Mass Transference Principles and Applications in Diophantine Approximation

Demi Denise Allen

PhD

University of York

Mathematics

December 2017

Abstract

This thesis is concerned with the Mass Transference Principle and its applications in Diophantine approximation. The Mass Transference Principle, proved by Beresnevich and Velani in 2006, is a powerful result allowing for the transference of Lebesgue measure statements for $\limsup$ sets arising from sequences of balls in $\mathbb{R}^k$ to Hausdorff measure statements. The significance of this result is especially prominent in Diophantine approximation, where many sets of interest arise naturally as $\limsup$ sets.

We establish a general form of the Mass Transference Principle for systems of linear forms conjectured by Beresnevich, Bernik, Dodson and Velani in 2009. This improves upon an earlier result in this direction due to Beresnevich and Velani from 2006. In addition, we present a number of applications of this "new" mass transference principle for linear forms to problems in Diophantine approximation, some of which were previously out of reach when using the result of Beresnevich and Velani. These include a general transference of Lebesgue measure Khintchine–Groshev type theorems to Hausdorff measure statements. The statements we obtain are applicable in both the homogeneous and inhomogeneous settings as well as allowing transference under any additional constraints on approximating integer points. In particular, we establish Hausdorff measure counterparts of some Khintchine–Groshev type theorems with primitivity constraints recently proved by Dani, Laurent and Nogueira.

Using a Hausdorff measure analogue of the inhomogeneous Khintchine–Groshev Theorem (established via the mass transference principle for linear forms), we give an alternative proof of most cases of a general inhomogeneous Jarník–Besicovitch Theorem which was originally proved by Levesley in 1998. We additionally show that without monotonicity Levesley's theorem no longer holds in general.

We conclude this thesis by discussing the concept of a mass transference principle for rectangles. In particular, we demonstrate how some known results may be extended using a slicing technique.

Contents

Acknowledgements

I have been truly blessed to spend the last few years in such an incredible place surrounded by some fantastic people and I am immensely grateful to the EPSRC for the funding which has enabled this. Inevitably, I cannot reasonably name everyone here who I am grateful to for the influence they have had on my journey to this point in my life. There are a few people, however, without whom this particular PhD journey would not have been possible at all, not least of all my amazing supervisors. I really have learnt an awful lot from them, for which I cannot thank them enough.

First and foremost, I would like to thank Victor for the continuous support and guidance offered throughout my PhD, for all the aptly timed words of encouragement and his remarkable patience. My words are insufficient to adequately express my deep gratitude. I thank Sanju also for his infectious enthusiasm. And, of course, I am grateful to both Sanju and Victor for proving the Mass Transference Principle — without which, this thesis really would not look the same!

I would like to thank Jonathan Fraser and Jason Levesley for agreeing to examine this thesis. I am also indebted to Jason for, perhaps inadvertently, making an innocent remark about eighteen months ago which ultimately led to the writing of Chapter 4.

I am grateful to all of my family for all of their love and encouragement. I really do appreciate their inquisitiveness and their earnest desire to understand what I do — I hope that one day I will be able to explain it to them better.

To my fellow residents of 66 Ambrose Street of the last couple of years — Mirjam, Merryn, Fabian, and Vicky — thanks for the fondue and the fun. In the Mathematics department in York, I am indebted to all of my office mates and the other PhD students and postdocs I have spent my time with. In particular, I must thank Faustin for his unlimited words of wisdom and for helping me to keep things in perspective through some of the tougher times of the past year. Further special thanks are due to Sky, Fabian and Sascha for reading various parts of this thesis.

I would like to thank everyone at York Vineyard for all of the love and support they have shown me. The time spent in York during my PhD has not only been an exciting academic journey but also an incredible spiritual one. Special thanks must, of course, go to Sue and David Heaton, Sam and Tim Abbey, Katie Evans, Alice Ovenden, and Richard and Nannette Gee, and everyone else in their homegroups I have been in. Helen and David — what can I say? — meeting you has been a true blessing.

Stuart — thank you for always being there and always believing in me.

I can do all things through Him who strengthens me.
Philippians 4:13, ESV

Author's Declaration

I declare that the work presented in this thesis, except where otherwise stated, is based on my own research carried out at the University of York and has not been submitted previously for any degree at this or any other university. Sources are acknowledged by explicit references.

Chapters 2 and 3 are essentially the contents of:

D. Allen, V. Beresnevich, *A mass transference principle for systems of linear forms and its applications*, Compos. Math. 154 (2018), no. 5, 1014–1047.

Chapters 4 and 5 are heavily based on, and the original results presented in these chapters appear in, respectively, Sections 3 and 4 of:

D. Allen, S. Troscheit, *The Mass Transference Principle: Ten Years On*, to appear in *Horizons of Fractal Geometry and Complex Dimensions*, AMS Contemporary Mathematics Series.

1 Introduction

In this chapter we gather background information and preliminaries which will be required throughout as well as give an overview of the contents of subsequent chapters. Much of this chapter is based on the surveys [4, 6]. For further information we also refer the reader to the many interesting and classical books on Diophantine approximation including, but not limited to, [11, 16, 30, 46, 47].

1.1 One Dimensional Approximation

Lying at the heart of Diophantine approximation is the question:

"How well can any given real number be approximated by rational numbers?"

It is well known that rationals are dense in the reals and so one answer to this question is essentially "as well as you like". Therefore, we refine our question and consider, for example, how well real numbers can be approximated by rationals with given denominators.

Trivially, given any real number x and natural number q we can always find $p \in \mathbb{Z}$ such that

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{2q}$$

or, equivalently,

$$|qx + p| \leq \frac{1}{2}. \tag{1.1}$$

For aesthetic reasons we shall typically favour this latter formulation.

In fact, this is rather weaker than what is actually always possible. A fundamental theorem of Dirichlet, the proof of which relies on the *pigeonhole principle*, provides us with a much stronger statement.

Theorem 1.1 (Dirichlet [22]). *For any $x \in \mathbb{R}$ and any $Q \in \mathbb{N}$, there exist integers p and q such that $1 \le q \le Q$ and*

$$|qx + p| < \frac{1}{Q}.$$

An immediate consequence of the above theorem is that if we replace the right-hand side of (1.1) with $\frac{1}{q}$ then, although it may not hold for *every* $q \in \mathbb{N}$, for any $x \in \mathbb{R}$ the inequality still holds infinitely often.

Theorem 1.2 (Dirichlet [22]). *For any $x \in \mathbb{R}$, there exist infinitely many pairs $(p, q) \in \mathbb{Z} \times \mathbb{N}$ such that*

$$|qx + p| < \frac{1}{q}. \tag{1.2}$$

From a slightly different viewpoint, we can also consider questions such as: what can be said about the set of $x \in \mathbb{R}$ for which (1.2) holds if we replace the right-hand side with a general function of q?

Given any function $\psi : \mathbb{N} \to \mathbb{R}^+$, which we call an *approximating function*, define

$$\mathcal{A}(\psi) := \{x \in \mathbb{I} : |qx + p| < \psi(q) \text{ for infinitely many pairs } (p, q) \in \mathbb{Z} \times \mathbb{N}\},$$

where $\mathbb{I}$ denotes the unit interval $[0, 1]$. Here, and throughout, we are using the notation $\mathbb{R}^+ := [0, \infty)$. The restriction of our attention to points in the unit interval here is purely for simplicity and causes no loss of generality since the approximation properties of real numbers are unaffected by integer translations. Similarly, when we consider approximation in higher dimensions we shall restrict our attention to points in the unit cube. We refer to the points in $\mathcal{A}(\psi)$ as *ψ-approximable points.* We will be interested throughout in the "size" of $\mathcal{A}(\psi)$ and other related sets. In particular, we will be concerned with Lebesgue measure, Hausdorff dimension and Hausdorff measures.

For monotonic ψ, a fundamental theorem of Khintchine gives us an elegant criterion for determining the Lebesgue measure of $\mathcal{A}(\psi)$. For a set $X \subset \mathbb{R}^k$ we will denote by $|X|$ the k-dimensional Lebesgue measure of X.

Theorem 1.3 (Khintchine [35]). *Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be an approximating function. Then*

$$|\mathcal{A}(\psi)| = \begin{cases} 0 & \textit{if} \quad \sum_{q=1}^{\infty} \psi(q) < \infty, \\ \\ 1 & \textit{if} \quad \sum_{q=1}^{\infty} \psi(q) = \infty \textit{ and } \psi \textit{ is monotonic.} \end{cases}$$

Remark. The above theorem is a modern improved version of Khintchine's original theorem (see, for example, [6]). In [35] the stronger condition that $q\psi(q)$ is monotonic was assumed.

An important observation, which will be central to most of what follows, is that $\mathcal{A}(\psi)$ and other sets we will be interested in can be expressed as *lim sup sets*.

Definition 1.4. Let $(A_i)_{i\in\mathbb{N}}$ be a collection of subsets of a set Y. Then

$$\limsup_{i\to\infty} A_i := \{x \in Y \,:\, x \in A_i \text{ for infinitely many } i \in \mathbb{N}\}.$$

Equivalently,

$$\limsup_{i\to\infty} A_i := \bigcap_{k=1}^{\infty}\bigcup_{i=k}^{\infty} A_i.$$

For lim sup sets, the following result from probability theory provides a sufficient condition for the set to have Lebesgue measure zero.

Lemma 1.5 (Borel–Cantelli Lemma). *Let (Ω, m) be a finite measure space and let $(A_i)_{i\in\mathbb{N}}$ be a sequence of m-measurable sets in Ω. If*

$$\sum_{i=1}^{\infty} m(A_i) < \infty,$$

then

$$m(\limsup_{i\to\infty} A_i) = 0.$$

To see this, suppose we are given an arbitrary $\varepsilon > 0$. Then, we can choose $N \in \mathbb{N}$ such that $\sum_{i\geq N} m(A_i) < \varepsilon$. Finally, since $\limsup_{i\to\infty} A_i \subset \bigcup_{i\geq N} A_i$, it follows by the subadditivity of measures that

$$m\left(\limsup_{i\to\infty} A_i\right) \leq m\left(\bigcup_{i\geq N} A_i\right) \leq \sum_{i\geq N} m(A_i) < \varepsilon.$$

Similar covering arguments to this one used to prove the Borel–Cantelli Lemma are fairly standard and appear relatively often in Diophantine approximation. They are often used for establishing the "convergence" part of statements like Khintchine's Theorem, as we shall now demonstrate. Indeed, we shall see this kind of argument appearing in a variety of settings throughout these pages.

Returning to the set $\mathcal{A}(\psi)$, let $B(x,r)$ denote a ball in $\mathbb{R}$ (i.e. an interval) of radius r centred at x. For each $(p,q) \in \mathbb{Z} \times \mathbb{N}$ with $B\left(\frac{p}{q}, \frac{\psi(q)}{q}\right) \cap \mathbb{I} \neq \emptyset$, define $B_{(p,q)}(\psi) := B\left(\frac{p}{q}, \frac{\psi(q)}{q}\right) \cap \mathbb{I}$. Then,

$$\mathcal{A}(\psi) = \limsup_{(p,q)} B_{(p,q)}(\psi).$$

Suppose, for the moment, that $\sum_{q=1}^{\infty} \psi(q) < \infty$. Thus, we must have $\psi(q) \to 0$ as $q \to \infty$ and, therefore, it follows that

$$\sum_{\substack{(p,q) \in \mathbb{Z} \times \mathbb{N} \\ B_{(p,q)}(\psi) \cap \mathbb{I} \neq \emptyset}} |B_{(p,q)}(\psi)| \ll \sum_{q=1}^{\infty} q \cdot 2\frac{\psi(q)}{q} = 2\sum_{q=1}^{\infty} \psi(q) < \infty.$$

Remark. Here, and throughout, we are using the standard Vinogradov notation. Thus, we write $A \ll B$ if $A \leq cB$ for some positive constant c and $A \gg B$ if $A \geq c'B$ for some positive constant c'. Finally, if $A \ll B$ and $A \gg B$ we write $A \asymp B$ and say that A and B are *comparable.*

The convergence part of Khintchine's Theorem follows on taking $\Omega = \mathbb{I}$ and m to be Lebesgue measure in the Borel–Cantelli Lemma. Notice that, in deriving the convergence part of Khintchine's Theorem, we have not required any monotonicity assumptions on the approximating function ψ. Statements like Khintchine's Theorem, so-called "zero-one" laws, with the measure of a set depending on the convergence or divergence of a certain sum, appear quite frequently for sets of interest in Diophantine approximation. It is often the case that the convergence parts of such statements follow from the Borel–Cantelli Lemma and require no monotonicity assumptions.

On the other hand, a counter-example constructed by Duffin and Schaeffer [25] shows that the monotonicity assumption is absolutely crucial in the divergence part of Khintchine's Theorem. They constructed a function $\theta : \mathbb{N} \to \mathbb{R}^+$ for which $\sum_{q=1}^{\infty} \theta(q) = \infty$, yet $|\mathcal{A}(\theta)| = 0$. At the same time, they also posed a conjecture on what should be true when considering general (not necessarily monotonic) approximating functions.

For an approximating function $\psi : \mathbb{N} \to \mathbb{R}^+$, let $\mathcal{A}'(\psi)$ denote the set of points $x \in \mathbb{I}$ for which the inequality

$$|qx + p| < \psi(q)$$

is satisfied for infinitely many pairs $(p,q) \in \mathbb{Z} \times \mathbb{N}$ with $\gcd(p,q) = 1$.

It follows from the Borel–Cantelli Lemma that

$$|\mathcal{A}'(\psi)| = 0 \quad \text{if} \quad \sum_{q=1}^{\infty} \varphi(q)\frac{\psi(q)}{q} < \infty,$$

where $\varphi(q)$ is the standard Euler function; recall that $\varphi : \mathbb{N} \to \mathbb{N}$ is defined by $\varphi(q) = \#\{1 \le p \le q : \gcd(p,q) = 1\}$.

Duffin and Schaeffer predicted that the complementary divergence statement should also be true.

Conjecture 1.6 (**Duffin–Schaeffer Conjecture** [25]). *Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be any approximating function and denote by $\varphi(q)$ the Euler function. If*

$$\sum_{q=1}^{\infty} \varphi(q)\frac{\psi(q)}{q} = \infty \quad \textit{then} \quad |\mathcal{A}'(\psi)| = 1.$$

Remark 1.7. Cassels [15] and Gallagher [27] have shown, respectively, that $|\mathcal{A}(\psi)|$ and $|\mathcal{A}'(\psi)|$ only take the values 0 or 1. As a consequence of this, in order to establish the Duffin–Schaeffer Conjecture, it would suffice to show that $|\mathcal{A}'(\psi)| > 0$ which seems more achievable than showing directly that $|\mathcal{A}'(\psi)| = 1$.

In the same paper, Duffin and Schaeffer proved their conjecture subject to an additional assumption.

Theorem 1.8 (**Duffin–Schaeffer Theorem** [25]). *Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be any approximating function and denote by $\varphi(q)$ the Euler function. Suppose that*

$$\sum_{q=1}^{\infty} \varphi(q)\frac{\psi(q)}{q} = \infty$$

and, additionally,

$$\limsup_{Q\to\infty} \frac{\sum_{q=1}^{Q} \varphi(q)\frac{\psi(q)}{q}}{\sum_{q=1}^{Q} \psi(q)} > 0. \tag{1.3}$$

Then, $|\mathcal{A}'(\psi)| = 1$.

1.2 Simultaneous Approximation

In higher dimensions, instead of rational numbers, we can consider how well points in $\mathbb{R}^m$ can be approximated by rational points, i.e. vectors in $\mathbb{R}^m$ where all of the entries are rational. In this case, given an approximating function $\psi : \mathbb{N} \to \mathbb{R}^+$, we let $\mathcal{A}_m(\psi)$ denote the set of points $\mathbf{x} \in \mathbb{I}^m$ such that

$$|q\mathbf{x} + \mathbf{p}| < \psi(q) \tag{1.4}$$

for infinitely many pairs $(\mathbf{p}, q) \in \mathbb{Z}^m \times \mathbb{N}$. Here, $|\cdot|$ is the supremum norm, i.e.

$$|q\mathbf{x} + \mathbf{p}| = \max_{1 \le i \le m} |qx_i + p_i|.$$

We refer to points in $\mathcal{A}_m(\psi)$ as *simultaneously ψ-approximable points.* Note that $\mathcal{A}_1(\psi) = \mathcal{A}(\psi)$.

In the setting of simultaneous approximation, we have the following analogue of Dirichlet's Theorem (Theorem 1.1) — see, for example, [6, 46].

Theorem 1.9 (**Higher-Dimensional Dirichlet** [22])**.** *For any $\mathbf{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m$ and $Q \in \mathbb{N}$, there exists $(\mathbf{p}, q) \in \mathbb{Z}^m \times \mathbb{N}$ with $1 \le q \le Q$ such that*

$$|q\mathbf{x} + \mathbf{p}| < \frac{1}{Q^{\frac{1}{m}}}.$$

In line with how Theorem 1.2 follows from Theorem 1.1, the next statement is a corollary to Theorem 1.9.

Theorem 1.10 (Dirichlet [22])**.** *For any $\mathbf{x} \in \mathbb{R}^m$, there exist infinitely many pairs $(\mathbf{p}, q) \in \mathbb{Z}^m \times \mathbb{N}$ such that*

$$|q\mathbf{x} + \mathbf{p}| < \frac{1}{q^{\frac{1}{m}}}.$$

In particular, $\mathcal{A}_m(q \mapsto q^{-\frac{1}{m}}) = \mathbb{I}^m$.

For more general approximating functions, Khintchine also extended his one-dimensional theorem to the setting of simultaneous approximation.

Theorem 1.11 (Khintchine [36])**.** *Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be an approximating function.*

Then

$$|\mathcal{A}_m(\psi)| = \begin{cases} 0 & \textit{if} \quad \sum_{q=1}^{\infty} \psi(q)^m < \infty, \\ \\ 1 & \textit{if} \quad \sum_{q=1}^{\infty} \psi(q)^m = \infty \textit{ and } \psi \textit{ is monotonic.} \end{cases}$$

Remark. As with the one dimensional case, Khintchine again had stronger monotonicity conditions on his original statement of this theorem in [36]. For this modern version see, for example, [6].

As in the one-dimensional case, the convergence part of Khintchine's Theorem is a consequence of the Borel–Cantelli Lemma and requires no monotonicity assumptions. Unlike in the one-dimensional case though, monotonicity is not needed at all in Theorem 1.11 when $m \geq 2$, not even for the divergence case. That this is the case is due to a result of Gallagher. To state Gallagher's result, let us denote by $\mathcal{A}'_m(\psi)$ the set of $\mathbf{x} \in \mathbb{I}^m$ such that (1.4) is satisfied for infinitely many $(\mathbf{p}, q) \in \mathbb{Z}^m \times \mathbb{N}$ such that $\gcd(p_1, p_2, \ldots, p_m, q) = 1$. Note that $\mathcal{A}'_1(\psi) = \mathcal{A}'(\psi)$.

Theorem 1.12 (Gallagher [28])**.** *Let $m \geq 2$. For any approximating function $\psi : \mathbb{N} \to \mathbb{R}^+$,*

$$|\mathcal{A}'_m(\psi)| = 1 \quad \textit{if} \quad \sum_{q=1}^{\infty} \psi(q)^m = \infty.$$

Note that $\mathcal{A}'_m(\psi) \subset \mathcal{A}_m(\psi)$. In particular, if $|\mathcal{A}'_m(\psi)| = 1$, then $|\mathcal{A}_m(\psi)| = 1$. Thus, combining Gallagher's Theorem with Khintchine's Theorem (Theorem 1.11) completely removes any monotonicity conditions from the latter whenever $m \geq 2$.

Theorem 1.13 (Khintchine + Gallagher)**.** *Let $m \geq 2$ and let $\psi : \mathbb{N} \to \mathbb{R}^+$ be an approximating function. Then*

$$|\mathcal{A}_m(\psi)| = \begin{cases} 0 & \textit{if} \quad \sum_{q=1}^{\infty} \psi(q)^m < \infty, \\ \\ 1 & \textit{if} \quad \sum_{q=1}^{\infty} \psi(q)^m = \infty. \end{cases}$$

We conclude this section by mentioning the analogue of the Duffin–Schaeffer Conjecture for simultaneous approximation. In order to do so, let us denote by $\mathcal{A}''_m(\psi)$ the set of points $\mathbf{x} \in \mathbb{I}^m$ for which the inequality (1.4) is satisfied for infinitely many $(\mathbf{p}, q) \in \mathbb{Z}^m \times \mathbb{N}$ which also have $\gcd(p_i, q) = 1$ for all $1 \leq i \leq m$. The following conjecture, which includes the Duffin–Schaeffer Conjecture (since $\mathcal{A}''_1(\psi) = \mathcal{A}'(\psi)$)

and naturally extends it to the setting of simultaneous approximation, was formulated by Sprindžuk [47, Chapter 1, Section 8].

Conjecture 1.14 (Higher-Dimensional Duffin–Schaeffer Conjecture [47]). *Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be any approximating function and denote by $\varphi(q)$ the Euler function. If*

$$\sum_{q=1}^{\infty} \varphi(q)^m \frac{\psi(q)^m}{q^m} = \infty \quad \text{then} \quad |\mathcal{A}''_m(\psi)| = 1.$$

For $m > 1$, Sprindžuk's conjecture (Conjecture 1.14) was proved in the affirmative by Pollington and Vaughan [41]. Meanwhile, the Duffin–Schaeffer Conjecture in the case of one-dimensional approximation still represents one of the most significant, unresolved, and long-standing conjectures in Diophantine approximation.

As well as being extended to higher dimensions, giving us the theory of simultaneous approximation, the fundamental theorems of Dirichlet and Khintchine have also been generalised in numerous other directions. In later chapters we will be particularly interested in generalisations of these theorems in two directions. Analogues of these results in the setting of approximation by linear forms will be mentioned in Chapter 3, and in Chapter 5 we shall provide some discussion of the theory of weighted simultaneous approximation. To some extent, in Chapter 3, we will also be interested in inhomogeneous approximation and approximation with restrictions imposed on the "approximating points".

Aside from those mentioned, there is a vast array of other directions in which the results from these first two sections have been generalised. For much more extensive surveys of such results we refer the reader to, for example, [4, 6] and references therein.

1.3 Limitation of Lebesgue Measure

In Diophantine approximation, there exist many elegant zero-one laws, such as Khintchine's Theorem, which give simple criteria — usually the convergence or divergence of a certain sum — for determining the Lebesgue measure of sets of interest. While these results are quite attractive, we will present in this section an illustrative example which demonstrates a limitation of such statements. In particular, such statements do not allow us to distinguish further between sets which have Lebesgue measure zero even when, intuitively, we may have good reason to believe that the sets under consideration are not the same "size". For example, let us consider the approximating function $\psi : \mathbb{N} \to \mathbb{R}^+$ defined by $\psi(q) = q^{-\tau}$ for some $\tau > 0$. In this case we write $\mathcal{A}(\tau)$ in place of $\mathcal{A}(\psi)$ and refer to the points in $\mathcal{A}(\tau)$ as *τ-approximable points*. By Khintchine's Theorem we see that if $\tau > 1$ then $|\mathcal{A}(\tau)| = 0$. However, for any $\tau_1 < \tau_2$ we have $\mathcal{A}(\tau_2) \subset \mathcal{A}(\tau_1)$ and intuitively one would expect that as τ increases the size of $\mathcal{A}(\tau)$ should decrease. Nevertheless, all that can be inferred from Khintchine's Theorem is that for any $\tau > 1$ the set of τ-approximable points has Lebesgue measure zero.

In order to distinguish such sets we have to appeal to a measure finer than Lebesgue measure. For this purpose we consider Hausdorff dimension and, more generally, Hausdorff measures. In the next section we will provide the definitions and some properties of Hausdorff measures and dimension. In the subsequent section we will provide statements which allow us to distinguish sets such as the τ-approximable points via Hausdorff measures and dimension.

1.4 Hausdorff Measures and Dimension

In this section we give a brief account of Hausdorff measures and dimension. Throughout, by a *dimension function* we shall mean a left continuous, non-decreasing function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(r) \to 0$ as $r \to 0$. We say that f is *doubling* if there exists a constant $\lambda > 1$ such that for $x > 0$ we have $f(2x) \leq \lambda f(x)$.

Given a ball $B := B(x, r)$ in $\mathbb{R}^k$ with respect to some norm $\|\cdot\|$ on $\mathbb{R}^k$, we define

$$V^f(B) := f(r)$$

and refer to $V^f(B)$ as the *f-volume of B*. Alternatively, we could consider balls with respect to a metric on $\mathbb{R}^k$ but here, for the most part, we will just be concerned with balls determined by norms. Note that if $|\cdot|$ is the k-dimensional Lebesgue measure, $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^k$, and $f(x) = |B(0,1)|x^k$, then V^f is simply the volume of B in the usual geometric sense; i.e. $V^f(B) = |B|$. In the case when $f(x) = x^s$ for some $s \geq 0$, we write V^s for V^f.

The Hausdorff f-measure with respect to the dimension function f will be denoted by $\mathcal{H}^f$ and is defined as follows. Suppose F is a subset of $\mathbb{R}^k$. For $\rho > 0$, a *ρ-cover for F* is a countable collection $\{B_i\}$ of balls in $\mathbb{R}^k$ with radii $r(B_i) \leq \rho$ for each i such that $F \subset \bigcup_i B_i$. Clearly such a cover exists for every $\rho > 0$. For a dimension function f, define

$$\mathcal{H}^f_\rho(F) := \inf\left\{\sum_i V^f(B_i) : \{B_i\} \text{ is a } \rho\text{-cover for } F\right\}.$$

The *Hausdorff f-measure*, $\mathcal{H}^f(F)$, of F with respect to the dimension function f is defined by

$$\mathcal{H}^f(F) := \lim_{\rho\to 0}\mathcal{H}^f_\rho(F) \;=\; \sup_{\rho>0}\mathcal{H}^f_\rho(F)\,.$$

Observe that, for a given f, this limit exists (but may be infinite) since the quantity $\mathcal{H}^f_\rho(F)$ is non-decreasing as $\rho \to 0$. This is because as ρ decreases the number of available ρ-covers also decreases.

We note that the precise value of $\mathcal{H}^f(F)$ may vary depending on the norm with which $\mathbb{R}^k$ is endowed. While it should be clear from the context which norm is being used, it is worth noting that we are usually interested in the supremum norm when discussing simultaneous approximation. For considering approximation by linear forms, in Chapter 3 we define another specific norm (3.1) which turns out to be particularly convenient. If we have the further assumption that the dimension function f is doubling, then the Hausdorff measure only varies by a constant when the underlying norm is changed. In particular, if we are dealing with sets which have either zero or infinite Hausdorff f-measure, then the underlying norm becomes essentially irrelevant in this case.

A simple consequence of the definition of $\mathcal{H}^f$ is the following useful fact (see, for example, [26]).

Lemma 1.15. *If f and g are two dimension functions such that the ratio $\frac{f(r)}{g(r)} \to 0$ as $r \to 0$, then $\mathcal{H}^f(F) = 0$ whenever $\mathcal{H}^g(F) < \infty$.*

Proof. Let $\varepsilon > 0$ and choose $\rho > 0$ such that $\frac{f(r)}{g(r)} < \varepsilon$ for all $r < \rho$. By definition, we have

$$\begin{aligned}
\mathcal{H}^f_\rho(F) &= \inf\left\{\sum_i V^f(B_i) : \{B_i\} \text{ is a } \rho\text{-cover for } F\right\}\\
&= \inf\left\{\sum_i \frac{V^f(B_i)}{V^g(B_i)} V^g(B_i) : \{B_i\} \text{ is a } \rho\text{-cover for } F\right\}\\
&< \varepsilon \inf\left\{\sum_i V^g(B_i) : \{B_i\} \text{ is a } \rho\text{-cover for } F\right\} = \varepsilon\mathcal{H}^g_\rho(F).
\end{aligned}$$

The result follows on letting $\rho \to 0$ and noting that ε was chosen to be arbitrary. $\square$

Often we are interested in Hausdorff dimension and the classical Hausdorff s-measure. The Hausdorff s-measure, which we denote by $\mathcal{H}^s$, can be obtained by letting $f(r) = r^s$ $(s \geq 0)$. The *Hausdorff dimension* of a set F, $\dim_{\mathrm{H}} F$, is then defined as

$$\dim_{\mathrm{H}} F = \inf \big\{ s > 0 \ : \ \mathcal{H}^s(F) = 0 \big\}.$$

One interesting property of Hausdorff measure is that, for subsets of $\mathbb{R}^k$, $\mathcal{H}^k$ is a constant multiple of the k-dimensional Lebesgue measure. Indeed, these constants are known explicitly — see [26]. However, it will suffice for us to know that $\mathcal{H}^k$ is comparable to the k-dimensional Lebesgue measure.

For Hausdorff s-measures, there exists a kind of analogue of the Borel–Cantelli Lemma, christened the *Hausdorff–Cantelli Lemma* by Bernik and Dodson [11], which gives a sufficient criterion for a set F, which is or is a subset of a $\limsup$ set, to have $\mathcal{H}^s(F) = 0$. This immediately also yields an upper bound for Hausdorff dimension.

Lemma 1.16 (Hausdorff–Cantelli Lemma). *Let $(E_i)_{i\in\mathbb{N}}$ be a sequence of subsets in $\mathbb{R}^k$ with diameter $d_i \to 0$ as $i \to \infty$. Suppose that, for some $s > 0$ we have that*

$$\sum_{i=1}^{\infty} d_i^s < \infty.$$

Then,

$$\mathcal{H}^s \left(\limsup_{i\to\infty} E_i \right) = 0.$$

In particular,

$$\dim_{\mathrm{H}} \left(\limsup_{i\to\infty} E_i \right) \leq s.$$

Proof. For each $i \in \mathbb{N}$, let B_i be a ball of radius d_i which covers E_i. Let $\varepsilon > 0$ and $\rho > 0$ be arbitrary.

Let $N \in \mathbb{N}$ be such that $\sum_{i\geq N} d_i^s < \varepsilon$ and $d_i < \rho$ for all $i \geq N$. This is possible due to the assumptions that $\sum_{i=1}^{\infty} d_i^s < \infty$ and $d_i \to 0$.

Note that $\bigcup_{i\geq N} B_i$ is a ρ-cover for $\limsup_{i\to\infty} E_i$. Hence, we have

$$\mathcal{H}^s_\rho(\limsup_{i\to\infty} E_i) \leq \sum_{i\geq N} r(B_i)^s = \sum_{i\geq N} d_i^s < \varepsilon.$$

Since $\varepsilon > 0$ and $\rho > 0$ were chosen arbitrarily it follows that

$$\mathcal{H}^s\left(\limsup_{i\to\infty} E_i\right) = 0.$$

Furthermore, by the definition of Hausdorff dimension it follows that

$$\dim_{\mathrm{H}}\left(\limsup_{i\to\infty} E_i\right) \leq s. \qquad \square$$

Remark. By the same argument one can prove similar statements for more general Hausdorff f-measures.

One of the advantages of Hausdorff dimension is that, in many cases, it allows us to distinguish sets of Lebesgue measure zero. Lebesgue null sets, i.e. sets X with $|X| = 0$, can still have intricate geometric structure and, as discussed, may not necessarily be of the same size. By appealing to the Hausdorff dimension of such sets we can often quickly get an indication of their relative size.

If we are faced with two sets, X and Y, with the same Hausdorff dimension, we can next appeal to their Hausdorff s-measure at the critical value $\dim_{\mathrm{H}} X = \dim_{\mathrm{H}} Y = s$ to see if that provides a way of distinguishing their size. However, it is worth noting that, computing Hausdorff s-measures is typically more complicated that determining Hausdorff dimension.

While Hausdorff dimension and Hausdorff s-measures help us to say much more about the size of sets than Lebesgue measure often can, sometimes we may desire an even sharper indication of the size or dimension of a set. This is where Hausdorff f-measures come into play. If we stumble across sets X and Y which, as well as having the same (trivial) Lebesgue measure, have the same Hausdorff dimension $\dim_{\mathrm{H}} X = \dim_{\mathrm{H}} Y = s$ and furthermore satisfy $\mathcal{H}^s(X) = \mathcal{H}^s(Y)$, there may still be a dimension function f such that X and Y can be distinguished via their Hausdorff f-measure. For an explicit example of two such sets, see [48, Chapter 1].

For further information regarding Hausdorff measures and dimension we refer the reader to [26, 40, 43].

1.5 Theorems of Jarník and Besicovitch

With the definitions of Hausdorff measures and dimension now at our disposal, we return to our example from Section 1.3. While Lebesgue measure proved to be insufficient for allowing us to distinguish between sets of τ-approximable points for values of $\tau > 1$, we have somewhat more luck if we consider the Hausdorff dimensions of these sets. The following theorem, proved independently by both Jarník and Besicovitch, indicates that our earlier intuition, that $\mathcal{A}(\tau)$ should get "smaller" as τ increases, is correct.

Theorem 1.17 (**Jarník–Besicovitch Theorem**, Jarník [33], Besicovitch [12])**.** *Let $\tau > 1$. Then*

$$\dim_{\mathrm{H}}(\mathcal{A}(\tau)) = \frac{2}{\tau + 1}.$$

As predicted, at least if we look at these sets from the perspective of Hausdorff dimension, the above theorem shows that as τ increases the size of $\mathcal{A}(\tau)$ decreases. In a further study, Jarník later proved a much stronger statement regarding the Hausdorff measures of more general sets of ψ-approximable points.

Theorem 1.18 (**Jarník's Theorem**, Jarník [34])**.** *Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be an approximating function and let f be a dimension function such that $r^{-m}f(r) \to \infty$ as $r \to 0$ and the function $r^{-m}f(r)$ is decreasing. Then*

$$\mathcal{H}^f(\mathcal{A}_m(\psi)) = \begin{cases} 0 & \text{if} \quad \sum_{q=1}^{\infty} q^m f\left(\frac{\psi(q)}{q}\right) < \infty, \\ \\ \infty & \text{if} \quad \sum_{q=1}^{\infty} q^m f\left(\frac{\psi(q)}{q}\right) = \infty \text{ and } \psi \text{ is monotonic.} \end{cases}$$

Remark 1.19. The statement we give here is a modern-day improvement on Jarník's original theorem, which required the additional hypotheses that $r\psi(r)^m$ is decreasing, $r\psi(r)^m \to 0$ as $r \to \infty$ and $r^{m+1}f\left(\frac{\psi(r)}{r}\right)$ is decreasing. However, in [5] it was shown that monotonicity of ψ suffices in Jarník's Theorem, thus leaving us with the above "cleaner" statement.

Now, considering the case when $m = 1$ and $f(r) = r^s$ for $s \in (0, 1)$, Jarník's

Theorem tells us that

$$\mathcal{H}^s(\mathcal{A}(\tau)) = \begin{cases} 0 & \text{if} \quad \sum_{q=1}^{\infty} q^{1-s\tau-s} < \infty, \\ \\ \infty & \text{if} \quad \sum_{q=1}^{\infty} q^{1-s\tau-s} = \infty. \end{cases}$$

From here we can easily recover the Jarník–Besicovitch Theorem. Furthermore, we gain the additional information that $\mathcal{H}^{\frac{2}{\tau+1}}(\mathcal{A}(\tau)) = \infty$, which the Jarník–Besicovitch Theorem alone does not yield. By a similar argument, Jarník's Theorem also allows for the easy inference of information about the Hausdorff dimension of sets of *simultaneously* τ*-approximable* points in higher dimensions. Let $\mathcal{A}_m(\tau) := \mathcal{A}_m(q \mapsto q^{-\tau})$ denote the set of τ-approximable points in $\mathbb{R}^m$. Then, the following is an immediate consequence of Jarník's Theorem.

Theorem 1.20 (Jarník [34])**.** *For* $\tau > \frac{1}{m}$*,*

$$\dim_{\mathrm{H}}(\mathcal{A}_m(\tau)) = \frac{m+1}{\tau+1} \quad \textit{and} \quad \mathcal{H}^{\frac{m+1}{\tau+1}}(\mathcal{A}_m(\tau)) = \infty.$$

Despite the generality of Jarník's Theorem, a consequence of the assumption that $r^{-m}f(r) \to \infty$ as $r \to 0$ is that Jarník's Theorem does not cover the natural case where $f(r) = r^m$. Nevertheless, following the improvements on Jarník's Theorem made in [5], the modern versions of the theorems of Khintchine (Theorem 1.11) and Jarník (Theorem 1.18) can be combined into the following unifying statement which can be thought of as the Hausdorff measure analogue of Khintchine's Theorem.

Theorem 1.21 (Khintchine–Jarník Theorem [5]**).** *Let* $\psi : \mathbb{N} \to \mathbb{R}^+$ *be an approximating function and let* f *be a dimension function such that* $r^{-m}f(r)$ *is monotonic. Then*

$$\mathcal{H}^f(\mathcal{A}_m(\psi)) = \begin{cases} 0 & \textit{if} \quad \sum_{q=1}^{\infty} q^m f\left(\frac{\psi(q)}{q}\right) < \infty, \\ \\ \mathcal{H}^f(\mathbb{I}^m) & \textit{if} \quad \sum_{q=1}^{\infty} q^m f\left(\frac{\psi(q)}{q}\right) = \infty \textit{ and } \psi \textit{ is monotonic.} \end{cases}$$

While, on the surface, it appears that the Khintchine–Jarník Theorem is a consequence of combining two independent results — Khintchine's Theorem for Lebesgue measure and Jarník's Theorem for Hausdorff measures — there is actually a much deeper connection. It turns out that Khintchine's Theorem implies Jarník's

Theorem and so, in fact, Khintchine's Theorem alone underpins the Khintchine–Jarník Theorem. That is, a statement about Lebesgue measure implies a seemingly more general statement about Hausdorff measures. This is especially surprising given that our motivation for considering Hausdorff measure in the first place was because we were finding that Lebesgue measure was not giving us sufficient information. This implication is just one of the surprising consequences of the *Mass Transference Principle*, which will be introduced in the next section.

As with the Lebesgue measure statements of Dirichlet and Khintchine, the Hausdorff measure results of Jarník and Besicovitch have also been generalised in various directions. Again, we will be interested in the extensions of these results to the setting of approximation by linear forms in Chapters 3 and 4, and in Chapter 5 we will encounter some results analogous to the Jarník–Besicovitch Theorem in the context of weighted simultaneous approximation.

1.6 The Mass Transference Principle

Originally discovered by Beresnevich and Velani in 2006, the Mass Transference Principle is a remarkable result which allows us to transfer a Lebesgue measure statement for a $\limsup$ set defined by a sequence of balls in $\mathbb{R}^k$ to a Hausdorff measure statement for a related $\limsup$ set. Over the intervening years since its initial discovery, the Mass Transference Principle has become an important tool in metric Diophantine approximation. This is largely because, as alluded to earlier, many sets of interest in Diophantine approximation arise as $\limsup$ sets.

In this section we present statements of the Mass Transference Principle and the more general theorem proved by Beresnevich and Velani in [7]. In Section 1.7 we will touch upon a couple of applications and some surprising consequences associated with the Mass Transference Principle.

We take this opportunity to remark that the Mass Transference Principle is the basis for the rest of what will be discussed in this thesis. More specifically, the main result of this thesis (Theorem 2.2) is an extension of the Mass Transference Principle to systems of linear forms. Furthermore, the proof of Theorem 2.2 presented in the next chapter is based heavily on the proof of the Mass Transference Principle given in [7]. In the subsequent two chapters, various applications of Theorem 2.2 will be discussed. The final chapter of this thesis will also be concerned with generalisations of the Mass Transference Principle in another direction, to rectangles, which is of interest for weighted simultaneous approximation.

1.6.1 *The Mass Transference Principle*

To state the Mass Transference Principle we first introduce a little extra notation. Given a dimension function $f : \mathbb{R}^+ \to \mathbb{R}^+$ and a ball $B := B(x, r)$ in $\mathbb{R}^k$ of radius r centred at x, let $B^f := B(x, f(r)^{\frac{1}{k}})$. We write B^s instead of B^f if $f(x) = x^s$ for some $s > 0$. In particular, we have $B^k = B$.

Theorem 1.22 (**Mass Transference Principle**, Beresnevich – Velani [7])**.** *Let $\{B_j\}_{j\in\mathbb{N}}$ be a sequence of balls in $\mathbb{R}^k$ with $r(B_j) \to 0$ as $j \to \infty$. Let f be a dimension function such that $x^{-k}f(x)$ is monotonic and let Ω be a ball in $\mathbb{R}^k$. Suppose that, for any ball B in Ω,*

$$\mathcal{H}^k\big(B \cap \limsup_{j\to\infty} B_j^f\big) = \mathcal{H}^k(B) \ .$$

Then, for any ball B in Ω,

$$\mathcal{H}^f\big(B \cap \limsup_{j\to\infty} B_j^k\big) = \mathcal{H}^f(B) \ .$$

Remark. Strictly speaking, the statement of the Mass Transference Principle given initially by Beresnevich and Velani, [7, Theorem 2], corresponds to the case where Ω is taken to be $\mathbb{R}^k$ in Theorem 1.22. The statement we have opted to give above is a consequence of [7, Theorem 2].

The Mass Transference Principle allows us to transfer a Lebesgue measure statement for a lim sup set of balls to a Hausdorff measure statement for a lim sup set of balls which are obtained by “shrinking” the original balls in a certain manner according to f. This is a remarkable result given that Lebesgue measure can be considered to be much “coarser” than Hausdorff measure.

The motivation which led to the Mass Transference Principle was a desire to find a general Hausdorff measure analogue of the Duffin–Schaeffer Conjecture (see Conjectures 1.6 and 1.14). We shall elaborate on this, together with some other applications of the Mass Transference Principle, in the next section.

1.6.2 *A more general mass transference principle*

In addition to the Mass Transference Principle, which is in itself a truly remarkable result, Beresnevich and Velani also record in [7] a natural generalisation which allows for the transference of $\mathcal{H}^g$ measure statements to $\mathcal{H}^f$ measure statements for lim sup sets of balls in a locally compact metric space. We now make this more precise.

Let (X, d) be a locally compact metric space and let g be a doubling dimension function. Recall that we say g is doubling if there exists a constant $\lambda > 1$ such that for $x > 0$ we have $g(2x) \leq \lambda g(x)$. Furthermore, suppose that there exist constants $0 < c_1 < 1 < c_2 < \infty$ and $r_0 > 0$ such that

$$c_1 g(r) \leq \mathcal{H}^g(B(x, r)) \leq c_2 g(r)$$

for any ball $B = B(x, r)$ with centre $x \in X$ and radius $r \leq r_0$. In this case, given a ball $B := B(x, r)$ and any dimension function f we define $B^{f,g} := B(x, g^{-1}f(r))$. Note that $B^{g,g} = B$.

Theorem 1.23 (Beresnevich – Velani [7])**.** *Let (X, d) be a locally compact metric space and let g be a doubling dimension function. Let $\{B_j\}_{j\in\mathbb{N}}$ be a sequence of balls in X with $r(B_j) \to 0$ as $j \to \infty$ and let f be a dimension function such that $\frac{f(x)}{g(x)}$ is monotonic. Suppose that, for any ball B in X,*

$$\mathcal{H}^g(B \cap \limsup_{j\to\infty} B_j^{f,g}) = \mathcal{H}^g(B).$$

Then, for any ball B in X, we have

$$\mathcal{H}^f(B \cap \limsup_{j\to\infty} B_j^{g,g}) = \mathcal{H}^f(B).$$

In the case that $g(x) = x^k$ and $X = \mathbb{R}^k$, Theorem 1.23 precisely matches the original statement of the Mass Transference Principle given by [7, Theorem 2]. However, Theorem 1.23 is applicable in more general settings. For example, by taking X to be the middle-third Cantor set in Theorem 1.23, Levesley, Salp, and Velani [39] have used Theorem 1.23 as a tool for proving an assertion of Mahler on the existence of very well approximable numbers in the middle-third Cantor set. We shall revisit this particular application of Theorem 1.23 in Section 1.7.4.

1.7 Some Consequences of the Mass Transference Principle

In this section we mention a few notable consequences of the Mass Transference Principle. First, we will discuss the Hausdorff measure analogue of the Duffin–Schaeffer Conjecture, which gave rise to the discovery of the Mass Transference Principle in the first place. Then we will show how the Mass Transference Principle can be used to deduce Jarník's Theorem given Khintchine's Theorem and how the Jarník–Besicovitch Theorem actually follows already from Dirichlet's Theorem. We will conclude this section by mentioning one application of the more general mass transference principle (Theorem 1.23) stated in the previous section.

1.7.1 *Hausdorff measure Duffin–Schaeffer Conjecture*

In [7], Beresnevich and Velani proposed a version of the Duffin–Schaeffer Conjecture for Hausdorff measures. Their statement naturally extends Conjecture 1.14 and, according to them, represents *"the 'real' problem and the truth of which yields a complete metric theory"*. The Mass Transference Principle further supports this view and was used to show that, in fact, the Duffin–Schaeffer Conjecture for Lebesgue measure gives rise to the more general analogous statement for Hausdorff measures.

Throughout this section, we will assume that any dimension function f satisfies the hypothesis that $r^{-m}f(r)$ is monotonic. Recall that $\mathcal{A}''_m(\psi)$ denotes the set of points $\mathbf{x} \in \mathbb{I}^m$ for which the inequality (1.4) is satisfied for infinitely many $(\mathbf{p}, q) \in \mathbb{Z}^m \times \mathbb{N}$ which also have $\gcd(p_i, q) = 1$ for all $1 \leq i \leq m$. Before we present the Duffin–Schaeffer Conjecture for Hausdorff measures, we first make the following observation.

Observation 1.24. *Denote by φ the standard Euler function. If*

$$\sum_{q=1}^{\infty} f\left(\frac{\psi(q)}{q}\right)\varphi(q)^m < \infty, \tag{1.5}$$

then

$$\mathcal{H}^f(\mathcal{A}''_m(\psi)) = 0.$$

Proof. This observation can be established via a relatively easy covering argument. We first note that, without loss of generality, we may assume that $\frac{\psi(q)}{q} \to 0$ as $q \to \infty$. Suppose that this is not the case, then there must exist some $c > 0$ such that $\frac{\psi(q)}{q} > c$

for infinitely many $q \in \mathbb{N}$. Therefore, in order for the sum in (1.5) to converge, there must be some $c' > 0$ such that $f(x) = 0$ for all $x < c'$. Consequently, we must have $\mathcal{H}^f(X) = 0$ for any set $X \subset \mathbb{R}^m$. In particular, $\mathcal{H}^f(\mathcal{A}''_m(\psi)) = 0$ in this case and so we may assume that $\frac{\psi(q)}{q} \to 0$ as $q \to \infty$, as claimed.

For each $(\mathbf{p}, q) \in \mathbb{Z}^m \times \mathbb{N}$ with $B\left(\frac{\mathbf{p}}{q}, \frac{\psi(q)}{q}\right) \cap \mathbb{I}^m \neq \emptyset$, let $B_{(\mathbf{p},q)}(\psi) := B\left(\frac{\mathbf{p}}{q}, \frac{\psi(q)}{q}\right)$ be the ball in $\mathbb{R}^m$ (with respect to the supremum norm) centred at $\frac{\mathbf{p}}{q}$ with radius $\frac{\psi(q)}{q}$.

Now, let $\rho > 0$ be arbitrary and let $Q(\rho) \in \mathbb{N}$ be such that $\frac{\psi(q)}{q} < \rho$ for every $q \geq Q(\rho)$. Observe that

$$\bigcup_{q \geq Q(\rho)} \bigcup_{\substack{\mathbf{p} \in \mathbb{Z}^m: \\ \gcd(p_i, q) = 1,\, 1 \leq i \leq m \\ B\left(\frac{\mathbf{p}}{q}, \frac{\psi(q)}{q}\right) \cap \mathbb{I}^m \neq \emptyset}} B_{(\mathbf{p},q)}(\psi)$$

is a ρ-cover for $\mathcal{A}''_m(\psi)$. Thus, remembering that we are only interested in balls which intersect $\mathbb{I}^m$, it follows that

$$\mathcal{H}^f_\rho(\mathcal{A}''_m(\psi)) \leq \sum_{q \geq Q(\rho)} \sum_{\substack{\mathbf{p} \in \mathbb{Z}^m: \\ \gcd(p_i, q) = 1,\, 1 \leq i \leq m \\ B\left(\frac{\mathbf{p}}{q}, \frac{\psi(q)}{q}\right) \cap \mathbb{I}^m \neq \emptyset}} f\left(\frac{\psi(q)}{q}\right) \ll \sum_{q \geq Q(\rho)} \varphi(q)^m f\left(\frac{\psi(q)}{q}\right).$$

Since we assumed that $\sum_{q=1}^{\infty} f\left(\frac{\psi(q)}{q}\right) \varphi(q)^m < \infty$, we can make the right-hand side of the above arbitrarily small by taking ρ to be sufficiently small. The claim then follows from the definition of Hausdorff measure. $\square$

In [7], Beresnevich and Velani proposed that the following corresponding opposite statement should also be true.

Conjecture 1.25 (Hausdorff Measure Duffin–Schaeffer Conjecture [7]**).** *Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be any approximating function and let f be a dimension function such that $r^{-m} f(r)$ is monotonic. If*

$$\sum_{q=1}^{\infty} \varphi(q)^m f\left(\frac{\psi(q)}{q}\right) = \infty \quad \textit{then} \quad \mathcal{H}^f(\mathcal{A}''_m(\psi)) = \mathcal{H}^f(\mathbb{I}^m).$$

Setting $f(r) = r^m$ in the above we see that we immediately recover Conjecture 1.14 and so Conjecture 1.25 really is a natural extension of the usual Duffin–Schaeffer Conjecture to Hausdorff measures. In fact, using the Mass Transference Principle, Beresnevich and Velani proved that Conjecture 1.14 implies Conjecture 1.25 and hence that they are actually equivalent. Before we proceed to give a proof of this statement, we first record another straightforward, but nevertheless convenient, observation.

Observation 1.26. *Let $c > 0$ and let $\mathbf{x} \in \mathbb{I}^m$. Then, for every sufficiently large $q \in \mathbb{N}$, there exists a $\mathbf{p} = (p_1, p_2, \ldots, p_m) \in \mathbb{Z}^m$ such that for each $1 \leq i \leq m$ we have*

$$|qx_i - p_i| < cq \quad \text{and} \quad \gcd(p_i, q) = 1.$$

Proof. Fix $1 \leq i \leq m$. Note that if we could find p_i with $\gcd(p_i, q) = 1$ in the interval

$$I^* := (qx_i - cq, qx_i + cq)$$

then we would be done. In fact, we will show that it is possible to find a suitable value of p_i in the subinterval

$$I := (\max\{qx_i - cq, 0\}, qx_i + cq) \subset I^*.$$

Observe that the number of prime divisors of q is less than $\log_2 q$.

Now, let us consider how many primes lie in the interval I. If $I = (qx_i - cq, qx_i + cq)$ then, by the Prime Number Theorem (see, for example, [3]), the number of primes in I is (or is possibly one less than)

$$\pi(qx_i + cq) - \pi(qx_i - cq) \sim \frac{qx_i + cq}{\log(qx_i + cq)} - \frac{qx_i - cq}{\log(qx_i - cq)},$$

where $\pi(x)$ is the number of primes less than or equal to x. As $q \to \infty$,

$$\frac{qx_i + cq}{\log(qx_i + cq)} - \frac{qx_i - cq}{\log(qx_i - cq)} \to \frac{2cq}{\log q}.$$

Furthermore, $\frac{2cq}{\log q} > \log_2 q$ for all large enough q and so, for each such q, there are primes contained in I which are not divisors of q. For each suitably large q we may just take p_i to be one such prime.

In the case that $I = (0, qx_i + cq)$ we use essentially the same argument. $\square$

Theorem 1.27 (Beresnevich – Velani [7])**.** *Conjecture 1.14 implies Conjecture 1.25.*

Proof. First note that we may assume without loss of generality that $\frac{\psi(q)}{q} \to 0$ as $q \to \infty$. Otherwise, in light of Observation 1.26, we see that $\mathcal{A}''_m(\psi) = \mathbb{I}^m$ and the result follows immediately. Also recall that we are given that $\sum_{q=1}^{\infty} f\left(\frac{\psi(q)}{q}\right)\varphi(q)^m = \infty$.

For each $(\mathbf{p}, q) \in \mathbb{Z}^m \times \mathbb{N}$ with $\gcd(p_i, q) = 1$ for $1 \leq i \leq m$ and either $B\left(\frac{\mathbf{p}}{q}, \frac{\psi(q)}{q}\right) \cap \mathbb{I}^m \neq \emptyset$ or $B\left(\frac{\mathbf{p}}{q}, f\left(\frac{\psi(q)}{q}\right)^{\frac{1}{m}}\right) \cap \mathbb{I}^m \neq \emptyset$, let us denote by $B_{(\mathbf{p},q)}(\psi)$ the ball in $\mathbb{R}^m$, with respect to the supremum norm, centred at $\frac{\mathbf{p}}{q}$ of radius $\frac{\psi(q)}{q}$. Notice that

$$\mathcal{A}''_m(\psi) = \mathbb{I}^m \cap \limsup_{(\mathbf{p},q)} B_{(\mathbf{p},q)}(\psi).$$

In order to use the Mass Transference Principle, we also consider the balls $B^f_{(\mathbf{p},q)}(\psi)$ of radius $f\left(\frac{\psi(q)}{q}\right)^{\frac{1}{m}}$ centred at $\frac{\mathbf{p}}{q}$ for the same pairs $(\mathbf{p}, q)$ in $\mathbb{Z}^m \times \mathbb{N}$ as before. We note that

$$\mathbb{I}^m \cap \limsup_{(\mathbf{p},q)} B^f_{(\mathbf{p},q)}(\psi) = \mathcal{A}''_m(\theta)$$

where $\theta(q) := qf\left(\frac{\psi(q)}{q}\right)^{\frac{1}{m}}$.

Now, assuming the validity of the Duffin–Schaeffer Conjecture (Conjecture 1.14) gives us that $|\mathcal{A}''_m(\theta)| = 1$ since, by assumption,

$$\sum_{q=1}^{\infty} \varphi(q)^m \frac{\theta(q)^m}{q^m} = \sum_{q=1}^{\infty} \varphi(q)^m q^m f\left(\frac{\psi(q)}{q}\right)\frac{1}{q^m} = \sum_{q=1}^{\infty} \varphi(q)^m f\left(\frac{\psi(q)}{q}\right) = \infty.$$

Thus, for any ball $B \subset \mathbb{I}^m$, we have

$$\mathcal{H}^m(B \cap \limsup_{(\mathbf{p},q)} B^f_{(\mathbf{p},q)}(\psi)) = \mathcal{H}^m(B).$$

By the Mass Transference Principle, it follows that for any ball $B \subset \mathbb{I}^m$ we have

$$\mathcal{H}^f(B \cap \limsup_{(\mathbf{p},q)} B_{(\mathbf{p},q)}(\psi)) = \mathcal{H}^f(B).$$

In particular, we have

$$\mathcal{H}^f(\mathcal{A}''_m(\psi)) = \mathcal{H}^f(\mathbb{I}^m \cap \limsup_{(\mathbf{p},q)} B_{(\mathbf{p},q)}(\psi)) = \mathcal{H}^f(\mathbb{I}^m),$$

as required. $\square$

Since Pollington and Vaughan [41] showed that Conjecture 1.14 holds for $m \geq 2$, an immediate consequence of the above equivalence is that Conjecture 1.25 also holds for $m \geq 2$ for general approximating functions ψ.

1.7.2 *Khintchine's Theorem implies Jarník's Theorem*

In this section we will indicate how Jarník's Theorem follows from Khintchine's Theorem as a corollary of the Mass Transference Principle. Strictly speaking, since the statements we have provided represent modern improved versions of Khintchine's Theorem and Jarník's Theorem, the argument we present below does not quite immediately let us extract Theorem 1.18 from Theorem 1.11 — some discrepancies arise between monotonicity conditions. However, we shall bypass these issues in this section by assuming, as Jarník originally did, that $q^{m+1}f\left(\frac{\psi(q)}{q}\right)$ is decreasing — see Remark 1.19.

We note that, without loss of generality, we may assume $\frac{\psi(q)}{q} \to 0$ as $q \to \infty$. To see this, suppose for the moment that this is not the case. Then, there exists some $c > 0$ such that $\frac{\psi(q)}{q} > c$ for infinitely many $q \in \mathbb{N}$. In particular, this means that $f\left(\frac{\psi(q)}{q}\right) > f(c)$ infinitely often. If $f(c) > 0$ then the assumption that $q^{m+1}f\left(\frac{\psi(q)}{q}\right)$ is decreasing is violated. Thus, we must have $f(x) = 0$ for all $x \leq c$. Consequently, for any set $X \in \mathbb{R}^m$ we have $\mathcal{H}^f(X) = 0$ and the desired result follows. Thus, we may assume without loss of generality that $\frac{\psi(q)}{q} \to 0$ as $q \to \infty$.

For each $(\mathbf{p}, q) \in \mathbb{Z}^m \times \mathbb{N}$ with $B\left(\frac{\mathbf{p}}{q}, \frac{\psi(q)}{q}\right) \cap \mathbb{I}^m \neq \emptyset$ or $B\left(\frac{\mathbf{p}}{q}, f\left(\frac{\psi(q)}{q}\right)^{\frac{1}{m}}\right) \cap \mathbb{I}^m \neq \emptyset$ let $B_{(\mathbf{p},q)}(\psi) := B\left(\frac{\mathbf{p}}{q}, \frac{\psi(q)}{q}\right)$ be the ball in $\mathbb{R}^m$, with respect to the supremum norm, centred at $\frac{\mathbf{p}}{q}$ with radius $\frac{\psi(q)}{q}$. Then,

$$\mathcal{A}_m(\psi) = \mathbb{I}^m \cap \limsup_{(\mathbf{p},q)} B_{(\mathbf{p},q)}(\psi) = \mathbb{I}^m \cap \bigcap_{Q=1}^{\infty} \bigcup_{q \geq Q} \bigcup_{\mathbf{p} \in \mathbb{Z}^m} B_{(\mathbf{p},q)}(\psi).$$

First of all, we will deal with the convergence part of Jarník's Theorem. This relies on a standard covering argument and does not utilise the Mass Transference Principle at all, nor does it require the additional monotonicity assumption we have imposed. To prove the convergence part let $\rho > 0$ and let $Q(\rho) \in \mathbb{N}$ be such that $\frac{\psi(q)}{q} < \rho$ for all $q \geq Q(\rho)$. Then,

$$\bigcup_{q \geq Q(\rho)} \bigcup_{\substack{\mathbf{p} \in \mathbb{Z}^m \\ B\left(\frac{\mathbf{p}}{q}, \frac{\psi(q)}{q}\right) \cap \mathbb{I}^m \neq \emptyset}} B_{(\mathbf{p},q)}(\psi)$$

is a ρ-cover for $\mathcal{A}_m(\psi)$. Since we are only concerned with balls which have non-empty intersection with $\mathbb{I}^m$ we see that

$$\mathcal{H}^f_\rho(\mathcal{A}_m(\psi)) \ll \sum_{q \geq Q(\rho)} q^m f\left(\frac{\psi(q)}{q}\right). \tag{1.6}$$

Since $\sum_{q=1}^{\infty} q^m f\left(\frac{\psi(q)}{q}\right)$ converges by assumption, we can make the sum on the right-hand side of (1.6) arbitrarily small by taking ρ to be sufficiently small. The result then follows from the definition of Hausdorff measure.

For the divergence part we are given $\sum_{q=1}^{\infty} q^m f\left(\frac{\psi(q)}{q}\right) = \infty$. We note that

$$B^f_{(\mathbf{p},q)}(\psi) = B\left(\frac{\mathbf{p}}{q}, f\left(\frac{\psi(q)}{q}\right)^{\frac{1}{m}}\right)$$

and

$$\mathbb{I}^m \cap \limsup_{(\mathbf{p},q)} B^f_{(\mathbf{p},q)}(\psi) = \mathcal{A}(\theta)$$

where $\theta(q) := qf\left(\frac{\psi(q)}{q}\right)^{\frac{1}{m}}$. By Khintchine's Theorem we have that $|\mathcal{A}_m(\theta)| = 1$ since

$$\sum_{q=1}^{\infty} \theta(q)^m = \sum_{q=1}^{\infty} \left[qf\left(\frac{\psi(q)}{q}\right)^{\frac{1}{m}}\right]^m = \sum_{q=1}^{\infty} q^m f\left(\frac{\psi(q)}{q}\right) = \infty$$

and, by our additional assumption, we have that $\theta(q)$ is decreasing. Thus,

$$|\mathbb{I}^m \cap \limsup_{(\mathbf{p},q)} B^f_{(\mathbf{p},q)}(\psi)| = |\mathbb{I}^m|.$$

So, applying the Mass Transference Principle with $\Omega = \mathbb{I}^m$ we conclude that for any ball B in $\mathbb{I}^m$ we have

$$\mathcal{H}^f(B \cap \limsup_{(\mathbf{p},q)} B_{p,q}(\psi)) = \mathcal{H}^f(B).$$

In particular,

$$\mathcal{H}^f(\mathcal{A}_m(\psi)) = \mathcal{H}^f(\mathbb{I}^m \cap \limsup_{(\mathbf{p},q)} B_{(\mathbf{p},q)}(\psi)) = \mathcal{H}^f(\mathbb{I}^m).$$

1.7.3 *Dirichlet's Theorem implies the Jarník–Besicovitch Theorem*

Another extraordinary consequence of the Mass Transference Principle is that the Jarník–Besicovitch Theorem (Theorem 1.17) actually follows from Dirichlet's Theorem. In fact, we will show here how the Jarník–Besicovitch Theorem follows from Theorem 1.2 and does not even require the full power of Theorem 1.1.

Recall that the consequence of Dirichlet's Theorem recorded in Theorem 1.2 states that for any $x \in \mathbb{R}$ there exist infinitely many pairs $(p,q) \in \mathbb{Z} \times \mathbb{N}$ such that $\left|x - \frac{p}{q}\right| < \frac{1}{q^2}$.

Consider $\tau > 1$. For a fixed $s > 0$, for each $(p,q) \in \mathbb{Z} \times \mathbb{N}$ with $B\left(\frac{p}{q}, \frac{1}{q^{\tau+1}}\right) \cap \mathbb{I} \neq \emptyset$ or $B\left(\frac{p}{q}, \frac{1}{q^{s(\tau+1)}}\right) \cap \mathbb{I} \neq \emptyset$, let $B_{(p,q)}(\tau) = B\left(\frac{p}{q}, \frac{1}{q^{\tau+1}}\right)$. Observe that

$$\mathcal{A}(\tau) = \mathbb{I} \cap \limsup_{(p,q)} B_{(p,q)}(\tau).$$

Now,

$$B^s_{(p,q)}(\tau) = B\left(\frac{p}{q}, \frac{1}{q^{s(\tau+1)}}\right)$$

and so, by Dirichlet's Theorem, if $s(\tau+1) \leq 2$ we have

$$\mathbb{I} \cap \limsup_{(p,q)} B^s_{(p,q)}(\tau) = \mathbb{I}.$$

That is, if $s \leq \frac{2}{\tau+1}$ then $|\mathbb{I} \cap \limsup_{(p,q)} B^s_{(p,q)}(\tau)| = |\mathbb{I}|$. In this case, we can apply the Mass Transference Principle with $\Omega = \mathbb{I}$ to conclude that

$$\mathcal{H}^s(\mathcal{A}(\tau)) = \mathcal{H}^s(\mathbb{I} \cap \limsup_{(p,q)} B_{(p,q)}(\tau)) = \mathcal{H}^s(\mathbb{I}) = \infty$$

when $s \leq \frac{2}{\tau+1}$. It follows that $\dim_{\mathrm{H}}(\mathcal{A}(\tau)) \geq \frac{2}{\tau+1}$.

The corresponding upper bound, $\dim_{\mathrm{H}}(\mathcal{A}(\tau)) \leq \frac{2}{\tau+1}$, can be obtained via a standard covering argument like the ones we have seen previously. We note that, for any $N \in \mathbb{N}$,

$$\mathcal{A}(\tau) \subset \mathbb{I} \cap \bigcup_{q \geq N} \bigcup_{\substack{p \in \mathbb{Z} \\ B\left(\frac{p}{q}, \frac{1}{q^{\tau+1}}\right) \cap \mathbb{I} \neq \emptyset}} B_{(p,q)}(\tau).$$

Let $\rho > 0$ and take $Q(\rho)$ to be such that $\frac{1}{q^{\tau+1}} < \rho$ for all $q \geq Q(\rho)$. Since we are only interested in balls intersecting $\mathbb{I}$, we see that

$$\mathcal{H}^s_\rho(\mathcal{A}(\tau)) \ll \sum_{q \geq Q(\rho)} q^{1-s(\tau+1)}.$$

Whenever $s > \frac{2}{\tau+1}$, the right-hand side can be made arbitrarily small by taking ρ to be sufficiently small. So, $\mathcal{H}^s(\mathcal{A}(\tau)) = 0$ when $s > \frac{2}{\tau+1}$ and, hence, $\dim_{\mathrm{H}}(\mathcal{A}(\tau)) \leq \frac{2}{\tau+1}$. Combining the upper and lower bounds yields $\dim_{\mathrm{H}} \mathcal{A}(\tau) = \frac{2}{\tau+1}$, as required.

1.7.4 *Mahler's assertion — an application of Theorem 1.23*

We end this section by also mentioning one example of a use of Theorem 1.23, the more general mass transference principle recorded in Section 1.6.2. As indicated there, we shall discuss how Levesley, Salp, and Velani [39] have used Theorem 1.23 as a tool for proving an assertion of Mahler on the existence of very well approximable numbers in the middle-third Cantor set. They were able to do this because, unlike Theorem 1.22, which just applies to lim sup sets in $\mathbb{R}^k$, Theorem 1.23 can be applied to lim sup sets in more general metric spaces. In this particular case, Levesley, Salp and Velani have made use of the fact that Theorem 1.23 can be used when X is the standard middle-third Cantor set. The *middle-third Cantor set*, which shall be denoted throughout by K, is the set of $x \in [0,1]$ which have a ternary expansion containing only 0s and 2s. Alternatively, the middle-third Cantor set can be obtained by removing the open middle-third from the unit interval and then subsequently repeatedly removing the open middle-third of every remaining interval. It is well known that

$$|K| = 0 \quad \text{and} \quad \dim_{\mathrm{H}} K = \frac{\log 2}{\log 3}.$$

As a result of Dirichlet's Theorem, we know that for any $x \in \mathbb{R}$ there exist infinitely many pairs $(p,q) \in \mathbb{Z} \times \mathbb{N}$ for which

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2}.$$

If the exponent in the denominator of the right-hand side of the above can be improved (i.e. increased) for some $x \in \mathbb{R}$ then x is said to be *very well approximable*; that is, a real number x is said to be very well approximable if there exists some $\varepsilon > 0$

such that

$$\left|x - \frac{p}{q}\right| < \frac{1}{q^{2+\varepsilon}} \tag{1.7}$$

for infinitely many pairs $(p, q) \in \mathbb{Z} \times \mathbb{N}$. We will denote the set of very well approximable numbers by $\mathcal{W}$. If, further, (1.7) is satisfied for every $\varepsilon > 0$ for some $x \in \mathbb{R} \setminus \mathbb{Q}$ then x is called a *Liouville number*. We will denote by $\mathcal{L}$ the set of all Liouville numbers.

It is known (see, for example, [14]) that

$$|\mathcal{W}| = 0, \ \dim_{\mathrm{H}}(\mathcal{W}) = 1,$$
$$|\mathcal{L}| = 0, \ \text{and} \ \dim_{\mathrm{H}}(\mathcal{L}) = 0.$$

Furthermore, both of the sets $\mathcal{W}$ and $\mathcal{L}$ are uncountable.

Regarding the intersection of $\mathcal{W}$ with the middle-third Cantor set, Mahler is attributed with having made the following claim.

Mahler's Assertion. *There exist very well approximable numbers, other than Liouville numbers, in the middle-third Cantor set; i.e.*

$$(\mathcal{W} \setminus \mathcal{L}) \cap K \neq \emptyset.$$

Remark. We refer the reader to [39] for discussion of the precise origin of this claim and also for some discussion regarding why it is natural/necessary to exclude Liouville numbers from Mahler's assertion.

Now, let $\mathcal{B} = \{3^n : n = 0, 1, 2, \dots\}$ and, given an approximating function $\psi : \mathbb{N} \to \mathbb{R}^+$, consider the set

$$\mathcal{A}_{\mathcal{B}}(\psi) := \left\{ x \in [0,1] : \left| x - \frac{p}{q} \right| < \psi(q) \text{ for infinitely many } (p,q) \in \mathbb{Z} \times \mathcal{B} \right\}.$$

In the case that $\psi(q) = q^{-\tau}$ for some $\tau > 0$, write $\mathcal{A}_{\mathcal{B}}(\tau)$ in place of $\mathcal{A}_{\mathcal{B}}(\psi)$.

Remark. Although it is slightly inconsistent with other notation used throughout, we have decided to use this particular notation here for clarity and since it is in keeping with the notation used in [39].

Levesley, Salp and Velani have used the general mass transference principle (Theorem 1.23) as a tool for establishing the following statement regarding Hausdorff measures of the set $\mathcal{A}_{\mathcal{B}}(\psi) \cap K$ in [39].

Theorem 1.28. *Let f be a dimension function such that $r^{-\frac{\log 2}{\log 3}}f(r)$ is monotonic. Then,*

$$\mathcal{H}^f(\mathcal{A}_{\mathcal{B}}(\psi)\cap K)=\begin{cases}0 & \text{if} \quad \sum_{n=1}^{\infty} f(\psi(3^n))\times(3^n)^{\frac{\log 2}{\log 3}}<\infty,\\ \\ \mathcal{H}^f(K) & \text{if} \quad \sum_{n=1}^{\infty} f(\psi(3^n))\times(3^n)^{\frac{\log 2}{\log 3}}=\infty.\end{cases}$$

Taking $f(r)=r^s$ in Theorem 1.28, and noting that $\mathcal{A}_{\mathcal{B}}(\tau)\cap K\subset\mathcal{W}\cap K$ for any $\tau>2$, we deduce that $\dim_{\mathrm{H}}(\mathcal{W}\cap K)\geq\frac{\log 2}{2\log 3}$. Combining this with the fact that $\dim_{\mathrm{H}}\mathcal{L}=0$, we obtain the following corollary to Theorem 1.28. For further details on the derivation of this corollary we refer the reader to [39].

Corollary 1.29 (Levesley – Salp – Velani [39])**.** *We have*

$$\dim_{\mathrm{H}}((\mathcal{W}\setminus\mathcal{L})\cap K)\geq\frac{\log 2}{2\log 3}.$$

The truth of Mahler's assertion follows immediately from this corollary.

1.8 Overview

While they are exceptional results with some astonishing applications, both the original Mass Transference Principle (Theorem 1.22) and its generalisation given by Theorem 1.23 concern lim sup sets arising from sequences of balls. Throughout this thesis, we will mostly be concerned with an extension of the Mass Transference Principle to the case where the lim sup sets of interest are defined by sequences of neighbourhoods of approximating planes, or linear forms, and some of the associated applications. We shall also briefly discuss some recent progress towards establishing a mass transference principle when the lim sup sets of interest are defined by sequences of rectangles.

In Chapter 2, we will present the statement and proof of a mass transference principle for systems of linear forms. In Chapter 3, we will discuss a number of applications of this theorem. Specifically, we develop a very general framework for transferring Lebesgue measure statements to Hausdorff measure statements in the context of approximation by linear forms. The material in Chapters 2 and 3 can be found in [1].

In Chapter 4, we shall discuss a general inhomogeneous Jarník–Besicovitch Theorem introduced by Levesley in [38]. In particular, we shall demonstrate how one of the consequences of the mass transference principle for linear forms can be used to provide an alternative proof of most of the cases of Levesley's result. Additionally, we will show that the monotonicity condition imposed in Levesley's theorem cannot be removed. The results from Chapter 4 appear in [2, Section 3.3].

In Chapter 5 we shall provide some discussion of weighted simultaneous approximation before mentioning some recent progress towards proving mass transference principles for rectangles. Chapter 5 is based on [2, Section 4] and the contributions made in Section 5.3 appear in [2, Section 4.2].

2 A Mass Transference Principle for Systems of Linear Forms

As mentioned previously, the findings of Khintchine, Jarník and Besicovitch described in Chapter 1 have been sharpened and generalised in numerous ways. One direction in which such results have been extended, and which we will be particularly interested in here and in subsequent chapters, is to involve problems concerning systems of linear forms.

As discussed in the previous chapter, many sets of interest in Diophantine approximation can be naturally expressed as $\limsup$ sets. There we were mostly concerned with sets expressible as $\limsup$ sets of balls in $\mathbb{R}^k$. In the setting of approximation by linear forms, which we will now be concerned with, many of the sets of interest can be expressed as $\limsup$ sets determined by sequences of neighbourhoods of "approximating planes", i.e. linear forms.

In this chapter we will be dealing with the extension of the Mass Transference Principle in this setting. This is not an entirely new direction of research. Indeed, such an extension has already been obtained in [8]. However, the mass transference principle result of [8] carries some technical conditions which arise as a consequence of the "slicing" technique that was used for the proof (for a simple demonstration of the idea of "slicing" see the proofs of Propositions 5.11 and 5.12 in Chapter 5). These conditions were conjectured to be unnecessary by Beresnevich, Bernik, Dodson and Velani in [4]. In this chapter we state and prove an extension of the Mass Transference Principle for systems of linear forms which verifies this conjecture and thereby improves upon the result obtained in this direction by Beresnevich and Velani in [8].

The mass transference principle for linear forms that we present in this chapter is the main result in [1] and constitutes the main result presented in this thesis. In the following two chapters, we will discuss various applications of this result.

The contents of this chapter appear as in [1] modulo minimal modifications made for the purposes of readability.

2.1 Statement of Main Result (Theorem 2.2)

Let $k, m \geq 1$ and $l \geq 0$ be integers such that $k = m + l$. Let $\mathcal{R} := (R_j)_{j \in \mathbb{N}}$ be a family of planes in $\mathbb{R}^k$ of common dimension l. For every $j \in \mathbb{N}$ and $\delta \geq 0$, define

$$\Delta(R_j, \delta) := \{\mathbf{x} \in \mathbb{R}^k : \operatorname{dist}(\mathbf{x}, R_j) < \delta\},$$

where $\operatorname{dist}(\mathbf{x}, R_j) = \inf\{\|\mathbf{x} - \mathbf{y}\| : \mathbf{y} \in R_j\}$ and $\|\cdot\|$ is any fixed norm on $\mathbb{R}^k$.

Let $\Upsilon : \mathbb{N} \to \mathbb{R} : j \mapsto \Upsilon_j$ be a non-negative real-valued function on $\mathbb{N}$ such that $\Upsilon_j \to 0$ as $j \to \infty$. Consider

$$\Lambda(\Upsilon) := \{\mathbf{x} \in \mathbb{R}^k : \mathbf{x} \in \Delta(R_j, \Upsilon_j) \text{ for infinitely many } j \in \mathbb{N}\}.$$

In [8], the following was established.

Theorem 2.1 (Beresnevich – Velani [8]). *Let $\mathcal{R}$ and Υ be as given above. Let V be a linear subspace of $\mathbb{R}^k$ such that $\dim V = m = \operatorname{codim} \mathcal{R}$,*

(i) *$V \cap R_j \neq \emptyset$ for all $j \in \mathbb{N}$, and*

(ii) *$\sup_{j \in \mathbb{N}} \operatorname{diam}(V \cap \Delta(R_j, 1)) < \infty$.*

Let f and $g : r \to g(r) := r^{-l} f(r)$ be dimension functions such that $r^{-k} f(r)$ is monotonic and let Ω be a ball in $\mathbb{R}^k$. Suppose that, for any ball B in Ω,

$$\mathcal{H}^k\left(B \cap \Lambda\left(g(\Upsilon)^{\frac{1}{m}}\right)\right) = \mathcal{H}^k(B).$$

Then, for any ball B in Ω,

$$\mathcal{H}^f(B \cap \Lambda(\Upsilon)) = \mathcal{H}^f(B).$$

Remark. In the case that $l = 0$, Theorem 2.1 coincides with the Mass Transference Principle (Theorem 1.22) stated earlier.

Remark. Conditions (i) and (ii) essentially say that V should intersect every plane and that the angle of intersection between V and each plane should be bounded away from 0. In other words, every plane R_n ought not to be parallel to V and should intersect V in precisely one place.

The conditions (i) and (ii) in Theorem 2.1 arise as a consequence of the particular proof strategy employed in [8]. However, it was conjectured [4, Conjecture E] that Theorem 2.1 should be true without conditions (i) and (ii). By adopting a different proof strategy — one similar to that used to prove the Mass Transference Principle in [7] rather than "slicing" — we are able to remove conditions (i) and (ii) and, consequently, prove the following.

Theorem 2.2. *Let $\mathcal{R}$ and Υ be as given above. Let f and $g : r \to g(r) := r^{-l}f(r)$ be dimension functions such that $r^{-k}f(r)$ is monotonic and let Ω be a ball in $\mathbb{R}^k$. Suppose that, for any ball B in Ω,*

$$\mathcal{H}^k\left(B \cap \Lambda\left(g(\Upsilon)^{\frac{1}{m}}\right)\right) = \mathcal{H}^k(B). \tag{2.1}$$

Then, for any ball B in Ω,

$$\mathcal{H}^f(B \cap \Lambda(\Upsilon)) = \mathcal{H}^f(B). \tag{2.2}$$

At first glance, conditions (i) and (ii) in Theorem 2.1 do not seem particularly restrictive. Indeed, there are a number of interesting consequences of this theorem — see [4, 8]. However, in the following chapter we present applications of Theorem 2.2 which may well be out of reach when using Theorem 2.1.

We proceed now by establishing the remaining necessary preliminaries and some auxiliary lemmas in Sections 2.2 and 2.3 before presenting the full proof of Theorem 2.2 in Section 2.4.

2.2 Preliminaries

Firstly, recall that, given a ball $B := B(x, r)$ in $\mathbb{R}^k$, with respect to a fixed norm $\|\cdot\|$ on $\mathbb{R}^k$, and a dimension function $f : \mathbb{R}^+ \to \mathbb{R}^+$ we write $V^f(B) := f(r)$ and refer to $V^f(B)$ as the *f-volume of B*. If $f(x) = x^s$ for some $s \geq 0$, we write V^s instead of V^f.

We observed earlier that if $|\cdot|$ is k-dimensional Lebesgue measure, $\|\cdot\|$ is the Euclidean norm, and $f(x) = |B(0,1)|x^k$, then V^f is simply the volume of B in the usual geometric sense; i.e. $V^f(B) = |B|$. In particular, for any ball B in $\mathbb{R}^k$ we have that $V^k(B)$ is comparable to $|B|$. Another observation we made earlier is that, for subsets of $\mathbb{R}^k$, $\mathcal{H}^k$ is also comparable to the k-dimensional Lebesgue measure. Combining these two facts, it follows that there are constants $0 < c_1 < 1 < c_2 < \infty$

such that for any ball B in $\mathbb{R}^k$ we have

$$c_1\, V^k(B) \le \mathcal{H}^k(B) \le c_2\, V^k(B). \tag{2.3}$$

A general and classical method for obtaining a lower bound for the Hausdorff f-measure of an arbitrary set F is the following *mass distribution principle*. This will play a central role in our proof of Theorem 2.2 in Section 2.4.

Lemma 2.3 (**Mass Distribution Principle**). *Let μ be a probability measure supported on a subset F of $\mathbb{R}^k$. Suppose there are positive constants c and r_o such that*

$$\mu(B) \le c\, V^f(B)$$

for any ball B with radius $r \le r_o$. If E is a subset of F with $\mu(E) = \lambda > 0$ then $\mathcal{H}^f(E) \ge \lambda/c$.

The above lemma is stated as it appears in [7] since this version is most useful for our current purposes.

We conclude this section by stating a basic, but extremely useful, covering lemma which we will use throughout. Let $B := B(x, r)$ be a ball in $\mathbb{R}^k$. For any $\lambda > 0$, we denote by λB the ball B scaled by a factor λ; i.e. $\lambda B := B(x, \lambda r)$.

Lemma 2.4 (**The $5r$-covering Lemma** [40]). *Every family $\mathcal{F}$ of balls of uniformly bounded diameter in $\mathbb{R}^k$ contains a disjoint subfamily $\mathcal{G}$ such that*

$$\bigcup_{B \in \mathcal{F}} B \subset \bigcup_{B \in \mathcal{G}} 5B.$$

2.3 The $K_{G,B}$ Covering Lemma

Our strategy for proving Theorem 2.2 is similar to that used for proving the Mass Transference Principle for balls in [7]. There are however various technical differences that account for the different shape of approximating sets. First of all we will require a covering lemma analogous to the $K_{G,B}$-lemma established in [7, Section 4]. This appears as Lemma 2.5 below. The balls obtained from Lemma 2.5 correspond to planes in the lim sup set $\Lambda(g(\Upsilon)^{\frac{1}{m}})$. Furthermore, for the proof of Theorem 2.2 it is necessary for us to obtain from each of these "larger" balls a collection of balls which correspond to the "shrunk" lim sup set $\Lambda(\Upsilon)$. The desired properties of this collection and the existence of such a collection are the contents of Lemma 2.7.

To save on notation, throughout let $\tilde{\Upsilon}_j := g(\Upsilon_j)^{\frac{1}{m}}$. For an arbitrary ball $B \in \mathbb{R}^k$ and for each $j \in \mathbb{N}$ define

$$\Phi_j(B) := \{B(\mathbf{x}, \tilde{\Upsilon}_j) \subset B : \mathbf{x} \in R_j\}.$$

Analogously to Lemma 5 from [7] we will require the following covering lemma.

Lemma 2.5. *Let $\mathcal{R}$, Υ, g and Ω be as in Theorem 2.2 and assume that* (2.1) *is satisfied. Then for any ball B in Ω and any $G \in \mathbb{N}$, there exists a finite collection*

$$K_{G,B} \subset \big\{(A; j) : j \geq G,\ A \in \Phi_j(B)\big\}$$

satisfying the following properties:

(i) *if $(A; j) \in K_{G,B}$ then $3A \subset B$;*

(ii) *if $(A; j), (A'; j') \in K_{G,B}$ are distinct then $3A \cap 3A' = \emptyset$; and*

(iii) $\mathcal{H}^k\left(\bigcup_{(A;j)\in K_{G,B}} A\right) \geq \frac{1}{4\times 15^k}\mathcal{H}^k(B)$.

Remark 2.6. Essentially, $K_{G,B}$ is a collection of balls drawn from the families $\Phi_j(B)$. We write $(A; j)$ for a generic ball from $K_{G,B}$ to "remember" the index j of the family $\Phi_j(B)$ that the ball A comes from. However, when we are referring only to the ball A (as opposed to the pair $(A; j)$) we will just write A. Keeping track of the associated j will be absolutely necessary in order to be able to choose the "right" collection of balls within A that at the same time lie in an Υ_j-neighbourhood of the relevant R_j. Indeed, for $j \neq j'$ we could have $A = A'$ for some $A \in \Phi_j(B)$ and $A' \in \Phi_{j'}(B)$.

Proof of Lemma 2.5. For each $j \in \mathbb{N}$, consider the collection of balls

$$\Phi_j^3(B) := \{B(\mathbf{x}, 3\tilde{\Upsilon}_j) \subset B : \mathbf{x} \in R_j\}.$$

By (2.1), for any $G \geq 1$ we have that

$$\mathcal{H}^k\left(\bigcup_{j\geq G}(\Delta(R_j, 3\tilde{\Upsilon}_j) \cap B)\right) = \mathcal{H}^k(B).$$

Observe that

$$\bigcup_{L \in \Phi_j^3(B)} L \quad \subset \quad \Delta(R_j, 3\tilde{\Upsilon}_j) \cap B$$

and that the difference of the two sets lies within $3\tilde{\Upsilon}_j$ of the boundary of B. Then, since $\Upsilon_j \to 0$, and consequently $\tilde{\Upsilon}_j \to 0$, as $j \to \infty$, we have that

$$\mathcal{H}^k\left(\bigcup_{j \geq G} \bigcup_{L \in \Phi_j^3(B)} L\right) \sim \mathcal{H}^k\left(\bigcup_{j \geq G} (\Delta(R_j, 3\tilde{\Upsilon}_j) \cap B)\right) = \mathcal{H}^k(B) \quad \text{as } G \to \infty.$$

In particular, there exists a sufficiently large $G' \in \mathbb{N}$ such that for any $G \geq G'$ we have

$$\mathcal{H}^k\left(\bigcup_{j \geq G} \bigcup_{L \in \Phi_j^3(B)} L\right) \geq \frac{1}{2}\mathcal{H}^k(B).$$

However, for any $G < G'$ we also have

$$\bigcup_{j \geq G} \bigcup_{L \in \Phi_j^3(B)} L \supset \bigcup_{j \geq G'} \bigcup_{L \in \Phi_j^3(B)} L.$$

Thus, for any $G \in \mathbb{N}$ we must have

$$\mathcal{H}^k\left(\bigcup_{j \geq G} \bigcup_{L \in \Phi_j^3(B)} L\right) \geq \frac{1}{2}\mathcal{H}^k(B). \tag{2.4}$$

In fact, using the same argument as above it is possible to show that for any $G \in \mathbb{N}$ we have $\mathcal{H}^k\left(\bigcup_{j \geq G} \bigcup_{L \in \Phi_j^3(B)} L\right) \geq (1-\varepsilon)\mathcal{H}^k(B)$ for any $0 < \varepsilon < 1$ and hence that we must have $\mathcal{H}^k\left(\bigcup_{j \geq G} \bigcup_{L \in \Phi_j^3(B)} L\right) = \mathcal{H}^k(B)$. However, (2.4) is sufficient for our purposes here.

By Lemma 2.4, there exists a disjoint subcollection

$$\mathcal{G} \subset \{(L; j) : j \geq G, \ L \in \Phi_j^3(B)\}$$

such that

$$\mathop{\dot{\bigcup}}_{(L;j) \in \mathcal{G}} L \subset \bigcup_{j \geq G} \bigcup_{L \in \Phi_j^3(B)} L \subset \bigcup_{(L;j) \in \mathcal{G}} 5L.$$

Now, let $\mathcal{G}'$ consist of all the balls from $\mathcal{G}$ but shrunk by a factor of 3; so the balls in $\mathcal{G}'$ will still be disjoint when scaled by a factor of 3. Formally,

$$\mathcal{G}' := \{(\tfrac{1}{3}L; j) : (L; j) \in \mathcal{G}\}.$$

Then, we have that

$$\mathop{\overset{\circ}{\bigcup}}_{(A;j)\in\mathcal{G}'} A \subset \bigcup_{j\geq G} \bigcup_{L\in\Phi_j^3(B)} L \subset \bigcup_{(A;j)\in\mathcal{G}'} 15A. \tag{2.5}$$

From (2.4) and (2.5) we have that

$$\begin{aligned}
\mathcal{H}^k\left(\bigcup_{(A;j)\in\mathcal{G}'} A\right) &= \sum_{(A;j)\in\mathcal{G}'} \mathcal{H}^k(A) \\
&= \sum_{(A;j)\in\mathcal{G}'} \frac{1}{15^k}\mathcal{H}^k(15A) \\
&\geq \frac{1}{15^k}\mathcal{H}^k\left(\bigcup_{(A;j)\in\mathcal{G}'} 15A\right) \\
&\geq \frac{1}{15^k}\mathcal{H}^k\left(\bigcup_{j\geq G}\bigcup_{L\in\Phi_j^3(B)} L\right) \\
&\geq \frac{1}{2\times 15^k}\mathcal{H}^k(B).
\end{aligned}$$

Next note that, since the balls in $\mathcal{G}'$ are disjoint and contained in B and $\tilde{\Upsilon}_j \to 0$ as $j \to \infty$, we have that

$$\mathcal{H}^k\left(\bigcup_{\substack{(A;j)\in\mathcal{G}' \\ j\geq N}} A\right) \to 0 \quad \text{as } N \to \infty.$$

Therefore, there exists a sufficiently large $N_0 \in \mathbb{N}$ such that

$$\mathcal{H}^k\left(\bigcup_{\substack{(A;j)\in\mathcal{G}' \\ j\geq N_0}} A\right) < \frac{1}{4\times 15^k}\mathcal{H}^k(B).$$

Taking $K_{G,B}$ to be the subcollection of $(A; j) \in \mathcal{G}'$ with $G \leq j < N_0$ ensures that $K_{G,B}$ is a finite collection of balls while still satisfying the required properties (i)–(iii). $\square$

Lemma 2.7. *Let $\mathcal{R}$, Υ, g, Ω and B be as in Lemma 2.5 and assume that (2.1) is satisfied. Furthermore, assume that $r^{-k}f(r) \to \infty$ as $r \to 0$. Let $K_{G,B}$ be as in Lemma 2.5. Then, provided that G is sufficiently large, for any $(A;j) \in K_{G,B}$ there exists a collection $\mathcal{C}(A;j)$ of balls satisfying the following properties:*

(i) *each ball in $\mathcal{C}(A;j)$ is of radius Υ_j and is centred on R_j;*

(ii) *if $L \in \mathcal{C}(A;j)$ then $3L \subset A$;*

(iii) *if $L, M \in \mathcal{C}(A;j)$ are distinct then $3L \cap 3M = \emptyset$;*

(iv) $\frac{1}{7^k}\mathcal{H}^k\big(\Delta(R_j,\Upsilon_j) \cap \tfrac{1}{2}A\big) \leq \mathcal{H}^k\left(\bigcup_{L\in\mathcal{C}(A;j)} L\right) \leq \mathcal{H}^k\big(\Delta(R_j,\Upsilon_j)\cap A\big)$*; and*

(v) *there exist some constants $d_1, d_2 > 0$, independent of G and j, such that*

$$d_1 \times \left(\frac{g(\Upsilon_j)^{\frac{1}{m}}}{\Upsilon_j}\right)^l \leq \#\mathcal{C}(A;j) \leq d_2 \times \left(\frac{g(\Upsilon_j)^{\frac{1}{m}}}{\Upsilon_j}\right)^l. \tag{2.6}$$

Proof. First of all note that, by the assumption that $r^{-k}f(r) \to \infty$ as $r \to 0$, we have that

$$\frac{\Upsilon_j}{\tilde{\Upsilon}_j} \to 0 \qquad \text{as } j \to \infty\,.$$

In particular we can assume that G is sufficiently large so that

$$6\Upsilon_j < \tilde{\Upsilon}_j \qquad \text{for any } j \geq G. \tag{2.7}$$

Let $\mathbf{x}_1, \ldots, \mathbf{x}_t \in R_j \cap \frac{1}{2}A$ be any collection of points such that

$$\|\mathbf{x}_i - \mathbf{x}_{i'}\| > 6\Upsilon_j \qquad \text{if } i \neq i' \tag{2.8}$$

and t is maximal possible. The existence of such a collection follows immediately from the fact that $R_j \cap \frac{1}{2}A$ is bounded and, by (2.8), the collection is discrete. Let

$$\mathcal{C}(A;j) := \{B(\mathbf{x}_1,\Upsilon_j), \ldots, B(\mathbf{x}_t,\Upsilon_j)\}\,.$$

Thus, property (i) is trivially satisfied for this collection $\mathcal{C}(A;j)$. Recall that, by construction, $A \in \Phi_j(B)$, which means that the radius of $\frac{1}{2}A$ is $\frac{1}{2}\tilde{\Upsilon}_j$. If $L \in \mathcal{C}(A;j)$, say $L := B(\mathbf{x}_i,\Upsilon_j)$, and A is centred at $\mathbf{x}_0$, then for any $\mathbf{y} \in 3L$ we have that $\|\mathbf{y}-\mathbf{x}_i\| < 3\Upsilon_j$ while $\|\mathbf{x}_i - \mathbf{x}_0\| \leq \frac{1}{2}\tilde{\Upsilon}_j$. Then, using (2.7) and the triangle inequality, we get that

$$\|\mathbf{y}-\mathbf{x}_0\| \leq \|\mathbf{y}-\mathbf{x}_i\| + \|\mathbf{x}_i-\mathbf{x}_0\| \leq 3\Upsilon_j + \tfrac{1}{2}\tilde{\Upsilon}_j < \tilde{\Upsilon}_j.$$

Hence, $3L \subset A$ whence property (ii) follows. Further, property (iii) follows immediately from condition (2.8).

By the maximality of the collection $\mathbf{x}_1, \ldots, \mathbf{x}_t$, for any $\mathbf{x} \in R_j \cap \frac{1}{2}A$ there exists an $\mathbf{x}_i$ from this collection such that $\|\mathbf{x}-\mathbf{x}_i\| \leq 6\Upsilon_j$. Hence,

$$\Delta(R_j, \Upsilon_j) \cap \tfrac{1}{2}A \;\subset\; \bigcup_{L \in \mathcal{C}(A;j)} 7L\,.$$

Thus,

$$\begin{aligned}
\mathcal{H}^k(\Delta(R_j, \Upsilon_j) \cap \tfrac{1}{2}A) &\leq \sum_{L \in \mathcal{C}(A;j)} \mathcal{H}^k(7L) \\
&\leq \sum_{L \in \mathcal{C}(A;j)} 7^k \mathcal{H}^k(L) \\
&= 7^k \mathcal{H}^k\left(\overset{\circ}{\bigcup_{L \in \mathcal{C}(A;j)}} L \right).
\end{aligned}$$

On the other hand, by property (ii), we have that

$$\overset{\circ}{\bigcup_{L \in \mathcal{C}(A;j)}} L \;\subset\; \Delta(R_j, \Upsilon_j) \cap A\,,$$

which together with the previous inequality establishes property (iv).

Finally, property (v) is an immediate consequence of property (iv) upon noting that

$$\mathcal{H}^k\big(\Delta(R_j, \Upsilon_j) \cap \tfrac{1}{2}A\big) \;\asymp\; \mathcal{H}^k\big(\Delta(R_j, \Upsilon_j) \cap A\big) \;\asymp\; \Upsilon_j^m \tilde{\Upsilon}_j^l$$

and

$$\mathcal{H}^k\left(\bigcup_{L \in \mathcal{C}(A;j)} L \right) \;=\; \#\mathcal{C}(A;j)\mathcal{H}^k(L) \;\asymp\; \#\mathcal{C}(A;j)\, \Upsilon_j^k\,,$$

where l is the dimension of R_j, $m = k - l$ and L is any ball from $\mathcal{C}(A;j)$. □

2.4 Proof of Theorem 2.2

As with the proof of the Mass Transference Principle given in [7] and the proof of Theorem 2.1 given in [8], we begin by noting that we may assume that $r^{-k}f(r) \to \infty$ as $r \to 0$. To see this we first observe that, by Lemma 1.15, if $r^{-k}f(r) \to 0$ as $r \to 0$ we have that $\mathcal{H}^f(B) = 0$ for any ball B in $\mathbb{R}^k$. Furthermore, since $B \cap \Lambda(\Upsilon) \subset B$, the result follows trivially.

Now suppose that $r^{-k}f(r) \to \lambda$ as $r \to 0$ for some $0 < \lambda < \infty$. In this case, $\mathcal{H}^f$ is comparable to $\mathcal{H}^k$ and so it would be sufficient to show that $\mathcal{H}^k(B \cap \Lambda(\Upsilon)) = \mathcal{H}^k(B)$. Since $r^{-k}f(r) \to \lambda$ as $r \to 0$ we have that the ratio $\frac{f(r)}{r^k}$ is bounded between positive constants for sufficiently small r. In turn, this implies that, in this case, the ratio of the values $g(\Upsilon_j)^{\frac{1}{m}}$ and Υ_j is uniformly bounded between positive constants. It then follows from [9, Lemma 4] that

$$\mathcal{H}^k\left(B \cap \Lambda\left(g(\Upsilon)^{\frac{1}{m}}\right)\right) = \mathcal{H}^k(B \cap \Lambda(\Upsilon)).$$

This together with (2.1) then implies the required result in this case.

Thus, for the rest of the proof we may assume without loss of generality that $r^{-k}f(r) \to \infty$ as $r \to 0$. With this assumption it is a consequence of Lemma 1.15 that $\mathcal{H}^f(B_0) = \infty$ for any ball B_0 in Ω, which we fix from now on. Therefore, our goal for the rest of the proof is to show that

$$\mathcal{H}^f(B_0 \cap \Lambda(\Upsilon)) = \infty.$$

To this end, for each $\eta > 1$, we will construct a Cantor subset $\mathbb{K}_\eta$ of $B_0 \cap \Lambda(\Upsilon)$ and a probability measure μ supported on $\mathbb{K}_\eta$ satisfying the condition that for any arbitrary ball D of sufficiently small radius $r(D)$ we have

$$\mu(D) \ll \frac{V^f(D)}{\eta}, \tag{2.9}$$

where the implied constant does not depend on D or η. By the Mass Distribution Principle (Lemma 2.3) and the fact that $\mathbb{K}_\eta \subset B_0 \cap \Lambda(\Upsilon)$, we would then have that $\mathcal{H}^f(B_0 \cap \Lambda(\Upsilon)) \geq \mathcal{H}^f(\mathbb{K}_\eta) \gg \eta$ and the proof is finished by taking η to be arbitrarily large.

2.4.1 *The desired properties of* $\mathbb{K}_\eta$

We will construct the Cantor set $\mathbb{K}_\eta := \bigcap_{n=1}^{\infty} \mathbb{K}(n)$ so that each level $\mathbb{K}(n)$ is a finite union of disjoint closed balls and the levels are nested, that is $\mathbb{K}(n) \supset \mathbb{K}(n+1)$ for $n \geq 1$. We will denote the collection of balls constituting level n by $K(n)$. As with the Cantor set in [7], the construction of $\mathbb{K}_\eta$ is inductive and each level $\mathbb{K}(n)$ will consist of local levels and sub-levels. So, suppose that the $(n-1)$th level $\mathbb{K}(n-1)$ has been constructed. Then, for every $B \in K(n-1)$ we construct the (n, B)-*local level*, $K(n, B)$, which will consist of balls contained in B. The collection of balls $K(n)$ will take the form

$$K(n) := \bigcup_{B \in K(n-1)} K(n, B).$$

Looking even more closely at the construction, each (n, B)-local level will consist of *local sub-levels* and will be of the form

$$K(n, B) := \bigcup_{i=1}^{l_B} K(n, B, i). \tag{2.10}$$

Here, $K(n, B, i)$ denotes the ith local sub-level and l_B is the number of local sub-levels. For $n \geq 2$ each local sub-level will be defined as the union

$$K(n, B, i) := \bigcup_{B' \in \mathcal{G}(n,B,i)} \ \bigcup_{(A;j) \in K_{G',B'}} \mathcal{C}(A;j)\,, \tag{2.11}$$

where B' will lie in a suitably chosen collection of balls $\mathcal{G}(n, B, i)$ within B, $K_{G',B'}$ will arise from Lemma 2.5 and $\mathcal{C}(A;j)$ will arise from Lemma 2.7. It will be apparent from the construction that the parameter G' becomes arbitrarily large as we construct levels. The set of all pairs $(A;j)$ that contribute to (2.11) will be denoted by $\tilde{K}(n, B, i)$. Thus,

$$\widetilde{K}(n, B, i) := \bigcup_{B' \in \mathcal{G}(n,B,i)} K_{G',B'} \qquad \text{and} \qquad K(n, B, i) = \bigcup_{(A;j) \in \widetilde{K}(n,B,i)} \mathcal{C}(A;j).$$

If additionally we start with $\mathbb{K}(1) := B_0$ then, in view of the definition of the sets $\mathcal{C}(A;j)$, the inclusion $\mathbb{K}_\eta \subset B_0 \cap \Lambda(\Upsilon)$ is straightforward. Hence the only real part of the proof will be to show the validity of (2.9) for some suitable measure supported on $\mathbb{K}_\eta$. This will require several additional properties which are now stated.

The properties of levels and sub-levels of $\mathbb{K}_\eta$

(P0) $K(1)$ consists of one ball, namely B_0.

(P1) For any $n \geq 2$ and any $B \in K(n-1)$ the balls

$$\{3L \ : \ L \in K(n,B)\}$$

are disjoint and contained in B.

(P2) For any $n \geq 2$, any $B \in K(n-1)$ and any $i \in \{1, \ldots, l_B\}$ the local sub-level $K(n,B,i)$ is a finite union of some collections $\mathcal{C}(A;j)$ of balls satisfying properties (i)–(v) of Lemma 2.7, where the balls $3A$ are disjoint and contained in B.

(P3) For any $n \geq 2$, $B \in K(n-1)$ and $i \in \{1, \ldots, l_B\}$ we have

$$\sum_{(A;j)\in\widetilde{K}(n,B,i)} V^k(A) \ \geq \ c_3 \, V^k(B)$$

where $c_3 := \frac{1}{2^{k+3}\times 5^k \times 15^k}\left(\frac{c_1}{c_2}\right)^2$ with c_1 and c_2 as defined in (2.3).

(P4) For any $n \geq 2$, $B \in K(n-1)$, any $i \in \{1, \ldots, l_B - 1\}$ and any $L \in K(n,B,i)$ and $M \in K(n,B,i+1)$ we have

$$f(r(M)) \leq \frac{1}{2}\, f(r(L)) \quad \text{and} \quad g(r(M)) \leq \frac{1}{2}\, g(r(L)).$$

(P5) The number of local sub-levels is defined by

$$l_B := \begin{cases} \left[\dfrac{c_2\,\eta}{c_3\,\mathcal{H}^k(B)}\right] + 1 \quad , & \text{if } B = B_0 := \mathbb{K}(1), \\[2ex] \left[\dfrac{V^f(B)}{c_3\,V^k(B)}\right] + 1 \quad , & \text{if } B \in K(n) \text{ with } n \geq 2, \end{cases}$$

and satisfies $l_B \geq 2$ for $B \in K(n)$ with $n \geq 2$. Here, for $x \in \mathbb{R}$, $[x]$ denotes the greatest integer less than or equal to x.

Properties **(P1)** and **(P2)** are imposed to make sure that the balls in the Cantor construction are sufficiently well separated. On the other hand, properties **(P3)** and

(P5) make sure that there are "enough" balls in each level of the construction of the Cantor set. Property **(P4)** essentially ensures that all balls involved in the construction of a level of the Cantor set are sufficiently small compared with balls involved in the construction of the previous level. All of the properties **(P1)**–**(P5)** will play a crucial role in the measure estimates we obtain in Section 2.4.4 and Section 2.4.5.

2.4.2 *The existence of* $\mathbb{K}_\eta$

In this section we show that it is possible to construct a Cantor set with the properties outlined in Section 2.4.1. In what follows we will use the following notation:

$$K_l(n,B) \ := \ \bigcup_{i=1}^{l} K(n,B,i) \qquad \text{and} \qquad \widetilde{K}_l(n,B) \ := \ \bigcup_{i=1}^{l} \widetilde{K}(n,B,i)\ .$$

Level 1. The first level is defined by taking the arbitrary ball B_0. Thus, $\mathbb{K}(1) := B_0$ and Property **(P0)** is trivially satisfied. We proceed by induction. Assume that the first $(n-1)$ levels $\mathbb{K}(1)$, $\mathbb{K}(2)$, ... , $\mathbb{K}(n-1)$ have been constructed. We now construct the nth level $\mathbb{K}(n)$.

Level n. To construct the nth level we will define local levels $K(n,B)$ for each $B \in K(n-1)$. Therefore, from now on we fix some ball $B \in K(n-1)$ and a sufficiently small constant $\varepsilon := \varepsilon(B) > 0$ which will be determined later. Recall that each local level $K(n,B)$ will consist of local sub-levels $K(n,B,i)$ where $1 \le i \le l_B$ and l_B is given by Property **(P5)**. Let $G \in \mathbb{N}$ be sufficiently large so that Lemmas 2.5 and 2.7 are applicable. Furthermore, suppose that G is large enough so that

$$3\Upsilon_j < g(\Upsilon_j)^{\frac{1}{m}} \qquad \text{whenever} \qquad j \ge G, \tag{2.12}$$

$$\frac{\Upsilon_j^k}{f(\Upsilon_j)} < \varepsilon\ \frac{r(B)^k}{f(r(B))} \qquad \text{whenever} \qquad j \ge G, \ \text{and} \tag{2.13}$$

$$\left[\frac{f(\Upsilon_j)}{c_3\,\Upsilon_j^k}\right] \ \ge \ 1 \qquad \text{whenever} \qquad j \ge G, \tag{2.14}$$

where c_3 is the constant appearing in Property **(P3)** above. Note that the existence of G satisfying (2.12)–(2.14) follows from the assumption that $r^{-k}f(r) \to \infty$ as $r \to 0$ and the condition that $\Upsilon_j \to 0$ as $j \to \infty$.

Sub-level 1. With B and G as above, let $K_{G,B}$ denote the collection of balls arising from Lemma 2.5. Define the first sub-level of $K(n,B)$ to be

$$K(n,B,1) := \bigcup_{(A;j)\in K_{G,B}} \mathcal{C}(A;j)\,,$$

thus

$$\widetilde{K}(n,B,1) = K_{G,B} \qquad \text{and} \qquad \mathcal{G}(n,B,1) = \{B\}\,.$$

By the properties of $\mathcal{C}(A;j)$ (Lemma 2.7), it follows that **(P1)** is satisfied within this sub-level. From the properties of $K_{G,B}$ (Lemma 2.5) and Lemma 2.7 it follows that **(P2)** and **(P3)** are satisfied for $i=1$.

Higher sub-levels. To construct higher sub-levels we argue by induction. For $l<l_B$, assume that the sub-levels $K(n,B,1),\ldots,K(n,B,l)$ satisfying properties **(P1)**–**(P4)** with l_B replaced by l have already been defined. We now construct the next sub-level $K(n,B,l+1)$.

As every sub-level of the construction has to be well separated from the previous ones, we first verify that there is enough "space" left over in B once we have removed the sub-levels $K(n,B,1),\ldots,K(n,B,l)$ from B. More precisely, let

$$A^{(l)} \;:=\; \tfrac{1}{2}B \setminus \bigcup_{L\in K_l(n,B)} 4L\;.$$

We will show that

$$\mathcal{H}^k\big(A^{(l)}\big) \geq \tfrac{1}{2}\,\mathcal{H}^k(\tfrac{1}{2}B)\;. \tag{2.15}$$

First, observe that

$$\begin{aligned}
\mathcal{H}^k\left(\bigcup_{L\in K_l(n,B)} 4L\right) &\leq \sum_{L\in K_l(n,B)} \mathcal{H}^k(4L)\\
&\overset{(2.3)}{\leq} 4^k c_2 \sum_{L\in K_l(n,B)} V^k(L)\\
&= 4^k c_2 \sum_{i=1}^{l}\sum_{L\in K(n,B,i)} V^k(L)\\
&= 4^k c_2 \sum_{i=1}^{l}\sum_{(A;j)\in\widetilde{K}(n,B,i)} \#\mathcal{C}(A;j)\times\Upsilon_j^k
\end{aligned}$$

$$
\begin{aligned}
&\overset{(2.6)}{\leq} 4^k c_2 d_2 \sum_{i=1}^{l} \sum_{(A;j)\in\widetilde{K}(n,B,i)} \left(\frac{g(\Upsilon_j)^{\frac{1}{m}}}{\Upsilon_j}\right)^{l} \Upsilon_j^k \\
&= 4^k c_2 d_2 \sum_{i=1}^{l} \sum_{(A;j)\in\widetilde{K}(n,B,i)} g(\Upsilon_j)^{\frac{l}{m}} \Upsilon_j^m \\
&= 4^k c_2 d_2 \sum_{i=1}^{l} \sum_{(A;j)\in\widetilde{K}(n,B,i)} g(\Upsilon_j)^{\frac{k}{m}} \frac{\Upsilon_j^m}{g(\Upsilon_j)} \\
&= 4^k c_2 d_2 \sum_{i=1}^{l} \sum_{(A;j)\in\widetilde{K}(n,B,i)} g(\Upsilon_j)^{\frac{k}{m}} \frac{\Upsilon_j^k}{f(\Upsilon_j)}.
\end{aligned}
$$

Hence, by (2.13), we get that

$$
\begin{aligned}
\mathcal{H}^k\left(\bigcup_{L\in K_l(n,B)} 4L\right) &\leq 4^k c_2 d_2 \varepsilon \frac{r(B)^k}{f(r(B))} \sum_{i=1}^{l} \sum_{(A;j)\in\widetilde{K}(n,B,i)} g(\Upsilon_j)^{\frac{k}{m}} \\
&= 4^k c_2 d_2 \varepsilon \frac{r(B)^k}{f(r(B))} \sum_{i=1}^{l} \sum_{(A;j)\in\widetilde{K}(n,B,i)} V^k(A) \\
&\overset{(2.3)}{\leq} 4^k \frac{c_2}{c_1} d_2 \varepsilon \frac{r(B)^k}{f(r(B))} \sum_{i=1}^{l} \sum_{(A;j)\in\widetilde{K}(n,B,i)} \mathcal{H}^k(A) \\
&\overset{\textbf{(P2)}}{\leq} 4^k \frac{c_2}{c_1} d_2 \varepsilon \frac{r(B)^k}{f(r(B))} l \mathcal{H}^k(B) \\
&\leq 4^k \frac{c_2}{c_1} d_2 \varepsilon \frac{r(B)^k}{f(r(B))} (l_B - 1) \mathcal{H}^k(B). \qquad (2.16)
\end{aligned}
$$

If $B = B_0$, set $\varepsilon = \varepsilon(B_0) := \dfrac{1}{2d_2}\left(\dfrac{c_1}{c_2}\right)^2 \dfrac{c_3}{2^k 4^k} \dfrac{f(r(B_0))}{\eta}$.
Otherwise, if $B \neq B_0$, set

$$
\varepsilon = \varepsilon(B) := \varepsilon(B_0) \times \frac{\eta}{f(r(B_0))} = \frac{1}{2d_2}\left(\frac{c_1}{c_2}\right)^2 \frac{c_3}{2^k 4^k}.
$$

Then, it follows from (2.16) combined with **(P5)** that

$$
\mathcal{H}^k\left(\bigcup_{L\in K_l(n,B)} 4L\right) \leq \tfrac{1}{2}\mathcal{H}^k\left(\tfrac{1}{2}B\right),
$$

thus verifying (2.15).

By construction, $K_l(n,B)$ is a finite collection of balls. Therefore, the quantity

$$d_{\min} := \min\{r(L) : L \in K_l(n,B)\}$$

is well-defined and positive. Let $\mathcal{A}(n,B,l)$ be the collection of all the balls of diameter $d_{\min}$ centred at a point in $A^{(l)}$. By the $5r$-covering lemma (Lemma 2.4), there exists a disjoint subcollection $\mathcal{G}(n,B,l+1)$ of $\mathcal{A}(n,B,l)$ such that

$$A^{(l)} \subset \bigcup_{B' \in \mathcal{A}(n,B,l)} B' \subset \bigcup_{B' \in \mathcal{G}(n,B,l+1)} 5B'.$$

The collection $\mathcal{G}(n,B,l+1)$ is clearly contained within B and, since the balls in this collection are disjoint and of the same size, it is finite. Moreover, by construction

$$B' \cap \bigcup_{L \in K_l(n,B)} 3L = \emptyset \qquad \text{for any } B' \in \mathcal{G}(n,B,l+1)\ ; \tag{2.17}$$

i.e. the balls in $\mathcal{G}(n,B,l+1)$ do not intersect any of the $3L$ balls from the previous sub-levels. It follows that

$$\mathcal{H}^k\left(\bigcup_{B' \in \mathcal{G}(n,B,l+1)} 5B'\right) \geq \mathcal{H}^k(A^{(l)}) \overset{(2.15)}{\geq} \tfrac{1}{2}\,\mathcal{H}^k(\tfrac{1}{2}B)\ .$$

On the other hand, since $\mathcal{G}(n,B,l+1)$ is a disjoint collection of balls we have that

$$\begin{aligned}
\mathcal{H}^k\left(\bigcup_{B' \in \mathcal{G}(n,B,l+1)} 5B'\right) &\leq \sum_{B' \in \mathcal{G}(n,B,l+1)} \mathcal{H}^k(5B') \\
&\overset{(2.3)}{\leq} 5^k\frac{c_2}{c_1} \sum_{B' \in \mathcal{G}(n,B,l+1)} \mathcal{H}^k(B') \\
&= 5^k\frac{c_2}{c_1}\mathcal{H}^k\left(\mathop{\mathring{\bigcup}}_{B' \in \mathcal{G}(n,B,l+1)} B'\right).
\end{aligned}$$

Hence,

$$\mathcal{H}^k\left(\mathop{\mathring{\bigcup}}_{B' \in \mathcal{G}(n,B,l+1)} B'\right) \geq \frac{c_1}{2c_2 5^k}\ \mathcal{H}^k(\tfrac{1}{2}B)\,. \tag{2.18}$$

Now we are ready to construct the $(l+1)$th sub-level $K(n,B,l+1)$. Let $G' \geq G+1$ be sufficiently large so that Lemmas 2.5 and 2.7 are applicable to every ball $B' \in \mathcal{G}(n,B,l+1)$ with G' in place of G. Furthermore, ensure that G' is sufficiently large so that for every $i \geq G'$,

$$f(\Upsilon_i) \leq \tfrac{1}{2} \min_{L \in K_l(n,B)} f(r(L)) \qquad \text{and} \qquad g(\Upsilon_i) \leq \tfrac{1}{2} \min_{L \in K_l(n,B)} g(r(L)). \tag{2.19}$$

Imposing the above assumptions on G' is possible since there are only finitely many balls in $K_l(n,B)$, $\Upsilon_j \to 0$ as $j \to \infty$, and f and g are dimension functions.

Now, to each ball $B' \in \mathcal{G}(n,B,l+1)$ we apply Lemma 2.5 to obtain a collection of balls $K_{G',B'}$ and define

$$K(n,B,l+1) := \bigcup_{B' \in \mathcal{G}(n,B,l+1)} \ \bigcup_{(A;j) \in K_{G',B'}} \mathcal{C}(A;j).$$

Consequently,

$$\widetilde{K}(n,B,l+1) = \bigcup_{B' \in \mathcal{G}(n,B,l+1)} K_{G',B'}\,.$$

Since $G' \geq G$, properties (2.12)–(2.14) remain valid. We now verify properties **(P1)**–**(P5)** for this sub-level.

Regarding **(P1)**, we first observe that it is satisfied for balls in

$$\bigcup_{(A;j) \in K_{G',B'}} \ \bigcup_{L \in \mathcal{C}(A;j)} L$$

by the properties of $\mathcal{C}(A;j)$ and the fact that the balls in $K_{G',B'}$ are disjoint. Next, since any balls in $K_{G',B'}$ are contained in B' and the balls $B' \in \mathcal{G}(n,B,l+1)$ are disjoint, it follows that **(P1)** is satisfied for balls L in $K(n,B,l+1)$. Finally, combining this with (2.17), we see that **(P1)** is satisfied for balls L in $K_{l+1}(n,B)$. That **(P2)** is satisfied for this sub-level is a consequence of Lemma 2.5 (i) and (ii) and the fact that the balls $B' \in \mathcal{G}(n,B,l+1)$ are disjoint.

To establish **(P3)** for $i = l+1$ note that

$$\begin{aligned}
\sum_{(A;j) \in \widetilde{K}(n,B,l+1)} V^k(A) &= \sum_{B' \in \mathcal{G}(n,B,l+1)} \ \sum_{(A;j) \in K_{G',B'}} V^k(A) \\
&\overset{(2.3)}{\geq} \frac{1}{c_2} \sum_{B' \in \mathcal{G}(n,B,l+1)} \ \sum_{(A;j) \in K_{G',B'}} \mathcal{H}^k(A).
\end{aligned}$$

Then, by Lemma 2.5 and the disjointness of the balls in $\mathcal{G}(n, B, l+1)$, we have that

$$\begin{aligned}
\sum_{(A;j)\in\widetilde{K}(n,B,l+1)} V^k(A) &\geq \frac{1}{c_2}\sum_{B'\in\mathcal{G}(n,B,l+1)} \frac{1}{4\times 15^k}\mathcal{H}^k(B') \\
&= \frac{1}{c_2\times 4\times 15^k}\mathcal{H}^k\left(\bigcup_{B'\in\mathcal{G}(n,B,l+1)} B'\right) \\
&\overset{(2.18)}{\geq} \frac{1}{c_2\times 4\times 15^k}\frac{c_1}{2\times c_2\times 5^k}\mathcal{H}^k\left(\tfrac{1}{2}B\right) \\
&\overset{(2.3)}{\geq} \frac{1}{2^{k+3}\times 5^k\times 15^k}\left(\frac{c_1}{c_2}\right)^2 V^k(B) \\
&= c_3 V^k(B).
\end{aligned}$$

Finally, **(P4)** is trivially satisfied as a consequence of the imposed condition (2.19) and **(P5)**, that $l_L \geq 2$ for any ball L in $K(n, B, l+1)$, follows from (2.14).

Hence, properties **(P1)**–**(P5)** are satisfied up to the local sub-level $K(n, B, l+1)$ thus establishing the existence of the local level $K(n, B) = K_{l_B}(n, B)$ for each $B \in K(n-1)$. In turn, this establishes the existence of the nth level $K(n)$ (and also $\mathbb{K}(n)$).

2.4.3 *The measure μ on $\mathbb{K}_\eta$*

In this section, we define a probability measure μ supported on $\mathbb{K}_\eta$. We will eventually show that the measure satisfies (2.9). For any ball $L \in K(n)$, we attach a weight $\mu(L)$ defined recursively as follows.

For $n = 1$, we have that $L = B_0 := \mathbb{K}(1)$ and we set $\mu(L) := 1$. For subsequent levels the measure is defined inductively.

Let $n \geq 2$ and suppose that $\mu(B)$ is defined for every $B \in K(n-1)$. In particular, we have that

$$\sum_{B\in K(n-1)} \mu(B) = 1\,.$$

Let L be a ball in $K(n)$. By construction, there is a unique ball $B \in K(n-1)$ such that $L \subset B$. Recall, by (2.10) and (2.11), that

$$K(n,B) := \bigcup_{(A;j)\in \widetilde{K}_{l_B}(n,B)} \mathcal{C}(A;j)$$

and so L is an element of one of the collections $\mathcal{C}(A';j')$ appearing in the right-hand side of the above. We therefore define

$$\mu(L) := \frac{1}{\#\mathcal{C}(A';j')} \times \frac{g(\Upsilon_{j'})^{\frac{k}{m}}}{\sum\limits_{(A;j)\in \widetilde{K}_{l_B}(n,B)} g(\Upsilon_j)^{\frac{k}{m}}} \times \mu(B)\,.$$

Thus μ is inductively defined on any ball appearing in the construction of $\mathbb{K}_\eta$. Furthermore, μ can be uniquely extended in a standard way to all Borel subsets F of $\mathbb{R}^k$ to give a probability measure μ supported on $\mathbb{K}_\eta$. Indeed, for any Borel subset F of $\mathbb{R}^k$,

$$\mu(F) := \mu(F \cap \mathbb{K}_\eta) \;=\; \inf \sum_{L\in\mathcal{C}(F)} \mu(L)\;,$$

where the infimum is taken over all covers $\mathcal{C}(F)$ of $F\cap\mathbb{K}_\eta$ by balls $L \in \bigcup\limits_{n\in\mathbb{N}} K(n)$. See [26, Proposition 1.7] for further details.

We end this section by observing that

$$\begin{aligned}\mu(L) &\leq \frac{1}{d_1\left(\frac{g(\Upsilon_{j'})^{\frac{1}{m}}}{\Upsilon_{j'}}\right)^l} \times \frac{g(\Upsilon_{j'})^{\frac{k}{m}}}{\sum\limits_{(A;j)\in \widetilde{K}_{l_B}(n,B)} g(\Upsilon_j)^{\frac{k}{m}}} \times \mu(B) \\ &= \frac{f(\Upsilon_{j'})}{d_1 \sum\limits_{(A;j)\in \widetilde{K}_{l_B}(n,B)} g(\Upsilon_j)^{\frac{k}{m}}} \times \mu(B). \qquad (2.20)\end{aligned}$$

This is a consequence of (2.6) and the relationship between f and g. In fact, the above inequality can be reversed if d_1 is replaced by d_2.

2.4.4 *The measure of a ball in the Cantor set construction*

The goal of this section is to prove that

$$\mu(L) \ll \frac{V^f(L)}{\eta} \tag{2.21}$$

for any ball L in $K(n)$ with $n \geq 2$. We will begin with the level $n = 2$. Fix any ball $L \in K(2) = K(2, B_0)$. Further let $(A'; j') \in \widetilde{K}_{l_{B_0}}(2, B_0)$ be such that $L \in \mathcal{C}(A'; j')$. Then, by (2.20), the definition of μ and the fact that $\mu(B_0) = 1$, we have that

$$\mu(L) \leq \frac{f(\Upsilon_{j'})}{d_1 \sum\limits_{(A;j)\in\widetilde{K}_{l_{B_0}}(2,B_0)} g(\Upsilon_j)^{\frac{k}{m}}} \quad . \tag{2.22}$$

Next, by properties **(P3)** and **(P5)** of the Cantor set construction, we get that

$$\begin{aligned}
\sum_{(A;j)\in\widetilde{K}_{l_{B_0}}(2,B_0)} g(\Upsilon_j)^{\frac{k}{m}} &= \sum_{(A;j)\in\widetilde{K}_{l_{B_0}}(2,B_0)} V^k(A) \\
&= \sum_{i=1}^{l_{B_0}} \sum_{(A;j)\in\widetilde{K}(2,B_0,i)} V^k(A) \\
&\overset{\textbf{(P3)}}{\geq} \sum_{i=1}^{l_{B_0}} c_3 V^k(B_0) \\
&= l_{B_0} c_3 V^k(B_0) \\
&\overset{(2.3)}{\geq} l_{B_0} \frac{c_3}{c_2} \mathcal{H}^k(B_0) \\
&\overset{\textbf{(P5)}}{\geq} \frac{c_2 \eta}{c_3 \mathcal{H}^k(B_0)} \frac{c_3}{c_2} \mathcal{H}^k(B_0) \;=\; \eta.
\end{aligned} \tag{2.23}$$

Combining (2.22) and (2.23) gives (2.21) as required since $f(\Upsilon_{j'}) = f(r(L)) = V^f(L)$.

Now let $n > 2$ and assume that (2.21) holds for balls in $K(n-1)$. Consider an arbitrary ball L in $K(n)$. Then there exists a unique ball $B \in K(n-1)$ such that $L \in K(n, B)$. Further let $(A'; j') \in \widetilde{K}_{l_B}(n, B)$ be such that $L \in \mathcal{C}(A'; j')$. Then it follows from (2.20) and our induction hypothesis that

$$\mu(L) \ll \frac{f(\Upsilon_{j'})}{d_1 \sum\limits_{(A;j)\in \widetilde{K}_{l_B}(n,B)} g(\Upsilon_j)^{\frac{k}{m}}} \times \frac{V^f(B)}{\eta}. \tag{2.24}$$

Now, we have that

$$\begin{aligned}
\sum_{(A;j)\in \widetilde{K}_{l_B}(n,B)} g(\Upsilon_j)^{\frac{k}{m}} &= \sum_{i=1}^{l_B} \sum_{(A;j)\in \widetilde{K}(n,B,i)} V^k(A) \\
&\overset{\textbf{(P3)}}{\geq} \sum_{i=1}^{l_B} c_3 V^k(B) \\
&= l_B c_3 V^k(B) \\
&\overset{\textbf{(P5)}}{\geq} \frac{V^f(B)}{c_3 V^k(B)} c_3 V^k(B) \\
&= V^f(B).
\end{aligned} \tag{2.25}$$

Since $V^f(L) = f(\Upsilon_{j'})$, combining (2.24) and (2.25) gives (2.21) and thus completes the proof of this section.

2.4.5 *The measure of an arbitrary ball*

Set $r_0 := \min\{r(B) : B \in K(2)\}$. Take an arbitrary ball D such that $r(D) < r_0$. We wish to establish (2.9) for D, i.e. we wish to show that

$$\mu(D) \ll \frac{V^f(D)}{\eta},$$

where the implied constant is independent of D and η. In accomplishing this goal the following lemma from [7] will be useful.

Lemma 2.8. *Let $A := B(x_A, r_A)$ and $M := B(x_M, r_M)$ be arbitrary balls such that $A \cap M \neq \emptyset$ and $A \setminus (cM) \neq \emptyset$ for some $c \geq 3$. Then $r_M \leq r_A$ and $cM \subset 5A$.*

A good part of the subsequent argument will follow the same reasoning as given in [7, Section 5.5]. However, there will also be obvious alterations to the proofs that arise from the different construction of a Cantor set used here. Recall that the measure μ is supported on $\mathbb{K}_\eta$. Without loss of generality, we will make the following two assumptions:

- $D \cap \mathbb{K}_\eta \neq \emptyset$;
- for every n large enough D intersects at least two balls in $K(n)$.

If the first of these were false then we would have $\mu(D) = 0$ as μ is supported on $\mathbb{K}_\eta$ and so (2.9) would trivially follow. If the second assumption were false then D would have to intersect exactly one ball, say L_{n_i}, from levels K_{n_i} with arbitrarily large n_i. Then, by (2.21), we would have $\mu(D) \leq \mu(L_{n_i}) \to 0$ as $i \to \infty$ and so, again, (2.9) would be trivially true.

By the above two assumptions, we have that there exists a maximum integer n such that

$$D \text{ intersects at least 2 balls from } K(n) \tag{2.26}$$

$$\text{and } D \text{ intersects only one ball } B \text{ from } K(n-1).$$

By our choice of r_0, we have that $n > 2$. If B is the only ball from $K(n-1)$ which has non-empty intersection with D, we may also assume that $r(D) < r(B)$. To see this, suppose to the contrary that $r(B) \leq r(D)$. Then, since $D \cap \mathbb{K}_\eta \subset B$ and f is increasing, upon recalling (2.21) we would have

$$\mu(D) \leq \mu(B) \ll \frac{V^f(B)}{\eta} = \frac{f(r(B))}{\eta} \leq \frac{f(r(D))}{\eta} = \frac{V^f(D)}{\eta},$$

and so we would be done.

Now, since $K(n, B)$ is a cover for $D \cap \mathbb{K}_\eta$, we have

$$\begin{aligned}
\mu(D) &\leq \sum_{i=1}^{l_B} \sum_{\substack{L \in K(n,B,i) \\ L \cap D \neq \emptyset}} \mu(L) \\
&= \sum_{i=1}^{l_B} \sum_{(A;j) \in \widetilde{K}(n,B,i)} \sum_{\substack{L \in \mathcal{C}(A;j) \\ L \cap D \neq \emptyset}} \mu(L) \\
&\overset{(2.21)}{\ll} \sum_{i=1}^{l_B} \sum_{(A;j) \in \widetilde{K}(n,B,i)} \sum_{\substack{L \in \mathcal{C}(A;j) \\ L \cap D \neq \emptyset}} \frac{V^f(L)}{\eta}.
\end{aligned} \tag{2.27}$$

To estimate the right-hand side of (2.27) we consider the following types of sub-levels:

Case 1 : Sub-levels $K(n, B, i)$ for which

$$\#\{L \in K(n, B, i) : L \cap D \neq \emptyset\} = 1.$$

Case 2 : Sub-levels $K(n, B, i)$ for which

$$\#\{L \in K(n, B, i) : L \cap D \neq \emptyset\} \geq 2 \quad \text{and}$$

$$\#\{(A; j) \in \widetilde{K}(n, B, i) \text{ with } D \cap L \neq \emptyset \text{ for some } L \in \mathcal{C}(A; j)\} \geq 2.$$

Case 3 : Sub-levels $K(n, B, i)$ for which

$$\#\{L \in K(n, B, i) : L \cap D \neq \emptyset\} \geq 2 \quad \text{and}$$

$$\#\{(A; j) \in \widetilde{K}(n, B, i) \text{ with } D \cap L \neq \emptyset \text{ for some } L \in \mathcal{C}(A; j)\} = 1.$$

Strictly speaking we also need to consider the sub-levels $K(n, B, i)$ for which

$$\#\{L \in K(n, B, i) : L \cap D \neq \emptyset\} = 0.$$

However, these sub-levels do not contribute anything to the sum on the right-hand side of (2.27).

Dealing with Case 1. Let $K(n, B, i^*)$ denote the first sub-level within Case 1 which has non-empty intersection with D. Then there exists a unique ball L^* in $K(n, B, i^*)$ such that $L^* \cap D \neq \emptyset$. By (2.26) there is another ball $M \in K(n, B)$ such that $M \cap D \neq \emptyset$. By Property **(P1)**, $3L^*$ and $3M$ are disjoint. It follows that $D \setminus 3L^* \neq \emptyset$. Therefore, by Lemma 2.8, we have that $r(L^*) \leq r(D)$ and so, since f is increasing,

$$V^f(L^*) \leq V^f(D). \tag{2.28}$$

By Property **(P4)** we have, for any $i \in \{i^* + 1, \ldots, l_B\}$ and any $L \in K(n, B, i)$, that

$$V^f(L) = f(r(L)) \leq 2^{-(i-i^*)}\, f(r(L^*)) = 2^{-(i-i^*)}\, V^f(L^*).$$

Using these inequalities and (2.28) we see that the contribution to the right-hand side of (2.27) from Case 1 is:

$$\sum_{i\in\text{Case 1}}\ \sum_{\substack{L\in K(n,B,i)\\ L\cap D\neq\emptyset}} \frac{V^f(L)}{\eta} \le \sum_{i\ge i^*} 2^{-(i-i^*)}\frac{V^f(L^*)}{\eta} \le 2\frac{V^f(L^*)}{\eta} \le 2\frac{V^f(D)}{\eta}. \tag{2.29}$$

Dealing with Case 2. Let $K(n,B,i)$ be any sub-level subject to the conditions of Case 2. Then there exist distinct balls $(A;j)$ and $(A';j')$ in $\widetilde{K}(n,B,i)$ and balls $L\in\mathcal{C}(A;j)$ and $L'\in\mathcal{C}(A';j')$ such that $L\cap D\neq\emptyset$ and $L'\cap D\neq\emptyset$. Since $L\cap D\neq\emptyset$ and $L\subset A$, we have that $A\cap D\neq\emptyset$. Similarly, $A'\cap D\neq\emptyset$. Furthermore, by Property **(P2)**, the balls $3A$ and $3A'$ are disjoint and contained in B. Hence, $D\setminus 3A\neq\emptyset$. Therefore, by Lemma 2.8, $r(A)\le r(D)$ and $A\subset 3A\subset 5D$. Similarly, $A'\subset 3A'\subset 5D$. Hence, on using (2.6), we get that the contribution to the right-hand side of (2.27) from Case 2 is estimated as follows

$$\begin{aligned}
\sum_{i\in\text{Case 2}}\ \sum_{(A;j)\in\widetilde{K}(n,B,i)}\ \sum_{\substack{L\in\mathcal{C}(A;j)\\ L\cap D\neq\emptyset}} \frac{V^f(L)}{\eta} &\le \sum_{i\in\text{Case 2}}\ \sum_{\substack{(A;j)\in\widetilde{K}(n,B,i)\\ A\subset 5D}} \#\mathcal{C}(A;j)\frac{f(\Upsilon_j)}{\eta}\\
&\overset{(2.6)}{\ll} \sum_{i\in\text{Case 2}}\ \sum_{\substack{(A;j)\in\widetilde{K}(n,B,i)\\ A\subset 5D}} \left(\frac{g(\Upsilon_j)^{\frac{1}{m}}}{\Upsilon_j}\right)^l \frac{f(\Upsilon_j)}{\eta}\\
&= \sum_{i\in\text{Case 2}}\ \sum_{\substack{(A;j)\in\widetilde{K}(n,B,i)\\ A\subset 5D}} \frac{g(\Upsilon_j)^{\frac{l}{m}}\Upsilon_j^{-l}\Upsilon_j^{l}g(\Upsilon_j)}{\eta}\\
&= \sum_{i\in\text{Case 2}}\ \sum_{\substack{(A;j)\in\widetilde{K}(n,B,i)\\ A\subset 5D}} \frac{g(\Upsilon_j)^{\frac{l}{m}+1}}{\eta}\\
&= \sum_{i\in\text{Case 2}}\ \sum_{\substack{(A;j)\in\widetilde{K}(n,B,i)\\ A\subset 5D}} \frac{g(\Upsilon_j)^{\frac{k}{m}}}{\eta}\\
&= \sum_{i\in\text{Case 2}}\ \sum_{\substack{(A;j)\in\widetilde{K}(n,B,i)\\ A\subset 5D}} \frac{V^k(A)}{\eta}.
\end{aligned}$$

Combining this with properties **(P2)** and **(P5)** we get

$$\begin{aligned}
\sum_{i \in \text{Case } 2} \sum_{(A;j) \in \widetilde{K}(n,B,i)} \sum_{\substack{L \in \mathcal{C}(A;j) \\ L \cap D \neq \emptyset}} \frac{V^f(L)}{\eta} &\overset{(2.3)}{\ll} \frac{1}{c_1 \eta} \sum_{i \in \text{Case } 2} \sum_{\substack{(A;j) \in \widetilde{K}(n,B,i) \\ A \subset 5D}} \mathcal{H}^k(A) \\
&\overset{\textbf{(P2)}}{=} \frac{1}{c_1 \eta} \sum_{i \in \text{Case } 2} \mathcal{H}^k \left(\bigcup_{\substack{(A;j) \in \widetilde{K}(n,B,i) \\ A \subset 5D}} A \right) \\
&\leq \frac{1}{c_1 \eta} \sum_{i \in \text{Case } 2} \mathcal{H}^k(5D) \\
&\leq \frac{1}{c_1 \eta} 5^k l_B \mathcal{H}^k(D) \\
&\overset{(2.3)}{\leq} \frac{c_2}{c_1 \eta} 5^k l_B V^k(D) \\
&\overset{\textbf{(P5)}}{\leq} \frac{c_2}{c_1 \eta} 5^k \left(\frac{2V^f(B)}{c_3 V^k(B)} \right) V^k(D).
\end{aligned}$$

Recalling our assumption that $r(D) < r(B)$ and the fact that $r^{-k}f(r)$ is decreasing, we obtain that

$$\begin{aligned}
\sum_{i \in \text{Case } 2} \sum_{(A;j) \in \widetilde{K}(n,B,i)} \sum_{\substack{L \in \mathcal{C}(A;j) \\ L \cap D \neq \emptyset}} \frac{V^f(L)}{\eta} &\ll \frac{c_2}{c_1 \eta} 5^k \frac{2}{c_3} \frac{V^f(D)}{V^k(D)} V^k(D) \\
&= \frac{2 c_2 5^k}{c_1 c_3} \frac{V^f(D)}{\eta} \\
&\ll \frac{V^f(D)}{\eta}.
\end{aligned} \tag{2.30}$$

Dealing with Case 3. First of all note that for each level i of Case 3 there exists a unique $(A_i; j_i) \in \widetilde{K}(n, B, i)$ such that D has a non-empty intersection with balls in $\mathcal{C}(A_i; j_i)$. Let $K(n, B, i^{**})$ denote the first sub-level within Case 3. Then there exists a ball L^{**} in $K(n, B, i^{**})$ such that $L^{**} \cap D \neq \emptyset$. By (2.26) there is another ball $M \in K(n, B)$ such that $M \cap D \neq \emptyset$. By Property **(P1)**, $3L^{**}$ and $3M$ are disjoint. It follows that $D \setminus 3L^{**} \neq \emptyset$ and therefore, by Lemma 2.8, we have that $r(L^{**}) \leq r(D)$ and so, since g is increasing, we have that

$$g(r(L^{**})) \leq g(r(D)). \tag{2.31}$$

Furthermore, by Property **(P4)**, for any $i \in \{i^{**}+1, \dots, l_B\}$ and any $L \in K(n,B,i)$ we have that

$$g(r(L)) \leq 2^{-(i-i^{**})}\, g(r(L^{**})).$$

Thus, the contribution to the sum (2.27) from Case 3 is estimated as follows

$$\begin{aligned}
\sum_{i\in\text{Case 3}} \sum_{(A;j)\in\widetilde{K}(n,B,i)} \sum_{\substack{L\in\mathcal{C}(A;j)\\ L\cap D\neq\emptyset}} \frac{V^f(L)}{\eta} &\leq \sum_{i\in\text{Case 3}} \sum_{\substack{L\in\mathcal{C}(A_i;j_i)\\ L\cap D\neq\emptyset}} \frac{V^f(L)}{\eta} \\
&= \sum_{i\in\text{Case 3}} \sum_{\substack{L\in\mathcal{C}(A_i;j_i)\\ L\cap D\neq\emptyset}} \frac{f(\Upsilon_{j_i})}{\eta} \\
&\ll \sum_{i\in\text{Case 3}} \left(\frac{r(D)}{\Upsilon_{j_i}}\right)^l \frac{f(\Upsilon_{j_i})}{\eta} \\
&= \sum_{i\in\text{Case 3}} r(D)^l \frac{g(\Upsilon_{j_i})}{\eta} \\
&\ll \frac{r(D)^l}{\eta} \sum_{i\in\text{Case 3}} \frac{g(\Upsilon_{j_{i^{**}}})}{2^{i-i^{**}}} \\
&\leq 2\frac{r(D)^l}{\eta} g(\Upsilon_{j_{i^{**}}}).
\end{aligned}$$

Noting that $\Upsilon_{j_{i^{**}}} = r(L^{**})$ and recalling (2.31) we see that

$$\sum_{i\in\text{Case 3}} \sum_{(A;j)\in\widetilde{K}(n,B,i)} \sum_{\substack{L\in\mathcal{C}(A;j)\\ L\cap D\neq\emptyset}} \frac{V^f(L)}{\eta} \ll 2\frac{r(D)^l}{\eta} g(r(D)) = 2\frac{f(r(D))}{\eta} \ll \frac{V^f(D)}{\eta}. \quad (2.32)$$

Finally, combining (2.29), (2.30) and (2.32) together with (2.27) gives $\mu(D) \ll \frac{V^f(D)}{\eta}$ and thus completes the proof of Theorem 2.2.

3 Hausdorff Measure Khintchine–Groshev Type Statements

In this chapter we highlight merely a few applications of Theorem 2.2 which we hope give an idea of the breadth of its consequences. In Section 3.1 we show that, using Theorem 2.2, with relative ease we are able to remove the last remaining monotonicity condition from a Hausdorff measure analogue of the classical Khintchine–Groshev Theorem — this is essentially the analogue of Khintchine's Theorem for approximation by linear forms. We also show how the same outcome may be achieved, albeit with a somewhat longer proof, by using Theorem 2.1 instead of Theorem 2.2. In Section 3.2 we obtain a Hausdorff measure analogue of the inhomogeneous version of the Khintchine–Groshev Theorem.

In Section 3.3 we present Hausdorff measure analogues of some recent results of Dani, Laurent and Nogueira [18]. They have established Khintchine–Groshev type statements in which the approximating points $(\mathbf{p}, \mathbf{q})$ are subject to certain primitivity conditions. We obtain the corresponding Hausdorff measure results. On the way to realising some of the results outlined above, in Section 3.2 and Section 3.3 we develop several more general statements which reformulate Theorem 2.2 in terms of transferring Lebesgue measure statements to Hausdorff measure statements for very general sets of Ψ-approximable points (see Theorems 3.10, 3.11 and 3.14). The recurring theme throughout this section is that given more-or-less any Khintchine–Groshev type statement, Theorem 2.2 can be used to establish the corresponding Hausdorff measure result.

The contents of this chapter are taken from [1, Section 2]. Although we have included minor additions and modifications in an attempt to improve clarity, most of the material appears here as it appears in [1].

3.1 The Khintchine–Groshev Theorem for Hausdorff Measures

Let $n \geq 1$ and $m \geq 1$ be integers. Denote by $\mathbb{I}^{nm}$ the unit cube $[0,1]^{nm}$ in $\mathbb{R}^{nm}$. Throughout this section we consider $\mathbb{R}^{nm}$ equipped with the norm $\|\cdot\| : \mathbb{R}^{nm} \to \mathbb{R}$ defined as follows:

$$\|\mathbf{x}\| = \sqrt{n} \max_{1\leq \ell \leq m} |\mathbf{x}_\ell|_2 \tag{3.1}$$

where $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_m)$ with each $\mathbf{x}_\ell$ representing a column vector in $\mathbb{R}^n$ for $1 \leq \ell \leq m$, and $|\cdot|_2$ is the usual Euclidean norm on $\mathbb{R}^n$. The role of the norm (3.1) will become apparent soon, namely through the proof of Theorem 3.2 below.

Given a function $\psi : \mathbb{N} \to \mathbb{R}^+$, let $\mathcal{A}_{n,m}(\psi)$ denote the set of $\mathbf{x} \in \mathbb{I}^{nm}$ such that

$$|\mathbf{q}\mathbf{x} + \mathbf{p}| < \psi(|\mathbf{q}|)$$

for infinitely many $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{\mathbf{0}\}$. Here, $|\cdot|$ denotes the supremum norm, $\mathbf{x} = (x_{i\ell})$ is regarded as an $n \times m$ matrix and $\mathbf{p}$ and $\mathbf{q}$ are regarded as a row vectors. Thus, $\mathbf{q}\mathbf{x}$ represents a point in $\mathbb{R}^m$ given by the system

$$q_1 x_{1\ell} + \cdots + q_n x_{n\ell} \quad (1 \leq \ell \leq m)$$

of m real linear forms in n variables. We will say that the points in $\mathcal{A}_{n,m}(\psi)$ are *ψ-approximable*. That $\mathcal{A}_{n,m}(\psi)$ satisfies an elegant zero-one law in terms of nm-dimensional Lebesgue measure when the function ψ is monotonic is the content of the classical Khintchine–Groshev Theorem. We opt to state here a modern improved version of this result which is best possible (see [10]).

As usual, $|X|$ will denote the k-dimensional Lebesgue measure of $X \subset \mathbb{R}^k$.

Theorem 3.1 (Beresnevich – Velani [10]). *Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be an approximating function and let $nm > 1$. Then*

$$|\mathcal{A}_{n,m}(\psi)| = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^{n-1}\psi(q)^m < \infty, \\ \\ 1 & \text{if } \sum_{q=1}^{\infty} q^{n-1}\psi(q)^m = \infty. \end{cases}$$

The earliest versions of this theorem were due to Khintchine [35] and Groshev [29] and included various extra constraints including monotonicity of ψ. A famous counterexample constructed by Duffin and Schaeffer [25], mentioned earlier in

Chapter 1, shows that, while Theorem 3.1 also holds when $m = n = 1$ and ψ is monotonic, the monotonicity condition cannot be removed when $m = n = 1$ and so it is natural to exclude this situation by letting $nm > 1$. In the latter case, the monotonicity condition has been removed completely, leaving Theorem 3.1. That monotonicity may be removed in the case $n = 1$ is due to a result of Gallagher [28] (see Theorems 1.12 and 1.13 in Chapter 1) and in the case where $n > 2$ it is a consequence of a result due to Schmidt [45]. Alternatively, for $n > 2$ this also follows from a more general result due to Sprindžuk [47, Chapter 1, Section 5] (see Theorem 3.6). For further details we refer the reader to [4] and references therein. The final unnecessary monotonicity condition to be removed was the $n = 2$ case. Formally stated as Conjecture A in [4], this case was resolved in [10].

Regarding the Hausdorff measure theory we shall show the following.

Theorem 3.2. *Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be any approximating function and let $nm > 1$. Let f and $g : r \to g(r) := r^{-m(n-1)}f(r)$ be dimension functions such that $r^{-nm}f(r)$ is monotonic. Then,*

$$\mathcal{H}^f(\mathcal{A}_{n,m}(\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^{n+m-1} g\left(\frac{\psi(q)}{q}\right) < \infty, \\ \\ \mathcal{H}^f(\mathbb{I}^{nm}) & \text{if } \sum_{q=1}^{\infty} q^{n+m-1} g\left(\frac{\psi(q)}{q}\right) = \infty. \end{cases}$$

Theorem 3.2 is not entirely new and was in fact previously obtained in [4] via Theorem 2.1 subject to ψ being monotonic in the case that $n = 2$. The deduction there was relying on a theorem of Sprindžuk (namely, Theorem 3.6) rather than Theorem 3.1 (which is what we shall use). In fact, with several additional assumptions imposed on ψ and f, the result was first obtained by Dickinson and Velani [21].

Theorem 3.3 (Dickinson – Velani [21])**.** *Let f be a dimension function such that $r^{-nm}f(r) \to \infty$ as $r \to 0$ and $r^{-nm}f(r)$ is non-increasing. Suppose $\psi : \mathbb{N} \to \mathbb{R}^+$ is an approximating function such that $\frac{\psi(q)}{q}$ is decreasing, $q^n\psi(q)^m \to 0$ as $q \to \infty$ and $q^n\psi(q)^m$ is non-increasing. Furthermore, suppose that $q^{n(m+1)}\psi(q)^{-m(n-1)}f\left(\frac{\psi(q)}{q}\right)$ is non-increasing. Then*

$$\mathcal{H}^f(\mathcal{A}_{n,m}(\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} f\left(\frac{\psi(q)}{q}\right)\psi(q)^{-m(n-1)}q^{n(m+1)-1} < \infty, \\ \\ \infty & \text{if } \sum_{q=1}^{\infty} f\left(\frac{\psi(q)}{q}\right)\psi(q)^{-m(n-1)}q^{n(m+1)-1} = \infty. \end{cases}$$

While the above theorem due to Dickinson and Velani constitutes the first Hausdorff measure result obtained for the sets $\mathcal{A}_{n,m}(\psi)$, the first Hausdorff dimension results were obtained even earlier by Bovey and Dodson [13].

Returning to Theorem 3.2, the convergence case of the proof makes use of standard covering arguments that, with little adjustment, can be drawn from [21]. For completeness we shall include this argument here.

Thereafter we shall give two proofs for the divergence case of Theorem 3.2, one using Theorem 2.1 and the other using Theorem 2.2. The reason for this is to show the advantage of using Theorem 2.2 on the one hand, and to explicitly exhibit obstacles in using Theorem 2.1 in other settings on the other hand. In the proofs we will use the following notation. For $(\mathbf{p},\mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{\mathbf{0}\}$ let

$$R_{\mathbf{p},\mathbf{q}} := \{\mathbf{x} \in \mathbb{R}^{nm} : \mathbf{q}\mathbf{x} + \mathbf{p} = \mathbf{0}\}.$$

Note that, throughout the proofs of Theorem 3.2, $(\mathbf{p},\mathbf{q})$ will play the role of the index j appearing in Theorem 2.1 and Theorem 2.2. Also note that for $\delta \geq 0$ we have

$$\Delta(R_{\mathbf{p},\mathbf{q}},\delta) = \{\mathbf{x} \in \mathbb{R}^{nm} : \mathrm{dist}(\mathbf{x}, R_{\mathbf{p},\mathbf{q}}) < \delta\},$$

where

$$\mathrm{dist}(\mathbf{x}, R_{\mathbf{p},\mathbf{q}}) = \inf_{\mathbf{z}\in R_{\mathbf{p},\mathbf{q}}} \|\mathbf{x}-\mathbf{z}\| = \frac{\sqrt{n}|\mathbf{q}\mathbf{x}+\mathbf{p}|}{|\mathbf{q}|_2}.$$

As with the results of Cassels and Gallagher mentioned in Remark 1.7, it is also the case that the sets of interest when we are approximating by systems of linear forms satisfy the dichotomy that their measures only take the values 0 or 1. In this setting the result is due to Beresnevich and Velani and will be useful shortly in deriving (3.3). In order that we might state the result of Beresnevich and Velani in its full generality, we first introduce a little more notation.

Given $\Psi : \mathbb{Z}^n \to \mathbb{R}^+$, let $\mathcal{A}_{n,m}(\Psi)$ be the set of $\mathbf{x} \in \mathbb{I}^{nm}$ for which

$$|\mathbf{q}\mathbf{x} + \mathbf{p}| < \Psi(\mathbf{q}) \tag{3.2}$$

is satisfied for infinitely many pairs $(\mathbf{p},\mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{\mathbf{0}\}$. We denote by $\mathcal{A}'_{n,m}(\Psi)$ the set of points $\mathbf{x} \in \mathbb{I}^{nm}$ for which (3.2) is satisfied for infinitely many pairs $(\mathbf{p},\mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{\mathbf{0}\}$ with $\gcd(\mathbf{p},\mathbf{q}) := \gcd(p_1,p_2,\ldots,p_m,q_1,q_2,\ldots,q_n) = 1$. Finally, let $\mathcal{A}''_{n,m}(\Psi)$ be the set of points $\mathbf{x} \in \mathbb{I}^{nm}$ for which (3.2) is satisfied for infinitely many pairs $(\mathbf{p},\mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{\mathbf{0}\}$ with $\gcd(p_i,\mathbf{q}) := \gcd(p_i, q_1, q_2, \ldots, q_n) = 1$ for each $i = 1, \ldots, m$. If $\Psi(\mathbf{q}) = \psi(|\mathbf{q}|)$ for some function $\psi : \mathbb{N} \to \mathbb{R}^+$ we write, respectively, $\mathcal{A}_{n,m}(\psi)$, $\mathcal{A}'_{n,m}(\psi)$ and $\mathcal{A}''_{n,m}(\psi)$ in place of $\mathcal{A}_{n,m}(\Psi)$, $\mathcal{A}'_{n,m}(\Psi)$ and $\mathcal{A}''_{n,m}(\Psi)$.

Theorem 3.4 (Beresnevich – Velani [9])**.** *For any $n, m \in \mathbb{N}$ and any approximating function $\Psi : \mathbb{Z}^n \to \mathbb{R}^+$, we have that*

$$|\mathcal{A}_{n,m}(\Psi)| \in \{0,1\}, \quad |\mathcal{A}'_{n,m}(\Psi)| \in \{0,1\}, \quad \textit{and} \quad |\mathcal{A}''_{n,m}(\Psi)| \in \{0,1\}.$$

Proof of convergence part of Theorem 3.2. We begin by noting that without loss of generality we may assume that $\frac{\psi(q)}{q} \to 0$ as $q \to \infty$. In particular, this means that $\psi(q) \ll q$.

Recall that, in the convergence case, we are given that $\sum_{q=1}^{\infty} q^{n+m-1} g\left(\frac{\psi(q)}{q}\right) < \infty$. Using the fact that $|\mathbf{q}| \leq |\mathbf{q}|_2$ we note that, for a fixed $(\mathbf{p},\mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{\mathbf{0}\}$,

$$\begin{aligned}
\{\mathbf{x} \in \mathbb{R}^{nm} : |\mathbf{q}\mathbf{x} + \mathbf{p}| < \psi(|\mathbf{q}|)\} &= \left\{\mathbf{x} \in \mathbb{R}^{nm} : \frac{\sqrt{n}|\mathbf{q}\mathbf{x}+\mathbf{p}|}{|\mathbf{q}|_2} < \frac{\sqrt{n}\psi(|\mathbf{q}|)}{|\mathbf{q}|_2}\right\} \\
&\subset \left\{\mathbf{x} \in \mathbb{R}^{nm} : \frac{\sqrt{n}|\mathbf{q}\mathbf{x}+\mathbf{p}|}{|\mathbf{q}|_2} < \frac{\sqrt{n}\psi(|\mathbf{q}|)}{|\mathbf{q}|}\right\} \\
&= \Delta\left(R_{\mathbf{p},\mathbf{q}}, \frac{\sqrt{n}\psi(|\mathbf{q}|)}{|\mathbf{q}|}\right).
\end{aligned}$$

Thus, for each $N \in \mathbb{N}$ we have

$$\mathcal{A}_{n,m}(\psi) \subset \bigcup_{q \geq N} \bigcup_{\substack{\mathbf{q} \in \mathbb{Z}^n \setminus \{\mathbf{0}\} \\ |\mathbf{q}| = q}} \bigcup_{\mathbf{p} \in \mathbb{Z}^m} \Delta\left(R_{\mathbf{p},\mathbf{q}}, \frac{\sqrt{n}\psi(q)}{q}\right) \cap \mathbb{I}^{nm}.$$

Observe that, for a fixed $(\mathbf{p},\mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{\mathbf{0}\}$ with $|\mathbf{q}| = q$, we may cover $\Delta\left(R_{\mathbf{p},\mathbf{q}}, \frac{\sqrt{n}\psi(q)}{q}\right) \cap \mathbb{I}^{nm}$ with a collection $\mathcal{C}_{\mathbf{p},\mathbf{q}}$ of balls of common radius $\frac{\psi(q)}{q}$ satisfying

$$\#\mathcal{C}_{\mathbf{p},\mathbf{q}} \ll \left(\frac{q}{\psi(q)}\right)^{m(n-1)}.$$

Furthermore, note that for a fixed $\mathbf{q}$ with $|\mathbf{q}| = q$,

$$\#\left\{\mathbf{p} \in \mathbb{Z}^m : \Delta\left(R_{\mathbf{p},\mathbf{q}}, \frac{\sqrt{n}\psi(q)}{q}\right) \cap \mathbb{I}^{nm} \neq \emptyset\right\} \ll q^m$$

and, also, that the number of vectors $\mathbf{q} \in \mathbb{Z}^n$ with $|\mathbf{q}| = q$ is $\ll q^{n-1}$.

Finally, since $\frac{\psi(q)}{q} \to 0$ as $q \to \infty$, for any $M \in \mathbb{N}$ there exists an $N_M \in \mathbb{N}$ such that for any $q \geq N_M$ we have $\frac{\psi(q)}{q} \leq \frac{1}{M}$. Thus, it follows from the definition of $\mathcal{H}^f$ that

$$\begin{aligned}\mathcal{H}^f_{\rho=\frac{1}{M}}(\mathcal{A}_{n,m}(\psi)) &\ll \sum_{q\geq N_M} f\left(\frac{\psi(q)}{q}\right) \times \left(\frac{\psi(q)}{q}\right)^{-m(n-1)} q^{m+n-1} \\ &= \sum_{q\geq N_M} g\left(\frac{\psi(q)}{q}\right) q^{m+n-1}.\end{aligned}$$

The proof is completed by letting $M \to \infty$ and noting that the term on the right-hand side of the above tends to 0. $\square$

Next, we turn our attention to proving the divergence part of Theorem 3.2 via the two routes outlined above. We note that if $\psi(r) \geq 1$ for infinitely many $r \in \mathbb{N}$, then $\mathcal{A}_{n,m}(\psi) = \mathbb{I}^{nm}$ and the divergence case of Theorem 3.2 is trivial. Hence, without loss of generality we may assume that $\psi(r) \leq 1$ for all $r \in \mathbb{N}$. First we show how

Theorem 2.1 and Theorem 3.1 imply the divergence case of Theorem 3.2. (3.3)

Proof. Recall that

$$\sum_{q=1}^{\infty} q^{n+m-1} g\left(\frac{\psi(q)}{q}\right) = \infty. \tag{3.4}$$

To use Theorem 2.1 we have to restrict the approximating integer points $\mathbf{q}$ in order to meet conditions (i) and (ii) of Theorem 2.1. We will use the same idea as in [4]; namely, we will impose the requirement that $|\mathbf{q}| = |q_K|$ for a fixed $K \in \{1, \ldots, n\}$. Sprindžuk's Theorem (Theorem 3.6) that is used in [4] allows for the introduction of this requirement almost instantly. Unfortunately, this is not the case when one is using Theorem 3.1 and hence we will need a new argument. For each $1 \leq i \leq n$ define the auxiliary functions $\Psi_i : \mathbb{Z}^n \setminus \{\mathbf{0}\} \to \mathbb{R}^+$ by setting

$$\Psi_i(\mathbf{q}) = \begin{cases} \psi(|\mathbf{q}|) & \text{if } |\mathbf{q}| = |q_i|, \\ \\ 0 & \text{otherwise.} \end{cases}$$

In what follows, similarly to $\mathcal{A}_{n,m}(\psi)$, we consider sets $\mathcal{A}_{n,m}(\Psi)$ of points $\mathbf{x} \in \mathbb{I}^{nm}$ such that

$$|\mathbf{qx} + \mathbf{p}| < \Psi(\mathbf{q})$$

for infinitely many pairs $(\mathbf{p},\mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{\mathbf{0}\}$, where $\Psi : \mathbb{Z}^n \setminus \{\mathbf{0}\} \to \mathbb{R}^+$ is a multivariable function. Since, by definition, $\Psi_i(\mathbf{q}) \leq \psi(|\mathbf{q}|)$ for each $1 \leq i \leq n$ and each $\mathbf{q} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$, it follows that

$$\mathcal{A}_{n,m}(\Psi_i) \subset \mathcal{A}_{n,m}(\psi) \qquad \text{for each } 1 \leq i \leq n. \tag{3.5}$$

By (3.5), to complete the proof of (3.3), it is sufficient to show that

$$\mathcal{H}^f(\mathcal{A}_{n,m}(\Psi_K)) = \mathcal{H}^f(\mathbb{I}^{nm}) \qquad \text{for some } 1 \leq K \leq n. \tag{3.6}$$

Without loss of generality we will assume that $K = 1$. Define

$$S := \{(\mathbf{p},\mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{\mathbf{0}\} : |\mathbf{q}| = |q_1| \text{ and } |\mathbf{p}| \leq M|\mathbf{q}|\},$$

where

$$M := \max\left\{2n, \sup_{r\in\mathbb{N}} \frac{2}{\sqrt{n}}\, g\left(\frac{\psi(r)}{r}\right)^{\frac{1}{m}}\right\}. \tag{3.7}$$

Note that, since g is increasing and $\psi(r) \leq 1$, the constant M is finite. Let $\Upsilon_{\mathbf{p},\mathbf{q}} := \dfrac{\Psi_1(\mathbf{q})}{|\mathbf{q}|}$ for each $(\mathbf{p},\mathbf{q}) \in S$. The purpose for introducing this auxiliary set S will become apparent later. Now, for each $(\mathbf{p},\mathbf{q}) \in S$,

$$\begin{aligned}
\Delta(R_{\mathbf{p},\mathbf{q}}, \Upsilon_{\mathbf{p},\mathbf{q}}) \cap \mathbb{I}^{nm} &= \left\{\mathbf{x} \in \mathbb{I}^{nm} : \frac{\sqrt{n}|\mathbf{q}\mathbf{x}+\mathbf{p}|}{|\mathbf{q}|_2} < \frac{\Psi_1(\mathbf{q})}{|\mathbf{q}|}\right\} \\
&= \left\{\mathbf{x} \in \mathbb{I}^{nm} : |\mathbf{q}\mathbf{x}+\mathbf{p}| < \frac{|\mathbf{q}|_2\Psi_1(\mathbf{q})}{\sqrt{n}|\mathbf{q}|}\right\} \\
&\subset \left\{\mathbf{x} \in \mathbb{I}^{nm} : |\mathbf{q}\mathbf{x}+\mathbf{p}| < \Psi_1(\mathbf{q})\right\},
\end{aligned}$$

since $|\mathbf{q}|_2 \leq \sqrt{n}\,|\mathbf{q}|$. It follows that $\Lambda(\Upsilon) \cap \mathbb{I}^{nm} \subset \mathcal{A}_{n,m}(\Psi_1) \subset \mathbb{I}^{nm}$, where

$$\Lambda(\Upsilon) = \limsup_{(\mathbf{p},\mathbf{q})\in S} \Delta(R_{\mathbf{p},\mathbf{q}}, \Upsilon_{\mathbf{p},\mathbf{q}})$$

and, in taking this limit, $(\mathbf{p},\mathbf{q}) \in S$ can be arranged in any order. Therefore, (3.6) will follow on showing that

$$\mathcal{H}^f(\Lambda(\Upsilon) \cap \mathbb{I}^{nm}) = \mathcal{H}^f(\mathbb{I}^{nm}). \tag{3.8}$$

Showing (3.8) will rely on Theorem 2.1. First of all observe that conditions (i) and (ii) are met with the m-dimensional subspace

$$V := \{\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m) \in \mathbb{R}^{nm} : x_{i\ell} = 0 \text{ for all } \ell = 1, \ldots, m \text{ and } i = 2, \ldots, n\}.$$

Indeed, regarding condition (i), we have that $R_{\mathbf{p},\mathbf{q}} \cap V$ consists of the single element

$$\begin{pmatrix} -\frac{p_1}{q_1} & -\frac{p_2}{q_1} & \ldots & -\frac{p_m}{q_1} \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix},$$

and so is non-empty. Regarding condition (ii), for $(\mathbf{p}, \mathbf{q}) \in S$ we have that

$$\begin{aligned} V \cap \Delta(R_{\mathbf{p},\mathbf{q}}, 1) &= \{\mathbf{x} \in V : \operatorname{dist}(\mathbf{x}, R_{\mathbf{p},\mathbf{q}}) < 1\} \\ &= \left\{\mathbf{x} \in V : \frac{\sqrt{n}|\mathbf{q}\mathbf{x} + \mathbf{p}|}{|\mathbf{q}|_2} < 1\right\} \\ &= \left\{\mathbf{x} \in \mathbb{R}^{nm} : \max_{1 \le \ell \le m} \frac{\sqrt{n}|q_1 x_{1,\ell} + p_\ell|}{|\mathbf{q}|_2} < 1 \quad \text{and} \quad x_{i\ell} = 0 \text{ for } i \neq 1\right\} \\ &\subset \left\{\mathbf{x} \in \mathbb{R}^{nm} : \max_{1 \le \ell \le m} \left|x_{1,\ell} + \frac{p_\ell}{q_1}\right| < 1 \quad \text{and} \quad x_{i\ell} = 0 \text{ for } i \neq 1\right\} \end{aligned}$$

since $|q_1| = |\mathbf{q}|$ and $|\mathbf{q}|_2 \le \sqrt{n}|\mathbf{q}|$. Hence $\operatorname{diam}(V \cap \Delta(R_{\mathbf{p},\mathbf{q}}, 1)) \le 2$ and we are done.

Now let $\theta : \mathbb{N} \to \mathbb{R}^+$ be given by

$$\theta(r) = \frac{r}{\sqrt{n}} g\left(\frac{\psi(r)}{r}\right)^{\frac{1}{m}}$$

and, for each $1 \le i \le n$, let $\Theta_i : \mathbb{Z}^n \setminus \{\mathbf{0}\} \to \mathbb{R}^+$ be given by

$$\Theta_i(\mathbf{q}) = \frac{|\mathbf{q}|}{\sqrt{n}}\, g\left(\frac{\Psi_i(\mathbf{q})}{|\mathbf{q}|}\right)^{\frac{1}{m}} \quad = \quad \begin{cases} \theta(|\mathbf{q}|) & \text{if } \ |\mathbf{q}| = |q_i|, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly to (3.5), we have that $\mathcal{A}_{n,m}(\Theta_i) \subset \mathcal{A}_{n,m}(\theta)$ for each $1 \le i \le n$. Furthermore,

$$\mathcal{A}_{n,m}(\theta) = \bigcup_{i=1}^{n} \mathcal{A}_{n,m}(\Theta_i). \tag{3.9}$$

Indeed, the "$\supset$" inclusion follows from the above. To show the converse, note that for any $\mathbf{x} \in \mathcal{A}_{n,m}(\theta)$ the inequality $|\mathbf{qx} + \mathbf{p}| < \theta(|\mathbf{q}|)$ is satisfied for infinitely many $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{\mathbf{0}\}$. Clearly, for each $\mathbf{q} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ we have that $\theta(|\mathbf{q}|) = \Theta_i(\mathbf{q})$ for some $1 \le i \le n$. Therefore, there is a fixed $i \in \{1, \ldots, n\}$ such that $|\mathbf{qx} + \mathbf{p}| < \theta(|\mathbf{q}|) = \Theta_i(\mathbf{q})$ is satisfied for infinitely many $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{\mathbf{0}\}$. This means that $\mathbf{x} \in \mathcal{A}_{n,m}(\Theta_i)$ for some i, thus verifying (3.9).

Next, observe that, by (3.4), the sum

$$\sum_{q=1}^{\infty} q^{n-1}\theta(q)^m = \sum_{q=1}^{\infty} \frac{q^{n+m-1}}{\sqrt{n}^m} g\left(\frac{\psi(q)}{q}\right) = \frac{1}{\sqrt{n}^m} \sum_{q=1}^{\infty} q^{n+m-1} g\left(\frac{\psi(q)}{q}\right)$$

diverges. Therefore, by Theorem 3.1, we have that $|\mathcal{A}_{n,m}(\theta)| = 1$. Hence, by (3.9), there exists some $1 \le K \le n$ such that $|\mathcal{A}_{n,m}(\Theta_K)| > 0$. By the zero-one laws in Theorem 3.4, we know that $|\mathcal{A}_{n,m}(\Theta_K)| \in \{0, 1\}$. Hence,

$$|\mathcal{A}_{n,m}(\Theta_K)| = 1. \tag{3.10}$$

Without loss of generality we will suppose that $K = 1$, the same as in (3.6).

Now, using the fact that $|\mathbf{q}| \le |\mathbf{q}|_2$, for $(\mathbf{p}, \mathbf{q}) \in S$ we have that

$$\begin{aligned}
\Delta(R_{\mathbf{p},\mathbf{q}}, g(\Upsilon_{\mathbf{p},\mathbf{q}})^{\frac{1}{m}}) \cap \mathbb{I}^{nm} &= \left\{ \mathbf{x} \in \mathbb{I}^{nm} : \frac{\sqrt{n}|\mathbf{qx} + \mathbf{p}|}{|\mathbf{q}|_2} < g\left(\frac{\Psi_1(\mathbf{q})}{|\mathbf{q}|}\right)^{\frac{1}{m}} \right\} \\
&= \left\{ \mathbf{x} \in \mathbb{I}^{nm} : |\mathbf{qx} + \mathbf{p}| < \frac{|\mathbf{q}|_2}{\sqrt{n}} g\left(\frac{\Psi_1(\mathbf{q})}{|\mathbf{q}|}\right)^{\frac{1}{m}} \right\} \\
&\supset \left\{ \mathbf{x} \in \mathbb{I}^{nm} : |\mathbf{qx} + \mathbf{p}| < \frac{|\mathbf{q}|}{\sqrt{n}} g\left(\frac{\Psi_1(\mathbf{q})}{|\mathbf{q}|}\right)^{\frac{1}{m}} \right\} \\
&= \{ \mathbf{x} \in \mathbb{I}^{nm} : |\mathbf{qx} + \mathbf{p}| < \Theta_1(\mathbf{q}) \}.
\end{aligned}$$

Furthermore, observe that if $\{\mathbf{x} \in \mathbb{I}^{nm} : |\mathbf{qx} + \mathbf{p}| < \Theta_1(\mathbf{q})\} \ne \emptyset$, then $|\mathbf{p}| \le M|\mathbf{q}|$ and so $(\mathbf{p}, \mathbf{q}) \in S$. To see this, suppose that $\mathbf{x} \in \{\mathbf{x} \in \mathbb{I}^{nm} : |\mathbf{qx} + \mathbf{p}| < \Theta_1(\mathbf{q})\}$ and note that if $|\mathbf{q}\mathbf{x}+\mathbf{p}| < \Theta_1(\mathbf{q})$ then

$$|\mathbf{q}\mathbf{x}_\ell + p_\ell| < \Theta_1(\mathbf{q}) \leq \frac{|\mathbf{q}|}{\sqrt{n}} g\left(\frac{\psi(|\mathbf{q}|)}{|\mathbf{q}|}\right)^{\frac{1}{m}} \quad \text{for each } 1 \leq \ell \leq m.$$

Using the reverse triangle inequality, it can be seen that for each $1 \leq \ell \leq m$,

$$\begin{aligned} |p_\ell| &\leq \frac{|\mathbf{q}|}{\sqrt{n}} g\left(\frac{\psi(|\mathbf{q}|)}{|\mathbf{q}|}\right)^{\frac{1}{m}} + |\mathbf{q}\mathbf{x}_\ell| \\ &= \frac{|\mathbf{q}|}{\sqrt{n}} g\left(\frac{\psi(|\mathbf{q}|)}{|\mathbf{q}|}\right)^{\frac{1}{m}} + \sum_{i=1}^{n} q_i x_{i\ell}. \end{aligned}$$

Since $\mathbf{x} \in \mathbb{I}^{nm}$, it follows that

$$|p_\ell| \leq \frac{|\mathbf{q}|}{\sqrt{n}} g\left(\frac{\psi(|\mathbf{q}|)}{|\mathbf{q}|}\right)^{\frac{1}{m}} + \sum_{i=1}^{n} |\mathbf{q}| = \frac{|\mathbf{q}|}{\sqrt{n}} g\left(\frac{\psi(|\mathbf{q}|)}{|\mathbf{q}|}\right)^{\frac{1}{m}} + n|\mathbf{q}|.$$

By the definition of M, we see that $|p_\ell| \leq M|\mathbf{q}|$ for each $1 \leq \ell \leq m$. Hence, $|\mathbf{p}| \leq M|\mathbf{q}|$ if $\{\mathbf{x} \in \mathbb{I}^{nm} : |\mathbf{q}\mathbf{x}+\mathbf{p}| < \Theta_1(\mathbf{q})\} \neq \emptyset$ and so, in this case, $(\mathbf{p},\mathbf{q}) \in S$.

Therefore,

$$\mathcal{A}_{n,m}(\Theta_1) \subset \Lambda(g(\Upsilon)^{\frac{1}{m}}) \cap \mathbb{I}^{nm} \subset \mathbb{I}^{nm}.$$

In particular, $|\Lambda(g(\Upsilon)^{\frac{1}{m}}) \cap \mathbb{I}^{nm}| = 1$ and so for any ball $B \subset \mathbb{I}^{nm}$ we have that $\mathcal{H}^{nm}(\Lambda(g(\Upsilon)^{\frac{1}{m}}) \cap B) = \mathcal{H}^{nm}(B)$. Hence, we may apply Theorem 2.1 with $k = nm$, $l = m(n-1)$ and m to conclude that, for any ball $B \subset \mathbb{I}^{nm}$, we have $\mathcal{H}^f(B \cap \Lambda(\Upsilon)) = \mathcal{H}^f(B)$. In particular, $\mathcal{H}^f(\Lambda(\Upsilon) \cap \mathbb{I}^{nm}) = \mathcal{H}^f(\mathbb{I}^{nm})$ and the proof is thus complete. $\square$

We now show how

Theorem 2.2 and Theorem 3.1 imply the divergence case of Theorem 3.2. (3.11)

Proof. As before, we are given the divergence condition (3.4). For each pair $(\mathbf{p},\mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{\mathbf{0}\}$ with $|\mathbf{p}| \leq M|\mathbf{q}|$, where M is given by (3.7), let

$$R_{\mathbf{p},\mathbf{q}} := \{\mathbf{x} \in \mathbb{R}^{nm} : \mathbf{q}\mathbf{x}+\mathbf{p} = \mathbf{0}\} \quad \text{and} \quad \Upsilon_{\mathbf{p},\mathbf{q}} := \frac{\psi(|\mathbf{q}|)}{|\mathbf{q}|}.$$

For such pairs $(\mathbf{p},\mathbf{q})$ we have that

$$\begin{aligned}\Delta(R_{\mathbf{p},\mathbf{q}},\Upsilon_{\mathbf{p},\mathbf{q}})\cap\mathbb{I}^{nm} &= \left\{\mathbf{x}\in\mathbb{I}^{nm} : \frac{\sqrt{n}|\mathbf{q}\mathbf{x}+\mathbf{p}|}{|\mathbf{q}|_2} < \frac{\psi(|\mathbf{q}|)}{|\mathbf{q}|}\right\}\\ &\subset \{\mathbf{x}\in\mathbb{I}^{nm} : |\mathbf{q}\mathbf{x}+\mathbf{p}|<\psi(|\mathbf{q}|)\}\end{aligned}$$

since $|\mathbf{q}|_2 \le \sqrt{n}\,|\mathbf{q}|$. Therefore

$$\Lambda(\Upsilon)\cap\mathbb{I}^{nm}\subset\mathcal{A}_{n,m}(\psi)\subset\mathbb{I}^{nm},$$

where the lim sup is taken over $(\mathbf{p},\mathbf{q})\in\mathbb{Z}^m\times\mathbb{Z}^n\setminus\{\mathbf{0}\}$ with $|\mathbf{p}|\le M|\mathbf{q}|$.

Consequently, if we could show that $\mathcal{H}^f(\Lambda(\Upsilon)\cap\mathbb{I}^{nm})=\mathcal{H}^f(\mathbb{I}^{nm})$ the divergence part of Theorem 3.2 would follow.

Define $\theta:\mathbb{N}\to\mathbb{R}^+$ by

$$\theta(r)=\frac{r}{\sqrt{n}}g\left(\frac{\psi(r)}{r}\right)^{\frac{1}{m}}$$

and note that

$$\begin{aligned}\Delta(R_{\mathbf{p},\mathbf{q}},g(\Upsilon_{\mathbf{p},\mathbf{q}})^{\frac{1}{m}})\cap\mathbb{I}^{nm} &= \left\{\mathbf{x}\in\mathbb{I}^{nm} : \frac{\sqrt{n}|\mathbf{q}\mathbf{x}+\mathbf{p}|}{|\mathbf{q}|_2} < g\left(\frac{\psi(|\mathbf{q}|)}{|\mathbf{q}|}\right)^{\frac{1}{m}}\right\}\\ &= \left\{\mathbf{x}\in\mathbb{I}^{nm} : |\mathbf{q}\mathbf{x}+\mathbf{p}| < \frac{|\mathbf{q}|_2}{\sqrt{n}}g\left(\frac{\psi(|\mathbf{q}|)}{|\mathbf{q}|}\right)^{\frac{1}{m}}\right\}\\ &\supset \left\{\mathbf{x}\in\mathbb{I}^{nm} : |\mathbf{q}\mathbf{x}+\mathbf{p}| < \frac{|\mathbf{q}|}{\sqrt{n}}g\left(\frac{\psi(|\mathbf{q}|)}{|\mathbf{q}|}\right)^{\frac{1}{m}}\right\}\\ &= \{\mathbf{x}\in\mathbb{I}^{nm} : |\mathbf{q}\mathbf{x}+\mathbf{p}|<\theta(|\mathbf{q}|)\},\end{aligned}$$

where this penultimate inclusion follows since $|\mathbf{q}| \le |\mathbf{q}|_2$. Observe that if $\{\mathbf{x}\in\mathbb{I}^{nm} : |\mathbf{q}\mathbf{x}+\mathbf{p}|<\theta(|\mathbf{q}|)\}\neq\emptyset$, then $|\mathbf{p}|\le M|\mathbf{q}|$. This can be seen using the same argument as the one beginning on page 70. It follows that

$$\mathcal{A}_{n,m}(\theta)\subset\Lambda(g(\Upsilon)^{\frac{1}{m}})\cap\mathbb{I}^{nm}.$$

Now, by Theorem 3.1 and the divergence condition (3.4), we know that

$|\mathcal{A}_{n,m}(\theta)| = 1$ since

$$\sum_{q=1}^{\infty} q^{n-1}\theta(q)^m = \sum_{q=1}^{\infty} \frac{q^{n+m-1}}{\sqrt{n}^m} g\left(\frac{\psi(q)}{q}\right) = \infty.$$

Hence, $|\Lambda(g(\Upsilon)^{\frac{1}{m}}) \cap \mathbb{I}^{nm}| = 1$ and so we may apply Theorem 2.2 with $k = nm$, $l = m(n-1)$ and m to conclude that, for any ball $B \subset \mathbb{I}^{nm}$, we have $\mathcal{H}^f(B \cap \Lambda(\Upsilon)) = \mathcal{H}^f(B)$. In particular, $\mathcal{H}^f(\Lambda(\Upsilon) \cap \mathbb{I}^{nm}) = \mathcal{H}^f(\mathbb{I}^{nm})$ and the proof is complete. □

Remark 3.5. Note that the proof of (3.11) is not only shorter and simpler than that of (3.3) but it also does not rely on the zero-one law of Theorem 3.4. This seemingly minor point becomes a substantial obstacle in trying to use the same line of argument as for (3.3) in other settings, for example, in inhomogeneous problems. The point is that, as of now, we do not have an inhomogeneous zero-one law similar to Theorem 3.4 — see [42] for partial results and further comments. The approach based on using Theorem 2.2, on the other hand, works with ease in the inhomogeneous and other settings.

3.2 Inhomogeneous Systems of Linear Forms

In this section we will be concerned with the inhomogeneous version of the Khintchine–Groshev Theorem presented in the previous section. Given an approximating function $\Psi : \mathbb{Z}^n \setminus \{\mathbf{0}\} \to \mathbb{R}^+$ and a fixed $\mathbf{y} \in \mathbb{I}^m$, we denote by $\mathcal{A}^{\mathbf{y}}_{n,m}(\Psi)$ the set of $\mathbf{x} \in \mathbb{I}^{nm}$ for which

$$|\mathbf{q}\mathbf{x} + \mathbf{p} - \mathbf{y}| < \Psi(\mathbf{q})$$

holds for infinitely many $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{\mathbf{0}\}$. In the case that $\Psi(\mathbf{q}) = \psi(|\mathbf{q}|)$ for some function $\psi : \mathbb{N} \to \mathbb{R}^+$ we write $\mathcal{A}^{\mathbf{y}}_{n,m}(\psi)$ for $\mathcal{A}^{\mathbf{y}}_{n,m}(\Psi)$.

For $n \geq 2$ we will represent by $P(\mathbb{Z}^n)$ the set of primitive vectors in $\mathbb{Z}^n$; that is, the non-zero integer vectors $(v_1, v_2, \ldots, v_n) \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ with $\gcd(v_1, v_2, \ldots, v_n) = 1$.

Regarding inhomogeneous Diophantine approximation, we have the following general statement due to Sprindžuk [47, Chapter 1, Section 5].

Theorem 3.6 (Sprindžuk [47])**.** *Let $m \geq 1$ and $n \geq 2$ be integers. Let $\Psi : \mathbb{Z}^n \to \mathbb{R}^+$ be an approximating function such that $\Psi(\mathbf{q}) = 0$ whenever $\mathbf{q} \notin P(\mathbb{Z}^n)$ and let $\mathbf{y} \in \mathbb{I}^m$*

be fixed. Then,

$$|\mathcal{A}^{\mathbf{y}}_{n,m}(\Psi)| = \begin{cases} 0 & \textit{if} \quad \sum_{\mathbf{q}\in\mathbb{Z}^n} \Psi^m(\mathbf{q}) < \infty, \\ \\ 1 & \textit{if} \quad \sum_{\mathbf{q}\in\mathbb{Z}^n} \Psi^m(\mathbf{q}) = \infty. \end{cases}$$

The following inhomogeneous version of the classical Khintchine–Groshev Theorem can be deduced as a corollary to Theorem 3.6 by restricting the approximating function Ψ so that it depends only on $|\mathbf{q}|$ (for further explanation of how see, for example, [4]). In the case that ψ is monotonic this statement also follows as a consequence of the ubiquity technique, see [5, Section 12.1].

Inhomogeneous Khintchine–Groshev Theorem. *Let $m, n \geq 1$ be integers and let $\mathbf{y} \in \mathbb{I}^m$. If $\psi : \mathbb{N} \to \mathbb{R}^+$ is an approximating function which is assumed to be monotonic if $n = 1$ or $n = 2$, then*

$$|\mathcal{A}^{\mathbf{y}}_{n,m}(\psi)| = \begin{cases} 0 & \textit{if} \quad \sum_{q=1}^{\infty} q^{n-1}\psi(q)^m < \infty, \\ \\ 1 & \textit{if} \quad \sum_{q=1}^{\infty} q^{n-1}\psi(q)^m = \infty. \end{cases}$$

The following is the Hausdorff measure version of the above statement.

Theorem 3.7. *Let $m, n \geq 1$ be integers, let $\mathbf{y} \in \mathbb{I}^m$, and let $\psi : \mathbb{N} \to \mathbb{R}^+$ be an approximating function. Let f and $g : r \to g(r) := r^{-m(n-1)} f(r)$ be dimension functions such that $r^{-nm} f(r)$ is monotonic. In the case that $n = 1$ or $n = 2$ suppose also that $rg\left(\frac{\psi(r)}{r}\right)^{\frac{1}{m}}$ is monotonic. Then,*

$$\mathcal{H}^f(\mathcal{A}^{\mathbf{y}}_{n,m}(\psi)) = \begin{cases} 0 & \textit{if} \quad \sum_{q=1}^{\infty} q^{n+m-1} g\left(\frac{\psi(q)}{q}\right) < \infty, \\ \\ \mathcal{H}^f(\mathbb{I}^{nm}) & \textit{if} \quad \sum_{q=1}^{\infty} q^{n+m-1} g\left(\frac{\psi(q)}{q}\right) = \infty. \end{cases}$$

Remark 3.8. Although the condition that $rg\left(\frac{\psi(r)}{r}\right)^{\frac{1}{m}}$ being monotonic when $n = 1$ or $n = 2$ is the one that we naturally arrive at upon combining Theorem 2.2 with the Inhomogeneous Khintchine–Groshev Theorem, it is worth noting here that this condition may be relaxed. In the case when $n = 2$, by appealing to the more general theorem of Sprindžuk (Theorem 3.6), we will show that it is possible to replace monotonicity of $rg\left(\frac{\psi(r)}{r}\right)^{\frac{1}{m}}$ in the statement of Theorem 3.7 with the more aesthetically pleasing assumption that ψ is monotonically decreasing. When $n = 1$ we believe it should be possible to make the same assumption replacement by using ideas from ubiquity (see [5, Section 12.1] and references within).

Proof of Theorem 3.7 — Convergence. We appeal to a standard covering argument, as in the proof of the convergence part of Theorem 3.2. We first note that we may assume without loss of generality that $\frac{\psi(q)}{q} \to 0$ as $q \to \infty$. For each pair $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{\mathbf{0}\}$ let

$$R_{\mathbf{p},\mathbf{q}} = \{\mathbf{x} \in \mathbb{R}^{nm} : \mathbf{qx} + \mathbf{p} - \mathbf{y} = \mathbf{0}\}.$$

Recall that for $\delta \geq 0$ we have

$$\Delta(R_{\mathbf{p},\mathbf{q}}, \delta) = \{\mathbf{x} \in \mathbb{R}^{nm} : \mathrm{dist}(\mathbf{x}, R_{\mathbf{p},\mathbf{q}}) < \delta\},$$

and, in this case,

$$\mathrm{dist}(\mathbf{x}, R_{\mathbf{p},\mathbf{q}}) = \inf_{\mathbf{z} \in R_{\mathbf{p},\mathbf{q}}} \|\mathbf{x} - \mathbf{z}\| = \frac{\sqrt{n}|\mathbf{qx} + \mathbf{p} - \mathbf{y}|}{|\mathbf{q}|_2}.$$

Since $|\mathbf{q}| \leq |\mathbf{q}|_2$, for any fixed pair $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{\mathbf{0}\}$ we have

$$\begin{aligned}
\{\mathbf{x} \in \mathbb{R}^{nm} : |\mathbf{qx} + \mathbf{p} - \mathbf{y}| < \psi(|\mathbf{q}|)\} &= \left\{\mathbf{x} \in \mathbb{R}^{nm} : \frac{\sqrt{n}|\mathbf{qx} + \mathbf{p} - \mathbf{y}|}{|\mathbf{q}|_2} < \frac{\sqrt{n}\psi(|\mathbf{q}|)}{|\mathbf{q}|_2}\right\} \\
&\subset \left\{\mathbf{x} \in \mathbb{R}^{nm} : \frac{\sqrt{n}|\mathbf{qx} + \mathbf{p} - \mathbf{y}|}{|\mathbf{q}|_2} < \frac{\sqrt{n}\psi(|\mathbf{q}|)}{|\mathbf{q}|}\right\} \\
&= \Delta\left(R_{\mathbf{p},\mathbf{q}}, \frac{\sqrt{n}\psi(|\mathbf{q}|)}{|\mathbf{q}|}\right).
\end{aligned}$$

Thus, for each $N \in \mathbb{N}$ we have

$$\mathcal{A}^{\mathbf{y}}_{n,m}(\psi) \subset \bigcup_{q \geq N} \bigcup_{\substack{\mathbf{q} \in \mathbb{Z}^n \setminus \{\mathbf{0}\} \\ |\mathbf{q}| = q}} \bigcup_{\mathbf{p} \in \mathbb{Z}^m} \Delta\left(R_{\mathbf{p},\mathbf{q}}, \frac{\sqrt{n}\psi(q)}{q}\right) \cap \mathbb{I}^{nm}.$$

The proof can now be completed by using the same covering argument used to prove the convergence part of Theorem 3.2. □

Remark 3.9. We note here that in both the Inhomogeneous Khintchine–Groshev Theorem and Theorem 3.7, the Hausdorff measure version we have just recorded, the monotonicity conditions on ψ when $n = 1$ or $n = 2$ are only required for the divergence cases. For both of these theorems the proofs of the convergence parts follow from standard covering arguments for which no monotonicity conditions need to be imposed.

The divergence part of the proof of Theorem 3.7 may be obtained directly using Theorem 2.2 — the argument is almost identical to that used for obtaining the divergence part of Theorem 3.2 via Theorem 2.2. However, by exploiting this argument a little further, we may actually use Theorem 2.2 to prove the following two more general statements from which both Theorems 3.2 and 3.7 follow as corollaries. Therefore, we shall postpone the proof of the divergence part of Theorem 3.7 until after we have established Theorems 3.10 and 3.11 below. In some sense Theorems 3.10 and 3.11 below are essentially reformulations of Theorem 2.2 in terms of, respectively, Ψ-approximable and ψ-approximable points.

Theorem 3.10. *Let $\Psi : \mathbb{Z}^n \setminus \{\mathbf{0}\} \to \mathbb{R}^+$ be an approximating function and let $\mathbf{y} \in \mathbb{I}^m$. Let f and $g : r \to g(r) := r^{-m(n-1)} f(r)$ be dimension functions such that $r^{-nm} f(r)$ is monotonic. Let*

$$\Theta : \mathbb{Z}^n \setminus \{\mathbf{0}\} \to \mathbb{R}^+ \qquad \textit{be defined by} \qquad \Theta(\mathbf{q}) = |\mathbf{q}|\, g\left(\frac{\Psi(\mathbf{q})}{|\mathbf{q}|}\right)^{\frac{1}{m}} .$$

Then

$$|\mathcal{A}^{\mathbf{y}}_{n,m}(\Theta)| = 1 \qquad \textit{implies} \qquad \mathcal{H}^f(\mathcal{A}^{\mathbf{y}}_{n,m}(\Psi)) = \mathcal{H}^f(\mathbb{I}^{nm}).$$

The proof of Theorem 3.10 is similar to that of (3.11). However, we shall omit this particular argument here. Instead, we shall explicitly deduce Theorem 3.10 from an even more general result which will be proved in Section 3.3, where the approximating function will be allowed to depend on $\mathbf{p}$ as well as $\mathbf{q}$ — see Theorem 3.14.

The following statement is a special case of Theorem 3.10 with $\Psi(\mathbf{q}) := \psi(|\mathbf{q}|)$.

Theorem 3.11. *Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be an approximating function, let $\mathbf{y} \in \mathbb{I}^m$ and let f and $g : r \to g(r) := r^{-m(n-1)} f(r)$ be dimension functions such that $r^{-nm} f(r)$ is monotonic. Let*

$$\theta : \mathbb{N} \to \mathbb{R}^+ \qquad \textit{be defined by} \qquad \theta(r) = r\, g\left(\frac{\psi(r)}{r}\right)^{\frac{1}{m}} .$$

Then

$$|\mathcal{A}^{\mathbf{y}}_{n,m}(\theta)| = 1 \qquad \textit{implies} \qquad \mathcal{H}^f(\mathcal{A}^{\mathbf{y}}_{n,m}(\psi)) = \mathcal{H}^f(\mathbb{I}^{nm}).$$

Proof. Define $\Psi : \mathbb{Z}^n \setminus \{\mathbf{0}\} \to \mathbb{R}^+$ by $\Psi(\mathbf{q}) = \psi(|\mathbf{q}|)$. Then, it is not too difficult to see that $\mathcal{A}^{\mathbf{y}}_{n,m}(\Psi) = \mathcal{A}^{\mathbf{y}}_{n,m}(\psi)$. Therefore, we may appeal to Theorem 3.10 which tells us that $\mathcal{H}^f(\mathcal{A}^{\mathbf{y}}_{n,m}(\Psi)) = \mathcal{H}^f(\mathbb{I}^{nm})$ if $|\mathcal{A}^{\mathbf{y}}_{n,m}(\Theta)| = 1$, where $\Theta : \mathbb{Z}^n \setminus \{\mathbf{0}\} \to \mathbb{R}^+$ is defined by $\Theta(\mathbf{q}) = |\mathbf{q}| g\left(\frac{\Psi(\mathbf{q})}{|\mathbf{q}|}\right)^{\frac{1}{m}}$. However, $\Theta(\mathbf{q}) = |\mathbf{q}| g\left(\frac{\Psi(\mathbf{q})}{|\mathbf{q}|}\right)^{\frac{1}{m}} = |\mathbf{q}| g\left(\frac{\psi(|\mathbf{q}|)}{|\mathbf{q}|}\right)^{\frac{1}{m}} = \theta(|\mathbf{q}|)$ and so $\mathcal{A}^{\mathbf{y}}_{n,m}(\Theta) = \mathcal{A}^{\mathbf{y}}_{n,m}(\theta)$. We are given that $|\mathcal{A}^{\mathbf{y}}_{n,m}(\theta)| = 1$. Hence it follows that $|\mathcal{A}^{\mathbf{y}}_{n,m}(\Theta)| = 1$ and the proof is thus complete. □

Theorem 3.7 now follows on combining the Inhomogeneous Khintchine–Groshev Theorem with Theorem 3.11. Furthermore, any progress in removing the monotonicity constraint on ψ from the Inhomogeneous Khintchine–Groshev Theorem can be instantly transferred into a Hausdorff measure statement upon applying Theorem 3.11. Indeed, we suspect that a full inhomogeneous analogue of Theorem 3.1 must be true. Recall that it is open only in the case when $n = 1$ or $n = 2$.

We shall conclude this section by providing further details of the proof of the divergence part of Theorem 3.7. Additionally we show that Theorem 3.7 holds when $n = 2$ under the more satisfying monotonicity assumption that ψ is monotonically decreasing.

Proof of Theorem 3.7 — Divergence. The result would follow from Theorem 3.11 provided that we could show that $|\mathcal{A}^{\mathbf{y}}_{n,m}(\theta)| = 1$ where $\theta : \mathbb{N} \to \mathbb{R}^+$ is defined by $\theta(r) = rg\left(\frac{\psi(r)}{r}\right)^{\frac{1}{m}}$. Assuming that $rg\left(\frac{\psi(r)}{r}\right)^{\frac{1}{m}}$ is monotonic when $n = 1$ or $n = 2$, we know by the Inhomogeneous Khintchine–Groshev Theorem that $|\mathcal{A}^{\mathbf{y}}_{n,m}(\theta)| = 1$ if

$$\sum_{q=1}^{\infty} q^{n-1}\theta(q)^m = \sum_{q=1}^{\infty} q^{n+m-1} g\left(\frac{\psi(q)}{q}\right) = \infty,$$

which is true by assumption.

In order to prove the statement subject to the condition that ψ is monotonically decreasing when $n = 2$ the argument is a little more complicated and relies on Theorem 3.6. So, suppose $n = 2$ and let us define $\Theta : \mathbb{Z}^2 \setminus \{\mathbf{0}\} \to \mathbb{R}^+$ by

$$\Theta(\mathbf{q}) = \begin{cases} \theta(|\mathbf{q}|) & \text{if } \mathbf{q} \in P(\mathbb{Z}^2), \\ \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\mathcal{A}^{\mathbf{y}}_{n,m}(\Theta) \subset \mathcal{A}^{\mathbf{y}}_{n,m}(\theta)$. Therefore, it would be sufficient for us to show that

$|\mathcal{A}^{\mathbf{y}}_{n,m}(\Theta)| = 1$. By Theorem 3.6, this would follow upon showing that

$$\sum_{\mathbf{q}\in P(\mathbb{Z}^2)} \Theta^m(\mathbf{q}) = \infty.$$

To do this, we will make use of the following two claims.

Claim 3.12. *Let $n = 2$. If $\frac{\psi(q)}{q}$ is monotonically decreasing then*

$$\sum_{q=1}^{\infty} q^{n+m-1} g\left(\frac{\psi(q)}{q}\right) \asymp \sum_{t=1}^{\infty} 2^{t(m+2)} g\left(\frac{\psi(2^t)}{2^t}\right).$$

Proof of Claim 3.12. Since $\frac{\psi(q)}{q}$ is monotonically decreasing, we may bound $\sum_{q=1}^{\infty} q^{n+m-1} g\left(\frac{\psi(q)}{q}\right)$ from below as follows,

$$\begin{aligned}
\sum_{q=1}^{\infty} q^{n+m-1} g\left(\frac{\psi(q)}{q}\right) &= \sum_{t=1}^{\infty} \sum_{2^{t-1}\leq q<2^t} q^{m+1} g\left(\frac{\psi(q)}{q}\right) \\
&\geq \sum_{t=1}^{\infty} \sum_{2^{t-1}\leq q<2^t} (2^{t-1})^{m+1} g\left(\frac{\psi(2^t)}{2^t}\right) \\
&= \sum_{t=1}^{\infty} (2^{t-1})^{m+2} g\left(\frac{\psi(2^t)}{2^t}\right) \\
&\gg \sum_{t=1}^{\infty} 2^{t(m+2)} g\left(\frac{\psi(2^t)}{2^t}\right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\sum_{q=1}^{\infty} q^{n+m-1} g\left(\frac{\psi(q)}{q}\right) &= \sum_{t=0}^{\infty} \sum_{2^{t}\leq q<2^{t+1}} q^{m+1} g\left(\frac{\psi(q)}{q}\right) \\
&\leq \sum_{t=0}^{\infty} \sum_{2^{t}\leq q<2^{t+1}} (2^{t+1})^{m+1} g\left(\frac{\psi(2^t)}{2^t}\right) \\
&= \sum_{t=0}^{\infty} 2^t (2^{t+1})^{m+1} g\left(\frac{\psi(2^t)}{2^t}\right) \\
&\ll \sum_{t=1}^{\infty} 2^{t(m+2)} g\left(\frac{\psi(2^t)}{2^t}\right).
\end{aligned}$$

The desired result follows on combining these upper and lower bounds. □

Claim 3.13. *Let $n = 2$. If $\frac{\psi(q)}{q}$ is monotonically decreasing then we have*

$$\sum_{\mathbf{q}\in P(\mathbb{Z}^2)} \Theta^m(\mathbf{q}) \gg \sum_{q=1}^{\infty} 2^{t(m+2)} g\left(\frac{\psi(2^t)}{2^t}\right).$$

Proof of Claim 3.13. We begin by observing that

$$\sum_{\mathbf{q}\in P(\mathbb{Z}^2)} \Theta^m(\mathbf{q}) = \sum_{\mathbf{q}\in P(\mathbb{Z}^2)} \theta^m(|\mathbf{q}|) = \sum_{q=1}^{\infty} \sum_{\substack{1\le p\le q: \\ \gcd(p,q)=1}} \theta^m(q). \tag{3.12}$$

As usual, let φ denote the Euler function. Remembering that m and n are constants, and using the monotonicity of $\frac{\psi(q)}{q}$, we have

$$\begin{aligned}
\sum_{q=1}^{\infty} \sum_{\substack{1\le p\le q: \\ \gcd(p,q)=1}} \theta^m(q) &= \sum_{q=1}^{\infty} \varphi(q)\theta^m(q) \\
&= \sum_{q=1}^{\infty} \varphi(q) \left(qg\left(\frac{\psi(q)}{q}\right)^{\frac{1}{m}}\right)^m \\
&= \sum_{q=1}^{\infty} \varphi(q) q^m g\left(\frac{\psi(q)}{q}\right) \\
&= \sum_{t=1}^{\infty} \sum_{2^{t-1}\le q<2^t} \varphi(q) q^m g\left(\frac{\psi(q)}{q}\right) \\
&\ge \sum_{t=1}^{\infty} \sum_{2^{t-1}\le q<2^t} \varphi(q) (2^{t-1})^m g\left(\frac{\psi(2^t)}{2^t}\right) \\
&\gg \sum_{t=1}^{\infty} 2^{tm} g\left(\frac{\psi(2^t)}{2^t}\right) \sum_{2^{t-1}\le q<2^t} \varphi(q).
\end{aligned} \tag{3.13}$$

We recall (see, for example, [3]) that, for real $x > 1$, we have

$$\sum_{q\le x} \varphi(q) = \frac{3}{\pi^2}x^2 + O(x\log x).$$

It follows from this that,

$$\sum_{2^{t-1}\le q<2^t} \varphi(q) \gg 2^{2t}.$$

Combining this fact with (3.12) and (3.13) yields

$$\sum_{\mathbf{q}\in P(\mathbb{Z}^2)} \Theta^m(\mathbf{q}) \gg \sum_{t=1}^{\infty} 2^{tm} g\left(\frac{\psi(2^t)}{2^t}\right) \sum_{2^{t-1}\leq q<2^t} \varphi(q) \gg \sum_{t=1}^{\infty} 2^{t(m+2)} g\left(\frac{\psi(2^t)}{2^t}\right),$$

as claimed. □

Now, recall that we are given $\sum_{q=1}^{\infty} q^{n+m-1} g\left(\frac{\psi(q)}{q}\right) = \infty$. Therefore, it follows from Claim 3.12 that $\sum_{t=1}^{\infty} 2^{t(m+2)} g\left(\frac{\psi(2^t)}{2^t}\right) = \infty$. In turn, by Claim 3.13, this implies that $\sum_{\mathbf{q}\in P(\mathbb{Z}^2)} \Theta^m(\mathbf{q}) = \infty$. Finally, in light of this it follows from Theorem 3.6 that $|\mathcal{A}^{\mathbf{y}}_{n,m}(\Theta)| = 1$, as required. □

As mentioned previously, Theorem 3.2 may also be derived as a corollary of Theorem 3.11. The argument for this is essentially the same as the first part of the above so we shall omit the details.

3.3 Approximation by Primitive Points and More

The key goal of this section is to present Hausdorff measure analogues of some recent results obtained by Dani, Laurent and Nogueira in [18]. The setup they consider assumes certain coprimality conditions on the $(m+n)$-tuple $(q_1, \dots, q_n, p_1, \dots, p_m)$ of approximating integers. To achieve our goal we will first prove a very general statement which further extends Theorems 3.10 and 3.11 and is of independent interest. In particular, we will allow for the approximating function to depend on $(\mathbf{p}, \mathbf{q})$ and will also introduce a "distortion" parameter Φ that allows certain flexibility within our framework. This allows us, for example, to incorporate the so-called "absolute value theory" [19, 31, 32].

Within this section $\Psi : \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{\mathbf{0}\} \to \mathbb{R}^+$ will be a function of $(\mathbf{p}, \mathbf{q})$, $\mathbf{y} \in \mathbb{I}^m$ will be a fixed point and $\Phi \in \mathbb{I}^{mm}$ will be a fixed $m \times m$ square matrix. Further, define $\mathcal{M}^{\mathbf{y},\Phi}_{n,m}(\Psi)$ to be the set of $\mathbf{x} \in \mathbb{I}^{nm}$ such that

$$|\mathbf{q}\mathbf{x} + \mathbf{p}\Phi - \mathbf{y}| < \Psi(\mathbf{p}, \mathbf{q})$$

holds for $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{\mathbf{0}\}$ with arbitrarily large $|\mathbf{q}|$. Based upon Theorem 2.2, we now state and prove the following generalisation of Theorems 3.10 and 3.11.

Theorem 3.14. *Let $\Psi : \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{\mathbf{0}\} \to \mathbb{R}^+$ be such that*

$$\lim_{|\mathbf{q}|\to\infty} \sup_{\mathbf{p}\in\mathbb{Z}^m} \frac{\Psi(\mathbf{p},\mathbf{q})}{|\mathbf{q}|} = 0\,, \tag{3.14}$$

and let $\mathbf{y} \in \mathbb{I}^m$ and $\Phi \in \mathbb{I}^{mm}$ be fixed. Let f and $g : r \to g(r) := r^{-m(n-1)}f(r)$ be dimension functions such that $r^{-nm}f(r)$ is monotonic. Let

$$\Theta : \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{\mathbf{0}\} \to \mathbb{R}^+ \qquad \text{be defined by} \qquad \Theta(\mathbf{p},\mathbf{q}) = |\mathbf{q}|\, g\left(\frac{\Psi(\mathbf{p},\mathbf{q})}{|\mathbf{q}|}\right)^{\frac{1}{m}}.$$

Then

$$|\mathcal{M}^{\mathbf{y},\Phi}_{n,m}(\Theta)| = 1 \qquad \text{implies} \qquad \mathcal{H}^f(\mathcal{M}^{\mathbf{y},\Phi}_{n,m}(\Psi)) = \mathcal{H}^f(\mathbb{I}^{nm})\,.$$

Proof. Let

$$M := \max\left\{3n, \sup_{(\mathbf{p},\mathbf{q})\in\mathbb{Z}^m\times\mathbb{Z}^n\setminus\{\mathbf{0}\}} \frac{3\Theta(\mathbf{p},\mathbf{q})}{\sqrt{n}|\mathbf{q}|}\right\}. \tag{3.15}$$

By the monotonicity of g and condition (3.14), we have that M is finite. Let

$$S := \{(\mathbf{p},\mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{\mathbf{0}\} : |\mathbf{p}\Phi| \le M|\mathbf{q}|\}$$

and let S_Φ be any fixed subset of S such that for each $(\mathbf{p}',\mathbf{q}) \in S$ there exists $(\mathbf{p},\mathbf{q}) \in S_\Phi$ such that

$$\mathbf{p}\Phi = \mathbf{p}'\Phi \qquad \text{and} \qquad \Theta(\mathbf{p}',\mathbf{q}) \le 2\Theta(\mathbf{p},\mathbf{q}). \tag{3.16}$$

Furthermore, let S_Φ be such that for all $(\mathbf{p},\mathbf{q}), (\mathbf{r},\mathbf{s}) \in S_\Phi$ we have

$$(\mathbf{p}\Phi,\mathbf{q}) \ne (\mathbf{r}\Phi,\mathbf{s}) \qquad \text{if} \qquad (\mathbf{p},\mathbf{q}) \ne (\mathbf{r},\mathbf{s}).$$

The existence of S_Φ is easily seen. For each $(\mathbf{p},\mathbf{q}) \in S_\Phi$, let

$$R_{\mathbf{p},\mathbf{q}} := \{\mathbf{x} \in \mathbb{R}^{nm} : \mathbf{q}\mathbf{x} + \mathbf{p}\Phi - \mathbf{y} = \mathbf{0}\} \qquad \text{and} \qquad \Upsilon_{\mathbf{p},\mathbf{q}} := \frac{\Psi(\mathbf{p},\mathbf{q})}{|\mathbf{q}|}\,.$$

For $(\mathbf{p},\mathbf{q}) \in S_\Phi$ we have that

$$\begin{aligned}\Delta(R_{\mathbf{p},\mathbf{q}}, \Upsilon_{\mathbf{p},\mathbf{q}}) \cap \mathbb{I}^{nm} &= \left\{\mathbf{x} \in \mathbb{I}^{nm} : \frac{\sqrt{n}|\mathbf{q}\mathbf{x} + \mathbf{p}\Phi - \mathbf{y}|}{|\mathbf{q}|_2} < \frac{\Psi(\mathbf{p},\mathbf{q})}{|\mathbf{q}|}\right\}\\ &\subset \{\mathbf{x} \in \mathbb{I}^{nm} : |\mathbf{q}\mathbf{x} + \mathbf{p}\Phi - \mathbf{y}| < \Psi(\mathbf{p},\mathbf{q})\}\end{aligned}$$

since $|\mathbf{q}|_2 \leq \sqrt{n}\,|\mathbf{q}|$. Also note that for each $\mathbf{q} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ there are only finitely many $\mathbf{p} \in \mathbb{Z}^m$ such that $(\mathbf{p}, \mathbf{q}) \in S_\Phi$ — indeed, the motivation for introducing the set S_Φ is to ensure such finiteness. Therefore

$$\Lambda(\Upsilon) \cap \mathbb{I}^{nm} \subset \mathcal{M}^{\mathbf{y},\Phi}_{n,m}(\Psi) \subset \mathbb{I}^{nm}, \tag{3.17}$$

where, when defining $\Lambda(\Upsilon)$, the lim sup is taken over $(\mathbf{p}, \mathbf{q}) \in S_\Phi$. Hence, by (3.17), it would suffice for us to show that

$$\mathcal{H}^f(\Lambda(\Upsilon) \cap \mathbb{I}^{nm}) = \mathcal{H}^f(\mathbb{I}^{nm}).$$

Consider $\Lambda(g(\Upsilon)^{\frac{1}{m}})$, where the lim sup is again taken over $(\mathbf{p}, \mathbf{q}) \in S_\Phi$. Take any $(\mathbf{p}', \mathbf{q}) \in S$ and let $(\mathbf{p}, \mathbf{q}) \in S_\Phi$ satisfy (3.16). Then, since $|\mathbf{q}| \leq |\mathbf{q}|_2$, we have that

$$\begin{aligned}
\Delta(R_{\mathbf{p},\mathbf{q}}, g(\Upsilon_{\mathbf{p},\mathbf{q}})^{\frac{1}{m}}) \cap \mathbb{I}^{nm} &= \left\{ \mathbf{x} \in \mathbb{I}^{nm} : \frac{\sqrt{n}|\mathbf{q}\mathbf{x} + \mathbf{p}\Phi - \mathbf{y}|}{|\mathbf{q}|_2} < g\left(\frac{\Psi(\mathbf{p},\mathbf{q})}{|\mathbf{q}|}\right)^{\frac{1}{m}} \right\} \\
&= \left\{ \mathbf{x} \in \mathbb{I}^{nm} : |\mathbf{q}\mathbf{x} + \mathbf{p}\Phi - \mathbf{y}| < \frac{|\mathbf{q}|_2}{\sqrt{n}} g\left(\frac{\Psi(\mathbf{p},\mathbf{q})}{|\mathbf{q}|}\right)^{\frac{1}{m}} \right\} \\
&\supset \left\{ \mathbf{x} \in \mathbb{I}^{nm} : |\mathbf{q}\mathbf{x} + \mathbf{p}\Phi - \mathbf{y}| < \frac{|\mathbf{q}|}{\sqrt{n}} g\left(\frac{\Psi(\mathbf{p},\mathbf{q})}{|\mathbf{q}|}\right)^{\frac{1}{m}} \right\} \\
&= \left\{ \mathbf{x} \in \mathbb{I}^{nm} : |\mathbf{q}\mathbf{x} + \mathbf{p}\Phi - \mathbf{y}| < \tfrac{1}{\sqrt{n}}\Theta(\mathbf{p},\mathbf{q}) \right\} \\
&\supset \left\{ \mathbf{x} \in \mathbb{I}^{nm} : |\mathbf{q}\mathbf{x} + \mathbf{p}'\Phi - \mathbf{y}| < \tfrac{1}{2\sqrt{n}}\Theta(\mathbf{p}',\mathbf{q}) \right\}.
\end{aligned}$$

Also observe that if $\left\{ \mathbf{x} \in \mathbb{I}^{nm} : |\mathbf{q}\mathbf{x} + \mathbf{p}'\Phi - \mathbf{y}| < \tfrac{1}{2\sqrt{n}}\Theta(\mathbf{p}',\mathbf{q}) \right\} \neq \emptyset$, then $|\mathbf{p}'\Phi| \leq M|\mathbf{q}|$. This can be shown by making suitable modifications to the argument beginning at the end of page 70. It follows that

$$\mathcal{M}^{\mathbf{y},\Phi}_{n,m}(\tfrac{1}{2\sqrt{n}}\Theta) \subset \Lambda(g(\Upsilon)^{\frac{1}{m}}) \subset \mathbb{I}^{nm}. \tag{3.18}$$

Recall that $|\mathcal{M}^{\mathbf{y},\Phi}_{n,m}(\Theta)| = 1$. Furthermore, in view of [9, Lemma 4], we have that $|\mathcal{M}^{\mathbf{y},\Phi}_{n,m}(\frac{1}{2\sqrt{n}}\Theta)| = 1$. Together with (3.18) this implies that $|\Lambda(g(\Upsilon)^{\frac{1}{m}}) \cap \mathbb{I}^{nm}| = 1$. Further, note that, by (3.14), $\Upsilon_{\mathbf{p},\mathbf{q}} \to 0$ as $|\mathbf{q}| \to \infty$. Therefore, Theorem 2.2 is applicable with $k = nm$, $l = m(n-1)$ and m and we conclude that for any ball $B \subset \mathbb{I}^{nm}$ we have that $\mathcal{H}^f(B \cap \Lambda(\Upsilon)) = \mathcal{H}^f(B)$. In particular, this means that $\mathcal{H}^f(\Lambda(\Upsilon) \cap \mathbb{I}^{nm}) = \mathcal{H}^f(\mathbb{I}^{nm})$, as required. □

Before we proceed to exhibit further applications of Theorem 3.14, we show how Theorem 3.10 in Section 3.2 follows as a corollary of Theorem 3.14. A consequence of this is that all of the Hausdorff measure results obtained so far in this chapter can be derived from Theorem 3.14. Furthermore, the rest of the Hausdorff measure results which will be presented in this chapter will also be deduced using Theorem 3.14. In short, Theorem 3.14 is an extremely versatile statement which can be used to easily extract Hausdorff measure statements from Lebesgue measure statements for a wide range of sets of interest in Diophantine approximation.

Proof of Theorem 3.10. Let Ψ be as in Theorem 3.10. First observe that if $\Psi(\mathbf{q}) \geq 1$ for infinitely many $\mathbf{q} \in \mathbb{Z}^n$, then $\mathcal{A}^{\mathbf{y}}_{n,m}(\Psi) = \mathbb{I}^{nm}$ and there is nothing to prove. Otherwise we obviously have that $\Psi(\mathbf{q})/|\mathbf{q}| \to 0$ as $|\mathbf{q}| \to \infty$. In this case, extending Ψ and Θ to be functions of $(\mathbf{p}, \mathbf{q})$ so that $\Psi(\mathbf{p}, \mathbf{q}) := \Psi(\mathbf{q})$ and $\Theta(\mathbf{p}, \mathbf{q}) := \Theta(\mathbf{q})$, we immediately recover Theorem 3.10 from Theorem 3.14. $\square$

Theorem 3.14 can be applied in various situations beyond what has already been discussed above. For example, divergence results of [20] can be obtained by using Theorem 3.14 with

$$\Phi := \begin{pmatrix} I_u & 0 \\ 0 & 0 \end{pmatrix}$$

where I_u is the identity matrix. In what follows we shall give applications of Theorem 3.14 in which the dependence of Ψ on both $\mathbf{p}$ and $\mathbf{q}$ becomes particularly useful. Namely, we shall extend the Lebesgue measure results of Dani, Laurent and Nogueira [18] to Hausdorff measures.

First we establish some notation. Recall that for any $d \geq 2$ we denote by $P(\mathbb{Z}^d)$ the set of points $\mathbf{v} = (v_1, \ldots, v_d) \in \mathbb{Z}^d$ such that $\gcd(v_1, \ldots, v_d) = 1$. For any subset $\sigma = \{i_1, \ldots, i_\nu\}$ of $\{1, \ldots, d\}$ with $\nu \geq 2$, let $P(\sigma)$ be the set of points $\mathbf{v} \in \mathbb{Z}^d$ such that $\gcd(v_{i_1}, \ldots, v_{i_\nu}) = 1$. Next, given a partition π of $\{1, \ldots, d\}$ into disjoint subsets π_ℓ of at least two elements, let $P(\pi)$ be the set of points $\mathbf{v} \in \mathbb{Z}^d$ such that $\mathbf{v} \in P(\pi_\ell)$ for all components π_ℓ of π.

Given an approximating function $\psi : \mathbb{N} \to \mathbb{R}^+$ and fixed $\Phi \in \mathbb{I}^{mm}$ and $\mathbf{y} \in \mathbb{I}^m$, let $\mathcal{M}^{\mathbf{y},\Phi}_{n,m}(\psi)$ be the set of $\mathbf{x} \in \mathbb{I}^{nm}$ such that

$$|\mathbf{q}\mathbf{x} + \mathbf{p}\Phi - \mathbf{y}| < \psi(|\mathbf{q}|) \tag{3.19}$$

holds for $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{\mathbf{0}\}$ with arbitrarily large $|\mathbf{q}|$. Also, given a partition π of $\{1, \ldots, m+n\}$, let $\mathcal{M}^{\pi,\mathbf{y},\Phi}_{n,m}(\psi)$ denote the set of $\mathbf{x} \in \mathbb{I}^{nm}$ for which (3.19) is satisfied for

$(\mathbf{p},\mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{\mathbf{0}\}$ with arbitrarily large $|\mathbf{q}|$ and with $(q_1,\dots,q_n,p_1,\dots,p_m) \in P(\pi)$. Specialising Theorem 3.14 for the approximating function

$$\Psi(\mathbf{p},\mathbf{q}) := \begin{cases} \psi(|\mathbf{q}|) & \text{if } (q_1,\dots,q_n,p_1,\dots,p_m) \in P(\pi), \\ \\ 0 & \text{otherwise,} \end{cases}$$

gives the following.

Theorem 3.15. *Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be an approximating function such that $\frac{\psi(q)}{q} \to 0$ as $q \to \infty$. Let π be any partition of $\{1,\dots,m+n\}$ and let $\Phi \in \mathbb{I}^{mm}$ and $\mathbf{y} \in \mathbb{I}^m$ be fixed. Let f and $g : r \to g(r) := r^{-m(n-1)} f(r)$ be dimension functions such that $r^{-nm} f(r)$ is monotonic and let $\theta : \mathbb{N} \to \mathbb{R}^+$ be defined by $\theta(q) = q\, g\left(\frac{\psi(q)}{q}\right)^{\frac{1}{m}}$. Then*

$$|\mathcal{M}^{\pi,\mathbf{y},\Phi}_{n,m}(\theta)| = 1 \qquad \textit{implies} \qquad \mathcal{H}^f(\mathcal{M}^{\pi,\mathbf{y},\Phi}_{n,m}(\psi)) = \mathcal{H}^f(\mathbb{I}^{nm}).$$

Now, let us turn our attention to the results of Dani, Laurent and Nogueira from [18]. For the moment, we will return to the homogeneous setting. Given a partition π of $\{1,\dots,m+n\}$ and an approximating function $\psi : \mathbb{N} \to \mathbb{R}^+$ we will denote by $\mathcal{A}^{\pi}_{n,m}(\psi)$ the set of $\mathbf{x} \in \mathbb{I}^{nm}$ such that

$$|\mathbf{q}\mathbf{x} + \mathbf{p}| < \psi(|\mathbf{q}|)$$

holds for $(\mathbf{p},\mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{\mathbf{0}\}$ with arbitrarily large $|\mathbf{q}|$ and with $(q_1,\dots,q_n,p_1,\dots,p_m) \in P(\pi)$. We note that in this case the inequality holds for $(\mathbf{p},\mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{\mathbf{0}\}$ with arbitrarily large $|\mathbf{q}|$ if and only if the inequality holds for infinitely many $(\mathbf{p},\mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{\mathbf{0}\}$. The notation $\mathcal{A}_{n,m}(\psi)$ will be used as defined in Section 3.1. The following statement is a consequence of [18, Theorem 1.2].

Theorem DLN1. *Let $n,m \in \mathbb{N}$ and let π be a partition of $\{1,\dots,m+n\}$ such that every component of π has at least $m+1$ elements. Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a function such that the mapping $x \to x^{n-1}\psi(x)^m$ is non-increasing. Then,*

$$|\mathcal{A}^{\pi}_{n,m}(\psi)| = \begin{cases} 0 & \textit{if} \ \ \sum_{q=1}^{\infty} q^{n-1}\psi(q)^m < \infty, \\ \\ 1 & \textit{if} \ \ \sum_{q=1}^{\infty} q^{n-1}\psi(q)^m = \infty. \end{cases}$$

The following Hausdorff measure analogue of Theorem DLN1 follows from Theorem 3.15.

Theorem 3.16. *Let $n, m \in \mathbb{N}$ and let π be a partition of $\{1, \ldots, m+n\}$ such that every component of π has at least $m+1$ elements. Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be an approximating function. Let f and $g : r \to g(r) := r^{-m(n-1)} f(r)$ be dimension functions such that the function $r^{-nm} f(r)$ is monotonic and $q^{n+m-1} g\left(\frac{\psi(q)}{q}\right)$ is non-increasing. Then,*

$$\mathcal{H}^f(\mathcal{A}^{\pi}_{n,m}(\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^{n+m-1} g\left(\frac{\psi(q)}{q}\right) < \infty, \\ \\ \mathcal{H}^f(\mathbb{I}^{nm}) & \text{if } \sum_{q=1}^{\infty} q^{n+m-1} g\left(\frac{\psi(q)}{q}\right) = \infty. \end{cases}$$

Proof. First note that in light of the fact that $q^{n+m-1} g\left(\frac{\psi(q)}{q}\right)$ is non-increasing we may assume without loss of generality that $\frac{\psi(q)}{q} \to 0$ as $q \to \infty$. To see this, suppose that $\frac{\psi(q)}{q} \nrightarrow 0$. Therefore, there must exist some $\varepsilon > 0$ such that $\frac{\psi(q)}{q} \geq \varepsilon$ infinitely often. In turn, since g is a dimension function, and hence non-decreasing, this means that $q^{n+m-1} g\left(\frac{\psi(q)}{q}\right) \geq q^{n+m-1} g(\varepsilon)$ infinitely often. However, since this expression is non-increasing, we must have that $g(\varepsilon) = 0$. In particular, this means that $g(r) = 0$ and, hence, also $f(r) = 0$ for all $r \leq \varepsilon$. Thus $\mathcal{H}^f(X) = 0$ for any $X \subset \mathbb{I}^{nm}$ and so the result is trivially true.

In view of the conditions imposed on π, we must have that $nm > 1$. Furthermore, since $\mathcal{A}^{\pi}_{n,m}(\psi) \subset \mathcal{A}_{n,m}(\psi)$, it follows from Theorem 3.2 that $\mathcal{H}^f(\mathcal{A}^{\pi}_{n,m}(\psi)) = 0$ when $\sum_{q=1}^{\infty} q^{n+m-1} g\left(\frac{\psi(q)}{q}\right) < \infty$. Alternatively, one can use a standard covering argument to obtain a direct proof of the convergence part of Theorem 3.16.

Regarding the divergence case, observe that $\mathcal{A}^{\pi}_{n,m}(\psi) = \mathcal{M}^{\pi;\mathbf{0},I_m}_{n,m}(\psi)$, where I_m represents the $m \times m$ identity matrix. Therefore, if $|\mathcal{M}^{\pi;\mathbf{0},I_m}_{n,m}(\theta)| = |\mathcal{A}^{\pi}_{n,m}(\theta)| = 1$ where $\theta : \mathbb{N} \to \mathbb{R}^+$ is defined by $\theta(q) = q\, g\left(\frac{\psi(q)}{q}\right)^{\frac{1}{m}}$, then it would follow from Theorem 3.15 that $\mathcal{H}^f(\mathcal{A}^{\pi}_{n,m}(\psi)) = \mathcal{H}^f(\mathcal{M}^{\pi;\mathbf{0},I_m}_{n,m}(\psi)) = \mathcal{H}^f(\mathbb{I}^{nm})$.

Now, by Theorem DLN1, $|\mathcal{A}^{\pi}_{n,m}(\theta)| = 1$ if $q \to q^{n-1}\theta(q)^m$ is non-increasing and $\sum_{q=1}^{\infty} q^{n-1}\theta(q)^m = \infty$. We have that $q^{n-1}\theta(q)^m = q^{n+m-1} g\left(\frac{\psi(q)}{q}\right)$ which is non-increasing by assumption. By our hypotheses, we also have

$$\sum_{q=1}^{\infty} q^{n-1}\theta(q)^m = \sum_{q=1}^{\infty} q^{n+m-1} g\left(\frac{\psi(q)}{q}\right) = \infty.$$

Hence the proof is complete. $\square$

If $\psi(q) := q^{-\tau}$ for some $\tau > 0$ let us write $\mathcal{A}^{\pi}_{n,m}(\tau) := \mathcal{A}^{\pi}_{n,m}(\psi)$. The following result regarding the Hausdorff dimension of $\mathcal{A}^{\pi}_{n,m}(\tau)$ is a corollary of Theorem 3.16.

Corollary 3.17. *Let $n, m \in \mathbb{N}$ and let π be a partition of $\{1, \ldots, m+n\}$ such that every component of π has at least $m+1$ elements. Then*

$$\dim_{\mathrm{H}}(\mathcal{A}^{\pi}_{n,m}(\tau)) = \begin{cases} m(n-1) + \frac{m+n}{\tau+1} & \textit{when} \quad \tau \geq \frac{n}{m}\,, \\ \\ nm & \textit{when} \quad \tau < \frac{n}{m}. \end{cases}$$

Proof. For $\tau \geq \frac{n}{m}$ the result follows on applying Theorem 3.16 with

$$f_\delta(r) := r^{s_0+\delta} \qquad \text{where} \qquad s_0 = m(n-1) + \frac{m+n}{\tau+1}\,.$$

Indeed, with δ sufficiently small, all the conditions of Theorem 3.16 are met and furthermore, letting $g_\delta(r) := r^{-m(n-1)} f_\delta(r)$, we have

$$\sum_{q=1}^{\infty} q^{n+m-1} g_\delta(q^{-\tau-1}) \begin{cases} < \infty & \text{if} \quad \delta > 0, \\ \\ = \infty & \text{if} \quad \delta \leq 0, \end{cases}$$

since $$\sum_{q=1}^{\infty} q^{n+m-1} g_\delta(q^{-\tau-1}) = \sum_{q=1}^{\infty} q^{n+m-1+(\tau+1)(m(n-1)-s_0-\delta)} = \sum_{q=1}^{\infty} q^{-1-\delta(\tau+1)}.$$

Thus, we have from Theorem 3.16 that

$$\mathcal{H}^{f_\delta}(\mathcal{A}^{\pi}_{n,m}(\tau)) = \begin{cases} 0 & \text{if} \quad \delta > 0, \\ \\ \mathcal{H}^{f_\delta}(\mathbb{I}^{nm}) & \text{if} \quad \delta \leq 0. \end{cases}$$

This means that $\mathcal{H}^{s_0+\delta}(\mathcal{A}^{\pi}_{n,m}(\tau)) = 0$ for $\delta > 0$ and $\mathcal{H}^{s_0+\delta}(\mathcal{A}^{\pi}_{n,m}(\tau)) = \mathcal{H}^{s_0+\delta}(\mathbb{I}^{nm})$ for $\delta \leq 0$. Therefore, if $s_0 \leq nm$ then $\dim_{\mathrm{H}}(\mathcal{A}^{\pi}_{n,m}(\tau)) = s_0$ since, in this case, $\mathcal{H}^{s_0+\delta}(\mathbb{I}^{nm}) = \infty$ whenever $\delta < 0$. Finally, note that $s_0 \leq nm$ if and only if $\tau \geq \frac{n}{m}$.

In the case where $\tau < \frac{n}{m}$ observe that $\mathcal{A}^{\pi}_{n,m}(\tau) \supset \mathcal{A}^{\pi}_{n,m}(\frac{n}{m})$ so

$$\dim_{\mathrm{H}}(\mathcal{A}^{\pi}_{n,m}(\tau)) \geq \dim_{\mathrm{H}}\left(\mathcal{A}^{\pi}_{n,m}\left(\frac{n}{m}\right)\right) = nm.$$

Combining this with the trivial upper bound gives $\dim_{\mathrm{H}}(\mathcal{A}^{\pi}_{n,m}(\tau)) = nm$ when $\tau < \frac{n}{m}$, as required. ☐

Next we consider two results of Dani, Laurent and Nogueira regarding inhomogeneous approximation. As before, for a fixed $\mathbf{y} \in \mathbb{I}^m$ we let $\mathcal{A}^{\mathbf{y}}_{n,m}(\psi)$ denote the set of points $\mathbf{x} \in \mathbb{I}^{nm}$ for which

$$|\mathbf{q}\mathbf{x} + \mathbf{p} - \mathbf{y}| < \psi(|\mathbf{q}|) \tag{3.20}$$

holds for infinitely many $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{\mathbf{0}\}$. Given a partition π of $\{1, \dots, m+n\}$, let $\mathcal{A}^{\pi;\mathbf{y}}_{n,m}(\psi)$ be the set of points $\mathbf{x} \in \mathbb{I}^{nm}$ for which (3.20) holds for infinitely many $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{\mathbf{0}\}$ with $(q_1, \dots, q_n, p_1, \dots, p_m) \in P(\pi)$.

Rephrasing it in a way which is more suitable for our current purposes, a consequence of [18, Theorem 1.1] reads as follows.

Theorem DLN2. *Let $n, m \in \mathbb{N}$ and let π be a partition of $\{1, \dots, m+n\}$ such that every component of π has at least $m+1$ elements. Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a function such that the mapping $x \to x^{n-1}\psi(x)^m$ is non-increasing. Then,*

(i) *if $\sum_{q=1}^{\infty} q^{n-1}\psi(q)^m = \infty$ then for almost every $\mathbf{y} \in \mathbb{I}^m$ we have $\left|\mathcal{A}^{\pi;\mathbf{y}}_{n,m}(\psi)\right| = 1$.*

(ii) *if $\sum_{q=1}^{\infty} q^{n-1}\psi(q)^m < \infty$ then for any $\mathbf{y} \in \mathbb{I}^m$ we have $\left|\mathcal{A}^{\mathbf{y}}_{n,m}(\psi)\right| = 0$.*

The corresponding Hausdorff measure statement we obtain in this case is:

Theorem 3.18. *Let $n, m \in \mathbb{N}$ and let π be a partition of $\{1, \dots, m+n\}$ such that every component of π has at least $m+1$ elements. Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be an approximating function. Let f and $g : r \to g(r) := r^{-m(n-1)}f(r)$ be dimension functions such that the function $r^{-nm}f(r)$ is monotonic and $q^{n+m-1}g\left(\frac{\psi(q)}{q}\right)$ is non-increasing. Then,*

(i) *if $\sum_{q=1}^{\infty} q^{n+m-1}g\left(\frac{\psi(q)}{q}\right) = \infty$ then for Lebesgue almost every $\mathbf{y} \in \mathbb{I}^m$ we have $\mathcal{H}^f\left(\mathcal{A}^{\pi;\mathbf{y}}_{n,m}(\psi)\right) = \mathcal{H}^f(\mathbb{I}^{nm})$.*

(ii) *if $\sum_{q=1}^{\infty} q^{n+m-1}g\left(\frac{\psi(q)}{q}\right) < \infty$ then for any $\mathbf{y} \in \mathbb{I}^m$ we have $\mathcal{H}^f(\mathcal{A}^{\mathbf{y}}_{n,m}(\psi)) = 0$.*

Remark. The proof of this result is similar to the proof of Theorem 3.16 with the only difference being the introduction of $\mathbf{y}$.

Proof. We note here that, by the same reasoning as that given in the proof of Theorem 3.16, the assumption that $q^{n+m-1}g\left(\frac{\psi(q)}{q}\right)$ is non-increasing means that we may assume without loss of generality that $\frac{\psi(q)}{q} \to 0$ as $q \to \infty$.

To prove statement (i) we first note that $\mathcal{A}^{\pi,\mathbf{y}}_{n,m}(\psi) = \mathcal{M}^{\pi,\mathbf{y},I_m}_{n,m}(\psi)$ for any approximating function. So, by Theorem 3.15 we have $\mathcal{H}^f(\mathcal{A}^{\pi,\mathbf{y}}_{n,m}(\psi)) = \mathcal{H}^f(\mathbb{I}^{nm})$ whenever $|\mathcal{M}^{\pi,\mathbf{y},I_m}_{n,m}(\theta)| = |\mathcal{A}^{\pi,\mathbf{y}}_{n,m}(\theta)| = 1$, where $\theta : \mathbb{N} \to \mathbb{R}^+$ is defined by $\theta(q) = qg\left(\frac{\psi(q)}{q}\right)^{\frac{1}{m}}$.

By Theorem DLN2, if $q \to q^{n-1}\theta(q)^m$ is non-increasing and $\sum_{q=1}^{\infty} q^{n-1}\theta(q)^m = \infty$ then $|\mathcal{M}^{\pi,\mathbf{y},I_m}_{n,m}(\theta)| = |\mathcal{A}^{\pi,\mathbf{y}}_{n,m}(\theta)| = 1$ for Lebesgue almost every $\mathbf{y} \in \mathbb{I}^m$. That these two conditions are satisfied can be verified by the same reasoning as in the proof of Theorem 3.16. Thus, for almost every $\mathbf{y} \in \mathbb{I}^m$ we have $|\mathcal{M}^{\pi,\mathbf{y},I_m}_{n,m}(\theta)| = 1$ and for each of these $\mathbf{y}$'s we also have $\mathcal{H}^f(\mathcal{A}^{\pi,\mathbf{y}}_{n,m}(\psi)) = \mathcal{H}^f(\mathcal{M}^{\pi,\mathbf{y},I_m}_{n,m}(\psi)) = \mathcal{H}^f(\mathbb{I}^{nm})$. This completes the proof of statement (i).

Statement (ii) follows from the convergence part of Theorem 3.7. □

Finally, let us re-introduce the parameter $\Phi \in \mathbb{I}^{mm}$. In this case, considering the sets $\mathcal{M}^{\pi,\mathbf{y},\Phi}_{n,m}(\psi)$ (as defined on page 83), it follows from [18, Theorem 1.3] that we have:

Theorem DLN3. *Let $n, m \in \mathbb{N}$ and let π be a partition of $\{1, \ldots, m+n\}$ such that every component of π has at least $m+1$ elements. Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a function such that the mapping $x \to x^{n-1}\psi(x)^m$ is non-increasing. Then, for any $\mathbf{y} \in \mathbb{I}^m$,*

(i) *if $\sum_{q=1}^{\infty} q^{n-1}\psi(q)^m = \infty$ then for almost every $\Phi \in \mathbb{I}^{mm}$ we have that $|\mathcal{M}^{\pi,\mathbf{y},\Phi}_{n,m}(\psi)| = 1$.*

(ii) *if $\sum_{q=1}^{\infty} q^{n-1}\psi(q)^m < \infty$ then for any $\Phi \in \mathbb{I}^{mm}$ we have $|\mathcal{M}^{\mathbf{y},\Phi}_{n,m}(\psi)| = 0$.*

Combining this with Theorem 3.15 we obtain the following Hausdorff measure statement.

Theorem 3.19. *Let $n, m \in \mathbb{N}$ and let π be a partition of $\{1, \ldots, m+n\}$ such that every component of π has at least $m+1$ elements. Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be an approximating function. Let f and $g : r \to g(r) := r^{-m(n-1)}f(r)$ be dimension functions such that the function $r^{-nm}f(r)$ is monotonic and $q^{n+m-1}g\left(\frac{\psi(q)}{q}\right)$ is non-increasing. Then, for any $\mathbf{y} \in \mathbb{I}^m$,*

(i) *if $\sum_{q=1}^{\infty} q^{n+m-1}g\left(\frac{\psi(q)}{q}\right) = \infty$ then for Lebesgue almost every $\Phi \in \mathbb{I}^{mm}$ we have that $\mathcal{H}^f(\mathcal{M}^{\pi,\mathbf{y},\Phi}_{n,m}(\psi)) = \mathcal{H}^f(\mathbb{I}^{nm})$.*

(ii) *if $\sum_{q=1}^{\infty} q^{n+m-1}g\left(\frac{\psi(q)}{q}\right) < \infty$ then, for any $\Phi \in \mathbb{I}^{mm}$, we have that $\mathcal{H}^f(\mathcal{M}^{\mathbf{y},\Phi}_{n,m}(\psi)) = 0$.*

Proof. The proof is essentially the same as the proof of Theorem 3.18 with the obvious modifications. Namely, we appeal to Theorem DLN3 rather than Theorem DLN2 in the divergence part of the proof and in the convergence part, for fixed $\mathbf{y} \in \mathbb{I}^m$ and $\Phi \in \mathbb{I}^{mm}$, we consider

$$R_{\mathbf{p},\mathbf{q}} = \{\mathbf{x} \in \mathbb{R}^{nm} : \mathbf{q}\mathbf{x} + \mathbf{p}\Phi - \mathbf{y} = \mathbf{0}\}$$

for each $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{\mathbf{0}\}$. $\square$

While all of the Hausdorff measure results established throughout this section could have been obtained by combining Theorem 2.2 with the relevant Lebesgue measure statement via a proof similar to that of (3.11), it can be seen that Theorem 3.14 provides an easier mechanism for transferring Lebesgue measure statements to their Hausdorff measure analogues. Moreover, the generality of Theorem 3.14 means that it is applicable in a vast range of settings including, as we have seen, homogeneous approximation, inhomogeneous approximation, and approximation with restrictions.

4 A General Inhomogeneous Jarník–Besicovitch Theorem

In this chapter we consider another indirect application of the mass transference principle for linear forms (Theorem 2.2). Namely, we show how the Hausdorff measure analogue of the Inhomogeneous Khintchine–Groshev Theorem (Theorem 3.7), obtained in the previous chapter as one of the consequences of Theorem 2.2, can be used to provide an alternative proof of most cases of a general inhomogeneous Jarník–Besicovitch Theorem due to Levesley [38].

Furthermore, inspired by this and the (lack of) monotonicity conditions required in Theorem 3.7, we investigate the necessity of the monotonicity condition imposed in Levesley's Theorem. We show that, in general, monotonicity cannot be removed from Levesley's Theorem. Aside from a few minor amendments and the addition of an explicit proof of Proposition 4.4, the material in this chapter appears here as it is presented in [2, Section 3.3].

4.1 A Theorem of Levesley

The Hausdorff dimension of $\mathcal{A}^{\mathbf{y}}_{n,m}(\psi)$, in the general inhomogeneous setting, was determined by Levesley in [38]. To state his result we first introduce one additional piece of notation. Given a function $f : \mathbb{N} \to \mathbb{R}^+$, the *lower order at infinity of* f, usually denoted by λ, is

$$\lambda(f) := \liminf_{q\to\infty} \frac{\log(f(q))}{\log(q)}.$$

Theorem 4.1 (Levesley, [38])**.** *Let $m, n \in \mathbb{N}$ and let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a monotonically decreasing function. Let λ be the lower order at infinity of $\frac{1}{\psi}$. Then, for any $\mathbf{y} \in \mathbb{I}^m$,*

$$\dim_{\mathrm{H}}(\mathcal{A}^{\mathbf{y}}_{n,m}(\psi)) = \begin{cases} m(n-1) + \frac{m+n}{\lambda+1} & \text{when} \quad \lambda \geq \frac{n}{m}\,, \\ \\ nm & \text{when} \quad \lambda < \frac{n}{m}. \end{cases}$$

Remark. In the homogeneous case, when $\mathbf{y} = \mathbf{0}$, this result was previously established by Dodson [23].

Levesley proved the above theorem by considering the cases of $n = 1$ and $n \geq 2$ separately. In both cases his argument uses ideas from ubiquitous systems. These are combined with ideas from uniform distribution in the former case and with a more statistical ("mean-variance") argument in the latter case.

Using Theorem 3.7, we can give an alternative proof of this theorem in the case that $n \geq 2$. That is, we will prove:

Theorem 4.2. *Let $m \geq 1$ and $n \geq 2$ be integers. Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a monotonically decreasing function and let λ be the lower order at infinity of $\frac{1}{\psi}$. Then, for any $\mathbf{y} \in \mathbb{I}^m$,*

$$\dim_{\mathrm{H}}(\mathcal{A}^{\mathbf{y}}_{n,m}(\psi)) = \begin{cases} m(n-1) + \frac{m+n}{\lambda+1} & \text{when} \quad \lambda \geq \frac{n}{m}\ , \\ \\ nm & \text{when} \quad \lambda < \frac{n}{m}. \end{cases}$$

4.2 An Alternative Proof of (most of) Levesley's Theorem

Recall that in Remark 3.8 we noted that it was sufficient in Theorem 3.7 to assume that ψ is monotonically decreasing in the case that $n = 2$. Throughout this section, we shall assume any mention of Theorem 3.7 refers to a statement including this nicer monotonicity condition for the $n = 2$ case.

To prove Theorem 4.2 using Theorem 3.7 we first establish a useful lemma.

Lemma 4.3. *Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be monotonic and bounded. Then,*

$$\liminf_{q\to\infty} \frac{-\log(\psi(q))}{\log q} = \liminf_{t\to\infty} \frac{-\log\left(\psi(2^t)\right)}{\log 2^t}.$$

Proof. Assume first that ψ is non-increasing. Note that $(2^t)_{t=1}^{\infty}$ is a subsequence of $(q)_{q=1}^{\infty}$ and so

$$\liminf_{q\to\infty} \frac{-\log(\psi(q))}{\log q} \leq \liminf_{t\to\infty} \frac{-\log\left(\psi(2^t)\right)}{\log 2^t}.$$

It remains to prove the reverse inequality. Suppose for now that $\psi(q) \geq 1$ for all $q \in \mathbb{N}$. In this case, since $\psi(q) \to c$ for some $c \geq 1$ by monotone convergence,

$$\liminf_{q\to\infty} \frac{-\log(\psi(q))}{\log q} = 0 = \liminf_{t\to\infty} \frac{-\log(\psi(2^t))}{\log 2^t}.$$

Thus, we may assume that $\psi(q) < 1$ for all sufficiently large q. Given $q \in \mathbb{N}$, set t_q to be the unique integer satisfying $2^{t_q} \leq q < 2^{t_q+1}$. Then $\psi(2^{t_q}) \geq \psi(q)$ and $\log(\psi(2^{t_q})) \geq \log(\psi(q))$. Since further $q < 2^{t_q+1}$ and so $\log q < \log 2^{t_q+1}$, we obtain

$$\begin{aligned}\liminf_{q\to\infty} \frac{-\log(\psi(q))}{\log q} &\geq \liminf_{q\to\infty} \frac{-\log\left(\psi(2^{t_q})\right)}{\log 2^{t_q+1}} \\ &= \liminf_{q\to\infty} \frac{-\log\left(\psi(2^{t_q})\right)}{\log 2^{t_q} + \log 2} \\ &= \liminf_{t\to\infty} \frac{-\log\left(\psi(2^{t})\right)}{\log 2^{t}},\end{aligned}$$

as required.

For non-decreasing ψ the proof is similar. By the same argument as above, it is again sufficient to show that

$$\liminf_{q\to\infty} \frac{-\log(\psi(q))}{\log q} \geq \liminf_{t\to\infty} \frac{-\log\left(\psi(2^{t})\right)}{\log 2^{t}}.$$

We note that if $\psi(q) \geq 1$ for all sufficiently large q then, since ψ is bounded,

$$\liminf_{q\to\infty} \frac{-\log(\psi(q))}{\log q} = 0 = \liminf_{t\to\infty} \frac{-\log\left(\psi(2^{t})\right)}{\log 2^{t}}.$$

Therefore, we may assume that $\psi(q) < 1$ for all $q \in \mathbb{N}$. Now, along the same lines as in the argument above, given $q \in \mathbb{N}$ let t'_q be the unigue integer for which $2^{t'_q-1} \leq q < 2^{t'_q}$. Thus, we have

$$\log 2^{t'_q} > \log q \quad \text{and} \quad \log \psi(2^{t'_q}) \geq \log\left(\psi(q)\right).$$

Hence, it follows that

$$\begin{aligned}\liminf_{q\to\infty} \frac{-\log\left(\psi(q)\right)}{\log q} &\geq \liminf_{q\to\infty} \frac{-\log\left(\psi(2^{t'_q})\right)}{\log 2^{t'_q}} \\ &= \liminf_{t\to\infty} \frac{-\log\left(\psi(2^{t})\right)}{\log 2^{t}},\end{aligned}$$

and the proof is thus complete. □

Proof of Theorem 4.2 using Theorem 3.7. To avoid confusion throughout the proof, for approximating functions $\psi : \mathbb{N} \to \mathbb{R}^+$ we will write λ_ψ to denote the lower order at infinity of $\frac{1}{\psi}$. However, when there is no ambiguity we will just write λ and omit the additional subscript.

We observe that, since ψ is assumed to be monotonically decreasing, we must have $\lambda_\psi \geq 0$. To see this, suppose that $\lambda_\psi < 0$. Then, by the definition of the lower order at infinity, it follows that for any $\varepsilon > 0$ we must have $\psi(q) \geq q^{-(\lambda_\psi+\varepsilon)}$ for infinitely many values of q. In particular, this is true for every $0 < \varepsilon < |\lambda_\psi|$ and so we conclude that ψ cannot be monotonically decreasing if $\lambda_\psi < 0$.

We will now show that if the result stated in Theorem 4.2 is true for approximating functions with $\lambda = \frac{n}{m}$, then this implies the validity of the result for approximating functions with $0 \leq \lambda < \frac{n}{m}$. We will then establish the result for approximating functions with $\lambda \geq \frac{n}{m}$.

For the time being, assume that the conclusion in Theorem 4.2 holds for any monotonically decreasing approximating function with $\lambda = \frac{n}{m}$ and let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a monotonically decreasing approximating function such that $\lambda_\psi < \frac{n}{m}$. Consider the function $\Psi : \mathbb{N} \to \mathbb{R}^+$ defined by $\Psi(q) = \min\{\psi(q), q^{-\frac{n}{m}}\}$. Note that Ψ is a monotonically decreasing function (since it is the minimum of two monotonically decreasing functions) and that $\Psi(q) \leq \psi(q)$ for all $q \in \mathbb{N}$. In particular, we have $\dim_{\mathrm{H}}(\mathcal{A}^{\mathbf{y}}_{n,m}(\Psi)) \leq \dim_{\mathrm{H}}(\mathcal{A}^{\mathbf{y}}_{n,m}(\psi))$. Next, note that it follows from the fact that $\Psi(q) \leq q^{-\frac{n}{m}}$ for all $q \in \mathbb{N}$ that $\lambda_\Psi \geq \frac{n}{m}$. On the other hand, since $\lambda_\psi < \frac{n}{m}$ we know that $\psi(q) \geq q^{-\frac{n}{m}}$ for infinitely many values of q. In particular, this implies that we must have $\Psi(q) = q^{-\frac{n}{m}}$ infinitely often and, consequently, that $\lambda_\Psi \leq \frac{n}{m}$. Hence, $\lambda_\Psi = \frac{n}{m}$ and so, by our assumption, we see that

$$\dim_{\mathrm{H}}(\mathcal{A}^{\mathbf{y}}_{n,m}(\psi)) \geq \dim_{\mathrm{H}}(\mathcal{A}^{\mathbf{y}}_{n,m}(\Psi)) = m(n-1) + \frac{n+m}{\lambda_\Psi + 1} = nm.$$

Combining this with the trivial upper bound we conclude that $\dim_{\mathrm{H}}(\mathcal{A}^{\mathbf{y}}_{n,m}(\psi)) = nm$, as required.

It remains to be shown that $\dim_{\mathrm{H}}(\mathcal{A}^{\mathbf{y}}_{n,m}(\psi)) = m(n-1) + \frac{n+m}{\lambda+1}$ for monotonically decreasing approximating functions $\psi : \mathbb{N} \to \mathbb{R}^+$ with $\lambda_\psi = \lambda \geq \frac{n}{m}$. To this end, suppose ψ is such an approximating function.

Let $s_0 = m(n-1) + \frac{m+n}{\lambda+1}$ and consider $f_\delta(r) = r^{s_0+\delta}$ where $-\frac{n+m}{\lambda+1} < \delta < \frac{n+m}{\lambda+1}$. We aim to show that

$$\mathcal{H}^{s_0+\delta}(\mathcal{A}^{\mathbf{y}}_{n,m}(\psi)) = \begin{cases} 0 & \text{if} \quad \delta > 0\,, \\ & \\ \mathcal{H}^{s_0+\delta}(\mathbb{I}^{nm}) & \text{if} \quad \delta < 0, \end{cases}$$

from which the result would follow.

Note that $f_\delta(r)$ is a dimension function and $r^{-nm}f_\delta(r)$ is monotonic. Let $g_\delta(r) = r^{-m(n-1)}f_\delta(r) = r^{-m(n-1)+s_0+\delta}$. Since $\delta > -\frac{n+m}{\lambda+1}$, and so $-m(n-1)+s_0+\delta > 0$, the function $g_\delta(r)$ is a dimension function. Thus f_δ and g_δ satisfy the hypotheses of Theorem 3.7.

It follows from the definition of the lower order at infinity that, for any $\varepsilon > 0$,

$$\psi(q) \leq q^{-(\lambda-\varepsilon)} \quad \text{for all large enough } q, \text{ and}$$

$$\psi(q) \geq q^{-(\lambda+\varepsilon)} \quad \text{for infinitely many } q \in \mathbb{N}. \tag{4.1}$$

Combining this with Lemma 4.3, we have

$$\psi(2^t) \leq 2^{-t(\lambda-\varepsilon)} \quad \text{for large enough } t, \text{ and}$$

$$\psi(2^t) \geq 2^{-t(\lambda+\varepsilon)} \quad \text{for infinitely many } t. \tag{4.2}$$

By Theorem 3.7 it follows that to determine $\mathcal{H}^{f_\delta}(\mathcal{A}^{\mathbf{y}}_{n,m}(\psi))$ we are interested in the behaviour of the sum

$$\sum_{q=1}^{\infty} q^{n+m-1} g_\delta\left(\frac{\psi(q)}{q}\right) = \sum_{q=1}^{\infty} q^{n+m-1}\left(\frac{\psi(q)}{q}\right)^{-m(n-1)+s_0+\delta}. \tag{4.3}$$

Observe that, by the conditions imposed on δ, $-m(n-1)+s_0+\delta > 0$ and also that, by (4.1), we have $\psi(q) \leq q^{-(\lambda-\varepsilon)}$ for sufficiently large q. Thus, (4.3) will converge if

$$\sum_{q=1}^{\infty} q^{n+m-1}(q^{-(\lambda-\varepsilon)-1})^{-m(n-1)+s_0+\delta} = \sum_{q=1}^{\infty} q^{n+m-1+(\lambda+1-\varepsilon)(m(n-1)-s_0-\delta)} < \infty. \tag{4.4}$$

This will be the case if

$$n+m-1+(\lambda+1-\varepsilon)(m(n-1)-s_0-\delta) < -1$$

which is true if and only if

$$\frac{n+m}{\lambda+1-\varepsilon} + m(n-1) < s_0 + \delta.$$

If $\delta > 0$ we can force the above to be true by taking ε to be sufficiently small. Thus we conclude that, for $\delta > 0$, (4.3) converges and consequently $\mathcal{H}^{s_0+\delta}(\mathcal{A}^{\mathbf{y}}_{n,m}(\psi)) = 0$.

Next we establish that (4.3) diverges when $-\frac{n+m}{\lambda+1} < \delta < 0$. First we note, since ψ is monotonically decreasing, that

$$\begin{aligned}\sum_{q=1}^{\infty} q^{n+m-1}\left(\frac{\psi(q)}{q}\right)^{-m(n-1)+s_0+\delta} &= \sum_{t=1}^{\infty}\sum_{2^{t-1}\le q<2^t} q^{n+m-1}\left(\frac{\psi(q)}{q}\right)^{-m(n-1)+s_0+\delta}\\ &\ge \sum_{t=1}^{\infty}\sum_{2^{t-1}\le q<2^t} (2^{t-1})^{n+m-1}\left(\frac{\psi(2^t)}{2^t}\right)^{-m(n-1)+s_0+\delta}\\ &= \sum_{t=1}^{\infty} 2^{t-1}(2^{t-1})^{n+m-1}\left(\frac{\psi(2^t)}{2^t}\right)^{-m(n-1)+s_0+\delta}\\ &= \frac{1}{2^{m+n}}\sum_{t=1}^{\infty} 2^{t(n+m)}\left(\frac{\psi(2^t)}{2^t}\right)^{-m(n-1)+s_0+\delta}. \end{aligned} \tag{4.5}$$

We proceed by showing that, when $\delta < 0$, we have for infinitely many t that

$$2^{t(m+n)}\left(\frac{\psi(2^t)}{2^t}\right)^{-m(n-1)+s_0+\delta} \ge 1. \tag{4.6}$$

For any $\delta < 0$ we can choose $\varepsilon > 0$ small enough such that

$$\frac{m+n}{\lambda+1+\varepsilon} + m(n-1) \ge s_0 + \delta.$$

Note that such an ε exists since we are assuming that δ is negative. Rearranging, this gives

$$m+n-(\lambda+\varepsilon+1)(-m(n-1)+s_0+\delta) \ge 0$$

and then, exponentiating,

$$2^{t(m+n)}\left(\frac{2^{-t(\lambda+\varepsilon)}}{2^t}\right)^{-m(n-1)+s_0+\delta} \ge 1.$$

Now, by (4.2) we have $\psi(2^t) \ge 2^{-t(\lambda+\varepsilon)}$ infinitely often and so (4.6) holds, thus proving the divergence of (4.5) and hence also the divergence of (4.3).

Hence, we have shown that

$$\mathcal{H}^{s_0+\delta}(\mathcal{A}^{\mathbf{y}}_{n,m}(\psi)) = \begin{cases} 0 & \text{if} \quad \delta > 0\,, \\ \\ \mathcal{H}^{s_0+\delta}(\mathbb{I}^{nm}) & \text{if} \quad \delta < 0. \end{cases}$$

If $s_0 \leq nm$ then $\mathcal{H}^{s_0+\delta}(\mathbb{I}^{nm}) = \infty$ whenever $\delta < 0$. From this it would follow that $\dim_{\mathrm{H}}(\mathcal{A}^{\mathbf{y}}_{n,m}(\psi)) = s_0$. The proof is completed upon noting that $s_0 \leq nm$ is equivalent to $\lambda \geq \frac{n}{m}$. □

4.3 The Necessity of Monotonicity in Levesley's Theorem

In Theorems 4.1 and 4.2 the approximating function ψ is assumed to be monotonic. However, the main tool in our proof of Theorem 4.2 is Theorem 3.7, which requires no monotonicity assumptions on ψ for $n \geq 3$. This leads immediately to the natural question of whether this monotonicity assumption is indeed necessary in Levesley's Theorem (Theorem 4.1).

In an attempt to address this question, let us consider general (not necessarily monotonic) approximating functions $\psi : \mathbb{N} \to \mathbb{R}^+$ with λ, the lower order at infinity of $\frac{1}{\psi}$, satisfying $\lambda > \frac{n}{m}$. Assuming no monotonicity conditions on ψ, and applying similar arguments to those which we have employed here to prove Theorem 4.2, we obtain the following bounds on the Hausdorff dimension of $\mathcal{A}^{\mathbf{y}}_{n,m}(\psi)$.

Proposition 4.4. *Let $m \geq 1$ and $n \geq 3$ be integers. If $\psi : \mathbb{N} \to \mathbb{R}^+$ is any function and λ is the lower order at infinity of $\frac{1}{\psi}$ then, for any $\mathbf{y} \in \mathbb{I}^m$, if $\lambda > \frac{n}{m}$ we have*

$$m(n-1) + \frac{m+n-1}{\lambda+1} \leq \dim_{\mathrm{H}}(\mathcal{A}^{\mathbf{y}}_{n,m}(\psi)) \leq m(n-1) + \frac{m+n}{\lambda+1}.$$

Proof. Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be any function with $\lambda > \frac{n}{m}$. We will use Theorem 3.7 to obtain upper and lower bounds for $\dim_{\mathrm{H}}(\mathcal{A}^{\mathbf{y}}_{n,m}(\psi))$. To this end, we will consider dimension functions $f(r) = r^s$ and, correspondingly, $g(r) = r^{-m(n-1)+s}$. We will be interested in establishing values of s for which the sum

$$\sum_{q=1}^{\infty} q^{n+m-1} g\left(\frac{\psi(q)}{q}\right) = \sum_{q=1}^{\infty} q^{n+m-1} \left(\frac{\psi(q)}{q}\right)^{-m(n-1)+s} \tag{4.7}$$

converges or diverges.

First of all we will consider when this sum converges. Note that it follows from the definition of lower order at infinity that for any $\varepsilon > 0$ we have

$$\psi(q) \leq q^{-(\lambda-\varepsilon)} \quad \text{for all sufficiently large } q \in \mathbb{N}.$$

In particular, this means that, for any $\varepsilon > 0$,

$$\begin{aligned} \sum_{q=1}^{\infty} q^{n+m-1} \left(\frac{\psi(q)}{q}\right)^{-m(n-1)+s} &\ll \sum_{q=1}^{\infty} q^{n+m-1} (q^{-(\lambda-\varepsilon)-1})^{-m(n-1)+s} \\ &= \sum_{q=1}^{\infty} q^{n+m-1-(s-m(n-1))(\lambda+1-\varepsilon)}. \end{aligned} \tag{4.8}$$

Thus, if the sum on the far right-hand side of (4.8) converges then (4.7) will also converge. Now, it can be seen that the sum on the far right-hand side of (4.8) converges if

$$s > m(n-1) + \frac{m+n}{\lambda+1-\varepsilon}.$$

Since (4.7) also converges for these values of s, it follows from Theorem 3.7 that

$$\mathcal{H}^s(\mathcal{A}_{n,m}(\psi)) = 0 \quad \text{for} \quad s > m(n-1) + \frac{m+n}{\lambda+1-\varepsilon}.$$

Furthermore, since the above holds for arbitrarily small $\varepsilon > 0$, it follows that

$$\dim_{\mathrm{H}}(\mathcal{A}^{\mathbf{y}}_{n,m}(\psi)) \leq m(n-1) + \frac{m+n}{\lambda+1}. \tag{4.9}$$

Now we turn our attention to investigating when (4.7) diverges. From the definition of lower order at infinity it follows that for any $\varepsilon > 0$ we have

$$\psi(q) \geq q^{-(\lambda+\varepsilon)} \quad \text{for infinitely many } q \in \mathbb{N}.$$

Consequently, for infinitely many $q \in \mathbb{N}$ we have

$$\begin{aligned} q^{n+m-1} \left(\frac{\psi(q)}{q}\right)^{-m(n-1)+s} &\geq q^{n+m-1} (q^{-(\lambda+\varepsilon)-1})^{-m(n-1)+s} \\ &= q^{n+m-1-(\lambda+\varepsilon+1)(s-m(n-1))}. \end{aligned}$$

Thus, if we had, for example,

$$q^{n+m-1-(\lambda+\varepsilon+1)(s-m(n-1))} \geq 1, \tag{4.10}$$

then the divergence of (4.7) would follow. It can be seen that (4.10) holds when

$$s \leq \frac{n+m-1}{\lambda+1+\varepsilon} + m(n-1).$$

Since $\varepsilon > 0$ was arbitrary, it follows from Theorem 3.7 that

$$\mathcal{H}^s(\mathcal{A}^{\mathbf{y}}_{n,m}(\psi)) = \mathcal{H}^s(\mathbb{I}^{nm}) \quad \text{when } s < \frac{n+m-1}{\lambda+1} + m(n-1).$$

Since $\lambda > \frac{n}{m}$ we have

$$\frac{n+m-1}{\lambda+1} + m(n-1) < nm$$

and, hence, it follows that

$$\mathcal{H}^s(\mathcal{A}^{\mathbf{y}}_{n,m}(\psi)) = \infty \quad \text{when } s < \frac{n+m-1}{\lambda+1} + m(n-1).$$

Thus, we conclude that

$$\dim_{\mathrm{H}}(\mathcal{A}^{\mathbf{y}}_{n,m}(\psi)) \geq m(n-1) + \frac{m+n-1}{\lambda+1}. \tag{4.11}$$

The proof of the proposition is complete upon combining the upper and lower bounds given by (4.9) and (4.11), respectively, for the Hausdorff dimension of $\mathcal{A}^{\mathbf{y}}_{n,m}(\psi)$. $\square$

We see that the upper and lower bounds we obtain for $\dim_{\mathrm{H}}(\mathcal{A}^{\mathbf{y}}_{n,m}(\psi))$ in Proposition 4.4 do not coincide. Interestingly, it turns out that these bounds are the best possible if one does not assume monotonicity of ψ — as we will now show. To the best of our knowledge the following result has not been considered before, even in the homogeneous setting.

Theorem 4.5. *Let $m, n \geq 1$ be integers. Let $\alpha > \frac{n}{m}$ be arbitrary and let s_0 be such that*

$$m(n-1) + \frac{m+n-1}{\alpha+1} < s_0 < m(n-1) + \frac{m+n}{\alpha+1}.$$

There exists an approximating function $\psi : \mathbb{N} \to \mathbb{R}^+$ such that for every $\mathbf{y} \in \mathbb{I}^m$ we have $\dim_{\mathrm{H}}(\mathcal{A}^{\mathbf{y}}_{n,m}(\psi)) = s_0$ and $\lambda_\psi = \alpha$ (where λ_ψ is the lower order at infinity of $\frac{1}{\psi}$).

Proof. Fix s_0 satisfying the inequality in the statement of the theorem and let $\mathbf{y} \in \mathbb{I}^m$ be arbitrary. Then, let $J := \{a_k : k \in \mathbb{N}\}$, where $a_k = \lceil k^{-\gamma} \rceil$,

$$\gamma := \frac{2}{n+m-1-(\alpha+1)\left(\frac{n+m}{\beta+1}\right)} \quad \text{and} \quad \beta := \frac{n+m}{s_0 - m(n-1)} - 1.$$

Note that $\gamma < 0$. Define $\psi : \mathbb{N} \to \mathbb{R}^+$ by

$$\psi(q) = \begin{cases} q^{-\alpha} & \text{if} \quad q \in J\,, \\ \\ q^{-\beta} & \text{if} \quad q \notin J. \end{cases}$$

We show that ψ is an approximating function which satisfies the desired properties of the theorem. First, note that

$$m(n-1) + \frac{n+m}{\alpha+1} > s_0,$$

which implies that

$$\frac{n+m}{s_0 - m(n-1)} - 1 > \alpha.$$

In turn, this implies that $\beta > \alpha$ and so $\liminf_{q\to\infty} \frac{-\log(\psi(q))}{\log(q)} = \alpha$, giving $\lambda_\psi = \alpha$, as required.

Recall that if $\lambda_\psi = \alpha$ then for any $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that $\psi(q) \leq q^{-(\alpha-\varepsilon)}$ for all $q \geq N$, and $\psi(q) \geq q^{-(\alpha+\varepsilon)}$ for infinitely many $q \in \mathbb{N}$.

To establish that the Hausdorff dimension is s_0 we note that

$$\dim_{\mathrm{H}}(\mathcal{A}^{\mathbf{y}}_{n,m}(\psi)) \geq \dim_{\mathrm{H}}(\mathcal{A}^{\mathbf{y}}_{n,m}(q \mapsto q^{-\beta}))$$

since $\psi(q) \geq q^{-\beta}$ for all q. Furthermore, since $q \to q^{-\beta}$ is a monotonic function with $\lambda_{(q\mapsto q^{-\beta})} = \beta$, by Theorem 4.1 we have

$$\dim_{\mathrm{H}}(\mathcal{A}^{\mathbf{y}}_{n,m}(q \mapsto q^{-\beta})) = m(n-1) + \frac{m+n}{\beta+1} = s_0.$$

Therefore, $\dim_{\mathrm{H}}(\mathcal{A}^{\mathbf{y}}_{n,m}(\psi)) \geq s_0$ and it remains to show that $\dim_{\mathrm{H}}(\mathcal{A}^{\mathbf{y}}_{n,m}(\psi)) \leq s_0$.

As a consequence of Theorem 3.7 (and Remark 3.9), we only need to verify that for all $\delta > 0$ we have

$$\sum_{q=1}^{\infty} q^{n+m-1}\left(\frac{\psi(q)}{q}\right)^{-m(n-1)+s_0+\delta} < \infty$$

since this would imply that $\mathcal{H}^{s_0+\delta}(\mathcal{A}^{\mathbf{y}}_{n,m}(\psi)) = 0$ and $\dim_{\mathrm{H}}(\mathcal{A}^{\mathbf{y}}_{n,m}(\psi)) \le s_0 + \delta$.

We note that

$$\begin{aligned}
&\sum_{q=1}^{\infty} q^{n+m-1}\left(\frac{\psi(q)}{q}\right)^{-m(n-1)+s_0+\delta} \\
&= \sum_{q\in J} q^{n+m-1}(q^{-\alpha-1})^{-m(n-1)+s_0+\delta} + \sum_{q\notin J} q^{n+m-1}(q^{-\beta-1})^{-m(n-1)+s_0+\delta} \\
&= \sum_{q\in J} q^{n+m-1-(\alpha+1)(s_0+\delta-m(n-1))} + \sum_{q\notin J} q^{n+m-1-(\beta+1)(s_0+\delta-m(n-1))}. \qquad (4.12)
\end{aligned}$$

We consider each of the terms on the right-hand side of (4.12) separately and show that each of them converges. We first consider the second sum on the right-hand side of (4.12). Since $\delta > 0$ we have $s_0 - m(n-1) < s_0 + \delta - m(n-1)$ and hence

$$n + m < \left(\frac{n+m}{s_0 - m(n-1)}\right)(s_0 + \delta - m(n-1)).$$

Recalling that

$$\beta = \frac{n+m}{s_0 - m(n-1)} - 1$$

it follows that

$$n + m - 1 - (\beta+1)(s_0 + \delta - m(n-1)) < -1$$

which is sufficient for the second sum on the right-hand side of (4.12) to converge.

For the first sum on the right-hand side of (4.12) we make the following observations. First of all notice that

$$n + m - 1 - (\alpha+1)\left(\frac{n+m}{\beta+1}\right) = n + m - 1 - (\alpha+1)(s_0 - m(n-1)).$$

Also note that

$$\frac{n+m-1}{\alpha+1} + m(n-1) < s_0 \quad \text{gives} \quad n + m - 1 - (\alpha+1)(s_0 - m(n-1)) < 0.$$

Thus, provided that δ is sufficiently small,

$$\begin{aligned}\sum_{q\in J} q^{n+m-1-(\alpha+1)(s_0+\delta-m(n-1))} &= \sum_{k=1}^{\infty} a_k^{n+m-1-(\alpha+1)(s_0+\delta-m(n-1))} \\ &= \sum_{k=1}^{\infty} \left\lceil k^{-\gamma}\right\rceil^{n+m-1-(\alpha+1)(s_0+\delta-m(n-1))} \\ &\leq \sum_{k=1}^{\infty} \left(k^{-\gamma}\right)^{n+m-1-(\alpha+1)(s_0+\delta-m(n-1))} \end{aligned} \tag{4.13}$$

as $n+m-1-(\alpha+1)(s_0+\delta-m(n-1))<0$ and $\gamma<0$.

Now, for $\delta>0$,

$$\begin{aligned}\frac{2}{\gamma} &= n+m-1-(\alpha+1)(s_0-m(n-1)) \\ &> n+m-1-(\alpha+1)(s_0+\delta-m(n-1)).\end{aligned}$$

Hence,

$$2<\gamma(n+m-1-(\alpha+1)(s_0+\delta-m(n-1)))$$

since $\gamma<0$. Therefore $(k^{-\gamma})^{n+m-1-(\alpha+1)(s_0+\delta-m(n-1))}<k^{-2}$ and so (4.13) converges. Consequently, since both the component sums converge, it follows that (4.12) converges, i.e.

$$\sum_{q=1}^{\infty} q^{n+m-1}\left(\frac{\psi(q)}{q}\right)^{-m(n-1)+s_0+\delta}<\infty,$$

and we conclude that $\dim_{\mathrm{H}}(\mathcal{A}^{\mathbf{y}}_{n,m}(\psi))\leq s_0+\delta$. The desired result follows upon noticing that $\delta>0$ can be taken to be arbitrarily small. $\square$

5 Mass Transference Principles for Rectangles

Another very natural situation, not covered by the setting of systems of linear forms, for which we might hope for some kind of mass transference principle, is when our $\limsup$ sets of interest are defined by sequences of rectangles. Recently some progress has been made in this direction by Wang, Wu and Xu [49]. Results of this kind are of interest, for example, when we consider weighted simultaneous approximation. Before presenting the results of Wang, Wu and Xu in Section 5.2, we will first survey some results in the theory of weighted simultaneous approximation.

We will conclude this chapter by discussing the problem of obtaining a general mass transference principle between $\limsup$ sets defined by rectangles. By combining the idea of "slicing" with either the Mass Transference Principle (Theorem 1.22) or a result of Wang, Wu, and Xu (Theorem 5.7), we will show how we may obtain partial results in this direction.

This chapter is heavily based on [2, Section 4]. In particular, the majority of the material presented in Sections 5.2 and 5.3, including Propositions 5.11 and 5.12 in Section 5.3, appears here as in [2, Section 4]. Any additions or modifications made here have only been done so to improve readability and comprehensiveness.

5.1 Weighted Simultaneous Approximation

Until now, we have been mainly concerned with simultaneous approximation and approximation by systems of linear forms. In other words, we have been interested so far in approximation by balls centred at rational points or approximation by planes. In this chapter we will consider *weighted simultaneous approximation*, that is, essentially, approximation by rectangles. Unlike in the classical simultaneous setting, we now consider approximation where we may require different levels of accuracy of approximation in different coordinate directions. In this setting there exist natural extensions of Dirichlet's Theorem and Khintchine's Theorem (Theorems 1.9 and 1.11).

5.1.1 *Weighted versions of Dirichlet's and Khintchine's Theorems*

Before we proceed to give any statements of theorems, we first establish some notation which will be used throughout this chapter. Suppose $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_k) \in \mathbb{R}^k$ and $\tau_i > 0$ for $1 \leq i \leq k$. We define $W_k(\boldsymbol{\tau})$ to be the set of points $\mathbf{x} \in (x_1, \ldots, x_k) \in \mathbb{I}^k$ such that

$$|qx_i + p_i| < q^{-\tau_i}, \quad 1 \leq i \leq k,$$

for infinitely many pairs $(\mathbf{p}, q) \in \mathbb{Z}^k \times \mathbb{N}$. If the vector $\boldsymbol{\tau}$ satisfies the further properties that

$$0 < \tau_i < 1 \quad \text{for each } 1 \leq i \leq k$$

$$\text{and} \quad \tau_1 + \cdots + \tau_k = 1,$$

then we shall refer to $\boldsymbol{\tau}$ as a *weight vector*. For $\tau > 0$ we will also define $\boldsymbol{\tau}_\tau := (\tau, \ldots, \tau) \in \mathbb{R}^k$. Thus, for example, $W_k(\boldsymbol{\tau}_\tau) = \mathcal{A}_k(\tau)$.

In the setting of weighted approximation, we have the following analogue of Dirichlet's Theorem.

Theorem 5.1 (Weighted Dirichlet's Theorem). *Let $\boldsymbol{\tau} \in \mathbb{R}^k$ be a weight vector. Then, for any $\mathbf{x} \in \mathbb{R}^k$ and $Q \in \mathbb{N}$, there exist $q \in \mathbb{N}$ with $1 \leq q \leq Q$ and $\mathbf{p} \in \mathbb{Z}^k$ such that*

$$|qx_i + p_i| < Q^{-\tau_i}, \quad 1 \leq i \leq k.$$

Remark. Notice that if $\boldsymbol{\tau} = \boldsymbol{\tau}_{\frac{1}{k}}$, then Theorem 5.1 reduces to the standard simultaneous version (Theorem 1.9).

This weighted version of Dirichlet's Theorem is a consequence of Minkowski's Linear Forms Theorem — for a statement of Minkowski's Linear Forms Theorem and details of how it implies Theorem 5.1 we refer the reader to, for example, [17, Chapter III] and [6, Section 1.4.1].

In much the same way as Theorem 1.2 follows from Theorem 1.1 and Theorem 1.10 follows from Theorem 1.9, the following corollary is a consequence of Theorem 5.1.

Corollary 5.2. *Let $\boldsymbol{\tau} \in \mathbb{R}^k$ be a weight vector. Then, for any $\mathbf{x} \in \mathbb{R}^k$ there exist infinitely many pairs $(\mathbf{p}, q) \in \mathbb{Z}^k \times \mathbb{N}$ such that*

$$|qx_i + p_i| < \frac{1}{q^{\tau_i}}, \quad 1 \leq i \leq k.$$

Hence, $W_k(\boldsymbol{\tau}) = \mathbb{I}^k$.

So far, the "weighted" sets we have been dealing with here have essentially been generalisations of the simultaneously τ-approximable points introduced in Section 1.5. We can also consider sets of *weighted ψ-approximable* points. Suppose we are given $\boldsymbol{\tau} \in \mathbb{R}^k$ and an approximating function $\psi : \mathbb{N} \to \mathbb{R}^+$. We will denote by $W_k(\psi, \boldsymbol{\tau})$ the set of points in $\mathbb{I}^k$ which satisfy

$$|qx_i + p_i| < \psi(q)^{\tau_i}, \quad 1 \leq i \leq k,$$

for infinitely many pairs $(\mathbf{p}, q) \in \mathbb{Z}^k \times \mathbb{N}$. Khintchine himself proved an extension of his simultaneous theorem (Theorem 1.11) to the setting of weighted simultaneous approximation.

Theorem 5.3 (Khintchine, [37])**.** *Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be any approximating function and let $\boldsymbol{\tau} \in \mathbb{R}^k$ be a weight vector. Then*

$$|W_k(\psi, \boldsymbol{\tau})| = \begin{cases} 0 & \text{if} \quad \sum_{q=1}^{\infty} \psi(q) < \infty, \\ \\ 1 & \text{if} \quad \sum_{q=1}^{\infty} \psi(q) = \infty \text{ and } \psi \text{ is monotonic.} \end{cases}$$

5.1.2 *The Hausdorff dimension of $W_k(\boldsymbol{\tau})$*

Here we record a general result due to Rynne relating to the Hausdorff dimension of the sets $W_k(\boldsymbol{\tau})$. Suppose Q is an arbitrary infinite set of natural numbers and, given $\boldsymbol{\tau} \in \mathbb{R}^k$ with $\tau_i > 0$ for $i = 1, \ldots, k$, let $W_k^Q(\boldsymbol{\tau})$ denote the set of points $\mathbf{x} \in \mathbb{I}^k$ for which the inequalities

$$|qx_i + p_i| < q^{-\tau_i}, \quad 1 \leq i \leq k, \tag{5.1}$$

hold for infinitely many pairs $(\mathbf{p}, q) \in \mathbb{Z}^k \times Q$, hence $W_k^{\mathbb{N}}(\boldsymbol{\tau}) = W_k(\boldsymbol{\tau})$. Define

$$\nu(Q) = \inf \left\{ \nu \in \mathbb{R} : \sum_{q \in Q} q^{-\nu} < \infty \right\}$$

and let $\sigma(\boldsymbol{\tau}) = \sum_{i=1}^{k} \tau_i$.

Theorem 5.4 (Rynne [44]). *Let* $\boldsymbol{\tau} = (\tau_1, \tau_2, \ldots, \tau_k) \in \mathbb{R}^k$ *be such that* $0 < \tau_1 \leq \tau_2 \leq \cdots \leq \tau_k$. *Let* Q *be an arbitrary infinite subset of* $\mathbb{N}$ *and suppose that* $\sigma(\boldsymbol{\tau}) \geq \nu(Q)$. *Then,*

$$\dim_{\mathrm{H}} W_k^Q(\boldsymbol{\tau}) = \min_{1 \leq j \leq k} \left\{ \frac{k + \nu(Q) + j\tau_j - \sum_{i=1}^{j} \tau_i}{1 + \tau_j} \right\}.$$

Sets such as $W_k(\boldsymbol{\tau})$ and variations on $W_k^Q(\boldsymbol{\tau})$ have been studied in some depth, with particular attention paid to the question of determining their Hausdorff dimension, even before the work of Rynne [44]. For example, consider $\tau \in \mathbb{R}$ for some $\tau > 1$. Then the set $W_1^{\mathbb{N}}(\tau) = W_1(\tau)$ coincides precisely with the set $\mathcal{A}(\tau)$ considered in the Jarník–Besicovitch Theorem (Theorem 1.17). For an overview of some other earlier results of this kind we direct the reader to the discussion given in [44] and references therein.

Given an approximating function $\psi : \mathbb{N} \to \mathbb{R}^+$, we write $W_k^Q(\psi, \boldsymbol{\tau})$ to denote the set of points in $\mathbb{I}^k$ which satisfy

$$|qx_i + p_i| < \psi(q)^{\tau_i}, \quad 1 \leq i \leq k,$$

for infinitely many pairs $(\mathbf{p}, q) \in \mathbb{Z}^k \times Q$. If the approximating function ψ satisfies a certain kind of limiting behaviour, then we can derive the Hausdorff dimension of the set $W_k^Q(\psi, \boldsymbol{\tau})$ as a corollary to Theorem 5.4.

Corollary 5.5 (Rynne [44]). *Let* Q *be an infinite set of positive integers, let* $\boldsymbol{\tau} \in \mathbb{R}^k$ *with* $\tau_i > 0$ *for each* $1 \leq i \leq k$, *and let* $\psi : \mathbb{N} \to \mathbb{R}^+$ *be an approximating function. Assume that the limit*

$$\lambda := \lim_{q \to \infty} \frac{-\log(\psi(q))}{\log q}$$

exists and is positive. Furthermore, suppose that $\sigma(\boldsymbol{\tau}) \geq \frac{\nu(Q)}{\lambda}$. *Then,*

$$\dim_{\mathrm{H}} W_k^Q(\psi, \boldsymbol{\tau}) = \min_{1 \leq j \leq k} \left\{ \frac{k + \nu(Q) + \lambda j\tau_j - \lambda \sum_{i=1}^{j} \tau_i}{1 + \lambda\tau_j} \right\}.$$

Proof. By the definition of λ, for any $\varepsilon > 0$ and each $1 \leq i \leq k$ we have

$$q^{-\lambda\tau_i - \varepsilon} \leq \psi(q)^{\tau_i} \leq q^{-\lambda\tau_i + \varepsilon}$$

for all sufficiently large $q \in Q$. It follows that

$$W_k^Q(\lambda\boldsymbol{\tau} + \boldsymbol{\tau}_\varepsilon) \subset W_k^Q(\psi, \boldsymbol{\tau}) \subset W_k^Q(\lambda\boldsymbol{\tau} - \boldsymbol{\tau}_\varepsilon).$$

On letting $\varepsilon \to 0$, we see that $\dim_{\mathrm{H}} W_k^Q(\psi, \boldsymbol{\tau}) = \dim_{\mathrm{H}} W_k^Q(\lambda\boldsymbol{\tau})$. Finally, taking $\lambda\boldsymbol{\tau}$ in place of $\boldsymbol{\tau}$ in Theorem 5.4, we obtain

$$\dim_{\mathrm{H}} W_k^Q(\lambda\boldsymbol{\tau}) = \min_{1\leq j\leq k} \left\{ \frac{k + \nu(Q) + \lambda j\tau_j - \lambda\sum_{i=1}^{j} \tau_i}{1 + \lambda\tau_j} \right\}$$

and the proof of the corollary is complete. □

5.2 A Mass Transference Principle from Balls to Rectangles

Throughout this section let $k \in \mathbb{N}$ and, as usual, denote by $\mathbb{I}^k$ the unit cube $[0,1]^k$ in $\mathbb{R}^k$. Given a ball $B = B(\mathbf{x}, r)$ in $\mathbb{R}^k$ of radius r centred at $\mathbf{x}$ and a k-dimensional real vector $\mathbf{a} = (a_1, a_2, \ldots, a_k)$ we will denote by $B^{\mathbf{a}}$ the rectangle with centre $\mathbf{x}$ and side-lengths $(r^{a_1}, r^{a_2}, \ldots, r^{a_k})$. Given a sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ of points in $\mathbb{I}^k$ and a sequence $(r_n)_{n\in\mathbb{N}}$ of positive real numbers such that $r_n \to 0$ as $n \to \infty$ we define

$$W_0 = \{\mathbf{x} \in \mathbb{I}^k : \mathbf{x} \in B_n := B(\mathbf{x}_n, r_n) \text{ for infinitely many } n \in \mathbb{N}\}.$$

For any $\mathbf{a} \in \mathbb{R}^k$ we will also write

$$W_{\mathbf{a}} = \{\mathbf{x} \in \mathbb{I}^k : \mathbf{x} \in B_n^{\mathbf{a}} \text{ for infinitely many } n \in \mathbb{N}\}.$$

In [49], Wang, Wu and Xu established the following mass transference principle.

Theorem 5.6 (Wang – Wu – Xu [49]). *Let $(\mathbf{x}_n)_{n\in\mathbb{N}}$ be a sequence of points in $\mathbb{I}^k$ and $(r_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers such that $r_n \to 0$ as $n \to \infty$. Let $\mathbf{a} = (a_1, a_2, \ldots, a_k) \in \mathbb{R}^k$ be such that $1 \leq a_1 \leq a_2 \leq \cdots \leq a_k$. Suppose that $|W_0| = 1$. Then,*

$$\dim_{\mathrm{H}} W_{\mathbf{a}} \geq \min_{1\leq j\leq k} \left\{ \frac{k + ja_j - \sum_{i=1}^{j} a_i}{a_j} \right\}.$$

Furthermore, if we have the additional constraint $a_d > 1$, Wang, Wu and Xu are also able to say something about the Hausdorff measure of $W_{\mathbf{a}}$ at the critical value

$$s := \min_{1 \le j \le k} \left\{ \frac{k + ja_j - \sum_{i=1}^{j} a_i}{a_j} \right\}. \tag{5.2}$$

Theorem 5.7 (Wang – Wu – Xu [49])**.** *Assume the same conditions as in Theorem 5.6. If the additional constraint that $a_d > 1$ holds, then*

$$\mathcal{H}^s(W_{\mathbf{a}}) = \infty.$$

Essentially, the results of Wang, Wu and Xu allow us to pass from a full Lebesgue measure statement for a lim sup set defined by a sequence of balls to a Hausdorff measure statement for a lim sup set defined by an associated sequence of rectangles. As an application, Wang, Wu and Xu demonstrate how Theorem 5.6 may be applied to obtain the Hausdorff dimension of the sets $W_k(\boldsymbol{\tau})$ of weighted simultaneously well-approximable points. The following is derived in [49] as a corollary to Theorem 5.6.

Corollary 5.8 (Wang – Wu – Xu [49])**.** *Let $\boldsymbol{\tau} = (\tau_1, \tau_2, \dots, \tau_k) \in \mathbb{R}^k$ be such that $\frac{1}{k} \le \tau_1 \le \tau_2 \le \dots \le \tau_k$, then*

$$\dim_{\mathrm{H}}(W_k(\boldsymbol{\tau})) = \min_{1 \le j \le k} \left\{ \frac{k + 1 + j\tau_j - \sum_{i=1}^{j} \tau_i}{1 + \tau_j} \right\}.$$

Proof. We first obtain an upper bound for the dimension of $W_k(\boldsymbol{\tau})$. For this we make use of a fairly standard covering argument and do not require Theorem 5.6. For $(\mathbf{p}, q) = (p_1, p_2, \dots, p_k, q) \in \mathbb{Z}^k \times \mathbb{N}$, let

$$E(\mathbf{p}, q) := \left\{ \mathbf{x} = (x_1, \dots, x_k) \in \mathbb{I}^k : \left| x_i - \frac{p_i}{q} \right| < \frac{1}{q^{\tau_i + 1}} \text{ for each } 1 \le i \le k \right\}.$$

Then,

$$W_k(\boldsymbol{\tau}) = \bigcap_{Q=1}^{\infty} \bigcup_{q \ge Q} \bigcup_{\substack{\mathbf{p} \in \mathbb{Z}^k: \\ E(\mathbf{p},q) \ne \emptyset}} E(\mathbf{p}, q).$$

Note that for a given $q \in \mathbb{N}$ there will be $\ll (q+1)^k$ rectangles $E(\mathbf{p}, q)$ which are non-empty.

Now, for a fixed $1 \leq j \leq k$, we can cover a rectangle $E(\mathbf{p}, q)$ using

$$\prod_{i=1}^{j} \frac{q^{-1-\tau_i}}{q^{-1-\tau_j}}$$

cubes of side-length $2q^{-1-\tau_j}$.

Thus, given $\rho > 0$ and letting $Q(\rho) \in \mathbb{N}$ be such that $q^{-1-\tau_j} < \rho$ for all $q \geq Q(\rho)$, we see that

$$\begin{aligned}
\mathcal{H}^s_\rho(W_k(\boldsymbol{\tau})) &\ll \sum_{q \geq Q(\rho)} (q+1)^k \prod_{i=1}^{j} \frac{q^{-1-\tau_i}}{q^{-1-\tau_j}} q^{-(1+\tau_j)s} \\
&\ll \sum_{q \geq Q(\rho)} q^{k+\sum_{i=1}^{j}(\tau_j - \tau_i) - s(1+\tau_j)} \\
&\ll \sum_{q=1}^{\infty} q^{k+j\tau_j - \sum_{i=1}^{j}\tau_i - s(1+\tau_j)}.
\end{aligned}$$

If $s > \frac{k+1+j\tau_j - \sum_{i=1}^{j}\tau_i}{1+\tau_j}$, the above sum converges and so, on letting $\rho \to 0$, for such values of s we see that $\mathcal{H}^s(W_k(\boldsymbol{\tau})) = 0$. Hence, $\dim_{\mathrm{H}} W_k(\boldsymbol{\tau}) \leq \frac{k+1+j\tau_j - \sum_{i=1}^{j}\tau_i}{1+\tau_j}$.

Finally, since the above argument holds with any choice of $1 \leq j \leq k$ we conclude that

$$\dim_{\mathrm{H}} W_k(\boldsymbol{\tau}) \leq \min_{1 \leq j \leq k} \left\{ \frac{k+1+j\tau_j - \sum_{i=1}^{j}\tau_i}{1+\tau_j} \right\}.$$

Next, we turn our attention to establishing a lower bound for the Hausdorff dimension of $W_k(\boldsymbol{\tau})$. For this we will appeal to Theorem 5.6. Let S be the set of pairs $(\mathbf{p}, q) \in \mathbb{Z}^k \times \mathbb{N}$ with $0 \leq p_i \leq q$ for $1 \leq i \leq k$. For each $(\mathbf{p}, q) \in S$, let

$$B_{(\mathbf{p},q)} = B\left(\frac{\mathbf{p}}{q}, \frac{1}{q^{1+\frac{1}{k}}}\right).$$

From Theorem 1.10 we see that

$$\left| \limsup_{(\mathbf{p},q) \in S} B_{(\mathbf{p},q)} \right| = |\mathcal{A}_k(q \mapsto q^{-\frac{1}{k}})| = 1.$$

If we take $\mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{R}^k$ to be the vector with components $a_i = \frac{k(1+\tau_i)}{1+k}$ for each $1 \leq i \leq k$ then

$$B^{\mathbf{a}}_{(\mathbf{p},q)} = \left\{ \mathbf{x} \in \mathbb{R}^k : \left| x_i - \frac{p_i}{q} \right| < \frac{1}{q^{1+\tau_i}} \text{ for each } 1 \leq i \leq k \right\}.$$

In particular,

$$\limsup_{(\mathbf{p},q)\in S} B^{\mathbf{a}}_{(\mathbf{p},q)} \subset W_k(\boldsymbol{\tau}).$$

Applying Theorem 5.6 with the vector $\mathbf{a}$ specified above we obtain

$$\begin{aligned}
\dim_{\mathrm{H}} W_k(\boldsymbol{\tau}) &\geq \dim_{\mathrm{H}} \left(\limsup_{(\mathbf{p},q)\in S} B^{\mathbf{a}}_{(\mathbf{p},q)} \right) \\
&\geq \min_{1\leq j\leq k} \left\{ \frac{k + ja_j - \sum_{i=1}^{j} a_i}{a_j} \right\} \\
&= \min_{1\leq j\leq k} \left\{ \frac{k + j\left(\frac{k(1+\tau_j)}{1+k}\right) - \sum_{i=1}^{j}\left(\frac{k(1+\tau_i)}{1+k}\right)}{\frac{k(1+\tau_j)}{1+k}} \right\} \\
&= \min_{1\leq j\leq k} \left\{ \frac{1 + k + j\tau_j - \sum_{i=1}^{j} \tau_i}{1+\tau_j} \right\}.
\end{aligned}$$

Finally, observing that this lower bound for the Hausdorff dimension of $W_k(\boldsymbol{\tau})$ coincides with the upper bound obtained earlier, we conclude that

$$\dim_{\mathrm{H}} W_k(\boldsymbol{\tau}) = \min_{1\leq j\leq k} \left\{ \frac{1 + k + j\tau_j - \sum_{i=1}^{j} \tau_i}{1+\tau_j} \right\},$$

as required. □

While the proof of Corollary 5.8 given in [49] is novel, and is a neat application of Theorem 5.6, the result itself was already previously known. In fact, Corollary 5.8 is a special case of the earlier more general theorem due to Rynne [44] cited in Section 5.1.2 — we may easily recover Corollary 5.8 by taking $Q = \mathbb{N}$ in Theorem 5.4 and noting that $\nu(\mathbb{N}) = 1$. Since the hypotheses of Corollary 5.8 demand that $\tau_i \geq \frac{1}{k}$ for all $1 \leq i \leq k$, we see that the condition $\sigma(\boldsymbol{\tau}) \geq \nu(Q)$ in Theorem 5.4 is also satisfied.

5.3 Mass Transference Principles from Rectangles to Rectangles

The original Mass Transference Principle (Theorem 1.22) allows us to transition from Lebesgue to Hausdorff measure statements when our original and "transformed" lim sup sets are defined by sequences of balls, i.e. it allows us to go from "balls to balls". Theorem 5.6 allows us to go from "balls to rectangles". Another goal which we might like to achieve, which is not covered by any of the frameworks mentioned so far, would be to prove a similar mass transference principle where we both start and finish with lim sup sets arising from sequences of rectangles, i.e. from "rectangles to rectangles".

Problem 5.9. *Does there exist a mass transference principle, similar to Theorem 1.22 or Theorem 5.6, where both the original and transformed* lim sup *sets are defined by sequences of rectangles?*

Although in the most general settings this problem remains open, we consider what can be said in a few special cases.

We first observe [1] that if we are in a metric space (X, d), satisfying the hypotheses of Theorem 1.23, where the balls with respect to d are actually rectangles, then Theorem 1.23 does gives us a kind of mass transference principle from rectangles to rectangles. The disadvantage with such a statement obtained in this way, though, is that the "shape" of the rectangles must be preserved between the associated original and transformed lim sup sets. Furthermore, the Hausdorff measure statement obtained will relate to Hausdorff measure with respect to the particular metric with which the space X is equipped.

In [8] Beresnevich and Velani employ a "slicing" technique, which uses a combination of a slicing lemma and the original Mass Transference Principle, to prove Theorem 2.1. We show how an appropriate combination of these two results can also be applied to considering the problem of proving a mass transference principle for rectangles. We proceed by stating the "Slicing Lemma" as given by Beresnevich and Velani in [8].

[1] The author is indebted to Henna Koivusalo for many conversations about rectangles while she was a postdoc in York and, in particular, thanks her for making the observation recorded above.

Lemma 5.10 (Slicing Lemma [8]). *Let $l, k \in \mathbb{N}$ be such that $l \leq k$ and let f and $g : r \to r^{-l} f(r)$ be dimension functions. Let $A \subset \mathbb{R}^k$ be a Borel set and let V be a $(k-l)$-dimensional linear subspace of $\mathbb{R}^k$. If for a subset S of $V^{\perp}$ of positive $\mathcal{H}^l$-measure*

$$\mathcal{H}^g(A \cap (V + b)) = \infty \quad \textit{for all } b \in S,$$

then $\mathcal{H}^f(A) = \infty$.

Suppose that $(\mathbf{x}_n)_n = (x_{n,1}, x_{n,2}, \ldots, x_{n,k})_n$ is a sequence of points in $\mathbb{I}^k$. Let $(r_n^1)_n, (r_n^2)_n, \ldots, (r_n^k)_n$ be sequences of positive real numbers and suppose that $r_n^1 \to 0$ as $n \to \infty$. Let

$$H_n = \prod_{i=1}^{k} B(x_{n,i}, r_n^i)$$

be a sequence of rectangles in $\mathbb{I}^k$, where $\prod_{i=1}^k A_i = A_1 \times A_2 \times \cdots \times A_k$ is the Cartesian product of subsets A_i of $\mathbb{R}^k$. Let $\alpha > 1$ be a real number and define another sequence of rectangles by

$$h_n = B(x_{n,1}, (r_n^1)^{\alpha}) \times \prod_{i=2}^{k} B(x_{n,i}, r_n^i).$$

So, h_n is essentially a "shrunk" rectangle corresponding to H_n from the original sequence. Note that in this case we only allow shrinking of the original rectangle in one direction. Then, we are able to establish the following.

Proposition 5.11. *Let the sequences H_n and h_n be as given above and further suppose that $|\limsup_{n\to\infty} H_n| = 1$. Then,*

$$\dim_{\mathrm{H}} \left(\limsup_{n\to\infty} h_n \right) \geq \frac{1}{\alpha} + k - 1.$$

Proof. Let $V = \{\mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{I}^k : x_i = 0 \text{ for all } i \neq 1\}$. Since $|\limsup_{n\to\infty} H_n| = 1$, for Lebesgue almost every

$$\mathbf{b} \in \{\mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{I}^k : x_1 = 0\}$$

we have

$$|(V + \mathbf{b}) \cap \limsup_{n\to\infty} H_n| = 1.$$

Let us fix a $\mathbf{b}$ for which this holds and let $W = V + \mathbf{b}$. Now, $\limsup_{n\to\infty} H_n \cap W$ can be written as the lim sup set of a sequence of balls $B_j = B(x_{n_j,1}, r_{n_j}^1)$ with radii $r_{n_j}^1$.

Note that $|\limsup_{j\to\infty} B_j \cap W| = 1$. For each j also let $b_j = B(x_{n_j,1}, (r^1_{n_j})^\alpha)$ and note that

$$\limsup_{j\to\infty} b_j \cap W = \limsup_{n\to\infty} h_n \cap W.$$

In accordance with our earlier notation, $b^s_j = B(x_{n_j,1}, (r^1_{n_j})^{\alpha s})$. Therefore, if $s \leq \frac{1}{\alpha}$ then $(r^1_{n_j})^{\alpha s} \geq r^1_{n_j}$ for sufficiently large j and so

$$b^s_j \supset B_j \quad \text{and} \quad |\limsup_{j\to\infty} b^s_j \cap W| = 1.$$

Thus, for any $s \leq \frac{1}{\alpha}$ we may use the Mass Transference Principle (Theorem 1.22) to conclude that for any ball $B \subset W$ we have

$$\mathcal{H}^s(\limsup_{j\to\infty} b_j \cap B) = \mathcal{H}^s(B).$$

In particular, since $s \leq \frac{1}{\alpha} < 1$, this means

$$\mathcal{H}^s(\limsup_{n\to\infty} h_n \cap W) = \mathcal{H}^s(W) = \infty.$$

Since this is the case for Lebesgue almost every $\mathbf{b} \in \{\mathbf{x} = (x_1, \ldots, x_k) : x_1 = 0\}$ we can use the Slicing Lemma (Lemma 5.10) to conclude that

$$\mathcal{H}^{s'}(\limsup_{n\to\infty} h_n) = \infty$$

for all $s' \leq \frac{1}{\alpha} + k - 1$. Therefore, it follows that

$$\dim_{\mathrm{H}}\left(\limsup_{n\to\infty} h_n\right) \geq \frac{1}{\alpha} + k - 1. \qquad \square$$

Using Theorem 5.7 in place of Theorem 1.22, we are actually able to extend this argument a little further. Again, let $(\mathbf{x}_n)_n = (x_{n,1}, x_{n,2}, \ldots, x_{n,k})_n$ be a sequence of points in $\mathbb{I}^k$ and let $(r^1_n)_n, (r^2_n)_n, \ldots, (r^k_n)_n$ be sequences of positive real numbers. Suppose that for some $1 \leq k_0 \leq k$ we have $r^1_n = r^2_n = \cdots = r^{k_0}_n$ for all $n \in \mathbb{N}$ and also that $r^1_n \to 0$ as $n \to \infty$. Let

$$H_n = \prod_{i=1}^{k} B(x_{n,i}, r^i_n)$$

be a sequence of rectangles in $\mathbb{I}^k$. Next, let $1 \leq a_1 \leq a_2 \leq \cdots \leq a_{k_0}$ be real numbers and suppose $a_{k_0} > 1$. For each rectangle H_n in our original sequence we define a corresponding "shrunk" rectangle

$$h_n = \prod_{i=1}^{k_0} B(x_{n,i}, (r_n^i)^{a_i}) \times \prod_{i=k_0+1}^{k} B(x_{n,i}, r_n^i).$$

In this case we are able to prove the following.

Proposition 5.12. *Let the sequences of rectangles H_n and h_n be as given above and further suppose that $|\limsup_{n\to\infty} H_n| = 1$. Then,*

$$\dim_{\mathrm{H}}\left(\limsup_{n\to\infty} h_n\right) \geq \min_{1\leq j\leq k_0}\left\{\frac{k_0 + ja_j - \sum_{i=1}^{j} a_i}{a_j} + k - k_0\right\}.$$

Proof. Let $V = \{\mathbf{x} = (x_1, x_2, \ldots, x_k) \in \mathbb{I}^k : x_i = 0 \text{ for all } i \geq k_0 + 1\}$. Since $|\limsup_{n\to\infty} H_n| = 1$, for almost every

$$\mathbf{b} \in \{\mathbf{x} = (x_1, x_2, \ldots, x_k) \in \mathbb{I}^k : x_i = 0 \text{ for all } i \leq k_0\}$$

we have

$$|(V + \mathbf{b}) \cap \limsup_{n\to\infty} H_n| = 1.$$

Let us fix a $\mathbf{b}$ for which this holds and let $W = V + \mathbf{b}$. As before, $\limsup_{n\to\infty} H_n \cap W$ can be written as a sequence of k_0-dimensional balls $B_j = B(\mathbf{x}_{n_j}^{k_0}, r_{n_j}^1)$ with radii $r_{n_j}^1 (= r_{n_j}^2 = \cdots = r_{n_j}^{k_0})$ and centres $\mathbf{x}_{n_j}^{k_0} := (x_{n_j,1}, x_{n_j,2}, \ldots, x_{n_j,k_0})$. Note that $|\limsup_{j\to\infty} B_j \cap W| = 1$.

This time, for each j let

$$b_j = \prod_{i=1}^{k_0} B(x_{n_j,i}, (r_{n_j}^i)^{a_i})$$

and note that

$$\limsup_{j\to\infty} b_j \cap W = \limsup_{n\to\infty} h_n \cap W.$$

By Theorem 5.7 it follows that

$$\mathcal{H}^s(\limsup_{n\to\infty} h_n \cap W) = \infty$$

where

$$s := \min_{1\leq j\leq k_0}\left\{\frac{k_0 + ja_j - \sum_{i=1}^{j} a_i}{a_j}\right\}.$$

Since this is the case for almost every

$$\mathbf{b} \in \{\mathbf{x} = (x_1, x_2, \ldots, x_k) \in \mathbb{I}^k : x_i = 0 \text{ for all } i \leq k_0\}$$

we may use Lemma 5.10 (with $l = k - k_0$) to conclude that

$$\mathcal{H}^{s'}(\limsup_{n\to\infty} h_n) = \infty$$

where

$$s' := \min_{1\leq j\leq k_0} \left\{ \frac{k_0 + ja_j - \sum_{i=1}^{j} a_i}{a_j} + k - k_0 \right\}.$$

Hence

$$\dim_{\mathrm{H}} \left(\limsup_{n\to\infty} h_n \right) \geq s',$$

as required. □

A disadvantage of using the "slicing" arguments above is that we have to impose quite strict conditions on both the original and transformed rectangles. Namely, the sides of the original rectangle which are permitted to "shrink" have to be of the same initial length (but can shrink at different rates). Meanwhile, the rest of the sides of the original rectangle are not allowed to shrink at all when passing to the corresponding transformed rectangle. We conclude this section by considering one more situation where all sides of the original rectangles may have different lengths and are all allowed to shrink in a specified manner. Let

$$H_n = \prod_{i=1}^{k} B(\mathbf{x}_{n,i}, r_n^{t_i})$$

be a sequence of rectangles in $\mathbb{I}^k$ with $1 \leq t_i$ for $1 \leq i \leq k$ and $r_n \to 0$.

Let the corresponding shrunk rectangles be defined as

$$h_n = \prod_{i=1}^{k} B(\mathbf{x}_{n,i}, r_n^{a_i t_i}),$$

where $1 \leq a_i$ for $1 \leq i \leq k$. Suppose without loss of generality that $1 \leq a_1 t_1 \leq a_2 t_2 \leq \cdots \leq a_k t_k$. Furthermore, suppose that

$$D := \inf \left\{ d \in \mathbb{R} : \sum_{n=1}^{\infty} r_n^d < \infty \right\}.$$

By using the "natural" covers of $\limsup_{n\to\infty} h_n$ we can get an upper bound for the Hausdorff dimension of this lim sup set.

Observation 5.13. *Let the sequence of rectangles $(h_n)_n$ be as defined above, then*

$$\dim_{\mathrm{H}}\left(\limsup_{n\to\infty} h_n\right) \le \min_{1\le j\le k}\left\{\frac{D + ja_jt_j - \sum_{i=1}^{j} a_it_i}{a_jt_j}\right\}. \tag{5.3}$$

Proof. To see this we first note that for a fixed $1 \le j \le k$ the rectangle h_n may be covered by

$$\prod_{i=1}^{j} \frac{r_n^{a_it_i}}{r_n^{a_jt_j}}$$

balls (i.e. cubes in this case) of radius $r_n^{a_jt_j}$.

Given $\rho > 0$, let $N(\rho) \in \mathbb{N}$ be such that $r_n^{a_jt_j} < \rho$ for all $n \ge N(\rho)$. It follows that

$$\mathcal{H}^s_\rho\left(\limsup_{n\to\infty} h_n\right) \le \sum_{n\ge N(\rho)}\left(\prod_{i=1}^{j}\frac{r_n^{a_it_i}}{r_n^{a_jt_j}} \times r_n^{a_jt_js}\right) = \sum_{n\ge N(\rho)} r_n^{\sum_{i=1}^{j}(a_it_i - a_jt_j) + a_jt_js}.$$

By the definition of D, the above sum converges if

$$s > \frac{D + ja_jt_j - \sum_{i=1}^{j} a_it_i}{a_jt_j}.$$

Thus, letting $\rho \to 0$, for such values of s we see that $\mathcal{H}^s(\limsup_{n\to\infty} h_n) = 0$. In particular, this means that

$$\dim_{\mathrm{H}}\left(\limsup_{n\to\infty} h_n\right) \le \frac{D + ja_jt_j - \sum_{i=1}^{j} a_it_i}{a_jt_j}.$$

Since this argument is valid for any $1 \le j \le k$, we conclude that

$$\dim_{\mathrm{H}}\left(\limsup_{n\to\infty} h_n\right) \le \min_{1\le j\le k}\left\{\frac{D + ja_jt_j - \sum_{i=1}^{j} a_it_i}{a_jt_j}\right\}. \qquad \square$$

This observation leads us to contemplate the following problem.

Problem 5.14. *Under what conditions do we get a lower bound for $\dim_{\mathrm{H}}(\limsup_{n\to\infty} h_n)$ which coincides with the upper bound given by (5.3)?*

List of References

[1] D. Allen, V. Beresnevich, *A mass transference principle for systems of linear forms and its applications*, Compos. Math. 154 (2018), no. 5, 1014–1047.

[2] D. Allen, S. Troscheit, *The Mass Transference Principle: Ten Years On*, to appear in *Horizons of Fractal Geometry and Complex Dimensions*, AMS Contemporary Mathematics Series.

[3] T. M. Apostol, *Introduction to analytic number theory*, Undergraduate Texts in Mathematics, Springer-Verlag, New York-Heidelberg, 1976.

[4] V. Beresnevich, V. Bernik, M. Dodson, S. Velani, *Classical metric Diophantine approximation revisited*, Analytic number theory, 38–61, Cambridge Univ. Press, Cambridge, 2009.

[5] V. Beresnevich, D. Dickinson, S. Velani, *Measure theoretic laws for lim sup sets*, Mem. Amer. Math. Soc. 179 (2006), no. 846, x+91 pp.

[6] V. Beresnevich, F. Ramírez, S. Velani, *Metric Diophantine Approximation: aspects of recent work*, Dynamics and Analytic Number Theory, 1–95, London Math. Soc. Lecture Note Ser., 437, Cambridge Univ. Press, Cambridge, 2016.

[7] V. Beresnevich, S. Velani, *A Mass Transference Principle and the Duffin–Schaeffer conjecture for Hausdorff measures*, Ann. of Math. (2) 164 (2006), no. 3, 971–992.

[8] V. Beresnevich, S. Velani, *Schmidt's theorem, Hausdorff measures, and slicing*, Int. Math. Res. Not. IMRN 2006, Art. ID 48794, 24 pp.

[9] V. Beresnevich, S. Velani, *A note on zero-one laws in metrical Diophantine approximation*, Acta Arith. 133 (2008), no. 4, 363–374.

[10] V. Beresnevich, S. Velani, *Classical metric Diophantine approximation revisited: the Khintchine–Groshev theorem*, Int. Math. Res. Not. IMRN 2010, no. 1, 69–86.

[11] V. I. Bernik, M. M. Dodson, *Metric Diophantine approximation on manifolds*, Cambridge Tracts in Mathematics, 137, Cambridge University Press, Cambridge, 1999.

[12] A. S. Besicovitch, *On the sum of digits of real numbers represented in the dyadic system*, Math. Ann. 110 (1935), no. 1, 321–330.

[13] J. D. Bovey, M. M. Dodson, *The Hausdorff dimension of systems of linear forms*, Acta Arith. 45 (1986), no. 4, 337–358.

[14] Y. Bugeaud, *Approximation by algebraic numbers*, Cambridge Tracts in Mathematics, 160, Cambridge Univ. Press, Cambridge, 2004.

[15] J. W. S. Cassels, *Some metrical theorems in Diophantine approximation. I.*, Proc. Cambridge Philos. Soc. 46 (1950), 209–218.

[16] J. W. S. Cassels, *An introduction to Diophantine approximation*, Facsimile reprint of the 1957 edition. Cambridge Tracts in Mathematics and Mathematical Physics, No. 45. Hafner Publishing Co., New York, 1972.

[17] J. W. S. Cassels, *An introduction to the geometry of numbers*, Corrected reprint of the 1971 edition. Classics in Mathematics. Springer-Verlag, Berlin, 1997.

[18] S. G. Dani, M. Laurent, A. Nogueira, *Multi-dimensional metric approximation by primitive points*, Math. Z. 279 (2015), no. 3-4, 1081–1101.

[19] H. Dickinson, *The Hausdorff dimension of systems of simultaneously small linear forms*, Mathematika 40 (1993), no. 2, 367–374.

[20] D. Dickinson, M. Hussain, *The metric theory of mixed type linear forms*, Int. J. Number Theory 9 (2013), no. 1, 77–90.

[21] D. Dickinson, S. Velani, *Hausdorff measure and linear forms*, J. Reine Angew. Math. 490 (1997), 1–36.

[22] G. L. Dirichlet, *Verallgemeinerung eines Satzes aus der Lehre von den Kettenbrüchen nebst einigen Anwendungen auf die Theorie der Zahlen*, S. B. Preuss. Akad. Wiss. (1842), 93–95.

[23] M. M. Dodson, *Hausdorff dimension, lower order and Khintchine's theorem in metric Diophantine approximation*, J. Reine Angew. Math. 432 (1992), 69–76.

[24] M. M. Dodson, B. P. Rynne, J. A. G. Vickers, *Diophantine approximation and a lower bound for Hausdorff dimension*, Mathematika 37 (1990), no. 1, 59–73.

[25] R. J. Duffin, A. C. Schaeffer, *Khintchine's problem in metric Diophantine approximation*, Duke Math. J. 8 (1941), 243–255.

[26] K. Falconer, *Fractal geometry: Mathematical foundations and applications*, Second edition, John Wiley & Sons, 2003.

[27] P. Gallagher, *Approximation by reduced fractions*, J. Math. Soc. Japan 13 (1961), 342–345.

[28] P. X. Gallagher, *Metric simultaneous diophantine approximation II*, Mathematika 12 (1965), 123–127.

[29] A. Groshev, *A theorem on a system of linear forms*, Dokl. Akad. Nauk SSSR, 19, (1938), 151–152, (in Russian).

[30] G. Harman, *Metric Number Theory*, London Mathematical Society Monographs. New Series, 18. The Clarendon Press, Oxford University Press, New York, 1998.

[31] M. Hussain, S. Kristensen, *Metrical results on systems of small linear forms*, Int. J. Number Theory 9 (2013), no. 3, 769–782.

[32] M. Hussain, J. Levesley, *The metrical theory of simultaneously small linear forms*, Funct. Approx. Comment. Math. 48 (2013), part 2, 167–181.

[33] V. Jarník, *Diophantischen Approximationen und Hausdorffsches Mass*, Mat. Sbornik, 36, (1929), 371–382.

[34] V. Jarník, *Über die simultanen diophantischen Approximationen*, Math. Z. 33 (1931), no. 1, 505–543.

[35] A. Khintchine, *Einige Sätze über Kettenbrüche, mit Anwendungen auf die Theorie der Diophantischen Approximationen*, Math. Ann. 92 (1924), 115–125.

[36] A. Khintchine, *Über die angenäherte Auflösung linearer Gleichungen in ganzen Zahlen*, Rec. Math. Soc. Moscou, 32 (1925), 203–218.

[37] A. Khintchine, *Zur metrischen Theorie der diophantischen Approximationen*, Math. Z. 24 (1926), no. 1, 706–714.

[38] J. Levesley, *A general inhomogeneous Jarnik–Besicovitch theorem*, J. Number Theory 71 (1998), no. 1, 65–80.

[39] J. Levesley, C. Salp, S. Velani, *On a problem of K. Mahler: Diophantine approximation and Cantor sets*, Math. Ann. 338 (2007), no. 1, 97–118.

[40] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces: Fractals and rectifiability*, Cambridge Studies in Advanced Mathematics, 44, Cambridge University Press, Cambridge, 1995.

[41] A. D. Pollington, R. C. Vaughan, *The k-dimensional Duffin and Schaeffer conjecture*, Mathematika 37 (1990), no. 2, 190–200.

[42] F. Ramírez, *Counterexamples, covering systems, and zero-one laws for inhomogeneous approximation*, Int. J. Number Theory 13 (2017), no. 3, 633–654.

[43] C. A. Rogers, *Hausdorff measures*, reprint of the 1970 original. Cambridge University Press, Cambridge, 1998.

[44] B. P. Rynne, *Hausdorff dimension and generalized simultaneous Diophantine approximation*, Bull. London Math. Soc. 30 (1998), no. 4, 365–376.

[45] W. M. Schmidt, *A metrical theorem in diophantine approximation*, Canad. J. Math. 12 (1960), 619–631.

[46] W. M. Schmidt, *Diophantine approximation*, Lecture Notes in Mathematics, 785, Springer, Berlin, 1980.

[47] V. G. Sprindžuk, *Metric theory of Diophantine approximations*, (translated by R. A. Silverman). Scripta Series in Mathematics, V. H. Winston & Sons, Washington D.C., 1979.

[48] R. Thorn, *Metric Number Theory: the good and the bad*, PhD Thesis (2005), Queen Mary, University of London.

[49] B. Wang, J. Wu, J. Xu, *Mass transference principle for limsup sets generated by rectangles*, Math. Proc. Cambridge Philos. Soc. 158 (2015), no. 3, 419–437.