Upper Triangular Matrices and Operations in Odd Primary Connective K-Theory

by

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Abstract

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Let $U_\infty\mathbb{Z}_p$ be the group of infinite invertible upper triangular matrices with entries in the p-adic integers. Also let $\mathrm{Aut}^0_{\text{left-}\ell\text{-mod}}(\ell \wedge \ell)$ be the group of left ℓ-module automorphisms of $\ell \wedge \ell$ which induce the identity on mod p homology, where ℓ is the Adams summand of the p-adically complete connective K-Theory spectrum. In this thesis we construct and prove there is an isomorphism between these two groups. We will then determine a specific matrix (up to conjugacy) which corresponds to the automorphism $1 \wedge \psi^q$ of $\ell \wedge \ell$ where ψ^q is the Adams operation and q is an integer which generates the p-adic units $\mathbb{Z}_p^\times$.

We go on to look at the map $1 \wedge \phi_n$ where $\phi_n = (\psi^q - 1)(\psi^q - r)\cdots(\psi^q - r^{n-1})$ and $r = q^{p-1}$ under a generalisation of the map which gave us the isomorphism. Lastly we use some of the ideas presented to give us a new way of looking at the ring of degree zero operations on the connective p-local Adams summand via upper triangular matrices.

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Contents

Introduction

This thesis consists of two main results. The first of these gives an isomorphism between a group of upper triangular matrices and a specific set of automorphisms of connective K-Theory. The second takes a particular automorphism and looks at which matrix this corresponds to under the isomorphism. Firstly let $U_\infty\mathbb{Z}_p$ be the group of infinite, invertible upper triangular matrices with entries in the p-adic integers under matrix multiplication. Let ku_p be the p-adically complete connective complex K-Theory spectrum in the stable homotopy category and let ℓ be the Adams summand relating to it. Denote by $\mathrm{Aut}^0_{\text{left-}\ell\text{-mod}}(\ell \wedge \ell)$ the group of left ℓ-module automorphisms of $\ell \wedge \ell$ which induce the identity on mod p homology.

Theorem 3.1.3. *There is an isomorphism of the form*

$$\Lambda_p : U_\infty\mathbb{Z}_p \to \mathrm{Aut}^0_{\textit{left-}\ell\textit{-mod}}(\ell \wedge \ell).$$

The other main theorem of the thesis determines an explicit matrix in the conjugacy class under this isomorphism of the automorphism $1 \wedge \psi^q : \ell \wedge \ell \to \ell \wedge \ell$. Here q is an integer which generates the p-adic units $\mathbb{Z}_p^\times$ and ψ^q is the Adams operation. Let $r = q^{p-1}$.

Theorem 6.4.2. *The isomorphism Λ_p can be chosen so that the automorphism $1 \wedge \psi^q$ corresponds to the matrix*

$$R = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & r & 1 & 0 & 0 & \cdots \\ 0 & 0 & r^2 & 1 & 0 & \cdots \\ 0 & 0 & 0 & r^3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

These two theorems are odd primary versions of theorems of Snaith, [Sna09, Theorem 3.1.2] and Snaith and Barker, [BS05, Theorem 1.1], which are both localised at the prime 2. The first of these goes as follows. Let $U_\infty\mathbb{Z}_2$ be the group of infinite, invertible upper triangular matrices with coefficients in the 2-adic integers, let ku_2 and ko_2 be the spectra in the stable homotopy category representing 2-adically complete complex and real connective K-Theory respectively. There is an isomorphism of the form

$$\Lambda_2 : U_\infty \mathbb{Z}_2 \to \mathrm{Aut}^0_{\text{left-}ku_2\text{-mod}}(ku_2 \wedge ko_2)$$

where $\mathrm{Aut}^0_{\text{left-}ku_2\text{-mod}}(ku_2 \wedge ko_2)$ is the group of left ku_2-module automorphisms of $ku_2 \wedge ko_2$ which induce the identity on mod 2 homology. The second of these results gives an explicit matrix in the conjugacy class of the automorphism $1 \wedge \psi^3 : ku_2 \wedge ko_2 \to ku_2 \wedge ko_2$, where ψ^3 is the Adams operation.

The isomorphisms Λ_p and Λ_2 are achieved in each case by virtue of the fact that both $\ell \wedge \ell$ and $ku_2 \wedge ko_2$ split as infinite wedges of smaller spectra. In the 2 primary case $ku_2 \wedge ko_2$ is only used instead of $ku_2 \wedge ku_2$ because the splitting of the former is easier to deal with, see [Sna09, Theorem 3.1.6]. In the odd primary case it is useful to split $ku_p \wedge ku_p$ into $p-1$ copies of ℓ on both sides of the smash product in order to avoid many copies of the same information appearing in the result.

This thesis is structured as follows. Chapter 1 introduces all the relevant background information, standard notation and brief introductions to the most useful tools which will be used in the rest of the thesis.

Chapter 2 contains an exposition of Kane's paper *'Operations in connective K-Theory'* ([Kan81]). The splitting of $\ell \wedge \ell$ into a infinite wedge of smaller spectra $\bigvee_{n \geqslant 0} \ell \wedge \mathcal{K}(n)$ is a fundamental aspect of the isomorphism Λ_p and it is very useful to understand how this splitting is constructed and what properties the 'pieces' have. In the paper, all Kane's results are statedly p-locally. It has been pointed out in other papers, e.g. [CDGM88], that what Kane asserts is only valid in a p-complete setting which we also discuss.

Once the splitting has been obtained, Chapter 3 establishes the construction and proof of Theorem 3.1.3. This is done by studying maps between different pieces of the splitting of $\ell \wedge \ell$, i.e. maps of the form $\iota_{m,n} : \ell \wedge \mathcal{K}(m) \to \ell \wedge \mathcal{K}(n)$. We use the concept of stable isomorphism classes, introduced by Adams, to determine the stable class of the mod p cohomology of all the pieces of the splitting. This then becomes the input data of an Adams Spectral Sequence converging to the p-adic completion of homotopy classes of maps from one piece of the splitting to another. This gives us the information that we have a map Λ_p from upper triangular matrices with units on the diagonal to our specific group of automorphisms of $\ell \wedge \ell$ of the form

$$\begin{aligned} &\Lambda_p : U_\infty \mathbb{Z}_p \to \mathrm{Aut}^0_{\text{left-}\ell\text{-mod}}(\ell \wedge \ell) \\ &\qquad X \mapsto \sum_{m \geqslant n} X_{n,m} \iota_{m,n} : \ell \wedge (\bigvee_{i \geqslant 0} \mathcal{K}(i)) \to \ell \wedge (\bigvee_{i \geqslant 0} \mathcal{K}(i)). \end{aligned}$$

We then establish this map as a group isomorphism.

Chapter 4 provides some material and calculations required for Chapter 5 which would otherwise break up the flow of the chapter. The main result is to establish the stable isomorphism class of $H^*(\ell;\mathbb{Z}/p)$ as a $B = \mathbb{Z}/p[Q_0, Q_1]/(Q_0^2, Q_1^2, Q_0Q_1 + Q_1Q_0)$-module where Q_0 and Q_1 are elements of the Milnor basis of the Steenrod Algebra, namely β and $\mathcal{P}^1\beta - \beta\mathcal{P}^1$ respectively. This is achieved by looking at a specific action of Q_0 and Q_1 on $H_*(\ell;\mathbb{Z}/p)$ and calculating their homologies. We finally prove the result by using and comparing the homologies of $H_*(\ell;\mathbb{Z}/p)$ under Q_0 and Q_1 with something called 'lightning flash modules', introduced by Adams.

Our aim then is to calculate which matrix, up to conjugacy, $1 \wedge \psi^q$ corresponds to under Λ_p. This is determined by looking at its action on $\pi_*(\ell \wedge \ell)$ modulo torsion. In Chapter 5 we find a p-adic basis for $\frac{\pi_*(\ell\wedge\ell)}{\text{Torsion}}$ using elements introduced in [CCW01] as a basis for $\pi_*(K \wedge ku) \otimes \mathbb{Z}_{(p)}$. We adapt this basis in an appropriate way following a method of Adams [Ada95, Chapter 17]. We then go on to explore various properties of this basis. These include how it relates to the splitting of $\ell\wedge\ell$ into $\bigvee_{n\geqslant 0} \ell\wedge\mathcal{K}(n)$, what each homotopy group $\frac{\pi_m(\ell\wedge\mathcal{K}(n))}{\text{Torsion}}$ is precisely and what degree of torsion is actually being quotiented out. Finally in this chapter we choose generators for some of the individual homotopy group of each piece of the splitting and find where they would be represented in a spectral sequence converging to the homotopy of that piece. We find how to express these generators in terms of our basis and what effect the induced maps on homotopy $(\iota_{m,n})_*$: $\pi_*(\ell \wedge \mathcal{K}(m)) \to \pi_*(\ell \wedge \mathcal{K}(n))$ have on them.

In Chapter 6 we investigate the effect of the induced map $(1 \wedge \psi^q)_*$ on our basis elements. Recalling the definition of Λ_p we look at the effect the induced maps of $1 \wedge \psi^q$ and $\sum_{m\geqslant n} X_{n,m}\iota_{m,n}$ have on the individual homotopy groups. Since we know how to express these in terms of our basis and what effect the maps $(\iota_{m,n})_*$ and $(1\wedge\psi^q)_*$ have on our basis elements we can equate coefficients and determine the form of the entries in the required matrix. We then show that this resultant matrix can be conjugated to obtain the matrix R in Theorem 6.4.2.

Finally Chapter 7 deals with using this knowledge to simplify the study of topological problems by translation into matrix algebra. The first application looks at the map

$$\phi_n = (\psi^q - 1)(\psi^q - r)\cdots(\psi^q - r^{n-1}) : \ell\wedge\ell \to \ell\wedge\ell$$

and the second uses the ideas of the thesis to present a new way of looking at the ring of degree zero operations on the connective p-local Adams summand $\ell^0(\ell)$ as a subring of the group of upper triangular matrices with entries in the p-local integers.

It turns out that there are a few small errors in the published version of the 2 primary case which I will point out. The splitting of $ku_2 \wedge ko_2$ used in [Sna09] and [BS05] is as follows,

$$\hat{L} : \bigvee_{k \geqslant 0} ku_2 \wedge \frac{F_{4k}}{F_{4k-1}} \to ku_2 \wedge ko_2,$$

where

$$\Omega^2 S^3 \simeq \bigvee_{k \geqslant 1} \frac{F_k}{F_{k-1}}$$

is the Snaith splitting [Sna74]. It turns out that this splitting is not actually correct. Instead of the pieces $\frac{F_{4k}}{F_{4k-1}}$, integral Brown-Gitler spectra for the prime 2 should be used, see [Mah81], [Shi84]. Any other discrepancies will be pointed out as and when they occur during the course of this thesis.

The odd primary case is not substantially different from the 2 primary case, the main story is basically the same in that we have analogous results. However there are many differences in the specifics, it is not simply a case of replacing 2 by p. This leads to the details of the algebra and proofs being different and this is where the original work in this thesis lies. Firstly, the pieces of the splitting are different in the odd primary case to the published version of the 2 primary case as I have detailed above. Integral Brown-Gitler spectra for odd primes are needed to split $\ell \wedge \ell$. Secondly, in the odd primary case it made sense to split the copy of ku on both sides of $ku \wedge ku$ into $p-1$ copies of ℓ to avoid dealing with many shifted copies of the same information.

Chapter 1

Background Material

The material in this section is a summary of the background material needed for later chapters. Throughout this thesis we will be working in the category used by Adams in [Ada95], Boardman's stable homotopy category. We will denote based homotopy classes of maps from a based space X to a based space Y by $[X, Y]$. Let SX denote the reduced suspension of a based space X. We will make it clear when we are working with not necessarily based spaces and will denote the set of unbased homotopy classes of unbased maps by $[X, Y]'$.

1.1 Spectra and localisation

We begin with some preliminary definitions regarding spectra from [Ada95, Part III] and [Rud98].

Let a *CW-spectrum* E be a sequence $\{E_n, s_n\}$ for $n \in \mathbb{Z}$ where each E_n is a CW-complex with a map $s_n : SE_n \to E_{n+1}$ such that $s_n(SE_n)$ is a subcomplex of E_{n+1}.

A *subspectrum* of E is a spectrum F such that F_n is a subcomplex of E_n for all n and the restriction of the structure maps s_n map SF_n into F_{n+1}.

A subspectrum $F \subset E$ is *cofinal* in E if for each cell $e \in E_n$ there exists m such that $S^m e$ is in F_{n+m}.

A *function* from a spectrum $\{E, s_n\}$ to a spectrum $\{F, t_n\}$ is a sequence of maps $f_n : E_n \to F_n$ such that $f_{n+1} \circ s_n = t_n \circ Sf_n$ for all n.

Definition 1.1.1. Consider the set of all cofinal subspectra $E' \subset E$ and functions $f' : E' \to F$. Two such functions $f' : E' \to F$ and $f'' : E'' \to F$ are equivalent if there exists a third cofinal subspectrum $E''' \subset E$ contained in E' and E'' such that the restrictions of f' and f'' to E''' coincide. This is an equivalence relation. A *map* from E to F is an equivalence class of such functions.

Definition 1.1.2. Let I^+ be the unit interval with a disjoint basepoint. For a spectrum $E = \{E_n, s_n\}$, define the *cylinder spectrum* to be have nth space $(\mathrm{Cyl}(E))_n = I^+ \wedge E_n$ and maps $(I^+ \wedge E_n) \wedge S^1 \xrightarrow{1 \wedge s_n} I^+ \wedge E_{n+1}$.

For a spectra E, F, two maps $f_0, f_1 : E \to F$ are *homotopic* if there is a map $h : \mathrm{Cyl}(E) \to F$ such that $f_0 = hi_0$ and $f_1 = hi_1$ where $i_0, i_1 : E \to \mathrm{Cyl}(E)$ are injections of E into the two ends of the cylinder.

This is an equivalence relation. We will denote homotopy classes of maps of degree n, i.e. maps which lower the degree by n, between spectra E and F by $[E, F]_n$.

Definition 1.1.3. A *stable cell* of a spectrum is a sequence of the form $\{e, Se, \cdots, S^k e, \cdots\}$ where a cell e of E_n is not the suspension of a cell in E_{n-1}. For such a cell, if e of E_n has dimension d then the dimension of the stable cell beginning with e has dimension $d - n$.

Definition 1.1.4. A CW-spectrum E is *finite* if it has finitely many stable cells.

This is equivalent to saying that a spectrum E is finite if it is equivalent to the (de)suspension of the suspension spectrum of a finite CW-complex X, i.e. $E = \Sigma^{-k}\Sigma^{\infty} X$ for some $k \in \mathbb{Z}$.

Definition 1.1.5. A spectrum E is of *finite type* if it has finitely many stable cells in each dimension.

There exists a similar concept of a smash product for spectra as for CW-complexes however it is rather tricky to define. For a complete construction of smash product of two spectra see either [Ada95, Part III, Chapter 4] or [Swi02, Chapter 13]. It is enough for us to know the properties of the smash product. For spectra E and F there exists another spectrum $E \wedge F$ such that $E \wedge F$ is a covariant functor in each argument, it is associative, commutative and has the sphere spectrum as a unit on each side up to natural equivalence.

Let G be an abelian group. There exists a spectrum MG, known as a *Moore spectrum* with the following properties.

$$\pi_i(MG) = \begin{cases} 0 & \text{if } i < 0 \\ G & \text{if } i = 0 \end{cases}$$

and

$$H_i(MG) = \begin{cases} G & \text{if } i = 0 \\ 0 & \text{if } i > 0. \end{cases}$$

Definition 1.1.6. We define the spectrum E with coefficients in G as

$$EG = E \wedge MG.$$

Given spectra E and F we define the E-homology and E-cohomology of F by

- $E_i(F) = [S, E \wedge F]_i$,
- $E^i(F) = [F, E]_{-i}$.

A spectrum E is a ring spectrum if it has a multiplication map $\mu : E \wedge E \to E$ and a unit map $\eta : S \to E$ such that the multiplication is associative and unital in the same way as a standard ring, i.e. the following diagrams commute up to homotopy.

$$\begin{tikzcd} E \wedge E \wedge E \arrow[r, "\mu\wedge 1"] \arrow[d, "1\wedge\mu"'] & E \wedge E \arrow[d, "\mu"] \\ E \wedge E \arrow[r, "\mu"'] & E \end{tikzcd} \qquad \begin{tikzcd} & E \wedge E \arrow[d, "\mu"'] & \\ S \wedge E \arrow[ur, "\eta\wedge 1"] \arrow[r, "\simeq"] & E & E \wedge S \arrow[ul, "1\wedge\eta"'] \arrow[l, "\simeq"'] \end{tikzcd}$$

For spectra $E = \{E_n, s_n\}$ and $F = \{F_n, t_n\}$, the wedge of the two spectra is a the spectrum $E \vee F$ with nth space $(E \vee F)_n = E_n \vee F_n$ and maps

$$(E\vee F)_n \wedge S^1 = (E_n \vee F_n) \wedge S^1 = (E_n \wedge S^1) \vee (F_n \wedge S^1) \xrightarrow{s_n \vee t_n} E_{n+1} \vee F_{n+1} = (E \vee F)_{n+1}.$$

The functor $\Sigma : [E, F] \to [\Sigma E, \Sigma F]$ is an equivalence. We can use this to define the concept of addition of maps between spectra.

Definition 1.1.7. Let E, F be spectra and let $\upsilon : S^2 \to S^2 \vee S^2$ be the pinch map. We can turn $[E, F]$ into an abelian group as follows

$$\begin{aligned} [E,F] \oplus [E,F] &= [\Sigma^2 E, \Sigma^2 F] \oplus [\Sigma^2 E, \Sigma^2 F] \\ &= [S^2 \wedge E, \Sigma^2 F] \oplus [S^2 \wedge E, \Sigma^2 F] \\ &= [(S^2 \wedge E) \vee (S^2 \wedge E), \Sigma^2 F] \\ &= [(S^2 \vee S^2) \wedge E, \Sigma^2 F] \\ &\xrightarrow{(\upsilon\wedge 1)_*} [S^2 \wedge E, \Sigma^2 F] = [E, F]. \end{aligned}$$

We will now go on to think about a certain type of spectrum known as a *connective* spectrum.

Theorem 1.1.8. *For a spectrum E there exists is a diagram, called its* Postnikov Tower*,*

$$\begin{tikzcd} \cdots \arrow[r, "1"] & E \arrow[r, "1"] \arrow[d, "\tau_{n+1}"'] & E \arrow[r, "1"] \arrow[d, "\tau_n"'] & E \arrow[r, "1"] \arrow[d, "\tau_{n-1}"'] & \cdots \\ \cdots \arrow[r] & E_{(n+1)} \arrow[r, "p_{n+1}"] & E_{(n)} \arrow[r, "p_n"] & E_{(n-1)} \arrow[r] & \cdots \end{tikzcd}$$

which commutes up to homotopy and for each $n \in \mathbb{Z}$

(i) $\pi_i(E_{(n)}) = 0$ *for* $i > n$,

(ii) $(\tau_n)_* : \pi_i(E) \to \pi_i(E_{(n)})$ *is an isomorphism for* $i \leqslant n$.

Every spectrum E has such a Postnikov Tower and each spectrum $E_{(n)}$ is unique up to equivalence (i.e. any other spectrum satisfying these conditions has a map to $E_{(n)}$ which induces an isomorphism on homotopy). Consider the cofibre sequence

$$F \xrightarrow{q} E \xrightarrow{\tau_n} E_{(n)}.$$

This gives us a long exact sequence in homotopy

$$\pi_*(F) \xrightarrow{q_*} \pi_*(E) \xrightarrow{(\tau_n)_*} \pi_*(E_{(n)})$$

and so $\pi_i(F) = 0$ for $i \leqslant n$.

Definition 1.1.9. A morphism $q : F \to E$ such that $\pi_i(F) = 0$ for $i \leqslant n$ and $q_* : \pi_i(F) \to \pi_i(E)$ is an isomorphism for $i > n$ is an *n-connective covering* of the spectrum E.

A *connective covering* is a (-1)-connective covering. An n-connective covering exists for every n and every spectrum E and these are unique up to equivalence.

In this thesis we will need certain spaces and spectra to be localised with respect to homology theories (mainly $H\mathbb{Z}/p_*$). This was defined by Bousfield in [Bou75] and [Bou79], as follows.

Theorem 1.1.10. *Given a generalised homology theory E_*, there exists a functor L_E from the homotopy category of based CW-complexes to itself and a map $\eta : 1 \to L_E$ such that*

(i) the map $\eta_X : X \to L_E(X)$ induces a homology isomorphism $E_*(X) \cong E_*(L_E(X))$ and

(ii) for any map $f : X \to Y$ inducing a homology isomorphism $E_*(X) \cong E_*(Y)$ there exists a unique map $r : Y \to L_E(X)$ with $rf = \eta_X$.

Theorem 1.1.11. *Given a spectrum E, there exists a functor L_E from the stable homotopy category of CW-spectra to itself and a map $\eta : 1 \to L_E$ such that*

(i) the map $\eta_A : A \to L_E(A)$ induces an E_*-homology isomorphism and

(ii) for any map $f : A \to B$ inducing an E_*-homology isomorphism there exists a unique map $r : B \to L_E(A)$ with $rf = \eta_A$.

Definition 1.1.12. These are called E_*-localisation functors.

It follows from the universal property satisfied by L_E that any other functor with this property is canonically equivalent to L_E.

We will need to be able to compute these localisations for certain connective spectra. In order to do this we will need a few more results from [Bou79].

Definition 1.1.13. A group G is uniquely p-divisible if for every element $g \in G$ the equation $px = g$ has exactly one solution for $x \in G$.

Definition 1.1.14. Two abelian groups G_1 and G_2 have the same *type of acyclicity* if

(i) G_1 is a torsion group if and only if G_2 is a torsion group,

(ii) For every prime p, G_1 is uniquely p-divisible if and only if G_2 is uniquely p-divisible.

Notation 1.1.15. Denote the p-local integers by $\mathbb{Z}_{(p)}$, the p-adic numbers by $\mathbb{Q}_p$ and the p-adic integers by $\mathbb{Z}_p$. The field of integers modulo p will be denoted by $\mathbb{Z}/p$.

Let E be a connective spectrum and let G be an abelian group which has the same type of acyclicity as $\oplus_n \pi_n(E)$. In our case we will be looking at $E = H\mathbb{Z}_{(p)}$ or $H\mathbb{Z}/p$ which have $\pi_*(H\mathbb{Z}_{(p)}) = \mathbb{Z}_{(p)}$ and $\pi_*(H\mathbb{Z}/p) = \mathbb{Z}/p$. Thus G will be either $\mathbb{Z}_{(p)}$ or $\mathbb{Z}/p$ respectively. The following theorem is [Bou79, Theorem 3.1].

Theorem 1.1.16. *For E and X connective spectra, $L_E(X) \simeq L_{MG}(X)$.*

The following are the two main examples we will need.

Proposition 1.1.17. *In the case of the p-local integers, localising with respect to the Moore spectrum $M\mathbb{Z}_{(p)}$ is an example of a* smashing localisation *i.e.*

$$L_{M\mathbb{Z}_{(p)}}(X) \simeq X \wedge L_{M\mathbb{Z}_{(p)}}(S) \simeq X \wedge M\mathbb{Z}_{(p)}$$

where S is the sphere spectrum. Also $\pi_*(L_{M\mathbb{Z}_{(p)}}(X)) \cong \mathbb{Z}_{(p)} \otimes X$.

A spectrum X is ${M\mathbb{Z}_{(p)}}_*$-local if and only if the groups $\pi_*(X)$ are uniquely q-divisible for all primes $q \neq p$.

Proposition 1.1.18. *Localisation of a spectrum X with respect to the Moore spectrum $M\mathbb{Z}/p$ is the function spectrum $F(\Sigma^{-1}M\mathbb{Z}/p^\infty, X)$. Here we denote by $\Sigma^{-1}M\mathbb{Z}/p^\infty$ the desuspension of the Moore spectrum $M\mathbb{Z}/p^\infty$. If the homotopy groups $\pi_*(X)$ are finitely generated then $\pi_*(L_{M\mathbb{Z}/p}(X)) \cong \mathbb{Z}_p \otimes X$.*

If a spectrum is localised with respect to $H\mathbb{Z}_{(p)}$ we will call it *p-local*, if a spectrum is localised with respect to $H\mathbb{Z}/p$ we will call it *p-complete*. The majority of this thesis will be in a p-complete setting, however there is a small section at the end which uses a p-local setting.

One important result for us will be the E_*-Whitehead Theorem, [Bou79, Lemma 1.2].

Theorem 1.1.19. *If spectra X and Y are E_*-local and $f : X \to Y$ is an E_*-equivalence then f is a homotopy equivalence.*

1.2 The Steenrod Algebra

The Steenrod Algebra is a well known object in Mathematics. The main references I have used are [Hat02], [MT68] and [Mil58].

Definition 1.2.1. An unstable *cohomology operation* of type (m, n) for ordinary cohomology with coefficients in a group G is a function

$$\theta_X : H^m(X; G) \to H^n(X; G)$$

for each topological space X, fixed integers m, n and group G, which is natural for any map of spaces $f : X \to Y$, i.e.

$$\begin{tikzcd} H^m(Y;G) \arrow[r, "\theta_Y"] \arrow[d, "f^*"] & H^n(Y;G) \arrow[d, "f^*"] \\ H^m(X;G) \arrow[r, "\theta_X"] & H^n(X;G) \end{tikzcd}$$

commutes. In other words, θ is a natural transformation from $H^m(-; G)$ to $H^n(-; G)$.

Some of the most important cohomology operations are Steenrod Squares and Powers. These are in fact stable cohomology operations, i.e. they satisfy a compatibility with suspension which will be detailed in the following definition. I will only define Steenrod Powers here as Steenrod Squares are the corresponding operations when $p = 2$ which we will not need.

Definition 1.2.2. There exist cohomology operations acting on ordinary mod p cohomology for p odd called *Steenrod Powers* of the form

$$\mathcal{P}^i : H^n(X; \mathbb{Z}/p) \to H^{n+2i(p-1)}(X; \mathbb{Z}/p)$$

for all $i \geqslant 0$ and defined for all n, which satisfy the following properties.

1. Additivity: $\mathcal{P}^i(x + y) = \mathcal{P}^i(x) + \mathcal{P}^i(y)$.
2. Cartan Formula: $\mathcal{P}^i(x \smile y) = \sum_j \mathcal{P}^j(x) \smile \mathcal{P}^{i-j}(y)$.
3. Stability: $\mathcal{P}^i(\sigma(x)) = \sigma(\mathcal{P}^i(x))$ where the map $\sigma : H^n(X; \mathbb{Z}/p) \to H^{n+1}(\Sigma X; \mathbb{Z}/p)$ is the suspension isomorphism.
4. $\mathcal{P}^i(x) = x^p = \overbrace{x \smile \cdots \smile x}^{p}$ if $2i = |x|$,
   $\mathcal{P}^i(x) = 0$ if $2i > |x|$.
5. $\mathcal{P}^0 = \text{id}$.

6. Adem Relations:

$$\mathcal{P}^a\mathcal{P}^b = \sum_j (-1)^{a+j}\binom{(p-1)(b-j)-1}{a-pj}\mathcal{P}^{a+b-j}\mathcal{P}^j$$

if $a < pb$, and

$$\begin{aligned}\mathcal{P}^a\beta\mathcal{P}^b = &\sum_j (-1)^{a+j}\binom{(p-1)(b-j)}{a-pj}\beta\mathcal{P}^{a+b-j}\mathcal{P}^j \\ &-\sum_j (-1)^{a+j+1}\binom{(p-1)(b-j)-1}{a-pj-1}\mathcal{P}^{a+b-j}\beta\mathcal{P}^j\end{aligned}$$

if $a \leqslant pb$, where β is the Bockstein homomorphism associated to the short exact coefficient sequence

$$0 \to \mathbb{Z}/p \xrightarrow{\times p} \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0.$$

We can define the Steenrod Algebra for each odd prime as follows.

Definition 1.2.3. The mod p *Steenrod Algebra* $\mathcal{A}_p$ is the non-commutative polynomial algebra over $\mathbb{Z}/p$ in $\beta, \mathcal{P}^1, \mathcal{P}^2, \ldots$ quotiented by the two-sided ideal generated by the Adem relations and the relation $\beta^2 = 0$. This is a graded algebra where $|\mathcal{P}^i| = 2i(p-1)$ and $|\beta| = 1$

Theorem 1.2.4. *For every space X, $H^*(X;\mathbb{Z}/p)$ is a graded left-module over $\mathcal{A}_p$ for all p.*

Theorem 1.2.5. *For any p, $\mathcal{A}_p$ is the algebra of all stable cohomology operations for ordinary mod p cohomology.*

Definition 1.2.6. An element $a \in \mathcal{A}_p$ is *decomposable* if it can be written in terms of operations in $\mathcal{A}_p$ of lower degree and is *indecomposable* otherwise.

The indecomposable elements of $\mathcal{A}_p$ are β and $\mathcal{P}^{p^k}$ for $k \geqslant 0$. So as an algebra $\mathcal{A}_p$ is generated by β and $\mathcal{P}^{p^k}$ for $k \geqslant 0$.

Definition 1.2.7. In $\mathcal{A}_p$ a sequence $\beta^{\varepsilon_1}\mathcal{P}^{i_1}\beta^{\varepsilon_2}\mathcal{P}^{i_2}\ldots$ is said to be *admissible* if $i_j \geqslant \varepsilon_{j+1} + pi_{j+1}$ for all $j \geqslant 0$.

Note that a sequence being admissible means that nowhere in the sequence does the left-hand side of an Adem relation appear, therefore, in effect, it cannot be simplified in any way.

Theorem 1.2.8. *The admissible monomials in $\mathcal{A}_p$ form an additive $\mathbb{Z}/p$ -basis for $\mathcal{A}_p$ for each prime p.*

Theorem 1.2.9. *The Steenrod Algebra $\mathcal{A}_p$ for each p is a Hopf algebra.*

This means that $\mathcal{A}_p$ has a comultiplication map $\Delta : \mathcal{A}_p \to \mathcal{A}_p \otimes \mathcal{A}_p$ which has the following effect on the operations

$$\Delta(\mathcal{P}^k) = \sum_{i+j=k} \mathcal{P}^i \otimes \mathcal{P}^j \quad \text{and} \quad \Delta(\beta) = \beta \otimes 1 + 1 \otimes \beta.$$

Corollary 1.2.10. *The dual Steenrod Algebra* $\mathcal{A}_p^* = \mathrm{Hom}_{\mathbb{Z}/p}(\mathcal{A}_p, \mathbb{Z}/p)$ *for each* p *is also a Hopf algebra.*

Although infinite, both $\mathcal{A}_p$ and $\mathcal{A}_p^*$ are finite-dimensional in each degree. Since taking the dual of either algebra is done degree-wise there are no problems with doing so.

So using that H_* and H^* are dual with field coefficients, the left action of $\mathcal{A}_p$ on $H^*(X;\mathbb{Z}/p)$ can give us a right action of $\mathcal{A}_p$ on $H_*(X;\mathbb{Z}/p)$ via

$$\langle xa, y\rangle = \langle x, ay\rangle$$

for $a \in \mathcal{A}_p$, $x \in H_*(X;\mathbb{Z}/p)$ and $y \in H^*(X;\mathbb{Z}/p)$. Here $\langle x, y\rangle = x(y)$ means evaluating the function x on the element y when both are in the same degree and zero otherwise. This gives us a map

$$\lambda_* : H_*(X;\mathbb{Z}/p) \otimes \mathcal{A}_p \to H_*(X;\mathbb{Z}/p)$$

which we can dualise to obtain a coaction map

$$\lambda^* : H^*(X;\mathbb{Z}/p) \to H^*(X;\mathbb{Z}/p)\hat{\otimes}\mathcal{A}_p^*$$

where $\hat{\otimes}$ denotes the completed tensor product, used to avoid difficulties with infinite sums. We will always in practice deal with finite complexes or complexes of finite type so will never need to worry about infinite sums.

In order to describe some elements of $\mathcal{A}_p^*$ we will consider the orbit space of $\mathbb{Z}/p$ acting freely on the unit sphere $S^{2n+1} \subset \mathbb{C}^n$ by rotating each factor of $\mathbb{C}$ in $\mathbb{C}^n$ by an angle of $\frac{2\pi}{p}$. This is known as a Lens space which we denote by $L_{n,p}$ and is a finite complex. This can be considered as the $(2n+1)$-skeleton of the Eilenberg-MacLane space $K(\mathbb{Z}/p, 1) = \frac{S^\infty}{\mathbb{Z}/p}$, we use the Lens space here to avoid dealing with infinite sums. The mod p cohomology ring of $K(\mathbb{Z}/p, 1)$ is given by

$$H^*(K(\mathbb{Z}/p, 1);\mathbb{Z}/p) = \Lambda(a) \otimes \mathbb{Z}/p[b]$$

where $|a| = 1$, $|b| = 2$ and $b = \beta a$ and Λ denotes an exterior algebra over $\mathbb{Z}/p$. The cohomology structure of the Lens space is the same but truncates at degree $2n+1$. Let $M_k \in (\mathcal{A}_p)_{2p^k-2}$ be $\mathcal{P}^{p^{k-1}}\mathcal{P}^{p^{k-2}} \cdots \mathcal{P}^p\mathcal{P}^1$, then $M_k b = b^{p^k}$. For any other monomial θ in the operations β and $\mathcal{P}^{p^i}$ for $i \geqslant 0$, $\theta b = 0$. Similarly $(M_k\beta)a = b^{p^k}$ but for θ any other monomial in the same operations, $\theta a = 0$.

The action of the map $\lambda^* : H^*(L_{n,p};\mathbb{Z}/p) \to H^*(L_{n,p};\mathbb{Z}/p) \otimes \mathcal{A}_p^*$ on the elements a and b is given by

$$\lambda^*(a) = a \otimes 1 + b \otimes \tau_0 + b^p \otimes \tau_1 + \cdots + b^{p^r} \otimes \tau_r$$

and

$$\lambda^*(b) = b \otimes 1 + b^p \otimes \xi_1 + \cdots + b^{p^r} \otimes \xi_r$$

where $p^r \leqslant n$ is the largest such power of p. This defines elements $\tau_k \in (\mathcal{A}_p^*)_{2p^k-1}$ and $\xi_k \in (\mathcal{A}_p^*)_{2p^k-2}$.

Theorem 1.2.11. $\mathcal{A}_p^* \cong \Lambda(\tau_0, \tau_1, \cdots) \otimes \mathbb{Z}/p[\xi_1, \xi_2, \cdots]$.

It can be shown that

$$\langle \xi_k, M \rangle = \begin{cases} 1 & \text{if } M = M_k \\ 0 & \text{if } M \text{ is any other admissible monomial,} \end{cases}$$

and

$$\langle \tau_k, M \rangle = \begin{cases} 1 & \text{if } M = M_k\beta \\ 0 & \text{if } M \text{ is any other admissible monomial.} \end{cases}$$

Standard sign conventions for graded algebras mean that the relations between the elements are as follows:

$$\begin{aligned} \xi_i\xi_j &= \xi_j\xi_i \\ \xi_i\tau_j &= \tau_j\xi_i \\ \tau_i\tau_j &= -\tau_j\tau_i. \end{aligned}$$

Let $R = (r_1, r_2, \cdots)$ be any infinite sequence of non-negative integers with only finitely many non-zero terms and let $E = (\varepsilon_0, \varepsilon_1, \cdots)$ be any infinite sequence of zeros and ones with only finitely many ones. Let $\xi^R = \xi_1^{r_1}\xi_2^{r_2}\ldots$ and $\tau^E = \tau_0^{\varepsilon_0}\tau_1^{\varepsilon_1}\ldots$.

Theorem 1.2.12. *The set $\{\tau^E\xi^R\}$ forms an additive $\mathbb{Z}/p$-basis for $\mathcal{A}_p^*$.*

Let $\rho(E,R) \in \mathcal{A}_p$ be the dual element to $\tau^E\xi^R$, i.e.

$$\langle \rho(E,R), \tau^{E'}\xi^{R'} \rangle = \begin{cases} 1 & \text{if } E = E' \text{ and } R = R' \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $\rho(0,(r,0,0,\cdots)) = \mathcal{P}^r$. Let Q_k be the element dual to τ_k, then $Q_0 = \rho((1,0,0,\cdots),0) = \beta$. The elements Q_k for $k > 0$ can be shown to satisfy the inductive property $Q_{k+1} = [\mathcal{P}^{p^k}, Q_k]$ where $[x,y] = xy - (-1)^{|x||y|}yx$ is the commutator.

Lemma 1.2.13. *The elements $\rho(E,R)$ form an additive basis for $\mathcal{A}_p$ dual to $\{\tau^E\xi^R\}$.*

The elements $Q_k \in (\mathcal{A}_p)_{2p^k-1}$ generate an exterior algebra.

Definition 1.2.14. Let $B \subseteq \mathcal{A}_p$ be the subalgebra generated by Q_0 and Q_1.

Definition 1.2.15. The left action of $\mathcal{A}_p$ on itself by using the multiplication, μ, gives us a right action of $\mathcal{A}_p$ on $\mathcal{A}_p^*$ given by

$$\langle fa, y\rangle = \langle f, ay\rangle$$

where $f \in \mathcal{A}_p^* = \mathrm{Hom}(\mathcal{A}_p; \mathbb{Z}/p)$ and $a, y \in \mathcal{A}_p$.

This can be expressed in many different formulae, one that will be useful to us later is the following. We use the notation $\Delta f = \Sigma f' \otimes f''$ for comultiplication in $\mathcal{A}_p^*$.

Proposition 1.2.16. *The right action of $\mathcal{A}_p$ on $\mathcal{A}_p^*$ given above can be expressed as*

$$\begin{aligned} \psi : \mathcal{A}_p^* \otimes \mathcal{A}_p &\to \mathcal{A}_p^* \\ f \otimes a &\mapsto \sum (-1)^{|f''||a|} \langle f', a\rangle f''. \end{aligned}$$

Proof. Using the fact that the comultiplication Δ in $\mathcal{A}_p^*$ is dual to the multiplication μ in $\mathcal{A}_p$, and that because $\mathcal{A}_p^*$ is finitely generated in each degree we have $\mathrm{Hom}(\mathcal{A}_p, \mathbb{Z}/p) \otimes \mathrm{Hom}(\mathcal{A}_p, \mathbb{Z}/p) \cong \mathrm{Hom}(\mathcal{A}_p \otimes \mathcal{A}_p, \mathbb{Z}/p)$, so the following diagram commutes.

$$\begin{tikzcd}
\mathcal{A}_p^* \otimes \mathcal{A}_p \otimes \mathcal{A}_p \arrow[rr, "1\otimes\mu"] \arrow[d, "\Delta\otimes 1\otimes 1"'] & & \mathcal{A}_p^* \otimes \mathcal{A}_p \arrow[d, "\langle\,,\rangle"] \\
\mathcal{A}_p^* \otimes \mathcal{A}_p^* \otimes \mathcal{A}_p \otimes \mathcal{A}_p \arrow[r, "\cong"] & (\mathcal{A}_p \otimes \mathcal{A}_p)^* \otimes \mathcal{A}_p \otimes \mathcal{A}_p \arrow[r, "\langle\,,\rangle"] & \mathbb{Z}/p
\end{tikzcd}$$

Using the upper route through the diagram sends $f \otimes a \otimes y$ to $\langle f, ay\rangle$, which is the definition of our right action of $\mathcal{A}_p$ on $\mathcal{A}_p^*$. We can express this in another equivalent formula by using the lower route through the commutative diagram, i.e. $\langle fa, y\rangle = \langle f, ay\rangle$ can be expressed as

$$\begin{aligned} f \otimes a \otimes y \mapsto \Delta(f) \otimes a \otimes y &= \sum f' \otimes f'' \otimes a \otimes y \\ \mapsto \langle \sum f' \otimes f'', a \otimes y\rangle &= \sum (-1)^{|f''||a|} \langle f', a\rangle \langle f'', y\rangle. \end{aligned}$$ □

This can also be viewed in another equivalent way which we will also make use of later.

Definition 1.2.17. There is a left action of $\mathcal{A}_p$ on $\mathcal{A}_p^*$ given in [Sch94, §2.5]

$$\begin{aligned} \phi : \mathcal{A}_p \otimes \mathcal{A}_p^* &\to \mathcal{A}_p^* \\ a \otimes f &\mapsto \sum (-1)^{|f''|(|f'|+|a|)} \langle f'', a\rangle f'. \end{aligned}$$

Let χ denote the canonical anti-automorphism of $\mathcal{A}_p^*$.

Proposition 1.2.18. *There is a left action of $\mathcal{A}_p$ on $\mathcal{A}_p^*$ given by*

$$\begin{array}{ccccccc}
\mathcal{A}_p \otimes \mathcal{A}_p^* & \xrightarrow{1\otimes\chi} & \mathcal{A}_p \otimes \mathcal{A}_p^* & \xrightarrow{\phi} & \mathcal{A}_p^* & \xrightarrow{\chi} & \mathcal{A}_p^* \\
a \otimes f & \longmapsto & a \otimes \bar{f} & \longmapsto & \sum(-1)^{|\bar{f}''|(|\bar{f}'|+|a|)}\langle \bar{f}'', a\rangle \bar{f}' & \longmapsto & \sum(-1)^{|\overline{f''}||a|}\langle \overline{f'}, a\rangle f'',
\end{array}$$

where the bar denotes the image of an element under the anti-automorphism χ of $\mathcal{A}_p^*$. This can be made into the right action of $\mathcal{A}_p$ on $\mathcal{A}_p^*$ given in Proposition 1.2.16 by using χ on $a \in \mathcal{A}_p$.

Proof. Using the anti-automorphism χ on $\mathcal{A}_p^*$ both before and after the left module action of ϕ still gives you a left module action. We also use the fact that

$$\sum \bar{f}' \otimes \bar{f}'' = \sum(-1)^{|\overline{f'}||\overline{f''}|}\overline{f''} \otimes \overline{f'}$$

so when we apply ϕ to $a \otimes \bar{f} \in \mathcal{A}_p \otimes \mathcal{A}_p^*$ we get

$$\begin{aligned}
\sum(-1)^{|f''|(|f'|+|a|)}\langle \bar{f}'', a\rangle \bar{f}' &= \sum(-1)^{|f''|(|f'|+|a|)+|\overline{f'}||\overline{f''}|}\langle \overline{f'}, a\rangle \overline{f''} \\
&= \sum(-1)^{|\overline{f''}||a|}\langle \overline{f'}, a\rangle \overline{f''}.
\end{aligned}$$

Since for any $a \in \mathcal{A}_p$ and $f \in \mathcal{A}_p^*$ we have $\langle \bar{a}, b\rangle = \langle a, \bar{b}\rangle$, the above left action can be simplified to

$$\begin{aligned}
&\mathcal{A}_p \otimes \mathcal{A}_p^* \to \mathcal{A}_p^* \\
&\quad a \otimes f \mapsto \sum(-1)^{|f''||a|}\langle f', \bar{a}\rangle f''.
\end{aligned}$$

Using the anti-automorphism to give us a right action of $\mathcal{A}_p$ on $\mathcal{A}_p^*$ we get

$$\begin{aligned}
&\mathcal{A}_p^* \otimes \mathcal{A}_p \to \mathcal{A}_p^* \\
&\quad f \otimes a \mapsto \sum(-1)^{|f''||a|}\langle f', \bar{\bar{a}}\rangle f'' = \sum(-1)^{|f''||a|}\langle f', a\rangle f''
\end{aligned}$$

which is precisely the action ψ of Proposition 1.2.16. $\square$

1.3 K-Theory and the Adams Splitting

K-Theory

The following material comes mainly from [Ati89] and [Hat04]. In this section X is a not necessarily based space, it will be made clear when X has a basepoint.

Definition 1.3.1. A complex *vector bundle* of dimension n is a topological space E together with a map $p : E \to B$ for a topological space B such that

$p^{-1}(b)$ is a finite dimensional complex vector space and the following local triviality condition is satisfied. For each $b \in B$ there exists an open neighbourhood U of b such that $E|_U$ is trivial, i.e. there exists a homeomorphism $h : p^{-1}(U) \to U \times \mathbb{C}^n$ which maps $p^{-1}(b)$ to $\{b\} \times \mathbb{C}^n$ by a linear map of vector spaces for all $b \in U$.

We call E the *total space*, B the *base space*, p the *projection map* and $p^{-1}(b)$ the *fibres* of the vector bundle.

An *isomorphism* of vector bundles $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$, denoted $\cong$, is a homeomorphism $h : E_1 \to E_2$ such that $p_2 h = p_1$ which maps $p_1^{-1}(b)$ to $p_2^{-1}(b)$ by a linear isomorphism for each $b \in B$.

We can form the *direct sum* of two vector bundles over B, $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$, to be the vector bundle over B with total space

$$E_1 \oplus E_2 = \{(v_1, v_2) \in E_1 \times E_2 : p_1(v_1) = p_2(v_2)\}$$

and obvious map $E_1 \oplus E_2 \to B$.

We can also take the *tensor product* of vector bundles. For $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$, the tensor product $E_1 \otimes E_2$ has total space the disjoint union of $p_1^{-1}(b) \otimes p_2^{-1}(b)$ for $b \in B$. The topologies of the two original vector bundles can be combined to give a coherent topology in $E_1 \otimes E_2$.

Given a vector bundle $p : E \to B$ and a map $f : A \to B$ there exists a unique bundle up to isomorphism $p' : f^*(E) \to A$, where $f^*(E)$ is the *pullback* of E by f, and a map $f' : f^*(E) \to E$ such that f' maps the fibre in $f^*(E)$ over a point $a \in A$ isomorphically onto the fibre in E over the image $f(a) \in B$. We can explicitly write down a pullback bundle as $f^*(E) = \{(a, v) \in A \times E : f(a) = p(v)\}$ and $p' : (a, v) \mapsto a$ is projection onto the first factor.

Definition 1.3.2. Let $\mathrm{Vect}(X)$ be the set of isomorphism classes of complex vector bundles over X.

The set $\mathrm{Vect}(X)$ is an abelian semi-group with addition given by the direct sum of vector bundles.

For any abelian semi-group A under $\oplus$ we can construct the *Grothendieck group* of A which is an abelian group. This is formed by taking the quotient $\frac{F(A)}{E(A)}$ where $F(A)$ is the free abelian group generated by the elements of A and $E(A) \subset F(A)$ is the subset generated by elements of the form $a + a' - (a \oplus a')$ for $a, a' \in A$.

Definition 1.3.3. Let X be compact Hausdorff. The group $K(X)$ is the Grothendieck group of $\mathrm{Vect}(X)$.

It can be shown that every element of $K(X)$ is of the form $[E] - [E']$, i.e. a formal difference of isomorphism classes of vector bundles over X. The zero element of this group is the class of $[E] - [E]$ for any vector bundle E and the inverse of the element $[E_1] - [E_2]$ is $[E_2] - [E_1]$.

Let the trivial vector bundle of dimension n over a based space X be denoted ε^n. This has as its total space $B \times \mathbb{C}^n$ and its map p is projection onto B. For every vector bundle E there exists a bundle E' such that $E \oplus E' \cong \varepsilon^n$ for some $n \in \mathbb{N}_0$. So for any element $[E_1] - [E_2] \in K(X)$ there exists a bundle E_3 such that $E_2 \oplus E_3 \cong \varepsilon^n$, this gives us that $[E_1] - [E_2] = [E_1] + [E_3] - ([E_2] + [E_3]) = [E_1 \oplus E_3] - [\varepsilon_n]$ and hence every element of $K(X)$ can be represented by a formal difference $[E] - [\varepsilon^n]$ for some n.

Two vector bundles E_1 and E_2 are said to be *stably equivalent*, denoted $E_1 \approx E_2$, if $E_1 \oplus \varepsilon^n \cong E_2 \oplus \varepsilon^n$ for some n. This is an equivalence relation. It can be shown that two bundles E_1 and E_2 represent the same element in $K(X)$ if and only if they are stably equivalent.

The tensor product of vector bundles can be extended to formal differences of vector bundles quite easily i.e.

$$([E_1] - [E_1'])([E_2] - [E_2']) = E_1 \otimes E_2 - E_1 \otimes E_2' - E_1' \otimes E_2 + E_1' \otimes E_2'$$

which gives $K(X)$ a commutative ring structure with identity ε^1.

A second equivalence relation of vector bundles is given by $E_1 \sim E_2$ if $E_1 \oplus \varepsilon^n \cong E_2 \oplus \varepsilon^m$ for some m, n.

Definition 1.3.4. Let X be a compact Hausdorff space, the set of equivalence classes of vector bundles over X under the relation $\sim$ forms an abelian group with respect to the direct sum of bundles known as the *reduced* K-Theory of X, denoted $\tilde{K}(X)$, with identity ε^0.

Unreduced K-Theory can be thought of as a contravariant functor from compact Hausdorff spaces to abelian groups using the pullback bundle. Given a map $f : X \to Y$, this induces a map $f^* : K(Y) \to K(X)$ by sending $[E] - [E']$ to $[f^*(E)] - [f^*(E')]$ for E and E' vector bundles over Y. Similarly reduced K-theory can be thought of as a functor from based, compact Hausdorff spaces to abelian groups in the same way.

There is a surjection $K(X) \to \tilde{K}(X)$ sending $E - \varepsilon^n$ to the class of E under $\sim$ whose kernel is $\{\varepsilon^m - \varepsilon^n : m, n \in \mathbb{Z}\} \cong \mathbb{Z}$. The inclusion of the basepoint $x_0 \hookrightarrow X$ induces a map $K(X) \to K(\{x_0\}) \cong \mathbb{Z}$ which becomes an isomorphism when restricted to $\ker(K(X) \to \tilde{K}(X))$ and so K-Theory splits as $K(X) \cong \tilde{K}(X) \oplus \mathbb{Z}$.

Theorem 1.3.5. *For compact Hausdorff spaces X it can be shown that $K(X) \cong [X, \mathbb{Z} \times BU]'$ where BU is the classifying space of the infinite unitary group. For based spaces we have that $\tilde{K}(X) \cong [X, \mathbb{Z} \times BU] \cong \mathrm{Map}(\pi_0(X), \mathbb{Z}) \times [X, BU]$. When X is path-connected this gives us that $K^0(X) = \mathbb{Z} \times [X, BU]$ and $\tilde{K}^0(X) = [X, BU]$.*

For X a more general space these should be taken as the definitions of $K(X)$ and $\tilde{K}(X)$.

Theorem 1.3.6 (Bott Periodicity)**.** *For any based topological space X there is an isomorphism*

$$\tilde{K}(X) \cong \tilde{K}(S^2 X).$$

This can be used to make K-Theory into a cohomology theory. Let (X, A) be a pair of based compact Hausdorff spaces, then the following exact sequence can be used to define cohomology groups

$$\tilde{K}(S^2 A) \to \tilde{K}(S(X/A)) \to \tilde{K}(SX) \to \tilde{K}(SA) \to \tilde{K}(X/A) \to \tilde{K}(X) \to \tilde{K}(A).$$

By letting $\tilde{K}(X) = \tilde{K}^0(X)$ and using the rules $\tilde{K}^{-n}(X) = \tilde{K}(S^n X)$ and $\tilde{K}^{-n}(X, A) = \tilde{K}(S^n(X/A))$ this sequence now becomes

$$\tilde{K}^{-2}(A) \to \tilde{K}^{-1}(X, A) \to \tilde{K}^{-1}(X) \to \tilde{K}^{-1}(A) \to \tilde{K}^0(X, A) \to \tilde{K}^0(X) \to \tilde{K}^0(A).$$

By Bott Periodicity we have that $\tilde{K}^0(X) \cong \tilde{K}^{-2}(X)$ and we can define the positive side of the theory in a similar way by letting $\tilde{K}^{2i}(X) = \tilde{K}^0(X)$ and $\tilde{K}^{2i+1}(X) = \tilde{K}^1(X)$. A similar process can be performed for unreduced K-Theory.

Periodic K-Theory is represented by the periodic K-Theory spectrum K, it has as every even space $\mathbb{Z} \times BU$ and every odd space U. This spectrum has coefficient groups

$$\pi_*(K) = \mathbb{Z}[u, u^{-1}]$$

where $|u| = 2$. If we look at the connective cover of K we get the spectrum ku representing connective complex K-Theory. This has coefficient groups

$$\pi_*(ku) = \mathbb{Z}[u].$$

The Adams Splitting

In this section we will explain that, when localised at an odd prime p, the spectra K and ku split into a wedge of suspensions of smaller spectra L and ℓ respectively. The following material comes from [Ada69].

Firstly let $\mu : BU \times BU \to BU$ be the H-Space multiplication coming from the direct sum of complex vector bundles and let $\pi_1, \pi_2 : BU \times BU \to BU$ be projection onto the first and second factor respectively. Consider the primitive elements of $\tilde{K}^0(BU) \cong [BU, BU]$, i.e. elements $a \in \tilde{K}^0(BU)$ such that

$$\mu^*(a) = \pi_1^*(a) + \pi_2^*(a).$$

Any such a is an operation on $\tilde{K}^0(X)$ by composition, looking at primitive elements guarantees the operations are additive.

We now turn our attention to K-Theory with coefficients in various subsets of the rational numbers.

Definition 1.3.7. Let $R \subset \mathbb{Q}$ and let

$$\tilde{A}(R) = \{\text{natural additive operations on } \tilde{K}^0(-; R) \text{ for based spaces}\},$$

$$A(R) = \{\text{natural additive operations on } K^0(-;R) \text{ for based spaces}\}$$

and

$$A(R)_0 = \{\alpha \in \tilde{A}(R) : \alpha = 0 \text{ on } \tilde{K}^0(S^0;R) \cong R\}.$$

Then because we are working with based spaces we have $K^0(X;R) \cong R \oplus \tilde{K}^0(X;R)$ so $A(R) = R \oplus \tilde{A}(R)$. It can also be shown that $\tilde{A}(R) \cong R \oplus A(R)_0$ where the copy of R comes from splitting off Ri where i is the identity operation.

Proposition 1.3.8. *There is a monomorphism*

$$\iota : A(R_1) \to A(R_2)$$

for $R_1 \subset R_2 \subset \mathbb{Q}$.

Proof. We can work out $K^0(BU;R_1)$ and $K^0(BU;R_2)$ explicitly as follows, see for example [Cla81]. We have that $K^0(BU(1)) = \mathbb{Z}[\![x]\!]$ where $x = \xi - 1$ for $\xi \in K^0(BU(1))$ the Hopf bundle. Let $1, \beta_1, \beta_2, \ldots$ be the dual basis in $K_0(BU(1))$ to the powers $1, x, x^2, \ldots$ in $K^0(BU(1))$. The elements $\beta_1, \beta_2, \ldots$ are polynomial generators for $K_0(BU)$, so $K_0(BU) = \mathbb{Z}[\beta_1, \beta_2, \ldots]$. Now let γ_i be the elements dual to ${\beta_1}^i$, then $K^0(BU)$ is a power series with generators γ_i, i.e. $K^0(BU) = \mathbb{Z}[\![\gamma_1, \gamma_2, \ldots]\!]$. So we see that

$$i_* : K^0(BU;R_1) = R_1[\![\gamma_1, \gamma_2, \ldots]\!] \to R_2[\![\gamma_1, \gamma_2, \ldots]\!] = K^0(BU;R_2)$$

is a monomorphism. The restricted map to the primitive elements $A(R_1)_0 = PK^0(BU;R_1) \to PK^0(BU;R_2) = A(R_2)_0$ will also be a monomorphism. Now we have

$$\begin{aligned} A(R_1) &\cong R_1 \oplus \tilde{A}(R_1) \\ &\cong R_1 \oplus R_1 \oplus A(R_1)_0 \\ &\cong R_1 \oplus R_1 \oplus PK^0(BU;R_1) \end{aligned}$$

and similarly for $A(R_2)$. The maps between each of the corresponding components of $A(R_1)$ and $A(R_2)$ are monomorphisms so ι is too. $\square$

This means we can look for operations in $A(\mathbb{Q})$ which split $K^0(-;\mathbb{Q})$ and show they also lie in $A(\mathbb{Z}_{(p)})$ hence splitting $K^0(-;\mathbb{Z}_{(p)})$ which we denote as $K^0_{(p)}$.

The Chern character gives us an isomorphism

$$ch : K^*(X;\mathbb{Q}) \cong \prod_n H^{2n}(X;\mathbb{Q}).$$

Let e_n be projection from $\prod_n H^{2n}(X;\mathbb{Q})$ onto the $2n$th factor

$$e_n(h^0, h^2, \cdots, h^{2n-2}, h^{2n}, h^{2n+2}, \cdots) = (0, \cdots, 0, h^{2n}, 0, \cdots)$$

where $h^i \in H^i(X;\mathbb{Q})$. This operation is clearly idempotent in $A(\mathbb{Q})$. Now for a fixed odd prime p we construct a similar object with one non-zero coefficient group every $2(p-1)$ dimensions. So for $\alpha \in \{0,1,\cdots,p-2\}$ and $n \equiv \alpha \mod p-1$ let

$$E_\alpha = \sum_n e_n.$$

This gives us that

$$E_\alpha(h^0,h^2,h^4,\cdots) = (k^0,k^2,k^4,\cdots)$$

where

$$k^{2n} = \begin{cases} h^{2n} & \text{if } n \in \alpha \\ 0 & \text{if } n \notin \alpha. \end{cases}$$

It can then be shown that the idempotents are defined p-locally.

Theorem 1.3.9. $E_\alpha \in A(\mathbb{Z}_{(p)})$.

These operations have the following properties:

(i) $E_\alpha^2 = E_\alpha$.

(ii) $E_\alpha E_\beta = 0$ if $\alpha \neq \beta$.

(iii) $\sum_\alpha E_\alpha = 1$.

(iv) For $x, y \in K^0_{(p)}(X)$ there is a Cartan formula

$$E_\alpha(xy) = \sum_{\beta+\gamma=\alpha} E_\beta(x)E_\gamma(y).$$

This all gives us pairwise orthogonal idempotent operations summing to 1, so we have a corresponding splitting:

$$K^0_{(p)}(X) \cong \sum_{\alpha=0}^{p-2} E_\alpha K^0_{(p)}(X).$$

Proposition 1.3.10. *The separate pieces have the following properties.*

(i) $E_\alpha K^0_{(p)}$ *is a representable functor by a space* BU_α*, i.e.* $E_\alpha K^0_{(p)}(-) \cong [-, BU_\alpha]$.

(ii) For $x \in E_\beta K^0_{(p)}(X)$ *and* $y \in E_\gamma K^0_{(p)}(X)$*, then* $xy \in E_{\beta+\gamma}K^0_{(p)}(X)$.

(iii) The coefficient groups are

$$E_\alpha \tilde{K}^0_{(p)}(S^n) = \begin{cases} \mathbb{Z}_{(p)} & \textit{if } \frac{n}{2} \in \alpha \\ 0 & \textit{otherwise.} \end{cases}$$

(iv) $\phi : E_\alpha \tilde{K}^0_{(p)}(X) \to E_{\alpha+1}\tilde{K}^0_{(p)}(S^2 \wedge X)$ *(the external product with the generator in* $E_1\tilde{K}^0_{(p)}(S^2)$*) is an isomorphism.*

We can iterate ϕ, the above map, $p-1$ times to get the isomorphism

$$E_\alpha \tilde{K}^0_{(p)}(X) \cong E_{\alpha+(p-1)}\tilde{K}^0_{(p)}(S^{2(p-1)} \wedge X) = E_\alpha \tilde{K}^0_{(p)}(S^{2(p-1)} \wedge X).$$

Using this periodicity we can extend this to a graded cohomology theory E^*_α for each α as follows. We define the reduced cohomology groups of this theory by

$$\begin{aligned}
\tilde{E}^0_\alpha(X) &= E_\alpha \tilde{K}^0_{(p)}(X) \\
\tilde{E}^{-1}_\alpha(X) &= E_\alpha \tilde{K}^0_{(p)}(S^1 \wedge X) \\
&\vdots \\
\tilde{E}^{-2(p-1)}_\alpha(X) &= E_\alpha \tilde{K}^0_{(p)}(S^{2(p-1)} \wedge X) \cong E_\alpha \tilde{K}^0_{(p)}(X).
\end{aligned}$$

We can then define the unreduced version by taking the reduced version on the desired space with a disjoint base-point, i.e.

$$E^n_\alpha(X) = \tilde{E}^n_\alpha(X_+).$$

So we see that $\tilde{E}^*_\alpha$ and E^*_α are periodic with period $2(p-1)$.

Proposition 1.3.10 *(ii)* implies that E^*_0 is represented by a ring spectrum.

Definition 1.3.11. We denote the spectrum representing E^*_0 by L and call this the Adams summand, so $E^*_0(X) \cong [X, L]_*$.

Periodic p-local K-theory has coefficient groups

$$\pi_*(K_{(p)}) = \mathbb{Z}_{(p)}[u, u^{-1}],$$

where $|u| = 2$ and the Adams summand has coefficient groups

$$\pi_*(L) = \mathbb{Z}_{(p)}[u^{p-1}, u^{-(p-1)}].$$

The other pieces in the splitting are just suspensions of L;

$$\tilde{E}^0_0(X) = E_0\tilde{K}^0_{(p)}(X) \cong E_1\tilde{K}^0_{(p)}(S^2 \wedge X) = \tilde{E}^{-2}_1(X)$$

which means $\tilde{E}^0_1(X) = \Sigma^2\tilde{E}^0_0(X)$ and so more generally

$$E^*_1(X) = \Sigma^2 E^*_0(X) \cong [X, \Sigma^2 L]_*.$$

Hence the spectrum splits:

$$K_{(p)} \cong \bigvee_{i=0}^{p-2} \Sigma^{2i} L.$$

We can also think of the same operations acting on connective K-theory so a similar result happens in the connective case where we have

$$ku_{(p)} \cong \bigvee_{i=0}^{p-2} \Sigma^{2i}\ell,$$

where ℓ is the connective Adams summand.

1.4 Adams Operations

Cohomology operations can be defined for any generalised cohomology theory E by replacing the Eilenberg-MacLane spectrum HG with the desired spectrum E in Definition 1.2.1. A main example of cohomology operations are Adams operations on K-Theory. These were first introduced by Adams in [Ada62] in order to show how many linearly independent vector fields exist on the sphere S^{n-1}, and subsequently to solve the Hopf invariant 1 problem. The following information has mostly come from [Ati89], [Hat04] and [Kar78].

Firstly we need to define symmetric and exterior powers on vector bundles.

Definition 1.4.1. Let V be a vector space. The *nth symmetric power* of V is defined to be

$$S^n(V) = V^{\otimes n}/\langle v_1 \otimes \cdots \otimes v_n - v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} : v_i \in V, \sigma \in \Sigma_n\rangle.$$

Definition 1.4.2. Let V be a vector space. The *nth exterior power* of V is defined to be

$$\Lambda^n(V) = V^{\otimes n}/\langle v_1 \otimes \cdots \otimes v_n - \mathrm{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} : v_i \in V, \sigma \in \Sigma_n\rangle.$$

These constructions pass to vector bundles and isomorphism classes of vector bundles easily. In order to think about these constructions in relation to K-Theory we need to work out what an exterior power of a formal difference of vector bundles would be. For vector bundles E and F we can write this as follows

$$\Lambda^n([E] - [F]) = \sum_{i=0}^{n} (-1)^i \Lambda^{n-i}(E) S^i(F).$$

We can use this to define the following power series in $K^0(X)[\![t]\!]$

$$\Lambda_t(x) = \sum_{k=0}^{\infty} \Lambda^k(x) t^k$$

for $x \in K^0(X)$. From this we can now define the *Adams Operations* using the equation

$$\psi_t(x) = \psi^0(x) - t\frac{\mathrm{d}}{\mathrm{d}t}(\log \Lambda_{-t}(x)) \in K^0(X)[\![t]\!]$$

where $\psi^0 : K^0(X) \to K^0(X)$ takes a formal difference $[E] - [F]$ to $[\varepsilon^{\dim E}] - [\varepsilon^{\dim F}]$ and the dimension of a vector bundle is just the dimension of a fibre of that bundle, which is locally constant. The kth Adams Operation $\psi^k(x)$ is the coefficient of t^k for $k \in \mathbb{Z}$.

This definition gives Adams operations on $K^0(X)$ for X a compact Hausdorff space. It is regrettable that a good reference could not be found for a construction of the operations ψ^k in more generality. All the major references on Adams operations construct them for compact Hausdorff spaces or finite CW-complexes such as [Ati89], [Ada62], [Ada63]. In [AHS71] it does not discuss the construction of such maps. It is mentioned in [Sul74] but is approached from a different point of view. These operations can be constructed for more general spaces and we will now outline a method for doing so.

For X a compact Hausdorff space, let $\mathrm{Vect}_n(X) \subseteq \mathrm{Vect}(X)$ be the subset of isomorphism classes of vector bundles of dimension n. Then

$$\mathrm{Vect}_n(X) \cong \varinjlim_m [X, G_n(\mathbb{C}^m)] \cong [X, BU(n)].$$

So $BU(n)$ is the representing space for complex vector bundles of dimension n. We define ψ^k on complex n-dimensional vector bundles as above in terms of exterior powers, this then has the property that if x is a line bundle over X then $\psi^k(x) = x^k$. By the Yoneda Lemma this gives us a map which we will denote $\psi^k_n : BU(n) \to BU(n)$. We have the inclusion $BU(n) \hookrightarrow BU(n+1)$ given by the addition of the trivial line bundle which is compatible with the maps ψ^k_n. This then gives us a compatible sequence of maps which gives us a map $\psi^k_{BU} : \cup_n BU(n) = BU \to BU$. We take $\psi^k : \mathbb{Z} \times BU \to \mathbb{Z} \times BU$ to be $\mathrm{id}_{\mathbb{Z}} \times \psi^k_{BU}$, since $\mathbb{Z} \times BU$ is the representing space for the functor $\tilde{K}^0$ on based spaces, see Theorem 1.3.5, this gives us the definition of ψ^k here. In a similar way to Definition 1.3.7 let

$$\tilde{A} = \{\text{natural additive operations on } \tilde{K}^0(-) \text{ for based spaces}\}$$

and let

$$A^+ = \{\text{natural additive operations on } K^0(-) \text{ for unbased spaces}\}.$$

Let X be an unbased space and let X_+ be the union of X with a disjoint basepoint. There is an isomorphism $\tilde{A} \to A^+$ as follows. Given an operation $\alpha \in \tilde{A}$, we have $\alpha_{X_+} : \tilde{K}(X_+) \to \tilde{K}(X_+)$. We have a natural identification $\tilde{K}^0(X_+) \cong K^0(X)$ which gives us an operation on $K^0(X)$. This then defines us ψ^k on the K-theory of unbased spaces.

These operations have the following properties

Proposition 1.4.3. *(i)* $\psi^k(x+y) = \psi^k(x) + \psi^k(y)$ *for all* k.

(ii) $\psi^k(xy) = \psi^k(x)\psi^k(y)$ *for all* k.

(iii) $\psi^k(\psi^l(x)) = \psi^{kl}(x)$ *for all* k, l.

(iv) If x *is a line bundle then* $\psi^k(x) = x^k$.

(v) For p *a prime,* $\psi^p(x) \equiv x^p \mod p$.

(vi) For any map $f : X \to Y$*, the operations are natural i.e.* $\psi^k(f^*(x)) = f^*(\psi^k(x))$.

(vii) For $u \in \tilde{K}(S^{2n})$*, we have* $\psi^k(u) = k^n u$ *for all* k.

These operations as defined above are *unstable* for nearly all $k \in \mathbb{Z}$, that is they are just natural transformations $\psi^k : K^0(X) \to K^0(X)$. A *stable* operation is a family of cohomology operations which commute with the suspension isomorphisms. The only Adams operations for integral K-Theory which are stable are $\psi^1 = \mathrm{id}$ and ψ^{-1} which is complex conjugation. If we introduce coefficients into K-Theory then this can allow other operations to become stable. In general for an operation ψ^k to be stable in $K^*(-; R)$, we need k to be a unit in R. If k is a unit in R, i.e. $\mathbb{Z}[\frac{1}{k}] \subseteq R \subseteq \mathbb{Q}$ then there is a unique ring spectrum map $\psi^k : KR \to KR$ such that the following diagram commutes for all spaces X.

$$\begin{tikzcd}
K^0(X) \otimes R \arrow[r, "\psi^k \otimes 1"] \arrow[d, "\mu"'] & K^0(X) \otimes R \arrow[d, "\mu"] \\
KR^0(X) \arrow[r, "(\psi^k)_*"'] & KR^0(X)
\end{tikzcd}$$

Here $KR^0(X) = [\Sigma^\infty X_+, KR]$ are based maps from the suspension spectrum of X to the spectrum KR. This process of making an operation ψ^k stable is discussed in more detail in [AHS71, Chapter 4].

1.5 The Adams Spectral Sequence

The Adams spectral sequence is a very useful gadget in calculating stable homotopy groups of spheres, and more generally stable homotopy classes of maps between spaces. It has many levels of complexity and much has been written about it over the years. I am not going to explain how it is constructed in detail as there are quite a few excellent accounts of this in existing literature, for example [Ada95, Part III, Chapter 15], [McC01], [Hat03] and [Koc96]. I will however give an idea of the construction, define it and outline how to use it, specifically in the way I will use it in later chapters. The material for this section has come from the above references.

The construction of the Adams spectral sequence involves producing a free $\mathcal{A}_p$-module resolution of $\tilde{H}^*(X;\mathbb{Z}/p)$, for a nice spectrum X. This is done by using wedges of Eilenberg-MacLane spectra as these are the nearest thing to having free $\mathcal{A}_p$-module cohomology. This data can then be used to construct an Adams resolution which is a geometric realisation of the free $\mathcal{A}_p$-module resolution of $\tilde{H}^*(X;\mathbb{Z}/p)$. Applying the functor $[Y,-]_t$ to the Adams resolution gives a staircase diagram which allows for the construction of a spectral sequence.

Theorem 1.5.1. *For X a spectrum of finite type and Y a finite spectrum there exists a spectral sequence of the form*

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}_p}^{s,t}(\tilde{H}^*(X;\mathbb{Z}/p), \tilde{H}^*(Y;\mathbb{Z}/p))$$

converging to the p-completion of

$$[Y,X]_{t-s}.$$

The differentials are of the form

$$d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}.$$

So if we let $Y = S$, the sphere spectrum, this specialises to

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}_p}^{s,t}(\tilde{H}^*(X;\mathbb{Z}/p), \mathbb{Z}/p) \Longrightarrow \pi_{t-s}(X) \otimes \mathbb{Z}_p.$$

To construct the doubly-graded Ext group first take a left $\mathcal{A}_p$-module M and form a projective resolution of M, i.e. a long exact sequence of the form

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

where each P_i is a projective $\mathcal{A}_p$-module for $i > 0$. Then delete M and apply the functor $\mathrm{Hom}_{\mathcal{A}_p}^*(-,N)$, where N is also a left $\mathcal{A}_p$-module and * denotes the degree of the homomorphism, to get the chain complex

$$\cdots \leftarrow \mathrm{Hom}_{\mathcal{A}_p}^*(P_2,N) \leftarrow \mathrm{Hom}_{\mathcal{A}_p}^*(P_1,N) \leftarrow \mathrm{Hom}_{\mathcal{A}_p}^*(P_0,N) \leftarrow 0.$$

Taking homology of this chain complex at stage s gives us $\mathrm{Ext}_{\mathcal{A}_p}^{s,*}(M,N)$.

An important property of these Ext groups is that they don't depend on the resolution taken.

To get from one page of a spectral sequence to another you take homology at each point with respect to the differential passing through that point. So each page and its differential determines the next page but not the next differential, some other information is usually needed to obtain this.

To extract information from a spectral sequence we usually want to know what the E_∞ page looks like. This can be done easily if the spectral sequence *collapses*, this means there exists a natural number R such that once the spectral sequence gets to the Rth page all further differentials are then zero for $r > R$. From there on taking homology has no effect on the terms of the spectral sequence so $E_R = E_{R+1} = \cdots = E_\infty$.

The information you can extract from the E_∞ page of this spectral sequence can be used to calculate the p-completion of $\pi_*(X)$ (in the simpler case). The terms on the E_∞ page are the quotient groups associated to a descending filtration of $\pi_*(X) \otimes \mathbb{Z}_p$, i.e.

$$E_\infty^{s,t} \cong \frac{F^s(\pi_{t-s}(X) \otimes \mathbb{Z}_p)}{F^{s+1}(\pi_{t-s}(X) \otimes \mathbb{Z}_p)}.$$

This information hopefully allows you to reconstruct the groups $\pi_{t-s}(X) \otimes \mathbb{Z}_p$ however there can be extension problems which might need knowledge about further structure in order to be solved.

1.6 Spanier-Whitehead Duality

The material in this section is derived and explained in more detail in [Ada95, Part III, Chapter 5], another good account can be found in [Rav92].

Definition 1.6.1. For a finite spectrum X, there exists a unique finite spectrum $D(X)$, called the *Spanier Whitehead dual* of X, such that

$$[X, Y]_* \cong [S, D(X) \wedge Y]_*$$

where S is the sphere spectrum and Y is any other spectrum. This is natural in both X and Y.

The concept of the Spanier-Whitehead dual is analogous of the concept of the linear dual of a vector space. If V is a vector space over a field K, then its linear dual is $V^* = \mathrm{Hom}_K(V, K)$. The defining property above corresponds to the property of vector spaces that for any other K-vector space W, $\mathrm{Hom}_K(V, W) \cong \mathrm{Hom}_K(K, V^* \otimes W) \cong V^* \otimes W$.

Example 1.6.2. The Spanier-Whitehead dual of the sphere spectrum is itself, i.e. $D(S) = S$.

There are other corresponding properties between Spanier-Whitehead duals of spectra and linear duals of vector spaces.

Lemma 1.6.3. *The dual of a dual is isomorphic to the spectrum itself*

$$D(D(X)) \cong X.$$

Lemma 1.6.4. *Taking duals commutes with the smash product*

$$D(X \wedge Y) \cong D(X) \wedge D(Y).$$

These two properties can be looked at as the analogues of the following two properties of finite dimensional vector space duals, $V^{**} \cong V$ and $(V \otimes W)^* \cong V^* \otimes W^*$.

There is another property of Spanier-Whitehead duality which generalises the concept of Alexander duality. For a finite CW-complex X, we can embed X into S^n for some $n \in \mathbb{N}$. Alexander duality then states that

$$H_k(X) \cong H^{n-1-k}(S^n \setminus X)$$

where $S^n \setminus X$ is the complement of X in S^n.

Because we are dealing with finite spectra we can view the spectrum X as the (de)suspension spectrum of a finite CW-complex X'. Following the method above we can embed X' in S^n and then $D(X)$ is just a suitably shifted version of the suspension spectrum of $S^n \setminus X'$. We get an analogous property to Alexander duality; for any generalised cohomology theory E,

$$E_k(X) \cong E^{-k}(D(X)).$$

We now look at how the action of the Steenrod algebra behaves with respect to Spanier-Whitehead duality.

Proposition 1.6.5. *For a finite spectrum X, an element $\alpha \in \mathcal{A}_p$ acts on $H^{-*}(D(X);\mathbb{Z}/p)$ as the dual of $\chi(\alpha)$ in $\mathcal{A}_p^*$ would act on $H_*(X;\mathbb{Z}/p)$, i.e. the following diagram commutes where $|\alpha| = a$.*

$$\begin{tikzcd}
H^{-n}(D(X);\mathbb{Z}/p) \arrow[r, "\alpha"] \arrow[d, "\cong"'] & H^{-n+a}(D(X);\mathbb{Z}/p) \arrow[d, "\cong"'] \\
H_n(X;\mathbb{Z}/p) \arrow[r, "\chi(\alpha)^*"'] & H_{n-a}(X;\mathbb{Z}/p)
\end{tikzcd}$$

We will only ever use this when looking at the action of the subalgebra B, see Definition 1.2.14. Since $\chi(Q_0) = -Q_0$ and $\chi(Q_1) = -Q_1$, we have that $H^{-*}(DX;\mathbb{Z}/p)$ is isomorphic as a left B-module to $H_*(X;\mathbb{Z}/p)$ where Q_0 and Q_1 act (up to sign) via their duals.

As stated in Definition 1.6.1 in order to have the Spanier-Whitehead dual of a spectrum X you need X to be finite. We will now look at a particular method of showing a spectrum is finite which we will use later on, this material comes from [BM04, Section 3].

Definition 1.6.6. A spectrum is *bounded below* if there exists $n \in \mathbb{Z}$ such that $\pi_i(X) = 0$ for $i < n$ (X is $(n+1)$-connected for some $n \in \mathbb{Z}$).

The main result we will need later is the following.

Proposition 1.6.7. *A bounded below p-complete spectrum with finitely generated mod p homology is the p-completion of a finite spectrum.*

It is stated in [BM04, Remarks 1.2(v)] that a finite type p-complete spectrum, i.e. one with mod p homology finitely generated in each degree, is the p-completion of a finite type p-local spectrum, i.e. one with p-local homology finitely generated in each degree. So if X is a finite type p-complete spectrum then $X = \mathcal{X}_p$ for a finite type p-local spectrum $\mathcal{X}$, where $\mathcal{X}_p$ denotes the p-completion of the spectrum $\mathcal{X}$. If the original spectrum X is bounded below then we can take $\mathcal{X}$ to be bounded below also.

Using the proof of [BM04, Theorem 3.3] it is possible to construct something known as a minimal spectrum Y and a homotopy equivalence $Y \to \mathcal{X}$ such that if $\mathcal{X}$ has finitely generated p-local homology, then Y will have finitely many p-local stable cells corresponding to the generators and relations in $H_*(\mathcal{X}; \mathbb{Z}_{(p)})$. Hence we have a model for $\mathcal{X}$ as a p-local finite spectrum.

We know from Proposition 1.1.17 that p-localisation is an example of a smashing localisation, so we can deduce that a spectrum built from finitely many p-local cells is the p-localisation of a finite spectrum. This means our finite p-local spectrum Y is the p-localisation of a finite spectrum Z and so we now have that X is equivalent to the p-completion of the finite spectrum Z.

1.7 Thom Spectra

The material in this section is mostly from [Rud98].

Notation 1.7.1. We will denote a homotopy from a space X to a space B as $g_t : X \to B$, which is a family of maps for $t \in I$ such that $g_t(x) = g(x,t)$ where $g : X \times I \to B$ is continuous.

Definition 1.7.2. A map $p : E \to B$ has the *homotopy lifting property* with respect to a space X if given a homotopy $g_t : X \to B$ and a map $\tilde{g}_0 : X \to E$ which lifts g_0, i.e. $p\tilde{g}_0 = g_0$, then there exists a homotopy $\tilde{g}_t : X \to E$ which lifts g_t.

Definition 1.7.3. A *fibration* is a map $p : E \to B$ which has the homotopy lifting property with respect to all spaces X. The spaces $p^{-1}(b) \subset E$ are called the fibres.

Example 1.7.4. The simplest example of a fibration is θ^n, the trivial fibration of rank n. This is given by the projection map onto the first factor $X \times \mathbb{C}^n \to X$. Here all the fibres are a copy of $\mathbb{C}^n$.

Definition 1.7.5. (i) An *F-fibration* for a topological space F is a fibration ξ such that all the fibres are homotopy equivalent to F.

(ii) An *$(F,*)$-fibration* is an F-fibration ξ with a section s_ξ such that the fibres $(F_x, s_\xi(x))$ are pointed homotopy equivalent to $(F,*)$ for all x in the base space of ξ.

Definition 1.7.6. A *universal F-fibration* is an F-fibration $\gamma_F = \{p_F : E_F \to B_F\}$ such that every F-fibration over a CW-complex X is equivalent to the pullback $f^*\gamma_F$ for some map $f : X \to B_F$, and two such maps $f, g : X \to B_F$ are homotopy equivalent if and only if $f^*\gamma_F$ and $g^*\gamma_F$ are equivalent.

The space B_F is the *classifying space for F-fibrations.*

It can be shown that for all spaces F, γ_F exists. Furthermore B_F can be chosen to be a CW-complex which is unique up to homotopy equivalence.

Definition 1.7.7. Let *$\mathcal{F}_n$-objects* be $(S^n, *)$-fibrations. These are classified by a space $B_{(S^n,*)}$ denoted $B\mathcal{F}_n$ and the universal $\mathcal{F}_n$-object is denoted $\gamma^n_{\mathcal{F}}$.

Let $\overline{B\mathcal{F}_n}$ be the telescope (the homotopy direct limit) of the finite sequence $\{B\mathcal{F}_1 \to \ldots \to B\mathcal{F}_n\}$, this is a subcomplex of $B\mathcal{F}$, the telescope of the infinite sequence $\{B\mathcal{F}_1 \to B\mathcal{F}_2 \to \ldots\}$. The set $\{\overline{B\mathcal{F}_n}\}$ gives an increasing filtration of $B\mathcal{F}$. Because $\overline{B\mathcal{F}_n}$ is homotopy equivalent to $B\mathcal{F}_n$, the universal $\mathcal{F}_n$-object $\gamma^n_{\mathcal{F}}$ is also an $\mathcal{F}_n$-object over $\overline{B\mathcal{F}_n}$.

Definition 1.7.8. A *stable $\mathcal{F}$-object* α over X is a map $f : X \to B\mathcal{F}$. The *stabilisation* of an $\mathcal{F}_n$-object $\alpha = \{f : X \to B\mathcal{F}_n\}$ is the map $X \xrightarrow{f} B\mathcal{F}_n \hookrightarrow B\mathcal{F}$.

Definition 1.7.9. Let $\alpha = \{p : Y \to X\}$ be an $\mathcal{F}_n$-object with section s. The *Thom space* of α is defined to be $T(\alpha) = Y/s(X)$.

Every morphism $\varphi : \alpha \to \beta$ of $\mathcal{F}_n$-objects induces a map $T(\varphi) : T(\alpha) \to T(\beta)$ so T is a functor from $\mathcal{F}_n$-objects to spaces.

Theorem 1.7.10 (Thom Isomorphism Theorem)**.** *In the case of spherical fibrations, for every abelian group G and every i there are isomorphisms*

$$H_i(X; G) \cong \tilde{H}_{i+n}(T(\alpha); G)$$

and

$$H^i(X; G) \cong \tilde{H}^{i+n}(T(\alpha); G)$$

where $\alpha = \{p : Y \to X\}$ is an $\mathcal{F}_n$-object.

For a CW-complex X, let $\alpha = \{f : X \to B\mathcal{F}\}$ be a stable $\mathcal{F}$-object. Let $\mathfrak{X} = \{X_0 \subset X_1 \subset \cdots \subset X_n \subset \cdots\}$ be a CW-filtration of the space X such that $\cup_n X_n = X$ and $f(X_n) \subset \overline{B\mathcal{F}_n}$. Let $f_n : X_n \to \overline{B\mathcal{F}_n}$ be the restriction of the map f, i.e. $f_n(x) = f(x)$ and let $\zeta^n = f_n^*(\gamma^n_{\mathcal{F}})$.

Proposition 1.7.11. *If $i_n : X_n \to X_{n+1}$ is the inclusion then $i_n^*(\zeta^{n+1}) = \zeta^n \oplus \theta^1$.*

Definition 1.7.12. We define the *Thom spectrum*

$$T(\mathfrak{X}, \alpha) = \{T(\zeta^n), s_n\},$$

the maps s_n are as follows

$$s_n = TI_{i_n, \zeta^{n+1}} : ST(\zeta^n) = T(\zeta^n \oplus \theta^1) \to T(\zeta^{n+1})$$

where $I_{i_n, \zeta^{n+1}} : i_n^*(\zeta^{n+1}) \to \zeta^{n+1}$ is the canonical map associated to the pullback.

The homotopy type of a Thom spectrum does not depend on the filtration used so this can be left out of the notation and we can just write $T(\mathfrak{X}, \alpha) \simeq T(\alpha)$.

If $\alpha = \{f : X \to B\mathcal{F}_k\}$ is an $\mathcal{F}_k$-object and α_{st} is its stabilisation then the nth term in $T(\alpha_{\mathrm{st}})$ is $T(\alpha \oplus \theta^{n-k})$ for $n \geqslant k$. This means that the nth term in the Thom spectrum $T_n(\alpha_{\mathrm{st}}) = \Sigma^{n-k}T(\alpha)$ and so there is an isomorphism

$$T(\alpha_{\mathrm{st}}) \cong \Sigma^{-k}\Sigma^{\infty}T(\alpha)$$

and we have that the Thom spectrum is equivalent to the (de)suspension of the suspension spectrum of a CW-complex.

1.8 Künneth Formulas

This is a general term for a formula which links the (co)homology of a product space to the (co)homology of the two factors.

In the case of ordinary singular homology we have the following result (see [Hat02, Theorem 3B.6]).

Theorem 1.8.1. *If X and Y are CW-complexes and R a principal ideal domain then there are natural short exact sequences*

$$0 \to H_*(X; R) \otimes H_*(Y; R) \to H_*(X \times Y; R) \to \mathrm{Tor}_1(H_*(X; R), H_*(Y; R)) \to 0$$

which split.

Corollary 1.8.2. *When the coefficients are taken to be a field F the* Tor *term is zero and so*

$$H_*(X; F) \otimes H_*(Y; F) \cong H_*(X \times Y; F).$$

We have a corresponding result in cohomology giving us that for F a field

$$H^*(X; F) \otimes H^*(Y; F) \cong H^*(X \times Y; F).$$

This style of formula can be generalised to other (co)homology theories, for example in [Ati89, Corollary 2.7.15], Atiyah gives a Künneth formula for K-Theory. The case for a general ring spectrum E can be found in [Ada95] and [Swi02, Theorem 13.75] and is as follows.

Theorem 1.8.3. *Let E be a ring spectrum and X and Y spectra.*

(i) If $E_*(X)$ is a flat right $\tilde{E}_*(S)$-module or $E_*(Y)$ a flat left $\tilde{E}_*(S)$-module then

$$E_*(X) \otimes_{\tilde{E}_*(S)} E_*(Y) \cong E_*(X \wedge Y).$$

(ii) If $E^*(X)$ is a finitely generated free right $\tilde{E}^*(S)$-module or $E^*(Y)$ a finitely generated free left $\tilde{E}^*(S)$-module then

$$E^*(X) \otimes_{\tilde{E}^*(S)} E^*(Y) \cong E^*(X \wedge Y).$$

As explained in [Ada95], the flatness condition is satisfied in the cases we will need it for e.g. $H\mathbb{Z}/p$, K, S.

Chapter 2

Kane's Splitting

2.1 Introduction

Here I will give an exposition of [Kan81] in order to provide the background material necessary for the following sections.

The main result of [Kan81] is the following splitting of $\ell \wedge \ell$ involving the finite spectra $K(n)$, where we use the notation ℓ to mean the p-complete Adams summand.

Theorem 2.1.1. *There is a homotopy equivalence of the form*

$$\ell \wedge \ell \simeq \ell \wedge \bigvee_{n \geqslant 0} \Sigma^{2n(p-1)} K(n).$$

The splitting is constructed by Kane in a p-local setting i.e. using Bousfield localisation with respect to the homology theory $H\mathbb{Z}_{(p)}$. During the construction of the spectra $K(n)$, Kane asserts that there is a filtration of the space $\Omega^2 S^3\langle 3\rangle$ which induces a certain filtration on its homology. It has been pointed out in [CDGM88] that they do not have a proof of this unless the space is localised with respect to mod p homology. We will proceed by localising spaces with respect to mod p homology where stated and interpret the splitting as a statement about p-complete spectra rather than p-local spectra. Where results from [Kan81] are quoted, we will actually be quoting the p-complete versions of these results. A different interpretation of the p-local splitting of $\ell \wedge \ell$ in terms of minimal Adams resolutions is given in [Lel84] however this relies on the result of Kane's splitting rather than being an alternative splitting proved from scratch.

2.2 Construction of the Spectra $K(n)$

The spectra $\{K(n)\}_{n \geqslant 0}$ are certain Thom spectra known as Integral Brown-Gitler spectra. The 2-primary versions were first introduced by Mahowald in [Mah81] for the 2-local splitting of $ko \wedge ko$. The odd primary versions of them are given in [Kan81] however a more explicit and detailed account of the construction of these spectra is given in [CDGM88] and [Kna97, Chapter 3].

We have an algebra isomorphism

$$H_*(\Omega^2 S^3; \mathbb{Z}/p) \cong \Lambda(a_0, a_1, \ldots) \otimes \mathbb{Z}/p[b_1, b_2, \ldots]$$

where $|a_i| = 2p^i - 1$, $|b_i| = 2p^i - 2$ and Λ denotes an exterior algebra. There is a weight function on monomials in $H_*(\Omega^2 S^3; \mathbb{Z}/p)$ given by

$$\mathrm{wt}(a_i) = \mathrm{wt}(b_i) = p^i \quad \text{and} \quad \mathrm{wt}(ab) = \mathrm{wt}(a) + \mathrm{wt}(b).$$

This filtration can be realised at space level giving an increasing filtration $F_n(\Omega^2 S^3)$ such that $H_*(F_n(\Omega^2 S^3); \mathbb{Z}/p) \subset H_*(\Omega^2 S^3; \mathbb{Z}/p)$ is the span of monomials of weight less than or equal to n.

Let $S^3\langle 3\rangle$ be the 3-connective cover of S^3 so that $\pi_i(S^3\langle 3\rangle) = 0$ for $i \leqslant 3$ and $\pi_i(S^3\langle 3\rangle) \cong \pi_i(S^3)$ for $i > 3$. There is a homotopy fibration

$$\Omega^2 S^3\langle 3\rangle \to \Omega^2 S^3 \to S^1$$

which splits as a product giving $\Omega^2 S^3 \simeq \Omega^2 S^3\langle 3\rangle \times S^1$. Since we have $H_*(\Omega^2 S^3; \mathbb{Z}/p) \cong H_*(\Omega^2 S^3\langle 3\rangle; \mathbb{Z}/p) \otimes H_*(S^1; \mathbb{Z}/p)$ and $\tilde{H}_*(S^1; \mathbb{Z}/p) \cong \mathbb{Z}/p$ in degree 1, we know that

$$H_*(\Omega^2 S^3\langle 3\rangle; \mathbb{Z}/p) \cong \Lambda(a_1, a_2, \ldots) \otimes \mathbb{Z}/p[b_1, b_2, \ldots]$$

which is the span of monomials of weight divisible by p. The filtration of $\Omega^2 S^3$ induces a filtration of $H_*(\Omega^2 S^3\langle 3\rangle; \mathbb{Z}/p)$ given by

$$F_n(H_*(\Omega^2 S^3\langle 3\rangle; \mathbb{Z}/p)) = H_*(F_n(\Omega^2 S^3); \mathbb{Z}/p) \cap H_*(\Omega^2 S^3\langle 3\rangle; \mathbb{Z}/p),$$

the span of monomials of weight divisible by p and less than or equal to n. In [CDGM88] it is stated that they do not know of an argument for this filtration being induced by an actual filtration of the space $\Omega^2 S^3\langle 3\rangle$, as Kane states this but never proves it. An argument is given in [CDGM88] for this happening when the spaces are localised with respect to mod p homology.

Recall that X_p denotes Bousfield localisation of a space X with respect to the homology theory $H\mathbb{Z}/p$ [Bou75].

Definition 2.2.1. Let the space A_n be defined by the homotopy fibration

$$A_n \to (F_{pn+1}(\Omega^2 S^3))_p \to S^1_p,$$

where the second map is the $H\mathbb{Z}/p$-localisation of the composite of the maps

$$F_{pn+1}(\Omega^2 S^3) \to \Omega^2 S^3 \to S^1.$$

By [Bou75, Proposition 12.7] this means that A_n is $H\mathbb{Z}/p$-local. The following theorem is proved in [CDGM88, Theorem 1.3].

Theorem 2.2.2. *The homotopy fibration of Definition 2.2.1 is equivalent to a product fibration and $H_*(A_n;\mathbb{Z}/p) \cong F_{pn}(H_*(\Omega^2 S^3\langle 3\rangle;\mathbb{Z}/p))$ hence $H_*(A_n;\mathbb{Z}/p)$ is the span of monomials of weight divisible by p and less than or equal to pn.*

In [Mah79, 2.6, 2.7] Mahowald constructs a spherical fibration ξ over $\Omega^2 S^3_p$ which, when the Thom space of ξ is pulled back to $\Omega^2 S^3\langle 3\rangle_p$ to form $T(\xi|\Omega^2 S^3\langle 3\rangle_p)$ is the Eilenberg-MacLane spectrum $H\mathbb{Z}_p$.

Definition 2.2.3. When the Thom space of ξ is further pulled back via the commutative diagram

$$\begin{tikzcd} A_n \arrow[r] \arrow[d, "i_n"'] & (F_{pn+1}(\Omega^2 S^3))_p \arrow[d] \\ \Omega^2 S^3\langle 3\rangle_p \arrow[r] & \Omega^2 S^3_p \end{tikzcd}$$

to the Thom spectrum $T(\xi|A_n)$, this is the nth integral Brown-Gitler spectrum $B_1(n)$.

The map $i_n : A_n \to \Omega^2 S^3\langle 3\rangle_p$ when Thomified gives a monomorphism in homology
$$T(i_n) : B_1(n) \to H\mathbb{Z}_p$$
sending a_i to $\chi(\tau_i)$ and b_i to $\chi(\xi_i)$ where

$$H_*(H\mathbb{Z}_p;\mathbb{Z}/p) \cong \Lambda(\chi(\tau_1), \chi(\tau_2), \ldots) \otimes \mathbb{Z}/p[\chi(\xi_1), \chi(\xi_2), \ldots].$$

Here ξ_i and τ_i are the elements of the dual Steenrod Algebra $\mathcal{A}_p^*$ defined in section 1.2 with $|\xi_i| = 2p^i - 2$ and $|\tau_i| = 2p^i - 1$ and χ is the canonical anti-automorphism. This monomorphism sends $H_*(B_1(n);\mathbb{Z}/p)$ into the span of monomials of weight less than or equal to pn in $H_*(H\mathbb{Z}_p;\mathbb{Z}/p)$ where

$$\mathrm{wt}(\chi(\tau_i)) = \mathrm{wt}(\chi(\xi_i)) = p^i \quad \text{and} \quad \mathrm{wt}(ab) = \mathrm{wt}(a) + \mathrm{wt}(b).$$

Proposition 2.2.4. *Each $B_1(n)$ is the p-completion of a finite spectrum.*

Proof. It is stated in [Rud98, Chapter IV, Theorem 5.23(i)] that for any stable $\mathcal{F}$-object α over a CW-complex X, we have $\pi_i(T\alpha) = 0$ for $i < 0$. $B_1(n)$ is a Thom spectrum produced in this way over the CW-complex A_n hence $B_1(n)$ is bounded below. It can be seen above that the mod p homology of $B_1(n)$ is finitely generated and so we can use Proposition 1.6.7 and the discussion following to show that $B_1(n)$ is the p-completion of a finite spectrum. □

There exist pairings of the spectra $B_1(n)$ of the form

$$B_1(n) \wedge B_1(m) \to B_1(n+m)$$

whose mod p homology homomorphism is compatible with the multiplication in $H_*(H\mathbb{Z}_p; \mathbb{Z}/p)$.

Definition 2.2.5. Let $K(pn) := B_1(n)$ and $K(pn+i) := K(pn)$ for $1 \leqslant i \leqslant p-1$.

The pairings on the spectra $B_1(n)$ give pairings on the spectra $K(n)$. For example, in the case where the p-adic expansions of $m = \sum_i \alpha_i p^i$ and $n = \sum_i \beta_i p^i$ satisfy $\alpha_i + \beta_i < p$ for all i, the spectra $\{K(n)\}_{n \geqslant 0}$ have multiplication maps between themselves

$$\mu_{m,n} : K(m) \wedge K(n) \to K(m+n)$$

such that the induced map in cohomology $\mu^*_{m,n} : H^*(K(m+n); \mathbb{Z}/p) \to H^*(K(m); \mathbb{Z}/p) \otimes H^*(K(n); \mathbb{Z}/p)$ is injective, see [Lel84, 1.2(iii)].

2.3 Properties of the Spectra $\mathcal{K}(n)$, $\mathcal{K}$ and ℓ

In this section we will look at the homology and cohomology of the spectra $K(n)$ as modules over a subalgebra of the Steenrod Algebra. We will then go on to define the spectra $\mathcal{K}(n)$ and hence $\mathcal{K}$ and give its homology as well as the homology of ℓ. This will set us up for the final section where we will construct the splitting of $\ell \wedge \ell$.

Consider $H^*(K(n); \mathbb{Z}/p)$ as a module over the Steenrod Algebra $\mathcal{A}_p$. Consider the left ideal of $\mathcal{A}_p$

$$\mathcal{I}(n) = \mathcal{A}_p \beta + \sum_{i > \frac{n}{p}} \mathcal{A}_p \chi(\mathcal{P}^i).$$

Proposition 2.3.1. *$H^*(K(n); \mathbb{Z}/p) \cong \mathcal{A}_p/\mathcal{I}(n)$ as an $\mathcal{A}_p$-module.*

Recall the Milnor elements of $\mathcal{A}_p$

$$Q_0 = \beta$$
$$Q_1 = \mathcal{P}^1\beta - \beta\mathcal{P}^1$$

where $|Q_0| = 1$, $|Q_1| = 2p-1$. Here $Q_0^2 = Q_1^2 = 0$ and $Q_0Q_1 = -Q_1Q_0$ so we can also look at $H^*(K(n); \mathbb{Z}/p)$ as a graded module over the exterior algebra $B = \Lambda(Q_0, Q_1)$.

Definition 2.3.2. Let L be the B-module with generators g_i, $i \in \mathbb{Z}$, where $|g_i| = 2i(p-1)$ and relations $Q_1(g_i) = Q_0(g_{i+1})$.

We can picture L in the following way where a g_i denotes a copy of $\mathbb{Z}/p$ generated by g_i and $Q_1 g_i$ also denotes a copy of $\mathbb{Z}/p$ with generator $Q_1 g_i$. An arrow denotes a non-trivial action of either Q_0 or Q_1 as indicated.

$$\begin{tikzcd}
& Q_1g_{-2} & Q_1g_{-1} & Q_1g_0 & Q_1g_1 & Q_1g_2 \\
{} \arrow[ur, dashed] & g_{-1} \arrow[u, "Q_0"] \arrow[ur, "Q_1"'] & g_0 \arrow[u, "Q_0"] \arrow[ur, "Q_1"'] & g_1 \arrow[u, "Q_0"] \arrow[ur, "Q_1"'] & g_2 \arrow[u, "Q_0"] \arrow[ur, "Q_1"'] & {} \arrow[u, dashed]
\end{tikzcd}$$

Definition 2.3.3. For $t \geqslant 0$, let $L(t)$ be the B-module quotient of L by the submodule generated by $\{g_i | i < 0 \text{ or } i > t\}$.

We can picture $L(t)$ as follows.

$$\begin{tikzcd}
& Q_1g_0 & Q_1g_1 & {} & & Q_1g_{t-1} \\
g_0 \arrow[ur, "Q_1"'] & g_1 \arrow[u, "Q_0"] \arrow[ur, "Q_1"'] & g_2 \arrow[u, "Q_0"] \arrow[ur, dashed] & \cdots & {} \arrow[ur, dashed] & g_t \arrow[u, "Q_0"]
\end{tikzcd}$$

Hence each $L(t)$ is finite dimensional over $\mathbb{Z}/p$.

Proposition 2.3.4.

$$H^*(K(n); \mathbb{Z}/p) \cong L(\nu) \oplus F$$

as B-modules where F is a free B-module of finite rank and $\nu = \nu_p(n!)$ where ν_p is the p-adic valuation function.

Definition 2.3.5. Let N be the $\mathbb{Z}/p$-linear dual of L. So N is the B-module with generators h_i, for $i \in \mathbb{Z}$, where $|h_i| = 2i(p-1)+1$ and $Q_0(h_i) = Q_1(h_{i+1})$.

Pictorially N looks like this:

$$\begin{tikzcd}
Q_1h_{-1} & Q_1h_0 & Q_1h_1 & Q_1h_2 & Q_1h_3 & {} \\
{} \arrow[u, dashed] & h_{-1} \arrow[u, "Q_0"'] \arrow[ul, "Q_1"'] & h_0 \arrow[u, "Q_0"'] \arrow[ul, "Q_1"'] & h_1 \arrow[u, "Q_0"'] \arrow[ul, "Q_1"'] & h_2 \arrow[u, "Q_0"'] \arrow[ul, "Q_1"'] & {} \arrow[ul, dashed]
\end{tikzcd}$$

Definition 2.3.6. Trivially let $N(0) = L(0) = \mathbb{Z}/p\{g_0\}$. For $t > 0$, we let $N(t)$ be the B-submodule of N generated by $\{h_i | 1 \leqslant i \leqslant t\}$.

So $N(t)$ is as follows.

$$\begin{tikzcd}
Q_1h_1 & Q_1h_2 & Q_1h_3 & {} & Q_0h_t \\
& h_1 \arrow[ul, "Q_1"'] \arrow[u, "Q_0"'] & h_2 \arrow[ul, "Q_1"'] \arrow[u, "Q_0"'] & {} \arrow[ul, dashed] \cdots & h_t \arrow[u, "Q_0"'] \arrow[ul, dashed]
\end{tikzcd}$$

Kane proves Proposition 2.3.4 so we will prove the dual statement here.

Proposition 2.3.7. *$H_*(K(n);\mathbb{Z}/p) \cong N(\nu) \oplus F'$ as B-modules where F' is a free B-module of finite rank.*

Proof. The Universal Coefficient Theorem tells us that $H^*(K(n);\mathbb{Z}/p) \cong \mathrm{Hom}_{\mathbb{Z}/p}(H_*(K(n);\mathbb{Z}/p),\mathbb{Z}/p)$ and we know that $H^*(K(n);\mathbb{Z}/p) \cong L(\nu) \oplus F$. We can show that the dual of $L(\nu)$ is $N(\nu)$. We can see that

$$L(t) = \mathbb{Z}/p\{g_0, g_1, \cdots, g_t, Q_1 g_0, Q_1 g_1, \cdots, Q_1 g_{t-1}\}$$

and

$$N(t) = \mathbb{Z}/p\{h_1, \cdots, h_t, Q_1 h_1, \cdots, Q_1 h_t, Q_0 h_t\}.$$

Both have dimension $2t+1$ over $\mathbb{Z}/p$. $L(t)$ is a left B-module, so naturally $\mathrm{Hom}_{\mathbb{Z}/p}(L(t),\mathbb{Z}/p)$ is a right B-module via the action $(f.b)(-) = f(b.-)$ for $b \in B$ and $f \in \mathrm{Hom}_{\mathbb{Z}/p}(L(t),\mathbb{Z}/p)$. We know that as a $\mathbb{Z}/p$-vector space, the dual of $L(t)$ is as follows

$$(L(t))^* = \mathbb{Z}/p\{g_0^*, g_1^*, \cdots, g_n^*, (Q_1 g_0)^*, (Q_1 g_1)^*, \cdots, (Q_1 g_{n-1})^*\},$$

where $*$ denotes the dual basis. So to define an isomorphism from $(L(t))^*$ to $N(t)$ we send

$$g_i^* \mapsto \begin{cases} Q_1 h_1 & \text{if } i = 0 \\ Q_1 h_{i+1} = Q_0 h_i & \text{if } 1 \leqslant i \leqslant t-1 \\ Q_0 h_t & \text{if } i = t \end{cases}$$

and

$$(Q_1 g_i)^* = (Q_0 g_{i+1})^* \mapsto h_{i+1} \quad \text{for } i = 0, \cdots, t-1.$$

This is a bijection so we just need to check it is consistent with the B-module actions, i.e. that $(Q_1 g_i)^* Q_0 = g_{i+1}^*$ and $(Q_1 g_i)^* Q_1 = g_i^*$ for $i = 0, \cdots, t-1$. But this is true because

$$\begin{aligned} ((Q_1 g_i)^* Q_0)(g_{i+1}) &= (Q_1 g_i)^*(Q_0 g_{i+1}) \\ &= (Q_1 g_i)^*(Q_1 g_i) \\ &= 1 \end{aligned}$$

and the evaluation of $(Q_1 g_i)^* Q_0$ on any other element of $L(t)$ is zero. Similarly we have $((Q_1 g_i)^* Q_1)(g_i) = (Q_1 g_i)^*(Q_1 g_i) = 1$ and the evaluation on any other element of $L(t)$ is zero. So we have shown that $(L(t))^* \cong N(t)$ as B-modules and hence

$$H_*(K(n);\mathbb{Z}/p) = N(\nu) \oplus F'.$$

□

Definition 2.3.8. Let

$$\mathcal{K}(n) = \Sigma^{2n(p-1)} K(n)$$

and let

$$\mathcal{K} = \bigvee_{n \geqslant 0} \mathcal{K}(n).$$

Now look at the algebra structure and the B-module structure of both $H_*(\mathcal{K};\mathbb{Z}/p)$ and $H_*(\ell;\mathbb{Z}/p)$.

The product maps $\mu_{m,n}$ for the spectra $K(n)$ give us an algebra structure on $H_*(\mathcal{K};\mathbb{Z}/p)$, and Kane shows that

$$H_*(\mathcal{K};\mathbb{Z}/p) = \Lambda(\alpha_2, \alpha_3, \dots) \otimes \mathbb{Z}/p[\beta_1, \beta_2, \dots]$$

as an algebra where $|\alpha_n| = 2p^n - 1$ and $|\beta_n| = 2p^n - 2$. Also the B-module structure is given by

$$Q_0(\alpha_n) = \beta_n, \qquad Q_1(\alpha_n) = \beta_{n-1}^p,$$

the action of both Q_0 and Q_1 on β_n is zero.

We know from [Ada95, Part III, Proposition 16.6] that $H^*(\ell;\mathbb{Z}/p) = \mathcal{A}_p/\mathcal{A}_pB$. The generator in $H^0(\ell;\mathbb{Z}/p) = [l, H\mathbb{Z}/p]_0$ gives a monomorphism

$$H_*(\ell;\mathbb{Z}/p) \to H_*(H\mathbb{Z}/p;\mathbb{Z}/p) = \Lambda(\chi(\tau_0), \chi(\tau_1), \dots) \otimes \mathbb{Z}/p[\chi(\xi_1), \chi(\xi_2), \dots].$$

Under this embedding we can identify the homology of ℓ as follows

$$H_*(\ell;\mathbb{Z}/p) = \Lambda(\chi(\tau_2), \chi(\tau_3), \dots) \otimes \mathbb{Z}/p[\chi(\xi_1), \chi(\xi_2), \dots]$$

with B action

$$Q_0(\chi(\tau_n)) = \chi(\xi_n), \qquad Q_1(\chi(\tau_n)) = \chi(\xi_{n-1})^p.$$

2.4 The Splitting

We now show that proving the splitting $\ell \wedge \mathcal{K} \simeq \ell \wedge \ell$ reduces to producing a map $f : \mathcal{K} \to \ell \wedge \ell$ with certain properties.

There is an algebra isomorphism

$$\begin{aligned} \Delta : H_*(\mathcal{K};\mathbb{Z}/p) &\to H_*(\ell;\mathbb{Z}/p) \\ \alpha_n &\mapsto \chi(\tau_n), \\ \beta_n &\mapsto \chi(\xi_n) \end{aligned}$$

which is also an isomorphism of B-modules.

This map Δ cannot be realised by a homotopy equivalence $\mathcal{K} \to \ell$, nor can

$$1 \otimes \Delta : H_*(\ell;\mathbb{Z}/p) \otimes H_*(\mathcal{K};\mathbb{Z}/p) \to H_*(\ell;\mathbb{Z}/p) \otimes H_*(\ell;\mathbb{Z}/p)$$

by a map $\ell \wedge \mathcal{K} \to \ell \wedge \ell$. So we aim to find an isomorphism closely related to $1 \otimes \Delta$ which can be realised by a homotopy equivalence.

Notice that $H_*(\ell;\mathbb{Z}/p) \otimes H_*(\mathcal{K};\mathbb{Z}/p)$ and $H_*(\ell;\mathbb{Z}/p) \otimes H_*(\ell;\mathbb{Z}/p)$ are $H_*(\ell;\mathbb{Z}/p)$-modules via $\ell \wedge \ell \wedge \mathcal{K} \xrightarrow{\mu\wedge 1} \ell \wedge \mathcal{K}$ and $\ell \wedge \ell \wedge \ell \xrightarrow{\mu\wedge 1} \ell \wedge \ell$.

Definition 2.4.1. We can define increasing filtrations $\{J_n\}$ on $H_*(\ell;\mathbb{Z}/p)\otimes H_*(\mathcal{K};\mathbb{Z}/p)$ and $\{G_n\}$ on $H_*(\ell;\mathbb{Z}/p)\otimes H_*(\ell;\mathbb{Z}/p)$ as follows.

$$\begin{aligned} J_n &= H_*(\ell;\mathbb{Z}/p)\otimes(\alpha_2,\ldots,\alpha_n,\beta_1,\ldots,\beta_n),\\ G_n &= H_*(\ell;\mathbb{Z}/p)\otimes(\chi(\tau_2),\ldots,\chi(\tau_n),\chi(\xi_1),\ldots,\chi(\xi_n)) \end{aligned}$$

where $(\alpha_2,\ldots,\alpha_n,\beta_1,\ldots,\beta_n)$ is the ideal of $H_*(\mathcal{K};\mathbb{Z}/p)$ generated by the set $\{\alpha_2,\ldots,\alpha_n,\beta_1,\ldots,\beta_n\}$ and similarly for G_n.

Proposition 2.4.2. *We can produce a map $f:\mathcal{K}\to\ell\wedge\ell$ such that*

$$\begin{aligned} f_*(\alpha_n) &= 1\otimes\chi(\tau_n) \mod G_{n-1}\\ f_*(\beta_n) &= 1\otimes\chi(\xi_n) \mod G_{n-1} \end{aligned} \tag{2.1}$$

and

$$f_* \text{ is multiplicative.} \tag{2.2}$$

This produces the required splitting via the map

$$\Omega:\ell\wedge\mathcal{K}\xrightarrow{1\wedge f}\ell\wedge\ell\wedge\ell\xrightarrow{\mu\wedge 1}\ell\wedge\ell. \tag{2.3}$$

The induced map $\Omega_*:H_*(\ell;\mathbb{Z}/p)\otimes H_*(\mathcal{K};\mathbb{Z}/p)\to H_*(\ell;\mathbb{Z}/p)\otimes H_*(\ell;\mathbb{Z}/p)$ is a map of left $H_*(\ell;\mathbb{Z}/p)$-modules. Also Ω_* preserves the filtrations i.e. $\Omega_*(J_n)\subseteq G_n$. So there is an induced map between the associated graded $H_*(\ell;\mathbb{Z}/p)$-modules $\frac{J_n}{J_{n-1}}\to\frac{G_n}{G_{n-1}}$ which is equal to the map induced by the isomorphism $1\otimes\Delta$ by properties (2.1) and (2.2). So Ω_* induces an isomorphism in $\mathbb{Z}/p$ homology. Since $\ell\wedge\ell$ and $\ell\wedge\mathcal{K}$ are both p-complete (i.e. $H\mathbb{Z}/p_*$-local) spectra, this map is a homotopy equivalence by Theorem 1.1.19.

Properties (2.1) and (2.2) reduce to easier to check conditions.

Property (2.1) reduces to

Condition 2.4.3.

$$f_*(\beta_1) = 1\otimes\chi(\xi_1)-\chi(\xi_1)\otimes 1.$$

Let $H_*(\mathcal{K}(n);\mathbb{Z}/p)=H(n)$, then $H_*(\mathcal{K};\mathbb{Z}/p)=\oplus_{n\geqslant 0}H(n)$ and the map $f_*:H_*(\mathcal{K};\mathbb{Z}/p)\to H_*(\ell\wedge\ell;\mathbb{Z}/p)$ becomes the collection

$$f(n):H(n)\to H_*(\ell\wedge\ell;\mathbb{Z}/p).$$

Let $E=(e_0,e_1,\ldots)$ be an exponential sequence, i.e. all the e_i are non-negative integers and only finitely many are non-zero, and let the last non-zero entry be e_s.

Definition 2.4.4. Define

$$H(E) = \left(\bigotimes_{i=1}^{e_0} H(1)\right) \otimes \cdots \otimes \left(\bigotimes_{i=1}^{e_s} H(p^s)\right)$$
$$f(E) : H(E) \to \bigotimes_{i=1}^{\sum_j e_j} H_*(\ell \wedge \ell; \mathbb{Z}/p)$$
$$f(E) = f(1)^{\otimes e_0} \otimes \cdots \otimes f(p^s)^{\otimes e_s}.$$

Let $n = \sum_{i=0}^{t} n_i p^i$ be the p-adic expansion of n, and let

$$E(n) = (n_0, n_1, \ldots, n_t, 0, 0, \ldots).$$

Property (2.2), f_* is multiplicative, reduces to the following two diagrams commuting.

Condition 2.4.5. If $n \neq p^s$, $s \geqslant 0$:

$$\begin{tikzcd} H(E(n)) \arrow[r, "f(E(n))"] \arrow[d] & \bigotimes_{i=1}^{\sum_j n_j} H_*(\ell \wedge \ell; \mathbb{Z}/p) \arrow[d] \\ H(n) \arrow[r, "f(n)"'] & H_*(\ell \wedge \ell; \mathbb{Z}/p) \end{tikzcd}$$

Condition 2.4.6. If $n = p^s$:

$$\begin{tikzcd} \bigotimes_{i=1}^{p} H(p^{s-1}) \arrow[r, "f(p^{s-1})^{\otimes p}"] \arrow[d] & \bigotimes_{i=1}^{p} H_*(\ell \wedge \ell; \mathbb{Z}/p) \arrow[d] \\ H(p^s) \arrow[r, "f(p^s)"'] & H_*(\ell \wedge \ell; \mathbb{Z}/p) \end{tikzcd}$$

The vertical maps in the above diagrams are repeated multiplication within $H_*(\mathcal{K}; \mathbb{Z}/p)$ and $H_*(\ell \wedge \ell; \mathbb{Z}/p)$.

A few background results about Eilenberg-MacLane spectra are needed. The first two of these are proved in [Mar74], the third follows.

- $[X, Y] \to \mathrm{Hom}_{\mathcal{A}_p}(H^*(Y; \mathbb{Z}/p), H^*(X; \mathbb{Z}/p))$ is an isomorphism if either $X = H\mathbb{Z}/p$ or $Y = H\mathbb{Z}/p$.
- Given an isomorphism $\alpha : H^*(X; \mathbb{Z}/p) \to N \oplus F$ where F is a free $\mathcal{A}_p$ module, there exist spectra Y and Z and a map $k : Y \vee Z \to X$ such that $H^*(Y; \mathbb{Z}/p) = N$, $H^*(Z; \mathbb{Z}/p) = F$ and $k^* = \alpha$.
- Let $H^*(X; \mathbb{Z}/p)$ be a free B-module. Then

$$\ell \wedge X = \bigvee_i \Sigma^{n_i} H\mathbb{Z}/p.$$

We now complete our exposition of Kane's paper by outlining the proofs of 2.4.3 and 2.4.5 to give the idea of the construction of the map f. To prove Condition 2.4.6 is similar to 2.4.5 but has the added problem that the equivalent object to $H^*(G;\mathbb{Z}/p)$ (defined in the proof) in that case is not a free B-module.

Outline proof of Condition 2.4.3. By definition $\mathcal{K} = \bigvee_{n\geqslant 0}\mathcal{K}(n)$ so we need to produce maps $f_n : \mathcal{K}(n) \to \ell\wedge\ell$. The first two of these are $f_0 : \mathcal{K}(0) = S^0 \to \ell\wedge\ell$ and $f_1 : \mathcal{K}(1) = S^{2p-2} \to \ell\wedge\ell$. Now $(f_1)_*(\beta_1)$ is spherical, i.e. in the image of the mod p Hurewicz map

$$h : \pi_*(X) \to H_*(X;\mathbb{Z}) \to H_*(X;\mathbb{Z}/p),$$

so up to a unit, $f_*(\beta_1) = 1\otimes\chi(\xi_1) - \chi(\xi_1)\otimes 1$ which is (2.4.3). $\square$

Outline proof of Condition 2.4.5. The rest of the f_ns are constructed by induction. Suppose f_n has been constructed for $n < N$ and construct f_n for $n = N$. Either $N = p^s$ for some $s \geqslant 0$ or $N \neq p^s$.

Assume $N \neq p^s$, so $N = N_0 + N_1p + \cdots + N_kp^k$ with $\sum_i N_i > 1$. For the sequence $E(N) = (N_0, N_1, \ldots, N_k, 0, 0, \ldots)$ let

$$\begin{aligned}
\mathcal{K}(E(N)) &= (\mathcal{K}(1)^{\wedge N_0}) \wedge\cdots\wedge (\mathcal{K}(p^k)^{\wedge N_k})\\
f(E(N)) &: \mathcal{K}(E(n)) \to (\ell\wedge\ell)^{\wedge\sum_i N_i}\\
f(E(N)) &= f(1)^{\wedge N_0}\wedge\cdots\wedge f(p^k)^{\wedge N_k}.
\end{aligned}$$

Also let $q : \mathcal{K}(E(N)) \to \mathcal{K}(N)$ and $r : (\ell\wedge\ell)^{\wedge\sum N_i} \to \ell\wedge\ell$ be repeated multiplication in $\mathcal{K}$ and ℓ respectively. Now we will explain how to construct f_N to make the following diagram commute

$$\begin{tikzcd}
\mathcal{K}(E(N)) \arrow[r, "f(E(N))"] \arrow[d, "q"'] & (\ell\wedge\ell)^{\wedge\sum_i N_i} \arrow[d, "r"] \\
\mathcal{K}(N) \arrow[r, "f_N"'] & \ell\wedge\ell
\end{tikzcd}$$

Because we are using mod p homology, the Künneth formula will ensure that (2.4.5) holds. The next map f_N is produced via the following diagram

$$\begin{tikzcd}
G \arrow[r, hook] \arrow[d, "j"'] & \ell\wedge G \arrow[d, "1\wedge j"'] & \\
\mathcal{K}(E(N)) \arrow[r, hook] \arrow[d, "q"'] & \ell\wedge\mathcal{K}(E(N)) \arrow[r, "1\wedge(r\circ f(E(N)))"] \arrow[d, "1\wedge q"'] & \ell\wedge\ell\wedge\ell \arrow[d, "\mu\wedge 1"] \\
\mathcal{K}(N) \arrow[r, hook] & \ell\wedge\mathcal{K}(N) & \ell\wedge\ell
\end{tikzcd}$$

where G is the fibre of the map q. Completing the diagram with the bottom right map will give us f_N. From the multiplication on $\mathcal{K}$ we have that $q^* : H^*(\mathcal{K}(N); \mathbb{Z}/p) \to H^*(\mathcal{K}(E(N)); \mathbb{Z}/p)$ is injective. We also know that

$$H^*(\mathcal{K}(N); \mathbb{Z}/p) = \Sigma^{2N(p-1)} L(\nu_p(N!)) \oplus F$$
$$H^*(\mathcal{K}(E(N)); \mathbb{Z}/p) = \Sigma^{2N(p-1)} L(\nu_p(N!)) \oplus F'$$

where F and F' are free B-modules. The map

$$q_1 : L(\nu_p(N!)) \hookrightarrow H^*(\mathcal{K}(N); \mathbb{Z}/p) \rightarrowtail H^*(\mathcal{K}(E(N)); \mathbb{Z}/p) \to L(\nu_p(N!))$$

is an isomorphism because q^* is an isomorphism in degree $2N(p-1)$ and the B-module structure of $L(\nu_p(N!))$ then means q_1 is an isomorphism. So $H^*(G; \mathbb{Z}/p) = \operatorname{coker} q^* = \frac{F}{F'}$ is free because any quotient of free B-modules of finite rank is free. This means $\ell \wedge G = \bigvee_i \Sigma^{n_i} H\mathbb{Z}/p$. Because $\ell \wedge G$ is the fibre of $1 \wedge q$, completing the square is the same as showing

$$\phi : \ell \wedge G \xrightarrow{1\wedge j} \ell \wedge \mathcal{K}(E(N)) \xrightarrow{1\wedge(r \circ f(E(N)))} \ell \wedge \ell \wedge \ell \xrightarrow{\mu \wedge 1} \ell \wedge \ell$$

is trivial and ϕ is trivial if and only if $\phi_* : H_*(\ell \wedge G; \mathbb{Z}/p) \to H_*(\ell \wedge \ell; \mathbb{Z}/p)$ is trivial. We can 'unsmash' one side of the composite ϕ with ℓ to get that ϕ_* is trivial if

$$G \xrightarrow{j} \mathcal{K}(E(N)) \xrightarrow{f(E(N))} (\ell \wedge \ell)^{\sum N_i} \xrightarrow{r} \ell \wedge \ell$$

is trivial in mod p homology. We know $\operatorname{im} j_* = \ker q_*$ so we just need that $\ker q_* = \ker r_* f(E(N))_*$. Let $p^k < N < p^{k+1}$ and let $\mathcal{A} = \Lambda(\alpha_2, \dots, \alpha_k) \otimes \mathbb{Z}/p[\beta_1, \dots, \beta_k]$. The f_n have been constructed for $n < N$ so let $\gamma_s = (f_{p^{s-1}})_*(\alpha_s)$ for $2 \leqslant s \leqslant k$ and $\Delta_t = (f_{p^{t-1}})_*(\beta_t)$ for $1 \leqslant t \leqslant k$. Let $\mathcal{B} \subset H^*(\ell \wedge \ell; \mathbb{Z}/p)$ be the free subalgebra generated by $\{\gamma_s\} \cup \{\Delta_t\}$, then $\mathcal{B} = E(\gamma_2, \dots, \gamma_k) \otimes \mathbb{Z}/p[\Delta_1, \dots, \Delta_k]$. There is an algebra isomorphism $\psi : \mathcal{A} \cong \mathcal{B}$. We also have embeddings $H(s) \subset \mathcal{A}$ for $s < N$ and ψ restricted to $H(s)$ is $(f_s)_* = f(s)$. Then the following diagram commutes;

$$\begin{tikzcd}
H(E(N)) \arrow[r] \arrow[d, "q_*"'] & \bigotimes_{i=1}^{\sum N_j} \mathcal{A} \arrow[r, "\cong"] \arrow[d] & \bigotimes_1^{\sum N_i} \mathcal{B} \arrow[r] \arrow[d] & \bigotimes_{i=1}^{\sum N_j} H_*(\ell \wedge \ell; \mathbb{Z}/p) \arrow[d, "r^*"] \\
H(N) \arrow[r] & \mathcal{A} \arrow[r, "\cong"] & \mathcal{B} \arrow[r] & H_*(\ell \wedge \ell; \mathbb{Z}/p)
\end{tikzcd}$$

the horizontal maps are injections and the composite of the top horizontal maps is $f(E(N))_*$, so $\ker q_* = \ker r_* f(E(N))_*$. □

Chapter 3

The Upper Triangular Group

3.1 Overview

The main aim of this chapter is to produce a p-local analogue of the 2-local theorem of Snaith, [Sna09, Theorem 3.1.2]. This provides an identification between p-adic infinite upper triangular matrices and certain operations on complex connective K-theory.

Let ku be the p-adic connective complex K-theory spectrum and let ℓ be the p-complete Adams summand. The smash product $\ell \wedge \ell$ is a left ℓ-module via the multiplication $\mu : \ell \wedge \ell \to \ell$ coming from the fact that ku and hence ℓ is a ring spectrum:

$$\ell \wedge \ell \wedge \ell \xrightarrow{\mu \wedge 1} \ell \wedge \ell.$$

Definition 3.1.1. Let $\mathrm{End}_{\text{left-}\ell\text{-mod}}(\ell \wedge \ell)$ be the ring of left ℓ-module endomorphisms of $\ell \wedge \ell$ of degree zero. Of these, the ones that can be inverted, i.e. the group of units of this ring, are the left ℓ-module automorphisms of $\ell \wedge \ell$ which we shall call $\mathrm{Aut}_{\text{left-}\ell\text{-mod}}(\ell \wedge \ell)$. These form a group under composition of functions. Because they are invertible these are homotopy classes of left ℓ-module homotopy equivalences. Finally denote by $\mathrm{Aut}^0_{\text{left-}\ell\text{-mod}}(\ell \wedge \ell)$ the subgroup of these homotopy equivalences consisting of those which induce the identity map in mod p homology i.e. $f \in \mathrm{Aut}^0_{\text{left-}\ell\text{-mod}}(\ell \wedge \ell)$ means that $f_* = \mathrm{id} : H_*(\ell \wedge \ell; \mathbb{Z}/p) \to H_*(\ell \wedge \ell; \mathbb{Z}/p)$.

Definition 3.1.2. Let $U_\infty \mathbb{Z}_p$ be the group of invertible infinite upper triangular matrices with entries in the p-adic integers.

The group structure of $U_\infty \mathbb{Z}_p$ is given by matrix multiplication; because these matrices are upper triangular each column is of finite height, so when multiplying, the sum for each entry of the product matrix is also finite. Any element of $U_\infty \mathbb{Z}_p$ is a matrix $X = (x_{i,j})$ for $i, j \in \mathbb{N}_0$, where all $x_{i,j} \in \mathbb{Z}_p$ and $x_{i,j} = 0$ for $i > j$. For an infinite upper triangular matrix with p-adic entries to be invertible it is necessary and sufficient for it to have p-adic units on the diagonal.

The main theorem of this chapter is as follows:

Theorem 3.1.3. *There is an isomorphism of groups of the form*

$$\Lambda : U_\infty \mathbb{Z}_p \xrightarrow{\cong} \mathrm{Aut}^0_{\text{left-}\ell\text{-mod}}(\ell \wedge \ell).$$

As explained in the previous chapter, from [Kan81] we have a p-adic splitting of the form

$$\ell \wedge \ell \simeq \ell \wedge \bigvee_{n \geqslant 0} \mathcal{K}(n).$$

We want to study left-ℓ-module maps of the form $\ell \wedge \ell \to \ell \wedge \ell$. Because there exists a splitting of $\ell \wedge \ell$ we only need to look at left-ℓ-module maps from any one piece of the splitting to any other piece, i.e. maps of the form

$$\phi_{m,n} : \ell \wedge \mathcal{K}(m) \to \ell \wedge \mathcal{K}(n)$$

for each m, $n \geqslant 0$.

We will use a suitable Adams spectral sequence to show that there exist particular maps

$$\iota_{m,n} : \ell \wedge \mathcal{K}(m) \to \ell \wedge \mathcal{K}(n)$$

which are represented by generators of certain groups on the E_2 page of the spectral sequence. This E_2 page consists of Ext groups which we will calculate using the theory of stable isomorphism classes. Once we have the maps $\iota_{m,n}$ we can define our isomorphism below.

Definition 3.1.4. Let the map Λ be as follows

$$\begin{aligned} \Lambda : U_\infty \mathbb{Z}_p &\to \mathrm{Aut}^0_{\text{left-}\ell\text{-mod}}(\ell \wedge \ell) \\ X &\mapsto \sum_{m \geqslant n} X_{n,m} \iota_{m,n} : \ell \wedge (\bigvee_{i \geqslant 0} \mathcal{K}(i)) \to \ell \wedge (\bigvee_{i \geqslant 0} \mathcal{K}(i)). \end{aligned}$$

The rest of the chapter will proceed in the following way. Section 3.2 covers the theory of stable isomorphism classes and results needed later concerning Ext groups. The main result of this section is Theorem 3.2.13 which identifies the stable isomorphism class of the mod p cohomology of $\mathcal{K}(n)$. Section 3.3 then sets up the required Adams spectral sequence. The results of the previous section are used to identify the E_2 term and then to show that the spectral sequence collapses at the E_2 term for dimensional reasons. We then pick generators of the groups on the E_2 page to give the maps $\iota_{m,n}$ used in the definition of Λ above. The spectral sequence is then further analysed to show, in Proposition 3.3.6, that Λ is bijective. Finally we show in Proposition 3.3.7 that the choice of the maps $\iota_{m,n}$ can be made in such a way that Λ is a group isomorphism.

3.2 Stable Isomorphisms and Ext Groups

A main ingredient in the proof of Theorem 3.1.3 is the idea of stable isomorphisms which were first introduced by Adams. The following definitions and theory come from [Ada95, Part III, Chapter 16] and are explored in more detail there. Another good reference for modules over Hopf algebras and their cohomology is [Mar83].

We start by working over a general Hopf algebra to introduce the general techniques before specialising to the subalgebra B of the Steenrod Algebra introduced in Definition 1.2.14.

Definition 3.2.1. Let A be a connected graded finite dimensional Hopf algebra over a field K. Two graded left A-modules M and N are *stably isomorphic*, which we will denote $M \approxeq N$, if there exist free A-modules F and G such that $M \oplus F \cong N \oplus G$.

Lemma 3.2.2. *Stable isomorphism is an equivalence relation.*

Proof.
- $M \oplus F \cong M \oplus F$ for any free A-module F, so $M \approxeq M$.
- If $M \approxeq N$ then $M \oplus F \cong N \oplus G$ for free A-modules F, G, this clearly works in both directions so $N \approxeq M$.
- If $M \approxeq N$ and $N \approxeq L$ then we know $M \oplus F \cong N \oplus G$ and $N \oplus H \cong L \oplus J$ for free A-modules F, G, H and J. So

$$\begin{aligned} M \oplus F \oplus H &\cong N \oplus G \oplus H \\ &\cong N \oplus H \oplus G \\ &\cong L \oplus J \oplus G \end{aligned}$$

  and since $F \oplus H$ and $J \oplus G$ are free A-modules we have $M \approxeq L$. $\square$

We can take tensor products of stable isomorphism classes so that for A-modules M and N, the tensor product of their stable classes is the stable class of $M \otimes N$. For this to be well-defined we note that the A-module $A \otimes N$ where we take A acting diagonally via the comultiplication is isomorphic as a left A-module to $A \otimes N$ where A acts by multiplication within the left factor of A.

Definition 3.2.3. Let '1' be the graded A-module with the ground field K in degree 0 and zero in other degrees. The action of A on 1 is trivial, i.e. for any $k \in K$ we have $a.k = 0$ for all $a \in A_n$ where $n > 0$ and $A_0 = K$ acts as multiplication within K.

Definition 3.2.4. An A-module M is *invertible* if there exists another class M' such that $M \otimes M' \approxeq 1$. Then we write $M^{-1} = M'$.

From here onwards the theory is for A the graded exterior algebra $K[x, y]$ where $|x| < |y|$ and $|x|$ and $|y|$ are both odd unless K has characteristic 2. This ensures that A is a Hopf algebra with x and y primitive, i.e. $\Delta(x) = x \otimes 1 + 1 \otimes x$ where Δ is the comultiplication on A and similarly for y. In our case we want $A = B = \Lambda[Q_0, Q_1]$, where $Q_0 = \beta$ has degree 1 and $Q_1 = \mathcal{P}^1\beta - \beta\mathcal{P}^1$ has degree $2p - 1$ and $K = \mathbb{Z}/p$ for an odd prime p.

Definition 3.2.5. Let Σ be the B-module with $\mathbb{Z}/p$ in degree 1, this is invertible with inverse Σ^{-1}, the B-module with $\mathbb{Z}/p$ in degree -1. Also $\Sigma^a = \underbrace{\Sigma \otimes \Sigma \otimes \cdots \otimes \Sigma}_{a}$ is the module with $\mathbb{Z}/p$ in degree a.

Definition 3.2.6. Let $\epsilon : B \to \mathbb{Z}/p$ be the augmentation map of B, i.e. if we think of B as the $\mathbb{Z}/p$-vector space $\mathbb{Z}/p\{1, Q_0, Q_1, Q_0Q_1\}$, ϵ is determined by taking the basis element $1 \in B$ to $1 \in \mathbb{Z}/p$ and all other basis elements Q_0, Q_1 and Q_0Q_1 to $0 \in \mathbb{Z}/p$. Let I be the augmentation ideal of B; that is $I = \ker(\epsilon)$. So $I = \mathbb{Z}/p\{Q_0, Q_1, Q_0Q_1\}$ as a $\mathbb{Z}/p$-vector space. Again $I^b = \underbrace{I \otimes I \otimes \cdots \otimes I}_{b}$.

Remark 3.2.7. For $s > 0$, $\mathrm{Ext}_B^{s,t}(M, K)$ only depends on the stable isomorphism class of M.

Adams gives a method for calculating stable isomorphism classes for the case where $p = 2$ which carries over to the odd prime case; the following result is in [Ada95, Part III, Theorem 16.3].

Theorem 3.2.8. *For a finite dimensional graded module M over an exterior algebra $K[x, y]$ as above with $H_*(M; x)$ and $H_*(M; y)$ both being of dimension 1 over K, M is invertible and stably isomorphic to $\Sigma^a I^b = \Sigma^a \otimes I^b$ for unique a, $b \in \mathbb{Z}$.*

In order to determine a and b Adams gives us the following formulae:

$$H_i(\Sigma^a I^b; x) = \begin{cases} K & \text{if } i = a + b|x| \\ 0 & \text{otherwise,} \end{cases}$$

and

$$H_j(\Sigma^a I^b; y) = \begin{cases} K & \text{if } j = a + b|y| \\ 0 & \text{otherwise.} \end{cases}$$

Because x and y are in different degrees, we can solve these equations for the values a and b which will be unique.

Here $H_*(M; x) = \dfrac{\ker x}{\operatorname{im} x}$ is taking homology with respect to the action of the element x and similarly for y.

Definition 3.2.9. For a B-module M, let M^* denote its $\mathbb{Z}/p$-linear dual i.e. $M^* = \mathrm{Hom}^*_{\mathbb{Z}/p}(M, \mathbb{Z}/p)$.

This will be graded in the following way; a homomorphism from something in degree m to $\mathbb{Z}/p$ in degree zero will have degree $-m$.

Adams proves in [Ada95, Part III, Lemma 16.2] that I is invertible with inverse $I^{-1} \approxeq I^*$. We know that $I = \mathbb{Z}/p\{Q_0, Q_1, Q_0Q_1\}$ so has a copy of $\mathbb{Z}/p$ in degrees 1, $2p-1$ and $2p$ and that Q_0 and Q_1 act by increasing the degree by 1 and $2p-1$ respectively. In contrast I^* has copies of $\mathbb{Z}/p$ in degrees -1, $1-2p$ and $-2p$ and Q_0 and Q_1 still act by increasing the degree by 1 and $2p-1$ respectively.

Lemma 3.2.10. $I^{-b} \approxeq (I^b)^* = \mathrm{Hom}^*_{\mathbb{Z}/p}(I^b, \mathbb{Z}/p)$.

Proof. Because I is free and finitely generated over $\mathbb{Z}/p$ we have that $(I \otimes I)^* \cong I^* \otimes I^*$, this means we have $(I^b)^* \cong (I^*)^b$, and stably we have $(I^*)^b \approxeq (I^{-1})^b$. We know that I^{-1} has Q_0 homology in degree -1 and Q_1 homology in degree $1-2p$ so by the Künneth formula $(I^{-1})^b$ will have Q_0 homology in degree $-b$ and Q_1 homology in degree $(1-2p)b$. By the criteria of Theorem 3.2.8 this is then stably isomorphic to I^{-b}. $\square$

Remark 3.2.11. In a similar way it is shown in [Ada95, Part III, Lemma 16.3(i)] that for any invertible B-module M, its linear dual M^* is its inverse stable isomorphism class.

The B-modules Σ and I give us a couple of dimension-shifting isomorphisms for Ext groups.

Lemma 3.2.12. *There exist isomorphisms of* Ext *groups of the form*

$$\mathrm{Ext}_B^{s,t}(I \otimes M, \mathbb{Z}/p) \cong \mathrm{Ext}_B^{s+1,t}(M, \mathbb{Z}/p)$$
$$\mathrm{Ext}_B^{s,t}(\Sigma^a M, \mathbb{Z}/p) \cong \mathrm{Ext}_B^{s,t-a}(M, \mathbb{Z}/p)$$

for $s > 0$ and M a B-module.

Proof. For a B-module M, from the short exact sequence

$$0 \to I \otimes M \to B \otimes M \to M \to 0$$

comes a long exact sequence of Ext groups

$$\cdots \to \mathrm{Ext}_B^{s,t}(B \otimes M, \mathbb{Z}/p) \to \mathrm{Ext}_B^{s,t}(I \otimes M, \mathbb{Z}/p) \to \mathrm{Ext}_B^{s+1,t}(M, \mathbb{Z}/p)$$
$$\to \mathrm{Ext}_B^{s+1,t}(B \otimes M, \mathbb{Z}/p) \to \cdots$$

and since $B \otimes M$ is a free B-module, $\mathrm{Ext}_B^{s,t}(B \otimes M, \mathbb{Z}/p) = 0$ for $s > 0$ and so

$$\mathrm{Ext}_B^{s,t}(I \otimes M, \mathbb{Z}/p) \xrightarrow{\cong} \mathrm{Ext}_B^{s+1,t}(M, \mathbb{Z}/p)$$

for all $s > 0$. It is also true from construction that $\mathrm{Ext}_B^{s,t}(\Sigma^a M, \mathbb{Z}/p) \cong \mathrm{Ext}_B^{s,t-a}(M, \mathbb{Z}/p)$. $\square$

Recall from Proposition 2.2.4 that the spectrum $\mathcal{K}(n)$ is equivalent to the p-completion of a finite spectrum, i.e. $\mathcal{K}(n) \simeq Y_p$ for some finite spectrum Y. With a slight abuse of notation, when we take the Spanier-Whitehead dual and write $D(\mathcal{K}(n))$ we really mean take the dual of the finite spectrum Y and complete later. This is not a problem as everything we are working with is in a p-complete setting so using the underlying finite spectrum and p-completing after will not make a difference.

We are now in a position to calculate the stable isomorphism classes of both $H^*(\mathcal{K}(n);\mathbb{Z}/p)$ and $H^*(D(\mathcal{K}(n));\mathbb{Z}/p)$. We will need both of these facts in the next section in order to simplify the spectral sequence we will construct there.

Theorem 3.2.13. *The stable isomorphism class of $H^*(\mathcal{K}(n);\mathbb{Z}/p)$ can be written as $\Sigma^{2n(p-1)-\nu_p(n!)}I^{\nu_p(n!)}$.*

Proof. In [Kan81, Lemma 8:3, Lemma 8:4], Kane provides the following facts:

$$\begin{aligned}&H(H^*(K(n);\mathbb{Z}/p);Q_0) = \mathbb{Z}/p \text{ in dimension } 0 \text{ and}\\ &H(H^*(K(n);\mathbb{Z}/p);Q_1) = \mathbb{Z}/p \text{ in dimension } 2\nu_p(n!)(p-1).\end{aligned}$$

Recall that $\mathcal{K}(n) = \Sigma^{2n(p-1)}K(n)$, so

$$\begin{aligned}H^*(\mathcal{K}(n);\mathbb{Z}/p) &= H^*(\Sigma^{2n(p-1)}K(n);\mathbb{Z}/p)\\ &= H^{*-2n(p-1)}(K(n);\mathbb{Z}/p),\end{aligned}$$

and thus for $s \in \mathbb{N}_0$,

$$\begin{aligned}&H(H^*(\mathcal{K}(n);\mathbb{Z}/p);Q_0) = \mathbb{Z}/p \text{ in dimension } 2n(p-1) \text{ and}\\ &H(H^*(\mathcal{K}(n);\mathbb{Z}/p);Q_1) = \mathbb{Z}/p \text{ in dimension } 2(p-1)(\nu_p(n!)+n).\end{aligned}$$

Using Theorem 3.2.8 we can then deduce that we have a stable isomorphism

$$H^*(\mathcal{K}(n);\mathbb{Z}/p) \approxeq \Sigma^{2n(p-1)-\nu_p(n!)}I^{\nu_p(n!)}. \qquad \square$$

Lemma 3.2.14. *There is a stable isomorphism*

$$H^*(D(\mathcal{K}(n));\mathbb{Z}/p) \approxeq \Sigma^{\nu_p(n!)-2n(p-1)}I^{-\nu_p(n!)}.$$

Proof. The Universal Coefficient Theorem gives us the following B-module isomorphism

$$H^*(\mathcal{K}(n);\mathbb{Z}/p) \cong \mathrm{Hom}^*_{\mathbb{Z}/p}(H_{-*}(\mathcal{K}(n);\mathbb{Z}/p),\mathbb{Z}/p).$$

We know from Remark 3.2.11 that for any B-module, its linear dual is its inverse stable isomorphism class. From Theorem 3.2.13 we have that $H^*(\mathcal{K}(n);\mathbb{Z}/p) \approxeq \Sigma^{2n(p-1)-\nu_p(n!)}I^{\nu_p(n!)}$, hence we have

$$H_{-*}(\mathcal{K}(n);\mathbb{Z}/p) \approxeq \Sigma^{\nu_p(n!)-2n(p-1)}I^{-\nu_p(n!)}.$$

Recall from Proposition 1.6.5 that Spanier-Whitehead duality gives us the B-module isomorphism

$$H^*(D(\mathcal{K}(n)); \mathbb{Z}/p) \cong H_{-*}(\mathcal{K}(n); \mathbb{Z}/p),$$

which proves the result. $\square$

One last result we will need in the next section is the following.

Lemma 3.2.15. $\operatorname{Ext}_B^{*,*}(\mathbb{Z}/p, \mathbb{Z}/p) = \mathbb{Z}/p[c, d]$ *where* $c \in \operatorname{Ext}_B^{1,1}$ *and* $d \in \operatorname{Ext}_B^{1,2p-1}$.[1]

Proof. This statement is proved in [Rav86, Lemma 3.1.9] with a minor mistake, so we shall prove the statement here also. Let $\Gamma = \Lambda(x)$ be the exterior Hopf algebra over $\mathbb{Z}/p$ on one generator x. We will first calculate $\operatorname{Ext}_\Gamma^{*,*}(\mathbb{Z}/p, \mathbb{Z}/p)$. Take an injective Γ-resolution of $\mathbb{Z}/p$ as follows

$$0 \to \mathbb{Z}/p \to \Gamma \xrightarrow{\partial} \Gamma \xrightarrow{\partial} \Gamma \xrightarrow{\partial} \cdots$$

where ∂ is the Γ-homomorphism where $\partial(1) = x$ and $\partial(x) = 0$. We can do this because Γ is a finite dimensional Hopf algebra and therefore self-injective. Now we apply the functor $\operatorname{Hom}_\Gamma(\mathbb{Z}/p, -)$ to get

$$0 \to \operatorname{Hom}_\Gamma(\mathbb{Z}/p, \Gamma) \xrightarrow{\partial_*} \operatorname{Hom}_\Gamma(\mathbb{Z}/p, \Gamma) \xrightarrow{\partial_*} \cdots .$$

We know that $\operatorname{Hom}_\Gamma(\mathbb{Z}/p, \Gamma) \cong \mathbb{Z}/p$ via the isomorphism which sends $f \in \operatorname{Hom}_\Gamma(\mathbb{Z}/p, \Gamma)$ to $\lambda \in \mathbb{Z}/p$ where $f(1) = \lambda x$. This must be the case as any such map f is determined by its value on 1 and we must have $f(1) = \lambda x$ for some $\lambda \in \mathbb{Z}/p$ otherwise f would not be a Γ-homomorphism. Now the maps ∂_* on a map f are obtained by post-composition with f and are still determined by their action on 1, i.e.

$$(\partial_* f)(1) = \partial \circ f(1) = \partial(\lambda x) = \lambda \partial(x) = 0.$$

Hence all the boundary maps are zero so when we take homology we get a copy of $\mathbb{Z}/p$ in every homological degree. Let the generator of the Ext^1 group be y. Because this map sends 1 to x it raises degree by $|x|$ and so lies in $\operatorname{Ext}_\Gamma^{1,|x|}(\mathbb{Z}/p, \mathbb{Z}/p)$. The composition product on Ext groups gives us that the generators for the Ext^2, Ext^3 groups and so on are y^2, y^3 respectively. Hence $\operatorname{Ext}_\Gamma^{*,*}(\mathbb{Z}/p, \mathbb{Z}/p) \cong \mathbb{Z}/p[y]$. Now our exterior algebra B is isomorphic to $\Lambda(Q_0) \otimes \Lambda(Q_1)$ and Ext groups come with an external pairing

$$\operatorname{Ext}_{\Lambda(Q_0)}^{*,*}(\mathbb{Z}/p, \mathbb{Z}/p) \otimes \operatorname{Ext}_{\Lambda(Q_1)}^{*,*}(\mathbb{Z}/p, \mathbb{Z}/p) \to \operatorname{Ext}_B^{*,*}(\mathbb{Z}/p, \mathbb{Z}/p),$$

which by the Künneth theorem gives us an isomorphism of vector spaces between the two. This is also an isomorphism of rings which gives us that $\operatorname{Ext}_B^{*,*}(\mathbb{Z}/p, \mathbb{Z}/p) \cong \mathbb{Z}/p[c] \otimes \mathbb{Z}/p[d] \cong \mathbb{Z}/p[c, d]$ where c and d are in the degrees stated above. $\square$

[1] A lot of authors would use the notation v_0 and v_1 instead of c and d.

3.3 The Spectral Sequence

We want to look at automorphisms of $\ell \wedge \ell$ which induce the identity on mod p homology. When looking at the separate pieces of the splitting we use the following lemma to translate this into a condition on the maps $\phi_{m,n}$.

Lemma 3.3.1. *Let E be a ring spectrum and F be spectra such that $F \simeq F_1 \vee F_2$. There is an isomorphism of groups of the form*

$$\mathrm{End}^0_{\text{left-}E\text{-mod}}(E \wedge F) \xrightarrow{\cong} \bigoplus_{1 \leqslant i,j \leqslant 2} \mathrm{Hom}^0_{\text{left-}E\text{-mod}}(E \wedge F_i, E \wedge F_j)$$
$$f \mapsto f_{11} \oplus f_{12} \oplus f_{21} \oplus f_{22}$$

where $\mathrm{Hom}^0_{\text{left-}E\text{-mod}}(E \wedge F_i, E \wedge F_j)$ means that if $i = j$ the morphism induces the identity on mod p homology and if $i \neq j$ the morphism induces the zero map on mod p homology, i.e. $(f_{11})_*$ and $(f_{22})_*$ are the identity maps and $(f_{12})_*$ and $(f_{21})_*$ are the zero maps.

Proof. An element of $\mathrm{End}_{\text{left-}E\text{-mod}}(E \wedge F)$ automatically splits into its component parts as homomorphisms between each of the pieces,

$$\mathrm{End}_{\text{left-}E\text{-mod}}(E \wedge F) \cong \bigoplus_{1 \leqslant i,j \leqslant 2} \mathrm{Hom}_{\text{left-}E\text{-mod}}(E \wedge F_i, E \wedge F_j).$$

We now restrict to the maps inducing the identity on mod p homology, i.e. given an element $f \in \mathrm{End}^0_{\text{left-}E\text{-mod}}(E \wedge F)$, this induces the identity map on $H_*(E \wedge F; \mathbb{Z}/p)$. So f must restrict to the identity maps on the homology of each piece $E \wedge F_i$, hence $(f_{11})_*$ and $(f_{22})_*$ must be the identity maps. We must also have $(f_{12})_*$ and $(f_{21})_*$ being the zero maps otherwise f_* would not then be the identity map. In other words;

$$\begin{aligned} f_* &= (f_{11} \oplus f_{12} \oplus f_{21} \oplus f_{22})_* \\ &= (f_{11})_* \oplus (f_{12})_* \oplus (f_{21})_* \oplus (f_{22})_* \\ &= 1 \oplus 0 \oplus 0 \oplus 1 = \mathrm{id}\,. \end{aligned}$$

□

We consider $\mathrm{Aut}^0_{\text{left-}\ell\text{-mod}}(\ell \wedge \ell) \subset \mathrm{End}^0_{\text{left-}\ell\text{-mod}}(\ell \wedge \ell)$ so, analogously to this lemma, we need to have $(\phi_{m,m})_* = \mathrm{id}$ for every $m \geqslant 0$ and $(\phi_{m,n})_*$ the zero map for every m, $n \geqslant 0$, $m \neq n$.

Since we are looking at left-ℓ-module maps, each map $\phi_{m,n}$ is determined by its restriction to $S^0 \wedge \mathcal{K}(m) \to \ell \wedge \mathcal{K}(n)$. This is an element of the homotopy group $[\mathcal{K}(m), \ell \wedge \mathcal{K}(n)]$. By Proposition 2.2.4 we know that $\mathcal{K}(m)$ is the p-completion of a finite spectrum, i.e. $\mathcal{K}(m) = Y_p$ for some finite spectrum Y. Since $\ell \wedge \mathcal{K}(n)$ is p-complete

$$[\mathcal{K}(m), \ell \wedge \mathcal{K}(n)] = [\mathcal{K}(m), \ell \wedge \mathcal{K}(n)_p] = [Y_p, \ell \wedge \mathcal{K}(n)]_p = [Y, \ell \wedge \mathcal{K}(n)]_p.$$

Now because Y is a finite spectrum we can take its Spanier-Whitehead dual so our homotopy group becomes $[S^0, \ell \wedge \mathcal{K}(n) \wedge D(Y)]_p$. We can take the p-completion of $D(Y)$ without changing the homotopy group and, as mentioned in the last section, we will refer to $D(Y)_p$ as $D(\mathcal{K}(m))$.

To study this homotopy group we are going to use the Adams spectral sequence whose E_2 term is as follows:

$$\begin{aligned} E_2^{s,t} &= \mathrm{Ext}^{s,t}_{\mathcal{A}_p}(H^*(D(\mathcal{K}(m)) \wedge \mathcal{K}(n) \wedge \ell; \mathbb{Z}/p), \mathbb{Z}/p) \\ &\cong \mathrm{Ext}^{s,t}_{\mathcal{A}_p}(H^*(D(\mathcal{K}(m)); \mathbb{Z}/p) \otimes H^*(\mathcal{K}(n); \mathbb{Z}/p) \otimes H^*(\ell; \mathbb{Z}/p), \mathbb{Z}/p) \\ &\cong \mathrm{Ext}^{s,t}_{\mathcal{A}_p}(H^*(D(\mathcal{K}(m)); \mathbb{Z}/p) \otimes H^*(\mathcal{K}(n); \mathbb{Z}/p) \otimes \mathcal{A}_p \otimes_B \mathbb{Z}/p, \mathbb{Z}/p) \\ &\cong \mathrm{Ext}^{s,t}_{\mathcal{A}_p}(H^*(D(\mathcal{K}(m)); \mathbb{Z}/p) \otimes H^*(\mathcal{K}(n); \mathbb{Z}/p) \otimes_B \mathcal{A}_p, \mathbb{Z}/p) \\ &\cong \mathrm{Ext}^{s,t}_{B}(H^*(D(\mathcal{K}(m)); \mathbb{Z}/p) \otimes H^*(\mathcal{K}(n); \mathbb{Z}/p), \mathbb{Z}/p), \end{aligned} \tag{3.1}$$

via the Künneth theorem. In [Ada95, Part III, Proposition 16.6], Adams gives us that $H^*(\ell; \mathbb{Z}/p) \cong \mathcal{A}_p \otimes_B \mathbb{Z}/p$, it is then shown in [Ada95, Part III, Proof of Proposition 16.1] that for an $\mathcal{A}_p$-module M we have an isomorphism of left $\mathcal{A}_p$-modules

$$(\mathcal{A}_p \otimes_B \mathbb{Z}/p) \otimes M \cong \mathcal{A}_p \otimes_B M$$

where $\mathcal{A}_p$ acts diagonally on the left-hand side by the comultiplication and on the right-hand side by multiplication within $\mathcal{A}_p$. We also use a standard change of rings isomorphism of the form

$$\mathrm{Ext}_{\mathcal{A}_p}(\mathcal{A}_p \otimes_B M, \mathbb{Z}/p) \cong \mathrm{Ext}_B(M, \mathbb{Z}/p).$$

We know that the sphere spectrum is finite and that the spectra $\mathcal{K}(n)$ and $D(\mathcal{K}(m))$ are also finite. Recall from section 2.3 that

$$H_*(\ell; \mathbb{Z}/p) \cong \Lambda(\chi(\tau_2), \chi(\tau_3), \ldots) \otimes \mathbb{Z}/p[\chi(\xi_1), \chi(\xi_2), \ldots]$$

where $\tau_i, \xi_j \in \mathcal{A}_p^*$ and χ is the canonical anti-automorphism. In a similar way to Proposition 1.6.7, a p-complete spectrum with mod p homology of finite type is the p-completion of finite type spectrum. We know ℓ is bounded below as it is connected and we can see that the mod p homology of ℓ is finitely generated in each degree as a $\mathbb{Z}/p$-vector space, so ℓ is of finite type and the conditions of the Adams spectral sequence are satisfied. Hence the spectral sequence converges to

$$E_\infty^{s,t} = [S^0, D(\mathcal{K}(m)) \wedge \mathcal{K}(n) \wedge \ell]_{t-s} \otimes \mathbb{Z}_p = \pi_{t-s}(D(\mathcal{K}(m)) \wedge \mathcal{K}(n) \wedge \ell) \otimes \mathbb{Z}_p.$$

Proposition 3.3.2. *For the spectral sequence above we have for $s > 0$,*

$$E_2^{s,t} \cong \mathrm{Ext}_B^{s+\nu_p(n!)-\nu_p(m!),\,t-2(n-m)(p-1)+\nu_p(n!)-\nu_p(m!)}(\mathbb{Z}/p, \mathbb{Z}/p).$$

Proof. Using Theorem 3.2.13, Lemma 3.2.14 and Lemma 3.2.12 from the previous section we get that for $s > 0$,

$$\begin{aligned}
E_2^{s,t} &\cong \mathrm{Ext}_B^{s,t}(\Sigma^{\nu_p(m!)-2m(p-1)} I^{-\nu_p(m!)} \otimes \Sigma^{2n(p-1)-\nu_p(n!)} I^{\nu_p(n!)}, \mathbb{Z}/p) \\
&\cong \mathrm{Ext}_B^{s,t}(\Sigma^{2(n-m)(p-1)+\nu_p(m!)-\nu_p(n!)} I^{\nu_p(n!)-\nu_p(m!)}, \mathbb{Z}/p) \\
&\cong \mathrm{Ext}_B^{s+\nu_p(n!)-\nu_p(m!),t-2(n-m)(p-1)+\nu_p(n!)-\nu_p(m!)}(\mathbb{Z}/p, \mathbb{Z}/p). \qquad \square
\end{aligned}$$

Lemma 3.3.3. *Our spectral sequence above collapses at the E_2 term and so $E_2^{*,*} = E_\infty^{*,*}$.*

Proof. From above we know that the E_2 term of our spectral sequence away from the line $s = 0$ is isomorphic to a sum of shifted copies of $\mathrm{Ext}_B^{*,*}(\mathbb{Z}/p, \mathbb{Z}/p) \cong \mathbb{Z}/p[c, d]$. Here, all non-zero terms are in even total degrees and the shifts given above have even total degrees so there are no non-trivial differentials when $s > 0$. Showing there are no non-trivial differentials when $s = 0$ is done for a more general case when $p = 2$ in [Ada95, Part III, Lemma 17.12]; however the method is the same. Consider an element $e \in E_2^{0,t}$ where $t \equiv 1 \mod 2$ (if $t \equiv 0 \mod 2$ then everything would be in even total degree and there would be no non-trivial differentials for degree reasons). We will proceed by induction. Suppose that $d_i = 0$ for $i < r$, then the spectral sequence would have $E_2^{s,t} \cong E_r^{s,t}$. For $c \in \mathrm{Ext}_B^{1,1}(\mathbb{Z}/p, \mathbb{Z}/p)$ we have $cd_r(e) = d_r(ce)$ because the spectral sequence we are looking at is one of modules over $\mathrm{Ext}_B^{*,*}(\mathbb{Z}/p, \mathbb{Z}/p)$. We know $ce = 0$ as this would be in odd total degree and there are no elements in odd total degree away from the $s = 0$ line, hence $d_r(ce) = 0$. Because, away from the $s = 0$ line, the $E_2 = E_r$ page of the spectral sequence reduces to a polynomial algebra with c as one of the generators, multiplication by c is a monomorphism on $E_r^{s,t}$ for $s > 0$. So if

$$cd_r(e) = d_r(ce) = 0,$$

then we must have $d_r(e) = 0$ which completes the induction. $\square$

Lemma 3.3.4.

$$\pi_0(D(\mathcal{K}(m)) \wedge \mathcal{K}(n) \wedge \ell) \otimes \mathbb{Z}_p = \begin{cases} 0 & \text{if } n > m \\ \mathbb{Z}_p & \text{if } n \leqslant m. \end{cases}$$

Proof. We want to study $\pi_0(D(\mathcal{K}(m)) \wedge \mathcal{K}(n) \wedge \ell) \otimes \mathbb{Z}_p$, which corresponds to the $s = t$ line of the E_∞ page of the spectral sequence, so we want to look at the groups $\{E_2^{s,s} | s \geqslant 0\}$. This information comes from $\mathrm{Ext}_B^{u,v}(\mathbb{Z}/p, \mathbb{Z}/p)$ where

$$u = s + \nu_p(n!) - \nu_p(m!) \quad \text{and} \tag{3.2}$$

$$v = s - 2(n-m)(p-1) + \nu_p(n!) - \nu_p(m!). \tag{3.3}$$

This gives us that $v-u=2(m-n)(p-1)$. If $n>m$ then $u>v$ and we know that all groups below the diagonal of $u=v$ are zero in $\mathrm{Ext}_B^{u,v}(\mathbb{Z}/p,\mathbb{Z}/p)$, hence $\pi_0(D(\mathcal{K}(m))\wedge\mathcal{K}(n)\wedge\ell)\otimes\mathbb{Z}_p=0$ if $n>m$.

If we let $n\leqslant m$, take for example $n=m=0$ (the general case is very similar), then we have $E_2^{*,*}=E_\infty^{*,*}=\mathrm{Ext}_B^{*,*}(\mathbb{Z}/p,\mathbb{Z}/p)=\mathbb{Z}/p[c,d]$ where $c\in\mathrm{Ext}_B^{1,1}$ and $d\in\mathrm{Ext}_B^{1,2p-1}$. More specifically, $E_2^{s,s}=E_\infty^{s,s}=\mathbb{Z}/p\{c^s\}$. For each s, this group is the filtration quotient $\frac{F^s}{F^{s+1}}$. Using the ring structure of the spectral sequence we know that the $E_2^{s,s}$ terms are a polynomial algebra on the variable c. When we pass to the E_∞ term, this is the following algebra

$$\pi_0(D(\mathcal{K}(m))\wedge\mathcal{K}(n)\wedge\ell)\otimes\mathbb{Z}_p\cong\left\{\sum_{i=0}^{\infty}x_ic^i : x_i\in\mathbb{Z}/p\right\}\cong\mathbb{Z}_p$$

which is filtered by ideals and where multiplication by c in the algebra corresponds to multiplication by p in $\mathbb{Z}_p$. □

Definition 3.3.5. Let $\iota_{m,n}:\ell\wedge\mathcal{K}(m)\to\ell\wedge\mathcal{K}(n)$ be a map which is represented in the spectral sequence by a choice of generator of

$$E_2^{(m-n)-\nu_p(n!)+\nu_p(m!),(m-n)-\nu_p(n!)+\nu_p(m!)}.$$

Also let $\iota_{m,m}$ be the identity on $\ell\wedge\mathcal{K}(m)$.

Recall in Definition 3.1.4 we defined the map

$$\begin{aligned}\Lambda : U_\infty\mathbb{Z}_p &\to \mathrm{Aut}^0_{\text{left-}\ell\text{-mod}}(\ell\wedge\ell)\\ X &\mapsto \sum_{m\geqslant n}X_{n,m}\iota_{m,n} : \ell\wedge(\bigvee_{i\geqslant 0}\mathcal{K}(i))\to\ell\wedge(\bigvee_{i\geqslant 0}\mathcal{K}(i)).\end{aligned}$$

Proposition 3.3.6. *The map Λ of Definition 3.1.4 is a bijection.*

Proof. It is clear that $\sum_{m\geqslant n}X_{n,m}\iota_{m,n}$ at least defines an endomorphism of $\ell\wedge\ell$. From the set up of the spectral sequence we were already limiting ourselves to left-ℓ-module maps so each of the $\iota_{m,n}$s must be. The maps will be invertible for the same reason the matrices are, the coefficients of the identity maps on each of the pieces are units.

Let $m\geqslant n$. Any non-zero $\mathrm{Ext}_B^{u,v}(\mathbb{Z}/p,\mathbb{Z}/p)$ group is isomorphic to $\mathbb{Z}/p$ generated by $c^{\frac{(2p-1)u-v}{2(p-1)}}d^{\frac{v-u}{2(p-1)}}$. If u and v are as in (3.2) and (3.3) then this group is generated by

$$c^{s+(n-m)+\nu_p(n!)-\nu_p(m!)}d^{m-n}.$$

We already have $m\geqslant n$ so all we need for this group to be non-zero is $s\geqslant(m-n)-\nu_p(n!)+\nu_p(m!)$.

Let $m>n$ and look at non-trivial homotopy classes of left-ℓ-module maps of the form

$$\phi_{m,n}:\ell\wedge\mathcal{K}(m)\to\ell\wedge\mathcal{K}(n)$$

which induce the zero map on mod p homology as stipulated in Lemma 3.3.1. These are represented in the spectral sequence as elements in $E_2^{s,s} = E_\infty^{s,s}$ with $s > 0$ as if $s = 0$ then

$$E_\infty^{0,*} = \mathrm{Ext}_{\mathcal{A}}^{0,*}(H^*(-;\mathbb{Z}/p), \mathbb{Z}/p) = \mathrm{Hom}_{\mathcal{A}}^*(H^*(-;\mathbb{Z}/p), \mathbb{Z}/p)$$

which, if non-trivial, means being detected by mod p homology.

We know $s \geqslant (m-n) - \nu_p(n!) + \nu_p(m!)$ so the map $\phi_{m,n}$ is represented in

$$E_\infty^{j+(m-n)-\nu_p(n!)+\nu_p(m!),j+(m-n)-\nu_p(n!)+\nu_p(m!)}$$

for some integer $j \geqslant 0$.

We can see from drawing the spectral sequence that if

$$E_\infty^{(m-n)-\nu_p(n!)+\nu_p(m!),(m-n)-\nu_p(n!)+\nu_p(m!)} = \mathbb{Z}/p\{x\}$$

then

$$E_\infty^{j+(m-n)-\nu_p(n!)+\nu_p(m!),j+(m-n)-\nu_p(n!)+\nu_p(m!)} = \mathbb{Z}/p\{c^j x\}.$$

From the ring structure of the spectral sequence, see Lemma 3.3.4, we see that multiplication by c in the spectral sequence corresponds to multiplication by p on $\pi_0(D(\mathcal{K}(m)) \wedge \mathcal{K}(n) \wedge \ell) \otimes \mathbb{Z}_p \cong \mathbb{Z}_p$, so we get that

$$\phi_{m,n} = \gamma p^j \iota_{m,n}$$

for some p-adic unit γ and integer $j \geqslant 0$.

If $m = n$ then we can use the same methods as above to look at terms of the form $E_2^{s,s} = E_\infty^{s,s}$ for $s > 0$. We cannot ignore the $E_2^{0,0}$ term this time though. However we find that

$$\begin{aligned} E_2^{0,0} &= \mathrm{Ext}_B^{0,0}(H^*(D(\mathcal{K}(m));\mathbb{Z}/p) \otimes H^*(\mathcal{K}(m);\mathbb{Z}/p), \mathbb{Z}/p) \\ &= \mathrm{Hom}_B(H^0(D(\mathcal{K}(m));\mathbb{Z}/p) \otimes H^0(\mathcal{K}(m);\mathbb{Z}/p), \mathbb{Z}/p) \\ &= \mathrm{Hom}_B(\mathbb{Z}/p, \mathbb{Z}/p) \cong \mathbb{Z}/p. \end{aligned}$$

So we similarly get that

$$\phi_{m,m} = \gamma p^j \iota_{m,m}$$

for a unit γ and some $j \geqslant 0$. The map $\phi_{m,m}$ induces the identity on mod p homology if and only if $j = 0$, which gives us p-adic units on the diagonal of our infinite matrices under the map Λ. We can see now that since we have $\phi_{m,n}$ inducing the zero map on homology and $\phi_{m,m}$ inducing the identity map, by Lemma 3.3.1, the resultant map on $\ell \wedge \ell$ once all the pieces have been put together will induce the identity on homology and hence lie in $\mathrm{Aut}^0_{\text{left-}\ell\text{-mod}}(\ell \wedge \ell)$.

We can see that Λ is surjective because once we have picked a generator, $\iota_{m,n}$, for the copy of $\mathbb{Z}/p$ corresponding to $\pi_0(D(\mathcal{K}(m)) \wedge \mathcal{K}(n) \wedge \ell) \otimes \mathbb{Z}_p$, all other elements are just multiples of this generator. We can also see that Λ is injective as the only matrix which maps to the identity on $\ell \wedge \ell$ under Λ will be the identity matrix as this corresponds to a single copy of the identity map on $\ell \wedge \mathcal{K}(m)$ for all m. Hence Λ is a bijection. $\square$

Proposition 3.3.7. *We can choose the maps $\iota_{m,n}$ in such a way that Λ is an isomorphism of groups. As before let $\iota_{m,m}$ be the identity map on $\ell \wedge \mathcal{K}(m)$, let $\iota_{m+1,m}$ be as already described, then let*

$$\iota_{m,n} = \iota_{n+1,n}\iota_{n+2,n+1}\cdots\iota_{m,m-1}$$

for all $m > n+1$. Then we have that

$$\iota_{m,n} \circ \iota_{k,l} = \begin{cases} \iota_{m,n} \circ \iota_{k,m} = \iota_{k,n} & \text{if } k \geqslant l = m \geqslant n, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We look at the relationship between the product $\iota_{m,n} \circ \iota_{k,m}$ and $\iota_{k,n}$. Let $s(m,n) = m - n - \nu_p(n!) + \nu_p(m!)$, then we know $\iota_{m,n}$ is represented by a generator of

$$\mathrm{Ext}_B^{s(m,n),s(m,n)}(\Sigma^{2(n-m)(p-1)+\nu_p(m!)-\nu_p(n!)} I^{\nu_p(n!)-\nu_p(m!)}, \mathbb{Z}/p),$$

$\iota_{k,m}$ is represented by a generator of

$$\mathrm{Ext}_B^{s(k,m),s(k,m)}(\Sigma^{2(m-k)(p-1)+\nu_p(k!)-\nu_p(m!)} I^{\nu_p(m!)-\nu_p(k!)}, \mathbb{Z}/p)$$

and $\iota_{k,n}$ is represented by a generator of

$$\mathrm{Ext}_B^{s(k,n),s(k,n)}(\Sigma^{2(n-k)(p-1)+\nu_p(k!)-\nu_p(n!)} I^{\nu_p(n!)-\nu_p(k!)}, \mathbb{Z}/p).$$

Each of these groups is a copy of $\mathbb{Z}/p$.

The product $\iota_{m,n} \circ \iota_{k,m}$ is represented by the product of the representatives under the pairing of Ext groups

$$\mathrm{Ext}^{s,s}(\Sigma^a I^b, \mathbb{Z}/p) \otimes \mathrm{Ext}^{s',s'}(\Sigma^{a'} I^{b'}, \mathbb{Z}/p) \to \mathrm{Ext}^{s+s',s+s'}(\Sigma^{a+a'} I^{b+b'}, \mathbb{Z}/p)$$

induced by the isomorphism $\Sigma^a I^b \otimes \Sigma^{a'} I^{b'} \cong \Sigma^{a+a'} I^{b+b'}$. We can identify this pairing using the following diagram:

$$\begin{tikzcd}
\mathrm{Ext}^{s,s}(\Sigma^a I^b, \mathbb{Z}/p) \otimes \mathrm{Ext}^{s',s'}(\Sigma^{a'} I^{b'}, \mathbb{Z}/p) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Ext}^{s+s',s+s'}(\Sigma^{a+a'} I^{b+b'}, \mathbb{Z}/p) \\
\mathrm{Ext}^{s+b,s-a}(\mathbb{Z}/p, \mathbb{Z}/p) \otimes \mathrm{Ext}^{s'+b',s'-a'}(\mathbb{Z}/p, \mathbb{Z}/p) \arrow[r] & \mathrm{Ext}^{s+s'+b+b',s+s'-a-a'}(\mathbb{Z}/p, \mathbb{Z}/p) \arrow[u, "\cong"']
\end{tikzcd}$$

The bottom pairing is the Yoneda splicing and is an isomorphism when all the groups are non-zero as any non-zero Ext group here is a copy of $\mathbb{Z}/p$. The vertical isomorphisms are the dimension shifting isomorphisms

$$\mathrm{Ext}_B^{s,t}(I \otimes M, \mathbb{Z}/p) \cong \mathrm{Ext}_B^{s+1,t}(M, \mathbb{Z}/p)$$

and

$$\mathrm{Ext}_B^{s,t}(\Sigma^a M, \mathbb{Z}/p) \cong \mathrm{Ext}_B^{s,t-a}(M, \mathbb{Z}/p)$$

from Lemma 3.2.12. Since $s(k,m) + s(m,n) = s(k,n)$ this diagram commutes in our case and so the top pairing is an isomorphism whenever the groups are non-zero. Hence up to a p-adic unit $u_{k,m,n}$ we have

$$\iota_{m,n} \circ \iota_{k,m} = u_{k,m,n} \iota_{k,n}.$$

Hence we can chose the maps $\iota_{m,n}$ in the way stated above.

Now Λ is a group isomorphism because

$$\begin{aligned}
\Lambda(X)\Lambda(Y) &= \left(\sum_{m \geqslant n} X_{n,m} \iota_{m,n} \right) \left(\sum_{k \geqslant l} Y_{l,k} \iota_{k,l} \right) \\
&= \sum_{k \geqslant l = m \geqslant n} X_{n,m} Y_{l,k} \iota_{m,n} \iota_{k,l} \\
&= \sum_{k \geqslant l \geqslant n} X_{n,l} Y_{l,k} \iota_{k,n} \\
&= \sum_{k \geqslant n} (XY)_{n,k} \iota_{k,n} \\
&= \Lambda(XY).
\end{aligned}$$

□

Hence we have now proved Theorem 3.1.3.

Chapter 4

Stable classes

4.1 Introduction

In order to prove a result in the next chapter and in an attempt not to break up its flow, I have decided to separate the material needed into this chapter.

The main result of this chapter is the following.

Proposition 4.1.1. *The stable isomorphism class of $H^*(\ell;\mathbb{Z}/p)$ as a B-module is*

$$\bigotimes_{i=1}^{\infty}\bigoplus_{j=0}^{p-1}\Sigma^{j(2p^i-2p^{i-1}-\pi_p(i-1))}I^{j(\pi_p(i-1))}$$

where $\pi_p(i)=\frac{p^i-1}{p-1}$.

In order to prove this we will look at the Q_0 and Q_1 homologies of $H^*(\ell;\mathbb{Z}/p)$. We will show that $H^*(\ell;\mathbb{Z}/p)$ decomposes as stable B-modules into a product of sums of smaller submodules such that the individual Q_0 and Q_1 homologies of the submodules are both one dimensional over $\mathbb{Z}/p$. We will then work out which degree this copy of $\mathbb{Z}/p$ is in. In Theorem 3.2.13 we identified the stable class of $H^*(\mathcal{K}(n);\mathbb{Z}/p)$ using results about its Q_0 and Q_1 homologies proved in [Kan81]. Here we will work out the Q_0 and Q_1 homologies of $H_{-*}(\ell;\mathbb{Z}/p)$ explicitly and work out its stable class and then dualise this statement to find the stable class of $H^*(\ell;\mathbb{Z}/p)$. Note here that although we have an infinite tensor product in the statement above, the expression is finite in each degree.

Recall from Proposition 1.2.16 that a right action of the Steenrod algebra $\mathcal{A}_p$ on its dual $\mathcal{A}_p^*$ is given by

$$\begin{aligned}\psi:\mathcal{A}_p^*\otimes\mathcal{A}_p&\to\mathcal{A}_p^*\\ f\otimes a&\mapsto\sum(-1)^{|f''||a|}\langle f',a\rangle f''.\end{aligned}$$

Let a bar over an element denote the image of that element under the anti-automorphism χ of $\mathcal{A}_p^*$. Recall also from Proposition 1.2.18 that the right action above can be obtained from the left action

$$\begin{array}{ccccccc} \mathcal{A}_p \otimes \mathcal{A}_p^* & \xrightarrow{1\otimes\chi} & \mathcal{A}_p \otimes \mathcal{A}_p^* & \xrightarrow{\phi} & \mathcal{A}_p^* & \xrightarrow{\chi} & \mathcal{A}_p^* \\ a \otimes f & \longmapsto & a \otimes \bar{f} & \longmapsto & \sum(-1)^{|\bar{f}''|(|\bar{f}'|+|a|)}\langle \bar{f}'', a\rangle \bar{f}' & \longmapsto & \sum(-1)^{|\overline{f''}||a|}\langle \overline{f'}, a\rangle f'', \end{array}$$

by using the anti-automorphism on $a \in \mathcal{A}_p$.

It is stated in [Kna95] that

$$\pi_*(\ell \wedge H\mathbb{Z}/p) \cong H_*(\ell; \mathbb{Z}/p) \cong \mathbb{Z}/p[\bar{\xi_1}, \bar{\xi_2}, \ldots] \otimes \Lambda(\bar{\tau_2}, \bar{\tau_3}, \ldots).$$

Recall that using the Universal Coefficient Theorem gives us

$$H^*(\ell; \mathbb{Z}/p) \cong \mathrm{Hom}^*_{\mathbb{Z}/p}(H_{-*}(\ell; \mathbb{Z}/p), \mathbb{Z}/p)$$

so we will be working with $\pi_{-*}(\ell \wedge H\mathbb{Z}/p)$ for the rest of this chapter. Our actions of Q_0 and Q_1 will still act by raising degrees by 1 and $2p-1$ respectively rather than lowering degrees. So we will be taking ξ_i to be in degree $2-2p^i$ and τ_i to be in degree $1-2p^i$ in $\pi_{-*}(\ell \wedge H\mathbb{Z}/p)$.

4.2 Calculating Homologies

Proposition 4.2.1. *The Q_0 homology of $\pi_{-*}(\ell \wedge H\mathbb{Z}/p)$ is isomorphic to $\mathbb{Z}/p[\bar{\xi_1}]$.*

Proof. The actions of Q_0 and Q_1 on the generators of $\pi_{-*}(\ell \wedge H\mathbb{Z}/p)$ are given in [Kna95, Equation 1.1] but we can work them out explicitly here. The elements Q_0 and Q_1 are derivations by [Mil58, Section 6] and we can work out the action of each of them on the generators of $\pi_{-*}(\ell \wedge H\mathbb{Z}/p)$ to calculate their homology. Using the description of the action of $\mathcal{A}_p$ on $\mathcal{A}_p^*$ given in Proposition 1.2.18, this will involve working out the effect of the left action of $\bar{Q}_0$ and $\bar{Q}_1$ under ϕ on the conjugates of the generators of $\pi_{-*}(\ell \wedge H\mathbb{Z}/p)$ and then conjugating the result. We can then use this to see the effect of the right action of Q_0 and Q_1 on $\pi_{-*}(\ell \wedge H\mathbb{Z}/p)$ by using χ on $\bar{Q}_0$ and $\bar{Q}_1$. We know the effect of the comultiplication on the conjugates of the generators above (see [Sch94, Theorem 1.10.2]);

$$\Delta\xi_k = \sum_{0\leqslant i\leqslant k} \xi_{k-i}^{p^i} \otimes \xi_i$$

and

$$\Delta\tau_k = \tau_k \otimes 1 + \sum_{0\leqslant i\leqslant k} \xi_{k-i}^{p^i} \otimes \tau_i.$$

Using the facts that Q_k is dual to τ_k and the anti-automorphism has the following effect; $\bar{Q}_k = -Q_k$, we then can conclude that $\langle -\tau_k, \bar{Q}_k\rangle = 1$ and the pairing of any other monomial with $\bar{Q}_k$ gives us zero for $k \in \{0,1\}$. From this it can be shown that the result of the left action of $\bar{Q}_0$ under ϕ on ξ_i is zero and on τ_i is $-\xi_i$. Hence when we include both conjugations and conjugate $\bar{Q}_0$ we have that the result of the right action of Q_0 under ψ on $\bar{\xi}_i$ is zero and on $\bar{\tau}_i$ is $-\bar{\xi}_i$.

We can then see that $\pi_{-*}(\ell \wedge H\mathbb{Z}/p)$ splits as complexes with differential Q_0 in the following way,

$$\pi_{-*}(\ell \wedge H\mathbb{Z}/p) \cong \mathbb{Z}/p[\bar{\xi}_1] \otimes \bigotimes_{j=2}^{\infty} \mathbb{Z}/p[\bar{\xi}_j] \otimes \Lambda(\bar{\tau}_j).$$

We can now use the Künneth formula to find the Q_0 homology of $\pi_{-*}(\ell \wedge H\mathbb{Z}/p)$, i.e.

$$\begin{aligned} H(\pi_{-*}(\ell \wedge H\mathbb{Z}/p); Q_0) &\cong H\left(\mathbb{Z}/p[\bar{\xi}_1] \otimes \bigotimes_{j=2}^{\infty} \mathbb{Z}/p[\bar{\xi}_j] \otimes \Lambda(\bar{\tau}_j); Q_0\right) \\ &\cong H(\mathbb{Z}/p[\bar{\xi}_1]; Q_0) \otimes \bigotimes_{j=2}^{\infty} H(\mathbb{Z}/p[\bar{\xi}_j] \otimes \Lambda(\bar{\tau}_j); Q_0). \end{aligned}$$

Since $H(\mathbb{Z}/p[\bar{\xi}_j] \otimes \Lambda(\bar{\tau}_j); Q_0) = 0$ for all j and $H(\mathbb{Z}/p[\bar{\xi}_1]; Q_0) = \mathbb{Z}/p[\bar{\xi}_1]$ we have

$$H(\pi_{-*}(\ell \wedge H\mathbb{Z}/p); Q_0) \cong \mathbb{Z}/p[\bar{\xi}_1]$$

as required. $\square$

Proposition 4.2.2. *The Q_1 homology of $\pi_{-*}(\ell \wedge H\mathbb{Z}/p)$ is isomorphic to* $\dfrac{\mathbb{Z}/p[\bar{\xi}_1, \bar{\xi}_2, \ldots]}{(\bar{\xi}_1^p, \bar{\xi}_2^p, \ldots)}$.

Proof. Using results stated in the previous proof we can see that the result of the left action of $\bar{Q}_1$ under ϕ on ξ_i is zero and on τ_i is $-\xi_{i-1}^p$. So the right action of Q_1 under ψ on $\bar{\xi}_i$ is zero and on $\bar{\tau}_i$ is $-\bar{\xi}_{i-1}^p$. Powers of any $\bar{\xi}_i$ which are less than p will not be in the image of Q_1, they also get sent to zero by Q_1 and so all such monomials appear as non-trivial homology classes.

In a similar way to the previous proof $\pi_{-*}(\ell \wedge \mathbb{Z}/p)$ splits as complexes with differential Q_1 in the following way,

$$\pi_{-*}(\ell \wedge \mathbb{Z}/p) \cong \bigotimes_{j=2}^{\infty} \mathbb{Z}/p[\bar{\xi}_{j-1}] \otimes \Lambda(\bar{\tau}_j).$$

Now we can use the Künneth formula to find the Q_1 homology of $\pi_{-*}(\ell \wedge \mathbb{Z}/p)$

i.e.

$$H(\pi_{-*}(\ell \wedge \mathbb{Z}/p); Q_1) \cong H\left(\bigotimes_{j=2}^{\infty} \mathbb{Z}/p[\bar{\xi}_{j-1}] \otimes \Lambda(\bar{\tau}_j); Q_1\right)$$
$$\cong \bigotimes_{j=2}^{\infty} H(\mathbb{Z}/p[\bar{\xi}_{j-1}] \otimes \Lambda(\bar{\tau}_j); Q_1).$$

We can see that $H(\mathbb{Z}/p[\bar{\xi}_{j-1}] \otimes \Lambda(\bar{\tau}_j); Q_1) = \frac{\mathbb{Z}/p[\bar{\xi}_{j-1}]}{(\bar{\xi}_{j-1}^p)}$ and so we get

$$H(\pi_{-*}(\ell \wedge \mathbb{Z}/p); Q_1) \cong \bigotimes_{j=2}^{\infty} \frac{\mathbb{Z}/p[\bar{\xi}_{j-1}]}{(\bar{\xi}_{j-1}^p)} \cong \frac{\mathbb{Z}/p[\bar{\xi}_1, \bar{\xi}_2, \ldots]}{(\bar{\xi}_1^p, \bar{\xi}_2^p, \ldots)}.$$

□

4.3 Lightning Flash Modules

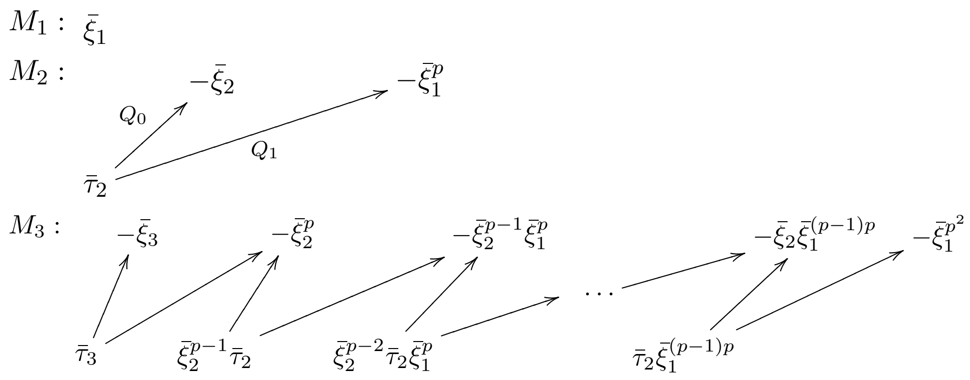
Before we can prove Proposition 4.1.1 we need to introduce one further element first given by Adams.

Definition 4.3.1. For $i \geq 1$, let M_i be a finite-dimensional submodule of $\pi_{-*}(\ell \wedge H\mathbb{Z}/p)$ such that

(i) $H(M_i; Q_0) \cong \mathbb{Z}/p$ generated by $\bar{\xi}_1^{p^{i-1}}$ and

(ii) $H(M_i; Q_1) \cong \mathbb{Z}/p$ generated by $\bar{\xi}_i$.

These are constructed following the method of [Ada95, Part III, Proposition 16.4]. We can view these submodules as so called 'lightning flash' modules, the first three of which are shown below. These diagrams are to be interpreted in the same way as those already introduced in section 2.3; an element stands for a copy of $\mathbb{Z}/p$ generated by that element and an arrow indicates a non-trivial action of either Q_0 or Q_1 (the more vertical of the arrows correspond to the action of Q_0 and the more horizontal of the arrows to the action of Q_1).



Using Theorem 3.2.8 we can work out the following.

Lemma 4.3.2. *The stable classes of the modules M_i are*

$$M_i \approxeq \Sigma^{\frac{-(2p^2-4p+1)p^{i-1}-1}{p-1}} I^{\frac{1-p^{i-1}}{p-1}}.$$

Lemma 4.3.3. *Let $M_i^k = M_i \otimes \cdots \otimes M_i$ be the tensor product of k copies of M_i. Then $H_*(M_i^k; Q_0) \cong \mathbb{Z}/p$ in degree $(2-2p)kp^{i-1}$ generated by $\bar{\xi}_1^{kp^{i-1}}$ and $H_*(M_i^k; Q_1) \cong \mathbb{Z}/p$ in degree $(2-2p^i)k$ generated by $\bar{\xi}_i^k$.*

Proof. The B-submodules M_i are all chain complexes of $\mathbb{Z}/p$-vector spaces, taking Q_0 as the differential (or a sum of chain complexes with differential Q_1 as explained in Proposition 4.2.2). So by the Künneth theorem

$$H_*(M_i^2; Q_0) \cong H_*(M_i; Q_0) \otimes H_*(M_i; Q_0)$$

and similarly for higher powers and for Q_1, the rest follows. $\square$

Proof of Proposition 4.1.1. We can put these submodules together in the following way

$$\bigotimes_{i=1}^{\infty} \bigoplus_{j=0}^{p-1} M_i^j = (1 + M_1 + M_1^2 + \cdots + M_1^{p-1})(1 + M_2 + M_2^2 + \cdots + M_2^{p-1}) \ldots$$

such that we have a bijection between the generators of the homology of $\otimes_{i=1}^{\infty} \oplus_{j=0}^{p-1} M_i^j$ and the homology of $\pi_{-*}(\ell \wedge H\mathbb{Z}/p)$ with respect to both Q_0 and Q_1.

For the Q_0 homology we need to show that every generator of the homology of $\pi_{-*}(\ell \wedge H\mathbb{Z}/p)$ can be expressed as a generator of the homology of $\otimes_{i=1}^{\infty} \oplus_{j=0}^{p-1} M_i^j$ and vice versa. The generators of the Q_0 homology of $\pi_{-*}(\ell \wedge H\mathbb{Z}/p)$ are all possible powers $\bar{\xi}_1^j$. The generators of the Q_0 homology of $\otimes_{i=1}^{\infty} \oplus_{j=0}^{p-1} M_i^j$ are all possible products $\prod_{k=0}^{\infty} \bar{\xi}_1^{\alpha_k p^k}$ for $\alpha_k \in \{0, \ldots, p-1\}$. Or alternatively, as formal power series, we can express this claim as

$$\prod_{i=1}^{\infty} (1 + \bar{\xi}_1^{p^i} + \bar{\xi}_1^{2p^i} + \cdots + \bar{\xi}_1^{(p-1)p^i}) = \sum_{j=0}^{\infty} \bar{\xi}_1^j.$$

There is a bijection between these two sets of generators because given any $j \geqslant 0$ we can use its p-adic expansion to express it uniquely as

$$j = \sum_{k=0}^{\infty} \alpha_k p^k$$

where $\alpha_k \in \{0, \ldots, p-1\}$ so the term $\bar{\xi}_1^j$ appears exactly once in the product as

$$\bar{\xi}_1^j = \bar{\xi}_1^{\sum_{k=0}^{\infty} \alpha_k p^k} = \prod_{k=0}^{\infty} \bar{\xi}_1^{\alpha_k p^k}.$$

For the Q_1 homology, the generators of the homology of $\pi_{-*}(\ell \wedge H\mathbb{Z}/p)$ are all products of the form $\bar{\xi}_1^{l_1}\bar{\xi}_2^{l_2}\dots\bar{\xi}_r^{l_r}$ where $l_s \in \{0, \cdots, p-1\}$ and $r \geqslant 0$. The generators of the Q_1 homology of $\otimes_{i=1}^{\infty} \oplus_{j=0}^{p-1} M_i^j$ are again products of this form. It is fairly clear to see that each term $\bar{\xi}_1^{l_1}\bar{\xi}_2^{l_2}\dots\bar{\xi}_r^{l_r}$ in $\pi_{-*}(\ell \wedge H\mathbb{Z}/p)$ appears exactly once in the product

$$\prod_{i=1}^{\infty}(1 + \bar{\xi}_i + \bar{\xi}_i^2 + \cdots + \bar{\xi}_i^{p-1}).$$

Because $\pi_{-*}(\ell \wedge H\mathbb{Z}/p)$ and all the M_is are B-modules, the map

$$(1+M_1+M_1^2+\cdots+M_1^{p-1})(1+M_2+M_2^2+\cdots+M_2^{p-1})\dots \to \pi_{-*}(\ell\wedge H\mathbb{Z}/p)$$

induces an isomorphism on Q_0 and Q_1 homology. Hence the two sides are stably isomorphic by [Ada95, Part III, Lemma 16.7]. We can now dualise this isomorphism and get that

$$H\mathbb{Z}/p^*(\ell) \approxeq (1+M_1^*+M_1^{*2}+\cdots+M_1^{*p-1})(1+M_2^*+M_2^{*2}+\cdots+M_2^{*p-1})\dots .$$

Recall from Remark 3.2.11 that for any B-module, its linear dual is its inverse stable isomorphism class. It then follows that

$$M_i^* \approxeq \Sigma^{\frac{(2p^2-4p+1)p^{i-1}+1}{p-1}} I^{\frac{p^{i-1}-1}{p-1}}$$

by Lemma 4.3.2 which gives the result. $\square$

Chapter 5

A Basis for $\frac{\pi_*(\ell\wedge\ell)}{\text{Torsion}}$

In this chapter we find a basis for the torsion free part of the homotopy groups $\pi_*(\ell \wedge \ell)$. To do this we follow methods introduced by Adams in [Ada95]. We then study some of the properties of this basis including how it relates to Kane's splitting and what order of p-torsion is present in $\pi_*(\ell \wedge \ell)$. In the last section we then explore its behaviour with relation to the Adams spectral sequence with the intention of assessing what effect the maps $(\iota_{m,n})_*$ of Definition 3.1.4 have on the individual homotopy groups. This will then allow us to compare this with the effect of $(1 \wedge \psi^q)_*$ in Chapter 6 and hence work out the matrix corresponding to $1 \wedge \psi^q$ under the isomorphism given in Definition 3.1.4.

It might have been interesting, given more time, to compare the basis that we find here with elements studied in [BR08, §9,10] of the torsion free part of $\ell_*\ell$. These are derived by different methods to the ones used here and are used to study the multiplicative structure of $\ell_*\ell$.

5.1 Finding a Basis

In [Ada95, Part III, Chapter 17] Adams studies the torsion free part of $\pi_*(ku \wedge ku)$ by looking at its image in $\pi_*(ku \wedge ku) \otimes \mathbb{Q} = \mathbb{Q}[u, v]$ where $u \in \pi_2(ku)$ and $v \in \pi_2(ku)$ are the generators for the two copies of ku. Here we carry out the analogous process for $\pi_*(\ell \wedge \ell)$. The main result is Theorem 5.1.10, which gives us a $\mathbb{Z}_p$-basis for the torsion free part of $\pi_*(\ell \wedge \ell)$. Returning to Adams' method, we are looking p-adically so we need to consider the image of $\pi_*(ku \wedge ku)$ in $\pi_*(ku \wedge ku) \otimes \mathbb{Q}_p = \mathbb{Q}_p[u, v]$. The following theorem is a p-adic version of [Ada95, Part III, Theorem 17.5]:

Theorem 5.1.1. *For $f(u, v) \in \mathbb{Q}_p[u, v]$ to be in the image of $\pi_*(ku \wedge ku)$ it is necessary and sufficient for f to satisfy the following two conditions.*

(i) $f(kt, lt) \in \mathbb{Z}_p[t]$ for all k, $l \in \mathbb{Z}_p^\times$.

(ii) $f(u,v)$ is in the subring additively generated by

$$\frac{u^i}{m(i)}\frac{v^j}{m(j)}$$

for $i,j \geqslant 0$ where $m(i) = p^{\left\lfloor \frac{i}{p-1} \right\rfloor}$ and similarly for j.

Proposition 5.1.2. *The subring required for the second condition above is*

$$\mathbb{Z}_p\left\{\frac{u^i}{p^{\left\lfloor \frac{i}{p-1} \right\rfloor}}.\frac{v^j}{p^{\left\lfloor \frac{j}{p-1} \right\rfloor}} : i,j \geqslant 0\right\} = \mathbb{Z}_p\left[u, v, \frac{u^{p-1}}{p}, \frac{v^{p-1}}{p}\right].$$

Proof. Each side is symmetric in u and v so we only need to look at one of these. It is clear we have the inclusion '$\subseteq$' as any monomial in u on the LHS can be made from generators on the RHS, i.e.

$$\frac{u^i}{p^{\left\lfloor \frac{i}{p-1} \right\rfloor}} = \frac{u^{k(p-1)+l}}{p^{\left\lfloor \frac{k(p-1)+l}{p-1} \right\rfloor}} = \left(\frac{u^{p-1}}{p}\right)^k u^l$$

for $k \in \mathbb{N}_0$, $0 \leqslant l \leqslant p-2$.

To show the opposite inclusion '$\supseteq$' we look at any monomial in u on the RHS which will be of the form

$$u^a\left(\frac{u^{p-1}}{p}\right)^b = \frac{u^{b(p-1)+a}}{p^b}$$

for $a, b \in \mathbb{N}_0$. If $0 \leqslant a \leqslant p-2$ this is obviously included in the LHS. If $a \geqslant p-1$ we can express a in the form $c(p-1)+d$, where $0 \leqslant d \leqslant p-2$, then

$$\frac{u^{b(p-1)+a}}{p^b} = \frac{u^{(b+c)(p-1)+d}}{p^b} = p^c\frac{u^{(b+c)(p-1)+d}}{p^{b+c}}$$

which is still included in the LHS. Hence we have the necessary equality. $\square$

We want an analogue of Theorem 5.1.1 for $\pi_*(\ell \wedge \ell)$. We know the following fact.

$$\pi_*(\ell \wedge \ell) \otimes \mathbb{Q}_p = \pi_*(ku \wedge ku) \cap \mathbb{Q}_p[u^{p-1}, v^{p-1}] \subseteq \mathbb{Q}_p[u,v]. \tag{5.1}$$

Hence we only need to consider powers of u and v of the form $u^{(p-1)k}$ and $v^{(p-1)l}$ for $k, l \in \mathbb{N}_0$. The analogue of Theorem 5.1.1 for $\pi_*(\ell \wedge \ell)$ is as follows.

Corollary 5.1.3. *For $f(u,v) \in \mathbb{Q}_p[u,v]$ to be in the image of $\pi_*(\ell \wedge \ell)$ it is necessary and sufficient for f to satisfy the following two conditions.*

(i) $f(kt, lt) \in \mathbb{Z}_p[t]$ for all k, $l \in \mathbb{Z}_p^\times$.

(ii) $f(u,v)$ is in the subring $\mathbb{Z}_p\left[\frac{u^{p-1}}{p}, \frac{v^{p-1}}{p}\right]$.

Proof. Using (5.1), the subring required for the second condition becomes

$$\mathbb{Z}_p\left[u, v, \frac{u^{p-1}}{p}, \frac{v^{p-1}}{p}\right] \cap \mathbb{Q}_p\left[u^{p-1}, v^{p-1}\right] = \mathbb{Z}_p\left[\frac{u^{p-1}}{p}, \frac{v^{p-1}}{p}\right]. \quad \square$$

A basis is given in [CCW01, Proposition 3] for $\pi_0(K \wedge ku) \otimes \mathbb{Z}_{(p)}$ consisting of $\{h_k(w) : k \geqslant 0\}$ where $w = u^{-1}v$ and

$$h_k(w) = \prod_{i=1}^{k} \frac{w - q^{i-1}}{q^k - q^{i-1}}.$$

Here $r = q^{p-1}$ for q an integer coprime to p which is a topological generator of $\mathbb{Z}_p^\times$ as explained in Proposition A.1. Now consider the elements

$$u^k h_k(w) = \prod_{i=1}^{k} \frac{v - q^{i-1}u}{q^k - q^{i-1}},$$

where we split up the factors of u and v and get rid of any negative powers of u. These are then elements of $\pi_*(ku \wedge ku) \otimes \mathbb{Z}_{(p)}$. We can multiply by u in this way because $\pi_*(ku \wedge X)$ is a left $\pi_*(ku)$-module for any spectrum X. Because we are looking for a basis for $\frac{\pi_*(\ell \wedge \ell)}{\text{Torsion}}$ rather than $\frac{\pi_*(ku \wedge ku)}{\text{Torsion}}$ we only want to consider $(p-1)$st powers of both u and v and of q. Hence the polynomials which we start with are given below.

Notation 5.1.4. I will now let $\hat{u} = u^{p-1}$, $\hat{v} = v^{p-1}$ and $\rho = 2(p-1)$ in order to simplify the algebra in the next two chapters.

Definition 5.1.5.

$$c_{\rho k} = \prod_{i=1}^{k} \frac{\hat{v} - r^{i-1}\hat{u}}{r^k - r^{i-1}} \in \mathbb{Q}_p[\hat{u}, \hat{v}]$$

where $r = q^{p-1}$ for q as above and $k \in \mathbb{N}$. Also let $c_0 = 1$.

Following the method of [BS05] we want to create out of these, elements which lie in the subring given in Corollary 5.1.3 (ii). This involves taking the elements $c_{\rho k}$ and multiplying them by exactly the right power of p so they lie in $\mathbb{Z}_p[\frac{\hat{u}}{p}, \frac{\hat{v}}{p}]$. This process brings us to the following polynomials.

Definition 5.1.6. Define

$$f_{\rho k} = p^{\frac{pk - S_p(k)}{p-1} - k} c_{\rho k} = p^{\nu_p(k!)} \prod_{i=1}^{k} \frac{\hat{v} - r^{i-1}\hat{u}}{r^k - r^{i-1}}$$

where $S_p(k)$ is the sum of the digits in the p-adic expansion of k and $\nu_p(k!) = \frac{k - S_p(k)}{p-1}$ as shown in Proposition A.4.

Proposition 5.1.7. *The elements $f_{\rho k}$ lie in $\mathbb{Z}_p[\frac{\hat{u}}{p}, \frac{\hat{v}}{p}]$ for all $k \in \mathbb{N}_0$.*

Proof. By Propositions A.3 and A.4 we know

$$\nu_p\left(\prod_{i=1}^{k}(r^k - r^{i-1})\right) = \nu_p(k!) + k$$

is the p-adic valuation of the denominator of $c_{\rho k}$. We also know that $f_{\rho k}$ can have at most k factors of p in the denominator to lie in the ring $\mathbb{Z}_p[\frac{\hat{u}}{p}, \frac{\hat{v}}{p}]$ since there are k factors. Hence the least power of p we needed to multiply $c_{\rho k}$ by is $(\nu_p(k!) + k) - k = \nu_p(k!)$. $\square$

We then produce elements of $\mathbb{Z}_p[\frac{\hat{u}}{p}, \frac{\hat{v}}{p}]$ which satisfy both conditions of Corollary 5.1.3 out of these $f_{\rho k}$s by following the method of [Ada95, Part III, Proposition 17.6]. He recommends taking each element $f_{\rho k}$, multiplying it by $\hat{u}^i$ for all non-negative values of i and then dividing by the largest power of p which will leave the resultant element satisfying both conditions of Corollary 5.1.3. There comes a stage for each k where past this you cannot divide by any more powers of p. The full list of elements we obtain is detailed in the following definition.

Definition 5.1.8.

$$F_{i,j,k} := \hat{u}^i \left(\frac{\hat{u}}{p}\right)^j f_{\rho k}$$

where $k \geqslant 0$, $0 \leqslant j \leqslant \nu_p(k!)$ and $i = 0$ if $j < \nu_p(k!)$ or $i \geqslant 0$ if $j = \nu_p(k!)$.

We know these elements lie in $\mathbb{Z}_p[\frac{\hat{u}}{p}, \frac{\hat{v}}{p}]$ and hence satisfy Theorem 5.1.1 condition (ii) so we now check that they satisfy Theorem 5.1.1 condition (i).

Proposition 5.1.9. *Let*

$$f(u, v) = \left(\frac{\hat{u}}{p}\right)^{\nu_p(k!)} f_{\rho k} \quad \text{and} \quad g(u, v) = \left(\frac{\hat{u}}{p}\right)^{\nu_p(k!)+1} f_{\rho k}.$$

Then $f(lt, mt) \in \mathbb{Z}_p[t, t^{-1}]$ for $l, m \in \mathbb{Z}_p^\times$ but there exists some $l, m \in \mathbb{Z}_p^\times$ such that $g(lt, mt) \notin \mathbb{Z}_p[t, t^{-1}]$.

Proof. Now

$$\begin{aligned} f(u, v) &= \left(\frac{\hat{u}}{p}\right)^{\nu_p(k!)} p^{\nu_p(k!)} \prod_{i=1}^{k} \frac{\hat{v} - r^{i-1}\hat{u}}{r^k - r^{i-1}} \\ &= \hat{u}^{\nu_p(k!)} \prod_{i=1}^{k} \frac{\hat{v} - r^{i-1}\hat{u}}{r^k - r^{i-1}}. \end{aligned}$$

So

$$
\begin{aligned}
f(lt, mt) &= (lt)^{\nu_p(k!)(p-1)} \prod_{i=1}^{k} \frac{(mt)^{p-1} - r^{i-1}(lt)^{p-1}}{r^k - r^{i-1}} \\
&= (lt)^{\nu_p(k!)(p-1)} \prod_{i=1}^{k} \frac{t^{p-1}(m^{p-1} - r^{i-1}l^{p-1})}{r^k - r^{i-1}},
\end{aligned}
$$

which we need to lie in $\mathbb{Z}_p[t, t^{-1}]$ for l, $m \in \mathbb{Z}_p^\times$. We know l has no factors of p so we can ignore the term $l^{\nu_p(k!)(p-1)}$ as this lies in $\mathbb{Z}_p^\times$. Turning our attention to the other factor we need

$$
\prod_{i=1}^{k} \frac{m^{p-1} - r^{i-1}l^{p-1}}{r^k - r^{i-1}} \in \mathbb{Z}_p.
$$

In other words we need the p-adic valuation of the numerator to be greater than or equal to the p-adic valuation of the denominator which is true by Propositions A.5 and A.3. Hence each element in the list above satisfies both the conditions of Corollary 5.1.3. In order to show $g(lt, mt) \notin \mathbb{Z}_p[t, t^{-1}]$ for some l, $m \in \mathbb{Z}_p^\times$ take $l = 1$ and $m = q^k$. Then

$$
\begin{aligned}
g(t, q^k t) &= \left(\frac{t^{p-1}}{p}\right)^{\nu_p(k!)+1} p^{\nu_p(k!)} \prod_{i=1}^{k} \frac{r^k t^{p-1} - r^{i-1}t^{p-1}}{r^k - r^{i-1}} \\
&= \frac{(t^{p-1})^{\nu_p(k!)+1}}{p} t^{k(p-1)} \notin \mathbb{Z}_p[t, t^{-1}]. \qquad \square
\end{aligned}
$$

By [Ada95, Part III, Chapter 17] we have a monomorphism

$$
\frac{\pi_*(\ell \wedge \ell)}{\text{Torsion}} \to \pi_*(\ell \wedge \ell) \otimes \mathbb{Q}_p \cong \mathbb{Q}_p[\frac{\hat{u}}{p}, \frac{\hat{v}}{p}].
$$

By Corollary 5.1.3 and the proceeding work we know that the list of elements in Definition 5.1.8 all lie in the image of $\frac{\pi_*(\ell\wedge\ell)}{\text{Torsion}}$ inside $\mathbb{Q}_p[\frac{\hat{u}}{p}, \frac{\hat{v}}{p}]$. It just remains to show that these form a basis. This is done using the following analogue of [Ada95, Part III, Proposition 17.6].

Theorem 5.1.10. *(a) The intersection of the subring satisfying condition (i) of Corollary 5.1.3 with* $\mathbb{Q}_p[\hat{u}, \hat{v}]$ *is free on the* $\mathbb{Z}_p[\hat{u}]$*-basis* $c_{\rho k}$ *for* $k \geqslant 0$.

(b) The following polynomials are a $\mathbb{Z}_p$*-basis for* $\frac{\pi_*(\ell\wedge\ell)}{\text{Torsion}}$

$$
F_{i,j,k} = \hat{u}^i \left(\frac{\hat{u}}{p}\right)^j f_{\rho k}
$$

where $k \geqslant 0$, $0 \leqslant j \leqslant \nu_p(k!)$ *and* $i = 0$ *if* $j < \nu_p(k!)$ *or* $i \geqslant 0$ *if* $j = \nu_p(k!)$.

Proof. To prove part (a) we should first notice that the elements $c_{\rho k}$ do satisfy condition (i) of Corollary 5.1.3 by Proposition 5.1.9 and so do $\mathbb{Z}_p[\hat{u}]$-linear combinations of them. It is clear that they are linearly independent. We now need to show that given any polynomial $f(u,v) \in \mathbb{Q}_p[\hat{u},\hat{v}]$ satisfying condition (i) of Corollary 5.1.3 we can write this as a $\mathbb{Z}_p[\hat{u}]$-linear combination of the $c_{\rho k}$s. We can separate homogeneous components and just consider f homogeneous of degree ρn. We can write f as

$$f(u,v) = \lambda_0 \hat{u}^n + \lambda_1 \hat{u}^{n-1} c_\rho + \lambda_2 \hat{u}^{n-2} c_{2\rho} + \cdots .$$

Assume as an inductive hypothesis that $\lambda_0, \lambda_1, \ldots, \lambda_{s-1}$ lie in $\mathbb{Z}_p$. Let the sum of the remaining terms be

$$g(u,v) = \lambda_s \hat{u}^{n-s} c_{\rho s} + \lambda_{s+1} \hat{u}^{n-s-1} c_{\rho(s+1)} + \cdots .$$

This sum must also satisfy condition (i) of Corollary 5.1.3. To determine λ_s let $u = t$ and $v = q^s t$, then

$$g(t, q^s t) = \lambda_s t^{(p-1)n}$$

and hence $\lambda_s \in \mathbb{Z}_p$. The initial case for λ_0 works in the same way and this completes the induction.

To prove part (b) we need another piece of notation; let

$$n_k := \text{ numerator of } c_{\rho k} = \prod_{i=1}^{k} (\hat{v} - r^{i-1}\hat{u}).$$

In any given degree ρk there are $k+1$ $\mathbb{Q}_p$-basis elements

$$n_k, \hat{u} n_{k-1}, \hat{u}^2 n_{k-2}, \ldots, \hat{u}^k.$$

In order to produce the elements $F_{i,j,k}$ we divided each of the above elements by the highest power of p which would leave it satisfying both conditions (i) and (ii) of Corollary 5.1.3. For an element $\hat{u}^i n_s$ the power of p leaving it satisfying the respective conditions is

(i) $p^{s+\nu_p(s!)}$,

(ii) p^{s+i}.

Hence the power of p we divided by in this case was $\min\{p^{s+\nu_p(s!)}, p^{s+i}\}$. Now going back to our elements $F_{i,j,k}$, we have shown that $F_{i,j,k}$ for the given range satisfy conditions (i) and (ii) of Corollary 5.1.3 and so do $\mathbb{Z}_p$-linear combinations of them. Now consider a general element $f(u,v) \in \mathbb{Q}_p[\hat{u},\hat{v}]$, homogeneous of degree ρk which satisfies conditions (i) and (ii) of Corollary 5.1.3. We can write f as

$$f(u,v) = \frac{\lambda_0}{p^{a_0}} \hat{u}^k + \frac{\lambda_1}{p^{a_1}} \hat{u}^{(k-1)} n_1 + \frac{\lambda_2}{p^{a_2}} \hat{u}^{(k-2)} n_2 + \cdots$$

where $\lambda_i \in \mathbb{Z}_p$ for $i \geqslant 0$. By part (a) we know that

$$a_s \leqslant \nu_p(\text{denominator of } c_{\rho s}) = s + \nu_p(s!)$$

by Proposition A.3. We also need to show that $a_s \leqslant (k-s)+s = k$ and then we have expressed f as a $\mathbb{Z}_p$-linear combination of the elements $F_{i,j,k}$. Let the inductive hypothesis for a downwards induction be that $a_{s'} \leqslant k$ for $s' > s$. Let the sum of the remaining terms be

$$g(u,v) = \frac{\lambda_0}{p^{a_0}}\hat{u} + \cdots + \frac{\lambda_s}{p^{a_s}}\hat{u}^{k-s}n_s,$$

which must also satisfy conditions (i) and (ii) of Corollary 5.1.3. The top coefficient $\frac{\lambda_s}{p^{a_s}}$ is the coefficient of $\hat{u}^{k-s}\hat{v}^s$ so because g satisfies condition (ii) of Corollary 5.1.3 we must have that $a_s \leqslant (k-s)+s = k$. The first step of the induction works in the same way on the top coefficient of f and the induction is complete. $\square$

5.2 Properties of the Basis

In this section we consider how the basis we have found above relates to Kane's splitting of $\ell \wedge \ell$ as given in Theorem 2.1.1.

Definition 5.2.1. Let

$$G_{m,n} = \frac{\pi_m(\ell \wedge \mathcal{K}(n))}{\text{Torsion}}.$$

Then we have

$$G_{*,*} = \bigoplus_{m,n} G_{m,n} \cong \frac{\pi_*(\ell \wedge \ell)}{\text{Torsion}}.$$

Proposition 5.2.2. *For each $n \geqslant 0$*

$$G_{m,n} = \begin{cases} \mathbb{Z}_p & \text{if } m \text{ is a multiple of } \rho \text{ and } m \geqslant \rho n \\ 0 & \text{otherwise.} \end{cases}$$

Proof. It is given in [Kan81, Proposition 9:2] that $\frac{\pi_*(\ell \wedge K(n))}{\text{Torsion}}$ is a $\mathbb{Z}_p[\hat{u}]$-module with generators

$$\{l_0, l_1, \ldots, l_\nu : \nu = \nu_p(n!), |l_j| = \rho j\}$$

and relations $\{pl_{j+1} = \hat{u}l_j : 0 \leqslant j < \nu_p(n!)\}$. If we take the relations into account, then, as a $\mathbb{Z}_p$-module,

$$\frac{\pi_*(\ell \wedge K(n))}{\text{Torsion}} \cong \mathbb{Z}_p\{l_0, l_1, \ldots, l_{\nu-1}, \hat{u}^i l_\nu : \nu = \nu_p(n!), i \geqslant 0\}.$$

This means there is a copy of $\mathbb{Z}_p$ in every degree of the form ρj for $j \geqslant 0$. Recall that

$$\mathcal{K}(n) = \Sigma^{\rho n} K(n),$$

so when we look at $G_{*,n}$ we find that there is a copy of $\mathbb{Z}_p$ in every degree of the form ρk for $k \geqslant n$. $\square$

Definition 5.2.3. For $m \geqslant l$, define the elements $g_{\rho m, \rho l} \in \mathbb{Z}_p[\frac{\hat{u}}{p}, \frac{\hat{v}}{p}]$ to be the element produced from $f_{\rho l}$ lying in degree ρm, i.e.

$$g_{\rho m, \rho l} = \begin{cases} F_{0, m-l, l} & \text{if } m \leqslant \nu_p(l!) + l, \\ F_{m-l-\nu_p(l!), \nu_p(l!), l} & \text{if } m > \nu_p(l!) + l. \end{cases}$$

Lemma 5.2.4. *The elements $\{g_{\rho m, \rho l} : 0 \leqslant l \leqslant m\}$ form a basis for $G_{\rho m, *}$.*

Proof. The elements $\{g_{\rho m, \rho l} : 0 \leqslant l \leqslant m\}$ are precisely all of the basis elements $F_{i,j,k}$ which lie in homotopy degree ρm. $\square$

I will now give an algebraic lemma which will be needed in the next section concerning how to express a power of $\frac{\hat{u}}{p}$ times a particular $g_{\rho m, \rho l}$ in term of our basis in Theorem 5.1.10.

Lemma 5.2.5. *For $0 \leqslant i \leqslant m-1$,*

$$\left(\frac{\hat{u}}{p}\right)^{\nu_p(m!)} g_{\rho m, \rho i} = \begin{cases} \frac{1}{p^{\nu_p(m!)+m-\nu_p(i!)-i}} \hat{u}^{\nu_p(m!)-\nu_p(i!)+m-i} \left(\frac{\hat{u}}{p}\right)^{\nu_p(i!)} f_{\rho i} & \text{if } m \leqslant \nu_p(i!) + i, \\ \frac{1}{p^{\nu_p(m!)}} \hat{u}^{\nu_p(m!)-\nu_p(i!)+m-i} \left(\frac{\hat{u}}{p}\right)^{\nu_p(i!)} f_{\rho i} & \text{if } m > \nu_p(i!) + i. \end{cases}$$

Proof. Using Definition 5.2.3

$$\begin{aligned} \left(\frac{\hat{u}}{p}\right)^{\nu_p(m!)} g_{\rho m, \rho i} &= \begin{cases} \left(\frac{\hat{u}}{p}\right)^{\nu_p(m!)+m-i} f_{\rho i} & \text{if } m \leqslant \nu_p(i!) + i, \\ \hat{u}^{m-i-\nu_p(i!)} \left(\frac{\hat{u}}{p}\right)^{\nu_p(i!)+\nu_p(m!)} f_{\rho i} & \text{if } m > \nu_p(i!) + i, \end{cases} \\ &= \begin{cases} \frac{\hat{u}^{\nu_p(m!)+m-i}}{p^{\nu_p(m!)+m-i}} f_{\rho i} & \text{if } m \leqslant \nu_p(i!) + i, \\ \frac{\hat{u}^{\nu_p(m!)+m-i}}{p^{\nu_p(m!)+\nu_p(i!)}} f_{\rho i} & \text{if } m > \nu_p(i!) + i. \end{cases} \end{aligned}$$

These can now be expressed as some power of p times a basis element (i.e. a power of $\hat{u}$ times a power of $\frac{\hat{u}}{p}$ times an element $f_{\rho i}$). In the case where $m > \nu_p(i!) + i$,

$$\frac{\hat{u}^{\nu_p(m!)+m-i}}{p^{\nu_p(m!)+\nu_p(i!)}} f_{\rho i} = \frac{1}{p^{\nu_p(m!)}} \hat{u}^{\nu_p(m!)-\nu_p(i!)+m-i} \left(\frac{\hat{u}}{p}\right)^{\nu_p(i!)} f_{\rho i}.$$

We can do this so long as the power of u we take out and put in the $\frac{\hat{u}}{p}$ factor is not more than the original power of u we had. So we need

$$\nu_p(i!)(p-1) \leqslant (\nu_p(m!) + m - i)(p-1),$$

i.e.

$$\nu_p(i!) + i \leq \nu_p(m!) + m$$

which is trivially true.

In the case $m \leqslant \nu_p(i!) + i$ we have

$$\frac{\hat{u}^{\nu_p(m!)+m-i}}{p^{\nu_p(m!)+m-i}} f_{\rho i} = \frac{1}{p^{\nu_p(m!)+m-\nu_p(i!)-i}} \hat{u}^{\nu_p(m!)-\nu_p(i!)+m-i} \left(\frac{\hat{u}}{p}\right)^{\nu_p(i!)} f_{\rho i}.$$

For this to be true we again only need

$$\nu_p(i!) + i \leq \nu_p(m!) + m. \qquad \square$$

Lemma 5.2.6. *Let $\tilde{G}_{m,n} = \pi_m(\ell \wedge \mathcal{K}(n))$, then $\tilde{G}_{m,n} \cong G_{m,n} \oplus W_{m,n}$ where $W_{m,n}$ is a finite elementary abelian p-group, i.e. $\tilde{G}_{m,n}$ contains no torsion of order larger than p.*

Proof. This is proved for the case $p = 2$ in [Ada95, Part III, Chapter 17], the odd primary analogue is similar. We require two conditions in order to apply the two results of Adams necessary to prove this. Firstly that $H_r(\ell \wedge \ell; \mathbb{Z})$ is finitely generated for each r which is true (see [Ada95, p.353]) and secondly that, as a B-module, $H^*(\ell \wedge \ell; \mathbb{Z}/p)$ is stably isomorphic to $\oplus_i \Sigma^{a(i,p)} I^{b(i,p)}$ where $b(i,p) \geqslant 0$ and $a(i,p) + b(i,p) \equiv 0 \mod 2$.

We can now prove the second of these conditions. We know the stable class of $H^*(\ell; \mathbb{Z}/p)$ from Proposition 4.1.1. Using the Künneth formula and Proposition 4.1.1

$$\begin{aligned} &H^*(\ell \wedge \ell; \mathbb{Z}/p) \cong H^*(\ell; \mathbb{Z}/p) \otimes H^*(\ell; \mathbb{Z}/p) \\ &\cong \bigotimes_{i=1}^{\infty} \bigoplus_{j=0}^{p-1} \Sigma^{j(2p^i - 2p^{i-1} - \pi_p(i-1))} I^{j(\pi_p(i-1))} \otimes \bigotimes_{i=1}^{\infty} \bigoplus_{j=0}^{p-1} \Sigma^{j(2p^i - 2p^{i-1} - \pi_p(i-1))} I^{j(\pi_p(i-1))} \end{aligned}$$

where $\pi_p(i) = \frac{p^i - 1}{p-1}$. When we look at $a(i,p) + b(i,p)$ for an individual $\Sigma^{j(2p^i - 2p^{i-1} - \pi_p(i-1))} I^{j(\pi_p(i-1))}$ we have

$$\frac{j((2p^2 - 4p + 1)p^{i-1} + 1)}{p-1} + \frac{j(p^{i-1} - 1)}{p-1} = \frac{j(2p^2 - 4p + 2)p^{i-1}}{p-1} = 2j(p-1)p^{i-1}$$

which is even. Since the assumption is true of all the pieces, the assumption is true of any product and so of $H^*(\ell \wedge \ell; \mathbb{Z}/p)$.

Given these assumptions we can now apply [Ada95, Part III, Lemma 17.1] which states that $H_*(ku \wedge \ell; \mathbb{Z})$ and hence $H_*(\ell \wedge \ell; \mathbb{Z})$ has no torsion of order higher than p. Then from [Ada95, Part III, Proposition 17.2(i)] we know that the Hurewicz homomorphism

$$h : \pi_*(ku \wedge \ell) \to H_*(ku \wedge \ell; \mathbb{Z})$$

is a monomorphism. Since this is true of $ku \wedge \ell$ it follows that the same is true of $\ell \wedge \ell$ and so the result follows. $\square$

Definition 5.2.7. We have a projection map

$$\ell \wedge \ell \simeq \bigvee_{n \geqslant 0} \ell \wedge \mathcal{K}(n) \to \ell \wedge \mathcal{K}(n)$$

which induces a map on homotopy

$$\pi_*(\ell \wedge \ell) \to \pi_*(\ell \wedge \mathcal{K}(n)).$$

Define P_n to be the map induced from this by tensoring with $\mathbb{Q}$ to annihilate torsion, i.e.

$$P_n : G_{*,*} \to G_{*,n}.$$

Lemma 5.2.8. *$P_n(g_{\rho n,\rho l}) = 0$ if $l < n$.*

Proof. Since $G_{m,n}$ is torsion free we can consider just whether $P_n(g_{\rho n,\rho l})$ is zero in $G_{*,n} \otimes \mathbb{Q}_p$. Let $l < n$, then for $\alpha(n,l) \in \mathbb{N}_0$

$$g_{\rho n,\rho l} = \frac{u^{\rho n - \rho l}}{p^{\alpha(n,l)}} f_{\rho l} \in u^{\rho n - \rho l} \frac{\pi_{\rho l}(\ell \wedge \ell)}{\text{Torsion}} \otimes \mathbb{Q}_p \subset \frac{\pi_{\rho n}(\ell \wedge \ell)}{\text{Torsion}} \otimes \mathbb{Q}_p.$$

We know P_n projects onto the $\mathcal{K}(n)$ piece so $P_n(g_{\rho n,\rho l})$ lies in $\frac{\pi_*(\ell \wedge \mathcal{K}(n))}{\text{Torsion}}$. We also know P_n commutes with multiplication by u so, from above, $P_n(g_{\rho n,\rho l})$ is $u^{\rho n - \rho l}$ times an element of $\frac{\pi_{\rho l}(\ell \wedge \mathcal{K}(n))}{\text{Torsion}}$. We know that the homotopy of $\ell \wedge \mathcal{K}(n)$ is trivial in degrees less than ρn by Proposition 5.2.2, hence the result follows. □

This lemma tells us that $g_{\rho n,\rho n}$ has a non-zero component in $G_{\rho n,n}$ and that all other elements $g_{\rho n,\rho l}$ for $l < n$ do not have a component in $G_{\rho n,n}$. In the next section we will define a generator for $G_{\rho n,n}$ and express it explicitly in terms of $g_{\rho n,\rho i}$ for $i \leqslant n$.

5.3 The Elements $z_{\rho m}$

Now we choose labels for some of the generators of certain homotopy groups.

Definition 5.3.1. Let $z_{\rho n}$ be a generator for $G_{\rho n,n} \cong \mathbb{Z}_p$ and let $\tilde{z}_{\rho n}$ be any element in $\tilde{G}_{\rho n,n} \cong G_{\rho n,n} \oplus W_{\rho n,n}$ where the first co-ordinate is $z_{\rho n}$.

Proposition 5.3.2. *In the Adams spectral sequence*

$$E_2^{s,t} \cong \operatorname{Ext}_{\mathcal{A}_p}^{s,t}(H^*(\ell \wedge \mathcal{K}(n); \mathbb{Z}/p), \mathbb{Z}/p) \Longrightarrow \pi_{t-s}(\ell \wedge \mathcal{K}(n)) \otimes \mathbb{Z}_p$$

the class of $\tilde{z}_{\rho n}$ is represented in either $E_2^{0,\rho n}$ or $E_2^{1,\rho n+1}$.

Proof. Consider the Adams spectral sequence

$$\begin{aligned} E_2^{s,t} &= \mathrm{Ext}^{s,t}_{\mathcal{A}_p}(H^*(\ell \wedge \mathcal{K}(n); \mathbb{Z}/p), \mathbb{Z}/p) \\ &\cong \mathrm{Ext}^{s,t}_{\mathcal{A}_p}(H^*(\ell; \mathbb{Z}/p) \otimes H^*(\mathcal{K}(n); \mathbb{Z}/p), \mathbb{Z}/p) \\ &\cong \mathrm{Ext}^{s,t}_{\mathcal{A}_p}(\mathcal{A}_p \otimes_B \mathbb{Z}/p \otimes H^*(\mathcal{K}(n); \mathbb{Z}/p), \mathbb{Z}/p) \\ &\cong \mathrm{Ext}^{s,t}_{\mathcal{A}_p}(\mathcal{A}_p \otimes_B H^*(\mathcal{K}(n); \mathbb{Z}/p), \mathbb{Z}/p) \\ &\cong \mathrm{Ext}^{s,t}_{B}(H^*(\mathcal{K}(n); \mathbb{Z}/p), \mathbb{Z}/p) \\ &\Longrightarrow \pi_{t-s}(\ell \wedge \mathcal{K}(n)) \otimes \mathbb{Z}_p. \end{aligned}$$

Here we have used the same isomorphisms mentioned to obtain equation (3.1) such as the Künneth formula, a change of rings isomorphism etc. We know from Theorem 3.2.13 that $H^*(\mathcal{K}(n); \mathbb{Z}/p) \approxeq \Sigma^{\rho n - \nu_p(n!)} I^{\nu_p(n!)}$. So

$$\begin{aligned} E_2^{s,t} &\cong \mathrm{Ext}^{s,t}_B(\Sigma^{\rho n - \nu_p(n!)} I^{\nu_p(n!)}, \mathbb{Z}/p) \\ &\cong \mathrm{Ext}^{s+\nu_p(n!), t-\rho n + \nu_p(n!)}_B(\mathbb{Z}/p, \mathbb{Z}/p). \end{aligned}$$

We see that the E_2 term is isomorphic to a shifted version of $\mathrm{Ext}^{*,*}_B(\mathbb{Z}/p, \mathbb{Z}/p) \cong \mathbb{Z}/p[c,d]$ with $c \in \mathrm{Ext}^{1,1}_B$ and $d \in \mathrm{Ext}^{1,2p-1}_B$ (see Lemma 3.2.15). Recall that

$$\tilde{z}_{\rho n} \in \tilde{G}_{\rho n, n} = \pi_{\rho n}(\ell \wedge \mathcal{K}(n)) \cong \mathbb{Z}_p \oplus W_{\rho n, n}.$$

The spectral sequence gives us information about the filtration of $\tilde{G}_{\rho n,n} = \pi_{\rho n}(\ell \wedge \mathcal{K}(n))$, i.e. we have a filtration

$$\cdots \subset F^i \subset \cdots F^2 \subset F^1 \subseteq F^0 = \pi_{\rho n}(\ell \wedge \mathcal{K}(n)) \otimes \mathbb{Z}_p.$$

Here $t - s = \rho n$ so $t = \rho n + s$, and the filtration gives us

$$\frac{F^i}{F^{i+1}}(\pi_{\rho n}(\ell \wedge \mathcal{K}(n)) \otimes \mathbb{Z}_p) \cong E_\infty^{i, \rho n + i} \cong E_2^{i, \rho n + i}$$

as the spectral sequence collapses because all non-zero elements are in even total degree. We know from the ring structure of the spectral sequence that multiplication by c in the spectral sequence corresponds to multiplication by p in homotopy groups. So in our filtration we have $pF^i \subseteq F^{i+1}$. Because we know from Lemma 5.2.6 that $W_{\rho n,n}$ is an elementary abelian p-group we know that $pW_{\rho n,n} = 0$ so $W_{\rho n,n}$ must be represented in $E_2^{0,\rho n}$.

Looking in more detail at the spectral sequence we know that each non-zero group is a copy of $\mathbb{Z}/p\{c^r d^s\}$ for some $r, s \in \mathbb{N}_0$, i.e. each filtration quotient is as follows

$$\frac{F^i}{F^{i+1}}(\pi_{\rho n}(\ell \wedge \mathcal{K}(n)) \otimes \mathbb{Z}_p) \cong \mathbb{Z}/p\{c^r d^s\}.$$

Multiplication by c gives us the next filtration quotient i.e.

$$\frac{F^{i+1}}{F^{i+2}}(\pi_{\rho n}(\ell \wedge \mathcal{K}(n)) \otimes \mathbb{Z}_p) \cong \mathbb{Z}/p\{c^{r+1}d^s\}$$

and this corresponds to multiplication by p within the homotopy groups. Solving the extension problems using the ring structure of the spectral sequence gives us that $pF^i = F^{i+1}$ for all $i = 1, 2, 3, \ldots$ and that $F^0 \cong F^1 \cong \mathbb{Z}_p$.

Assume that the generator $\tilde{z}_{\rho n}$ is represented in $E_2^{j,\rho n+j}$ for $j \geqslant 2$, then we must have that $\tilde{z}_{\rho n} \in F^j$. Because to pass from one filtration group to the next involves multiplication by p we must have had some generator $\tilde{z}'_{\rho n} \in F^1$ such that $p^j \tilde{z}'_{\rho n}$ is a generator for F^{j+1}. Inside F^{j+1} it must be true that there exists some $\gamma \in \mathbb{Z}_p$ such that

$$p^j \gamma \tilde{z}'_{\rho n} = p\tilde{z}_{\rho n}.$$

By taking out a factor of p we have

$$p(p^{j-1}\gamma \tilde{z}'_{\rho n} - \tilde{z}_{\rho n}) = 0$$

hence we must have $p^{j-1}\gamma \tilde{z}'_{\rho n} - \tilde{z}_{\rho n} \in W_{\rho n,n}$ because nothing else has any torsion. This implies that in $G_{\rho n,n}$, $z_{\rho n}$ has a factor of p which contradicts that fact that we chose $z_{\rho n}$ to be a generator of $G_{\rho n,n} \cong \mathbb{Z}_p$. □

We can now give a more explicit description of the generators $z_{\rho n}$ in terms of our basis elements $g_{\rho m,\rho l}$.

Proposition 5.3.3. *The generators $z_{\rho m} \in \frac{\pi_{\rho m}(\ell \wedge \ell)}{Torsion}$ have the following form*

$$z_{\rho m} = \sum_{i=0}^{m} p^{\beta(m,i)} \lambda_{\rho m,\rho i} g_{\rho m,\rho i}$$

where $\lambda_{s,t} \in \mathbb{Z}_p$ if $s \neq t$, $\lambda_{s,s} \in \mathbb{Z}_p^{\times}$ and

$$\beta(m,i) = \begin{cases} \nu_p(m!) & \text{if } m > \nu_p(i!) + i, \\ \nu_p(m!) + m - \nu_p(i!) - i & \text{if } m \leqslant \nu_p(i!) + i. \end{cases}$$

Proof. Because $\{g_{\rho m,\rho l} : 0 \leqslant l \leqslant m\}$ form a basis for $G_{\rho m,*}$ by Lemma 5.2.4, we can express our element $z_{\rho m}$ in terms of this basis as follows

$$z_{\rho m} = \lambda_{\rho m,\rho m} g_{\rho m,\rho m} + \tilde{\lambda}_{\rho m,\rho(m-1)} g_{\rho m,\rho(m-1)} + \cdots + \tilde{\lambda}_{\rho m,0} g_{\rho m,0} \qquad (5.2)$$

where $\lambda_{\rho m,\rho m}, \tilde{\lambda}_{\rho m,\rho l} \in \mathbb{Z}_p$. When the projection map $P_m : G_{\rho m,*} \to G_{\rho m,m}$ is applied, this acts as the identity on $z_{\rho m}$ and as the zero map on all $g_{\rho m,\rho l}$ where $m \neq l$ by Lemma 5.2.8. Hence

$$\begin{aligned} z_{\rho m} = P_m(z_{\rho m}) &= P_m(\lambda_{\rho m,\rho m} g_{\rho m,\rho m} + \cdots + \tilde{\lambda}_{\rho m,0} g_{\rho m,0}) \\ &= \lambda_{\rho m,\rho m} P_m(g_{\rho m,\rho m}). \end{aligned}$$

This shows us that the coefficient $\lambda_{\rho m,\rho m}$ must be a unit otherwise $z_{\rho m}$ would have a factor of p and this contradicts the fact that we chose it to be a generator of $G_{\rho m,m} \cong \mathbb{Z}_p$. We know by definition that for all $m \geqslant 0$, $g_{\rho m,\rho m} = f_{\rho m}$ hence the above equation gives us

$$z_{\rho m} = \lambda_{\rho m,\rho m} P_m(f_{\rho m})$$

in $G_{\rho m,m}$. We can now multiply by the largest power of $\frac{\hat{u}}{p}$ possible to leave the result still lying in $G_{*,m}$ and we get

$$\left(\frac{\hat{u}}{p}\right)^{\nu_p(m!)} z_{\rho m} = \lambda_{\rho m,\rho m} P_m\left(\left(\frac{\hat{u}}{p}\right)^{\nu_p(m!)} f_{\rho m}\right)$$

which lies in $G_{\rho(\nu_p(m!)+m),m}$. By multiplying equation (5.2) by $\left(\frac{\hat{u}}{p}\right)^{\nu_p(m!)}$ we now have the following relation in $G_{\rho(\nu_p(m!)+m),m} \otimes \mathbb{Q}_p$

$$\left(\frac{\hat{u}}{p}\right)^{\nu_p(m!)} z_{\rho m} = \left(\frac{\hat{u}}{p}\right)^{\nu_p(m!)} \lambda_{\rho m,\rho m} f_{\rho m} + \sum_{i=0}^{m-1} \tilde{\lambda}_{\rho m,\rho i} \left(\frac{\hat{u}}{p}\right)^{\nu_p(m!)} g_{\rho m,\rho i}. \tag{5.3}$$

We already know the left hand side of this equation lies in $G_{\rho(\nu_p(m!)+m),m}$ so now we just need to know how many factors of p each $\tilde{\lambda}_{\rho m,\rho i}$ must have to ensure that, once the right hand side is expressed in terms of the basis in Theorem 5.1.10, all the coefficients are p-adic integers.

Using Lemma 5.2.5 we can see that if $m \leqslant \nu_p(i!) + i$ we have

$$\left(\frac{\hat{u}}{p}\right)^{\nu_p(m!)} g_{\rho m,\rho i} = \frac{1}{p^{\nu_p(m!)+m-\nu_p(i!)-i}} \hat{u}^{\nu_p(m!)-\nu_p(i!)+m-i} \left(\frac{\hat{u}}{p}\right)^{\nu_p(i!)} f_{\rho i}$$

and so in equation (5.3) we need our coefficient $\tilde{\lambda}_{\rho m,\rho i}$ to be divisible by $p^{\nu_p(m!)+m-\nu_p(i!)-i}$ in $\mathbb{Z}_p$, hence we choose $\tilde{\lambda}_{\rho m,\rho i} = p^{\beta(m,i)} \lambda_{\rho m,\rho i}$ for $\lambda_{\rho m,\rho i} \in \mathbb{Z}_p$ as in the statement of the Proposition.

Similarly when $m > \nu_p(i!) + i$ we have

$$\left(\frac{\hat{u}}{p}\right)^{\nu_p(m!)} g_{\rho m,\rho i} = \frac{1}{p^{\nu_p(m!)}} \hat{u}^{\nu_p(m!)-\nu_p(i!)+m-i} \left(\frac{\hat{u}}{p}\right)^{\nu_p(i!)} f_{\rho i}$$

and so in equation (5.3) we need $\tilde{\lambda}_{\rho m,\rho i}$ to be divisible by $p^{\nu_p(m!)}$ in $\mathbb{Z}_p$, hence we choose $\tilde{\lambda}_{\rho m,\rho i} = p^{\beta(m,i)} \lambda_{\rho m,\rho i}$ for $\lambda_{\rho m,\rho i} \in \mathbb{Z}_p$ as in the statement of the Proposition. □

Proposition 5.3.4. *In Proposition 5.3.2, $\tilde{z}_{\rho n}$ is actually represented in $E_2^{0,\rho n}$.*

Proof. We will assume that $\tilde{z}_{\rho n}$ is represented in $E_2^{1,\rho n+1}$ in the spectral sequence and obtain a contradiction. From Lemma 5.3.2 we know that the spectral sequence in question collapses and the E_2 page is obtained as follows,

$$E_2^{s,t} \cong \mathrm{Ext}_B^{s+\nu_p(n!),t-\rho n+\nu_p(n!)}(\mathbb{Z}/p, \mathbb{Z}/p).$$

We know that this is a shifted version of $\mathrm{Ext}_B^{*,*}(\mathbb{Z}/p, \mathbb{Z}/p) \cong \mathbb{Z}/p[c, d]$ (see Lemma 3.2.15) so we can work out that on the line $s = 1$ the non-zero groups are $E_2^{1,\rho n+1}$, $E_2^{1,\rho(n+1)+1}$, ..., $E_2^{1,\rho(n+\nu_p(n!)+1)+1}$ and each of these is a single copy of $\mathbb{Z}/p$. Using the multiplicative structure of the spectral sequence we know that if there is a class $w \in E_2^{j,\rho(n+j)+1}$ and the group $E_2^{j,\rho(n+j+1)+1}$ is non-zero then there exists a class $w' \in E_2^{j,\rho(n+j+1)+1}$ such that $pw' = \hat{u}w$. In other words if w is represented by $c^x d^y$ in the spectral sequence where $x \geqslant 1$, $y \geqslant 0$ then w' is represented by $c^{x-1}d^{y+1}$. We can apply this theory to $\tilde{z}_{\rho n} \in E_2^{1,\rho n+1}$, since we know that $E_2^{1,\rho(n+\nu_p(n!)+1)+1}$ is non-zero there must exist a class $w \in E_2^{1,\rho(n+\nu_p(n!)+1)+1}$ such that

$$\hat{u}^{1+\nu_p(n!)}\tilde{z}_{\rho n} = p^{1+\nu_p(n!)}w.$$

This implies that $\hat{u}^{1+\nu_p(n!)}\tilde{z}_{\rho n}$ is divisible by $p^{1+\nu_p(n!)}$ in $G_{*,*}$ however this contradicts the proof of Proposition 5.3.3, hence $\tilde{z}_{\rho n}$ must be represented in $E_2^{0,\rho n}$. □

Lemma 5.3.5. *In the spectral sequence*

$$E_2^{s,t} \cong \mathrm{Ext}_B^{s,t}(H^*(\mathcal{K}(n); \mathbb{Z}/p), \mathbb{Z}/p) \Longrightarrow \pi_{t-s}(\ell \wedge \mathcal{K}(n)) \otimes \mathbb{Z}_p,$$

up to multiplication by a unit, $\left(\frac{\hat{u}}{p}\right)^i (p\tilde{z}_{\rho n})$ *is represented by* $c^{\nu_p(n!)+1-i}d^i$ *for* $0 \leqslant i \leqslant \nu_p(n!)$ *and* $\hat{u}^j \left(\frac{\hat{u}}{p}\right)^{\nu_p(n!)} (\tilde{z}_{\rho n})$ *is represented by* $d^{\nu_p(n!)+j}$ *for* $j \geqslant 1$.

Proof. From Proposition 5.3.4 we know that in the spectral sequence

$$E_2^{s,t} \cong \mathrm{Ext}_B^{s,t}(H^*(\mathcal{K}(n); \mathbb{Z}/p), \mathbb{Z}/p) \Longrightarrow \pi_{t-s}(\ell \wedge \mathcal{K}(n)) \otimes \mathbb{Z}_p$$

$\tilde{z}_{\rho n}$ is represented in $E_2^{0,\rho n}$. By the multiplicative structure of the spectral sequence this means that $p\tilde{z}_{\rho n}$ is represented in $E_2^{1,\rho n+1}$. From the proof of Proposition 5.3.2 we know that $E_2^{s,t} \cong \mathrm{Ext}_B^{s+\nu_p(n!),t-\rho n+\nu_p(n!)}(\mathbb{Z}/p, \mathbb{Z}/p)$ so

$$\begin{aligned} E_2^{1,\rho n+1} &\cong \mathrm{Ext}_B^{1+\nu_p(n!),1+\nu_p(n!)}(\mathbb{Z}/p, \mathbb{Z}/p) \\ &\cong \mathbb{Z}/p\langle c^{1+\nu_p(n!)}\rangle. \end{aligned} \tag{5.4}$$

We know from [Ada95, Part III, Lemma 17.11] that in our spectral sequence, multiplication by c and d correspond to multiplication by p and $\hat{u}$ respectively on homotopy groups. We list below some homotopy elements of

$\pi_*(\ell \wedge \mathcal{K}(n))$ with a choice of corresponding representatives in the spectral sequence.

Homotopy element	Representative
$p\tilde{z}_{\rho n}$	$c^{1+\nu_p(n!)}$
$\left(\frac{\hat{u}}{p}\right)(p\tilde{z}_{\rho n})$	$c^{\nu_p(n!)}d$
$\left(\frac{\hat{u}}{p}\right)^2(p\tilde{z}_{\rho n})$	$c^{\nu_p(n!)-1}d^2$
$\vdots$	$\vdots$
$\left(\frac{\hat{u}}{p}\right)^{\nu_p(n!)}(p\tilde{z}_{\rho n})$	$cd^{\nu_p(n!)}$
$\hat{u}\left(\frac{\hat{u}}{p}\right)^{\nu_p(n!)}(\tilde{z}_{\rho n})$	$d^{\nu_p(n!)+1}$
$\hat{u}^2\left(\frac{\hat{u}}{p}\right)^{\nu_p(n!)}(\tilde{z}_{\rho n})$	$d^{\nu_p(n!)+2}$
$\vdots$	$\vdots$

From this table it is clear to see that the descriptions given in the statement of the Lemma are correct. $\square$

Recall from Definition 3.3.5 the maps

$$\iota_{m,n} : \ell \wedge \mathcal{K}(m) \to \ell \wedge \mathcal{K}(n)$$

which were maps represented in the spectral sequence

$$\begin{aligned} E_2^{s,t} &\cong \mathrm{Ext}_B^{s,t}(H^*(D(\mathcal{K}(m)); \mathbb{Z}/p) \otimes H^*(\mathcal{K}(n); \mathbb{Z}/p), \mathbb{Z}/p) \\ &\Longrightarrow \pi_{t-s}(D(\mathcal{K}(m)) \wedge \mathcal{K}(n) \wedge \ell) \otimes \mathbb{Z}_p \end{aligned}$$

by a choice of generator of

$$E_2^{(m-n)-\nu_p(n!)+\nu_p(m!),(m-n)-\nu_p(n!)+\nu_p(m!)}.$$

These were crucial in defining the isomorphism Λ of Definition 3.1.4 as for any given automorphism in $\mathrm{Aut}^0_{\text{left-}\ell\text{-mod}}(\ell \wedge \ell)$ their coefficients determined the entries in the matrix corresponding to that automorphism.

Proposition 5.3.6. *For $m > n$, the map induced in the (ρm)th homotopy group*

$$(\iota_{m,n})_* : G_{\rho m,m} \to G_{\rho m,n}$$

satisfies the following condition

$$(\iota_{m,n})_*(z_{\rho m}) = \mu_{\rho m,\rho n} p^{\nu_p(m!)-\nu_p(n!)} \hat{u}^{(m-n)} z_{\rho n}$$

for some p-adic unit $\mu_{\rho m,\rho n}$.

Proof. We know that $\tilde{z}_{\rho m}$ is any element in $G_{\rho m,m} \oplus W_{\rho m,m}$ whose first co-ordinate is $z_{\rho m}$, we also know that $W_{\rho m,m}$ has torsion of order p at the highest by Lemma 5.2.6. In order to forget about the torsion we will prove the analogous result for the element $p\tilde{z}_{\rho m} = pz_{\rho m}$; then by linearity the required result will be true for $z_{\rho m}$.

We already know from equation (5.4) in the proof of Lemma 5.3.5 that in the spectral sequence

$$E_2^{s,t} \cong \mathrm{Ext}_B^{s,t}(H^*(\mathcal{K}(m);\mathbb{Z}/p),\mathbb{Z}/p) \Longrightarrow \pi_{t-s}(\ell \wedge \mathcal{K}(m)) \otimes \mathbb{Z}_p,$$

$p\tilde{z}_{\rho m}$ is represented in $E_2^{1,\rho m+1} \cong \mathrm{Ext}_B^{1+\nu_p(m!),1+\nu_p(m!)}(\mathbb{Z}/p,\mathbb{Z}/p)$, up to a unit, by $c^{1+\nu_p(m!)}$.

Recall from Chapter 3 that in the spectral sequence

$$\begin{aligned} E_2^{s,t} &\cong \mathrm{Ext}_B^{s,t}(H^*(D(\mathcal{K}(m));\mathbb{Z}/p) \otimes H^*(\mathcal{K}(n);\mathbb{Z}/p),\mathbb{Z}/p) \\ &\Longrightarrow \pi_{t-s}(D(\mathcal{K}(m)) \wedge \mathcal{K}(n) \wedge \ell) \otimes \mathbb{Z}_p \end{aligned}$$

the maps $\iota_{m,n} : \ell \wedge \mathcal{K}(m) \to \ell \wedge \mathcal{K}(n)$ described there are represented in

$$\begin{aligned} E_2^{m-n-\nu_p(n!)+\nu_p(m!),m-n-\nu_p(n!)+\nu_p(m!)} &\cong \mathrm{Ext}_B^{m-n,(m-n)(\rho+1)}(\mathbb{Z}/p,\mathbb{Z}/p) \\ &\cong \mathbb{Z}/p\langle d^{m-n}\rangle. \end{aligned}$$

Using the pairing of Ext groups described in the proof of Proposition 3.3.7

$$\mathrm{Ext}^{s,t}(\Sigma^a I^b,\mathbb{Z}/p) \otimes \mathrm{Ext}^{s',t'}(\Sigma^{a'} I^{b'},\mathbb{Z}/p) \to \mathrm{Ext}^{s+s',t+t'}(\Sigma^{a+a'} I^{b+b'},\mathbb{Z}/p)$$

we get an induced pairing on the E_2 pages of the respective Adams spectral sequences. Since in all cases the spectral sequences collapse for degree reasons this passes to the E_∞ pages too. This pairing also respects the filtration on each of the spectral sequences as the Ext group pairing is, in essence, reducing everything to a splicing of $\mathrm{Ext}_B(\mathbb{Z}/p,\mathbb{Z}/p) \cong \mathbb{Z}/p[c,d]$ with itself, which is just polynomial multiplication. So the Ext group pairing passes to a pairing of spectral sequences giving us a map

$$\begin{aligned} \mathrm{Ext}_B^{s,t}(H^*(D(\mathcal{K}(m));\mathbb{Z}/p) \otimes H^*(\mathcal{K}(n);\mathbb{Z}/p),\mathbb{Z}/p) \otimes \mathrm{Ext}_B^{s',t'}(H^*(\mathcal{K}(m);\mathbb{Z}/p),\mathbb{Z}/p) \\ \to \mathrm{Ext}_B^{s+s',t+t'}(H^*(\mathcal{K}(n);\mathbb{Z}/p),\mathbb{Z}/p). \end{aligned}$$

This shows that $(\iota_{m,n})_*(p\tilde{z}_{\rho m})$ is represented in the spectral sequence

$$E_2^{s,t} \cong \mathrm{Ext}_B^{s,t}(H^*(\mathcal{K}(n);\mathbb{Z}/p),\mathbb{Z}/p) \Longrightarrow \pi_{t-s}(\ell \wedge \mathcal{K}(n)) \otimes \mathbb{Z}_p$$

and by adding together the respective bidegrees, we see it is represented by a generator of

$$
\begin{aligned}
&E_2^{1+m-n-\nu_p(n!)+\nu_p(m!),\rho m+1+m-n-\nu_p(n!)+\nu_p(m!)}\\
&\quad\cong \mathrm{Ext}_B^{1+m-n-\nu_p(n!)+\nu_p(m!),\rho m+1+m-n-\nu_p(n!)+\nu_p(m!)}(H^*(\mathcal{K}(n);\mathbb{Z}/p),\mathbb{Z}/p)\\
&\quad\cong \mathrm{Ext}_B^{1+m-n-\nu_p(n!)+\nu_p(m!),\rho m+1+m-n-\nu_p(n!)+\nu_p(m!)}(\Sigma^{\rho n-\nu_p(n!)}I^{\nu_p(n!)},\mathbb{Z}/p)\\
&\quad\cong \mathrm{Ext}_B^{1+m-n+\nu_p(m!),\rho m+1+m-n+\nu_p(m!)-\rho n}(\mathbb{Z}/p,\mathbb{Z}/p)\\
&\quad\cong \mathrm{Ext}_B^{1+m-n+\nu_p(m!),1+(\rho+1)(m-n)+\nu_p(m!)}(\mathbb{Z}/p,\mathbb{Z}/p)\\
&\quad\cong \mathbb{Z}/p\langle c^{1+\nu_p(m!)}d^{m-n}\rangle
\end{aligned}
$$

Thus $(\iota_{m,n})_*(p\tilde{z}_{\rho m})$ is, up to a unit, represented by $c^{1+\nu_p(m!)}d^{m-n}$ and all that remains is to express this element in terms of $p\tilde{z}_{\rho n}$.

Using Lemma 5.3.5 we can see that we have two cases for $(\iota_{m,n})_*(p\tilde{z}_{\rho m})$, either the power of d in its representative is at least $\nu_p(n!)+1$ (and hence the power of c in its representative is zero) or not.

In the first case we have $m-n \geqslant \nu_p(n!)+1$. Then we can use Lemma 5.3.5 to see that d^{m-n} represents

$$\left(\frac{\hat{u}}{p}\right)^{\nu_p(n!)}\hat{u}^{m-n-\nu_p(n!)}\tilde{z}_{\rho n} = p^{-\nu_p(n!)}\hat{u}^{m-n}\tilde{z}_{\rho n}.$$

This implies that up to a p-adic unit, $(\iota_{m,n})_*(p\tilde{z}_{\rho m})$, which is represented by $c^{1+\nu_p(m!)}d^{m-n}$, is equal to

$$p^{1+\nu_p(m!)}p^{-\nu_p(n!)}\hat{u}^{m-n}\tilde{z}_{\rho n} = p^{\nu_p(m!)-\nu_p(n!)}\hat{u}^{m-n}(p\tilde{z}_{\rho n}).$$

In the second case we have $m-n < \nu_p(n!)+1$. Hence we can see from Lemma 5.3.5 that the representative involving d^{m-n} is $c^{1+\nu_p(n!)-m+n}d^{m-n}$ and this represents the homotopy element

$$\left(\frac{\hat{u}}{p}\right)^{m-n}(p\tilde{z}_{\rho n}).$$

This gives us that up to a p-adic unit, $(\iota_{m,n})_*(p\tilde{z}_{\rho m})$, which is represented by $c^{1+\nu_p(m!)}d^{m-n}$, is equal to

$$p^{1+\nu_p(m!)-(1+\nu_p(n!)-m+n)}\left(\frac{\hat{u}}{p}\right)^{m-n}(p\tilde{z}_{\rho n}) = p^{\nu_p(m!)-\nu_p(n!)}\hat{u}^{m-n}(p\tilde{z}_{\rho n}). \quad \square$$

Now we have an expression for the effect of $(\iota_{m,n})_*$ on $z_{\rho m}$ we can, in the next chapter, look at the specific automorphism $1\wedge\psi^q$ and determine which matrix it corresponds to under the isomorphism Λ of Definition 3.1.4. This will involve working out what effect the induced map $(1\wedge\psi^q)_*$ has on the elements $z_{\rho m}$ and comparing coefficients to determine the entries in the matrix.

Chapter 6

The Matrix

6.1 Introduction

In this chapter we will determine the coefficients of the matrix corresponding to the map $1 \wedge \psi^q : \ell \wedge \ell \to \ell \wedge \ell$ under the isomorphism Λ of Definition 3.1.4. We already know what effect the maps $(\iota_{m,n})_*$ have on our basis elements $z_{\rho m}$ by Proposition 5.3.6. If we now work out what effect the induced map $(1 \wedge \psi^q)_*$ has on the same basis elements we can then compare the two using the construction of the isomorphism in Definition 3.1.4 to work out the necessary coefficients of the matrix.

This particular Adams operation is important; because powers of q are dense in $\mathbb{Z}_p$, the ring of operations $\ell^0(\ell)$ on the complex connective p-complete Adams summand is generated as a power series over $\mathbb{Z}_p$ by $\psi^q - 1$ (see [CCW05, Theorem 5.1]). [1] We can now show that ψ^q induces the identity map on mod p homology and therefore $1 \wedge \psi^q$ is an element of $\mathrm{Aut}^0_{\text{left-}\ell\text{-mod}}(\ell \wedge \ell)$.

Proposition 6.1.1. *The map*

$$\psi^q_* : H_*(\ell; \mathbb{Z}/p) \to H_*(\ell; \mathbb{Z}/p)$$

is the identity map.

Proof. We have an augmentation map $\varepsilon : \ell \to H\mathbb{Z}/p$ such that $\varepsilon_* : \pi_*(\ell) \to \pi_*(H\mathbb{Z}/p)$ sends $\hat{u} \in \pi_*(\ell) = \mathbb{Z}_p[\hat{u}]$ to zero in $\pi_*(H\mathbb{Z}/p)$ and sends any number $a \in \mathbb{Z}_p$ to its reduction mod p. Since ${\psi^q}_*(\hat{u}) = q^{p-1}\hat{u} = r\hat{u}$, and ${\psi^q}_*$ has no effect on the coefficients, we see that $(\varepsilon \circ \psi^q)_* = \varepsilon_*$. If we now apply the functor $- \wedge H\mathbb{Z}/p$ to the composition $\varepsilon \circ \psi^q$ we get

$$\ell \wedge H\mathbb{Z}/p \xrightarrow{\psi^q \wedge 1} \ell \wedge H\mathbb{Z}/p \xrightarrow{\varepsilon \wedge 1} H\mathbb{Z}/p \wedge H\mathbb{Z}/p.$$

[1] We follow [CCW05] in denoting this Adams operation as ψ^q; some authors write ψ^r where $r = q^{p-1}$ for this operation.

Taking the induced maps in homotopy and using the standard identification $\pi_*(E \wedge F) \cong F_*(E)$ for spectra E, F, we get the maps

$$H_*(\ell; \mathbb{Z}/p) \xrightarrow{\psi^q{}_*} H_*(\ell; \mathbb{Z}/p) \xrightarrow{\varepsilon_*} \mathcal{A}_p^*$$

where ε_* is the standard inclusion of the subalgebra

$$H_*(\ell; \mathbb{Z}/p) = \Lambda(\chi(\tau_2), \chi(\tau_3), \dots) \otimes \mathbb{Z}/p[\chi(\xi_1), \chi(\xi_2), \dots]$$

into the dual Steenrod Algebra. Since the composite $(\varepsilon \circ \psi^q)_*$ is equal to the inclusion ε_*, we see that $\psi^q{}_*$ must be the identity map. $\square$

6.2 The Effect of $1 \wedge \psi^q$ on the Basis

Recall the elements $g_{\rho m, \rho n}$ given in Definition 5.2.3.

$$g_{\rho m,\rho n} = \begin{cases} \left(\frac{\hat{u}}{p}\right)^{m-n} f_{\rho n} & \text{if } m \leqslant \nu_p(n!) + n, \\ \hat{u}^{m-n-\nu_p(n!)} \left(\frac{\hat{u}}{p}\right)^{\nu_p(n!)} f_{\rho n} & \text{if } m > \nu_p(n!) + n. \end{cases}$$

In this section we will look at what happens to our basis elements $g_{\rho m,\rho n}$ under the map $(1 \wedge \psi^q)_*$ where ψ^q is the Adams operation. These elements are defined in terms of the polynomials $f_{\rho k} = p^{\nu_p(k!)} c_{\rho k}$ given in Definition 5.1.6, so we first we need to look at the effect of $(1 \wedge \psi^q)_*$ on the elements $f_{\rho k}$.

Lemma 6.2.1.

$$(1 \wedge \psi^q)_*(f_{\rho m}) = r^m f_{\rho m} + p^{\nu_p(m)} \hat{u} f_{\rho(m-1)}$$

for $m \geqslant 1$.

Proof. Recall the polynomials

$$c_{\rho k} = \prod_{i=1}^{k} \frac{\hat{v} - r^{i-1}\hat{u}}{r^k - r^{i-1}} \in \mathbb{Q}_p[\hat{u}, \hat{v}]$$

given in Definition 5.1.5. The map $(1 \wedge \psi^q)_*$ fixes u, multiplies v by q, and so $\hat{v}$ by $r = q^{p-1}$, and is additive and multiplicative. So we have

$$\begin{aligned} r^m c_{\rho m} + \hat{u} c_{\rho(m-1)} &= r^m \prod_{i=1}^{m} \frac{\hat{v} - r^{i-1}\hat{u}}{r^m - r^{i-1}} + \hat{u} \prod_{i=1}^{m-1} \frac{\hat{v} - r^{i-1}\hat{u}}{r^{m-1} - r^{i-1}} \\ &= \prod_{i=1}^{m} \frac{\hat{v} - r^{i-1}\hat{u}}{r^{m-1} - r^{i-2}} + \hat{u} \prod_{i=1}^{m-1} \frac{\hat{v} - r^{i-1}\hat{u}}{r^{m-1} - r^{i-1}} \end{aligned}$$

$$
\begin{aligned}
&= \prod_{i=1}^{m-1} \frac{\hat{v} - r^{i-1}\hat{u}}{r^{m-1} - r^{i-1}} \left(\frac{\hat{v} - r^{m-1}\hat{u}}{r^{m-1} - r^{-1}} + \hat{u} \right) \\
&= \prod_{i=1}^{m-1} \frac{r\hat{v} - r^{i}\hat{u}}{r^{m} - r^{i}} \left(\frac{\hat{v} - r^{-1}\hat{u}}{r^{m-1} - r^{-1}} \right) \\
&= \prod_{i=1}^{m-1} \frac{r\hat{v} - r^{i}\hat{u}}{r^{m} - r^{i}} \left(\frac{r\hat{v} - \hat{u}}{r^{m} - 1} \right) \\
&= \prod_{i=0}^{m-1} \frac{r\hat{v} - r^{i}\hat{u}}{r^{m} - r^{i}} \\
&= \prod_{j=1}^{m} \frac{r\hat{v} - r^{j-1}\hat{u}}{r^{m} - r^{j-1}} \\
&= (1 \wedge \psi^q)_*(c_{\rho m}).
\end{aligned}
$$

Now

$$
\begin{aligned}
(1 \wedge \psi^q)_*(f_{\rho m}) &= p^{\nu_p(m!)}(1 \wedge \psi^q)_*(c_{\rho m}) \\
&= p^{\nu_p(m!)} r^m c_{\rho m} + p^{\nu_p(m!)+\nu_p((m-1)!)-\nu_p((m-1)!)} \hat{u} c_{\rho(m-1)} \\
&= r^m f_{\rho m} + p^{\nu_p(m!)-\nu_p((m-1)!)} \hat{u} f_{\rho(m-1)}.
\end{aligned}
$$

Because $\nu_p(m!) - \nu_p((m-1)!) = \nu_p\left(\frac{m!}{(m-1)!}\right) = \nu_p(m)$ we then have that

$$
(1 \wedge \psi^q)_*(f_{\rho m}) = r^m f_{\rho m} + p^{\nu_p(m)} \hat{u} f_{\rho(m-1)}
$$

for $m \geqslant 1$ as stated. $\square$

Proposition 6.2.2. *The action of $(1 \wedge \psi^q)_*$ on our basis elements is as follows*

$$
(1 \wedge \psi^q)_*(g_{\rho m, \rho m}) = \begin{cases} r^m g_{\rho m,\rho m} + p^{\nu_p(m)+1} g_{\rho m,\rho(m-1)} & \text{if } m > p, \\ r^m g_{\rho m,\rho m} + p g_{\rho m,\rho(m-1)} & \text{if } m = p, \\ r^m g_{\rho m,\rho m} + g_{\rho m,\rho(m-1)} & \text{if } 1 \leqslant m \leqslant p-1, \\ g_{0,0} & \text{if } m = 0. \end{cases}
$$

In the 2 primary case the result [BS05, Proposition 3.3] should read as follows

$$
(1 \wedge \psi^3)_*(g_{4k,4k}) = \begin{cases} 9^k g_{4k,4k} + 2^{\nu_2(k)+3} g_{4k,4(k-1)} & \text{if } k \geqslant 3, \\ 9^2 g_{8,8} + 2^3 g_{8,4} & \text{if } k = 2, \\ 9 g_{4,4} + 2 g_{4,0} & \text{if } k = 1, \\ g_{0,0} & \text{if } k = 0. \end{cases}
$$

Proof. Using Lemma 6.2.1 and Definition 5.2.3 we get that

$$\begin{aligned}(1 \wedge \psi^q)_*(g_{\rho m,\rho m}) &= r^m f_{\rho m} + p^{\nu_p(m)+1}\left(\frac{\hat{u}}{p}\right) f_{\rho(m-1)} \\ &= r^m g_{\rho m,\rho m} + p^{\nu_p(m)+1} g_{\rho m,\rho(m-1)}\end{aligned}$$

for $m > p$. If $m = p$ then from Definition 5.2.3 we have

$$\begin{aligned}(1 \wedge \psi^q)_*(g_{\rho m,\rho m}) &= r^m f_{\rho m} + p^{\nu_p(m)} \hat{u} f_{\rho(m-1)} \\ &= r^m g_{\rho m,\rho m} + p g_{\rho m,\rho(m-1)}\end{aligned}$$

and

$$(1 \wedge \psi^q)_*(g_{\rho m,\rho m}) = r^m g_{\rho m,\rho m} + g_{\rho m,\rho(m-1)}$$

for $1 \leqslant m \leqslant p-1$. ☐

Proposition 6.2.3. *When $m > n$*

$$\begin{aligned}&(1 \wedge \psi^q)_*(g_{\rho m,\rho n}) \\ &= \begin{cases} r^n g_{\rho m,\rho n} + g_{\rho m,\rho(n-1)} & \text{if } m > \nu_p(n!) + n, \\ r^n g_{\rho m,\rho n} + p^{\nu_p(n!)+n-m} g_{\rho m,\rho(n-1)} & \text{if } \nu_p((n-1)!) + n - 1 < m \leqslant \nu_p(n!) + n, \\ r^n g_{\rho m,\rho n} + p^{\nu_p(n)+1} g_{\rho m,\rho(n-1)} & \text{if } m \leqslant \nu_p((n-1)!) + n - 1. \end{cases}\end{aligned}$$

In the 2 primary case the result [BS05, Proposition 3.4] should read as follows. When $k > l$

$$\begin{aligned}&(1 \wedge \psi^3)_*(g_{4k,4l}) \\ &= \begin{cases} 9^l g_{4k,4l} + g_{4k,4(l-1)} & \text{if } 4l - \alpha(l) \leqslant 2k, \\ 9^l g_{4k,4l} + 2^{4l-\alpha(l)-2k} g_{4k,4(l-1)} & \text{if } 4l - \alpha(l) - \nu_2(l) - 3 \leqslant 2k < 4l - \alpha(l), \\ 9^l g_{4k,4l} + 2^{3+\nu_2(k)} g_{4k,4(l-1)} & \text{if } 2k < 4l - \alpha(l) - \nu_2(l) - 3 < 4l - \alpha(l). \end{cases}\end{aligned}$$

Proof. For the first case let's take $m > \nu_p(n!) + n$. Then using Lemma 6.2.1 we have

$$\begin{aligned}(1 \wedge \psi^q)_*(g_{\rho m,\rho n}) &= (1 \wedge \psi^q)_*\left(\hat{u}^{m-n-\nu_p(n!)}\left(\frac{\hat{u}}{p}\right)^{\nu_p(n!)} f_{\rho n}\right) \\ &= \hat{u}^{m-n-\nu_p(n!)}\frac{\hat{u}^{\nu_p(n!)}}{p^{\nu_p(n!)}}(r^n f_{\rho n} + p^{\nu_p(n)}\hat{u} f_{\rho(n-1)}) \\ &= r^n \frac{\hat{u}^{m-n}}{p^{\nu_p(n!)}} f_{\rho n} + p^{\nu_p(n)}\frac{\hat{u}^{m-n+1}}{p^{\nu_p(n!)}} f_{\rho(n-1)} \\ &= r^n g_{\rho m,\rho n} + \frac{\hat{u}^{m-n+1}}{p^{\nu_p(n!)-\nu_p(n)}} f_{\rho(n-1)}.\end{aligned}$$

Since

$$\begin{aligned}\nu_p((n-1)!) + n - 1 &= \nu_p(n!) + n - \nu_p(n!) + \nu_p((n-1)!) - 1 \\ &= \nu_p(n!) + n - \nu_p(n) - 1 \\ &< m - \nu_p(n) - 1 < m\end{aligned}$$

we have

$$\begin{aligned}g_{\rho m,\rho(n-1)} &= \hat{u}^{m-n+1-\nu_p((n-1)!)}\left(\frac{\hat{u}}{p}\right)^{\nu_p((n-1)!)} f_{\rho(n-1)} \\ &= \frac{\hat{u}^{m-n+1}}{p^{\nu_p((n-1)!)}} f_{\rho(n-1)} \\ &= \frac{\hat{u}^{m-n+1}}{p^{\nu_p(n!)-\nu_p(n)}} f_{\rho(n-1)}.\end{aligned}$$

This gives us that

$$(1 \wedge \psi^q)_*(g_{\rho m,\rho n}) = r^n g_{\rho m,\rho n} + g_{\rho m,\rho(n-1)}.$$

Now let $m \leqslant \nu_p(n!) + n$, then

$$\begin{aligned}(1 \wedge \psi^q)_*(g_{\rho m,\rho n}) &= (1 \wedge \psi^q)_*\left(\left(\frac{\hat{u}}{p}\right)^{m-n} f_{\rho n}\right) \\ &= \frac{\hat{u}^{m-n}}{p^{m-n}}(r^n f_{\rho n} + p^{\nu_p(n)}\hat{u} f_{\rho(n-1)}) \\ &= r^n g_{\rho m,\rho n} + \frac{\hat{u}^{m-n+1}}{p^{m-n-\nu_p(n)}} f_{\rho(n-1)}.\end{aligned}$$

Now one of the two following cases will apply

(i) $\nu_p((n-1)!) + n - 1 < m \leqslant \nu_p(n!) + n$,

(ii) $m \leqslant \nu_p((n-1)!) + n - 1$.

Case (i): Here

$$\begin{aligned}g_{\rho m,\rho(n-1)} &= \hat{u}^{m-n+1-\nu_p((n-1)!)}\left(\frac{\hat{u}}{p}\right)^{\nu_p((n-1)!)} f_{\rho(n-1)} \\ &= \frac{\hat{u}^{m-n+1}}{p^{\nu_p((n-1)!)}} f_{\rho(n-1)}.\end{aligned}$$

Substituting this back into the above equation we have

$$\begin{aligned}(1 \wedge \psi^q)_*(g_{\rho m,\rho n}) &= r^n g_{\rho m,\rho n} + p^{\nu_p((n-1)!)+\nu_p(n)-m+n} g_{\rho m,\rho(n-1)} \\ &= r^n g_{\rho m,\rho n} + p^{\nu_p(n!)-m+n} g_{\rho m,\rho(n-1)}.\end{aligned}$$

Case (ii): Here

$$g_{\rho m,\rho(n-1)} = \left(\frac{\hat{u}}{p}\right)^{m-n+1} f_{\rho(n-1)}.$$

When substituted back into the above equation we have

$$(1 \wedge \psi^q)_*(g_{\rho m,\rho n}) = r^n g_{\rho m,\rho n} + p^{\nu_p(n)+1} g_{\rho m,\rho(n-1)}.$$

□

6.3 The Coefficients of the Matrix

Let $A \in U_\infty \mathbb{Z}_p$ be a matrix such that under the isomorphism in Definition 3.1.4

$$\Lambda(A) = 1 \wedge \psi^q.$$

The rest of this subsection will be devoted to proving the following result on the form of A.

Proposition 6.3.1. *The matrix A corresponding to the map $1 \wedge \psi^q : \ell \wedge \ell \to \ell \wedge \ell$ under the isomorphism Λ has the following form:*

$$A = \begin{pmatrix} 1 & \upsilon_0 & a_{0,2} & a_{0,3} & a_{0,4} & \cdots \\ 0 & r & \upsilon_1 & a_{1,3} & a_{1,4} & \cdots \\ 0 & 0 & r^2 & \upsilon_2 & a_{2,4} & \cdots \\ 0 & 0 & 0 & r^3 & \upsilon_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $r = q^{p-1}$ for q a topological generator of the p-adic units, $\upsilon_i \in \mathbb{Z}_p^\times$ for all $i \geqslant 0$ and $a_{i,j} \in \mathbb{Z}_p$ for all $i, j \geqslant 0$.

Before we can prove this we need the following lemma concerning expressing $\hat{u}^{m-n} g_{\rho n,\rho i}$ in terms of $g_{\rho m,\rho i}$.

Lemma 6.3.2.

$$\hat{u}^{m-n} g_{\rho n,\rho i} = \begin{cases} p^{m-n} g_{\rho m,\rho i} & \text{if } n \leqslant m \leqslant \nu_p(i!) + i, \\ p^{\nu_p(i!)-n+i} g_{\rho m,\rho i} & \text{if } n \leqslant \nu_p(i!) + i < m, \\ g_{\rho m,\rho i} & \text{if } \nu_p(i!) + i < n \leqslant m. \end{cases} \tag{6.1}$$

Proof. From Definition 5.2.3 we know that

$$\begin{aligned} \hat{u}^{m-n} g_{\rho n,\rho i} &= \begin{cases} \hat{u}^{m-n} \left(\frac{\hat{u}}{p}\right)^{n-i} f_{\rho i} & \text{if } n \leqslant \nu_p(i!) + i, \\ \hat{u}^{m-n+n-i-\nu_p(i!)} \left(\frac{\hat{u}}{p}\right)^{\nu_p(i!)} f_{\rho i} & \text{if } n > \nu_p(i!) + i, \end{cases} \\ &= \begin{cases} \frac{\hat{u}^{m-i}}{p^{n-i}} f_{\rho i} & \text{if } n \leqslant \nu_p(i!) + i, \\ \frac{\hat{u}^{m-i}}{p^{\nu_p(i!)}} f_{\rho i} & \text{if } n > \nu_p(i!) + i. \end{cases} \end{aligned}$$

Comparing this with the original definition

$$g_{\rho m,\rho i} = \begin{cases} \frac{\hat{u}^{m-i}}{p^{m-i}} f_{\rho i} & \text{if } m \leqslant \nu_p(i!) + i, \\ \frac{\hat{u}^{m-i}}{p^{\nu_p(i!)}} f_{\rho i} & \text{if } m > \nu_p(i!) + i, \end{cases}$$

we get the formulas in the statement. □

Recall the following expression from Proposition 5.3.3:

$$z_{\rho m} = \sum_{i=0}^{m} p^{\beta(m,i)} \lambda_{\rho m,\rho i}(g_{\rho m,\rho i})$$

where $\lambda_{s,t} \in \mathbb{Z}_p$ if $s \neq t$, $\lambda_{s,s} \in \mathbb{Z}_p^\times$ and

$$\beta(m,i) = \begin{cases} \nu_p(m!) & \text{if } m > \nu_p(i!) + i, \\ \nu_p(m!) + m - \nu_p(i!) - i & \text{if } m \leqslant \nu_p(i!) + i. \end{cases}$$

Proof of Proposition 6.3.1. Using Definition 3.1.4 we have that

$$\begin{aligned} \sum_{n \leqslant m} A_{n,m}(\iota_{m,n})_*(z_{\rho m}) = (1 \wedge \psi^q)_*(z_{\rho m}) & \\ = \sum_{i=0}^{m} p^{\beta(m,i)} \lambda_{\rho m,\rho i}(1 \wedge \psi^q)_*(g_{\rho m,\rho i}). \end{aligned}$$

Also by expanding out the left hand side of this equation and using Proposition 5.3.3 and Proposition 5.3.6 we get that

$$\begin{aligned} &\sum_{n \leqslant m} A_{n,m}(\iota_{m,n})_*(z_{\rho m}) \\ &\quad = A_{m,m} z_{\rho m} + \sum_{n<m} A_{n,m} \mu_{\rho m,\rho n} p^{\nu_p(m!)-\nu_p(n!)} \hat{u}^{m-n} z_{\rho n} \\ &\quad = A_{m,m} \sum_{i=0}^{m} p^{\beta(m,i)} \lambda_{\rho m,\rho i} g_{\rho m,\rho i} \\ &\qquad + \sum_{n<m} \sum_{i=0}^{n} A_{n,m} \mu_{\rho m,\rho n} p^{\nu_p(m!)-\nu_p(n!)} \hat{u}^{m-n} p^{\beta(n,i)} \lambda_{\rho n,\rho i} g_{\rho n,\rho i} \end{aligned}$$

where $\mu_{\rho m,\rho n} \in \mathbb{Z}_p^\times$. Hence we have

$$\begin{aligned} \sum_{i=0}^{m} p^{\beta(m,i)} \lambda_{\rho m,\rho i}(1 \wedge \psi^q)_*(g_{\rho m,\rho i}) = A_{m,m} \sum_{i=0}^{m} p^{\beta(m,i)} \lambda_{\rho m,\rho i} g_{\rho m,\rho i} & \\ + \sum_{n<m} \sum_{i=0}^{n} A_{n,m} \mu_{\rho m,\rho n} p^{\nu_p(m!)-\nu_p(n!)} \hat{u}^{m-n} p^{\beta(n,i)} \lambda_{\rho n,\rho i} g_{\rho n,\rho i}. & \quad (6.2) \end{aligned}$$

We want to determine the $A_{n,m}$s by equating coefficients in Equation (6.2) above. Firstly let $m = 0$, then by Proposition 5.3.3 we know $z_0 = \lambda_{0,0} g_{0,0} = \lambda_{0,0} \in \mathbb{Z}_p^\times$. Then

$$z_0 = (1 \wedge \psi^q)_*(z_0) = A_{0,0}(\iota_{0,0})_*(z_0) = A_{0,0} z_0$$

which means $A_{0,0} = 1$.

We will now split the rest of the proof into three cases. Case (i): Let $1 \leqslant m \leqslant p-1$. From Equation (6.2) we can use Proposition 6.2.2 and equate the coefficient of $g_{\rho m, \rho m}$:

$$r^m \lambda_{\rho m,\rho m} = A_{m,m} \lambda_{\rho m,\rho m}$$

which gives us that $A_{m,m} = r^m$.

Looking at the terms which will contribute to the coefficient of $g_{\rho m,\rho(m-1)}$, we can use the first case given in Proposition 6.2.3 on the left hand side of Equation (6.2). We find that

$$\begin{aligned}
&(p^{\beta(m,m)} \lambda_{\rho m,\rho m} + p^{\beta(m,m-1)} \lambda_{\rho m,\rho(m-1)} r^{m-1}) g_{\rho m,\rho(m-1)} \\
&\ = A_{m,m} p^{\beta(m,m-1)} \lambda_{\rho m,\rho(m-1)} g_{\rho m,\rho(m-1)} \\
&\ + A_{m-1,m} \mu_{\rho m,\rho(m-1)} p^{\nu_p(m!)-\nu_p((m-1)!)} p^{\beta(m-1,m-1)} \lambda_{\rho(m-1),\rho(m-1)} \hat{u} g_{\rho(m-1),\rho(m-1)}.
\end{aligned}$$

From Lemma 6.3.2 we know that in this case $\hat{u} g_{\rho(m-1),\rho(m-1)} = g_{\rho m,\rho(m-1)}$ and from Proposition 5.3.3 we know that $\beta(m, m-1) = 0$ and $\beta(m,m) = 0$, hence the coefficient of $g_{\rho m,\rho(m-1)}$ is given by

$$\lambda_{\rho m,\rho m} + \lambda_{\rho m,\rho(m-1)} r^{m-1} = r^m \lambda_{\rho m,\rho(m-1)} + A_{m-1,m} \mu_{\rho m,\rho(m-1)} \lambda_{\rho(m-1),\rho(m-1)}$$

which gives us that

$$A_{m-1,m} = \mu^{-1}_{\rho m,\rho(m-1)} \lambda^{-1}_{\rho(m-1),\rho(m-1)} ((r^{m-1} - r^m) \lambda_{\rho m,\rho(m-1)} + \lambda_{\rho m,\rho m})$$

which is a p-adic unit.

Case (ii): Let $m = p$. From Equation (6.2) and Proposition 6.2.2 we have that the coefficient of $g_{\rho m,\rho m}$ on each side is given by

$$r^m \lambda_{\rho m,\rho m} = A_{m,m} \lambda_{\rho m,\rho m},$$

hence we have $A_{m,m} = r^m$ as before.

We can look at the terms which will contribute to the coefficient of $g_{\rho m,\rho(m-1)}$. Using the second case given in Proposition 6.2.3 on the left hand side of equation (6.2) we get

$$\begin{aligned}
&(p^{\beta(m,m)} \lambda_{\rho m,\rho m} p + p^{\beta(m,m-1)} \lambda_{\rho m,\rho(m-1)} r^{m-1}) g_{\rho m,\rho(m-1)} \\
&\ = A_{m,m} p^{\beta(m,m-1)} \lambda_{\rho m,\rho(m-1)} g_{\rho m,\rho(m-1)} \\
&\ + A_{m-1,m} \mu_{\rho m,\rho(m-1)} p^{\nu_p(m!)-\nu_p((m-1)!)} p^{\beta(m-1,m-1)} \lambda_{\rho(m-1),\rho(m-1)} \hat{u} g_{\rho(m-1),\rho(m-1)}.
\end{aligned}$$

From Lemma 6.3.2 we know that in this case $\hat{u}g_{\rho(m-1),\rho(m-1)} = g_{\rho m,\rho(m-1)}$. We also know from Proposition 5.3.3 that $\beta(m, m-1) = \nu_p(p!) = 1$ and $\beta(m, m) = 0$. This gives us that the coefficient of $g_{\rho m,\rho(m-1)}$ is given by

$$\lambda_{\rho m,\rho m}p+p\lambda_{\rho m,\rho(m-1)}r^{m-1} \\ = r^m p\lambda_{\rho m,\rho(m-1)} + A_{m-1,m}\mu_{\rho m,\rho(m-1)}p\lambda_{\rho(m-1),\rho(m-1)}$$

which gives us that

$$A_{m-1,m} = \mu^{-1}_{\rho m,\rho(m-1)}\lambda^{-1}_{\rho(m-1),\rho(m-1)}((r^{m-1} - r^m)\lambda_{\rho m,\rho(m-1)} + \lambda_{\rho m,\rho m})$$

which is a p-adic unit.

Case (iii): Now assume $m > p$. We find that $A_{m,m} = r^m$ in the same way as given in the other two cases.

Using the third case given in Proposition 6.2.3 on the left hand side of equation (6.2), we can look at all the terms which will contribute to the coefficient of $g_{\rho m,\rho(m-1)}$ and we get

$$(p^{\beta(m,m)}\lambda_{\rho m,\rho m}p^{\nu_p(m)+1} + p^{\beta(m,m-1)}\lambda_{\rho m,\rho(m-1)}r^{m-1})g_{\rho m,\rho(m-1)} \\ = A_{m,m}p^{\beta(m,m-1)}\lambda_{\rho m,\rho(m-1)}g_{\rho m,\rho(m-1)} \\ + A_{m-1,m}\mu_{\rho m,\rho(m-1)}p^{\nu_p(m!)-\nu_p((m-1)!)}p^{\beta(m-1,m-1)}\lambda_{\rho(m-1),\rho(m-1)}\hat{u}g_{\rho(m-1),\rho(m-1)}.$$

From Lemma 6.3.2 we have that $g_{\rho m,\rho(m-1)} = \frac{\hat{u}}{p}g_{\rho(m-1),\rho(m-1)}$, and we can work out that

$$\beta(m, m-1) = \nu_p(m!) + m - \nu_p((m-1)!) - (m-1) = \nu_p(m) + 1.$$

Hence the coefficient of $g_{\rho m,\rho(m-1)}$ is given by

$$\lambda_{\rho m,\rho m}p^{\nu_p(m)+1} + p^{\nu_p(m)+1}\lambda_{\rho m,\rho(m-1)}r^{m-1} \\ = r^m p^{\nu_p(m)+1}\lambda_{\rho m,\rho m-1} + A_{m-1,m}\mu_{\rho m,\rho(m-1)}p^{\nu_p(m)+1}\lambda_{\rho(m-1),\rho(m-1)},$$

which gives us that

$$A_{m-1,m} = \mu^{-1}_{\rho m,\rho(m-1)}\lambda^{-1}_{\rho(m-1),\rho(m-1)}((r^{m-1} - r^m)\lambda_{\rho m,\rho(m-1)} - \lambda_{\rho m,\rho m})$$

which is a p-adic unit. □

6.4 Conjugation

In this subsection we prove the odd primary analogue of [BS05, Theorem 4.2]. The proof in [BS05] is incomplete; however a finishing argument appears in [Sna09, §5.4.6] which completes the proof. A more succinct proof from an idea suggested by Francis Clarke also appears as [Sna09, Theorem 5.4.3] however with typographical errors. The proof we will give here follows the argument suggested by Clarke.

From Proposition 6.3.1 we know A has the form

$$A = \begin{pmatrix} 1 & \upsilon_0 & a_{0,2} & a_{0,3} & a_{0,4} & \cdots \\ 0 & r & \upsilon_1 & a_{1,3} & a_{1,4} & \cdots \\ 0 & 0 & r^2 & \upsilon_2 & a_{2,4} & \cdots \\ 0 & 0 & 0 & r^3 & \upsilon_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $r = q^{p-1}$ as before for q a topological generator of the p-adic units, $\upsilon_i \in \mathbb{Z}_p^\times$ for all $i \geqslant 0$ and $a_{i,j} \in \mathbb{Z}_p$ for all $i, j \geqslant 0$.

Let

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \upsilon_0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \upsilon_0\upsilon_1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \upsilon_0\upsilon_1\upsilon_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

lying in $U_\infty\mathbb{Z}_p$, then its inverse is

$$E^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \upsilon_0^{-1} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \upsilon_0^{-1}\upsilon_1^{-1} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \upsilon_0^{-1}\upsilon_1^{-1}\upsilon_2^{-1} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and we can conjugate A by E to get

$$EAE^{-1} = C = \begin{pmatrix} 1 & 1 & c_{0,2} & c_{0,3} & c_{0,4} & \cdots \\ 0 & r & 1 & c_{1,3} & c_{1,4} & \cdots \\ 0 & 0 & r^2 & 1 & c_{2,4} & \cdots \\ 0 & 0 & 0 & r^3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

for some $c^{i,j} \in \mathbb{Z}_p$.

We want to know if we can turn C into a more desirable form, i.e. get rid of all the terms above the superdiagonal and produce a matrix

$$R = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & r & 1 & 0 & 0 & \cdots \\ 0 & 0 & r^2 & 1 & 0 & \cdots \\ 0 & 0 & 0 & r^3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Theorem 6.4.1. *There exists a matrix $U \in U_\infty \mathbb{Z}_p$ such that $UCU^{-1} = R$. Moreover one is given by $U = (u_{i,j})_{i,j \geqslant 0} \in U_\infty \mathbb{Z}_p$ where the first row is chosen to be*

$$u_{0,j} = \begin{cases} 1 & \textit{if } j = 0 \\ 0 & \textit{otherwise} \end{cases}$$

and the next row is defined recursively from the previous one as follows;

$$u_{i+1,j} = \left(\sum_{s=i}^{j-2} u_{i,s} c_{s,j} \right) + u_{i,j-1} + (r^j - r^i) u_{i,j}.$$

Proof. Let U be the matrix defined recursively above. We need to show that $U \in U_\infty \mathbb{Z}_p$. It is clear that $u_{i,j} \in \mathbb{Z}_p$ for $i, j \geqslant 0$. It can be shown that $u_{i,j} = 0$ if $i > j$ by induction on i. It is true from the formula that $u_{1,0} = 0$. Now assume that $u_{i-1,j} = 0$ for all $j < i-1$. By the formula above for $i > j$,

$$u_{i,j} = u_{i-1,j-1} + (r^j - r^{i-1}) u_{i-1,j}.$$

Now one of two cases will apply. Firstly we have $j < i-1$, in which case both $u_{i-1,j-1}$ and $u_{i-1,j}$ are zero by assumption. Or in the second case $j = i-1$, in which case $u_{i-1,j-1}$ is still zero but now $u_{i-1,j}$ may not be zero however its coefficient is $(r^{i-1} - r^{i-1}) = 0$ and so the induction is complete.

We are left needing to show $u_{i,i} \in \mathbb{Z}_p^\times$ for all $i \geqslant 0$, then U will be invertible. We will do this by induction. Clearly $u_{0,0} = 1$ is in $\mathbb{Z}_p^\times$. Now assume that $u_{i,i} \in \mathbb{Z}_p^\times$, we show that means $u_{i+1,i+1} \in \mathbb{Z}_p^\times$ too. We know

$$u_{i+1,i+1} = u_{i,i} + (r^{i+1} - r^i) u_{i,i+1}$$

from the definition. Fermat's Little Theorem tells us that $r - 1 \equiv 0 \mod p$ so $r^{i+1} - r^i = r^i(r-1) \equiv 0 \mod p$, and by assumption $u_{i,i} \in \mathbb{Z}_p^\times$. Hence $u_{i+1,i+1}$ is a unit plus something divisible by p and so is also a unit. So by induction $u_{i,i} \in \mathbb{Z}_p^\times$ for all $i \geqslant 0$ and hence U is invertible.

Now we just need to show that $UCU^{-1} = R$ so we will compare entries $(UC)_{i,j}$ and $(RU)_{i,j}$. Diagonally $(UC)_{i,i} = r^i u_{i,i} = (RU)_{i,i}$. Now let $j > i$, the entries of UC and RU are given as follows:

$$\begin{aligned} (UC)_{i,j} &= u_{i,i} c_{i,j} + u_{i,i+1} c_{i+1,j} + \cdots + u_{i,j-2} c_{j-2,j} + u_{i,j-1} + r^j u_{i,j} \\ &= \left(\sum_{s=i}^{j-2} u_{i,s} c_{s,j} \right) + u_{i,j-1} + r^j u_{i,j} \end{aligned}$$

and

$$(RU)_{i,j} = r^i u_{i,j} + u_{i+1,j}.$$

From our recurrence relation for the entries $u_{i,j}$ we know that

$$\left(\sum_{s=i}^{j-2} u_{i,s} c_{s,j} \right) + u_{i,j-1} + r^j u_{i,j} = u_{i+1,j} + r^i u_{i,j}$$

and so we get that

$$(UC)_{i,j} = (RU)_{i,j}.$$

Hence $(UC)_{i,j} = (RU)_{i,j}$ for all $i, j \geqslant 0$ and $j \geqslant i$. $\square$

Our chosen U is just one example of a matrix which will work, any coefficients can be chosen for the first row of the matrix providing $u_{0,0} = 1$ or any unit, we just choose all of the others to be zero to simplify things.

So in summary we have shown the following result.

Theorem 6.4.2. *Under the isomorphism Λ the automorphism $1 \wedge \psi^q$ corresponds to a matrix in the conjugacy class of*

$$R = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & r & 1 & 0 & 0 & \cdots \\ 0 & 0 & r^2 & 1 & 0 & \cdots \\ 0 & 0 & 0 & r^3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Chapter 7

Applications

7.1 The Map $1 \wedge \phi_n$ and the Matrix X_n.

In this chapter we will use Theorem 6.4.2 to obtain and study further information on the map

$$1 \wedge \phi_n = 1 \wedge (\psi^q - 1)(\psi^q - r) \ldots (\psi^q - r^{n-1}) : \ell \wedge \ell \to \ell \wedge \ell.$$

The analogous map was first studied by Milgram in [Mil75] in relation to real connective K-Theory ko localised at the prime 2. The method used here is the method used in [BS05, Theorem 5.4] to produce the 2-local analogue of Theorem 7.1.8. In the 2 primary case, the formulae in [BS05, Theorem 5.4(iv)] for finding entries in the analogous 2 primary version of the matrix X_n (as defined below) are incorrect. The formulae I have stated in Theorem 7.1.8(iii) with rs replaced by 9s will hold true (as will the proof) for the prime 2.

Recall that we let $U_\infty \mathbb{Z}_p$ be infinite upper triangular matrices with entries in $\mathbb{Z}_p$ which are invertible. In practice this means they must have p-adic units on the diagonal. This is the multiplicative group of units of the ring $\tilde{U}_\infty \mathbb{Z}_p$ of upper triangular matrices with entries in the p-adic integers. Generalising the group isomorphism Λ of Theorem 3.1.3 we can construct the following diagram

$$\begin{tikzcd} U_\infty \mathbb{Z}_p \arrow[rr, "\Lambda", "\cong"'] \arrow[d, "\cap"'] & & \mathrm{Aut}^0_{\text{left-}\ell\text{-mod}}(\ell \wedge \ell) \arrow[d, "\cap"'] \\ \tilde{U}_\infty \mathbb{Z}_p \arrow[rr, "\lambda"'] & & \mathrm{End}_{\text{left-}\ell\text{-mod}}(\ell \wedge \ell) \end{tikzcd}$$

where $\lambda_{|U_\infty \mathbb{Z}_p} = \Lambda$. The map Λ was constructed by sending a matrix $A \in U_\infty \mathbb{Z}_p$ to $\Lambda(A) = \sum_{m \geqslant n} A_{n,m} \iota_{m,n}$. The same process can be applied to a matrix $A' \in \tilde{U}_\infty \mathbb{Z}_p$ by letting

$$\lambda(A') = \sum_{m \geqslant n} A'_{n,m} \iota_{m,n}$$

to obtain a left-ℓ-module endomorphism of $\ell \wedge \ell$. This is a multiplicative map by the same argument given for Λ in the proof of Proposition 3.3.7.

By moving from $U_\infty \mathbb{Z}_p$ to $\tilde{U}_\infty \mathbb{Z}_p$ it is now possible to use the additive structure given by matrix addition. The concept of addition in the group $\mathrm{End}_{\text{left-}\ell\text{-mod}}(\ell \wedge \ell)$ was given in Definition 1.1.7.

Now for $A, B \in \tilde{U}_\infty \mathbb{Z}_p$ we have

$$\begin{aligned}\lambda(A+B) &= \sum_{m \geqslant n} (A+B)_{n,m} \iota_{m,n} \\ &= \sum_{m \geqslant n} A_{n,m} \iota_{m,n} + \sum_{m \geqslant n} B_{n,m} \iota_{m,n} \\ &= \lambda(A) + \lambda(B).\end{aligned}$$

From Theorem 6.4.2 we know that the map $1 \wedge \psi^q$ corresponds under Λ to an element in the conjugacy class of the matrix R, where, for $r = q^{p-1}$,

$$R = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & r & 1 & 0 & 0 & \cdots \\ 0 & 0 & r^2 & 1 & 0 & \cdots \\ 0 & 0 & 0 & r^3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This means that there exists a matrix $H \in U_\infty \mathbb{Z}_p$ such that

$$1 \wedge \psi^q = \lambda(HRH^{-1}).$$

We also have that $1 \wedge 1 = \lambda(I)$ where I is the infinite identity matrix, hence

$$1 \wedge r^i = \lambda(r^i I).$$

Definition 7.1.1. Let

$$\phi_n = (\psi^q - 1)(\psi^q - r) \cdots (\psi^q - r^{n-1})$$

and let $R_n = R - r^{n-1} I \in \tilde{U}_\infty \mathbb{Z}_p$ and $X_n = R_1 R_2 \cdots R_n \in \tilde{U}_\infty \mathbb{Z}_p$.

Proposition 7.1.2. $1 \wedge \phi_n = \lambda(H X_n H^{-1})$

Proof. Using the definition above and previous discussion

$$\begin{aligned}1 \wedge (\psi^q - r^{n-1}) &= (1 \wedge \psi^q) - (1 \wedge r^{n-1}) \\ &= \lambda(HRH^{-1}) - \lambda(r^{n-1} I) \\ &= \lambda(HRH^{-1} - r^{n-1} I) \\ &= \lambda(HRH^{-1} - H r^{n-1} I H^{-1}) \\ &= \lambda(H(R - r^{n-1} I) H^{-1}) \\ &= \lambda(H R_n H^{-1}).\end{aligned}$$

Hence when we look at $1 \wedge \phi_n$ we see that

$$\begin{aligned}
1 \wedge \phi_n &= 1 \wedge (\psi^q - 1)(\psi^q - r) \cdots (\psi^q - r^{n-1}) \\
&= (1 \wedge (\psi^q - 1))(1 \wedge (\psi^q - r)) \cdots (1 \wedge (\psi^q - r^{n-1}) \\
&= \lambda(HR_1H^{-1})\lambda(HR_2H^{-1}) \cdots \lambda(HR_nH^{-1}) \\
&= \lambda(HR_1H^{-1}HR_2H^{-1} \cdots HR_nH^{-1}) \\
&= \lambda(HR_1R_2 \cdots R_nH^{-1}) \\
&= \lambda(HX_nH^{-1}). \qquad \square
\end{aligned}$$

Before we can prove our main result Theorem 7.1.8, we first need to introduce Gaussian polynomials.

Definition 7.1.3. A *Gaussian polynomial* is of the form

$$\begin{bmatrix} n \\ i \end{bmatrix} = \prod_{j=0}^{i-1} \frac{1 - x^{n-j}}{1 - x^{j-i}}$$

where $n, i \in \mathbb{N}_0$.

We will need the value of this polynomial when $x = r$ which will be denoted $\begin{bmatrix} n \\ i \end{bmatrix}_r$. The lemma below can be used to show inductively that the Gaussian polynomials are indeed polynomials.

Lemma 7.1.4. *The following analogue of Pascal's identity holds for Gaussian polynomials:*

$$\begin{bmatrix} n \\ i \end{bmatrix}_r = r^{n-i} \begin{bmatrix} n-1 \\ i-1 \end{bmatrix}_r + \begin{bmatrix} n-1 \\ i \end{bmatrix}_r.$$

Proof.

$$\begin{aligned}
r^{n-i} \begin{bmatrix} n-1 \\ i-1 \end{bmatrix}_r + \begin{bmatrix} n-1 \\ i \end{bmatrix}_r &= r^{n-i} \prod_{j=0}^{i-2} \frac{1 - r^{n-1-j}}{1 - r^{i-1-j}} + \prod_{j=0}^{i-1} \frac{1 - r^{n-1-j}}{1 - r^{i-j}} \\
&= \prod_{j=0}^{i-3} \frac{1 - r^{n-1-j}}{1 - r^{i-1-j}} (1 - r^{n-i+1}) \left(\frac{r^{n-i}}{1 - r} + \frac{1 - r^{n-i}}{(1 - r)(1 - r^i)} \right) \\
&= \prod_{j=0}^{i-3} \frac{1 - r^{n-1-j}}{1 - r^{i-1-j}} \frac{(1 - r^{n-i+1})(1 - r^n)}{(1 - r)(1 - r^i)} \\
&= \prod_{j=0}^{i-1} \frac{1 - r^{n-j}}{1 - r^{i-j}} = \begin{bmatrix} n \\ i \end{bmatrix}_r.
\end{aligned}$$

$\square$

Another tool we will use in the proof of Theorem 7.1.8 is splitting up the matrix R, as detailed below, in order to make calculating powers of it much easier.

Definition 7.1.5. Let

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & r & 0 & 0 & 0 & \cdots \\ 0 & 0 & r^2 & 0 & 0 & \cdots \\ 0 & 0 & 0 & r^3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then $R = D + S$.

The three facts in the following lemma are easy to prove.

Lemma 7.1.6.

$$D^i = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & r^i & 0 & 0 & 0 & \cdots \\ 0 & 0 & r^{2i} & 0 & 0 & \cdots \\ 0 & 0 & 0 & r^{3i} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$S^j = \begin{pmatrix} \overbrace{0 \quad \cdots \quad 0}^{j} & 1 & 0 & 0 & 0 & \cdots \\ 0 \quad \cdots \quad 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 \quad \cdots \quad 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 \quad \cdots \quad 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots \quad \cdots \quad \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and $SD = rDS$*.*

Lemma 7.1.7. *For two matrices* $D, S \in \tilde{U}_\infty \mathbb{Z}_p$ *such that* $SD = rDS$ *and any* $n \in \mathbb{N}_0$ *we have*

$$(D+S)^n = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_r D^i S^{n-i}. \tag{7.1}$$

Proof. We can show this by induction. In the case where $n = 1$ we have $(D+S)^1 = \begin{bmatrix}1\\0\end{bmatrix}_r S + \begin{bmatrix}1\\1\end{bmatrix}_r D = S + D$ which is obviously true. Now assume the statement is true for $(D+S)^{n-1}$, then we have

$$\begin{aligned}(D+S)^n &= \left(\sum_{i=0}^{n-1} \begin{bmatrix}n-1\\i\end{bmatrix}_r D^i S^{n-1-i}\right)(D+S)\\ &= \sum_{i=0}^{n-1} \begin{bmatrix}n-1\\i\end{bmatrix}_r D^i S^{n-1-i} D + \sum_{i=0}^{n-1} \begin{bmatrix}n-1\\i\end{bmatrix}_r D^i S^{n-i}\\ &= \sum_{i=0}^{n-1} \begin{bmatrix}n-1\\i\end{bmatrix}_r r^{n-1-i} D^{i+1} S^{n-1-i} + \sum_{i=0}^{n-1} \begin{bmatrix}n-1\\i\end{bmatrix}_r D^i S^{n-i}\end{aligned}$$

We can now reindex the first sum by letting $j = i+1$ and obtain

$$\begin{aligned}&\sum_{j=1}^{n} \begin{bmatrix}n-1\\j-1\end{bmatrix}_r r^{n-j} D^j S^{n-j} + \sum_{i=0}^{n-1} \begin{bmatrix}n-1\\i\end{bmatrix}_r D^i S^{n-i}\\ &\quad= \sum_{j=1}^{n-1} \left(\begin{bmatrix}n-1\\j-1\end{bmatrix}_r r^{n-j} + \begin{bmatrix}n-1\\j\end{bmatrix}_r\right) D^j S^{n-j} + \begin{bmatrix}n-1\\0\end{bmatrix}_r S^n + \begin{bmatrix}n-1\\n-1\end{bmatrix}_r D^n\\ &\quad= \sum_{j=1}^{n-1} \begin{bmatrix}n\\j\end{bmatrix}_r D^j S^{n-j} + D^n + S^n\\ &\quad= \sum_{j=0}^{n} \begin{bmatrix}n\\j\end{bmatrix}_r D^j S^{n-j}\end{aligned}$$

as required, using Lemma 7.1.4. □

Now we can state and prove the main theorem of this section which gives us more detailed and specific information on how the map ϕ_n relates to the pieces of the splitting. We will also use a p-local version of part (i) in Theorem 7.2.2 in the next section. Let Ω denote the homotopy equivalence giving Kane's splitting, i.e.

$$\Omega : \bigvee_{n\geq 0} \ell \wedge \mathcal{K}(n) \to \ell \wedge \ell.$$

Theorem 7.1.8. *(i) The first n columns of X_n are trivial.*

(ii) Let C_n be the mapping cone of the restriction of Ω to the first n 'pieces' of the splitting of $\ell \wedge \ell$, i.e.

$$C_n = Cone\left(\Omega_| : \bigvee_{0\leqslant m\leqslant n-1} \ell \wedge \mathcal{K}(m) \to \ell \wedge \ell\right)$$

which is also a left ℓ-module. Then in the p-complete stable homotopy category there exists a commutative diagram of left ℓ-module spectra of the form

$$\begin{tikzcd} \ell\wedge\ell \arrow[rr, "1\wedge\phi_n"] \arrow[dr, "\pi_n"'] & & \ell\wedge\ell \\ & C_n \arrow[ur, "\hat{\phi}_n"'] & \end{tikzcd}$$

where π_n is the cofibre of $\Omega_|$ and $\hat{\phi}_n$ is determined up to homotopy by the diagram.

(iii) For $n \geqslant 1$ we have

$$(X_n)_{s,s+c} = 0 \textit{ if } c < 0 \textit{ or } c > n$$

and for $0 \leqslant c \leqslant n$ we have

$$(X_n)_{s,s+c} = \sum_{i=c}^{n} (-1)^{n-i} r^{\binom{n-i}{2}+(s-1)(i-c)} \begin{bmatrix} n \\ i \end{bmatrix}_r \begin{bmatrix} i \\ i-c \end{bmatrix}_r .$$

Proof. (i) The result is certainly true of $X_1 = R_1$. We will proceed to prove the result for all $n \geqslant 1$ by induction. Assume that the first n columns of X_n are trivial, i.e. $(X_n)_{i,j} = 0$ if $j \leqslant n$. By definition $X_{n+1} = X_n R_{n+1}$. We also know that $(R_{n+1})_{i,j} = 0$ unless $(i,j) = (s,s)$ or $(s, s+1)$ and that $(R_{n+1})_{n+1,n+1} = 0$ also. Now

$$(X_{n+1})_{i,j} = (X_n)_{i,j-1}(R_{n+1})_{j-1,j} + (X_n)_{i,j}(R_{n+1})_{j,j}$$

which is zero if $j \leqslant n$ because $(X_n)_{i,j-1}, (X_n)_{i,j} = 0$. If $j = n+1$,

$$(X_{n+1})_{i,n+1} = (X_n)_{i,n}(R_{n+1})_{n,n+1} + (X_n)_{i,n+1}(R_{n+1})_{n+1,n+1}$$

which is zero because $(X_n)_{i,n}, (R_{n+1})_{n+1,n+1} = 0$.

(ii) From Lemma 7.1.2 we know that $1 \wedge \phi_n = \lambda(HX_nH^{-1})$. In order for $1\wedge\phi_n$ to factor via C_n (and the diagram to commute) we need to show that HX_nH^{-1} corresponds under λ to a left ℓ-module endomorphism of $\vee_{m\geqslant 0}\ell\wedge\mathcal{K}(m)$ which is trivial on each piece $\ell\wedge\mathcal{K}(m)$ where $m \leqslant n-1$. The map $\lambda(HX_nH^{-1})$ acts trivially on pieces $\ell\wedge\mathcal{K}(m)$ where $m \leqslant n-1$ if each map

$$\iota_{m,k} : \ell\wedge\mathcal{K}(m) \to \ell\wedge\mathcal{K}(k)$$

has coefficient zero when $m \leqslant n-1$ in the explicit description of $\lambda(HX_nH^{-1})$. This gives us that $(HX_nH^{-1})_{k,m}$ needs to equal zero when $m \leqslant n-1$. This is true though because we know from part (i) that the first n columns of X_n are trivial, and since H is upper triangular and invertible, this means the first n columns of HX_nH^{-1} are also trivial.

(iii) For the first part, we know all the R_is are upper triangular matrices so X_n will be too, hence $(X_n)_{s,s+c} = 0$ if $c < 0$. We can show that $(X_n)_{s,s+c} = 0$ if $c > n$ by induction on n. The initial case for the induction is X_1 where this clearly holds. Assume that $(X_{n-1})_{s,s+c} = 0$ if $c > n-1$. As in part (i), we know that $X_n = X_{n-1}R_n$ and that $(X_n)_{i,j} = (X_{n-1})_{i,j-1}(R_n)_{j-1,j} + (X_{n-1})_{i,j}(R_n)_{j,j}$. Now let $j > n$, then

$$(X_n)_{s,s+j} = (X_{n-1})_{s,s+j-1}(R_n)_{s+j-1,s+j} + (X_{n-1})_{s,s+j}(R_n)_{s+j,s+j}$$

and this is zero because both $(X_{n-1})_{s,s+j-1}$ and $(X_{n-1})_{s,s+j}$ are zero by the inductive hypothesis, hence the induction is complete.

In order to prove the second part we are going to first consider the matrix R rather than X_n. Recall from Definition 7.1.5 that we can think of R as the matrix sum $D + S$. Then we have that

$$R^n = (D+S)^n.$$

Using Lemma 7.1.6 we can see that any product of the form D^iS^j can be expressed as the matrix

$$D^iS^j = \begin{array}{l} \quad\overbrace{\hphantom{0 \quad \dots \quad 0}}^{j} \\ \begin{pmatrix} 0 & \dots & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & \cdots & 0 & 0 & r^i & 0 & 0 & \cdots \\ 0 & \cdots & 0 & 0 & 0 & r^{2i} & 0 & \cdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & r^{3i} & \cdots \\ \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array}.$$

In other words a single entry can be written as

$$(D^iS^j)_{s,t} = \begin{cases} r^{(s-1)i} & \text{if } t = s+j \\ 0 & \text{otherwise.} \end{cases}$$

Recall that

$$SD = rDS$$

so, although the multiplication is non-commutative, there is a simple relation between the two orderings. Now to be able to find any entry in the matrix R^n, using Lemma 7.1.7, we have a formula in terms of the matrices D^iS^j, equation (7.1), for which we know all the entries. Hence

$$(R^n)_{s,s+c} = ((D+S)^n)_{s,s+c} = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_r (D^iS^{n-i})_{s,s+c}.$$

For any particular value of c at most one term in this sum is non-zero, namely the $i = n-c$ term if $0 \leqslant c \leqslant n$. So we have that

$$(R^n)_{s,s+c} = \begin{bmatrix} n \\ n-c \end{bmatrix}_r (D^{n-c}S^c)_{s,s+c} = \begin{bmatrix} n \\ n-c \end{bmatrix}_r r^{(s-1)(n-c)}.$$

We can now use this information to produce a formula for the required entries in the matrix X_n. By [CCW01, Proposition 8] we know

$$\begin{aligned} X_n &= (R-1)(R-r)\cdots(R-r^{n-1}) \\ &= \sum_{i=0}^{n}(-1)^{n-i} r^{\binom{n-i}{2}} \begin{bmatrix} n \\ i \end{bmatrix}_r R^i. \end{aligned}$$

As before, to produce the entry in the $(s, s+c)$th place in X_n we just need to know the entries in the same place in all the powers of R in the above sum, hence

$$\begin{aligned} (X_n)_{s,s+c} &= \sum_{i=0}^{n}(-1)^{n-i} r^{\binom{n-i}{2}} \begin{bmatrix} n \\ i \end{bmatrix}_r (R^i)_{s,s+c} \\ &= \sum_{i=c}^{n}(-1)^{n-i} r^{\binom{n-i}{2}} \begin{bmatrix} n \\ i \end{bmatrix}_r \begin{bmatrix} i \\ i-c \end{bmatrix}_r r^{(s-1)(i-c)}. \end{aligned}$$

The final sum has been reduced from a sum starting from 0 to a sum starting from c as the second Gaussian polynomial for $i \leqslant c$ is zero.

□

7.2 K-Theory Operations

The next application we will consider provides us with another way of viewing a ring of operations on p-local complex connective K-Theory. We will work in this chapter in the p-local stable homotopy category. In a slight abuse of notation let ℓ now denote the Adams summand of p-local complex connective K-Theory (rather than the p-complete version) for the rest of this section. Let $\tilde{U}_\infty\mathbb{Z}_{(p)}$ be the ring of upper triangular matrices with entries in the p-local integers and using standard notation let $E^*(F) = [F, E]_{-*}$ for spectra E, F.

For the following application we will need to use the theory of filtered topological rings. A good reference for this is [Nor68, Chapter 9]. Recall that a *decreasing filtration* on a ring R is a family of two-sided ideals $\{R_n\}_{n\geqslant 0}$ such that $R_{n+1} \subseteq R_n$ for all $n \geqslant 0$. This gives the ring a filtration topology. The identity map induces a map for each $n \geqslant 0$

$$R_{n+1} \to R_n$$

which together give us the following sequence of maps

$$\frac{R}{R_0} \leftarrow \frac{R}{R_1} \leftarrow \cdots \leftarrow \frac{R}{R_{n-1}} \leftarrow \frac{R}{R_n} \leftarrow \frac{R}{R_{n+1}} \leftarrow \cdots.$$

Let $\hat{R} = \varprojlim_n \frac{R}{R_n}$ be the inverse limit of this system. The ring is *complete* if the obvious map $R \to \hat{R}$ is an isomorphism.

Definition 7.2.1. We can define a filtration on $\tilde{U}_\infty \mathbb{Z}_{(p)}$ by letting the first n columns be zero, i.e. for $n \in \mathbb{N}$, let

$$U_n = \{X \in \tilde{U}_\infty \mathbb{Z}_{(p)} : x_{i,j} = 0 \text{ if } j \leqslant n\}.$$

This gives us a decreasing filtration

$$\tilde{U}_\infty \mathbb{Z}_{(p)} = U_0 \supset U_1 \supset U_2 \supset \cdots$$

where each U_n is a two-sided ideal of $\tilde{U}_\infty \mathbb{Z}_{(p)}$.

Filtering by columns gives us a two-sided ideal because our matrices are upper triangular, this would not be the case if we filtered by rows. This can be regarded as the natural filtration on $\tilde{U}_\infty \mathbb{Z}_{(p)}$. Also $\tilde{U}_\infty \mathbb{Z}_{(p)}$ is complete with respect to this topology.

Theorem 7.2.2. *The ring of degree zero operations of the Adams summand of complex connective p-local K-Theory, $\ell^0(\ell)$, is isomorphic as a topological ring to the completion of the subring of $\tilde{U}_\infty \mathbb{Z}_{(p)}$ generated by R.*

Proof. Define a map

$$\alpha : \ell^0(\ell) \to \tilde{U}_\infty \mathbb{Z}_{(p)}$$

by saying α is the continuous ring homomorphism determined by sending $\psi^q \in \ell^0(\ell)$ to $R \in \tilde{U}_\infty \mathbb{Z}_{(p)}$. Recall that $\phi_n = (\psi^q - 1)(\psi^q - r)\cdots(\psi^q - r^{n-1})$. Because of the definition of α we know that $\alpha(\phi_n) = (R-1)(R-r)\cdots(R-r^{n-1}) = X_n$.

We have the following description of $\ell^0(\ell)$ from [CCW05, Theorem 4.4]

$$\ell^0(\ell) = \left\{ \sum_{n=0}^{\infty} a_n \phi_n : a_n \in \mathbb{Z}_{(p)} \right\}.$$

This is complete in the filtration topology when filtered by ideals

$$(\ell^0(\ell))_m = \left\{ \sum_{n=m}^{\infty} a_n \phi_n : a_n \in \mathbb{Z}_{(p)} \right\}.$$

We know $\alpha(\phi_n) = X_n$ and by Theorem 7.1.8 (i) we know that the first n columns of X_n are trivial hence $\alpha(\phi_n) \in U_n$. This tells us that α respects the filtration and so when applied to infinite sums $\alpha\left(\sum_{n=0}^{\infty} a_n \phi_n\right) = \sum_{n=0}^{\infty} a_n X_n$ is well defined (each entry in the matrix is a finite sum).

Let $S = \left\{\sum_{n=0}^{N} a_n R^n : a_n \in \mathbb{Z}_{(p)}, N \in \mathbb{N}_0\right\}$. It is clear that $S \subseteq \operatorname{im}(\alpha)$. Because α is continuous and $\tilde{U}_\infty \mathbb{Z}_{(p)}$ is complete it follows that the completion of S is precisely the image of α. Finally because we know $\ker \alpha = \{\sum_{n=0}^{\infty} a_n \phi_n : a_n = 0 \text{ for all } n\}$ it is clear that α is injective. $\square$

This gives us quite a nice way of looking at $\ell^0(\ell)$ which is in its own right a quite complex ring. Because the filtration on $\tilde{U}_\infty\mathbb{Z}_{(p)}$ is the most natural one to pick this in turn makes the filtration on $\ell^0(\ell)$ appear more natural.

In exactly the same way we can obtain a description of ${ku_{(p)}}^0(ku_{(p)})$ as the completion of the subring of $\tilde{U}_\infty\mathbb{Z}_{(p)}$ generated by

$$R' = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & q & 1 & 0 & 0 & \cdots \\ 0 & 0 & q^2 & 1 & 0 & \cdots \\ 0 & 0 & 0 & q^3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The proof follows as above using the description of ${ku_{(p)}}^0(ku_{(p)})$ given in [CCW05, Theorem 2.2] as

$${ku_{(p)}}^0(ku_{(p)}) = \left\{ \sum_{n=0}^{\infty} a_n \varphi'_n : a_n \in \mathbb{Z}_{(p)} \right\}$$

where $\varphi'_n = (\psi^q - 1)(\psi^q - q) \cdots (\psi^q - q^{n-1})$.

We can obtain a similar description of a ring of 2-local operations on real connective K-Theory. In a similar way to above define $\tilde{U}_\infty\mathbb{Z}_{(2)}$ to be the ring of upper triangular matrices with entries in the 2-local integers and let

$$R_{(2)} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 9 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 9^2 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 9^3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Theorem 7.2.3. *The ring of degree zero operations of real connective 2-local K-Theory ${ko_{(2)}}^0(ko_{(2)})$ is isomorphic as a topological ring to the completion of the subring of $\tilde{U}_\infty\mathbb{Z}_{(2)}$ generated by $R_{(2)}$.*

Proof. Using the description of ${ko_{(2)}}^0(ko_{(2)})$ given in [CCW05, Theorem 9.3(1)] as

$${ko_{(2)}}^0(ko_{(2)}) = \left\{ \sum_{n=0}^{\infty} a_n \varphi_n : a_n \in \mathbb{Z}_{(2)} \right\}$$

where $\varphi_n = (\psi^3 - 1)(\psi^3 - 9) \cdots (\psi^3 - 9^{n-1})$, the proof follows in exactly the same way as the previous Theorem. $\square$

The topological bialgebra $\ell^0(\ell)$ is $\mathbb{Z}_{(p)}$-dual to $L_0(\ell)$, where L is the periodic Adams summand. We have a basis for $L_0(\ell)$ given in [CCW05, Proposition 4.2] of $\{\hat{f}_n(z) : n \geqslant 0\}$ where $z = (u^{-1}v)^{p-1}$ and

$$\hat{f}_n(z) = \prod_{i=1}^{n} \frac{z - r^{i-1}}{r^n - r^{i-1}}.$$

The action of $\ell^0(\ell)$ on $L_0(\ell)$ is determined by the action of ψ^q which multiplies v by q and acts as the identity on u hence $\psi^q \hat{f}(z) = \hat{f}(rz)$. We look at the action of ψ^q on the basis elements and find that

$$\psi^q \hat{f}_n(z) = r^n \hat{f}_n(z) + \hat{f}_{n-1}(z).$$

This means that the action of ψ^q on the basis $\{\hat{f}_n(z) : n \geqslant 0\}$ of $L_0(\ell)$ is given by the matrix R.

This is not a coincidence but just two different ways of getting to the same answer. This way involves looking at the action of ψ^q on the basis for $L_0(\ell)$ which gives you exactly R and the other way as demonstrated in Chapter 5 involves looking at the basis for the whole graded object $\ell_*(\ell)$ and, via maps between the pieces of the splitting, looking at the action of ψ^q again but not keeping track what happens to the particular basis elements in any degree. This ambiguity over what happens to specific basis elements is taken care of by the fact that the matrix you produce is not precisely R but something in the conjugacy class of R.

In a similar way ${ku_{(p)}}^0(ku_{(p)})$ has as its dual ${K_{(p)}}_0(ku_{(p)})$. A basis for ${K_{(p)}}_0(ku_{(p)})$ is given by [CCW01, Proposition 3] as $\{h_n(w) : n \geqslant 0\}$ where $w = u^{-1}v$ and

$$h_n(w) = \prod_{i=1}^{n} \frac{w - q^{i-1}}{q^n - q^{i-1}}.$$

The action is again determined by that of ψ^q and we have that

$$\psi^q h_n w = q^n h_n(w) + h_{n-1}(w).$$

So the action is given by the matrix R'.

A similar statement is also true of the action of ${ko_{(2)}}^0(ko_{(2)})$ on its dual ${KO_{(2)}}_0(ko_{(2)})$. A basis for ${KO_{(2)}}_0(ko_{(2)})$ is given in [CCW05, Proposition 9.2(1)] of $\{g_n(x) : n \geqslant 0\}$ where $x = (u^{-1}v)^2$ and

$$g_n(x) = \prod_{i=1}^{n} \frac{x - 9^{i-1}}{9^n - 9^{i-1}}.$$

In this case the action is determined by ψ^3 which acts on the basis elements as

$$\psi^3 g_n(x) = 9^n g_n(x) + g_{n-1}(x).$$

So the action is given by the matrix $R_{(2)}$.

Appendix A

Results about p-adic valuations

Let p be an odd prime and let ν_p be the p-adic valuation function. The two primary versions of the first four of these statements appear in [BS05, Proposition 2.3 - Proposition 2.6].

Proposition A.1. *For any integer $n \geqslant 0$,*

$$\nu_p(r^{p^n} - 1) = n + 1,$$

where $r = q^{p-1}$ and $q \in \mathbb{Z}_p^\times$ is a topological generator (i.e. q generates a dense subgroup of $\mathbb{Z}_p^\times$).

Proof. We know

$$r^{p^n} - 1 = (r-1)(r^{p^n-1} + r^{p^n-2} + \cdots + r^2 + r + 1). \tag{A.1}$$

We can choose q to be a particular integer coprime to p which generates a dense subset of $\mathbb{Z}_p^\times$. This happens if q is primitive modulo p^2 that is if $\overline{q}$ is a generator for $(\mathbb{Z}/p^2)^\times$. So we can assume $(q,p) = 1$.

Let's first deal with the second factor of equation (A.1).

$$\begin{aligned} &r^{p^n-1} + r^{p^n-2} + \cdots + r + 1 \\ &\qquad = (r^{p-1} + r^{p-2} + \cdots + r + 1)(r^{p^n-p} + r^{p^n-2p} + \cdots + r^p + 1) \\ &\qquad = \prod_{k=0}^{n-1} (r^{p^k(p-1)} + r^{p^k(p-2)} + \cdots + r^{p^k} + 1). \end{aligned} \tag{A.2}$$

Each of the n brackets of the form $(r^{p^k(p-1)} + r^{p^k(p-2)} + \cdots + r^{p^k} + 1)$ for $k \in \mathbb{N}_0$, has one and only one factor of p as follows. We know $r \equiv 1 \mod p$ by Fermat's Little Theorem so $r^s \equiv 1 \mod p$ for all $s \in \mathbb{N}_0$. There are p terms in each bracket all congruent to 1 mod p so

$$r^{p^k(p-1)} + r^{p^k(p-2)} + \cdots + r^{p^k} + 1 \equiv p \mod p.$$

Hence we know p divides each bracket. We also know that $q^{\varphi(p^2)} \equiv 1 \mod p^2$ where $\varphi(p^2) = (p-1)p$ is Euler's totient function. Hence $q^{(p-1)p} \equiv 1 \mod p^2$ i.e. $r^p \equiv 1 \mod p^2$. Each summand in all but the first bracket of equation (A.2) is of the form $r^{p.p^{k-1}m}$ for $m = 0, 1, \ldots, p-1$, so all terms are congruent to 1 mod p^2 and hence each bracket bar the first is congruent to p mod p^2 and hence does not have a factor of p^2. If we take the first bracket of equation (A.2) and combine it with the remaining factor in equation (A.1) we have

$$(r-1)(r^{p-1} + r^{p-2} + \cdots + r + 1) = r^p - 1.$$

We know each bracket has a factor of p so $p^2 | r^p - 1$, we just need that $p^3 \nmid r^p - 1$ then the statement will follow.

We have that $\overline{q}$ generates $(\mathbb{Z}/p^2)^\times$, this means $\overline{q}$ generates $(\mathbb{Z}/p^3)^\times$ also. We know $q^{\varphi(p^3)} \equiv 1 \mod p^3$ by Euler's Theorem. Since $\overline{q}$ generates $(\mathbb{Z}/p^3)^\times$ and there are $\varphi(p^3)$ elements in $(\mathbb{Z}/p^3)^\times$, this is the lowest power of q which is congruent to 1 mod p^3 hence no lower power of q can be congruent to 1 mod p^3. But $p(p-1) < p^2(p-1) = \varphi(p^3)$ so $q^{p(p-1)} \not\equiv 1 \mod p^3$, hence $p^3 \nmid r^p - 1$. □

Proposition A.2. *For any integer $l \geqslant 1$,*

$$\nu_p(r^l - 1) = \nu_p(l) + 1.$$

Proof. We can factorise

$$(r^l - 1) = (r-1)(r^{l-1} + r^{l-2} + \cdots + r + 1).$$

We know that $\nu_p(r-1) = 1$ as follows. Firstly $p|r-1$ by Fermat's Little Theorem. Also $\overline{q}$ generates $(\mathbb{Z}/p^2)^\times$ so $q^{\varphi(p^2)}$ is the lowest power of q which is congruent to 1 mod p^2. Since $p - 1 < p(p-1) = \varphi(p^2)$, r cannot be congruent to 1 mod p^2 and so $p^2 \nmid r - 1$.

If $p|l$ we can factorise further:

$$(r^{l-1} + r^{l-2} + \cdots + 1) = (r^{p-1} + r^{p-2} + \cdots + 1)(r^{l-p} + r^{l-2p} + \cdots + r^p + 1).$$

If $p|\frac{l}{p}$ we can continue this process until there are no more factors of p, i.e. $p \nmid \frac{l}{p^k}$ for some $k \in \mathbb{N}_0$. So we have factorised out $\nu_p(l)$ brackets. From Proposition A.1 we know the p-adic valuation of each of the brackets is 1. The remaining term is a sum of $\frac{l}{p^k}$ terms each of the form r^m for some $m \in \mathbb{N}_0$. By Fermat's Little Theorem we know each of these $r^m \equiv 1 \mod p$ so the whole remaining term is congruent to $\frac{l}{p^k} \mod p \not\equiv 0 \mod p$. □

Proposition A.3. *For any integer $l \geqslant 1$,*

$$\nu_p\left(\prod_{i=1}^{l}(r^l - r^{i-1})\right) = \nu_p(l!) + l.$$

Proof. Expanding the product gives

$$\begin{aligned}\prod_{i=1}^{l}(r^l - r^{i-1}) &= (r^l-1)(r^l-r)(r^l-r^2)\dots(r^l-r^{l-1})\\ &= r^{\frac{l(l-1)}{2}}(r^l-1)(r^{l-1}-1)(r^{l-2}-1)\dots(r-1).\end{aligned}$$

So when we take the valuation,

$$\begin{aligned}\nu_p\left(\prod_{i=1}^{l}(r^l-r^{i-1})\right) &= \nu_p(r^{\frac{l(l-1)}{2}}) + \nu_p(r^l-1) + \nu_p(r^{l-1}-1) + \dots + \nu_p(r-1)\\ &= 0 + (\nu_p(l)+1) + (\nu_p(l-1)+1) + \dots + (\nu_p(1)+1)\\ &= \nu_p(l(l-1)(l-2)\dots 1) + l\\ &= \nu_p(l!) + l.\end{aligned}$$

□

Proposition A.4. *For any integer $l \geqslant 1$,*

$$\nu_p(l!) + l = \frac{pl - S_p(l)}{p-1},$$

where $S_p(l)$ is the sum of the digits of l in its base p expansion, i.e. if $l = l_0 + l_1 p + l_2 p^2 + \cdots$ with $l_i \in \{0, 1, \dots, p-1\}$ then $S_p(l) = l_0 + l_1 + l_2 + \cdots$.

Proof. Let $l = \sum_{i=0}^{N} l_i p^i$. We have

$$\nu_p(l!) = \sum_{j=1}^{\infty}\left\lfloor \frac{l}{p^j}\right\rfloor.$$

Also $\left\lfloor \frac{l}{p}\right\rfloor = \sum_{i=0}^{N-1} l_{i+1}p^i$, $\left\lfloor \frac{l}{p^2}\right\rfloor = \sum_{i=0}^{N-2} l_{i+2}p^i$ and so on. Combining these we get

$$\begin{aligned}l &= l_0 + p\left\lfloor \frac{l}{p}\right\rfloor\\ \left\lfloor \frac{l}{p}\right\rfloor &= l_1 + p\left\lfloor \frac{l}{p^2}\right\rfloor\\ &\vdots\\ \left\lfloor \frac{l}{p^j}\right\rfloor &= l_j + p\left\lfloor \frac{l}{p^{j+1}}\right\rfloor.\end{aligned}$$

Adding these equations we get

$$l + \nu_p(l!) = l_0 + l_1 + \dots + p\nu_p(l!) = S_p(l) + p\nu_p(l!)$$

and rearranging gives the result. □

Proposition A.5.

$$\nu_p\left(\prod_{i=1}^{k}(m^{p-1}-r^{i-1}l^{p-1})\right)\geqslant \nu_p(k!)+k$$

for all l, $m\in\mathbb{Z}_p^\times$ and $k\in\mathbb{N}$.

Proof. Let $f_k:\mathbb{Q}_p\times\mathbb{Q}_p\to\mathbb{Q}_p$ be given by

$$f_k(l,m)=\frac{\prod_{i=1}^{k}(m^{p-1}-r^{i-1}l^{p-1})}{p^{\nu_p(k!)+k}}.$$

The proposition is equivalent to saying that $f_k(\mathbb{Z}_p^\times,\mathbb{Z}_p^\times)\subseteq\mathbb{Z}_p$.

This can be checked on a dense subset of $\mathbb{Z}_p^\times$. Since f_k is continuous and $\mathbb{Z}_p$ is closed in $\mathbb{Q}_p$ the result is then true for $\mathbb{Z}_p^\times$. The dense subset we will use is $D=\{q^j|j\geqslant 0\}$.

Let $l=q^t$ and $m=q^s$. Then

$$\prod_{i=1}^{k}((q^s)^{p-1}-r^{i-1}(q^t)^{p-1})=\prod_{i=1}^{k}(r^s-r^{i-1}r^t).$$

So we just need that $\nu_p(\prod_{i=1}^{k}(r^s-r^{i-1}r^t))\geqslant\nu_p(k!)+k$.

Firstly we consider the case where $|s-t|>k-1$. Then

$$\begin{aligned}\nu_p\left(\prod_{i=1}^{k}(r^s-r^{i-1}r^t)\right)&=\nu_p\left(\prod_{i=1}^{k}(r^{s-t-i+1}-1)\right)\\&=\sum_{i=1}^{k}\nu_p(r^{s-t-i+1}-1)\\&=\sum_{i=1}^{k}(\nu_p(s-t-i+1)+1)\\&=\nu_p\left(\prod_{i=1}^{k}(s-t-i+1)\right)+k\\&=\nu_p\left(\frac{(|s-t|)!}{(|s-t-k|)!}\right)+k\end{aligned}$$

using Proposition A.2. So we need

$$\nu_p\left(\frac{(|s-t|)!}{(|s-t-k|)!}\right)+k\geqslant\nu_p(k!)+k$$

i.e.

$$\nu_p\left(\frac{(|s-t|)!}{(|s-t-k|)!}\right)\geqslant\nu_p(k!).$$

Now the highest power of p less than k will be a factor of $k!$ and so will every power of p less than this. Since $\frac{(|s-t|)!}{(|s-t-k|))!}$ is the product of a run of k consecutive elements of $\mathbb{Z}$, like $k!$, a multiple of each of the powers of p featuring in $k!$ will feature in $\frac{(|s-t|)!}{(|s-t-k|)!}$ too plus possibly some higher powers of p. Hence the p-adic valuation of $\frac{(|s-t|)!}{(|s-t-k|)!}$ will be equal or greater than that of $k!$.

Finally, let $|s-t| \leqslant k-1$, then there will be a factor of $(r^{|s-t|}-r^{|s-t|}) = 0$ in the product, hence the whole product will be zero and since $\nu_p(0)$"$=$"∞ the proposition will be trivially true. $\square$

Glossary

$\mathbb{Q}_p$ The p-adic numbers. 5

$\mathbb{Z}_p$ The p-adic integers. 5

$\mathbb{Z}_{(p)}$ The p-local integers. 5

$\mathcal{A}_p$ The mod p Steenrod Algebra. 7

$\mathcal{A}_p^*$ The dual mod p Steenrod Algebra. 8

B The exterior algebra $\Lambda(Q_0, Q_1)$. 10

β The Bockstein homomorphism. 7

BU The classifying space for the group U. 13

$c_{\rho k}$ $\prod_{i=1}^{k} \frac{\hat{v}-r^{i-1}\hat{u}}{r^k-r^{i-1}}$. 61

$D(X)$ The Spanier-Whitehead dual of a spectrum X. 22

$f_{\rho k}$ $p^{\nu_p(k)} c_{\rho k}$. 61

$g_{\rho m,\rho l}$ The element of $G_{m,l}$ produced from $f_{\rho l}$ lying in degree ρm. 66

$G_{m,n}$ $\frac{\pi_m(\ell \wedge \mathcal{K}(n)) \otimes \mathbb{Z}_p}{\text{Torsion}}$. 65

$\tilde{G}_{m,n}$ $G_{m,n} \oplus W_{m,n}$ where $W_{m,n}$ is a finite elementary abelian p-group. 67

$H\mathbb{Z}/p$ The mod p Eilenberg-MacLane spectrum. 4

$\iota_{m,n}$ A map from $\ell \wedge \mathcal{K}(m)$ to $\ell \wedge \mathcal{K}(n)$. 49

K The periodic K-theory spectrum. 14

$\mathcal{K}(n)$ The nth 'piece' in Kane's splitting. 33

ku The connective K-theory spectrum. 18

ℓ The connective Adams summand. 18

Λ An exterior algebra. 8

ν_p The p-adic valuation function. 99

$\mathcal{P}^i$ Steenrod Power i. 6

ψ^k The kth Adams operation. 19

q An integer which is primitive modulo p. 99

Q_k The kth Milnor element of $\mathcal{A}_p$. 9

r q^{p-1}. 99

R The matrix $\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & r & 1 & 0 & 0 & \cdots \\ 0 & 0 & r^2 & 1 & 0 & \cdots \\ 0 & 0 & 0 & r^3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$. 85

ρ $2(p-1)$. 61

$S_p(l)$ The sum of the digits of l in its base p expansion. 101

$\hat{u}$ u^{p-1}. 61

$z_{\rho n}$ A generator for $G_{\rho n,n}$. 68

$\tilde{z}_{\rho n}$ A generator for $\tilde{G}_{\rho n,n}$. 68

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