^*\{s_k^i a_k \mid 0 \leq i\}$$

and δ_k *is cogenerated under the coLeibniz rule for the* σ^**-coalgebra structure by*

$$\delta_k(s_k^i a_k) \sim_0 \begin{cases} u^{p^{n(k+v)}-1} s_k^{\bar{N}_k+i} & \text{if } j = k, i \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\delta_k(s_j^{\bar{N}_{j-1}+i}) = 0.$$

Remark 10.4.14. By 10.3.1, the Kronecker pairing makes the pages $E^r_{*,*}(B\mathcal{V}_*)$ and $E_r^{*,*}(B\mathcal{V}_*)$ dual to each other. Under this duality, the statements in propositions 10.4.12 and 10.4.13 are dual, so it suffices to prove only one statement of each pair. In fact, to avoid coalgebras in favour of algebras, we will use arguments in both spectral sequences to prove both propositions together.

Proof of 10.4.12 and 10.4.13. The spectral sequences are concentrated in even K_*-grading, which means that differentials can only be non-trivial on odd-numbered pages. Apart from this observation, the K_*-grading will serve essentially no purpose, so to simplify the discussion, we will ignore it. Moreover, as K_* is a field, the differentials to/from the 0th K_*-grading determine all the other differentials, so we write $\sim$ to denote the relation of K_*-unit multiples. The exponents of u appearing in the stated differentials will follow from what we prove by considering the grading of the differentials.

We will inductively prove that $E^r_{*,*}(B\mathcal{V}_*)$ and $E_r^{*,*}(B\mathcal{V}_*)$ are as stated for $r \geq 2$. The rest of the statements will follow from the proof of this. The base case is true by 10.2.4 and 10.2.5, so we assume that our hypothesis is true for $E^r_{*,*}(B\mathcal{V}_*)$ and $E_r^{*,*}(B\mathcal{V}_*)$ and aim to prove it for $E^{r+1}_{*,*}(B\mathcal{V}_*)$ and $E_{r+1}^{*,*}(B\mathcal{V}_*)$. It is straightforward to see that there exists a unique $k \geq 0$ such that $r_{k-1} \leq r < r_k$.

By our induction hypothesis, as a K_*-vector space, the dimensions of $E^r_{*,*}(B\mathcal{V}_m)$ for $m < p^k$ are equal to the dimensions of the mth graded part

$$P[\{b_{i,j} \mid j < k, 0 \leq i < N_j\}]_m.$$

If we consider the truncated Hilbert series

$$f_r(t) := \sum_{m=0}^{p^k-1} \dim_{K_*}(E^r_{*,*}(B\mathcal{V}_m))t^m$$

then by the induction hypothesis we have

$$f_r(t) = \prod_{j=0}^{k-1}(1-t^{p^j})^{-N_j} \mod t^{p^k}.$$

However, by 5.2.20, we see that

$$\dim_{K_*}(E^r_{*,*}(B\mathcal{V}_m)) = \dim_{K_*}(K_*(B\mathcal{V}_m)) = \dim_{K_*}(E^\infty_{*,*}(B\mathcal{V}_m)).$$

It follows that there can be no more non-trivial differentials in $\mathcal{V}_*$-grading less than p^k.

By our induction hypothesis, $E^r_{*,*}(B\mathcal{V}_{p^k})$ has infinite dimension, but we know that $K_*(B\mathcal{V}_{p^k})$ has finite dimension, and hence so does the E^∞-page, so there must be further non-trivial differentials.

Suppose the differential d^r on $E^r_{*,*}(B\mathcal{V}_{p^k})$ is non-trivial (and so r is odd). Let

$$V := K_*\{b_{i,j} \mid j < k, 0 \leq i < N_j\} \oplus K_*\{b_{i,k} \mid 0 \leq i\} \oplus K_*\{e_{i,k} \mid 0 \leq i\}.$$

This is a generating vector space for $E^r_{*,*}(B\mathcal{V}_*)$, so by 10.4.7, it suffices for us to determine the composition

$$f : V \rightarrowtail E^r_{*,*}(B\mathcal{V}_*) \xrightarrow{d} E^r_{*,*}(B\mathcal{V}_*).$$

As d^r is trivial for $\mathcal{V}_*$-grading less that p^k, we have

$$K_*\{b_{i,j} \mid j < k, 0 \leq i < N_j\} \subseteq Ker(d^r).$$

By 10.3.4 and 10.3.8, we have a map of differential graded algebras

$$\begin{aligned} T_* : E^r_{*,*}(B\mathcal{V}_*) &\to E^r_{*,*}(B\bar{\mathcal{V}}_*) \\ b_{i,j} &\mapsto b^{p^j}_{\bar{N}_{j-1}+i} \\ e_{i,k} &\mapsto 0 \end{aligned}$$

where the differential on the codomain is trivial. This implies that

$$K_*\{e_{i,k} \mid 0 \leq i\} = Ker(T_*) \supseteq Im(d^r).$$

Combining these two observations, we see that f factors as

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, two heads] & E^r_{*,*}(B\mathcal{V}_*) \\ K_*\{b_{i,k}\}_{i\geq 0} \arrow[r] & K_*\{e_{i,k}\}_{i\geq 0} \arrow[u, tail] \end{tikzcd} .$$

By the Leibniz rule, d^r must send monomials in $\{b_{i,j} \mid j < k, 0 \leq i < N_j\}$ to zero. It follows that the differential on $E^r_{*,*}(B\mathcal{V}_{p^k})$ factors as

$$\begin{tikzcd} E^r_{*,*}(B\mathcal{V}_{p^k}) \arrow[r, "d^r"] \arrow[d, two heads] & E^r_{*,*}(B\mathcal{V}_{p^k}) \\ K_*\{b_{i,k}\}_{i\geq 0} \arrow[r] & K_*\{e_{i,k}\}_{i\geq 0} \arrow[u, tail] \end{tikzcd}$$

where the left hand map sends the set

$$P[\{b_{i,j} \mid j < k, 0 \leq i < N_j\}]_{p^k} \oplus K_*\{e_{i,k}\}_{i\geq 0}$$

to zero.

We now dualise and work with $E_r^{*,*}(B\mathcal{V}_{p^k})$ to utilise the cup product structure to determine d_r. By dualising the previous diagram we have a commutative diagram

$$\begin{tikzcd} E_r^{*,*}(B\mathcal{V}_{p^k}) \arrow[r, "d_r"] \arrow[d, two heads] & E_r^{*,*}(B\mathcal{V}_{p^k}) \\ K^*\{s_k^i a_k\}_{i\geq 0} \arrow[r] & K^*\{s_k^{\bar{N}_{k-1}+i}\}_{i\geq 0} \arrow[u, tail] \end{tikzcd}$$

where the left hand map sends the duals of all monomials in $\{b_{i,j}\}_{j\leq k}$ to zero. Recall that by 10.4.6, s_k^i is dual to $b_i^{p^k}$, so $d_r(s_k^i) = 0$.

By the Leibniz rule for the cup product (10.3.4), it follows that

$$d_r(s_k^i a_k) = s_k^i d_r(a_k)$$

and so d_r is determined by its value on a_k. By non-triviality of d_r, this must be non-zero. As d_r is homologically graded, we must have

$$d_r(a_k) \sim s_k^{\bar{N}_{k-1}+m}$$

for some $m \geq 0$. Dually, we must have

$$d_r(b_{i+m,k}) = e_{i,k}$$

for all $i \geq 0$, and so the next page is the subquotient

$$E^{r+1}_{*,*}(B\mathcal{V}_{p^k}) = P[\{b_{i,j} \mid j < k, 0 \leq i < N_j\}]_{p^k} \oplus K_*\{b_{i,k}\}_{i=0}^{m-1}.$$

This is concentrated exclusively in even bidegree and so the spectral sequence for $B\mathcal{V}_{p^k}$ must collapse here. But by 5.2.20, we already know the dimension of the E^∞-page to be

$$\dim(P[\{b_{i,j} \mid j < k, 0 \leq i < N_j\}]_{p^k}) + N_k$$

so we must have $m = N_k$. In particular, by considering the length of the differential, we see that we must have $r = r_k - 1$ in order to have a non-trivial differential, and when $r = r_k - 1$, the differential must be as stated.

If $r = r_k - 1$, then it remains for us to identify the E^{r_k}-page. By above, applying the Leibniz rule, we see that the differential restricts to give sub-chain complexes

$$P[\{b_{i,j} \mid j < k, 0 \leq i < N_j\}] \otimes P[b_{0,k}, \ldots, b_{N_k-1,k}]$$

$$P[b_{N_k+i,k}] \otimes E[e_{i,k}]$$

for $i \geq 0$, where the differential on the first chain complex is zero. If we define $B_i := P[b_{N_k+i,k}] \otimes E[e_{i,k}]$, then the inclusions of these chain complexes into E^r induce an isomorphism of chain complexes

$$\begin{aligned} E^{r_k-1}_{*,*}(B\mathcal{V}_*) = P[\{b_{i,j} \mid j < k, 0 \leq i < N_j\}] & \\ \otimes_{K_*} P[b_{0,k}, \ldots, b_{N_k-1,k}] \otimes_{K_*} (\bigotimes_{i\geq 0} B_i). \end{aligned}$$

By the Leibniz rule, we have

$$d_r(b^s_{N_k+i,k}) \sim s b^{s-1}_{N_k+i,k} e_{i,k}$$

$$d_r(b^s_{N_k+i,k} e_{i,k}) = 0$$

so as we are working in characteristic p, it's easy to see that

$$H_*(B_i) = P[b^p_{N_k+i,k}] \otimes E[b^{p-1}_{N_k+i,k} e_{i,k}] = P[b_{i,k+1}] \otimes E[e_{i,k+1}].$$

It follows by the Künneth theorem that the homology of the tensor product is the tensor product of the homology, so

$$\begin{aligned}
E^{r_k}_{*,*}(B\mathcal{V}_*) &= H_*(P[\{b_{i,j} \mid j < k, 0 \leq i < N_j\}] \\
&\qquad \otimes_{K_*} P[b_{0,k}, \ldots, b_{N_k-1,k}]) \otimes_{K_*} (\bigotimes_{i \geq 0} H_*(B_i)) \\
&= P[\{b_{i,j} \mid j < k+1, 0 \leq i < N_j\}] \\
&\qquad \otimes_{K_*} P[b_{0,k+1}, b_{1,k+1}, \ldots] \otimes E[e_{0,k+1}, e_{1,k+1}, \ldots]
\end{aligned}$$

as required. Proposition 10.4.13 follows dually.

□

Corollary 10.4.15. *As a K_*-algebra under σ_*, the E^∞-page is then the free commutative algebra generated by*

$$K_*\{b_{i,k} \mid 0 \leq k, 0 \leq i < N_k\}$$

i.e. we have

$$E^\infty_{*,*}(B\mathcal{V}_*) = P[\{b_{i,k} \mid 0 \leq k, 0 \leq i < N_k\}].$$

Dually, as a coalgebra under σ^*, the E_∞-page is the cofree cocommutative coalgebra cogenerated by

$$K_*\{s_k^{\bar{N}_{k-1}+i} \mid 0 \leq k, 0 \leq i < N_k\}.$$

Proof. This follows because for $i < p^{k+1}$, by 10.4.12 and 10.4.13 we have that $E^{r_k}_{*,*}(B\mathcal{V}_i)$ and $E_{r_k}^{*,*}(B\mathcal{V}_i)$ are the degree i parts of the stated free algebra and cofree coalgebra respectively. By 10.4.12 and 10.4.13 again, there are no more differentials for $i < p^{k+1}$, so these are the ∞-pages as well.

□

10.5 Algebra and Coalgebra Structures

In this section, we make various deductions from the spectral sequences of the previous section. We firstly deduce a lower bound on the nilpotence of the top Chern class in $K^0 B\mathcal{V}_{p^k}$. We then show that the commutative algebra and cocommutative coalgebra structures induced by σ and $\sigma^!$ on $K_0 B\mathcal{V}_*$, $K^0 B\mathcal{V}_*$, $E_0^\vee B\mathcal{V}_*$, and $E^0 B\mathcal{V}_*$ are free and cofree respectively.

Definition 10.5.1. If we have a normalised $\bar{\mathbb{F}}$-orientation x, for $k \geq 0$, $R = E, K$, we write $s_k := c_{p^k} \in R^0 B\bar{\mathcal{V}}_{p^k}$, where c_{p^k} is the p^kth normalised Chern class. We also write s_k for the image of this element in $R^0 B\mathcal{V}_{p^k}$ under the map T^*.

Lemma 10.5.2. *In $K^0 B\mathcal{V}_*$ for all $k \geq 0$, we have $s_k^{\bar{N}_k - 1} \neq 0$.*

Proof. If we fix a periodic element u and a normalised $\bar{\mathbb{F}}$-orientation $x \in K^0 B\bar{\mathcal{V}}_1$, then we can define $y \in E_\infty^{*,*}(B\bar{\mathcal{V}}_1)$ to be the non-zero image of ux. By the Atiyah-Hirzebruch spectral sequence for $B\bar{\mathcal{V}}_1$, we find that

$$E_\infty^{*,*}(B\bar{\mathcal{V}}_1) = K^*[y].$$

It follows that in the Atiyah-Hirzebruch spectral sequence for $B\bar{\mathcal{V}}_1^{p^k}$, $x^i \otimes \cdots \otimes x^i$ is a lift of the element

$$u^{-p^k} y^i \otimes \cdots \otimes y^i \in E_\infty^{*,*}(B\bar{\mathcal{V}}_1^{p^k}) = E_\infty^{*,*}(B\bar{\mathcal{V}}_1)^{\otimes p^k}.$$

Now σ induces an injection

$$K^* B\bar{\mathcal{V}}_{p^k} \rightarrowtail K^* B\bar{\mathcal{V}}_1^{p^k}$$

where s_k^i maps to $x^i \otimes \cdots \otimes x^i$. Analogously, by identifying the E_∞-page with the E_2-page and thus with the ordinary cohomology, σ induces an injection

$$E_\infty^{*,*}(B\bar{\mathcal{V}}_{p^k}) \rightarrowtail E_\infty^{*,*}(B\bar{\mathcal{V}}_1^{p^k})$$

where s_k^i is the preimage of $y^i \otimes \cdots \otimes y^i$. It follows by naturality of the Atiyah-Hirzebruch filtration that $s_k^i \in K^* B\bar{\mathcal{V}}_{p^k}$ is a lift of the element

$$u^{-p^k} s_k^i \in E_\infty^{*,*}(B\bar{\mathcal{V}}_{p^k}).$$

The element $s_k^i \in K^* B\mathcal{V}_{p^k}$ is the image of the element $s_k^i \in K^* B\bar{\mathcal{V}}_{p^k}$ under the map

$$T^* : K^* B\bar{\mathcal{V}}_{p^k} \to K^* B\mathcal{V}_{p^k}.$$

Analogously, by considering the E_2-page and thus the ordinary cohomology, the element $s_k^i \in E_\infty^{*,*}(B\mathcal{V}_{p^k})$ is the image of the element $s_k^i \in E_\infty^{*,*}(B\bar{\mathcal{V}}_{p^k})$ under the map

$$T^* : E_\infty^{*,*}(B\bar{\mathcal{V}}_{p^k}) \to E_\infty^{*,*}(B\mathcal{V}_{p^k}).$$

It follows by naturality of the Atiyah-Hirzebruch filtration, that $s_k^i \in K^* B\mathcal{V}_{p^k}$ is a lift of $s_k^i \in E_\infty^{*,*}(B\mathcal{V}_{p^k})$.

In particular, by 10.4.15, $s_k^{\bar{N}_k - 1} \in E_\infty^{*,*}(B\mathcal{V}_{p^k})$ is non-zero, so it follows that our particular lift must also be non-zero.

□

Remark 10.5.3. This will be useful for determining the primitives later.

Theorem 10.5.4. *The graded algebras* $(K_0 B\mathcal{V}_*, \sigma_*)$ *and* $(K^0 B\mathcal{V}_*, \sigma_!)$ *are free commutative (i.e. polynomial). Dually, the graded coalgebras* $(K_0 B\mathcal{V}_*, \sigma^!)$ *and* $(K^0 B\mathcal{V}_*, \sigma^*)$ *are cofree cocommutative.*

Proof. The homological Atiyah-Hirzebruch spectral sequence is strongly convergent by [4, Theorem 12.2]. In particular, the filtration is exhaustive, Hausdorff, and complete.

As part of the Atiyah-Hirzebruch spectral sequence there is a natural isomorphism

$$E^\infty_{*,*}(B\mathcal{V}_*) = gr_F K_*(B\mathcal{V}_*)$$

where F is the Atiyah-Hirzebruch filtration. By 10.4.15, this is a polynomial algebra. If we let $\{y_i\}$ be generators for the polynomial algebra $gr_F(K_* B\mathcal{V}_*)$, then for each y_i, we can choose a lift $x_i \in K_* B\mathcal{V}_*$. We filter the polynomial algebra $P[\{x_i\}]$ by letting x_i have the same filtration as it has in $K_* B\mathcal{V}_*$ and by multiplicatively extending to monomials. By the multiplicativity of the Atiyah-Hirzebruch spectral sequence, this implies that the map

$$P[\{x_i\}] \xrightarrow{f} K_* B\mathcal{V}_*$$

is filtered. On the associated graded vector spaces, this becomes an isomorphism of algebras by construction, so by A.0.19, this implies that f must be an isomorphism and thus $K_* B\mathcal{V}_* = K_* \otimes_{K_0} K_0 B\mathcal{V}_*$ must be a polynomial algebra. By multiplying by powers of u, we may assume that all generators are elements of $K_0 B\mathcal{V}_*$. It follows that $K_0 B\mathcal{V}_*$ is a polynomial algebra.

Dually, we must have that $(K^0 B\mathcal{V}_*, \sigma^*)$ is a cofree cocommutative coalgebra. The other two statements then follow by 6.2.12.

□

Theorem 10.5.5. *The graded algebras* $(E_0^\vee B\mathcal{V}_*, \sigma_*)$ *and* $(E^0 B\mathcal{V}_*, \sigma_!)$ *are free commutative (i.e. polynomial). Dually, the graded coalgebras* $(E_0^\vee B\mathcal{V}_*, \sigma^!)$ *and* $(E^0 B\mathcal{V}_*, \sigma^*)$ *are cofree cocommutative.*

Proof. Firstly, we prove that $E_0^\vee B\mathcal{V}_*$ is a polynomial algebra. If we let $\{x_i\}$ be a set of generators for $K_0 B\mathcal{V}_*$, then we can choose a set of liftings $\{z_i\}$ in $E_0^\vee B\mathcal{V}_*$.

This determines a map

$$P[\{z_i\}] \xrightarrow{f} E_0^\vee B\mathcal{V}_*.$$

By 10.5.4, this is an isomorphism after tensoring along $E_0 \to K_0$, so by A.0.9, f is an isomorphism.

Dually, we must have that $(E^0 B\mathcal{V}_*, \sigma^*)$ is a cofree cocommutative coalgebra. The other two statements then follow by 6.2.12.

□

Chapter 11

Identifying Indecomposables

We now know by 10.5.5 that the algebra structures $(\sigma_!, \eta_!)$ on $E^0 B\mathcal{V}_*$ and $K^0 B\mathcal{V}_*$ are polynomial. To complete our understanding of this structure, it remains to determine the associated indecomposable quotient, and identify a set of polynomial generators. Recall from 9.1.23 that for A a graded faux Hopf ring, $Q_d A$ forms a ring under $\bullet$.

In this chapter, we complete the proof of theorem 1.

Theorem 1. *$E^0 B\mathcal{V}_*$ and $K^0 B\mathcal{V}_*$ are polynomial algebras under the algebra structure $(\sigma_!, \eta_!)$ generated over E^0 and K^0 respectively by the set*

$$\{s_k^i \mid 0 \leq k, 0 \leq i < N_k\}$$

where s_k is the top $\bar{\mathbb{F}}$-Chern class of $B\mathcal{V}_{p^k}$.

This is proved in 11.3.6.

We also prove theorem 2.

Theorem 2. *The indecomposables of $E^0 B\mathcal{V}_*$ and $K^0 B\mathcal{V}_*$ are concentrated in pth power degrees. For $k \geq 0$, we have*

$$QE^0 B\mathcal{V}_{p^k} = E^0[s_k]/h_k(s_k)$$

$$QK^0 B\mathcal{V}_{p^k} = K^0[s_k]/s_k^{N_k}$$

where $h_k(s_k)$ is a Weierstrass polynomial of degree N_k.

This follows from 11.3.3 and 11.3.5.

In the first section, we use the results from the previous chapter to determine some restrictions on the form of the indecomposables and primitives.

In the second section, we use the character theory of $B\mathcal{V}_*$ from chapter 5 to make an educated guess at what the indecompsables should look like. We use the Galois theory from chapter 4 to study our candidate, D^{Γ}_{k+v}.

In the third section, we use character theory, the results from chapters 9 and 10, and the results from the second section of this chapter to determine the indecomposables of $E^0B\mathcal{V}_*$ and $K^0B\mathcal{V}_*$. We then use our description of the indecomposables to give a canonical set of generators for the algebra structures.

11.1 Indecomposables and Primitives

In this section, we use the results from the previous chapter to determine some coarse behaviour of the indecomposables and primitives.

Remark 11.1.1. Recall from 9.2.15 that for the faux Hopf ring $A = K^0B\mathcal{V}_*/E^0B\mathcal{V}_*$, and the A-modules $M = K^0B\mathcal{V}_*/E^0B\mathcal{V}_*$ and $N = K_0B\mathcal{V}_*/E_0^{\vee}B\mathcal{V}_*$ respectively, K-local duality gives an isomorphism of A-modules

$$M \cong N.$$

In particular, this isomorphism induces isomorphisms of the indecomposables and primitives as QA-modules. As such, in this section we restrict ourselves to making statements about $E^0B\mathcal{V}_*$ and $K^0B\mathcal{V}_*$.

Proposition 11.1.2. *For the faux Hopf ring $A = E^0B\mathcal{V}_*/K^0B\mathcal{V}_*$ over E^0/K^0, we have isomorphisms of QA-modules*

$$P(A) \cong Q(A)^{\vee}.$$

Proof. By 9.2.20, we have an isomorphism of QA-modules

$$P(A^{\vee}) = Q(A)^{\vee}.$$

However, by 9.2.15, A is self-dual as an A-module, so

$$P(A^{\vee}) = P(A).$$

□

Proposition 11.1.3. *For $A = K^0B\mathcal{V}_*$, QA and PA have dimensions*

$$\dim_{K^0}(QA_d) = \dim_{K^0}(PA_d) = \begin{cases} N_k & \text{if } d = p^k \\ 0 & \text{otherwise} \end{cases}.$$

Proof. As $K^0B\mathcal{V}_*$ is a polynomial algebra by 10.5.4, the dimension of the indecomposables in each degree is equal to the number of polynomial generators in that degree. By 5.2.20, $K^0B\mathcal{V}_* \cong K_0B\mathcal{V}_*$ has Hilbert series

$$\sum_{i\geq 0} \dim_{K^0}(K^0B\mathcal{V}_i)t^i = \prod_{j\geq 0}(1-t^{p^j})^{-N_j}$$

so $K^0B\mathcal{V}_*$ only has generators in pth power degrees, and in particular, has N_j generators in degree p^j. This implies the dimensions of the indecomposables.

The statement for the primitives follows dually by 11.1.2.

□

Corollary 11.1.4. *For $A = E^0B\mathcal{V}_*$, QA and PA are free graded E^0-modules. Moreover, we have*

$$\mathrm{rk}_{E^0}(QA_d) = \mathrm{rk}_{E^0}(PA_d) = \begin{cases} N_k & \text{if } d = p^k \\ 0 & \text{otherwise} \end{cases}.$$

Proof. As $E^0B\mathcal{V}_*$ is an E^0-polynomial algebra by 10.5.5, we must have that the indecomposables are free over E^0, and the rank of the indecomposables in each degree is equal to the number of polynomial generators in that degree.

By 5.2.20, $E^0B\mathcal{V}_* \cong E_0^\vee B\mathcal{V}_*$ has Hilbert series

$$\sum_{i\geq 0} \mathrm{rk}_{E^0}(E^0B\mathcal{V}_i)t^i = \prod_{j\geq 0}(1-t^{p^j})^{-N_j}$$

so $E^0B\mathcal{V}_*$ only has generators in pth power degrees, and in particular, has N_j generators in degree p^j. This implies the dimensions of the indecomposables.

The statement for the primitives follows dually by 11.1.2.

□

Definition 11.1.5. For $A = E^0B\mathcal{V}_*/K^0B\mathcal{V}_*$ define the *module of decomposables of order p^k*, $I_{p^k}(A)$, to be the kernel of the map

$$A_{p^k} \twoheadrightarrow QA_{p^k}.$$

Corollary 11.1.6. *For $A = E^0B\mathcal{V}_*/K^0B\mathcal{V}_*$, $I_{p^k}(A)$ is a summand in A_{p^k}, and QA_{p^k} is a retract.*

Proof. We have an exact sequence

$$0 \to I_{p^k}(A) \to A_{p^k} \to QA_{p^k} \to 0.$$

The result follows because QA_{p^k} is free over E^0/K^0 and thus we can take a section.

□

Corollary 11.1.7. *For $A = E^0B\mathcal{V}_*, K^0B\mathcal{V}_*$, and for all $k \geq 0$, we have exact sequences*

$$A_{p^{k-1}}^{\otimes p} \xrightarrow{\sigma_!} A_{p^k} \to QA_{p^k} \to 0$$

$$0 \to PA_{p^k} \to A_{p^k} \xrightarrow{\sigma^*} A_{p^{k-1}}^{\otimes p}.$$

Proof. We know that $A = S(V_*)$ is a free commutative graded algebra, with V_* concentrated in pth power degrees. In particular, it is easy to see that $I_{p^k}(A) = S(V_{\leq p^{k-1}})_{p^k}$. We need to show that we have a surjection

$$S(V_{\leq p^{k-1}})_{p^{k-1}}^{\otimes p} \xrightarrow{*} S(V_{\leq p^{k-1}})_{p^k}.$$

It will suffice to prove the following claim

Claim: For all $k, m \geq 0$ we have a surjective map

$$S(V_{\leq p^k})_{p^k}^{\otimes m} \to S(V_{\leq p^k})_{mp^k}.$$

We prove this by induction on k. For the base case, it is clearly true because $S(V_1)_m = V_1^{\otimes m}/\Sigma_m$. For the inductive step, observe that we have a commutative diagram

$$\begin{tikzcd}
\bigoplus_{i=0}^m V_{p^k}^{\otimes i} \otimes S(V_{\leq p^{k-1}})_{p^{k-1}}^{\otimes(m-i)} \arrow[r, "id \otimes *"] \arrow[d] & \bigoplus_{i=0}^m V_{p^k}^{\otimes i} \otimes S(V_{\leq p^{k-1}})_{(m-i)p^{k-1}} \arrow[d, "*"] \\
S(V_{\leq p^k})_{p^k}^{\otimes m} \arrow[r, "*"'] & S(V_{\leq p^k})_{mp^k}
\end{tikzcd}.$$

By definition $S(V_{\leq p^k})$ is the symmetric quotient of $T(V_{\leq p^k})$. We have

$$\begin{aligned}
S(V_{\leq p^k})_{mp^k} &= [\bigoplus_{i=0}^{\infty} V_*^{\otimes i}/\Sigma_i]_{mp^k} \\
&= \bigoplus_{\sum i_a = mp^k} [V_{i_1} \otimes \cdots \otimes V_{i_j}]/\Sigma_j \\
&= \bigoplus_{i=0}^{m} (\bigoplus_{\underline{i}} [V_{p^k}^{\otimes i} \otimes V_{i_1} \otimes \cdots \otimes V_{i_j}]/\Sigma_{i+j})
\end{aligned}$$

where the sum inside the brackets on the last line is over the set

$$\{\underline{i} = (i_1, \ldots, i_j) \mid \sum i_a = (m-i)p^k, i_a \leq p^{k-1}\}.$$

Thus, the right hand map is surjective. By the induction hypothesis, the top map is surjective, so the composite is surjective, and in particular the bottom map is surjective, as required.

The second statement follows by dualising.

$\square$

Proposition 11.1.8. *For $k \geq 0$, we have an isomorphism of homogeneous K^0-algebras*

$$QK^0B\mathcal{V}_* = K^0 \otimes_{E^0} QE^0B\mathcal{V}_*.$$

Proof. Write $A = E^0B\mathcal{V}_*$. By 9.1.16, we have an isomorphism of faux Hopf rings over K^0

$$K^0 \otimes_{E^0} E^0B\mathcal{V}_* = K^0B\mathcal{V}_*.$$

We have exact sequences

$$IA^{\otimes 2} \to IA \to QA \to 0 \tag{11.1}$$

$$K^0 \otimes_{E^0} (IA^{\otimes 2}) \to K^0 \otimes_{E^0} IA \to K^0 \otimes_{E^0} QA \to 0 \tag{11.2}$$

$$(K^0 \otimes_{E^0} IA)^{\otimes 2} \to K^0 \otimes_{E*} IA \to Q(K^0 \otimes_{E^0} A) \to 0 \tag{11.3}$$

where 11.2 is 11.1 tensored along $E^0 \to K^0$. There is a map of exact sequences from 11.1 to 11.3, and thus from 11.2 to 11.3. It is easy to see that

$$K^0 \otimes_{E^0} (IA^{\otimes 2}) = (K^0 \otimes_{E^0} IA)^{\otimes 2}$$

and so this map is an isomorphism of exact sequences. In particular, we get an isomorphism

$$K^0 \otimes_{E^0} QA = Q(K^0 \otimes_{E^0} A).$$

$\square$

Proposition 11.1.9. *We have an isomorphism of homogeneous $QK^0B\mathcal{V}_{p^k}$-modules*

$$PK^0B\mathcal{V}_* = K^0 \otimes_{E^0} PE^0B\mathcal{V}_*.$$

Proof. Write $A = E^0 B\mathcal{V}_*$. Then we have $K^0 B\mathcal{V}_* = K^0 \otimes_{E^0} A$. We have

$$\begin{aligned} P(K^0 \otimes_{E^0} A) &= \mathrm{Hom}_{K^0}(Q(K^0 \otimes_{E^0} A), K^0) \\ &= \mathrm{Hom}_{K^0}(K^0 \otimes_{E^0} Q(A), K^0) \\ &= \mathrm{Hom}_{E^0}(QA, K^0) \\ &= K^0 \otimes_{E^0} \mathrm{Hom}_{E^0}(QA, E^0) \\ &= K^0 \otimes PA \end{aligned}$$

where the first line follows by 11.1.2, the second line follows by 11.1.8, the third line follows by the hom-tensor adjunction, the fourth line follows by freeness of QA over E^0, and the fifth line follows by 11.1.2.

□

11.2 Γ-Orbits in $\mathbb{H}_E$

By 11.1.3 and 11.1.4 we have $Q_d E^0 B\mathcal{V}_* = 0$ for all d that aren't powers of p. In this section, we identify a candidate for the algebra of indecomposables, $Q_{p^k} E^0 B\mathcal{V}_*$, and give an algebraic description of it.

Recall that in the character theory of $\mathcal{V}_*$, the indecomposables of $L^0 B\mathcal{V}_*$ were given by the quotient ring

$$\mathrm{Map}(\mathrm{Irr}(\Theta^*; \mathbb{F}), L) = \mathrm{Map}(\Phi_\Gamma, L).$$

Recall that in the proof of 5.2.11, the irreducible $\mathbb{F}$-linear Θ^*-representations of dimension p^k were constructed from 1-dimensional $\mathbb{F}(p^k)$-linear Θ^*-representations via the map

$$U : \mathcal{V}(p^k)_1 \to \mathcal{V}_{p^k}.$$

By 3.3.8, we have

$$\begin{aligned} U^* : E^0 B\mathcal{V}_{p^k} &\to E^0 B\mathcal{V}(p^k)_1 \\ c_i &\mapsto \sigma_i(x, [q](x), \ldots, [q^{p^{k-1}}](x)) \end{aligned}$$

and by 4.4.21, $\mathrm{spf}(U^*)$ is naturally isomorphic to the map

$$\mathbb{H}(p^{k+v}) \to \mathrm{Div}_{p^k}(\mathbb{H})^\Gamma.$$

By 5.2.18, we have $\mathrm{Irr}(\Theta^*; \mathbb{F})_d = 0$ unless $d = p^k$ for some k in which case

$$\mathrm{Irr}(\Theta^*; \mathbb{F})_{p^k} = \mathrm{Ord}_{p^{k+v}}(\Phi)_\Gamma = (\Phi(p^{k+v}) \setminus \Phi(p^{k+v-1}))_\Gamma.$$

Character theory suggests that we should look for a meaningful scheme with $\mathbb{H} = \mathbb{H}_E$ in place of Φ. By 4.6.13, the map

$$E^0 B\mathcal{V}(p^k)_1 \to E^0 B\mathcal{V}(p^k)_1/I_{tr}$$

corresponds geometrically to the closed inclusion

$$\mathrm{Ord}_{p^{k+v}}(\mathbb{H}) \rightarrowtail \mathbb{H}(p^{k+v}).$$

If we write $D_{k+v} := E^0 B\mathcal{V}(p^k)_1/I_{tr}$ then this suggests that we should consider D^{Γ}_{k+v} as a candidate for $QE^0 B\mathcal{V}_{p^k}$.

Lemma 11.2.1. *The action of Γ on $\mathbb{H}$ induces an action of Γ on the closed subscheme $\mathrm{Ord}_{p^{k+v}}(\mathbb{H})$ over $\mathrm{spf}(E^0)$. Equivalently, the action of Γ on $E^0 B\bar{\mathcal{V}}_1$ induces an action of Γ on the quotient E^0-algebra D_{k+v}.*

Proof. We use the description of D_{k+v} from 4.6.13. For $m \geq 1$, the action of Γ on $\bar{\mathcal{V}}_1$ restricts to an action on $\mathcal{V}(m)_1$. For $\gamma \in \Gamma$, we have a homotopy pullback square

$$\begin{tikzcd}
\mathcal{V}(p^{k-1})_1 \arrow[r, "i"] \arrow[d, "\gamma"'] & \mathcal{V}(p^k)_1 \arrow[d, "\gamma"] \\
\mathcal{V}(p^{k-1})_1 \arrow[r, "i"'] & \mathcal{V}(p^k)_1
\end{tikzcd}$$

because γ induces equivalences. It follows by the Mackey property 2.3.9 that $\gamma i^! = i^! \gamma$. In particular, the map

$$i_! : E^0 B\mathcal{V}(p^{k-1})_1 \to E^0 B\mathcal{V}(p^k)_1$$

is Γ-equivariant, and so we get an induced action on the quotient D_{k+v}. Moreover, it is easy to see that the map $\mathcal{V}(p^k)_1 \to *$ is Γ-invariant, and thus the composition

$$E^0 \to E^0 B\mathcal{V}(p^k)_1 \to D_{k+v}$$

is Γ-invariant as well.

□

Remark 11.2.2. If we fix a coordinate x on $\mathbb{H}$, then by 4.6.11 we have

$$D_{k+v} = E^0[\![x]\!]/\langle p\rangle([p^{k+v-1}](x))$$

and by 1.7.1, we have $F_q(x) = [q](x)$.

From a scheme theoretic perspective, for an element $a \in \mathrm{Ord}_{p^k}(\mathbb{H})$, $F_q(a) = q.a$.

Proposition 11.2.3. *The map*

$$D_{k+v}^{\Gamma} \to D_{k+v}$$

is a Galois extension of degree $|\Gamma_{p^k}| = p^k$. Moreover, D_{k+v}^{Γ} is a complete Noetherian regular local E^0-algebra of dimension n and the map

$$E^0 \to D_{k+v}^{\Gamma}$$

is free of degree N_k.

Proof. In this proof, let $D := D_{k+v}$. As D is a local ring with maximal ideal $\mathfrak{m}$, it is straightforward to check that the units in $D^{\Gamma} = D^{\Gamma_{p^k}}$ are precisely the elements not contained in the maximal ideal $\mathfrak{m}^{\Gamma_{p^k}}$. It follows that D^{Γ} is a local ring. Moreover, D^{Γ} is a limit, but completion is also a limit. Γ acts continuously, and thus the action of Γ descends to actions on $D/\mathfrak{m}^i$ for all $i \geq 0$, so as limits commute with limits, and D is complete, so is D^{Γ}.

Let $a \in \mathbb{H}(D)$ be an element of exact order p^{k+v} over $\mathrm{Ord}_{p^{k+v}}(\mathbb{H})$. This is a universal example of such an element, and we let $x := x_a$ be the corresponding generator for D over E^0. Define the Γ-orbit divisor

$$[S] := \sum_{i=0}^{p^k} [q^i.a]$$

over $\mathrm{Ord}_{p^{k+v}}(\mathbb{H})$, which has degree p^k. Then by definition, we see that $[S]$ is Γ-invariant and thus can be defined over the subring $D^{\Gamma} \subset D$. As $a \in S$, we see that $f_S(x) = 0$, and so we have a surjective map

$$g : D^{\Gamma}[x]/f_S(x) \twoheadrightarrow D.$$

Let $K := Frac(D)$. If we tensor along $D^{\Gamma} \rightarrowtail Frac(D^{\Gamma}) = K^{\Gamma}$ (by A.0.7), we get a surjective map by A.0.6

$$K^{\Gamma}[x]/f_S(x) \twoheadrightarrow K^{\Gamma} \otimes_{D^{\Gamma}} D = K.$$

This shows that

$$[K : K^{\Gamma}] \leq |S| = |\Gamma_{p^k}| \leq |\mathrm{Aut}_{K^{\Gamma}}(K)|$$

and so the extension K/K^{Γ} is Galois with Galois group Γ_{p^k}, and all of these inequalities are equalities. In particular, g is also injective and thus an isomorphism. As $f_S(x)$ is a monic polynomial, it follows that the extension D/D^{Γ}

is finite and free, so D^Γ is Noetherian by A.0.8. As D is regular local, A.0.17 implies that D^Γ is a regular local ring.

By A.0.14, D and D^Γ are integrally closed, and applying 4.2.6 shows that the extension D/D^Γ is also Galois with Galois group Γ_{p^k}. This is a finite, free extension and so must be integral. By A.0.13 this implies that $\dim(D^\Gamma) = \dim(D) = n$.

By 11.2.1, the map

$$E^0 \rightarrowtail D$$

is Γ-invariant, so it factors through a map

$$E^0 \rightarrowtail D^\Gamma$$

which is necessarily injective, and finite by A.0.4. We know that the first map has degree $|\mathrm{Ord}_{p^{k+v}}(\Phi)|$ by 4.6.10, so the final statement follows by the first statement and 4.2.5.

$\square$

Remark 11.2.4. In particular, 11.2.3 implies that there is a well-defined smooth formal scheme $\mathrm{Ord}_{p^k}(\mathbb{H})_\Gamma$ over $\mathrm{spf}(E^0)$ and that the map

$$\mathrm{Ord}_{p^{k+v}}(\mathbb{H}) \to \mathrm{Ord}_{p^{k+v}}(\mathbb{H})_\Gamma$$

is a Galois covering of degree $|\Gamma_{p^k}| = p^k$. Moreover, it implies that $\mathrm{Ord}_{p^{k+v}}(\mathbb{H})_\Gamma$ is smooth of dimension n and the map

$$\mathrm{Ord}_{p^{k+v}}(\mathbb{H})_\Gamma \to \mathrm{spf}(E^0)$$

is free of degree N_k.

Remark 11.2.5. Recall that

$$N_k = |\mathrm{Ord}_{p^k}(\Phi)_\Gamma| = |\mathrm{Ord}_{p^{k+v}}(\Phi)|/|\Gamma_{p^k}|.$$

Remark 11.2.6. By 6.1.3, this also implies that $K^0 \otimes_{E^0} D^\Gamma_{k+v}$ is Gorenstein. As it is finite dimensional over K^0, it must have dimension zero.

Definition 11.2.7. If x is an $\bar{\mathbb{F}}$-orientation, define a Γ-invariant element

$$y := \sigma_{p^k}(x, [q](x), \ldots, [q^{p^k-1}](x)) = \prod_{i=0}^{p^k-1} [q^i](x) \in D^\Gamma_{k+v}.$$

Corollary 11.2.8. *We have*

$$(E^0 B\mathcal{V}(p^k)_1/I_{tr})^\Gamma = D^\Gamma_{k+v} = E^0[y]/h_k(y)$$

where h_k is a Weierstrass polynomial of degree N_k such that

$$h_k(y) \sim \langle p\rangle([p^{k+v-1}](x)).$$

Moreover, we have

$$K^0 \otimes_{E^0} D^\Gamma_{k+v} = K^0[y]/(y^{N_k}).$$

Proof. In this proof, let $D := D_{k+v}$.

Let $[S]$ be the divisor from 11.2.3 and let R be the subring of D^Γ generated by the coefficients of the equation of $[S]$. Then $[S]$ is defined over R and we can run through the proof of 11.2.3 with R instead of D^Γ. Consequently, we deduce that $R = D^\Gamma$. In other words, the set of symmetric polynomials

$$\{\sigma_i(x, [q](x), \dots, [q^{p^k-1}](x))\}$$

generates D^Γ.

We can actually do better than this. Recall that $K^0 \otimes_{E^0} D = K^0[x]/(x^{p^{k+v}-p^{k+v-1}})$, so

$$y = \prod_{i=0}^{p^k-1} (q^i x + O(x^2)) \sim x^{p^k} \mod \mathfrak{m}_{E^0}.$$

In particular, $y^{N_k} = 0$, and the composite

$$K^0\{y^i \mid 0 \le i < N_k\} \to K^0 \otimes_{E^0} D^\Gamma \rightarrowtail K^0 \otimes_{E^0} D$$

is an injection, and hence the first map is also an injection. By dimension count, it is an isomorphism, and thus the map

$$E^0\{y^i \mid 0 \le i < N\} \to D^\Gamma$$

must be surjective. As this is a surjection of free E^0-modules of the same rank, it is an isomorphism. Consequently, we have an isomorphism

$$E^0[y]/h_k(y) \to D^\Gamma$$

for h_k of degree N_k and with

$$h_k(y) \equiv y^{N_k} \mod \mathfrak{m}_{E^0}.$$

By comparing degree in x we must have that $h_k(y) \sim \langle p \rangle([p^{k+v-1}](x))$ in D.

$\square$

Corollary 11.2.9. *We have a closed inclusion of formal schemes over* $\mathrm{spf}(E^0)$

$$\mathrm{spf}(U^*) : \mathrm{Ord}_{p^{k+v}}(\mathbb{H})_\Gamma \to \mathrm{Div}_{p^k}(\mathbb{H})^\Gamma.$$

Equivalently, we have a surjective map

$$U^* : E^0 B\mathcal{V}_{p^k} \to (E^0 B\mathcal{V}(p^k)_1 / I_{tr})^\Gamma = D^\Gamma_{k+v}.$$

Proof. By 3.3.8, we have

$$E^0 B\mathcal{V}_{p^k} \to D_{k+v}$$

$$c_i \mapsto \sigma_i(x_1, [q](x_1), \ldots, [q^{d-1}](x_1), x_2, [q](x_2), \ldots, [q^{d-1}](x_m))$$

which clearly lands in the Γ-invariants. In particular, we have

$$s_k \mapsto y$$

and by 11.2.8, y generates D^Γ_{k+v}, so the map to D^Γ_{k+v} is surjective.

$\square$

11.3 Indecomposables and Generators

In this section, we construct a map from the degree p^k indecomposables to D^Γ_{k+v}, and show that it is an isomorphism. We then use this result to construct generators for the algebra structures on $E^0 B\mathcal{V}_*$ and $K^0 B\mathcal{V}_*$.

Proposition 11.3.1. *We have a natural isomorphism*

$$L^0 \otimes_{E^0} D_{k+v} = \mathrm{Map}(\mathrm{Ord}_{p^{k+v}}(\Phi), L^0).$$

Proof. By 4.6.13, we have an exact sequence

$$0 \to E^0 B\mathcal{V}(p^{k-1})_1 \xrightarrow{f_!} E^0 B\mathcal{V}(p^k)_1 \to D_{k+v} \to 0$$

where $f_!$ is the transfer along the inclusion $f : \mathcal{V}(p^{k-1})_1 \rightarrowtail \mathcal{V}(p^k)_1$. By the flatness of $E^0 \to L^0$ (5.1.4), we also get an exact sequence

$$0 \to L^0 B\mathcal{V}(p^{k-1})_1 \xrightarrow{f_!} L^0 B\mathcal{V}(p^k)_1 \to L^0 \otimes_{E^0} D_{k+v} \to 0.$$

By character theory (5.1.4), this is isomorphic to the exact sequence

$$0 \to \mathrm{Map}(\Phi(p^{k+v-1}), L^0) \xrightarrow{f_!} \mathrm{Map}(\Phi(p^{k+v}), L^0) \to L^0 \otimes_{E^0} D_{k+v} \to 0.$$

By freeness of the terms over L^0, this is dual to the exact sequence

$$0 \to Ker(f^!) \to L_0\{\Phi(p^{k+v})\} \xrightarrow{f^!} L_0\{\Phi(p^{k+v-1})\} \to 0.$$

If we consider the map

$$f_* : \mathcal{H}(\Theta^*, \mathcal{V}(p^{k-1})_1) \to \mathcal{H}(\Theta^*, \mathcal{V}(p^k)_1)$$

then it is π_0-injective. If ϕ_1, ϕ_2 are objects of $\mathcal{H}(\Theta^*, \mathcal{V}(p^k)_1)$, then by 6.5.4

$$\mathrm{Isom}_{\Theta^*}(\phi_1, \phi_2) \cong GL_1(\mathbb{F}(p^k))$$

so we deduce that f_* is faithful and as $|GL_1(\mathbb{F}(p^k))| = q^{p^k} - 1$, the transfer behaves as

$$f^! : \phi \mapsto \begin{cases} \frac{(q^{p^k}-1)}{(q^{p^{k-1}}-1)}\phi & \text{if } p^{k+v-1}\phi = 0 \\ 0 & \text{otherwise} \end{cases}$$

by 6.4.3. It is easy to see that

$$Ker(f^!) = L_0\{\mathrm{Ord}_{p^{k+v}}(\Phi)\}$$

so we have an isomorphism

$$D_{k+v} \cong \mathrm{Map}(\mathrm{Ord}_{p^{k+v}}(\Phi), L^0)$$

□

Lemma 11.3.2. *For $k \geq 1$, the compositions*

$$E^0 B\mathcal{V}^p_{p^{k-1}} \xrightarrow{\sigma_!} E^0 B\mathcal{V}_{p^k} \xrightarrow{U^*} D^\Gamma_{k+v}$$

$$K^0 B\mathcal{V}^p_{p^{k-1}} \xrightarrow{\sigma_!} K^0 B\mathcal{V}_{p^k} \xrightarrow{U^*} K^0 \otimes_{E^0} D^\Gamma_{k+v}$$

are zero. In particular, we have induced surjections

$$QE^0 B\mathcal{V}_{p^k} \to D^\Gamma_{k+v}$$

$$QK^0 B\mathcal{V}_{p^k} \to K^0 \otimes_{E^0} D^\Gamma_{k+v}.$$

Proof. The statements for K-theory follow by tensoring the statements for E-theory along the map $E^0 \to K^0$, so we focus on the E-theory case.

To prove that, it suffices to show that the composition

$$E^0 B\mathcal{V}^p_{p^{k-1}} \xrightarrow{\sigma_!} E^0 B\mathcal{V}_{p^k} \xrightarrow{U^*} D_{k+v}$$

is zero. As all of the terms are free E^0-modules, it suffices to check that this is zero in character theory. By 5.1.4, 6.4.3, after tensoring along $E^0 \to L^0$, the composition is isomorphic to the dual of

$$L_0\{\mathrm{Ord}_{p^{k+v}}(\Phi)\} \xrightarrow{U_*} L_0\{\mathrm{Div}_{p^k}(\Phi)^\Gamma\} \xrightarrow{\sigma^!} L_0\{(\mathrm{Div}_{p^{k-1}}(\Phi)^\Gamma)^p\}.$$

Given an element $[a] \in \mathrm{Ord}_{p^{k+v}}(\Phi)$, U_* sends $[a]$ to

$$\sum_{i=0}^{p^k}[q^i a] = \sum_{j=0}^{p^k}[(1+jp^v).a]$$

and $\sigma^!$ sends divisors $D \in \mathrm{Div}_{p^k}(\Phi)^\Gamma$ to

$$\sum_{D_1+\cdots+D_p=D} \frac{|\mathrm{Aut}_{\Theta^*}(D)|}{\prod_i |\mathrm{Aut}_{\Theta^*}(D_i)|}(D_1,\ldots,D_p)$$

where the sum is over p-tuples of degree p^{k-1} divisors, and $\mathrm{Aut}_{\Theta^*}(D)$ refers to the automorphism group of the corresponding $\mathbb{F}$-linear Θ^* representation. By 5.2.18, U_* factors through the degree p^k-irreducibles

$$L_0\{\mathrm{Irr}(\Theta^*;\mathbb{F})_{p^k}\} = L_0\{\mathrm{Ord}_{p^{k+v}}(\Phi)_\Gamma\}$$

of $\mathrm{Rep}(\Theta^*;\mathbb{F})_* = \mathrm{Div}_*(\Phi)^\Gamma$, so

$$\sum_{i=0}^{p^k}[q^i a]$$

cannot be split up as a sum of degree p^{k-1} divisors, and thus the composition is zero.

The induced maps follow by factoring the maps U^* through the cokernels of the maps $\sigma_!$. By 11.2.9, these must be surjective.

∎

Theorem 11.3.3. *For $k \geq 1$, the maps*

$$QE^0 B\mathcal{V}_{p^k} \to D^{\Gamma}_{k+v}$$

$$QK^0 B\mathcal{V}_{p^k} \to K^0 \otimes_{E^0} D^{\Gamma}_{k+v}$$

are isomorphisms. For $k = 0$, we have isomorphisms

$$E^0 B\mathcal{V}_1 \to QE^0 B\mathcal{V}_1$$

$$K^0 B\mathcal{V}_1 \to QK^0 B\mathcal{V}_1.$$

Geometrically, for $k \geq 1$, the complete Noetherian local rings $QE^0 B\mathcal{V}_{p^k}$ and $QK^0 B\mathcal{V}_{p^k}$ classify points of order p^{k+v} on $\mathbb{H}_E$ and $\mathbb{H}_K$ respectively. For $k = 0$, they classify points of order $\leq p^v$ on $\mathbb{H}_E$ and $\mathbb{H}_K$ respectively.

Proof. The case for $k = 0$ follows because by grading considerations, everything in degree 1 must be indecomposable.

For $k \geq 1$, the stated maps are surjective by 11.3.2. However, by 11.1.3, 11.1.4 and 11.2.8, they are maps of free E^0/K^0-modules of the same rank/dimension (i.e. N_k). By standard theory they must be isomorphisms.

□

Definition 11.3.4. Define $h_0(x)$ to be the unique Weierstrass polynomial of degree N_0 that is a unit multiple of the power series $[p^v](x)$.

Theorem 11.3.5. *For all $k \geq 0$*

$$QE^0 B\mathcal{V}_{p^k} = E^0[s_k]/h_k(s_k)$$

$$QK^0 B\mathcal{V}_{p^k} = K^0[s_k]/s_k^{N_k}.$$

Proof. For $k = 0$, by 11.3.3, $QE^0 B\mathcal{V}_1 = E^0 B\mathcal{V}_1$ and $QK^0 B\mathcal{V}_1 = K^0 B\mathcal{V}_1$. By 3.1.4 and finite field theory, $\mathcal{V}_1 \simeq GL_1(\mathbb{F}) = \mathbb{F}^\times \cong C_{q-1}$, and $v_p(q-1) = p^v$ by assumption, so we have the result by 1.4.2.

For $k \geq 1$, recall from 3.3.8 that under the maps

$$E^0 B\mathcal{V}_{p^k} \to D^{\Gamma}_{k+v}$$

$$K^0 B\mathcal{V}_{p^k} \to K^0 \otimes_{E^0} D^{\Gamma}_{k+v}$$

s_k maps to y. Then by 11.3.3, the coimages of these maps correspond to the indecomposables, and by 11.2.8, these are of the form stated.

□

Theorem 11.3.6. *As an algebra under $(\sigma_!, \eta_!)$, we have isomorphisms*

$$E^0 B\mathcal{V}_* \cong Sym_{E^0}(\bigoplus_{i\geq 0} D^\Gamma_{k+v})$$

$$K^0 B\mathcal{V}_* \cong Sym_{K^0}(\bigoplus_{i\geq 0} K^0 \otimes_{E^0} D^\Gamma_{k+v}).$$

More specifically, we have

$$E^0 B\mathcal{V}_* = E^0[\{s^i_k \mid 0 \leq k, 0 \leq i < N_k\}]$$

$$K^0 B\mathcal{V}_* = K^0[\{s^i_k \mid 0 \leq k, 0 \leq i < N_k\}].$$

Proof. The map

$$E^0 B\mathcal{V}_* \to Q(E^0 B\mathcal{V}_*) = \bigoplus_{k\geq 0} D^\Gamma_{k+v}$$

sends s^i_k to y^i, so sending y^i to s^i_k defines a section of E^0-modules with image

$$E^0\{s^i_k \mid 0 \leq k, 0 \leq i < N_k\}.$$

As we already know that $E^0 B\mathcal{V}_*$ is polynomial by 10.5.5, it follows that this submodule generates it as a symmetric algebra. The same argument works for $K^0 B\mathcal{V}_*$ using 10.5.4.

□

Chapter 12

Identifying Primitives

Recall from 9.2.19 that the primitives of a graded faux Hopf ring form a homogeneous module over the indecomposables. In this chapter, we study the primitives of the graded faux Hopf rings $E^0B\mathcal{V}_*$ and $K^0B\mathcal{V}_*$. In particular, we compute generators for the primitives in K-theory.

By the end of this chapter, we will have completed the proof of theorem 3.

Theorem 3. *The primitives of $E^0B\mathcal{V}_*$ and $K^0B\mathcal{V}_*$ form free homogeneous modules of rank 1 over their respective indecomposables, such that*

$$PK^0B\mathcal{V}_* = K^0 \otimes_{E^0} PE^0B\mathcal{V}_*.$$

Moreover, for $k \geq 0$, $PK^0B\mathcal{V}_{p^k}$ is generated over $QK^0B\mathcal{V}_{p^k}$ by $s_k^{\bar{N}_{k-1}}$.

This follows from 12.1.5, 11.1.9, and 12.3.1.

In the first section, we show that the rings of indecomposables in each degree are Frobenius algebras. We then use this result to show that the primitives form a free module of rank 1 over the indecomposables.

In the second section, we use the Frobenius algebra structues to determine a criterion for a primitive element to be a generator of $PK^0B\mathcal{V}_{p^k}$. We also note that this gives a description of the socle of $K^0B\mathcal{V}_{p^k}$ in terms of a generator for the primitives.

In the last section, we implement the criterion in conjunction with the lower bound on the nilpotence of s_k deduced from the Atiyah-Hirzebruch spectral sequence. We use this inductively to determine generators for $PK^0B\mathcal{V}_{p^k}$ and prove theorem 3. Along the way, we also observe that this work enables us to determine a generator for the socle of $K^0B\mathcal{V}_{p^k}$.

At the end of the last section, we conclude with some remarks explaining what we understand of the primitives in E-theory.

12.1 Application of Self-Duality

In this section, we show that the rings of indecomposables in each degree are Frobenius algebras. We then use the self-duality of the faux-Hopf rings $K^0B\mathcal{V}_*$ and $E^0B\mathcal{V}_*$ as modules over themselves to show that in both cases the primitives form a free module over the indecomposables of rank 1.

Proposition 12.1.1. *For $A = K^0B\mathcal{V}_*, E^0B\mathcal{V}_*$, we have isomorphisms of QA-modules*

$$P(A) \cong Q(A)^\vee$$

$$P(A)^\vee \cong Q(A).$$

Proof. The first statement is 11.1.2. By 11.1.4, $Q(A)$ and $P(A)$ are free over K^0/E^0 of finite type, so both are strongly dual. Thus dualising the first statement gives the second statement.

□

Remark 12.1.2. So in order to understand the primitives, by 12.1.1 we must understand the dual of the quotient map $IA \twoheadrightarrow Q(A)$, i.e. the inclusion $Q(A)^\vee \rightarrowtail IA$.

Lemma 12.1.3. *For all $k \geq 0$, $QE^0B\mathcal{V}_{p^k}$ is a Frobenius algebra over E^0, and $QK^0B\mathcal{V}_{p^k}$ is a Frobenius algebra over K^0.*

Proof. For $k = 0$, $QE^0B\mathcal{V}_1 = E^0B\mathcal{V}_1$ and $QK^0B\mathcal{V}_1 = K^0B\mathcal{V}_1$, so this statement is true by 6.2.7.

For $k \geq 1$, $QE^0B\mathcal{V}_{p^k} = D^\Gamma_{k+v}$ and $QK^0B\mathcal{V}_{p^k} = K^0 \otimes_{E^0} D^\Gamma_{k+v}$.

By 11.2.8, D^Γ_{k+v} is a regular local ring of dimension n, and thus an E^0-algebra that is Gorenstein. But by 6.1.11, this implies that D^Γ_{k+v} is a Frobenius algebra over E^0.

By the freeness of D^Γ_{k+v}, we can tensor the Frobenius E^0-algebra structure of D^Γ_{k+v} along $E^0 \to K^0$, to give a Frobenius K^0-algebra structure on $K^0 \otimes_{E^0} D^\Gamma_{k+v}$.

□

Remark 12.1.4. If R is a local ring and $S = R[y]/f(y)$ where f is a Weierstrass polynomial of degree m, then S has a basis $\{1, y, \ldots, y^{m-1}\}$ and the S-module map

$$S \to \mathrm{Hom}_R(S, R)$$

defined by sending 1 to

$$\delta_{m-1} : \sum_{i=0}^{m-1} a_i y^i \mapsto a_{m-1}$$

is an isomorphism.

Recall that $N_k = p^{n(k+v)-k} - p^{n(k+v-1)-k}$.

Proposition 12.1.5. *For all $k \geq 0$, and for $A = E^0 B\mathcal{V}_* / K^0 B\mathcal{V}_*$, PA_{p^k} is a free, rank 1 QA_{p^k}-module.*

In particular, as an ideal of A_{p^k}, PA_{p^k} is principal. Moreover, if t_{p^k} is a generator of PA_{p^k}, then the set $\{t_{p^k} s_k^i \mid 0 \leq i < N_k\}$ is a basis for PA_{p^k}.

Proof. By 12.1.1 and 12.1.3, as $QE^0 B\mathcal{V}_{p^k}$-modules we have

$$PE^0 B\mathcal{V}_{p^k} = (QE^0 B\mathcal{V}_{p^k})^\vee = QE^0 B\mathcal{V}_{p^k}$$

and as $QK^0 B\mathcal{V}_{p^k}$-modules we have

$$PK^0 B\mathcal{V}_{p^k} = (QK^0 B\mathcal{V}_{p^k})^\vee = QK^0 B\mathcal{V}_{p^k}.$$

The image t of $1 \in QA_{p^k}$ under these isomorphisms clearly generates PA_{p^k} as a QA_{p^k}-module, and thus as an A_{p^k}-module, so PA_{p^k} is a principal ideal of A_{p^k}. Recalling the structure of QA_{p^k} in 11.3.5, we see that it has a basis

$$\{s_k^i \mid 0 \leq i < N_k\}$$

and so PA_{p^k} has a basis

$$\{t s_k^i \mid 0 \leq i < N_k\}.$$

If t' is a different generator, then $t' = ut$ for some unit $u \in QA_{p^k}$, and it's easy to see that

$$\{u t s_k^i \mid 0 \leq i < N_k\}$$

is also a basis for PA_{p^k}, which gives the last statement.

□

Lemma 12.1.6. *An element $t \in PE^0B\mathcal{V}_{p^k}$ is a $QE^0B\mathcal{V}_{p^k}$-module generator iff its image $\bar{t} \in PK^0B\mathcal{V}_{p^k}$ is a $QK^0B\mathcal{V}_{p^k}$-module generator.*

Proof. By 12.1.5, t is a generator iff the map

$$f_t : QE^0B\mathcal{V}_{p^k}\{t\} \to PE^0B\mathcal{V}_{p^k}$$

is an isomorphism. By 12.1.5, $\bar{t}$ is a generator iff the map

$$f_{\bar{t}} : QK^0B\mathcal{V}_{p^k}\{\bar{t}\} \to PK^0B\mathcal{V}_{p^k}$$

is an isomorphism. By 11.1.8 and 11.1.9, $f_{\bar{t}} = K^0 \otimes_{E^0} f_t$, and as $QE^0B\mathcal{V}_{p^k}$ and $PE^0B\mathcal{V}_{p^k}$ are free over E^0, by A.0.9, f_t is an isomorphism iff $f_{\bar{t}}$ is an isomorphism.

□

Proposition 12.1.7. *For all $k \geq 0$, and for $A = E^0B\mathcal{V}_{p^k}/K^0B\mathcal{V}_{p^k}$, $I_{p^k}(A)$ and PA_{p^k} are summands in A_{p^k}. Moreover, $ann(I_{p^k}(A)) = PA_{p^k}$, and $ann(PA_{p^k}) = I_{p^k}(A)$.*

Proof. We have a short exact sequence

$$0 \to I_{p^k}(E^0B\mathcal{V}_{p^k}) \to E^0B\mathcal{V}_{p^k} \to QE^0B\mathcal{V}_{p^k} \to 0.$$

This splits because $QE^0B\mathcal{V}_{p^k}$ is a free E^0-module by 11.1.4, and so $I_{p^k}(E^0B\mathcal{V}_{p^k})$ is a summand.

By 12.1.1, the split short exact sequence

$$0 \to I_{p^k}(A) \to A_{p^k} \to QA_{p^k} \to 0$$

is dual to the exact sequence

$$0 \to PA_{p^k} \to A_{p^k} \to A_{p^k}/P_{p^k} \to 0$$

which must also be split, making PA_{p^k} a summand. But by 6.1.13, the first exact sequence is also dual to the exact sequence

$$0 \to ann(I_{p^k}(A)) \to A_{p^k} \to A_{p^k}/ann(I_{p^k}(A)) \to 0$$

so $PA_{p^k} = ann(I_{p^k}(A))$. Now 6.1.15 tells us that $ann(ann(I_{p^k}(A))) = I_{p^k}(A)$, which gives the final statement.

□

12.2 Generator Criterion

In order to completely understand the primitives, the only remaining problem is to find a generator t_{p^k} for PA_{p^k} as a QA_{p^k}-module for each $k \geq 0$.

In this section, we determine a simple criterion for a primitive element to be a generator of $PK^0B\mathcal{V}_{p^k}$.

Proposition 12.2.1. *An element t in $PK^0B\mathcal{V}_{p^k}$ is a $QK^0B\mathcal{V}_{p^k}$-module generator iff $ts_k^{N_k-1} \neq 0 \in K^0B\mathcal{V}_{p^k}$.*

Proof. Write A for $K^0B\mathcal{V}_*$. We know that QA_{p^k} is a local ring with maximal ideal $\mathfrak{m}$. By 12.1.5, $PA_{p^k} \cong QA_{p^k}$ as a QA_{p^k}-module, and thus also as an A_{p^k}-module, so it suffices to prove the criterion for QA_{p^k}. It is straightforward that $t \in QA_{p^k}$ is a generator iff the image of t in

$$QA_{p^k}/\mathfrak{m}$$

is non-zero where $\mathfrak{m}$ is the maximal ideal of QA_{p^k}. By 12.1.3, QA_{p^k} is a Frobenius algebra, so by 6.1.12, the self-duality isomorphism induces an isomorphism

$$QA_{p^k}/\mathfrak{m} \to \mathrm{Hom}(soc(QA_{p^k}), K^0)$$
$$a \mapsto \theta(a.-)$$

and we see that $a \neq 0$ iff $\theta(a.-)$ is non-zero on $soc(QA_{p^k})$. By 11.3.5, it is easy to see that $soc(QA_{p^k}) = K^0\{s_k^{N_k-1}\}$, so $\theta(a.-) \neq 0$ iff $\theta(as_k^{N_k-1}) \neq 0$. As $soc(QA_{p^k})$ is one-dimensional, for $a \in soc(QA_{p^k})$, we have $\theta(a) \neq 0$ iff $a \neq 0$. Combining this, we see that $t \in QA_{p^k}/\mathfrak{m}$ is non-zero iff $ts_k^{N_k-1}$ is non-zero.

□

Remark 12.2.2. We could also prove this criterion by writing an element $t \in QA_{p^k} = K^0[s_k]/s_k^{N_k}$ as

$$t = \sum_{i=0}^{N_k-1} a_i s_k^i$$

and observing that t is a generator iff $t \sim 1$ iff $a_0 \neq 0$ iff $ts_k^{N_k-1} \neq 0$.

Corollary 12.2.3. *If t is a generator for $PK^0B\mathcal{V}_{p^k}$, then $ts_k^{N_k-1}$ generates the socle in $K^0B\mathcal{V}_{p^k}$.*

Proof. By 6.1.17, $soc(K^0B\mathcal{V}_{p^k})$ is 1-dimensional over K^0, so it suffices to show that $ts_k^{N_k-1} \in soc(K^0B\mathcal{V}_{p^k}) \setminus \{0\}$. By 12.2.1, we know that $ts_k^{N_k-1} \neq 0$. The action of $K^0B\mathcal{V}_{p^k}$ on $PK^0B\mathcal{V}_{p^k}$ factors through an action of $QK^0B\mathcal{V}_{p^k}$, so it suffices to prove that $ts_k^{N_k-1}$ annihilates the maximal ideal of $QK^0B\mathcal{V}_{p^k}$, but by 11.3.5, the maximal ideal is generated by s_k, which is annihilated because $s_k^{N_k} = 0$.

□

12.3 Identification of the Primitives

In this section, we describe a generator $t_{p^k} \in PK^0B\mathcal{V}_{p^k}$ in terms of Chern classes using the criterion from the previous section, and the nilpotence lower bound of s_k.

Recall that we have

$$N_0 = p^{nv}$$

$$N_k = p^{n(k+v)-k} - p^{n(k+v-1)-k}$$

for $k > 0$, and that we have inductive relations

$$\bar{N}_k := \sum_{i=0}^{k} N_i$$

$$\bar{N}_k = N_k + \bar{N}_{k-1}.$$

We also let

$$\bar{N}_{-1} := 0.$$

Theorem 12.3.1. *For all $k \geq 0$, the element $s_k^{\bar{N}_{k-1}}$ generates $PK^0B\mathcal{V}_{p^k}$ as a module over $QK^0B\mathcal{V}_{p^k}$.*

Proof. Write A for $K^0B\mathcal{V}_*$. We prove this by induction on k. It is clearly true for $k = 0$ by grading considerations. If we assume that the statement is true for $k-1$, then

$$PA_{p^{k-1}} = QA_{p^{k-1}}.\{s_{k-1}^{\bar{N}_{k-2}}\}.$$

By 11.3.5

$$QA_{p^k} = K^0[s_k]/s_k^{N_k}.$$

In particular,

$$s_{k-1}^{\bar{N}_{k-1}} = s_{k-1}^{N_{k-1}}.s_{k-1}^{\bar{N}_{k-2}} = 0.$$

But we also have

$$\sigma_p^*(s_k^i) = (s_{k-1}^i)^{\otimes p} \in A_{p^{k-1}}^{\otimes p}$$

and so by 11.1.7 and the induction hypothesis, $s_k^{\bar{N}_{k-1}}$ is primitive in $K^0 B\mathcal{V}_{p^k}$. By 10.5.2,

$$s_k^{N_k - 1}.s_k^{\bar{N}_{k-1}} = s_k^{\bar{N}_k - 1} \neq 0$$

so by 12.2.1, $s_k^{\bar{N}_{k-1}}$ must be a generator of PA_{p^k}.

□

Corollary 12.3.2. *The element* $s_k^{\bar{N}_k - 1}$ *generates the socle of* $K^0 B\mathcal{V}_{p^k}$.

Proof. This follows immediately from 12.2.3.

□

Remark 12.3.3. The case for the E-theory primitives in degree p^k appears to be more complicated (ignoring the trivial case $k = 0$). We do know by 11.1.9 and 12.3.1 that an element $t_k \in PE^0 B\mathcal{V}_{p^k}$ is a generator iff

$$t_k \sim s_k^{\bar{N}_{k-1}} \mod \mathfrak{m}_{E^0}.$$

However, there is no obvious primitive candidate in terms of Chern classes with the right image in $K^0 B\mathcal{V}_{p^k}$. One can show that $\psi^{p^{k+v-1}}(s_k)$ is primitive. When $k = 1$, we find that $\bar{N}_0 = p^{nv}$ and thus $\psi^{p^v}(s_1)$ is a generator of $PE^0 B\mathcal{V}_p$. This was proved in [24, Corollary 6.84] using different techniques.

But in general, we have a telescopic sum

$$p^{n(k+v)} = \sum_{i=0}^{k} p^i N_i$$

and in particular, when $k > 0$, we see that $\bar{N}_k < p^{n(k+v)}$, and thus $\psi^{p^{k+v-1}}(s_k)$ cannot be a generator.

The conclusion seems to be that in order to identify a generator for $PE^0 B\mathcal{V}_{p^k}$, one probably needs to form a better understanding of the rings D_{k+v}^{Γ}, and the Weierstrass polynomials $h_k(y)$, which itself probably requires a better understanding of how the action of Γ interacts with $\langle p \rangle([p^{k+v-1}](x))$.

Appendix A

Commutative Algebra

In this appendix, we recall a variety of statements from commutative algebra that will be important in this thesis for a variety of reasons.

Lemma A.0.1. *For p an odd prime, $q \in \mathbb{Z}$ not divisible by p, if a is the order of q in $(\mathbb{Z}/p)^\times$, then*

$$v_p(q^d - 1) = \begin{cases} v_p(q^a - 1) + v_p(d) & \text{if } a|d \\ 0 & \text{otherwise.} \end{cases}$$

Proof. [24, Proposition 2.3.3].

□

Definition A.0.2. Let $(R, \mathfrak{m})$ be a complete Noetherian local ring. Then if $f(x) = \sum_{i=0}^{\infty} a_i x^i \in R[\![x]\!]$, we define the *Weierstrass degree* of f to be the smallest m such that $a_i \in \mathfrak{m}$ for $i < m$.

We say that a polynomial $f(x) = \sum_{i=0}^{m} a_i x^i$ is a *Weierstrass polynomial of degree* m if $a_m = 1$ and $a_i \in \mathfrak{m}$ for $i < m$.

Lemma A.0.3. *Let $(R, \mathfrak{m})$ be a complete Noetherian local ring and $f(x) \in R[\![x]\!]$ be a power series with Weierstass degree $m < \infty$. Then there exists a unique invertible power series $u(x) \in R[\![x]\!]$ and a unique Weierstrass polynomial $g(x)$ of degree m such that*

$$f(x) = u(x)g(x).$$

Proof. [16, Theorem A.3.2].

□

Lemma A.0.4. *If R is a Noetherian ring and M is a finitely generated R-module, then every submodule of M is finitely generated over R.*

Proof. [12, Proposition 1.4].

□

Lemma A.0.5. *If $k \to S$ is a map of rings such that k is a field, and S is a finite dimensional integral domain over k, then S is also a field.*

Proof. This is standard. We apply the rank-nullity theorem applied to the k-linear maps

$$(x \bullet -) : S \to S$$

for each x. As these maps are injective, they are also surjective. Then x^{-1} is the preimage of 1.

□

Lemma A.0.6. *If $R \rightarrowtail S$ is an extension of integral domains, and S is a finitely generated R-module, then the natural map*

$$S \otimes_R Frac(R) \rightarrowtail Frac(S)$$

is an isomorphism.

Proof. This is standard. By A.0.5, $S \otimes_R Frac(R) \subseteq Frac(S)$ is a field extension of $Frac(R)$, but it is a field containing S, so also must contain $Frac(S)$.

□

Lemma A.0.7. *If G acts on a domain D, then*

$$Frac(D)^G = Frac(D^G).$$

Proof. By A.0.6

$$Frac(D)^G = (D \otimes_{D^G} Frac(D^G))^G = D^G \otimes_{D^G} Frac(D^G) = Frac(D^G)$$

because $Frac(D^G)$ is flat over D^G (it is a localisation), and thus commutes with limits.

□

Lemma A.0.8. *If S is a Noetherian ring, $R \subseteq S$ is a subring, and S is finitely generated as an R-module, then R is Noetherian.*

Proof. [25, Theorem 3.7].

□

Lemma A.0.9. *Let R be a local Noetherian ring and $M \xrightarrow{f} N$ be a map of finitely generated R-modules such that $x_1, \ldots, x_m$ is a regular sequence on both M and N that generates the maximal ideal of R. Then f is an isomorphism iff*

$$\bar{f} : M/(x_1, \ldots, x_n)M \to N/(x_1, \ldots, x_n)N$$

is an isomorphism.

Proof. This is standard. One way is obvious, and the other way follows inductively from [12, Proposition 21.13].

□

Remark A.0.10. In particular, if R is a regular local ring and M is a finitely generated R-module such that $\mathfrak{m}_R$ acts regularly on M, then M is a free module.

Lemma A.0.11. *For a local Noetherian ring $(R, \mathfrak{m})$, and a finitely generated R-module M, we have*

$$\hat{M} = M \otimes_R \hat{R}$$

where $\hat{-}$ refers to $\mathfrak{m}$-adic completion. In particular, if R is complete, then M is complete as well.

Proof. [25, Theorem 8.7].

□

Lemma A.0.12. *If R is a local Noetherian ring and S is a local Noetherian R-algebra that is finitely generated over R, then the $\mathfrak{m}_R$-adic and $\mathfrak{m}_S$-adic topologies on S cocincide.*

Moreover, if R is complete, then S is also complete.

Proof. As $R \to S$ is a map of local rings, it is clear that $\mathfrak{m}_R S \subseteq \mathfrak{m}_S$, and so

$$\mathfrak{m}_R^i S \subseteq \mathfrak{m}_S^i$$

for all $i \geq 0$.

If we consider $k \otimes_R S = S/\mathfrak{m}_R S$, then it is easy to see that this is a finite dimensional local k-algebra, and it is thus Artinian. In particular, the maximal ideal is nilpotent, with nilpotency r, say. This implies that $\mathfrak{m}_S^r \subseteq \mathfrak{m}_R S$, so

$$\mathfrak{m}_S^{ri} \subseteq \mathfrak{m}_R^i S$$

for all $i \geq 0$. Thus the topologies are equivalent. The last statement follows from A.0.11.

$\square$

Lemma A.0.13. *If $R \rightarrowtail S$ is an integral extension of rings, then* $\dim(R) = \dim(S)$.

Proof. This is a consequence of the going up theorem and follows from [12, Proposition 9.2].

$\square$

Lemma A.0.14. *Regular local rings are unique factorisation domains, and therefore integrally closed.*

Proof. This is the Auslander-Buchsbaum theorem. The first statement follows from [25, Theorem 20.3]. The second statement follows from [25, p. 65].

$\square$

Lemma A.0.15. *A finite extension of regular Noetherian semi-local rings $R \to S$ is flat.*

In particular, a finite epimorphism of smooth formal schemes is flat.

Proof. By 6.1.2, S is Cohen-Macaulay, so this follows by [6, Theorem 2.2.11] or [12, Corollary 19.15].

$\square$

Remark A.0.16. In particular, a finite extension of regular local rings is free.

Lemma A.0.17. *If $R \to S$ is a finite, flat extension of Noetherian semi-local rings such that S is regular, then R is regular as well.*

In particular, for formal schemes X and Y, if X is smooth and $X \xrightarrow{f} Y$ is finite, flat, and epi, then Y is smooth.

Proof. [6, Theorem 2.2.12].

$\square$

Lemma A.0.18. *Let L/K be an extension of fields, and A be a finite dimensional K-algebra. If M, N are finitely generated A-modules, then $M \cong N$ as A-modules iff $L \otimes_K M \cong L \otimes_K N$ as $L \otimes_K A$-modules.*

Proof. [8, Theorem 29.11].

□

Lemma A.0.19. *For a map $M \xrightarrow{f} N$ of complete, filtered k-modules, if $gr_F(f)$ is an injection/surjection/isomorphism, then f is also an injection/surjection/ isomorphism respectively.*

Proof. e.g. [27, Proposition 20.1.1] or [11].

□

Bibliography

[1] J. F. Adams. *Stable homotopy and generalised homology.* Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1995. Reprint of the 1974 original.

[2] I.T. Adamson. *Introduction to Field Theory.* Cambridge University Press, 1982.

[3] Matthew Ando. Isogenies of formal group laws and power operations in the cohomology theories E_n. *Duke Math. J.*, 79(2):423–485, 1995.

[4] J. Michael Boardman. Conditionally convergent spectral sequences. In *Homotopy invariant algebraic structures (Baltimore, MD, 1998)*, volume 239 of *Contemp. Math.*, pages 49–84. Amer. Math. Soc., Providence, RI, 1999.

[5] A. K. Bousfield. The localization of spectra with respect to homology. *Topology*, 18(4):257–281, 1979.

[6] W. Bruns and H.J. Herzog. *Cohen-Macaulay Rings.* Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1998.

[7] J.F. Carlson, L. Townsley, L. Valero-Elizondo, and M. Zhang. *Cohomology Rings of Finite Groups: With an Appendix: Calculations of Cohomology Rings of Groups of Order Dividing 64.* Algebra and Applications. Springer Netherlands, 2013.

[8] C.W. Curtis and I. Reiner. *Representation Theory of Finite Groups and Associative Algebras.* AMS Chelsea Publishing Series. Interscience, 1966.

[9] Daniel Dugger. Multiplicative structures on homotopy spectral sequences i, 2003.

[10] Daniel Dugger. Multiplicative structures on homotopy spectral sequences ii, 2003.

[11] Samuel Eilenberg and John C. Moore. Limits and spectral sequences. *Topol-*

ogy, 1:1–23, 1962.

[12] David Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[13] E. M. Friedlander. *Etale Homotopy of Simplical Schemes*. Annals of Mathematics Studies. Princeton University Press, 1982.

[14] Eric M. Friedlander and Guido Mislin. Cohomology of classifying spaces of complex Lie groups and related discrete groups. *Comment. Math. Helv.*, 59(3):347–361, 1984.

[15] P. G. Goerss and M. J. Hopkins. Moduli spaces of commutative ring spectra. In *Structured ring spectra*, volume 315 of *London Math. Soc. Lecture Note Ser.*, pages 151–200. Cambridge Univ. Press, Cambridge, 2004.

[16] Michiel Hazewinkel. *Formal groups and applications*. AMS Chelsea Publishing, Providence, RI, 2012. Corrected reprint of the 1978 original.

[17] Michael J. Hopkins, Nicholas J. Kuhn, and Douglas C. Ravenel. Generalized group characters and complex oriented cohomology theories. *J. Amer. Math. Soc.*, 13(3):553–594, 2000.

[18] Mark Hovey and Neil P. Strickland. Morava K-theories and localisation. *Mem. Amer. Math. Soc.*, 139(666):viii+100, 1999.

[19] John Hunton. The Morava K-theories of wreath products. *Math. Proc. Cambridge Philos. Soc.*, 107(2):309–318, 1990.

[20] C. U. Jensen. *Les foncteurs dérivés de* $\varprojlim$ *et leurs applications en théorie des modules*. Lecture Notes in Mathematics, Vol. 254. Springer-Verlag, Berlin-New York, 1972.

[21] S.O. Kochman. *Bordism, Stable Homotopy, and Adams Spectral Sequences*. Fields Institute for Research in Mathematical Sciences Toronto: Fields Institute monographs. American Mathematical Society, 1996.

[22] Igor Kriz. Morava K-theory of classifying spaces: some calculations. *Topology*, 36(6):1247–1273, 1997.

[23] Peter S. Landweber. Homological properties of comodules over $M\mathrm{U}_*(M\mathrm{U})$ and $\mathrm{BP}_*(\mathrm{BP})$. *Amer. J. Math.*, 98(3):591–610, 1976.

[24] S. J. Marsh. *The Morava E-theories of finite general linear groups*. PhD thesis, University of Sheffield, 2009.

[25] H. Matsumura and M. Reid. *Commutative Ring Theory*. Cambridge Studies

in Advanced Mathematics. Cambridge University Press, 1989.

[26] J. P. May. *A concise course in algebraic topology.* Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1999.

[27] J. P. May and K. Ponto. *More concise algebraic topology.* Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2012. Localization, completion, and model categories.

[28] Masayoshi Nagata. *Local rings.* Interscience Tracts in Pure and Applied Mathematics, No. 13. Interscience Publishers a division of John Wiley & Sons New York-London, 1962.

[29] Daniel Quillen. On the cohomology and K-theory of the general linear groups over a finite field. *Ann. of Math. (2)*, 96:552–586, 1972.

[30] Douglas C. Ravenel. Morava K-theories and finite groups. In *Symposium on Algebraic Topology in honor of José Adem (Oaxtepec, 1981)*, volume 12 of *Contemp. Math.*, pages 289–292. Amer. Math. Soc., Providence, R.I., 1982.

[31] Douglas C. Ravenel. *Complex cobordism and stable homotopy groups of spheres*, volume 121 of *Pure and Applied Mathematics.* Academic Press, Inc., Orlando, FL, 1986.

[32] Douglas C. Ravenel. *Nilpotence and periodicity in stable homotopy theory*, volume 128 of *Annals of Mathematics Studies.* Princeton University Press, Princeton, NJ, 1992. Appendix C by Jeff Smith.

[33] Douglas C. Ravenel and W. Stephen Wilson. The Morava K-theories of Eilenberg-Mac Lane spaces and the Conner-Floyd conjecture. *Amer. J. Math.*, 102(4):691–748, 1980.

[34] N. P. Strickland. Morava E-theory of symmetric groups. *Topology*, 37(4):757–779, 1998.

[35] N. P. Strickland. Products on MU-modules. *Trans. Amer. Math. Soc.*, 351(7):2569–2606, 1999.

[36] N. P. Strickland. $K(N)$-local duality for finite groups and groupoids. *Topology*, 39(4):733–772, 2000.

[37] Neil P. Strickland. Finite subgroups of formal groups. *J. Pure Appl. Algebra*, 121(2):161–208, 1997.

[38] Neil P. Strickland. Formal schemes and formal groups. In *Homotopy invariant algebraic structures (Baltimore, MD, 1998)*, volume 239 of *Contemp. Math.*, pages 263–352. Amer. Math. Soc., Providence, RI, 1999.

[39] R.M. Switzer. *Algebraic topology–homotopy and homology.* Grundlehren der mathematischen Wissenschaften. Springer-Verlag, 1975.

[40] Michimasa Tanabe. On Morava K-theories of Chevalley groups. *Amer. J. Math.*, 117(1):263–278, 1995.

[41] Paul R. Turner. Dyer-Lashof operations in the Hopf ring for complex cobordism. *Mathematical Proceedings of the Cambridge Philosophical Society*, 114(3):453–460, 1993.

[42] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1994.