Binomial Rings and their Cohomology

Shadman Rahman Kareem

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Supervisor: Prof. Sarah Whitehouse

University of Sheffield

Abstract

A binomial ring is a $\mathbb{Z}$-torsion free commutative ring R, in which all the binomial operations $\binom{r}{n} = \dfrac{r(r-1)(r-2)\cdots(r-(n-1))}{n!} \in R\otimes\mathbb{Q}$, actually lie in R, for all r in R and $n \geq 0$. It is a special type of λ-ring in which the Adams operations on it all are the identity and the λ-operations are given by the binomial operations. This thesis studies the algebraic properties of binomial rings, considers examples from topology and begins a study of their cohomology. The first two chapters give an introduction and some background material.

In Chapter 3 and Chapter 4 we study the algebraic structure and properties of binomial rings, focusing on the notion of a binomial ideal in a binomial ring. We study some classes of binomial rings. We show that the ring of integers $\mathbb{Z}$ is a binomially simple ring. We give a characterisation of binomial ideals in the ring of integer valued-polynomials $\mathrm{Int}(\mathbb{Z}^{\{x\}})$. We apply this to prove that $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ is a binomially principal ring and rings of polynomials that are integer valued on a subset of the integers are also binomially principal rings. Also, we prove that the ring $\mathrm{Int}(\mathbb{Z}^{\{x,y\}})$ of integer-valued polynomials on two variables is a binomially Noetherian ring.

The ring $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ and its dual appear as certain rings of operations and cooperations in topological K-theory. We give some non-trivial examples of binomial rings that come from topology such as stably integer-valued Laurent polynomials $\mathrm{SLInt}(\mathbb{Z}^{\{x\}})$ on one variable and stably integer-valued polynomials $\mathrm{SInt}(\mathbb{Z}^{\{x\}})$ on one variable. We study generalisations of these rings to a set X of variables. We show that in the one variable case both rings are binomially principal rings and in the case of finitely many variables both are binomially Noetherian rings. As a main result we give new descriptions of these examples.

In Chapter 5 and Chapter 6 we define cohomology of binomial rings as an example of a cotriple cohomology theory on the category of binomial rings. To do so, we study binomial modules and binomial derivations. Our cohomology has coefficients given by the contravariant functor $\mathrm{Der}_{\mathrm{Bin}}(-, M)$, of binomial derivations to a binomial module M. We give some examples of binomial module structures and calculate derivations for these examples. We define homomorphisms connecting the cohomology of binomial rings to the cohomology of λ-rings and to the André-Quillen cohomology of the underlying commutative rings.

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Contents

Chapter 1

Introduction

The notion of a binomial ring was originally introduced by Hall [29] in connection with his work in the theory of nilpotent groups. A binomial ring is a commutative ring R with unit whose additive group is $\mathbb{Z}$-torsion free such that all the binomial operations

$$\binom{r}{n} = \frac{r(r-1)(r-2)\cdots(r-(n-1))}{n!} \in R \otimes \mathbb{Q}$$

actually in R, for all r in R and $n \geq 0$. There is another important type of ring called a λ-ring which is a commutative ring R with unit equipped with a sequence of functions

$$\lambda^n : R \longrightarrow R,$$

for all $n \geq 0$, called λ-operations, which satisfy certain relations that are satisfied by the binomial operations. These rings were originally introduced in algebraic geometry by Grothendieck [28] in his work in Riemann-Roch theory. The λ-operations are not group homomorphisms. Their action on sums is given by

$$\lambda^n(x+y) = \sum_{i+j=n} \lambda^i(x)\lambda^j(y),$$

for all x, y in a λ-ring. Adams [2] introduced other operations on commutative rings to study vector fields on spheres, $\psi^n : R \longrightarrow R$, for $n \geq 1$, which are called Adams operations. The Adams operations also exist on a λ-ring R.

Binomial rings have several applications. For example Hall [29] uses a member of a binomial ring to determine a type of generalised exponentiation of an element of any nilpotent group. Wilkerson [55] shows that a binomial ring is a special type of λ-ring in which all Adams operations are equal to the identity. The λ-operations are then given by the binomial operations

$$\lambda^n(r) = \binom{r}{n},$$

for r in the ring and $n \geq 1$. Yau [57] consider filtered λ-ring, which is a λ-ring together with a decreasing sequence of λ-ideals. He shows that for a binomial ring R, the set

$\lambda(R[x]/(x^2))$ of isomorphism classes of filtered λ-ring structures on the ring $(R[x]/(x^2))$ is uncountable. The ring of integer-valued polynomials on a set X of variables, is the set of polynomials with coefficients in $\mathbb{Q}$ that are integer-valued over integers. This is denoted by

$$\mathrm{Int}(\mathbb{Z}^X) = \{f \in \mathbb{Q}[X] : f(\mathbb{Z}^X) \subset \mathbb{Z}\},$$

where $\mathbb{Z}^X = \mathrm{Hom}(X, \mathbb{Z})$ see [15]. It is an example of a binomial ring. This plays an important role in our thesis. We show that the ring of integer-valued polynomials over a subset $K \subseteq \mathbb{Z}$, which is denoted by

$$\mathrm{Int}(K^X, \mathbb{Z}) = \{f \in \mathbb{Q}[X] : f(K^X) \subset \mathbb{Z}\},$$

where $K^X = \mathrm{Hom}(X, K)$ is also a binomial ring. Elliott [25] shows that $\mathrm{Int}(\mathbb{Z}^X)$ is the free binomial ring on the set X. He also defines a right adjoint to the inclusion functor from the category of binomial rings to the category of λ-rings. Using Adams operations on λ-rings we define a left adjoint Q_λ to this inclusion functor.

Indeed, Adams operations give another type of ring closely related to λ-rings. This is a commutative ring R with unit equipped with a sequence of ring homomorphisms

$$\psi^n : R \longrightarrow R,$$

for all $n \geq 1$. They are required to satisfy $\psi^1(r) = r$ and $\psi^i(\psi^j(r)) = \psi^{ij}(r)$. Such a ring is called a ψ-ring. Wilkerson in [55] shows that there exists a λ-ring structure on a $\mathbb{Z}$-torsion free ψ-ring R satisfying the condition

$$\psi^p(r) \equiv r^p \pmod{pR},$$

for r in R and prime p, whose Adams operations are given by the ψ-ring structure on R. Our results by applying Wilkerson's theorem and Theorem 2.7.1 we show that binomial rings are preserved under localization and completion.

Theorem 2.9.5 Let S be a multiplicative closed subset of the binomial ring R. Then the localization $S^{-1}R$ is a binomial ring.

Theorem 2.9.19 Let d be a metric on a binomial ring R. Then the ring $\hat{R}_d$ is a binomial ring.

We study the algebraic structure of binomial rings. We start with the notion of a binomial ideal of a binomial ring. An ideal I of a binomial ring R is called a binomial ideal if it is closed under the binomial operations; that is

$$\binom{a}{n} \in I,$$

for a in I and $n \geq 1$. There is not much work on binomial ideals. Xantcha [56] gives a short survey on binomial ideals in his work on binomial rings: axiomatisation, transfer and classification. This encouraged us to investigate classes of binomial rings by properties of their binomial ideals. At the beginning, we show that the quotient ring of a binomial ring by a binomial ideal is also a binomial ring. This will be a very useful tool in our work. A well-known example of a non-Noetherian commutative ring is Int $(\mathbb{Z}^X)$. We introduce the notion of a binomially principal ring and a binomially Noetherian ring. We show the followings.

Theorem 3.6.13 The binomial ring Int $(\mathbb{Z}^{\{x\}})$ is a binomially principal ring.

Theorem 3.7.7 The ring Int $(\mathbb{Z}^{\{x,y\}})$ on two variables x and y is a binomially Noetherian ring.

The complex topological K-theory built out from vector bundle of a space X by apply the Grothendieck construction to the semi-ring Vect (X) with addition operation the direct sum $\oplus$ and multiplication the tenser product $\otimes$ on the equivalence classes of vector bundles over X. The origin of K-theory goes back to Grothendieck in algebraic geometry in his first work on the Riemann-Roch theorem [28]. Atiyah and Hirzebruch in [7] published the first work on K-theory in algebraic topology which is called topological K-theory. This is an extraordinary cohomology theory. For a good space X, (for example para-compact Hausdorff space) $K^0(X)$ is a λ-ring with λ-operations given by exterior powers on vector bundles E over X,

$$\lambda^n(E) = \Lambda^n E.$$

Knutson in [38] shows that for a binomial ring R which comes with a particular type of generating subset, there is an isomorphism

$$R \cong \mathbb{Z}.$$

He applies this result to K-theory. It shows that if $K^0(X)$ for a good space X is a binomial ring, then

$$K^0(X) \cong \mathbb{Z}.$$

However non-trivial examples of binomial rings do arise in relation to topological K-theory. The ring Int $(\mathbb{Z}^{\{x\}})$ and its dual appear as various types of operations and cooperations in topological K-theory. Some nice works in this direction can be found in [4, 17, 18, 20]. Bases of this kind of ring of cooperations in K-theory are given in [19]. Bases of the dual of this ring, related to operations in K-theory, can be found in [50]. We use $K_0(X)$ as a dual to $K^0(X)$ for a good space X and the properties of the ring Int $(\mathbb{Z}^{\{x\}})$, to give some non-trivial examples of binomial rings arising from topology. Our main results give new descriptions of these examples. The most important one is

Theorem 4.6.9 Let Int $(\mathbb{Z}^{\{x,y\}})$ be the ring of integer-valued polynomials over two x, y variables and let Int $((Z^{\{x\}})[x^{-1}]$ be the localization of the ring Int $(\mathbb{Z}^{\{x\}})$ with respect to the multiplicatively closed set $\{x^n : n \in \mathbb{N}\}$. Then we have an isomorphism of binomials rings,

$$\frac{\text{Int}(\mathbb{Z}^{\{x,y\}})}{((xy-1))} \cong \text{Int}(\mathbb{Z}^{\{x\}})[x^{-1}].$$

Simplicial methods were introduced by Dold and Kan around 1950. They played an important part in the development of homological algebra and led to non-abelian derived functors. The simplicial method provides a way to define cohomology in a categorical setting. The concept of a triple on a category traces back to Godement [26] and cotriple to Huber [35] as a dual of triples. It is well known that a cotriple $\mathbb{C} = (C, \varepsilon, \delta)$ in a category yields a simplicial object in this category, built out of iterating C, with face and degeneracy maps determined by ε and δ. Cohomology theories have been defined in different areas of abstract algebra. For example for associative algebras a cohomology theory is defined via the theory of Hochschild [33], for groups via the theories of Eilenberg and Mac Lane [23] and for Lie algebras via the theories of Chevalley and Eilenberg [16].

Barr and Beck [22] use a cotriple that comes from an adjoint pair of functors, to introduce a cohomology theory which is called the cotriple cohomology theory. André [6] and Quillen [46] separately introduced a cohomology theory on the category of commutative algebras using a cotriple on the category of commutative algebras that comes from the composite of a free functor and a forgetful functor. This is now called André-Quillen cohomology theory.

Robinson in [48] introduced the cohomology of λ-rings with coefficients in the contravariant functor $\mathrm{Der}_\lambda(-, M)$, which is the set of all λ-derivations with values in a λ-module M over the λ-ring. This is an example of a cotriple cohomology theory on the category of λ-rings. We apply Robinson's notions of λ-module and λ-derivation to binomial rings to introduce the cohomology of binomial rings as another example of a cotriple cohomology theory on the category of binomial rings, with values in the contravariant functor $\mathrm{Der}_{\mathrm{Bin}}(-, M)$, which is the set of all λ-derivations with values in a λ-module M over the λ-ring. our main result

Theorem 6.3.16 Let R be a binomial ring and let M be a binomial module over R with module structure given by $\varphi_n^M = \frac{(-1)^{n-1}}{n} Id_M$. Then

$$\mathrm{Der}(R, M) = \mathrm{Der}_{\mathrm{Bin}}(R, M).$$

Theorem 6.4.8 Let R be a binomial ring and let M be a binomial module over R. Then there exists an R-module homomorphism, for each $n \geq 0$

$$\varrho_n : H^n_{\mathrm{Bin}}(R, M) \longrightarrow H^n_\lambda(I_{\mathrm{Bin}} R, I_{\mathrm{Bin}} M).$$

This thesis is organized as follows. In **chapter 2** we provide an overview about special classes of commutative rings which are called binomial rings and λ-rings. In general we show that for $K \subseteq \mathbb{Z}$, $\mathrm{Int}(K^X, \mathbb{Z})$ is a binomial ring. Then we introduce the notion of Adams operation on a λ-ring. It is shown that a binomial ring is a special type of λ-ring in which all Adams operations are the identity and the λ-operations are given by the binomial operations.

In §2.8, we introduce the functor Q_λ from the category of λ-rings to the category of binomial rings. We show that it is left adjoint to the inclusion functor $\mathrm{I}_{\mathrm{Bin}}$ from the category of binomial rings to the category of λ-rings. At the end of this chapter we show that binomial rings are preserved by localization and completion.
In **chapter 3** we focus on the notion of a binomial ideal of a binomial ring. We start with the definition alongside some examples and properties. The proof that the quotient ring of a binomial ring by a binomial ideal is a binomial ring is given in §3.3. Then we introduce the notion of a principal binomial ideal. We use this to give some examples

of quotients of $\mathrm{Int}(\mathbb{Z}^{\{x\}})$, (Theorem 3.4.14 and Theorem 3.6.21). The relation with usual principal ideals is given.

As a main aim of this chapter we introduce some classes of binomial rings by properties of their binomial ideals. First, we introduce the notion of binomially simple ring. We show that the ring of integers $\mathbb{Z}$ is a binomially simple ring. Then we introduce the notion of binomially principal ring. As the first step we give the characterisation of binomial ideals in $\mathrm{Int}(\mathbb{Z}^{\{x\}})$. We use it and the fact that $\mathbb{Q}[x]$ is a principal integral domain to show that $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ is a binomially principal ring. Finally, we define the notion of binomially Noetherian ring. We use a characterisation of binomial ideals in $\mathrm{Int}(\mathbb{Z}^{X})$ on a set X of variables and the fact that $\mathbb{Q}[x, y]$ is a Noetherian ring to show that $\mathrm{Int}(\mathbb{Z}^{\{x,y\}})$ is a binomially Noetherian ring. In the last section of this chapter we define the notion of binomially filtered rings. We show that the power series ring

$$\mathbb{Z}[\![\binom{x}{1}, \binom{x}{2}, \binom{x}{3}, \cdots]\!]$$

is a binomial ring. Bhargava [12], for $S \subseteq \mathbb{Z}$, gives a regular basis of the ring $\mathrm{Int}(S^{\{x\}}, \mathbb{Z})$. We use this in the case where S has a p-ordering simultaneously for all primes p to give a description of a particular completion of this ring.

Chapter 4 is devoted to giving some non-trivial examples of binomial rings arising from topology. We start with the construction of K-theory geometrically in terms of classes of vector bundles over the space X, $\mathrm{Vect}(X)$ and some basic results on it. Then we introduce the spectrum $\mathbf{K}$ associated with the spaces $BU \times \mathbb{Z}$ and U, which defines a cohomology theory called complex K-theory. The various types of cohomology operations and related cooperations are given. We also give all the necessary background on Hopf algebras. In §4.5, we start with discussion of stably integer-valued Laurent polynomials. We show that the ring

$$\mathrm{SLInt}(\mathbb{Z}^{\{x\}}) = \{f(x) \in \mathbb{Q}[x, x^{-1}] : \ z^m f(z) \in \mathbb{Z} \text{ for all } z \in \mathbb{Z} \text{ and some } m \geq 0\}$$

is a binomial ring. Also, we introduce the ring of stably integer-valued polynomials

$$\mathrm{SInt}(\mathbb{Z}^{\{x\}}) = \{f(x) \in \mathbb{Q}[x] : \ z^m f(z) \in \mathbb{Z} \text{ for all } z \in \mathbb{Z} \text{ and some } m \geq 0\}.$$

We show that it is a binomial ring. At the end of this chapter, we explain how these examples of binomial rings come from topology. The main new results in this chapter are Theorem 4.6.4 and Theorem 4.6.9, giving new descriptions of these examples.

In **chapter 5** we provide background material on cotriple cohomology theory. Also we give an overview of André-Quillen cohomology theory for commutative algebras as an example of cotriple cohomology. At the end of this chapter, we give a summary of cohomology of λ-rings. There is no original work in this chapter.

Chapter 6 is devoted to introducing the cohomology of binomial rings as another example of a cotriple cohomology theory, on the category of binomial rings. We introduce the notion of a binomial module over a binomial ring by applying the notion of a λ-module to the special case of a binomial ring. We give examples of different binomial module structures. In the same way we apply the notion of a λ-derivation of a λ-ring with values in a λ-module to introduce the notion of a binomial derivation of a binomial ring with values in a binomial module M. We look at derivations of the binomial polynomials. We use this to understand derivations of the integer-valued polynomial ring. We investigate binomial derivations on a binomial ring with different binomial module structures. At the end of this chapter we define the cohomology of binomial rings using cotriple cohomology with values in the contravariant functor $\mathrm{Der}_{\mathrm{Bin}}(-, M)$. We show that for a binomial ring R and binomial modules M over R, with a particular binomial module structure, given by $\varphi_n^M = (-1)^{n-1}\mathrm{Id}_M$, we have

$$\mathrm{Der}_{\mathrm{Bin}}(R, M) = 0.$$

As a consequence for this kind of module structure, we get

$$H^n_{\mathrm{Bin}}(R, M) = 0,$$

for all $n \geq 0$. However, other binomial module structures give non-zero cohomology at lest in degree zero see Proposition 6.4.2. We define homomorphisms connecting the cohomology of binomial rings to the cohomology of λ-rings and to the André-Quillen cohomology of the underlying commutative rings.

Chapter 2

Binomial rings

§ 2.1 Introduction

The main purposes of this chapter are as follows.

1. To give the definitions and review some basic properties and well known results about special classes of rings which are called binomial rings and λ-rings.

2. To investigate the relationship between them using Adams operations on λ-rings.

Our main result is to construct the functor denoted by Q_λ from the category of λ-rings to the category of binomial rings. We show that this functor is left adjoint to the inclusion functor from the category of binomial rings to the category of λ-rings (Theorem 2.8.12). We extend well known results about λ-rings to binomial rings. It is shown that binomial rings are closed under localization (Theorem 2.9.5) and completion (Theorem 2.9.20).

In §2.2 we give a short introduction to binomial rings. The definition and some basic properties of binomial rings used throughout the whole thesis alongside some examples are given in §2.3. In §2.4 we discuss the ring of integer-valued polynomials on a set X of variables, $\mathrm{Int}(\mathbb{Z}^X)$. We show that the ring $\mathrm{Int}(\mathbb{Z}^X)$ is a binomial ring. We use the ring $\mathrm{Int}(\mathbb{Z}^X)$ to give another description of binomial rings: a $\mathbb{Z}$-torsion free ring which is the homomorphic image of the ring $\mathrm{Int}(\mathbb{Z}^X)$ is a binomial ring and all binomial rings are of this form. In §2.5 the definition of λ-rings is presented along with some examples that will be referred to later in this thesis. The notion of Adams operations on λ-rings given in §2.6.

The proof that the binomial ring is special type of λ-rings in which all Adams operations are the identity maps as binomial rings (Theorem 2.7.1) given in §2.7. In §2.8 we introduce the category of binomial rings whose objects are binomial rings and morphisms are ring homomorphisms. We construct the left adjoint functor to the inclusion functor from the category of binomial rings to the category of λ-rings. We use Theorem 2.7.1 to show that binomial rings are closed under localisation and completion.

In particular the facts that p-local integer ring $\mathbb{Z}_{(p)}$ and p-adic integers $\widehat{\mathbb{Z}}_p$ both are binomial rings is the subject of §2.9.

§ 2.2 Introduction to binomial rings

The concept of binomial ring was originally introduced by Hall [29] in connection with his groundbreaking work in the theory of nilpotent groups. His original definition is as follows. Let R be a commutative ring with unity. It is a binomial ring if it is $\mathbb{Z}$-torsion-free and closed under the binomial operations

$$\binom{r}{n} = \frac{r(r-1)(r-2)\cdots(r-(n-1))}{n!} \in R$$

for every $r \in R$ and $n \geq 1$.

Alongside the original reference today there are three other basic references for binomial rings.

The most recent one is the book [57] by Donald Yau. In Chapter 5 of this book he gives an elementary introduction to binomial rings with a few examples, basic properties and theorems. He explains that the universal λ-ring on a binomial ring R is isomorphic to the necklace ring $\mathrm{Nr}(R)$ of R, where $\mathrm{Nr}(R)$ of R is the ring with underlying set $\mathrm{Nr}(R) = \prod_{n=1}^{\infty} R$. Finally he introduces the concept of a filtered λ-ring. He shows that for a binomial ring R the set $\lambda(R[x]/(x^2))$ of isomorphism classes of filtered λ-ring structures on the ring $(R[x]/(x^2))$ with the x-adic filtration is uncountable.

The second basic reference is the paper [25] by Elliott. The main theme of this paper is to elucidate the connection between binomial rings and λ-rings. He defines the free binomial ring on the set X via the integer-valued polynomial ring $\mathrm{Int}(\mathbb{Z}^X)$ on the set X. He applies this to give another characterisation of binomial rings, that they are homomorphic images of the rings of integer-valued polynomials that are $\mathbb{Z}$-torsion free rings.

More generally he introduce the notion of "quasi binomial" as a homomorphic image of a binomial ring to describe another characterisation of a binomial ring. Furthermore, he constructs both left and right adjoint functors to the inclusion functor from binomial rings to rings and from the point of view of Adams operations, he describes a right adjoint for the inclusion functor from binomial rings to λ-rings.

The third basic reference is the paper [55] by Wilkerson. In this paper from the point of view of Adams operations he shows that a binomial ring is equivalent to a λ-ring in which all Adams operations are the identity.

There is also a nice paper [38] by Knutson, which applies the Adams operations to show that the triviality of Adams operations in group representation rings and topological K-theory of spaces lead to triviality of the whole ring. It shows that if the ring $R(G)$ of a finite group is a binomial ring then necessarily $G = \{e\}$. On the other hand, later in other chapters we will see some examples of non-trivial binomial rings arising from topology.

§ 2.3 Binomial rings

Since this thesis deals with binomial rings a lot, we begin with a section on them including the definition, some basic properties and examples. Let us begin with the $\mathbb{Z}$-torsion free property.

One of the conditions for any ring to be a binomial ring is that it should be $\mathbb{Z}$-torsion free as a $\mathbb{Z}$-module. Thus first we give the definition and some examples of $\mathbb{Z}$-torsion free rings.

Definition 2.3.1. An element r in a ring R is called a *$\mathbb{Z}$-torsion element*, if $nr = 0$ for some $n \in \mathbb{Z}^+$.

Example 2.3.2. The ring $\mathbb{Z}_3 = \{\overline{0}, \overline{1}, \overline{2}\}$ of integers modulo 3 has three $\mathbb{Z}$-torsion elements.

Definition 2.3.3. A ring R is called a *$\mathbb{Z}$-torsion free ring*, if 0 is the only $\mathbb{Z}$-torsion element in R.

Some examples of $\mathbb{Z}$-torsion free rings include binomial rings, polynomial rings over $\mathbb{Z}$ and any subring of the rationals $\mathbb{Q}$.

Let R be a ring. Consider the ring homomorphism

$$R \longrightarrow R \otimes \mathbb{Q}$$

given by

$$r \mapsto r \otimes 1.$$

The property of R be $\mathbb{Z}$-torsion free means exactly that this ring homomorphism is injective.

Definition 2.3.4. A *binomial ring* is a commutative ring R with unit whose additive group is $\mathbb{Z}$-torsion free and that contains all the binomial operations

$$\binom{r}{n} = \frac{r(r-1)(r-2)\cdots(r-(n-1))}{n!} \in R \otimes \mathbb{Q}$$

actually in R for every $r \in R$ and $n \geq 0$, where $\binom{r}{0} = 1$.

In other words, a $\mathbb{Z}$-torsion free commutative ring R is a binomial ring if and only if it is closed under taking binomial operations.

Note that the binomial ring structure on a $\mathbb{Z}$-torsion free commutative ring R is unique (if it exists). But later we will see when we define the notion of binomial module over binomial ring the structure of binomial module is not unique.

Example 2.3.5. Some examples of binomial rings are the following.

1. The simplest binomial ring is the ring of integers $\mathbb{Z}$. It is clear $\mathbb{Z}$ is a $\mathbb{Z}$-torsion free ring. Since the binomial operations are integers this implies that the ring $\mathbb{Z}$ is preserved by binomial operations.

2. Any field R of characteristic 0. Since R has no zero-divisors and its characteristic is 0 this implies R is a $\mathbb{Z}$-torsion free ring. Since R is a field, every non-zero element in R is invertible. This implies that the binomial operations lie in R for every element in R.

3. Any $\mathbb{Q}$-algebra is a binomial ring.

Other examples of binomial rings will appear in the following sections when we introduce the ring of integer-valued polynomials and consider Adams operations on λ-rings and in following chapters when we define the concept of binomial ideal in a binomial ring. Here is a preview of some of them.

Example 2.3.6. Every ring of integer-valued polynomials $\mathrm{Int}(\mathbb{Z}^X)$ on a set X of variables is a binomial ring (Theorem 2.4.7).

Example 2.3.7. Every ring of stable integer-valued Laurent polynomials $\mathrm{SLInt}(\mathbb{Z}^X)$ on a set X of variables is a binomial ring (Theorem 4.5.6).

Example 2.3.8. In general every ring of stable integer-valued polynomials $\mathrm{SInt}(\mathbb{Z}^X)$ on a set X of variables is a binomial ring (Theorem 4.5.20).

Example 2.3.9. The ring $\mathbb{Z}_{(p)}$ of p-local integers (Corollary 2.9.6) and the ring $\widehat{\mathbb{Z}}_p$ of p-adic integers (Corollary 2.9.21) both are binomial rings.

Example 2.3.10. In general every ring of the integer-valued polynomial rings over a subset $K \subseteq \mathbb{Z}$, $\mathrm{Int}(K^X, \mathbb{Z})$ on a set X of variables is a binomial ring (Theorem 2.4.11) and the generalization of an integer-valued polynomial ring on a binomial domain R with quotient field F, $\mathrm{Int}(R)$ is a binomial ring (Proposition 2.9.13).

Example 2.3.11. Any λ-ring whose Adams operations all are the identity is a binomial ring (Theorem 2.7.1).

Example 2.3.12. The quotient ring R/I of a binomial ring R by a binomial ideal I is a binomial ring (Theorem 3.3.1).

Example 2.3.13. The power series ring $\mathbb{Z}[\![\binom{x}{1}, \binom{x}{2}, \binom{x}{3}, \cdots]\!]$ is a binomial ring (Proposition 3.8.9).

We now state without proof some basic properties of binomial operations, which follow from standard facts about binomial coefficients.

Theorem 2.3.14. *[56] Let R be a binomial ring. For all $a, b \in R$ and $m, n, k \in \mathbb{N}$ the following hold.*

1. $$\binom{a+b}{n} = \sum_{n=p+q} \binom{a}{p}\binom{b}{q}.$$

2. $$\binom{ab}{n} = \sum_{m=0}^{n} \binom{a}{m} \sum_{q_1+q_2+\cdots+q_m=n} \binom{b}{q_1} \cdots \binom{b}{q_m}.$$

3. $\binom{a}{m}\binom{a}{n} = \sum_{k=0}^{n} \binom{a}{m+k}\binom{m+k}{n}\binom{n}{k}$.

4. $\binom{1}{n} = 0 \qquad \textit{when } n \geq 2.$

5. $\binom{a}{1} = a.$

Next here are some good properties of binomial rings.

Proposition 2.3.15. *Let R and K be binomial rings.*

1. *The direct product ring $R \times K$ is a binomial ring with binomial operations given by*
$$\binom{(r,k)}{n} = \left(\binom{r}{n}, \binom{k}{n}\right),$$
for $r \in R, k \in K$ and $n \geq 0$. So, if $R_1, \ldots, R_m$ are binomial rings, the product ring $\prod_{i=1}^{m} R_i$ is a binomial ring.

2. *The tensor product ring $R \otimes K$ is a binomial ring with binomial operations determined by*
$$\binom{r \otimes 1}{n} = \binom{r}{n} \otimes 1,$$
$$\binom{1 \otimes k}{n} = 1 \otimes \binom{k}{n},$$
for $r \in R, k \in K$ and $n \geq 0$.

3. *The intersection $R \cap K$ is a binomial ring. More generally, if $\{R_i\}_{i \in I}$ is a family of binomial rings then the ring $\cap_i R_i$ is a binomial ring.*

Proof. Property 1 is clear. We are going to prove property 2. We can write
$$\binom{r \otimes k}{n} = \binom{(r \otimes 1)(1 \otimes k)}{n}.$$
Since R and K both are binomial rings, by Theorem 2.3.14(2) and above formula this implies that
$$\binom{r \otimes k}{n} \in R \otimes K.$$
For Property 3 see [25]. □

Definition 2.3.16. The *Stirling number of the first kind*, denoted by $\begin{bmatrix} n \\ i \end{bmatrix}$ for $n \geq 0$ and $i \in \mathbb{N}$, is defined as the number of ways to permute n elements into exactly i cycles.

Proposition 2.3.17. *The Stirling number of the first kind* $\begin{bmatrix} n \\ i \end{bmatrix}$ *for* $n \geq 0$ *and* $i \in \mathbb{N}$*, satisfy the linear recurrence,*

$$\begin{bmatrix} n \\ i \end{bmatrix} = (n-1)\begin{bmatrix} n-1 \\ i \end{bmatrix} + \begin{bmatrix} n-1 \\ i-1 \end{bmatrix}, \tag{2.1}$$

with initial conditions $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1$ *and for* $i < 0$*,* $\begin{bmatrix} n \\ i \end{bmatrix} = 0.$

Proof. For proof see [27, p. 247]. □

We let

$$x^{\underline{n}} = x(x-1)(x-2)\ldots.(x-(n-1)), \tag{2.2}$$

for $n \geq 0$, be the n-th falling power of x and

$$x^{\overline{n}} = x(x+1)(x+2)\ldots.(x+(n-1)), \tag{2.3}$$

be the nth rising powers of x.

Actually, the Stirling number of the first kind $\begin{bmatrix} n \\ i \end{bmatrix}$ can be expressed in many equivalent different ways.

They appear as the coefficients in $x^{\underline{n}}$.

Proposition 2.3.18. *[27, p. 249] For* $n \geq 0$ *we have*

$$x^{\underline{n}} = \sum_{i=0}^{n} (-1)^{n-i} \begin{bmatrix} n \\ i \end{bmatrix} x^i. \tag{2.4}$$

Proof. We will prove this by induction on n. Suppose that $n = 0$. Then both sides are equal 1. Assume that the result holds for $n-1$. Then we have

$$x^{\underline{n-1}} = \sum_{i=0}^{n-1} (-1)^{(n-1)-i} \begin{bmatrix} n-1 \\ i \end{bmatrix} x^i. \tag{2.5}$$

We will prove it for n. We have

$$x^{\underline{n}} = x^{\underline{n-1}}(x-(n-1)) = xx^{\underline{n-1}} - (n-1)x^{\underline{n-1}}.$$

Then by our assumption we obtain

$$
\begin{aligned}
x^{\underline{n}} &= x\sum_{i=1}^{n-1}(-1)^{(n-1)-i}\begin{bmatrix} n-1\\ i\end{bmatrix}x^i-(n-1)\sum_{i=1}^{n-1}(-1)^{(n-1)-i}\begin{bmatrix} n-1\\ i\end{bmatrix}x^i, \quad \text{by (2.5)}\\
&= \sum_{i=1}^{n-1}(-1)^{(n-1)-i}\begin{bmatrix} n-1\\ i\end{bmatrix}x^{i+1}-(n-1)\sum_{i=1}^{n-1}(-1)^{(n-1)-i}\begin{bmatrix} n-1\\ i\end{bmatrix}x^i,\\
&= \sum_{i=2}^{n}(-1)^{(n-i)}\begin{bmatrix} n-1\\ i-1\end{bmatrix}x^{i}-(n-1)\sum_{i=1}^{n-1}(-1)^{(n-1)-i}\begin{bmatrix} n-1\\ i\end{bmatrix}x^i,\\
&= \sum_{i=0}^{n}(-1)^{(n-i)}\begin{bmatrix} n\\ i\end{bmatrix}x^{i} \qquad \text{by (2.1).}
\end{aligned}
$$

□

Proposition 2.3.19. *For $n \geq 0$ we have*

$$x^{\overline{n}} = \sum_{i=0}^{n}\begin{bmatrix} n\\ i\end{bmatrix}x^i.$$

Proof. The proof similar to the proof of Proposition 2.5. □

We use the expression of Stirling numbers of the first kind $\begin{bmatrix} n\\ i\end{bmatrix}$ in (2.4) to expand $\binom{x}{n}$. This implies that

$$\binom{x}{n} = \frac{1}{n!}\left(\sum_{i=0}^{n}(-1)^{n-i}\begin{bmatrix} n\\ i\end{bmatrix}x^i\right). \tag{2.6}$$

In combinatorics see [40, p. 56], the Stirling number of the first kind can be expressed as the sum over $(c_1, \dots, c_n)$ of the number of permutations of type $[c_1, c_2, \cdots, c_n]$

$$\begin{bmatrix} n\\ i\end{bmatrix} = (-1)^{n+i}\sum_{\substack{i=c_1+c_2+\cdots+c_n\\ n=c_1+2c_2+\dots.+nc_n}}\frac{n!}{1^{c_1}2^{c_2}\dots.n^{c_n}c_1!c_2!\dots c_n!}. \tag{2.7}$$

Lemma 2.3.20. *For a prime p, the Stirling number of the first kind $\begin{bmatrix} p\\ i\end{bmatrix}$ is divisible by p for $1 < i < p$.*

Proof. From (2.7) we have,

$$\begin{bmatrix} p\\ i\end{bmatrix} = (-1)^{n+i}\sum_{\substack{i=c_1+c_2+\cdots+c_p\\ p=c_1+2c_2+\cdots+pc_p}}\frac{p!}{1^{c_1}2^{c_2}\dots p^{c_p}c_1!c_2!\dots c_p!}.$$

The numerator is divisible by p. Since $1 < i < p$ and we have $c_p = 0$ each factor in the denominator is less than p and so the prime p is not canceled in the numerator. Therefore $\begin{bmatrix} p\\ i\end{bmatrix}$ is divisible by p. □

Proposition 2.3.21. *In a commutative ring R we have,*

$$r^p - r \equiv r(r-1)(r-2)\dots(r-(p-1)) \pmod{pR}, \tag{2.8}$$

for $r \in R$ and a prime p.

Proof. From (2.4) we have

$$r(r-1)(r-2)\cdots.(r-(p-1)) = \sum_{i=1}^{p}(-1)^{n-i}\begin{bmatrix}p\\i\end{bmatrix}r^i.$$

And by Lemma 2.3.20, $\begin{bmatrix}p\\i\end{bmatrix}$ is divisible by p for $1 < i < p$. We obtain

$$\begin{bmatrix}p\\i\end{bmatrix} \equiv 0 \pmod{pR}.$$

Also by Wilson's theorem see ([30, p. 85]) we have

$$\begin{bmatrix}p\\1\end{bmatrix} = (p-1)! \equiv -1 \pmod{p},$$

and $\begin{bmatrix}p\\p\end{bmatrix} = 1.$ □

Proposition 2.3.22. *[57, Lemma 5.5]In a binomial ring R the congruence condition*

$$r^p \equiv r \pmod{pR} \tag{2.9}$$

holds for all $r \in R$ and p prime.

Proof. By (2.8) we have

$$\begin{aligned} r^p - r &\equiv r(r-1)(r-2)\cdots.(r-(p-1)) \pmod{pR} \\ &= p!\binom{r}{p} \text{ by (2.6)} \\ &\equiv 0 \pmod{pR} \end{aligned}$$

Therefore

$$r^p \equiv r \pmod{pR}.$$

□

§ 2.4 Integer-valued polynomials

Most of the examples in this thesis are related to rings of integer-valued polynomials on a set X of variables. So we will begin with a section on rings of integer-valued polynomials. This means rings of polynomials with rational coefficients that are integer-valued on integers. We prove that the ring $\mathrm{Int}(\mathbb{Z}^X)$ on a set X of variables is a binomial ring (Theorem 2.4.7). Precisely later in §2.8 we will show that the ring $\mathrm{Int}(\mathbb{Z}^X)$ on a set X of variables is the free binomial ring on the set X. The result is given in [25]. Later we introduce the notion of integer-valued polynomials $\mathrm{Int}(K^{\{x\}}, \mathbb{Z})$ over a subset $K \subseteq \mathbb{Z}$. As a result we show that $\mathrm{Int}(K^X, \mathbb{Z})$, is a binomial ring (Theorem 2.4.11). For a more thorough description of integer-valued polynomials we refer to [15]. We begin with the definition of the ring of integer-valued polynomials on a set X of variables.

Definition 2.4.1. Let $\mathbb{Q}[X]$ be the ring of polynomials on a set X of variables with rational coefficients. We define the set of *integer-valued polynomials* on X by

$$\mathrm{Int}(\mathbb{Z}^X) = \{f \in \mathbb{Q}[X] \ : \ f(\mathbb{Z}^X) \subset \mathbb{Z}\}.$$

This is a subring of $\mathbb{Q}[X]$ and it is called the ring of integer-valued polynomials on X, where $\mathbb{Z}^X = \mathrm{Hom}(X, \mathbb{Z})$, which is the set of functions $\underline{n}(x)$. We computing f at any $\underline{n}$ by replacing each $x \in X$ with integer $\underline{n}(x)$ in f. Then the condition $f(\mathbb{Z}^X) \subset \mathbb{Z}$ means that $f(\underline{n}) \in \mathbb{Z}$. In particular we have

$$\mathrm{Int}(\mathbb{Z}^{\{x\}}) = \{f \in \mathbb{Q}[x] \ : \ f(\mathbb{Z}) \subset \mathbb{Z}\}, \tag{2.10}$$

the ring of integer-valued polynomials in one variable x.

Definition 2.4.2. The *binomial polynomial* in one variable x is defined by

$$\binom{x}{n} = \frac{x(x-1)(x-2)\cdots(x-(n-1))}{n!} \in \mathbb{Q}[x].$$

for all $n \geq 0$, where $\binom{x}{0} = 1$.

Notation 2.4.3. *For some non-empty set X of variables, we define a multi-index to be*

$$J = (j_x)_{x \in X} \in \bigoplus_{x \in X} \mathbb{Z}_{\geq 0}. \tag{2.11}$$

With this multi-index J we define the generalized binomial polynomial to be

$$\binom{X}{J} = \prod_{x \in X} \binom{x}{j_x}. \tag{2.12}$$

Note that for another multi-index $J = (j_x)$ the binomial operation define by,

$$\binom{J}{I} = \prod_{x \in X} \binom{j_x}{i_x} \in \mathbb{Z}. \tag{2.13}$$

Notation 2.4.4. *Let* $I = (j_1, \ldots, i_n) \in \bigoplus_{k=1}^{n} \mathbb{Z}_{\geq 0}$. *and let* $J = (j_1, \ldots, j_n)$ *be another multi-index. Then we mean by* $I > J$ *if and only if* $i_1 = j_1, \ldots, i_k = j_k$ *and* $i_{k+1} > j_{k+1}$ *for some* k *with* $0 \leq k \leq n-1$.

In particular, for any multi-indexes $J_1 < J_2 < J_3 < \cdots < J_n$, we have

$$\binom{J_t}{J_k} = \begin{cases} 1 & \text{if } t = k, \\ 0 & \text{if } t < k. \end{cases} \tag{2.14}$$

Lemma 2.4.5. *For* $n \geq 0$ *the binomial polynomial* $\binom{x}{n}$ *is an integer-valued polynomial in one variable* x.

In general the product of binomial polynomials each in one variable

$$\binom{x_1}{n_1}\binom{x_2}{n_2}\cdots\binom{x_i}{n_i}, \tag{2.15}$$

in the polynomial ring $\mathbb{Q}[x_1, x_2, \ldots, x_i]$ *in* i *variables for* $n_1, n_2, \ldots, n_i \geq 0$ *is an integer-valued polynomial in* i *variables.*

□

Theorem 2.4.6. *[57]The generalized binomial polynomials in a set of variables* X

$$\left\{\binom{X}{J} : J = (j_x) \in \bigoplus_{x \in X} \mathbb{Z}_{\geq 0}\right\}, \tag{2.16}$$

is a $\mathbb{Z}$*-basis of the ring Int*$(\mathbb{Z}^X)$. *In particular, the polynomials* $\binom{x}{n}$, *for* $n \geq 0$, *form a* $\mathbb{Z}$*-module basis of the ring Int*$(\mathbb{Z}^{\{x\}})$ *in one variable, and the set*

$$\left\{\binom{x_1}{n_1}\cdots\binom{x_i}{n_i} : n_1, \ldots, n_i \geq 0\right\} \tag{2.17}$$

is a $\mathbb{Z}$*-module basis of the ring Int*$(\mathbb{Z}^{\{x_1, \cdots, x_i\}})$ *in* i *variables.*

Proof. First the set

$$\left\{\binom{X}{J} : J = (j_x) \in \bigoplus_{x \in X} \mathbb{Z}_{\geq 0}\right\},$$

is a $\mathbb{Q}$-vector space basis of the polynomial ring $\mathbb{Q}[X]$ by [57, Proposition 5.31]. Then for $f \in \text{Int}(\mathbb{Z}^X)$, with $f \neq 0$, we can write

$$f = \sum_{t=1}^{n} a_t \binom{X}{J_t}$$

for some $n \geq 1$ with $a_t \in \mathbb{Q}$ and some multi-indexes $\{J_t\}_{t\in T}$. We obtain

$$f(J) = \sum_{t=1}^{n} a_t \binom{J}{J_t}.$$

In the same way in (2.14) when we compute f for ordering the indexes $J_1 < J_2 < \cdots < J_k$ at $X = J_1$, we obtain $f(J_1) = a_1 \in \mathbb{Z}$. So by induction on k assume that $a_1, a_2 \cdots, a_k \in \mathbb{Z}$. Now we are going to show that $a_{k+1} \in \mathbb{Z}$, to see that we calculate f at $X = J_{k+1}$, also by (2.14) we have

$$\begin{aligned} f(J_{k+1}) &= \sum_{t=1}^{k+1} a_t \binom{J_{k+1}}{J_t} \\ &= a_1 \binom{J_{k+1}}{J_1} + \cdots + a_k \binom{J_{k+1}}{J_k} + a_{k+1}. \end{aligned}$$

We know that $f(J_{k+1})$ is an integer. So by the induction we conclude that a_{k+1} is also an integer. This shows that the generalized binomial polynomials in a set X of variables spans $\mathrm{Int}(\mathbb{Z}^X)$ over $\mathbb{Z}$. Also we know from [57, Proposition 5.31] that the generalized binomial polynomials in the set of variables X are a $\mathbb{Q}$-vector space basis of the polynomial ring $\mathbb{Q}[X]$. So they are linearly independent over $\mathbb{Q}$. Hence they are also linearly independent over $\mathbb{Z}$. □

Here is the main purpose of this section, which shows that the ring $\mathrm{Int}(\mathbb{Z}^X)$ on a set X of variables is a binomial ring.

Theorem 2.4.7. *[57] The ring Int($\mathbb{Z}^X$) on a set X of variables is a binomial ring.*

Proof. First we need to show that $\mathrm{Int}(\mathbb{Z}^X)$ is $\mathbb{Z}$-torsion free, which is clear since $\mathrm{Int}(\mathbb{Z}^X)$ is a subring of $\mathbb{Q}[X]$. To see the other condition of a binomial ring, consider $f \in \mathrm{Int}(\mathbb{Z}^X)$. We have

$$\binom{f}{n} \in \mathbb{Q}[X].$$

Then for an $\underline{m} \in \mathbb{Z}^X$, we have $f(\underline{m}) \in \mathbb{Z}$. Notice

$$\binom{f}{n}(\underline{m}) = \binom{f(\underline{m})}{n}.$$

Then by Lemma 2.4.5,

$$\binom{f(\underline{m})}{n} \in \mathbb{Z}.$$

So

$$\binom{f}{n} \in \mathrm{Int}(\mathbb{Z}^X).$$

□

Now we turn attention to a ring of integer-valued polynomials over a subset.

Definition 2.4.8. For a subset $K \subseteq \mathbb{Z}$, we say that a polynomial $f \in \mathbb{Q}[X]$ on a set X of variables which satisfies that $f(K^X) \in \mathbb{Z}$ is an *integer-valued polynomial over subset* K, where $K^X = \mathrm{Hom}(X, K)$, which is the set of functions $\underline{n}$ as in Definition 2.4.1. We computing f at any $\underline{n}$ by replacing each $x \in X$ with $k \in K$, $\underline{n}(x)$ in f. Then the condition $f(K^X) \subset \mathbb{Z}$ means that $f(\underline{n}) \in \mathbb{Z}$.

$$\mathrm{Int}(K^X, \mathbb{Z}) = \{f \in \mathbb{Q}[X] \; : \; f(K^X) \subseteq \mathbb{Z}\}. \tag{2.18}$$

This is a subring of $\mathbb{Q}[X]$ and it is called the ring of integer-valued polynomials over K on set X. In particular we have

$$\mathrm{Int}(K^{\{x\}}, \mathbb{Z}) = \{f(x) \in \mathbb{Q}[x] \; : \; f(K) \subset \mathbb{Z}\}, \tag{2.19}$$

called the ring of integer-valued polynomials over subset K in one variable x.

Note that the integer-valued polynomial ring $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ is integer-valued over $\mathbb{Z}$, that is

$$\mathrm{Int}(\mathbb{Z}^{\{x\}}) = \mathrm{Int}(\mathbb{Z}^{\{x\}}, \mathbb{Z}).$$

So we have inclusion

$$\mathbb{Z}[x] \subset \mathrm{Int}(\mathbb{Z}^{\{x\}}) \subseteq \mathrm{Int}(K^{\{x\}}, \mathbb{Z}) \subseteq \mathbb{Q}[x]. \tag{2.20}$$

Example 2.4.9. In particular from the ring of integer-valued polynomials over $\{0\}$ on one variable x we have

$$\mathrm{Int}(\{0\}^{\{x\}}, \mathbb{Z}) = \{f(x) \in \mathbb{Q}[x] \; : \; f(0) \in \mathbb{Z}\}.$$

In other words, the ring $\mathrm{Int}(\{0\}^{\{x\}}, \mathbb{Z})$ is the set of all polynomials in $\mathbb{Q}[x]$ with constant term is an integer.

In example 2.4.9 for each non-zero subset $K \subset \mathbb{Z}$ we have

$$\mathbb{Z}[x] \subset \mathrm{Int}(\mathbb{Z}^{\{x\}}) \subseteq \mathrm{Int}(K^{\{x\}}, \mathbb{Z}) \subset \mathrm{Int}(\{0\}^{\{x\}}) \subset \mathbb{Q}[x]. \tag{2.21}$$

Here is our main result of this section.

Theorem 2.4.10. *For $K \subseteq \mathbb{Z}$, the ring Int$(K^{\{x\}}, \mathbb{Z})$ is a binomial ring.*

Proof. We know from (2.20) that $\mathrm{Int}(K^{\{x\}}, \mathbb{Z})$ is a subring of $\mathbb{Q}[x]$, so it is clearly a $\mathbb{Z}$-torsion free ring. To see the other condition of a binomial ring, pick an element $g(x) \in \mathrm{Int}(K^{\{x\}}, \mathbb{Z})$. We have

$$\binom{g(x)}{n} = \frac{g(x)(g(x)-1)(g(x)-2)\cdots(g(x)-(n-1))}{n!} \in \mathbb{Q}[x],$$

for $n \geq 0$. Then

$$\begin{aligned}\binom{g(x)}{n}(k) &= \frac{g(k)(g(k)-1)(g(k)-2)\cdots(g(k)-(n-1))}{n!}\\ &= \binom{g(k)}{n} \in \mathbb{Z} \text{ by Lemma 2.4.5,}\end{aligned}$$

for $k \in K$. Therefore $\binom{g(x)}{n} \in \mathrm{Int}(K^{\{x\}}, \mathbb{Z})$ as desired. □

Theorem 2.4.11. *For a subset $K \subseteq \mathbb{Z}$ the ring of integer-valued polynomials over subset K, Int$(K^X, \mathbb{Z})$ on a set X of variables is a binomial ring.*

Proof. The proof is analogues to the proof of Theorem 2.4.10. □

§ 2.5 λ-rings

In this section we discuss λ-rings, we give some well known results on them and some of their properties. A λ-ring is a commutative ring R with identity equipped with a sequence of functions $\lambda^i : R \to R$ for $i \geq 0$ which are called λ-operations, satisfying certain relations that are satisfied by the binomial operations. The λ-rings were first introduced in algebraic geometry by Grothendieck [28] under the name special λ-ring. They have been shown to play important roles in various field of mathematics.

For example in group theory, the paper [7] used λ-rings to study group representations. In algebraic topology, the K-theory of a good space is a λ-ring. In both cases the λ-operations are induced by exterior powers of vector spaces. Knutson in [37] used λ-rings to study representations of the symmetric group. In pure algebra, Donald Yau published a book under the name λ-rings [57]. The main aim of this is to study λ-rings purely algebraically. For example if R is a commutative ring with unit, Then the ring $W(R)$ of big witt vectors on R has canonical λ-ring structure.

Also the notion of λ-ring R uses the classical essential theorem of symmetric functions to describe the action of λ-operations on a product $\lambda^n(r_1 r_2)$ and the composition of λ-operations $\lambda^n \lambda^m(r_1)$ for $r_1, r_2 \in R$.

We begin with the definition of λ-rings.

Definition 2.5.1. A λ*-ring* is a commutative with unit ring R together with a sequence of functions $\lambda^n : R \to R$ (called λ-operations) for each $n \geq 0$ such that the following axioms are satisfied.

1. $\lambda^0(x) = 1$,
2. $\lambda^1(x) = x$,
3. $\lambda^n(1) = 0$ for $n \geq 2$,
4. $\lambda^n(x + y) = \sum_{i+j=n} \lambda^i(x)\lambda^j(y)$,

5. $\lambda^n(xy) = P_n(\lambda^1(x), \cdots, \lambda^n(x); \lambda^1(y), \cdots, \lambda^n(y))$,

6. $\lambda^n(\lambda^m(x)) = P_{i,j}(\lambda^1(x), \cdots, \lambda^{nm}(x))$,

for all $x, y \in R$ and $n, m \geq 0$.

The polynomials P_n and $P_{n,m}$ which describe the action of λ-operations on products and the composition of λ-operations are described below.

Definition 2.5.2. For a ring R, we consider the ring $R[x_1, x_2, \cdots, x_n]$ of polynomials in n independent variables $x_1, x_2, \cdots, x_n$. The polynomial $f \in R[x_1, x_2, \cdots, x_n]$ is called a *symmetric function* if it is unaltered under every permutation of the variables. That is, we have

$$f(x_1, x_2, \cdots, x_n) = f(x_{\pi(1)}, x_{\pi(2)}, \cdots, x_{\pi(n)}),$$

for every permutation π on the set $\{1, 2, \cdots, n\}$. We say that the polynomial $g \in R[x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n]$ in $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ independent variables is a symmetric function if it is unaltered under every permutation of variables. That is, we have

$$f(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n) = f(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)}; y_{\tau(1)}, y_{\tau(2)}, \cdots, y_{\tau(n)}),$$

for every part of permutations π and τ on the set $\{1, 2, \cdots, n\}$.

Example 2.5.3. For each $1 \leq k \leq n$, an important symmetric function is the kth symmetric polynomial $s_k \in R[x_1, x_2, \cdots, x_n]$ which is the sum of all products of monomial of length k. That is we have

$$s_k = \sum_{1 < i_1 < \cdots < i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

In particular, we have

$$s_1 = x_1 + x_2 + \cdots + x_n,$$
$$s_2 = x_1 x_2 + x_1 x_3 + \cdots + x_{n-1} x_n,$$
$$s_n = x_1 x_2 \cdots x_n.$$

Another way to obtain the kth elementary symmetric function $s_k(x_1, x_2, \cdots, x_n)$ in n variables is by considering the formula, where t is an extra variable,

$$f(t) = \sum_{k=0}^{n} s_k t^k = \prod_{i=1}^{n} (1 + t x_i), \tag{2.22}$$

Theorem 2.5.4. *[57, p. 3]Any symmetric function f in $R[x_1, x_2, \cdots, x_n]$ can be written as a polynomial in the elementary symmetric functions $s_1, s_2, \cdots, s_n$ with coefficients in R and it is unique.*

The polynomials $P_{n,m}$ and P_n that appeared in the definition of λ-ring are called universal polynomials. The polynomial $P_n(s_1, s_2, \cdots, s_n; \alpha_1, \alpha_2, \cdots, \alpha_n)$ is the coefficient of t^n in the polynomial

$$f(t) = \prod_{i,j=1}^{n} (1 + x_i y_j t),$$

where each of s_i and α_i are i^{th} elementary symmetric functions in $x_1, \cdots, x_n$ and in $y_1, \cdots, y_n$ respectively.
The polynomial $P_{n,m}(s_1, s_2, \cdots, s_{nm})$ is the coefficient of t^n in the polynomial

$$f(t) = \prod_{1 \le i_1 < \cdots < i_m \le nm} (1 + x_{i_1} x_{i_2} \cdots x_{i_m} t).$$

Example 2.5.5. $P_{n,1}(s_1, s_2, \cdots, s_n)$ is the coefficient of t^n in the polynomial

$$f(t) = \prod_{1 \le i \le n} (1 + x_i t) = 1 + s_1 t \cdots + s_n t^n,$$

so

$$P_{n,1}(s_1, s_2, \cdots, s_n) = s_n$$

and $P_{1,m}(s_1, s_2, \cdots, s_m)$ is coefficient of t in the polynomial

$$f(t) = \prod_{1 \le i_1 < \cdots < i_m \le m} (1 + x_{i_1} x_{i_2} \cdots x_{i_m} t) = 1 + x_1 x_2 \cdots x_m t$$

so

$$P_{1,m}(s_1, s_2, \cdots, s_m) = s_m.$$

For more detail on symmetric functions see [37, chapter 1] and for the universal polynomials see [57, chapter 1]. In general $P_{n,m} \neq P_{m,n}$ as λ-operations do not commute. For more detail and calculation see [34]. There the author gives several forms for $P_{n,m}$ and calculates P_n up to $n = 10$.
Here are some small values of both universal polynomials.

1. $P_0 = 1$.
2. $P_1(s_1, \alpha_1) = s_1 \alpha_1$.
3. $P_2(s_1, s_2; \alpha_1, \alpha_2) = s_1^2 \alpha_2 - 2 s_2 \alpha_2 + s_2 \alpha_1^2$.
4. $P_3(s_1, s_2, s_3; \alpha_1, \alpha_2, \alpha_3) = s_1^3 \alpha_3 - 3 s_1 s_2 \alpha_3 + s_1 s_2 \alpha_1 \alpha_2 - 3 s_3 \alpha_1 \alpha_2 + s_3 \alpha_1^3 + 3 s_3 \alpha_3$.
5. $P_{0,m} = 1, \quad$ for all $m \in \mathbb{N}$.
6. $P_{1,0} = 1$.
7. $P_{n,0} = 0, \quad$ for all $n \ge 2$.
8. $P_{1,1}(s_1) = s_1$.

9. $P_{2,2}(s_1, s_2, s_3, s_4) = s_1 s_3 - s_4$.

10. $P_{2,3}(s_1, s_2, s_3, s_4, s_5, s_6) = s_6 + s_2 s_4 - s_1 s_5$.

11. $P_{3,2}(s_1, s_2, s_3, s_4, s_5, s_6) = s_6 - 2s_2 s_4 - s_1 s_5 + s_1^2 s_4 + s_3^2 - s_1 s_5$.

12. $P_{3,3}(s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9) = s_9 - s_1 s_8 - s_4 s_5 - s_2 s_7 + s_1^2 s_7 + s_1^2 s_4 + 3s_3 s_6 - 2s_1 s_3 s_5 - s_1 s_2 s_6$.

Definition 2.5.6. Let R_1 and R_2 be λ-rings. A ring homomorphism $f : R_1 \longrightarrow R_2$ is called a λ*-homomorphism* if it commutes with the λ-operations that is

$$\lambda^n(f(r)) = f(\lambda^n(r)),$$

for all $r \in R$ and $n \geq 0$. We write $\mathbf{Ring}_\lambda$ for the category of λ-rings, whose objects are λ-rings and morphisms are λ-homomorphisms.

Definition 2.5.7. We call R a *pre* λ*-ring* if only the first four axioms of Definition 2.5.1 are satisfied.

Consider a (pre) λ-ring R together with the homomorphism λ_t from the additive group of R into the multiplicative group of power series in t with constant term 1, defined by

$$\lambda_t(r) = \sum_{n=0}^{\infty} \lambda^n(r) t^n \ \in R[[t]]. \tag{2.23}$$

Now we can use (2.23) to write addition of λ-operations as

$$\lambda_t(r_1 + r_2) = \lambda_t(r_1).\lambda_t(r_2) \tag{2.24}$$

Example 2.5.8. The ring $\mathbb{Z}$ of integers is a pre λ-ring with

$$\lambda_t(r) = (1+t)^r = \sum_{n=0}^{r} \binom{r}{n} t^n. \tag{2.25}$$

So

$$\lambda^n(r) = \binom{r}{n}.$$

It is clear by Theorem 2.3.14 $\binom{r}{n}$. is satisfy all axioms.

Definition 2.5.9. If $\lambda_t(x)$ is a polynomial of degree n, then we say that x has dimension n. If each $r \in R$ is difference of finite dimension elements, then we say that R is finite dimensional.

Example 2.5.10. [57, p. 9]Some examples of λ-rings are the following.

1. The simplest finite dimensional λ-ring is the ring of integers $\mathbb{Z}$ with the λ-operations defined by the binomial operations $\lambda^n(r) = \binom{r}{n}$. that is

$$\lambda_t(n) = \text{coefficientsof} t^i \text{in} (1+t)^n = \sum_{i=0}^{n} \binom{r}{i} t^i.$$

So

$$\lambda^n(r) = \binom{r}{n}.$$

These are also the coefficients of t^n in above power series.

2. One can get a λ-ring structure on the representation ring for a group G, in which λ^n is induced from the n^{th} exterior power on representations of the group G, $\lambda^n(V) = \Lambda^n(V)$ for V in rep(G).

3. The topological K-theory $K(X)$ of any good space X (para-compact Hausdorff space). This is a λ-ring, in which λ^n is induced from the n^{th} exterior power,

$$\lambda^n(B) = \Lambda^n(B),$$

for a vector bundle B over X. In particular the K-theory of a point is $K(pt) \cong \mathbb{Z}$ with the structure of λ-operations given in example 1.

§ 2.6 Adams operations

The aim of this section is to introduce the notion of Adams operations on λ-ring. The λ-operations have complicated axioms. It can be difficult to construct a λ-ring structure on some kinds of commutative ring and λ-operations are not group homomorphisms. So in [2] Adams introduced the ψ^n-operations to study vector fields on spheres from the λ-operations on a ring R. We will use it in the next section.

In fact, ψ^n-operations give us another type of ring, which is a commutative ring R, equipped with a sequence of functions

$$\psi^n : R \to R,$$

for all $n \geq 1$, satisfying certain properties. These are called ψ-rings.

The ψ-rings are much easier to deal with and sometimes we will need to pass to them to execute some calculations for λ-rings and to construct λ-ring structures on some particular types of rings. For example Wilkerson in [55] explains that constructing ψ-operations on a $\mathbb{Z}$-torsion free ring R that satisfy the axiom

$$\psi^p(r) \equiv r^p \pmod{pR}$$

for every $r \in R$ and every prime p, is sufficient to construct a λ-ring structure on R (Theorem 2.6.10). He also considers the Adams operations to show that a λ-ring whose ψ^n-operations all are the identity for all $n \geq 0$ is a binomial ring (Theorem 2.7.1).

The Adams operations on a λ-ring R, $\psi^n : R \longrightarrow R$ for $n \geq 1$, are defined by using the λ-operations on R. We construct the group homomorphism

$$\lambda_t : R \longrightarrow R[[t]].$$

We know from (2.24) that the addition formula is given by

$$\lambda_t(r_1 + r_2) = \lambda_t(r_1).\lambda_t(r_2),$$

for $r_1, r_2 \in R$. In order to obtain an additive homomorphism $\psi_t : R \longrightarrow R$ a natural idea is to apply the logarithm to (2.24). Precisely, Since $\lambda_t(r)$ has constant equal to 1, we apply the power series formula

$$\log(1+x) = \sum_{i=1}^{\infty}(-1)^{i+1}\frac{x}{i},$$

to $\log(\lambda_t(r))$ which is means we add denominators into R, say by tensoring with $\mathbb{Q}$. We obtain group homomorphism with coefficients of power series of t of $\log(\lambda_t(r))$ which take value in $R \otimes \mathbb{Q}$. Now by applying the operator $\frac{d}{dr}$ to this we eliminate $\mathbb{Q}$. Therefore we obtain from the above information this generating function,

$$\psi_{-t}(r) = -t\frac{d}{dt}(\log \lambda_t(r)) = \frac{-t\lambda_t(r)'}{\lambda_t(r)}. \tag{2.26}$$

Definition 2.6.1. Let R be a λ-ring. We define the n^{th} *Adams operations* on R $\psi^n : R \to R$ by considering the generating function (2.26) for all $n \geq 1$ and $r \in R$, where

$$\psi_t(r) = \sum_{i \geq 1} \psi^i(r)t^i. \tag{2.27}$$

On other words, $\psi^i(r)$ is the coefficient of $(-t)^i$ in ψ_{-t}.

Example 2.6.2. We know from the previous section that the ring $\mathbb{Z}$ of integers is a λ-ring with λ-operations given by $\lambda_t(a) = (1+t)^a$ for $a \in \mathbb{Z}$. Then the Adams operations in the ring $\mathbb{Z}$ are given by

$$\psi_{-t}(a) = -t\frac{d}{dt}(log(1+t)^a) = \frac{-at}{1+t}.$$

This implies that

$$\psi_t(a) = \frac{at}{1-t} = a(t + t^2 + \cdots).$$

So for all $i \geq 1$, $\psi^i(a) = a$.

Later we will show that the same thing holds in all binomial rings.

The Adams operations satisfy the following properties.

Proposition 2.6.3. *[2]Let R be a λ-ring. For fixed $i, j \geq 1$ and $r_1, r_2 \in R$, the following properties hold in R.*

1. *$\psi^i : R \longrightarrow R$ is a ring homomorphism.*
2. *$\psi^1 =$Id.*
3. *$\psi^i\psi^j = \psi^{ij} = \psi^j\psi^i$.*
4. *$\psi^p(r) \equiv r^p \pmod{pR}$ for all prime numbers p.*

The Adams operations are connected to the λ-operations by the following formula, which is known as Newton's Formula (which is quite closely related to Newton's formula for symmetric functions see [57, Theorem 3.9], but recursive rather that closed formula relating Adams operations and λ-operations).

Theorem 2.6.4. *[57, Theorem 3.10]The following equality holds in a λ-ring R.*

$$\psi^n(r) = \lambda^1(r)\psi^{n-1}(r) - \lambda^2(r)\psi^{n-2}(r) + \cdots + (-1)^n\lambda^{n-1}(r)\psi^1(r) + (-1)^{n+1}n\lambda^n(r), \tag{2.28}$$

for $r \in R$ and $n \geq 1$.

Proof. For a proof see [57, Theorem 3.10]. □

In other words, Newton's Formula gives a recursive formula for the Adams operations in terms of λ-operations. So we can calculate Adams operations recursively in terms of λ-operations.
Here are the values of Adams operations for some small values of n, in terms of λ-operations.

1. $\psi^1(r) = \lambda^1(r) = r$.
2. $\psi^2(r) = r^2 - 2\lambda^2(r)$.
3. $\psi^3(r) = r^3 - 3r\lambda^2(r) + 3\lambda^3(r)$.
4. $\psi^4(r) = r^4 - 4r^2\lambda^2(r) + 4r\lambda^3(r) - 4r\lambda^4(r) + 2(\lambda_2(r))^2$.
5. $\psi^5(r) = r^5 - 5r^3\lambda^2(r) + 5r^2\lambda^3(r) - 5r\lambda^4(r) + 5\lambda_5(r) + 5(\lambda_2(r))^2 - 5\lambda_2(r)\lambda_3(r)$.

Theorem 2.6.5. *Let R be a λ-ring, then every Adams operation on R, $\psi^n : R \to R$ for $n \geq 1$, is a λ-homomorphism.*

Proof. For a proof see [57, Theorem 3.6]. □

Next we introduce another type of ring closely related to λ-rings, which is known as a ψ-ring.

Definition 2.6.6. A *ψ-ring* is a commutative ring R with unit, together with a sequence of ring homomorphisms $\psi^n : R \to R$, for all $n \geq 1$, which are called ψ-operations such that the following axioms are satisfied.

1. $\psi^1 =$Id,

2. $\psi^i\psi^j = \psi^{ij} = \psi^i\psi^j$, for all $r \in$ R and $i, j \geq 1$.

We say that a ψ-ring R is special if it also satisfies the axiom

$$\psi^p(r) \equiv r^p \pmod{pR} \tag{2.29}$$

for each prime p.

Example 2.6.7. The ring of integers $\mathbb{Z}$ is a ψ-ring with ψ-operations given by $\psi^i(n) = n$ for all $n \in \mathbb{Z}$ and $i \geq 0$.

Example 2.6.8. In general every commutative ring R with unit is a ψ-ring with ψ-operations given by $\psi^i(r) = r$ for all $r \in R$ and $i \geq 0$.

Definition 2.6.9. Let R_1 and R_2 be ψ-rings, then a ring homomorphism $f : R_1 \to R_2$ is called *ψ-homomorphism* if it commutes with the ψ-operations that is

$$\psi^n(f(r)) = f(\psi^n(r)),$$

for all $r \in R_1$ and $n \geq 0$. We write the set of all ψ-homomorphisms by $\mathrm{Hom}_\psi(R_1, R_2)$. We write $\mathfrak{Ring}_\psi$ for the category of ψ-rings whose objects are ψ-rings and morphisms are ψ-homomorphisms.

We know from the previous section that the λ-operations are neither additive nor multiplicative and a λ-ring R has some complicated axioms. Thus it can be hard to construct a λ-ring structure on some types of rings.

However the ψ-ring axioms are easier to deal with. So Wilkerson in [55] showed that to construct a special ψ-ring structure on a $\mathbb{Z}$-torsion free ring R, it is enough to construct a λ-ring structure on R from it is Adams operation the ones that given the ψ-ring structure. We will use it in the coming section when we show that binomial rings are preserved by localization and completion.

Theorem 2.6.10. *Let R be a $\mathbb{Z}$-torsion free special ψ-ring. Then the Adams operations on R which give the ψ-ring structure on R determine a λ-ring structure on R, related λ-operation to Adams operation by Newton's Formula as in Theorem 2.6.4.*

Proof. For a proof see [57, Theorem 3.54]. □

§ 2.7 Binomial rings as λ-ring structure

The purpose of this section is to introduce a special class of λ-ring structures from the point of view of Adams operations. The result is due to Wilkerson [55] who shows that a λ-ring R whose Adams operations all are the identity on R is a binomial ring (Theorem 2.7.1). Later in this section we use this result to give another description of binomial rings (Proposition 2.7.4).

Here is the main aim of this subsection. This will be a very useful tool in the next section and coming chapters in this thesis.

Theorem 2.7.1. *A λ-ring R whose Adams operations all are the identity map on R, is a binomial ring, in which the λ-operations are given by the binomial operations*

$$\lambda^n(r) = \binom{r}{n},$$

for all $r \in R$ and $n \geq 1$.

To prove theorem 2.7.1 we require the following.

Lemma 2.7.2. *[57, Lemma 5.6]For a λ-ring R whose Adams operations are the identity map the following equality holds in R*

$$r(r-1)\cdots(r-(n-1)) = n!\lambda^n(r),$$

for all $r \in R$ and $n \geq 1$.

Proposition 2.7.3. *[57, Propostion 5.10]Let R be a λ-ring whose Adams operations all are the identity map on R. Then R is a $\mathbb{Z}$-torsion free ring.*

Proof. of Theorem 2.7.1 First by Proposition 2.7.3 R is a $\mathbb{Z}$-torsion free ring. To see the second condition of a binomial ring, pick an element $r \in R$. Then by Lemma 2.7.2 we have

$$r(r-1)\cdots(r-(n-1)) = n!\lambda^n(r).$$

So by the $\mathbb{Z}$-torsion free property of R we obtain

$$R \ni \lambda^n(r) = \binom{r}{n},$$

for $n \geq 1$. □

Here is an application of Theorem 2.7.1, which gives another description of binomial rings.

Proposition 2.7.4. *Let R be a $\mathbb{Z}$-torsion free ring. Then R is a binomial ring if and only if the congruence condition*

$$r^p \equiv r \pmod{pR}, \tag{2.30}$$

for any prime p and all $r \in R$.

Proof. For if part, first for $n \geq 1$, we define the Adams operation on R, $\psi^n : R \longrightarrow R$, by $\psi^n(r) = r$. Consequently R is a $\mathbb{Z}$-torsion free ψ-ring and by hypothesis satisfies the axiom (2.30). So R is special ψ-ring. Now by Theorem 2.6.10 R is a λ-ring in which all the Adams operations are the identity map on R. Therefore by Theorem 2.7.1 R is a binomial ring. The only if part follows by Proposition 2.3.22. □

Next we will see that actually the converse of Theorem 2.7.1 is also true.

Proposition 2.7.5. *[57] Let R be a binomial ring. Then R has a unique λ-ring structure whose Adams operations all are the identity on R. The λ-operations are given by the binomial operations,*

$$\lambda^n(r) = \binom{r}{n}$$

for all $r \in R$ and $n \geq 1$.

Proof. The first part is the analogue of the proof of Proposition 2.7.4 and the second part follows by Lemma 2.7.2. □

We know from Theorem 2.4.11 that for $K \subseteq \mathbb{Z}$ every ring $\mathrm{Int}(K^X, \mathbb{Z})$ on a set X of variables is a binomial ring.

Corollary 2.7.6. *Every ring of $Int(K^X, \mathbb{Z})$ is a λ-ring in which all the Adams operations are the identity and with λ-operations given by the binomial operations.*

From the point of view of Adams operations, we know that the binomial rings are a special class of λ-rings (Theorem 2.7.1).

Proposition 2.7.7. *Every λ-ring R contains a λ-subring S defined by*

$$S = \{r \in R : \psi^n(r) = r \text{ for all } n \geq 1\},$$

which is a binomial ring.

Proof. First it is clear by Proposition 2.6.3(1) S is subring of R. We know from Theorem 2.6.5 that the Adams operations are λ-homomorphisms. This implies that

$$\lambda^k(\psi^n(x)) = \psi^n(\lambda^k(x)).$$

Therefore S is λ-subring of R. Finally by Theorem 2.7.1 it is clear S is a binomial ring. □

Proposition 2.7.8. *Let R be a binomial ring. Then R satisfies the following condition for all $n, m \in \mathbb{N}$ and $r, s \in R$.*

1. $\binom{\binom{r}{m}}{n} = P_{n,m}\left(\binom{r}{1}, \binom{r}{2}, \cdots, \binom{r}{mn}\right).$
2. $\binom{rs}{n} = P_n\left(\binom{r}{1}, \binom{r}{2}, \cdots, \binom{r}{n}; \binom{s}{1}, \binom{s}{2} \cdots \binom{s}{m}\right)$

where $P_n, P_{m,n}$ are the universal polynomials, in Definition 2.5.1.

Proof. We know from Proposition 2.7.5 the binomial ring R has a unique λ-ring structure given by

$$\lambda^n(r) = \binom{r}{n}$$

for all $r \in R$ and $n \geq 1$. Then the coefficients $\binom{r}{n}$ satisfy all axioms in Definition 2.5.1. □

§ 2.8 Category of binomial rings

From the previous section we know that binomial rings are a special class of λ-ring, in which all the Adams operations are the identity. So in this section we use this result and Proposition 2.7.7, first to define a functor Q_λ from the category of the λ-rings to the category of binomial rings. We show that the functor Q_λ is left adjoint to the inclusion functor $\mathrm{I}_{\mathrm{Bin}}$ from the category of binomial rings to the category of λ-rings (Theorem 2.8.12).

First we introduce the category of binomial rings $\mathfrak{BinRing}$ whose objects are binomial rings and morphisms are ring homomorphisms. We show that the binomial ring $\mathrm{Int}(\mathbb{Z}^X)$ on a set X of variables is the free binomial ring on the set X (Proposition 2.8.13). The result is due to Elliott [25]. We use the ring $\mathrm{Int}(\mathbb{Z}^X)$ to give another description of binomial rings (Theorem 2.8.14).

Let R and K be binomial rings and let f be a ring homomorphism $f : R \longrightarrow K$. Then, as K is a $\mathbb{Z}$-torsion free ring, we have

$$\begin{aligned} n!f\binom{r}{n} &= f((r(r-1)\cdots(r-(n-1))) \\ &= f(r)(f(r)-1)\cdots(f(r)-(n-1))) \\ &= n!\binom{f(r)}{n}, \end{aligned}$$

for all $r \in R$ and $n \geq 1$. This implies that the ring homomorphism preserves binomial operations. Thus we mean by binomial homomorphism a ring homomorphism between binomial rings.

Now we give characterizations of binomial rings in terms of homomorphic images.

Proposition 2.8.1. *Let R be a binomial ring. Any $\mathbb{Z}$-torsion free homomorphic image ring K of R is a binomial ring.*

Proof. Let φ be a ring homomorphism from the binomial ring R onto a $\mathbb{Z}$-torsion free ring K. To show that K is a binomial ring we need to show that K is closed under the binomial operations. To see that, pick an element $k \in K$. Since φ is onto, we have $\varphi(r) = k$ for some $r \in R$. By Definition 2.3.4 we have $\binom{r}{n} \in R$. Therefore

$$\varphi\binom{r}{n} = \binom{\varphi(r)}{n} = \binom{k}{n} \in K$$

as desired. □

Proposition 2.8.2. *Let X be a non-empty set, let R be a ring and let $K = map(X, R)$ be the set of all maps from X to R. In symbols,*

$$K = map(X, R) = \{f | f : X \to R\}.$$

Then

1. *K is a ring with the usual operations on functions, that is point-wise addition and multiplication.*
2. *If R is a binomial ring then K is a binomial ring.*

Proof. The first part is obvious. We are going to prove second part. To see that K is closed under taking binomial operations, let $f \in K$. Since $f(x) \in R$ for $x \in X$ and R is a binomial ring,

$$\binom{f}{n}(x) = \binom{f(x)}{n} \in R$$

for $n \geq 0$. This implies that

$$\binom{f}{n} \in K.$$

On the other hand to show that K is $\mathbb{Z}$-torsion free, suppose that $nf = 0$ for $n \in \mathbb{Z}$ and $f \in K$, $f \neq 0$. Then $nf(x) = 0$ for all $x \in X$. Since R is a binomial ring, it is $\mathbb{Z}$-torsion free. This implies that $n = 0$ as desired. □

Definition 2.8.3. The *category of binomial rings* is the category whose objects are binomial rings and morphisms are binomial homomorphisms (ring homomorphism between binomial rings). We denote it by $\mathfrak{BinRing}$ and morphisms by Hom_{Bin}.

Thus, the category of binomial rings $\mathfrak{BinRing}$ is a full subcategory of the category of commutative rings $\mathfrak{CRing}$.
We know from Definition 2.5.6 that λ-rings form a category $\mathfrak{Ring}_\lambda$ with λ-homomorphisms as morphisms.

There is a functor $\mathrm{I}_{Bin} : \mathfrak{BinRing} \to \mathfrak{Ring}_\lambda$, which assigns to a binomial ring the λ-ring with λ-operations given by

$$\lambda^n(x) = \binom{x}{n}$$

and I_{Bin} sends a ring homomorphism to the same map viewed as a λ-homomorphism.

In fact, it is an inclusion functor.

The notion of adjoint functors was first introduced by Kan [36] to compare categories. He studied the adjunction between the Hom functor and the tensor product functor. The functor F is called left adjoint to the functor G and G right adjoint to F if there exists a natural bijection

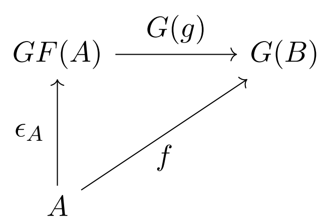
$$\beta : \mathrm{Hom}(F(-), -) \longrightarrow \mathrm{Hom}(-, G(-)).$$

In mathematics we usually use isomorphisms as a way to compare objects with each other. In category theory we use the notion of adjunction between two functors to say two categories are related.

Definition 2.8.4. An *adjunction* between categories $\mathfrak{A}$ and $\mathfrak{B}$ consists of the following.

1. Functors $F : \mathfrak{A} \to \mathfrak{B}$ and $G : \mathfrak{B} \to \mathfrak{A}$.
2. A natural transformation $\epsilon : I_{\mathfrak{A}} \to GF$.

These satisfy the following property. For any $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$ and $f : A \to G(B)$ in $\mathfrak{A}$ there exists a unique $g : F(A) \to B$ in $\mathfrak{B}$, such that the diagram



is commutative. Then we call F the *left adjoint* of G and G the *right adjoint* of F and ϵ the *unit* of the adjunction.

Example 2.8.5. (In the category of the topological spaces). There is a functor $G : \mathfrak{Top} \to \mathfrak{Set}$ taking a topological space to the set of its elements forgetting the topology and taking a continuous function to the corresponding function between sets. Such a functor is called a forgetful functor. This is a right adjoint functor to the free functor $F : \mathfrak{Set} \to \mathfrak{Top}$ which is the functor giving each set the discrete topology.

Example 2.8.6. Here is a more important example for the work in this thesis (in the category of commutative rings). There is a forgetful functor $G : \mathfrak{CRing} \to \mathfrak{Set}$, which takes a commutative ring to the set of its elements forgetting the ring structure and takes a ring homomorphism to the corresponding function between sets. This is a right adjoint functor to the free functor $F : \mathfrak{Set} \to \mathfrak{CRing}$ taking each set to the free commutative ring (polynomial ring) generated by this set.

The definition of adjoint functor is useful for getting an axiomatic understanding of adjunction. But it is useful to consider the adjunction as a natural isomorphism between Hom-sets. The following is equivalent to Definition 2.8.4. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two categories. An *adjunction* from $\mathfrak{A}$ to $\mathfrak{B}$ is a triple (G, F, ρ) such that

1. $F : \mathfrak{A} \to \mathfrak{B}$ and $G : \mathfrak{B} \to \mathfrak{A}$ are functors.
2. There is a bijection ρ for each pair of objects $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$

$$\rho = \rho_{A,B} : \mathrm{Hom}_{\mathfrak{B}}(F(A), B) \cong \mathrm{Hom}_{\mathfrak{A}}(A, G(B))$$

   which is natural in both A and B.

Have adjunction means that for all $f : A \to C$ in $\mathfrak{A}$ and $g : B \to D$ in $\mathfrak{B}$ the following diagram is commutative

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathfrak{B}}(F(C), B) & \xrightarrow{Ff^*} & \mathrm{Hom}_{\mathfrak{B}}(F(A), B) & \xrightarrow{g_*} & \mathrm{Hom}_{\mathfrak{B}}(F(A), D) \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
\mathrm{Hom}_{\mathfrak{A}}(C, G(B)) & \xrightarrow{f^*} & \mathrm{Hom}_{\mathfrak{A}}(A, G(B)) & \xrightarrow{Gg_*} & \mathrm{Hom}_{\mathfrak{A}}(A, G(D))
\end{array}$$

Then F is a left adjoint functor of G and G is a right adjoint functor of F.

Elliott in [25] defined a right adjoint to inclusion functor I_{Bin}. The main result of this section, is to defined a left adjoint Q_λ to the functor I_{Bin}.

Before we give the construction of the functor Q_λ, we will require the following from [57].

Definition 2.8.7. Let R be a λ-ring. An ideal I in R is called a λ*-ideal* if

$$\lambda^n(a) \in I,$$

for all $a \in I$, and $n \geq 1$.

Proposition 2.8.8. *[57, Proposition 1.28]Let R be a λ-ring. Then the usual ideal J in R generated by the set $A = \{s_t\}_{t\in T}$, is a λ-ideal if and only if $\lambda^n(s_t) \in J$ for $n \geq 1$ and $t \in T$.*

Proposition 2.8.9. *[57, Proposition 1.27]Let R be a λ-ring, and let I be a λ-ideal of R. Then the quotient ring R/I is a λ-ring, with λ-operations given by*

$$\lambda^n(r + I) = \lambda^n(r) + I,$$

for all $n \geq 1$ and $r \in R$. □

Let us record the following results before we give the construction of the functor Q_λ from the category of λ-rings to the category of binomial rings.

Proposition 2.8.10. *Let R be a λ-ring and let*

$$S_R = \{\psi^k(r) - r :\ r \in R \text{ and } k \geq 1\}.$$

Then the ideal I generated by S_R is a λ-ideal in R.

Proof. By Proposition 2.8.8 we need to show that if $s \in S_R$ then $\lambda^n(s) \in I$ for $n \geq 1$. Now let $s = \psi^k(r) - r$ for some $r \in R$ and some $k \geq 1$. Then by Definition 2.5.1 (4), we have

$$0 = \lambda^n(r - r) = \sum_{n=i+j} \lambda^i(r)\lambda^j(-r). \tag{2.31}$$

Also

$$\lambda^n(s) = \lambda^n(\psi^k(r) - r) = \sum_{n=i+j} \lambda^i\Big(\psi^k(r)\Big)\lambda^j(-r). \tag{2.32}$$

Then from (2.32)-(2.31) we obtain that

$$\begin{aligned}\lambda^n(s) &= \sum_{j=0}^{n}\Big(\lambda^{n-j}(\psi^k(r)) - \lambda^{n-j}(r)\Big)\lambda^j(-r),\\ &= \sum_{j=0}^{n}\Big(\psi^k(\lambda^{n-j}(r)) - \lambda^{n-j}(r)\Big)\lambda^j(-r) \text{ by Theorem 2.6.5.}\end{aligned}$$

This implies that $\lambda^n(s) \in I$, as desired. □

We recall from Theorem 2.7.1 that a λ-ring whose Adams operations all are the identity is a binomial ring.

Theorem 2.8.11. *Let R be a λ-ring, let $S_R = \{\psi^k(r) - r : for\, r \in R\ and\, k \geq 1\}$ and let I be the ideal in R generated by S_R. Then the quotient ring R/I is a binomial ring.*

Proof. First by Proposition 2.8.10, I is a λ-ideal of R. By Proposition 2.8.9 the quotient ring R/I is a λ-ring.
Finally, by Theorem 2.7.1 need to show that all the Adams operations in R/I are the identity. To see that, pick an element $r + I$ of R/I for $r \in R$, then $\psi^k(r) - r \in I$ and this implies that

$$\psi^k(r) + I = r + I.$$

Therefore

$$\psi^k(r + I) = r + I,$$

as desired. □

We are now in the right position to construct the left adjoint functor to the functor I_{Bin}, which we denote by

$$\mathrm{Q}_\lambda : \mathfrak{Ring}_\lambda \to \mathfrak{BinRing}.$$

It will be described as follows. For any λ-ring R let

$$\mathrm{Q}_\lambda(R) = R/I,$$

where

$$S_R = \{\psi^k(r) - r : \text{for } r \in R \text{ and } k \geq 1\}$$

and I is the ideal of R generated by S_R. We know from Proposition 2.8.9 that I is a λ-ideal of R and by Theorem 2.8.11, R/I is a binomial ring. Then Q_λ defines a functor from the category of λ-rings to the category of binomial rings by taking a λ-homomorphism

$$f : R \to K$$

to the induced binomial homomorphism

$$\widetilde{f} : R/I \to K/J,$$

where J is ideal generated by S_K. We get such an induced map since every λ-homomorphism is a ring homomorphism and

$$f(I) \subseteq J.$$

Now we can give the main result of this section, which shows that the functor Q_λ is left adjoint to the inclusion functor I_{Bin}.

Theorem 2.8.12. *Let R be a λ-ring and let $S_R = \{\psi^k(r) - r : \ r \in R \text{ and } k \geq 1\}$. Then the functor Q_λ from λ-rings to binomial rings is left adjoint to the functor I_{Bin} from binomial rings to λ-rings: in other words there is a natural bijection,*

$$Hom_{Bin}(\mathrm{Q}_\lambda R, K) \cong \ Hom_\lambda(R, I_{Bin}K),$$

for all $R \in \mathfrak{Ring}_\lambda$ and $K \in \mathfrak{BinRing}$.

Proof. First we define the map

$$\theta : \mathrm{Hom}_\lambda(R, \mathrm{I}_{Bin}K) \to \mathrm{Hom}_{\mathrm{Bin}}(\mathrm{Q}_\lambda R, K)$$

by $(\theta(g))[r] = g(r)$ for $g \in \mathrm{Hom}_\lambda(R, I_{\mathrm{Bin}}K)$ and $r \in R$. Then to show that $\theta(g)$ is well defined it is sufficient to show that for the ideal I of R generated by S_R, $g(I) = 0$. To see this, let $a \in I$. By construction of S_R,

$$a = \sum_{i,k} s_{i,k}(\psi^k(r_{i,k}) - r_{i,k})$$

for some $s_{i,k}$, $r_{i,k} \in R$. This implies that,

$$\begin{aligned} g(a) &= \sum_{i,k} g(s_{i,k})g(\psi^k(r_{i,k}) - r_{i,k}) \\ &= \sum_{i,k} g(s_{i,k})(\psi^k g(r_{i,k}) - g(r_{i,k})) = 0, \end{aligned}$$

as K is a binomial ring. Now to see that θ is injective, suppose that $\theta(g) = \theta(h)$ for $g, h \in \mathrm{Hom}_\lambda(R, \mathrm{I}_{\mathrm{Bin}}K)$. This implies that $(\theta(g))[r] = (\theta(h))[r]$ for all $r \in R$, therefore $g(r) = h(r)$. Finally we show that θ is surjective. Let

$$h \in \mathrm{Hom}_{\mathrm{Bin}}(\mathrm{Q}_\lambda R, K).$$

We want to show that $h = \theta(g)$ for some $g \in \mathrm{Hom}_\lambda(R, \mathrm{I}_{\mathrm{Bin}}K)$. Consider the following commutative diagram

$$\begin{array}{ccc} R/I & \xrightarrow{\quad h \quad} & K \\ \uparrow{\scriptstyle \pi} & \nearrow{\scriptstyle g} & \\ R & & \end{array}$$

Using this we can define the free functor . This is left adjoint to the forgetful functor

Here h is a λ-ring homomorphism and we define $g = h \circ \pi$. As π is a λ-ring homomorphism so is g and $\theta(g) = h$. □

We recall from Theorem 2.4.7 that the ring $\mathrm{Int}(\mathbb{Z}^X)$ on a set X of variables is a binomial ring. Next we show that the ring $\mathrm{Int}(\mathbb{Z}^X)$ is the free binomial ring on the set X.

Proposition 2.8.13. *[25] Let X be a non-empty set of variables and R be a binomial ring. The free functor $F_{Bin} : \mathfrak{Set} \to \mathfrak{BinRing}$, which takes a set X to the free binomial ring generated by this set and takes functions between sets to the corresponding ring homomorphisms is left adjoint to the forgetful functor $G_{Bin} : \mathfrak{BinRing} \to \mathfrak{Set}$ which sends a binomial ring to its underlying set and takes a ring homomorphism to the underlying function between sets. This means that there is a natural bijection*

$$Hom_{Bin}(Int(\mathbb{Z}^X), R) \cong Hom(X, R).$$

In other words, the ring $Int(\mathbb{Z}^X)$ is the free binomial ring on a set X of variables.

Proof. First we define a map

$$\theta : \mathrm{Hom}_{\mathrm{Bin}}(\mathrm{Int}(\mathbb{Z}^X), R) \longrightarrow \mathrm{Hom}(X, R),$$

by $\theta(\gamma)(x) = \gamma(x)$ for $\gamma \in \mathrm{Hom}_{\mathrm{Bin}}(\mathrm{Int}(\mathbb{Z}^X), R)$ and $x \in X$, that is by restriction. Then for any $g \in \mathrm{Hom}(X, R)$, by Definition 2.3.4 and Theorem 2.4.6 it follows that the ring homomorphism

$$\xi : \mathbb{Q}[X] \longrightarrow R \otimes_{\mathbb{Z}} \mathbb{Q}$$

determined by $\xi(x) = g(x)$ for each $x \in X$, restricts to a ring homomorphism

$$\bar{\xi} : \mathrm{Int}(\mathbb{Z}^X) \longrightarrow R$$

determined by

$$x \longmapsto g(x).$$

This proves that θ is surjective. Again because ring homomorphisms respect binomial operations, we have

$$\gamma\binom{X}{I} = \prod_{x \in X} \binom{\gamma(x)}{i_x},$$

for I a multi-index. Therefore the value on any polynomial in the variables in X is totally determined by γ. Therefore θ is injective. □

Now we use Proposition 2.8.13 to describe general binomial rings in relation to the rings $\mathrm{Int}(\mathbb{Z}^X)$ as homomorphic images.

Theorem 2.8.14. *[57] A $\mathbb{Z}$-torsion free ring R is a binomial ring if and only if R is the homomorphic image of a binomial ring Int$(\mathbb{Z}^X)$ on a set X of variables.*

Proof. First suppose that R is $\mathbb{Z}$-torsion free and there exists a surjective ring homomorphism

$$\varphi : \mathrm{Int}(\mathbb{Z}^X) \longrightarrow R.$$

We know from Theorem 2.4.7 that the ring $\mathrm{Int}(\mathbb{Z}^X)$ is a binomial ring. Then by Proposition 2.8.1, R is a binomial ring.

Conversely suppose that R is a binomial ring. We need to see that there exists a surjective ring homomorphism

$$\varphi : \mathrm{Int}(\mathbb{Z}^X) \longrightarrow R.$$

Consider $X = \{x_r : \ r \in R\}$ and the function

$$f : X \longrightarrow R,$$

given by

$$x_r \longmapsto r.$$

Then by Proposition 2.8.13 we can extend f to the surjective ring homomorphism

$$\varphi : \mathrm{Int}(\mathbb{Z}^X) \longrightarrow R,$$

determined by $\varphi(x_r) = f(x_r) = r$. □

§ 2.9 Localization and completion of binomial rings

The goal of this section is to show that binomial rings are preserved under localization (Theorem 2.9.5) and completion (Theorem 2.9.20). We use fact that λ-rings are closed under localization and completion and the point of view of Adams operations on λ-rings Theorem 2.7.1.

Applying this result to the binomial ring $\mathbb{Z}$ of integers, it will follow that the p-local integers $\mathbb{Z}_{(p)}$ is a binomial ring and the p-adic integers $\hat{\mathbb{Z}}_p$ is also a binomial ring.
Finally more generally we turn our attention to generalizations of the notion of integer-valued polynomials. In particular the ring of integer-valued polynomials on an integral domain D with its quotient field F, $\mathrm{Int}(D)$. We also show that the ring $\mathrm{Int}(D)$ over a binomial domain D is a binomial ring (Proposition 2.9.13). For more detail on localization and completion see [51] and [24].

2.9.1 Localization of binomial rings

First we started by describing localization of commutative rings which is a generalisation of the idea of constructing a fraction field from an integral domain. We also give some examples of it.

A multiplicatively closed subset of commutative ring R with unit is a subset $S \subset R\setminus\{0\}$, which is closed under multiplication and such that $1 \in S$. We define a relation on $R \times S$ by

$$r/s \sim k/t \text{ if and only if } x(rt - ks) = 0,$$

for some $x \in S$. This is an equivalence relation. Then the localization of R at S is defined to be the set of equivalence classes of symbols r/s with $r \in R$ and $s \in S$. It is denoted by $S^{-1}R$ and it has a ring structure corresponding to the usual addition and multiplication laws of fractions,

$$\left[\frac{r}{s}\right] + \left[\frac{k}{t}\right] = \left[\frac{rt+ks}{st}\right], \tag{2.33}$$

$$\left[\frac{r}{s}\right].\left[\frac{k}{t}\right] = \left[\frac{rk}{st}\right]. \tag{2.34}$$

There is a canonical ring homomorphism $\theta_S : R \to S^{-1}R$ defined by $\theta(r) = r/1$ for all $r \in R$ with

$$Ker(\theta_S) = \{r \in R : tr = 0 \text{ for some } t \in S\}.$$

Example 2.9.1. Let $S = \mathbb{Z}\setminus\{0\}$ which is a multiplicatively closed subset of the integer ring $\mathbb{Z}$. Then $S^{-1}\mathbb{Z} = \mathbb{Q}$.

Example 2.9.2. Let p be prime. Then $S_{(p)} = \mathbb{Z}\setminus p\mathbb{Z}$ is a multiplicative closed subset of the integer ring $\mathbb{Z}$. Then the localization $\mathbb{Z}_{(p)} = S^{-1}\mathbb{Z}$ is a ring which is called the $p-local$ integer ring. This is given by,

$$\mathbb{Z}_{(p)} = \left\{\frac{a}{b} : a, b \in \mathbb{Z}, p \nmid b\right\},$$

which is a subring of the ring $\mathbb{Q}$.

Before we give the main result of this subsection we will require the following from [57].

Theorem 2.9.3. *Let S be a multiplicatively closed subset of a λ-ring R without zero-divisors, and such that $\psi^n(S) \subseteq S$ for all $n \geq 1$. Then the localization $S^{-1}R$ has a λ-ring structure from the ψ-ring structure on $S^{-1}R$ given by*

$$\psi^n\left(\frac{r}{s}\right) = \frac{\psi^n(r)}{\psi^n(s)}$$

for all $r \in R$, $s \in S$ and $n \geq 1$.

Proof. For a proof see [57, p. 74]. □

We need the following preliminary result for the construction of $S^{-1}R$ as a binomial ring where R is a binomial ring.

Lemma 2.9.4. *Let S be a multiplicative closed subset of the $\mathbb{Z}$-torsion free ring R. Then the localization $S^{-1}R$ is a $\mathbb{Z}$-torsion free ring.*

Proof. Pick an element $\frac{r}{s} \in S^{-1}R$ for $r \in R$ and $s \in S$. Suppose that $n.\frac{r}{s} = 0$ for $n \in \mathbb{Z}$. Then there exists $t \in S$ such that $tnr = 0$. So either $n = 0$ or $tr = 0$ but if $tr = 0$ this implies that

$$\frac{r}{s} = \frac{tr}{ts} = \frac{0}{ts} = 0$$

as desired. □

Here is our main result of this subsection, by applying Theorem 2.9.3 to spacial case shows that the binomial property is preserved by localization.

Theorem 2.9.5. *Let S be a multiplicative closed subset of the binomial ring R. Then the localization $S^{-1}R$ is a binomial ring.*

Proof. We know from Proposition 2.7.5 that the binomial ring R is a λ-ring, whose Adams operations all are the identity on R. This implies that $\psi^n(S) \subseteq S$. Also by Lemma 2.9.4 the ring $S^{-1}R$ is $\mathbb{Z}$-torsion free. Therefore by Theorem 2.9.3 the localization $S^{-1}R$ is a λ-ring.
Now to show that $S^{-1}R$ is a binomial ring it is sufficient to show that all Adams operations on $S^{-1}R$ are the identity. To see this, pick an element $a \in S^{-1}R$ such that $a = \frac{r}{s}$ for $r \in R$ and $s \in S$. Then

$$\psi^n\left(\frac{r}{s}\right) = \frac{\psi^n(r)}{\psi^n(s)} = \frac{r}{s}$$

as desired. □

Corollary 2.9.6. *The localization of the integers $\mathbb{Z}$ at a set of primes is a binomial ring. In particular, the p-local integer ring $\mathbb{Z}_{(p)}$ is a binomial ring.* □

Proposition 2.9.7. *Let R be a binomial ring and let S be a multiplicatively closed subset of R. Then*

$$f : R \longrightarrow S^{-1}R$$

is binomial homomorphism. □

The following results are consequences of Theorem 2.9.5. First for a prime p, we define the p-localized integer-valued polynomials by

$$\mathrm{Int}(\mathbb{Z}^X)_{(p)} = \mathrm{Int}(\mathbb{Z}^X) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}.$$

Proposition 2.9.8. *For a prime p, we have*

$$Int(\mathbb{Z}^X)_{(p)} = \{f \in \mathbb{Q}[X] \mid f(\mathbb{Z}_{(p)})^X \subseteq \mathbb{Z}_{(p)}\},$$

where $(\mathbb{Z}_{(p)})^X = Hom(X, \mathbb{Z}_{(p)})$.

Proof. First we need to show that the binomial operations take $(\mathbb{Z}_{(p)})^X$ to $\mathbb{Z}_{(p)}$. This is clear by Corollary 2.9.6.

Conversely, suppose $f \in \mathbb{Q}[X]$ of degree n satisfies $f(\mathbb{Z}_{(p)})^X \subseteq \mathbb{Z}_{(p)}$. Then by [57, Proposition 5.31], f can be written by

$$f = \sum_{j=0}^{n} r_j \binom{X}{J_i},$$

for $r_j \in \mathbb{Q}$ and J multi-index as in (2.16). To show that $r_j \in \mathbb{Z}_{(p)}$, the proof is analogous to the proof of Theorem 2.4.6. □

Corollary 2.9.9. *The ring Int*$(\mathbb{Z}^X)_{(p)}$ *is a binomial ring.* □

Now we turn attention to rings of integer-valued polynomials over a subset of $\mathbb{Z}_{(p)}$.

Definition 2.9.10. For $K \subseteq \mathbb{Z}_{(p)}$, we say that a polynomial $f \in \mathbb{Q}[X]$ which satisfies $f(K^X) \in \mathbb{Z}_{(p)}$ for all $k \in K$ is an *integer-valued polynomial on* $\mathbb{Z}_{(p)}$ *over the subset* K, where $K^X = \text{Hom}(X, K)$, which is the set of functions $\underline{n}$ as in Definition 2.4.1. We computing f at any $\underline{n}$ by replacing each $x \in X$ with $k \in K$. Then the condition $f((K^X) \subseteq (\mathbb{Z}_{(p)}$ means that $f(\underline{n}) \in \mathbb{Z}_{(p)}$. We define the set of all *integer-valued polynomials on* $\mathbb{Z}_{(p)}$ *over the subset* K as,

$$\text{Int}(K^X, \mathbb{Z}_{(p)}) = \{f \in \mathbb{Q}[X] \mid f(K^X) \subseteq \mathbb{Z}_{(p)}\}. \tag{2.35}$$

This is a subring of $\mathbb{Q}[X]$ and it is called the ring of integer-valued polynomials on $\mathbb{Z}_{(p)}$ over K on a set X of variables. In particular we have

$$\text{Int}(K^{\{x\}}, \mathbb{Z}_{(p)}) = \{f(x) \in \mathbb{Q}[x] \ : \ f(K) \subset \mathbb{Z}_{(p)}\}, \tag{2.36}$$

called the ring of integer-valued polynomials of $\mathbb{Z}_{(p)}$ over the subset K in one variable x.

Note that the ring $\text{Int}(\mathbb{Z}_{(p)}^X)$ is integer-valued on $\mathbb{Z}_{(p)}$ over $\mathbb{Z}_{(p)}$, that is

$$\text{Int}(\mathbb{Z}_{(p)}^X) = \text{Int}(\mathbb{Z}_{(p)}^X, \mathbb{Z}_{(p)}).$$

So for $K \subseteq \mathbb{Z}_{(p)}$, we have inclusion

$$\mathbb{Z}_{(p)}[X] \subset \text{Int}(\mathbb{Z}_{(p)}^X) \subseteq \text{Int}(K^X, \mathbb{Z}_{(p)}) \subseteq \mathbb{Q}[X]. \tag{2.37}$$

Corollary 2.9.11. *For* $K \subseteq \mathbb{Z}_{(p)}$ *the ring Int*$(K^X, \mathbb{Z}_{(p)})$ *is a binomial ring.* □

Next we focus on the generalisation of integer-valued polynomials on an integral domain. In [15] the authors consider an ring of integral domain D with quotient field F. Let

$$\text{Int}(D^X) = \{f \in F[X] : f(D^X) \subseteq D\},$$

where $D^X = \text{Hom}(X, D)$. It can easily be seen that the ring $\text{Int}(D^X)$ is a D-module, it contains $D[X]$ and it is a subring of $F[X]$. Therefore we have the following inclusions,

$$D[X] \subseteq \text{Int}(D^X) \subseteq F[X]. \tag{2.38}$$

Definition 2.9.12. A binomial domain is a $\mathbb{Z}$-torsion free commutative integral domain D with unit, which is closed under the binomial operations

$$\binom{d}{n} = \frac{d(d-1)(d-2)\cdots(d-(n-1))}{n!},$$

for every $d \in D$ and $n \geq 0$, where $\binom{d}{0} = 1$.

We will use Theorem 2.9.5 to show that the ring $\mathrm{Int}(D^X)$ for a binomial domain D is a binomial ring.

Proposition 2.9.13. *Let D be a binomial domain with quotient field F. Then the ring $Int(D^X)$ is a binomial ring.*

Proof. First by Theorem 2.9.5 F is a binomial ring. It is clear by (2.38), that $\mathrm{Int}(D^X)$ is $\mathbb{Z}$-torsion free. The proof of the other condition of a binomial ring is analogous to the proof of Theorem 2.4.7. □

Corollary 2.9.14. *Let S be a multiplicative closed subset of the binomial domain D. Then the ring $S^{-1}Int(D^X)$ is a binomial ring.* □

2.9.2 Completion of binomial rings

First we start with some background on completion of rings. We say that a topological space (X, τ) is induced by a metric space (X, d) if the open balls in (X, d) form a basis of the topology τ.

By topological ring we mean a commutative ring R with unit equipped with metric d on R such that the ring operations on R are continuous regarding the metric topology.

Let R be a ring equipped with a filtration

$$R = I_0 \supset I_1 \supset I_2 \supset \cdots$$

by ideals I_n in R with

$$\bigcap_{i=1}^{\infty} I_i = \{0\}.$$

Then for fixed $e \in \mathbb{R}$, $e > 1$, we define a metric on R using these ideals by

$$d(x,y) = \begin{cases} e^{-k} & \text{for} \quad x \neq y, \\ 0 & \text{for} \quad x = y. \end{cases}$$

where

$$x - y \in I_k \text{ but } x - y \notin I_{k+1}.$$

Example 2.9.15. Fix a prime p, the ring of integers $\mathbb{Z}$ with

$$(p) \supset (p^2) \supset \cdots \tag{2.39}$$

is a topological ring with respect to the metric $d : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ defined by $d(x, y) = 2^{-k}$ if and only if

$$x - y \equiv 0 \pmod{p^k} \text{ but } x - y \not\equiv 0 \pmod{p^{k+1}}$$

for a p prime.

Let

$$R = I_0 \supset I_1 \supset I_2 \supset \cdots$$

be a decreasing sequence of ideals in R. A sequence $\{x_i\} = \{x_1, x_2, \cdots\}$ of elements in R converges to $x \in R$ with respect to the metric d on R if for all n there exists N such that $x_i - x \in I_n$ for all $i \geq N$. A sequence $\{x_i\}$ is called a Cauchy sequence if for all n there exists N such that $x_i - x_j \in I_n$ for all $i, j \geq N$.

Note that in general not every Cauchy sequence converges. We say that the metric d on R is complete if every Cauchy sequence $\{x_i\}$ converges with respect to the metric d on R.

Definition 2.9.16. For a ring R with metric d, we define the *completion* of R with respect to d denoted by $\hat{R}_d$. The elements in $\hat{R}_d$ are represented by the Cauchy sequences in R. The Cauchy sequence $\{x_n\}$ is equivalent to $\{x_m\}$ if

$$d(\{x_n\}, \{x_m\}) = \lim_{n,m \to \infty} d(x_n, x_m) = 0.$$

Then the d-completion $\hat{R}_d$ becomes a topological ring with respect to the ring operations on equivalence classes in $\hat{R}_d$ defined by

$$[\{x_n\}] + [\{y_n\}] = [\{x_n + y_n\}]$$

$$[\{x_n\}].[\{y_n\}] = [\{x_n.y_n\}].$$

There is a ring homomorphism

$$\varphi : R \longrightarrow \hat{R}_d$$

given by

$$r \longrightarrow [\{r, r, r, \cdots\}].$$

Example 2.9.17. The completion of the ring of integers $\mathbb{Z}$ with respect to the descending ideals (2.39) and metric in Example 2.9.15 is the ring called the p-adic integers, denoted by $\hat{\mathbb{Z}}_p$.

Proposition 2.9.18. *[5, p. 498] Let R be a ring with a descending sequence*

$$R = I_0 \supset I_1 \supset I_2 \supset \cdots$$

of ideals in R. Then there is a ring isomorphism

$$\hat{R}_I \cong \varprojlim R/I_n.$$

Before we give the main result of this subsection we require the following from [57].

Theorem 2.9.19. *[57, p. 77]Let d be a metric on a $\mathbb{Z}$-torsion free λ-ring R and let Adams operations $\psi^k : \hat{R}_d \to \hat{R}_d$ be defined by $\psi^k(\{X_n\}) = \{\psi^k(X_n)\}$. Then the ring $\hat{R}_d$ is a λ-ring, where the λ-ring structure is induced from the Adams operation on $\hat{R}_d$.*

Here is our main result of this subsection, by applying Theorem 2.9.19 to spacial case shows that the binomial property is preserved by completion.

Theorem 2.9.20. *Let d be a metric on a binomial ring R. Then the ring $\hat{R}_d$ is a binomial ring.*

Proof. First by definition of binomial ring R is $\mathbb{Z}$-torsion free. This implies that the ring $\hat{R}_d$ is also $\mathbb{Z}$-torsion free. We know from Proposition 2.7.5, that a binomial ring R is a λ-ring in which all Adams operations are the identity on R. Then by Theorem 2.9.19, $\hat{R}_d$ is a λ-ring. Now to show that $\hat{R}_d$ is a binomial ring it is sufficient to show that all Adams operations on it are the identity. To see this let $\{x_n\} \in \hat{R}_d$. Then

$$\psi^n(\{x_n\}) = \{\psi^n(x_n)\} = \{x_n\},$$

as desired. □

Corollary 2.9.21. *The ring $\hat{\mathbb{Z}}_p$ of p-adic integers is a binomial ring.* □

Chapter 3

Binomial ideals of binomial rings

§ 3.1 Introduction

The main purpose of this chapter is to study some classes of a binomial rings by using properties of their binomial ideals. We define a binomial ideal to be an ideal of a binomial ring preserved by binomial operations. In §3.2 we give the definition of a binomial ideal alongside some examples and proving some properties. The proof that the quotient ring of a binomial ring by a binomial ideal is also a binomial ring (Theorem 3.3.1) is given in §3.3. By example we show that an ideal generated by a set is not a binomial ideal in general. In §3.4 we introduce the notion of a binomial ideal generated by a set.

In the following sections we study some classes of a binomial rings. In §3.5 we start with the notion of a binomially simple ring. We show that the ring of integers is a binomially simple ring (Proposition 3.5.2). We know from Theorem 2.4.7, that $\mathrm{Int}(\mathbb{Z}^X)$ is a binomial ring and we will see in Example 3.7.13 that the ring $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ is non-Noetherian ring. In §3.6 we introduce the notion of a binomially principal ring. We give the characterization of a binomial ideal in $\mathrm{Int}(\mathbb{Z}^X)$ on set X of variables, we use it to show that $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ is a binomially principal ring (Theorem 3.6.13). In §3.7 we introduce the notion of a binomially Noetherian ring. In the same way we use the characterization of the binomial ideals in $\mathrm{Int}(\mathbb{Z}^X)$ to show that $\mathrm{Int}(\mathbb{Z}^{\{x,y\}})$ is a binomially Noetherian ring (Theorem 3.7.7). The notion of a binomially filtered ring, which is a binomial ring equipped with descending sequence of binomial ideals given in §3.8. We show that the power series ring

$$\mathbb{Z}[\![\binom{x}{1}, \binom{x}{2}, \binom{x}{3}, \cdots]\!]$$

is a binomial ring (Proposition 3.8.9).

§ 3.2 Binomial ideals

In this section we introduce the concept of a binomial ideal for a binomial ring, with some examples and prove some basic properties. In the same way that as Lie algebras have Lie ideals and λ-rings have λ-ideals, similarly Xantcha gives a short survey of binomial ideals in [56]. We start with the definition of a binomial ideal of a binomial ring.

Definition 3.2.1. Let R be a binomial ring. An ideal I of R is called a *binomial ideal* if

$$\binom{a}{n} \in I$$

for all $a \in I,\ n \geq 1$.

Before we look at some theory of binomial ideals, we will examine this concept by means of several specific examples, which illustrate the definition.

Example 3.2.2. In $\mathrm{Int}(\mathbb{Z}^{\{x\}})$, we define the set

$$I = \{f(x) \in \mathrm{Int}(\mathbb{Z}^{\{x\}})\ :\ f(x) = xg(x) \text{ for some } g(x) \in \mathbb{Q}[x]\}.$$

First it is clear I is a usual ideal in $\mathrm{Int}(\mathbb{Z}^{\{x\}})$. To see that I is closed under taking binomial operations, pick an element $f(x) \in I$ and an integer $n \geq 1$. Then

$$\begin{aligned}\binom{f}{n}(x) &= \binom{f(x)}{n} \\ &= \frac{f(x)(f(x)-1)\cdots(f(x)-(n-1))}{n!} \\ &= \frac{xg(x)(xg(x)-1)\cdots(xg(x)-(n-1))}{n!} \in x\mathbb{Q}[X].\end{aligned}$$

Thus $\binom{f}{n}(x) \in I$ as desired.

A generalisation of the previous example is the following.

Example 3.2.3. Let $h(x) \in \mathrm{Int}(\mathbb{Z}^{\{x\}})$, we define the set

$$I_{h(x)} = \{f(x) \in \mathrm{Int}(\mathbb{Z}^{\{x\}})\ :\ f(x) = h(x)g(x) \text{ for some } g(x) \in \mathbb{Q}[x]\}.$$

Then $I_{h(x)}$ is a binomial ideal in $\mathrm{Int}(\mathbb{Z}^{\{x\}})$.

Example 3.2.4. Let $K \subseteq \mathbb{Z}$. Then for a fixed integer $k \in K$, let

$$I_k = \{f(x) \in \mathrm{Int}(K^{\{x\}}, \mathbb{Z})\ :\ f(k) = 0\}.$$

Then I_k is a binomial ideal in $\mathrm{Int}(K^{\{x\}}, \mathbb{Z})$.

In the same way we give examples of binomial ideals in the binomial ring $\mathrm{Int}(\mathbb{Z}^X)$ on a set X of variables.

Example 3.2.5. Let $x \in X$. In $\mathrm{Int}(\mathbb{Z}^X)$ we define the set

$$I_x = \{f \in \mathrm{Int}(\mathbb{Z}^X) : \ f = xg \text{ for } \ g \in \mathbb{Q}[X]\}.$$

Then I_x is a binomial ideal in $\mathrm{Int}(\mathbb{Z}^X)$.

Here are some properties of binomial ideals.

Proposition 3.2.6. *Let I and J be binomial ideals of a binomial ring R. Then the product of binomial ideals IJ is also a binomial ideal of R. In particular I^n is a binomial ideal of R for $n \geq 1$.*

Proof. First consider an element in IJ of the form ab for $a \in I$ and $b \in J$. We must show that $\binom{ab}{n} \in IJ$ for $n \geq 1$. This follows by applying Theorem 2.3.14(2). Now the general element of IJ is a finite-linear combination of the form

$$Y = \{\sum_{t=1}^{m} a_t b_t : \ a_t \in I, b_t \in J \text{ and } m \in \mathbb{N}\}.$$

Then by applying Theorem 2.3.14(1), we get that $\binom{Y}{n}$ is a finite sum of products of the form

$$\binom{a_1 b_1}{p_1}\binom{a_2 b_2}{p_2}\cdots\binom{a_m b_m}{p_m},$$

where $p_1 + p_2 + \cdots + p_m = n$. This implies that $\binom{Y}{n} \in IJ$ as desired. □

Proposition 3.2.7. *Let I and J be binomial ideals of a binomial ring R. Then the sum of binomial ideals $I + J$ is also a binomial ideal of R. For a collection of binomial ideals $\{I_\alpha\}_{\alpha \in \Lambda}$, the sum is a binomial ideal of R.*

Proof. Consider an element in $I + J$ of the form $a + b$ for $a \in I$ and $b \in J$. Then we must show that $\binom{a+b}{n} \in I + J$ for $n \geq 1$. This follows by applying Theorem 2.3.14(1). □ □

Proposition 3.2.8. *Let I and J be binomial ideals of a binomial ring R. Then the intersection of binomial ideals $I \cap J$ is also a binomial ideal of R. For a collection of binomial ideals $\cap_n I_n$, is a binomial ideal of R.* □

Proposition 3.2.9. *Let R and K be binomial rings and let $f : R \to K$ be a binomial homomorphism. Then the kernel of f is a binomial ideal in R.*

Proof. Let $I = Ker(f)$. First it is clear that I is an ideal in R. To see that I is closed under taking binomial operations, pick an element $a \in I$. Since f is a binomial homomorphism, we have

$$f(\binom{a}{n}) = \binom{f(a)}{n} = \binom{0}{n} = 0,$$

for all $n \geq 1$. This implies that

$$\binom{a}{n} \in I,$$

as desired. □

Next will study the relation between the binomial ideals of a binomial ring R and the binomial ideals of its localization $S^{-1}R$ where S is a multiplicative closed subset of R. We will follow the same argument as shows that the localization $S^{-1}R$ is a binomial ring (Theorem 2.9.5), to see that $S^{-1}I$ is a binomial ideal of $S^{-1}R$ for a binomial ideal I of R.

Proposition 3.2.10. *Let I be a binomial ideal of a binomial ring R and let S be a multiplicative closed subset of R. Then $S^{-1}I$ is a binomial ideal of the binomial ring $S^{-1}R$. Actually every binomial ideal of $S^{-1}R$ is the extension of a binomial ideal of R.*

Proof. Our proof is an adaptation of Theorem 2.9.5 with modification done for satisfying the condition of Theorem 2.9.3. First by Lemma 2.9.4, $S^{-1}I$ is $\mathbb{Z}$-torsion free. To see that $S^{-1}I$ is closed under taking binomial operations, we will show that $S^{-1}I$ satisfies the condition of Theorem 2.6.10. Now

$$\begin{aligned} \frac{a}{s} - \left(\frac{a}{s}\right)^p &= \frac{a}{s} - \frac{a^p}{s^p} \\ &= \frac{s^p(a - a^p) + a^p(s^p - s)}{s^{p+1}}, \end{aligned}$$

for $a \in I$, $s \in S$ and p prime. Since $a^p - a$ and $s^p - s \in pR$, it follows that

$$a^p(s^p - s) \in pI \text{ and } s^p(a - a^p) \in pI.$$

Therefore

$$\frac{a}{s} - \left(\frac{a}{s}\right)^p \text{ lies in } pS^{-1}I.$$

Now by applying Theorem 2.9.3, we obtain a λ-structure on $S^{-1}I$ induced from that on R. Finally by Theorem 2.7.1, it is sufficient to show that all Adams operations on $S^{-1}I$ are the identity. This is clear because I is a binomial ideal of R.

For the proof of the extension property, let J be a binomial ideal of $S^{-1}R$ and let $I = J \cap R$. Then by Proposition 3.2.8 I is a binomial ideal of R. And as for usual ideals, we have $J = S^{-1}I$. □

§ 3.3 Quotient binomial rings

In the same way that a ring R can be factored by an ideal I to build a new ring R/I which is known as the quotient ring, we consider binomial ideals of binomial rings and their corresponding quotient rings to build new binomial rings R/I (Theorem 3.3.1). This will lead us to some interesting theoretical results in this chapter and the next chapters.

We begin with the main result of this section, which says that the quotient ring R/I of a binomial ring R by a binomial ideal I of R is a binomial ring.

Theorem 3.3.1. *If I is a binomial ideal of the binomial ring R, then the quotient ring R/I is a binomial ring.*

To give the proof of Theorem 3.3.1, we need the following useful results. We begin with the $\mathbb{Z}$-torsion free property in the quotient ring R/I.

Proposition 3.3.2. *Let R be a binomial ring, and let I be a binomial ideal of R. Then the quotient ring R/I is $\mathbb{Z}$-torsion free.*

Proof. Consider an element $r+I$ in R/I for $r \in R$. If $m(r+I) = 0+I$ for some $m \in \mathbb{Z}$, $m \neq 0$, it follows that $mr \in I$. We choose $n = |m|$ with $n \in \mathbb{Z}$ such that $n! = m.\bar{m}$ with $\bar{m} \in \mathbb{Z}$, $\bar{m} \neq 0$. Then $n!r \in I$. We want to show that

$$r + I = I,$$

equivalently $r \in I$. Since $n!r \in I$ and I is a binomial ideal of R, by Definition 3.2.1,

$$\binom{n!r}{n} \in I.$$

Now

$$\begin{aligned} \binom{n!r}{n} &= \frac{n!r(n!r-1)(n!r-2)\cdots(n!r-(n-1))}{n!} \\ &= r(n!r-1)(n!r-2)\cdots(n!r-(n-1)). \end{aligned}$$

Thus

$$(-1)^{n-1}(n-1)!r = \binom{n!r}{n} - n!r \cdot x,$$

for some $x \in R$. Since $n!r.x \in I$, it follows that

$$(n-1)!r \in I.$$

Then from $(n-1)!r \in I$ in the same way as above by takeing $n > |m|$, we obtain $(n-2)!r \in I$ and so on. This implies that $r \in I$ as desired. □

Next we will show that the binomial operations on the quotient ring R/I are well defined.

Remark 3.3.3. For a binomial ring R and a binomial ideal I of R, the binomial operations on the quotient ring R/I by multiplication operation structure on the quotient ring are given by

$$\binom{r+I}{n} = \binom{r}{n} + I, \tag{3.1}$$

for $r \in R$ and $n \geq 1$.

Theorem 3.3.4. *Let R be a binomial ring and let I be a binomial ideal of R. Then the binomial structure on the quotient ring R/I given in (3.1) is well defined.*

Proof. Let $r+I$ and $k+I$ be two elements of R/I for $r, k \in R$ such that

$$r + I = k + I.$$

We need to show that

$$\binom{r+I}{n} = \binom{k+I}{n},$$

equivalently

$$\binom{r}{n} + I = \binom{k}{n} + I.$$

This is equivalent to proving

$$\binom{r}{n} - \binom{k}{n} \in I.$$

In Theorem 2.3.14(1), let $a = r - k$ and $b = k$. We have

$$\begin{aligned} \binom{(r-k)+k}{n} &= \sum_{n=p+q} \binom{r-k}{p}\binom{k}{q} \\ &= \binom{k}{n} + \sum_{p=1}^{n} \binom{r-k}{p}\binom{k}{n-p}. \end{aligned}$$

Thus

$$\binom{r}{n} - \binom{k}{n} = \sum_{p=1}^{n} \binom{r-k}{p}\binom{k}{n-p}. \tag{3.2}$$

Since $r - k \in I$ and I is a binomial ideal, by Definition 3.2.1

$$\binom{r-k}{p} \in I$$

for all $p \geq 1$. Then the right hand side of (3.2) is in I. So the left hand side is too. Therefore

$$\binom{r}{n} - \binom{k}{n} \in I$$

as desired. □

If R is a binomial ring and I is an ideal of R, then the quotient ring R/I is not a binomial ring in general. The example below illustrates that.

Example 3.3.5. The ring of integers $\mathbb{Z}$ is a binomial ring, and $2\mathbb{Z}$ is an ideal in $\mathbb{Z}$ generated by 2 but not a binomial ideal of $\mathbb{Z}$. Let $n \in \mathbb{Z}$ and assume n is odd. Then $2n \in 2\mathbb{Z}$, but, for example

$$\binom{2n}{2} = \frac{2n(2n-1)}{2} = n(2n-1) \notin 2\mathbb{Z}.$$

So $2\mathbb{Z}$ is not a binomial ideal. Also, the quotient ring $\mathbb{Z}/2\mathbb{Z}$ is not a binomial ring because it is not $\mathbb{Z}$-torsion free.It is true for all ideals $m\mathbb{Z}$ of $\mathbb{Z}$ by

$$\binom{m\mathbb{Z}}{m\mathbb{Z}} = 1.$$

We are now in the right position to give the proof of Theorem 3.3.1.

Proof. {Theorem 3.3.1} First by Proposition 3.3.2, R/I is $\mathbb{Z}$-torsion free.
In Theorem 3.3.4 we checked that R/I is closed under taking binomial operations. This implies that R/I is a binomial ring. □

Proposition 3.3.6. *Let I be a binomial ideal in a binomial ring R. Then for any binomial ideal J of R the quotient J/I is a binomial ideal of the binomial ring R/I.*

Proof. First by Proposition 3.3.2, J/I is $\mathbb{Z}$-torsion free. To see that J/I is closed under taking binomial operations, pick an element $a + I \in J/I$ for $a \in J$ and let $n \geq 1$. We want to show that

$$\binom{a+I}{n} \in J/I.$$

Since J is binomial ideal,

$$\binom{a+I}{n} = \binom{a}{n} + I \in J/I,$$

as desired. □

Proposition 3.3.7. *Let I be an ideal in the ring R. If I and the quotient ring R/I are both $\mathbb{Z}$-torsion free, then R is also $\mathbb{Z}$-torsion free.*

Proof. Consider $r \in R$ with $r \neq 0$ and suppose that $nr = 0$ for some $n \in \mathbb{Z}$. So

$$n(r + I) = I,$$

that is $n[r] = 0$ in R/I. Since by hypothesis R/I is $\mathbb{Z}$-torsion free, if $[r] \neq 0$, this implies $n = 0$.
On the other hand, if $[r] = 0$, we have $r \in I$ and $nr = 0$. But, since also by hypothesis I is $\mathbb{Z}$-torsion free this also implies $n = 0$ as required. □

Proposition 3.3.8. *Let I be an ideal in the commutative ring R. If I and the quotient ring R/I are both closed under binomial operation, then R is also closed under binomial operation.*

Proof. Let $r \in R$. Then $r + I \in R/I$. By our hypothesis that I is closed under binomial operations, by (3.1),

$$\binom{r+I}{n} = \binom{r}{n} + I.$$

Holds in R/I. This implies that $\binom{r}{n} \in R$. □

Proposition 3.3.9. *If I is a binomial ideal of the binomial ring R, then the quotient map*

$$\varphi : R \to R/I$$

is a binomial homomorphism. □

§ 3.4 Principal binomial ideals

We know by Example 3.3.5 that in a binomial ring the (usual) ideal generated by a set is not a binomial ideal in general. The aim of this section is to introduce the notion of the binomial ideal generated by a set X. We employ the symbol $((X))$ to denote the binomial ideal generated by a set X. In addition we employ the symbol $((a))$ to denote the principal binomial ideal generated by an element a. This will be used in the coming sections when some classes of binomial rings are presented.

Later the relation between $((X))$, where X is a finite set and a usual ideal is described (Proposition 3.4.6). At the end of the section, some examples of binomial rings arising from such binomial ideals in $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ are given (Theorem 3.4.14).

A usual ideal in a ring R is often characterized by a set of generators. To examine whether an ideal J is a binomial ideal in the binomial ring R, we should check whether the ideal J is closed under the binomial operations for an element in the generator set.

Proposition 3.4.1. *Let R be a binomial ring. Then the usual ideal J in R generated by the set $A = \{a_i\}_{i \in I}$ is a binomial ideal if and only if $\binom{a_i}{n} \in J$ for all $i \in I$ and $n \geq 1$.*

Proof. The only if part follows from Definition 3.2.1. For the other direction, consider an element of J of the form ra_i for $r \in R$, $a_i \in A$ and $i \in I$. We must show that

$$\binom{ra_i}{n} \in J,$$

for $n \geq 1$. This holds by applying Theorem 2.3.14(2). Now a general element of J is of the form

$$y = \sum_{t=1}^{m} r_t a_{i_t}.$$

We need to show that $\binom{y}{n} \in J$ for $n \geq 1$. By applying Theorem 2.3.14(1), we get that $\binom{y}{n}$ is a finite sum of products of the form

$$\binom{r_1 a_{i_1}}{p_1}\binom{r_2 a_{i_2}}{p_2}\cdots\binom{r_m a_{i_m}}{p_m},$$

where $p_1 + p_2 + \cdots + p_m = n$. This implies that $\binom{y}{n} \in J$ as required. □

Next we will give a description of a binomial ideal of a binomial ring in terms of a generator set. Later we apply this result to present a characterization of binomial ideals of binomial rings on a set of generators.

Proposition 3.4.2. *Let R be a binomial ring and let $x_i \in R$ for $i = 1, 2, \ldots, k$. Consider the set $A = \left\{ \binom{x_i}{m} \;:\; m \geq 1 \text{ and } i = 1, 2, \ldots, k \right\}$. Then the ideal I generated by A is a binomial ideal in R.*

Proof. First a general element in I is a linear combination

$$Y = \sum_{i=1}^{k} \sum_{m=1}^{M_i} r_{i,m} \binom{x_i}{m},$$

with coefficients $r_{i,m}$ in R. To show that I is a binomial ideal, by Proposition 3.4.1 it is sufficient to show that $\binom{Y}{n} \in I$ for $n \geq 1$. By applying Theorem 2.3.14(1) we get that $\binom{Y}{n}$ is a finite sum of products of the form

$$\binom{r_1 \binom{x_{i_1}}{m_1}}{p_1}\binom{r_2 \binom{x_{i_2}}{m_2}}{p_2}\cdots\binom{r_m \binom{x_{i_t}}{m_t}}{p_t},$$

where $p_1 + p_2 + \cdots + p_t = n$. By Proposition 2.7.8

$$\binom{\binom{x_{i_s}}{m_s}}{p_s} = P_{n_s,m_s}(\binom{x_{i_s}}{1}, \binom{x_{i_s}}{2}, \cdots, \binom{x_{i_s}}{p_s m_s}),$$

where the constant term of P_{p_s,m_s} is zero, so this is a finite sum of terms, each one containing a factor

$$\binom{x_{i_s}}{k} \in I,$$

for various s and k. So $\binom{\binom{x_{i_s}}{m_s}}{p_s}$ lies in I as well. Therefore by Theorem 2.3.14(2), $\binom{r_s\binom{x_{i_s}}{m_s}}{p_s} \in I$. Consequently $\binom{Y}{n} \in I$ as desired. □

A usual ideal in a ring R often comes with a set of generators. We denoted by (X) the ideal generated by X for $X \subset R$.

Example 3.4.3. In $\mathrm{Int}(\mathbb{Z}^{\{w\}})$, let $I = (w)$. So $w \in I$, but

$$\binom{w}{2} = \frac{w(w-1)}{2!} = \frac{1}{2}w(w-1) \notin I,$$

because $\frac{(w-1)}{2}$ is not an element of $\mathrm{Int}(\mathbb{Z}^{\{w\}})$. So I is not a binomial ideal in $\mathrm{Int}(\mathbb{Z}^{\{w\}})$.

We have seen in Example 3.4.3 that a principal ideal I in a binomial ring is not a binomial ideal in general. Hence, we introduce the notion of a binomial ideal generated by a set.

Definition 3.4.4. Let R be a binomial ring. We mean by the *binomial ideal in R generated by the set X* for $X \subseteq R$, the intersection of all binomial ideals in R containing X. We denote it by $((X))$.

Definition 3.4.5. Let R be a binomial ring. We mean by *principal binomial ideal* in R, a binomial ideal $I = ((a))$ generated by a for some element $a \in R$.

Here is the main result of this section, which is partially motivated by our main result in the next sections, when we show that the binomial ring $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ is a binomially principal ring (Theorem 3.6.13).

Proposition 3.4.6. *Let R be a binomial ring and let*

$$I = \left(\left\{\binom{x_i}{m} \;:\; m \geq 1 \textit{ and } i = 1, 2, \ldots, n\right\}\right),$$

for $x_i \in R$. Then I is the binomial ideal of R generated by $\{x_1, x_2, \ldots, x_n\}$.

Proof. First by Proposition 3.4.2, I is a binomial ideal. Since $x_1, x_2, \cdots, x_n \in I$, by Definition 3.4.4, this implies that

$$((x_1, x_2, \ldots, x_n)) \subseteq I.$$

On the other hand, by Definition 3.2.1, we have

$$\binom{x_i}{m} \in ((x_1, x_2, \ldots, x_n)),$$

for all $m \geq 1$, $i = 1, 2, \ldots, n$. Since $((x_1, x_2, \ldots, x_n))$ is also an ideal in R, this implies that

$$I \subseteq ((x_1, x_2, \ldots, x_n)).$$

□

Proposition 3.4.7. *Let R and K be binomial rings and let φ be a binomial homomorphism from R onto K. Then*

$$\varphi((r_1, r_2, \cdots, r_n)) = ((\varphi(r_1), \varphi(r_2), \cdots, \varphi(r_n))).$$

for $r_1, r_2, \cdots, r_n \in R$.

Proof. By Proposition 3.4.6, we have

$$((r_1, r_2, \ldots, r_n)) = \left(\left\{\binom{r_i}{m} \;:\; i = 1, \cdots, n \text{ and } m \geq 1\right\}\right).$$

Then

$$\begin{aligned}
\varphi\Big(((r_1, r_2, \ldots, r_n))\Big) &= \varphi\left(\left\{\binom{r_i}{m} \;:\; i = 1, \ldots, n \textit{ and } m \in \mathbb{N}\right\}\right) \\
&= \left(\left\{\varphi\binom{r_i}{m} \;:\; i = 1, \ldots, n \textit{ and } m \in \mathbb{N}\right\}\right) \\
&\quad \text{since } \varphi \text{ is onto} \\
&= \left(\left\{\binom{\varphi(r_i)}{m} \;:\; i = 1, \ldots, n \textit{ and } m \in \mathbb{N}\right\}\right) \\
&\quad \text{since } \varphi \text{ is a binomial homomorphism} \\
&= ((\varphi(r_1), \varphi(r_2), \ldots, \varphi(r_n))) \text{ by Proposition 3.4.6.}
\end{aligned}$$

□

Proposition 3.4.8. *Let R and K be binomial rings, let φ be a binomial homomorphism from R onto K and let I be a binomial ideal in K. Then $\varphi^{-1}(I)$ is a binomial ideal in R.*

Proof. First it is standard that $\varphi^{-1}(I)$ is an ideal in R. Now to see that $\varphi^{-1}(I)$ is a binomial ideal in R, we need to show that $\varphi^{-1}(I)$ is closed under binomial operations. To see that, pick an element $a \in \varphi^{-1}(I)$. Then $\varphi(a) \in I$. By our hypothesis it follows that $\binom{\varphi(a)}{n} \in I$ for $n \geq 1$. Since

$$\binom{\varphi(a)}{n} = \varphi\left(\binom{a}{n}\right),$$

this implies that $\binom{a}{n} \in \varphi^{-1}(I)$. □

Next we present examples of the integer-valued polynomial ring $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ quotiented by various binomial ideals. First we recall from Theorem 3.3.1 that the quotient ring R/I of the binomial ring R, factored by a binomial ideal I in R is a binomial ring.

Example 3.4.9. In $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ we have a binomial isomorphism

$$\frac{\mathrm{Int}(\mathbb{Z}^{\{x\}})}{((n))} \cong 0$$

for each $n \geq 1$, since

$$\binom{n}{n} = 1 \in ((n)).$$

We recall a $\mathbb{Z}$-module basis of the ring $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ from Theorem 2.4.6. We can write $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ in the form

$$\mathrm{Int}(\mathbb{Z}^{\{x\}}) = \left\{ \sum_{i\geq 0}^{\text{finite}} a_i \binom{x}{i} \,:\, a_i \in \mathbb{Z} \right\}.$$

In order to prove the main result of this section, the following needs to presented.

We begin with a formula for the multiplication of elements in $\mathrm{Int}(\mathbb{Z}^{\{x\}})$.

Proposition 3.4.10. *[47, p. 15]The multiplication of general elements in Int$(\mathbb{Z}^{\{x\}})$ is given by*

$$\left(\sum_{i=0}^{n} a_i \binom{x}{i}\right)\left(\sum_{j=0}^{m} b_j \binom{x}{j}\right) = \sum_{t=0}^{n+m} \left(\sum_{\substack{0\leq i\leq n \\ 0\leq j\leq m}} a_i b_j \binom{t}{j}\binom{j}{t-i} \right)\binom{x}{t}$$

for $a_i, b_j \in \mathbb{Z}$ *and* $i, j \geq 0$.

Proof. First we have

$$\begin{aligned}
\left(\sum_{i\geq 0}^{n} a_i \binom{x}{i}\right)\left(\sum_{j\geq 0}^{m} b_j \binom{x}{j}\right) &= \sum_{\substack{0\leq i\leq n \\ 0\leq j\leq m}} a_i b_j \binom{x}{i}\binom{x}{j} \\
&\quad \text{by Proposition 2.3.14(3)} \\
&= \sum_{\substack{0\leq i\leq n \\ 0\leq j\leq m}} a_i b_j \left(\sum_{k=0}^{j} \binom{x}{i+k}\binom{i+k}{j}\binom{j}{k} \right) \\
&= \sum_{t=0}^{n+m} \left(\sum_{\substack{0\leq i\leq n \\ 0\leq j\leq m}} a_i b_j \binom{t}{j}\binom{j}{t-i} \right)\binom{x}{t}.
\end{aligned}$$

□

For example, the coefficient of $\binom{x}{2}$ is

$$a_0b_2 + a_2b_0 + 2a_1b_1 + 2a_1b_2 + 2a_2b_1 + a_2b_2.$$

Also we need to define a non-standard multiplication operation on the abelian group $\mathbb{Z}^{\oplus n}$ for $n \geq 2$, in order to identify certain quotient binomial rings of $\mathrm{Int}(\mathbb{Z}^{\{x\}})$.

Definition 3.4.11. We define the *operation* $*$ on the abelian group $\mathbb{Z}^{\oplus n}$ for $n \geq 2$, by

$$(a_0, a_1, \dots, a_{n-1}) * (b_0, b_1, \dots, b_{n-1}) = (k_0, k_1, \dots, k_{n-1}),$$

where

$$k_m = \left(\sum_{\substack{0 \leq i \leq n \\ 0 \leq j \leq n}} a_i b_j \binom{m}{j} \binom{j}{m-i} \right). \tag{3.3}$$

$$= \left(\sum_{\substack{0 \leq i \leq n \\ 0 \leq j \leq n}} a_i b_j \binom{m}{i} \binom{i}{m-j} \right). \tag{3.4}$$

for $0 \leq m \leq n-1$.

Proposition 3.4.12. *The set* $(\mathbb{Z}^{\oplus n}, +, *)$ *is a commutative ring with multiplicative identity* $(1, 0, 0, \cdots)$.

To prove Proposition 3.4.12 we need to present the following result.

Lemma 3.4.13. *For* $i, j, k, t, l \geq 0$, *we have the following equality of binomial coefficients.*

$$\sum_t \binom{t}{j} \binom{j}{t-i} \binom{l}{k} \binom{k}{l-t} = \sum_t \binom{t}{j} \binom{j}{t-k} \binom{l}{i} \binom{i}{l-t}.$$

Proof.

$$\begin{aligned}
\left(\binom{x}{i} \binom{x}{j} \right) \binom{x}{k} &= \sum_{m=0}^{j} \binom{i+m}{j} \binom{j}{m} \binom{x}{i+m} \binom{x}{k} \quad \text{by Theorem 2.3.14(3)} \\
&= \sum_{t=0}^{i+j} \binom{t}{j} \binom{j}{t-i} \left(\binom{x}{t} \binom{x}{k} \right) \\
&= \sum_{t=0}^{i+j} \binom{t}{j} \binom{j}{t-i} \sum_{s=0}^{t+k} \binom{s}{k} \binom{k}{s-t} \binom{x}{s} \Bigg) \\
&= \sum_t \sum_s \binom{t}{j} \binom{j}{t-i} \binom{s}{k} \binom{k}{s-t} \binom{x}{s}.
\end{aligned}$$

On the other hand: $\binom{x}{i} \left(\binom{x}{j} \binom{x}{k} \right) = \sum_t \sum_s \binom{t}{j} \binom{j}{t-i} \binom{s}{i} \binom{i}{s-t} \binom{x}{s}$. Since polynomial multiplication is associative, this implies by equating coefficients of $\binom{x}{s}$ on the bases, we obtain

$$\sum_t \binom{t}{j} \binom{j}{t-i} \binom{s}{k} \binom{k}{s-t} = \sum_t \binom{t}{j} \binom{j}{t-k} \binom{s}{i} \binom{i}{s-t}.$$

□

Proof. {Proposition 3.4.12} Clearly $\mathbb{Z}^{\oplus n}$ is an abelian group under usual addition with additive zero $(0,0,\ldots,0)$. It is easy to check that multiplication distributes over addition and that multiplication is commutative. So the only property of a ring we need to prove is the associativity of $*$. To see this, pick elements $(a_0,a_1,\cdots,a_{n-1}),(b_0,b_1,\cdots,b_{n-1})$ and $(c_0,c_1,\cdots,c_{n-1})$ in $\mathbb{Z}^{\oplus n}$ for a_i,b_i and $c_i\in\mathbb{Z}$. Then
$\Big((a_0,a_1,\ldots,a_{n-1})*(b_0,b_1,\ldots,b_{n-1})\Big)*(c_0,c_1,\ldots,c_{n-1})=(s_0,s_1,\ldots,s_{n-1})*(c_0,c_1,\ldots,c_{n-1})$
where s_m is as in (3.4). Then let

$$\begin{aligned}
A_n &= (s_0,s_1,\ldots,s_{n-1})*(c_0,c_1,\ldots,c_{n-1})\\
&= \sum_{\substack{0\le m\le n\\0\le k\le n}} s_m c_k\binom{n}{m}\binom{m}{n-k}, \text{ by (3.4)}\\
&= \sum_{\substack{0\le m\le n\\0\le k\le n}}\left(\sum_{\substack{0\le i\le n\\0\le j\le n}} a_ib_j\binom{m}{i}\binom{i}{m-j}\right)c_k\binom{n}{m}\binom{m}{n-k}\\
&= \sum_{\substack{0\le i\le n\\0\le j\le n\\0\le k\le n}} a_ib_jc_k\sum_{\substack{n-k\le m\le i+j\\i,j\le m\le n}}\binom{m}{i}\binom{i}{m-j}\binom{n}{m}\binom{m}{n-k}.
\end{aligned}$$

On the other hand:
$(a_0,a_1,\cdots,a_{n-1})*\Big((b_0,b_1,\ldots,b_{n-1})*(c_0,c_1,\cdots,c_{n-1})\Big)=(a_0,a_1,\ldots,a_{n-1})*(l_0,l_1,\ldots,l_{n-1})$
where l_m is as in (3.3). Then let

$$\begin{aligned}
B_n &= (a_0,a_1,\ldots,a_{n-1})*(l_0,l_1,\ldots,l_{n-1})\\
&= \sum_{\substack{0\le i\le n\\0\le m\le n}} a_il_m\binom{n}{m}\binom{m}{n-i}, \text{ by (3.3)}\\
&= \sum_{\substack{0\le i\le n\\0\le m\le n}} a_i\left(\sum_{\substack{0\le j\le n\\0\le k\le n}} b_jc_k\binom{m}{k}\binom{k}{m-j}\right)\binom{n}{m}\binom{m}{n-i}\\
&= \sum_{\substack{0\le i\le n\\0\le j\le n\\0\le k\le n}} a_ib_jc_k\sum_{\substack{n-i\le m\le j+k\\j,k\le m\le n}}\binom{m}{k}\binom{k}{m-j}\binom{n}{m}\binom{m}{n-i}.
\end{aligned}$$

Then by Lemma 3.4.13 we obtain $A_n=B_n$. □

Now we fix $n\ge 1$. In $\mathrm{Int}(\mathbb{Z}^{\{x\}})$, let $I=((\binom{x}{n}))$. So I is the binomial ideal in $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ generated by $\binom{x}{n}$ and the quotient ring

$$\frac{Int(\mathbb{Z}^{\{x\}})}{((\binom{x}{n}))}$$

is a binomial ring by Theorem 3.3.1.

Theorem 3.4.14. *Fix* $n \geq 1$*. In* $Int(\mathbb{Z}^{\{x\}})$*, let* $I = ((\binom{x}{n}))$*. Then we have an isomorphism of binomial rings*

$$\frac{Int(\mathbb{Z}^{\{x\}})}{((\binom{x}{n}))} \cong (\mathbb{Z}^{\oplus n}, +, *)$$

which comes from a surjective binomial homomorphism

$$\epsilon : Int(\mathbb{Z}^{\{x\}}) \to \mathbb{Z}^{\oplus n},$$

given by

$$\sum_{i \geq 0}^{finite} a_i \binom{x}{i} \mapsto (a_0, a_1, \cdots, a_{n-1})$$

for $a_i \in \mathbb{Z}$*.*

Before we give the proof of Theorem 3.4.14, the following preliminary result is needed.

Lemma 3.4.15. *Fix* $m \geq 1$*. In* $Int(\mathbb{Z}^{\{x\}})$ *the binomial ideal* $I = ((\binom{x}{m}))$ *contains all binomial operations* $\binom{x}{n}$ *for* $n \geq m$*.*

Proof. Let $n \geq m$. Then

$$\begin{aligned}
\binom{x}{n} &= \frac{x(x-1)(x-2)\cdots(x-(n-1))}{n!} \\
&= \frac{x(x-1)(x-2)\cdots(x-(m-1))}{m!} . \frac{(x-m)(x-(m+1))\cdots(x-(n-1))}{\frac{n!}{m!}} \\
&= \binom{x}{m} . \frac{f(x)}{\frac{n!}{m!}} .
\end{aligned}$$

Since $f(x) \in \mathbb{Z}[x] \subseteq \text{Int}(\mathbb{Z}^{\{x\}})$ and

$$\frac{n!}{m!}\binom{x}{n} = \binom{x}{m} . f(x) \in I,$$

by Proposition 3.3.2, this implies that

$$\binom{x}{n} \in I.$$

□

Proof. {Proof of Theorem 3.4.14} First it is clear ϵ is additive and preserves the additive zero $(0, 0, \cdots, 0)$. To show that ϵ is multiplicative observe that,

$$\begin{aligned}
\epsilon\left(\sum_{i\geq 0}^{finite} a_i \binom{x}{i} \sum_{i\geq 0}^{finite} b_i \binom{x}{i}\right) &= \epsilon\left(k_0 + k_1 x + k_2 \binom{x}{2} + \cdots + k_{n-1}\binom{x}{n-1}\right) \\
&= \left(k_0, k_1, k_2, \ldots, k_{n-1}\right) \; k_i \text{ as in Definition 3.4.11} \\
&= (a_0, a_1, a_2, \cdots, a_{n-1}) * (b_0, b_1, b_2, \cdots, b_{n-1}) \\
&= \epsilon\left(\sum_{i\geq 0}^{finite} a_i \binom{x}{i}\right) * \epsilon\left(\sum_{i\geq 0}^{finite} b_i \binom{x}{i}\right).
\end{aligned}$$

It is clear that ϵ preserves the multiplicative identity.
Next to see that ϵ is surjective, let $(a_0, a_1, a_2, \cdots, a_{n-1}) \in \mathbb{Z}^{\oplus n}$ for $a_i \in \mathbb{Z}$.
Then

$$(a_0, a_1, a_2, \cdots, a_{n-1}) = \epsilon\left(a_0 + a_1\binom{x}{1} + a_2\binom{x}{2} + \cdots + a_{n-1}\binom{x}{n-1}\right).$$

So ϵ is surjective. Finally with reference to the first isomorphism theorem, we need to show that

$$\mathrm{Ker}(\epsilon) = ((\binom{x}{n})).$$

We have

$$\begin{aligned}
\mathrm{Ker}(\epsilon) &= \left\{\sum_{i\geq 0}^{finite} a_i\binom{x}{i} \;:\; a_0 = a_1 = \cdots = a_{n-1} = 0\,,\; a_i \in \mathbb{Z}\right\} \\
&= \left\{\sum_{i\geq n}^{finite} a_i\binom{x}{i} \;:\; a_i \in \mathbb{Z}\right\}.
\end{aligned}$$

By Lemma 3.4.15,

$$\left\{\sum_{i\geq n}^{finite} a_i\binom{x}{i}\right\} \subseteq ((\binom{x}{n})),$$

so

$$\mathrm{Ker}(\epsilon) \subseteq ((\binom{x}{n})).$$

In the other direction, by Proposition 3.2.9, $\mathrm{Ker}(\epsilon)$ is a binomial ideal and $\mathrm{Ker}(\epsilon)$ contains $\binom{x}{n}$. Therefore

$$((\binom{x}{n})) \subseteq \mathrm{Ker}(\epsilon),$$

as desired. $\square$

§ 3.5 Binomially simple rings

The main purpose of this section and coming sections is to characterize binomial rings by the properties of their binomial ideals. We begin with a section on the notion of binomially simple ring, which is a binomial ring which has no non-trivial binomial ideal.

We show that the ring $\mathbb{Z}$ of integers is a binomially simple ring (Proposition 3.5.2) In the same way we show that the p-local ring $\mathbb{Z}_{(p)}$ is also a binomially simple ring (Proposition 3.5.3).

Definition 3.5.1. A non-trivial binomial ring R is called a *binomially simple ring* if R and 0 are the only binomial ideals in R.

Proposition 3.5.2. *The binomial ring $\mathbb{Z}$ of integers is a binomially simple ring.*

Proof. All usual ideals in the ring $\mathbb{Z}$ are principal ideals of the form $I = (n)$ for $n \in \mathbb{Z}$. Since, $\binom{n}{n} = 1$ and for $n \geq 1$, $\binom{n}{n} = 1$. Then I is a binomial ideal only for $n = 0$ and $n = 1$. This implies that $\mathbb{Z}$ has only trivial binomial ideals of the form $0 = ((0))$ and $\mathbb{Z} = ((1))$. □

Proposition 3.5.3. *The ring $\mathbb{Z}_{(p)}$ of p-local integers is a binomially simple ring.*

Proof. All non-zero usual ideals in the ring $\mathbb{Z}_{(p)}$ are principal ideals of the form $I = (p^n)$ for $n \in \mathbb{Z}$. In the same way as the previous example we have,

$$\binom{p^n}{p^n} = 1.$$

So (p^n) is a binomial ideal only for $n = 0$. This implies that $\mathbb{Z}_{(p)}$ has only trivial binomial ideals of the form $0 = ((0))$ and $\mathbb{Z}_{(p)} = ((1))$. □

Proposition 3.5.4. *Let R be a binomially simple ring. Then any $\mathbb{Z}$-torsion free homomorphic image ring K of R is a binomially simple ring.*

Proof. Let $\varphi : R \to K$ be a ring homomorphism of a binomially simple ring R onto a $\mathbb{Z}$-torsion free ring K. Then by Proposition 2.8.1 K is a binomial ring. Now to see K is a binomially simple ring, consider a binomial ideal I of K. Then $\varphi^{-1}(I)$ by Proposition 3.4.8 is a binomial ideal of R. So by our hypothesis $\varphi^{-1}(I) = ((0))$ or $\varphi^{-1}(I) = R$. Thus $\varphi((0)) = I$ or $\varphi((R)) = I$. Consequently

$$I = ((0)) \text{ or } I = K,$$

as desired. □

Proposition 3.5.5. *If R is a binomially simple ring and S is any multiplicatively closed subset of R then the localization $S^{-1}R$ is a binomially simple ring.*

Proof. By Theorem 2.9.5 $S^{-1}R$ is a binomial ring and we know from Proposition 3.2.10 that every binomial ideal in $S^{-1}R$ is an extended binomial ideal in R. Then let I be a binomial ideal in $S^{-1}R$ and let

$$J = \{a \in R : \frac{a}{s} \in I \text{ for } s \in S\}. \tag{3.5}$$

Then J is a binomial ideal in R. Thus by our hypothesis $J = ((0))$ or $J = R$. Consequently

$$I = ((0)) \text{ or } I = S^{-1}R.$$

□

§ 3.6 Binomially principal rings

The aim of this section is to introduce another class of binomial rings characterized by properties of their binomial ideals which is called the class of binomially principal rings. For the first step to the main result of this section, a characterization of the binomial ideals in $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ is given (Theorem 3.6.9). As a main result of this section we use this and the fact that the ring $\mathbb{Q}[x]$ is a principal ideal domain to show that the binomial ring $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ is a binomially principal ring (Theorem 3.6.13). By the same argument we show that for $K \subseteq \mathbb{Z}$ the ring $\mathrm{Int}(K^{\{x\}}, \mathbb{Z})$ is also a binomially principal ring (Theorem 3.6.15). We end this section by giving a bijection between the set of all binomial ideals in $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ and usual ideals in the ring $\mathbb{Q}[x]$ (Theorem 3.6.22).

An important type of ring is a principal ideal domain in which all ideals are principal ideals. Similarly here we shall introduce the notion of a binomially principal ring in which all binomial ideals are binomially principal ideals.

Definition 3.6.1. A binomial ring R is said to be a *binomially principal ring* if every binomial ideal I in R is a principal binomial ideal (see Definition 3.4.5).

Obviously, every binomially simple ring is a binomially principal ring. So the ring of integers $\mathbb{Z}$ and the p-local integers ring $\mathbb{Z}_{(p)}$ are both binomially principal rings.

The following properties hold in any binomially principal ring.

Proposition 3.6.2. *Let R be a binomially principal ring. Then any $\mathbb{Z}$-torsion free homomorphic image ring K of R is a binomially principal ring.*

Proof. Let $\varphi : R \to K$ be a ring homomorphism of a binomially principal ring R onto a $\mathbb{Z}$-torsion free ring K. Then by Proposition 2.8.1 K is a binomial ring. Now to see K is a binomially principal ring, we consider a binomial ideal I of K. Then by Proposition 3.4.8 $\varphi^{-1}(I)$ is a binomial ideal of R and by hypothesis $\varphi^{-1}(I) = ((a))$ for some $a \in R$. Thus $\varphi((a)) = I$. By Proposition 3.4.7, $((\varphi(a))) = \varphi((a))$. So this implies that

$$((\varphi(a))) = I,$$

as desired. □

Corollary 3.6.3. *Let R be a binomially principal ring. If I is a binomial ideal of R, then the quotient ring R/I is a binomially principal ring.* □

Proposition 3.6.4. *If R is a binomially principal ring and S is any multiplicatively closed subset of R then the localization $S^{-1}R$ is a binomially principal ring.*

Proof. By Theorem 2.9.5 $S^{-1}R$ is a binomial ring and we know from Proposition 3.2.10 that every binomial ideal in $S^{-1}R$ is an extended binomial ideal in R. Then let I be a binomial ideal in $S^{-1}R$ and let

$$J = \{a \in R \,:\, \frac{a}{s} \in I \text{ for } s \in S\}.$$

Then J is a binomial ideal in R and $I = S^{-1}J$. Thus by hypothesis $J = ((r))$ for $r \in R$. We claim that $I = ((\frac{r}{1}))$. First the inclusion $((\frac{r}{1})) \subseteq I$ is clear. Now to establish the inclusion $I \subseteq ((\frac{r}{1}))$, pick an element $\frac{k}{s} \in I$ for $k \in J$ and $s \in S$. By hypothesis

$$k = \sum_{i=0}^{finite} r_i \binom{r}{i}$$

for $r_i \in R$. So

$$\frac{k}{s} = \sum_{i=0}^{finite} \frac{r_i}{s} \binom{\frac{r}{1}}{i}.$$

This implies that

$$\frac{k}{s} \in ((\frac{r}{1})).$$

□

From Proposition 2.9.13, we know that the ring Int$(D^{\{x\}})$ for D a binomial integral domain is a binomial ring. One way to study usual ideals in the ring Int$(D^{\{x\}})$ is by taking the set

$$I(a) = \{f(a) :\ f(x) \in I\}, \tag{3.6}$$

where $a \in D$ and I is an ideal in Int$(D^{\{x\}})$, this is an ideal in D. We call it the value ideal of I at a, see [13].

Next we will discuss the value ideals of a binomial domain D in terms of binomial ideals in Int$(D^{\{x\}})$. We call it the binomial value ideal of J at a.

Proposition 3.6.5. *Let D be a binomial domain with quotient field F and let J be a binomial ideal in Int$(D^{\{x\}})$. Then the set*

$$J(a) = \{f(a) :\ f(x) \in J\},$$

for $a \in D$ is a binomial ideal in D.

Proof. First it is clear $J(a)$ is an ideal in D. To see that $J(a)$ is a binomial ideal in D, consider $f(a) \in J(a)$. So $f(x) \in J$. By hypothesis J is a binomial ideal. So we have

$$\binom{f(x)}{n} \in J$$

for $n \geq 1$. Therefore

$$\begin{aligned} \binom{f(x)}{n}(a) &= \frac{f(a)(f(a)-1)\dots(f(a)-(n-1))}{n!} \\ &= \binom{f(a)}{n} \in J(a). \end{aligned}$$

□

It well known that every ideal in $\text{Int}(D^{\{x\}})$ cannot be characterized by using value ideals in D. For example every ideal in $\text{Int}(\mathbb{Z}^{\{x\}})$ cannot characterized by using value ideals in $\mathbb{Z}$, see [39].

There is one question arising, is it possible to characterize all binomial ideals in $\text{Int}(D^{\{x\}})$ in terms of their binomial value ideals in D. The answer is no. Later we will explain that by Example 3.6.10.

To answer the above question and as a first step to give the main result of this section Theorem 3.6.13, we will give a characterization of binomial ideals of $\text{Int}(\mathbb{Z}^{\{x\}})$ in terms of polynomials in $\mathbb{Q}[x]$. Consider,

$$I = J \cap \text{Int}(\mathbb{Z}^{\{x\}}), \tag{3.7}$$

where J is an ideal in $\mathbb{Q}[x]$. In Theorem 3.6.9, we will show that all binomial ideals in $\text{Int}(\mathbb{Z}^{\{x\}})$ are of this form and by Example 3.6.7, we see that all usual ideals in $\text{Int}(\mathbb{Z}^{\{x\}})$ are not of this form.

Proposition 3.6.6. *Let J be an ideal in $\mathbb{Q}[x]$ and let $I = J \cap Int(\mathbb{Z}^{\{x\}})$. Then I is a binomial ideal in $Int(\mathbb{Z}^{\{x\}})$.*

Proof. First it is easy to see that I is a usual ideal in $\text{Int}(\mathbb{Z}^{\{x\}})$. To see that I is a binomial ideal in $\text{Int}(\mathbb{Z}^{\{x\}})$, we need to show that I is closed under binomial operations. To see that, let $f(x) \in I$ and $n \geq 1$. Then

$$\begin{aligned} \binom{f(x)}{n} &= \frac{f(x)(f(x)-1)\cdots(f(x)-(n-1))}{n!} \\ &= f(x).\Big(\frac{(f(x)-1)\cdots(f(x)-(n-1))}{n!}\Big) \in J \end{aligned}$$

because

$$\frac{(f(x)-1)\cdots(f(x)-(n-1))}{n!} \in \mathbb{Q}[x],$$

and J is an ideal in $\mathbb{Q}[x]$. Clearly,

$$\binom{f(x)}{n} \in \text{Int}(\mathbb{Z}^{\{x\}}).$$

Thus

$$\binom{f(x)}{n} \in I,$$

as desired. $\square$

Example 3.6.7. This example is given to show that every ideal in $\text{Int}(\mathbb{Z}^{\{x\}})$ is not of the form in (3.7). Let

$$I = \left(\frac{x(x-1)(x-2)}{2}\right),$$

an ideal in $\text{Int}(\mathbb{Z}^{\{x\}})$.
Suppose that

$$I = J \cap \text{Int}(\mathbb{Z}^{\{x\}}),$$

for some ideal J in $\mathbb{Q}[x]$.
Since

$$\frac{x(x-1)(x-2)}{2}.\frac{1}{3} \in J,$$

we have

$$\frac{x(x-1)(x-2)}{6} = \binom{x}{3} \in J \cap \text{Int}(\mathbb{Z}^{\{x\}}).$$

But

$$\frac{x(x-1)(x-2)}{6} \notin I.$$

Example 3.6.8. This example is given to illustrate that if $I = ((df(x)))$ a binomial ideal in $\text{Int}(\mathbb{Z}^{\{x\}})$ for $d \in \mathbb{Z}$, then $f(x) \in I$. So $I = ((f(x)))$, which is a particular case of the our result already proved in Proposition 3.3.2. Here we give a particular calculation to illustrate this.

Let $I = ((3\binom{x}{3}))$ in $\text{Int}(\mathbb{Z}^{\{x\}})$. We will show that $\binom{x}{3} \in I$.

Now

$$2.3\binom{x}{3} = x(x-1)(x-2) \in I.$$

Since I is a binomial ideal, by Definition 3.2.1

$$\binom{x(x-1)(x-2)}{3} \in I.$$

Now

$$\binom{x(x-1)(x-2)}{3} = \binom{x}{3}\Big(x(x-1)(x-2)-1\Big)\Big(x(x-1)(x-2)-2\Big), \tag{3.8}$$

and on the other hand

$$\Big(x(x-1)(x-2)\Big)\Big(x(x-1)(x-2)-3\Big)\binom{x}{3} \in I. \tag{3.9}$$

Then from $(3.8)-(3.9)$ we get

$$2\binom{x}{3} \in I.$$

So

$$3\binom{x}{3} + (-2)\binom{x}{3} = \binom{x}{3} \in I.$$

Next we are going to show that actually all binomial ideals in $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ can be characterized by usual ideals in $\mathbb{Q}[x]$ in the form (3.7).

Theorem 3.6.9. *If I is a binomial ideal of $Int(\mathbb{Z}^{\{x\}})$, then $I = J \cap Int(\mathbb{Z}^{\{x\}})$, for the ideal $J = I \otimes \mathbb{Q}$ in $\mathbb{Q}[x]$.*

Proof. The plan to prove the theorem is to start by letting $J = I \otimes \mathbb{Q}$. Then we are going to show that J is an ideal in $\mathbb{Q}[x]$. To see that, let $f(x) \in J$ and $g(x) \in \mathbb{Q}[x]$. We need to show that $f(x)g(x) \in J$. We can write $f(x)$ as

$$f(x) = \frac{\bar{f}(x)}{d} \text{ for } \bar{f}(x) \in I \text{ and } d \in \mathbb{Z}\backslash\{0\}.$$

Also,

$$g(x) = \frac{\bar{g}(x)}{\bar{d}},$$

for some $\bar{g}(x) \in \mathbb{Z}[x]$ and $\bar{d} \in \mathbb{Z}\backslash\{0\}$. Then

$$\begin{aligned} f(x)g(x) &= \frac{\bar{f}\bar{g}}{d\bar{d}} \\ &= \bar{f}(x)\bar{g}(x) \otimes \frac{1}{d\bar{d}} \in I \otimes \mathbb{Q} = J. \end{aligned}$$

Thus J is an ideal in $\mathbb{Q}[x]$. Next we are going to show that

$$I = J \cap \mathrm{Int}(\mathbb{Z}^{\{x\}}).$$

The inclusion

$$I \subseteq J \cap \mathrm{Int}(\mathbb{Z}^{\{x\}})$$

is clear. To establish the inclusion

$$J \cap \mathrm{Int}(\mathbb{Z}^{\{x\}}) \subseteq I,$$

let $f(x) \in J \cap \mathrm{Int}(\mathbb{Z}^{\{x\}})$. Then,

$$f(x) = \frac{\bar{f}(x)}{d}$$

for $\bar{f}(x) \in I$ and some $d \in \mathbb{Z}\backslash\{0\}$. But I is a binomial ideal, and we know that $df(x) \in I$ with $f(x) \in \mathrm{Int}(\mathbb{Z}^{\{x\}})$. Since by Proposition 3.3.2 the binomial quotient ring is $\mathbb{Z}$-torsion free as in Example 3.6.8, this implies that $f(x) \in I$. □

Example 3.6.10. We give this example to show that binomial ideals in $\mathrm{Int}(D^{\{x\}})$ cannot be characterized using binomial value ideals in D. For this purpose we consider the ring $\mathrm{Int}(\mathbb{Z}^{\{x\}})$. We know from Proposition 3.5.2 that the ring $\mathbb{Z}$ of integers is a binomially simple ring. Therefore all binomial ideals in $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ characterized in the form (3.7) it cannot be characterized by using binomial value ideals.

Theorem 3.6.11. *If I is a binomial ideal of $Int(\mathbb{Z}^X)$, for a set X of variables, then*

$$I = J \cap Int(\mathbb{Z}^X)$$

for the ideal $J = I \otimes \mathbb{Q}$ in $\mathbb{Q}[X]$.

Proof. The proof is analogous to the proof of Theorem 3.6.9. □

We know from Theorem 2.4.11, that for $K \subseteq \mathbb{Z}$ the ring $\mathrm{Int}(K^X, \mathbb{Z})$ on a set X of variables is a binomial ring.

Theorem 3.6.12. *If I is a binomial ideal in $Int(K^X, \mathbb{Z})$, for a finite set X of variables, then*

$$I = J \cap Int(K^X, \mathbb{Z})$$

for the ideal $J = I \otimes \mathbb{Q}$ in $\mathbb{Q}[X]$.

Proof. The proof is analogous to the proof of Theorem 3.6.9. □

Now we are in the right position to state the main result of this section, which shows that the binomial ring $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ is a binomially principal ring.

Theorem 3.6.13. *The binomial ring $Int(\mathbb{Z}^{\{x\}})$ is a binomially principal ring.*

Proof. Let I be a binomial ideal in $\mathrm{Int}(\mathbb{Z}^{\{x\}})$. We need to show that I is a principal binomial ideal. Since $\mathbb{Q}[x]$ is a principal ideal domain, any ideal J in $\mathbb{Q}[x]$ is of the form

$$J = (f(x)) \quad \text{for some } f(x) \in \mathbb{Q}[x].$$

Hence, by Theorem 3.6.9, we can write I in the form

$$I = J \cap \mathrm{Int}(\mathbb{Z}^{\{x\}}) = (f(x)) \cap \mathrm{Int}(\mathbb{Z}^{\{x\}}).$$

Now choose minimal $n \in \mathbb{N}$ such that

$$nf(x) \in \mathrm{Int}(\mathbb{Z}^{\{x\}}).$$

Let $g(x) = nf(x)$. So

$$J = (g(x)) \text{ and } I = (g(x)) \cap \mathrm{Int}(\mathbb{Z}^{\{x\}}).$$

Finally we claim that $I = ((g(x)))$. First by hypothesis I is a binomial ideal of $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ and I contains $g(x)$. So by Definition 3.4.4

$$((g(x))) \subseteq I.$$

To establish the inclusion $I \subseteq ((g(x)))$, suppose that

$$I \nsubseteq ((g(x))).$$

So, there exists another generator polynomial $h(x) \in \text{Int}(\mathbb{Z}^{\{x\}})$ such that

$$((g(x), h(x))) \subseteq I \text{ and } ((g(x), h(x))) \nsubseteq ((g(x))).$$

This implies that $g(x), h(x) \in J$. So

$$h(x) = f(x)a(x) = g(x)\frac{a(x)}{n},$$

for $a(x) \in \mathbb{Q}[x]$. There exist $\bar{a}(x) \in \mathbb{Z}[x]$ and $d \in \mathbb{Z} \setminus \{0\}$ such that

$$a(x) = \frac{\bar{a}(x)}{d}.$$

So,

$$h(x) = g(x)\frac{\bar{a}(x)}{nd} \in \text{Int}(\mathbb{Z}^{\{x\}}),$$
$$(nd)h(x) = g(x).\bar{a}(x) \in \text{Int}(\mathbb{Z}^{\{x\}}).$$

Then $(nd)h(x)$ is in the usual ideal in $\text{Int}(\mathbb{Z}^{\{x\}})$ generated by $g(x)$. Therefore by Definition 3.2.1

$$(nd)h(x) \in ((g(x))).$$

Consequently by Proposition 3.3.2

$$h(x) \in ((g(x))).$$

So we have a contradiction and thus $I = ((g(x)))$. □

Remark 3.6.14. The proof of Theorem 3.6.13 shows that if J is an ideal in $\mathbb{Q}[x]$ generated by $f(x)$, for some $f(x) \in \mathbb{Q}[x]$, where $g(x) = nf(x)$ for minimal $n \in \mathbb{N}$ such that $nf(x) \in \text{Int}(\mathbb{Z}^{\{x\}})$, then the principal binomial ideal $((g(x)))$ in $\text{Int}(\mathbb{Z}^{\{x\}})$, can be written in the form

$$((g(x))) = (f(x)) \cap \text{Int}(\mathbb{Z}^{\{x\}}),$$

Since $\text{Int}(\mathbb{Z}^{\{x\}})$ is integer-valued over $\mathbb{Z}$, we obtain the following for $K \subseteq \mathbb{Z}$:

$$\mathbb{Z}[x] \subset \text{Int}(\mathbb{Z}, \mathbb{Z}^{\{x\}}) = \text{Int}(\mathbb{Z}^{\{x\}}) \subset \text{Int}(K^{\{x\}}, \mathbb{Z}) \subseteq \mathbb{Q}[x]. \tag{3.10}$$

This leads to the generalization of Theorem 3.6.13.

Theorem 3.6.15. *For every $K \subseteq \mathbb{Z}$, the binomial ring Int$(K^{\{x\}}, \mathbb{Z})$ is a binomially principal ring.*

Proof. First by Theorem 3.6.12 we can write every binomial ideal I in $\text{Int}(K^{\{x\}}, \mathbb{Z})$ as

$$I = J \cap \text{Int}(K^{\{x\}}, \mathbb{Z}),$$

for some ideal J in $\mathbb{Q}[x]$. The rest of the proof is analogous to the proof of Theorem 3.6.13. □

Proposition 3.6.16. *Every binomial principal ideal domain R is a binomially principal ring.*

Proof. It is clear since every binomial ideal in R is an ideal in R. □

The converse of Proposition 3.6.16 is not true in general.

Example 3.6.17. The ring $\text{Int}(\mathbb{Z}^{\{x\}})$ is an example of a binomially principal ring which is not a principal ideal domain. By Theorem 3.6.13 it is binomially principal ring. To see that $\text{Int}(\mathbb{Z}^{\{x\}})$ is not a principal ideal domain, we will see that $\text{Int}(\mathbb{Z}^{\{x\}})$ contains an ideal which is not principal. First we recall from Theorem 2.4.6 that $\binom{x}{n}$ for all $n \geq 0$ is a $\mathbb{Z}$-module basis of $\text{Int}(\mathbb{Z}^{\{x\}})$. So we can write $\text{Int}(\mathbb{Z}^{\{x\}})$ in the form

$$\text{Int}(\mathbb{Z}^{\{x\}}) = \Big\{ \sum_{i=0}^{n} a_i \binom{x}{i} \;:\; a_i \in \mathbb{Z} \text{ and } n \in \mathbb{N} \Big\}.$$

Let p be a prime. Then the ideal $\left(\left\{ \binom{x}{m} \;:\; \text{for } 1 \leq m < p \right\} \right)$ does not contain $\binom{x}{p}$. This can be seen from the formula for

$$\binom{x}{m} (\sum_{i=0}^{n} a_i \binom{x}{i}), \tag{3.11}$$

when written in the term of the basis given in Proposition 3.4.10. The coefficient of $\binom{x}{p}$ is given by

$$\sum_{i=0}^{n} a_i \binom{p}{m} \binom{m}{p-i}. \tag{3.12}$$

Then $\binom{p}{m}$ is divisible by p for $1 \leq m < p$. So the coefficient of $\binom{x}{p}$ is a multiple of p. This implies that $\binom{x}{p}$ is not in the ideal $\left(\left\{ \binom{x}{m} \;:\; \text{for } 1 \leq m < p \right\} \right)$. Now to show that $\text{Int}(\mathbb{Z}^{\{x\}})$ is not a principal ideal domain, we consider the ideal $(\binom{x}{2}, \binom{x}{3})$. Let $f(x) \in \text{Int}(\mathbb{Z}^{\{x\}})$ such that

$$(\binom{x}{2}, \binom{x}{3}) = (f(x)).$$

Since $\binom{x}{2} \in (f(x))$, there exist $g(x) \in \text{Int}(\mathbb{Z}^{\{x\}})$ such that $f(x)g(x) = \binom{x}{2}$. So $\deg(f) \leq 2$. Also, $f(x), g(x)$ have not constant term because $\binom{x}{2}$ does not. So

$f(x) = a_1\binom{x}{1} + a_2\binom{x}{2}$, for some $a_1, a_2 \in \mathbb{Z}$. Then

$$f(x) = \left(\left\{\binom{x}{m}\right) : \text{ for } 1 \leq m < 3\right\}\right).$$

So by above $\binom{x}{3} \notin (f(x))$. So the ideal $(\binom{x}{2}, \binom{x}{3})$ is not principal in Int$(\mathbb{Z}^{\{x\}})$. Therefore Int$(\mathbb{Z}^{\{x\}})$ is not a principal ideal domain.

In the next section the ring Int$(\mathbb{Z}^{\{x,y\}})$ on two variables x, y will be our object of study when we introduce the notion of binomially Noetherian ring. First recall from Theorem 2.4.7, that the ring Int$(\mathbb{Z}^X)$ on a set X of variables is a binomial ring.

Example 3.6.18. The ring Int$(\mathbb{Z}^{\{x,y\}})$ on two variables is not a binomially principal ring. Consider the binomial ideal $((x, y))$ in Int$(\mathbb{Z}^{\{x,y\}})$. Since (x, y) is not a principal ideal of $\mathbb{Z}[x, y]$, it is clear $((x, y))$ is not a principal binomial ideal of Int$(\mathbb{Z}^{\{x,y\}})$ also.

We give another example of integer-valued polynomials quotiented by the principal binomial ideal $((x^2))$. We need to give another non-standard multiplication operation on the abelian group $\mathbb{Z} \oplus \mathbb{Q}$.

Definition 3.6.19. We define the *operation* $\circledast$ on the abelian group $\mathbb{Z} \oplus \mathbb{Q}$, by

$$(a_0, a_1) \circledast (b_0, b_1) = (l_0, l_1),$$

where

$$l_m = \sum_{i=0}^{m} a_i b_{m-i}, \tag{3.13}$$

for $0 \leq m \leq 1$.

Proposition 3.6.20. *The set* $(\mathbb{Z} \oplus \mathbb{Q}, +, \circledast)$ *is a commutative ring with multiplicative identity* $(1, 0)$.

Proof. Since $\circledast$ is induced from multiplication of polynomials, it is clear that $(\mathbb{Z} \oplus \mathbb{Q}, +, \circledast)$ is a commutative ring. □

Theorem 3.6.21. *In Int*$(\mathbb{Z}^{\{x\}})$*, let* $I = ((x^2))$*. Then there is a surjective ring homomorphism*

$$\varphi : Int(\mathbb{Z}^{\{x\}}) \to \mathbb{Z} \oplus \mathbb{Q},$$

given by

$$\sum_{i \geq 0}^{finite} a_i x^i \mapsto (a_0, a_1)$$

for $a_0 \in \mathbb{Z}$ *and* $a_1 \in \mathbb{Q}$*. This has kernel* $((x^2))$ *and thus there is an isomorphism of binomial rings*

$$\frac{Int(\mathbb{Z}^{\{x\}})}{((x^2))} \cong (\mathbb{Z} \oplus \mathbb{Q}, +, \circledast).$$

Proof. First it is clear φ is additive. Also it is clear that φ preserves the multiplicative identity. Thus it remains to show that φ is multiplicative. To see that φ is multiplicative observe that,

$$\begin{aligned}\varphi\left(\sum_{i\geq 0}^{\text{finite}} a_i x^i \sum_{i\geq 0}^{\text{finite}} b_i x^i\right) &= \varphi\left(\sum_{i\geq 0}^{\text{finite}} l_i x^i\right) \text{ by (3.13)}\\ &= (l_0, l_1)\\ &= (a_0, a_1) \circledast (b_0, b_1)\\ &= \varphi\left(\sum_{i\geq 0}^{\text{finite}} a_i x^i\right) \circledast \varphi\left(\sum_{i\geq 0}^{\text{finite}} b_i x^i\right).\end{aligned}$$

Next to see that φ is surjective, let $(a_0, a_1) \in \mathbb{Z} \oplus \mathbb{Q}$ for $a_0 \in \mathbb{Z}$ and $a_1 \in \mathbb{Q}$. We consider $a_0 + (-1)^{q-1} p\binom{x}{q} \in \mathrm{Int}(\mathbb{Z}^{\{x\}})$ for $q, p \in \mathbb{Z}$ and $p \nmid q$. Then

$$\begin{aligned}\varphi(a_0 + (-1)^{q-1} p\binom{x}{q}) &= (a_0, \frac{p}{q}) \quad \left(\text{because the coefficient of } x \text{ in } \binom{x}{q} \text{ is } (\frac{(-1)^{q-1}}{q})\right)\\ &= (a_0, a_1),\end{aligned}$$

This proves that φ is surjective.

Finally with reference to the first isomorphism theorem, we need to show that $\mathrm{Ker}(\varphi) = ((x^2))$. Since

$$\begin{aligned}\mathrm{Ker}(\varphi) &= \left\{\sum_{i\geq 0}^{finite} a_i x^i \ : \ a_0 = a_1 = 0\right\}\\ &= \left\{\sum_{i\geq 2}^{finite} a_i x^i\right\}\\ &= x^2\mathbb{Q}[x] \cap \mathrm{Int}(\mathbb{Z}^{\{x\}})\\ &= ((x^2)) \text{ by Remark 3.6.14.}\end{aligned}$$

□

We close this section by giving a bijection between the set of all binomial ideals in $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ and the set of usual ideals in the ring $\mathbb{Q}[x]$. First we denote by $\mathrm{BinIds}(\mathrm{Int}(\mathbb{Z}^{\{x\}}))$ the set of all binomial ideals in $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ and by $\mathrm{Ids}(\mathbb{Q}[x])$ the set of all usual ideals in $\mathbb{Q}[x]$.

Theorem 3.6.22. *There is a bijection*

$$\theta : (I, Int(\mathbb{Z}^{\{x\}}))_{Bin} \to (J, \mathbb{Q}[x])$$

defined by

$$\theta(I) = I \otimes \mathbb{Q}$$

that sends a binomial ideal I in $Int(\mathbb{Z}^{\{x\}})$ to the ideal $I \otimes \mathbb{Q}$ in $\mathbb{Q}[x]$.

Proof. To see that θ is a bijection, we define the map

$$\alpha : (J, \mathbb{Q}[x]) \to (I, \mathrm{Int}(\mathbb{Z}^{\{x\}}))_{\mathrm{Bin}},$$

by

$$\alpha(J) = J \cap \mathrm{Int}(\mathbb{Z}^{\{x\}}).$$

We will show that α is the inverse of θ. By Theorem 3.6.9 it is clear that $\alpha\theta = Id$.

So we need to verify $\theta\alpha = Id$. We start with an ideal J in $\mathbb{Q}[x]$.

Consider $K := J \cap \mathrm{Int}(\mathbb{Z}^{\{x\}})$. Clearly $K \otimes \mathbb{Q} \subseteq J$. To establish the inclusion $J \subseteq K \otimes \mathbb{Q}$, if $f(x) \in J$, there exists $n \in \mathbb{Z}$ such that

$$nf(x) \in J \cap \mathrm{Int}(\mathbb{Z}^{\{x\}}).$$

But

$$f(x) = nf(x) \otimes \frac{1}{n} \in K \otimes \mathbb{Q},$$

as desired. □

§ 3.7 Binomially Noetherian rings

The aim of this section is to present the notion of binomially Noetherian ring. These satisfy a certain finiteness condition, namely, that every binomial ideal of the binomial ring should be finitely generated as a binomial ideal. As a main result of this section we show that the binomial ring $\mathrm{Int}(\mathbb{Z}^{\{x,y\}})$ on two variables x, y is a binomially Noetherian ring (Theorem 3.7.7). In general in the same way for finitely many variables $x_1, x_2, \ldots, x_i$ the binomial ring $\mathrm{Int}(\mathbb{Z}^{\{x_1,x_2,\ldots,x_i\}})$ and the ring $\mathrm{Int}(K^{\{x_1,x_2,\ldots,x_i\}}, \mathbb{Z})$ for $K \subseteq \mathbb{Z}$ both are also binomially Noetherian rings (Theorem 3.7.8 and Theorem 3.7.11 respectively).

Definition 3.7.1. By a *binomially Noetherian ring* we mean a binomial ring R such that every binomial ideal I in R is finitely generated as a binomial ideal.

Obviously, every binomially principal ring is a binomially Noetherian ring. So the ring $\mathbb{Z}$ of integers, the ring $\mathbb{Z}_{(p)}$ of p-local integers, the ring $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ and the ring $\mathrm{Int}(K^{\{x\}}, \mathbb{Z})$ for $K \subseteq \mathbb{Z}$ all are binomially Noetherian rings.

Theorem 3.7.2. *Let R be a binomial ring. Then R is a binomially Noetherian ring if and only if every ascending sequence of binomial ideals in R stabilizes.*

Proof. The proof is the same as the usual case, for example see [14, Theorem 11.1]. □

The following properties hold in any binomially Noetherian ring.

Proposition 3.7.3. *Let R be a binomially Noetherian ring. Then any $\mathbb{Z}$-torsion free homomorphic image ring K of R is a binomially Noetherian ring.*

Proof. The proof is analogous to the proof of Proposition 3.6.2. □

Corollary 3.7.4. *Let R be a binomially Noetherian ring. If I is a binomial ideal of R, then the quotient ring R/I is a binomially Noetherian ring.* □

Proposition 3.7.5. *If R is binomially Noetherian ring and S is any multiplicatively closed subset of R, then the localization $S^{-1}R$ is a binomially Noetherian ring.*

Proof. From Proposition 2.9.7 we have the binomial homomorphism

$$\theta : R \to S^{-1}R.$$

Suppose I is a binomial ideal in $S^{-1}R$. By Proposition 3.4.8 $\theta^{-1}(I)$ is a binomial ideal in R. Therefore by hypothesis $\theta^{-1}(I)$ is finitely generated. It follows that

$$\theta^{-1}(I)(S^{-1}R)$$

is finitely generated as a binomial ideal in $S^{-1}R$, where by Proposition 3.4.7 the generators are the images of the generators of $\theta^{-1}(I)$. We claim that

$$\theta^{-1}(I)(S^{-1}R) = I.$$

First the inclusion

$$\theta^{-1}(I)(S^{-1}R) \subseteq I.$$

is clear. To establish the inclusion

$$I \subseteq \theta^{-1}(I)(S^{-1}R),$$

let $\frac{r}{s} \in I$, then $r \in \theta^{-1}(I)$. So

$$\frac{r}{s} = r(\frac{1}{s}) \in \theta^{-1}(I)(S^{-1}R).$$

Thus I is finitely generated. □

Proposition 3.7.6. *Let I be a binomial ideal in a binomial ring R. If I and the quotient ring R/I both are binomially Noetherian, R is also a binomially Noetherian ring.*

Proof. To show that R is binomially Noetherian, by Theorem 3.7.2, we need to show that every ascending sequence of binomial ideals in R stabilizes. To see that, let

$$J_1 \subseteq J_2 \subseteq \cdots \subseteq J_n \subseteq \ldots,$$

be any ascending sequence of binomial ideals in R. Then by Proposition 3.2.8 it follows that

$$J_1 \cap I \subseteq J_2 \cap I \subseteq \cdots \subseteq J_n \cap I \subseteq \ldots,$$

is an ascending sequence of binomial ideals in I and by Proposition 3.3.6 also

$$J_1/I \subseteq J_2/I \subseteq \cdots \subseteq J_n/I \subseteq \cdots,$$

is an ascending sequence of binomial ideals in R/I. Since by hypothesis I and R/I are both binomially Noetherian. There exist N_1 such that for all $n, m \geq N_1$

$$J_m \cap I = J_n \cap I$$

and N_2 such that for all $n, m \geq N_2$

$$J_m/I = J_n/I.$$

Consequently by [14, Lemma p. 225], for $N = \max\{N_1, N_2\}$ $J_m = J_n$ for all $n, m \geq N$. □

The proof of Theorem 3.6.13 is based on the fact that the ring $\mathbb{Q}[x]$ is a principal ideal domain. The ring $\mathbb{Q}[x, y]$ is a Noetherian ring. This will be our motivation when we present the main result of this section. This says that the integer-valued polynomial ring on two variables $\mathrm{Int}(\mathbb{Z}^{\{x,y\}})$ is a binomially Noetherian ring.

Theorem 3.7.7. *The ring Int($\mathbb{Z}^{\{x,y\}}$) is a binomially Noetherian ring.*

Proof. First by Theorem 2.4.7, $\mathrm{Int}(\mathbb{Z}^{\{x,y\}})$ is a binomial ring. Let I be a binomial ideal in $\mathrm{Int}(\mathbb{Z}^{\{x,y\}})$. We are going to show that I is finitely generated as a binomial ideal. By Theorem 3.6.11 we can write I in the form

$$I = J \cap \mathrm{Int}(\mathbb{Z}^{\{x,y\}})$$

for an ideal J in $\mathbb{Q}[x, y]$. Since $\mathbb{Q}[x, y]$ is a Noetherian ring, every ideal in $\mathbb{Q}[x, y]$ is finitely generated. So suppose that $J = (f_1, f_2, \ldots, f_n)$ for $f_i \in \mathbb{Q}[x, y]$. This implies that

$$I = (f_1, f_2, \ldots, f_n) \cap \mathrm{Int}(\mathbb{Z}^{\{x,y\}}).$$

Now choose $b_i \in \mathbb{N}$ minimal such that

$$b_i f_i \in \mathrm{Int}(\mathbb{Z}^{\{x,y\}}).$$

Let $g_i = b_i f_i$. So

$$J = (g_1, g_2, \ldots, g_n)$$

and

$$I = (g_1, g_2, \ldots, g_n) \cap \mathrm{Int}(\mathbb{Z}^{\{x,y\}}).$$

We claim that

$$I = ((g_1, g_2, \ldots, g_n)).$$

By hypothesis I is a binomial ideal and contains each g_i. Then by Definition 3.4.4,

$$((g_1, g_2, \ldots, g_n)) \subseteq I.$$

Now to establish the inclusion

$$I \subseteq ((g_1, g_2, \cdots .g_n)),$$

by contradiction we suppose

$$I \not\subseteq ((g_1, g_2, \ldots . g_n)).$$

So there exists another generator $h \in \mathrm{Int}(\mathbb{Z}^{\{x,y\}})$ such that $((g_1, g_2, \ldots, g_n, h)) \subseteq I$ and

$$((g_1, g_2, \ldots, g_n, h)) \not\subseteq ((g_1, g_2, \ldots, g_n)).$$

This implies that $g_1, g_2, \ldots, g_n$ and $h \in J$. It follows that

$$h = \sum_{i=0}^{n} f_i p_i = \sum_{i=0}^{n} g_i \frac{p_i}{b_i},$$

for some $p_i \in \mathbb{Q}[x, y]$. There exists $\bar{p}_i \in \mathbb{Z}[x, y]$ and $d_i \in \mathbb{Z} \setminus \{0\}$ such that

$$p_i = \frac{\bar{p}_i}{d_i}.$$

Then

$$h = \sum_{i=0}^{n} g_i \frac{\bar{p}_i}{b_i d_i} \in \mathrm{Int}(\mathbb{Z}^{\{x,y\}}).$$

So we obtain

$$Nh = \sum_{i=0}^{n} \alpha_i g_i \bar{p}_i \in \mathrm{Int}(\mathbb{Z}^{\{x,y\}}) \text{ for } \alpha_i \in \mathbb{Z},$$

for N the least common multiple of $\{b_1 d_1, \ldots, b_n d_n\}$. Then Nh is in the usual ideal in $\mathrm{Int}(\mathbb{Z}^{\{x,y\}})$ generated by $\{g_1, g_2, \ldots, g_n\}$. Therefore by Definition 3.2.1

$$Nh \in ((g_1, g_2, \ldots, g_n)).$$

Consequently by Proposition 3.3.2

$$h \in ((g_1, g_2, \ldots, g_n)).$$

So we have a contradiction and we conclude that $I = ((g_1, g_2, \ldots, g_n))$. □

Theorem 3.7.8. *The binomial ring* $\mathrm{Int}(\mathbb{Z}^{\{x_1, x_2, \ldots, x_i\}})$ *on finitely many variables* $x_1, x_2, \ldots, x_i$ *is a binomially Noetherian ring.*

Proof. First it is a fact that the ring $\mathbb{Q}[x_1, x_2, \ldots, x_i]$ on finitely many variables $x_1, x_2, \ldots, x_i$ is a Noetherian ring and by Theorem 3.6.11 we can write every binomial ideal I in $\mathrm{Int}(\mathbb{Z}^{\{x_1, x_2, \ldots, x_i\}})$ by

$$I = J \cap \mathrm{Int}(\mathbb{Z}^{\{x_1, x_2, \ldots, x_i\}}),$$

for some ideal J in $\mathbb{Q}[x_1, x_2, \ldots, x_i]$. So the rest of the proof is analogous to the proof of Theorem 3.7.7. □

Remark 3.7.9. The proof of Theorem 3.7.8, shows that if J an ideal in $\mathbb{Q}[x_1, x_2, \ldots, x_i]$ generated by $f_1, f_2, \ldots, f_n$ for some $f_i \in \mathbb{Q}[x_1, x_2, \ldots, x_i]$, then the binomial ideal $((g_1, g_2, \ldots, g_n))$ in $\mathrm{Int}(\mathbb{Z}^{\{x_1,x_2,\ldots,x_i\}})$, can be written in a form

$$((g_1, g_2, \ldots, g_n)) = (f_1, f_2, \ldots, f_n) \cap \mathrm{Int}(\mathbb{Z}^{\{x_1,x_2,\ldots,x_i\}}),$$

for minimal $b_i \in \mathbb{N}$ such that $b_i f_i \in \mathrm{Int}(\mathbb{Z}^{\{x_1,x_2,\ldots,x_i\}})$.

Example 3.7.10. This example is given to show that the ring $\mathrm{Int}(\mathbb{Z}^X)$ on an infinite set X of variables is not binomially Noetherian. To see that, we consider the binomial ideal

$$((x_1, x_2 \ldots, x_{i-1}, x_i, \ldots))$$

of $\mathrm{Int}(\mathbb{Z}^X)$, for $x_1, x_2, \cdots \in X$. It is clear that $x_i \notin ((x_1, x_2, \ldots, x_{i-1}))$, so this is not finitely generated as a binomial ideal. Thus $\mathrm{Int}(\mathbb{Z}^X)$ is not a binomially Noetherian ring.

The following result is a generalisation of Theorem 3.7.7.

Theorem 3.7.11. *For $K \subseteq \mathbb{Z}$, the binomial ring Int$(K^{\{x_1,x_2,\ldots,x_i\}}, \mathbb{Z})$ on finitely many variables $x_1, x_2, \ldots, x_i$ is a binomially Noetherian ring.*

Proof. First by Theorem 3.6.12 we can write every binomial ideal I in $\mathrm{Int}(K^{\{x_1,x_2,\ldots,x_i\}}, \mathbb{Z})$ in the form

$$I = J \cap \mathrm{Int}(K^{\{x_1,x_2,\ldots,x_i\}}, \mathbb{Z}),$$

for some ideal J in $\mathbb{Q}[x_1, x_2, \ldots, x_i]$. The rest of the proof is analogous to the proof of Theorem 3.7.7. □

Proposition 3.7.12. *Every Noetherian binomial ring is a binomially Noetherian ring.*

Proof. This is clear since every binomial ideal is a usual ideal. □

The converse of Proposition 3.7.12 is not true in general.

Example 3.7.13. We give an example of a binomially Noetherian ring which is non-Noetherian ring. For this purpose we consider the ring $\mathrm{Int}(\mathbb{Z}^{\{x\}})$. First $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ is a binomially Noetherian ring. To see that $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ is non-Noetherian ring, we will see that $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ contains an ideal which is not finitely generated. Consider the ideal

$$I = \left(\left\{\binom{x}{p} : \text{ for all primes } p\right\}\right), \tag{3.14}$$

in $\mathrm{Int}(\mathbb{Z}^{\{x\}})$. We know from (3.11) that the polynomial $\binom{x}{p}$ is not in the ideal $J = (\binom{x}{r})$ for $0 \leq r < p$. There are infinitely many primes. So I is not finitely generated in $\mathrm{Int}(\mathbb{Z}^{\{x\}})$. This implies that $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ is not a Noetherian ring.

Example 3.7.14. Consider the subset $\{0\}$ of $\mathbb{Z}$. Then by Theorem 3.7.11, $\mathrm{Int}(\{0\}^{\{x\}}, \mathbb{Z})$ is a binomially Noetherian ring. To show that $\mathrm{Int}(\{0\}^{\{x\}}, \mathbb{Z})$ is a non-Noetherian ring, Strickland in a commutative algebra course [45, Exsample 18.3], claims that the ideal $I = x\mathbb{Q}[x]$ is not finitely generated in $\mathrm{Int}(\{0\}^{\{x\}}, \mathbb{Z})$. To see that, set

$$f : I \longrightarrow \mathbb{Q},$$

given by

$$\sum_{i>0} d_i x^i \longmapsto d_1.$$

Let J be an ideal in $\mathrm{Int}(\{0\}^{\{x\}}, \mathbb{Z})$, generated by finitely many elements $h_0, h_1, \ldots, h_{k-1} \in I$. There exists $n > 0$ such that $nf(h_i) \in \mathbb{Z}$ for all i. Let $b \in J$, then $b = \sum_i h_i g_i$, for $g_i \in \mathrm{Int}(\{0\}^{\{x\}}, \mathbb{Z})$. Therefore $nf(b) = \sum_i g_i(0).nf(h_i) \in \mathbb{Z}$. So we obtain $x/2n \in I \setminus J$. This implies that $J \neq I$.

§ 3.8 Binomially filtered rings

The aim of this section is to present the notion of a binomially filtered ring. We show that the power series ring $\mathbb{Z}[\![\binom{x}{1}, \binom{x}{2}, \binom{x}{3}, \cdots]\!]$ is a binomial ring (Proposition 3.8.9). Recall from Theorem 2.4.11 that for a subset $S \subseteq \mathbb{Z}$, $\mathrm{Int}(S^X, \mathbb{Z})$ is a binomial ring. We give a description of a particular completion of the ring $\mathrm{Int}(S^X, \mathbb{Z})$, when S has a p-ordering $\{a_i\}_{i=0}^{\infty}$ of elements of S for all primes p.

First we start with the concept of a filtered ring. Later we will introduce the concept of a binomially filtered ring with a filtration by binomial ideals.

Definition 3.8.1. Let R be a commutative ring with unit. We call R a *filtered ring* if it is equipped with a decreasing sequence of ideals I_n in R. The ideals are called filtration ideals. A ring homomorphism $\theta : R \to K$ between two filtered rings is called a filtered ring homomorphism if it preserves the filtration ideals. We denote the filtered ring by (R, I_n).

Example 3.8.2. The ring of integers $\mathbb{Z}$ is a filtered ring together with filtration ideals $I_n = p^n\mathbb{Z}$ for $n \geq 0$ and p prime. This is called the p-adic filtration.

We define the concept of binomially filtered ring with filtration by binomial ideals.

Definition 3.8.3. Let R be a binomial ring. We call R a *binomially filtered ring* if it is equipped with a decreasing sequence of binomial ideals I_n in R. The binomial ideals are called binomial filtration ideals. A ring homomorphism $\theta : R \to K$ between two binomially filtered rings is called a binomially filtered ring homomorphism, if it preserves the binomial filtration ideals. We denote the binomially filtered ring by $(R, I_n)_{\mathrm{Bin}}$.

Example 3.8.4. The ring $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ is a binomially filtered ring which is equipped with the binomial filtration ideals

$$((\binom{x}{1})) \supset ((\binom{x}{2})) \supset ((\binom{x}{3}))\cdots . \tag{3.15}$$

Now we turn our attention to special kind of rings over binomial polynomials.

Proposition 3.8.5. *Let R be a commutative ring with unit. The set of all linear combinations of the binomial polynomials*

$$a_0\binom{x}{1}+a_1\binom{x}{2}+\cdots+a_n\binom{x}{n},$$

for $n\geq 0$ with coefficients a_i in R, which is denoted by

$$R[\binom{x}{1},\binom{x}{2},\binom{x}{3},\ldots]$$

is a ring with usual addition and multiplication of binomial operations as in Proposition 3.4.10. □

Proposition 3.8.6. *If R is a binomial ring, then the ring $R[\binom{x}{1},\binom{x}{2},\binom{x}{3},\ldots]$ is a binomial ring.*

Proof. The ring $R[\binom{x}{1},\binom{x}{2},\binom{x}{n},\cdots]$ is isomorphic to $R\otimes \mathrm{Int}(\mathbb{Z}^{\{x\}})$. Since by Theorem 2.4.7 the ring $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ is a binomial ring, by Proposition 2.3.15(3), the ring $R\otimes \mathrm{Int}(\mathbb{Z})$ is also a binomial ring. □

In particular, consider the ring

$$\mathbb{Z}[\binom{x}{1},\binom{x}{2},\binom{x}{3},\ldots],$$

which contains the set of all polynomials $a_0+a_1\binom{x}{1}+a_2\binom{x}{2}+a_3\binom{x}{3},\cdots+a_n\binom{x}{n}$, for $a_0,a_1,a_2,\cdots,a_n\in\mathbb{Z}$, for any $n\geq 0$

On the other hand, we know from Theorem 2.4.6 that the binomial polynomials $\binom{x}{n}$ for $n\geq 0$, form a $\mathbb{Z}$-module basis for $\mathrm{Int}(\mathbb{Z}^{\{x\}})$. Therefore, we have

$$\mathbb{Z}[\binom{x}{1},\binom{x}{2},\binom{x}{3},\cdots]=\mathrm{Int}(\mathbb{Z}^{\{x\}}).$$

To give the main result of this section we record the following from [44].

Proposition 3.8.7. *For $m,n\geq 2$, we have*

$$\binom{\binom{x}{n}}{m}=\sum_{k=n+1}^{mn}a_k\binom{x}{k},$$

where $a_k=\frac{k!}{m!}\sum_{j=1}^{m}\frac{1}{(n!)^j}\left[{m \atop j}\right]\sum_{1\leq l_j\leq n}\left[{n \atop l_1}\right]\left[{n \atop l_2}\right]\cdots\left[{n \atop l_j}\right]\left\{{l_1+l_1+\cdots+l_j \atop m}\right\}$, where $\left\{ {k \atop m} \right\}$ is a Stirling number of the second kind.

Theorem 3.8.8. *Let R be a commutative ring with unit. The set of all power series in the binomial polynomials*

$$a_0\binom{x}{1} + a_1\binom{x}{2} + a_3\binom{x}{3} + \dots$$

with coefficients a_i in R, which is denoted by

$$R[\![\binom{x}{1}, \binom{x}{2}, \binom{x}{3}, \cdots]\!]$$

is a ring with usual addition and multiplication of binomial operations as in Proposition 3.4.10.

Proof. For a proof see [44]. □

The main result of this section is that this ring is a binomial ring. First we know from Example 3.8.4 that $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ is a binomially filtered ring with binomial filtration ideals (3.15). We recall from Theorem 2.9.20 that binomial rings are preserved by completion.

Proposition 3.8.9. *For $n \geq 0$, we have an isomorphism.*

$$\mathbb{Z}[\![\binom{x}{1}, \binom{x}{2}, \binom{x}{3}, \dots]\!] \cong \varprojlim \frac{Int(\mathbb{Z}^{\{x\}})}{(((\binom{x}{n})))} \cong Int(\mathbb{Z}^{\{x\}})_{(((\binom{x}{n})))}.$$

given by

$$\sum_{i=0}^{\infty} a_i\binom{x}{i} \longmapsto \Big(\sum_{i=0}^{n-1} a_i\binom{x}{i} + ((\binom{x}{n}))\Big)_{n\geq 1}.$$

Proof. First part is easy to show the given map is surjective and its Ker $= 0$. Then, by Theorem 2.9.18

$$\mathbb{Z}[\![\binom{x}{1}, \binom{x}{2}, \binom{x}{3}, \dots]\!],$$

is a binomial ring. □

Remark 3.8.10. Similarly, for any binomial ring R

$$R[\![\binom{x}{1}, \binom{x}{2}, \binom{x}{3}, \cdots]\!],$$

is a binomial ring.

We know from Theorem 2.4.10 that the ring $\mathrm{Int}(S^{\{x\}}, \mathbb{Z})$, for $S \subseteq \mathbb{Z}$ is a binomial ring. In the same way we will give the structure of a binomial filtration on $\mathrm{Int}(S^{\{x\}}, \mathbb{Z})$. First recall the following information from [12].

Definition 3.8.11. Consider a non-empty subset $S \subseteq \mathbb{Z}$, and a prime p. We mean by *p-ordering* of S, a sequence $\{a_i\}_{i=0}^{\infty}$ of elements of S, which are selected as follows.

1. Select any number $a_0 \in S$.
2. Select any number $a_1 \in S$, such that the highest power of p that divides $(a_1 - a_0)$, is minimum.
3. Select any number $a_2 \in S$, such that the highest power of p that divides $(a_2 - a_0)(a_2 - a_1)$ is minimum.
4. The procedure is continued.

From the above construction for $k \in \mathbb{N}$, we get an increasing sequence $\{V_k(S,p)\}_{k=0}^{\infty}$, of powers of p, where $V_k(S,p)$ is the highest power of p that divides

$$(a_k - a_0)(a_k - a_1)\cdots(a_k - a_{k-1}).$$

This is called the associated p-sequence of S.

Now we can define the generalized factorial of $k \in \mathbb{Z}$, associated to S as follows.

Definition 3.8.12. Let $S \subseteq \mathbb{Z}$. For $k \in \mathbb{Z}$, we define the *generalized factorial of k associated to S* as follows.

$$k!_S = \prod_p V_k(S,p). \tag{3.16}$$

In particular if for all primes p, $\{a_i\}_{i=0}^{\infty}$ is a p-ordering of S, then we have

$$k!_S = \mid (a_k - a_0)(a_k - a_1)\cdots(a_k - a_{k-1}) \mid . \tag{3.17}$$

Example 3.8.13. If $S = \mathbb{Z}$, then $k!_S = k!$.

Example 3.8.14. Let $S = 2\mathbb{Z}$. For all primes p, the sequence $0, 2, 4, 6, 8.\dots$ in $2\mathbb{Z}_{\geq 0}$ is a p-ordering of $2\mathbb{Z}$. Then by (3.17) we have

$$k!_S = (2k-0)(2k-2)\cdots(2k-(2k-2)) = 2^k k!. \tag{3.18}$$

For $S \subseteq \mathbb{Z}$, we define the polynomial

$$A_{k,S}(x) = (x - a_{0,k})(x - a_{1,k})\dots(x - a_{k-1,k}), \tag{3.19}$$

where for each k, $\{a_{i,k}\}_{i=0}^{\infty}$ is a sequence in $\mathbb{Z}$ and is termwise congruent modulo $V_k(S,p)$ to some p-ordering of S, for each prime p dividing $k!_S$. We call it the global falling factorial. In particular, if S has a p-ordering $\{a_i\}_{i=0}^{\infty}$ for all primes p, then a global falling factorial is given by

$$B_{k,S}(x) = (x - a_0)(x - a_1)\dots(x - a_{k-1}), \tag{3.20}$$

Bhargava in [12] gives a $\mathbb{Z}$-basis of the ring $\mathrm{Int}(S^{\{x\}}, \mathbb{Z})$, for $S \subseteq \mathbb{Z}$.

Theorem 3.8.15. *Let $S \subseteq \mathbb{Z}$. Then the polynomials*

$$P_{k,S} = \frac{A_{k,S}(x)}{k!_S} = \frac{(x-a_{0,k})(x-a_{1,k})\cdots(x-a_{k-1,k})}{k!_S}, \tag{3.21}$$

form a $\mathbb{Z}$-basis of the ring Int($S^{\{x\}}, \mathbb{Z}$), where $A_{k,S}(x)$ is as in (3.19).
In particular, if S has a p-ordering for all primes p, then the polynomials

$$P_{k,S} = \frac{B_{k,S}(x)}{k!_S} = \frac{(x-a_0)(x-a_1)\cdots(x-a_{k-1})}{k!_S}, \tag{3.22}$$

is a $\mathbb{Z}$-basis of the ring Int($S^{\{x\}}, \mathbb{Z}$), where $B_{k,S}(x)$ is as in (3.20).

If S has a p-ordering for all primes p, by Theorem 3.6.12, we have the binomial filtration ideals,

$$((P_{0,S})) \supseteq ((P_{1,S})) \supseteq ((P_{2,S})) \supseteq \ldots, \tag{3.23}$$

in $\mathrm{Int}(S^{\{x\}}, \mathbb{Z})$.

Example 3.8.16. For $S \subseteq \mathbb{Z}$, $\mathrm{Int}(S^{\{x\}}, \mathbb{Z})$, is a binomially filtered ring with binomial filtration ideals (3.23).

So for $S \subseteq \mathbb{Z}$, if S has a p-ordering for all primes p, we have

$$\mathrm{Int}(S^{\{x\}}, \mathbb{Z})_{((P_{n,S}))} \cong \varprojlim \frac{\mathrm{Int}(S^{\{x\}}, \mathbb{Z})}{((P_{n,S}))} =: \mathbb{Z}[\![((P_{0,S})), ((P_{1,S})), ((P_{2,S})), \ldots]\!].$$

By Theorem 2.9.20 this is a binomial ring.

Chapter 4

Binomial rings arising in topology

§ 4.1 Introduction

The ring $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ is interesting in topological complex K-theory. Roughly speaking, in topological complex K-theory the ring $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ and its dual appear as various types of operations and cooperations. Knutson in [38] shows that any binomial ring with a particular type of generator subset is isomorphic to $\mathbb{Z}$. Applying this result to the ring $K^0(X)$, if the ring $K^0(X)$ for a good space X is a binomial ring, then it leads to

$$K^0(X) \cong \mathbb{Z}. \tag{4.1}$$

We know from Theorem 2.4.7 that the ring $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ is a binomial ring. In contrast the purpose of this chapter is using $K_0(X)$ for some good spaces X and the properties of the ring $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ to give some non-trivial examples of binomial rings coming from topology.

In §4.2 The ring $K^0(X)$ is described in terms of classes of vector bundles using the Grothendieck construction. Later the spectrum $\mathbf{K}$ associated with the spaces $BU \times \mathbb{Z}$ and U is introduced. The basic definition and necessary background on coalgebras, bialgebras, comodules and Hopf algebras which will be needed later in this chapter is given in § 4.3

Let E be a spectrum which has a ring structure (Definition 4.2.22). The functors $E^*(-)$ and $E_*(-)$ take spaces to graded π_*E-modules, where $E^*(-)$ is a generalized cohomology theory defined by $E^*(-) = [-, E]$. The homology theory $E_*(-)$ dual to $E^*(-)$ is defined by $E_*(-) = \pi_*(E \wedge -)$ see [54]. In §4.4 the various types of operations defined on $E^*(-)$ and corresponding cooperations on $E_*(-)$. For a ring spectrum E, $E_*(E)$ is a $\pi_*(E)$-coalgebra (Theorem 4.4.8) and for every spaces X, $E_*(X)$ is an $E_*(E)$-comodule (Proposition 4.4.9).

§4.5 begins with the discussion of certain kinds of rings of polynomials closely related to the ring $\mathrm{Int}(\mathbb{Z}^{\{x\}})$. These are called the ring of stably integer-valued Laurent polynomials and the ring of stably integer-valued polynomials. We show that both rings are binomial rings (Theorem 4.5.2 and Theorem 4.5.13). Therefore both rings are λ-rings in which all Adams operations are the identity. Some known results on the ring $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ extend to both rings. Precisely, we show that both rings are binomially principal rings (Theorem 4.5.5 and Theorem 4.5.19).

In §4.6 we explain how these examples of binomial rings coming from topology using property of $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ and spectrum $\mathbf{K}$ in K-theory, starting with the well known $\mathrm{Int}(\mathbb{Z}^{\{x\}}) \cong K_0(CP^\infty)$, (Theorem 4.6.3). The main results give new descriptions of those examples (Theorem 4.6.4 and Theorem 4.6.9). For this chapter we mean by good space a para compact Hausdorff apace.

§ 4.2 K-Theory

The material of this section is well known. The main purpose of this section is to recollect the necessary background for K-theory. We begin by describing geometrically the construction of K-theory in terms of the semigroup $\mathrm{Vect}(X)$ of equivalence classes of complex vector bundles over X by applying the Grothendieck construction. In the second place the construction of K-theory is explained in terms of the spectrum $\mathbf{K}$ associated with the spaces $BU \times \mathbb{Z}$ and U. Some general references for K-theory are [7] and [31].

First we discuss the structure of topological K-theory. We begin this section by defining the notion of a vector bundle over a space X, which is a family of finite dimensional vector spaces formed by attaching them to each point of X and linking them together in an appropriate way.

Definition 4.2.1. Let X be a topological space. An n-dimensional *complex vector bundle* over X is a topological space B with the following.

1. A continuous surjective map $\pi : E \to X$.
2. For each $x \in X$, the space $B_x = \pi^{-1}(x)$, has a complex vector space structure of dimension n such that the following local triviality condition is satisfied. For all $b \in B$, there exists an open neighborhood U_β of b and a homeomorphism
$$f(U_\beta) : \pi^{-1}(U_\beta) \to U_\beta \times \mathbb{C}^n$$
which takes $\pi^{-1}(b)$ into $\{b\} \times \mathbb{C}^n$ by a linear map of vector spaces for each $b \in U_\beta$.

We call X the *base space*, B the *total space* and B_x the *fibre* over $x \in X$.

Here we give a list of some important examples of vector bundles needed later in this chapter.

Example 4.2.2. The trivial complex vector bundle of dimension n over space X, $B = X \times \mathbb{C}^n$ is defined by the projection on the first factor.

Example 4.2.3. Let $\mathbb{C}P^n$ be the base space, thought of as the space of lines through the origin in $\mathbb{C}^{n+1}$. We define

$$B = \{(L, v) \in \mathbb{C}P^n \times \mathbb{C}^{n+1} : \ v \in L\}.$$

Then B is the line bundle defined by the projection

$$\pi : B \to \mathbb{C}P^n,$$

given by

$$(L, v) \longmapsto L.$$

Example 4.2.4. Let $G_n(\mathbb{C}^k)$ be the complex Grassmannian space of n-planes in $\mathbb{C}^k$ through the origin. We define

$$\varepsilon_{n,k} = \{(T, v) \in G_n(\mathbb{C}^k) \times \mathbb{C}^k : \ v \in T\}.$$

Thus $\varepsilon_{n,k}$ is the universal n-dimensional bundle defined by the projection

$$\pi : \varepsilon_{n,k} \to G_n(\mathbb{C}^k)$$

given by

$$(T, v) \longmapsto T.$$

Next the construction of new vector bundles built from old ones is considered.

Example 4.2.5.

1. The direct sum of two vector bundles B_1 and B_2 over the same space X, has total space

$$B_1 \oplus B_2 = \{(b_1, b_2) \in B_1 \times B_2 : \pi_1(b_1) = \pi_2(b_2)\}$$

and projection map

$$\pi : B_1 \oplus B_2 \longrightarrow X$$

given by

$$(b_1, b_2) \longmapsto \pi_1(b_1) = \pi_2(b_2).$$

2. The tensor product of two vector bundles E_1 and E_2 over the same space X has total space

$$B_1 \otimes B_2 = \bigcup_{x \in X} \pi_1^{-1}(x) \otimes \pi_2^{-1}(x)$$

and

$$\pi : B_1 \otimes B_2 \longrightarrow X,$$

sending an element in $\pi_1^{-1}(x) \otimes \pi_2^{-1}(x)$ to $x \in X$.

The topology of the two original vector bundles can be combined to give a coherent topology in $B_1 \otimes B_2$ see [31, p. 13].

Note that to give the local triviality homeomorphism on $B_1 \otimes B_2$, we choose triviality homeomorphisms

$$f_i : \pi_i^{-1}(U_\beta) \longrightarrow U_\beta \times \mathbb{C}^{n_i},$$

for an open neighborhood $U_\beta \subseteq X$, and use

$$f_1 \otimes f_2 : \pi_1^{-1}(U_\beta) \otimes \pi_2^{-1}(U_\beta) \longmapsto U_\beta \times (\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2}).$$

The topology of the two original vector bundles can be combined to give a coherent topology in $B_1 \otimes B_2$.

3. Let $f : X \to Y$ be a continuous map and let B be a vector bundle over Y. Then the pullback vector bundle of B by f defined over X has total space

$$f^*B = \{(x, b) \in X \times E : f(x) = \pi(b)\},$$

with the projection

$$\tilde{\pi} : f^*B \to X,$$

given by

$$(x, b) \longmapsto x.$$

Definition 4.2.6. A *morphism of vector bundles* B_1 and B_2 over base spaces X_1 and X_2 respectively consists of a pair of continuous maps $\theta : B_1 \to B_2$ and $g : X_1 \to X_2$, such that the diagram

$$\begin{array}{ccc} B_1 & \xrightarrow{\theta} & B_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ X_1 & \xrightarrow[g]{} & X_2 \end{array}$$

is commutative and the restriction

$$\theta : \pi_1^{-1}(x) \to \pi_2^{-1}(g(x))$$

is linear for each $x \in X$.

Definition 4.2.7. An *isomorphism between two vector bundles* B_1 and B_2 over the same base X is a homeomorphism $f : B_1 \to B_2$, such that the restriction

$$f : \pi_1^{-1}(x) \to \pi_2^{-1}(x)$$

is a linear isomorphism on each fibre for all $x \in X$.

At this point, we can talk about the set of all isomorphism classes of vector bundles E over a base space X. We denote by $\mathrm{Vect}(X)$ the set of all isomorphism classes of vector bundles over the space X. The set $\mathrm{Vect}(X)$ forms an abelian semigroup with addition given by direct sum (Whitney sum) of vector bundles as in Example 4.2.5 (1) and with identity given by the class of the trivial 0-dimension bundle on X.

There is a classical construction of an abelian group $K(H)$ associated with a semigroup H called the Grothendieck group (completion group), with a homomorphism of the underlying semigroup

$$\theta : H \to K(H),$$

having the following universal properties.

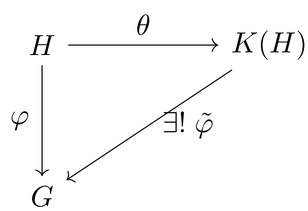
For any abelian group G and homomorphism of the underlying semigroup H,

$$\varphi : H \to G,$$

there is a unique group homomorphism

$$\tilde{\varphi} : K(H) \to G,$$

such that the diagram



is commutative. There are several possible ways to construct $K(H)$. We define $K(H)$ as the quotient of $H \times H$ by the equivalence relation

$$(a,b) \sim (c,d) \Leftrightarrow \text{there exists } h \in H \text{ such that}$$

$$a + d + h = b + c + h$$

and the map from H to $K(H)$ is given by

$$h \longmapsto (h, 0).$$

The lift of a map $\varphi : H \longrightarrow G$ to a map from $K(H)$ to G is given by

$$[(a,b)] \longmapsto \varphi(a) - \varphi(b).$$

Also

$$[(h_1,h_2)] = [(h_1,0) + (0,h_2)] = [(h_1,0)] - [(h_2,0)] = [h_1] - [h_2],$$

for $h_1, h_2 \in H$.

Thus, we represent the elements in $K(H)$ by $[h_1] - [h_2]$

Example 4.2.8. A standard example of the Grothendieck construction is the construction of the group $\mathbb{Z}$ from the semigroup $\mathbb{N}$.

The construction of the Grothendieck group leads to the idea of K-theory, by applying these ideas to the abelian semigroup Vect(X) over X.

Definition 4.2.9. Let X be a good space (compact Hausdorff space). Then $K^0(X)$ is the Grothendieck group of the semigroup Vect(X). An element of $K^0(X)$ is represented by $[B]-[D]$, where $[B]$ and $[D]$ are isomorphism classes of vector bundles over the base space X. The addition operation is defined by

$$([B_1]-[D_1]) \oplus ([B_2]-[D_2]) = [B_1 \oplus B_2] - [D_1 \oplus D_2]. \tag{4.2}$$

The class $[B]-[B]$ is the zero element of this group and $[D]-[B]$ represents the inverse of the element $[B]-[D]$.

The tensor product operation on vector bundles passes to isomorphism classes, so Vect(X) has the multiplication operation given by the tensor product as in Example 4.2.5 (2). This makes Vect(X) into a semiring with the trivial line bundle as multiplicative identity. Then by extending the Grothendieck construction to the semiring Vect(X), we obtain a ring $K^0(X)$ under the multiplication defined by

$$([B_1]-[D_1])([B_2]-[D_2]) = [B_1 \otimes B_2] - [B_1 \otimes D_2] - [D_1 \otimes B_2] + [D_1 \otimes D_2]. \tag{4.3}$$

Example 4.2.10. A complex vector bundle over a point is determined up to isomorphism by its dimension. So the dimension map gives an isomorphism $\text{Vect(pt)} \cong \mathbb{N}$. By Example 4.2.8 the Grothendieck group of $\mathbb{N}$ is $\mathbb{Z}$ and it induces an isomorphism $K(\text{pt}) \cong \mathbb{Z}$.

Now for any base point $x_0 \in X$, consider the inclusion map

$$f : \{x_0\} \to X,$$

which induces a surjective ring homomorphism

$$f^* : K^0(X) \to K^0(x_0),$$

by pullback properties on vector bundles as in Example 4.2.5 (3). This leads to the reduced K-theory ring.

Definition 4.2.11. For any base point $x_0 \in X$, the *reduced K-theory* $\tilde{K}^0(X)$ of a based space X is defined to be the kernel of f^*.

Remark 4.2.12. Let $x_0 \in X$. For the collapsing map $f : X \to \{x_0\}$. Since the map f^* is surjective, the inclusion induces a splitting.

$$0 \longrightarrow \tilde{K}^0(X) \longrightarrow K^0(X) \rightleftarrows K^0(X) \longrightarrow 0.$$

Then by Example 4.2.10 and basic homological algebra we obtain

$$K^0(X) \cong \tilde{K}^0(X) \oplus K^0(x_0) \cong \tilde{K}^0(X) \oplus \mathbb{Z}.$$

So the element of $\tilde{K}^0(X)$ can be written in the form $[E]-n$, where E is a vector bundle over X of dimension n, for $n \in \mathbb{N}$.

At this stage the construction of topological K-theory in terms of the semigroup Vect(X) of vector bundles over a space X described only geometrically. Next we will give another construction of topological K-theory in terms of the space BU.

Definition 4.2.13. The *unitary group* of order n, $U(n)$, is the topological group of $n \times n$ unitary matrices where the group operation is the usual matrix multiplication.

Remark 4.2.14. For each $n \geq 1$ we have an inclusion

$$\iota_n : U(n) \hookrightarrow U(n+1)$$

defined by

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

We mean by S^1 the circle taken as follows

$$S^1 = \{z \in \mathbb{C} : \ |z| = 1\} \subset \mathbb{C} \cong \mathbb{R}^2,$$

which is given the subspace topology and is a subgroup of $\mathbb{C}$.

Example 4.2.15.

$$\begin{array}{rcl} U(1) & = & \{z \in \mathbb{C} : \ z\bar{z} = 1\} \\ & = & \{z \in \mathbb{C} : \ |z| = 1\} \\ & = & S^1. \end{array}$$

Note that from [41, p. 197] the classifying space of $U(n)$ can be written as

$$BU(n) = G_n(\mathbb{C}^\infty) = \bigcup_{k \geq 1} G_n(\mathbb{C}^k).$$

From the bundles $\varepsilon_{n,k}$ as in Example 4.2.4, we obtain the tautological vector bundle ε_n over $BU(n)$

Example 4.2.16. $BU(1) = \mathbb{C}P^\infty$.

The pullback preserve the isomorphism classes. So that given a continuous map

$$f : X \to Y$$

we define

$$f^* : \mathrm{Vect}(Y) \to \mathrm{Vect}(X)$$

by sending $[B]$ to $[f^*B]$. We have the following commutative diagram

$$\begin{array}{ccc} f^*B & \xrightarrow{f^*} & B \\ {\scriptstyle \tilde{\pi}} \downarrow & & \downarrow {\scriptstyle \pi} \\ X & \xrightarrow[f]{} & Y \end{array}$$

Indeed the pullback construction with vector bundles B_1 and B_2 over a space X induces a ring homomorphism

$$f^* : K^0(Y) \to K^0(X)$$

given by

$$[B_1] - [B_2] \to [f^*(B_1)] - [f^*(B_2)].$$

This implies that $K^0(-)$ is a contravariant functor from the category of topological spaces to the category of commutative rings.

Proposition 4.2.17. *For each $n \geq 0$ and good space X, we have an isomorphism*

$$[X, BU(n)] \cong Vect_n(X),$$

given by

$$[f] \longmapsto [f^*(\varepsilon_n)].$$

□

On the other hand, there is a map

$$B\iota_n : BU(n) \to BU(n+1)$$

for $n \geq 1$, and we let

$$BU = \lim_{n \to \infty} BU(n),$$

with the topology of the limit.

Actually from the construction of the space BU one can show that the functor $K^0(-)$ is a representable functor.

Proposition 4.2.18. *[41, chapter 24] For a compact and connected based space X, we have an isomorphism of rings*

$$\tilde{K}^0(X) \xrightarrow{\cong} [X, BU]$$

given by

$$[E] - n \mapsto [i_n f],$$

where E is a vector bundle of dimenstion n over X, defined by the map $f : X \longrightarrow BU(n)$ and $i_n : BU(n) \hookrightarrow BU$ is the inclusion. □

If the space X is not necessarily connected, a vector bundle over X may have a different dimension over each connected component. Such a bundle corresponds to the homotopy class of map $X \longrightarrow \mathbb{Z} \times BU$, where $\mathbb{Z}$ is given the discrete topology.

Then for a general based space X we define $K^0(X)$ to be

$$K^0(X) := [X, \mathbb{Z} \times BU]. \tag{4.4}$$

Definition 4.2.19. An Ω-*spectrum* $E = \{E_i \text{ for } i \in \mathbb{Z}\}$ is a sequence of based spaces E_n together with structure maps

$$\eta_i : E_i \to \Omega E_{i+1},$$

that are homotopy equivalences. If the homotopy groups $\pi_i(E) = 0$ for all negative i, then we call E a *connective spectrum.*

Next we will give the structure of addition and product in the space BU. First recall that $BU(n) = G_n(\mathbb{C}^\infty)$. So $BU(n+m) = G_{n+m}(\mathbb{C}^\infty)$ and from an isomorphism $(\mathbb{C}^\infty \oplus \mathbb{C}^\infty) \cong \mathbb{C}^\infty$, we obtain a homotopy equivalence $G_{n+m}(\mathbb{C}^\infty) \simeq G_{n+m}(\mathbb{C}^\infty \oplus \mathbb{C}^\infty)$. There is an induced classifying map

$$\rho_{n.m} : BU(n) \times BU(m) \longrightarrow BU(n+m).$$

given by

$$(x, y) \longmapsto x \oplus y,$$

for an n-plane x and an m-plane y in $\mathbb{C}^\infty$.

Also consider the map

$$\oplus : BU \times BU \to BU$$

induced from the map $\rho_{n.m}$ by passage to colimit over n and m. This leads to an additive H-space structure on the space $\mathbb{Z} \times BU$.

Next in the same way we consider the tensor product map

$$P_{n,m} : BU(n) \times BU(m) \longrightarrow BU(nm)$$

given by

$$(x, y) \longmapsto x \otimes y.$$

We need to consider the bilinearity of $\otimes$. By an elaborate argument one can pass to the colimit over n and m to have a product map on the space BU,

$$\wedge : BU \wedge BU \to BU.$$

These two constructions correspond to the direct sum and tensor product of vector bundles. These maps $\oplus$ and $\wedge$ give the additive and multiplicative H-space structure on the space $\mathbb{Z} \times BU$.

There is a homotopy equivalence,

$$\Omega(\mathbb{Z} \times BU) \simeq U, \tag{4.5}$$

as additive H-spaces. On the other hand we have Bott periodicity.

Theorem 4.2.20. *[Bott periodicity] There is a homotopy equivalence*

$$\mathbb{Z} \times BU \simeq \Omega U.$$

Now one can construct the Ω-spectrum K, in which for all $i \in \mathbb{Z}$ the spaces are $K_i = \mathbb{Z} \times BU$ for i even, and $K_i = U$ for i odd.

If i is even the structure map

$$K_i \to \Omega K_{i+1},$$

is given by the Bott map

$$\mathbb{Z} \times BU \longmapsto \Omega^2(\mathbb{Z} \times BU),$$

and if i is odd the structure is given by composing the identity with the above homotopy equivalence

$$U \to U \simeq \Omega(\mathbb{Z} \times BU).$$

Next we want to extend K-theory into a cohomology theory. To do this we use suspension to define

$$\tilde{K}^{-n}(X) \cong \tilde{K}^0(\Sigma^n X)$$

for $n \in \mathbb{N}$ and by Bott periodicity there are isomorphisms of groups

$$K^n(X) \cong K^{n-2}(X) \text{ for all } n \in \mathbb{Z}.$$

Thus we have construction of $K^n(X)$ for $n \in \mathbb{Z}$.
In fact $K^*(-)$ is a contravariant functor from a spaces to graded commutative rings.

Remark 4.2.21. [31, 2.2] All the properties of a generalised cohomology theory are satisfied by the functor $K^*(-)$.

The information on the space BU mostly came from [41, Chapter 23,24].

Definition 4.2.22. A *ring spectrum* is a spectrum E together with a product map

$$\mu : E \wedge E \longrightarrow E \tag{4.6}$$

and identity

$$\iota : S^0 \longrightarrow E \tag{4.7}$$

such that the following diagrams

$$\begin{array}{ccc}
E \wedge E \wedge E & \xrightarrow{Id_E \wedge \mu} & E \wedge E \\
\Big\downarrow{\scriptstyle \mu \wedge Id_E} & & \Big\downarrow{\scriptstyle \mu} \\
E \wedge E & \xrightarrow{\quad\mu\quad} & E
\end{array}$$

$$\begin{array}{ccccc}
 & & E & & \\
 & \nearrow{\scriptstyle \simeq} & \Big\uparrow{\scriptstyle \mu} & \nwarrow{\scriptstyle \simeq} & \\
S^0 \wedge E & \xrightarrow{\iota \wedge Id_E} & E \wedge E & \xleftarrow{Id_E \wedge \iota} & E \wedge S^0
\end{array}$$

commute up to homotopy.
The spectrum K defines the K-cohomology $K^*(X)$ of a space X by

$$K^*(X) = [X, K]. \tag{4.8}$$

We also obtain a K-homology theory $K_*(X)$, where is define by

$$K_*(X) = \pi_*(X \wedge K). \tag{4.9}$$

The coefficient ring is given by

$$\pi_*(K) \cong \mathbb{Z}[u, u^{-1}], \tag{4.10}$$

where $u \in \pi_2(K)$.

§ 4.3 Hopf algebras

A Hopf algebra is a well known object in algebraic topology and can be found in many text books such as [1] and [43]. The examples we will give in §4.6, have a Hopf algebra structure and $K_*(X)$ for a good space X will be a comodule over the coalgebra which will construct in §4.5. The main aim of this section is to communicate the necessary background on Hopf algebras. All the tensor products in the chapter will be taken over R, $\otimes = \otimes_R$, unless otherwise stated.

Definition 4.3.1. Let R be a commutative ring with unit. An R-*algebra* is an R-module A together with R-linear maps,

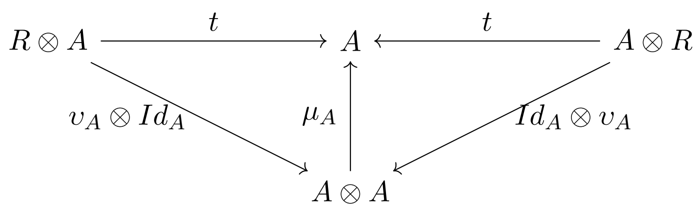
1. $\mu_A : A \otimes A \longrightarrow A$ (product)
2. $\upsilon_A : R \longrightarrow A$ (unit)

such that

1. $\mu_A \circ (\mu_A \otimes Id_A) = \mu_A \circ (Id_A \otimes \mu_A)$, (associativity law)
2. $\mu_A \circ (\upsilon_A \otimes Id_A) = \mu_A \circ (Id_A \otimes \upsilon_A) = Id_A$. (unit property)

In the language of diagrams it means that the following diagrams

$$\begin{array}{ccc}
A \otimes A \otimes A & \xrightarrow{Id_A \otimes \mu_A} & A \otimes A \\
{\scriptstyle \mu_A \otimes Id_A}\downarrow & & \downarrow{\scriptstyle \mu_A} \\
A \otimes A & \xrightarrow{\mu_A} & A
\end{array}$$



are commutative, where t is the isomorphism given by scalar multiplication.

For an R-module C the R-linear maps given by

$$c \mapsto c \otimes 1 \quad \text{and} \quad c \mapsto 1 \otimes c,$$

for $c \in C$ lead to R-isomorphisms

$$C \cong C \otimes R \quad \text{and} \quad C \cong R \otimes C$$

respectively. These isomorphisms are inverse to the t once above. So by identifying $C \otimes R$ and $R \otimes C$ with C and dualizing Definition 4.3.1 we obtain the notion of R-coalgebra.

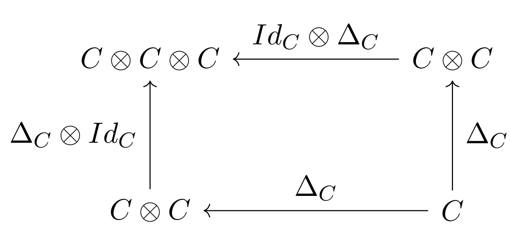
Definition 4.3.2. Let R be a commutative ring with unit. An R-coalgebra is an R*-module* C together with R-linear maps,

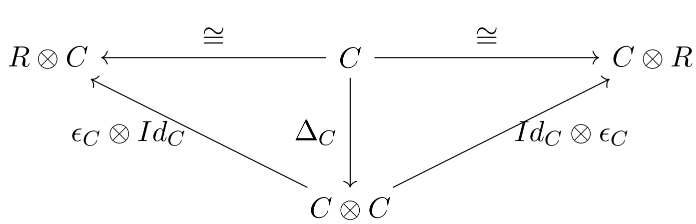
1. $\Delta_C : C \longrightarrow C \otimes C\,,$ (coproduct)
2. $\epsilon_C : C \longrightarrow R\,,$ (counit)

such that

1. $(\Delta_C \otimes Id_C) \circ \Delta_C = (Id_C \otimes \Delta_C) \circ \Delta_C\,,$ (coassociativity law)
2. $(\epsilon_C \otimes Id_C) \circ \Delta_C = (Id_C \otimes \epsilon_C) \circ \Delta_C = Id_C\,.$ (counit property)

In the language of diagrams it means that the following diagrams





are commutive.

Example 4.3.3. Let S be any set and let KS be the free K-module on the set S. Then KS is a K-coalgebra with coproduct map

$$\Delta : KS \longrightarrow KS \otimes KS$$

determined by

$$s \longmapsto s \otimes s$$

and counit map

$$\epsilon : KS \longrightarrow K$$

determined by

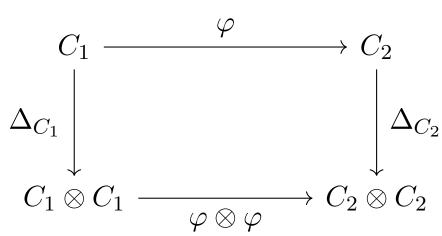
$$s \longmapsto 1,$$

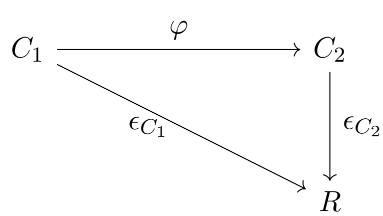
for $s \in S$ and extending linearly.

Definition 4.3.4. Let C_1 and C_2 be two R-coalgebras. An R-linear map $\varphi : C_1 \longrightarrow C_2$ satisfying the following condition

$$\Delta_{C_2} \circ \varphi = (\varphi \otimes \varphi) \circ \Delta_{C_1} \quad \text{and} \quad \epsilon_{C_1} = \epsilon_{C_2} \circ \varphi,$$

is called an *R-coalgebra morphism.* In the language of diagrams it means that the following diagram





are commutive.

Definition 4.3.5. Let R be a commutative ring. An *R-bialgebra* is an R-module B such that

1. (B, μ_B, υ_B) is an R-algebra structure,
2. $(B, \Delta_B, \epsilon_B)$ is an R-coalgebra structure,

and one of the following (equivalent) conditions holds

1. μ_B and υ_B are R-coalgebra morphisms,
2. Δ_B and ϵ_B are R-algebra morphisms.

Example 4.3.6. The rational polynomial algebra $\mathbb{Q}[w]$ is a $\mathbb{Q}$-bialgebra with the following structure. The coproduct is given by the algebra map

$$\Delta_w : \mathbb{Q}[w] \longrightarrow \mathbb{Q}[w] \otimes \mathbb{Q}[w]$$

determined by

$$w^n \mapsto w^n \otimes w^n,$$

with counit by

$$\epsilon_w : \mathbb{Q}[w] \longrightarrow \mathbb{Q}$$

determined by

$$w^n \mapsto 1,$$

for $n \geq 0$. Since this gives a coalgebra structure and Δ_w and ϵ_w are algebra maps, $\mathbb{Q}[w]$ is a $\mathbb{Q}$-bialgebra.

Proposition 4.3.7. *[32, Proposition 3.1.7]Int($\mathbb{Z}^{\{w\}}$) is a $\mathbb{Z}$-subbialgebra of $\mathbb{Q}[w]$ with coproduct inherited from $\mathbb{Q}[w]$, given in terms of basis elements by*

$$\Delta\left(\binom{w}{n}\right) = \sum_{i,j\leq n}\sum_{k=0}^{i}\sum_{t=0}^{j}(-1)^{(i+j)-(k+t)}\binom{i}{k}\binom{j}{t}\binom{kt}{n}\binom{w}{i}\otimes\binom{w}{j}$$

for $n \geq 0$.

Definition 4.3.8. Let B_1 and B_2 be two R-bialgebras. An R-linear map $\varphi : B_1 \longrightarrow B_2$ is called an *R-bialgebra morphism* if it is both an R-algebra and an R-coalgebra morphism. We denote by $\mathbf{Bia}(R)$ the category of R-bialgebras whose objects are R-bialgebras and whose morphisms are R-bialgebras morphisms.

Let B_1 and B_2 be two R-bialgebras. Then recall that $B_1 \otimes B_2$ has an R-algebra structure. The product is given by the composite R-linear map

$$\mu_{B_1\otimes B_2} : B_1 \otimes B_2 \otimes B_1 \otimes B_2 \xrightarrow{Id_{B_1}\otimes\tau\otimes Id_{B_2}} B_1 \otimes B_1 \otimes B_2 \otimes B_2 \xrightarrow{\mu_{B_1}\otimes\mu_{B_2}} B_1 \otimes B_2,$$

where $\tau : B_2 \otimes B_1 \longrightarrow B_1 \otimes B_2$ is the twist R-linear map and unit is given by the composite R-linear map

$$\upsilon_{B_1\otimes B_2} : R \cong R \otimes R \xrightarrow{\upsilon_{B_1}\otimes\upsilon_{B_2}} B_1 \otimes B_2.$$

Similarly, $B_1 \otimes B_2$ has an R-coalgebra structure with coproduct is given by the composite R-linear map

$$\Delta_{B_1\otimes B_2} : B_1 \otimes B_2 \xrightarrow{\Delta_{B_1}\otimes\Delta_{B_2}} B_1 \otimes B_1 \otimes B_2 \otimes B_2 \xrightarrow{Id_{B_1}\otimes\tau\otimes Id_{B_2}} B_1 \otimes B_2 \otimes B_1 \otimes B_2,$$

and counit is given by the composite R-linear map

$$\upsilon_{B_1\otimes B_2} : B_1 \otimes B_2 \xrightarrow{\epsilon_{B_1}\otimes\epsilon_{B_2}} R \otimes R \cong R.$$

For R-modules C and A, $\mathrm{Mod}_R(C, A)$ is an abelian group under addition of maps and it has R-linear structure

$$R \times \mathrm{Mod}_R(C, A) \longrightarrow \mathrm{mod}_R(C, A),$$

given by

$$(rf)(c) = rf(c)$$

for $r \in R$, $c \in C$ and $f \in \mathrm{Mod}_R(C, A)$.

Next, if C is an R-coalgebra and A is an R-algebra we define a special product on $\mathrm{Mod}_R(C, A)$ and we show that $\mathrm{Mod}_R(C, A)$ becomes an R-algebra with this product.

Definition 4.3.9. Let C be an R-coalgebra and let A be an R-algebra, then we define a product on $\mathrm{Mod}_R(C, A)$ by

$$f * g : C \xrightarrow{\Delta_C} C \otimes C \xrightarrow{f\otimes g} A \otimes A \xrightarrow{\mu_A} A$$

for f and $g \in \mathrm{Mod}_R(C, A)$. This is called the *convolution* of f and g. Also we define the unit of $*$ by the composite of R-linear maps

$$\upsilon_A\epsilon_C : C \xrightarrow{\epsilon_C} R \xrightarrow{\upsilon_A} A.$$

Proposition 4.3.10. *Let C be an R-coalgebra and let A be an R-algebra, then $Mod_R(C, A)$ is an R-algebra with convolution product $*$.* □

Proof. Let f, g and $h \in \mathrm{Mod}_R(C, A)$, then $(f * g) * h$ is defined by the R-linear map

$$C \xrightarrow{\Delta_C} C \otimes C \xrightarrow{\Delta_C\otimes Id_C} C \otimes C \otimes C \xrightarrow{f\otimes g\otimes h} A \otimes A \otimes A \xrightarrow{\mu_A\otimes 1} A \otimes A \xrightarrow{\mu_A} A.$$

Then by coassociativity and associativity of C and A respectively we have

$$(f * g) * h = f * (g * h).$$

This implies that $*$ is associative. To show that $\upsilon_A\epsilon_C$ is a left identity element, let $c \in C$ then

$$\begin{aligned}
\upsilon_A\epsilon_C * f(c) &= \mu_A \circ (\upsilon_A\epsilon_C \otimes f) \circ \Delta_C \\
&= \mu_A \circ (\upsilon_A \otimes \mathrm{Id}_A)(f \otimes \mathrm{Id}_R)(\epsilon_C \otimes \mathrm{Id}_C) \circ \Delta_C \\
&= f(c).
\end{aligned}$$

Similarly it is a right identity. So $*$ has two sided unit $\upsilon_A\epsilon_C$. □

Notation 4.3.11. *Let C be an R-coalgebra. Then for any $c \in C$ we write*

$$\Delta_C(c) = \sum c_i^{(1)} \otimes c_i^{(2)}.$$

Then we can write the convolution product for $f, g \in Mod_R(C, A)$ by

$$(f * g)(c) = \sum_i f(c_i^{(1)}) g(c_i^{(2)}).$$

When the identity R-linear map Id_H of an R-bialgebra H has an inverse $\chi \in \mathrm{Hom}_R(H, H)$ under convolution product, then χ is called the antipode of H.

In other words, the antipode χ is an element which satisfies the following

$$\mu_H \circ (\chi \otimes Id_H) \circ \Delta_H = \mu_H \circ (Id_H \otimes \chi) \circ \Delta_H = \upsilon_H \epsilon_H.$$

Definition 4.3.12. By *Hopf algebra* we mean an R-bialgebra H together with an antipode χ.

Example 4.3.13. The rational Laurent polynomial algebra $\mathbb{Q}[w, w^{-1}]$ is a $\mathbb{Q}$-Hopf algebra by extending the coproduct of $\mathbb{Q}[w]$, for $n \geq 0$ such that

$$w^{-n} \mapsto w^{-n} \otimes w^{-n}$$

and with counit determined by

$$w^{-n} \mapsto 1.$$

The antipode is determined by

$$\chi(w^n) = w^{-n},$$

for $n \in \mathbb{Z}$.

Definition 4.3.14. An R-linear map $\psi : H_1 \longrightarrow H_2$ between R-Hopf algebras is called an *R-Hopf algebra morphism*, if it is an R-bialgebra morphism and commutes with the antipodes

$$\psi \circ \chi_{H_1} = \chi_{H_2} \circ \psi,$$

for χ_{H_1} and χ_{H_2} the antipodes of H_1 and H_2 respectively.

Also we use $\mathrm{Mod}_R(C, R)$ to define the dual of an R-coalgebra.

For any R-coalgebra C, set $C^* = \mathrm{Mod}_R(C, R)$, the R-linear dual of C. Define a bilinear form

$$\langle -, - \rangle : C^* \otimes C \longrightarrow R$$

by

$$\langle f, c \rangle \longmapsto f(c)$$

for $f \in C^*$ and $c \in C$. If $\psi : M \longrightarrow N$ is an R-linear map, then the induced R-linear map $\psi^* : N^* \longrightarrow M^*$ is defined by

$$\psi^*(f)(c) := f(\psi(c)).$$

for $f \in N^*$ and $c \in M$. For any R-module M we have an R-linear map

$$\varphi_M : M^* \otimes M^* \longrightarrow (M \otimes M)^*$$

defined by

$$\langle \varphi_M(g_1 \otimes g_2), m_1 \otimes m_2 \rangle = \langle g_1, m_1 \rangle \langle g_2, m_2 \rangle = g_1(m_1) g_2(m_2).$$

For an R-coalgebra C, we use φ_M to define a product on C^* as the composite R-linear map

$$\mu^* : C^* \otimes C^* \xrightarrow{\varphi_C} (C \otimes C)^* \xrightarrow{\Delta^*} C^*.$$

In a similar way we use the isomorphism $R \cong R^*$ to define a unit on C^* as the composite R-linear map

$$\upsilon^* : R \xrightarrow{\cong} R^* \xrightarrow{\epsilon^*} C^*.$$

As a result we have the following.

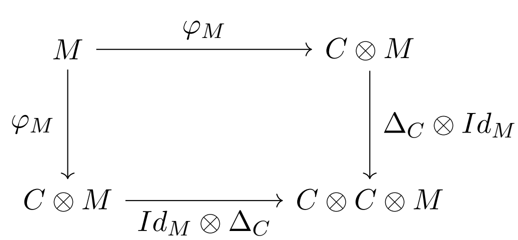
Proposition 4.3.15. *Let C be an R-coalgebra, then C^* is an R-algebra with product map $\Delta^*_{C^*}$ and unit map $\upsilon^*_{C^*}$.*

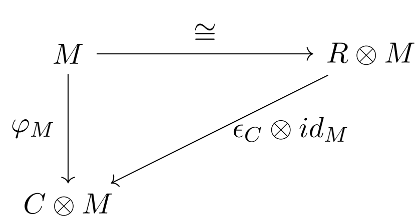
Next in the same way that for an R-algebra A, we have A-modules, we define for an R-coalgebra C the notion of C-comodule.

Definition 4.3.16. For an R-coalgebra C, a *left C-comodule* is an R-module with an R-linear map $\varphi_M : M \longrightarrow C \otimes M$, satisfying the following conditions,

1. $(\Delta_C \otimes Id_M) \circ \varphi_M = (Id_M \otimes \Delta_C) \circ \varphi_M$,
2. $(\epsilon_C \otimes Id_M) \circ \varphi_M = Id_M$.

In the language of diagrams this means that the following diagrams





commute. Then we call φ_M a left C-*coaction* on M.

Also there is a notion of right C-comodule defined by a coaction $\psi_M : M \longrightarrow M \otimes C$.

Definition 4.3.17. Let M and N be left C-comodules together with coactions ψ_M and ψ_N of M and N respectively. An R-linear map $g : M \longrightarrow N$ is called a *comodule morphism* if it commutes with ψ_M and ψ_N. In the language of diagrams this means the following diagram

$$\begin{array}{ccc} M & \xrightarrow{\quad g \quad} & N \\ {\scriptstyle \psi_M}\downarrow & & \downarrow{\scriptstyle \psi_N} \\ C \otimes M & \xrightarrow[Id_C \otimes g]{} & C \otimes N \end{array}$$

commutes. We denote by ${}_C\mathbf{Comod}$ the category of left C-comodules whose objects are left C-comodules and whose morphisms are C-comodule morphisms.

Recall from Notation 4.3.9 that for $c \in C$,

$$\Delta(c) = \sum_i c_i^{(1)} \otimes c_i^{(2)}.$$

Similarly for a left C-comodule M we can write the coaction as

$$\psi_M(m) = \sum_i c_i \otimes m_i$$

for $m, m_i \in M$ and $c_i \in C$.

Lemma 4.3.18. *Let M be a left C-comodule, then M is an right C^*-module with action $m.f = \sum c_i \langle f, m_i \rangle$, for $m, m_i \in M$, $c_i \in C$ and $f \in C^*$.*

Lemma 4.3.19. *Let M be an R-module, then $C \otimes M$ is a left C-comodule with coaction $\psi_{C \otimes M} = \Delta_C \otimes Id_M$.*

If M and N are R-modules, then $C \otimes M$ and $C \otimes N$ are C-comodules by Lemma 4.3.19. For any R-linear map $g : M \longrightarrow N$ we have a C-comodule map

$$Id_C \otimes g : C \otimes M \longrightarrow C \otimes N.$$

§ 4.4 Cohomology operations and homology cooperations

For a cohomology theory it is not adequate to study only cohomology groups. Also we should look at the natural operations on the cohomology groups. Recall that a ring spectrum E yields E-cohomology theory given by $E^* = [-, E]$. For a space X, $[X, E]$ is a graded group of maps. The spectrum also determines E-homology theory given by $E_* = \pi_*(E \wedge -)$. In this section the various types of operations on the general cohomology theory $E^*(-)$ and the corresponding cooperations on the general homology theory $E_*(-)$ are given. For a nice introduction to $E^*(-)$ operations and $E_*(-)$ cooperations I refer to [26, ch. 14 and 15].

Definition 4.4.1. A *cohomology operation* θ of type (n, m) on the cohomology theory $E^*(-)$ for fixed n, m is a natural transformation $\beta : E^n(-) \to E^m(-)$ between the functors from spaces to sets. Thus such an unstable operation, for each space X, consists of a family of a maps of sets

$$\beta_X : E^n(X) \to E^m(X).$$

This is natural in X, meaning that for every continuous map $g : X \to Y$ the diagram

$$\begin{array}{ccc} E^n(Y) & \xrightarrow{g_n^*} & E^n(X) \\ {\scriptstyle \beta_Y}\downarrow & & \downarrow{\scriptstyle \beta_X} \\ E^m(Y) & \xrightarrow{g_m^*} & E^m(X) \end{array}$$

commutes.

Definition 4.4.2. A cohomology operation θ of type (n, m) on the cohomology theory $E^*(-)$ for fixed n, m is an *additive unstable operation*, if it is an unstable operation and $\beta_X : E^n(X) \to E^m(X)$ is a group homomorphism such that for any continuous map $g : X \to Y$ the following diagram of abelian groups

$$\begin{array}{ccc} E^n(Y) & \xrightarrow{g_n^*} & E^n(X) \\ {\scriptstyle \beta_Y}\downarrow & & \downarrow{\scriptstyle \beta_X} \\ E^m(Y) & \xrightarrow{g_m^*} & E^m(X) \end{array}$$

commutes. In other words the functor $E^n(-)$ is viewed as from spaces to abelian groups.

Definition 4.4.3. A *stable operation* of degree r is a collection of operations, which for each space X

$$\beta_X : E^n(X) \longrightarrow E^{n+r}(X),$$

is natural in X and which commutes with suspension. In the language of diagrams this means that for each space X and each $n \in \mathbb{Z}$ the following diagram,

$$\begin{array}{ccc} E^n(X) & \xrightarrow{\beta_X^n} & E^{n+r}(X) \\ {\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle \cong} \\ E^{n+1}(\Sigma X) & \xrightarrow{\beta_{\Sigma X}^{n+1}} & E^{n+r+1}(\Sigma X) \end{array}$$

commute.

Remark 4.4.4. By the Yoneda lemma, all stable operations from $E^*(-)$ to $E^*(-)$ can identified with $E^*(E) = [E, E]$.

Example 4.4.5. An important example of unstable cohomology operations are λ-operations on K-theory. For $n \geq 0$, is the type $(0, 0)$ operation, $\lambda^n : K^0(X) \to K^0(X)$, is induced from the nth exterior power on vector bundles B over X, $\lambda^n(B) = \Lambda^n B$. This leads $K^0(X)$ to be a λ-ring as in Example 2.5.10(3).

We also considered λ-ring B over X as a formal power series on variable t, that is $\lambda^n(B)$ is the coefficient of t^n in the power series

$$\lambda_t(B) = \sum_{i=0}^{\infty} (\lambda^i B) t^i = 1 + [B]t + [\lambda^2 B]t^2 + \cdots \in K^0[[t]].$$

The λ-operations are not group homomorphisms. On sums we have

$$\lambda^n(B_1 + B_2) = \sum_{i+j=n} \lambda^i(B_1)\lambda^j(B_2).$$

As a formal power series in t with coefficients in R, we can rewrite this as

$$\lambda_t(B_1 + B_2) = \lambda_t(B_1)\lambda_t(B_2).$$

For more detail on λ-operations see Section 2.5.

Example 4.4.6. Important examples of additive unstable cohomology operations are the Adams operations on K-theory, $\psi^n : K^0(X) \to K^0(X)$. These are defined using the λ-operations of K-theory. They are constructed from the group homomorphism

$$\psi_t : K^0(X) \to K^0(X)[[t]],$$

where ψ_{-t} is defined by

$$\psi_{-t}(x) = -t\frac{d}{dt}(\log \lambda_t(x)) = \sum_{i=0}^{\infty} (-1)^i \psi^i(x) t^i,$$

for $x \in K^0(x)$. So $\psi^n(x)$ is the coefficient of t^n in $\psi_t(x)$.

For more detail on Adams operations see Section 2.6.

Next we are going to give construction of coalgebra built out from cohomology operations. Precisely we show that $E_*(E)$ with suitable ring spectrum E is a $\pi_*(E)$-coalgebra. All information in this section is mostly are based on [3, part III 12] and [52, Chapter 13].

Given maps $S \xrightarrow{g} E$ and $S \xrightarrow{h} E \wedge E$, where S is the sphere spectrum the class $([g \wedge h]) \in \pi_*(E \wedge E)$ is represented by the map

$$S \simeq S \wedge S \xrightarrow{g \wedge h} E \wedge E \wedge E \xrightarrow{\mathrm{Id}_E \otimes \mu} E \wedge E.$$

This induces a map

$$\pi_*(S) \xrightarrow{\pi_*(g\wedge h)} \pi_*(E\wedge E\wedge E) \xrightarrow{\pi_*(\mathrm{Id}_E\otimes\mu)} \pi_*(E\wedge E).$$

We also get a right action on $E_*(E)$,

$$\varphi_E : E_*(E)\otimes\pi_*(E) \longrightarrow E_*(E). \tag{4.11}$$

This action can be described as follows. Given maps $S \xrightarrow{g} E\wedge E$ and $S \xrightarrow{h} S\wedge E$, the class $([g\wedge h]) \in \pi_*(E\wedge E)$ is represented by the map

$$S \simeq S\wedge S \xrightarrow{g\wedge h} E\wedge E\wedge E \xrightarrow{\mathrm{Id}_E\otimes\mu} E\wedge E.$$

In the same way from the map

$$S \simeq S\wedge S \xrightarrow{g\wedge h} E\wedge E\wedge E \xrightarrow{\mu\otimes\mathrm{Id}_E} E\wedge E,$$

we induce a left action on $E_*(E)$

$$\vartheta_E : \pi_*(E)\otimes E_*(E) \longrightarrow E_*(E). \tag{4.12}$$

Proposition 4.4.7. *For a ring spectrum E, $E_*(E)$ is a $\pi_*(E)$-module with φ_E as action map.*

Theorem 4.4.8. *[3, p. 281]Suppose E is a ring spectrum such that $E_*(E)$ is flat as $\pi_*(E)$-module. Then $E_*(E)$ is a $\pi_*(E)$-coalgebra.*

Proof. First by Proposition 4.4.7, $E_*(E)$ is a $\pi_*(E)$-module. For a spectrum X we have the product map

$$\wedge_{E,X} : E_*(E)\otimes_{\pi_*(E)} E_*(X) \longrightarrow E_*(X\wedge E), \tag{4.13}$$

induced by the composite of maps

$$\pi_*(E\wedge E)\otimes_{\pi_*(E)}\pi_*(E\wedge X) \longrightarrow \pi_*(E\wedge E\wedge E\wedge X) \xrightarrow{\mathrm{Id}_E\otimes\mu\otimes\mathrm{Id}_X} \pi_*(E\wedge E\wedge X).$$

On the other hand, we have

$$X \simeq X\wedge S^0 \xrightarrow{Id_X\otimes\epsilon} X\wedge E,$$

inducing a map

$$\eta_X : E_*(X) \to E_*(E\wedge X). \tag{4.14}$$

By a assumption $E_*(E)$ is flat, then the map $\wedge_{E,X}$ is an isomorphism by [52, Theorem 13.75] Therefore by composing η_X with $\wedge_{E,X}^{-1}$ we get a map

$$\theta_X : E_*(X) \to E_*(X\wedge E) \xrightarrow{\wedge_{E,X}^{-1}} E_*(X)\otimes_{\pi_*(E)} E_*(E). \tag{4.15}$$

Now by taking $X = E$ we obtain the coproduct map of $E_*(E)$

$$\Delta_E : E_*(E) \to E_*(E) \otimes_{\pi_*(E)} E_*(E), \tag{4.16}$$

where the associativity property of μ leads to the coassociativity of Δ_E. Finally we define a counit map of $E_*(E)$

$$\epsilon : E_*(E) \to \pi_*(E), \tag{4.17}$$

which induced by the product map on a spectrum (4.6) as in Definition 4.2.22. □

Proposition 4.4.9. *For a spectrum E and spectrum X, $E_*(X)$ is an $E_*(E)$-comodule with $\theta_X : E_*(X) \to E_*(X) \otimes_{\pi_*(E)} E_*(E)$ as coaction map.*

§ 4.5 Special kinds of rings of polynomials

We will give constructions of some examples of binomial rings coming from topology in §4.6. For the first step in that section we begin our discussion of special kind of rings of polynomials closely related to the ring $\mathrm{Int}(\mathbb{Z}^{\{x\}})$. We describe their relation with the ring $\mathrm{Int}(\mathbb{Z}^{\{x\}})$. Our results use this relation to show that the rings considered are binomial rings (Theorem 4.5.2 and Theorem 4.5.13). Therefore their localization rings are also binomial rings. For more detail on both rings see [20].

4.5.1 Rings of stably integer-valued Laurent polynomials

The main aim of this subsection is to introduce the ring of stably integer-valued Laurent polynomials. First we start our discussion of stably integer-valued polynomials. We show that the ring $\mathrm{SLInt}(\mathbb{Z}^{\{x\}})$ is a binomial ring (Theorem 4.5.2). More precisely we use the result in Theorem 3.6.13 to show that it is a binomially principal ring. We show that also the ring of stably integer-valued Laurent polynomials over a subset $\mathrm{SLInt}(K^{\{x\}}, \mathbb{Z})$ for $K \subseteq \mathbb{Z}$ is a binomial ring (Theorem 4.5.9). At the end of this chapter we will use this result to give some examples of binomial rings coming from topology.

Definition 4.5.1. For $f(x) \in \mathbb{Q}[x]$, $f(x)$ is a stably integer-valued polynomial if for some $m \geq 0$

$$x^m f(x) \in \mathrm{Int}(\mathbb{Z}^{\{x\}}). \tag{4.18}$$

Let $\mathbb{Q}[x, x^{-1}]$ denote the ring of Laurent polynomials in one variable x with rational coefficients. Define the set of stably integer-valued polynomials in $\mathbb{Q}[x, x^{-1}]$ as

$$\mathrm{SLInt}(\mathbb{Z}^{\{x\}}) = \{f(x) \in \mathbb{Q}[x, x^{-1}] : z^m f(z) \in \mathbb{Z} \text{ for some } m \geq 0 \text{ and all } z \in \mathbb{Z}\}.$$

It is a subring of $\mathbb{Q}[x, x^{-1}]$ and is called the *ring of stably integer-valued Laurent polynomials* on one variable x.

Now we give new example of a binomial ring.

Theorem 4.5.2. *The ring SLInt*($\mathbb{Z}^{\{x\}}$) *is a binomial ring.*

Proof. Note that any polynomial $f(x) \in \mathrm{SLInt}(\mathbb{Z}^{\{x\}})$ can be written as

$$f(x) = x^{-M}g(x) \quad \text{for } g(x) \in \mathrm{Int}(\mathbb{Z}^{\{x\}}) \text{ and some } M \in \mathbb{N}_0.$$

This lies in the localization of $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ at the multiplicatively closed subset $S = \{x, x^2, \cdots, x^i, \cdots\}$.

Conversely, consider an element $h(x) \in \mathrm{Int}(\mathbb{Z}^{\{x\}})[x^{-1}]$. Then there exists $g(x) \in \mathrm{Int}(\mathbb{Z}^{\{x\}})$ and $i \in \mathbb{N}$, such that

$$h(x) = x^{-i}g(x).$$

So

$$x^i h(x) = g(x) \in \mathrm{Int}(\mathbb{Z}^{\{x\}}).$$

This implies that $h(x) \in \mathrm{SLInt}(\mathbb{Z}^{\{x\}})$. So we have

$$\mathrm{SLInt}(\mathbb{Z}^{\{x\}}) = \mathrm{Int}(\mathbb{Z}^{\{x\}})[x^{-1}]. \tag{4.19}$$

We know from Theorem 2.4.7 that $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ is a binomial ring. Therefore by Theorem 2.9.5, $\mathrm{SLInt}(\mathbb{Z}^{\{x\}})$ is a binomial ring. □

Clarke and Whitehouse in [20] give another characterization of the ring $\mathrm{SLInt}(\mathbb{Z}^{\{x\}})$ by an integrality condition.

Proposition 4.5.3. *SLInt*($\mathbb{Z}^{\{x\}}$) $= \{f(x) \in \mathbb{Q}[x, x^{-1}] \mid f(k) \in \mathbb{Z}[\frac{1}{k}]$ *for all* $k \in \mathbb{Z}\backslash\{0\}\}$.

Proof. It is clear by Theorem 4.5.2 $\mathrm{SLInt}(\mathbb{Z}^{\{x\}})$ satisfies the given condition. Conversely let $f(x) \in \mathbb{Q}[x, x^{-1}]$ satisfy the given condition. Take $j > 0$ and $m > 0$, setting

$$g(x) = mx^j f(x),$$

such that $g(x) \in \mathbb{Z}[x]$. Now we set $\theta(x) = x^{j+i}f(x)$ for i the maximum exponent of any prime that divides m. Now we are going to show that $\theta(x) \in \mathrm{Int}(\mathbb{Z}^{\{x\}})$. To see this first note that $\theta(0) = 0$. Now pick $k \in \mathbb{Z}/\{0\}$. If p does not divide k then by hypothesis

$$\theta(k) \in \mathbb{Z}[\frac{1}{k}] \subset \mathbb{Z}_{(p)}$$

and if p divided k then

$$\theta(k) = \frac{k^i}{n}g(k),$$

this implies that $\theta(k) \in \mathbb{Z}_{(p)}$. So $\theta(k)\mathbb{Z}_{(p)}$ for all primes p. Therefore $\theta(k) \in \mathbb{Z}$. So $f(x) \in \mathrm{SLInt}(\mathbb{Z}^{\{x\}})$. □

Proposition 4.5.4. *Let T be a multiplicatively closed subset of the ring SLInt*($\mathbb{Z}^{\{x\}}$). *Then the localization T^{-1}SLInt*($\mathbb{Z}^{\{x\}}$) *is a binomial ring.*

Proof. It is follows by Theorem 2.9.5. □

We know from Proposition 2.9.8 that

$$\mathrm{Int}(\mathbb{Z}_{(p)}^{\{x\}}) = \{f(x) \in \mathbb{Q}[x] : \ f(\mathbb{Z}_{(p)}) \subseteq \mathbb{Z}_{(p)}\}. \tag{4.20}$$

Then we obtain

$$\mathrm{SLInt}(\mathbb{Z}_{(p)}^{\{x\}}) = \mathrm{Int}(\mathbb{Z}_{(p)}^{\{x\}})[x^{-1}]. \tag{4.21}$$

Using Proposition 4.5.3 we have

$$\mathrm{SLInt}(\mathbb{Z}_{(p)}^{\{x\}}) = \{f(x) \in \mathbb{Q}[x, x^{-1}] : \ f(\mathbb{Z}_{(p)}^{\times}) \subseteq \mathbb{Z}_{(p)}\}, \tag{4.22}$$

where $\mathbb{Z}_{(p)}^{\times} = \{\frac{a}{b} \in \mathbb{Z}_{(p)} : \ p \nmid a\}$ is the group of units of $\mathbb{Z}_{(p)}$.

For $f(x) \in \mathrm{SLInt}(\mathbb{Z}^{\{x\}})$, pick m such that $h(x) = x^m f(x)$ is integer-valued and set

$$\Delta(h(x)) = \sum h_i^{(1)}(x) \otimes h_i^{(2)}(x),$$

then

$$\Delta(f(x)) = \sum x^{-m} h_i^{(1)}(x) \otimes x^{-m} h_i^{(2)}(x). \tag{4.23}$$

The ring $\mathrm{SLInt}(\mathbb{Z}^{\{x\}})$ is then a $\mathbb{Z}$-coalgebra. In fact, the ring $\mathrm{SLInt}(\mathbb{Z}^{\{x\}})$ is a $\mathbb{Z}$-sub Hopf algebra of $\mathbb{Q}[x, x^{-1}]$.

We know from Theorem 3.6.13 that the ring $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ is a binomially principal ring. Next we will use this result to show that the ring $\mathrm{SLInt}(\mathbb{Z}^{\{x\}})$ is also a binomially principal ring.

Note that, the ring $\mathrm{SLInt}(\mathbb{Z}^{\{x\}})$ is not a principal ideal domain or even a Noetherian ring. Clarke and Whitehouse in [20] show that the ideal of $\mathrm{SLInt}(\mathbb{Z}^{\{x\}})$,

$$\mu_{p,a} = \{h(x) \in \mathrm{SLInt}(\mathbb{Z}^{\{x\}}) \ : \ h(a) \in p\mathbb{Z}_p\}$$

for $a \in \mathbb{Z}_p^{\times}$ is not finitely generated. Here we are viewing polynomials as uniformly continuous functions on the completion $\mathbb{Z}_{(p)}$. (See [15, Subsection III. 2]).

Theorem 4.5.5. *The ring SLInt*$(\mathbb{Z}^{\{x\}})$ *is a binomially principal ring.*

Proof. By Theorem 4.5.2 $\mathrm{SLInt}(\mathbb{Z}^{\{x\}})$ is a localization of the binomial ring $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ and by Theorem 3.6.13 $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ is a binomially principal ring. Therefore by Proposition 3.6.4, $\mathrm{SLInt}(\mathbb{Z}^{\{x\}})$ is a binomially principal ring. □

We could also consider stably integer-valued polynomials over a set X of variables. For $f \in \mathbb{Q}[X]$, f is a stably integer-valued polynomial if

$$x_1^{m_1}.x_2^{m_2} \ldots x_i^{m_i} \ldots f \in \mathrm{Int}(\mathbb{Z}^X), \tag{4.24}$$

for some $m_i \geq 0$ and $x_i \in X$. Let SLInt$(\mathbb{Z}^X)$ be the set of stably integer-valued Laurent polynomials. It is a subring of $\mathbb{Q}[X, X^{-1}]$ and is called the ring of stably integer-valued Laurent polynomials on a set X of variables.

By the same argument as for Theorem 4.5.2 we have

$$\mathrm{SLInt}(\mathbb{Z}^X) = \mathrm{Int}(\mathbb{Z}^X)[X^{-1}]. \tag{4.25}$$

Theorem 4.5.6. *The ring SLInt$(\mathbb{Z}^X)$ is a binomial ring.*

Proof. This follows from Theorem 2.9.5. □

Theorem 4.5.7. *SLInt$(\mathbb{Z}^{\{x_1,x_2,\ldots,x_n\}})$, on finitely many variables $x_1, x_2, \ldots x_i$ is a binomially Noetherian ring..*

Proof. This follows from Theorem 3.7.8. □

Now we consider the notion of stably integer-valued polynomials over a subset as follows.

Definition 4.5.8. Consider $K \subseteq \mathbb{Z}$. Let $f \in \mathbb{Q}[X]$ for a set X of variables. Then f is called a stably integer-valued polynomial over the subset K if

$$x_1^{m_1}.x_2^{m_2}\ldots x_i^{m_i}\ldots f \in \mathrm{Int}(K^X, \mathbb{Z}), \tag{4.26}$$

for some $m_i \geq 0$ and $x_i \in X$. Let SLInt$(K^X, \mathbb{Z})$ be the set of stably integer-valued Laurent polynomials over the subset K. It is a subring of $\mathbb{Q}[X, X^{-1}]$ and is called *the ring of stably integer-valued Laurent polynomials over the subset* K on the set X of variables.

In particular, we have

$$\mathrm{SLInt}(K^{\{x\}}, \mathbb{Z}) = \{f(x) \in \mathbb{Q}[x, x^{-1}] : \text{ there is some } m \geq 0,\ k^m f(k) \in \mathbb{Z} \text{ for all } k \in K\}.$$

This is called the ring of stably integer-valued Laurent polynomials on one variable x over the subset K.

Note that the ring SLInt$(\mathbb{Z}^X)$ is the ring of stably integer-valued Laurent polynomials over $\mathbb{Z}$, that is

$$\mathrm{SLInt}(\mathbb{Z}^X) = \mathrm{SLInt}(\mathbb{Z}^X, \mathbb{Z}). \tag{4.27}$$

So we have

$$\mathbb{Z}[X, X^{-1}] \subset \mathrm{SLInt}(\mathbb{Z}^X) \subseteq \mathrm{SLInt}(K^X, \mathbb{Z}) \subseteq \mathbb{Q}[X, X^{-1}]. \tag{4.28}$$

Theorem 4.5.9. *For $K \subseteq \mathbb{Z}$, SLInt$(K^X, \mathbb{Z})$ is a binomial ring.*

Proof. As in Theorem 4.5.2, we have

$$\mathrm{SLInt}(K^X, \mathbb{Z}) = \mathrm{Int}(K^X, \mathbb{Z})[X^{-1}]. \tag{4.29}$$

So by Theorem 2.4.11, SLInt$(K^X, \mathbb{Z})$ is a binomial ring. □

Finally we are going to extend some properties of the ring $\mathrm{Int}(K^{\{x\}},\mathbb{Z})$ to the ring $\mathrm{SLInt}(K^{\{x\}},\mathbb{Z})$.

Theorem 4.5.10. *For $K \subseteq \mathbb{Z}$, the ring SLInt$(K^{\{x\}},\mathbb{Z})$ is a binomially principal ring.*

Proof. This follows from Theorem 3.6.15. □

Theorem 4.5.11. *For $K \subseteq \mathbb{Z}$, the ring SLInt$(K^{\{x_1,x_2,\ldots,x_i\}},\mathbb{Z})$ on finitely many variables $x_1, x_2, \ldots, x_i$ is a binomially Noetherian ring.*

Proof. This follows from Theorem 3.7.11. □

4.5.2 Rings of stably integer-valued polynomials

The main aim of this subsection is to introduce another kind of ring of polynomials closely related to the ring $\mathrm{Int}(\mathbb{Z}^{\{x\}})$, which is called the stably integer-valued polynomial ring $\mathrm{SInt}(\mathbb{Z}^{\{x\}})$. We show that it is a binomial ring (Theorem 4.5.13). We show that also the ring of stably integer-valued polynomials over a subset $K \subseteq \mathbb{Z}$ of $\mathrm{SInt}(K^{\{x\}},\mathbb{Z})$ is a binomial ring (Theorem 4.5.23). We will use rings of stably integer-valued polynomials to give some examples of binomial rings that come from topology in §4.6.

Definition 4.5.12. Define the set of stably integer-valued polynomials on one variable x as a subring of the ring $\mathbb{Q}[x]$ by

$$\mathrm{SInt}(\mathbb{Z}^{\{x\}}) = \{f(x) \in \mathbb{Q}[x] : \text{ there is some } m \geq 0 \text{ such that } z^m f(z) \in \mathbb{Z} \text{ for all } z \in \mathbb{Z}\}.$$

It is a subring of $\mathbb{Q}[x]$ and is called t*he ring of stably integer-valued polynomials* in one variable x.

In other words, we can express $\mathrm{SInt}(\mathbb{Z}^{\{x\}})$ by,

$$\mathrm{SInt}(\mathbb{Z}^{\{x\}}) = \mathrm{SLInt}(\mathbb{Z}^{\{x\}}) \cap \mathbb{Q}[x]. \tag{4.30}$$

Theorem 4.5.13. *The ring SInt$(\mathbb{Z}^{\{x\}})$ is a binomial ring.*

Proof. We know from Theorem 4.5.2 that $\mathrm{SLInt}(\mathbb{Z}^{\{x\}})$ is a binomial ring. Then by (4.30) $\mathrm{SInt}(\mathbb{Z}^{\{x\}})$ is the intersection of two binomial subrings of the binomial ring $\mathbb{Q}[x,x^{-1}]$. Therefore by Proposition 2.3.15 (4), $\mathrm{SInt}(\mathbb{Z}^{\{x\}})$ is a binomial ring. □

Proposition 4.5.14. *Let T be a multiplicatively closed subset of SInt$(\mathbb{Z}^{\{x\}})$. Then any localization T^{-1}SInt$(\mathbb{Z}^{\{x\}})$ is a binomial ring.* □

Also, we have

$$\mathrm{SInt}(\mathbb{Z}_{(p)}^{\{x\}}) = \{f(x) \in \mathbb{Q}[x] : \ f(\mathbb{Z}_{(p)}^{\times}) \subseteq \mathbb{Z}_{(p)}\}. \tag{4.31}$$

It is easy to verify that $\mathrm{SLInt}(\mathbb{Z}^{\{x\}})$ is a localization of $\mathrm{SInt}(\mathbb{Z}^{\{x\}})$.

Proposition 4.5.15. *SLInt*$(\mathbb{Z}^{\{x\}}) =$ *SInt*$(\mathbb{Z}^{\{x\}})[x^{-1}]$. □

Corollary 4.5.16. *Let J be a binomial ideal in SLInt$(\mathbb{Z}^{\{x\}})$. Then $J[x^{-1}]$ is a binomial ideal in SLInt$(\mathbb{Z}^{\{x\}})$.* □

Corollary 4.5.17. *Let I be a binomial ideal in SLInt$(\mathbb{Z}^{\{x\}})$. Then $I \cap \mathbb{Q}[x]$ is a binomial ideal in SInt$(\mathbb{Z}^{\{x\}})$.* □

For $f(x) \in \mathrm{SInt}(\mathbb{Z}^{\{x\}})$, pick m such that $h(x) = x^m f(x)$ is integer-valued. If we write,

$$\Delta(h(x)) = \sum h_i^{(1)}(x) \otimes h_i^{(2)}(x),$$

then, let

$$\Delta(f(x)) = \sum x^{-m} h_i^{(1)}(x) \otimes x^{-m} h_i^{(2)}(x). \tag{4.32}$$

This makes the ring $\mathrm{SInt}(\mathbb{Z}^{\{x\}})$ into a $\mathbb{Z}$-coalgebra. Actually, the ring $\mathrm{SInt}(\mathbb{Z}^{\{x\}})$ is $\mathbb{Z}$-subbialgebra of $\mathbb{Q}[x]$. We have,

$$\mathrm{Int}(\mathbb{Z}^{\{x\}}) \subseteq \mathrm{SInt}(\mathbb{Z}^{\{x\}}) \subset \mathbb{Q}[x]. \tag{4.33}$$

In the same way as in $\mathrm{Int}(\mathbb{Z}^{\{x\}})$, we can give a characterization of all binomial ideals in $\mathrm{SInt}(\mathbb{Z}^{\{x\}})$ by usual ideals in $\mathbb{Q}[x]$.

Theorem 4.5.18. *If I is a binomial ideal of SInt$(\mathbb{Z}^{\{x\}})$, then $I = J \cap$ SInt$(\mathbb{Z}^{\{x\}})$, for the ideal $J = I \otimes \mathbb{Q}$ in $\mathbb{Q}[x]$.*

Proof. The proof is analogous to the proof of Theorem 3.6.9. □

Now we can state the main result of this subsection, which says that $\mathrm{SInt}(\mathbb{Z}^{\{x\}})$ is a binomially principal ring.

Theorem 4.5.19. *The ring SInt$(\mathbb{Z}^{\{x\}})$ of stably integer-valued polynomials is a binomially principal ring.*

Proof. First by Theorem 4.5.18, we can write every binomial ideal I in $\mathrm{SInt}(\mathbb{Z}^{\{x\}})$ in the form $I = J \cap \mathrm{SInt}(\mathbb{Z}^{\{x\}})$, where J is an ideal in $\mathbb{Q}[x]$. Then the rest of the proof is analogous to the proof of Theorem 3.6.13. □

Let $\mathrm{SInt}(\mathbb{Z}^X)$ be the set of stably integer-valued polynomials (4.24) over a set X of variables. It is a subring of $\mathbb{Q}[X]$ and is called the *ring of stably integer-valued polynomials* on the set X of variables.

In other words, we can express $\mathrm{SInt}(\mathbb{Z}^X)$ by

$$\mathrm{SInt}(\mathbb{Z}^X) = \mathrm{SLInt}(\mathbb{Z}^X) \cap \mathbb{Q}[X]. \tag{4.34}$$

Theorem 4.5.20. *The ring SInt$(\mathbb{Z}^X)$ is a binomial ring.*

Proof. The proof is clear by Theorem 4.5.6. □

By the same argument as in Theorem 4.5.18, we can write all binomial ideals I in $\mathrm{SInt}(\mathbb{Z}^X)$ by the form

$$I = J \cap \mathrm{SInt}(\mathbb{Z}^X), \tag{4.35}$$

for the ideal $J = I \otimes \mathbb{Q}$ in $\mathbb{Q}[X]$.

Theorem 4.5.21. *The ring* $SInt(\mathbb{Z}^{\{x_1,x_2\ldots x_i\}})$ *on finitely many variables* $x_1, x_2, \ldots x_i$ *is a binomially Noetherian ring.*

Proof. First by (4.35), we can write every binomial ideal I in $\mathrm{SInt}(\mathbb{Z}^{\{x\}})$ in the form $I = J \cap \mathrm{SInt}(\mathbb{Z}^{\{x_1,x_2\ldots,x_i\}})$, where J is an ideal in $\mathbb{Q}[X]$. Then the rest of the proof is analogous to the proof of Theorem 3.7.7. □

Definition 4.5.22. Let $\mathrm{SInt}(K^X, \mathbb{Z})$ be the set of stably integer-valued polynomials over a subset $K \subseteq \mathbb{Z}$ on a set X of variables in $\mathbb{Q}[X]$, it is a subring of $\mathbb{Q}[X]$ and is called *the ring of stably integer-valued polynomials over the subset* K, on the set X of variables.

In particular, we have

$\mathrm{SInt}(K^{\{x\}}, \mathbb{Z}) = \{f(x) \in \mathbb{Q}[x] :$ there is some $m \geq 0$ such that $k^m f(k) \in \mathbb{Z}$ for all $k \in K\}$.

This is called the ring of stably integer-valued polynomials on one variable x over the subset K.

In other words, we can express the ring $\mathrm{SInt}(K^X, \mathbb{Z})$ by

$$\mathrm{SInt}(K^X, \mathbb{Z}) = \mathrm{SLInt}(K^X, \mathbb{Z}) \cap \mathbb{Q}[X]. \tag{4.36}$$

Note that the ring $\mathrm{SInt}(\mathbb{Z}^X)$ is stably integer-valued polynomial over $\mathbb{Z}$, that is

$$\mathrm{SInt}(\mathbb{Z}^X) = \mathrm{SInt}(\mathbb{Z}^X, \mathbb{Z}). \tag{4.37}$$

We have

$$\mathbb{Z}[X] \subset \mathrm{SInt}(\mathbb{Z}^X) \subseteq \mathrm{SInt}(K^X, \mathbb{Z}) \subseteq \mathbb{Q}[X]. \tag{4.38}$$

Theorem 4.5.23. *For* $K \subseteq \mathbb{Z}$ *the ring* $SInt(K^X, \mathbb{Z})$ *is a binomial ring.*

Proof. The proof is clear by (4.36). □

For $K \subseteq \mathbb{Z}$, We have inclusions of binomial rings

$$\mathrm{Int}(K^X, \mathbb{Z}) \subseteq \mathrm{SInt}(K^X, \mathbb{Z}) \subset \mathbb{Q}[X]. \tag{4.39}$$

Also, we can give a characterization of all binomial ideals in $\mathrm{SInt}(K^X, \mathbb{Z})$ by usual ideals in $\mathbb{Q}[X]$. Let I be a binomial ideal in $\mathrm{SInt}(K^X, \mathbb{Z})$. Then

$$I = J \cap \mathrm{SInt}(K^X, \mathbb{Z}), \tag{4.40}$$

for the ideal $J = I \otimes \mathbb{Q}$ in $\mathbb{Q}[X]$.

Theorem 4.5.24. *For $K \subseteq \mathbb{Z}$, the ring SInt$(K^{\{x\}}, \mathbb{Z})$ is a binomially principal ring.*

Proof. First by (4.40), we can write every binomial ideal I in $\mathrm{SInt}(K^{\{x\}}, \mathbb{Z})$ in the form $I = J \cap \mathrm{SInt}(K^{\{x\}}, \mathbb{Z})$, where J is an ideal in $\mathbb{Q}[x]$. Then the rest of the proof is analogous to the proof of Theorem 3.6.13. □

Theorem 4.5.25. *For $K \subseteq \mathbb{Z}$, the ring SInt$(K^{\{x_1,x_2,\ldots,x_i\}}, \mathbb{Z})$ on finitely many variables $x_1, x_2, \ldots, x_i$ is a binomially Noetherian ring.*

Proof. First by (4.40), we can write every binomial ideal I in $\mathrm{SInt}(K^{\{x_1,x_2,\ldots,x_i\}}, \mathbb{Z})$ in the form $I = J \cap \mathrm{SInt}(K^{\{x_1,x_2,\ldots,x_i\}}, \mathbb{Z})$, where J is an ideal in $\mathbb{Q}[x_1, x_2, \ldots, x_i]$. Then the rest of the proof is analogous to the proof of Theorem 3.7.7. □

§ 4.6 Some topologically derived binomial rings

Knutson in [38] proves that a binomial ring R with a particular type of generating subset leads to an isomorphism $R \cong \mathbb{Z}$. One application of this result is to topological K-theory. Actually, it means that if $K^0(X)$ for a good space X is a binomial ring then it is isomorphic to the ring of integers $\mathbb{Z}$. In contrast the purpose of this section is to use $K_0(X)$ to give some non-trivial examples of binomial rings which come from topology. The main results give new descriptions of these examples (Theorem 4.6.4 and Theorem 4.6.9).

Theorem 4.6.1. *[Knutson] [57, p. 126] Let R be a binomial ring with a subset S which satisfies the following.*

1. *S generates R as an abelian group.*
2. *Each s in S has finite dimension (where $\lambda_t(s)$ is a polynomial whose degree is called the dimension of s Definition 2.5.9).*
3. *$\lambda^n(s) \in S$ for each s in S and $n \geq 1$.*
4. *If s is one dimension it is insertable in R.*

Then $R \cong \mathbb{Z}$.

We recall from Theorem 2.7.1 that a binomial ring is a special type of λ-ring whose Adams operations all are identity.

Also, $K^0(X)$ for a good space X is a λ-ring with λ operations defined by the exterior powers on vector bundles B over X, $\lambda^n(B) = \Lambda^n(B)$ (See Example 2.5.10(3)).

Here is a consequence of Knutson's Theorem 4.6.1 in $K^0(X)$.

Corollary 4.6.2. *suppose $K^0(X)$ for a good space X is a binomial ring. Then*

$$K^0(X) \cong \mathbb{Z}.$$

In contrast, next we use $K_0(-)$ for good spaces and spectra to construct some non-trivial examples of binomial rings derived from topology. We will start with a well-known one.

Before giving the construction of the first example of a binomial ring arising in topology, the following preliminary information is needed.

First recall from Theorem 2.4.7 that the ring $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ is a binomial ring. We know from Theorem 2.4.6 that the polynomials $\binom{x}{n}$, for $n \geq 0$, form a $\mathbb{Z}$-module basis of $\mathrm{Int}(\mathbb{Z}^{\{x\}})$.

On the other hand it is well known that $K^0(\mathbb{C}P^\infty) = \mathbb{Z}[[T]]$ for $T = L - \tilde{1}$ where L is the universal line bundle and, $\tilde{1}$ is the trivial line bundle. The K-cohomology of $\mathbb{C}P^\infty$ is $\mathbb{Z}$-linear dual to the K-homology see [17], that is we have an isomorphism,

$$K^0(\mathbb{C}P^\infty) \cong \mathrm{Hom}(K_0(\mathbb{C}P^\infty), \mathbb{Z}), \tag{4.41}$$

As a free abelian group $K_0(\mathbb{C}P^\infty)$ has a basis $\{\alpha_n\}$ such that $\{T^n\}$ is dual to $\{\alpha_n\}$.

Now this brings us to the right position to give the first example. Clarke in [17] defines a ring isomorphism using the above basis as follows.

Theorem 4.6.3. *[17] From the map*

$$\varphi : Int(\mathbb{Z}^{\{x\}}) \to K_0(\mathbb{C}P^\infty), \tag{4.42}$$

determined by

$$\binom{x}{n} \mapsto \alpha_n$$

on the basis elements, we obtain a ring isomorphism,

$$Int(\mathbb{Z}^{\{x\}}) \cong K_0(CP^\infty). \tag{4.43}$$

Proof. It is clear φ is additive by extending it linearly.

Thus, it remains to show that φ preserves multiplication. First we know from Proposition 3.4.10 that $\binom{x}{i}\binom{x}{j}$ in $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ is given in terms of the basis by

$$\binom{x}{i}\binom{x}{j} = \sum_{n \geq 0} \binom{n}{j}\binom{j}{n-i}\binom{x}{n}.$$

On the other hand, $K^0(\mathbb{C}P^\infty) = \mathbb{Z}[[T]]$, with the coproduct determined by

$$\Delta T = T \otimes 1 + 1 \otimes T + T \otimes T. \tag{4.44}$$

Hence,

$$\begin{aligned} \Delta(T^n) &= (T \otimes 1 + 1 \otimes T + T \otimes T)^n, \\ &= ((T+1) \otimes (T+1) - 1 \otimes 1)^n. \end{aligned}$$

Then by applying the binomial theorem we get

$$\Delta(T^n) = \sum_{t=0}^{n}\sum_{i=0}^{t}\sum_{j=0}^{t}(-1)^{n+t}\binom{n}{t}\binom{t}{i}\binom{t}{j}T^i \otimes T^j. \tag{4.45}$$

As T^i is dual to α_i, we obtain

$$\alpha_i\alpha_j = \sum_{n\geq 0}\left[\sum_{t\geq i,j}(-1)^{n+t}\binom{n}{t}\binom{t}{i}\binom{t}{j}\right]\alpha_n. \tag{4.46}$$

From the identity

$$\sum_{t\geq i,j}(-1)^{n+t}\binom{n}{t}\binom{t}{i}\binom{t}{j} = \binom{n}{j}\binom{j}{n-i},$$

given in [47, p. 15] we obtain,

$$\alpha_i\alpha_j = \sum_{n\geq 0}\binom{n}{j}\binom{j}{n-i}\alpha_n. \tag{4.47}$$

This implies that φ is multiplicative. □

Next recall from Theorem 3.3.1 that the quotient ring R/I for a binomial ideal I of a binomial ring R is also a binomial ring and from Definition 3.4.5 $((x-y))$ is a principal binomial ideal in $\mathrm{Int}(\mathbb{Z}^{\{x,y\}})$ generated by $x-y$. So now we can state the main result of this section, which gives a new description of the binomial ring $K_0(\mathbb{C}P^\infty)$.

Theorem 4.6.4. *Let $I = ((x-y))$ in the ring $Int(\mathbb{Z}^{\{x,y\}})$. Then we have an isomorphism of binomial rings,*

$$\frac{Int(\mathbb{Z}^{\{x,y\}})}{((x-y))} \cong Int(\mathbb{Z}^{\{t\}}). \tag{4.48}$$

Proof. First consider the map

$$\theta : \mathrm{Int}(\mathbb{Z}^{\{x,y\}}) \longrightarrow \mathrm{Int}(\mathbb{Z}^{\{t\}})$$

given by

$$f(x,y) \longmapsto f(t,t),$$

for $f(x,y) \in \mathrm{Int}(\mathbb{Z}^{\{x,y\}})$. It is easy to see that θ is an onto ring homomorphism. With reference to the first isomorphism theorem we need to show that

$$\mathrm{Ker}(\theta) = ((x-y)).$$

First pick an element $h \in ((x-y))$. Then by Proposition 3.4.6 rewrite $((x-y))$ using,

$$((x-y)) = \left(\left\{\binom{x-y}{i}, i \geq 1\right\}\right). \tag{4.49}$$

So,

$$h = \sum_{i \geq 1}^{finite} \binom{x-y}{i} g_i, \text{ for } g_i \in \mathrm{Int}(\mathbb{Z}^{\{x,y\}}).$$

Therefore

$$\theta(h) = \sum_{i \geq 1}^{finite} \binom{t-t}{i} g_i = 0.$$

This implies that $h \in \mathrm{Ker}(\theta)$.

Conversely suppose that $f(x,y) \in \mathrm{Ker}(\theta)$. Notice that can rewrite it as

$$f(x,y) = f_1(x) \mod (x-y)\mathbb{Q}[x,y]$$

where $f_1(x)$ is a polynomial in x. This is because

$$x^n y^m = \sum_{i-1}^{m} (y-x) x^{n-1+i} y^{m-i} + x^{n+m}.$$

Since $\theta(f(x,y)) = 0$, we see that $f_1(x)$ is zero. Therefore

$$f(x,y) = 0 \mod (x-y)\mathbb{Q}[x,y].$$

Hence,

$$f(x,y) \in (x-y)\mathbb{Q}[x,y] \cap \mathrm{Int}(\mathbb{Z}^{\{x,y\}}).$$

By Remark 3.6.14, this implies that $f(x,y) \in ((x-y))$.

□

Corollary 4.6.5. *There is an isomorphism of binomial rings,*

$$K_0(\mathbb{C}P^\infty) \cong \frac{Int(\mathbb{Z}^{\{x,y\}})}{((x-y))}.$$

□

In section 4.2, the spectrum K was described via the spaces U and BU. This spectrum yields K-cohomology operations, which are defined by $K^*(K) = [K,K]$ and K-homology cooperations defined by $K_*(K) = \pi_*(K \wedge K)$. By [52, 16.33] $K_*(BU)$ is torsion free. So by passing to the limit, $K_*(K)$ is also torsion free. Therefore the map,

$$\varphi : K_*(K) \to K_*(K) \otimes \mathbb{Q}, \tag{4.50}$$

is injective. On the other hand, it is easy to verify that both $\mathbb{Q}$ and $\pi_*(K) \otimes \mathbb{Q}$ are flat $\mathbb{Z}$-modules. This implies that the functors $\pi_*(K \wedge -) \otimes \mathbb{Q}$ and $\pi_*(-) \otimes \pi_*(K) \otimes \mathbb{Q}$ are homology theories. Also by [52, 17.19], the map $\wedge \otimes 1$ induces an isomorphism

$$K_*(K) \otimes \mathbb{Q} \cong \pi_*(K) \otimes \pi_*(K) \otimes \mathbb{Q}. \tag{4.51}$$

We know from (4.10) in Section 4.2 that the coefficient ring $\pi_*(K)$ of K is isomorphic to the Laurent polynomial ring $\mathbb{Z}[u, u^{-1}]$. As a result the following isomorphism of rings is obtained for K,

$$K_*(K) \otimes \mathbb{Q} \cong \mathbb{Q}[u, v, u^{-1}, v^{-1}], \tag{4.52}$$

for $u, v \in K_2(K)$. Now the description of $K_*(K)$ can be given in terms of its image in $\mathbb{Q}[u, v, u^{-1}, v^{-1}]$.

Theorem 4.6.6. *[4] The map*

$$\varphi : K_*(K) \to \mathbb{Q}[u, v, u^{-1}, v^{-1}]$$

induces an isomorphism,

$$K_*(K) \cong \left\{ g(u, v) \in \mathbb{Q}[u, v, u^{-1}, v^{-1}] : g(ax, bx) \in \mathbb{Z}\left[\frac{1}{ab}, x, x^{-1}\right] \textit{ for all } a, b \in \mathbb{Z} \setminus \{0\} \right\}. \tag{4.53}$$

We are now in the right position to give the second example of a binomial ring coming from topology. Let w be the degree zero element in $K_0(K)$ given by $w = u^{-1}v$. As a result of Theorem 4.6.6, we obtain the following.

Corollary 4.6.7. $K_0(K) = \{g(w) \in \mathbb{Q}[w, w^{-1}] : g(a) \in \mathbb{Z}[\frac{1}{a}]$ *for all* $a \in \mathbb{Z} \setminus \{0\}\}$. □

On the other hand, by Proposition 4.5.3 the ring $\mathrm{SLInt}(\mathbb{Z}^{\{x\}})$ satisfies the integrality condition.

Corollary 4.6.8. *There is an isomorphism of binomial rings,*

$$\mathit{SLInt}(\mathbb{Z}^{\{x\}}) \cong K_0(K).$$

□

Here is our main result of this section, which gives another description of the binomial ring $\mathrm{SLInt}(\mathbb{Z}^{\{x\}})$

Theorem 4.6.9. *Let $Int(\mathbb{Z}^{\{x,y\}})$ be the ring of integer-valued polynomials over two x, y variables and let $Int((Z^{\{x\}})[x^{-1}]$ be the localization of the ring $Int(\mathbb{Z}^{\{x\}})$ with respect to the multiplicatively closed set $\{x^n : n \in \mathbb{N}\}$. Then we have an isomorphism of binomials rings,*

$$\frac{Int(\mathbb{Z}^{\{x,y\}})}{((xy-1))} \cong Int(\mathbb{Z}^{\{x\}})[x^{-1}]. \tag{4.54}$$

Proof. Define the map

$$\rho : \mathrm{Int}(\mathbb{Z}^{\{x,y\}}) \longrightarrow \mathrm{Int}(\mathbb{Z}^{\{x\}})[x^{-1}]$$

given by

$$f(x, y) \to f(x, x^{-1}),$$

for $f(x,y) \in \mathrm{Int}(\mathbb{Z}^{\{x,y\}})$. It is easy to see that ρ is a ring homomorphism. Consider a general element $x^{-n}f(x)$ in $\mathrm{Int}(\mathbb{Z}^{\{x\}})[x^{-1}]$, for $n \in \mathbb{N}$ and $f(x) \in \mathrm{Int}(\mathbb{Z}^{\{x\}})$.

We see that

$$\rho(y^n f(x)) = x^{-n} f(x) \text{ and } y^n f(x) \in \mathrm{Int}(\mathbb{Z}^{\{x,y\}}),$$

so ρ is an onto homomorphism. To finish the proof, we need to show that

$$\mathrm{Ker}(\rho) = ((xy-1)).$$

As in Theorem 4.6.4 it is clear that $((xy-1)) \subseteq \mathrm{Ker}(\rho)$. Conversely consider $f(x,y)$ in $\mathrm{Ker}(\rho)$. We claim that $f(x,y) = f_1(x) + f_2(y) \mod (xy-1)\mathbb{Q}[x,y]$.This can be seen, using

$$x^n y^m = y^{m-n} + \sum_{i=0}^{n-1} (xy-1)x^i y^{m-n+i},$$

for $0 < n \leq m$. Then, since $\rho(f(x,y)) = 0$, we see that $f_1(x) = f_2(y) = 0$. Therefore

$$f(x,y) = 0 \mod (xy-1)\mathbb{Q}[x,y].$$

Hence

$$f(x,y) \in (xy-1)\mathbb{Q}[x,y] \cap \mathrm{Int}(\mathbb{Z}^{\{x,y\}}).$$

So by Remark 3.7.9, $f(x,y) \in ((xy-1))$. □

Corollary 4.6.10. *There is an isomorphism of binomial rings.*

$$K_0(K) \cong \frac{Int(\mathbb{Z}^{\{x,y\}})}{((xy-1))}.$$

In order to give a third example, the following information is needed. First we begin with the notion of a connective spectrum. A spectrum E is called a connective spectrum if $\pi_n(E) = 0$ for all $n < 0$. A connective spectrum e which has a map $e \longrightarrow E$ for a spectrum E such that it is universal amongst maps $\bar{e} \longrightarrow E$ in which $\bar{e}$ is connective, is unique up to homotopy equivalence. This is called the connective cover of E.

We let k be the connective cover (connective K-theory spectrum) of K. The coefficient ring $\pi_*(k)$ of k is isomorphic to $\mathbb{Z}[t]$. By applying the functors $\pi_*(K \wedge -) \otimes \mathbb{Q}$ and $\pi_*(-) \otimes \pi_*(K) \otimes \mathbb{Q}$ to k as in (4.51) we obtain

$$\begin{aligned} \pi_*(k) \otimes \pi_*(K) \otimes \mathbb{Q} &\cong K_*(k) \otimes \mathbb{Q}, \\ &= \mathbb{Q}[u, u^{-1}, v]. \end{aligned}$$

Proposition 4.6.11. *For the connective K-theory spectrum k, we have*

$$K_0(k) = K_0(K) \cap \mathbb{Q}[x].$$

Proof. Apply Proposition 17.2 (iii) and Theorem 17.4 of [3] to k. □

On the other hand, recall from Theorem 4.5.13 that the ring $\mathrm{SInt}(\mathbb{Z}^{\{x\}})$ is a binomial ring.

Corollary 4.6.12. *There is an isomorphism of binomial rings*

$$SInt(\mathbb{Z}^{\{x\}}) \cong K_0(k).$$

□

From Theorem 4.6.9 we can give a new description of $K_0(k)$.

Corollary 4.6.13. *There is an isomorphism of binomial rings*

$$K_0(k) \cong \frac{Int(\mathbb{Z}^{\{x,y\}})}{((xy-1))} \cap \mathbb{Q}[x].$$

□

Finally. we know from Theorem 2.9.5 that binomial rings are preserved by localization. From Proposition 2.9.8 and Theorem 4.6.8, we have the following.

Example 4.6.14. There is an isomorphism of binomial rings,

$$\mathrm{Int}(\mathbb{Z}_{(p)}^{\{x\}}) \cong (K_{(p)})_0(\mathbb{C}P^\infty).$$

From (4.22) and Corollary 4.6.8, we have the following.

Example 4.6.15. There is an isomorphism of binomial rings,

$$\mathrm{SLInt}(\mathbb{Z}_{(p)}^{\{x\}}) \cong (K_{(p)})_0(K_{(p)}).$$

Also from (4.31) and Corollary 4.6.12, we have the following.

Example 4.6.16. There is an isomorphism of binomial rings,

$$\mathrm{SInt}(\mathbb{Z}_{(p)}^{\{x\}}) \cong (K_{(p)})_0(k_{(p)}).$$

For more examples in this direction see [50].

Chapter 5

Cotriple Cohomology

§ 5.1 Introduction

Huber [35] realised that a cotriple $\mathbb{C} = (C, \varepsilon, \delta)$ in a category $\mathfrak{A}$ defined using the iterates of C, with face operations $C^{n+1}(A) \to C^n(A)$ and degeneracy operations $C^n(A) \to C^{n+1}(A)$, constructed by using ε and δ, for $A \in \mathfrak{A}$ yields a simplicial object in $\mathfrak{A}$.

Barr and Beck used a cotriple from an adjoint pair of functors, to introduce a cohomology theory, by using the Huber simplicial structure. This is called cotriple cohomology theory [10]. Furthermore, Beck in his PhD dissertation [11], gave more details of the cotriple cohomology theory and interpreted the 0-th and first cohomology groups. The main tools of cotriple cohomology theory and some necessary background on it are given in §5.2.

André in [6] described a cohomology theory arising from a specific complex and at the same time Quillen in [46] described a cohomology theory with regards to abelianization. Both theories turn out to compute the same cohomology theory on the category of commutative algebras. This is called the André-Quillen cohomology theory. It is an example of a cotriple cohomology theory in the category of commutative algebras, where the cotriple comes from the composite of a free functor and a forgetful functor. It will be discussed in §5.3.

Robinson in his thesis [48], defines the cohomology of λ-rings with coefficients in a contravariant functor $\mathrm{Der}_\lambda(-, M)$. This is the set of all λ-derivations with values in a λ-module M. In §5.4, a summary of this theory is presented. In the next chapter, we will apply Robinson's concepts to binomial rings to define cohomology of binomial rings as another example of cotriple cohomology.

§ 5.2 Cotriple cohomology

A cotriple yields an augmented simplicial object which can be used to construct a cochain complex and hence a resolution in the sense of homological algebra. The cotriple cohomology was originally defined by Barr and Beck, where they used a cotriple to construct a cotriple resolution by using simplicial methods. André and Quillen define a cohomology theory in the category of commutative algebras as an example of cotriple cohomology theory. Before defining cotriple cohomology, in this section, we present all necessary background material.

5.2.1 Simplicial objects

Definition 5.2.1. A *simplicial object* in a category $\mathfrak{A}$ is given by a system $X_* = (X_n, d_n^i, t_n^i)$ consisting of a sequence of objects $X_0, X_1 \cdots, X_n, \cdots$ with two families of arrows of $\mathfrak{A}$

$$d_n^i : X_n \to X_{n-1}, \quad \text{for} \quad 0 \le i \le n \quad \text{and} \quad 1 \le n < \infty, \tag{5.1}$$

$$t_n^i : X_n \to X_{n+1}, \quad \text{for} \quad 0 \le i \le n \quad \text{and} \quad 1 \le n < \infty, \tag{5.2}$$

where d_n^i is called a face operation and t_n^i is called a degeneracy operation. They satisfy the following relations (called simplicial identities).

$$d_n^i \circ d_{n+1}^j = d_n^{j-1} \circ d_{n+1}^i \quad \text{for} \quad 0 \le i < j \le n+1,$$

$$t_n^j \circ t_{n-1}^i = t_n^i \circ t_{n-1}^{j-1} \quad \text{for} \qquad 0 \le i < j \le n.$$

$$d_{n+1}^i \circ t_n^j = \begin{cases} t_{n-1}^{j-1} \circ d_n^i, & \text{for} \quad 0 \le i < j \le n, \\ 1, & \text{for} \quad 0 \le i = j \le n \quad \textit{or} \quad 0 \le i-1 = j \le n, \\ t_{n-1}^j \circ d_n^{i-1}, & \text{for} \quad 0 \le j < i-1 \le n. \end{cases}$$

We write X_* for the simplicial object as above.

There is an equivalent description of a simplicial object X_* in a category $\mathfrak{A}$. Let Δ be the category whose objects are finite totally ordered sets and whose morphisms are the maps preserving order. Then giving a simplicial object X_* in $\mathfrak{A}$ is equivalent to giving covariant functor

$$X_* : \Delta^{\text{op}} \longrightarrow \mathfrak{A}.$$

We mean by an *augmented simplicial object* in a category $\mathfrak{A}$ a simplicial object X_* in $\mathfrak{A}$ together with an additional object X_{-1} and a morphism $\epsilon : X_0 \to X_{-1}$ such that $\epsilon \circ d_1^0 = \epsilon \circ d_1^1$. We express it by the diagram

$$\cdots \vdots X_{n+1} \vdots \cdots \rightrightarrows X_1 \rightrightarrows X_0 \xrightarrow{\epsilon} X_{-1}.$$

Suppose that X_* is a simplicial object in an additive category $\mathfrak{C}$. Then we have the structure of a complex

$$\dots \xrightarrow{d} X_{n+1} \xrightarrow{d} X_n \xrightarrow{d} X_{n-1} \xrightarrow{d} \cdots \xrightarrow{d} X_0 \to 0, \tag{5.3}$$

with differential $d = \sum_{i=0}^{n} (-1)^i d_n^i : X_n \longrightarrow X_{n-1}$.

Proposition 5.2.2. *[8, Proposition 4.1]We have* $d \circ d = 0$.

So the complex in (5.3) is a chain complex associated to X_*, and we write it as $C(X_*)$.

All information on simplicial objects comes from [8, Chapter 3] and [53, Chapter 8]. For more details about simplicial methods we refer to [42].

5.2.2 Triples and Cotriples

Any endofunctor which is defined on a category $\mathfrak{A}$ has composites. In particular $T^2 = T \circ T : \mathfrak{A} \to \mathfrak{A}$ is also an endofunctor. So we can consider a natural transformation $\beta : T^2 \to T$. On the other hand every category has an identity functor $I_{\mathfrak{A}} : \mathfrak{A} \to \mathfrak{A}$. Therefore we can consider a natural transformation $\eta : I_{\mathfrak{A}} \to T$. From the following data in any category we can define a triple (it is an alternative name monad). More detail on this material can be found in [53] and [11].

Definition 5.2.3. Let $\mathfrak{A}$ be a category, a *triple* (monad) $\mathbb{T} = (T, \eta, \beta)$ on $\mathfrak{A}$ consists of the following.

1. An endofunctor $T : \mathfrak{A} \to \mathfrak{A}$.
2. A unit natural transformation $\eta : Id_{\mathfrak{A}} \to T$.
3. A multiplication natural transformation $\beta : T^2 \to T$.

These are such that the following diagrams

$$\begin{array}{ccc} T^3 & \xrightarrow{T\beta} & T^2 \\ {\scriptstyle \beta T}\downarrow & & \downarrow{\scriptstyle \beta} \\ T^2 & \xrightarrow[\beta]{} & T \end{array}$$

$$\begin{array}{ccccc} T & \xrightarrow{\eta T} & T^2 & \xleftarrow{T\eta} & T \\ & {\scriptstyle Id_T}\searrow & \downarrow{\scriptstyle \beta} & \swarrow{\scriptstyle Id_T} & \\ & & T & & \end{array}$$

are commutative. These are called the associativity law, left and right unit respectively.

The notion of a cotriple on a category $\mathfrak{A}$ is defined dually. That is, a cotriple (comonad) in $\mathfrak{A}$ is a triple $\mathbb{T}$, on the category $\mathfrak{A}^{\mathrm{op}}$.

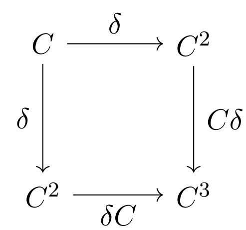
Definition 5.2.4. A *cotriple* (comonad) $\mathbb{C} = (C, \varepsilon, \delta)$ on the category $\mathfrak{A}$ consists of the following.

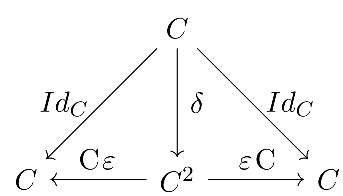
1. An endofunctor $C : \mathfrak{A} \to \mathfrak{A}$.

2. A counit natural transformation $\varepsilon : C \to Id_{\mathfrak{A}}$.

3. A natural transformation $\delta : C \to C^2$.

These are such that the following diagrams





are commutative.

Proposition 5.2.5. *Let $F : \mathfrak{A} \to \mathfrak{B}$ be a left adjoint functor to the functor $G : \mathfrak{B} \to \mathfrak{A}$ between two categories $\mathfrak{A}$ and $\mathfrak{B}$. Let $\eta : Id_{\mathfrak{A}} \to GF$ be the unit morphism of the adjunction and $\varepsilon : FG \to Id_{\mathfrak{B}}$ be the counit morphism of the adjunction. Then $(FG, \varepsilon, F\eta G)$ is a cotriple on $\mathfrak{A}$.*

Proof. First to show the associativity law, we have

$$F\eta G \circ F\eta GFG = F(\eta G \circ \eta GFG) = F(\eta G \circ GF\eta G) = F\eta G \circ FGF\eta G.$$

In the language of diagrams this means that the diagram

$$\begin{array}{ccc} FG & \xrightarrow{F\eta G} & (FG)^2 \\ {\scriptstyle F\eta G}\downarrow & & \downarrow{\scriptstyle FGF\eta G} \\ (FG)^2 & \xrightarrow[F\eta GFG]{} & (FG)^3 \end{array}$$

is commutative. Second to show the left and right unit laws, we have

$$F\eta G \circ FG\varepsilon = F(\eta G \circ G\varepsilon) = F(Id) = Id$$

and

$$F\eta G \circ \varepsilon FG = F\eta G \circ FG\varepsilon = F(\eta G \circ G\varepsilon) = F(Id) = Id.$$

That is, the diagram,

$$\begin{array}{ccccc} & & FG & & \\ & \swarrow^{Id_{FG}} & \downarrow{\scriptstyle F\eta G} & \searrow^{Id_{FG}} & \\ FG & \xleftarrow{\mathrm{FG}\varepsilon} & (FG)^2 & \xrightarrow{\varepsilon\ \mathrm{FG}} & FG \end{array}$$

is commutative. □

5.2.3 Cotriple cohomology

Let $\mathbb{C} = (C, \varepsilon, \delta)$ be a cotriple on the category $\mathfrak{A}$. For an object A in $\mathfrak{A}$ we have the following augmented simplicial object in $\mathfrak{A}$.

$$\ldots \vdots\ C^{m+1}(A) \vdots\ \ldots \ \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} C^2(A) \rightrightarrows C(A) \xrightarrow{\varepsilon} A \tag{5.4}$$

We denote it by $C_*(A) \longrightarrow A$, with the face and degeneracy operations defined by

$$d_n^i = C^i \varepsilon C^{n-i} : C^{n+1}(A) \to C^n(A) \quad \text{for } 0 \le i \le n, \tag{5.5}$$

$$t_n^i = C^i \delta C^{n-i} : C^{n+1}(A) \to C^{n+2}(A) \quad \text{for } 0 \le i \le n. \tag{5.6}$$

The simplicial identities hold for d_n^i and t_n^i see [22, p.187], with augmentation

$$\varepsilon : C(A) \longrightarrow A,$$

where $C_*(A)$ is the simplicial object with

$$(C_*(A))_n = C \circ C \cdots \circ C(A) = C^{n+1}(A),$$

with $C^0(A) = A$ and for any $n \geq -1$.

We call $C_*(A)$ the cotriple resolution of A in $\mathfrak{A}$.

Definition 5.2.6. Let $D : \mathfrak{A} \to \mathfrak{C}$ be a contravariant functor from a category $\mathfrak{A}$ to an abelian category $\mathfrak{C}$. Then by applying D to $C_*(A)$, we get an augmented cosimplicial object $DA \longrightarrow D(C_*(A))$ in $\mathfrak{C}$. Then we obtain the *cotriple cohomology groups* of the object A with coefficients in D. We write this theory as $H^n_{\mathbb{C}}(A, D)$, in which

$$H^n_{\mathbb{C}}(A, D) =: H^n(D(C_*(A))).$$

In other words, $H^n_{\mathbb{C}}(A, D)$ is the n^{th} cohomology group of the object A with coefficients in D with respect to the cotriple $\mathbb{C}$. This is the cohomology associated to the cochain complex

$$0 \to D(C(A)) \xrightarrow{\sigma_1} D(C^2(A)) \xrightarrow{\sigma_2} D(C^3(A)) \xrightarrow{\sigma_3} \cdots$$

where $\sigma_n = \sum_{i=0}^{n} (-1)^i d_n^i D$.

Proposition 5.2.7. *[10] Suppose $D : \mathfrak{A} \to \mathfrak{C}$ transforms*

$$C^2(A) \rightrightarrows C(A) \xrightarrow{\varepsilon} A \tag{5.7}$$

into an equalizer diagram in $\mathfrak{C}$ and let $A \in \mathfrak{A}$, then

$$H^0_{\mathbb{C}}(A, D) \cong D(A).$$

Proposition 5.2.8. *Let $X \in \mathfrak{C}$ and let $A = C(X)$. Then*

$$H^n_{\mathbb{C}}(A, D) \cong 0.$$

for all $n \geq 1$.

Proof. There exists a contracting homotopy map $\varsigma_n : C^{n+2} \longrightarrow C^{n+3}$, for $n \geq -1$ given by $\varsigma_n = C^{n+1}\delta$. Since $\epsilon\varsigma_{-1} = Id$, $\varepsilon_{n+1}\varsigma_n = Id$, $\varepsilon_0\varsigma_0 = \varsigma_{-1}\epsilon$ and $\varepsilon_i\varsigma_n = \varsigma_{n-1}\varepsilon_i$, this implies that $H^n_{\mathbb{C}}(A, D) \cong 0$. □

Now we will consider the case of cotriple cohomology for the category of commutative rings $\mathfrak{CRing}$. By Proposition 5.2.5, we can consider a cotriple $\mathbb{C} = (FG, \varepsilon, F\eta G)$ coming from the composite of a pair of adjoint functors.

$$\mathfrak{Set} \xrightarrow{F} \mathfrak{CRing} \xrightarrow{G} \mathfrak{Set},$$

where F is free functor and G is forgetful functor as in Example 2.8.6. Our coefficients will we given be a derivation functor.

The product rule for differentiation,

$$\frac{d(fg)}{dx} = f\frac{dg}{dx} + g\frac{df}{dx},$$

leads to the idea of a derivation of a ring.

Definition 5.2.9. A *derivation of a commutative ring* R is an additive homomorphism $d : R \longrightarrow R$ satisfying the condition

$$d(xy) = xd(y) + d(x)y, \tag{5.8}$$

for all $x, y \in R$.

For an R-module M, an additive homomorphism $d : R \to M$ that satisfies the derivation condition (5.8) is called a *derivation of* R *with values in* M. As sets we denote by $\mathrm{Der}(R)$ the set of all derivations of R and by $\mathrm{Der}(R, M)$ the set of all derivations of R with values in M.

Note that the maps $(d_1 + d_2)$ and (ad) given by

$$(d_1 + d_2)(x) = d_1(x) + d_2(x)$$

and

$$(ad)(x) = ad(x),$$

for d_1, d_2 and d be derivations of R and $x, a \in R$, are also derivations of R. So $\mathrm{Der}(R)$ is an R-module. Similarly $\mathrm{Der}(R, M)$, is also an R-module.

Proposition 5.2.10. *Let* $d : R \longrightarrow M$ *be a derivation of a commutative ring* R *with unit with values in an* R*-module* M*, then*

1. $d(1) = 0$,
2. $d(r^n) = nr^{n-1}d(r)$ *for* $r \in R$ *and* $n \geq 1$.

Proof. The first property follows from (5.8) by taking $x = y = 1$. We are going to prove property 2 by induction on n. For $n = 1$ it is clear. Assume the statement is true for n. For $n + 1$ we have

$$\begin{aligned} d(r^{n+1}) &= rd(r^n) + r^n d(r) \\ &= rnr^{n-1}d(r) + r^n d(r) \\ &= (n+1)r^n d(r). \end{aligned}$$

□

For commutative rings R and S, let M be a R-module and let $f : S \longrightarrow R$ be a ring-homomorphism. Then it is clear by the action given by $(s, m) \longmapsto f(s)m$, M is also an S-module. Thus we define the category of commutative rings over R, whose objects are ring-homomorphisms $f_i : S_i \longrightarrow R$, from a commutative ring S_i and whose morphisms are given by the commutative diagrams of ring-homomorphisms,

$$\begin{array}{ccc} S_1 & \xrightarrow{f_1} & R \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle Id_R} \\ S_2 & \xrightarrow[f_2]{} & R \end{array}$$

We denote this category by $\mathfrak{CRing}/R$.

Now for a commutative ring R and an R-module M the derivation functor $\mathrm{Der}(-, M)$ give us a contravariant functor

$$\mathrm{Der}(-, M) : \mathfrak{CRing}/R \longrightarrow \mathfrak{Ab}.$$

There is a canonical homomorphism from $C^n(R)$ to R given by composite of maps in (5.4). Therefore M becomes a $C^n(R)$-module. This leads to the following.

Definition 5.2.11. Let R be a commutative ring and let M be an R-module. Then we define *the cohomology of R with coefficients in M* to be the cotriple cohomology of R with coefficients in $\mathrm{Der}(-, M)$, that is

$$H^n(R, M) = H^n_{\mathbb{C}}(R, M) = H^n_{\mathbb{C}}(R, \mathrm{Der}(-, M)).$$

Proposition 5.2.12. *Let M be an R-module. Then*

$$H^0_{\mathbb{C}}(R, M) \cong Der(R, M).$$

□

Proposition 5.2.13. *Let R be a free commutative ring and let M be an R-module, then*

$$H^n_{\mathbb{C}}(R, M) \cong 0.$$

for all $n \geq 1$.

Proof. The proof follow by applying Proposition 5.2.8 with $C = FG$. □

Dually if $U : \mathfrak{A} \to \mathfrak{C}$ is a functor from a category $\mathfrak{A}$ to an abelian category $\mathfrak{C}$ and $\mathbb{T} = (T, \eta, \beta)$ is a triple in $\mathfrak{A}$, in the same way as above we obtain the triple cohomology groups of an object A with coefficients in U. We write this theory as $H^n_{\mathbb{T}}(A, U)$, for $n \geq 0$, where

$$H^n_{\mathbb{T}}(A, U) = H^n(U(T_*(A))).$$

In other words, $H^n_{\mathbb{T}}(A, U)$ is the n^{th} cohomology group of the object A with coefficients in U with respect to the triple $\mathbb{T}$. This is the cohomology associated to the cochain complex

$$0 \to U(T(A)) \xrightarrow{\sigma_1} U(T^2(A)) \xrightarrow{\sigma_2} U(T^3(A)) \xrightarrow{\sigma_3} \cdots$$

where $\sigma_n = \sum(-1)^i \eta_i U$.

For more details about cotriple cohomology we refer to [9] and [21].

§ 5.3 André-Quillen cohomology

There is a cohomology theory for commutative algebras, which was independently introduced by André and Quillen. André describes the cohomology theory arising from a specific complex and Quillen describes the cohomology theory with regard to abelianization. Both theories compute the same cohomology. This theory is now called André-Quillen cohomology theory of commutative algebras over a commutative ring K. In this section we will present this theory.

For a fixed commutative ring K, we consider the category $\mathfrak{Commalg}$ whose objects are commutative K-algebras and whose morphisms are K-linear algebra maps $f : R \longrightarrow S$, between two commutative K-algebras R and S. The free functor F_K that sends a set X to the polynomial algebra $K[X]$, is left adjoint to the forgetful functor $G_K : \mathfrak{Commalg} \longrightarrow \mathfrak{Set}$ [46]. Then by Proposition 5.2.5 from $F_K G_K$ we obtain a cotriple $\mathbb{C}_K$ on $\mathfrak{Commalg}$. In an obvious way as a result we obtain a cotriple resolution $(\mathbb{C}_*)_K(R)$ for each object $R \in \mathfrak{Commalg}$.

Definition 5.3.1. Let R be an K-algebra and let M be an R-module, we mean by K-derivation a K-module map $d_K : R \longrightarrow M$, such that satisfying the following condition

$$d_K(rs) = r(ds) + (dr)s,$$

for all $r, s \in R$.

Then in the same way of commutative ring all K-derivations $\mathrm{Der}_K(R, M)$, with value in M is an R-module.

For K-algebras R and S, let M be a R-module and let $f : S \longrightarrow R$ be a K-homomorphism. Then by the same way of commutative ring it is clear M is also an S-module. Thus we define the category of K-algebra over R whose objects are K-homomorphisms $f_i : S_i \longrightarrow R$, from a K-algebra S_i and whose morphisms given by the following commutative diagrams of K-algebras

$$\begin{array}{ccc} S_1 & \xrightarrow{f_1} & R \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle Id_R} \\ S_2 & \xrightarrow[f_2]{} & R \end{array}$$

for a K-homomorphism g. We denote this category by $\mathfrak{Commalg}/R$.

Now for a K-algebra R and an R-module M, we define the contravariant functor $\mathrm{Der}_K(-, M) : \mathfrak{Commalg}/R \longrightarrow \mathfrak{Ab}$.

There is a canonical map from $(C^n)_K(R)$ to R. given by $\epsilon d_2^0 \dots d_{n-2}^0 d_{n-1}^0$ (or equivalently any sequence of maps in (5.4), M becomes a module over $(C^n)_K(R)$ for all $n \geq 1$.

Definition 5.3.2. Let R be a K-algebra and let M be a R-module. By applying the functor $\mathrm{Der}_K(-,M)$ to the cotriple resolution $(C_*)_K(R)$ on $\mathfrak{Commalg}$, we define the *André-Quillen cohomology groups* of R with coefficients in $\mathrm{Der}_K(-,M)$, we write this theory as $H^n_{AQ}(R/K,M)$ in which

$$H^n_{AQ}(R/K,M) = H^n_{\mathbb{C}_K}(R/K,M) = H^n_{\mathbb{C}_K}(R,\mathrm{Der}_K(-,M)). \tag{5.9}$$

In other words, $H^n_{AQ}(R/K,M)$ is the n^{th} cotriple cohomology group of R with coefficients in $\mathrm{Der}_K(-,M)$ with respect the to cotriple $\mathbb{C}_K$ on $\mathfrak{Commalg}$.

Thus André-Quillen cohomology on $\mathfrak{Commalg}$ is a particular example of cotriple cohomology.

Proposition 5.3.3. *Let R be an K-algebra and let M be an R-module. Then*

$$H^0_{AQ}(R,M) \cong Der_K(R,M).$$

□

Proposition 5.3.4. *Let R be a free commutative K-algebra and let M be an R-module. Then*

$$H^n_{AQ}(R,M) \cong 0.$$

for all $n \geq 1$.

Proof. The proof follow from Proposition 5.2.8 □

§ 5.4 Cohomology of λ-rings

Robinson in [48], gives a cohomology theory for λ-rings, with values in the contravariant functor $\mathrm{Der}_\lambda(-,M)$ which is the set of all λ-derivations with values in a λ-module M over the λ-ring. This is another example of cotriple cohomology. In this section we will summarize Robinson Construction of cohomology of λ-rings.

First we start with Robinson's notion of a λ-module over a λ-ring, which will supply us with the coefficients of cohomology of λ-rings.

Definition 5.4.1. Let R be a λ-ring. An R-module M is called a λ-*module* over R, if there is a sequence of abelian group homomorphisms $\Omega^M_n : M \longrightarrow M$, satisfying the following conditions.

1. $\Omega^M_1(a) = a$,
2. $\Omega^M_n(ra) = \psi^n(r)\Omega^M_n(a)$,
3. $\Omega^M_{nm}(a) = (-1)^{(n+1)(m+1)}\Omega^M_n\Omega^M_m(a)$,

for all $a \in M,\ r \in R$ and $n,m \geq 1$.

Definition 5.4.2. Let (M, Ω_n^M) and (N, Ω_n^N) be two λ-modules over a λ-ring R. An R-homomorphism $f : M \longrightarrow N$ is called a *λ-module homomorphism* if it preserves the λ-module structure. That is, the following diagram

$$\begin{array}{ccc} M & \xrightarrow{\ f\ } & N \\ {\scriptstyle \Omega_n^M}\big\downarrow & & \big\downarrow{\scriptstyle \Omega_n^N} \\ M & \xrightarrow[\ f\ ]{} & N \end{array}$$

commutes for $n \geq 1$. We denote this category by R-$\mathfrak{Mod}_\lambda$ the category of λ-modules over the λ-ring R whose objects are λ-modules over the λ-ring R and whose morphisms are λ-module homomorphisms.

Proposition 5.4.3. *Every λ-ring R whose Adams operations all are the identity map on R is a λ-module over itself with module structure given by*

$$\Omega_n^M = (-1)^{(n+1)} Id_M,$$

for $n \geq 1$. □

Example 5.4.4. The ring of integers $\mathbb{Z}$ is a λ-module over itself with module structure given by

$$\Omega_n^{\mathbb{Z}} = (-1)^{(n+1)} \mathrm{Id}_{\mathbb{Z}},$$

for $n \geq 1$.

Definition 5.4.5. Let R be a λ-ring and M a λ-module over R. Then a derivation of R with values in M, $d : R \longrightarrow M$, is called a *λ-derivation* if it satisfies the following condition,

$$d(\lambda^n(r)) = \sum_{i=0}^{n-1} \Omega_{n-i}^M(d(r))\lambda^i(r), \tag{5.10}$$

for all $r \in R$ and $n \geq 1$.

We know from Proposition 5.2.5, that an adjoint pair of functors leads to a cotriple.

We give the construction of the free λ-ring on one generator as a step towards constructing a cotriple resolution on $\mathfrak{Ring}_\lambda$. Let Λ_y be a free λ-ring in one generator y. Then by Definition 2.5.1, Λ_y should contain all $\lambda^n(y)$ for $n \geq 1$. This implies that Λ_y contains all polynomials in $y, \lambda(y), \lambda^2(y), \ldots, \lambda^n(y), \ldots$ with integer coefficients.

Proposition 5.4.6. *The free λ-ring on one generator y is*

$$\Lambda_y = \mathbb{Z}[y_1, y_2, \cdots],$$

where $\lambda^n(y_1) = y_n$ for all $n \geq 1$.

Proof. For a proof see [57, Proposition 1.38]. □

Example 5.4.7. Let M be a λ-module over Λ_x, then we have

$$\mathrm{Der}_\lambda(\Lambda_x, M) \cong M,$$

for any λ-derivation $d \in \mathrm{Der}_\lambda(\Lambda_x, M)$, determined by

$$d(x_1) = m,$$

$$d(x_i) = \sum_{j=1}^{i} \Omega_j^M x_{i-j},$$

for $m \in M$ and $x_0 = 1$.

In the same way we can construct the free λ-ring on a set of generators. Let $y_1, y_2, \ldots, y_n \in \Lambda_{y_1,y_2,\ldots,y_n}$ such that for any λ-ring R there exists a λ-homomorphism

$$f : \Lambda_{y_1,y_2,\ldots,y_n} \longrightarrow R$$

determined by

$$f(y_i) = r_i,$$

for $1 \leq i \leq n$ and $r_i \in R$. This implies that the ring $\Lambda_{y_1,y_2,\ldots,y_n}$ is the polynomial ring over $\mathbb{Z}$ generated by the $\lambda^n(y_i)$. So

$$\Lambda_{y_1,y_2,\ldots,y_n} = \Lambda_{y_1} \otimes \Lambda_{y_2} \otimes \cdots \otimes \Lambda_{y_n}.$$

On can also construct the free Λ-ring on an arbitrary set.

So we get

$$\mathfrak{Set} \xrightarrow{F_\lambda} \mathfrak{Ring}_\lambda \xrightarrow{G_\lambda} \mathfrak{Set},$$

where F_λ is the free functor taking a set X to the free λ-ring generated by this set and G_λ is the forgetful functor. As a result we obtain a cotriple $\mathbb{C}_\lambda$ on $\mathfrak{Ring}_\lambda$. For $R \in \mathfrak{Ring}_\lambda$ we have a cotriple resolution $(C_*)_\lambda(R)$ on $\mathfrak{Ring}_\lambda$.

For λ-rings R and S, let M be a λ-module over R and let $f : S \longrightarrow R$ be a λ-homomorphism. Then it is clear M is also a λ-module over S. Thus we define the category of λ-rings over R, whose objects are λ-homomorphisms $f_i : S_i \longrightarrow R$ from a λ-rings S_i and whose morphisms are given by the following commutative diagrams of λ-rings

$$\begin{array}{ccc} S_1 & \xrightarrow{f_1} & R \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle Id_R} \\ S_2 & \xrightarrow[f_2]{} & R \end{array}$$

for a λ-homomorphism g. We denote this category by $\mathfrak{Ring}_\lambda/R$.

Let R be a λ-ring and let M be a λ-module over R. Then we define the contravariant functor $\mathrm{Der}_\lambda(-, M) : \mathfrak{Ring}_\lambda/R \longrightarrow \mathfrak{Ab}$

Definition 5.4.8. Let R be a λ-ring and let M be a λ-module over R. By applying the functor $\mathrm{Der}_\lambda(-, M)$ to the cotriple resolution $(C_*)_\lambda(R)$ of $R \in \mathfrak{Ring}_\lambda$, we define the *cohomology of* λ*-ring* R with coefficients in M by

$$H^n_\lambda(R, M) = H^n_{\mathbb{C}_\lambda}(R, M) = H^n_{\mathbb{C}_\lambda}(R, \mathrm{Der}_\lambda(-, M)). \tag{5.11}$$

In other words, $H^n_\lambda(R, M)$ is the n^{th} cotriple cohomology group of R with coefficients in $\mathrm{Der}_\lambda(-, M)$ with respect to the cotriple $\mathbb{C}_\lambda$ on $\mathfrak{Ring}_\lambda$.

In the next chapter we will introduce cohomology of binomial rings as another example of cotriple cohomology.

Proposition 5.4.9. *Let* R *be a* λ*-ring and let* M *be a* λ*-module over* R*. Then*

$$H^0_\lambda(R, M) \cong Der_\lambda(R, M).$$

Proof. For proof see [48, Theorem 7.6]. □

Proposition 5.4.10. *Let* R *be a free* λ *ring and let* M *be a* λ*-module over* R *Then*

$$H^n_\lambda(R, M) = 0,$$

for all $n \geq 1$*.*

Proof. The proof follow from Proposition 5.2.8. □

Chapter 6

Cohomology of binomial rings

We know from Chapter 5 that the concept of a cotriple provides a simplicial method to define cohomology in a categorical setting. The most familiar example is a cotriple that comes from a composite of adjoint functors such as a free functor and a forgetful functor. This section will start with free binomial rings as a step to construct a cotriple on $\mathfrak{BinRing}$.

§ 6.1 Free binomial rings

We know from Proposition 2.8.13 that the integer-valued polynomial ring over a set X of variables,

$$\mathrm{Int}(\mathbb{Z}^X) = \{f \in \mathbb{Q}[X] : f(\mathbb{Z}^X) \subseteq \mathbb{Z}\}$$

(Definition 2.4.1) is the free binomial ring on the set X.

Now by taking the composite of this pair of adjoint functors

$$C = F_{\mathrm{Bin}} G_{\mathrm{Bin}} : \mathfrak{BinRing} \to \mathfrak{BinRing},$$

we obtain the cotriple $\mathbb{C}_{\mathrm{Bin}}$ on $\mathfrak{BinRing}$. Thus for any binomial ring $R \in \mathfrak{BinRing}$ we can take the associated cotriple resolution $(C_*)_{\mathrm{Bin}}(R)$, where each component of $(C_*)_{\mathrm{Bin}}(R)$ is a free binomial ring.

To define cohomology of binomial rings by using cotriple cohomology theory, we need first to define the notion of a binomial module over a binomial ring R, in order have coefficients for the theory.

§ 6.2 Binomial modules

When we define the cohomology of a usual ring its modules supply us with coefficients. Quillen in [46], through the concept of modules and derivations on commutative algebras, defined a cohomology theory on commutative algebras. To introduce the concept of a binomial module over a binomial ring, we apply the construction of Robinson for λ-modules over λ-rings in Definition 5.4.1 to a binomial ring R. We get the following.

Definition 6.2.1. Let R be a binomial ring. An R-module M is called a *binomial module* over R, if there is a sequence of R-homomorphisms, $\varphi_n^M : M \to M$, for $n \geq 1$ satisfying the following.

1. $\varphi_1^M(m) = m$,
2. $\varphi_{ij}^M(m) = (-1)^{(i+1)(j+1)}\varphi_i^M(\varphi_j^M(m))$,

for all $m \in M$ and $i, j \geq 1$.

Proposition 6.2.2. *Let R be a binomial ring. Then every R-module M is a binomial module over R, with binomial module structure $\varphi_n^M : M \to M$, for $n \geq 1$, given by*

$$\varphi_n^M = (-1)^{n+1} Id_M. \tag{6.1}$$

Proof. 1. $\varphi_1^M(m) = m$.

2.

$$\begin{aligned}(-1)^{(i+1)(j+1)}\varphi_i^M(\varphi_j^M(m)) &= (-1)^{ij+i+j+1}\varphi_i^M((-1)^{j+1}m)\\ &= (-1)^{ij+i}\varphi_i^M(m)\\ &= (-1)^{ij+i}(-1)^{i+1}m\\ &= (-1)^{ij+1}m\\ &= \varphi_{ij}^M(m).\end{aligned}$$

□

Here are some examples of binomial modules.

Example 6.2.3. Every binomial ring R is a binomial module over itself with binomial module structure (6.1).

Example 6.2.4. For $m \geq 0$, consider the set of stably integer-valued polynomials

$$\mathrm{SInt}^m(\mathbb{Z}^{\{x\}}) = \{f(x) \in \mathbb{Q}[x] :\ x^m f(x) \in \mathrm{Int}(\mathbb{Z}^{\{x\}})\}.$$

Then for fixed m each set $\mathrm{SInt}^m(\mathbb{Z}^{\{x\}})$ is a binomial module over the binomial ring $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ with binomial module structure (6.1).

Indeed we have a filtration on the binomial ring $\mathrm{SInt}(\mathbb{Z}^{\{x\}})$ by binomial modules $\mathrm{SInt}^m(\mathbb{Z}^{\{x\}})$ for $m \geq 0$ given by

$$\mathrm{Int}(\mathbb{Z}^{\{x\}}) = \mathrm{SInt}^0(\mathbb{Z}^{\{x\}}) \subset \mathrm{SInt}^1(\mathbb{Z}^{\{x\}}) \subset \cdots \subset \mathrm{SInt}^n(\mathbb{Z}^{\{x\}}) \subset \cdots \subset \mathrm{SInt}(\mathbb{Z}^{\{x\}}).$$

Note that each $\mathrm{SInt}^m(\mathbb{Z}^{\{x\}})$ is not a ring for $m \geq 1$ because it is not closed under multiplication. But

$$\mathrm{SInt}(\mathbb{Z}^{\{x\}}) = \bigcup_{m \geq 0} \mathrm{SInt}^m(\mathbb{Z}^{\{x\}}).$$

is a ring (see Definition 4.5.12).

Example 6.2.5. Note that $ev : \text{Int}(\mathbb{Z}^{\{x\}} \longrightarrow \mathbb{Z}$ is ring homomorphism. Then $\mathbb{Z}$ is a binomial module over $\text{Int}(\mathbb{Z}^{\{x\}})$ with the action

$$\text{Int}(\mathbb{Z}^{\{x\}}) \times \mathbb{Z} \longrightarrow \mathbb{Z},$$

given by

$$(f, n) \longmapsto nf(0) \tag{6.2}$$

for $f \in \text{Int}(\mathbb{Z}^{\{x\}})$ and $n \in \mathbb{Z}$, and with the binomial module structure (6.1).

To give another binomial module structure, recall from Proposition 2.7.5 that Binomial ring is a spacial; type of λ-ring. We start with Robinson [48] observation for λ-ring that is in general a λ-ring R is not a λ-module over itself unless the multiplication in R is trivial.

First we present the following result. We mean by square zero binomial ideal I that the multiplication in I is trivial.

Lemma 6.2.6. *Let R be a binomial ring and suppose that I is a square zero binomial ideal in R, then I is a rational vector space.*

Proof. Let $a \in I$, then by the definition of binomial ideal we have $\binom{a}{n} \in I$ for all $n \geq 1$. Then by our hypotheses we get,

$$\begin{aligned} \binom{a}{n} &= \frac{a(a-1)\cdots(a-(n-1))}{n!} \\ &= (-1)^{n-1}\frac{a}{n} \in I. \end{aligned}$$

So, for each $n \geq 1$ there exists $b_n \in I$, for which $b_n = \frac{(-1)^{n-1}a}{n}$. So I is divisible. Since I is also $\mathbb{Z}$-torsion free, by [49, Lemma 19.2.1], I is a rational vector space. □

Proposition 6.2.7. *Let I be a square zero binomial ideal in a binomial ring R. Then I is a binomial module over R with binomial module structure $\varphi_n^I : I \to I$ given by,*

$$\varphi_n^I(a) = (-1)^{n+1}\frac{a}{n}, \tag{6.3}$$

for $a \in I$ and $n \geq 1$.

Proof. First it is clear I is an R-module. By Lemma 6.2.6, I is a rational vector space, so φ_n is well-defined. Then we have the following.

1. $\varphi_1^I(a) = a$.

2.

$$\begin{aligned}(-1)^{(i+1)(j+1)}\varphi_i^I(\varphi_j^I(a)) &= (-1)^{ij+i+j+1}\varphi_i^I((-1)^{j+1}\frac{a}{j}) \\ &= (-1)^{ij+i}\varphi_i^I(\frac{a}{j}) \\ &= (-1)^{ij+i}(-1)^{i+1}\frac{a}{ij} \\ &= (-1)^{ij+1}\frac{a}{ij} \\ &= \varphi_{ij}^I(a).\end{aligned}$$

□

Example 6.2.8. The ring of rationals $\mathbb{Q}$ is a binomial module over $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ with the action given in Example 6.2.5 with binomial module structure (6.3).

Note that in Example 6.2.8, $\mathbb{Q}$ is also a binomial module over $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ with binomial module structure (6.1). So we can have different binomial module structures on the same underlying R-module.

Proposition 6.2.9. *Let R be a binomial ring. Let $r_1 = 1$ in R and for each prime p, pick an element r_p in R. If $n = p_1.p_2.\dots.p_m$, a product of primes, let $r_n = r_{p_1}r_{p_2}\dots.r_{p_m}$. Define*

$$\varphi_n^R : R \longrightarrow R,$$

by

$$\varphi_n^R(r) = (-1)^{n+1}r_n.r.$$

Then R is a binomial module over itself with module structure given by φ_n^R for $n \geq 1$.

Proof. 1. $\varphi_1^R(r) = r_1.r = r.$

2.

$$\begin{aligned}(-1)^{(i+1)(j+1)}\varphi_i^R(\varphi_j^R(r)) &= (-1)^{ij+i+j+1}(-1)^{i+1}r_i\varphi_j^R(r) \\ &= (-1)^{ij+i}(-1)^{i+1}r_i.r_j.r \\ &= (-1)^{ij+1}r_{ij}.r \\ &= \varphi_{ij}^R(r).\end{aligned}$$

□

Example 6.2.10. Similarly for any R-module M over a binomial ring R, M is a binomial module over R with module structure

$$\varphi_n^M : M \to M$$

determined by

$$m \mapsto (-1)^{n+1}r_n.m,$$

for a sequence of elements $r_n \in R$ as in Proposition 6.2.9.

Definition 6.2.11. Let (M, φ_n^M) and (N, φ_n^N) be two binomial modules over a binomial ring R. An R-homomorphism $f: M \to N$ is called a *binomial R-homomorphism* if it preserves the binomial module structure, that is for $n \geq 1$ and $m \in M$ we have

$$\varphi_n^N(f(m)) = f(\varphi_n^M(m)).$$

We denote by $R - \mathfrak{Mod}_{\text{Bin}}$ the category of binomial modules over R whose objects are binomial modules over R and whose morphisms are binomial R-homomorphisms.

§ 6.3 Binomial derivations

In this section we will introduce the notion of a binomial derivation of a binomial ring R with values in a binomial module M over R. For this purpose in the same way as for binomial modules we apply the Robinson notion of λ-derivation of λ-rings with values in a λ-modules as in Definition 5.4.5, to the special case of binomial modules over binomial rings.

Notation 6.3.1. *We will use the binomial symbols $\binom{x}{i}$ and their derivatives frequently in this section. For convenience in formulas now we introduce the notational device $b_i(x)$ for $\binom{x}{i}$ and $d_j(x)$ for $\frac{d}{dx}\binom{x}{j}$, for $i \geq 0$ and $j \geq 1$.*

Definition 6.3.2. Let R be a binomial ring and let M be a binomial module over R, with binomial module structure φ_n, $n \geq 1$. Then a derivation of R with values in M, $d: R \longrightarrow M$, is called a *binomial derivation* if it satisfies the following condition,

$$d(b_n(r)) = \sum_{i=0}^{n-1} \varphi_{n-i}^M(d(r))b_i(r), \tag{6.4}$$

for all $r \in R$ and $n \geq 1$.

We denote by $\text{Der}_{\text{Bin}}(R, M)$ the set of all binomial derivations of the binomial ring R with values in the binomial module M over R.

For a binomial ring R and $r \in R$ by (2.6) we have

$$b_n(r) = \frac{1}{n!}\left(\sum_{i=1}^{n}(-1)^{n-i}\begin{bmatrix} n \\ i \end{bmatrix} r^i\right).$$

Thus by Proposition 5.2.10 (2) for a usual derivation d of R we obtain

$$d(b_n(r)) = \frac{1}{n!}\left(\sum_{i=1}^{n}(-1)^{n-i}\begin{bmatrix} n \\ i \end{bmatrix} i r^{i-1} d(r)\right). \tag{6.5}$$

On the other hand also by applying (2.6) to (6.4), we obtain,

$$d(b_n(r)) = \sum_{i=0}^{n-1} \varphi_{n-i}^M(d(r))\left(\frac{1}{i!}\left(\sum_{k=1}^{i}(-1)^{i-k}\begin{bmatrix} i \\ k \end{bmatrix} r^k\right)\right). \tag{6.6}$$

We are going to explore how Robinson's λ-derivations behave in the special case of binomial rings and binomial modules.

First we give differentiation of polynomials as an example of a derivation. We will explain what happens for derivations of integer-valued polynomial rings. We start by exploring differentiation.

Definition 6.3.3. Let $D := \mathrm{Im}(d : \mathrm{Int}(\mathbb{Z}^{\{x\}}) \to \mathbb{Q}[x])$ where d is differentiation on $\mathrm{Int}(\mathbb{Z}^{\{x\}})$.

Example 6.3.4. This example is given to show that D is not an $\mathrm{Int}(\mathbb{Z}^{\{x\}})$-module. Consider the elements $d_2(x) = x - \frac{1}{2} \in D$ and $x \in \mathrm{Int}(\mathbb{Z}^{\{x\}})$. Suppose

$$xd_2(x) = x\left(x - \frac{1}{2}\right) = \sum_{i=1}^{N} a_i\, d_i(x) \qquad \text{with } a_i \in \mathbb{Z}.$$

Then for degree reasons $a_i = 0$ for $i > 3$ and

$$a_1 d_1(x) + a_2 d_2(x) + a_3 d_3(x) = x(x - \frac{1}{2}).$$

where $d_1(x) = 1$ and $d_3(x) = (\frac{3x^2-6x+2}{6})$. So by the equality of the coefficients we get $a_3 = 2$ and then $a_2 = \frac{3}{2} \notin \mathbb{Z}$. Therefore D is not an $\mathrm{Int}(\mathbb{Z}^{\{x\}})$-module.

Next we are going to find the smallest $\mathrm{Int}(\mathbb{Z}^{\{x\}})$-module M such that

$$D \subset M \subseteq \mathbb{Q}[x].$$

Definition 6.3.5. Let M be the smallest $\mathrm{Int}(\mathbb{Z}^{\{x\}})$-module such that $D \subset M \subseteq \mathbb{Q}[x]$.

Then

$$M = \{\sum_{ij}^{\text{finite}} a_{ij}\, b_i(x)\, d_j(x) : \text{ for } a_{ij} \in \mathbb{Z}\}. \tag{6.7}$$

The product rule for differentiation gives the following.

Lemma 6.3.6. *For $n \geq 1$, we have,*

$$d_n(x) = \frac{1}{n!} \sum_{i=0}^{n-1} \prod_{j=0, j\neq i}^{n} (x - j).$$

□

Next we need to present the following result.

Proposition 6.3.7. *For all non-negative integers n, we have the equality*

$$d_n(x) = \sum_{j=1}^{n} (-1)^{j-1}\, \frac{1}{j}\, b_{n-j}(x). \tag{6.8}$$

Proof. We use induction on n. The statement is true for $n = 1$, since $d_1(x) = 1 = b_0(x)$. We assume it is true for n. Then

$$d_n(x) = \sum_{j=1}^{n} (-1)^{j-1} \frac{1}{j} \, b_{n-j}(x).$$

To prove it for $n + 1$, we start with

$$b_{n+1}(x) = \frac{b_n(x).(x-n)}{n+1}.$$

Then by taking the derivative of both sides we get,

$$d_{n+1}(x) = \frac{1}{n+1}(d_n(x)(x-n) + b_n(x)).$$

So

$$\begin{aligned}
d_{n+1}(x) &= \frac{1}{n+1}\left(\left(\sum_{j=1}^{n}(-1)^{j-1}\frac{1}{j}b_{n-j}(x)\right)(x-n) + b_n(x)\right) \\
&= \frac{1}{n+1}\left(\left(\sum_{j=1}^{n}(-1)^{j-1}\frac{1}{j}b_{n-j}(x)(x-n-j+j)\right) + b_n(x)\right) \\
&= \frac{1}{n+1}\left(\left(\sum_{j=1}^{n}(-1)^{j-1}\frac{1}{j}(n-j+1)b_{n+1-j}(x) + (-1)^j b_{n-j}(x)\right) + b_n(x)\right) \\
&= \sum_{j=1}^{n}(-1)^{j-1}\frac{1}{j}b_{n+1-j}(x) + \frac{1}{n+1}\sum_{j=1}^{n}(-1)^j b_{n+1-j}(x) \\
&\qquad + \frac{1}{n+1}\sum_{k=2}^{n+1}(-1)^{k-1}b_{n+1-k}(x) + \frac{b_n(x)}{n+1} \\
&= \sum_{j=1}^{n}(-1)^{j-1}\frac{1}{j}b_{n+1-j}(x) - \frac{1}{n+1}b_n(x) + \frac{1}{n+1}(-1)^n b_0(x) + \frac{b_n(x)}{n+1} \\
&= \sum_{j=1}^{n+1}(-1)^{j-1}\frac{1}{j}b_{n+1-j}(x).
\end{aligned}$$

□

In the same way the following proposition provides an equality regarding derivative of rising power of x see equation (2.3) in §2.4.

Proposition 6.3.8. *For all $n \geq 1$, we have equality of rational polynomials of degree $n - 1$,*

$$\sum_{i=0}^{n-1} \frac{x^{\overline{i}}}{(n-i)i!} = \frac{d\left(\frac{x^{\overline{n}}}{n!}\right)}{dx}. \tag{6.9}$$

Proof. This can be proved by induction on n, in the same way as Proposition 6.3.7, using

$$\frac{x^{\overline{n}}}{n!} = \frac{x^{\overline{n-1}}(x+n-1)}{n(n-1)!}.$$

□

Corollary 6.3.9. *For all $n \geq 1$ and $0 \leq j \leq n-1$, we have*

$$\sum_{i=j}^{n-1} \frac{\left[{i \atop j}\right]}{(n-i)i!} = \frac{(j+1)}{n!}\left[{n \atop j+1}\right]. \tag{6.10}$$

Proof. For fixed j, the statement follows by taking the coefficient of x^j in (6.9), using Proposition 2.3.19. □

Recall from Definition 6.3.5 that we mean by M a smallest $\mathrm{Int}(\mathbb{Z}^{\{x\}})$-module such that $D \subset M \subseteq \mathbb{Q}[x]$.

Theorem 6.3.10. *For all non-negative integers n, we have $\frac{1}{n} \in M$.*

Proof. We use induction on $n \geq 1$. Certainly $1 = d_1 = b_0 \in M$. For the induction assumption we suppose $\frac{1}{j} \in M$ for $1 \leq j \leq n-1$. Then by Definition 6.3.5,

$$\frac{1}{j} = \sum_{l,k}^{j} a_{lk}^{j}\, b_l(x)\, d_k(x) \ \text{ for some } a_{lk}^{j} \in \mathbb{Z}.$$

We need to show that $\frac{1}{n} \in M$. From Proposition 6.3.7 we have the equality

$$d_n(x) = \sum_{j=1}^{n} (-1)^{j-1}\, \frac{1}{j}\, b_{n-j}(x)$$

for all $n \geq 1$. Then by rearranging we get

$$\begin{aligned} \frac{(-1)^n}{n} &= d_n(x) + \sum_{j=1}^{n-1} (-1)^j\, \frac{1}{j}\, b_{n-j}(x) \\ &= d_n(x) + \sum_{j=1}^{n-1} (-1)^j \left(\sum_{lk} a_{lk}^{j} b_l d_k \right) b_{n-j}(x). \end{aligned}$$

Since by Theorem 2.4.6, the product of $b_k(x)$ and $b_l(x)$ is a $\mathbb{Z}$-linear combination of the $b_l(x)$, this implies that $\frac{1}{n} \in M$. □

To illustrate Theorem 6.3.10 we present the special case for $n = 3$.

Example 6.3.11. For $n = 3$ by Proposition 6.3.7 we get

$$d_3(x) = b_2(x) - \frac{1}{2}b_1(x) + \frac{1}{3}b_0(x).$$

This implies that

$$\frac{1}{3} = d_3(x) - b_2(x) + \frac{1}{2}b_1(x).$$

Again by Proposition 6.3.7 we have

$$\frac{1}{2} = -d_2(x) + b_1(x).$$

By Theorem 2.4.6, we can write $b_1(x)b_1(x)$ as a linear combination of basis elements

$$b_1(x)b_1(x) = 2b_2(x) + b_1(x).$$

Then

$$\frac{1}{3} = d_3(x)b_0(x) + d_1(x)b_2(x) - d_2(x)b_1(x) - d_0(x)b_1(x).$$

Therefore $\frac{1}{3} \in M$.

Corollary 6.3.12. *We have* $M = \mathbb{Q}[x]$.

Proof. Let $f(x) \in \mathbb{Q}[x]$. We write $f(x) = \frac{g(x)}{n}$, where $g(x) \in \mathbb{Z}[x]$ and $n \geq 1$. Then $g(x) \in \text{Int}(\mathbb{Z}^{\{x\}})$. Since M is an $\text{Int}(\mathbb{Z}^{\{x\}})$-module and by Theorem 6.3.10, $\frac{1}{n} \in M$, this implies that $\frac{g(x)}{n} \in M$. □

Proposition 6.3.13. *There is an isomorphism of Int*$(\mathbb{Z}^{\{x\}})$*-modules*

$$Der(Int(\mathbb{Z}^{\{x\}}), \mathbb{Q}) \cong \mathbb{Q}.$$

Proof. First we define the map

$$\varphi : \text{Der}(\text{Int}(\mathbb{Z}^{\{x\}}), \mathbb{Q}) \to \mathbb{Q}$$

given by

$$d \longmapsto d(x),$$

for $d \in \text{Der}(\text{Int}(\mathbb{Z}^{\{x\}}), \mathbb{Q})$.

It is clear φ is an $\text{Int}(\mathbb{Z}^{\{x\}})$-module map. Next, if $\varphi(d) = 0$, then $d(x) = 0$, therefore $d = 0$. This shows that φ is injective.

Finally to show that φ is surjective, let $a \in \mathbb{Q}$. We want to show that $a = d(x) = \varphi(d)$ for some $d \in \text{Der}(\text{Int}(\mathbb{Z}^{\{x\}}), \mathbb{Q})$. This is true by letting d be the derivation determined by $d(x) = a$. □

It is well known that the binomial symbols are integer-valued but their derivatives are not.

Example 6.3.14. Let $f(x) = d_3 x$, then $\dfrac{df(x)}{dx} = \dfrac{3x^2 - 6x + 2}{6}$. So $\dfrac{df(1)}{dx} \notin \mathbb{Z}$.

We know from Example 6.2.5 that $\mathbb{Z}$ is an $\mathrm{Int}(\mathbb{Z}^{\{x\}})$-module.

Proposition 6.3.15. *We have*

$$Der(Int(\mathbb{Z}^{\{x\}}), \mathbb{Z}) \cong 0.$$

Proof. Let $f \in \mathrm{Der}(\mathrm{Int}(\mathbb{Z}^{\{x\}}), \mathbb{Z})$ and consider $b_n(x) \in \mathrm{Int}(\mathbb{Z}^{\{x\}})$, for $n \geq 0$. Then since

$$n! b_n(x) = x(x-1)\dots(x-(n-1)),$$

we have

$$\begin{aligned} n! f(b_n(x)) &= \sum_{i=1}^{n} (-1)^{n-i} \begin{bmatrix} n \\ i \end{bmatrix} i x^{i-1} f(x) \text{ by (6.5)} \\ &= (-1)^{n-1}(n-1)! f(x) \text{ by (6.2).} \end{aligned}$$

So $f(x) = (-1)^{n-1} n f(b_n(x)) \in n\mathbb{Z}$. Since this holds for all n, this implies that $f(x) = 0$. So $f = 0$. □

Here is the main result of this section.

Theorem 6.3.16. *Let R be a binomial ring and let M be a binomial module over R with module structure given by $\varphi_n^M = \frac{(-1)^{n-1}}{n} Id_M$. Then*

$$Der(R, M) = Der_{Bin}(R, M).$$

Proof. To prove the equality we need to show that the condition (6.4) of compatibility with the binomial module structure follows from the usual derivation condition (5.8). In other words, we need to show for each $n \geq 1$

$$\sum_{i=0}^{n-1} (-1)^{(n-i)-1} \frac{d(r)}{(n-i)i!} \left(\sum_{k=1}^{i} (-1)^{i-k} \begin{bmatrix} i \\ k \end{bmatrix} r^k \right) = \frac{1}{n!} \left(\sum_{i=1}^{n} (-1)^{n-i} \begin{bmatrix} n \\ i \end{bmatrix} i r^{i-1} d(r) \right). \tag{6.11}$$

Since both sides of (6.11) are polynomials in r, we compare coefficients. For fixed j the coefficient of r^j on the right hand side is

$$(-1)^{n-j+1} \frac{d(r)}{n!} (j+1) \begin{bmatrix} n \\ j+1 \end{bmatrix},$$

and on the left hand side it is

$$\sum_{i=j}^{n-1} (-1)^{n-j+1} \frac{d(r)}{(n-i)i!} \begin{bmatrix} i \\ j \end{bmatrix}.$$

By Corollary 6.3.9 both sides are equal. □

Proposition 6.3.17. *Let R be a binomial ring and let M be a binomial module over R with module structure given by $\varphi_n^M = (-1)^{n-1} Id_M$. Then we have*

$$Der_{Bin}(R, M) = 0.$$

Proof. For $\varphi_n^M = (-1)^{n-1} Id_M$ in (6.11), we have

$$\sum_{i=0}^{n-1} (-1)^{(n-i)-1} \frac{d(r)}{i!} \left(\sum_{k=1}^{i} (-1)^{i-k} \begin{bmatrix} i \\ k \end{bmatrix} r^k \right) = \frac{1}{n!} \left(\sum_{i=1}^{n} (-1)^{n-i} \begin{bmatrix} n \\ i \end{bmatrix} i r^{i-1} d(r) \right). \quad (6.12)$$

Then the coefficient of r on the right hand side of (6.12) is

$$(-1)^n \frac{d(r)}{n!} (2) \begin{bmatrix} n \\ 2 \end{bmatrix},$$

and on the left hand side it is

$$\sum_{i=1}^{n-1} (-1)^n \frac{d(r)}{(n-i)i!} \begin{bmatrix} i \\ 1 \end{bmatrix}.$$

By Corollary 6.3.9 both sides are equal, for all $n \geq 1$. Then for $n = 3$, we have

$$\begin{aligned} \tfrac{-d(r)}{2} - \tfrac{d(r)2}{2!} &= \tfrac{-d(r)(2)(3)}{3!} \\ \tfrac{-3d(r)}{2} &= -d(r). \end{aligned}$$

Then we get $d(r) = 0$. □

Proposition 6.3.18. *Let M be a binomial module over $Int(\mathbb{Z}^{\{x\}})$ with module structure given by $\varphi_n^M = \frac{(-1)^{n-1}}{n} Id_M$. Then we have an isomorphism of $Int(\mathbb{Z}^{\{x\}})$-modules*

$$Der_{Bin}(Int(\mathbb{Z}^{\{x\}}), M) \cong M.$$

Proof. For any binomial derivation $d \in \mathrm{Der}_{\mathrm{Bin}}(\mathrm{Int}(\mathbb{Z}^{\{x\}}), M)$, suppose that $d(x) = m$, for $m \in M$. We define the map

$$\theta : \mathrm{Der}_{\mathrm{Bin}}(\mathrm{Int}(\mathbb{Z}^{\{x\}}), M) \longrightarrow M,$$

given by

$$d \longmapsto d(x) = m.$$

First to show that θ is an injection, we know from Theorem 2.4.6 that a general element in $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ is of the form $\sum_{i=1}^{m} a_i b_i(x)$ for $a_i \in \mathbb{Z}$. Then

$$\begin{aligned} d(\sum_{i=1}^{m} a_i b_i(x)) \sum_{i=1}^{m} a_i d(b_i(x)) & \\ &= \sum_{i=1}^{m} a_i \left(\frac{1}{i!} \left(\sum_{j=1}^{i} (-1)^{i-j} \begin{bmatrix} i \\ j \end{bmatrix} j x^{j-1} d(x) \right) \text{ by (6.5)}, \right. \\ &= \sum_{i=1}^{m} a_i \left(\frac{1}{i!} \left(\sum_{j=1}^{i} (-1)^{i-j} \begin{bmatrix} i \\ j \end{bmatrix} j x^{j-1} m \right) \in M. \right. \end{aligned}$$

Thus θ is totally determined by m. This implies that θ is injective. Since setting $d(x) = m$ determines a derivation, it is clear that θ is surjective. □

By Theorem 6.3.16 for a binomial module M over R with module structure $\varphi_n^M = \frac{(-1)^{n-1}}{n} Id_M$ as in Proposition 6.3.18, $\mathrm{Der}(R, M) = \mathrm{Der}_{\mathrm{Bin}}(R, M)$. Note that an ordinary derivation d is determined by $d(x)$ and the divisibility property of such an M means that any choice of $d(x) \in M$ extends to a derivation.

Proposition 6.3.19. *Let M be a binomial module over $Int(\mathbb{Z}^{\{x,y\}})$ with module structure given by $\varphi_n^M = \frac{(-1)^{n-1}}{n} Id_M$. Then we have an isomorphism of $Int(\mathbb{Z}^{\{x,y\}})$-modules*

$$Der_{Bin}(Int(\mathbb{Z}^{\{x,y\}}), M) \cong M \oplus M.$$

Proof. For any binomial derivation $d \in \mathrm{Der}_{\mathrm{Bin}}(\mathrm{Int}(\mathbb{Z}^{\{x,y\}}), M)$, suppose that $d(x, y) = (m_1, m_2)$ for $m_1, m_2 \in M$. Then the rest of the proof is analogs to the proof of Proposition 6.3.18 □

Recall from Theorem 4.6.9 that

$$\frac{\mathrm{Int}(\mathbb{Z}^{\{x,y\}})}{((xy-1))} \cong \mathrm{Int}(\mathbb{Z}^{\{x\}})[x^{-1}].$$

Next similarly as in Proposition 6.3.19 we compute the $\mathrm{Der}_{\mathrm{Bin}}$ on non free binomial ring.

Example 6.3.20. Let $\mathbb{Q}$ be a binomial module over $\mathrm{SLInt}(\mathbb{Z}^{\{x\}})$ with module structure given by $\varphi_n^{\mathbb{Q}} = \frac{(-1)^{n-1}}{n} Id_{\mathbb{Q}}$. Then we have an isomorphism of $\mathrm{SLInt}(\mathbb{Z}^{\{x\}})$-modules

$$\mathrm{Der}_{\mathrm{Bin}}(\mathrm{SLInt}(\mathbb{Z}^{\{x\}}), \mathbb{Q}) \cong \mathbb{Q} \oplus \mathbb{Q},$$

with derivation given by

$$d(x, y) = (q_1, q_2)$$

for $q_1, q_2 \in \mathbb{Q}$ and $y = x^{-1}$.

We know from section 5.3 that the set $\mathrm{Der}(R, M)$ for an R-module M is an R-module. Similarly we are going to show that the set $\mathrm{Der}_{\mathrm{Bin}}(R, M)$ is a binomial module over R.

Proposition 6.3.21. *Let R be a binomial ring and let M be a binomial module over R with structure maps φ_n^M. Then the set $D = Der_{Bin}(R, M)$ is also a binomial module over R with module structure $\varphi_n^D : D \longrightarrow D$, defined by setting*

$$\varphi_n^D(d)(r) = \varphi_n^M(d(r)),$$

for $d \in D$, $r \in R$ and $n \geq 1$.

Proof. First we are going to show that $\varphi_n^D(d) : R \longrightarrow M$ is a binomial derivation for $d \in D$. To see that for $r, s \in R$ we have, 1.

$$\begin{aligned}(\varphi_n^D(d))(rs) &= \varphi_n^M(d(rs))\\ &= \varphi_n^M(rd(s)+sd(r))\\ &= r.\varphi_n^M(d(s))+s.\varphi_n^M(d(r))\\ &= r.\varphi_n^D(d)(s)+s.\varphi_n^D(d)(r)\\ &= r.(\varphi_n^D(d))(s)+s.(\varphi_n^D(d))(r).\end{aligned}$$

2.

$$\begin{aligned}(\varphi_n^D(d(b_m(r))) &= \varphi_n^M(d(b_m(r)))\\ &= \varphi_n^M(\sum_{i=0}^{m-1}\varphi_{m-i}^M(d(r))b_i(r))\\ &= \sum_{i=0}^{m-1}\varphi_n^M(\varphi_{m-i}^M(d(r))b_i(r))\\ &= \sum_{i=0}^{m-1}\varphi_{m-i}^M(\varphi_n^M(d(r))b_i(r))\\ &= \sum_{i=0}^{m-1}\varphi_{m-i}^M(\varphi_n^D(d(r)))b_i(r)).\end{aligned}$$

Next we need to show that D is a binomial module over R.

1. $(\varphi_1^D(d))(r)=\varphi_1^M(d(r))=d(r)$.

2.

$$\begin{aligned}(\varphi_{ij}^D(d))(r) &= \varphi_{ij}^M(d(r))\\ &= (-1)^{(i+1)(j+1)}\varphi_i^M(\varphi_j^M(d(r)))\\ &= (-1)^{(i+1)(j+1)}\varphi_i^M((\varphi_j^D d)(r))\\ &= (-1)^{(i+1)(j+1)}(\varphi_i^D\varphi_j^D(d))(r).\end{aligned}$$

for all $i,j\geq 1$. □

§ 6.4 Cohomology of binomial rings

The main aim of this section is to introduce the cohomology of binomial rings as another example of cotriple cohomology theory, on $\mathfrak{BinRing}$ with coefficients in the contravariant functor $\mathrm{Der}_{\mathrm{Bin}}(-,M)$ for a binomial module M.

First, for binomial rings R and S, let M be a binomial module over R and let $f : S \longrightarrow R$ be a ring homomorphism. Then it is clear M is also a binomial module over S. We define the category of binomial rings over R, whose objects are ring homomorphisms $f : S \longrightarrow R$ from a binomial ring S and whose morphisms are given by the following commutative diagrams

$$\begin{array}{ccc} S_1 & \xrightarrow{f_1} & R \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle Id_R} \\ S_2 & \xrightarrow[f_2]{} & R \end{array}$$

for a ring homomorphism g. We denote this category by $\mathfrak{BinRing}/R$.

Then for a binomial module M over R, we define the contravariant functor

$$\mathrm{Der}_{\mathrm{Bin}}(-, M) : \mathfrak{BinRing}/R \longrightarrow \mathfrak{Ab}.$$

Recall from Subsection 5.2.2 the simplicial object $C^n(R)$ that came from an adjoint pair of functors. The binomial module M over binomial ring R becomes a binomial module over $(C^n(R))_{\mathrm{Bin}}$ for all $n \geq 1$ by the canonical map from $(C^n(R))_{\mathrm{Bin}}$ to R.

Definition 6.4.1. Let R be a binomial ring and let M be a binomial module over R. By applying the functor $\mathrm{Der}_{\mathrm{Bin}}(-, M)$ to the cotriple resolution $(C_*)_{\mathrm{Bin}}(R)$ of an object R in $\mathfrak{BinRing}$, we define the *cohomology of the binomial ring* R with coefficients in M by

$$H^n_{\mathrm{Bin}}(R, M) = H^n_{\mathbb{C}_{\mathrm{Bin}}}(R, M) := H^n_{\mathbb{C}_{\mathrm{Bin}}}(R, \mathrm{Der}_{\mathrm{Bin}}(-, M)). \tag{6.13}$$

In other words, $H^n_{\mathrm{Bin}}(R, M)$ is the n^{th} cotriple cohomology group of R with coefficients in $\mathrm{Der}_{\mathrm{Bin}}(-, M)$ with respect to the cotriple $\mathbb{C}_{\mathrm{Bin}}$ on $\mathfrak{BinRing}$.

Proposition 6.4.2. *Let R be a binomial ring and let M be a binomial module over R. Then*

$$H^0_{Bin}(R, M) \cong Der_{Bin}(R, M).$$

Proof. Applying $\mathrm{Der}_{\mathrm{Bin}}(-, M)$ to the bottom of the augmented simplicial object (5.4), we obtain the following diagram

$$\mathrm{Der}_{\mathrm{Bin}}(R, M) \xrightarrow{\varepsilon^*} \mathrm{Der}_{\mathrm{Bin}}(CR, M) \underset{(\varepsilon_1^1)^*}{\overset{(\varepsilon_1^0)^*}{\rightrightarrows}} \mathrm{Der}_{\mathrm{Bin}}(C^2R, M).$$

It is clear that ε^* is injective and since $\varepsilon\varepsilon_1^0 = \varepsilon\varepsilon_1^1$,

$$\mathrm{Im}\varepsilon^* \subseteq \mathrm{Ker}((\varepsilon_1^1)^* - (\varepsilon_1^0)^*).$$

Now let $f \in \mathrm{Ker}((\varepsilon_1^1)^* - (\varepsilon_1^0)^*)$. Then, we writing $\star$ for the product in CR, it follows that $f(a \star b) = f(ab)$. Thus we can define $g : R \longrightarrow M$ by $g(a) = f(a)$. Then g is a derivative because f is and g is compatible with the module structure of M as f is. So $f = g\varepsilon = \varepsilon^*(g) \in \mathrm{Im}\varepsilon^*$. Thus

$$\mathrm{Der}_{\mathrm{Bin}}(R, M) \cong \mathrm{Im}\varepsilon^* = \mathrm{Ker}((\varepsilon_1^1)^* - (\varepsilon_1^0)^*) = \mathrm{H}^0_{\mathrm{Bin}}(R, M).$$

□

Proposition 6.4.3. *Let R be a free binomial ring and let M be a binomial module over R. Then*

$$H^n_{Bin}(R, M) = 0,$$

for all $n \geq 1$.

Proof. The Proposition follows from Proposition 5.2.8. □

Proposition 6.4.4. *Let R be a binomial ring and let M be a binomial module over R with module structure given by $\varphi_n^M = (-1)^{n-1} Id_M$. Then we have*

$$H^n_{Bin}(R, M) = 0,$$

for all $n \geq 1$.

Proof. The proof is clear by Proposition 6.3.17. □

We end this thesis by defining homomorphisms between binomial cohomology, λ-cohomology and André-Quillen cohomology of the underlying commutative ring.

We know from Proposition 2.7.5 that a binomial ring has a unique λ-ring structure given by binomial operations on R, $\lambda^n(r) = \binom{r}{n}$ and whose Adams operations all are the identity on R. We are going to show that there exist homomorphisms from cohomology of λ-rings to cohomology of binomial rings for all n. In order to define these homomorphisms, first we present the following results.

Proposition 6.4.5. *In the free λ-ring, Λ_y, let*

$$S_{\Lambda_y} = \{\psi^n(h) - h \;:\; h \in \Lambda_y \; and \, n \geq 1\}.$$

Then we have an isomorphism of binomial rings,

$$\frac{\Lambda_y}{I} \cong Int(\mathbb{Z}^{\{x\}}),$$

where $I = (S_{\Lambda_y})$.

Proof. First we know from Proposition 2.8.10 that the ideal I is a λ-ideal of Λ_y. So by Proposition 2.8.9, $\frac{\Lambda_y}{I}$ is a Λ-ring. Since Λ_y is a free λ-ring and by Proposition 2.7.5, $\mathrm{Int}(\mathbb{Z}^{\{x\}})$ is a λ-ring, by the universal property we can define a λ-homomorphism

$$\theta : \Lambda_y \longrightarrow \mathrm{Int}(\mathbb{Z}^{\{x\}}),$$

determined by

$$y \longmapsto x.$$

Also by Proposition 2.7.5, we have

$$\theta(\lambda^n(y)) = \lambda^n(x) = \binom{x}{n}.$$

Since $\binom{x}{n} \in \text{Im}(\theta)$ for all n and by Theorem 2.4.6 these span $\text{Int}(\mathbb{Z}^{\{x\}})$, θ is surjective. Now pick an element $p \in I$, then

$$p = \sum_{i=1}^{r} (\psi^{n_i}(h_i) - h_i) g_i,$$

for some $h_i, g_i \in \Lambda_y$ and some $n_i \geq 1$. Then

$$\begin{aligned} \theta(p) &= \sum_{i=1}^{r} \theta((\psi^{n_i}(h_i) - h_i) g_i) \\ &= \sum_{i=1}^{r} \left(\psi^{n_i}(\theta(h_i)) - \theta(h_i) \right) \theta(g_i) \qquad \text{by Proposition 2.6.5} \\ &= 0 \qquad \text{by Proposition 2.7.5.} \end{aligned}$$

Therefore $I \subseteq \text{Ker}\,\theta$ and θ induces a ring homomorphism

$$\bar{\theta} : \frac{\Lambda_y}{I} \longrightarrow \text{Int}(\mathbb{Z}^{\{x\}}).$$

On the other hand we define the $\mathbb{Z}$-linear map

$$\alpha : \text{Int}(\mathbb{Z}^{\{x\}}) \longrightarrow \frac{\Lambda_y}{I}$$

determined on basis elements by

$$\begin{aligned} \alpha(\binom{x}{n}) &= \lambda^n(y + I) \\ &= \lambda^n(y) + I \\ &= \binom{y}{n} + I \qquad \text{by Theorem 2.7.5.} \end{aligned}$$

Then α is a ring homomorphism.

Then

$$\bar{\theta}\alpha(\binom{x}{n}) = \bar{\theta}(\lambda^n(y) + I) = \theta(\lambda^n(y)) = \binom{x}{n}$$

and

$$\alpha\bar{\theta}(\lambda^n(y) + I) = \alpha(\binom{x}{n}) = \lambda^n(y) + I.$$

This implies that α is an inverse to $\bar{\theta}$. □

Recall from §2.8 that the functor Q_λ from the category of λ-rings to the category of binomial rings is left adjoint to the inclusion functor I_{Bin} from the category of binomial rings to the category of λ-rings (Theorem 2.8.12), from §5.4 that $\mathbb{C}_\lambda = F_\lambda G_\lambda$ is a cotriple on $\mathfrak{Ring}_\lambda$ where F_λ is the free functor taking a set X to the free λ-ring generated by this set and G_λ is the forgetful functor and from §6.1 that $\mathbb{C}_\lambda = F_{Bin}G_{Bin}$ is a cotriple on $\mathfrak{BinRing}$ where F_{Bin} is the free functor taking a set X to the free Binomial ring generated by this set and G_{Bin} is the forgetful functor.

Remark 6.4.6. From Proposition 6.4.5 we have isomorphism of binomial ring

$$\mathrm{Q}_\lambda F_\lambda(\{y\}) \cong F_{Bin}(\{y\}).$$

Similarly, we have

$$\mathrm{Q}_\lambda F_\lambda(X) \cong F_{Bin}(X),$$

for any set X.

Proposition 6.4.7. *Let R be a binomial ring. For $n \geq 0$ there exists a binomial ring homomorphism,*

$$\eta_n^R : \mathrm{Q}_\lambda \mathbb{C}_\lambda^n I_{Bin} R \longrightarrow \mathbb{C}_{Bin}^n R.$$

Proof. We will prove it by induction on n. For $n = 1$. Since $G_\lambda I_{\mathrm{Bin}} = G_{Bin}$, by Remark 6.4.6 and Proposition 6.4.5, we obtain

$$\mathrm{Q}_\lambda \mathbb{C}_\lambda I_{\mathrm{Bin}} R \cong \mathbb{C}_{\mathrm{Bin}} R$$

which is clear that is natural in R.

For $n = 2$, for a λ-ring A and a binomial ring $\mathrm{Q}_\lambda A$ in Theorem 2.8.12, we have

$$\mathrm{Hom}_{\mathrm{Bin}}(\mathrm{Q}_\lambda A, \mathrm{Q}_\lambda A) \cong \mathrm{Hom}_\lambda(A, I_{\mathrm{Bin}} \mathrm{Q}_\lambda A).$$

So from the identity map on $\mathrm{Q}_\lambda A$ we obtain a λ-homomorphism

$$\zeta : A \longrightarrow I_{\mathrm{Bin}} \mathrm{Q}_\lambda A, \tag{6.14}$$

which is natural in A. Then we obtain the homomorphism

$$\mathrm{Q}_\lambda \mathbb{C}_\lambda^2 I_{\mathrm{Bin}} R = \mathrm{Q}_\lambda \mathbb{C}_\lambda \mathrm{Id} \mathbb{C}_\lambda I_{\mathrm{Bin}} R \xrightarrow{\zeta} \mathrm{Q}_\lambda \mathbb{C}_\lambda I_{\mathrm{Bin}} \mathrm{Q}_\lambda \mathbb{C}_\lambda I_{\mathrm{Bin}} R \cong \mathbb{C}_{\mathrm{Bin}}^2 R. \tag{6.15}$$

Assume that for $n = k$ we have a binomial ring homomorphism

$$\mathrm{Q}_\lambda \mathbb{C}_\lambda^k I_{\mathrm{Bin}} R \xrightarrow{\eta_k^R} \mathbb{C}_{\mathrm{Bin}}^k R. \tag{6.16}$$

Then for $n = k + 1$, by (6.16), we define η_{k+1} by the composite

$$\mathrm{Q}_\lambda \mathbb{C}_\lambda^{k+1} I_{\mathrm{Bin}} R = \mathrm{Q}_\lambda \mathbb{C}_\lambda^k Id_R \mathbb{C}_\lambda I_\lambda R \xrightarrow{\eta_{k+1}^R} \mathrm{Q}_\lambda \mathbb{C}_\lambda^k I_{\mathrm{Bin}} R \mathrm{Q}_\lambda \mathbb{C}_\lambda I_{\mathrm{Bin}} R \cong \mathbb{C}_{\mathrm{Bin}}^{k+1} R. \tag{6.17}$$

□

Theorem 6.4.8. *Let R be a binomial ring and let M be a binomial module over R. Then there exists an R-module homomorphism, for each $n \geq 0$*

$$\varrho_n : H^n_{Bin}(R, M) \longrightarrow H^n_\lambda(I_{Bin}R, I_{Bin}M).$$

Proof. Consider the cotriple resolution $(\mathbb{C}_*)_{\text{Bin}}(R)$ of R. We know $I_{\text{Bin}}R$ is a λ-ring. Then consider the cotriple resolution $(\mathbb{C}_*)_\lambda$ of $I_{\text{Bin}}R$. Then by Proposition 6.4.7, we have the ring homomorphism

$$\mathrm{Q}_\lambda \mathbb{C}^*_\lambda I_{\text{Bin}}R \xrightarrow{\eta^R} \mathbb{C}^*_{\text{Bin}}R.$$

By applying the functor $\text{Der}_{\text{Bin}}(-, M)$, we obtain an R-module homomorphism

$$\text{Der}_{\text{Bin}}(\mathbb{C}^*_{\text{Bin}}R, M) \xrightarrow{(\eta^R)^*} \text{Der}_{\text{Bin}}(\mathrm{Q}_\lambda \mathbb{C}^*_\lambda I_{\text{Bin}}R, M). \tag{6.18}$$

And by Definition 6.3.2 and Definition 6.2.1, we have an isomorphism of R-modules

$$\text{Der}_{\text{Bin}}(\mathrm{Q}_\lambda \mathbb{C}^*_\lambda I_{\text{Bin}}R, M) \cong \text{Der}_\lambda(I_{\text{Bin}}\mathrm{Q}_\lambda \mathbb{C}^*_\lambda I_{\text{Bin}}R, I_{\text{Bin}}M). \tag{6.19}$$

On the other hand, also by applying $I_{\text{Bin}}\mathrm{Q}_\lambda$ we obtain the λ-homomorphism,

$$\mathbb{C}^*_\lambda I_{\text{Bin}}R \xrightarrow{\gamma} I_{\text{Bin}}\mathrm{Q}_\lambda \mathbb{C}^*_\lambda I_{\text{Bin}}R. \tag{6.20}$$

Now by applying the functor $\text{Der}_\lambda(-, I_{\text{Bin}}M)$, we get an R-module homomorphism

$$\text{Der}_\lambda(I_{\text{Bin}}\mathrm{Q}_\lambda \mathbb{C}^*_\lambda I_{\text{Bin}}R, I_{\text{Bin}}M) \xrightarrow{\gamma^*} \text{Der}_\lambda(\mathbb{C}^*_\lambda I_{\text{Bin}}R, I_{\text{Bin}}M). \tag{6.21}$$

Finally this gives us

$$\varrho_n : H^n_{\text{Bin}}(R, M) \xrightarrow{(\gamma^*(\eta^R)^*)^*} H^n_\lambda(I_{\text{Bin}}R, I_{\text{Bin}}M). \tag{6.22}$$

□

Example 6.4.9. This example is given to show that the above homomorphism is non-trivial. We consider binomial ring $\text{Int}(\mathbb{Z}^{\{x\}})$ and let M be a binomial module over $\text{Int}(\mathbb{Z}^{\{x\}})$ with module structure given by $\varphi^M_n = \frac{(-1)^{n-1}\text{Id}_M}{n}$.

Then for zero degree $n = 0$ we have $H^0_{\text{Bin}}(\text{Int}(\mathbb{Z}^{\{x\}}), M) \cong \text{Der}_{\text{Bin}}(\mathbb{Z}^{\{x\}}), M)$ by Proposition 6.4.2 and $\text{Der}_{\text{Bin}}(\mathbb{Z}^{\{x\}}), M) \cong M$ by Proposition 6.3.18, with derivation d given by $d \longmapsto d(x) = m$, for $m \in M$.

Similarly $H^0_\lambda(\text{Int}(\mathbb{Z}^{\{x\}}), I_{\text{Bin}}M) \cong \text{Der}_\lambda(\mathbb{Z}^{\{x\}}), I_{\text{Bin}}M) \cong I_{\text{Bin}}M$ [48]. Therefor ϱ_0 is identity map on M.

From the construction of the free λ-ring (Proposition 5.4.6) it is clear that the forgetful functor from the category of λ-rings to the category of commutative rings takes a free λ-ring to a free commutative ring. For a λ-ring R and a λ-module M over R, Robinson [48, Lemma 7.7] defines the homomorphism

$$\gamma_n : H^n_\lambda(R, M) \longrightarrow H^n_{AQ}(\underline{R}, \underline{M}).$$

Then for a binomial ring R and a binomial module M over R we have

$$\xi_n : H^n_{\mathrm{Bin}}(R, M) \longrightarrow H^n_{\mathrm{AQ}}(\underline{I_{\mathrm{Bin}} R}, \underline{I_{\mathrm{Bin}} M}),$$

where

$$\xi_n = \gamma_n \circ \varrho_n.$$

Thus we have composition maps between the cohomology of binomial rings, the cohomology of λ-rings and the André-Quillen cohomology of the underlying commutative rings.

Next steps to complete this work. We will investigate degree one cohomology of binomial ring $H^1_{\mathrm{Bin}}(R, M)$ and degree two cohomology of binomial ring $H^2_{\mathrm{Bin}}(R, M)$ for a binomial ring R and a binomial module M over R with module structure $\varphi^M_n = \frac{(-1)^{n-1}}{n} Id_M$. we expect to be equivalence classes of binomial extensions of R by M and connected components of the category of crossed binomial extension of R by M respectively (with suitable definition of binomial extensions and crossed binomial extension) similar to Robinson's [48] definitions for λ-rings.

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