The Role of Housing in Household Decision Making

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Abstract

This thesis studies the role of housing in household decision making from both the theoretical and empirical point of view. Chapter 2 uses the British Household Panel Survey (BHPS) from 1997-2008 and studies the impact of local authority district house prices on labour supply of couples via a bivariate probit model. The two-equation system not only enhances efficiency of estimation but also makes the estimation of marginal effect of house prices on the marginal, joint and conditional probability of the couple's labour supply possible while a univariate model only gives information on marginal probability of individual's labour supply. We find gender and age differences in labour participation when house price changes. Chapter 3 is motivated by the theory that different preferences and circumstances (cash-on-hand) generate alternative portfolio regimes that reflect different degrees of proximity to borrowing constraints. It fits a multivariate Gaussian mixture model via a censored data expectation-maximisation (EM) algorithm on data from Wealth and Asset Survey (WAS) to classify the four solution regimes implied by the theory. Based on these classification results, Chapter 4 estimates the marginal propensity to consume out of wealth (MPC) for heterogeneous older homeowners by minimising the difference between model predicted consumption and imputed consumption from the data for older homeowners. The results suggest a stimulus is most effective for the borrowing constrained, low net-worth households since their consumption is more sensitive to a wealth shock. Chapter 5 analyses a life cycle model with three financial assets (one is mortgage debt) and housing with borrowing constraints and uncertain asset return and labour income. We find conditions on general preferences and constraint parameters for the solutions in a period to involve different sets of binding constraints. We derive closed form solutions and simulate life time paths for different realisations of uncertainty with specialised preferences.

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Author's declaration

I declare that this thesis is a presentation of original work. Chapter 5 is joint work with Prof. Peter Simmons, it may be attributed to both authors equally. I am the sole author for other chapters. This work has not previously been presented for an award at this, or any other, University. All sources are acknowledged as References.

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Chapter 1

Introduction

In essence, the household decision problem stems from the household objective and constraints. The objective is in general set in an intertemporal framework. There are three types of decisions. The first type of decision only affects the current period utility, e.g. consumption of perishable goods and labour supply (labour income only supplements the current income). The second type of decision only affects the future utility, e.g. saving in a safe asset, investment in a risky asset. The third type of decision affects both the current and the future utility, e.g. housing purchase. This is because one can derive current utility from the house and can also enjoy/suffer the benefit/loss brought by appreciation/depreciation of housing in the future. A rational decision maker will aim to maximise the intertemporal utility. The most popular model is time additive expected utility. On the other hand, the intertemporal utility maximisation problem is subject to some constraints including financial and market constraints. Without the perfect financial market, one cannot borrow as much as one would like. Moreover, the initial wealth restricts the feasible set of decisions. For example, some are born rich while others are born poor. This will constrain people's behaviour especially in their early stages of life as the poor may not be able to accumulate enough wealth to pay for their education or fund the deposit of their first house. In reality, households can differ in their preferences and constraints which are not totally observable. The unobservable constraints together with the dual nature of housing as both a consumption and investment good makes the analysis more complicated. Understanding the heterogeneous household decisions is important for policy makers who may try to affect the labour market, financial and real estate markets.

This thesis analyses the role of housing in household decision making from both a theoretical and empirical point of view. Empirically, one key issue is to identify the exogenous shock of housing on decision variables such as labour participation. Theoretically, it is important to consider the dual nature of housing as both a consumption and investment good. This thesis studies some short run decisions where housing is fixed or predetermined: labour supply conditional on housing (Chapter 2), classifying distribution of portfolio choices with fixed housing quantity (Chapter 3), consumption response to wealth shocks conditional on housing (Chapter 4). Some long run decisions are also considered: joint consumption and housing and portfolio choice (Chapter 5).

In Chapter 2 we study the impact of house prices on household labour supply. Housing is a major component of wealth for a typical household in the UK. In a life-cycle model, individuals will reallocate their resources over time once they get new information related to their lifetime resource constraint such as a change in their housing wealth. This not only affects the house owners but also the renters who plan to buy houses in the future. The change of house prices thus has a redistributional effect among heterogeneous households. For example, when house prices rise those who own houses and plan to downsize their houses would be better off while those who will become first time house buyers or plan to upsize their houses will be worse off. Apart from the wealth effect channel, housing affects household decisions via the collateral channel. An increase of house prices would improve the possibility for the borrowing constrained homeowners to borrow against the housing equity and allow them increase current consumption and possibly decrease current labour supply. By using the British Household Panel Survey (BHPS) from 1997-2008 we study the impact of local authority district house prices on labour supply of couples. Potential wages for non-workers cannot be observed. So first we impute potential wages for both workers and non-workers using the Heckman selectivity approach. To determine labour participation, we use a bivariate probit model including the predicted wages as regressors. The use of local authority district house prices has the virtue of being disaggregate but not right down to the individual level and this avoids the potential simultaneity/endogeneity bias from using self-reported house prices (this and labour supply may both be affected by unobserved individual heterogeneity).

Moreover, if people can change local authority district where they live and these changes are not random, then the estimation of the effect of house prices could be still biased. For this reason, we restrict our estimation sample to non-movers. We control for local authority fixed effects and time effects to correct for the potential endogeneity bias caused by unobserved macro-economic factors which may drive both house prices and household decisions.

The two-equation system not only enhances efficiency of estimation but also makes the estimation of the marginal effect of house prices on the marginal, joint and conditional probability of the couple's labour supply possible while a univariate model only gives information on effects on the marginal probability of individual's labour supply. We find gender and age differences in labour participation when house prices change. To be specific, we find some limited wealth effect of housing among young and middle aged male partners. There is no evidence of a wealth effect of housing on labour participation of young and middle aged female partners. For older households, the wealth effect is small, if present at all, and probably offset by other effects such as a strong bequest motive.

My second research question is about how to classify heterogeneous households into different solution regimes in terms of financial asset and housing allocations (Chapter 3). This is motivated by the theory that different preferences and circumstances (cash-on-hand) generate alternative portfolio regimes that reflect different degrees of proximity to mortgage borrowing constraints and no-short-selling constraint in risky asset. Empirically we cannot directly observe which households are constrained in safe, risky or housing finance and housing. We fit a multivariate Gaussian mixture model via a censored data expectation-maximisation (EM) algorithm on data from Wealth and Asset Survey (WAS) to classify the four solution regimes implied by the theory. Estimation results reveal that on average about 80% of the households are no-short-selling constrained in risky asset investment and with low net worth. Among other things, we find that households who are younger, less educated with lower income are more likely to be no-short-selling constrained in risky asset investment and with lower net worth. Our predicted regime classification is aligned to those of the theory model.

Chapter 4 studies the marginal propensity to consume out of wealth (MPC) for heterogeneous older homeowners based on the classification results from Chapter 3. It considers two aspects of heterogeneity, i.e. different preferences and different circumstances (cash-on-hand). We estimate the structural parameters by minimising the difference between the model predicted consumption and imputed consumption from the data for older homeowners. The estimation results show that households who are closest to the borrowing constraints (group 1) have highest MPC (close to 1 on average), i.e. they behave in a hand-to-mouth way. This may suggest a stimulus is most effective for the borrowing constrained, low net-worth households since their consumption is more sensitive to a wealth shock. I also find MPC declines with total wealth, which is in line with the existing literature. The estimated average MPC (0.86) is bigger than in existing literature (Jappelli and Pistaferri, 2014; Sahm et al., 2010), which may be a result of a shorter planning horizon and less risk faced by the older homeowners I use for estimation.

Finally, Chapter 5 tries to improve upon the canonical life cycle model that includes only one asset and one consumption good. It takes a life cycle model with three financial assets (one is mortgage debt) and housing. The financial assets have borrowing constraints (for the mortgage a loan to value and loan to income constraint). Asset returns and labour income are uncertain. We find conditions on general preferences and constraint parameters for the solutions in a period to involve different sets of binding constraints. Finally we provide closed form solutions and simulated life time paths for different realisations of uncertainty with specialised preferences. Our simulations show that some households are rationed out of owner occupation for their whole life, although they may invest in financial asset.

Finally, Chapter 6 concludes the thesis and shows a future research agenda.

Chapter 2

Estimating the Impact of House Prices on Household Labour Participation in the UK

2.1 Introduction

Housing is both a consumption and an investment good, which constitutes a large proportion of wealth for most households in the UK. Figure 2.1 shows that in 2010-2012, net property wealth is the second largest proportion of aggregate total wealth, accounting for 37% of total wealth in Great Britain. There is an extensive literature on the impact of house prices on consumption based on both macro and micro data. A gap in the literature (except for one paper by Disney and Gathergood (2017)) is the impact of housing wealth changes due to variations of house prices on household labour supply, which is of interest to both policy makers trying to affect the labour market and academics trying to understand household decision making.

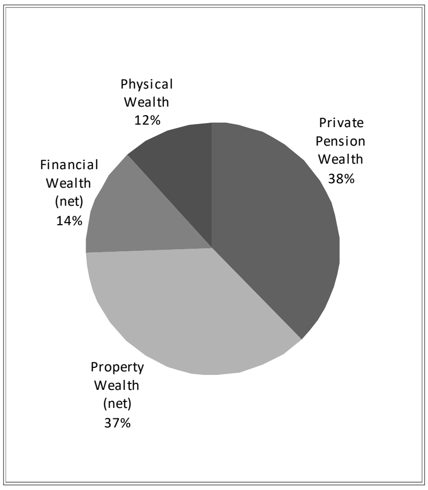


Figure 2.1: Break down of aggregate total wealth in Great Britain 2010-2012

Source: Wealth and Assets Survey - Office for National Statistics

As a major component of wealth for a typical household in the UK, housing wealth is an important source of wealth effects on household consumption of non-housing goods and leisure. In a life-cycle model, individuals will reallocate their resources over time once they get new information related to their life-time resource constraint such as a change in their housing wealth. Considering the dual nature of housing as both a component of financial assets and a consumption good, households who plan to purchase new houses or trade up their current houses can be thought of as "short" in housing, i.e. for them the fundamental value of housing they own is smaller than the present discounted value of their planned future consumption of housing services (Buiter, 2008). On the other hand, households who plan to downsize their houses are "long" in housing. As Campbell and Cocco (2007) point out, in the absence of instruments that can insure these short and long positions, unexpected shocks to house prices have a redistributive wealth effect. To be specific, given the sequences of future income and interest rates, we should expect to see those "short" in housing cut their consumption or increase labour supply and those "long" in housing increase their consumption or decrease labour supply when house prices rise. These effects are expected to be significant due to the magnitude of the housing wealth as well as the volatility of house prices. Existing literature in support of this hypothesis includes Case et al. (2005), Campbell and Cocco (2007), Carroll et al. (2011) and Disney and Gathergood (2017). However, an increase of house prices does not necessarily mean an increase in homeowners' real wealth. Homeowners with long expected tenure of their houses are hedged against the fluctuations in house prices, i.e. increasing house prices compensate for the increase of the (implicit) price of their future housing needs. In this case, increasing house prices have no real wealth effect in household consumption (Sinai and Souleles, 2005). This means we can't simply attribute the correlations between house prices and household non-housing consumption and labour supply to a pure housing wealth effect without further analysis (Campbell and Cocco, 2007; Browning et al., 2013). One alternative mechanism is the role of the housing asset as collateral available to homeowners. An increase of house prices would improve the possibility for the borrowing constrained homeowners to borrow against the housing equity and allow them increase consumption and decrease labour supply. Ortalo-Magne and Rady (2006), Lustig and Van Nieuwerburgh (2006) and Campbell and Cocco (2007) find evidence for the collateral effect of housing.

To disentangle the effect of location choice (migration choice) and the effect of local authority house prices on labour supply decision, our main estimation is confined to a non-mover sample although we subsequently check the impact of adding movers to the sample. To circumvent the possibility that some common macroeconomic factors may drive house prices and consumption/ labour supply simultaneously, we control for the local authority fixed effect and time effect.

In our study, the effect of house prices on labour supply is estimated for three age groups separately (the young households (aged 18-39), middle aged households (aged 40-54) and old households (aged 55-75)) and all age as a whole sample. Such grouping is based on the distinct features of these three age groups that would probably be the main drivers of the correlation between house prices and labour supply. Young households are more likely to face borrowing constraints, be renters and plan to upsize their houses. For them an increase in house prices raises the need for income to finance a new/upgraded purchase so labour supply/participation should increase. Old households are likely to plan downsizing so they are better off from increased house prices. So just the wealth channel implies an increase in house prices sees young renting households increase their labour supply while old home owning households decrease labour supply or choose to retire, ceteris paribus. Meanwhile, we expect that some homeowners do not plan to move and house price increases are neutral (they are hedged) and have no real wealth effects. For all ages, precautionary saving motives and bequest targets, can also affect labour supply adjustments in response to a house price change. Finally habit formation (preference for the current household labour supply pattern) can lead to inertia. Each channel may impact differentially on the male and female partners in the joint household labour supply determination. We also distinguish the house prices effect between owners and renters by using interaction of owner dummy and house price as a regressor. If the wealth mechanism is dominating, then a rise in house prices is not expected to decrease renter's labour supply.

We use the British Household Panel Survey (BHPS) from 1997-2008 and study the impact of local authority district house prices on the labour participation of couples via a bivariate probit model after imputing potential wages for both workers and non-workers using the Heckman selectivity approach. In the process of estimation, we treat the panel data as cross sections of household-year observations, but allow for heteroskedasticity and general correlation over time for the same household in computing standard errors. In this chapter we neglect the unobserved individual time-invariant heterogeneity in the error terms and do not take the selection of tenure choice into consideration. In other words, we just investigate the households' behaviour given their home ownership at a point in time [1]. In order to disentangle the decision of location (migration) choice between local authority districts and the decision of labour participation, we only estimate the model on the subsample (88% of the whole sample) who do not move between local authority districts during the periods covered by the data set. As a robustness check, we estimate the model on the whole sample and find including movers does change the result, which means that localities in which movers live at different times cannot be treated as randomly assigned.

A recent paper (Disney and Gathergood, 2017) also uses BHPS to study the impact of house prices on labour supply decisions. My chapter differs from Disney and Gathergood (2017) in the following aspects. First, while Disney and Gathergood (2017) estimate a linear probability model for labour participation, we consider the nature of the binary choice variable of labour participation by using a probit model. Second, by using a system estimator (bivariate probit) we are able to compute the marginal effect of house price on joint probability of labour participation and conditional probability of participation as well as the marginal probability of participation for each partner in the household, while Disney and Gathergood (2017) model male and female separately so that can only study the marginal probability of male and female labour participation.

[1] The perfect intertemporal life cycle model considers labour supply, consumption, saving in each period as well as changes of house finance and tenure over time. However, given the fact that change of tenure is infrequent and our use of pooled data, we don't model the tenure choice with labour supply in this paper.

In contrast to a number of empirical studies that model the labour supply of married female treating the husband's labour supply as predetermined (Blundell and Walker, 1982), we allow for the interdependent nature of the couple's labour supply decisions and enhance efficiency of estimation by exploiting the structure in the error terms of the two-equation system. The rationale for this is that the shocks to labour supply of male and female in the same family should be correlated via some common unobserved factors[2]. If decisions of partners are interrelated this is essential to gain efficiency.

The local authority district house prices cover 333 Districts and are assumed exogenous to the individual[3]. This avoids the potential simultaneity bias from using self-reported house prices (this and labour supply may both be affected by unobserved individual heterogeneity).

Another important advantage of our study is the use of individual self-reported financial expectations as an independent variable to control for income expectations as Disney and Gathergood (2017) do [4], which avoids the reliance of a parametric specification of the income process to control for income expectations in previous literature (Campbell and Cocco, 2007; Browning et al., 2013).

We first concentrate on finding the heterogeneous responses of labour supply from different household types to house prices as well as the joint decision making of spouses/partners within a household on labour supply. Our results give interpretations on the role of the different channels. Controlling for tenure, number of children, age, individual subjective financial expectations and other demographics, we find heterogeneous effects of house prices on labour supply at the extensive margin which differ by gender and age of partners.

The remainder of this paper is organised as follows: Section 2 describes the data we use. Section 3 discusses the econometric specification and estimation strategy. Section 4 discusses the estimation results. Finally, Section 5 concludes.

[2] Amemiya (1974) states that the multivariate regression system can be viewed as a reduced form of the simultaneous equation model. In the same paper, Amemiya derives the estimator for a simultaneous-equation tobit model and points out that this model can be applied in modelling the joint determination of work hours of husband and wife. However, the identification is subject to restrictions of parameters that are not necessarily satisfied in the context of household labour supply.

[3] In the whole sample, only 12% of households ever moved between local authority districts they live in. This means the majority of households stay in the same local authority districts through the period covered by our sample.

[4] As argued by Browning et al. (2013), a possible drawback of using the self-reported financial expectation is that it may include expectation of house prices.

2.2 The data

We use micro data from the twelve waves in British Household Panel Survey (BHPS) from 1997-2008. This survey is based on a representative sample of more than 5000 households, where individuals aged above 16 years old are interviewed. All the individuals included in the survey are interviewed successively across years, and their new household members will also be included.

We restrict the sample to observations satisfying the following criteria:

(1) Nobody is self-employed.

(2) All the individuals are spouses/live-in partners who stayed together through the period observed (They are either married or never married).

(3) All the couples include people with different genders[5].

(4) All the individuals are aged 18-75.

After such selection, we have a sample of 25294 household-year observations, with each observation containing the information of male and female partners and corresponding household characteristics.

In addition to the BHPS, we also include some macro variables from other sources:

(1) Regional average earnings from a report by DCLG which was based on Annual Survey of Hours and Earnings (ASHE) data for average earnings.

(2) Regional claimant count rates calculated using claimant count and claimant denominators from ONS.

(3) Local authority district (LAD) house prices (mean) from Land Registry[6].

(4) Retail Prices Index from ONS.

Tables 2.1 and 2.2 show the summary statistics of the whole sample for individual level and

[5] The number of live-in partners with the same gender is very small compared to the sample size. The purpose of excluding them is to investigate the possible gender difference in family labour supply.

[6] This data is very disaggregate. In our sample there are 333 individual local authorities in England and Wales. The data excludes sales at less than market price (e.g. Right To Buy), sales below £1000 and sales above £20m.

household level variables, respectively. The age range of 18-75 means there are some retired couples in the sample who are benefit receivers and they may voluntarily choose not to work if they are satisfied with the benefit.

Table 2.1: Summary statistics of individual level variables

		Male		Female	
Individual level variables	Observation	Mean	Standard deviation	Mean	Standard deviation
Employed	18789	0.73	0.45	0.62	0.49
Weekly hours of work	18649	28.04	19.21	17.28	16.19
Hourly gross pay (nominal)	11987	13.76	16.95	9.71	13.86
Age	18789	48.26	14.63	46.05	14.33
Negative financial expectation	18789	0.11	0.32	0.10	0.30
Positive financial expectation	18789	0.27	0.45	0.24	0.43
Married	18789	0.89	0.31	0.89	0.31
Never married	18789	0.11	0.31	0.11	0.31
Degree	18789	0.21	0.41	0.18	0.38
Hnd	18789	0.21	0.40	0.15	0.36
A-level	18789	0.23	0.42	0.30	0.46
Gcse	18789	0.31	0.46	0.34	0.47
Excellent health status	18789	0.25	0.43	0.22	0.41
Good health status	18789	0.47	0.50	0.48	0.50
Poor health status	18789	0.06	0.23	0.07	0.25
Very poor health status	18789	0.01	0.11	0.01	0.12

Notes: The question asked in the BHPS questionnaire about financial expectations: "Looking ahead, how do you think you will be financially a year from now?" And interviewees can choose from the four answers :"(1) Better off; (2) Worse off than now; (3) About the same; (4) Don't know."

Table 2.2: Summary statistics of household level variables

Household level variables	Observation	Mean	Standard deviation
Owner	18789	0.84	0.36
Number of children	18789	0.78	1.04
Annual household non-labour income (nominal)	18789	6926.97	8879.95

As we can see from Table 2.3, there are discrepancies in terms of participation patterns among the three age groups. In particular, most of households aged under 54 have both partners working while most households aged above 54 have neither partner working. Among the three age groups, the middle aged couples (aged 40-54) have the highest percentage of both working and lowest percentage of neither working. For all age groups, it is very rare to find that only the female works in a household.

Table 2.3: Participation patterns in the sample

	All age	Aged 18-39	Aged 40-54	Aged 55-75
Both work	10528 (56%)	4128 (71.4%)	3422 (80.6%)	864 (15.1%)
Only male works	3112(16.6%)	1280 (22.1%)	630 (14.8%)	662 (11.6%)
Only female works	1049 (5.6%)	139 (2.4%)	120 (2.8%)	588 (10.3%)
Neither work	4100 (21.8%)	237 (4.1%)	75 (1.8%)	3602 (63.1%)
Total	18789 (100%)	5784 (100%)	4247 (100%)	5706 (100%)

Figure 2.2 shows the scatterplot of weekly working hours for male and female in the whole sample, the vertical axis shows hours of work for female, while the horizontal axis shows hours of work for male. The red line is the 45 degree reference line. As we can see from the scatter, the majority of the observations are to the right of the red line. This suggests that in most households, male partners have more hours of work than female. The most intensive area of observations (the darkest area) are the "band" with males working around 40 hours a week and females working less than 40 hours a week.

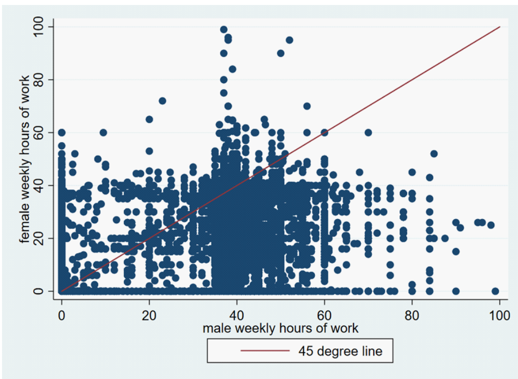


Figure 2.2: Scatter plot of weekly working hours for male and female partners

We are also interested in the destinations when one exits the labour market in each age group, i.e. the job status of non-workers in each age group (Table 2.4). As we can see from Table 2.4, there are some gender differences in destinations in each age group. For those under 40 and aged 40-54, most male non-workers are unemployed while most female non-workers engage in family care. And for those aged above 54, most of the non-workers, either male or female, are retired. Therefore, in the older group, the decisions of not to work are more likely to be permanent than in other age groups, though we don't exclude the possibility that one will want to re-enter the labour market at some point after retirement.

Table 2.4: Tabulation of job status of non-workers in the sample

age	gender	unemployed	retired	family care	government training	other
aged 18-39	male	86.97%	0.80%	6.12%	1.60%	4.52%
	female	8.83%	0.13%	89.06%	0.13%	1.85%
aged 40-54	male	49.74%	36.41%	10.26%	0.51%	3.08%
	female	10.64%	5.39%	81.28%	0.14%	2.55%
aged 55-75	male	1.22%	97.90%	0.21%	0.02%	0.64%
	female	14.03%	0.38%	76.63%	8.59%	0.36%
All age	male	10.84%	86.58%	1.32%	0.17%	1.09%
	female	8.83%	0.13%	89.06%	0.13%	1.85%

As to house prices, as an example, here we pick up 4 cities and 4 rural areas from 333 local authority districts and show the real house prices overtime for each district. Of course, in the estimation the full sample contains 333 local authority districts. As is shown in Figure 2.3, the house prices in different localities all have a growing trend over time. It should be noticed that in London, it has a higher mean.

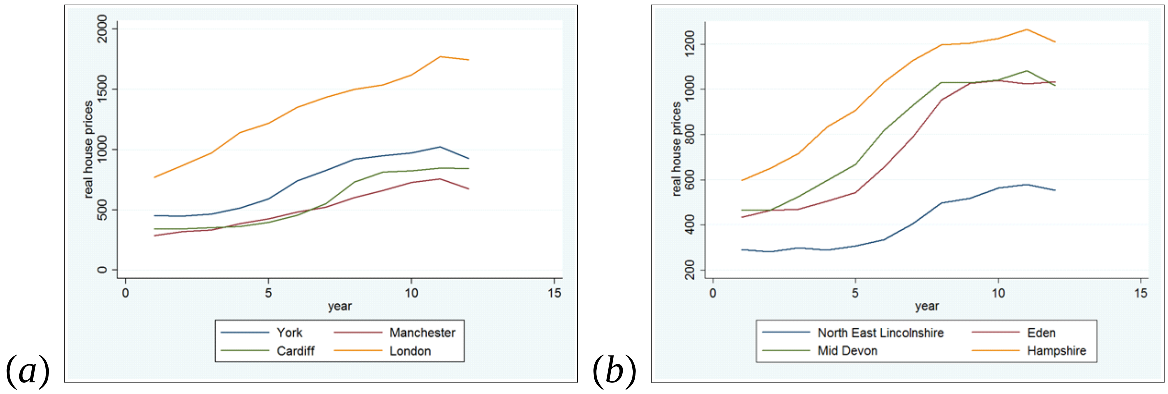


Figure 2.3: Three examples of Local Authority District real house prices across years

(a) Four cities: London, York, Cardiff and Manchester

(b) Four rural areas: Hampshire, Mid Devon, Eden and North East Lincolnshire

2.3 Econometric specification and estimation strategy

2.3.1 Econometric specification

In a family with two workers or potential workers, the family utility maximisation problem ignoring intertemporal effects yields optimal hours of work of the two persons conditional on real market wages of the two persons and household non-labour income.

We assume a single mechanism to decide the behaviour of labour supply both on the extensive margin (participation) and intensive margin (hours of work), ignoring monetary or time fixed costs of work (Cogan,1981).

If both persons in the household work, then the interior solution of labour supply (constant time endowment minus leisure demand) of each person i can be written as a reduced form function of two real wages and household non-labour income. We specify the labour supply of person i as follows:

$$h^*_{hilt} = \alpha_i + \beta_{1i} \ln w_{hilt} + \beta_{2i} X_{hilt} + \beta_{3i} \ln w_{hjlt} + \delta_l + \tau_t + \varepsilon_{hilt}, i, j = m, f \tag{2.1}$$

where m and f denote the male and female partners in the household, respectively. h^*_{hilt} is the desired hours of work of person i in household h living in district l at time t, w_{hilt} is real wages . X_{hilt} are a set of exogenous observable variables determining labour supply behaviours, ε_{hilt} is individual heterogeneity in tastes for work[7], δ_l is the local authority fixed effect, τ_t is the time effect (year dummies), α_i, β_{1i}, β_{2i} and β_{3i} are individual preference parameters and the subscripts m and f denote male partner and female partner, respectively. This specification assumes linearity of hours of work in $\ln w_{hilt}$ and other exogenous regressors.

If we allow for the possibility that the family utility is higher with person i not working and consider the fact that negative hours of work is infeasible, then the optimal labour supply of person i becomes

$$h_{hilt} = \max\{h^*_{hilt}, 0\}, i = m, f$$

where h_{hilt} indicates observed hours of work of person i, h^*_{hilt} indicates desired hours of work of person i. $h_{hilt} = 0$ is the corner solution where desired hours of work are non-positive and actual hours are zero. Basing the analysis on the subsample of workers will lead to sample selection bias with respect to the population distribution of desired hours of work given by the labour supply function. In particular, existing empirical work shows the elasticity of wage and income on hours of work would be misleading if we only consider the group of workers.

The two-equation censored model for participation is:

$$\begin{aligned} I_{hmlt} &= \begin{cases} 1 \text{ if } \alpha_m + \beta_{1m} \ln w_{hmlt} + \beta_{2m} X_{hmlt} + \beta_{3m} \ln w_{hflt} + \delta_l + \tau_t + \varepsilon_{hmlt} > 0 \\ 0 \quad \text{otherwise} \end{cases} \\ I_{hflt} &= \begin{cases} 1 \text{ if } \alpha_f + \beta_{1f} \ln w_{hflt} + \beta_{2f} X_{hflt} + \beta_{3m} \ln w_{hmlt} + \delta_l + \tau_t + \varepsilon_{hflt} > 0 \\ 0 \quad \text{otherwise} \end{cases} \end{aligned} \tag{2.2}$$

where I_{hmlt} and I_{hflt} are observed labour participation decisions for male and female partners, respectively; X_{hilt} $(i = m, f)$ includes interaction of owner dummy and log of real house prices, renter dummy and log of real house prices, age, age squared, dummy of worse financial expectation, dummy of better financial expectation, dummies of being married, dummies of highest education qualifications including degree, hnd, alevel and gcse, number of children in the household, health status dummies including 4 categories, log of real annual household non-labour income. δ_l is the local authority fixed effect, τ_t is the time effect (year dummies).

[7] We treat the data as pooled cross section data but correct for the autocorrelation of the same household over time and the heteroskedasticity across households in estimating the standard errors of coefficients.

The error terms ε_{hm} and ε_{fm} are assumed to be bivariate normally distributed with zero means and covariance matrix:

$$\sum = \begin{bmatrix} \sigma^2_{\varepsilon_{hmlt}} & \sigma_{\varepsilon_{hmlt}\varepsilon_{hflt}} \\ \sigma_{\varepsilon_{hmlt}\varepsilon_{hflt}} & \sigma^2_{\varepsilon_{hflt}} \end{bmatrix}$$

At the same time, we also assume the wage equations for person i (i=f,m) in the household as follows:

$$\ln w_{hilt} = \gamma_i Z_{hilt} + \eta_{hilt} \tag{2.3}$$

where Z_{hilt} are a set of variables to determine real wages including dummies of highest education qualifications including degree, hnd, alevel and gcse, age, age squared, year dummies, regional claimant count rate, log of real regional average earnings (Note that regional claimant count rate, log of real regional average earnings only appear in the wage equation but not in the selection equation.); η_{hilt} is the error term which is normally distributed. Assume $E(\eta_{hilt}) = 0$, the error terms in the wage equations for each partner are independent, i.e. $E(\eta_{hilt}\eta_{hjlt}) = 0$,and allow for the correlation between ε_{hilt} and η_{hilt}. In other words, the covariance matrix of the error terms of equation systems (2.2) and (2.3) is:

$$\Omega = \begin{bmatrix} \sigma^2_{\varepsilon_{hmlt}} & \sigma_{\varepsilon_{hmlt}\varepsilon_{hflt}} & \sigma_{\varepsilon_{hmlt}\eta_{hflt}} & 0 \\ \sigma_{\varepsilon_{hmlt}\varepsilon_{hflt}} & \sigma^2_{\varepsilon_{hflt}} & 0 & \sigma_{\varepsilon_{hflt}\eta_{hflt}} \\ \sigma_{\varepsilon_{hmlt}\eta_{hmlt}} & 0 & \sigma^2_{\eta_{hmlt}} & 0 \\ 0 & \sigma_{\varepsilon_{hflt}\eta_{hflt}} & 0 & \sigma^2_{\eta_{hflt}} \end{bmatrix}$$

2.3.2 Estimation strategy

2.3.2.1 Imputation of wages

Before estimating the participation and hours of work decisions for households, an important issue to deal with is the missing wages for non-workers. The information on wage is required if we take the selection problem into account. However, there is still no consensus on particular solutions to wage-imputation problem (Heckman, 1993). Here we adopt the Heckman-style selectivity adjusted method. The idea underlying this method is the reservation wage condition, i.e. we assume that people decide not to work because the potential in-work wages are lower than for comparable workers. In other words, when people make the participation decisions, they simply consider the "average" wages of workers with the same observable characteristics as themselves. Compared to the entry wage measures, the Heckman selectivity approach captures the "long term" equilibrium wage for certain types of workers as long as they are long-lived enough to realise these "long run" wages (Myck and Reed, 2005).

In particular, we impute wages for everyone by the following procedures: (Hereafter we omit the subscript h indicating household, l indicating district and t indicating time for notational convenience.)

In the first step, we substitute the wage equations to the desired hours of work equation to have a reduced form for hours of work for each partner $(i = m, f)$

$$h_i = \begin{cases} \pi_i q_i + v_i & \text{if } RHS > 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.4}$$

where $\pi_i = (\alpha, \beta_{1i}\gamma, \beta_{2i}, \beta_{3i}\gamma_j), q_i' = (1, Z_i, X_i, Z_j), v_i = \varepsilon_i + \beta_{1i}\eta_i + \beta_{3i}\eta_j$

Then we run a univariate probit model on equation (2.5) for each partner and find λ_i to correct for the selection bias.

$$I_i = \begin{cases} 1 \text{ if } \pi_i q_i + v_i & > 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.5}$$

In the second step, we use λ_{hi} as an extra regressor to predict ln(wage) for everyone:

$$\ln w_i = \gamma_i Z_i + \delta_i \lambda_i + \eta_i$$

where the extra term $\lambda_i = \frac{\phi_{vi}(\widehat{\pi}_i q_i)}{\Phi_{vi}(\widehat{\pi}_i q_i)}$ is the predicted inverse Mills ratio from equation (2.5). As to the implementation of estimation, the idea is to assume η_{hi} and v_{hi} are bivariate normal, and it follows $E(\eta_i|v_i) = \delta_i v_i$, which allows the use of the individual inverse Mills ratio.

The justification of using the individual inverse Mills ratio is as follows.

If we ignore the possible endogeneity of wages and assume ε_i and η_i are independent of each other, then we can simply impute the wages for all non-workers using the fitted values of the wage equation as if wages are observed and correct for the standard errors. However, wages could be endogenous because of possible correlation between unobservables affecting tastes for work and unobservables affecting productivity hence wages (Blundell et al, 2007). The two step Heckman sample selection approach allows for the endogeneity of wages by considering the joint distribution for ε_i and η_i.

Substitute equation (2.3) to equation (2.1) to have a reduced form for desired hours of work for each person:

$$\begin{aligned} h_i^* &= \alpha_i + \beta_{1i}(\gamma_i Z_i + \eta_i) + \beta_{2i} X_i + \beta_{3i}(\gamma_j Z_j + \eta_j) + \varepsilon_i \\ &= \alpha_i + \beta_{1i}\gamma_i Z_i + \beta_{2i} X_i + \beta_{3i}\gamma_j Z_j + (\varepsilon_i + \beta_{1i}\eta_i + \beta_{3i}\eta_j) \\ &= \pi_i q_i + v_i \end{aligned} \tag{2.6}$$

where $\pi_i = (\alpha, \beta_{1i}\gamma, \beta_{2i}, \beta_{3i}\gamma_j), q_i' = (1, Z_i, X_i, Z_j), v_i = \varepsilon_i + \beta_{1i}\eta_i + \beta_{3i}\eta_j$

Equations (2.3) and (2.6) constitute a triangular system to completely describe labour supply and wages (Blundell et al., 2007).

Wages are only observed if person i participates, i.e.if desired hours of work for person i are positive.

The probability of person i participating whatever his/her partner/spouse person j's participation decision (i,j=1,2), i.e. marginal probability of person i participating is:

$\Pr(h_i^* > 0|X_i, Z_i, h_j) = \Pr(v_i > -\pi_i q_i, h_j > 0) + \Pr(v_i > -\pi_i q_i, h_j \leq 0) = \Pr(v_i > -\pi_i q_i, v_j > -\pi_j q_j) + \Pr(v_i > -\pi_i q_i, v_j \leq -\pi_j q_j)$

$= \Pr(v_i > -\pi_i q_i) = \Phi(\pi_i q_i) = \Phi(\alpha + \beta_{1i}\gamma_i Z_i + \beta_{2i} X_i + \beta_{3i}\gamma_j Z_j)$

where $\Phi(.)$ is the marginal distribution of v_i.

The mean of log of wages for person i given he/she works is

$$\begin{gathered} E(\ln w_i|h_1^* > 0, X_i, Z_i) \\ = E(\gamma_i Z_i + \eta_i|v_i > -\pi_i q_i) \\ = E(\gamma_i Z_i + \eta_i|\varepsilon_i + \beta_{1i}\eta_i + \beta_{3i}\eta_j > -\alpha - \beta_{1i}\gamma Z_i - \beta_{2i}X_i - \beta_{3i}\gamma_j Z_j) \\ = E(\gamma_i Z|\varepsilon_i + \beta_{1i}\eta_i + \beta_{3i}\eta_j > -\alpha - \beta_{1i}\gamma Z_i - \beta_{2i}X_i - \beta_{3i}\gamma_j Z_j) \\ +E(\eta_i|\varepsilon_i + \beta_{1i}\eta_i + \beta_{3i}\eta_j > -\alpha - \beta_{1i}\gamma Z_i - \beta_{2i}X_i - \beta_{3i}\gamma_j Z_j) \end{gathered} \tag{2.7}$$

The correlation between the error term of reduced form participation equation and the error term of the wage equation for the same person is:

$$\begin{aligned} cov(v_i,\eta_i) &= cov((\varepsilon_i+\beta_{1i}\eta_i+\beta_{3i}\eta_j),\eta_i) \\ &= E((\varepsilon_i+\beta_{1i}\eta_i+\beta_{3i}\eta_j)\eta_i) \\ &= E(\varepsilon_i\eta_i)+E(\beta_{1i}\eta_i^2)+E((\beta_{3i}\eta_j)\eta_i) \end{aligned} \tag{2.8}$$

Assuming $E(\varepsilon_i)=0$, $E(\eta_i)=0$ and $E(\eta_i\eta_j)=0$[8], equation (2.8) becomes

$$cov(v_i,\eta_i)=cov(\varepsilon_i,\eta_i)+\beta_{1i}\sigma^2_{\eta_i} \tag{2.9}$$

If either the covariance of ε_i and η_i is non-zero or variance of η_i is non-zero (which is for sure unless wages are non-random), the covariance of v_i and η_i is non-zero.

On the other hand, the mean of v_i is

$$E(v_i)=E(\varepsilon_i+\beta_{1i}\eta_i)=0$$

Assume v_i and η_i are jointly normal distributed:

$$\begin{bmatrix}\eta_i\\ v_i\end{bmatrix} \sim N\left[\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}\sigma^2_{\eta_i} & \sigma_{\eta_i v_i}\\ \sigma_{\eta_i v_i} & \sigma^2_{v_i}\end{bmatrix}\right]$$

where $\sigma_{\eta_i v_i}=\sigma_{\varepsilon_i\eta_i}+\beta_{1i}\sigma^2_{\eta_i}$ and

$\sigma^2_{v_i}=\sigma^2_{\varepsilon_i}+\beta^2_{1i}\sigma^2_{\eta_i}+\beta^2_{3i}\sigma^2_{\eta_j}+2\beta_{1i}\sigma_{\varepsilon_i\eta_i}+2\beta_{1i}\beta_{3i}\sigma_{\eta_i\eta_j}+2\beta_{3i}\sigma_{\varepsilon_i\eta_j}=\sigma^2_{\varepsilon_i}+\beta^2_{1i}\sigma^2_{\eta_i}+\beta^2_{3i}\sigma^2_{\eta_j}+2\beta_{1i}\sigma_{\varepsilon_i\eta_i}+2\beta_{3i}\sigma_{\varepsilon_i\eta_j}$

Since the conditional expectation of η_i given v_i is

$$E(\eta_i|v_i)=\frac{\sigma_{\eta_i v_i}}{\sigma^2_{v_i}}v_i$$

it follows that

[8] Assume the error terms in the wage equations for each partner are independent.

$$
\begin{aligned}
\eta_i &= \frac{\sigma_{\eta_i v_i}}{\sigma^2_{v_i}} v_i + \zeta \\
&= \frac{\sigma_{\varepsilon_i \eta_i} + \beta_{1i}\sigma^2_{\eta_i}}{\sigma^2_{\varepsilon_i} + \beta^2_{1i}\sigma^2_{\eta_i} + \beta^2_{3i}\sigma^2_{\eta_j} + 2\beta_{1i}\sigma_{\varepsilon_i\eta_i} + 2\beta_{3i}\sigma_{\varepsilon_i\eta_j}} v_i + \zeta
\end{aligned}
\tag{2.10}
$$

Using equation (2.10) and assuming ζ and $v_i = \varepsilon_i + \beta_{1i}\eta_i + \beta_{3i}\eta_j$ are independent, equation (2.7) becomes

$$
\begin{gathered}
E(\ln w_i | h^*_i > 0, X_i, Z_i, Z_j) \\
= E(\gamma_i Z_i | \varepsilon_i + \beta_{1i}\eta_i + \beta_{3i}\eta_j > -\alpha - \beta_{1i}\gamma Z_i - \beta_{2i}X_i - \beta_{3i}\gamma_j Z_j) + \\
E((\frac{\sigma_{\varepsilon_i \eta_i} + \beta_{1i}\sigma^2_{\eta_i}}{\sigma^2_{\varepsilon_i} + \beta^2_{1i}\sigma^2_{\eta_i} + \beta^2_{3i}\sigma^2_{\eta_j} + 2\beta_{1i}\sigma_{\varepsilon_i\eta_i} + 2\beta_{3i}\sigma_{\varepsilon_i\eta_j}}) \\
(\varepsilon_i + \beta_{1i}\eta_i + \beta_{3i}\eta_j) + \zeta) \\
|\varepsilon_i + \beta_{1i}\eta_i + \beta_{3i}\eta_j > -\alpha - \beta_{1i}\gamma Z_i - \beta_{2i}X_i - \beta_{3i}\gamma_j Z_j) \\
= \gamma_i Z_i + (\frac{\sigma_{\varepsilon_i \eta_i} + \beta_{1i}\sigma^2_{\eta_i}}{\sigma^2_{\varepsilon_i} + \beta^2_{1i}\sigma^2_{\eta_i} + \beta^2_{3i}\sigma^2_{\eta_j} + 2\beta_{1i}\sigma_{\varepsilon_i\eta_i} + 2\beta_{3i}\sigma_{\varepsilon_i\eta_j}}) \\
E(\varepsilon_i + \beta_{1i}\eta_i + \beta_{3i}\eta_j | \varepsilon_i + \beta_{1i}\eta_i + \beta_{3i}\eta_j > -\alpha - \beta_{1i}\gamma Z_i - \beta_{2i}X_i - \beta_{3i}\gamma_j Z_j) \\
= \gamma_i Z_i + (\frac{\sigma_{\varepsilon_i \eta_i} + \beta_{1i}\sigma^2_{\eta_i}}{\sigma^2_{\varepsilon_i} + \beta^2_{1i}\sigma^2_{\eta_i} + \beta^2_{3i}\sigma^2_{\eta_j} + 2\beta_{1i}\sigma_{\varepsilon_i\eta_i} + 2\beta_{3i}\sigma_{\varepsilon_i\eta_j}}) \\
\frac{\varphi(-\alpha - \beta_{1i}\gamma Z_i - \beta_{2i}X_i - \beta_{3i}\gamma_j Z_j)}{1 - \Phi(-\alpha - \beta_{1i}\gamma Z_i - \beta_{2i}X_i - \beta_{3i}\gamma_j Z_j)}
\end{gathered}
\tag{2.11}
$$

where $\Phi(.)$ and $\varphi(.)$ are the marginal distribution and marginal density of v_i, respectively.

By symmetry of normal distribution,

$$
\frac{\varphi(-\alpha - \beta_{1i}\gamma Z_i - \beta_{2i}X_i - \beta_{3i}\gamma_j Z_j)}{1 - \Phi(-\alpha - \beta_{1i}\gamma Z_i - \beta_{2i}X_i - \beta_{3i}\gamma_j Z_j)} = \frac{\varphi(\alpha + \beta_{1i}\gamma Z_i + \beta_{2i}X_i + \beta_{3i}\gamma_j Z_j)}{\Phi(\alpha + \beta_{1i}\gamma Z_i + \beta_{2i}X_i + \beta_{3i}\gamma_j Z_j)} = \lambda_i
$$

Therefore equation (2.11) becomes:

$$
\begin{gathered}
E(\ln w_i | h^*_1 > 0, X_i, Z_i, Z_j) = \\
\gamma_i Z_i + (\frac{\sigma_{\varepsilon_i \eta_i} + \beta_{1i}\sigma^2_{\eta_i}}{\sigma^2_{\varepsilon_i} + \beta^2_{1i}\sigma^2_{\eta_i} + \beta^2_{3i}\sigma^2_{\eta_j} + 2\beta_{1i}\sigma_{\varepsilon_i\eta_i} + 2\beta_{3i}\sigma_{\varepsilon_i\eta_j}})\lambda_i
\end{gathered}
$$

To be specific, the Heckman model corrects for the selection bias by adding an extra term to the wage model:

$$\ln w_i = \gamma_i Z_i + \delta_i \lambda_i + \eta_i \tag{2.12}$$

where the extra term $\lambda_i = \frac{\phi(\widehat{\pi}_i q_i)}{\Phi(\widehat{\pi}_i q_i)}$ is the predicted inverse Mills ratio from the reduced form participation/selection equation as derived above.

2.4 Estimation results

In the estimations of household participation, the three age groups are treated separately and the results are compared among them and with the all age result. In the estimations, we pooled the panel as if it is cross sectional, but allow heteroskedasticity and general correlation over time for the same household, while independence over households is still assumed. Since the imputed wage is a generated variable in the participation equations, the conventional standard errors will be biased. We correct for the bias by bootstrapping the standard errors in the participation equations. The number of bootstrap replications is 200 for each age group. In each replication we re-estimate imputed wages and the bivariate probit for participation.

2.4.1 Estimation results for wage equations

Table 2.5: The wage equation estimates (non-movers)

	All age (non-movers)	
Independent variables	male	female
Degree	-0.029	0.067*
	(-1.30)	(2.37)
Hnd	-0.30***	-0.29***
	(-13.49)	(-10.39)
A level	-0.37***	-0.38***
	(-16.52)	(-13.87)
Gcse	-0.56***	-0.58***
	(-25.00)	(-20.59)
Age	0.070***	0.023***
	(20.85)	(6.52)
Age^2	-0.00074***	-0.00021***
	(-18.69)	(-5.08)
Regional claimant count rate	0.81	3.34***
	(1.43)	(5.45)
Ln (regional average earnings)	0.32***	0.45***
	(9.55)	(12.70)
Inverse mills ratio	-0.17***	-0.12***
	(-8.77)	(-7.26)
N	15570	16244

Note: Dependent variables: natural log of real wage of male and female partners. Additional independent variables not shown in the table: year dummies, local authority districts dummies.

t statistics in parentheses. * p<0.05,** p<0.01, *** p<0.001.

Table 2.5 shows the wage equation (the second step of Heckman selection approach) estimates[9]. The inverse Mills ratio is statistically significant for both male and female partners, showing

[9] The first step of Heckman selection approach (two univariate probit estimates) are shown in the appendix.

strong evidence of sample selection.

2.4.2 Estimation results for household participation

The estimation of the structural bivariate probit model is done for the non-movers[10]. It is estimated for all ages together and three age groups separately. Tables 2.6 and 2.7 show the estimated coefficients for the household labour participation for the non-movers. The sign of the estimated coefficients is the sign of the marginal effect of each regressor. However, the marginal effect is not equal to the coefficient because of the nonlinearity of the model. Table 2.8 show the marginal effects of house prices for an average renting/home owning household. Let Pr(0,0), Pr(1,0), Pr(0,1) and Pr(1,1) denote the probabilities of neither work, only male works, only female works and both work, respectively. Let Pr(demploy=1) and Pr(sp_demploy=1) denote the marginal probabilities of male works and female works respectively. Let Pr(demploy=1|sp_demploy=1) denote the probability of male works conditional on his female partner works. Let Pr(sp_demploy=1|demploy=1) denotes the probability of female works conditional on her male partner works.

[10] Non-movers are those households who never move between loacal authority districts in the years covered by the sample.

Table 2.6: Estimated coefficients for household labour participation (non-movers) (to be continued on the next page)

	Aged 18-39		Aged 40-54		Aged 55-75		All age	
Independent variables	Male	Female	Male	Female	Male	Female	Male	Female
Renter*ln(real house price)	0.44	-0.30	0.80	-0.84	0.31	1.23**	0.050	0.028
	(0.55)	(-0.79)	(0.73)	(-1.56)	(0.65)	(2.75)	(0.20)	(0.13)
Owner*ln(real house price)	0.58	-0.23	0.96	-0.74	0.36	1.24**	0.16	0.081
	(0.74)	(-0.60)	(0.87)	(-1.36)	(0.76)	(2.76)	(0.66)	(0.37)
Age	0.18	0.10	0.99	0.55	-0.69	-0.39	0.27	0.28***
	(0.87)	(0.62)	(0.66)	(0.99)	(-0.91)	(-0.38)	(1.57)	(11.09)
Age^2	-0.0031	-0.00089	-0.011	-0.0062	0.0038	0.0018	-0.0035	-0.0034***
	(-1.31)	(-0.37)	(-0.67)	(-0.99)	(0.82)	(0.24)	(-1.89)	(-13.57)
Worse financal expectation	-0.13	-0.045	-0.23	-0.038	-0.050	0.090	-0.061	0.045
	(-0.56)	(-0.41)	(-1.07)	(-0.30)	(-0.56)	(1.12)	(-1.09)	(0.93)
Better financal expectation	-0.27*	0.057	-0.45**	0.046	-0.086	0.026	-0.15**	0.078*
	(-2.41)	(0.98)	(-2.84)	(0.49)	(-0.87)	(0.25)	(-3.12)	(2.29)
Married	0.36**	0.020	-4.96***	0.35	0.77	-1.63	0.33***	-0.091
	(2.66)	(0.26)	(-11.98)	(1.12)	(0.45)	(-1.92)	(3.31)	(-1.58)
Degree	0.36	0.24	-6.56***	0.16	-0.73	0.38	-0.24	0.18
	(1.03)	(1.04)	(-7.59)	(0.25)	(-0.79)	(0.47)	(-1.66)	(1.47)
Hnd	0.15	-0.066	-7.60*	-0.91	-0.99	-0.37	-0.50	-0.68*
	(0.16)	(-0.27)	(-2.21)	(-0.25)	(-0.48)	(-0.05)	(-0.68)	(-2.24)
A level	-0.094	0.23	-7.77	-1.34	-1.24	-0.90	-0.60	-0.86*
	(-0.09)	(0.58)	(-1.81)	(-0.36)	(-0.48)	(-0.11)	(-0.67)	(-2.27)
Gcse	-0.27	-0.10	-8.97	-2.50	-1.90	-1.85	-1.07	-1.69**
	(-0.16)	(-0.17)	(-1.40)	(-0.47)	(-0.57)	(-0.16)	(-0.79)	(-2.97)

Note: Dependent variables: participation dummies of male and female partners. Additional independent variables not shown in the table: year dummies, local authority districts dummies. t statistics in parentheses. * $p<0.05$,** $p<0.01$, *** $p<0.001$.

Table 2.7: Estimated coefficients for household labour participation (non-movers) (continued)

	Aged 18-39		Aged 40-54		Aged 55-75		All age	
Independent variables	Male	Female	Male	Female	Male	Female	Male	Female
Number of children	0.17**	-0.35***	0.070	-0.20***	0.34*	-0.56**	0.079**	-0.23***
	(3.17)	(-9.63)	(1.01)	(-4.59)	(2.24)	(-2.60)	(3.04)	(-14.49)
Excellent health status	0.26	0.038	-0.12	0.47***	0.31**	0.50***	0.23***	0.24***
	(1.74)	(0.46)	(-0.60)	(4.12)	(3.03)	(5.86)	(4.07)	(5.92)
Good health status	0.13	0.077	0.084	0.34***	0.38***	0.30***	0.24***	0.16***
	(1.24)	(1.08)	(0.44)	(3.98)	(5.42)	(4.22)	(5.10)	(5.52)
Poor health status	-0.44*	-0.059	-0.35	-0.41**	-0.72***	-0.12	-0.48***	-0.23***
	(-2.35)	(-0.50)	(-1.21)	(-2.84)	(-3.75)	(-0.99)	(-5.20)	(-4.03)
Very poor health status	-0.94*	-0.29	-1.26**	-0.98**	-0.85	-0.42*	-0.86***	-0.44***
	(-2.16)	(-1.25)	(-2.60)	(-2.65)	(-1.45)	(-2.07)	(-5.01)	(-3.66)
Ln (real non-labour income)	-0.55***	-0.31***	-0.53***	-0.18***	-0.77***	-0.33***	-0.61***	-0.28***
	(-8.45)	(-8.86)	(-6.37)	(-6.08)	(-11.73)	(-12.59)	(-23.37)	(-21.54)
Ln (wage)	-0.52	0.55	-4.29	-3.26	-1.95	-3.43	-1.37	-2.28*
	(-0.14)	(0.40)	(-0.42)	(-0.47)	(-0.32)	(-0.24)	(-0.56)	(-2.36)
Ln (partner's wage)	0.88**	-0.041	-0.60	0.10	-0.98***	-0.11	-0.35***	-0.093
	(2.67)	(-0.23)	(-1.77)	(0.58)	(-6.57)	(-0.74)	(-3.50)	(-1.38)
Rho12	0.25***		0.078		0.28***		0.21***	
	(3.52)		(0.72)		(5.37)		(7.32)	
N	4687		3814		5302		16512	

Note: Dependent variables: participation dummies of male and female partners. Additional independent variables not shown in the table: year dummies, local authority districts dummies. Rho12 is the correlation of the residuals from the participation equations of male and female partners. t statistics in parentheses. * $p<0.05$,** $p<0.01$, *** $p<0.001$..

Table 2.8: Marginal effects of house price on labour participation (non-movers)

		aged 18-39	aged 40-54	aged 55-75	All age
Pr(demploy=1)	renter	0.001	6.93e-06	0.03	0.01
	owner	0.00007	0.0002	0.06	0.02
Pr(sp_demploy=1)	renter	-0.07	-0.003	0.15	0.01
	owner	-0.03	-0.0003	0.16	0.03
Pr(1,1)	renter	-0.07	-0.003	0.02	0.02
	owner	-0.03	-0.0003	0.04	0.04
Pr(1,0)	renter	0.07	0.003	0.01	-0.00003
	owner	0.03	-0.0001	0.03	-0.02
Pr(0,1)	renter	-0.0009	-2.52e-07	0.13	-0.004
	owner	-0.0005	-2.86e-07	0.12	-0.01
Pr(0,0)	renter	-0.0004	-1.05e-11	-0.16	-0.01
	owner	-0.00002	-3.05e-15	-0.19	-0.01
Pr(demploy=1|sp_demploy=1)	renter	0.001	-0.01	0.005	0.01
	owner	-0.003	-0.03	0.02	0.02
Pr(sp_demploy=1|demploy=1)	renter	-0.07	-0.003	0.28	0.01
	owner	-0.03	-0.0003	0.27	0.03
N		4687	3814	5302	16512

The Wald tests performed after estimation for the three age groups and all age all reject the null hypotheses that $cov(\varepsilon_{hmlt}, \varepsilon_{hflt}) = 0$ except for the middle aged group, which shows evidence that cross equation correlations matter, i.e. the decisions of participation of both partners are interdependent via some unobserved household heterogeneity and/or the shocks to each partner's participation are likely to be mutually correlated.

As Table 2.9 shows, the predicted mean probabilities of the four participation patterns for each age group exhibit great differences among age groups. On average, the predicted probability of both working is very high for middle aged households but low for the older group. On the other hand, the pattern of neither working dominates for the older households in general. The estimated histogram of participation patterns for older households are shown in Figure 2.4. These findings are consistent with the life cycle profile of the participation decision.

Table 2.9: Predicted mean of probabilities of four participation patterns (non-mover)

	All age	Aged 18-39	Aged 40-54	Aged 55-75
Pr(1,1)	0.54	0.71	0.81	0.14
Pr(1,0)	0.16	0.22	0.14	0.12
Pr(0,1)	0.07	0.03	0.03	0.11
Pr(0,0)	0.23	0.04	0.01	0.64

Note: * p<0.05,** p<0.01, *** p<0.001

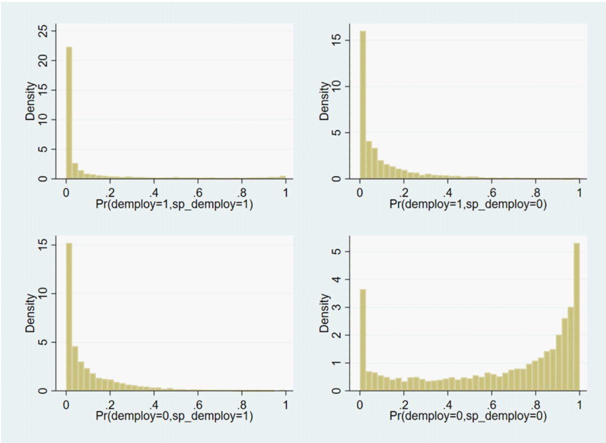


Figure 2.4: Estimated histogram of participation patterns for households aged 55-75

Table (2.8) is computed for an average household[11]. In line with the existing Disney and Gathergood (2017), these marginal effects at the extensive margin of labour supply are biggest for younger and older women and older men. Conditional on older men working, there is a high marginal effect of house prices on the probability that older women work. For younger and middle aged households, a rise in house price will insignificantly encourage the male partner to participate in the labour market, which can be explained by the fact that most of them plan to purchase a new house or upgrade their houses thus are "short" in housing and for this reason would be worse off and decrease their consumption of leisure when house prices rise. This wealth effect could be reinforced by the possibility that the younger households are borrowing constrained. This greater magnitude of effect for younger renters could be due to the possibility that renters are more borrowing constrained and more "short" in housing than owners who

[11] By "an average household", we refer to a household whose charateristics are the mean values of the corresponding sample, i.e., with the mean male/female age, mean number of children, mean wages, mean household non-labour income, etc.

want to upsize their houses[12]. For older households and all age, both male and female partners will be more likely to participate in the labour force given an increase in house prices. For example, a 10% increase in house price is associated with an increase of 0.2 in the probability of both working for older renter couples. This effect is bigger for their owner counterparts. Since the older home owning households are more likely to downsize their houses ("long" housing), this positive marginal effect cannot be explained by a pure wealth effect. Moreover, an increase of house prices should make them better off and might allow some of them to achieve their bequest targets earlier and possibly bring forward their retirement timing. Thus this positive effect could be attributed to a portfolio effect which makes old people want to invest more in financial assets to balance the increased weight of housing in the portfolio.

The house price effect is nowhere significant in Table 2.8. The young and middle aged are at prime working ages with increasingly stable careers so that their participation decisions are irrelevant to house price changes. Alternatively, the insignificant response could be explained by our finding in the appendix that house prices are stationary and the shocks of house prices would not have a lasting effect and for this reason those households with rational expectations that are not about to exit the housing market are not likely to respond to the temporary shocks of house prices (Browning et al., 2013). For older people, the insignificance could be due to the prevalence of retirement.

In Table 2.6 the better financial expectation dummy is significantly negative for the labour participation of young and middle aged males. It is possible that some of the housing appreciation expectation is contained in the better financial expectation dummy and it explains part of the wealth effect of housing. Unfortunately this effect cannot be extracted. Household non-labour income is significantly negative for the participation of males and females in all age groups and the whole sample. Interestingly an individual's wage generally has an insignificant negative effect on the individual's participation but his/her partner's wage has a more variable sign effect and is significant for some age groups (Table 2.7). One interpretation of the impact of non-labour income and wages could be in terms of income and substitution effects on the demand for leisure. If the labour supply is backward bending, the wage could have a negative effect on participation, while the partner's wage and real non-labour income could both act as household income sources which only have an income effect on participation. Poor health

[12] This is not necessarily true because we don't have information either on the renters' house purchase plan or on the owners' house upsizing plan.

status and very poor health status are significantly negative for both males and females in the whole sample. Number of children in the household is significantly negative for females at all ages and the whole sample. On the other hand, it has a positive mainly significant effect on male participation at all ages.

It should also be noticed that the marginal effects differ when evaluated at different values of regressors. For example, for an average young female, the probability of labour participation conditional on the male partner works are negative and the absolute value declines with the number of children in the household for both renters and owners (Figure 2.5).

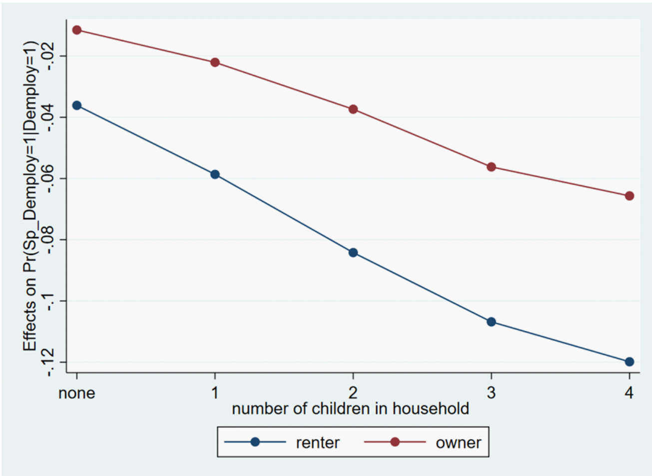


Figure 2.5: Marginal effect of house prices on conditional labour participation of young female

2.4.3 Robustness check

To check whether including movers between local authority districts, we re-estimate the model using the whole sample with movers and non-movers. Table 2.10 shows the wage equation estimates for the whole sample. Tables 2.11 and 2.12 show the estimated coefficients of household labour participation while Table 2.13 shows the marginal effect of house price on labour participation probability. The result is in general qualitatively different from the result with the non-movers sample. This means including movers in the sample does change the result.

Table 2.10: The wage equation estimates (movers and non-movers)

	All age (whole sample)	
Independent variables	male	female
Degree	-0.043*	-0.0068
	(-2.10)	(-0.27)
Hnd	-0.32***	-0.35***
	(-15.65)	(-13.59)
A level	-0.39***	-0.43***
	(-19.29)	(-17.78)
Gcse	-0.58***	-0.62***
	(-28.24)	(-24.47)
Age	0.073***	0.019***
	(22.46)	(5.76)
Age^2	-0.00078***	-0.00017***
	(-20.36)	(-4.31)
Regional claimant count rate	0.56	3.78***
	(1.04)	(6.61)
Ln (regional average earnings)	0.41***	0.44***
	(13.24)	(13.54)
Inverse mills ratio	-0.16***	-0.15***
	(-8.35)	(-9.82)
N	17733	18428

Note: Dependent variables: natural log of real wage of male and female partners. Additional independent variables not shown in the table: year dummies, local authority districts dummies.

t statistics in parentheses. * $p<0.05$,** $p<0.01$, *** $p<0.001$.

Table 2.11: Estimated coefficients for household labour participation (movers and non-movers) (to be continued on the next page)

	Aged 18-39		Aged 40-54		Aged 55-75		All age	
Independent variables	Male	Female	Male	Female	Male	Female	Male	Female
Renter*ln(real house price)	-0.026	0.051	1.07	-0.51	0.11	1.29**	-0.11	0.11
	(-0.04)	(0.15)	(1.20)	(-1.03)	(0.24)	(3.28)	(-0.40)	(0.54)
Owner*ln(real house price)	0.11	0.12	1.21	-0.43	0.14	1.29**	-0.012	0.15
	(0.16)	(0.34)	(1.35)	(-0.86)	(0.30)	(3.25)	(-0.04)	(0.77)
Age	0.12	0.21	0.95	0.61	-0.56	-0.38	0.22	0.27***
	(0.54)	(1.29)	(0.87)	(1.55)	(-0.76)	(-0.50)	(1.48)	(13.49)
Age^2	-0.0021	-0.0021	-0.011	-0.0070	0.0029	0.0018	-0.0029	-0.0033***
	(-0.82)	(-0.93)	(-0.88)	(-1.58)	(0.61)	(0.29)	(-1.84)	(-16.00)
Worse financal expectation	-0.025	-0.047	-0.27	-0.051	-0.025	0.11	-0.055	0.041
	(-0.13)	(-0.51)	(-1.32)	(-0.43)	(-0.32)	(1.53)	(-1.08)	(0.92)
Better financal expectation	-0.26**	0.023	-0.47**	-0.00089	-0.012	-0.047	-0.15***	0.030
	(-2.85)	(0.43)	(-3.29)	(-0.01)	(-0.12)	(-0.56)	(-3.37)	(1.08)
Married	0.39***	-0.085	-4.87***	0.24	0.67	-1.65	0.34***	-0.16***
	(3.67)	(-1.38)	(-10.80)	(0.47)	(0.39)	(-1.41)	(4.26)	(-3.35)
Degree	0.27	0.15	-0.41	-0.048	-0.46	0.033	-0.14	0.013
	(1.02)	(1.00)	(-0.18)	(-0.08)	(-0.48)	(0.04)	(-0.99)	(0.14)
Hnd	0.0049	-0.73*	-1.64	-0.50	-0.62	-0.91	-0.17	-0.76*
	(0.01)	(-2.30)	(-0.48)	(-0.18)	(-0.32)	(-0.36)	(-0.27)	(-2.41)
A level	-0.24	-0.83	-1.83	-0.69	-0.86	-1.62	-0.24	-1.01**
	(-0.20)	(-1.60)	(-0.46)	(-0.25)	(-0.41)	(-0.62)	(-0.32)	(-2.61)
Gcse	-0.44	-1.57*	-3.18	-1.41	-1.46	-2.67	-0.56	-1.79**
	(-0.26)	(-2.30)	(-0.60)	(-0.37)	(-0.53)	(-0.72)	(-0.49)	(-3.26)

Note: Dependent variables: participation dummies of male and female partners. Additional independent variables not shown in the table: year dummies, local authority districts dummies.

t statistics in parentheses. * $p<0.05$,** $p<0.01$, *** $p<0.001$.

Table 2.12: Estimated coefficients for household labour participation (movers and non-movers) (continued)

	Aged 18-39		Aged 40-54		Aged 55-75		All age	
Independent variables	Male	Female	Male	Female	Male	Female	Male	Female
Number of children	0.11*	-0.37***	0.026	-0.20***	0.29	-0.43*	0.071**	-0.25***
	(2.32)	(-12.11)	(0.37)	(-5.12)	(1.90)	(-2.12)	(3.14)	(-15.53)
Excellent health status	0.17	0.12	0.012	0.46***	0.37***	0.46***	0.25***	0.27***
	(1.32)	(1.59)	(0.06)	(4.99)	(4.07)	(5.70)	(5.10)	(7.07)
Good health status	0.11	0.075	0.19	0.31***	0.41***	0.28***	0.26***	0.15***
	(1.13)	(1.12)	(1.12)	(3.99)	(6.00)	(4.70)	(6.17)	(4.59)
Poor health status	-0.43*	-0.099	-0.30	-0.43***	-0.67***	-0.077	-0.44***	-0.23***
	(-2.45)	(-0.94)	(-1.01)	(-3.38)	(-3.55)	(-0.70)	(-6.13)	(-4.38)
Very poor health status	-0.92*	-0.38	-1.18**	-1.00**	-0.80	-0.43*	-0.83***	-0.48***
	(-2.25)	(-1.67)	(-2.99)	(-3.05)	(-1.94)	(-2.27)	(-5.32)	(-4.35)
Ln (real non-labour income)	-0.41***	-0.31***	-0.51***	-0.20***	-0.78***	-0.31***	-0.56***	-0.28***
	(-7.86)	(-12.01)	(-7.55)	(-6.93)	(-12.62)	(-12.08)	(-21.69)	(-22.61)
Ln (wage)	-0.60	-2.12	-4.81	-1.28	-1.39	-4.32	-0.49	-2.33**
	(-0.17)	(-1.43)	(-0.67)	(-0.26)	(-0.30)	(-0.83)	(-0.25)	(-2.62)
Ln (partner's wage)	0.76*	-0.13	-0.64*	0.017	-0.84***	-0.00092	-0.34***	-0.082
	(2.52)	(-0.75)	(-2.05)	(0.10)	(-5.99)	(-0.01)	(-3.52)	(-1.35)
Rho12	0.25***		0.17		0.32***		0.24***	
	(3.69)		(1.88)		(7.51)		(9.82)	
N	5781		4247		5706		18789	

Note: Dependent variables: participation dummies of male and female partners. Additional independent variables not shown in the table: year dummies, local authority districts dummies.

Rho12 is the correlation of the residuals from the participation equations of male and female partners. t statistics in parentheses. * $p<0.05$, ** $p<0.01$, *** $p<0.001$.

Table 2.13: Marginal effects of house price on labour participation (movers and non-movers)

		aged 18-39	aged 40-54	aged 55-75	All age
Pr(demploy=1)	renter	-0.00008	0.00006	0.02	-0.03
	owner	0.00001	0.0003	0.03	-0.002
Pr(sp_demploy=1)	renter	0.01	-0.004	0.2	0.04
	owner	0.02	-0.0008	0.18	0.05
Pr(1,1)	renter	0.01	-0.004	0.03	0.02
	owner	0.02	-0.0008	0.04	0.05
Pr(1,0)	renter	-0.01	0.004	-0.02	-0.05
	owner	-0.02	0.0007	-0.01	-0.05
Pr(0,1)	renter	0.00006	-5.97e-08	0.17	0.02
	owner	-7.72e-06	-8.97e-07	0.14	0.005
Pr(0,0)	renter	0.00001	-2.91e-07	-0.18	0.007
	owner	-5.25e-06	-8.97e-10	-0.17	-0.004
Pr(demploy=1|sp_demploy=1)	renter	-0.00006	-0.009	-0.06	-0.03
	owner	0.002	0.00003	-0.06	-0.004
Pr(sp_demploy=1|demploy=1)	renter	0.01	-0.004	0.37	0.05
	owner	0.02	-0.0008	0.33	0.05
N		5781	4247	5706	18789

2.5 Conclusion

This chapter attempts to analyse the impact of house prices on couples' labour supply at the extensive margin (participation). We impute wages for both workers and non-workers by adopting the Heckman selection model. We prove that with some specification of error structure for the wage equation and the participation equation, we can use the individual inverse Mills ratio to estimate wages. With the information of predicted wages, we further estimate the participation equation considering the interdependent nature of couple's labour supply with a bivariate probit model. Our setting of the two-equation system is an improvement on previous literature that apply a single-equation model to estimate individual labour supply, which may suffer from lack of behavioural appeal in the context of household decision making and loss of estimation efficiency and cannot predict joint participation and conditional participation.

The estimation results show that overall house prices do not have a significant effect on labour participation although the signs of marginal effects are heterogeneous for different age groups. The sensitivity of labour supply to house prices depends on homeownership, age and gender. More precisely, at the extensive margin, all age groups except for the young and middle aged females show increased participation in response to an increase of house prices. This is consistent with an increase of house prices leading to negative wealth effect on consumption of leisure for those young household who are likely to be "short" in housing with borrowing constraints. On the other hand, the increased participation probabilities for old households who are likely to be "long" in housing imply the wealth effect is very small, if present at all, and probably offset by other effects such as a strong bequest motive. Therefore, old people with a very strong preference for work may choose to work when they can find a job. In this case, the correlation between house prices and labour supply does not reflect a wealth effect of housing. We find that the house price process is stationary, which implies the shocks of house prices tend to fade away in the long run and for this reason households who have rational expectations on house prices and do not plan to exit the housing market immediately may not be affected by house prices variations.

Gender difference is present. First, there is no evidence of a wealth effect of housing on labour participation of young and middle aged female partners while the response of labour participation of their male partners reflects a wealth effect of housing. Second, the number of children in the household have significantly decreased likelihood to work facing a rise in house prices, which reflects the possibility that more of the commitments of family care are undertaken by female partners.

There are some limitations and potential extensions in this chapter. First, by specifying the two-equation system for male and female partners' labour supply without cross-equation restriction, we allow for different parameters associated with the two genders. But to what extent these parameters differ between equations for male and female needs to be further tested. Second, our modelling and estimation method are static in that we do not control for the individual time-invariant heterogeneity and the selection of housing tenure choice. On the other hand, while the house prices data on local authority district level exhibits rich variation across localities and over time, the period covered by our sample (12 years) may not be long enough to capture long term fluctuations of house prices which may be non-stationary and might affect household labour supply more significantly and differently in each age group. Finally, the implicit ad hoc assumption that every non-worker is voluntarily unemployed might be problematic if the labour market condition is not good enough to accommodate everyone who is willing to work. One solution is to model another hurdle (Blundell and Meghir, 1987).

Chapter 3

Housing and Financial Asset Allocations of Heterogeneous Homeowners

3.1 Introduction

Recognition of the theoretical and empirical importance of market constraints (especially borrowing and no-short-selling constraints, on household portfolio choices has stimulated research (Attanasio et al., 2012). The role of housing and housing finance in household portfolios is of special importance given the relative size, illiquidity and transaction costs involved (Campbell, 2006). The study of household finance is challenging because household behaviour is difficult to measure and complicated to model. Compared to corporate finance, household finance has some special features such as planning over long but finite horizons, having nontradable human capital and illiquid housing assets, and facing borrowing constraints. There is some empirical literature based on the framework of Merton (1973) where agents plan for the long term with time-varying investment opportunities (Campbell et al., 2003; Kim and Omberg, 1996), which emphasised the distinction between real and nominal returns in the long horizon models. But the Merton framework assumes wealth is liquid and tradable, which is in contradiction with nontradable human capital and illiquid housing. Moreover, as the biggest component of wealth for most households, human capital is nontradable because much of the labour income risk is idiosyncratic it is generally unhedgeable (Campbell, 2006). It represents a background risk which could make households more risk averse and invest more cautiously in other risky assets if the correlation between returns on these assets and labour income is positive (Heaton and Lucas, 2000). Or conversely if the correlation is negative. In addition, as an important largely indivisible asset for homeowners, illiquid housing may discourage investment in risky assets by homeowners leading to a crowding-out effect (Cocco, 2005). Adding borrowing constraints makes it even more complicated especially if it is not possible to exactly observe such constraints so that in a household survey we just do not know a priori if a household is constrained or not. Portfolio decisions (and consumption decisions) will generally differ between households which are borrowing constrained or unconstrained but also for other reasons. The estimation problem is that from the data we often cannot directly see who is constrained. External evidence suggests that borrowing constraints are typically more important for younger households who have not accumulated sufficient savings and have little or no housing wealth as collateral. Therefore there are some life cycle effects in financial strategies as households age and accumulate wealth.

The existing literature considers the complications of household finance in different ways. One branch of the literature derives numerical solutions to the housing and portfolio decisions in a life cycle framework with such constraints by calibration and simulation (Attanasio et al., 2012; Cocco, 2005; Yao and Zhang, 2005). However, the calibration of state variables is based on some dataset as a whole without considering possible heterogeneity among different groups of household. Here the calibration includes initial wealth as well as the parameters of stochastic processes (e.g. income process, house price process, risky asset return process)(Carroll, 2012). A second branch of the literature estimates the structural parameters (preference parameters) using Euler equations from a theoretical model with and without liquidity constraints (Zeldes, 1989; Whited and Wu, 2006). A third branch of the literature applies reduced form models to find empirical evidence about the impact of individual characteristics (e.g. financial illiteracy (Rooij et al., 2011) and income hedging motives (Bonaparte, et al., 2014)) on household portfolio choice.

A common feature of most of the literature stated above is the modelling of a typical household. That is, they analyse the average behaviour of the population. An exception is Zeldes (1989). Zeldes (1989) a priori selects a set of families that he believes to be not liquidity constrained in terms of wealth to income ratio. In his terms this subsample has all interior solutions to consumption, thus the Euler equation holds as an equality and the preference parameters can be estimated from the Euler equation. However, his analysis strongly relies on the ad hoc criterion used to split the sample into constrained and unconstrained groups: households with wealth to income ratio above a certain threshold are not liquidity constrained and vice versa. In comparison, our paper does not assume any certain criterion to split the sample. Instead, we try to see if the data on the multiple assets holdings gives probabilistic splits of the sample and then explore the behaviour of households derived from the probabilistic split. Another exception that considers the heterogeneous intrinsic nature of subsamples is King and Leape (1998), who estimate the joint discrete and continuous choice of household portfolios by a switching regression model. In their model, both the discrete choice of owning particular combinations of assets and the continuous choice of asset demand system conditional on ownership are parametrised by a set of household characteristics. Besides the different focuses of research, there are two main differences between their model and ours. First, our model studies the asset allocation behaviour at both the extensive and intensive margins at the same time via a censored data EM algorithm, while their model studies the extensive and intensive margins in two steps. Second, as opposed to their fully parametric model, our paper is only semi-parametric in the sense that the classification of households in terms of asset allocations is unconditional, which has the advantage of being more flexible and circumventing possible endogeneity brought by covariates.

Specifically, this paper aims to investigate housing and financial asset allocations decisions (hereafter, asset allocations) by heterogeneous homeowners with a flexible model motivated by economic theory. We find distinct patterns of unconditional housing and financial assets allocation among homeowners by fitting a multivariate Gaussian mixture model via a censored data expectation-maximisation (EM) algorithm. Considering the choices of different assets are made simultaneously, the Gaussian mixture model we fit has a multivariate nature. The existence of a no-short-selling constraint on risky asset motivates the use of the censored data EM algorithm. The assumptions in our mixture model are minimal in the sense that we only assume a multinomial distribution for the component membership indicators and a mixture of multivariate normal distributions for housing and the two other assets although we estimate the unknown component density parameters. That is, neither mixing weights nor the mean of each asset is parametrised. This allows flexibility for the data to talk by avoiding possible spurious inclusion of covariates and subsequent endogeneity bias. The choice of the number of components is based on the economic intuition from a theoretical model presented in Section

2. After finding the chances of a household being in different regimes (mixture components) we want to understand which households are assigned to which mixture components (regimes) and how this aligns with the theoretical regimes. Descriptive statistics are presented to describe the features of each group. Then a linear regression is implemented to find the determinants of group membership. The results are encouraging: we use the number of components/regimes suggested by the theory and find quite strong sample separation into these, the no-short-selling constraint on risky financial assets clearly binds in the poorer lower two components but is slack in the two richer components. Within the components where risky financial asset constraints either do or do not bind, there is evidence that the subdivision into two further components (somewhat weakly) supports households who are mortgage borrowing constrained or unconstrained. Our model is within the broader field of latent class models[1] where the discrete and finite latent variables in our model can be interpreted as heterogeneity in initial wealth and preferences (e.g. different marginal utility, expectation, risk aversion, etc.), and household idiosyncratic shocks.

Our empirical results contribute to the literature on structural models on household finance. The idea is to use the state variable values of the component such as the initial wealth to calibrate the model for simulation. On the other hand, it sheds light on empirical work that tries to estimate heterogeneous household behaviour by giving a basis for classifying observations into different latent classes. This paper also makes a preliminary attempt to understand the household characteristics that may be required to apply the structural model to capture the observed differences in portfolio allocation among households. The rest of the paper is organised as follows. Section 2 shows the theoretical motivation of this paper. Section 3 describes the Wealth and Asset Survey (WAS) data we use. Section 4 presents the econometric model, estimation method and algorithm we use to estimate the model. Section 5 reports the estimation results. Finally, Section 6 concludes.

[1] Latent class models are also referred to as unsupervised learning in the field of machine learning.

3.2 Theoretical motivation

3.2.1 A model for stable homeowners

Considering the infrequency of housing purchase observed from the data (Section 3), we focus on the behaviour of stable homeowners who own their main residence only[2] and don't move during the timespan we study. For these stable homeowners, housing consumption is constant across time. For this reason the tenure choice of housing (whether to be a renter or homeowner) and the decision to upsize or downsize the house are beyond the scope of this paper. We formulate the model as follows.

Families are treated as forward looking. Families can access financial markets. There are three: there is a safe asset with a known one period return on asset r_{at}; there is a risky asset with an return $r_{f,t+1}$ that is only realised at the end of period t (at the beginning of period $t+1$) after the investment in period t is made. And there is also a housing mortgage debt. In period t, families face uncertainty in general about future income y_{t+1}, house prices p_{t+1}, return on the risky asset $r_{f,t+1}$. $y_{t+1}, r_{f,t+1}, p_{t+1}$ are random and only realised at the start of the next period $t+1$. On the other hand, the return on safe asset $r_{a,t+1}$ is time-varying but non-random and known by the households[3]. Family utility in period t depends on a composite consumption c_t and utility derived from their present housing quantity H_t. Figure 3.1 shows the timeline of the model.

The holding of the safe asset at t is X_{st}. Holdings of equities in t are $F_t \geq 0$ since borrowing in equity is infeasible (short selling is not allowed). The mortgage interest rate $r_{m,t+1}$ is realised after the mortgage of period t has been taken. There are two borrowing constraints associated with the mortgage: the loan-to-value ratio constraint and the loan-to-income ratio constraints:

$$M_t \leq min[\tau_1 p_t H_t, \tau_2 y_t]$$

For simplicity, now suppose we can assume $r_{mt} = r_{at}$ (Attanasio et al., 2012) and borrowing in the safe asset is only possible via a mortgage. Define net safe assets $A_t = X_{st} - M_t$ where $X_{st} \geq 0, M_t \leq min[\tau_1 p_{bt} H_t, \tau_2 y_t]$ which means that $A_t \geq \max[-\tau_1 p_t H_t, -\tau_2 y_t]$[4].

[2] We exclude households owning buy-to-let properties. Therefore households cannot get any rental income from the houses they own.

[3] We assume households have rational expectations.

[4] $A_t \geq \max[-\tau_1 p_{bt} H_t, -\tau_2 I_t]$ is the necessary but not sufficient condition of $M_t \leq min[\tau_1 p_{bt} H_t, \tau_2 I_t]$.

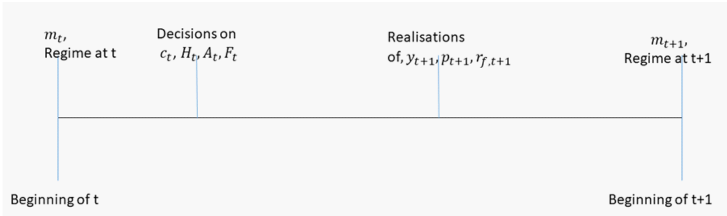


Figure 3.1: Timeline of the model

Suppose families live for T periods. For $t \leq T$, families have a time additive expected utility life cycle objective

$$\Sigma_t \beta^t E_t U_t(c_t, H_t)$$

where $U_t(.)$ is the per-period utility function at time t, β is the constant rate of time preference, the expectations operator E_t is taken when any of future house prices, asset returns and income flows are uncertain.

We assume a composite consumption good c_t with the price normalised to 1 in each period t. As a result, in each period, monetary variables including return on assets, labour income, rent and house prices are expressed as a ratio of the consumption price. In other words, every monetary variable is in real terms.

We write the household value function in period t in recursive form from the Bellman equation as:

$$V_t(m_t) = \max_{\{A_t, F_t\}} U_t(c_t, H_t) + \beta E_t V_{t+1}(m_{t+1})$$

subject to

$$\begin{aligned} m_t &= c_t + A_t + F_t + p_t H_t \\ m_{t+1} &= y_{t+1} + (1 + r_{a,t+1}) A_t + (1 + r_{f,t+1}) F_t + p_{t+1} H_t \\ F_t &\geq 0 \\ A_t &\geq \max[-\tau_1 p_t H_t, -\tau_2 y_t] \end{aligned}$$

Notice the stable homeowners are "locked" in housing consumption in the sense that they are making decisions as if housing consumption H_t is not a choice variable for them. In other words, their decisions for asset allocations are conditional on their unchanged housing consumption $H_t = \overline{H}$. On the other hand, the values of their total housing wealth $p_t H_t$ could well change through time due to the change of house prices p_t. And via A_t mortgage debt can change.

Forming the KuhnTucker Lagrangian

$$\begin{aligned} L &= U_t[(m_t - A_t - F_t - p_t H_t), H_t] + \\ & \beta E_t V_{t+1}[y_{t+1} + (1 + r_{a,t+1})A_t + (1 + r_{ft+1})F_t + p_{t+1}H_t] \\ & + \lambda_{1t}(A_t + \tau_1 p_t H_t) + \lambda_{2t}(A_t + \tau_2 y_t) + \lambda_{3t} F_t \end{aligned}$$

The Envelope Theorem gives (Carroll, 2017)

$$\begin{aligned} V_t'(m_t) &= \frac{\partial U_t}{\partial c_t} \\ V_{t+1}'(m_{t+1}) &= \frac{\partial U_{t+1}}{\partial c_{t+1}} \end{aligned}$$

The first order conditions are

$$F_t : -\frac{\partial U_t}{\partial c_t} + \beta E_t(1 + r_{ft+1})\frac{\partial U_{t+1}}{\partial c_{t+1}} + \lambda_{3t} = 0 \tag{3.1}$$

$$A_t : -\frac{\partial U_t}{\partial c_t} + \beta(1 + r_{a,t+1})E_t\frac{\partial U_{t+1}}{\partial c_{t+1}} + \lambda_{1t} + \lambda_{2t} = 0 \tag{3.2}$$

3.2.1.1 Evolution and cross-sectional variation of assets allocations

For notational convenience, hereafter we denote the real value of housing $p_t H_{it}$ for household i at period t as G_{it}:

$$G_{it} = p_t H_{it}$$

G_{it} is H_{it} scaled by the house price p_t which is assumed to be the same for every household in each period (Law of one price). This assumption is made along with the usual assumption that housing quantity H_{it} not only represents the physical size of the house but also the quality of the house[5]. In other words, p_t is just a universal conversion factor to translate H_{it} into the observable monetary value G_{it}.

Notice that the argument of the value function $V_{t+1}(m_{t+1})$ is

$$\begin{aligned} m_{t+1} &= y_{t+1} + (1 + r_{a,t+1})A_t + (1 + r_{f,t+1})F_t + p_{t+1}H_t \\ &= y_{t+1} + (1 + r_{a,t+1})A_t + (1 + r_{f,t+1})F_t + \frac{p_{t+1}}{p_t}G_{it} \end{aligned}$$

[5]Here the quality of the house includes the location etc.

which suggests that the direct effect of the current period t on the next period value function V_{t+1} is not from c_t or m_t, but from (G_{it}, A_t, F_t) (Carroll, 2012). That is, for each household i, the vector (G_{it}, A_{it}, F_{it}) is a sufficient statistic which captures all the information from the current period t that is needed to solve the intertemporal maximisation problem in future periods. Thus the evolution of (G_{it}, A_{it}, F_{it})[6] may reflect a combination of the changes in planning horizon[7], updates of random state variables $y_{i,t+1}$, $r_{f,t+1}$, p_{t+1},changes of expectations about the future y_{t+1}, $r_{f,t+1}$, p_{t+1}[8] and even preference parameters such as risk aversion and subsistence level of consumption[9].

Households have different (G_{it}, A_{it}, F_{it}) because of household-specific (τ_{1i}, τ_{2i}) associated with the borrowing constraints, individual idiosyncratic income process, different initial wealth m_{it}, different ages and hence different planning horizons and different preference parameters.

Putting this theoretical setting into a statistical framework, given the timespan in our data is short (only 3 waves, i.e. 6 years), it means the cross sectional variation of (G_{it}, A_{it}, F_{it}) at each period t is likely to be more significant than the time series variation for each household. This is why we analyse the data by wave later in this paper[10].

3.2.1.2 Possible asset allocation regimes

From equations (3.1) and (3.2), we can see that there are four possible solution regimes in which different sets of constraints bind or are slack for stable homeowners (Table 3.1). For a particular household, the graph of regimes are shown in Figure 3.2.

It is tempting to identify the signs of λ_1 and λ_2 (Lagrangian multipliers associated with loan-to-value ratio constraint and loan-to-income ratio constraint) so that we can divide the sample into regimes and use proper econometric specification. However, if we just rely on the first order conditions from the theoretical model, then it is hard to distinguish the error terms of

[6] The evolution of (G_{it}, A_{it}, F_{it}) for stable homeowners where $G_{it} = p_t H_{it}$ can be seen as the evolution of (A_{it}, F_{it}) with $H_{it} = \overline{H_i}$ being time-invariant. However, we can also view H_{it} as endogenous and rational and the optimal choice is not to change H_{it} in the timespan (3 waves) we consider here.

[7] In a finite horizon setting, the policy functions change through time. But in our data which only covers 3 waves (6 years), this effect for a household should be less important.

[8] This might be due to the change of the general macro economic environment.

[9] The change of subsistence level of consumption may happen when there are new children born in the family or when children grow up and leave the family.

[10] There is also the consideration of meeting the assumption that datapoints are i.i.d. for the Gaussian mixture model.

Table 3.1: Possible solution regimes for stable homeowners

	No-short-selling constrained ($\lambda_{3t} > 0$)	Borrowing constrained ($\lambda_{1t} > 0$ or $\lambda_{2t} > 0$)
Regime 1	Yes	Yes
Regime 2	Yes	No
Regime 3	No	Yes
Regime 4	No	No

the moment conditions and λ_1 and/or λ_2 unless by making very strong parametric assumptions for λ_1 and λ_2 (Whited and Wu, 2006) and assuming that λ_1 and λ_2 are independent of the errors of the moment conditions. In this paper, instead of trying to identify the signs of λ_1 and λ_2, we use Taylor approximation to obtain non-parametric solutions for each regime and try to identify the corresponding joint distribution of (G_i, A_i, F_i) for each regime, allowing for household heterogeneity which explains the cross-sectional variations of asset allocations across households. Without loss of generality, let the decision rules for F_i and A_i in Regime x for household i at each period be[11].

$$A_i = A_x(\boldsymbol{\varepsilon}_i)$$

$$F_i = F_x(\boldsymbol{\varepsilon}_i)$$

where $\boldsymbol{\varepsilon}_i = [\varepsilon_{1i}, \varepsilon_{2i}, \varepsilon_{3i}]$, ε_{1i} is heterogeneity in initial wealth, ε_{2i} is heterogeneity in preferences, ε_{3i} is household idiosyncratic shocks.

One challenge to link the empirical model with the theoretical model is that τ_1, τ_2 may be partially individual specific[12] and are only partially observable to the econometrician. In other words, looking at the data, we have no exact idea whether the borrowing constraints (either one of the loan-to-value and loan-to-income constraints or both) are binding. For a particular household, we can see from Figure 3.4 (Section 3) that there is some boundary (upper limit) for the loan-to-value ratio. And in reality, choice of loan value affected by the loan-to-value constraint may affect the loan-to-income constraint and such restrictions are imposed differently by different lenders[13]. This complicated relationship between loan-to-value and

[11] The time subscript t is omitted here since we consider the cross-sectional variation in the same period.

[12] The values of τ_1 and τ_2 depend on occupation, income, age, credit record, etc., but the information in the dataset is not sufficient to reflect exact τ_1, τ_2.

[13] One example is in 2015, the loan-to-income restriction imposed by Barclays is initially 4.5 for all, later relaxed to 4.5 if loan value <£300000 and 5 if >£30000. (Bank of England, July 2015, Financial Stability Report)

loan-to-income constraints contributes to the unknown nature of borrowing constraint faced by heterogenous households in the sense that τ_2 is not only unobservable but also "endogenous" depending on personal preference for lenders and choice of mortgage. For this reason we change the borrowing constraint in the optimisation problem to

$$A_i \geq B_x(\boldsymbol{\varepsilon}_i)$$

where B_x is a function of household heterogeneity and shocks whose functional form is regime specific[14].

Assume the expectation over observations who belong to regime x $E_x(\boldsymbol{\varepsilon}_i) = 0$. Taking a first order Taylor expansion around $E_x(\boldsymbol{\varepsilon}_i) = 0$ yields

$$\begin{aligned} A_i|\text{Regime x} &\sim A_x(E_x(\boldsymbol{\varepsilon}_i)) + [\boldsymbol{\varepsilon}_i - E_x(\boldsymbol{\varepsilon}_i)]\nabla A_x(E_x(\boldsymbol{\varepsilon}_i)) \\ &= A_x(m_i, E_x(\boldsymbol{\varepsilon}_i)) + \boldsymbol{\varepsilon}_i \nabla A_x(E_x(\boldsymbol{\varepsilon}_i)) \end{aligned}$$

$$\begin{aligned} F_i|\text{Regime x} &\sim F_x(E_x(\boldsymbol{\varepsilon}_i)) + [\boldsymbol{\varepsilon}_i - E_x(\boldsymbol{\varepsilon}_i)]\nabla F_x(E_x(\boldsymbol{\varepsilon}_i)) \\ &= F_x(E_x(\boldsymbol{\varepsilon}_i)) + \boldsymbol{\varepsilon}_i \nabla F_x(E_x(\boldsymbol{\varepsilon}_i)) \end{aligned}$$

where $\nabla A_x(E_x(\boldsymbol{\varepsilon}_i)) = \begin{bmatrix} \frac{\partial A_x}{\partial \varepsilon_1}|_{\varepsilon_1 = E_x(\varepsilon_1)} \\ \frac{\partial A_x}{\partial \varepsilon_2}|_{\varepsilon_2 = E_x(\varepsilon_2)} \\ \frac{\partial A_x}{\partial \varepsilon_3}|_{\varepsilon_3 = E_x(\varepsilon_3)} \end{bmatrix}, \nabla F_x(E_x(\boldsymbol{\varepsilon}_i)) = \begin{bmatrix} \frac{\partial F_x}{\partial \varepsilon_1}|_{\varepsilon_1 = E_x(\varepsilon_1)} \\ \frac{\partial F_x}{\partial \varepsilon_2}|_{\varepsilon_2 = E_x(\varepsilon_2)} \\ \frac{\partial F_x}{\partial \varepsilon_3}|_{\varepsilon_3 = E_x(\varepsilon_3)} \end{bmatrix}.$

Note that in Regimes 1 and 2, the decision rules for F_i are reduced to

$$F_i|\text{Regime x} = 0$$

where x=1,2.

Notice here the observed F_i in regimes 1 and 2 is a mass point at 0, while the latent counterpart F_i^* is the solution obtained as if the no-short-selling constraint $F_i \geq 0$ is not present, i.e.

$$F_i^*|\text{Regime 3} \sim F_{3i}(E_x(\boldsymbol{\varepsilon}_i)) + \boldsymbol{\varepsilon}_i \nabla F_{3i}(E_x(\boldsymbol{\varepsilon}_i)) \leq 0$$

$$F_i^*|\text{Regime 4} \sim F_{4i}(E_x(\boldsymbol{\varepsilon}_i)) + \boldsymbol{\varepsilon}_i \nabla F_{4i}(E_x(\boldsymbol{\varepsilon}_i)) \leq 0$$

On the other hand, in Regimes 1 and 3, the decision rules for A_i are reduced to

[14] The regime-specific functional form B_x is a result of normalisation of $E_x(\varepsilon_i) = 0$ for all regimes.

Table 3.2: Approximation of decision rules in each solution regime

	A_i	F_i	F_i^*
Regime 1	$B_1(E_x(\varepsilon_i)) + \varepsilon_i \nabla B_1(E_x(\varepsilon_i))$	0	$F_1(E_x(\varepsilon_i))$ $+\varepsilon_i \nabla F_1(E_x(\varepsilon_i)) \leq 0$
Regime 2	$A_2(E_x(\varepsilon_i)) + \varepsilon_i \nabla A_2(E_x(\varepsilon_i))$	0	$F_2(E_x(\varepsilon_i))$ $+\varepsilon_i \nabla F_2(E_x(\varepsilon_i)) \leq 0$
Regime 3	$B_3(E_x(\varepsilon_i)) + \varepsilon_i \nabla B_3(E_x(\varepsilon_i))$	$F_3(E_x(\varepsilon_i)) + \varepsilon_i \nabla F_3(E_x(\varepsilon_i))$	$F_3(E_x(\varepsilon_i))$ $+\varepsilon_i \nabla F_3(E_x(\varepsilon_i)) > 0$
Regime 4	$A_4(E_x(\varepsilon_i)) + \varepsilon_i \nabla A_4(E_x(\varepsilon_i))$	$F_4(E_x(\varepsilon_i)) + \varepsilon_i \nabla F_4(E_x(\varepsilon_i))$	$F_4(E_x(\varepsilon_i))$ $+\varepsilon_i \nabla F_4(E_x(\varepsilon_i)) > 0$

Note: For regime x, $G_x \sim G_x(E_x(\varepsilon_i)) + \varepsilon_i \nabla G_x(E_x(\varepsilon_i))$

$$A_i|\text{Regime x} \sim B_x(E_x(\boldsymbol{\varepsilon}_i)) \ + \boldsymbol{\varepsilon}_i \nabla B_x(E_x(\boldsymbol{\varepsilon}_i))$$

$$\text{where x=1,3,} \nabla B_x(E_x(\boldsymbol{\varepsilon}_i)) = \begin{bmatrix} \frac{\partial B_x}{\partial \varepsilon_1}|_{\varepsilon_1 = E_x(\varepsilon_1)} \\ \frac{\partial B_x}{\partial \varepsilon_2}|_{\varepsilon_2 = E_x(\varepsilon_2)} \\ \frac{\partial B_x}{\partial \varepsilon_3}|_{\varepsilon_3 = E_x(\varepsilon_3)} \end{bmatrix}.$$

In summary, the approximation of decision rules in each solution regime are shown in Table 3.2. Meanwhile, though housing quantity stays constant for each household in this model, housing wealth varies across households. To capture sufficient information of a household in a period, we further assume the self-reported housing wealth is a function of $\boldsymbol{\varepsilon}_i$ with functional form G_x which varies with regime x[15].

$$G_x(\boldsymbol{\varepsilon}_i) \sim G_x(E_x(\boldsymbol{\varepsilon}_i)) + \boldsymbol{\varepsilon}_i \nabla G_x(E_x(\boldsymbol{\varepsilon}_i))$$

$$\text{where } \nabla G_x(E_x(\boldsymbol{\varepsilon}_i)) = \begin{bmatrix} \frac{\partial G_x}{\partial \varepsilon_1}|_{\varepsilon_1 = E_x(\varepsilon_1)} \\ \frac{\partial G_x}{\partial \varepsilon_2}|_{\varepsilon_2 = E_x(\varepsilon_2)} \\ \frac{\partial G_x}{\partial \varepsilon_3}|_{\varepsilon_3 = E_x(\varepsilon_3)} \end{bmatrix}.$$

[15] The regime-specific functional form G_x is a result of normalisation of $E_x(\varepsilon_i) = 0$ for all regimes.

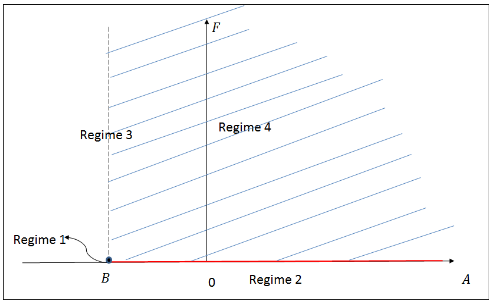


Figure 3.2: Possible solution regimes in the A-F space

Note: The point where A=B and F=0 is Regime 1. The bold line is Regime 2. The dotted line is Regime 3. The shaded area is Regime 4.

In order to derive the joint distribution of $(G_x, A_x, F_x^*)^T$ in each solution regime x, we first assume the joint distribution of household heterogeneity and shocks.

Assume the 3×1 vector $\boldsymbol{\varepsilon}_i^T$ of heterogeneity effects and shocks for each household i is drawn from a multivariate normal distribution

$$\boldsymbol{\varepsilon}_i^T \tilde{} N(\mathbf{0}, \Omega) \tag{3.3}$$

where $\Omega = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_3^2 \end{bmatrix}$ is the covariance matrix for the heterogeneity and shocks.

Based on the normality assumption (3.3) and the approximation of decision rules in each solution regime as shown in Table 3.2, the joint distribution of $(G_x, A_x, F_x^*)^T$ for regime x is

$$\begin{bmatrix} G_x \\ A_x \\ F_x^* \end{bmatrix} \tilde{} N(\boldsymbol{\mu}_x, \Sigma)$$

where $\boldsymbol{\mu}_x = [G_x(E_x(\boldsymbol{\varepsilon}_i)), A_x(E_x(\boldsymbol{\varepsilon}_i)), F_x^*(E_x(\boldsymbol{\varepsilon}_i))]^T$,

$$\boldsymbol{\alpha}_x = \begin{bmatrix} \nabla G_x(E_x(\boldsymbol{\varepsilon}_i)) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \nabla A_x(E_x(\boldsymbol{\varepsilon}_i)) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \nabla F_x(E_x(\boldsymbol{\varepsilon}_i)) \end{bmatrix},$$

$$\Sigma = \boldsymbol{\alpha}_x^T \Phi \boldsymbol{\alpha}_x, \Phi = \begin{bmatrix} \Omega & \Omega & \Omega \\ \Omega & \Omega & \Omega \\ \Omega & \Omega & \Omega \end{bmatrix},$$

Note that if $x = 1, 3$, then $A_x(E_x(\boldsymbol{\varepsilon}_i)) = B_x(E_x(\boldsymbol{\varepsilon}_i))$ and $\nabla A_x(E_x(\boldsymbol{\varepsilon}_i)) = \nabla B_x(E_x(\boldsymbol{\varepsilon}_i))$.

3.3 Data

The data applied in this paper is from Wealth and Asset Survey (WAS). WAS is a longitudinal survey, including information on holdings of various types of assets (savings, share investments, property wealth, mortgage, pension, etc.), and different sources of income flows (labour income, benefit income, pension income, etc.) of households in Great Britain. Currently there are three waves of data available (2006-2012) with each wave covering two years. In addition, demographic variables such as age, education qualification, household characteristics are available in the dataset. The advantage of this dataset is the comprehensive information on multiple asset stocks. This makes it feasible to study household finance decisions. As we don't study the tenure decisions in this paper, we select only homeowners for analysis.

3.3.1 Conceptual definition

For purpose of estimation, we group some important financial assets into the following 2 categories.

1. Net risk-free asset (A):

The risk-free asset includes household value of cash ISA, household value of national savings Product, household value of savings accounts, and household value of current accounts in credit. Net risk-free asset is defined as risk-free asset net of mortgage.

2. Risky asset (F):

Risky asset includes household value of Investment ISA, household value of UK Shares, household value of employee shares, household value of fixed term investment bonds, and household value of unit investment trusts.

All the monetary variables are converted to real values by Retail Price Index (RPI) setting the year 2006 as the base year.

3.3.2 Infrequency of home purchase decisions

Previous literature suggests an average time between house purchases is 20 to 30 years with a conservative estimate of transaction cost of 5% of value of the house sold (Grossman and Laroque, 1990). Statistical evidence shows that the annual average turnover of housing stock in the UK has fallen from over 12% in 1980s to 4.5% in 2010s. This means, on average, houses changed hands once every 8 years in 1980s and every 23 years now[16]. Flavin and Yamashita (2002) argue that though housing purchase decision is endogenous and rational, it is infrequent due to transaction cost. And in our data, the majority of the homeowners (over 97%) are non-movers.

3.3.3 Sample selection, household wealth and demographics

Table 3.3 reports the summary statistics for key variables in the sample of all the homeowners, which is a balanced panel tracing 8067 homeowners across three waves (24201 observations in total). Table B.1 in the Appendix shows a detailed description of all these variables. The age range for all the homeowners is quite big, ranging from 21 to 101. To exclude the impact of pension income on asset allocations and focus on behaviours of non-movers, we further select households aged under 65, not retired without pension income and did not move homes during the three waves, which is a balanced panel tracing 2593 homeowners across three waves (7779 observations in total). Among the stable homeowners under 65, only 19% own their homes outright while the majority (81%) have mortgages, compared to 58% owning their homes outright and 42% having mortgages among all the homeowners. About 37% of the observations have risky assets. This participation rate in risky asset is highest in wave 2 (39%) and lowest in wave 3 (35%), while Table 3.3 shows the 50th percentile of risky asset holding in the whole sample of homeowners is positive. This means for the stable homeowners under 65 both the borrowing constraints and the no-short-selling constraint probably have a more important role to play compared with older households. Table 3.4 presents the summary statistics for these households. Figure 3.3 shows the histograms of housing wealth, net safe asset and risky asset for these households[17]. The distribution of housing value, net safe asset and risky asset are all skewed with long tails. Table 3.5 reports the correlations among age, education and

[16] Source: The new 'normal'- one year on (Is the march back to a sustainable market on track?), April 2015, Intermediary Mortgage Lenders Associations (imla) Report

[17] For the figure to be more presentable, we exclude the top 10% for each histogram.

asset holdings. Figure 3.4 is a scatter plot of mortgage against house value, where the red line represents the combinations of mortgage and house values with the loan-to-value ratio equal to 90%. We can see that while the loan-to-value ratio varies among households, most of the observations are below the red line. This is consistent with the financial practice. The correlation coefficients show that as households age, all the asset holdings rise. Housing wealth is positively correlated with both safe and risky assets, but negatively correlated with the net safe asset, which is probably due to the bigger positive correlation between housing wealth and mortgage. Having a degree is positively correlated with housing wealth and risky asset.

Table 3.3: Summary statistics for all the homeowners from 2006-2012

			Coefficient of		Percentiles	
Variable	Observations	Mean	variation	25th	50th	75th
employ	24201	3.61	0.80	1.00	2.00	7.00
nkids	24201	0.43	1.96	0.00	0.00	0.00
degree	24201	0.30	1.53	0.00	0.00	1.00
quali	24201	0.52	0.96	0.00	1.00	1.00
Age	24201	58.41	0.25	47.00	59.00	70.00
marital	24201	2.08	0.75	1.00	1.00	3.00
totHval	24201	259205.80	0.97	134240.10	191771.50	294791.70
A	24201	-333.58	-380.92	-41354.22	5072.39	32924.44
cash	24201	37809.20	2.45	3211.29	13144.87	40000.00
mortgage	24201	38142.78	2.10	0.00	0.00	52568.43
risky	24201	38672.60	3.97	0.00	88.60	22248.01
hhNetFin	24201	83106.23	3.07	4313.18	24850.68	82907.62
GrossEmploy	24201	20778.50	1.53	0.00	9794.81	33808.89
GrossSE	24201	1940.26	5.70	0.00	0.00	0.00
Invest	24201	1822.34	5.91	0.00	48.73	664.47
income	24201	25024.18	1.47	0.00	14400.00	39918.80
lvratio	24201	0.17	1.56	0.00	0.00	0.29
hhsize	24201	2.30	0.51	2.00	2.00	3.00
bedrooms	24201	3.14	0.31	3.00	3.00	4.00
hsetype	24201	1.37	1.55	1.00	2.00	2.00

Table 3.4: Summary statistics for stable homeowners under 65 from 2006-2012

			Coefficient of		Percentiles	
Variable	Observations	Mean	variation	25th	50th	75th
employ	7779	1.32	0.76	1.00	1.00	1.00
nkids	7779	0.97	1.10	0.00	1.00	2.00
degree	7779	0.33	1.42	0.00	0.00	1.00
quali	7779	0.58	0.86	0.00	1.00	1.00
Age	7779	44.65	0.20	38.00	45.00	51.00
marital	7779	1.95	0.77	1.00	1.00	3.00
totHval	7779	200661.00	0.66	120000.00	166860.10	239714.40
A	7779	-44660.75	-1.82	-84316.98	-41041.96	12.24
cash	7779	17803.36	2.44	1052.83	4897.40	16651.17
mortgage	7779	62464.11	1.03	14318.45	49613.59	92225.33
risky	7779	12483.43	5.53	0.00	0.00	2876.57
hhNetFin	7779	35762.21	2.65	221.14	8840.79	37642.45
GrossEmploy	7779	35502.52	0.87	17664.38	31833.13	48008.09
GrossSE	7779	2276.80	4.13	0.00	0.00	0.00
Invest	7779	540.72	5.43	0.00	9.59	185.30
income	7779	41678.97	0.82	22200.00	37104.00	54700.13
lvratio	7779	0.35	0.86	0.08	0.31	0.56
hhsize	7779	2.93	0.44	2.00	3.00	4.00
bedrooms	7779	3.13	0.29	3.00	3.00	4.00
hsetype	7779	1.58	1.34	1.00	2.00	3.00
north_esat	7779	0.03	5.30	0.00	0.00	0.00
north_west	7779	0.14	2.49	0.00	0.00	0.00
yorkshire_humb	7779	0.12	2.69	0.00	0.00	0.00
east_mid	7779	0.10	2.96	0.00	0.00	0.00
west_mid	7779	0.10	2.99	0.00	0.00	0.00
east_england	7779	0.12	2.66	0.00	0.00	0.00
london	7779	0.09	3.26	0.00	0.00	0.00
south_east	7779	0.16	2.25	0.00	0.00	0.00
south_west	7779	0.06	3.93	0.00	0.00	0.00
wales	7779	0.07	3.75	0.00	0.00	0.00

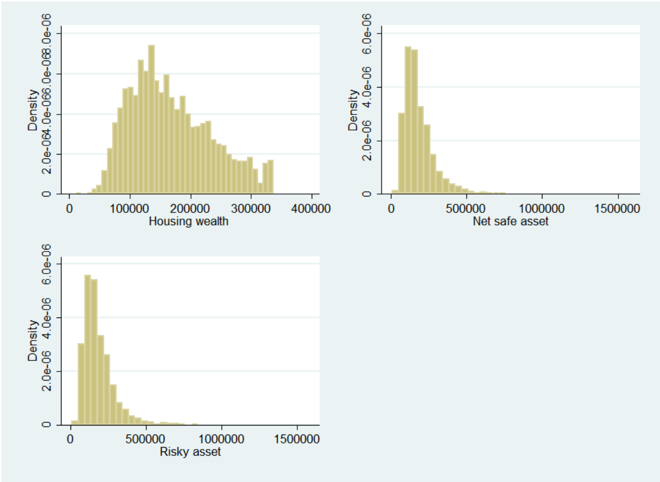


Figure 3.3: Histograms of housing wealth, net safe asset and risky asset for stable homeowners under 65

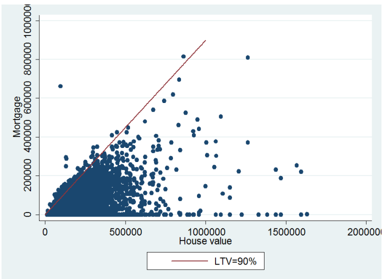


Figure 3.4: Scatter plot of mortgage vs. house value

Table 3.5: Correlation matrix for stable homeowners under 65

Variables	Age	Age2	degree	risky	totHval	A	cash	mortgage	income	lvratio
Age	1.00									
Age2	0.99	1.00								
degree	-0.08	-0.09	1.00							
risky	0.11	0.11	0.09	1.00						
totHval	0.14	0.13	0.25	0.25	1.00					
A	0.38	0.38	-0.05	0.18	-0.08	1.00				
cash	0.17	0.18	0.15	0.26	0.36	0.62	1.00			
mortgage	-0.36	-0.37	0.16	-0.06	0.35	-0.85	-0.11	1.00		
income	0.01	0.00	0.25	0.13	0.40	-0.08	0.23	0.25	1.00	
lvratio	-0.53	-0.52	0.03	-0.13	-0.18	-0.68	-0.24	0.70	0.03	1.00

3.4 The econometric model

3.4.1 A multivariate Gaussian mixture model for asset allocation patterns

We aim to estimate the asset allocation patterns. Specifically, we try to fit the data on asset holdings with a multivariate Gaussian mixture model via a censored data EM algorithm. We proceed with the assumption for a multivariate Gaussian mixture in a clustering context that any nonnormal features in the data result from some underlying group structure (McLachlan and Peel, 2000). We will illustrate the necessity of using a censored data EM algorithm rather than the widely applied standard EM algorithm.

3.4.1.1 A standard EM algorithm for a multivariate Gaussian mixture model

Let $\boldsymbol{y} = (\boldsymbol{y}_1, \boldsymbol{y}_2, ..., \boldsymbol{y}_N)$ be a set of independently and identically distributed (i.i.d.) observations on a d-dimensional space R^d. In our case, $d = 3$ and

$$\boldsymbol{y}_n = (G_n, A_n, F_n^*)^T$$

where G_n, A_n are observed housing wealth and net safe asset for household n, and F_n^* is the latent counterpart of observed risky asset F_n with the observation rule as follows[18].

$$\begin{aligned} F_n &= F_n^* \text{ if } F_n^* > 0 \\ &= 0 \text{ otherwise} \end{aligned} \tag{3.4}$$

In this paper, we use capital letters $\boldsymbol{Y}_n$ and $\boldsymbol{Z}_n$ to represent random variables and the corresponding lower letters $\boldsymbol{y}_n$ and $\boldsymbol{z}_n$ to denote the realisations of them, respectively. The subscript n here denotes the n-th data point. When there is no subscript n, both the capital letters and lower letters represent the entire sample. The probability density function of an observation under a K-component Gaussian mixture model is written in parametric form as

$$f(\boldsymbol{y}_n; \Psi) = \sum_{k=1}^{K} \pi_k f_k(\boldsymbol{y}_n; \theta_k) \tag{3.5}$$

[18] This is due to the no-short-selling constraint in risky asset investment. Here F_n^* represent the optimal amount of investment in risky asset for household n. In the discussion of the standard EM algorithm, we assume that the latent F_n^* is observable.

where π_k are scalars of positive mixing proportions summing to unity[19], f_k are multivariate normal density functions for component k parametrised by θ_k, and $\Psi = (\pi_1, ..., \pi_K, \theta_1, ..., \theta_K)$ is the vector containing all unknown parameters in the mixture model, $\theta_k = (\boldsymbol{\mu}_k, \Sigma_k)$ with $\boldsymbol{\mu}_k$ being the vector of means and Σ_k being the covariance matrix of component k.We use maximum likelihood (ML) to fit this mixture model via a widely applied approach, EM algorithm.

The EM algorithm is first introduced by the seminal paper by Dempster et al. (1977), which aims to find the maximum likelihood estimate from incomplete data. It is useful in incomplete data problems where algorithms such as the Newton-Raphson method may be more complicated. In our case, the Gaussian mixture model can be viewed as a model for the joint distribution of elements of $\boldsymbol{Y}_n$ depending on some unobservable (latent) vector $\boldsymbol{Z}_n$ which indicates the membership of observation n belonging to one of the K components for each observation, i.e. the complete data is

$$\boldsymbol{y_c} = (\boldsymbol{y}^T, \boldsymbol{z}^T)^T$$

where $\boldsymbol{z} = (\boldsymbol{z}_1, ..., \boldsymbol{z}_n)$ and $\boldsymbol{z}_n$ is a K-dimensional component-label vector with its k-th element $z_n^k = 1$ if $\boldsymbol{y}_n$ is generated from component k and 0 otherwise. In our case, the missing data $\boldsymbol{z}$ is the membership indicator to the regimes that we conjecture in Section 2.

The missing Z_n can be thought of as one draw from K categories with probabilities $\pi_1, ..., \pi_K$.

That is, we assume Z_n is distributed according to a multinomial distribution:

$$\mathbf{Z}_n \tilde{\ } Mult_K(1, \boldsymbol{\pi})$$

where $\boldsymbol{\pi} = (\pi_1, ..., \pi_K)^T$.

The complete-data log likelihood function is

$$L_c(\Psi) = \sum_{n=1}^{N} \sum_{k=1}^{K} z_n^k [\ln \pi_k + \ln f_k(\boldsymbol{y}_n; \theta_k)] \tag{3.6}$$

On the other hand, the incomplete-data log likelihood function is[20]

[19] Since $\sum_{k=1}^{K} \pi_k = 1$, one of the mixing proportions π_k is redundant.

[20] $L(\Psi)$ can be seen as the log of joint density when marginalising out the unknown $\mathbf{Z}$.

$$
\begin{aligned}
L(\Psi) &= \sum_{n=1}^{N} \ln f(\boldsymbol{y}_n; \Psi) \\
&= \sum_{n=1}^{N} \ln[\sum_{k=1}^{K} \pi_k f_k(\boldsymbol{y}_n; \theta_k)]
\end{aligned} \tag{3.7}
$$

The standard EM algorithm proceeds iteratively in two steps, E (for expectation) and M (for maximisation). Let $\Psi^{(0)}$ be the initial value for Ψ and $\Psi^{(p)}$ be the value of Ψ after the p-th EM iteration.

In the (p+1)-th iteration, the E-step estimates the complete-data sufficient statistics, which is the conditional expectation of $L_c(\Psi)$ given $\boldsymbol{y}$ using $\Psi^{(p)}$ for Ψ.

$$
Q(\Psi; \Psi^{(p)}) = E[\ln L_c(\Psi) | \boldsymbol{y}, \Psi^{(p)}] \tag{3.8}
$$

In order to get $Q(\Psi; \Psi^{(p)})$, we need to compute $E(Z_n^k | \mathbf{y}, \Psi^{(p)})$ as follows.

$$
\begin{aligned}
E(Z_n^k | y, \Psi^{(p)}) &= \Pr(Z_n^k = 1 | \boldsymbol{y}, \Psi^{(p)}) \\
&= \frac{\pi_k^{(p)} f_k(\boldsymbol{y}_n; \theta_k^{(p)})}{\sum_{j=1}^{K} \pi_j^{(p)} f_j(\boldsymbol{y}_n; \theta_j^{(p)})} \\
&= w_n^k(\Psi^{(p)})
\end{aligned} \tag{3.9}
$$

where we denote $E(Z_n^k | \mathbf{y}, \Psi^{(p)})$ by $w_n^k(\Psi^{(p)})$, the posterior probability that the n-th observation of the sample belongs to the k-th component of the mixture.

Hence the conditional expectation of the complete data likelihood can be written as[21]

$$
Q(\Psi; \Psi^{(p)}) = \sum_{n=1}^{N} \sum_{k=1}^{K} w_n^k(\Psi^{(p)}) [\ln \pi_k + \ln f_k(\boldsymbol{y}_n; \theta_k)] \tag{3.10}
$$

The M-step of the (p+1)-th iteration involves maximising equation (3.10) with respect to Ψ. Here the update rule for $\pi_k^{(p+1)}$ is computed independently of the updated estimates $\theta^{(p+1)}$. The update rules in the M-step are in closed form:

$$
\pi_k^{(p+1)} = \frac{1}{N} \sum_{n=1}^{N} w_n^k(\Psi^{(p)})
$$

$$
\boldsymbol{\mu}_k^{(p+1)} = \frac{\sum_{n=1}^{N} w_n^k(\Psi^{(p)}) \boldsymbol{y}_n}{\sum_{n=1}^{N} w_n^k(\Psi^{(p)})}
$$

[21] $Q(\Psi; \Psi^{(p)})$ is obtained by replacig the unknown z_n^k in $L_c(\Psi)$ by its expected value $w_n^k(\Psi^{(p)})$.

$$\Sigma_k^{(p+1)} = \frac{\sum_{n=1}^{N} w_n^k(\Psi^{(p)})(\boldsymbol{y}_n - \boldsymbol{\mu}_k^{(p+1)})(\boldsymbol{y}_n - \boldsymbol{\mu}_k^{(p+1)})^T}{\sum_{n=1}^{N} w_n^k(\Psi^{(p)})}$$

The E-step and the M-step are alternated until convergence. Dempster et al. (1977) show the monotonicity of the EM algorithm; that is,

$$L(\Psi^{(p+1)}) \geq L(\Psi^{(p)})$$

3.4.1.2 The censored data EM algorithm for a multivariate Gaussian mixture model

In the application of standard EM algorithm, the data points $\boldsymbol{y}_n$ are all fully observed and the only missing data is the component memberships $\boldsymbol{z}_n$. However, as we mentioned above, the risky asset F_n is censored according to the observation rule (equation (3.4)). That is, one coordinate (F_n^*) of our data $\boldsymbol{y}_n = (G_n, A_n, F_n^*)^T$ is not fully observable because of the no-short-selling constraint. To summarise, all individuals face the exogenous constraint $F_n \geq 0$. For A_n, individuals face heterogeneous constraints. So the observed variation in A_n could result from constrained behaviour with function B or from some individuals being constrained and others unconstrained. The aim is to get four distributions that capture different degrees of proximity to the constraints. But after that we still cannot tell for sure exactly who is constrained. If we pretended $F_n = F_n^*$ all the time, the model would be misspecified. For this reason we apply the censored data EM algorithm introduced by Lee and Scott (2012) where they apply this algorithm to synthetic and flow cytometry data and use simulations to show that their algorithm outperforms the standard EM algorithm when there is truncation and censoring. The censored data EM algorithm deals with both the missing component memberships and the loss of exact values of the censored data. The missing component memberships are formulated as missing data as in the standard EM algorithm, while the censoring problem is addressed by integrating out the density of unknown latent values of F_n^* in the likelihood function.

We can express our observed data $\boldsymbol{x}_n$ in the following form:

$$\begin{aligned} \boldsymbol{x}_n &= \boldsymbol{y}_n \text{ if } F_n^* > 0 \\ &= \boldsymbol{x}_{mn} \text{ otherwise} \end{aligned} \tag{3.11}$$

where $\boldsymbol{y}_n = (G_n, A_n, F_n^*)^T$ denote the fully observed observations that preserve their latent values of positive F_n^* while $\boldsymbol{x}_{mn} = (G_n, A_n, 0)^T$ denote the observations with corner solutions of $F_n = 0$.

The additional complication of censoring added to the standard EM algorithm is dealt with by identifying whether the pattern of each observation is $\boldsymbol{y}_n$ or $\boldsymbol{x}_{mn}$ according to equation (3.11) and then modifying the likelihood contribution of an observation as opposed to equation (3.5). That is, now the likelihood contribution of the observed $\boldsymbol{x}_n$ is:

$$\begin{aligned} f(\boldsymbol{x}_n;\Psi) &= \sum_{k=1}^{K} \pi_k f_k(\boldsymbol{x}_n;\theta_k) \text{ if } F_n^* > 0 \\ &= \int_{-\infty}^{0} [\sum_{k=1}^{K} \pi_k f_k(\boldsymbol{x}_n;\theta_k)] dF_n \text{ otherwise} \end{aligned}$$

The posterior probability is

$$w_n^k(\Psi^{(p)}) = \frac{\pi_k^{(p)} f_k(\boldsymbol{x}_n;\theta_k^{(p)})}{\sum_{j=1}^{K} \pi_j^{(p)} f_j(\boldsymbol{x}_n;\theta_j^{(p)})}$$

Applying the EM machinery, the update rule in the M-step changes from the standard EM algorithm accordingly. These are analysed in detail in the work by Lee and Scott (2012).

$$\pi_k^{(p+1)} = \frac{1}{N} \sum_{n=1}^{N} w_n^k(\Psi^{(p)})$$

$$\boldsymbol{\mu}_k^{(p+1)} = \frac{\sum_{n=1}^{N} w_n^k(\Psi^{(p)})[1(F_n > 0)\boldsymbol{x}_n + 1(F_n = 0) \begin{bmatrix} G_n \\ A_n \\ E(F_n^*|\boldsymbol{x}_n, z_n^k = 1) \end{bmatrix}]}{\sum_{n=1}^{N} w_n^k(\Psi^{(p)})}$$

$$\Sigma_k^{(p+1)} = \frac{\sum_{n=1}^{N} w_n^k(\Psi^{(p)}) S_n^k}{\sum_{n=1}^{N} w_n^k(\Psi^{(p)})}$$

where

$$
\begin{aligned}
S_n^k &= \{[1(F_n > 0)\boldsymbol{x}_n + 1(F_n = 0) \begin{bmatrix} G_n \\ A_n \\ E(F_n^*|\boldsymbol{x}_n, z_n^k = 1) \end{bmatrix}] - \boldsymbol{\mu}_k^{(p+1)}\} \\
& \quad \{[1(F_n > 0)\boldsymbol{x}_n + 1(F_n = 0) \begin{bmatrix} G_n \\ A_n \\ E(F_n^*|\boldsymbol{x}_n, z_n^k = 1) \end{bmatrix}] - \boldsymbol{\mu}_k^{(p+1)}\}^T \\
& \quad + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ 0 & R_n^k \end{bmatrix}
\end{aligned}
$$

$$
R_n^k = 1(F_n = 0)\{E(F_n^{*2}|\boldsymbol{x}_n, z_n^k = 1) - E(F_n^*|\boldsymbol{x}_n, z_n^k = 1)[E(F_n^*|\boldsymbol{x}_n, z_n^k = 1)]^T\}
$$

In our application, to choose the initial parameters $\Psi^{(0)}$, we implement k-means clustering algorithm 5 times with different starting points and choose the set of mixture model parameters from k-means that gives the maximum complete-data log likelihood. We terminate the algorithm when the increase of the complete-data log likelihood between two successive iterations is smaller than the tolerance parameter we set (we use 1e-10), or when the number of iterations reaches 300.

3.5 Main empirical results

Tables 3.6, 3.7 and 3.8 show the estimation results for the multivariate Gaussian mixture model for asset allocation patterns on each wave of data and on pooled data, respectively. Here the labels of the components are assigned by ascending order of housing wealth. The means of the four components show that on average, net worth $(A_n + G_n + F_n)$ rises when housing wealth rises. Thus the first component is the poorest while the fourth is the richest. The estimated mixing proportions suggest that the unconditional probability of belonging to component 1 is the highest, while the unconditional probability of belonging to component 4 is the lowest.

Since we use the censored data EM algorithm for estimation, the estimated parameters are associated with the latent data $\boldsymbol{y}_n = (G_n, A_n, F_n^*)^T$. For example, the estimated mean of risky assets for each component is the mean of F_n^* as if the observation rule (3.11) is not present and we could always observe the optimal risky asset holdings F_n^*. From the estimation results,

Table 3.6: Estimated mixing proportions and means from censored data EM algorithm by wave

Mixing proportions									
Component	Wave 1	(N=2593)		Wave 2	(N=2593)		Wave 3	(N=2593)	
	π			π			π		
1	0.41			0.45			0.42		
2	0.42			0.30			0.39		
3	0.14			0.20			0.15		
4	0.02			0.05			0.04		
Means									
Component	Wave 1	(N=2593)		Wave 2	(N=2593)		Wave 3	(N=2593)	
	E(G)	E(A)	E(F*)	E(G)	E(A)	E(F*)	E(G)	E(A)	E(F*)
1	133317	-47985	-3055	126399	-45112	-2448	111619	-40104	-3342
2	219935	-58148	-3065	194802	-35403	-16783	182992	-34850	-14071
3	358255	-52165	21455	292327	-73757	1980	326184	-40461	5918
4	653977	-39814	154306	508513	13728	104694	526712	7118	170426

Table 3.7: Estimated covariance matrices from censored data EM algorithm by wave

Covariance										
Component		Wave 1	(N=2593)		Wave 2	(N=2593)		Wave 3	(N=2593)	
1	Σ_1	G	A	F*	G	A	F*	G	A	F*
	G	1390807049	-4.1E+08	21133465	1.36E+09	-3.6E+08	33646917	1.02E+09	-3E+08	31272845
	A	-408607665	1.44E+09	18321741	-3.6E+08	1.5E+09	13601821	-3E+08	1.31E+09	6837128
	F*	21133464	18321741	10635291	33646917	13601821	12580937	31272845	6837128	13321620
2	Σ_2	G	A	F*	G	A	F*	G	A	F*
	G	4317414631	-6.4E+08	-6.9E+07	3.42E+09	1.18E+08	2.19E+08	2.8E+09	-6.8E+08	-3.4E+08
	A	-637002894	4.38E+09	2.03E+08	1.18E+08	4.23E+09	1.72E+09	-6.8E+08	3.51E+09	1.03E+09
	F*	-69146830	2.03E+08	2.02E+08	2.19E+08	1.72E+09	2.4E+09	-3.4E+08	1.03E+09	1.34E+09
3	Σ_3	G	A	F*	G	A	F*	G	A	F*
	G	17409770683	-4.8E+09	-2.2E+09	1.08E+10	-2E+09	2.73E+08	9.57E+09	-2.1E+09	5.89E+08
	A	-4790325601	1.72E+10	2.54E+09	-2E+09	9.64E+09	4.32E+08	-2.1E+09	1.39E+10	1.54E+09
	F*	-2210681489	2.54E+09	2.86E+09	2.73E+08	4.32E+08	1.85E+08	5.89E+08	1.54E+09	1.18E+09
4	Σ_4	G	A	F*	G	A	F*	G	A	F*
	G	1.28286E+11	1.06E+10	-2.5E+10	8.65E+10	-1.9E+10	2.8E+09	1.17E+11	-8.2E+09	-2.8E+10
	A	10587840343	7.64E+10	2.5E+10	-1.9E+10	4.84E+10	1.51E+10	-8.2E+09	5.53E+10	1.77E+10
	F*	-24513449300	2.5E+10	1.17E+11	2.8E+09	1.51E+10	7.21E+10	-2.8E+10	1.77E+10	1.77E+11

Table 3.8: Estimated parameters by censored data EM algorithm on pooled data

Pooled data (N=7779)				
Mixing proportions				
	π_1	π_2	π_3	π_4
	0.48	0.31	0.17	0.042
Means				
	$E(G)$	$E(A)$	$E(F^*)$	
μ_1	128888	-44189	-3515	
μ_2	206786	-39257	-19295	
μ_3	311812	-67574	3244	
μ_4	534198	1075	125359	
Covariance				
Σ_1	G	A	F^*	
G	1.56E+09	-3.9E+08	51069621	
A	-3.9E+08	1.53E+09	16220325	
F^*	51069621	16220325	23127503	
Σ_2	G	A	F^*	
G	4.14E+09	-2.2E+08	1.91E+08	
A	-2.2E+08	4.43E+09	1.8E+09	
F^*	1.91E+08	1.8E+09	2.84E+09	
Σ_3	G	A	F^*	
G	1.31E+10	-2.4E+09	1.84E+08	
A	-2.4E+09	1.2E+10	4.56E+08	
F^*	1.84E+08	4.56E+08	2.25E+08	
Σ_4	G	A	F^*	
G	9.96E+10	-1E+10	-1E+10	
A	-1E+10	5.47E+10	1.7E+10	
F^*	-1E+10	1.7E+10	1.13E+11	

we can see in all the three waves and the pooled data, the first two components both have negative means of F_n^*, which suggests that on average the first two components are no-short-selling constrained in risky asset investment. The mixing proportions π_1 and π_2 add up to about 80%, which suggests about 80% of the households are no-short-selling constrained on average.

Comparing the estimated means from wave 1 and wave 2 data, the mean of latent risky asset increases for the first two components while the housing wealth decreases for all the components. The drop in housing wealth could reflect the drop in house prices due to the financial crisis in wave 2 (2008-2009), while the increase in risky asset investment may show substitution effect of risky asset for housing for the first two components. For the third and fourth components, on the other hand, the mean of risky asset investment drops by about 91% and 32% in wave 2, which may show lack of confidence in risky asset in the face of financial crisis. In wave 3, however, all components except component 1 increase holdings of the risky asset. This may reflect the recovery of confidence on the risky asset for the second, third and fourth components, while the first component (the poorest) is less resilient in the post-crisis period[22].

In the results on wave 1, the estimated covariances of housing wealth and risky asset are negative for the first component, but positive for other components. This suggests for the poorest component, housing wealth and risky asset investment move in the same direction. However, for the other components, the more housing wealth owned the less investment in risky asset, which is consistent with the argument that house price risk crowds out stockholdings (Cocco, 2005).

This analysis is soft clustering where for each observation n we obtain the posterior probability of belonging to the k-th component conditional on the data, w_n^k. For the purpose of visualising the results of soft clustering, we label each observation with the label of the component with the highest posterior probability. For example, if the n-th observation has the posterior probabilities $w_n^1 = 0.8, w_n^2 = 0.03, w_n^3 = 0.07, w_n^4 = 0.1$, then this observation is labelled as belonging to group 1. That is, the group labels we assign here is in the context of hard clustering. Figures 3.5, 3.6, 3.7 and 3.8 show the histograms of posterior probabilities of belonging to the corresponding component with which each observation is labelled. The

[22] One should bear in mind that the above comparisons between waves are based on the average behaviour of each cluster rather than the average behaviour of a particular group of people.

peaks around 1 in these histograms means the maximum of w_n^k is near to one for most of the observations $\boldsymbol{y}_n$, which shows evidence that the components are well separated. Figures 3.9 and 3.10 show the distributions of housing wealth, net safe asset for each group (hard cluster) for each wave of the data and the pooled data, where the solid curves are fitted kernel densities. In Figures 3.9 and 3.10, the distributions of housing wealth and net safe asset in each hard cluster are quite symmetric and are most concentrated in the first group and most dispersed in the fourth group. This is consistent with the estimated variances of housing wealth and net safe asset in each component. With the labels of component membership, we find most of the observations do not change the label across the three waves using the results from both the pooled data and each wave of the data. This means most of the poor households stay poor while most of the rich households stay rich in the timespan we cover (3 waves), reflecting the inertia.

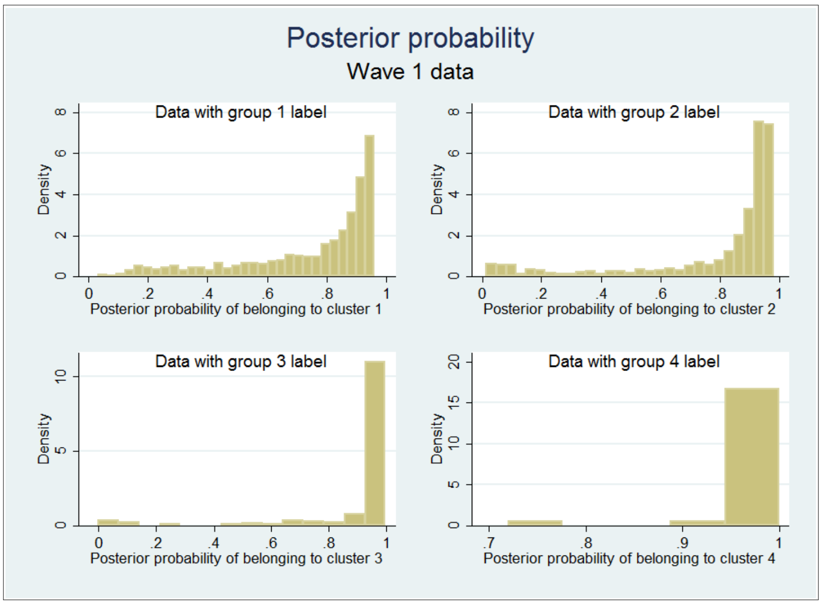


Figure 3.5: Histogram of posterior probabilities in each group for wave 1 data

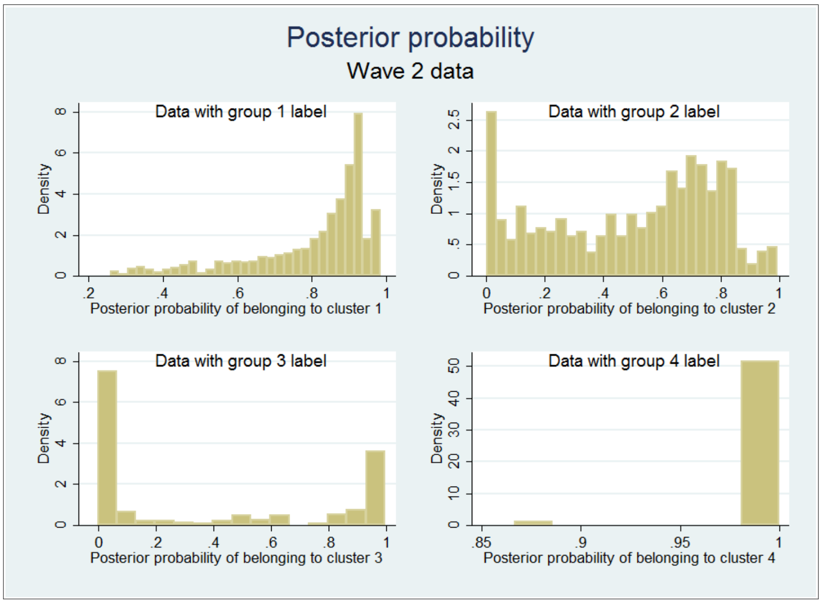


Figure 3.6: Histogram of posterior probabilities in each group for wave 2 data

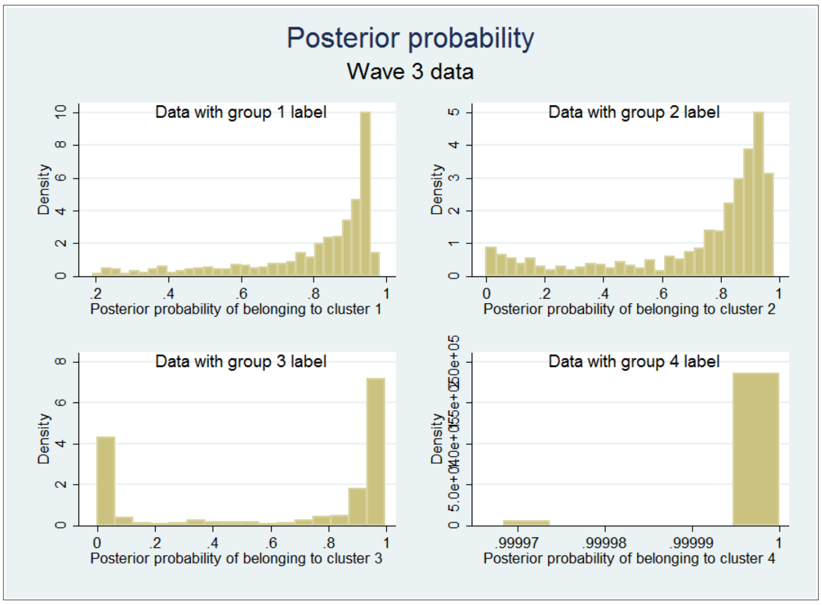


Figure 3.7: Histogram of posterior probabilities in each group for wave 3 data

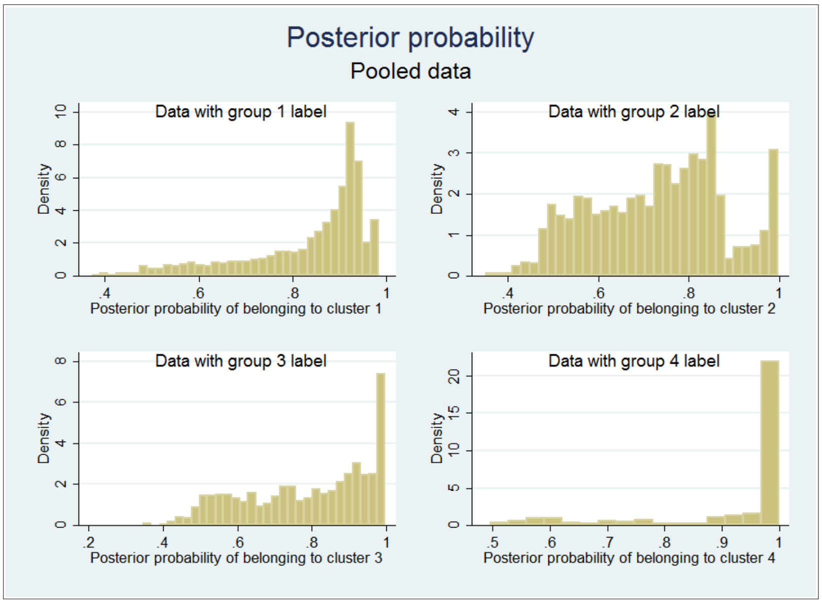


Figure 3.8: Histogram of posterior probabilities in each group for pooled data

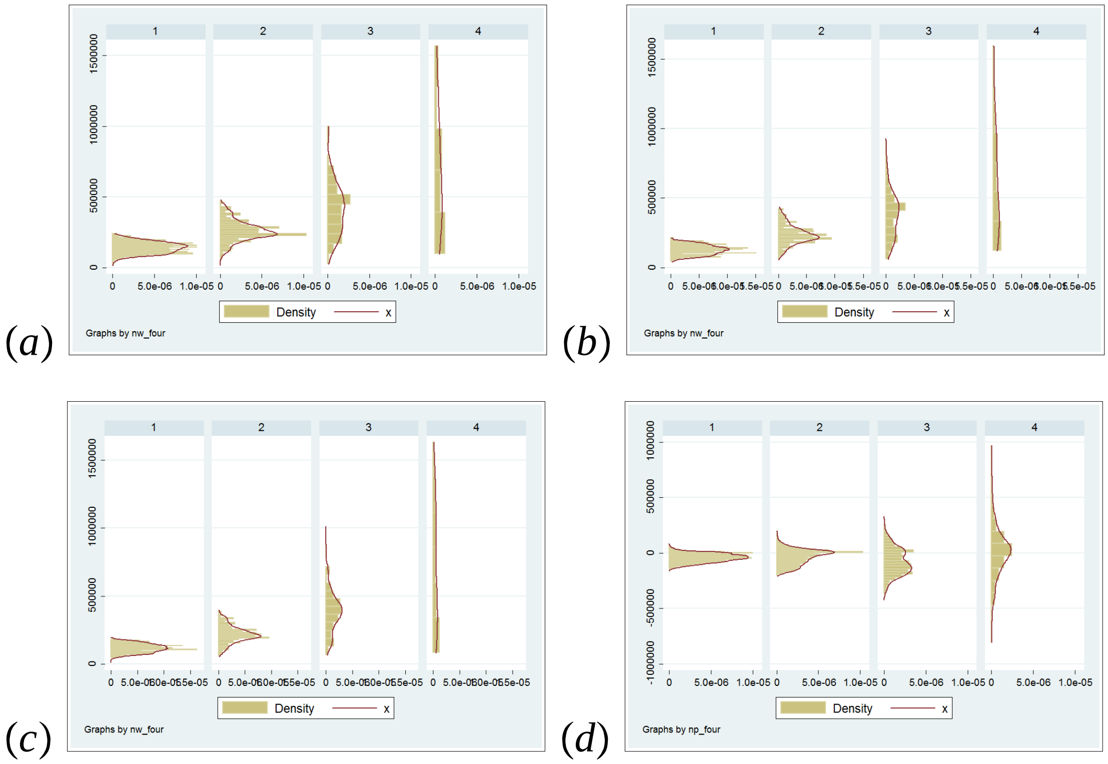


Figure 3.9: Histograms of housing wealth for each group

Note: (a) Wave 1 data; (b) Wave 2 data; (3) Wave 3 data; (d) Pooled data.

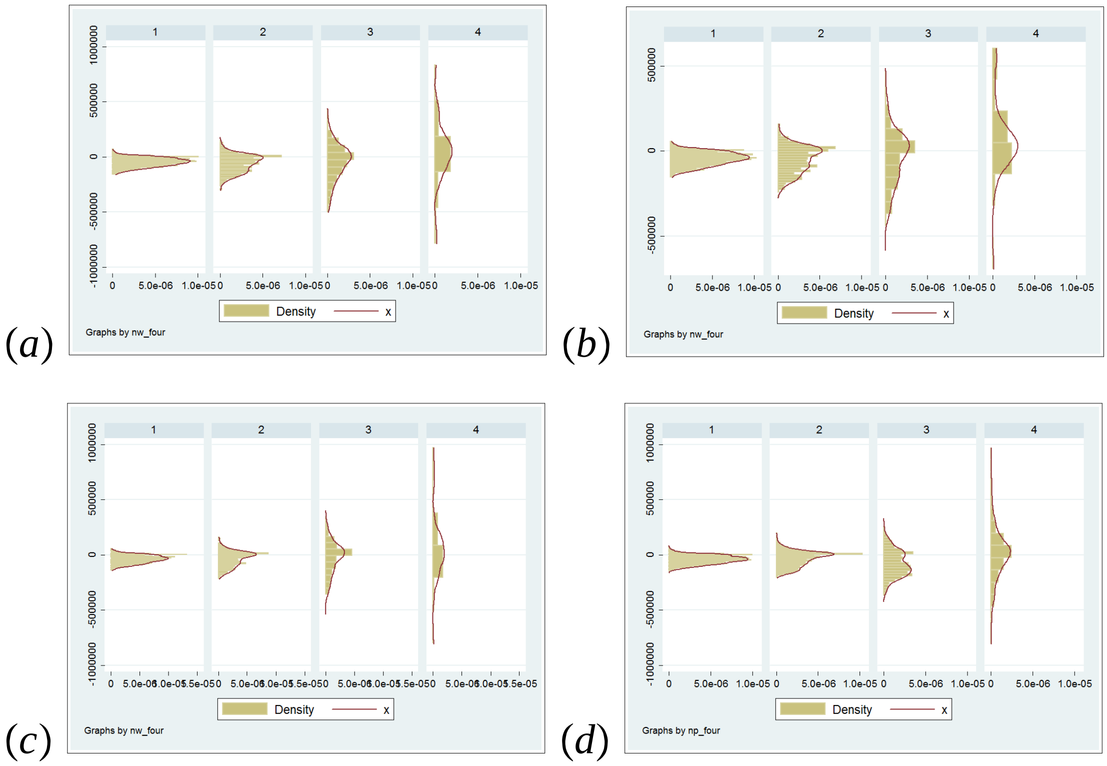


Figure 3.10: Histograms of net safe asset for each group

Note: (a) Wave 1 data; (b) Wave 2 data; (c) Wave 3 data; (d) Pooled data.

3.5.1 Determinants of component membership

We study the determinants of component membership by regressing the posterior probability of component membership on demographics and regional dummies. Since there are four components, there are four posterior probabilities correspondingly. Therefore we use a multivariate regression model, i.e. a system of linear models (SUR).

$$w_n^k = \boldsymbol{\beta}_k \mathbf{X}_n + \varepsilon_{nk}$$

where w_n^k is the estimated posterior probability that observation n belongs to component k obtained from the estimation of the multivariate Gaussian mixture model; $\mathbf{X}$ is a column vector of demographic variables including age, age squared, income, degree dummy, number of children under 18, and regional dummies; $\boldsymbol{\beta}_k$ is the row vector of coefficients on $\mathbf{X}$ for the k-th equation. Here the base category of regional dummies is Wales. This model is equivalent to doing k OLS regressions for each equation separately (in our model 4 regressions), except that between-equation covariances of the residuals is also estimated using this system estimator[23].

Tables 3.9 and 3.10 report the signs and statistical significance of the coefficients on each wave of data and the pooled data. The detailed results of regressions are shown in Tables B.2, B.3, B.4 and B.5 in the Appendix. The definitions of the variables are shown in Table B.1 in the Appendix. With this joint estimator instead of regressing each equation separately, we can estimate the between-equation covariance. The Breusch–Pagan test rejects the null hypothesis that the residuals of the four equations are independent of each other on all the datasets we estimate. The F test shows that all the regressors as a whole are strongly significant on all the data sets we use. The results show that households who are younger, less educated with lower income are more likely to belong to component 1, which is on average no-short-selling constrained and with lower net worth. There is also a regional effect. In general, households living in regions outside Wales are less likely to belong to the poorest component (component 1). Households living in London are more likely to belong to the richest component (component 4), compared to households living in Wales.

[23] Each equation has identical regressors.

Table 3.9: The signs and statistical significance of coefficients in multivariate regressions by wave

	Wave 1 (N=2593)				Wave 2 (N=2593)				Wave 3 (N=2593)			
	w^1	w^2	w^3	w^4	w^1	w^2	w^3	w^4	w^1	w^2	w^3	w^4
Age	- **	+ *	+ n.s.	- n.s.	- *	+ n.s.	+ **	- n.s.	- ***	+ ***	+ **	- n.s.
Age2	+ n.s.	- *	+ n.s.	+ *	+ n.s.	- n.s.	- *	+ **	+ **	- **	- n.s.	+ *
income	- ***	+ n.s.	+ ***	+ ***	- ***	+ n.s.	+ ***	+ ***	- ***	- n.s.	+ ***	+ ***
degree	- ***	+ n.s.	+ ***	+ n.s.	- ***	+ n.s.	+ ***	+ ***	- ***	+ n.s.	+ ***	+ ***
nkids	- ***	+ *	+ **	+ ***	- ***	- n.s.	+ ***	+ **	- ***	- n.s.	+ ***	+ n.s.
northeast	- n.s.	+ n.s.	+ n.s.	- n.s.	+ n.s.	+ n.s.	- n.s.	- n.s.	+ n.s.	+ n.s.	- n.s.	- n.s.
northwest	- n.s.	+ n.s.	+ n.s.	- n.s.	- n.s.	+ n.s.	- n.s.	- n.s.	+ n.s.	+ n.s.	- n.s.	- n.s.
yorkshirehumb	- **	+ *	+ n.s.	- n.s.	- n.s.	+ *	- n.s.	+ n.s.	- n.s.	+ n.s.	+ n.s.	- n.s.
eastmid	- **	+ **	+ n.s.	- *	- *	+ **	- n.s.	- n.s.	- n.s.	+ n.s.	+ n.s.	- n.s.
westmid	- ***	+ **	+ n.s.	- n.s.	- **	+ **	+ n.s.	- n.s.	- **	+ *	+ n.s.	- n.s.
eastengland	- ***	+ ***	+ **	- n.s.	- ***	+ ***	+ ***	+ n.s.	- ***	+ ***	+ ***	+ n.s.
london	- ***	+ ***	+ ***	+ **	- ***	+ **	+ ***	+ ***	- ***	+ **	+ ***	+ ***
southeast	- ***	+ ***	+ ***	+ n.s.	- ***	+ ***	+ ***	+ n.s.	- ***	+ ***	+ ***	+ *
southwest	- ***	+ ***	+ ***	- n.s.	- ***	+ **	+ ***	- n.s.	- ***	+ ***	+ ***	+ n.s.
cons	+ ***	- n.s.	- **	+ n.s.	+ ***	- n.s.	- ***	+ n.s.	+ ***	- *	- ***	+ n.s.

Note: Dependent variable: the estimated posterior probability of component membership. n.s. not significant, *p<0.05, **p<0.01, *** p<0.001.

Table 3.10: The signs and statistical significance of coefficients in multivariate regressions on pooled data

Pooled data (N=7779)				
	w^1	w^2	w^3	w^4
Age	-	+	+	-
	***	***	***	**
Age2	+	-	-	+
	n.s.	*	**	***
income	-	+	+	+
	***	n.s.	***	***
degree	-	+	+	+
	***	**	***	***
nkids	-	+	+	+
	***	n.s.	***	***
northeast	-	+	+	-
	n.s.	n.s.	n.s.	n.s.
northwest	-	+	+	-
	n.s.	n.s.	n.s.	n.s.
yorkshirehumb	-	+	+	-
	**	**	n.s.	n.s.
eastmid	-	+	+	-
	**	***	n.s.	n.s.
westmid	-	+	+	-
	***	***	n.s.	n.s.
eastengland	-	+	+	+
	***	***	***	n.s.
london	-	+	+	+
	***	***	***	***
southeast	-	+	+	+
	***	***	***	**
southwest	-	+	+	-
	***	***	***	n.s.
cons	+	-	-	+
	***	n.s.	***	n.s.

Note: Dependent variable: the estimated posterior probability of component membership. n.s. not significant, *p<0.05, **p<0.01, *** p<0.001.

3.5.2 Alignment in theoretical regimes and empirical components

Figure 3.11 shows the non-participation rate in risky asset and the percentage of mortgage holders for each group. We can see that on all the datasets, the non-participation rate decreases from Group 1 to Group 4 monotonically. The majority of households with the Group 1 label (about 80%) do not invest in any risky asset, while about half (waves 1, 2, and 3 data) or more than half (pooled data) of households with Group 2 label do not participate in the risky asset investment. On the other hand, only a small proportion of the households with Group 3 and Group 4 labels do not hold the risky asset. This suggests that Groups 1 and 2 are likely to be no-short-selling constrained, i.e. their optimal holding of risky asset would be non-positive if the no-short-selling constraint is not present. This is consistent with the signs of estimated means of F^* in Tables 3.6 and 3.8.

The percentage of mortgage holders decreases from Group 1 to Group 4 on all the datasets except for the pooled data (Figure 3.11). For the pooled data the percentage of mortgage holders in Group 3 (79%) is slightly higher than in Group 2 (72%). Similarly, in the estimation in each wave, the percentage of mortgage holders is the highest in Group 1 and lowest in Group 4. To investigate whether households in each group are borrowing constrained or not, we show the means and coefficients of variation of loan-to-value ratio for each group in Table 3.11. Note that only the sample with a loan-to-value ratio within the reasonable range (0,1) are included in Table 3.11, since a loan-to-value ratio outside this range could be a result of measurement error in the data and does not serve as a sensible signal of borrowing constraint. Table 3.11 shows that on average Group 1 has the highest loan-to-value ratio with smallest variation while Group 4 has the lowest loan-to-value ratio with big variation. This indicates that households with the Group 1 label are most likely to be mortgage borrowing constrained since they borrow the highest proportion of the house value on average and this proportion tends to concentrate around some maximum limit. As discussed in Section 3 the maximum borrowing limit is unobserved and can be individual-specific. However, the relative clustering of loan-to-value ratio in Group 1 compared to other groups may result from the fact that the majority of Group 1 mortgage holders borrow as much as they can to finance their houses. By contrast, households with Group 4 label are least likely to be borrowing constrained since they borrow the least proportion of the house value with the biggest variation of loan-to-value ratio. According to Figure 3.11 and Table 3.11, on the pooled data, Group 3 has a higher percentage of mortgage holders and a higher mean of loan-to-value ratio with bigger variation than Group 2, which may serve as evidence of Group 3 being more borrowing constrained than Group 2. However, this evidence is not as strong as the ones supporting Group 1 and Group 4, because using the predicted characteristics from estimation by wave, Group 2 looks more constrained in both assets.

In summary, there is strong evidence to support that components 1 and 4 are in line with the theoretical regimes 1 and 4. However, the evidence is weaker to align components 2 and 3 with the corresponding theoretical regimes.

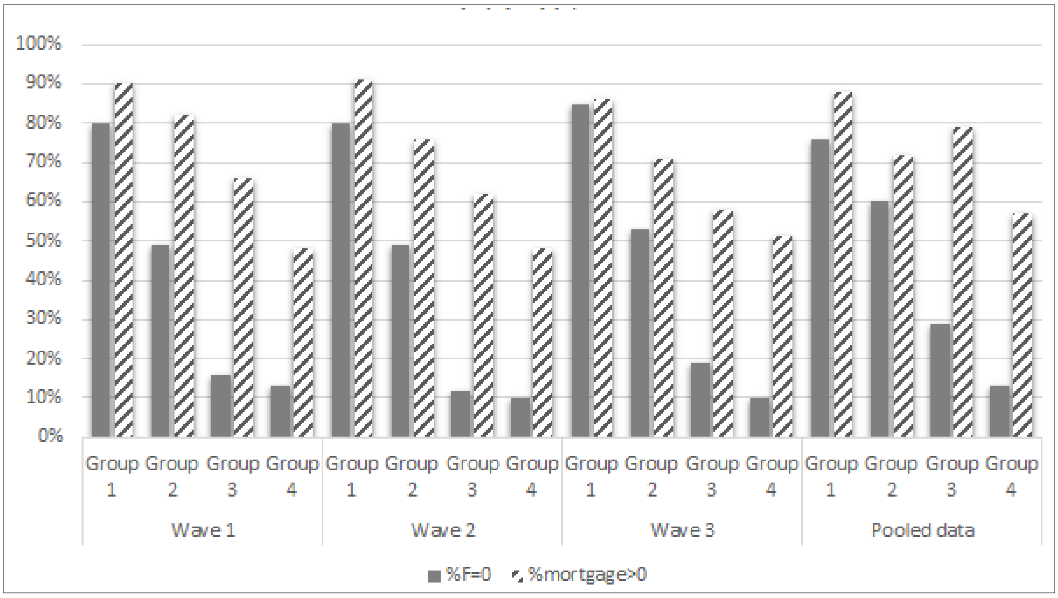


Figure 3.11: Non-participation rate in risky asset and percentage of mortgage holders for each group

Table 3.11: The means and coefficients of variation of loan-to-value ratio for mortgage holders in each group

Wave 1		
Group	Mean	Coefficient of variation
1	0.44	0.54
2	0.37	0.64
3	0.33	0.71
4	0.27	0.87
Wave 2		
Group	Mean	Coefficient of variation
1	0.46	0.53
2	0.39	0.62
3	0.36	0.69
4	0.24	0.78
Wave 3		
Group	Mean	Coefficient of variation
1	0.45	0.53
2	0.37	0.64
3	0.32	0.74
4	0.32	0.81
Pooled data		
Group	Mean	Coefficient of variation
1	0.45	0.53
2	0.36	0.67
3	0.40	0.60
4	0.30	0.79

Note: The sample selected for this table is with loan-to-value ratio within the (0,1) interval.

3.6 Conclusion

This paper starts with a theoretical model and derives four theoretical regimes of asset allocations depending on the degrees of proximity to the mortgage borrowing constraint and no-short-selling constraint. The theoretical model gives a steer to setting the number of components in the empirical work. Considering the complication of identifying the theoretical regime membership in the data, a censored data EM algorithm is used to estimate the multivariate Gaussian mixture model. Estimation results show distinct patterns of asset allocations across homeowners using the WAS data from the UK. The estimated parameters reveal that on average about 80% of the households are no-short-selling constrained in risky asset investment and with low net worth. A system of linear models are estimated to find determinants of component membership. Among other things, we find that households who are younger, less educated with lower income are more likely to be no-short-selling constrained in risky asset investment and with lower net worth. These findings reflect a life-cycle effect as well as an education effect on asset allocation. The education effect could work through changing life cycle human capital and/or improving financial literacy. The estimation results and indicative evidence from loan-to-value ratio variation between components strongly suggest that the first empirical component is aligned with the first theoretical regime, while the fourth empirical component is in line with the fourth theoretical regime. There is weaker evidence to indicate that the other two components match the corresponding theoretical regimes. Apart from unobservable borrowing constraints and no-short-selling constraint, potentially, some random factors that are modeled in the theoretical model could account for the split of empirical components: heterogeneity in initial wealth and preferences (e.g. different marginal utility, expectation, risk aversion, etc.), and household idiosyncratic shocks.

The analysis in this paper is semi-parametric in the sense that the mixing proportions and component means are not parametrised. This enables the data to talk in a more flexible way than the fully parametric model. Nevertheless, it would be interesting to extend our work by parametrising mixing proportions and component means and compare the estimation results with this chapter.

We have used the Gaussian mixture model with pooled panel data. In a sense this pushes some of the individual heterogeneity into the probabilities of belonging to different components. An alternative would be to use individual fixed and idiosyncratic effects to account for individual heterogeneity. An interesting exercise would be to combine these two approaches. It would lead towards a panel model in which the sample splits into components; within component the fixed effects are a random sample from a particular distribution but in other components the fixed effects are a sample from a different distribution.

Chapter 4

Heterogeneous Wealth Effects on Consumption for Older Homeowners with Unobservable Borrowing Constraints

4.1 Introduction

Understanding how the consumption of heterogeneous households responds to wealth shocks- the marginal propensity to consume out of wealth (MPC)- is important for policymakers who want to implement economic stimulus. To be specific, the macroeconomic impact of redistributive policy (e.g. tax or welfare reform) on spending will depend on the microeconomic heterogeneity of MPC and the distribution of different types of households in the economy. Existing literature estimates MPC with different approaches (Jappelli and Pistaferri, 2010; Mian et al., 2013).

The canonical consumption/saving model usually only considers one single asset. However, total net wealth does not capture enough information needed to understand the dynamics of consumption, given different expected returns for different asset classes, the correlation among these returns and the associated borrowing constraints on them. Recognition of the theoretical and empirical importance of borrowing constraints (especially mortgage borrowing and no-short-selling constraints) on household portfolio choices led to a new research direction.

Portfolio decisions (and consumption decisions) will generally differ between households which are borrowing constrained or unconstrained. Adding borrowing constraints makes the model more complicated especially if it is not possible to exactly observe such constraints. The estimation problem is that from the survey data we often cannot directly see who is closer to the borrowing constraints because the individual-specific borrowing constraints are unobservable. On the other hand, heterogeneous preferences also play a role in different patterns of portfolio decision. A gap in the literature is modelling individual households with heterogeneous preferences and including richer information on the household balance sheet in the presence of borrowing constraints.

This chapter aims to investigate heterogeneous wealth effects on consumption for older homeowners in the presence of unobservable borrowing constraints. We develop a structural model where homeowners are allowed to have different types of preferences and face individual-specific borrowing constraints associated with housing and risky financial asset investment. Homeowners are forward-looking and derive utility from both housing and non-housing consumption. Different types of preferences and different circumstances (cash-on-hand) generate alternative portfolio regimes that reflect different degrees of proximity to borrowing constraints.

The structural estimation is based on the result from the previous chapter- the classification of regimes by fitting a multivariate Gaussian mixture model via a censored data expectation-maximisation (EM) algorithm on data from Wealth and Asset Survey (WAS). In this chapter, we estimate the structural parameters by minimising the difference between the model predicted consumption and the imputed consumption from the data for older homeowners based on the classification results. As there is no direct measure of consumption in WAS data, we utilise the detailed wealth and income information in WAS to impute consumption from an intertemporal budget constraint. Estimation results show that homeowners who are closer to borrowing constraints have higher MPC on average. Households who are closer to the borrowing constraints (group 1) have highest MPC (close to 1 on average), i.e. they behave in a hand-to-mouth way. This may suggest a stimulus is most effective for the borrowing constrained, low net-worth households since their consumption is more sensitive to a wealth shock. We also find MPC declines with total wealth, which is in line with the existing literature, although our estimated average MPC is bigger than existing literature (Jappelli and Pistaferri, 2014; Sahm et al., 2010).

This paper contributes to the existing literature in three aspects. First, it takes a step forward compared to the benchmark single-asset model (Carroll et al., 2014). Allowing for the existence of multiple assets not only enriches the model but also makes the borrowing constraints associated with different asset classes possible. Second, it introduces a novel way to split the sample for structural estimation. Some existing literature split the sample for either structural estimation or reduced-form estimation. Zeldes (1989) a priori selects a set of families that he believes to be not liquidity constrained in terms of wealth to income ratio and estimates the preference parameters from the Euler equation. Arrondel et al. (2015) estimates the MPC across wealth distribution by including dummy variables of percentiles of wealth in their regressions. In comparison, we do not assume any certain criterion to split the sample. Instead, we split the sample by the component membership indicated by the Gaussian mixture model estimation in Chapter 3. Third, it contrasts with most of the structural literature that imposes homogeneous preference on households. Our classification on the sample is a result of different preferences (different values of parameters in the utility function) and circumstances. For this reason, we allow for different types of preferences among homeowners and make the model more flexible.

The rest of the chapter is organised as follows. Section 2 shows how to predict consumption from the theoretical model. Section 3 shows the way to impute consumption data. Section 4 reports the results of structural estimation on MPC. Finally, Section 5 concludes.

4.2 Predicted non-housing consumption from the theoretical model

In the structural estimation, we will focus on the behaviour of older homeowners and use the theory model in Chapter 3. Recall that in that model it is assumed that $R_{At} = R_{Mt}$ so that A_t measures the safe asset holding net of mortgage. In addition we assume that a household will continue living in the same house in the terminal period of life (or a house that provides equivalent housing service) because of inertia. That means while the housing wealth changes when the house price changes, the consumption of housing is the same through time. And households only choose non-housing consumption, net safe asset, risky asset investment. Households will be able to withdraw the equity of their housing at the beginning of the terminal period and just pay a fraction of the house value as rent or take a lifetime mortgage in which the equity in the house is generally surrendered to the lender in exchange for the right to live in the house until death (empirically financial details of the lifetime mortgage arrangement vary (Wikipedia, 2012)).

Now assume the functional form of the utility is

$$U_t(c_t, H_t) = c_t^{1-\rho} H_t^{\rho}$$

Define

$$a_t = A_t + F_t + p_t H_t$$

$$\begin{aligned} R_{At} &= 1 + r_{At} \\ R_{Ft} &= 1 + r_{Ft} \\ R_{Ht} &= 1 + \frac{p_t - p_{t-1}}{p_{t-1}} \end{aligned}$$

For $a_t \neq 0$

$$\begin{aligned} s_{At} &= \frac{A_t}{a_t} \\ s_{Ft} &= \frac{F_t}{a_t} \\ s_{Ht} &= \frac{p_t H_t}{a_t} \end{aligned}$$

$$\begin{aligned} R_{p,t+1} &= (1 + r_{A,t+1}) s_{At} + (1 + r_{F,t+1}) s_{Ft} + (1 + r_{Ht+1}) s_{Ht} \\ &= R_{A,t+1} + (R_{F,t+1} - R_{A,t+1}) s_{Ft} + (R_{H,t+1} - R_{A,t+1}) s_{Ht} \end{aligned}$$

In a two-period model, suppose households are somehow 'locked' in housing consumption, so that $H_t = \overline{H}$. The Bellman equation is

$$V_{T-1}(m_t) = \max_{\{c_{T-1}, s_{AT-1}, s_{FT-1}\}} U_{T-1}(c_{T-1}, \overline{H}) + \beta E_{T-1} V_T(m_T)$$

s.t.

$$m_{T-1} = c_{T-1} + a_{T-1} \tag{4.1}$$

$$m_T = y_T + R_{p,T} a_{T-1} \tag{4.2}$$

$$\begin{aligned} s_{AT-1} &\geq B \\ 0 &\leq s_{FT-1} \leq 1 \end{aligned}$$

where $-B$ is the household-specific upper limit of borrowing amount.

In the final period T, suppose households would have the lifetime mortgage (reverse mortgage) and live in the same house as in period T-1, but they need to pay some interest in period T for the lifetime mortgage. For simplicity, we assume households would pay back any previous mortgage at the beginning of T and pay a lump sum of interest which is a fixed proportion of house value at the beginning of T, $\gamma p_T\overline{H}$. Alternatively, this can be interpreted as household selling the house at the beginning of period T but continue living in the same house with the cost being $\gamma p_T\overline{H}$. Since there is no bequest motive in this model, households would consume everything they have in the final period. That is, the utility maximisation problem in period T is

$$V_T(m_T) = \max_{c_T} c_T^{1-\rho}\overline{H}^{\rho}$$

s.t.

$$m_T = c_T + \gamma p_T\overline{H}$$

So

$$c_T^* = m_T - \gamma p_T\overline{H}$$

$$\begin{aligned}
V_T(m_T) &= (m_T - \gamma p_T\overline{H})^{1-\rho}\overline{H}^{\rho} \\
&= [(y_T + R_{p,T}(m_{T-1} - c_{T-1})) - \gamma p_T\overline{H}]^{1-\rho}\overline{H}^{\rho}
\end{aligned}$$

The FOC of the Bellman equation wrc c_{T-1} is

$$\frac{\partial U_{T-1}(c_{T-1}, \overline{H})}{\partial c_{T-1}} + \beta E_{T-1}\frac{\partial V_T}{\partial c_{T-1}} = 0$$

$$(1-\rho)c_{T-1}^{-\rho}\overline{H}^{\rho} - \beta(1-\rho)\overline{H}^{\rho}E_{T-1}R_{pT}[(y_T + R_{p,T}(m_{T-1} - c_{T-1})) - \gamma p_T\overline{H}]^{-\rho} = 0$$

So

$$\begin{aligned}
c_{T-1} &= \beta^{-\frac{1}{\rho}}E_{T-1}R_{pT}^{-\frac{1}{\rho}}[(y_T + R_{p,T}(m_{T-1} - c_{T-1})) - \gamma p_T\overline{H}] \\
&= \beta^{-\frac{1}{\rho}}E_{T-1}R_{pT}^{-\frac{1}{\rho}}[(y_T + R_{p,T}a_{T-1}) - \gamma p_T\overline{H}] \\
&= \beta^{-\frac{1}{\rho}}E_{T-1}R_{pT}^{-\frac{1}{\rho}}[(y_T + R_{p,T}a_{T-1}) - \gamma R_{HT}p_{T-1}\overline{H}] \\
&= \beta^{-\frac{1}{\rho}}E_{T-1}R_{pT}^{-\frac{1}{\rho}}[(y_T + R_{p,T}a_{T-1}) - \gamma R_{HT}s_{HT-1}a_{T-1}]
\end{aligned}$$

where

$$R_{p,T} = R_{A,T} + (R_{F,T} - R_{A,T})s_{FT-1} + (R_{H,T} - R_{A,T})s_{HT-1}$$

Notice that in this model, all the households are not constrained in non-housing consumption, since they can always liquidate their house and move to a rented accommodation if they want more non-housing consumption. This would be affordable because the equity of the house is typically higher than the rental cost of housing of the same size. Therefore, though households may be different in terms of proximity to the borrowing constraints, the FOC holds for every household. So once we know $a_{T-1}, s_{HT-1}, s_{FT-1}$, and assume some distribution for stochastic state variables in period T, we can work out what c_{T-1} was chosen. The expectation is computed by numerical integration (Gauss quadrature).

We can use the above equation to compute predicted values of c_{T-1}.

4.3 Imputed non-housing consumption

In the data, non-housing consumption is not directly observable. To impute consumption, we utilise the detailed information of wealth and income from the data and work out consumption from an intertemporal budget constraint. To be specific, I define non-housing consumption as "net financial wealth at the start of the period- mortgage at the start of period+ labour income+investment income+household pension income+household benefit income+gift received - hire purchase instalment-loan instalment- mail order instalment-mortgage instalment-(net financial wealth at the end of the period-mortgage at the end of the period)". Net financial wealth is a derived variable in the data set. Here labour income, household pension income, household benefit income, gift received, hire purchase instalment, loan instalment, mail order instalment are obtained by summing up the corresponding individual variables in the same household. It should be noted that we need net financial wealth and mortgage debt at the start of the period as well as at the end of the period, so we can only generate consumption for waves 2 and 3 (wave 1 does not have lagged period).

4.4 Structural estimation on MPC for older homeowners

Assume the mean of measurement error of imputed non-housing consumption[1] is 0.

$$E(c_{imputed} - c_{predicted}) = 0$$

where $c_{imputed}$ is non-housing consumption imputed from the data and $c_{predicted}$ is non-housing consumption predicted by the theoretical model.

The corresponding sample moment is

$$g(\rho) = \sum_i (c_{imputed,i} - c_{predicted,i})$$

This sample moment is a function of ρ because $c_{predicted}$ is a function of ρ. In this paper, the discount factor β is calibrated because the discount factor in dynamic models is typically not identified without strong restrictions (Rust, 1994). Table 4.1 shows the values of calibrated parameters.

Table 4.1: Values of calibrated parameters

Description	Parameters	Value
Discount factor	β	0.99
Cost of housing in T	γ	0.1
Return of risk-free asset	R_A	1.01
Mean of $\ln(R_F)$	$E(\ln(R_F))$	0.09
Mean of $\ln(R_H)$	$E(\ln(R_H))$	0.04
S.D. of $\ln(R_F)$	$\sigma_{\ln(R_F)}$	0.14
S.D. of $\ln(R_H)$	$\sigma_{\ln(R_H)}$	0.1
Correlation of $\ln(R_F)$ and $\ln(R_H)$	$\sigma_{\ln(R_F),\ln(R_H)}$	0.71

The sample we choose for the structural estimation is those aged above 50 in wave 3 who are likely to start planning for their later period of life. In addition to estimating the

[1] From the data it is clear that there is measurement error since some imputed consumptions are negative. But there may also be error associated with assuming the FOC holds as an equality. If this is so, there is a composite error.

preference parameter ρ, we also compute the marginal propensity to consume (MPC) out of wealth according to the following equation.

$$MPC = \frac{\partial c_{T-1}}{\partial m_{T-1}} = \beta^{-\frac{1}{\rho}} E_{T-1} R_{pT}^{1-\frac{1}{\rho}}$$

The estimation result is shown in Table 4.2. It shows that homeowners who are closer to borrowing constraints have higher MPC on average. Households who are closest to the borrowing constraints (group 1) have highest MPC (close to 1 on average), i.e. they behave in a hand-to-mouth way. This may suggest a stimulus is most effective for the borrowing constrained, low net-worth households since their consumption is more sensitive to a wealth shock. We also find MPC declines with total wealth, which is in line with the existing literature. The estimated average MPC (0.86) is bigger than existing literature (Jappelli and Pistaferri, 2014; Sahm et al., 2010), which may be a result of shorter planning horizon and less risk faced by the older homeowners we use for estimation.

Table 4.2: Results of structural estimation

Sample	Obs	ρ	Average MPC out of wealth
Group 1 aged above 50	325	0.73	0.97
Group 2 aged above 50	364	0.24	0.82
Group 3 aged above 50	143	0.22	0.81
Group 4 aged above 50	62	0.14	0.66

Figure 4.1 shows the scatter plot of estimated MPC versus total wealth. It suggests that in general MPC decreases with total wealth, which is consistent with the finding in existing literature. A regression of MPC on total wealth and other household characteristics also confirms this (Table 4.3).

Figure 4.1: Scatter plot of estimated MPC vs. total wealth

Table 4.3: Regression of estimated MPC on household characteristics

	MPC
Total wealth	-2.30e-07***
	(1.05e-08)
Age	-0.000724
	(0.000640)
Income	2.94e-08
	(4.23e-08)
Degree	-0.0199***
	(0.00577)
Constant	0.971***
	(0.0362)
Observations	894
R-squared	0.420

Note: Dependent variable: estimated MPC. Standard errors in parentheses. *p<0.05, **p<0.01, *** p<0.001.

4.5 Conclusion

This chapter studies the MPC for heterogeneous older homeowners. It is based on the result from the previous chapter which starts with a theoretical model and derives four theoretical regimes of asset allocations depending on whether the borrowing constraint and no-short-selling constraint are binding or not. In the previous chapter, the theoretical model gives a steer to setting the number of solution patterns that result from heterogeneous preferences and circumstances and a censored data EM algorithm is used to estimate the multivariate Gaussian mixture model. Based on the classification results, I estimate the structural parameters by matching the model predicted consumption with imputed consumption from the data for older homeowners. The estimation results show that households who are closest to the borrowing constraints (group 1) have highest MPC (close to 1 on average), i.e. they behave in a hand-to-mouth way. This may suggest a stimulus is most effective for the borrowing constrained, low net-worth households since their consumption is more sensitive to a wealth shock. We also find MPC declines with total wealth, which is in line with the existing literature. The estimated average MPC (0.86) is bigger than existing literature (Jappelli and Pistaferri, 2014; Sahm et al., 2010), which may be a result of shorter planning horizon and less risk faced by older homeowners I use for estimation. One caveat is that our calculations of MPC are performed assuming no bequest motive. Hence, my estimated MPC are likely to be an upper bound to the true MPC of the older homeowners.

In general the idea of identifying either the optimisation error or measurement error in the data can be applied to any variable in the data set, so the above exercise can be repeated for different assets in the sample.

Chapter 5

A Life Cycle Model with Housing Tenure, Constrained Mortgage Finance and a Risky Asset under Uncertainty

5.1 Introduction

The chapter examines individual housing tenure, housing finance and financial portfolio decisions in a life cycle framework in which utility each period depends on both consumption c_t and housing services h_t (the pleasure of living in a house). There are imperfect financial markets. Allowing for the tenure choice is theoretically important. First renting and buying with or without a mortgage have different risks (Deng et al., 2000); Sinai and Souleles (2005) find that empirically the volatility of rent can exceed the risk in house prices. Vigdor (2006) points out that housing finance constraints can also distort the relation between house prices and house rents, depressing the former. Moreover in recent times in the UK buy to let housing has become increasingly important as the percentage of renters in the population has increased, from 1999-2015 the number of new mortgages for buy to let more than doubled (Council of Mortgage Lenders, 2017). This is seen as the joint result of inequality in the income distribution (and the intergenerational wealth distributions-wealthy parents can help children into the housing market) and a rising real house price that makes it even more difficult for lower income individuals to afford the downpayment (i.e. meet the loan to income constraint). In the UK there has also been a move towards more prudent lending policies with tighter control (Financial Services Authority, 2009) in contrast to the big increase in the supply of mortgage finance in the US prior to the subprime mortgage crisis (Mian and Sufi, 2009). In the UK the percentage of houseowners fell from around 71% in 2003 to around 64% in 2016, this aggregate figure also masks large regional differentials. On the other hand the share of renters rose from about 18% in 2003 to around 27% in 2016. In 2000 only about 27% of UK households had any direct participation in the stock market (Guiso et al., 2003), although there is indirect participation via pension schemes. Taking all these facts together the problem is of high policy importance in the UK. Indeed since 2010 there have increasingly been fiscal changes to reduce the return on buy to let by raising property taxes and conversely financial subsidies of various kinds to slacken the initial loan to income ratio constraint on first entry to owner occupied housing (Tucker, 2013).

In European and American households, a typical life cycle pattern of asset ownership between housing and financial assets (safe or risky) which arises partly because borrowing is allowed only against real assets like housing and not against future income is that in the young adult epoch households are renters. After saving from labour income or informal loans to finance the transaction cost, in middle age households become houseowners but with a finite term mortgage (typically 20-25 years). In this epoch households are typically also financing pre-employment children through education, financing mortgage debt and possibly elderly relatives. In the later epoch the financial demands on a households income have fallen: children have established their own households; elderly relatives are no more and the mortgage M_t has matured. If all this happens after retirement or a permanent income fall, the disposable income for savings may fall too. Thus typically we expect different stages of the life cycle to choose very different asset portfolios. But since idiosyncratic shocks are heterogeneous, there will be a variance within each life cycle group. A particular issue with UK households concerns the long term economics and demographics since 1945. The wars of 1914-18 and 1939-45 gave a shift in the income distribution and also especially after 1945 a shift in publicly provided services like education, health even some forms of income insurance. Households of this period for the first time could access the owner occupied housing market with mortgage finance. There was also a baby boom roughly 1945-1950. The wartime households have mainly died, passed some limited assets to their children. Hirsch (2017) finds that the inequality of wealth by age particularly has risen.

Of course with any constrained optimisation under uncertainty, the realisations of random income variables like labour income or asset returns or of debt variables like the mortgage interest rate may be so bad that future feasibility is impossible and the individual must default on his obligations. There is a large literature on mortgage default on housing partly inspired by the subprime mortgage crisis in the US. Major advances have been made to understand voluntary and involuntary US mortgage default. A key study here is Campbell and Cocco (2015). But in the UK even with the impact of the global financial crisis, mortgage default has been found less important. The data shows that the arrears rate more or less hovers around 1% of mortgage loan debt (Building Societies Association) and Aron and Muellbauer (2016) reinforce both this and why it is so. Consequently we abstract from these problems by essentially requiring that there is always a portfolio which has a zero probability of default. For example, this holds if the return on housing is always above the interest cost of the mortgage. The reason is that we primarily want to see how the life cycle environment (basically the past of a household and its expected future) and the imperfect asset market restrictions affect the housing and financial asset portfolio choices of the household as it ages.

We assume financial markets are imperfect: there is a safe return asset, a risky purely financial asset, housing mortgage debt. The only way of borrowing is through the mortgage which must be associated with house purchase but the amount that can be borrowed on the mortgage is the lower of a loan to house value ratio constraint and a loan to labour income constraint. Returns on these assets are uncertain over time, so are house prices and labour income. All random variables can be correlated.

Housing is basically measured in quality adjusted square meters. The individual can rent and/or buy units of housing. He can also buy housing to rent out. There is a minimum size of house that can be purchased, e.g. one cannot purchase one square meter of housing but it is possible to rent it. Housing and the financial asset portfolio are readjusted each period of time in light of changes in expectations and in the realisations of past labour income and asset returns (including housing). The constraint structure is relatively complicated and as usual for general time additive strictly concave preferences the optimal decisions over time are not analytically soluble. However the framework is rich in including most of the important empirical features of housing: it includes the housing tenure choice (rent or buy), the possibility of buy to let, part renting and part buying (e.g. shared ownership systems) and financial borrowing constraints especially the loan to value and loan to income constraints on the mortgage. We separate the consumption and investment sides of housing by specifying two variables, one for housing investment H_t and one for housing consumption h_t.

We measure all returns and the mortgage interest rate in real terms. This is consistent with the standard UK mortgage contract having an adjustable rate of interest[1](FCA, 2016). Each period of life, the consumer can costlessly readjust their portfolio and select nonhousing consumption. Thus mortgage refinancing is allowed within the loan to value (LTV) and loan to income (LTI) constraints. This allows for financing either nonhousing consumption or other asset investment via equity withdrawal. It also allows the mortgage to be used within these limits to hedge income shocks. Empirical evidence shows that both of these are important in the US (Chen et al., 2013; Mian and Sufi, 2011). There is also some theoretical backing for using an adjustable rate mortgage with flexible refinancing and housing retraded every period (Piskorski and Tchistyi, 2010).

There are broadly two approaches to deriving the optimal decisions in such a framework. The first is to fix the preferences and the main parameters involved (utility parameters, the discount rate, the joint distribution of uncertain asset returns and labour income, the rental income and the parameters in the mortgage constraints). Then numerically solve for the time path of optimal decisions. This approach is quite widely used by Cocco (2005), Attanasio et al. (2012). But even in these papers the utility used is quite special e.g. if c_t and h_t refer to nonhousing consumption and housing services respectively, in Cocco (2005) lifetime utility for t<T is $\Sigma_t \beta^t \frac{(c_t^{1-\theta} h_t^{\theta})^{1-\gamma}}{1-\gamma}$ which we can rewrite with a linear transformation as

$$\Sigma_t \beta^t \frac{(c_t^{1-\theta} h_t^{\theta})^{1-\gamma}}{1-\gamma} = \frac{1-\delta}{1-\gamma} \left(\frac{\Sigma_t \beta^t c_t^{1-\rho} h_t^{\alpha}}{1-\delta} \right)$$

where $1-\rho = (1-\theta)(1-\gamma), \alpha = \theta(1-\gamma)$. So it is homothetic.

A key issue is the balance between generality of the optimal solutions and generality of the model. With a very general model, analytical solutions are impossible and so one is forced to use numerical/simulation solution which inevitably depends on precise parameter assumptions. On the other hand, a narrower model can generate analytical solutions which are universal for all parameter values. We compromise by using a general preference framework

[1] In the UK, although around 50% of mortgage loans are termed fixed rate, in fact the rate is only fixed for a limited time period, usually two years.

which can establish some analytical insights into the best solution method and the nature of possible solutions. We then specialise this to preferences from which closed form analytical solutions can be calculated but in which the dynamic time path of solutions still depends on realisations of random variables and so has to be simulated. Both these models have no borrowing constraints in any asset except the mortgage backed by collateral from the house purchase, although mortgage debt is constrained by LTV and LTI ratios and no-short-selling constraint on risky asset.

Net worth a_t is defined as the sum of the total value of assets (safe, risky and housing) net of mortgage debt. To find the solution we first determine the optimal portfolio holdings for the safe and risky assets and for mortgage finance for given total savings a_t, housing investment H_t and consumption levels c_t and h_t. These only have an effect on the future value function. Next we determine the split of spending between c_t and h_t for given values of a_t and H_t. Conditional on a_t and H_t, these only affect the current utility. This reduces the value function to a function of a_t, H_t, cash on hand m_t and other state variables. Finally we solve for optimal values of a_t and H_t in terms of m_t and other state variables in turn. The result is the value function at any date and the optimal decisions at any date. This allows explicit characterisation of the solution path in a framework which allows for constraints preventing borrowing in all assets except the mortgage, mortgage constraints (based on loan to value and loan to income) and housing tenure choices including renting, buy to let, part renting part buying, or owning (with or without mortgage finance).

We start with quite unrestricted preferences and derive some key properties of the optimal decisions under our assumptions on asset returns. The main one is that we assume that the expected marginal value of returns on the risky asset dominates that on the safe asset and the cost of the mortgage. A main finding is that so long as it is optimal to participate in housing it will be financed with a maximum possible mortgage so that one of the mortgage constraints will bind and there will be no investment in the safe asset. Given that we can reduce the decision problem from one with six choice variables to one with only two variables: net worth in a period and investment in housing. Still with general preferences we characterise the range of qualitative types of solution which can emerge e.g. zero saving, save all current cash on hand, specialise any investment in either the risky asset or housing or diversify between the two. There may also be an interior solution. However, these results are just qualitative and not in a closed form. So next we specialise the preferences taking within period utility to be a special case of the Cocco class which does allow closed form solutions. We derive these.

The results are with general preferences and the return assumptions we use:

(i) so long as there is investment in housing, there is always a maximum constrained mortgage.

(ii) whether the mortgage is income or value constrained depends on planned housing investment at t reflected in H_t. The higher the planned housing investment is the more likely that the mortgage is loan to income constrained.

(iii) investing in the safe asset is not worthwhile.

(iv) there is a tradeoff between investing in the risky asset or housing which partly is conditional on the assumed marginal value of asset returns.

(v) if the relevant future (stochastically) discounted marginal utility of the future (taking the optimal portfolio of assets into account) is low enough, then it is optimal to consume all cash on hand today and save nothing for the future. Conversely if the current marginal utility of spending (after optimal allocation between housing and non-housing consumption) is low enough, despite the Inada conditions on current consumption forms, it is optimal to transfer all possible cash on hand into the future.

(vi) even if the stochastically discounted risky asset return is higher than that on housing, it may still be optimal to invest in housing and even to invest all savings in housing if the return to buy to let/ savings on rental income are high enough.

(vii) in general the higher is planned net worth the more likely that there is buy to let. But with low planned net worth, the optimum may involve both purchase of some housing and in additional renting of some housing. With special preferences we can derive analytical conditions for buy to let to occur.

With special preferences (Cobb-Douglas utility within a period) optimal behaviour can only be at corners of the constraint set and we can derive explicit equations for the value function and the solutions.

Our paper is related to a set of literature that includes housing consumption and housing investment in the life cycle model. Attanasio et al. (2012) numerically solves a life-cycle model for households choosing consumption, saving and housing when they face uncertainties on both income and house prices with mortgage borrowing constraints. But there are only three types of housing: renting, owning a flat and owning a house. And there is only one asset in their model. This implies the mortgage interest rate and safe saving interest rate are exactly the same, which is a restriction on modeling the interaction between different asset classes. Cocco (2005) studies the portfolio choice of homeowners by numerically solving a life cycle model with continuous housing, one riskless asset, one risky asset and mortgage debt. But he does not study the tenure choice (buy and/or rent). Brueckner (1997) also focuses on the behaviour of homeowners only. While including multiple assets and allowing for buy to let behaviour, he does not separate the mortgage and risky asset and does not solve the model explicitly. If the mortgage rate is identified with the safe rate, then there is no mortgage debt interest risk. Our paper differs from the existing literature in the following aspects. We derive some analytical properties for general preferences. Moreover, instead of solving the model numerically, we derive closed form solutions for the special preferences. This avoids the impreciseness of solution caused by interpolation and extrapolation. Second, we distinguish among a safe asset, mortgage debt, and a risky asset, which allows us to simulate the impact of uncertain asset returns for different assets on the individual choices. Third, we do not impose any restriction on the relative magnitude of housing consumption and housing investment, which allows for many different choices including renting only, owner occupation, buy to let, partly renting and partly owning (e.g. shared ownership scheme in the UK).

The rest of the paper is organised as follows. Section 2 introduces the general framework. Section 3 specialises on the model with Cobb-Douglas utility and proves the linearity of the value function under some assumptions. Closed form solutions are derived for this model. Section 4 simulates the life cycle paths of consumption and asset allocation with stochastic income and asset return processes. Finally, Section 5 concludes.

5.2 The general framework

In each period t utility depends on consumption c_t and the use of housing h_t. It is strictly concave and increasing in these variables.

$$u(c_t, h_t)$$

Cash on hand at period start is m_t measured in units of consumption used for investing (in safe asset A_t, risky asset F_t) or the purchase of housing units $p_t H_t$ (p_t is the price of a unit of housing measured in units of consumption, H_t is the units purchased) or consumption c_t. If housing is purchased the consumer can take out a mortgage in amount M_t in units of consumption. This gives a budget constraint on the use of cash on hand at t.

$$m_t = c_t + F_t + A_t + p_t H_t - M_t + y_{rt}(h_t - H_t)$$

Dynamics of cash on hand: $H_t - h_t$ is rented housing, $p_t(H_t - h_t)$ is rented housing in units of consumption which generates a rental income $y_{rt}(H_t - h_t)$. For each unit of housing rented, rental income/cost is y_r in units of consumption. Define the rate of return on housing owned by $r_{Ht} = (p_{t+1} - p_t)/p_t$ so we can write $p_{t+1}H_t = (1 + r_{Ht})p_t H_t$. Then next period's cash on hand evolves from the decisions at t according to

$$\begin{aligned} m_{t+1} &= y_{t+1} + (1 + r_{At+1})A_t + (1 + r_{Ft+1})F_t - (1 + r_{Mt+1})M_t + p_{t+1}H_t \\ &= y_{t+1} + R_{At+1}A_t + R_{Ft+1}F_t - R_{Mt+1}M_t + R_{Ht+1}p_t H_t \end{aligned} \tag{5.1}$$

where $r_{At+1}, r_{Ft+1}, r_{Mt+1}, r_{Ht+1}$ are the realised real interest rates on the various assets and $R_{At+1}, R_{Ft+1}, R_{Mt+1}, R_{Ht+1}$ are defined as the gross returns on safe asset, risky asset, mortgage and housing wealth ($R_{it} = 1 + r_{it}$); y_t is labour income of period t. At date t the returns on assets $F_{t+1}, H_{t+1}, M_{t+1}$, the unit cost of rental y_{rt+1} and labour income in the future are uncertain. However the future interest rate on the safe asset is certain. And current house price and rental cost are known at the start of the period.

It is convenient to define the net worth in period t, $a_{t,}$ as the sum of the values of safe asset A_t, risky asset F_t, mortgage M_t, and housing wealth $p_t H_t$ owned by households, where p_t is the house price in period t in terms of units of c_t and H_t is the size of housing owned by households.

$$a_t = A_t + F_t - M_t + p_t H_t$$

Then we can write

$$m_t = c_t + a_t + y_{rt}(h_t - H_t) \tag{5.2}$$

The constraints on the possible mortgage level reflect the facts that a mortgage can only be taken against the value of housing purchased but the upper limit on the amount borrowed is the smaller of some % τ_1 ($\tau_1 < 1$) of the house value purchased and some multiple τ_2 of current labour income. In addition $M_t \geq 0$ and since $\tau_1 < 1$, $M_t < p_t H_t$. Equation (5.2) is the within-period budget constraint describing how resources are allocated among housing and non-housing consumption, and housing wealth. Equation (5.1) is the intertemporal budget constraint describing how wealth accumulates through time. The time line of the model is shown in Figure 5.1.

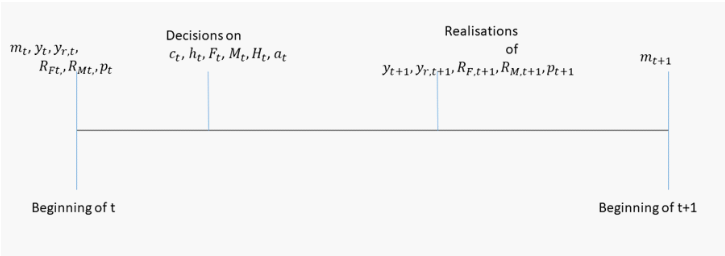


Figure 5.1: Timeline of the model

Here we distinguish between two types of state variables, some are purely exogenous but cash on hand m_t is not purely exogenous. We define the purely exogenous variables at t in our model as $S_t = (y_t, y_{rt}, R_{Ft}, R_{Mt}, p_t)$.

Note we don't have any constraint on the relative values of H_t and h_t, which has the following implications. When the housing owned is bigger than the housing consumed, i.e. $H_t > h_t$, household would rent out $(H_t - h_t)$ (buy-to-let) and gain the rental income $y_{rt}(H_t - h_t)$. This can be interpreted as either renting out the extra rooms in the same house consumed or a household having some buy-to-let housing and at the same time living in a rented house which is smaller than the buy-to-let house. When the housing owned is smaller than the housing consumed, i.e. $H_t < h_t$, the household would need to pay rent $y_{rt}(H_t - h_t)$.This can be interpreted as either an equity sharing scheme in the UK or a household having some buy-to-let housing and at the same time living in a different rented house which is bigger than the buy-to-let house. Finally, if the housing owned is the same as the housing consumed, i.e. $H_t = h_t$, then the household has neither rental income nor rental expenditure. This makes the model more general than Brueckner (1997) and Henderson and Ioannides (1983) that require housing owned to be bigger than housing consumed.

The constraints on asset variables (H_t, A_t, F_t, M_t) reflect the borrowing constraints. All the assets must be nonnegative. The only borrowing possible is in the mortgage but this is constrained by both the loan to value constraint and the loan to income constraint so

$M_t \leq \min(\tau_1 p_t H_t, \tau_2 y_t)$, and can only be accessed if housing is purchased.

We assume that there is a minimum size of house H_t^* that can be purchased, which creates a threshold for owner occupation. That is if $H_t > 0$ then $H_t \geq H_t^*$. But houses of any divisible above H_t^* can be purchased.

The asset constraints are

$$
\begin{aligned}
M_t &\leq \min(\tau_1 p_t H_t, \tau_2 y_t) &\quad (5.3a)\\
H_t, A_t, F_t, M_t &\geq 0 &\quad (5.3b)\\
H_t &\geq H_t^* \text{ if } H_t > 0 &\quad (5.3c)
\end{aligned}
$$

The overall optimisation problem is

$$
\begin{aligned}
& \max_{c_t, a_t, h_t, H_t F_t, M_t, A_t} \Sigma\beta^t u(c_t, h_t), t < T \\
& \text{s.t. } m_t = c_t + a_t + y_{rt}(h_t - H_t) \\
m_{t+1} &= y_{t+1} + R_{At+1}A_t + R_{Ft+1}F_t - R_{Mt+1}M_t + R_{Ht+1}p_t H_t \\
a_t &= A_t + F_t - M_t + p_t H_t \\
& \text{the asset constraints (5.3)}
\end{aligned}
$$

The general Bellman equation is

$$
v_t(m_t) = \max_{c_t, h_t, M_t, H_t, A_t, a_t} u(c_t, h_t) + \beta E_t v_{t+1}(m_{t+1}, S_{t+1}), t < T
$$

s.t.

$$
m_t = c_t + y_{rt}(h_t - H_t) + a_t \quad (5.4)
$$

$$
\begin{aligned}
m_{t+1} &= y_{t+1} + R_{At+1}A_t + R_{Ft+1}F_t - R_{Mt+1}M_t + R_{Ht+1}p_t H_t &\quad (5.5)\\
& \text{the asset constraints (5.3)} &\quad (5.6)
\end{aligned}
$$

Here $v(m_t)$ is the value function, of course implicitly it also depends on the probability distribution of the future uncertain variables $y_{rt+1}, y_{t+1}, R_{it+1}$. Given the properties of $u()$, it is strictly increasing and strictly concave in m_t (Bobenrieth et al., 2012).

5.2.1 The final period

Since there is no bequest motive, optimally $m_{T+1} = 0$ and so in the final period it is always better to rent than buy. At T, the only choices are of c_T, h_T which are chosen within the budget constraint $c_T + y_{rT}h_T = m_T$ to maximise final period utility.

$$\begin{aligned} & \max_{c_T . h_T} u(c_T, h_T) \\ \text{s.t. } c_T + y_{rT}h_T &= m_T \end{aligned}$$

Since $u(.)$ is strictly concave and strictly increasing in each variable, its indirect utility $u^*(m_T, y_{rT})$ is also strictly concave and strictly increasing in m_T (Bobenrieth et al., 2012)

By definition,

$$a_{T-1} = A_{T-1} + F_{T-1} + p_{T-1}H_{T-1} - M_{T-1}$$

The only situation where $a_{T-1} = 0$ is when $A_{T-1} = F_{T-1} = p_{T-1}H_{T-1} = M_{T-1} = 0$. This is because $A_{T-1}, F_{T-1} \geq 0, p_{T-1}H_{T-1} + M_{T-1} > 0$ for any non-zero H_{T-1} and M_{T-1} and any $0 < \tau_1 < 1$.

5.2.2 Generic period t

For a generic period t the optimisation problem is

$$v_t(m_t) = \max_{c_t, h_t, F_t, M_t, H_t, A_t, a_t} u(c_t, h_t) + \beta E_t v_{t+1}(m_{t+1}, S_{t+1}), t < T$$

s.t.

$$m_t = c_t + y_{rt}(h_t - H_t) + a_t \tag{5.7}$$

$$m_{t+1} = y_{t+1} + R_{At+1}A_t + R_{Ft+1}F_t - R_{Mt+1}M_t + R_{Ht+1}p_tH_t \tag{5.8}$$

$$\begin{aligned} M_t &\leq \min(\tau_1 p_t H_t, \tau_2 y_t) &(5.9) \\ H_t, A_t, F_t, M_t &\geq 0 \\ H_t &\geq H_t^* \text{ if } H_t > 0 &(5.10) \end{aligned}$$

5.2.3 Optimal conditional portfolio allocation

Fix a_t, H_t and think of the optimal A_t, F_t, M_t. These must solve

$$\begin{aligned} & \max_{A_t, F_t, M_t} E_t v_{t+1}(m_{t+1}, S_{t+1}) \\ = & \max_{A_t, F_t, M_t} E_t v_{t+1}(y_{t+1} + R_{At+1}A_t + R_{Ft+1}F_t - R_{Mt+1}M_t + R_{Ht+1}p_tH_t, S_{t+1}) \end{aligned}$$

$s.t.$

$$\begin{aligned} M_t &\leq \min(\tau_1 p_t H_t, \tau_2 y_t) \\ A_t, F_t, M_t &\geq 0 \end{aligned}$$

Assume

$$E_t \frac{\partial v_{t+1}}{\partial m_{t+1}}(R_{Ft+1} - R_{Mt+1}) > 0 \tag{5.11a}$$

$$E_t \frac{\partial v_{t+1}}{\partial m_{t+1}}(R_{Ft+1}) - R_{At+1} E \frac{\partial v_{t+1}}{\partial m_{t+1}} > 0 \tag{5.11b}$$

This means that the covariation of the marginal value of m_{t+1} with the risky asset return exceeds its covariation with either the mortgage rate or the safe asset rate. Since $E_t R_{Ft+1} > E_t R_{Mt+1}, R_{At+1}$ is a weak assumption, the overall assumption holds if $cov(\frac{\partial v_{t+1}}{\partial m_{t+1}}, R_F) >$

$\max(0, cov(\frac{\partial v_{t+1}}{\partial m_{t+1}}, R_M))$. That is variations in the risky rate of return have a bigger impact on the marginal future value than variations in the mortgage rate.

A sufficient condition for this is $R_{Ft+1} > \max(R_{Mt+1}, R_{At+1})$ with probability 1, i.e. always the realised risky return is above the mortgage and the safe rate. Then just so long as $\frac{\partial v_{t+1}}{\partial m_{t+1}} > 0$ with probability 1 (which is a weak assumption) it follows that $E_t \frac{\partial v_{t+1}}{\partial m_{t+1}} R_{Ft+1} > \max(E_t \frac{\partial v_{t+1}}{\partial m_{t+1}} R_{Mt+1}, R_{At+1} E_t \frac{\partial v_{t+1}}{\partial m_{t+1}})$.

Then

(i) optimally $A_t = 0$. If $A_t > 0$ it would raise utility to reduce A_t a little and use this reduction to further invest in the risky asset[2]

(ii) A higher mortgage and using the extra funds to invest in the risky asset must raise expected value[3]. This process can continue until M_t reaches its upper bound

So we know that conditional on given values of a_t and H_t, the optimum for general preferences must have

$$\begin{aligned} M_t &= \min(\tau_1 p_t H_t, \tau_2 y_t) \\ A_t &= 0 \end{aligned}$$

[2] Variations satisfying $dF_t + dA_t - dM_t$ are feasible. So setting $dF_t = -dA_t > 0$ will move to another feasible point which generates higher value.

[3] Similarly if the mortgage is not yet constrained choosing variations $dF_t = -dM_t > 0$ will raise the value but retain feasibility.

This also means that $F_t = a_t + \min(\tau_1 p_t H_t, \tau_2 y_t) - p_t H_t$.

5.2.4 The optimal c_t, h_t conditional on a_t, H_t

Still with a_t, H_t fixed at levels such that $m_t + y_{rt} H_t - a_t \geq 0$ we can calculate the optimal c_t, h_t. This gives an indirect utility in general of $u^*(m_t + y_{rt} H_t - a_t)$. Note that $u^*()$ is concave. But if $w_t = m_t + y_{rt} H_t - a_t = 0$ the only feasible solution is $c_t, h_t = 0$. In general the solutions when $m_t + y_{rt} H_t - a_t > 0$ allow us to determine the sign of $h_t - H_t$ (i.e. if there is buy to let or if a house owner rents additional housing). Write the solution for h_t as $h_t = f(w_t, y_{rt})$; he does neither if $H_t = f(w_t, y_{rt}) = f(m_t + y_{rt} H_t - a_t, y_{rt})$. This gives us a locus in a_t, H_t space whose slope is defined by

$$\frac{dH_t}{da_t} = -\frac{\partial f / dw_t}{1 - y_{rt} \partial f / dw_t}$$

If both c_t, h_t are normal goods then $\partial f / dw_t > 0$ and also $\partial f / dc_t > 0$. Since $dc_t / dw_t + y_{rt} dH_t / dw_t = 1$ this means that $1 - y_{rt} \partial f / dw_t > 0$ and hence the zero BTL locus is downward sloping; it also requires $H_t = 0$ when $a_t = m_t$.

5.2.5 The optimal a_t, H_t

The remaining part of the optimisation is to determine a_t, H_t

$$\begin{aligned} v_t(m_t) &= \max_{H_t, a_t} u^*(m_t + y_{rt} H_t - a_t) \\ &\quad + \beta E_t v_{t+1}(y_{t+1} + R_{F,t+1}(a_t + \min(\tau_1 p_t H_t, \tau_2 y_t) - p_t H_t) - R_{M,t+1} \min(\tau_1 p_t H_t, \tau_2 y_t)) \\ &\quad + p_{t+1} H_t) \end{aligned}$$

s.t.

$$\min(\tau_1 p_t H_t, \tau_2 y_t) \geq p_t H_t - a_t \tag{5.12}$$

$$m_t + y_{rt} H_t - a_t \geq 0 \tag{5.13}$$

$$F_t \geq 0 \tag{5.14}$$

$$a_t, H_t \geq 0 \tag{5.15}$$

$$H_t \geq H_t^* \text{ if } H_t > 0 \tag{5.16}$$

Since from the definition of net worth

$$F_t = a_t + \min(\tau_1 p_t H_t, \tau_2 y_t) - p_t H_t \geq 0$$

(5.12) and (5.14) coincide so the constraints are

$$\begin{aligned} F_t &= a_t + \min(\tau_1 p_t H_t, \tau_2 y_t) - p_t H_t \geq 0 && (5.17) \\ m_t + y_{rt} H_t - a_t &\geq 0 \\ a_t, H_t &\geq 0 \\ H_t &\geq H_t^* \text{ if } H_t > 0 && (5.18) \end{aligned}$$

Since we know optimally there is a maximal mortgage we can write the objective function in piecewise form

$$\begin{aligned} v_t(m_t) &= V_{1t}() = \max_{H_t, a_t} u^*(m_t + y_{rt} H_t - a_t) + \beta E_t v(y_{t+1} + R_{F,t+1}(a_t + \tau_1 p_t H_t - p_t H_t) \\ & \quad -R_{M,t+1} \tau_1 p_t H_t + p_{t+1} H_t) \\ \text{if } \tau_1 p_t H_t &> \tau_2 y_t \\ v_t(m_t) &= V_{2t}() = \max_{H_t, a_t} u^*(m_t + y_{rt} H_t - a_t) + \beta E_t v(y_{t+1} + R_{F,t+1}(a_t + \tau_2 y_t - p_t H_t) \\ & \quad -R_{M,t+1} \tau_2 y_t \ + p_{t+1} H_t) \\ \text{if } \tau_1 p_t H_t &< \tau_2 y_t \end{aligned}$$

Define the marginal rates of substitution of the two piecewise parts of v by

$$MRS_{1t} = -\frac{\partial V_{1t}/\partial a_t}{\partial V_{1t}/\partial H_t}, MRS_{2t} = -\frac{\partial V_{2t}/\partial a_t}{\partial V_{2t}/\partial H_t}$$

These will be useful in the sequel.

5.2.6 Feasible set of $v_t(a_t, H_t)$

There are two basic forms of the main part of the feasible set. We can see these most clearly if $H_t^* = 0$:

(1) if the maximum size house that can be purchased when all savings are spent on house purchase is higher than the highest house size possible with a LTV constrained mortgage ($H_t = \frac{a_t}{(1-\tau_1)p_t}$), then it would be possible for the individual to purchase a still larger house with the aid of a larger LTI constrained mortgage. We call this condition 1: $\frac{m_t}{1 - y_{rt}\frac{1}{(1-\tau_1)p_t}} > \frac{\tau_2 y_t (1-\tau_1)}{\tau_1}$. Basically here the feasible set is a convex polygon, it has a boundary kink where $F_t = 0$ and the individual switches from a LTV to LTI constrained mortgage.

(2) if the maximum size house that can be purchased when all savings are spent on house purchase is lower than the highest house possible with a LTV constrained mortgage then the

LTI constraint is irrelevant. We call this condition 2: $\frac{m_t}{1-y_{rt}\frac{1}{(1-\tau_1)p_t}} < \frac{\tau_2 y_t(1-\tau_1)}{\tau_1}$. basically this generates a triangular feasible set.

If $H_t^* = 0$ these define the two types of feasible set. But if $H_t^* > 0$ then the feasible set splits into two parts: a convex polygon or triangle bounded below by H_t^* and then (since for non-purchasers with $H_t = 0, H_t^*$ plays no role), the feasible part of the axis $H = 0$. In all cases the feasible set has boundary kinks where different parts of its linear boundary segments meet (and at the ends of the feasible $H_t = 0$ axis).

The set can be shown graphically. In the diagrams below of the feasible sets the upper boundary of the LTV locus ($H_t = \frac{a_t}{p_t(1-\tau_1)}$ gives all combinations of a_t, H_t which give $F_t = a - (1-\tau_1)p_t H_t = 0$ when the LTV mortgage constraint binds). The upper boundary in the LTI region ($H_t = \frac{a_t+\tau_2 y_t}{p_t}$) similarly shows all combinations of a_t, H_t which give $F_t = a_t + \tau_2 y_t - p_t H_t = 0$ when the LTV mortgage constraint binds. To the southeast of these loci $F_t > 0$. The right hand boundary of the feasible set ($H = \frac{a_t - m_t}{y_{rt}}$) shows all combinations of a_t, H_t giving $c_t = h_t = 0$. To the northwest of this locus some resources are consumed on c_t, h_t in the current period.

The corners of the feasible set will be especially useful in the sequel (Table 5.1). For all possible levels of H_t^* and consequently all types of feasible set there are 10 corners in all, we label these by subscript i ($i = 1, ..., 10$) and Table 5.1 gives the values of a_t, H_t at each corner together with the level of H_t^* in relation to levels of H_t which give boundaries between different binding housing or housing finance constraints. In terms of these a_{2t} and a_{3t} are values determining if an LTI mortgage is possible.

Table 5.1: Solutions of net worth and housing investment (at kinks) and corresponding H* for general preference

Kink	H_t^*	C1,C2	a_t	H_t
1	any	any	$a_{1t} = 0$	$H_{1t} = 0$
2	$[0, H_{2t}]$	2	$a_{2t} = \frac{m_t}{1-y_{rt}\frac{1}{(1-\tau_1)p_t}}$	$H_{2t} = \frac{1}{p_t(1-\tau_1)} \frac{m_t}{(1-y_{rt}\frac{1}{(1-\tau_1)p_t})}$
3	$[0, H_{3t}]$	1	$a_{3t} = \frac{\tau_2 y_t(1-\tau_1)}{\tau_1}$	$H_{3t} = \frac{1}{p_t}\frac{\tau_2 y_t}{\tau_1}$
4	$[0, H_{4t}]$	1	$a_{4t} = \frac{m_t+\frac{y_{rt}y_t\tau_2}{p_t}}{1-\frac{y_{rt}}{p_t}}$	$H_{4t} = \frac{m_t+\frac{\tau_2 y_{rt} y_t}{p_t}}{p_t-y_{rt}} + \frac{\tau_2 y_t}{p_t}$
5	any	any	$a_{5t} = 0$	$H_{5t} = m_t$
6	$[0, H_{3t}]$	1	$a_{6t} = m_t + \frac{\tau_2 y_{rt} y_t}{\tau_1 p_t}$	$H_{6t} = \frac{\tau_2 y_t}{\tau_1 p_t}$
7	(H_{3t}, H_{4t})	1	$a_{7t} = p_t H_t^* - \tau_2 y_t$	$H_{7t} = H_t^*$
8	(H_{3t}, H_{4t})	1	$a_{8t} = H_t^* y_{rt} + m_t$	$H_{8t} = H_t^*$
9	$(0, H_{2t})$	2	$a_{9t} = H_t^* p_t(1-\tau_1)$	$H_{9t} = H_t^*$
10	$(0, H_{2t})$	2	$a_{10t} = H_t^* y_{rt} + m_t$	$H_{10t} = H_t^*$

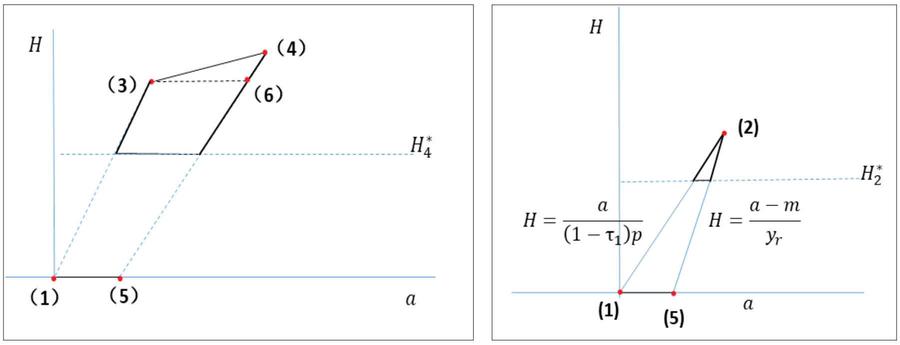
For the purpose of illustration, we define the notations C1 and C2 in Figure 5.1 as

condition 1 (C1): $a_{2t} < 0$ or $a_{3t} < a_{2t}$ and $a_{2t} > 0$

condition 2 (C2): $a_{3t} \geq a_{2t}$ and $a_{2t} > 0$

At this stage the feasible set is defined in the space of a_t, H_t. We describe the different possible shapes and positions of the feasible set. Figure 5.2 shows just two examples (the full set of geometric shapes is in the Appendix). Within each possible feasible set there is also a BTL locus along which optimal behaviour leads to $H_t = h_t$. We know this locus is downward sloping and passes through $H_t = 0, a_t = m_t$ but we do not know any more about its shape[4]. For H_t below the locus, the household is renting additional space to the housing owned. Above the locus the household is engaged in BTL.

[4]For this reason we do not show the BTL locus in the diagram. Later when we specialise the preference, the algebraic is known and so we add in the BTL locus in that case.



Condition 1 Condition 2

Figure 5.2: Two examples of feasible sets

Of course depending on whether $H_t^* \gtreqless H_{3t}, H_{4t}, H_{2t}$ a loan to income constrained mortgage may or may not be feasible, and indeed any purchase of housing may be impossible if H_t^* is high enough (above H_{4t}, H_{2t})[5]. Under condition 1 the feasible region is the convex polygon bounded by the kinked straight lines on the left which is the locus along which $F_t = 0$, the right hand positively sloped line along which all disposable income is saved ($c_t = h_t = 0$) and part of the horizontal axis. This feasible region is divided into the upper part where $\tau_1 p_t H_t > \tau_2 y_t$ so the effective mortgage constraint is the loan to income ratio (LTI) and the lower part where $\tau_1 p_t H_t < \tau_2 y_t$ and the effective mortgage constraint is loan to value (LTV). Under condition 2 there is no feasible point with a LTI constrained mortgage.

Next we show that the overall objective is concave in a_t, H_t. To show this we know that each V_{it} $(i = 1, 2)$ is concave and that $v_t = V_{1t}$ for $H_t < \frac{\tau_2 y_t}{\tau_1 p_t}; v_t = V_{2t}$ for $H_t > \frac{\tau_2 y_t}{\tau_1 p_t}$. Moreover $V_{2t} = V_{1t}$ if $H_t = \frac{\tau_2 y_t}{\tau_1 p_t}$. We also know that at any point $\frac{\partial V_{1t}(a_t,H_t)}{\partial a_t} = \frac{\partial V_{2t}(a_t,H_t)}{\partial a_t}; \frac{\partial V_{1t}(a_t,H_t)}{\partial H_t} > \frac{\partial V_{2t}(a_t,H_t)}{\partial H_t}$. Since the denominator of MRS_{1t} of V_{1t} is larger than that of MRS_{2t} of V_{2t}, it follows that whatever the sign pattern of the partial derivatives of V_{1t}, V_{2t}, the overall utility function v_t is concave.

For all directions of change of $V_{it}()$ $(i = 1, 2)$ (i.e. for all combinations of signs of the four partial derivatives of V_{1t}, V_{2t} with respect to their arguments a_t, H_t this implies at any switch point a_t, H_t, there is an increase in the degree of concavity of $v_t(a_t, H_t)$. The full six key examples of how this works are in the following diagrams (Figure 5.3) which show how the marginal rate of substitutions of V_{1t} and V_{2t} are related at switchpoints.

[5]The full variety of feasible sets is shown in the appendix.

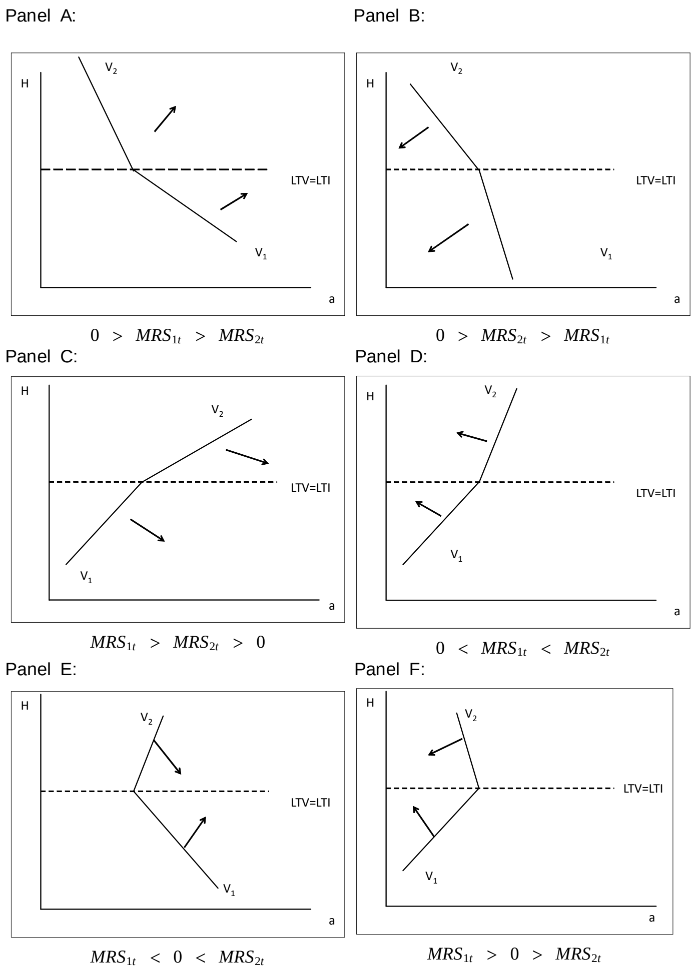


Figure 5.3: Slope of indifference curves at switchpoints

From Figure 5.3 (See Appendix C.3 for more details), in Panel A the indifference curves in both LTI and LTV regions are negatively sloped and increasing to the northeast. In Panel B the indifference curves in the LTV region are negatively sloped but are positively sloped in the LTI region and increasing to the southwest. In Panel C the indifference curves in both LTI and LTV regions are positively sloped and increasing to the southeast. In Panel D the indifference curves in both LTI and LTV regions are positively sloped and increasing to the northwest. In Panel E the indifference curves are positively sloped in the LTI region increasing to the southeast but negatively sloped in the LTV region increasing to the northeast. In Panel F the indifference curves are negatively sloped in the LTI region increasing to the southwest but positively sloped in the LTV region increasing to the northwest.

Examining the possible feasible set, in total they consist of a convex set or the union of a convex set with linear boundaries and a line segment along the horizontal axis. To characterise the optimum notice that there may be optima at kink points between two adjacent boundary segments, a tangency to one of the boundary segments or in the interior of the feasible set. We take these in turn starting with the boundary kinks.

Table 5.2 summarises the first ten corner solutions (at kinks) and corresponding conditions for the general preference case.

Table 5.2: Conditions for corner solutions (at kinks) 1-10 with general preference

Corner	condition	$\frac{\partial V_{1t}(a_t,H_t)}{\partial a_t}$	$\frac{\partial V_{1t}(a_t,H_t)}{\partial H_t}$	$\frac{\partial V_{2t}(a_t,H_t)}{\partial H_t}$	H_t^*	MRS
1	any	<0	<0		any	$MRS_{1t}(0,0) < 0$
1	C1	<0	any		$H_t^* > H_{4t}$	any
1	C2	<0	any		$H_t^* > H_{2t}$	any
1	C2	<0	>0		$0 \leq H_t^* \leq H_{2t}$	$MRS_{1t}(0,0) > \frac{1}{(1-\tau_1)p_t}$
1	C1	<0	>0		$0 \leq H_t^* \leq H_{3t}$	$MRS_{1t}(0,0) > \frac{1}{(1-\tau_1)p_t}$
2	C2	>0	>0		$0 \leq H_t^* \leq H_{2t}$	$MRS_{1t}(a_{2t},H_{2t}) < 0$
2	C2	<0	>0		$0 \leq H_t^* \leq H_{2t}$	$MRS_{1t}(a_{2t},H_{2t}) < \frac{1}{(1-\tau_1)p_t}$
2	C2	>0	<0		$0 \leq H_t^* \leq H_{2t}$	$MRS_{1t}(a_{2t},H_{2t}) > \frac{1}{y_{rt}}$
3	C1	<0	>0	<0	$0 \leq H_t^* \leq H_{3t}$	$MRS_{1t}(a_{3t},H_3) > 0$ and $MRS_{2t}(a_{3t},H_3) < 0$
3	C1	<0	$> \frac{\partial V_{2t}(a_{3t},H_{3t})}{\partial H_t}$	>0	$0 \leq H_t^* \leq H_{3t}$	$MRS_{1t}(a_{3t},H_{3t}) < \frac{1}{p_t} < \frac{1}{(1-\tau_1)p_t}$ and $MRS_{2t}(a_{3t},H_{3t}) > \frac{1}{(1-\tau_1)p_t} > \frac{1}{p_t}$
4	C1	>0		>0	$0 \leq H^* \leq H_{4t}$	$MRS_{2t}(a_{4t},H_{4t}) < 0$
4	C1	<0		>0	$0 \leq H^* \leq H_{4t}$	$MRS_{2t}(a_{4t},H_{4t}) < \frac{1}{p_t}$
4	C1	>0		<0	$0 \leq H^* \leq H_{4t}$	$MRS_{2t}(a_{4t},H_{4t}) < \frac{1}{y_{rt}}$
5	any	>0	<0		any	$MRS_{1t}(a_{5t},H_{5t}) < \frac{1}{y_{rt}}$
5	C1	>0	>0		$H^* > H_{4t}$	$MRS_{1t}(a_{5t},H_{5t}) < 0$
5	C2	>0	>0		$H^* > H_{2t}$	$MRS_{1t}(a_{5t},H_{5t}) < 0$
6	C1	>0	<0	<0	$0 \leq H^* \leq H_{3t}$	$MRS_{1t}(a_{6t},H_{6t}) > \frac{1}{y_{rt}}$ and $MRS_{2t}(a_{6t},H_{6t}) < \frac{1}{y_{rt}}$
6	C1	>0	>0	<0	$0 \leq H^* \leq H_{3t}$	$MRS_{2t}(a_{6t},H_{6t}) < \frac{1}{y_{rt}}$
7	C1	<0		>0	$H_{3t} < H^* < H_{4t}$	$MRS_{2t}(a_{7t},H_{7t}) < \frac{H^*}{p_t H_7 - \tau_2 y_t}$
8	C1	>0		<0	$H_{3t} < H^* < H_{4t}$	$MRS_{2t}(a_{8t},H_{8t}) < \frac{1}{y_{rt}}$
9	C2	<0	>0		$0 \leq H^* \leq H_{2t}$	$MRS_{1t}(a_{9t},H_{9t}) \leq \frac{1}{(1-\tau_1)p_t}$
10	C2	>0	>0		$0 \leq H^* \leq H_{2t}$	$MRS_{1t}(a_{10t},H_{1t}) > \frac{1}{y_{rt}}$

(1) For the origin to be optimal there are various possibilities on the relationship between

the feasible set and preferences. If $H_t^* > 0$ then for the origin to be optimal it must dominate all points on the $H_t = H_t^*$ line unless housing is infeasible anyway ($H_t^* > H_{4t}, H_{2t}$ depending on the relevant condition). We also need it to dominate all points on the $H_t = 0$ feasible locus: this is ensured if $\frac{\partial v_t(0,0)}{\partial a_t} < 0$. Combined these conditions require that the indifference curves of V_{1t} are sloped so that either it's best to reduce both a_t, H_t or to reduce a_t but increase H_t (in this case the V_{1t} indifference curve is positively sloped) and we have to check that the origin dominates any point on the next upper level of the feasible set e.g. the H_t^* line if $H_t^* < H_{3t}$. Because of the concavity of V_{1t} this will certainly hold if the slope of the indifference curve of V_{1t} is greater than $\frac{1}{p_t(1-\tau_1)}$ which is the slope of the line joining the origin and the next feasible case with $H_t > 0$. But if this fails to hold we need to make a comparison of the utility level between $H_t = 0$ and the utility at the next feasible lowest value of H_t.

(2) For regime (2) to be optimal, we need condition 2 to hold. Given all the possible combinations of indifference curve shapes, it is optimal to choose regime 2 either if utility increases with each of a_t and H_t; or when the marginal utility of a_t and H_t are of opposite signs so that the indifference curves are positively sloped, if MRS_{1t} at point 2 is less than $\frac{1}{(1-\tau_1)p_t}$ (when $\frac{\partial v_t(a_{2t},H_{2t})}{\partial a_t} < 0$) .

(3) For regime (3) to be optimal, we need condition 1 to hold. We need to consider both V_{1t} and V_{2t} here. Whenever it is attainable, it requires V_{1t} and V_{2t} are both decreasing in a_t while V_{1t} is increasing in H_t and V_{2t} is decreasing in H_t. Alternatively, if both V_{1t} and V_{2t} are increasing in H_t and decreasing in a_t, regime (3) competes with regime (1); if MRS_{1t} at point 3 of the positively sloped indifference curve of V_{1t} is less than $\frac{1}{(1-\tau_1)p_t}$ point 3 dominates point 1 (and any point on the feasible part of the $H_t = 0$ axis.

(4) For regime (4) to be optimal, we need condition 1 to hold. Whenever it is attainable and V_{2t} is increasing in both a_t and H_t, it is optimal to choose regime (4). Alternatively, if V_{2t} is increasing in H_t and decreasing in a_t, regime (4) competes with regime (3), so we need MRS_{2t} at 4 to be lower than the relevant boundary slope $1/p_t$. Similarly if V_{2t} is decreasing in H_t and increasing in a_t, regime (4) competes with regime (6), so we need MRS_{2t} at 4 to be greater than the relevant boundary slope $1/y_{rt}$.

(5) Regime (5) is feasible in all cases. Whenever V_{1t} is decreasing in H_t and a_t at point 5 regime (5) is preferred to anywhere above it (also if any house purchase is infeasible). Alternatively, if V_{1t} is decreasing in H_t and increasing in a_t with MRS_{1t} at 5 less than $1/y_{rt}$, then

regime (5) is the optimum.

(6) For regime (6) to be optimal, we need condition 1 to hold. As for regime (3), we need to consider both V_{1t} and V_{2t} here. Whenever it is attainable ($H_t^* \geq \frac{\tau_2 y_t}{\tau_1 p_t}$) and V_{1t} and V_{2t} are both increasing in a_t and decreasing in H_t, it competes with regimes (4) and (5),it will dominate any other feasible point if $MRS_{2t} < 1/y_r < MRS_{1t}$ at the point 6. Alternatively, when V_{1t} is increasing in H_t and V_{2t} is decreasing in H_t, it competes with regime (4) and with an MRS condition regime(6) can be optimal.

(7) For regime (7) to be optimal, we need condition 1 to hold. When regime (3) is not attainable ($H_t^* > H_{3t}$) and V_{2t} is increasing in H_t and decreasing in a_t, it competes with regime (1) in particular. If MRS_{2t} at point 7 is less than $\frac{H^*}{p_t H^* - \tau_2 y_t}$, point 7 will dominate point 1.

(8) For regime (8) to be optimal, we need condition 1 to hold. When regime (6) is not attainable and V_{2t} is decreasing in H_t and increasing in a_t, it competes with regime (5). If MRS_{2t} at point 8 is greater than $1/y_{rt}$ point 8 will dominate point 5.

(9) For regime (9) to be optimal, we need condition 2 to hold. When V_{1t} is increasing in H_t and decreasing in a_t, it competes with regime (1), If MRS_{2t} at point 9 is less than $1/p_t$ point 9 will dominate point 1.

(10) For regime (10) to be optimal, we need condition 2 to hold. When V_{1t} is increasing in H_t and a_t, it competes with regime (5). If MRS_{2t} at point 10 is greater than $1/y_{rt}$ point 10 will dominate point 5.

For each segment of the boundary we can get relevant conditions for a tangency solution by finding the conditions which make the marginal rate of substitution of the relevant preferences above or below the slopes of the linear boundary segments at their corners equal to the relevant slope of the segment in question[6]. The possibilities are shown in Table 5.3.

[6] In addition there may also be an interior solution, so we will sometimes have to compare a proposed boundary solution in one part of the feasible set with the relevant value of the objective at corners in the nonempty interior part of the feasibe set.

Table 5.3: Conditions for corner solutions (tangencies) 11-16 with general preference

Label	Segment	MRS_{1t}	MRS_{2t}	H^*	H	
11	$F_t = 0, LTI$		$\frac{1}{p_t}$	$0 < H_t^* < H_{4t}$		$\frac{\partial V_1(0,a_t)}{\partial H_t} > 0$, some $a_t \in [0, m_t]$
12	$F_t = 0, LTV$	$\frac{1}{(1-\tau_1)p_t}$		$0 < H_t^* < H_{3t}$		
13	$H_t = H_t^*$	0		$H_t^* \leq H_{3t}$	H_t^*	$\frac{\partial V_1(0,a_t)}{\partial H_t} > 0$, some $a_t \in [0, m_t]$
14	$H_t = H_t^*$		0	$H_{3t} < H_t^* < H_{4t}$	H_t^*	$\frac{\partial V_2(H_{3t},a_t)}{\partial H_t} > 0$, some $a_t \in [0, m_t]$
15	$c_t = h_t = 0, H_t > H_t^*$	$\frac{1}{y_{rt}}$		$\neq \tau_2 y/\tau_1 p$ $0 < H_t^* < H_{4t}$		
16	$H_t = 0$	0		any	$H_t = 0$	

(11) describes a tangency on the $F_t = 0$ locus in the LTI area. MRS_{2t} must be positive and the direction of increase of utility is towards the north west. It is the optimal form of solution if $\partial V_{2t}(a_{4t}, H_{4t})/\partial a_t < 0$, and $0 < MRS_{2t}(a_{4t}, H_{4t}) > 1/p_t$ and $0 < MRS_{2t}(a_{3t}, H_{3t}) < 1/p_t$ if $H_t^* < \frac{\tau_2 y_t}{\tau_1 p_t}$ but $0 < MRS_{2t}(pH^* + \tau_2 y_t, H_t^*)/\partial H_t < 1/p_t$ if $H_t^* > \frac{\tau_2 y_t}{\tau_1 p_t}$

(12) similarly describes a tangency on the $F_t = 0$ locus but in the LTV area. Again the direction of increase of utility is towards the north west. It is the optimal form of solution when $0 < H_t^*, H_{3t}$ if $0 < MRS_{1t}((1-\tau_1)p_t H_t^*, H_t^*) > 1/p_t(1-\tau_1)$ and $0 < MRS_{1t}(a_{3t}, H_{3t}) < 1/p_t(1-\tau_1)$

(13) describes a tangency on the $H_t = H_t^*$ horizontal, it needs either $MRS_{1t}(a_t, H_t^*) = 0$ if $H_t^* < H_{3t}$ or $MRS_{2t}(a_t, H_t^*) = 0$ if $H_{4t} > H_t^* > H_{3t}$. in addition is must dominate the feasible part of the H_t^* axis which it can do if for example $V_{it}(a_t, 0)/\partial H_t > 0$ for $0 \leq a_t \leq m_t$

(14) describes a tangency along the $c_t = h_t = 0$ segment in the LTV section. It is the optimal form if V_{1t} increases towards the south east, $H_t^* < \frac{\tau_2 y_t}{\tau_1 p_t}$ and $MRS_{1t}((1-\tau_1)p_t H_t^*, H_t^*) > 0$ but $MRS_{1t}(m_t + y_{rt} H_t^*, H_t^*) < 0$.

(15) describes a tangency along the $c_t = h_t = 0$ segment in the LTI section. It is optimal if V_{2t} increases towards the south east, $\frac{\tau_2 y_t}{\tau_1 p_t} < H_t^* < H_{4t}$ and $MRS_{2t}(p_t H_t^* + \tau_2 y_t, H_t^*) > 0$ but

$MRS_{2t}(p_t H_t^* + \tau_2 y_t, H_t^*) < 0$

(16) describes a tangency along the $H_t = 0$ segment of the feasible set. This needs $\partial V_{1t}(a_t, 0)/\partial H_t < 0$ and $\partial V_{1t}(a_t, 0)/\partial a_t = 0$.

In addition to these cases there may be interior solutions when $\frac{\partial v_t(a_t,H_t)}{\partial a_t} = 0, \frac{\partial v_t(a_t,H_t)}{\partial H_t} = 0$. Note that since the utility values $V_{1t} = V_{2t}$ along the line $H_t = \frac{\tau_2 y_t}{\tau_1 p_t}$, if V_{1t} has a bliss point on this line, then also $\frac{\partial V_{2t}(a_t,H_t)}{\partial a_t} = 0$ at this point and hence the indifference curve of V_{2t} must be tangent to the $H_t = \frac{\tau_2 y_t}{\tau_1 p_t}$ line. Then if $\frac{\partial V_{2t}(a_t,H_t)}{\partial H_t} < 0$ at this point, V_{2t} must be increasing above the line and so any bliss point in V_{2t} must also be above the line (if it were below V_{2t} is unattainable). Similarly if V_{2t} has a bliss point on the line then if V_{1t} has an attainable bliss point it must be below the line.

So with general preferences we can identify conditions on constraints and preferences (expressed in terms of the MRS and the signs of marginal utilities of $V_{1t}()$ and $V_{2t}()$ determining which of 10 types of optimal solution will occur. Since v_t is concave the optimum is unique (it is strictly concave except along the line $H_t = \frac{\tau_2 y_t}{\tau_1 p_t}$). These conditions are implicit functions of the decision variables. With boundary corner solutions we can also write down closed form expressions for all the choice variables. But we cannot do so for tangency solutions.

Without specialising preferences we cannot go further in either generating closed form life cycle paths or simulating actual realised optimal life cycle paths. We also cannot clearly distinguish the solutions in which buy to let occurs. So next we specialise preferences.

5.3 Closed form solutions for special preferences

Cocco's preferences (2005) are equivalent to $u(c_t, h_t) = c_t^{\alpha} h_t^{\rho}, \alpha > 0, \rho > 0, \alpha + \rho < 1$. Generally these do not yield closed form solutions for the value function. But if we specialise the form to Cobb-Douglas $u(c_t, h_t) = c_t^{1-\rho} h_t^{\rho}$ we can derive closed form solutions essentially because the utility function is not only homothetic but also homogeneous of degree one.

For these preferences we use the standard solution method in dynamic programming, argument by induction: assume a form for the value function $v_{t+1}(m_{t+1})$, use Bellman equation to solve the problem at t and then verify that the value function does indeed have the assumed form. We conjecture that the value function at t is linear in m_t for $t \leq T$

$$v_t(m_t) = B_{1t} m_t + B_{2t}$$

where both B_{1t}, B_{2t} are realisations at t of random functions. First we consider the final period and then recursively move backward in time.

5.3.1 The Final Period

As above at period T, the only choices are of c_T, h_T which are chosen within the budget constraint $c_T + y_{rT} h_T = m_T$ to maximise final period utility. The solution is to divide m_T into a constant share of spending on each, which results in a maximal level of final period utility $v_T(m_T)$:

$$\begin{aligned} c_T &= (1-\rho) m_T \\ h_T &= \frac{\rho}{y_{rT}} m_T \\ v_T &= (1-\rho)^{1-\rho} (\frac{\rho}{y_{rT}})^{\rho} m_T = B_{1T} m_T \end{aligned}$$

Since these preferences are just a special case of our general preferences we know from the general analysis that $A_{T-1} = 0, M_{T-1} = \min(\tau_1 p_{T-1} H_{T-1}.\tau_2 y_{T-1}), F_{T-1} = a_{T-1} + M_{T-1}$

Hence we can write

$$\begin{aligned} v_T(m_T) &= B_{1T}[y_T + R_{F,T}(a_{T-1} + \min(\tau_1 p_{T-1} H_{T-1}.\tau_2 y_{T-1}) - p_{T-1} H_{T-1}) \\ &\quad -R_{M,T} \min(\tau_1 p_{T-1} H_{T-1}.\tau_2 y_{T-1}) \ + p_T H_{T-1}] + B_{2T} && (5.19) \\ &= B_{1T}[y_T + R_{F,T} a_t + (R_{FT} - R_{M,T}) \min(\tau_1 p_{T-1} H_{T-1}, \tau_2 y_{T-1}) && (5.20) \\ &\quad + (R_{HT} - R_{F,T}) p_{T-1} H_{T-1}] + B_{2T} && (5.21) \\ B_{1T} &= (1-\rho)^{1-\rho} (\frac{\rho}{y_{rT}})^{\rho} \\ E_{T-1} v_T(m_T) &= E_{T-1} B_{1T}[y_T + p_{T-1}(R_{F,T}(\frac{a_t}{p_{T-1}} + \tau_1 H_{T-1} - H_{T-1}) && (5.22) \\ &\quad -R_{M,T} \tau_1 H_{T-1} \ + H_{T-1})] && (5.23) \end{aligned}$$

The value function v_T is linear in $m_T : v_T = B_{1T} m_T + B_{2T}$ where $B_{1T} = (1-\rho)^{1-\rho} (\frac{\rho}{y_{rT}})^{\rho}, B_{2T} = 0$. Note that B_{1T} is random as at date $T-1$. This confirms the conjecture for T.

5.3.2 Generic Period t

Again we can use the result derived above for general preferences, optimally $A_t = 0, M_t = \min(\tau_1 p_t H_t.\tau_2 y_t), F_t = a_t + M_t - p_t H_t$. Also

$$\begin{aligned} c_t &= (1-\rho)(m_t + y_{rt} H_t - a_t) \\ h_t &= \frac{\rho}{y_{rt}}(m_t + y_{rt} H_t - a_t) \end{aligned}$$

Using these solutions $v_t(a_t, H_t)$ becomes

$$\begin{aligned} V_{1t} = \quad & (1-\rho)^{1-\rho}(\tfrac{\rho}{y_{rt}})^{\rho}(m_t + y_{rt}H_t - a_t) + \beta E_t B_{1t+1}(y_{t+1} + R_{F,t+1}a_t + (R_{Ft+1} - R_{M,t+1})\tau_1 p_t H_t \\ & +(R_{Ht+1} - R_{F,t+1})p_t H_t \ + B_{2t+1}) \text{ if } \tau_1 p_t H_t < \tau_2 y_t \\ V_{2t} = \quad & (1-\rho)^{1-\rho}(\tfrac{\rho}{y_{rt}})^{\rho}(m_t + y_{rt}H_t - a_t) + \beta E_t B_{1t+1}(y_{t+1} + R_{F,t+1}a_t + (R_{Ft+1} - R_{M,t+1})\tau_2 y_t \\ & +(R_{Ht+1} - R_{F,t+1})p_t H_t) + B_{2t+1}) \text{ if } \tau_1 p_t H_t > \tau_2 y_t \end{aligned}$$

Along the common boundary between the LTI and LTV areas, we have $\tau_1 p_t H_t = \tau_2 y_t$ and so at any point (a_t, H_t) on this boundary $V_1 = V_2$.

The derivatives are

$$\begin{aligned} \frac{\partial v_t(a_t,H_t)}{\partial a_t} &= \frac{\partial V_{1t}(a_t,H_t)}{\partial a_t} = \frac{\partial V_{2t}(a_t,H_t)}{\partial a_t} \\ &= \beta E B_{1t+1} R_{Ft+1} - (1-\rho)^{1-\rho}\rho^{\rho} y_{rt}^{-\rho} \\ \frac{\partial V_{1t}(a_t,H_t)}{\partial H_t} &= (1-\rho)^{1-\rho}\rho^{\rho} y_{rt}^{1-\rho} \\ & \quad + p_t \beta E_t B_{1t+1}(R_{Ht+1} - R_{Ft+1} + \tau_1(R_{F,t+1} - R_{M,t+1})) \\ \frac{\partial V_{2t}(a_t,H_t)}{\partial H_t} &= (1-\rho)^{1-\rho}\rho^{\rho} y_{rt}^{1-\rho} + p_t \beta E_t B_{1t+1}(R_{Ht+1} - R_{Ft+1}) \end{aligned}$$

All these are constants independent of a_t, H_t so the indifference curves are just linear exactly as in the diagrams for the MRS with general preferences but with constant MRS's. But the signs of all of them are ambiguous. Since in all cases the indifference curves are linear then if always the slope of the indifference curves are not equal to the slope of any part of the boundary of the feasible set, all solutions must be at one of the corners of the boundary of the feasible set except for one configuration, where a quasitangency (with a jump in the marginal rate of substitution) is possible. Which corner depends on the derivatives of V_{1t}, V_{2t}

and the value of H_t^*.Table 5.4 indicates behaviours at the 8 possible optimal solutions with special preferences.

Table 5.4: The description for different regimes for special preference

Regime	Solution
1	spend everything today on c and h
2	LTV constrained M; max. H; zero F; zero c,h
3	LTV/LTI constrained M; some H; zero F; some c,h
4	LTI constrained M; max. H; zero F; zero c,h
5	spend everything today on F
6	LTV/LTI constrained M; some H; some F; zero c,h
7	LTI constrained M; H at threshold; zero F; some c,h
8	LTI constrained M; H at threshold; some F; zero c,h

We can characterise the conditions on the feasible set and parameters of preferences under which each feasible corner kink is optimal[7]. This involves comparing the constant MRS with the slopes of the boundary segments of the different feasible set configurations and the direction of increase of utility. The conditions define if the feasible set is a polygon or triangle, i.e. conditions (1) and (2) the relative preference and constraint slopes and finally the value of H^*.

For each corner we know the values of all the variables and so can deduce the form of the maximal value function $v_t(m_t)$ as in the Table 5.5.

In each regime, irrespective of H^* the value function at t<T is linear in m_t. Hence by

[7] With $H_t^* > 0$, the feasible set of a_t, H_t is not convex and consists of the union of a nonempty interior convex set and part of the line segment ($H_t = 0$). But we rule out the cases when the indifference curves coincide with lines that join kink points on the upper half of the feasible and the end points of the feasible part of the line segment. These cases are with probability 0.

Table 5.5: Value functions for different regimes for special preference

Regime	B_{1t}	B_{2t}	v_t
1	$(1-\rho)^{1-\rho}(\frac{\rho}{y_{rt}})^{\rho}$	$\beta E_t(B_{1t+1}y_{t+1}+B_{2t+1})$	$(1-\rho)^{1-\rho}(\frac{\rho}{y_{rt}})^{\rho}m_t$ $+\beta E_t(B_{1t+1}y_{t+1}+B_{2t+1})$
2	$\beta E_t B_{1t+1}([\frac{(R_{Ht+1}-\tau_1 R_{Mt+1})}{1-\tau_1}\frac{1}{(1-y_{rt}\frac{1}{(1-\tau_1)p_t})}])$	$\beta E_t[B_{1t+1}y_{t+1}+B_{2t+1}]$	$\beta E_t[B_{1t+1}(y_{t+1}+[\frac{(R_{Ht+1}-\tau_1 R_{Mt+1})}{1-\tau_1}\frac{m_t}{(1-y_{rt}\frac{1}{(1-\tau_1)p_t})}])+B_{2t+1}]$
3	$(1-\rho)^{1-\rho}(\frac{\rho}{y_{rt}})^{\rho}$	$(1-\rho)^{1-\rho}(\frac{\rho}{y_{rt}})$ $^{\rho}[-\frac{\tau_2 y_t(1-\tau_1)}{\tau_1}(1-\frac{y_{rt}}{p_t(1-\tau_1)})]$ $+\beta E_t[B_{1t+1}(y_{t+1}+\frac{R_{Ht+1}-\tau_1 R_{Mt+1}}{1-\tau_1}$ $\frac{\tau_2 y_t(1-\tau_1)}{\tau_1})+B_{2t+1}]$	$(1-\rho)^{1-\rho}(\frac{\rho}{y_{rt}})^{\rho}[m_t-$ $\frac{\tau_2 y_t(1-\tau_1)}{\tau_1}(1-\frac{y_{rt}}{p_t(1-\tau_1)})]$ $+\beta E_t[B_{1t+1}(y_{t+1}+\frac{R_{Ht+1}-\tau_1 R_{Mt+1}}{1-\tau_1}$ $\frac{\tau_2 y_t(1-\tau_1)}{\tau_1})+B_{2t+1}]$
4	$\beta E_t B_{1t+1}R_{Ht+1}\frac{1}{(1-\frac{y_{rt}}{p_t})}$	$\beta E_t[B_{1t+1}(y_{t+1}+R_{Ht+1}\frac{\frac{\tau_2 y_t y_{rt}}{p_t}}{(1-\frac{y_{rt}}{p_t})}$ $+(R_{Ht+1}-R_{Mt+1})\tau_2 y_t)+B_{2t+1}]$	$\beta E_t[B_{1t+1}(y_{t+1}+R_{Ht+1}\frac{m_t+\frac{\tau_2 y_t y_{rt}}{p_t}}{(1-\frac{y_{rt}}{p_t})}$ $+(R_{Ht+1}-R_{Mt+1})\tau_2 y_t)+B_{2t+1}]$
5	$\beta E_t(B_{1t+1}R_{Ft+1})$	$\beta E_t(B_{1t+1}y_{t+1}+B_{2t+1})$	$\beta E_t[B_{1t+1}(y_{t+1}+R_{Ft+1}m_t)+B_{2t+1}]$
6	$\beta E_t(B_{1t+1}R_{Ft+1})$	$\beta E_t[B_{1t+1}(y_{t+1}+R_{Ft+1}(\frac{\tau_2 y_{rt}y_t}{\tau_1 p_t}$ $-\frac{\tau_2 y_t(1-\tau_1)}{\tau_1})+R_{Mt+1}(-\tau_2 y_t)$ $+R_{Ht+1}\frac{\tau_2 y_t}{\tau_1})+B_{2t+1}]$	$\beta E_t[B_{1t+1}(y_{t+1}+R_{Ft+1}(m_t$ $+\frac{\tau_2 y_{rt}y_t}{\tau_1 p_t}-\frac{\tau_2 y_t(1-\tau_1)}{\tau_1})+$ $R_{Mt+1}(-\tau_2 y_t)+$ $R_{Ht+1}\frac{\tau_2 y_t}{\tau_1})+B_{2t+1}]$
7	$(1-\rho)^{1-\rho}(\frac{\rho}{y_{rt}})^{\rho}$	$(1-\rho)^{1-\rho}(\frac{\rho}{y_{rt}})^{\rho}(y_{rt}H_t^*-$ $p_t H_t^*+\tau_1 y_t)+\beta E_t[B_{1t+1}(y_{t+1}$ $+R_{Ht+1}H_t^*-R_{Mt+1}\tau_2 y_t)+B_{2t+1}]$	$(1-\rho)^{1-\rho}(\frac{\rho}{y_{rt}})^{\rho}(m_t+$ $y_{rt}H_t^*-p_t H_t^*+\tau_1 y_t)$ $+\beta E_t[B_{1t+1}(y_{t+1}+R_{Ht+1}H_t^*-$ $R_{Mt+1}\tau_2 y_t)+B_{2t+1}]$
8	$\beta E_t(B_{1t+1}R_{Ft+1})$	$\beta E_t[B_{1t+1}(y_{t+1}+R_{Ht+1}H_t^*$ $+R_{Ft+1}(H_t^*(y_{rt}-p_t)+\tau_2 y_t)$ $-R_{Mt+1}\tau_2 y_t)+B_{2t+1}]$	$\beta E_t[B_{1t+1}(y_{t+1}+R_{Ht+1}H_t^*+$ $R_{Ft+1}(H_t^*(y_{rt}-p_t)+m_t$ $+\tau_2 y_t)-R_{Mt+1}\tau_2 y_t)+B_{2t+1}]$

Table 5.6: The solutions for different regimes for special preference

Regime	a_t	H_t	F_t	M_t	c_t	h_t
1	a_{1t}	H_{1t}	0	0	$(1-\rho)m_t$	$\rho m_t \frac{1}{y_{rt}}$
2	a_{2t}	H_{2t}	0	$\frac{\tau_1}{1-\tau_1} \frac{m_t}{(1-y_{rt}\frac{1}{(1-\tau_1)p_t})}$	0	0
3	a_{3t}	H_{3t}	0	$\tau_2 y_t$	$(1-\rho)(m_t - \frac{\tau_2 y_t(1-\tau_1)}{\tau_1} + \frac{y_{rt} y_t \tau_2}{p_t \tau_1})$	$\frac{\rho}{y_{rt}}(m_t - \frac{\tau_2 y_t(1-\tau_1)}{\tau_1} + \frac{y_{rt} y_t \tau_2}{p_t \tau_1})$
4	a_{4t}	H_{4t}	0	$\tau_2 y_t$	0	0
5	a_{5t}	H_{5t}	m_t	0	0	0
6	a_{6t}	H_{6t}	$m_t - (1-\tau_1 - \frac{y_{rt}}{p_t})\frac{\tau_2 y_t}{\tau_1}$	$\tau_2 y_t$	0	0
7	a_{7t}	H_{7t}	0	$\tau_2 y_t$	$(1-\rho)^{1-\rho}(\frac{\rho}{y_{rt}})^{\rho}(m_t + y_{rt} H_t^* - p_t H_t^* + \tau_1 y_t)(1-\rho)$	$(1-\rho)^{1-\rho}(\frac{\rho}{y_{rt}})^{\rho}(m_t + y_{rt} H_t^* - p_t H_t^* + \tau_1 y_t)(1-\rho)\frac{1}{y_{rt}}$
8	a_{8t}	H_{8t}	$H_t^*(y_{rt} - p_t) + m_t + \tau_2 y_t$	$\tau_2 y_t$	0	0

backward recursion, this verifies our conjecture that the value function at $t+1$ is linear in m_{t+1}

$$E_t v_{t+1}(m_{t+1}) = E_t[B_{1t+1}m_{t+1} + B_{2t+1}]$$

Theorem 1 *For the preference $u(c_t, h_t) = c_t^{1-\rho} h_t^{\rho}$,if Assumption (5.11) holds, then the value function is linear in cash on hand, i.e. $v_t(m_t) = B_{1t}(S_t)m_t + B_{2t}(S_t)$*

From this we can deduce the explicit form of the BTL=0 locus. Its slope of the BTL=0 locus is

$$\frac{dH_t}{da_t} = -\frac{\rho}{y_{rt}(1-\rho)}$$

Below the locus $H_t = \frac{\rho}{y_{rt}(1-\rho)}(m_t - a_t)$ the homeowner rents additional space; above the locus there is buy to let activity.

We can also examine the interaction between the BTL locus and the relevant feasible corners. Under condition 1 if

$$H_{3t} = \frac{\tau_2 y_t}{\tau_1 p_t} = \frac{\rho(m_t - a_{3t})}{y_{rt}(1-\rho)} = \frac{\rho(m_t - \tau_2 y_t(1-\tau_1)/\tau_1)}{y_{rt}(1-\rho)}$$

then the BTL=0 locus intersects the $F = 0$ locus at the kink H_{3t}, a_{3t}. But if

$$\frac{\tau_2 y_t}{\tau_1 p_t} < \frac{\rho(m_t - \tau_2 y_t(1-\tau_1)/\tau_1)}{y_{rt}(1-\rho)}$$

then BTL>0 ($H_t > h_t$) while if

$$\frac{\tau_2 y_t}{\tau_1 p_t} > \frac{\rho(m_t - \tau_2 y_t(1-\tau_1)/\tau_1)}{y_{rt}(1-\rho)}$$

a homeowner rents additional housing ($H_t < h_t$). Figure 5.4 shows three example of feasible sets with the BTL locus (the orange line is the BTL locus in each of the feasible sets).

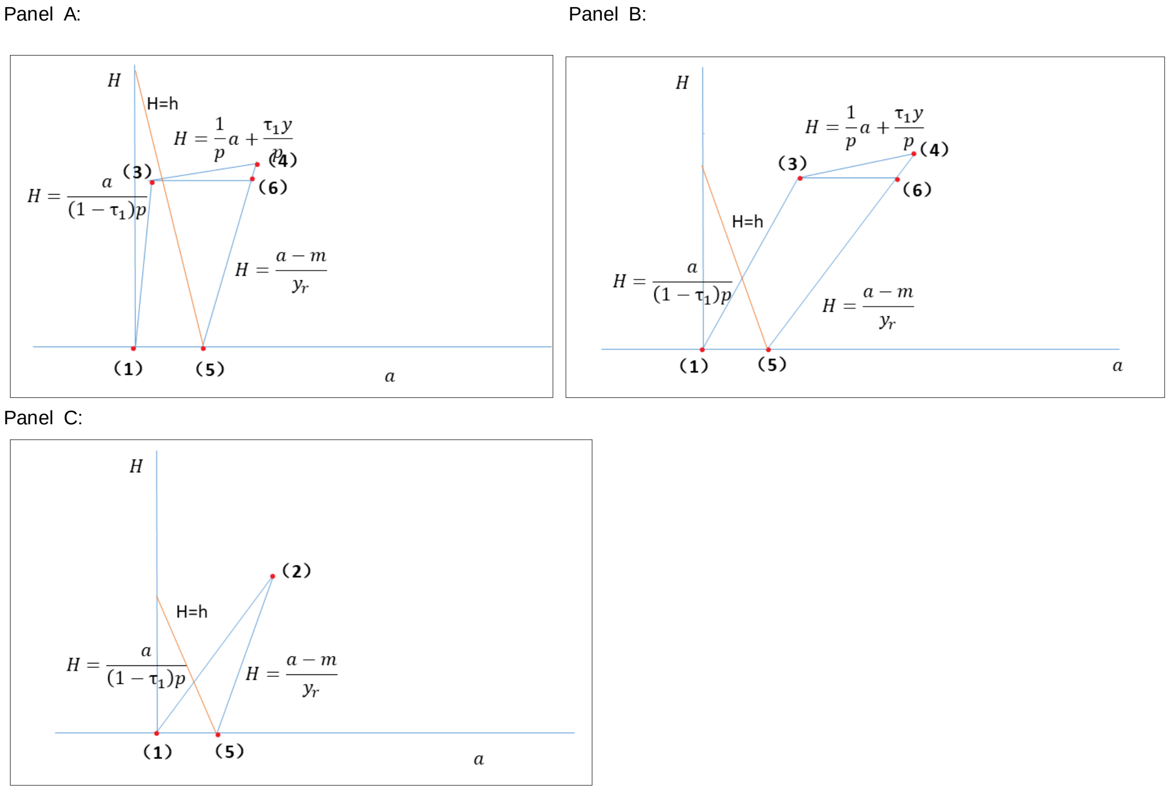


Figure 5.4: Three examples of feasible sets with BTL locus

Note: These three examples are with H*=0, so there is no regimes 7 and 8.

Why is this interesting? It means that always (so long as some housing is affordable) BTL will be feasible with either an LTI or LTV constrained mortgage (See examples in Figure 5.4). But if $\frac{\tau_2 y_t}{\tau_1 p_t} < \frac{\rho(m_t - \tau_2 y_t(1-\tau_1)/\tau_1)}{y_{rt}(1-\rho)}$ there will be no homeowners who rent additional space with only an income constrained mortgage. Combine this with the 7 possible optimal regimes under condition 1. If $\frac{\tau_2 y_t}{\tau_1 p_t} = \frac{\rho(m_t - \tau_2 y_t(1-\tau_1)/\tau_1)}{y_{rt}(1-\rho)}$ regimes $4, 6, 7, 8$ have positive BTL at the optimum but regimes $1, 3, 5$ do not (Panel B). Also if $\frac{\tau_2 y_t}{\tau_1 p_t} = \frac{\rho(m_t - \tau_2 y_t(1-\tau_1)/\tau_1)}{y_{rt}(1-\rho)}$ there is no optimum with a homeowner also renting extra space. If $\frac{\tau_2 y_t}{\tau_1 p_t} < \frac{\rho(m_t - \tau_2 y_t(1-\tau_1)/\tau_1)}{y_{rt}(1-\rho)}$ then also there is BTL in regime 3 (Panel A). The remaining case is condition 2 with a_{2t}, H_{2t}. Again there is no BTL or additional renting for a homeowner in $1, 5$ but 2 will always involve some BTL (Panel C).

5.3.3 The Life Cycle Path

The life cycle path diagram with no entry threshold on house ownership is shown in Figure (5.5). Working backwards from the final period, at $T-1$ the individual must choose the best solution form given the feasible set at $T-1$ given the cash on hand m_{T-1} (which depends on realisations of labour income, asset returns at $T-1$) to maximise $u(c_{T-1}, h_{T-1}) + \beta E_{T-1}[B_{1T} m_T + B_{2T}]$ within his $T-1$ budget constraint $m_{T-1} + y_{rT-1} H_{T-1} - a_{T-1}$ and knowing that $m_T = y_T +$asset income is uncertain. This results in selection of a regime at T-1 and the optimal decision variables within that regime. It also gives his $T-1$ value function $v_{T-1}(m_{T-1}) = B_{1T-1} m_{T-1} + B_{2T-1}$. Coming back to $T-2$ he repeats the process, maximising

$$\begin{aligned} & u(c_{T-2}, h_{T-2}) + \beta E_{T-2} v_{T-1}(m_{T-1}) \\ = \quad & u(c_{T-2}, h_{T-2}) + \beta E_{T-2} \max[u(c_{T-1}, h_{T-1}) + \beta E_{T-1}(B_{1T} m_T + B_{2T})] \end{aligned}$$

Continuing this way, he traces out his alternative life cycle paths (including the choice of regimes) conditional on what future realisations of random state variables may occur.

We can schematically represent the life cycle path in the diagram below. His starting point at any $t+k$ will depend on the realisations of random variables between t and $t+k$.

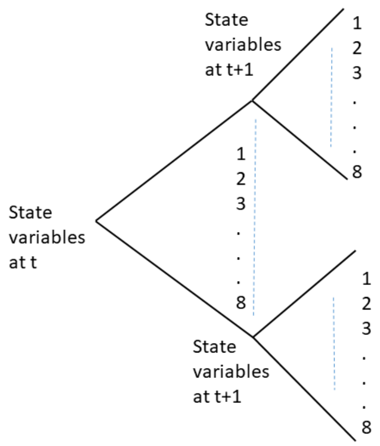


Figure 5.5: Life cycle path diagram with no entry threshold on house ownership

5.4 Simulation of the special preference model

We simulate behaviour for heterogeneous households. Heterogeneity is reflected in two aspects. First, there are different expectations and realisations of labour income among individuals. To be specific, we specify different stochastic processes for people with different education attainment. Individuals with the same education attainment can differ in the history of the individual shocks since in general they receive different draws from the same shock distribution. Expectation of income varies with age and education attainment as specified in the income process. Second, we let initial cash on hand differ among people. Other than these two aspects, people are identical; they have the same preferences and share the same expectations and realisation of asset returns, house prices. The planning horizon is also the same for everyone in each period.

To be specific, for one realisation of the aggregate shocks[8], we generate realisations for the shocks to the labour income process for 100 individuals who differ in education qualification and initial cash on hand. Then we compute the optimal consumption and investment decisions for these 100 individuals. We repeat this process for 80 paths for the aggregate variables, each path with 100 individuals. This gives 8000 different paths in total for each of the four sets of calibrated parameters (models (i)-(iv) as shown below).

5.4.1 Stochastic processes

In the model, there are five sources of uncertainty: the risky asset return, the mortgage rate, house prices, rental and labour income. We assume all of them to be independent of each other.

We select stochastic processes for each of them based on UK data (see Table 5.7). The house price follows a random walk with a deterministic upward trend ($\gamma > 0$).

$$\ln(p_t) = \gamma + \ln(p_{t-1}) + \varepsilon_{pt}$$

where $\varepsilon_{pt} \tilde{} N(0, \sigma^2_{\varepsilon p})$.

The labour income process is assumed to be i.i.d. with a hump shape. The coefficients of age, the intercept and the distribution of the shocks are different for different education groups.

For people with higher education, the income process is

$$\ln(y_h) = \alpha_{0h} + \alpha_{1h} age + \alpha_{2h} age^2 + \varepsilon_{yh}$$

where $\varepsilon_{yh} \tilde{} N(0, \sigma^2_{\varepsilon yh})$.

[8]Here aggregate shocks include the shocks for house prices, rental, mortgage interest rate and risky asset return, which are assumed to be common for everyone.

For people with lower education, the income process is

$$\ln(y_l) = \alpha_{0l} + \alpha_{1l} age + \alpha_{2l} age^2 + \varepsilon_{yl}$$

where $\varepsilon_{yl}\tilde{}N(0, \sigma^2_{\varepsilon yl})$.

Rental income follows an AR(1) process.

$$\ln(y_{rt}) = \eta_0 + \eta_1 \ln(y_{rt-1}) + \varepsilon_{y_r t}$$

where $\varepsilon_{yrt}\tilde{}N(0, \sigma^2_{\varepsilon yr})$.

The risky asset return is assumed to be i.i.d. (in real terms) and log normally distributed over time.

$$\ln(R_F)\tilde{}N(\mu_{RF}, \sigma^2_{RF})$$

The mortgage interest rate is assumed to be an AR(1) process.

$$\ln(R_{Mt}) = \delta_0 + \delta_1 \ln(R_{Mt-1}) + \varepsilon_{RMt}$$

where $\varepsilon_{RMt}\tilde{}N(0, \sigma^2_{\varepsilon RM})$.

5.4.2 Calibration

The calibrated parameters are shown in Table 5.7. Note that all the monetary values are in real terms. All the parameters in the calibration are from data.

Table 5.7: Calibrated parameters for simulation

Parameter	Value	Source
Utility parameter ρ	0.5	
LTV ratio τ_1	0.9	
LTI ratio τ_2	3	
Initial cash on hand range	$[15000, 100000]$	
House price process		
Initial house price p_1	281032.8	ONS
γ	0.024	ONS
σ_p	0.052	ONS
Income process		
α_{0h}	7.96	ONS
α_{1h}	0.11	ONS
α_{2h}	-0.001	ONS
$\sigma_{\varepsilon_{yh}}$	0.057	ONS
α_{0l}	8.65	ONS
α_{1l}	0.06	ONS
α_{2l}	-0.0007	ONS
$\sigma_{\varepsilon_{yl}}$	0.023	ONS
Rental process		
Initial rent y_{r1}	9930.41	
η_0	0.195	VOA
η_1	0.981	VOA
ε_{yr}	0.03	VOA
Risky asset return distribution		
μ_{RF}	0.086	FTSE
σ_{RF}	0.152	FTSE
Mortgage interest rate distribution		
Initial mortgage interest rate R_{M1}	0.99	BSA
δ_0	-0.005	BSA
δ_1	0.71	BSA
ε_{RM}	0.0192	BSA

Note that this leaves two parameters that we can vary β, H^* to see how the optimal life cycle path varies with each of these. Obviously β affects the intertemporal MRS whilst H^* is an attempt to capture indivisibility in the house ownership market in a model in which housing is otherwise treated as continuous.

5.4.3 The method

We simulate solution of the optimal life cycle path calibrated to fit UK data for three epochs of 15 years each with different samples of realisations of shocks. There are two broad reasons for working with epochs of the young (age range 21-35), the middle aged (age range 36-50), and the old (age range 51-65).

First it brings the theoretical framework closer to the real world. In the theory the financial markets have no transaction costs in housing or financial assets (including housing debt) markets and in the rental market. In reality there are costs in most of these markets e.g. stampduty and legal costs in house purchase; mortgage fees in mortgage markets and in equity markets spreads and broker fees. In the rental markets there are security deposit bonds and contract costs to pay. In theory the household adjusts its portfolio every period but in reality housing tenure, ownership of a particular house and taking a particular mortgage are adjusted less frequently than this, presumably partly because of these transaction costs. In the UK owner occupiers change their house or mortgage about every 14 years on average in the latest data (ONS, Social Trends, 2011).

Moreover many of the perceived life cycle patterns and policy problems of the UK especially in the housing dimension are related to epochs. For the young there is a problem of lack of affordability of housing especially to buy (to do this needs a cash deposit of usually around 10% of the house cost). In recent years this has led to various UK policies like subsidised mortgages, most recently no stamp duty for first time buyers to try to improve access of the young to house purchase. For the middle aged of all tenures a common pattern is to try to upsize housing as the need for more space becomes critical in families with children[9]. Older owner occupiers often withdraw housing equity by downsizing once the children have left home,

[9] The needs of space for children in middle age are not reflected in the current version of simulation. But as an extension in the future, we are thinking of using a Stone-Geary utility function to include the subsistence levels of non-housing and housing consumption so that these needs can be captured by the subsistence level calibration. (See Appendix C.4)

using the proceeds to supplement income, which is often pension income by this stage of life.

A second reason for concentrating on just 3 epochs rather than say a $40-70$ year life cycle path is the computational complexity involved. The theory model has 8 regimes each period because of the market and other constraints. The choice of optimal regime in a period depends on the cash on hand at the start of that period, in turn this depends on the outcome of exogenous random variables in that period and past savings/portfolio decisions in the preceding period. Thinking of the possible sequences of time paths of optimal regimes through the future from any period onwards, we have to find the optimal future regime for each realisation of future random variables and each level of saving today in order to decide the optimal savings portfolio today. With 8 regimes, T periods and N random realisations this means comparing payoffs between $8^{T-1}N^{T}$ paths of possible outcomes. Other studies (Carroll, 2012) overcome this problem by using interpolation of the solution path for choice variables between points of a finite grid. This obviously introduces some additional approximation errors.

We solve for B_{1t} and B_{2t} by backward induction using their recurrence relations from the closed form solutions in the previous section. This allows us to simulate data for different people. For any period t an individual starts with cash on hand and exogenous state variables which reflect their own past history of realisations. At t they have to make decisions which maximise their current utility and their future expected utility, taking into account that in the future they will replan depending on how the future realisations work out. But the future B_{1t+1} and B_{2t+1} are determined recursively backwards from T and depend on what turns out to be the optimal decision at each future date. So we first solve for the best decisions in the final period T for given cash on hand and other exogenous variables at T. Then come back to $T-1$. At $T-1$ the individual knows that his period T realisations will affect his best choices at T, so at $T-1$ he takes expectations over these maximal utilities at T to determine his best choices at $T-1$. Similarly at any prior period t, the future expected value function depends on the expected future course of optimal choices conditional on future realisations of random variables.

To be specific, in our code solving the three epoch model ($T=3$), random exogenous state variables denoted S_t are realised at the start of period t; the available cash on hand m_t is also known. The decision variables at t are denoted x_t and the objective at t is the current utility u_t^* plus the discounted expected future value. The latter depends on which of the possible 8 regimes (corners) it is optimal to take at $t+1$ and the optimal decisions within that regime;

in turn these depend on the realisations of the state variables S_{t+1} and the optimal choices each period ahead. The available cash on hand depends on decisions of the preceding period about asset accumulation and on the current realisation of labour income and asset returns. The final period is special since there are no corners, the only choice is a static one of c_T, h_T. To work out simulations of the optimal life cycle path we need to calculate the future expected values. The main idea here is to use Monte Carlo simulation to compute the expectation of the future utility over the the future state variables $E_t(B_{1t+1}(S_{t+1})m_{t+1} + B_{2t+1}(S_{t+1}))$ i.e. we use the mean of simulated function values to approximate the expectation of the function. We use Matlab to do this. To find the optimal corner for any given realisations at a period, we compare the objective (value) functions for each corner.

5.4.3.1 The final period

In the final period, as there is no future and no bequest motive by construction, it is optimal to spend everything on housing rental and non-housing consumption. Thus it is like solving a static problem of utility maximisation. As the within period utility is Cobb-Douglas, the optimal expenditure on housing and non-housing consumption at T depends on ρ and the relative price of housing (y_{rT}). At the same time, investment in all other assets and housing is zero.

5.4.3.2 The periods before the final period

In any period $t < T$, the overall utility not only depends on the current utility but also the expected future utility. The expected future utility is affected by both the expected random shocks (the purely exogenous variables like house prices, asset returns, etc.) and the current decisions of how much resources to carry forward to the next period. Any positive housing investment this period H_t not only affects the future period utility but also the current period utility via buy-to-let income or rental saving. This is due to the dual role of housing as both a consumption and an investment good. At period $T-1$, as there is only one regime in the next period T, the parameter B_{1T} in the current period value function can be computed for each simulated state given the current choice of F_t, H_t, M_t. Having these $B_{1T}'s$ we can compute the value function at $T-1$ for each possible regime choice and choose the regime that gives us the highest value function. For periods t before $T-1$, the idea is the same except that now the regime in the next period can be different for different realisations of random variables in future periods. But these regimes are computed in the previous iteration at $t+1$ and can be used to compute value functions for each realisation at t; taking the expectation over these we can get the value function at t and choose the one that gives us the highest overall value at t.

5.4.4 Simulated result

We simulate the special preference model with $T = 3$. Assuming rational expectations, expectations of future realisations are computed by Monte Carlo integration. We simulate the model for four different combinations of parameters (H^* and β). We do it for 8000 different paths for each of the four sets of calibrated parameters. In our implementation, 50% of the households are high educated and 50% are low educated in each path. Within each of the education groups, the initial cash on hand is uniformly distributed over the same range.

5.4.4.1 Impact of minimum house to purchase and time preference on decision

One of the aims is to investigate the impact of these two parameters on household decision making, especially housing purchase behaviour. Table 5.8 shows the simulated life cycle decisions for different combinations of H^* and β (models (i)-(iv))[10]. When any house size can be purchased ($H^* = 0$), all the households purchase housing in the first two epochs of their life since it is a profitable investment and can supplement income by saving rental cost and/or earning rental income from buy to let. This is true even if people discount the future heavily with $\beta = 0.7$. In comparison, a minimum house size to buy ($H^* = 0.4$) discourages all households from holding the housing asset in the first epoch while some households enter homeownership in the second epoch (models (iii) and (iv)). Whenever households buy housing, they borrow as much mortgage debt as they can. In our simulation the loan to income ratio constraint is binding for everyone who takes a mortgage[11]. This implies that whenever households decide to buy housing, the exogenous parameters always result in the polygon feasible set of a_t and H_t which was discussed previously. As we set the income in the first epoch to be the same for all the households, everyone with $H^* = 0$ borrows the same amount of mortgage in the first epoch. For this reason those who buy more housing in the first epoch will have more

[10] Note that the average LTV and LTI are computed for mortgage borrowers only. And the average housing wealth are computed for homeowners only.

[11] With $H^* = 0, \beta = 0.7$, both loan to value and loan to income ratio constraints bind for mortgage borrowers.

housing equity and get more capital gain from housing if the house price increases. Since our simulated house price does rise over time, we can expect to see the initially richer households (with higher initial cash on hand) who can afford a bigger house size to be richer in the second epoch in terms of capital gains compared with the poorer counterpart if we ignore the different income processes due to education. The transition of social status will be discussed in the next subsection.

Less patient households ($\beta = 0.7$) invest less on housing compared to their more patient counterparts ($\beta = 0.95$) because they derive relatively more utility from current consumption rather than future consumption. Average housing wealth of house buyers is the highest when $H^* = 0$ and $\beta = 0.95$.

The intertemporal marginal rate of substitution (IMRS) is defined as

$$IMRS = \frac{\beta E v'_{t+1}(m_{t+1})}{u_t^{*\prime}(m_t - a_t + y_{rt} H_t)} = \frac{\beta E B_{1t+1}}{(1-\rho)^{1-\rho}(\frac{\rho}{y_{rt}})^{\rho}}$$

IMRS measures the tradeoff of marginal utility between today and tomorrow, i.e. how much the expected future marginal utility of income rises at $t+1$ relative to the fall in the current marginal utility of cash on hand if we increase current savings. The IMRS is in general higher for more patient people with $\beta = 0.95$ as they care about the future more.

The reason why the non-housing and housing consumption, c_t and h_t are sometimes zero is because of the Cobb-Douglas utility we assume which leads to the linear value function as is shown in Thoerem 1 from the previous section. In essence, c_t and h_t should be thought of as deviations from their subsistence levels. In our case, the subsistence levels for c_t and h_t are both zero.

Table 5.8: Simulated life cycle decisions for different parameter combinations

minimum housing to buy	$H^* = 0$		$H^* = 0.4$	
Time discount factor	$\beta = 0.95$	$\beta = 0.7$	$\beta = 0.95$	$\beta = 0.7$
	model (i)	model (ii)	model (iii)	model (iv)
Homeownership in epoch 1	100%	100%	0	0
Homeownership in epoch 2	100%	100%	8%	8%
Average housing wealth for home owners in epoch 1	105201.5	49215.9	N/A	N/A
Average housing wealth for home owners in epoch 2	236844.1	99831.4	230779.3	134157.2
Average BTL in epoch 1	105201.5	-787114.4	-265386.5	-787201.2
Average BTL in epoch 2	236844.1	-813726.2	-186137.3	-261523.6
Average LTV in epoch 1	0.45	0.9	N/A	N/A
Average LTV in epoch 2	0.38	0.9	0.53	0.9
Average LTI in epoch 1	3	3	NA	NA
Average LTI in epoch 2	3	3	3	3
Average risky asset holding in epoch 1	0	0	40309.9	6509.9
Average risky asset holding in epoch 2	0	0	53893.5	19676.2
Average non-housing consumption in epoch 1	0	27086.2	8595.1	25495
Average non-housing consumption in epoch 2	0	25431.3	6009.7	8355.1
Average IMRS in epoch 1	1.45	0.61	0.85	0.61
Average IMRS in epoch 2	0.86	0.63	0.86	0.63

5.4.4.2 Heterogeneous households

Figure 5.6 shows the distribution of loan to value ratio for mortgage borrowers for model (i) in the first two epochs. Although everyone in model (i) purchase houses in the first two epochs and are loan to income constrained in mortgage, their equity in housing is different.

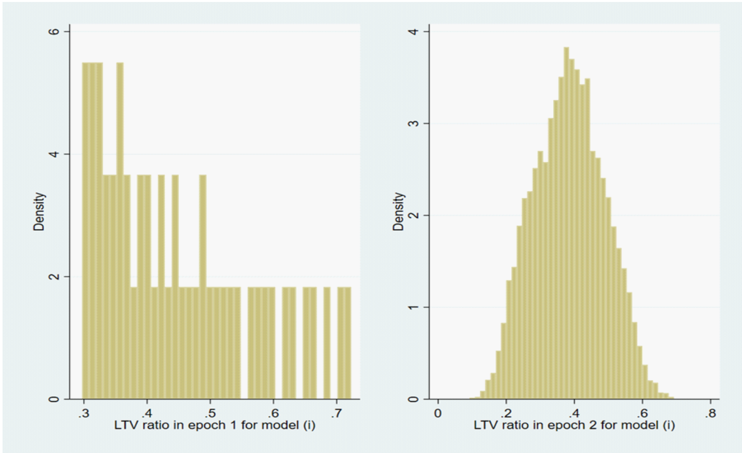


Figure 5.6: Simulated distribution of loan to value ratios for mortgage borrowers for model (i) in the first two epochs

Note: Left panel: LTV ratio in epoch 1; right panel: LTV ratio in epoch 2

Tables 5.9, 5.10, 5.11, 5.12 show the transition of education attainment and cash on hand combinations through time for each model. In the tables there are four notations: HH denotes high education and high cash on hand, HL denotes high education and low cash on hand, LH denotes low education and high cash on hand, LL denotes low education and low cash on hand. The number (1 and 2) following these notations means the epoch of these states. To define membership of the high or low cash on hand group, we use the median of cash on hand in the corresponding epoch as the threshold. i.e. if the cash on hand is less than or equal to the median, cash on hand is defined as low, otherwise it is defined as high. Since education attainment and initial cash on hand is calibrated the same way as stated above for each model, each of the categories HH1, HL1, LH1, LL1 must account for the same percentage (25% each in epoch 1). As education attainment is assumed to be constant through time for a particular household, some elements such as (HL1, LH2) must be zero. As there are 8000 households in each of the model simulations, the sum of all the elements for each transition matrix must be 8000. From the tables we can see that the most common pattern is transition from HH to HH, i.e. if one household is high educated with high cash on hand in the current epoch, then it is likely to remain in the high cash on hand group in the next epoch. Although some transition between high and low cash on hand happens, it is interesting to notice that for every model in the last epoch, most of the households are in either HH or LL group, which means in the end the high educated tend to get rich and the low educated tend to get poor.

Table 5.9: Transition of education attainment and cash on hand combinations through time for model (i)

epoch 1 to epoch 2				
	HL1	HH1	LH1	LL1
HL2	1482 (18.5%)	243 (3%)	0	0
HH2	518 (6.6%)	1757 (22%)	0	0
LH2	0	0	1493 (18.7%)	232 (3%)
LL2	0	0	507 (6.3%)	1768 (22.1%)
epoch 2 to epoch 3				
	HL2	HH2	LH2	LL2
HL3	1054 (13.2%)	255 (3.2%)	0	0
HH3	671 (8.4%)	2020 (25.3%)	0	0
LH3	0	0	1099 (13.7%)	210 (2.6%)
LL3	0	0	626 (7.8%)	2065 (25.8%)

Table 5.10: Transition of education attainment and cash on hand combinations through time for model (ii)

epoch 1 to epoch 2				
	HL1	HH1	LH1	LL1
HL2	488 (6.1%)	472 (5.9%)	0	0
HH2	1512 (18.9%)	1528 (19.1%)	0	0
LH2	0	0	487 (6.1%)	473 (6%)
LL2	0	0	1513 (18.9%)	1527 (19.1%)
epoch 2 to epoch 3				
	HL2	HH2	LH2	LL2
HL3	139 (1.7%)	254 (3.2%)	0	0
HH3	821 (10.3%)	2786 (34.8%)	0	0
LH3	0	0	136 (1.7%)	257 (3.2%)
LL3	0	0	824 (10.3%)	2783 (34.8%)

Table 5.11: Transition of education attainment and cash on hand combinations through time for model (iii)

epoch 1 to epoch 2				
	HL1	HH1	LH1	LL1
HL2	756 (9.6%)	0	0	0
HH2	1244 (15.6%)	2000 (25%)	0	0
LH2	0	0	337 (4.2%)	419 (5.2%)
LL2	0	0	1663 (20.8%)	1581 (19%)
epoch 2 to epoch 3				
	HL2	HH2	LH2	LL2
HL3	6 (0.08%)	0	0	0
HH3	750 (9.4%)	3244 (40.6%)	0	0
LH3	0	0	3 (0.04%)	3 (0.04%)
LL3	0	0	753 (9.4%)	3241 (40.5%)

Table 5.12: Transition of education attainment and cash on hand combinations through time for model (iv)

epoch 1 to epoch 2				
	HL1	HH1	LH1	LL1
HL2	5 (0.06%)	42 (0.5%)	0	0
HH2	1995 (25%)	1958 (24.5%)	0	0
LH2	0	0	19 (0.24%)	28 (0.4%)
LL2	0	0	1981 (24.8%)	1972 (24.7%)
epoch 2 to epoch 3				
	HL2	HH2	LH2	LL2
HL3	0	8 (0.1%)	0	0
HH3	47 (0.6%)	3945 (49.3%)	0	0
LH3	0	0	3 (0.04%)	5 (0.06%)
LL3	0	0	44 (0.6%)	3948 (49.4%)

5.5 Conclusion

There are some stylised facts for UK housing and financial asset decisions:

(i) Mortgage constraints combined with a minimum scale of house purchase can ration the cash poor (especially the young) out of house ownership. A related phenomenon is the growth of parental cash contributions to their offspring to facilitate initial house purchase but only when the parents can afford and choose to do this.

(ii) On average UK households own a relatively small proportion of their wealth in financial assets compared with the US (Banks et al., 2002).

(iii) In recent decades buy to let has become increasingly important.

To try to understand why these patterns arise and especially the underlying forces which lead to heterogeneity across the population in the life cycle paths followed in housing and financial asset decisions, we set up a life cycle model where individuals derive utility from housing and non-housing consumption, and make decisions about consumption and investment under uncertainty. The constraint set is relatively complicated, including mortgage borrowing constraint (LTV and LTI ratio constraints), and no-short-selling constraints for the safe and risky asset and also a fixed minimum level of house purchase. Applying life cycle models under uncertainty with investment constraints is difficult. With a general setting often just the first order conditions can be characterised but explicit closed form solutions cannot be found analytically. Thus in general it is difficult to answer questions such as which market constraints hurt the most or which types of individual (with heterogeneous life cycle income profiles, varying within period and intertemporal preferences, varying initial endowments) are going to be constrained in particular ways or specialise their portfolio in particular directions. An alternative in much of the literature is to abandon the search for general analytical characterisations and with specific parametrised but still quite general preferences and uncertainty, derive numerical solutions through simulation. Here we take a compromise between these. In the first half of the chapter we take quite general concave preferences, additive over time and with within period utility depending on housing and non-housing consumption. We find a solution strategy which allows us to solve for the portfolio allocation and the current period allocations between consumption and housing services just within one period problems. These solutions are conditional on the variables with intertemporal effects: the housing stock H_t and net worth a_t. In fact we show that conditional on net worth in a period and investment in house purchase, the remaining decisions can be found in two independent blocks: current period decisions on consumption and housing consumption; current decisions on the investment portfolio of nonhousing assets. With general concave preferences we sketch properties of the optimal solution patterns and conditions under which different solution patterns can hold. However in this part we impose quite strong restrictions on the distributions of asset returns: one (overstrong) interpretation of these is that the distributions are such that with probability one the housing mortgage rate is always below the returns on housing and risky financial assets and then that again with probability one the return on the safe investment is always below that on the risky financial asset. Another interpretation is that the expected marginal value function return on the risky asset is always above that on either the safe asset or the mortgage. A consequence is that the optimal portfolio choice has some clear strong features: the safe asset is never held, if there is investment in home ownership then it is always with a maximum possible mortgage (which can be either LTV or LTI constrained). This leaves the choice of net worth and house purchase both of which have intertemporal utility effects. We can characterise the set of possible optimal choices if housing and net worth including which, if any, market constraints bind. The solutions may show zero marginal value in these two variables i.e. intertemporal smoothing of marginal values is achieved or inequalities in the FOC's due to the constraints. But in this approach we cannot explicitly derive optimal life cycle consumption and asset paths except in a highly conditional way[12] so they would have limited interpretability.

So in the second half of the chapter we add a functional form on preferences (concave but homogenous degree one within a period) which simplifies the problem enormously since it makes the intertemporal MRS (each of the current marginal utility of total current spending and the expected future marginal value of spending) independent of the levels of current spending or wealth carried forward.

Specialising the preferences, we derive closed form solutions, special because it generates a value function linear in cash on hand. The main competing investments are housing and the risky asset. While housing investment will save the rent or bring rental income for the current period, the risky asset can only bring income for the future period. For this reason, even if the return of the risky asset is higher than housing, it can be optimal to hold some housing asset.

[12] For example we could derive conditions for optimal choice in period t involving inequalities on all future period returns, intertemporal MRS's and initial cash on hand).

Depending on the minimum available size of house to buy, some poor people can be rationed out of buying housing over life.

Value functions linear in cash on hand imply that the intertemporal MRS is constant independent of wealth. This is strong but allows us to generate clear solutions. In particular we can derive explicit conditions on preference parameters, means and covariances of asset returns under which different regimes with their constraints are optimal. We can also identify in which situations buy to let will occur or households will choose to rent rather than purchase any housing. One implication of the constant intertemporal MRS is that it may be optimal to transfer all resources to the future, spending zero today, or conversely consume all available cash on hand today, transferring no wealth into the future. This is somewhat unpalatable, to offset it we can easily accommodate a subsistence level in nonhousing and housing consumption at each date and the qualitative results on when it is optimal to choose alternative sets of constraints to bind will be unaffected (see appendix).

Combining the special preferences, the basic distributional assumptions on asset returns with calibrated preference parameters and empirically estimated stochastic processes and initial conditions for all the exogenous random variables (three asset returns, labour income, housing rent, house prices) we can numerically solve for optimal life cycle paths for different realisations of the random variables. The stochastic processes matter both in determining how agents form their expected future value functions and, for a given realisation of the random variables at time t, the cash on hand available at t. In this part we maintain the assumptions on the distribution of asset returns referred to previously so that it is still true that the safe asset is dominated by the risky financial asset and potentially housing and that if there is investment in home ownership, it is always financed with a maximum mortgage. We find that in general in any period there are eight possible configurations of optimal decisions with different sets of binding constraints. How the optimal life cycle switches between these depends on the distributions of random variables and their realisations, the preference parameters especially the rate of time preference and then key parameters in the constraints like the minimum house size available for purchase, the maximal mortgage loan to value and to labour income ratios. Within a given framework of key constraint parameter values and time preference rate (we take four alternative frameworks for these) we take 100 individuals with varying education (50 high and 50 low level individuals) with varying initial cash on hand. For each individual we take 80 realisation paths of the future uncertain variables and determine the optimal life cycle profile regimes for each set of realisations and individual. Comparing these our main simulation findings are:

- Social status switching (education and cash on hand effect): In our simulation, almost all the high educated people with high initial cash on hand remain in the high cash on hand group. Some of the high educated with low initial cash on hand switch to the richer group, but luck is important for them. On the other hand, almost all the low educated with low initial cash on hand remain in the low cash on hand group all through their life. The low educated with high initial cash on hand tend to switch to the poorer group either in the second or final epoch of their life. Those who climb up the housing ladder are all high educated.

- The minimum house purchase size effect: With no minimum, every household invests in housing in the first two epochs. On the other hand, when the minimum size is 0.4, no household invests in housing in the first epoch. This is either because they cannot afford the minimum housing to buy, or because the existence of the positive minimum changes their expectation about the future which makes being a renter the most desirable. But in the second epoch, some households manage to climb up the housing ladder by becoming a house owner.

- The time preference effect: The more patient house owners tend to be pure rentiers while the less patient tend to enjoy shared ownership (both buy some housing and also rent some housing). With a zero minimum house purchase size and low discounting of the future ($\beta = 0.95$), everyone is a pure rentier ($H > 0, h = 0$) in the first two epochs. But with a zero minimum house purchase size and low discounting of the future ($\beta = 0.7$), all households buy housing and rent some part of it but live in the remaining part $0 < h < H$ in the first two epochs . In the second epoch with $H^* = 0.4$ and $\beta = 0.95$, $0 < h < H$ for homeowners and with $H^* = 0.4$ and $\beta = 0.7$, $H > 0, h = 0$ for homeowners.

- Consumption is not smoothed over time partly because of the constraints, partly because the composite interest rate depends on the current portfolio allocation and so that its relationship to the time preference rate is unclear (it depends on the asset composition of the portfolio).

But using these preferences, we could in fact avoid imposing dominance type restrictions on the distributions of returns. Then in some circumstances the safe asset could dominate.

We also ignore the possibility of mortgage default (essentially by assuming the housing return always exceeds the mortgage rate). In fact in the data mortgage arrears and house repossessions by lenders are now below their 2007 level and have been falling since the financial crisis (BoE, 2017).

There are some possible extensions to this chapter. One immediate possibility is to simulate for more than 3 epochs/periods. We have worked with reallocating the portfolio and housing tenure/ownership each period and effectively with one period adjustable rate mortgages. There are no transaction costs of changing tenure or portfolio in the approach here. An obvious extension is to allow for these. Similarly we could add a bequest motive at T or a random time horizon (date of death). Within our framework results could also be presented in different respects e.g. in the life cycle pattern we could compute the probability of each regime being chosen at any date t for given m_t and hence from this the Markov chain for cash on hand as regimes switch between adjacent periods. Furthermore, other parameters such as the maximal LTI or LTV could be varied between simulations to generate further comparative static results. As this is an individual decision model, the house price and interest rates are taken as given, but using the features here for the demand side (net demand if there is no new build or demolition) model and aggregating the net demand over individuals we could try to determine equilibrium house purchase and rental prices. The range of decision variables could be extended e.g. labour income is partly determined by an individual's choice of working hours. And finally in principle simulated and numerically solved paths could be calculated for a general preference case as in Cocco (2005).

Chapter 6

Concluding Remarks

This thesis studies household behaviour with the focus on considering the role of housing empirically and theoretically both as a choice in itself (tenure, financing purchase, the amount of housing to consume) and as a conditioning variable on other short run decisions.

Chapter 2 analyses the impact of house prices on couples' labour supply at the extensive margin (participation) after imputing wages for both workers and non-workers by adopting the Heckman selection model. We find a suitable way to correct for bivariate selectivity in a Heckman style. We estimate the participation equation with the imputed wage for both spouses as regressors, considering the interdependent nature of couple's labour supply with a bivariate probit model. We find age effects and differences in gender effects. The age effects mirror popular caricatures of young and middle aged males who increase participation in response to house price increase. On the other hand, young and middle aged females have a negative response to house price increases. However these are not significant. The labour participation response of older people does not show wealth effect of housing as well. We find that the house price process is stationary, which implies the shocks of house prices tend to fade away in the long run and for this reason households who have rational expectations on house prices and do not plan to exit the housing market immediately may not be affected by house price variations.

In Chapter 3 we use the number of solution regimes from a theoretical model to guide choice of the number of components in a sample. Each component is designed to capture varying degrees of proximity of different households to the mortgage borrowing constraints and no-short-selling constraint. We use a censored data EM algorithm to estimate the multivariate Gaussian mixture model on the UK WAS data. The estimated component members seem to be ordered by wealth. Estimating a system of linear models with dependent variable the posterior probability of component membership, we find that households who are younger, less educated with lower income are more likely to be no-short-selling constrained in risky asset investment and with lower net worth. By contrast on the whole members of the top component are not financially constrained either in the mortgage or the risky asset.

In Chapter 4 we study the MPC for heterogeneous older homeowners. It is based on the result from Chapter 3 which derives four theoretical regimes of asset allocations depending on the extent to which the borrowing constraint and no-short-selling constraint are binding or not. To estimate the structural parameters in the FOC's of the underlying theoretical model, we minimise the difference between the model predicted consumption and imputed consumption from the data for older homeowners. The estimated MPC is the highest for households who are closest to the borrowing constraints (group 1) (close to 1 on average), i.e. they behave in a hand-to-mouth way. It follows that subsidy policy will have the greatest multiplier effect if aimed at the borrowing constrained, low net-worth older households since their consumption is more sensitive to a wealth shock. My estimated MPC falls with total wealth, which is in line with the existing literature. But the value of the estimated average MPC (0.86) is higher. One caveat is that my calculations of MPC are performed assuming no bequest motive. Hence, my estimated MPC's are likely to be an upper bound to the true MPC of the older homeowners.

Chapter 5 uses a life cycle model with two consumption goods and four assets. Asset returns and labour income are uncertain. Each period the household chooses its consumption bundle and asset portfolio subject to borrowing constraints on the mortgage and the risky asset. There is also a fixed minimum level of house purchase reflecting basic indivisibility. Initially we take quite general concave preferences, additive over time. We find conditions on preferences and constraints which lead to different types of constrained optimal choices. In the second half of the chapter we add a functional form on preferences (concave but homogenous degree one within a period) which generates a constant intertemporal MRS. Adding some assumptions on the distribution on asset returns and preference parameters we derive closed form solutions. This is possible because the value function becomes linear in cash on hand. Since the asset returns, labour income and house prices are uncertain, we estimate stochastic processes for each of these. Using other calibrated parameters of preferences and constraints we then run 8000 optimal life cycle paths for 80 realisations of the uncertain variables and 100 households for each of four sets of calibrated parameters. Of course this allows for changes in optimal regimes over life and between households.

Each of the chapters has some questions that remain open, some of which we list below as directions for future research.

In Chapter 2, by specifying the two-equation system for male and female partners' labour supply without cross-equation restrictions, we allow for different parameters associated with the two genders. But to what extent these parameters differ across equations for males and females needs to be further tested. Finally, the implicit ad hoc assumption that every non-worker is voluntarily unemployed might be problematic if the labour market condition is not good enough to accommodate everyone who is willing to work. One solution is to model another hurdle (Blundell and Meghir, 1987). Chapter 2 also uses pooled panel data and focuses on joint labour participation. To allow for individual effects a bivariate panel probit model is an obvious extension. The labour supply side can also be extended to consider hours of work as well as participation again in a panel context. So far we have also treated the non-wage regressors in Chapter 2 as exogenous but an obvious extension will be to test for exogeneity of some regressors.

While one advantage of Chapter 3 is to let the data talk without restricting any the functional forms of the mixing proportions and component means of the assets and housing wealth, it would be interesting to extend our work by parametrising mixing proportions and component means and comparing the estimation results with this chapter. We have used the Gaussian mixture model with pooled panel data. In a sense this pushes some of the individual heterogeneity into the probabilities of belonging to different components. An alternative would be to use time-invariant individual fixed/random and idiosyncratic effects to account for individual heterogeneity. With panel data, we can potentially classify the data into different regimes after de-meaning or first differencing if we assume fixed effect and idiosyncratic effect are additive.

As a possible extension to Chapter 4, the idea of identifying either the optimisation error or measurement error in the data can be applied to any variable in the data set, so the exercise of estimating the preference parameters by matching predicted and actual/imputed data can be repeated for different assets in the sample.

One immediate possible extension to Chapter 5 is to simulate for more than 3 epochs/periods. We have worked with reallocating the portfolio and housing tenure/ownership each period and effectively with one period adjustable rate mortgages. There are no transaction costs of changing tenure or portfolio in the approach here. An obvious extension is to allow for these. Similarly we could add a bequest motive at T or a random time horizon (date of death). Within our framework results could also be presented in different respects e.g. in the life cycle pattern we could compute the probability of each regime being chosen at any date t for given cash on hand and hence from this, the Markov chain for cash on hand as regimes switch between adjacent periods. Furthermore, other parameters such as the maximal LTI or LTV could be varied between simulations to generate further comparative static results. As this is an individual household decision model, the house price and interest rates are taken as given, but using the features here for the demand side (net demand if there is no new build or demolition) model and aggregating the net demand over individuals we could try to determine equilibrium house purchase and rental prices. The range of decision variables could be extended e.g. labour income is partly determined by an individual's choice of working hours. Finally, in principle, simulated and numerically solved paths could be calculated for a general preference case as in Cocco (2005).

To summarise, the housing market raises issues in consumer behaviour that are diverse. Empirically it is a "big" decision affecting all households (everyone has to live somewhere) and also absorbs a high proportion of the budget of most households. Buying housing also involves related financial transactions like securing a mortgage. This then raises the issue of financing constraints like loan to value and loan to income constraints on mortgages and the difficulty of borrowing elsewhere without collateral. Because housing is durable, subject to indivisibility and also trading housing has transaction costs, including time and non-financial market costs, typically households do not move every period. Indeed now houses change hands in the UK once every 23 years on average (Intermediary Mortgage Lenders Associations, 2015). It follows that some household decisions are likely to be taken in the context of their current fixed housing state. In Chapter 2 we explore the role of house prices on household labour participation. Since the change of house prices brings a redistributional effect of wealth for households with different housing tenure and purchasing/liquidating plans, this effect of house price on labour participation is expected to be different for households with different positions in housing. There is evidence that the mortgage constraints facing individual households in terms of loan to value or loan to income limits vary with individual characteristics such as occupation. The individual specific borrowing limits together with heterogeneous preferences and circumstances would generate different patterns of housing and asset allocation. For homeowners with a fixed housing quantity, Chapter 3 identifies these patterns from the data, which forms the basis for the estimation of heterogeneous wealth effect on consumption in Chapter 4. At other times when changing house, there is joint interdependence between housing, consumption, work and other financial asset decisions. The intertemporal nature of the choices and the high share of owned housing in consumer wealth means that past housing choices and outcomes of past returns on housing strongly condition the current spending power of consumers. Chapter 5 shows theoretically that there is a variety of optimal behaviour and binding constraints according to households' initial position in any period.

The thesis has demonstrated a lot of heterogeneity in decisions of consumption, labour participation and asset portfolio choices in the presence of housing but there are many directions to take it forward. Some of these are empirical and some theoretical.

Appendix A

Appendices to Chapter 2

A.1 The house price process

If we assume that households are forward-looking and have rational expectaions, then investigating the house price process is important in understanding how households predict their housing wealth in the future. We assume that the house prices follow a first order autoregressive (AR1) model with a deterministic part a, an individual effect e_l and an idiosyncratic error θ_{lt}:

$$\ln(P_{lt}) = a + \rho \ln(P_{l,t-1}) + e_l + \theta_{lt} \tag{A.1}$$

,where $\ln(P_{lt})$ is the log of real house price; the subscripts l denotes local authority districts and t denotes time.

The unit root hypothesis $\rho = 1$ in equation (A.1) is rejected using Levin-Lin-Chu test[1]. Therefore we conclude that real house prices are stationary and the shocks to house prices do not have a lasting effect. Households who have rational expectations of current house prices based on their observations of house prices in the past in the local authorities they live in should not be affected by variation of house prices. This finding will serve as evidence to explain the insignificant response of house prices in Section 4 where we present the estimation results.

[1] $l = 1, ..., 332. t = 1, ..., 12$. The period covered by our sample (12 years) may not be long enough to capture long term fluctuations of house prices which may be non-stationary.

A.2 Reduced form labour participation estimates

Table A.1: Reduced form univariate probit (the first step of wage equation estimation) (to be continued on the next page)

	Non-movers all age		Whole sample all age	
Independent variables	male	female	male	female
Renter*ln(real house price)	0.11	-0.19	-0.019	-0.13
	(0.35)	(-0.86)	(-0.07)	(-0.62)
Owner*ln(real house price)	0.22	-0.14	0.075	-0.081
	(0.71)	(-0.63)	(0.27)	(-0.40)
Age	0.17***	0.23***	0.18***	0.23***
	(14.59)	(27.40)	(16.51)	(30.45)
Age^2	-0.0024***	-0.0030***	-0.0025***	-0.0030***
	(-20.83)	(-32.41)	(-23.11)	(-35.52)
Worse financal expectation	-0.093	0.039	-0.077	0.042
	(-1.56)	(0.84)	(-1.41)	(1.01)
Better financal expectation	-0.14**	0.066*	-0.16***	0.015
	(-3.02)	(2.02)	(-3.78)	(0.51)
Married	0.33***	-0.11	0.36***	-0.17***
	(3.91)	(-1.94)	(4.70)	(-3.41)
Degree	-0.20	0.023	-0.087	0.0069
	(-1.74)	(0.26)	(-0.85)	(0.09)
Hnd	-0.074	0.027	0.040	0.046
	(-0.63)	(0.30)	(0.39)	(0.56)
A level	-0.10	0.064	-0.023	0.055
	(-0.88)	(0.74)	(-0.22)	(0.70)
Gcse	-0.26*	-0.30***	-0.21*	-0.29***
	(-2.33)	(-3.39)	(-2.09)	(-3.69)

Note: To be continued on the next page. Dependent variables: participation dummies of male and female partners. Additional independent variables not shown in the table: year dummies, local authority districts dummies.

t statistics in parentheses. * p<0.05,** p<0.01, *** p<0.001.

Table A.2: Reduced form univariate probit (the first step of wage equation estimation) (continued)

	Non-movers all age		Whole sample all age	
Independent variables	male	female	male	female
Number of children	0.075**	-0.25***	0.072***	-0.27***
	(3.19)	(-15.25)	(3.38)	(-17.94)
Excellent health status	0.17**	0.28***	0.20***	0.30***
	(3.00)	(7.03)	(3.99)	(8.06)
Good health status	0.20***	0.20***	0.23***	0.18***
	(4.20)	(5.89)	(5.16)	(5.93)
Poor health status	-0.46***	-0.23***	-0.41***	-0.23***
	(-5.79)	(-4.28)	(-5.60)	(-4.42)
Very poor health status	-0.94***	-0.41***	-0.92***	-0.46***
	(-6.17)	(-3.44)	(-6.19)	(-4.12)
Ln (real non-labour income)	-0.63***	-0.28***	-0.58***	-0.28***
	(-32.76)	(-24.96)	(-34.06)	(-26.54)
Regional claimant count rate	-21.2**	-17.3***	-22.0***	-18.9***
	(-3.16)	(-3.63)	(-3.55)	(-4.30)
Ln (regional average earnings)	-0.21	-0.51	-0.31	-0.37
	(-0.27)	(-0.93)	(-0.44)	(-0.72)
N	15570	16244	17733	18428

Note: Dependent variables: participation dummies of male and female partners. Additional independent variables not shown in the table: year dummies, local authority districts dummies.

t statistics in parentheses. * p<0.05,** p<0.01, *** p<0.001.

Appendix B

Appendices to Chapter 3

B.1 Definition of variables

Table B.1: Variable definitions

Variable	Definition
employ	Employment Status of household representative person or partner. (1 if Employee, 2 if self-employed,3 if unemployed, 4 if student, 5 if looking after family, 6 if sick or disabled, 7 if retired, 8 if other.)
nkids	Number of children under 18.
degree	1 if have a degree or above and 0 otherwise.
quali	1 if have qualification lower than the degree level and 0 otherwise.
Age	Age of the household repersentative person or partner.
Age2	Age squared.
marital	Marital status of household representative person or partner. (1 if married, 2 if cohabiting, 3 if single, 4 if widowed, 5 if divorced , 6 if separated, 7 if same sex couple, 8 if civil partner, 9 if former separated civil partner.)
totHval	Real value of the house owned.
A	Real net safe asset.
cash	Real safe asset.
mortgage	Real total mortgage on main residence.
risky	Real risky asset.
hhNetFin	Real household net financial wealth.
GrossEmploy	Real gross annual employee payment.
GrossSE	Real gross annual income from self employment.
Invest	Real total investment income.
income	The sum of GrossEmploy and GrossSE.
lvratio	Loan to value ratio calculated by mortgage devided by totHval.
hhsize	Number of people in household.
bedrooms	Number of bedrooms.
hsetype	Type of house (1 if detached, 2 if semi-detached, 3 if terraced).
northeast	1 if live in North East and 0 otherwise.
northwest	1 if live in North West and 0 otherwise.
yorkshirehumb	1 if live in Yorkshire and the Humber and 0 otherwise.
eastmid	1 if live in East Midlands and 0 otherwise.
westmid	1 if live in West Midlands and 0 otherwise.
eastengland	1 if live in East of England and 0 otherwise.
london	1 if live in London and 0 otherwise.
southeast	1 if live in South East and 0 otherwise.
southwest	1 if live in South West and 0 otherwise.
wales	1 if live in Wales and 0 otherwise.

B.2 Regression of the estimated posterior probability on household characteristics

Table B.2: Multivariate regression results on wave 1 data

	w^1	w^2	w^3	w^4
Age	-0.021**	0.019*	0.0055	-0.0039
	(-2.95)	(2.55)	(1.02)	(-1.52)
Age2	0.00013	-0.00020*	0.0000042	0.000064*
	(1.63)	(-2.31)	(0.07)	(2.12)
income	-0.0000034***	0.00000049	0.0000022***	0.00000071***
	(-14.41)	(1.94)	(12.00)	(8.19)
degree	-0.14***	0.0046	0.13***	0.0094
	(-9.58)	(0.29)	(11.09)	(1.74)
nkids	-0.044***	0.017*	0.017**	0.0094***
	(-6.51)	(2.41)	(3.27)	(3.82)
northeast	-0.068	0.080	0.010	-0.022
	(-1.55)	(1.69)	(0.30)	(-1.33)
northwest	-0.040	0.036	0.025	-0.021
	(-1.27)	(1.07)	(1.01)	(-1.82)
yorkshirehumb	-0.083**	0.086*	0.0093	-0.012
	(-2.59)	(2.51)	(0.37)	(-1.03)
eastmid	-0.094**	0.096**	0.023	-0.026*
	(-2.84)	(2.74)	(0.91)	(-2.15)
westmid	-0.13***	0.11**	0.039	-0.015
	(-3.94)	(3.03)	(1.50)	(-1.25)
eastengland	-0.24***	0.18***	0.073**	-0.0093
	(-7.65)	(5.31)	(2.92)	(-0.79)
london	-0.38***	0.18***	0.16***	0.034**
	(-10.97)	(4.99)	(6.02)	(2.64)
southeast	-0.29***	0.17***	0.12***	0.00074
	(-9.46)	(5.05)	(5.21)	(0.07)
southwest	-0.29***	0.19***	0.12***	-0.019
	(-7.74)	(4.78)	(4.06)	(-1.42)
cons	1.43***	-0.16	-0.31**	0.034
	(9.94)	(-1.01)	(-2.76)	(0.65)

Note: Dependent variable: the estimated posterior probability of component membership. t statistics in parentheses. *p<0.05, **p<0.01, *** p<0.001.

Table B.3: Multivariate regression results on wave 2 data

	w^1	w^2	w^3	w^4
Age	-0.016*	0.0079	0.016**	-0.0072
	(-2.26)	(1.30)	(2.81)	(-1.87)
Age2	0.000069	-0.000045	-0.00014*	0.00012**
	(0.84)	(-0.65)	(-2.26)	(2.72)
income	-0.0000037***	0.00000014	0.0000024***	0.0000012***
	(-14.87)	(0.66)	(12.31)	(9.06)
degree	-0.12***	0.0054	0.081***	0.036***
	(-8.34)	(0.43)	(7.14)	(4.63)
nkids	-0.047***	-0.00046	0.037***	0.010**
	(-6.89)	(-0.08)	(7.11)	(2.77)
northeast	0.0054	0.021	-0.0057	-0.021
	(0.12)	(0.57)	(-0.17)	(-0.89)
northwest	-0.0061	0.025	-0.0059	-0.013
	(-0.19)	(0.95)	(-0.24)	(-0.79)
yorkshirehumb	-0.061	0.066*	-0.0087	0.0041
	(-1.91)	(2.43)	(-0.35)	(0.24)
eastmid	-0.065*	0.079**	-0.0052	-0.0088
	(-1.96)	(2.82)	(-0.20)	(-0.50)
westmid	-0.092**	0.073**	0.021	-0.0019
	(-2.78)	(2.60)	(0.82)	(-0.11)
eastengland	-0.21***	0.10***	0.094***	0.014
	(-6.48)	(3.71)	(3.79)	(0.79)
london	-0.38***	0.083**	0.19***	0.098***
	(-10.86)	(2.83)	(7.30)	(5.33)
southeast	-0.26***	0.087***	0.15***	0.029
	(-8.62)	(3.33)	(6.28)	(1.79)
southwest	-0.23***	0.10**	0.13***	-0.0033
	(-6.04)	(3.18)	(4.47)	(-0.16)
cons	1.42***	-0.038	-0.42***	0.043
	(9.10)	(-0.29)	(-3.53)	(0.51)

Note: Dependent variable: the estimated posterior probability of component membership. t statistics in parentheses. *p<0.05, **p<0.01, *** p<0.001.

Table B.4: Multivariate regression results on wave 3 data

	w^1	w^2	w^3	w^4
Age	-0.036***	0.027***	0.015**	-0.0064
	(-4.78)	(3.58)	(2.63)	(-1.67)
Age2	0.00026**	-0.00025**	-0.00010	0.00010*
	(3.15)	(-3.08)	(-1.67)	(2.45)
income	-0.0000017***	-0.00000018	0.0000012***	0.00000071***
	(-10.55)	(-1.06)	(9.66)	(8.41)
degree	-0.14***	0.0034	0.11***	0.028***
	(-9.31)	(0.22)	(9.57)	(3.57)
nkids	-0.039***	-0.0012	0.034***	0.0061
	(-5.43)	(-0.17)	(6.24)	(1.68)
northeast	0.031	0.0062	-0.018	-0.019
	(0.70)	(0.14)	(-0.53)	(-0.84)
northwest	0.0031	0.0090	-0.0018	-0.010
	(0.10)	(0.28)	(-0.07)	(-0.63)
yorkshirehumb	-0.053	0.048	0.014	-0.0087
	(-1.62)	(1.44)	(0.56)	(-0.52)
eastmid	-0.048	0.045	0.022	-0.018
	(-1.43)	(1.31)	(0.84)	(-1.07)
westmid	-0.092**	0.075*	0.023	-0.0069
	(-2.69)	(2.17)	(0.90)	(-0.40)
eastengland	-0.25***	0.13***	0.093***	0.027
	(-7.75)	(4.03)	(3.72)	(1.63)
london	-0.40***	0.10**	0.23***	0.069***
	(-11.26)	(2.83)	(8.47)	(3.83)
southeast	-0.31***	0.11***	0.16***	0.039*
	(-9.90)	(3.60)	(6.59)	(2.42)
southwest	-0.26***	0.14***	0.10***	0.012
	(-6.69)	(3.61)	(3.57)	(0.61)
cons	1.83***	-0.38*	-0.50***	0.054
	(10.76)	(-2.19)	(-3.90)	(0.63)

Note: Dependent variable: the estimated posterior probability of component membership. t statistics in parentheses. *p<0.05, **p<0.01, *** p<0.001.

Table B.5: Multivariate regression results on pooled data

	w^1	w^2	w^3	w^4
Age	-0.017***	0.012***	0.012***	-0.0062**
	(-4.28)	(-3.37)	(-4.11)	(-3.13)
Age2	0.000088	-0.000088*	-0.00010**	0.00010***
	(-1.93)	(-2.24)	(-3.15)	(-4.55)
income	-0.0000025***	1.5E-07	0.0000015***	0.00000090***
	(-21.44)	(-1.52)	(-17.65)	(-15.52)
degree	-0.15***	0.023**	0.094***	0.030***
	(-17.04)	(-3.09)	(-15.4)	(-7.06)
nkids	-0.043***	0.0043	0.029***	0.0099***
	(-10.88)	(-1.27)	(-10.31)	(-5.04)
northeast	-0.0088	0.03	0.00093	-0.022
	(-0.34)	(-1.34)	(-0.05)	(-1.72)
northwest	-0.014	0.023	0.005	-0.015
	(-0.75)	-1.47	-0.39	(-1.61)
yorkshirehumb	-0.052**	0.052**	0.0056	-0.0057
	(-2.77)	(-3.23)	(-0.42)	(-0.62)
eastmid	-0.061**	0.071***	0.0044	-0.014
	(-3.16)	(-4.23)	(-0.32)	(-1.44)
westmid	-0.11***	0.089***	0.025	-0.0072
	(-5.48)	(-5.29)	(-1.84)	(-0.76)
eastengland	-0.23***	0.12***	0.096***	0.0096
	(-12.05)	(-7.47)	(-7.23)	(-1.04)
london	-0.39***	0.13***	0.19***	0.073***
	(-19.35)	(-7.59)	(-13.02)	(-7.32)
southeast	-0.29***	0.12***	0.14***	0.027**
	(-16.02)	(-7.63)	(-11.19)	(-3.1)
southwest	-0.24***	0.14***	0.098***	-0.0019
	(-10.85)	(-7.53)	(-6.3)	(-0.17)
cons	1.41***	-0.13	-0.33***	0.043
	(-16.26)	(-1.69)	(-5.35)	(-1.02)

Note: Dependent variable: the estimated posterior probability of component membership. t statistics in parentheses. *p<0.05, **p<0.01, *** p<0.001.

Appendix C

Appendices to Chapter 5

C.1 Feasible Sets

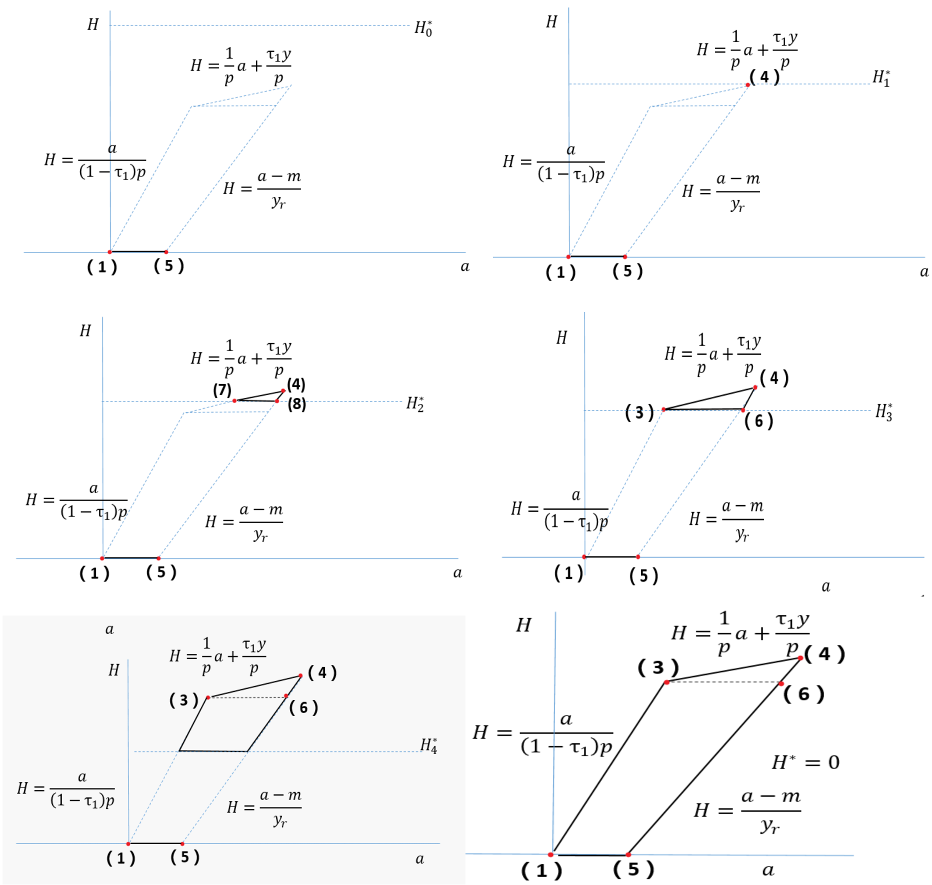


Figure C.1: Possible shapes of feasible sets under Condition 1 for different values of H*

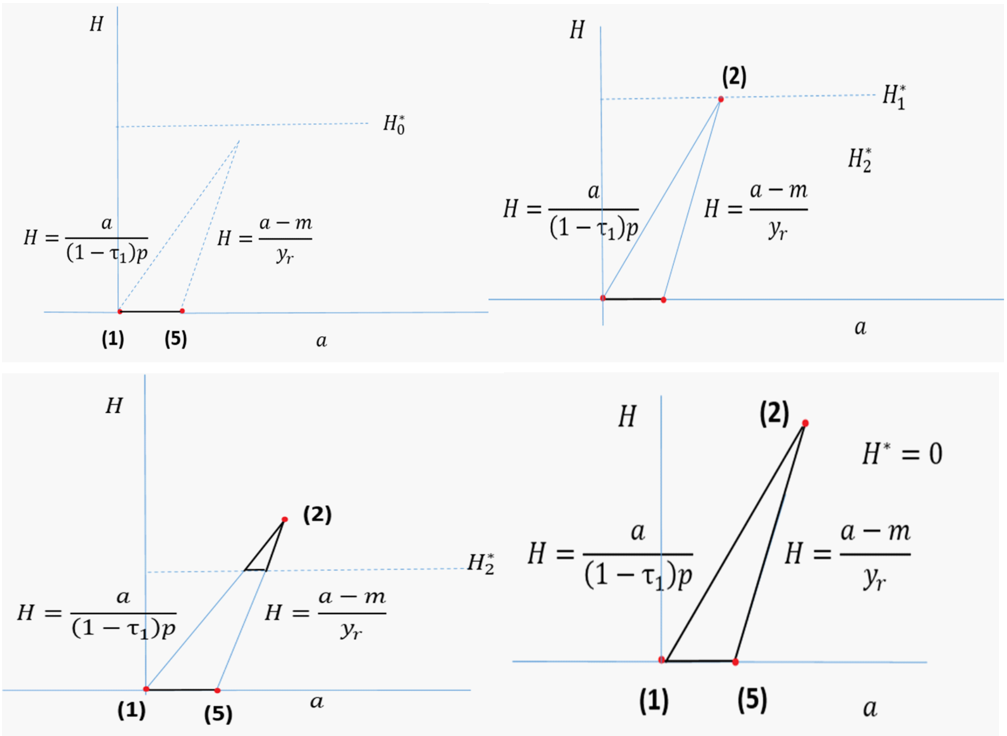


Figure C.2: Possible shapes of feasible sets under Condition 2 for different values of H*

C.2 Concavity of $v_t(a_t, H_t)$

We have assumed that $u(c_t, h_t)$ is strictly increasing and strictly concave. This means that $u^*(x)$ is also strictly concave in x, and also that the value function in the Bellman equation is strictly $t+2$.

$$\begin{aligned} v_t(m_t) &= \max_{H_t, a_t} u^*(m_t + y_{rt}H_t - a_t) \\ &\quad + \beta E_t v_{t+1}(y_{t+1} + R_{F,t+1}(a_t + \min(\tau_1 p_t H_t, \tau_2 y_t) - p_t H_t) - R_{M,t+1}) \min(\tau_1 p_t H_t, \tau_2 y_t) \\ &\quad + p_{t+1} H_t) \end{aligned}$$

The constraints are

$$\begin{aligned} F_t &= a_t + \min(\tau_1 p_t H_t, \tau_2 y_t) - p_t H_t \geq 0 \qquad \text{(C.1)} \\ m_t + y_{rt}H_t - a_t &> 0 \\ a_t, H_t &\geq 0 \end{aligned}$$

But the objective function could be increasing or decreasing in each of a, H at any point in the a, H space.

The two cases of the objective are

$$V_1 = u^*(m_t + y_{rt}H_t - a_t) + \beta E_t v_{t+1}(y_{t+1} + R_{F,t+1}(a_t + \tau_1 p_t H_t - p_t H_t)$$
$$-R_{M,t+1}\tau_1 p_t H_t \ + p_{t+1}H_t) \text{ if } \tau_1 pH < \tau_2 y$$
$$V_2 = u^*(m_t + y_{rt}H_t - a_t) + \beta E_t v_{t+1}(y_{t+1} + R_{F,t+1}(a_t + \tau_2 y_t - p_t H_t)$$
$$-R_{M,t+1}\tau_2 y_t \ + p_{t+1}H_t) \text{ if } \tau_1 pH > \tau_2 y$$

So the derivatives are

$$\begin{aligned}
\frac{\partial V_1}{\partial H} &= \frac{\partial u^*}{\partial x}y_r + \beta E_t \frac{\partial v_{t+1}}{\partial m_{t+1}}(-R_{F,t+1}(1-\tau_1)p_t - R_{M,t+1}\tau_1 p_t + p_{t+1}) \\
\frac{1}{p_t}\frac{\partial V_1}{\partial H} &= \frac{\partial u^*}{\partial x}\frac{y_r}{p_{tt}} + \beta E_t \frac{\partial v_{t+1}}{\partial m_{t+1}}(R_{Ht+1} - R_{F,t+1} + \tau_1(R_{F,t+1} - R_{M,t+1})) \\
\frac{\partial V_1}{\partial a} &= -\frac{\partial u^*}{\partial x} + \beta E_t \frac{\partial v_{t+1}}{\partial m_{t+1}} R_{F,t+1}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial V_2}{\partial H} &= \frac{\partial u^*}{\partial x}y_r + \beta E_t \frac{\partial v_{t+1}}{\partial m_{t+1}}(-R_{F,t+1}p_t + p_{t+1}) \\
\frac{1}{p_t}\frac{\partial V_2}{\partial H} &= \frac{\partial u^*}{\partial x}\frac{y_r}{p_{tt}} + \beta E_t \frac{\partial v_{t+1}}{\partial m_{t+1}}(R_{Ht+1} - R_{F,t+1} \\
\frac{\partial V_2}{\partial a} &= -\frac{\partial u^*}{\partial x} + \beta E_t \frac{\partial v_{t+1}}{\partial m_{t+1}} R_{F,t+1}
\end{aligned}$$

since $R_{Ht+1} = p_{t+1}/p_t$. So

$$\begin{aligned}
\frac{\partial V_1}{\partial a} &= \frac{\partial V_2}{\partial a} \\
\frac{\partial V_2}{\partial H} &< \frac{\partial V_1}{\partial H}
\end{aligned}$$

if $\frac{\partial v_{t+1}}{\partial m_{t+1}} > 0$ and $E_t \frac{\partial v_{t+1}}{\partial m_{t+1}}(R_{F,t+1} - R_{M,t+1}) > 0$

The slopes of contours of the objectives are

$$\frac{dH}{da}|_1 = -\frac{\partial V_1}{\partial a}/\frac{\partial V_1}{\partial H}, \frac{dH}{da}|_2 = -\frac{\partial V_2}{\partial a}/\frac{\partial V_1}{\partial H}$$

The signs of the derivatives can vary with the point a, H at which they are evaluated because in general $\frac{\partial u^*}{\partial x}$ and $\frac{\partial v_{t+1}}{\partial m_{t+1}}$ vary with a, H. Generally with concavity of u^*, v, $\frac{\partial u^*}{\partial x}$ is decreasing in H, and increasing in a, $\frac{\partial v_{t+1}}{\partial m_{t+1}}$ decreasing in a but ambiguous in H so $\frac{\partial V_i}{\partial a}$ decreasing in a but of ambiguous effect in H- could increase or fall with H. $\frac{\partial V_i}{\partial H}$ ambiguous in both a, H.

The general concavity of objective in a, H : we know $u^*(m + yrH - a)$ and $v(m(t+1))$ are $t+2$ in their single arguments; so substituting in x and m_{t+1} to get these as functions of

a, H we have that both x and m_{t+1} are linear functions of a, H. So to save writing say we have $x = \alpha_1 a + a_2 H; m_{t+1} = \beta_1 a + \beta_2 H$

Then we want to look at 2nd derivatives of u^*, v wrt a, H to see if the objective function is $t+2$ in a, H.We have

$\frac{\partial^2 u^*}{\partial a^2} = \alpha_1^2 u^{*\prime\prime} < 0; \frac{\partial^2 v}{\partial H^2} = \beta_1^2 v'' < 0; \frac{\partial^2 u^*}{\partial H^2} = \alpha_2^2 u^{*\prime\prime} < 0; \frac{\partial^2 v}{\partial a^2} = \beta_2^2 v'' < 0; \frac{\partial^2 u^x}{\partial a \partial H} = \alpha_1 \alpha_2 u^{x\prime\prime}; \frac{\partial^2 v}{\partial a \partial H} = \beta_1 \beta_2 v''$. So the Hessian is

$$u^{*\prime\prime} \begin{bmatrix} \alpha_1^2 & \alpha_1\alpha_2 \\ \alpha_1\alpha_2 & \alpha_2^2 \end{bmatrix} + \beta E v'' \begin{bmatrix} \beta_1^2 & \beta_1\beta_2 \\ \beta_1\beta_2 & \beta_2^2 \end{bmatrix}$$

The diagonals are negative but the determinant is zero. Hence as a function of a, the objective is concave but not strictly concave. Any strictly concave function defined on a convex set has a unique maximum. With concavity this is less clear but if there were two maxima like a, H and a', H' then any convex combination of a, H and a', H' would have to yield at least as large a value of the objective and so would be maximal too. Hence the whole line segment connecting two potential optima would also have to yiled the same optimal value which is only possible if the value function is linear in a, H. But we know that it is not if each of u^*, v are strictly concave.

C.3 Slope of the indifference curve for the Cobb-Douglas utility

The slope of the indifference curves is

$$\begin{aligned} MRS_{1t} &= -\frac{\partial V_{1t}/\partial a_t}{\partial V_{1t}/\partial H_t} \\ MRS_{2t} &= -\frac{\partial V_{2t}/\partial a_t}{\partial V_{2t}/\partial H_t} \end{aligned}$$

Notice $\frac{\partial V_{1t}(a_t,H_t)}{\partial H_t} > \frac{\partial V_{2t}(a_t,H_t)}{\partial H_t}$ since $p_t \beta E_t B_{1t+1} \tau_1 (R_{F,t+1} - R_{M,t+1}) > 0$. So the there are only three possible rankings of $\frac{\partial V_{1t}(a_t,H_t)}{\partial H_t}$, $\frac{\partial V_{2t}(a_t,H_t)}{\partial H_t}$, 0. i.e.

(1) $\frac{\partial V_{1t}(a_t,H_t)}{\partial H_t} > \frac{\partial V_{2t}(a_t,H_t)}{\partial H_t} > 0$

(2) $0 > \frac{\partial V_{1t}(a_t,H_t)}{\partial H_t} > \frac{\partial V_{2t}(a_t,H_t)}{\partial H_t}$

(3) $\frac{\partial V_{1t}(a_t,H_t)}{\partial H_t} > 0 > \frac{\partial V_{2t}(a_t,H_t)}{\partial H_t}$

Remember $\frac{\partial v_t(a_t,H_t)}{\partial a_t} = \frac{\partial V_{1t}(a_t,H_t)}{\partial a_t} = \frac{\partial V_{2t}(a_t,H_t)}{\partial a_t}$. Depending on the sign of $\frac{\partial v_t(a_t,H_t)}{\partial a_t}$, there are six different shapes of indifference curve. The arrows in the Figure 5.3 show the direction of increase of utility.

(Panel A) $\frac{\partial v_t(a_t,H_t)}{\partial a_t} > 0, \frac{\partial V_{1t}(a_t,H_t)}{\partial H_t} > \frac{\partial V_{2t}(a_t,H_t)}{\partial H_t} > 0 \Longrightarrow 0 > MRS_{1t} > MRS_{2t}$

(Panel B) $\frac{\partial v_t(a_t,H_t)}{\partial a_t} < 0, 0 > \frac{\partial V_{1t}(a_t,H_t)}{\partial H_t} > \frac{\partial V_{2t}(a_t,H_t)}{\partial H_t} \Longrightarrow 0 > MRS_{2t} > MRS_{1t}$

(Panel C) $\frac{\partial v_t(a_t,H_t)}{\partial a_t} > 0, 0 > \frac{\partial V_{1t}(a_t,H_t)}{\partial H_t} > \frac{\partial V_{2t}(a_t,H_t)}{\partial H_t} \Longrightarrow MRS_{1t} > MRS_{2t} > 0$

(Panel D) $\frac{\partial v_t(a_t,H_t)}{\partial a_t} < 0, \frac{\partial V_{1t}(a_t,H_t)}{\partial H_t} > \frac{\partial V_{2t}(a_t,H_t)}{\partial H_t} > 0 \Longrightarrow 0 < MRS_{1t} < MRS_{2t}$

(Panel E) $\frac{\partial v_t(a_t,H_t)}{\partial a_t} > 0, \frac{\partial V_{1t}(a_t,H_t)}{\partial H_t} > 0 > \frac{\partial V_{2t}(a_t,H_t)}{\partial H_t} \Longrightarrow MRS_{1t} < 0 < MRS_{2t}$

(Panel F) $\frac{\partial v_t(a_t,H_t)}{\partial a_t} < 0, \frac{\partial V_{1t}(a_t,H_t)}{\partial H_t} > 0 > \frac{\partial V_{2t}(a_t,H_t)}{\partial H_t} \Longrightarrow MRS_{1t} > 0 > MRS_{2t}$

C.4 Subsistence levels for housing and non-housing consumption

We can easily add subsistence levels for housing and non-housing consumption, this makes the interpretation of transferring all resources into the future become retaining just enough today to finance the subsistence levels easier to understand. For example, with special preferences, let K_t, L_t be the respective subsistence levels so that utility becomes

$$(c_t - K_t)^{1-\rho}(h_t - L_t)^{\rho}$$

The within period solution should be

$$\begin{aligned} c_t &= K_t + (1-\rho)(m_t + y_{rt}H_t - a_t - K_t - y_{rt}L_t) \\ h_t &= L_t + \rho\frac{(m_t + y_{rt}H_t - a_t - K_t - y_{rt}L_t)}{y_{rt}} \end{aligned}$$

and the within period indirect utility still linear

$$(1-\rho)^{1-\rho}(\frac{y_{rt}}{\rho})^{\rho}(m_t + y_{rt}H_t - a_t - K_t - y_{rt}L_t)$$

so there is still a linear overall value function.

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