Essays on the Secular Stagnation Hypothesis and the Cross-Section of Assets Returns

XIAOGE ZHANG

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Abstract

This thesis studies the relations between growth and cross-sectional assets prices. I develop four discrete-time models in both the exchange and the production economy. Chapter 3 introduces the model with two Lucas trees and studies the interactions between two trees in terms of their price dividend ratios and returns. Chapter 4 explores a production economy with multiple balanced growth paths. The model shows that pessimistic beliefs may trigger persistent slumps, low interest rates and high risk premia. Chapter 5 extends the model used in chapter 4 to the Epstein and Zin framework and calibrates the model to match the historical data moments. Chapter 6 considers a model with two parallel sectors in the production economy and examines the cross-sectional co-movements between growth and asset returns.

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Declaration

I declare that except where specific reference is made to the work of others, the contents of this thesis are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. All sources are acknowledged as references.

Chapter 1

Introduction

1.1 Origins

The relations between macroeconomic fundamentals and cross-sectional assets prices are popular issues in the study of economics. Intuitively, economic performance should have a strong relation to the production and the consumption sectors. In addition, one important, if not the most important, linkage between these two sectors is the financial market. The classic assumptions are that firms finance their production by issuing securities and that consumers smooth their consumption by trading these assets. The economy achieves general equilibrium when agents on both sides reach their optimal. This thesis attempts to make some contributions to the existing literature based on three pillars: investments, growth and asset prices.

To understand these, it is necessary to connect firms' stock prices and cash flows to macroeconomic fundamentals. More explicitly, we should consider aspects including but not limited to: (1) firms' production and technologies used in production, (2) the efficiency and strategy of firms' investments, (3) the macro-economic conditions such as aggregate consumption and its growth and (4) the variables of the financial market, namely the risk-free rate and risk premium. In this thesis, I choose models that describe firms' production, investment decisions and its assets' prices.

The origin of the asset pricing model, as introduced in the textbook by Cochrane (2009), ties together the asset's dividends D and its prices P by the discount factor π and the rational expectation as in following equation.

$$P_t = E_t\left[\frac{\pi_{t+1}}{\pi_t}\left(D_{t+1} + P_{t+1}\right)\right] \tag{1.1}$$

If the intertemporal conditions do not change, we can iterate the basic pricing equation to obtain,

$$P_t = E_t\left[\sum_{\tau=t+1}^{T}\frac{\pi_\tau}{\pi_t}D_\tau\right] + E_t\left[\frac{\pi_T}{\pi_t}P_T\right] \tag{1.2}$$

In the mainstream related literature, the discount factor is determined by the marginal utility of consumption. I follow this convention not only because it is a frequently used approach but also since it serves as the bridge between consumption and asset returns.

There are two typical categories for models involving asset pricing. The model in the exchange economy takes the assets as pre-existing and assumes that the dividends fall from “heaven”. As a result, we can filter out other factors and focus on the relations between the dynamics of dividends, discount factors and asset returns. I consider this model in chapter 3. The production economy incorporates a production sector and thus fully endogenises the investment behaviours of firms, production and its growth, assets cash flows, stochastic discount factors and consumption. In this framework, asset prices and macroeconomic fundamentals are determined endogenously and thus depend on a set of structural parameters. Chapter 4, 5 and 6 adopt this approach.

1.2 Motivations and Contributions of Chapter 3

The basic pricing equation (1.2) shows that the price of an asset is equal to the optimal forecast of its discounted cash flows, conditional on all available information at the time. Therefore, theoretically, asset prices should reflect the best information about fundamental values. The analysis and extensions of this framework construct the well-known efficient markets theory.

However, the theory has been criticised in recent decades. Among many others, the risk premium puzzle raised by Mehra and Prescott (1985) reports that empirical observations of risk premium are significantly larger than the predictions yielded by those traditional pricing models. In addition, LeRoy and Porter (1981) show that the discount factor used in the studies is relatively stable in reality through time. However, the stock prices observed are significantly more volatile. This suggests that there is excess volatility in the aggregate stock market, relative to the present value implied by the efficient markets model.

Chapter 3 of this thesis explores a possible explanation of this issue by introducing multiple assets into the basic pricing equation (1.2). Ideally, the expectation $E(\pi D)$ can be decomposed into $E(\pi)E(D)+Cov(\pi, D)$. The criticisms suggest the first two terms are stable and that the dynamics of the last term cannot support the excess volatility in reality. Yet, if the dynamics of the dividend D of a specific asset interacts with the movements of the discount factor π, we yield more complex behaviours in the asset price P.

Specifically, I introduce a two Lucas trees model in chapter 3. The two trees have independent cash flows D_1 and D_2. Accordingly, the discount factor is affected by the dynamics of two cash flows. Roughly speaking, the expectation $E(\pi D_1)$ of asset 1 becomes $E[\pi(D_1, D_2)]E[D_1]+Cov[\pi(D_1, D_2), D_1]$. The model shows that the dynamics in cash flow D_1, in some conditions, are amplified by the function $\pi(D_1, D_2)$.

The contributions are mainly methodological. I consider a discrete time model which has not been developed before. The model, to the best of my knowledge, does not have a closed form solution. I use a second order Taylor expansion to derive an approximated analytical solution. With the help of this analytical solution and the computational environment, chapter 3 is able to identify the economic meaning of each part and to understand the rationale beneath the pricing mechanism in this framework.

1.3 Motivations and Contributions of Chapter 4 and Chapter 5

I turn to the production economy models in chapter 4, 5 and 6. The general equilibrium model nests aspects such as investment, growth, consumption and asset prices. The model attempts to use this to account for the weak recovery and the trapped risk-free rate that occurred in the US after the great recession of 2008.

Among many others, Fernald and Jones (2014) review the modern growth theory and document how economic growth in the US is decelerating along with growth in educational attainment, R&D intensity and population. Especially after the great recession, whether we are in a period of "secular stagnation" is controversial after people have observed some evidence of sick recovery.

Hansen (1939) introduced the term "secular stagnation" to describe a long-lasting period of slow growth in an economy. At the time, the world had experienced the most severe recession ever. He warned people about the possibility that the economy would be stagnant for a long time and wrote:

> "This is the essence of secular stagnation - sick recoveries which die in their infancy and depressions which feed on themselves and leave a hard and seemingly immovable core of unemployment."

Nonetheless, in a sense, the huge demand that resulted from the world war pulled the economy out of its downturn. The discussion of this hypothesis was diminished. In fact, this hypothesis challenged the classic theory of the real business cycle (RBC) that stated that real macroeconomic fundamental variables generally tend to recover to a "natural" level after exogenous shocks.

With several decades of development in macroeconomics and economic growth theory, we still face the same debate. Nowadays, the representative researchers of these discussions are Gordon (2015) and Summers (2015a). Gordon (2014) argues that growth in the 25 to 40 years after 2007 will be much slower than before. He reckons that the primary causes of this growth slowdown are the worsening of demographic shifts, educational attainment, inequality and debt to GDP rato. All of these are problems in the supply sector of the economy.

Summers (2015a) relates the weak performance after 2008 to the zero lower bound on nominal interest rates and the lack of investment. He suggests that many factors contribute to an increase of savings and a decrease in demand for investment. As a result, we are likely in a position where the equilibrium real interest rates is negative. Coupled with low rates of inflation, it is impossible to achieve due to the zero-lower bound. Hence, we are not able to achieve an adequate aggregate demand and full employment as well.

Chapter 4 builds an endogenous growth model, which synthesises two sides. The main contribution is that the parsimonious model manages to generate arbitrarily long period of suppressed growth accompanied by trended decreases in the risk-free rate and counter-cyclicality in the risk premium. This model features characteristics such as "AK" linear production, endogenised productivity and extrinsic randomness. The "AK" production offers a balanced growth path (BGP). I assume a co-move relation between the technology scale factor and investment. Specifically, technology is a threshold function of the investment-capital ratio. A negative demand shock on investment leads to low productivity. On the other hand, firms set the optimal investment according to their productivity to maximise their values. This structure generates multiple BGPs. Furthermore, sunspots can alter beliefs and activate shifts among multiple balanced growth paths (BGPs) in the economy. With these setups, the economy shifts between a healthy path and stagnation based on beliefs. Basically, if the economy dwells in pessimism, the stagnation is prolonged.

Theoretically, the model in chapter 4 is able to account for phenomena such as slow economic growth, low investment levels, trended decreases in the risk-free rate and the counter-cyclicality of the risk premium. Nonetheless, when I collect historical data from the US data on these variables, the calibrated model cannot fully generate those data moments. Instead, it introduces a trade-off. Those calibrated parameters that can generate correct macro-fundamental moments cannot offer reasonable moments of financial variables and vice versa. Hence, I update the model in chapter 5.

Chapter 5 inherits most of the setups of chapter 4 yet uses the Epstein and Zin (1989) (EZ) utility. In the standard constant relative risk aversion (CRRA) utility, the intertemporal elasticity of substitution (IES) is fixed to be reciprocal of the risk aversion parameter. However, this assumption is unreasonable in many cases in reality. The EZ framework separates the risk aversion parameter from the parameter of IES. With the extra degree of freedom on IES, the new model improves the calibration in chapter 4. With reasonable parameterisations, the regime switching model with EZ preference can accommodate all 10 historical data moments. Moreover, calibration suggests that the model needs highly persistent regimes and a high level of IES to generate the historical data moments.

Basically, the model suggests that the long-term cycles in growth, investment and risk-free rate might be the result of persistent regimes and dynamic switches in the agents' beliefs.

1.4 Motivations and Contributions of Chapter 6

I combine the model structure used in chapter 4 with the analysis of the heterogeneity in chapter 3. In other words, chapter 6 extends the multiple assets model in chapter 3 to the production economy.

Empirical studies in this field mainly focus on the cross-sectional dynamics of firms' growth and asset returns. For growth and sales, it is rather obvious that cross-sectional firms tend to co-move. The general equilibrium model can nest this part without difficulty. For asset prices, Fama and French (1993, 1995, 1996, 2006) wrote a series of papers discussing the cross-sectional properties of asset returns. They show that average returns on common stocks are related to firm characteristics like size, earnings-to-price ratio, cash-flow-to-price ratio, book-to-market equity and so forth.

Although the model used in chapter 6 is not going to cover all of these factors, I want to establish a simple theoretical framework of co-movements between sectors in terms of growth, investment and asset prices in first stage.

The model in chapter 6 has (1) two parallel productive sectors, (2) constant return to scale in production function and (3) spillovers and complementarities in productivity. Once again, the "AK" framework assures the endogenous growth of the firms. In addition, I assume there is a shared component in the technology scales of two sectors. This shared part is determined by the investment level of two sectors. By this means, the model endogenises the technology scale. There are 2 channels for cross-sectoral interactions in the model. The first is the common component inherited in the productivity of both sectors. The second is the stochastic discount factor (SDF) just like the "bridge" between two assets in chapter 3. The unique SDF is formed by the investor's consumption choice across states of natures. Since there are two parallel sectors, the SDF is constructed on the basis of the dividend growth rate from 2 sectors. The idiosyncratic shocks to one sector spread through these two channels and generates various patterns in investments, growth and assets returns of two sectors.

1.5 Main Structure of This Thesis

Next chapter reviews the related literature to provide background information. The main body starts by examining the multiple assets problem in the exchange economy in chapter 3. It draws attentions to the interactions between different assets with independent cash flows. It is a natural experiment to study the properties of heterogeneities in the exchange economy.

Next, I extend the exchange economy to the production economy and turn the spotlight to relations between investment, endogenous growth and asset returns. Chapter 4 and 5 inherit the asset pricing equation from chapter 3.

Chapter 6 uses the model in the production economy yet refocuses on heterogeneities. The model combines the properties from chapter 3 and 4, and is able to explore the impact of the idiosyncratic shock on either the firms' investment behaviours or the assets returns.

Chapter 7 makes concluding remarks and points out the main limitations of the thesis.

Chapter 2

Literature Review

To provide background information, I survey the literature related to this thesis around 4 main aspects, namely: asset pricing, secular stagnation, economic growth and heterogeneities.

2.1 Literature on Asset Pricing

One common feature of finance models in the 1970s is that they often use rational expectations to relate assets prices to other economic variables.

For instance, Merton (1973) extends the capital asset pricing model (CAPM) to an intertemporal general equilibrium model. Compared to previous models, its new feature is that asset demands are affected by the uncertainties in assets' future performance. Later, Lucas (1978) shows that rational asset prices should be related to the aggregate consumption in a rational expectations general equilibrium model. The literature often refer this as the "Lucas tree" framework.

The pricing function derived from the Euler equation of Lucas (1978) is given by,

$$P_t U'(C_t) = E_t \left[\beta U'(C_{t+1}) (P_{t+1} + D_{t+1}) \right] \tag{2.1}$$

The representative consumer balances between the utilities scarified for buying one asset and the expected present value of the utilities obtained from that asset. In the one-tree case, the market clears at $C_t = D_t$ in the equilibrium. The price is dominated by two opposing forces. The income effect drives an asset to be more attractive when its future cash flows increase. The substitution effect drives an asset to be less attractive when the marginal utility is lowered by increases of the future dividends.

Mehra and Prescott (1985) report the equity premium puzzle. They solve for the one-tree case and find that the risk premium predicted by the model is considerably smaller than the empirical observations. Theoretically, if the model is adjusted to match the observation, one needs to raise the risk aversion parameter to a ridiculously high level. In fact, this is consistent with the findings of Grossman and Shiller (1981) that the model cannot predict excess volatilities in reality.

Cochrane et al. (2008) extend the solution of the original model to incorporate two trees and find one possible source of the excess volatility. As reviewed in Sargent (1987), the Euler equation for the multiple trees model is an analogue of the one-tree model as,

$$P_{i,t}U'\left(D_t\right) = E_t\left[\beta\left(P_{t+1} + D_{i,t+1}\right)U'\left(D_{t+1}\right)\right] \tag{2.2}$$

$$D_t = \sum_{i=1}^{N} D_{i,t} \tag{2.3}$$

The difficulty comes in when we attempt to solve this difference equation. Cochrane et al. (2008) solve the corresponding continuous time model, which allows the dividends to follow two independent geometric Brownian motion. They find that two trees' prices interact with each other. The price dividend ratio generally decreases when a tree's share increases. In their model, the price-dividend ratio for one asset is at the highest level when it takes a small share. Additionally, the expected return of a specific asset varies along with the dynamics of its share.

Martin (2013) extends the model to a N-tree model. Indeed, the model becomes more complicated to solve. With the help of the cumulant generating function, the Fourier transform and contour integral in the complex analysis, the paper gives a close solution. The behaviours of the assets prices and returns are similar to what has been shown in Cochrane et al. (2008). The effect of the share s on risk premium is numerically evaluated. However, due to the complexity of the method, it is subjects to the curse of dimensionality and is computationally intractable as N increases.

With all the convenience of the continuous time model, the multiple trees model can be solved in a closed form. However, to the best of my knowledge there has been no attempt to study the solution in discrete time. The purpose of chapter 3 is to solve the discrete time model to explore more economic rationale and intuition since the mathematical solutions of previous papers are rather difficult to understand.

2.2 Literature on Secular Stagnation

Besides the series of paper by Gordon (2014, 2015) and Summers (2015a,b), the majority of models on secular stagnation follows the arguments of Summers. As surveyed by Gourinchas et al. (2016), there are many studies on interest rate and the zero lower bound, including Pescatori and Turunen (2016), Gruber and Kamin (2016), Favero et al. (2016), Sajedi and Thwaites (2016) and Eggertsson et al. (2017). Essentially, they formalise the arguments of Summers (2015a) in Neokeynesian overlapping generation (OLG) models. In these models, factors such as a slowdown in population growth, an increase in life expectancy, an increase in income inequality and a fall in the price of investment goods, can reduce the natural interest rate.

Additionally, Blanchard et al. (2017) run regression on data of forecasts of economy growth and the corresponding forecast errors. They argue that low expectation of long-run productivity growth can affect output and inflation in the short run. Theoretically, in thier model, both consumers and firms tend to revise their behaviour when the economy is in a recession. Consumers modify their expectations of future consumption and firms adjust their investment strategy. A downturn in the economy is reinforced by the pessimism.

Eichengreen (2017) studies similarities and differences of the Great Recession in the 1930s and the recent one in 2008. He concludes that the recovery after the 1930s was faster than today's. He argues that the diffusion of the new technologies such as electricity and a national highway system serves as cure to the recession. On the other hand, the large demand for those technologies from the world war II also simulated the innovation and hence productivity. However, the world does not have analogue demands today.

Another related paper comes from Acemoglu and Restrepo (2017). Since many papers have discussed the importance of demographic change and aging, Acemoglu and Restrepo (2017) test the effects of aging on economic growth. Generally, they show that there is no negative relation between aging and GDP per capita. Interestingly, in some specifications, the association is positive. They defend this by showing that those countries with an aging problem are more likely to develop labour replacement technology such as robots. They also establish a model to demonstrate that a lack of labour can lead to the adoption of automation technologies and further increase productivity and output.

2.3 Literature on Endogenous Growth

The traditional growth models represented by Solow (1956) predict that even for countries with different endowments, their growth rate should converge. This is mainly due to the assumption of the diminishing returns to scale in production. Basically, the assumption ensures that when capital is deficient, the returns to investment are high.

However, the empirical studies show that many developing countries, to a large extent, suffer from the "poverty trap". Among many other examples, Sachs et al. (2004) point out Mali as representative of this. They describe how Mali is relatively well governed and has "free" conditions. Yet, it is profoundly subjected to poverty.

Under this circumstance, some researchers started to consider the increasing returns to scale (IRS) in production. Romer (1986) initiated a new era for the endogenous growth theory. The paper introduces a growth model with increasing marginal productivity. Importantly, knowledge is an input in production and exhibits increasing marginal product. In addition, knowledge has positive externalities since it cannot be perfectly patented. Due to the IRS, a rise in input lowers the average cost for both the specific firm and the entire economy, which further stimulates the production. A cycle like this can generates either self-reinforcing growth or poverty. With this framework, the model is capable of explaining long historical growth and the non-convergence property of the cross-country growth in the data.

In terms of the methodology of endogenised technology used in chapter 4 and 6, it was mainly borrowed from Romer (1986) and Azariadis and Drazen (1990). The key feature is that the productivity parameter A is a function of investment to capital ratio I/K. The intuition in Romer (1986) is that private investment produces new knowledge. Knowledge enters the production process of all firms in the economy. Similarly, in Azariadis and Drazen (1990), they consider the spillovers from human capital accumulation processes. Durlauf (1993) and Matsuyama (1997) also provide a micro-foundation to the framework by introducing a system with complementarity and show how investment may feed back to specialisation and productivity.

2.4 Literature on Heterogeneity

The studies of co-movements and heterogeneities at the firm level mainly focus on cross-sectional dynamics of firms' growth and asset returns.

For growth or sales, Bachmann and Bayer (2014) and Higson et al. (2002) examine the cross-sectional behaviours of firms in terms of their growth, investments and sales. Higson et al. (2002) found a negative correlation between the rate of growth of GDP and the cross-sectional variance in growth rates of sales. Similarly, Bachmann and Bayer (2014) show that the cross-sectional standard deviation of firm-level investment is significantly pro-cyclical.

Holly et al. (2013) systematically examine the relation between the cross-sectional dynamics of a firm's growth and aggregate business cycle. Their evidence confirms that the distribution of firm growth is more skewed and leptokurtic in an economic slowdown. In addition, they build a model and account for the asymmetry in density of firm growth by financial constraints and asymmetric information in the capital market.

In terms of the microfoundations of the theoretical works in this area, the vast majority are around the Bertola–Caballero–Engel model in the papers by Caballero and Eduardo M. R. A. Engel (1991) and Bertola and Caballero (1990). It is an extension to the (S, s) model and can be dated back to Arrow et al. (1951). Basically, this strand of model reckons that the heterogeneity at the firm level originated from the fact that fixed costs make small adjustment impractical. Like in Caballero (1993), a shock causes some firms to adjust their investment plan and leaves others to stay.

For the dynamics of cross-sectional asset prices, the well-known Fama and French (1993) model sorts firms with different characteristics and uses this to explain the cross-sectional differences in expected stock returns. Eugene Fama was one of the three laureates who were awarded the Nobel Memorial Prize in Economic Sciences in 2013 for their outstanding works on the empirical study of asset pricing. Basically, the "Fama–French three factor model" uses firm size, book-market ratio and return of a market portfolio as predictors of the expected asset return. Later Fama and French (2015) added two factors namely the profitability of the firm and the rate of investment. They find that, in general, smaller firms, value firms (with higher book to market ratio), firms that are more profitable and firms that invest less earn higher average returns. There are many studies following this stream. Some illustrative works include but are not limited to Berk et al. (1999), Campbell and Vuolteenaho (2004), Hansen et al. (2008), Lettau and Wachter (2007), Parker and Julliard (2003) and Yogo (2006).

The model in chapter 6 is definitely not going to cover all these characteristics, but I want to establish a parsimonious theoretical framework with co-movements between sectors in terms of growth, investment and asset prices in this stage.

Chapter 3

Asset Pricing in the Forest

3.1 Abstract

This paper studies assets prices in an exchange economy with two Lucas trees. Two assets with independent cash flows interact with each other in terms of their price dividend ratios and expected returns. Explicitly, the share of a specific asset in the aggregate consumption plays a significant role in the asset pricing mechanism. An idiosyncratic shock to one asset affects the shares of both assets and their individual assets prices. I decomposed the pricing equation and find that the "precautionary saving" effect and the "market β" effect are the major forces driving the pricing mechanism.

3.2 Introduction

Mehra and Prescott (1985) describe the risk premium puzzle, which states that the empirical observation of risk premium is significantly larger than the theoretical prediction. Furthermore, Grossman and Shiller (1981) shows that the present value of dividends discounted by the marginal rates of substitution in consumption has only a moderate relation to actual stock prices. Additionally, the present value of dividends is not volatile enough to justify the price's movements unless the risk aversion coefficient is set to extremely high levels. These studies are conducted on an aggregate level of the stock market. On the other hand, Jung and Shiller (2002) show that, cross-sectionally, a firm's price dividend ratio serves as a strong predictor of the long-term changes in their future dividends. Samuelson et al. (1998) summarise these by stating that the stock market is "micro efficient but macro inefficient". The empirical difference between the aggregate market and the individual firms draws attention to the cross-sectional heterogeneities of the firms.

Some of the following studies introduce the multiple assets framework to replace the single asset model. The frontier of theoretical research includes two trees by Cochrane et al. (2008) and Lucas orchard by Martin (2013). These models have two major features. Firstly, the assets' prices interact with each other even though they have independent cash flows. This phenomenon is mainly caused by feature of general equilibrium itself. On average, the investor must hold the market portfolio. When a shock hits one asset, the investors rebalance their portfolio, which affects the prices of all assets. The model in this framework theoretically predicts that the volatility of the asset's return exceeds the volatility of the underlying dividends in certain situations. Further, Chabakauri (2013) includes portfolio constraints in this framework. He finds a positive relation between the amount of leverage in the economy and volatilities of the stock return.

This paper uses a discrete time model in the exchange economy with two assets. The model develops an approximated analytical solution. To some extent, it is the discrete time version of the model used in Cochrane et al. (2008). Nevertheless, the contributions are mainly methodological. With the help of computational software, I find that different components play different roles in the pricing mechanism. Nonetheless,an analytical result is developed in this paper when the computational power is insufficient.

The model follows a simple setup. The dividends growths of two assets follow independent and identical log-normal distribution. The representative investor, with log-utility preference, consumes the dividends from two assets. I solve the price dividend ratio and the expected return for the specific asset and the market portfolio.

I define s as the share of one asset's dividends in the aggregate consumption. The share s, as a state variable, plays a significant role in the pricing mechanism. The varies of the share s are directly related to the dynamics of the prices of both two assets. Generally, a positive dividend shock to one asset, which levels up its share s, decreases its price dividend ratio. In the meantime, the shock naturally depresses the share s of the other asset. Therefore, the price dividend ratio of the other asset increases.

In fact, the idiosyncratic shock spreads through the pricing kernel. This paper follows the basics of consumption-based capital asset pricing model (CCAPM). With the no-arbitrage condition and the complete markets assumption, there is an unique stochastic discount factor (SDF) in the market. In theory, it is determined by the marginal rates of substitution in consumption. Investors use the SDF as the pricing kernel. Therefore, once the SDF is affected by some factors, all assets' prices are adjusted accordingly.

In terms of the underlying rationales, the model finds that the substitution effects in the one tree case can be decomposed into "market precautionary saving" effects and the "market β" effect. The former is due to the rebalancing of the market portfolio. Specifically, a dividend shock to any asset changes the composition of the aggregate consumption. Since the proportions of assets in the market portfolio are changed, now investors diversify their consumption differently. Therefore, investors change their behaviours due to the precautionary saving effect, which affects the prices of all assets. The latter arises from the interaction between a specific asset and the market portfolio. By the definition of the substitution effect, an increase of the future dividend decreases the future marginal utility of consumption, which makes the asset less attractive. Here, as s increases, the connection between the asset dividend and the aggregate consumption becomes closer. A larger s leads to a stronger substitution effect, which lowers the price of the asset in the current period.

Along with the movement of the share s, these two forces take it in turns to dominate the pricing mechanism. To summarise, this parsimonious two-asset model enables us to explore the features that are not captured by the single-asset model.

The rest of this chapter consists of the following parts. Section 3.3 introduces the model set-ups and gives the numerical and approximate analytical solution to this model. Section 3.4 focuses on understanding the pricing mechanism and decomposes the factors. Section 3.5 concludes the chapter.

3.3 Two-Asset Model in Discrete Time

This section introduces the model of the two trees. Time t discretely runs from 0 to infinity. The representative agent consumes dividends from two trees. For simplicity of later calculation, I only consider the log utility for consumers. As in the standard model, in equilibrium, the representative investor holds the market portfolio. Each tree yields independent dividends $D_{i,t}$ in period t where $i = 1$ or 2. I assume the growth rate of the dividend independently follows log normal distribution.

$$\frac{D_{i,t+1}}{D_{i,t}} \equiv G_{i,t+1} \sim LN\left(\mu_i,\, \sigma_i^2\right) \tag{3.1}$$

In period t, let $P_{1,t}$ and $P_{2,t}$ denote the price of asset 1 and 2 respectively. C_t is the aggregate consumption of period t. Since consumers own the two trees, the market clears when $C_t = D_{1,t} + D_{2,t}$.

The Euler equation of this model is well established.[1] In fact, in this model, there is no difference between the Euler equations of one tree and two trees. Key intuition is that consumers use their aggregate consumption to independently price each asset in the market. I provide derivation and explanation of the standard CCAPM in this two-trees case in appendix 3.6.1.

$$U'(C_t)\, P_{i,t} = \beta E\left[\left(P_{i,t+1} + D_{i,t+1}\right) U'(C_{t+1})\right] \tag{3.2}$$

where the prime $'$ denotes the derivatives.

[1] See details in book of Cochrane (2009)

With log utility, it can be iterated to obtain,

$$P_{i,t} = C_t \sum_{\tau=0}^{\infty} \beta^{\tau} E_t \left(\frac{D_{i,t+\tau}}{C_{t+\tau}} \right) \tag{3.3}$$

I define the state variable $s_t \equiv D_{1,t}/C_t$ which is the share of the asset 1's dividends in aggregate consumption. Additionally, the dynamics of s_t and the price dividend ratio for asset 1 are given by,

$$s_{t+1} = \frac{s_t G_{1,t+1}}{s_t G_{1,t+1} + (1 - s_t) G_{2,t+1}} \tag{3.4}$$

$$\frac{P_{1,t}}{D_{1,t}} = \frac{1}{s_t} \sum_{\tau=0}^{\infty} \beta^{\tau} E_t (s_{t+\tau}) \tag{3.5}$$

From now on, I take the asset 1 as the study object and focus on its price dividend ratio. Later, I handle the solution of asset return. All findings can be applied to asset 2 due to the symmetry. Equation (3.5) shows that the price dividend ratio is a function of the current value of the share s_t and the expectation of the geometric sum of the process $\{s_t\}_t^{\infty}$. If we can solve the expectation in equation (3.5) the price dividend ratio is nothing but a function of the state variable s_t. Unfortunately, the solution to equation (3.5) is much less elegant than itself. To the best of my knowledge, there is no closed form solution. Hence, firstly the next section studies the behaviour of the stochastic process $\{s_t\}_t^{\infty}$. Then I introduce a numerical method to calibrate the expectation. Later, the Taylor expansion helps to obtain an approximate analytical solution. This enables us to analyse the asset return and the market portfolio return.

3.3.1 The Stochastic Process of the Share s

Since the process of the share $\{s_t\}_0^{\infty}$ is key to the pricing function, this section offers a description and simulation of $\{s_t\}_0^{\infty}$. Following the dynamics of s_t in equation

(3.4), the process of s_t can be rewritten as,

$$s_t = \frac{s_0}{s_0 + (1 - s_0) \prod_{\tau=1}^{t} (G_{2,\tau}/G_{1,\tau})} \tag{3.6}$$

It is straight forward that I should firstly study the product of G's.

$$\prod_{\tau=1}^{t} \left(\frac{G_{2,\tau}}{G_{1,\tau}} \right) = \frac{G_{2,1} \times G_{2,2} \cdots \times G_{2,t}}{G_{1,1} \times G_{1,2} \cdots \times G_{1,t}} \tag{3.7}$$

$G_{i,\tau}$ is log-normally distributed with parameter $\left(\mu_i, \sigma_i^2\right)$. Due to the property of log normal distribution, $G_{2,\tau}/G_{1,\tau}$ follows log normal distribution with parameter $\left(\mu_2 - \mu_1,\ \sigma_2^2 + \sigma_1^2\right)$.[2] Moreover, the product as a unity follows log normal distribution with parameter $\left(t\,(\mu_2 - \mu_1)\,,\ t\left(\sigma_2^2 + \sigma_1^2\right)\right)$.

The stochastic process of $\{s_t\}_0^\infty$, according to equation (3.6), is a function of the products of the log normal random variables. Intuitively, s_t has 3 possible behaviours.

If $\mu_2 > \mu_1$, the accumulation of products of G_2/G_1 makes s_t converging to 0.

If $\mu_2 < \mu_1$, because the product converges to 0, s_t converges to 1.

If $\mu_2 = \mu_1$, s_t has a drift yet it will not converge quickly.[3]

Nonetheless, I am not able to analyse the moments of s_t itself since it is formed from a non-linear transform of the random variable $\prod (G_2/G_1)$. Hence I resort to a simulation to verify these predictions.

Figure 3.1 simulates the stochastic process s_t. The functions converging to 0 and 1 are the simulations for $\{s_t\}_0^\infty$ with G_2/G_1 that has drift -0.04 and 0.04 respectively. The process in the middle is the simulation of the symmetric case. For the clearance of the figure, I specify the simulation in the middle with smaller variance. However, the trend does not change if I assign identical variance to 3 simulations. Similar to the predictions, 3 simulations have 3 different directions.

[2]See the discussion in appendix 3.6.2 for details.

[3]See the discussion in appendix 3.6.3 for details.

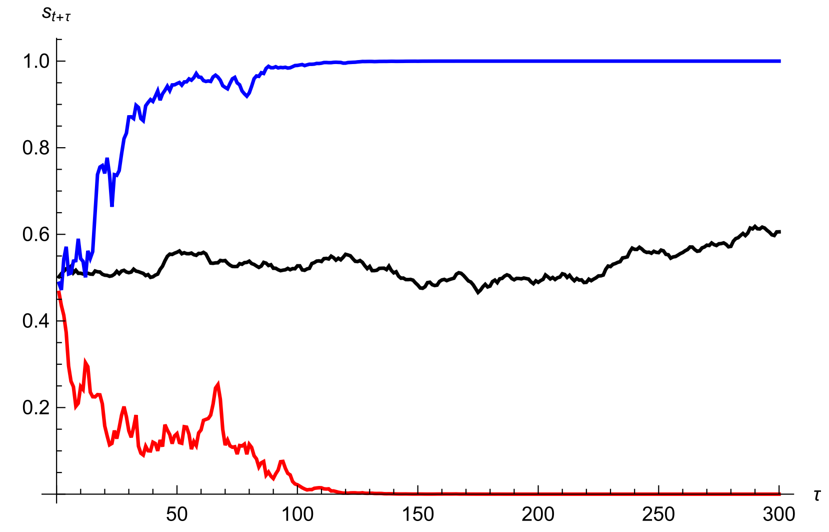


Fig. 3.1 Simulation of s_t
Red: $G_2/G_1 \sim LN(0.04, 0.2)$, Black: $G_2/G_1 \sim LN(0, 0.02)$, Blue: $G_2/G_1 \sim LN(-0.04, 0.2)$

The uneven case collapses to the 1 tree model quickly when s_t converge to either 1 or 0, which is less interesting since the aim is to study the interactions between 2 assets. Therefore, the rest of this chapter only studies the model with the symmetric drifts.

3.3.2 Numerical solution

I tackle the pricing equation with two methods. Firstly, the expectation is nothing but an integral of an adjusted form of the probability density function. Therefore proposition 3.1 solves the equation (3.5) as an integral function. Then computational software can calculate the result numerically. I use Mathemetica as the numerical computational environment. It samples a sequence of points to evaluate a integral numerically.

Proposition 3.1 *The pricing equation for price dividend ratio of Equation* (3.5), *under the assumptions of* $\mu_1 = \mu_2$ *and* $\sigma_1^2 = \sigma_2^2 = \sigma^2$, *can be solved as a integral function,*

$$\frac{P_{1,t}}{D_{1,t}} = \sum_{\tau=0}^{\infty} \beta^{\tau} \int_{-\infty}^{\infty} \frac{1}{s_t + (1-s_t)\exp(g)} \frac{\exp\left(-\frac{g^2}{2\tau\sigma^2}\right)}{\sqrt{2\pi\tau\sigma^2}} \, dg \tag{3.8}$$

where $\exp(g) \equiv G$*. The log-normal variable* G *is replaced by the normally distributed variable* g *because it is more computational friendly.*

Proof. See proof in the appendix 3.6.5.

Additionally, lemma 3.1 shows that this expression converges.

Lemma 3.1 *The function for pricing dividend ratio in proposition 3.1 converges.*

Proof. The integral itself is noting but the expectation of s_t.

$$E_t(s_{t+\tau}) = \int_{-\infty}^{\infty} \frac{1}{s_t + (1-s_t)\exp(g)} \frac{\exp\left(-\frac{g^2}{2\tau\sigma^2}\right)}{\sqrt{2\pi\tau\sigma^2}} \, dg \tag{3.9}$$

The first term in the integral is bounded by 0 and 1. If $\exp(g) \to \infty$, it goes to 0. If $\exp(g) \to 0$, it goes to 1. The second term in the integral is the corresponding density assigned to the function. Therefore the expectation is also bounded by 0 and 1. Since β is smaller than 1, the function (3.8) itself converges.

Q.E.D. ■

With the parameterisations as $\sigma^2 = 0.04$ and $\beta = 0.96$, figure 3.2 shows the calibration of the price dividend ratio which varies with s_t. The state variable s plays an important role in the pricing function. Due to the symmetry, the price dividend ratio of asset 2 (yellow curve) mirrors the price dividend ratio of asset 1 (blue curve). It is clear that s affects the price dividend ratio of asset 2 as well. In other words, a shock to s, no matter from which asset, affects both assets even though the cash flows of two assets are independent. Explicitly, if we have a positive shock to asset 1's dividend growth, the share s becomes larger. In general, the increase of s lowers the price dividend ratio of asset 1 as shown in figure 3.2. Meanwhile, this increase of s raises the price dividend ratio of asset 2. Two assets with independent cash flows co-move with the state variable s.

However, the expression in proposition 3.1 is rather inconvenient to use to later derivation for other variables such as asset return and market portfolio return. Hence the approximated closed form solution for the price dividend ratio is the main target of the next subsection.

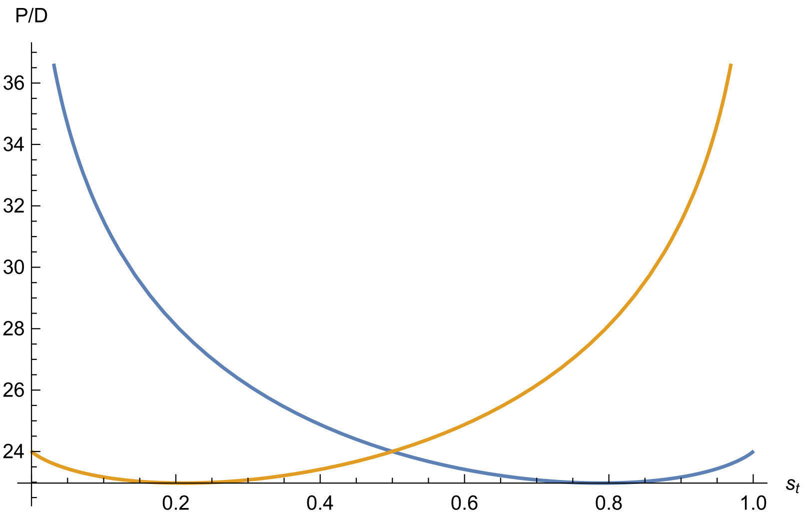


Fig. 3.2 Price-Dividend Ratio
Parameterisation: $\sigma^2 = 0.04$ and $\beta = 0.96$.
Blue line represents the P/D of asset 1.
Yellow line represents the P/D of asset 2.

3.3.3 Approximated Analytical Solution

The function in proposition 3.1 cannot offer economic intuition in the pricing mechanism. Hence, I provide the following proposition as an approximate analytic solution for price dividend ratio.[4]

Proposition 3.2 *An approximate analytic solution for the price-dividend ratio of asset 1 is given by,*

$$\frac{P_{1,t}}{D_{1,t}} \approx \frac{\beta}{1-\beta} + \frac{1}{2}\sigma^2 (1-s_t)(1-2s_t)\frac{\beta}{(1-\beta)^2} \tag{3.10}$$

Proof. First of all, because the derivations involve many notations of sum $\sum$ and product $\prod$ I set the current time at 0 or t interchangeably to suit the particular situation. Here, for simplifying the notations, this proof solves for $P_{1,0}/D_{1,0}$. The price dividend ratio in equation (3.5) is given by

$$\frac{P_{1,0}}{D_{1,0}} = \frac{1}{s_0}\sum_{\tau=0}^{\infty} \beta^{\tau} E_0(s_\tau) \tag{3.11}$$

[4]See an detailed study of equation (3.6) in appendix 3.6.4

Generally speaking, the solution of $E_0(s_\tau)$ is of interest. The second order Taylor expansion helps to solve this approximately. Starting from s_1,

$$s_1 = \frac{s_0}{s_0 + (1 - s_0)\exp(g_1)} \tag{3.12}$$

Applying the second order Taylor expansion around the point $g_1 = E(g_1) = 0$, I have,

$$s_1 = s_1|_0 + \left.\frac{\partial s_1}{\partial g_1}\right|_0 \cdot g_1 + \frac{1}{2}\left.\frac{\partial s_1^2}{\partial^2 g_1}\right|_0 \cdot g_1^2 + \frac{1}{3!}\left.\frac{\partial s_1^3}{\partial^3 g_1}\right|_\xi \cdot g_1^3 \tag{3.13}$$

Where ξ is some real number between g_1 and $E(g_1)$. According to the Taylor theorem, the third term in the right hand side represents the remainder in Lagrange form.

By definition of the little o notation, as $g_1 \to 0$, I have

$$s_1 = s_1|_0 + \left.\frac{\partial s_1}{\partial g_1}\right|_0 \cdot g_1 + \frac{1}{2}\left.\frac{\partial s_1^2}{\partial^2 g_1}\right|_0 \cdot g_1^2 + o\left[g_1^2\right] \tag{3.14}$$

With the expectation operator, the second term vanishes since $E(g_1) = 0$. The third term is the variance $Var(g_1) = \sigma^2$ of the random variable g_1. Basic algebra offers

$$E(s_1) = s_0 + \frac{1}{2}s_0\sigma^2(1 - 2s_0)(1 - s_0) + E\left(o\left[g_1^2\right]\right) \tag{3.15}$$

For s_2,

$$s_2 = \frac{s_0}{s_0 + (1 - s_0)\exp(g_1 + g_2)} \tag{3.16}$$

We can either apply the second order Taylor expansion for two variables or take the $g = g_1 + g_2$ as a unity. In fact, the latter simplifies the process. Accordingly, s_2

has a similar results to s_1.

$$s_2 = s_0 + \left.\frac{\partial s_2}{\partial g}\right|_0 (g_1+g_2) + \frac{1}{2}\left.\frac{\partial s_2^2}{\partial^2 g}\right|_0 (g_1+g_2)^2 + o\left[(g_1+g_2)^2\right] \tag{3.17}$$

With the i.i.d assumption in g's, I have $Var(g_1+g_2) = 2\sigma^2$.

$$E(s_2) = s_0 + \frac{1}{2}s_0 2\sigma^2 (1-2s_0)(1-s_0) + E\left(o\left[g^2\right]\right) \tag{3.18}$$

We can calculate the expansion for s_3 and s_4 so on and so forth. The pattern is clear.

$$E(s_t) = s_0 + \frac{t}{2}s_0\sigma^2 (1-2s_0)(1-s_0) + E\left(o\left[g^2\right]\right) \tag{3.19}$$

Sum up all the terms, the price dividend ratio is given by,

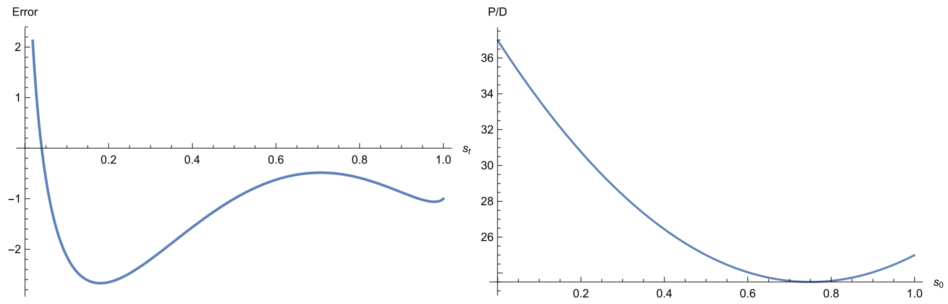
$$\frac{P_{1,0}}{D_{1,0}} = \frac{1}{s_0}\sum_{\tau=0}^{\infty}\beta^\tau\left(s_0 + \frac{\tau}{2}s_0\sigma^2(1-2s_0)(1-s_0) + E\left(o\left[g^2\right]\right)\right) \tag{3.20}$$

$$= \frac{\beta}{1-\beta} + \frac{1}{2}\sigma^2(1-s_0)(1-2s_0)\frac{\beta}{(1-\beta)^2} + E\sum_{\tau=1}^{\infty}\beta^\tau o\left[g^2\right] \tag{3.21}$$

Q.E.D. ■

Due to the property of the little o notation, the approximation for the price dividend ratio behaves well in the local area. However, the last term in equation (3.21) shows that the approximation error accumulates over time. To measure the accuracy of the approximation, I apply the same numerical method as before to obtain a numerical calculation of the approximate error. Explicitly, the computational environment offers the difference between equation (3.8) and equation (3.10).

$$NumerError = \left[\sum_{\tau=0}^{\infty}\beta^\tau\int_{-\infty}^{\infty}\frac{1}{s_t+(1-s_t)\exp(g)}\frac{\exp\left(-\frac{g^2}{2\tau\sigma^2}\right)}{\sqrt{2\pi\tau\sigma^2}}dg\right] - \left[\frac{\beta}{1-\beta} + \frac{1}{2}\sigma^2(1-s_t)(1-2s_t)\frac{\beta}{(1-\beta)^2}\right] \tag{3.22}$$



(a) Approximate Error **(b)** P/D Taylor Approximation

Fig. 3.3 Numerical Error and Approximated Price-Dividend Ratio

Figure (3.3a) plots the numerical calculation of the approximation error. Figure (3.3b) plots the approximated solution for the price dividend ratio namely equation (3.10). Comparing figure (3.2) with figure (3.3b), I believe the error in the most of area between 0 and 1 is acceptable. Nonetheless, when s approaches 0, the approximate error expands to a significant level. Fortunately, this extreme case can be analysed separately. Lemma 3.2 illustrates this extreme situation.

Lemma 3.2 *The price dividend ratio in equation* (3.5) *approaches infinity when s is close to 0.*

Proof. Recall the expression in equation (3.5)

$$\frac{P_{1,t}}{D_{1,t}} = \frac{1}{s_t} \sum_{\tau=0}^{\infty} \beta^{\tau} E_t \left(s_{t+\tau} \right) \tag{3.23}$$

Follow similar procedures in the proof of lemma 3.1, one can show that the expectation is bounded by 0 and 1. Hence, the polynomial is finite. When s_t approaches 0, the price dividend ratio itself goes to infinity.

Q.E.D. ■

Together with the separately analysed case of $s_t \to 0$, the approximated solution in equation (3.10) can capture the main character of the pricing function. Generally, the price dividend ratio declines with the increase of the share s_0. However, the function shows a slight U shape around the point $s_0 = 0.75$. It is clear that the share s plays a significant role in the pricing function, especially when the share s is small.

The next section attempts to decompose the pricing equation and help us understand the rationale of the pricing mechanism.

3.4 Understanding the Price-Dividend Ratio

To understand the economic intuition of the pricing function, I firstly rearrange the equation (3.3) into.

$$\frac{P_{1,0}}{D_{1,0}} = C_0 \sum_{t=1}^{\infty} \beta^t E\left(\frac{D_{1,t}}{D_{1,0}C_t}\right) \tag{3.24}$$

$$= C_0 \sum_{t=1}^{\infty} \beta^t \left[E\left(\prod_{\tau=1}^{t} G_{1,\tau}\right) E\left(C_t^{-1}\right) + Cov\left(\prod_{\tau=1}^{t} G_{1,\tau},\ C_t^{-1}\right)\right] \tag{3.25}$$

The second equality decomposes the expectation of the product of $1/C_t$ and $D_{1,t}/D_{1,0}$. Clearly, the state variable s_0 enters the term $E\left(C_t^{-1}\right)$ and the covariance term to affect the dividend yield. Respectively, I call them the "market precautionary saving" effect and the "market β" effect of s. In the following subsections, the two channels are examined explicitly.

3.4.1 The First Channel of the Share s

To understand the first channel namely the interaction between s and C^{-1}, I take the first period as an example.

$$E\left(C_1^{-1}\right) = C_0^{-1} E\left[\left(s_0 G_{1,1} + (1-s_0)\, G_{2,1}\right)^{-1}\right] \tag{3.26}$$

$$= C_0^{-1} E\left[\left(s_0 \exp\left(g_{1,1}\right) + (1-s_0) \exp\left(g_{2,1}\right)\right)^{-1}\right] \tag{3.27}$$

Again, the second order Taylor approximation can expand this function around $g_{1,0} = E(g_{1,0}) = 0$ and $g_{2,0} = E(g_{2,0}) = 0$ and offers

$$E\left(C_1^{-1}\right) \approx C_0^{-1} E_0 \left[\frac{1}{2}(2s_0 - 1)\left(g_{1,1}^2 s_0 + g_{2,1}^2 (s_0 - 1)\right) + 1\right] \tag{3.28}$$

$$= C_0^{-1}\left[\sigma^2 (2s_0 - 1)^2 + 1\right] \tag{3.29}$$

Figure 3.4 shows the relation between $E\left(C_1^{-1}\right)$ and s_0. The expectation reaches the lowest level at $s_0 = 0.5$. The parameterisations follow the symmetric assumption with $\mu_1 = \mu_2 = 0$ and $\sigma_1^2 = \sigma_2^2 = 0.02$ in the previous case. Although this example only presents the relation between s_0 and $E\left(C_1^{-1}\right)$, it is straight forward to generalise this analysis to any relation between s_0 and $E\left(C_t^{-1}\right)$.

In fact, the U-shape in the figure corresponds to the changes of variance in the aggregate consumption. To elaborate, point $s_0 = 0.5$ indicates that the aggregate consumption is fully diversified. The two assets have independent cash flows. Hence the market portfolio with two evenly distributed assets carries the smallest risks. The less risk the investor faces, the less attractive the asset is. This is due to the precautionary saving effect. Holding other effects constant, the closer the share s to 0.5, the lower is the price dividend ratio of the particular asset. To distinguish this intuition from the standard precautionary saving effect in one tree case, I name this channel "market precautionary saving" effect of the s.

3.4.2 The Second Channel of the Share s

In the meantime, s_0 also enters the covariance term in the equation (3.25). It is the covariance between the asset's dividend growth and the marginal utility of the aggregate consumption. Although it cannot fit into the standard definition of the market β, here I slightly abuse the terminology and name it market β effect. To

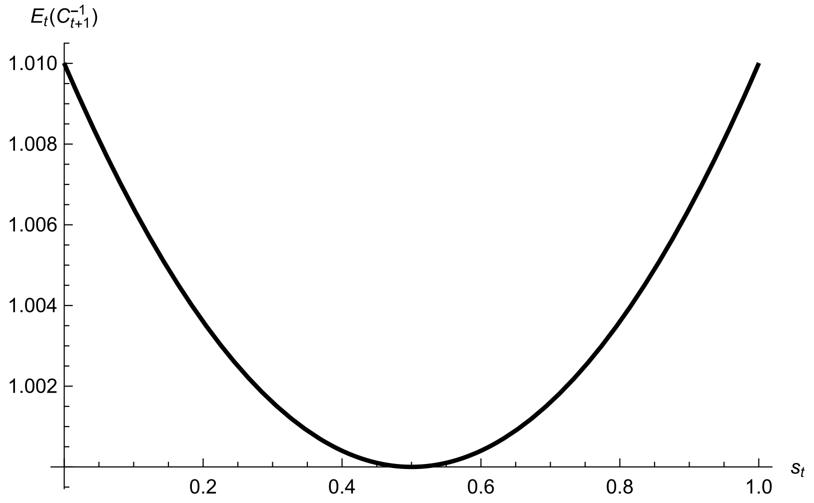


Fig. 3.4 Expectation of Marginal Utility
Parameterisations: $\sigma_1^2 = \sigma_2^2 = 0.02$.

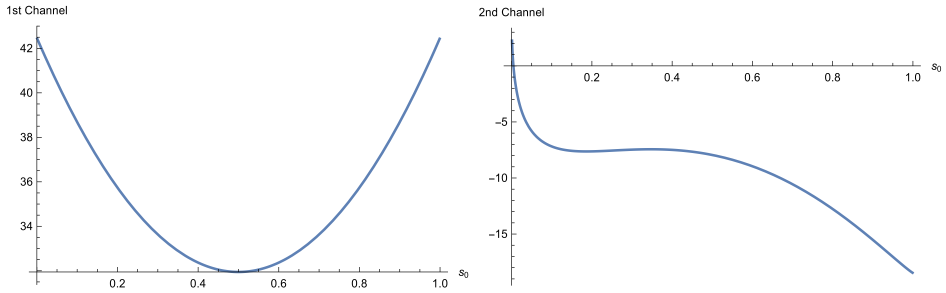
understand this effect, firstly, the C_t^{-1} can be rearranged into

$$C_t^{-1} = C_0^{-1} \left(s_0 \prod_{\tau=1}^{t} G_{1,\tau} + (1 - s_0) \prod_{\tau=1}^{t} G_{2,\tau} \right)^{-1} \tag{3.30}$$

The covariance between C_t^{-1} and $\prod G_{1,\tau}$ is of interest. As s_0 increase from 0 to 1, C_t^{-1} gains a stronger linkage to $\prod G_{1,\tau}$ while $\prod G_{2,\tau}$ has a smaller weight. The term is raised to the power of -1. Therefore, the covariance is negative. With these arguments, I know that the covariance term declines when s_0 move from 0 to 1 holding other effects constant. To explain, the negative linkage between the asset 1's dividends' growth and marginal utility of aggregate consumption becomes stronger as s increases. In this case, every unit of increase of the dividend leads to more units decreases in the marginal utility of aggregate consumption. I count this as the second factor of s to affect the price dividend ratio.

3.4.3 Comparison Between Two Channels

This section keeps exploring the two channels. According to the last section, through the first channel, the effect of s on price dividend ratio forms a U-shape. The turning point is 0.5. Through the second channel, the price dividend ratio is decreasing with



(a) 1st Channel **(b)** 2nd Channel

Fig. 3.5 P/D Decomposition

the share s. In this section, I separately calibrate the two forces and compare them. The equation (3.25) can be rearranged into,

$$\frac{P_{1,0}}{D_{1,0}} = \sum_{t=1}^{\infty} \beta^t E\left[\prod_{\tau=1}^{t} G_{1,\tau}\right] E\left[\frac{C_0}{C_t}\right] + \sum_{t=1}^{\infty} \beta^t Cov\left[\prod_{\tau=1}^{t} G_{1,\tau}, \frac{C_0}{C_t}\right] \tag{3.31}$$

The product of G is easy to handle since the product of log-normal variables is log normally distributed. Equation (3.31) enables us to calibrate two channels separately.

Figure 3.5 shows the two channels. Roughly speaking, the first half of s_0 the price-dividend ratio experiences rapid decreasing due to the decline of both two components. In the second half, the power from the first channel completely changes its direction since it goes beyond the fully diversification point. Additionally, the market β effect puts pressure on the price-dividend ratio and overcomes the up-pushing force from the first channel. Figure 3.5a verifies the prediction about the first channel. Figure 3.5b generally verifies that the market β effect is negative and decreasing with s.

In the figure 3.5b, we have a relative stable area in the middle of s. The covariance between C_t^{-1} and $\prod G$ is not very responsive to the change of s in this area. Intuitively, this is due to the fact that in this area $E\left(C_t^{-1}\right)$ itself is in a low level. By definition, the covariance not only captures the dependence between two variables but also is affected by the scale of the variables. The overall decreasing behaviour in the figure 3.5b reflects the increasingly tightened relation between marginal utility and individual asset's dividend growth when s moves from 0 to 1. However, when s is around 0.5, the low level of marginal utility itself moderates the decreasing speed.

This multi-assets model has a key finding which is the interaction between assets with independent cash flows. The intuition is that once there is a shock on one assets' dividend, its share s in the market portfolio must change accordingly. The movement of s alters the investors preference by changing the expectation of the marginal consumption. Further, the investor rebalances their portfolio. Due to the general equilibrium, all assets prices are affected.

I summarize the explanation of the interaction between the share s and the price dividend ratio to conclude this section. Roughly, the path of s from 0 to 1 can be divided into 3 phases.

In the first phase, s is small. The increase of s is along with the diversification the market portfolio. The risk of the holding market portfolio is decreasing. Investors are less willing to delay their consumption by buying assets. On the other hand investors realise that when s is large, the asset 1 takes more share in the market portfolio. The investor's consumption relies more on asset 1, due to the decreasing property of the marginal utility, the asset 1 become less attractive.

In the second phase, when s is in the middle, the precautionary saving effect becomes moderate when the investors have fully diversified their portfolio. The power from the market β effect also becomes weak because overall the marginal utility of the aggregate consumption is relatively small. Hence, the function ceases decline around the point $s = 0.75$.

In the third phase, when s is large and close to 1, the market portfolio enters a process of anti-diversification. The asset 1 becomes attractive in the sense that, with more risk, investors want to delay their consumption by holding more assets. Nonetheless the market β effect becomes strong again and makes the asset 1 less attractive to overcome the precautionary saving effect.

3.4.4 Asset Returns

This section studies the relation between share s and the asset returns. Although the price dividend ratio can reveal the relation to some extent, it is not simply the reciprocal of the asset return. By definition, $E_0(R_{1,1}) = E_0[(P_{1,1}+D_{1,1})/P_{1,0}]$. With the similar methods used before, I can analyse the impact of s on the expected return and the return variance of the asset. The identity of the expected return can be rearranged into,

$$E_0(R_{1,1}) = E_0\left[\frac{D_{1,0}G_{1,1}}{P_{1,0}}\right] + E_0\left[\frac{D_{1,0}}{P_{1,0}}\frac{P_{1,1}}{D_{1,1}}G_{1,1}\right] \tag{3.32}$$

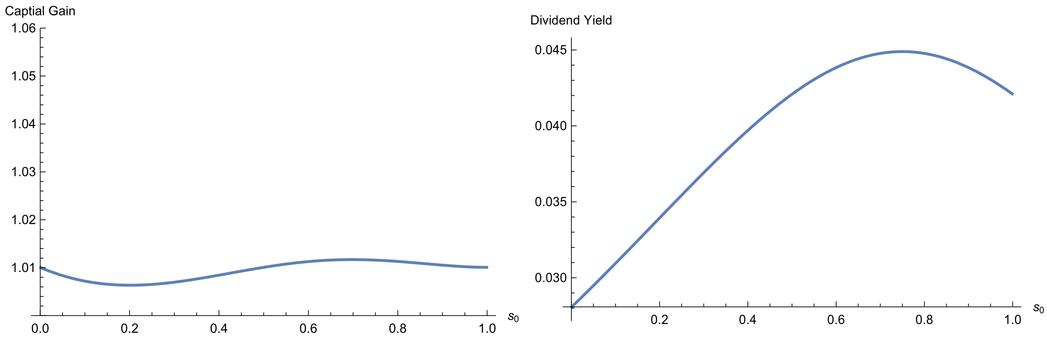
The equation (3.32) decomposes the asset return into the expectation of dividend yield and capital gain respectively. Again, the purpose is to express it as a function of s_0. The dividend yield simply represents the product of the reciprocal of the price dividend ratio and $E(G_{1,1})$. The proposition 3.2 has derived the former. The latter is the expectation of the log-normal variable, which is a constant. For capital gain, I forward the approximated analytical solution equation (3.10) for one period and substitute the dynamics of s in equation (3.4) into it.

$$\begin{aligned}\frac{D_{1,0}}{P_{1,0}}\frac{P_{1,1}}{D_{1,1}}G_{1,1} = &\left[\frac{\beta}{1-\beta}+\frac{1}{2}\sigma^2(1-s_0)(1-2s_0)\frac{\beta}{(1-\beta)^2}\right]^{-1}\times \\ &\left[\frac{\beta}{1-\beta}+\frac{1}{2}\sigma^2(1-s_1)(1-2s_1)\frac{\beta}{(1-\beta)^2}\right]G_{1,1}\end{aligned} \tag{3.33}$$

$$s_1 = \frac{s_0G_{1,1}}{s_0G_{1,1}+(1-s_0)G_{2,1}} \tag{3.34}$$

Substitute equation (3.34) back into equation (3.33), the capital gain is a function of s_0, $G_{1,1}$ and $G_{2,1}$. The second order Taylor expansion enables us to calculate the expectation. Admittedly, this derivation applies an approximation on another approximation. I believe it is acceptable to use for understanding the intuition.

Figure (3.6) plots this function of s_0 separately for capital gain and dividend yield. It is not surprising that overall the expected return mirrors the price dividend ratio



(a) Capital Gain **(b)** Dividend Yield

Fig. 3.6 Expected Return Decomposition

function. In addition, the dividend yield in figure (3.6b) dominates the interaction between the share s and asset return. Moreover, it generally mirrors the behaviour of price dividend ratio which is explored in the last section.

3.4.5 Market Portfolio

In the last part of the analysis, the attention is on the market portfolio in this economy. In the first thinking, one might think that the market portfolio in this multiple assets economy is exactly identical to the one tree case. In fact, the argument is correct about the price dividend ratio. In equilibrium, the market portfolio pays the aggregate consumption as its "dividends". According to the pricing equation (3.3), the price consumption ratio simply collapse to,

$$P_{M,0} = C_0 \sum_{t=0}^{\infty} \beta^{\tau} E_0 \left(\frac{C_t}{C_t} \right) \tag{3.35}$$

$$\frac{P_{M,0}}{C_0} = \frac{\beta}{1-\beta} \tag{3.36}$$

which is identical to the one tree case. However, the expected asset return and the variance of the asset return are different from the one tree case. Proposition 3.3 illustrates this situation.

Proposition 3.3 *In the modelled economy, the expected return and the variance of the asset return of the market portfolio are given by*

$$E(R_{M,1}) = \frac{1}{\beta}\left[s_0 E(G_{1,1}) + (1-s_0) E(G_{2,1})\right] \tag{3.37}$$

$$Var(R_{M,1}) = \frac{1}{\beta^2}\left(s_0^2\sigma_1^2 + (1-s_0)^2\sigma_2^2\right) \tag{3.38}$$

Proof. See proof in the appendix 3.6.6.

Indeed, the price consumption ratio is constant and irrelevant to our state variable s. Nonetheless, the expected return is slight different. If two asset's dividend growth is symmetric, the s is cancelled out in the equation (3.37). If two assets are asymmetric, when the share s moves from 0 to 1 the weights assigned to two assets rebalance, which leads to a change of expected return of the market portfolio.

Moreover, for the variance of the market portfolio return, figure 3.7 shows the return's variance as a function of the share s. An intuitive explanation of this behaviour is the diversification effect. There are only 2 independent assets in the market. The market bear the minimal risk when there is fully diversification in $s_0 = 0.5$. Even in the symmetric case of $\sigma_1^2 = \sigma_2^2$ the share s plays a relative significant role. In the asymmetric case, the share s has more significant impact on the market volatility as seen in the yellow line in the figure 3.7. Additionally, in the uneven case, the market has the smallest volatility when the relative stable asset dominates the portfolio.

The additional dynamics and volatilities are raised from the non-linear structure of the pricing function. In fact, if I alter the assumption of log utility to power utility the function structure becomes more non-linear. With CRRA preference, even the dividend yield is no longer a constant. It becomes

$$\frac{P_{M,0}}{C_0} = \sum_{t=1}^{\infty}\beta^t E\left(\left[s_0\prod_{\tau=0}^{t-1}G_{1,\tau} + (1-s_0)\prod_{\tau=0}^{t-1}G_{2,\tau}\right]^{1-\gamma}\right) \tag{3.39}$$

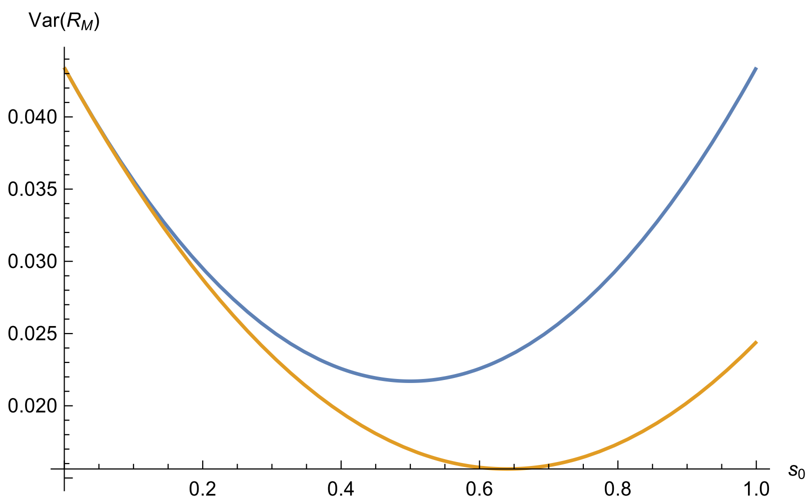


Fig. 3.7 Variance of the Market Portfolio Return
Blue Curve: symmetric case $\sigma_1^2 = \sigma_2^2 = 0.04$ and $\beta = 0.96$
Yellow Curve: asymmetric case $\sigma_1^2 = 0.01, \sigma_2^2 = 0.04$ and $\beta = 0.96$

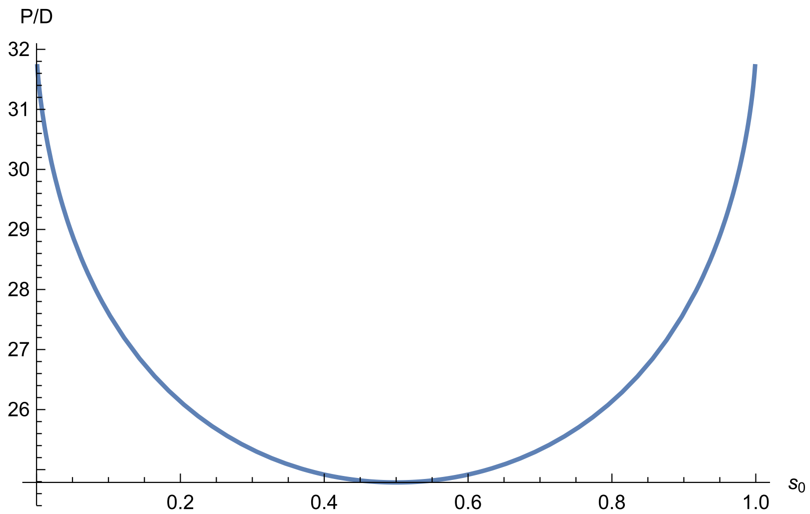


Fig. 3.8 Price-Consumption Ratio in CRRA Utility Model
$\sigma_1^2 = \sigma_2^2 = 0.04$, $\gamma = 2$ and $\beta = 0.96$

Again, the computational software offers a numerical solution for this function. Figure 3.8 plots the price consumption ratio for the market portfolio. Unlike the log utility framework, the price-consumption ratio varies over s when I set risk aversion parameter to 2. Imaginably, higher value of risk aversion parameter brings larger dynamics and volatility to not only the price consumption ratio but also the return and return variance.

Generally, this section shows that the market portfolio in the multiple assets case is not equivalent to the one tree in the standard model. Even in case which assets are with symmetric dividend distribution, the share s plays a significant role in the pricing mechanism due to the diversification in the market.

3.5 Concluding Remarks

This paper presents a two assets Lucas tree model. I introduce the state variable s as the share of one asset's dividend in aggregate consumption. The impact of the movement of this share s on the pricing mechanism of the assets is of interest. The purpose is to unveil the pricing mechanism of the investor in this exchange economy with multiple assets.

Beyond the one Lucas tree model, this simple model finds the price dividend ratio of one asset depends on various of factors. First of all, the discount factor varies over time as the aggregate volatility moves with the share s. Thus the pricing mechanism to a large extend is affected by the market precautionary saving effect.

Another important factor is the market β effects, which captures the covariance between the dividend growth of the asset and the marginal utility of the aggregate consumption. In the one tree model, we know that the dividend of asset is negatively related to the marginal utility. Correspondingly, the market β effect captures the relation between the share s and and this substitution effects. The larger is the s, the stronger is this substitution effect.

The further research can join this multi assets framework with the rational bubble model. Froot and Obstfeld (1989) and Lansing (2010) show that the model with intrinsic bubble enables us to add a bubble component in the pricing function. In the framework of this chapter, presumably the bubble term in one asset spreads into the whole market through the channels identified.

Another research direction is to extend the exchange economy into a production economy. Some progress have been made in the asset pricing in production economy like shown in Jermann (1998) and Campbell (1986). Heterogeneities lead to a different structure of the consumption smoothing process. Chapter 6 follows this idea and extends the framework into the production economy.

3.6 Appendix

3.6.1 Euler Equation in CCAPM

The representative consumer faces a standard infinite horizon utility maximization problem given by,

$$J_t = \underset{C}{Max} \quad E_t \left[\sum_{\tau=t}^{\infty} \beta^{\tau-t} U(C_t) \right] \tag{3.40}$$

subject to budget constrain

$$S_{1,t+1}P_{1,t} + S_{2,t+1}P_{2,t} = S_{1,t}(P_{1,t} + D_{1,t}) + S_{2,t}(P_{2,t} + D_{2,t}) - C_t \tag{3.41}$$

$$C_t > 0, \quad S_t > 0 \tag{3.42}$$

$$P_t > 0, \quad D_t > 0 \tag{3.43}$$

S's are the shares of two assets holding by the representative consumer. Since S_1 and S_2 are symmetric, I arbitrarily reckon S_1 as the state variable. S_2 and C are the choice variables.

Further, in this general set-up, the Bellman equation can be written as

$$J_t(S_{1,t}) = \underset{C}{Max} \quad U(C_t) + \beta E_t(J_{t+1}(S_{1,t+1})) \tag{3.44}$$

FOC of consumption C_t gives,

$$U'(C_t) = \beta E_t \left(\frac{\partial J_{t+1}}{\partial S_{1,t+1}} \frac{1}{P_t} \right) \tag{3.45}$$

The analogous FOC of $S_{2,t}$ gives,

$$\frac{\partial U(C_t)}{\partial S_{2,t}} + \beta E_t \left[\frac{\partial J_{t+1}(S_{1,t+1})}{\partial S_{1,t+1}} \frac{\partial S_{1,t+1}}{\partial S_{2,t}} \right] = 0 \tag{3.46}$$

For this simple setup, the model does not have enough information to determine the optimal choose for $S_{2,t}$ since there is only one equation of budget constrain. However, this does not prevent us to obtain the Euler equation for asset one.

Envelope theorem of S_1 offers

$$\frac{\partial J_t}{\partial S_{1,t}} = \beta E_t \left[\frac{\partial J_{t+1}}{\partial S_{1,t+1}} \left(\frac{P_t + D_t}{P_t} \right) \right] \tag{3.47}$$

Combine the two, yield

$$\frac{\partial J_t}{\partial S_{1,t}} = U'(C_t)(P_t + D_t) \tag{3.48}$$

Further, I forward 1 period and substitute back into (3.47). I obtain

$$U'(C_t)(P_t + D_t) = \beta E_t \left[U'(C_{t+1})(P_{t+1} + D_{t+1}) \left(\frac{P_t + D_t}{P_t} \right) \right] \tag{3.49}$$

$$U'(C_t) P_t = \beta E_t \left[U'(C_{t+1})(P_{t+1} + D_{t+1}) \right] \tag{3.50}$$

which is the Euler equation for asset one. Due to the symmetry, it is straight forward to obtain the Euler equation for asset two.

3.6.2 Proof of Normally Distributed Growth Ratio $G_{2,t}/G_{1,t}$

Proof. This section offers a discussion of product of $\prod (G_2/G_1)$. Firstly, I show that $G_{2,t}/G_{1,t}$ follows log normal distribution.

By definition, I have

$$G_{i,t} \sim LN\left(\mu_i,\ \sigma_i^2\right) \tag{3.51}$$

It is equivalent to

$$\log\ G_{i,t} \sim N\left(\mu_i,\ \sigma_i^2\right) \tag{3.52}$$

Naturally, I have

$$\log G_{2,t} - \log G_{1,t} \sim N\left(\mu_2 - \mu_1,\ \sigma_2^2 + \sigma_1^2\right) \tag{3.53}$$

$$\log\left(\frac{G_{2,t}}{G_{1,t}}\right) \sim N\left(\mu_2 - \mu_1,\ \sigma_2^2 + \sigma_1^2\right) \tag{3.54}$$

$$\frac{G_{2,t}}{G_{1,t}} \sim LN\left(\mu_2 - \mu_1,\ \sigma_2^2 + \sigma_1^2\right) \tag{3.55}$$

Further, if I assume $\mu_1 = \mu_2$, I have

$$\prod_{\tau=0}^{t-1}\left(\frac{G_{2,\tau}}{G_{1,\tau}}\right) \sim LN\left(0,\ t\left(\sigma_2^2 + \sigma_1^2\right)\right) \tag{3.56}$$

Q.E.D. ■

3.6.3 $\{s_t\}_0^\infty$ Under the Symmetric Assumption

By definition of the expectation of log normal variable, I have

$$E_0\left[\prod_{\tau=0}^{t-1}\left(\frac{G_{2,\tau}}{G_{1,\tau}}\right)\right] = \exp\left(\frac{t\left(\sigma_2^2 + \sigma_1^2\right)}{2}\right) > 1 \tag{3.57}$$

Accordingly, the process of s_t,

$$s_t = \frac{s_0}{s_0 + (1 - s_0)\prod_{\tau=0}^{t-1}\left(G_{2,\tau}/G_{1,\tau}\right)}, \tag{3.58}$$

will not be mean reverting.

3.6.4 Remarks on the equation (3.6)

This section points out the difficulties to analytically solve the expectation of equation (3.6). To simplify the calculation, I take the first period in the equation (3.6) for example.

$$s_1 = \frac{s_0}{s_0 + (1 - s_0)(G_{2,1}/G_{1,1})} \tag{3.59}$$

By definition and the property of log-normal distribution, we have

$$\frac{G_{2,1}}{G_{1,1}} \sim LN\left(\mu_2 - \mu_1,\ \sigma_2^2 + \sigma_1^2\right) \tag{3.60}$$

To simplify notation, I use the transform of variables $G_{2,1}/G_{1,1} \equiv G$, $\mu_2 - \mu_1 \equiv \mu$ and $\sigma_2^2 + \sigma_1^2 \equiv \sigma^2$. Accordingly, equation (3.59) becomes

$$s_1 = \frac{s_0}{s_0 + (1 - s_0)\, G} \tag{3.61}$$

Here, s_0 is a parameter and s_1 is a function of log-normally distributed random variable G. It is straight forward that s_1 is also a random variable with its own density function. In fact, this density function is relatively painless to obtain.

Firstly, since $G \in (0,\ \infty)$, we have $s_1 \in (0,\ 1)$. As a result, we have $Prob\,[s_1 > 1] = 0$ and $Prob\,[s_1 < 0] = 0$.

Secondly, I deal with the density within range $s_1 \in (0,\ 1)$. According to equation (3.61), we have

$$G = \frac{s_0}{1 - s_0}\left(\frac{1}{s_1} - 1\right) \tag{3.62}$$

Clearly, s_1 is negatively related to G. Hence we have

$$Prob\,[s < s_1] = Prob\left[g > \frac{s_0}{1 - s_0}\left(\frac{1}{s_1} - 1\right)\right] \tag{3.63}$$

$$= 1 - Prob\left[g \leq \frac{s_0}{1 - s_0}\left(\frac{1}{s_1} - 1\right)\right] \tag{3.64}$$

According to the probability density function of the log-normal distribution, we can write the cumulative distribution function for s_1,

$$Prob\left[s < s_1\right] = 1 - \int_0^{\frac{s_0}{1-s_0}\left(\frac{1}{s_1}-1\right)} \frac{1}{g\sigma\sqrt{2\pi}} \exp\left[-\frac{(\ln g - \mu)^2}{2\sigma^2}\right] dg \tag{3.65}$$

Finally, due to the relation between PDF and CDF, we have the PDF of random variables s_1 as

$$\begin{aligned} & d\left\{1 - \int_0^{\frac{s_0}{1-s_0}\left(\frac{1}{s_1}-1\right)} \frac{1}{g\sigma\sqrt{2\pi}} \exp\left[-\frac{(\ln g - \mu)^2}{2\sigma^2}\right] dg\right\} \Big/ ds_1 \\ & = \frac{1}{s_1\left(1-s_1\right)\sigma\sqrt{2\pi}} \exp\left[-\frac{\left[\ln\left(\frac{s_0}{1-s_0}\left(\frac{1}{s_1}-1\right)\right)-\mu\right]^2}{2\sigma^2}\right] \end{aligned} \tag{3.66}$$

Unfortunately, when comes to the expectation of this random variable s_1, the best we can do is to implement the definition of the mathematical expectation as

$$E\left[s_1\right] = \int_0^1 s_1 \times \frac{1}{s_1\left(1-s_1\right)\sigma\sqrt{2\pi}} \exp\left[-\frac{\left[\ln\left(\frac{s_0}{1-s_0}\left(\frac{1}{s_1}-1\right)\right)-\mu\right]^2}{2\sigma^2}\right] ds_1 \tag{3.67}$$

To the best of my knowledge, there is no closed form solution for this integral, which leave the expectation unsolvable for the analytical solution. Therefore, I resort to the approximated solution of this expectation.

3.6.5 Proof of Proposition 3.1

This section provides proof to proposition 3.1.

Proof. With the assumption of symmetric drift, for simplification, I define,

$$\exp(g_t) \equiv \frac{G_{2,t}}{G_{1,t}} \tag{3.68}$$

$$\sigma^2 \equiv \sigma_1^2 + \sigma_2^2 \tag{3.69}$$

Accordingly, I have $g_t \sim N(0,\ \sigma^2)$. By definition of expectation $E_t(s_{t+1})$ is

$$E_t(s_{t+1}) = E\left(\frac{s_t}{s_t + (1 - s_t)\exp(g_t)}\right) \tag{3.70}$$

$$= \int_{-\infty}^{\infty} \frac{s_t}{s_t + (1 - s_t)\exp(g)} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{g^2}{2\sigma^2}\right) dg \tag{3.71}$$

In some context, this also called law of the unconscious statistician. The last term in equation (3.71) is the density function of the normal variable g_t.

Since the expectation of future period s_t involves the accumulation of the i.i.d random variable, I have

$$E_t(s_{t+\tau}) = E_t\left(\frac{s_t}{s_t + (1 - s_t)\exp\sum_{k=0}^{\tau} g_k}\right) \tag{3.72}$$

$$= \int_{-\infty}^{\infty} \frac{s_t}{s_t + (1 - s_t)\exp(g)} \frac{\exp\left(-\frac{g^2}{2\tau\sigma^2}\right)}{\sqrt{2\pi\tau\sigma^2}} dg \tag{3.73}$$

The final term in (3.73) is the density function of the sum of τ normal variables. Obviously, this sum is normally distributed with 0 mean and variance $t\sigma^2$. Thus, the function of price-dividend ratio, as a geometric decayed sum, can be expressed as follows.

$$\frac{P_{1,t}}{D_{1,t}} = \sum_{\tau=0}^{\infty} \beta^\tau \int_{-\infty}^{\infty} \frac{1}{s_t + (1 - s_t)\exp(g)} \frac{\exp\left(-\frac{g^2}{2\tau\sigma^2}\right)}{\sqrt{2\pi\tau\sigma^2}} dg \tag{3.74}$$

Q.E.D. ■

3.6.6 Proof of Proposition 3.3

This section provides proof to proposition 3.3.

Proof. For the expectation we have

$$
\begin{aligned}
E\left(R_{M,1}\right) &= E\left[\frac{C_1+P_{M,1}}{P_{M,0}}\right] && (3.75)\\
&= E\left[\frac{C_0\left[s_0G_{1,1}+\left(1-s_0\right)G_{2,1}\right]}{P_{M,0}}+\frac{C_0}{P_{M,0}}\frac{P_{M,1}}{C_1}\left[s_0G_{1,1}+\left(1-s_0\right)G_{2,1}\right]\right]\\
&= \frac{1}{\beta}\left[s_0E\left(G_{1,1}\right)+\left(1-s_0\right)E\left(G_{2,1}\right)\right] && (3.76)
\end{aligned}
$$

due to the fact that

$$
\begin{aligned}
C_1 &= C_0\left[s_0G_{1,1}+\left(1-s_0\right)G_{2,1}\right] && (3.77)\\
\frac{P_{M,0}}{C_0} &= \frac{P_{M,1}}{C_1}=\frac{\beta}{1-\beta} && (3.78)
\end{aligned}
$$

Similarly, for the variance we have

$$
\begin{aligned}
Var\left(R_{M,1}\right) &= \left(\frac{C_0}{P_{M,0}}\right)^2 Var\left[\left[s_0G_{1,1}+\left(1-s_0\right)G_{2,1}\right]\left(1+\frac{P_{M,1}}{C_1}\right)\right] && (3.79)\\
&= \frac{1}{\beta^2}\left(s_0^2\sigma_1^2+\left(1-s_0\right)^2\sigma_2^2\right) && (3.80)
\end{aligned}
$$

Q.E.D. ■

Chapter 4

Beliefs Driven Secular Stagnation

4.1 Abstract

This chapter constructs an endogenous growth model to study the US economy before the 2008 great recession and the recovery period that followed. In particular, it explores the secular stagnation hypothesis and its implications for asset pricing. The model features constant return to scale in capital and extrinsic randomness, which imply multiple perfect foresight balanced growth paths. In this setup a change in agents' expectations may trigger persistent slumps, low interest rates and elevated risk premia, consistent with the recent US experience.

4.2 Introduction

After the great recession in 2008, there is increasing evidence of a decline in global long-run growth. For example, Fernald and Jones (2014) document that the economic growth in the US is decelerating along with growth in educational attainment, R&D intensity and population. Antolin-Diaz et al. (2017) extend this narrative to other countries. This persistent reduction in long-run growth rates has led to discussion of the secular stagnation hypothesis. For example, Gordon (2015) indicates that the economic engine on the supply side is gradually flaming out. He argues that the pace of innovation is slowing down and the labour force participation rate is decreasing permanently. Meanwhile, Summers (2015a) focuses on the demand side. He points out that the natural interest rate has declined but cannot be implemented because of the zero lower bound. This leads to a deviation of employment from the full employment level.

This chapter builds an endogenous growth model in which secular stagnation may occur as a result of beliefs-driven self-fulfilling equilibria. The key setups in the model are (1) “AK” linear production, (2) the assumption that investment can feed back to productivity and (3) extrinsic randomness. In theory, the model is able to account for phenomena such as persistent slow growth in the economy, low investment levels, trended decreases in the risk-free rate and counter-cyclicality of the risk premium.

Firstly, since the aim is to study long-run growth, I allow for constant return to scale in capital to assure endogenous growth in the economy. Secondly, to study secular stagnation, I need a model which generates self-reinforcing slow growth. The economic growth literature on the “poverty traps” is inspiring. For example, Azariadis and Stachurski (2005) survey many models with multiple equilibria. A widely used setup is a direct linkage between capital and productivity. The investment feeds back into productivity through channels like complementarities and externalities. Azariadis and Drazen (1990) consider the spillovers from human capital accumulation processes. Hence, they assume that productivity is a function of capital, which consists of both human and physical capital. Here, I assume that technology is a threshold function of the investment-capital ratio. Technology jumps to a new level when the investment capital ratio reaches the “threshold”. On the contrary, a negative demand shock on investment leads to weak productivity. On the other hand, the firms set their investment according to productivity to maximise their values. Like in the literature, this assumption offers multiplicity.

Lastly, to enable the shift between different equilibria, I introduce regime switching sunspots. They alter the beliefs and activate shifts among multiple balanced growth paths (BGPs) in the economy. The transition is governed by an exogenous

Markov chain. Similar setups are found in Benigno and Fornaro (2016) and Christiano and Harrison (1999). The former studies a Keynesian growth model with nominal rigidities, zero lower bound in interest rate and confidence shocks. The difference between their and my paper is that this one considers the real variables without nominal rigidities and balanced growth paths instead of the steady state equilibrium in their model. The latter studies the implication of sunspot equilibria, cyclical and chaotic equilibria in a real business cycle model for automatic stabilizer tax systems. However, this chapter does not consider governance.

The rest of the setups are standard in the models of a production economy with complete markets. The model takes a general equilibrium approach. The firms are owned by households. I assume that the firm faces the adjustment costs of investment. The optimal level of investment pins down the balanced growth path (BGP). On the consumer side, the model follows the standard consumption-based capital asset pricing model (CCAPM). As usual, under the complete market assumption and no arbitrage condition, the household's problem determines the unique stochastic discount factor (SDF).

Importantly, the model has implications for movements of the risk-free rate and asset prices. Roughly speaking, in the US, the risk-free rate drops dramatically in recent decades. Since there is no significant change in the risk asset return, the risk premia expand accordingly. Section 4.3 elaborately describes the observations in these variables. Nonetheless, the standard real business cycle (RBC) model has difficulties in explaining this. Among many others, Gourio (2012) and Gabaix (2012) consider the massive "disaster" shock in the RBC model and successfully generate the time-varying risk premia. My results are similar to theirs in term of the large downward shift in economic growth. Likewise, my model can account for the movement of risk premia and the risk-free rate.

Intuitively, stagnant period in the modelled economy can be described as follows. The confidence shock hit the economy, therefore investors and firms are pessimistic about economic performance. Firms cut their investments and hence investment cannot maintain productivity at a healthy level through the linkage between investment and technology. In return, this reinforces the firms' low investment strategy. The growth is trapped. Due to pessimism, consumers cannot expect any high growth in consumption in the future. Through the pricing mechanism, the asset return is low. To summarise, the model generates a dynamic system with (1) sustainable low growth and depressing investment, (2) persistent fall in productivity and (3) trended decrease in the risk-free rate and (4) widening risk premia in the corresponding period.

There are two groups of literature related to this paper. Firstly, researches on secular stagnation are expanding in recent years. Besides Benigno and Fornaro (2016), there are many studies on the behaviours of interest rates and zero lower bound, including Eggertsson et al. (2017), Pescatori and Turunen (2016), Gruber and Kamin (2016), Favero et al. (2016) and Sajedi and Thwaites (2016). These models mainly focus on the phenomenon that a low or negative natural rate of interest leads to a chronically binding zero lower bound (ZLB). Essentially, they formalise the arguments of Summers (2015a) in New Keynesian overlapping generation (OLG) models. They show the mechanisms that enable factors such as a slowdown in population growth, an increase in life expectancy, an increase in income inequality and a fall in the price of investment goods to reduce the natural interest rate. In contrast, this chapter does not consider the zero lower bound since I only consider the real variables. More precisely, the model in this paper examines the trended long-run decreases in the real risk-free rate, yet without explaining the bounded period of the nominal risk-free rate. Blanchard et al. (2017) argue that a low expectation of long-run productivity growth can affect output and inflation in the short run. Empirically, they run regression on data of forecasts of economic growth and forecast errors. Theoretically, their paper uses similar logics to my model. Both consumers and firms, when they are pessimistic about future growth, tend to revise their behaviour. Consumers modify their expectations of permanent income and firms change their investment plans. Hence, the downturn in the economy is self-reinforcing. However, their model mainly explores the mechanism in terms of the unemployment, which is different from this paper. Eichengreen (2017) makes a comparison between the Great Recession of the 1930s and the recent one in 2008. He compares the real GDP level of the year of crisis and 8 years afterwards and concludes that the recovery after the 1930s was faster than today's. He argues that this is due to the fast adoption of new technologies such as electricity and a national highway system and the large demand to those technologies due to the incoming war. However, we do not see analogous processes today.

Next, this paper follows the model structure used in the literature in many aspects. Firstly, researches studying constant return to scale (CRS) model are abundant. Benhabib and Farmer (1994) explores two sources to generate CRS namely input externalities and monopolistic competition. Their model features indeterminate equilibria. On one hand, the paper establish theoretical foundations for successive studies which reckon social technology is linear in capital. On the other hand, in a broad sense, it is related to this paper since its model is consistent with the existence of equilibria that are driven by shocks in agents beliefs. A recent work in this framework comes from Bambi and Venditti (2016). They consider a neoclassical growth model to study endogenous fluctuations and sunspot equilibria based on self-fulfilling expectations. Importantly, their model admits sunspot fluctuations around an unique deterministic BGP without transitional dynamics. The condition of absence of transitional dynamics is similar with this chapter. However, their discussion particularly centres on the government's consumption taxation policy and its relation to endogenous fluctuations. Methodologically, their paper considers sunspots equilibria which asymptotically converge to the unique balanced growth path. Meanwhile, the sunspots in this paper are regime-switching sunspots.

Secondly, many studies, such as Cochrane (1991), Jermann (1998), Boldrin et al. (2001) and Campbell (2003), have examined firms' investment behaviour in the production economy. These studies pay attention to the relations between firms' investments, stock returns and macroeconomic fluctuations. The general equilibrium structure of this paper follows the literature in this area. The firms are owned by the households. In the equilibrium, the investor holds all the stock and consumes the dividends. Given the pricing kernel, the firm decides on the investment for next period. Kogan and Papanikolaou (2012) survey the research in this field. In terms of methodology, some closely related papers are Kogan (2001) and Eberly and Wang (2009). They solve the central planner's problem in the continuous time framework. There is also literature on endogenous growth such as Fatas (2000) and Azariadis and Drazen (1990). The former considers the AK framework with cyclical shocks. It is, in some sense, a version of my model without investment adjustment costs. However, the paper focuses on explaining the persistent fluctuations in the output. The latter offers us the inspiration of the assumption between technology and investment. Nonetheless, it mainly inspects the "poverty traps" by linking human capital accumulation processes to productivity. Regarding sunspots, Benhabib and Farmer (1999) survey the literature that use various structures of the production function, nonlinear accumulation of capital and extrinsic randomness to handle the multiple equilibria.

This paper is composed as follows. The next section describes the empirical observations that this paper wants to account for. Section 4.4 presents the baseline model and its solution. In section 4.5, I introduce the assumption that endogenises the technology. Section 4.6 constructs sunspot equilibria and illustrates the implications in terms of growth and asset pricing. Section 4.7 shows the calibration and indicates the limits of this model. Section 4.8 concludes the paper.

4.3 Motivating Observations

This section offers observations for which this model mainly wants to account.[1] Figures 4.1 and 4.2 display the historical data on risk premium, risk-free rate and per capita real GDP growth in the US. In figure 4.1, the dark bars are risk-free rates. The grey bars on top indicate the risk premium. Two parts add up to the risky asset return. Figure 4.2 shows the per capita real growth rate in the US in past decades. In addition, I construct the investment to GDP ratio using the difference between 1 and the aggregate consumption's share in GDP. Figure 4.3 plots this

[1]See data descriptions in 4.9.10

constructed investment to GDP ratio in recent decades. The red shapes in the three figures roughly indicate the patterns that the model attempts to account for. Those phenomena are: stagnant growth accompanied by weak investments, a dramatic fall in the risk-free rate and an expansion of the risk premia.

Roughly, I split the movements of these variables into 2 periods by the year 2000. Before 2000, growth rate on average is approximately around 2.5%. In addition, the risk premium stays relatively stable. In the 2000s, the growth rate reduced to around 1%, especially when we look at the period after the crisis. In figure 4.3, there is a relative clear decrease trend in the investment to GDP ratio. Visually, it also seems there is a structure break around 2001. In the model, these could be generated by a switch between two balanced growth paths (BGPs) due to the confidence shock, or in other words, sunspots shock. The confidence shock hit the economy, and agents are pessimistic about growth in the future. Accordingly the best strategy for them is to reduce their investment. Therefore, by assumption, this downturn in investment affects productivity. In return, low productivity confirms that low investment plan is optimal. Hence, the weak growth in the future is actually reinforced.

Meanwhile, the risk-free rate falls noticeably and the risk premium expands. These observations directly link to the well-known risk premium puzzle and risk-free rate puzzle, which can be dated back to Mehra and Prescott (1985). The conventional model predicts that the risk-free rate is the reciprocal of the expected intertemporal marginal rate of substitution of consumers' utility, namely $E_t\left[U'\left(C_{t+1}\right)/U'\left(C_t\right)\right]$. Since the consumption is relatively stable in the data, the traditional model does not expect large volatility in the risk-free rate. In the model, these could also be the result of pessimism. Since consumers cannot expect any high growth in consumption in the future, the pricing mechanism indicates that the risk-free will be low. In addition, the adjustment cost of investment and the structure of the stochastic regimes switching framework also helps to explain the expansion of the risk premia.

Specifically, I use the period between 1992 to 2001 as a high growth period with relatively high risk-free rate and small risk premium. By contrast, the years 2005 to 2014 are used as the low growth period with low risk-free rate and large risk

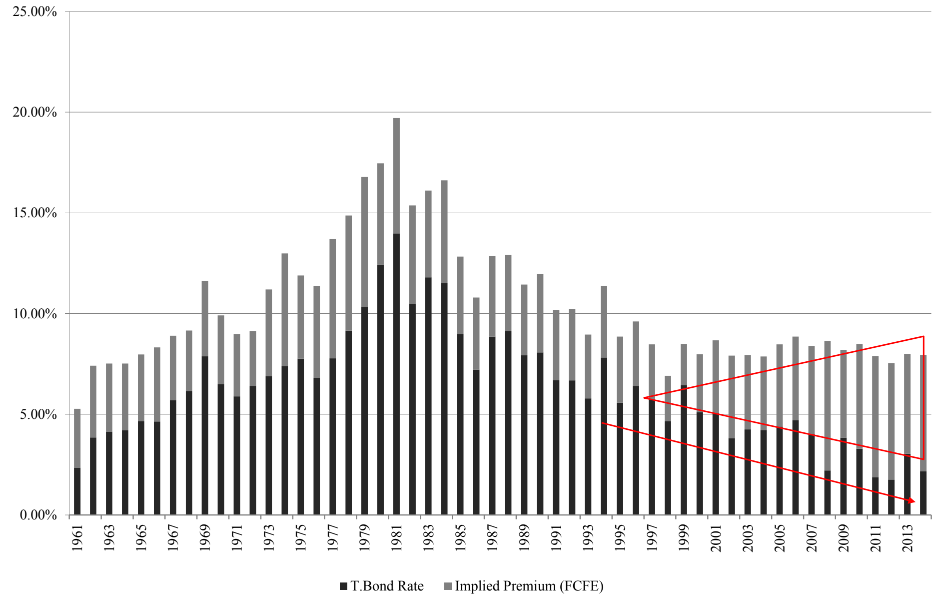


Fig. 4.1 Historical Risk Premium and risk-free Rate in the US

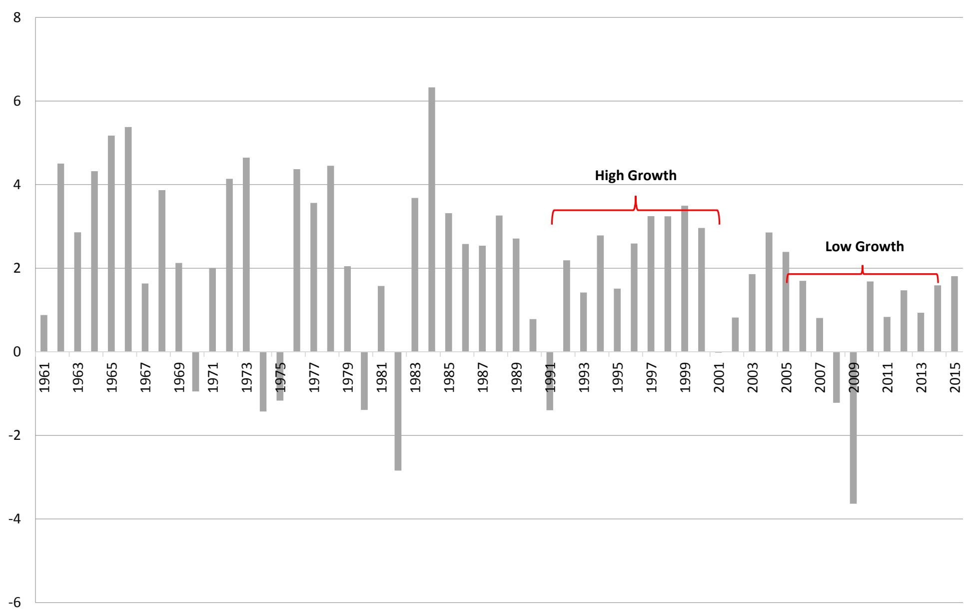


Fig. 4.2 Historical per capita GDP growth in the US

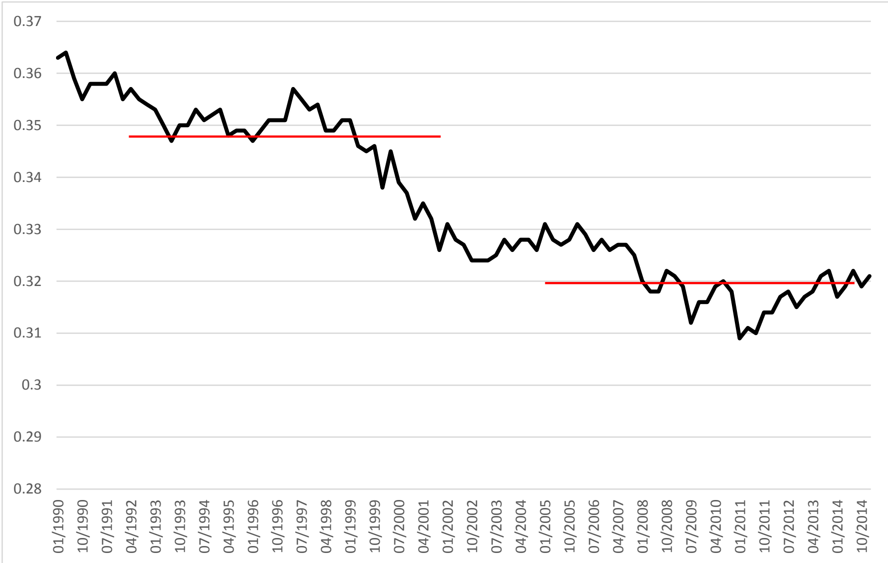


Fig. 4.3 Constructed Investment to GDP Ratio in the US

premium. Later, calibrations are carried out based on the data collected from these two periods.

4.4 Baseline Model and Its Solution

In this section, I solve the baseline model to establish the relation between a firm's investment and technology. The model describes an economy with one productive sector. Time t discretely runs from 0 to infinity. There are a large number of identical firms and consumers. I consider a state variable s_t. Let $s^t = (s_0, s_1, \ldots, s_t)$ be the notation of the history of the state variable. The probability of history s^t is denoted by $\mu(s^t)$. All endogenous variables introduced later are functions of the histories s^t. The production function is linear as $Y(s^t) = A(s^t)K(s^t)$. Y, K and A denote the output, the capital stock and the technology scale factor respectively. A firm uses its operation profit to pay the dividends as $D(s^t) \equiv A(s^t)K(s^t) - I(s^t)$, where I is the investment. Additionally, I restrict dividends to be positive, namely $D(s^t) > 0$. The representative firm maximises its stock value, represented by the discounted cash flows,

$$V\left(s^{t}\right)=\underset{I}{Max} \quad E_{t}\left[\sum_{\tau=t}^{\infty} \beta^{\tau-t} \frac{\Lambda\left(s^{\tau}\right)}{\Lambda\left(s^{t}\right)} D\left(s^{\tau}\right)\right] \tag{4.1}$$

subject to the constrains

$$\frac{K\left(s^{t+1}\right)}{K\left(s^{t}\right)}=1-\delta+\phi\left(i\left(s^{t}\right)\right) \tag{4.2}$$

$$A\left(s^{t}\right)>0, \quad K\left(s^{t}\right)>0, \quad \Lambda\left(s^{t}\right)>0 \tag{4.3}$$

$$V\left(s^{t}\right)>0, \quad I\left(s^{t}\right)>0 \tag{4.4}$$

$$K\left(s^{0}\right) \text{ is given.} \tag{4.5}$$

Where E_t is the mathematical expectation based on information in time t. $\beta \in (0,\ 1)$ is the time preference parameter. $\delta \in (0,\ 1)$ is the capital-depreciation rate. For simplicity, I use the notation for investment capital ratio $i \equiv I/K$. β and Λ together constitute the discount factor. The firm is a price taker. It takes the discount factor as given. The equation (4.2) is the capital accumulation condition. The function $\phi(\cdot)$ captures the effectiveness in converting investment to capital inputs. For later reference, I name this function the efficiency function of investment. To understand this function, a good example is the extreme case of $\phi(i) = i$. The capital accumulation condition becomes $K_{t+1} = (1-\delta)K_t + I_t$. Here, the investment has no adjustment costs and is completely efficient. However, I restrain the function by $\phi(i) > 0$, $1 > \phi'(i) > 0$ and $\phi''(i) \leq 0$. These constrains capture the convexity of adjustment costs, which follows the convention in the literature.[2] For the firms, the more they invest, the more they cost. I assume that the efficiency function is homogeneous of degree one in I and K to follow the proposition of Hayashi (1982). This proposition simplifies the model and helps to derive the findings of asset prices.

[2]See details appendix 4.9.1.

There is a broad range of literature discussing the linear production and constant return to scale. Here I adapt intuition used in Azariadis and Drazen (1990). There is a distinction between the private and public factors in production as introduced by Romer (1986). The private factor is controlled by individual firms. The public factor is not controlled by any specific producer. In the production process, there are spillovers from the private capital factor to the public capital factor. In the aggregate level, the productivity scale A consists of both two factors.

On the other hand, the households own the firms and face the consumer's problem. Specifically, the shares of the stock are normalised to unity. In this case, the representative consumer faces a standard infinite horizon utility maximisation problem given by,

$$J\left(s^t\right) = \underset{C}{Max} \quad E_t\left[\sum_{\tau=t}^{\infty} \beta^{\tau-t} U\left(C\left(s^\tau\right)\right)\right] \tag{4.6}$$

subject to budget constrain

$$S\left(s^{t+1}\right) P\left(s^t\right) = S\left(s^t\right)\left[P\left(s^t\right) + D\left(s^t\right)\right] - C\left(s^t\right) \tag{4.7}$$

where S is the stock shares holding by the consumers. P is the asset price. C is the consumption.

The model is closed by the resource constrain,

$$C\left(s^t\right) = D\left(s^t\right) \tag{4.8}$$

For the consumers, the model is a standard consumption-based capital asset pricing model (CCAPM), which can be dated back to Lucas (1978). In equilibrium, the investor holds the single asset and consumes its dividends. The asset price P maps to the stock value V in the firms problem. However, they are slightly different from each other. To follow the frequently used notation, here P stands for the ex-dividends price, which fulfils $V_t = P_t + D_t$.

The general equilibrium exists between two sides. Firstly, I solve for the first order conditions on both sides. Next, the combination of the first order conditions and the market clearing condition offers the general equilibrium.

4.4.1 General Equilibrium Condition

The standard method of dynamic programming can derive the optimal conditions. Proposition 4.1 provides the key results to the firms' problem. For the simplicity, from now on, I use $X\left(s^t\right)$, X_t and the simplified notation X interchangeably when there is no ambiguity.

Proposition 4.1 *The firms' problem shown in the last subsection has the following first order conditions (FOCs) and Euler equation.*

The first order conditions with respect to investment I and capital K are

$$E_t\left[\beta\frac{\Lambda_{t+1}}{\Lambda_t}\frac{\partial V_{t+1}}{\partial K_{t+1}}\right]=\frac{1}{\phi'\left(i_t\right)} \tag{4.9}$$

$$\frac{V_t}{K_t}=A_t-i_t+\frac{\phi\left(i_t\right)+1-\delta}{\phi'\left(i_t\right)} \tag{4.10}$$

The Euler equation is

$$E_t\left[\beta\frac{\Lambda_{t+1}}{\Lambda_t}\left(A_{t+1}-i_{t+1}+\frac{\phi\left(i_{t+1}\right)+1-\delta}{\phi'\left(i_{t+1}\right)}\right)\phi'\left(i_t\right)\right]=1 \tag{4.11}$$

Proof. See appendix 4.9.2.

The prime $'$ denotes the derivatives such as $\phi'(i)\equiv\partial\phi(i)/\partial i$ and $\phi''(i)\equiv\partial^2\phi(i)/\partial i^2$. I follow the convention and name the marginal price of capital on the left-hand side of FOC (4.9) "marginal q". I also use the "average Q" to indicate the average price of capital V/K. This FOC shows that the firm's value is maximised when the investment is chosen to balance the marginal gain and the marginal lost to the firm value.

The Hayashi (1982) proposition helps the model to obtain the FOC (4.10) of capital K. For a problem like this, the Hayashi (1982) proposition confirms that the "marginal q" $(\partial V/\partial K)$ equals to the "average Q" (V/K).[3] Equation (4.10) shows that, given the predetermined capital stock K_t, the firm's value V is not in a monotonic relation with the investment I. Firstly, $A-i$ represents the dividends in current period. High investment tends to decrease the current dividends payment. Therefore, the asset is less attractive. Next, the investment I enters into the second term. This term is constructed by the growth rate of capital $\phi(i)+1-\delta$ over $\phi'(i)$ which is the marginal effectiveness of the investment. It is obvious that the second term is positively related to the investment-capital ratio i. As a result, there is a trade-off between two terms. A firm with higher i has higher growth. Yet it dose not necessarily have higher Q. However the implication for the asset return is not clear until the model is solved for the general equilibrium.

Furthermore, the Euler equation (4.11) is a stochastic difference equation that defines the path of the firm's optimal investment behaviour given the process of discount factor Λ and the technology scale A.

On the other hand, the consumer's problem is standard and provides the well-known Euler equation given by

$$P_t U'(C_t) = E_t\left[\beta U'(C_{t+1})(P_{t+1}+D_{t+1})\right] \tag{4.12}$$

The Euler equation of the consumer side offers the discount factor that $\Lambda = U'(C)$.[4] Hence, the model is ready to define an general equilibrium for the economy. Hereby, I offer the following definition.

Definition 4.4.1 *An equilibrium is a set of sequences $K^*(s^t)$, $I^*(s^t)$, $i^*(s^t)$, $D^*(s^t)$, $C^*(s^t)$, $S^*(s^t)$, $A(s^t)$, $\Lambda(s^t)$, $V(s^t)$, such that:*

1. *$C^*(s^t)$ and $S^*(s^t)$ solve the household's optimisation problem (4.6), given $V(s^t)$ and $D^*(s^t)$.*

2. *$K^*(s^t)$, $I^*(s^t)$, $i^*(s^t)$ and $D^*(s^t)$ solve the firm's problem (4.1), given $A(s^t)$, $\Lambda(s^t)$ and the initial capital stock K_0.*

[3] See proof in appendix 4.9.3.

[4] See appendix 4.9.4 for derivation.

3. *$\Lambda(s^t)$ is the unique discount factor that satisfies $\Lambda(s^t) = U'(C^*(s^t))$.*

4. *Markets clear: $C^*(s^t) = D^*(s^t)$.*

5. *Transversality condition holds.*[5]

As usual, I combine the optimal conditions on both sides to obtain the general equilibrium condition. For simplicity, this paper considers the log-utility case which makes $\Lambda = C^{-1}$. With the market clearing condition $C = D = AK - I$, I rearrange equation (4.11) into

$$E_t\left[\beta\left(\frac{(A_{t+1}-i_{t+1})K_{t+1}}{(A_t-i_t)K_t}\right)^{-1}\left(A_{t+1}-i_{t+1}+\frac{\phi(i_{t+1})+1-\delta}{\phi'(i_{t+1})}\right)\phi'(i_t)\right]=1 \tag{4.13}$$

This is the core stochastic difference equation governing the dynamics of firm's optimal decisions. Ideally, with the initial values, the firm solves this equation recursively to follow the optimal path. By solving it, I obtain several findings on macroeconomic fundamentals and asset prices in this economy.

4.4.2 Perfect Foresight Equilibrium

This section offers three propositions to illustrate the perfect foresight equilibrium. The proposition 4.2 shows that there is no transitional dynamics in the model. Based on this, proposition 4.3 unveils the possible interactions between the exogenous technology scale A and the investment to capital ratio i. Later, proposition 4.4 develops the solutions to the risk premium, the expected risk asset return and the risk-free rate. Firstly, I study the deterministic model which ignores the stochastic parts and the expectation operator in the model. Accordingly, the difference equation

[5] See appendix 4.9.5 for details.

(4.13) is shown by,

$$\beta\left[\left(\frac{A-i_{t+1}}{A-i_t}\right)(\phi(i_t)+1-\delta)\right]^{-1}\left(A-i_{t+1}+\frac{\phi(i_{t+1})+1-\delta}{\phi'(i_{t+1})}\right)\phi'(i_t)=1 \tag{4.14}$$

The following propositions are constructed based on this difference equation. Firstly, lemma 4.1 offers the transversality condition in this deterministic model. It helps to develop the first proposition, which states that the deterministic equilibrium has no transitional dynamics.

Lemma 4.1 *The transversality condition for the deterministic version of the baseline model is given by*

$$\lim_{t\to\infty}\left[\beta^t\frac{\phi(i_t)+1-\delta}{(A-i_t)\phi'(i_t)}\right]=0 \tag{4.15}$$

Proof. See proof in appendix 4.9.5.

Proposition 4.2 *With the condition*

$$\frac{\phi(0)+1-\delta}{A\phi'(0)}<\frac{\beta}{1-\beta}, \tag{4.16}$$

the model has the following proposition.

In all the paths which governed by difference equation (4.14), the only feasible path for the model is the one in the fixed point where the investment to capital ratio $\bar{i}$ solves

$$\frac{\phi(\bar{i})+1-\delta}{\phi'(\bar{i})(A-\bar{i})}=\frac{\beta}{1-\beta} \tag{4.17}$$

Namely, there is no transitional dynamics towards equilibrium.

Proof. Firstly I rearrange the difference equation (4.14) into,

$$\frac{\phi(i_{t+1})+1-\delta}{(A-i_{t+1})\phi'(i_{t+1})}=\frac{1}{\beta}\frac{\phi(i_t)+1-\delta}{(A-i_t)\phi'(i_t)}-1 \tag{4.18}$$

An change of variable makes the difference equation clearer. I define

$$F_t(i_t)\equiv\frac{\phi(i_t)+1-\delta}{(A-i_t)\phi'(i_t)} \tag{4.19}$$

Immediately, the difference equation (4.18) is

$$F_{t+1}=\frac{1}{\beta}F_t-1 \tag{4.20}$$

Next, I show that the F function is monotonically increasing with investment to capital ratio i. Once this relation is established, the solution of the difference equation (4.20) can be generalised to difference equation (4.18).

It is straight forward that

$$F'(i)=\frac{\phi'(i)^2(A-i)+[\phi'(i)-\phi''(i)(A-i)][\phi(i)+1-\delta]}{[(A-i)\phi'(i)]^2}>0 \tag{4.21}$$

where I applied the assumptions that (1) $D>0$ therefore $A-i>0$, (2) $\phi(i)>0$, (3) $\phi'(i)>0$, (4) $1-\delta>0$ and (5) $\phi''(i)<0$.

Finally, with the first condition (4.16) in the proposition, the solutions of the difference equation (4.20) have three kinds of potential paths.

1. If $F_0<\beta/(1-\beta)$ and correspondingly $i_0<\bar{i}$, then F_t and i_t decrease over time and eventually break the assumption of $i>0$.

2. If $F_0>\beta/(1-\beta)$ and correspondingly $i_0>\bar{i}$, then F_t and i_t increase over time and eventually break the transversality condition. This is shown by the

general solution of the difference equation (4.20).

$$F_t = \frac{1}{\beta^t} F_0 - \frac{\beta}{\beta - 1}\left(1 - \frac{1}{\beta^t}\right) \tag{4.22}$$

$$\lim_{t\to\infty} \beta^t F_t = F_0 - \frac{\beta}{1-\beta} \tag{4.23}$$

The transversality condition in the last equality can not be 0 if $F_0 > \beta/(1-\beta)$.

3. If $F_0 = \beta/(1-\beta)$ correspondingly $i_0 = \bar{i}$, then F_t and i_t stay at this fixed point meanwhile transversality condition holds.

$$\lim_{t\to\infty} \beta^t F_t = F_0 - \frac{\beta}{1-\beta} = 0 \tag{4.24}$$

Additionally, the variable i is restricted by the range $(0, A)$ since $I > 0$ and $A - i > 0$. Hence, the value of the function $F(i)$ falls into

$$F(i) \in \left(\frac{\phi(0) + 1 - \delta}{A\phi'(0)}, \infty\right) \tag{4.25}$$

Therefore, the first condition (4.16) in this proposition guarantees that the fixed point $\bar{i}$ is feasible.

In all, in our model, there is no transitional dynamics. The only feasible path is the one that initiates the economy at $i = \bar{i}$.

Q.E.D. ■

Since the model has the only feasible path in the fixed point, in some sense, the model is static. In the deterministic difference equation (4.14), technology A is a constant. Nonetheless, it is straight forward to generalise the key equation in proposition 4.2 to the case allowing A to be a stochastic process.

$$\frac{\phi(\bar{i}_t) + 1 - \delta}{\phi'(\bar{i}_t)(A_t - \bar{i}_t)} = \frac{\beta}{1-\beta} \tag{4.26}$$

Overall, when the firm observes current realisation of technology scale A, it solves the equation (4.26) and places its investment to capital ratio i to $\bar{i}_t$.[6] In this situation, i shadows the dynamics of the technology process A. Moreover, in this BGP, the consumption also grows at the rate of capital growth since $C_{t+1}/C_t = (A_{t+1} - i_{t+1}) K_{t+1} / (A_t - i_t) K_t$. Yet, the functional form of the non-linear efficiency function $\phi(\cdot)$ is not well established. Hence, I cannot solve the equation (4.26) explicitly. However, proposition 4.3 to illustrates the relation between investment to capital ratio i and technology A in this implicit function. In addition, for later reference, I use $i[A]$ to denote the solution of a given A.

Proposition 4.3 *With the identical conditions in proposition 4.2, the implicit function between A and i, namely the equation* (4.26)*, has the features that* $1 > \partial i/\partial A > 0$ *and* $\partial^2 i/\partial A^2 < 0$.

Proof. See proof in the middle part of appendix 4.9.6.

Proposition 4.3 shows that if the technology scale A increases, the investment-capital ratio i increases. The firm chooses a higher investment to cope with a higher productivity parameter A. Moreover, this boosts the growth rate of the economy since the growth is determined by $\phi(i) + 1 - \delta$. Nonetheless, $\partial^2 i/\partial A^2 < 0$ tells that the impact of technology A on economy growth becomes less and less efficient in the model. This is because the efficiency function $\phi(i)$ is concave. Intuitively, with the increases of technology A, the firm wants to raise investment to obtain its optimal value. However, the adjustment costs are high when investment level is high. Roughly speaking, a considerable amount of investments is "wasted" and cannot be transferred into capital inputs.

Moreover, with the solution $i[A]$, the model yields the expressions for finance related variables in proposition 4.4.

Proposition 4.4 *When the technology scale A follows a stochastic process, condition on the current state s^t, the stochastic version of the baseline model has the*

[6] See the first part of appendix 4.9.6 for the discussion of the root.

following solutions for the expected risk premium $E_t(RP_{t+1})$, expected risky asset return $E_t(R_{t+1})$ and the risk-free rate r_t^f.

$$E_t\left(R_{t+1}|s^t\right) = \frac{\phi'(i[A_t])}{1-\beta} E_t\left[A_{t+1} - i[A_{t+1}]|s^t\right] \tag{4.27}$$

$$r_t^f = \frac{\phi'(i[A_t])}{1-\beta}\left[E_t\left(\frac{1}{A(s^{t+1}) - i[A(s^{t+1})]}\middle| s^t\right)\right]^{-1} \tag{4.28}$$

$$E_t\left(RP_{t+1}|s^t\right) = \frac{\phi'(i[A_t])}{1-\beta}\left\{E_t\left[A_{t+1} - i[A_{t+1}]|s^t\right] - \left[E_t\left(\frac{1}{A(s^{t+1}) - i[A(s^{t+1})]}\middle| s^t\right)\right]^{-1}\right\} \tag{4.29}$$

Proof. See proof in appendix 4.9.7.

More proposition 4.4 offers the solutions to 3 finance-related variables. All three expressions are functions of the stochastic process A_t when $i[A]$ is substituted by the root of the equation (4.26). In the deterministic case, one can ignore the expectation operator and thus the risky asset return and risk-free rate collapse to one unified expression.

Importantly, solution to the risk premium reveals that the risk premium can move counter-cyclically in the modelled economy. It consists of two parts, namely $\phi'(i[A_t])/(1-\beta)$ and the term in the bracket. For now, I consider the effect of the first part only and leave the second for later analysis. Due to the concavity of the efficiency function $\phi(i)$, the model has the assumption $\phi''(i) < 0$. Therefore, in the economy with higher investment to capital ratio i and higher growth, the term $\phi'(i[A_t])/(1-\beta)$ is actually smaller. For example, if s^t is independent and identically distributed, the term in the brace of function (4.29) is a constant. Therefore, we observe a higher risk premium in the economy with slow growth in this situation. In fact, in this example, the counter-cyclical risk premium is caused by the adjustment costs of investment. The extreme case demonstrates this. When $\phi(i) \to i$, there is no cost for investment. Accordingly, $\phi'(i) \to 1$. Here, the risk premium becomes

$$RP\left(s^{t}\right) = \frac{1}{1-\beta}\left\{E_{t}\left[A_{t+1} - i\left[A_{t+1}\right] \middle| s^{t}\right] - \left[E\left(\frac{1}{A\left(s^{t+1}\right) - i\left[A\left(s^{t+1}\right)\right]}\middle| s^{t}\right)\right]^{-1}\right\} \tag{4.30}$$

which is a constant under the i.i.d assumption. The counter-cyclicality of the risk premium disappears.

Besides that, the expectation terms in the solutions are of interested. In later sections, I release the i.i.d assumption and study the second parts in the solutions. If the distribution of A_{t+1} relates to A_t, the movement of risk premium is affected by the expectations in the bracket. That could be the second source to account for the counter-cyclical behaviour.

4.5 Endogenous Productivity and Multiple Equilibria

The baseline model shows the conventional logics of the economic growth. The firms decided on their investments according to productivity. Furthermore, level of investment pins down growth rate. In this section, I endogenies the technology. This methodology originally comes from the growth literature that studies the poverty traps. Azariadis and Stachurski (2005) have a good survey on this field. Often, those models assume that the technology scale A is a function of capital K or investment I. By doing that, the convex neoclassical growth model generates multiple BGPs. This, to a extent, can explain the self-reinforced poverty shown in many developing countries. In addition, many research attempt to provide micro-foundation to the relation between technology and investment. Roughly speaking, related models show the micro-foundation can be obtained from imperfect competition and/or complementarity. Among many others, Matsuyama (1997) summarises a series of papers discussing the feeding back of investment to externalities and productivity.

However, the structures in those models complicate the parsimonious model in here. Like in the Azariadis and Drazen (1990), I leave aside the micro-foundation and assume a simple relation between technology and investment in assumption 4.1. Explicitly, there is a discontinuous relation between the productivity scale factor A and the investment-capital ratio i.

Assumption 4.1 *Technology scale factor A is a discontinuous function of i given by*

$$A(i_t) = \begin{cases} A_H;\ if\ i_t \geq i^*,\ or\ equivalently\ s^t \geq s^* \\ \\ A_L;\ if\ i_t < i^*,\ or\ equivalently\ s^t < s^* \end{cases} \tag{4.31}$$

I define i_H and i_L to be the corresponding solution to the equation (4.17) *with A_H and A_L respectively i.e. $i_H = i[A_H]$ and $i_L = i[A_L]$.*

The assumption describes a threshold relation. The investment-capital ratio should be maintained above a certain level, i^*, to assure a high performance of technology level A_H. Otherwise, the technology is trapped in a relatively low level A_L. Clearly, this assumption ensures two solutions to equation (4.17) namely $i[A_H]$ and $i[A_L]$, given A_H and A_L. There are two equilibria such as $\{A_H,\ i_H\}$ and $\{A_L,\ i_L\}$ for the firms to select.

To explain, the more the firm invest, the more spillovers to the public factor in the production. Hence, the model has a higher productivity A. In return, a high technology scale A urges the firm to investment more. For instance, if all firms choose the high investment-capital ratio i_H, this generates a high technology scale A_H. In return, A_H confirms that i_H is the optimal choice for an individual firm. Since we have endogenised the technology, it is of little interest to distinguish which one pins down the other. The co-movement assumed is clear. In the story like this, the technology A stands for the supply side and the investment stands for the demand side. They are intertwined with each other.

With the assumption 4.1, the baseline model exhibits multiple BGPs. More importantly, these two BGPs correspond to different growth rates, risk-free rates and risk premia. Obviously, the BGP with the low investment-capital ratio i_L grows at a slower pace. The economy is trapped permanently if there is no "shifting device" in the economy.

Furthermore, I join the findings in proposition 4.4 with assumption 4.1, which yields

$$E_t\left(R_{t+1}|\, i_t = i_L\right) = \frac{\phi'\left(i_L\right)}{1-\beta} E_t\left[A_{t+1} - i\left[A_{t+1}\right]|\, i_t = i_L\right] \tag{4.32}$$

$$E_t\left(R_{t+1}|\, i_t = i_H\right) = \frac{\phi'\left(i_H\right)}{1-\beta} E_t\left[A_{t+1} - i\left[A_{t+1}\right]|\, i_t = i_H\right] \tag{4.33}$$

$$\left[r_t^f \middle|\, i_t = i_L\right] = \frac{\phi'\left(i_L\right)}{1-\beta}\left[E\left(\frac{1}{A\left(s^{t+1}\right) - i\left[A\left(s^{t+1}\right)\right]}\middle|\, i_t = i_L\right)\right]^{-1} \tag{4.34}$$

$$\left[r_t^f \middle|\, i_t = i_H\right] = \frac{\phi'\left(i_H\right)}{1-\beta}\left[E\left(\frac{1}{A\left(s^{t+1}\right) - i\left[A\left(s^{t+1}\right)\right]}\middle|\, i_t = i_H\right)\right]^{-1} \tag{4.35}$$

As predicted by proposition 4.4, with the i.i.d assumption we have

$$E_t\left(R_{t+1}|\, i_t = i_L\right) > E_t\left(R_{t+1}|\, i_t = i_H\right) \tag{4.36}$$

$$\left[r^f \middle|\, i_t = i_L\right] > \left[r^f \middle|\, i_t = i_H\right] \tag{4.37}$$

To explain, when the technology scale A is high the firms invests more to optimise its stock value. High investment promises a good growth in the future. This makes the asset more attractive. Nonetheless, the high adjustment costs harm the consumption growth and the asset prices. In this specific simple-structured setup, the second force overcomes the first one. Hence, the model yields low asset return in the high growth state.

Clearly, these intuitions are not consistent with the observations shown in figure 4.1. In the years after 2008, we have relatively low growth companied with the drop of the risk-free rate. Yet, the effects of the expectation term have not been explored. In next section, I relax the i.i.d assumption. In the end, the risk premium in this model is driven by the interactions between the efficiency function $\phi\left(i\right)$ and the expectation of the difference between the technology A and the investment to capital ratio i.

From now on, I consider s_t as the non-fundamental extrinsic shocks. Following the conventions, this chapter uses the term “sunspots”. Intuitively, s^t serves as a selecting device to kick off the endogenous chain reactions. The next subsection introduces the sunspots in a formal way.

4.6 Sunspots Equilibria

This section defines the sunspots and studies the results of the model with them. The sunspots defined below are in different environment compare to the mainstream literature. Among many others, Woodford (1986) explores a continuum of sunspots equilibria which asymptotically converge to the BGP. Here, the sunspots equilibria are not construed near to the indeterminate equilibrium. They are similar to those used in Benigno and Fornaro (2016) and Christiano and Harrison (1999). In this framework, sunspots are signals shown in each period to help the firm choose the equilibrium for current period. As indicated by Christiano and Harrison (1999), a proper name for this kind of extrinsic randomness is regime switching sunspots. Specifically, I provide assumption 4.2.

Assumption 4.2 *s_t is an extrinsic random variable, which governs the system and acts like a selection device in the model. $\{s_t\}_0^\infty$ is a Markov chain with state space $\{s_L,\ s_H\}$ and transitional matrix $\mathbf{P}_s$ given by*

$$\begin{pmatrix} p_L & 1-p_L \\ 1-p_H & p_H \end{pmatrix} \tag{4.38}$$

For example, if $s_t = s_L$, the probability of shifting to the other state s_H is

$$Prob\,(s_{t+1} = s_H | \, s_t = s_L) = 1 - p_L \tag{4.39}$$

Explicitly, given s_t, we draw s_{t+1}. The realisation of s_{t+1} determines which equilibrium the economy stays, $\{A_H,\, i_H\}$ or $\{A_L,\, i_L\}$. Apparently, two equilibria represent two BGPs with high growth rate and low growth rate respectively. Similarly, Gourio (2012) discusses the situation of two states and state-shifting. Nonetheless, in his paper, two states are assigned with two different technology shock processes. Here, two states are defined by two distinct equilibria.

In fact, s_t can represents a symbol of beliefs. At the beginning of each period, firms observe a signal s_L or s_H. Accordingly, they choose $\{A_H,\, i_H\}$ or $\{A_L,\, i_L\}$. The economy grows at rate $\phi\,(i_H)+1-\delta$ or $\phi\,(i_L)+1-\delta$ accordingly. The appearance of s_L or s_H is exogenous and governed by Markovian property. With this setup, the economy switches between two BGPs. This parsimonious structure generates all kinds of growth patterns. For example, if p_L and p_H have relatively high values, the economy has a high probability of lingering in its current state.

In last section, the model's predictions under the i.i.d. assumption are actually against the observations. Here, the i.i.d. assumption is replaced by the assumption 4.2. Correspondingly, the solutions of the expected risk return and the risk-free rate in this framework are given by

$$E\left[R_{t+1}|\, s_t = s_L\right] = \frac{\phi'\,(i_L)}{1-\beta}\left[AiH - p_L\,(AiH - AiL)\right] \tag{4.40}$$

$$E\left[R_{t+1}|\, s_t = s_H\right] = \frac{\phi'\,(i_H)}{1-\beta}\left[AiL + p_H\,(AiH - AiL)\right] \tag{4.41}$$

$$\left[r_t^f \middle|\, s_t = s_L\right] = \frac{\phi'\,(i_L)}{1-\beta}\left[\frac{1}{AiH} - p_L\left(\frac{1}{AiH} - \frac{1}{AiL}\right)\right]^{-1} \tag{4.42}$$

$$\left[r_t^f \middle|\, s_t = s_H\right] = \frac{\phi'\,(i_H)}{1-\beta}\left[\frac{1}{AiL} + p_H\left(\frac{1}{AiH} - \frac{1}{AiL}\right)\right]^{-1} \tag{4.43}$$

where $AiL \equiv A_L - i_L$ and $AiH \equiv A_H - i_H$.[7]

If p_L equals $1-p_H$, the results back to the i.i.d sunspots case. With different specifications of AiH, AiL, p_H and p_L, these Markovian sunspots generate different

[7] See appendix 4.9.8 for derivation.

kinds of risk premiums patterns. I present the proposition 4.5 to illustrate the possible outcomes.

Proposition 4.5 *With assumptions 4.1 and 4.2, if the p_L, p_H and/or $AiH - AiL$ are sufficiently large, it is possible to obtain*

(1) pro-cyclical expected risk return that $E\left[R_{t+1}|\, s_H\right] > E\left[R_{t+1}|\, s_L\right]$,

(2) pro-cyclical risk-free rate that $\left[r_t^f \middle| s_H\right] > \left[r_t^f \middle| s_L\right]$,

simultaneously.

Proof. Before the main body of the proof, I state two points. The first is that the condition of $AiH > AiL > 0$ has been proofed in the last part of appendix 4.9.6. Hence, $(1/AiH) - (1/AiL)$ is negative. This condition is repeatedly used. The second is that the term "second term" used in this proof refer to the term in the equation (4.40), (4.41), (4.42) and (4.43) excluding $\phi'(i)/(1-\beta)$. For example, the second term in equation (4.40) is $[AiH - p_L(AiH - AiL)]$.

I hold the $AiH - AiL$ as constant and discuss the probability parameter p_L and p_H. When p_L increases, with the condition that $(1/AiH) - (1/AiL)$ is negative, the second terms in the equation (4.40) and the equation (4.42) decrease. For p_H, the opposite is true in equation (4.41) and equation (4.43).

Admittedly, in equation (4.40) and equation (4.41) have $\phi'(i_L) > \phi'(i_H)$. Despite this, if the p_L and p_H are sufficient high, it could be case that the second terms dominates and makes $E\left[R_{t+1}|\, s_H\right] > E\left[R_{t+1}|\, s_L\right]$.

In equation (4.42) and equation (4.43), since $(1/AiH) - (1/AiL)$ is negative, it is also true that if the p_L and p_H are sufficient high, we have

$$\left[\frac{1}{AiH} - p_L\left(\frac{1}{AiH} - \frac{1}{AiL}\right)\right]^{-1} < \left[\frac{1}{AiL} + p_H\left(\frac{1}{AiH} - \frac{1}{AiL}\right)\right]^{-1} \tag{4.44}$$

Again, if this effect overcomes the effect of $\phi'(i_L) > \phi'(i_H)$, the model results in $\left[r_t^f \middle| s_H\right] > \left[r_t^f \middle| s_L\right]$.

Secondly, $AiH - AiL$ offers similar conclusion if the probability parameter p_L and p_H are hold as constant. For equation (4.40) and equation (4.41), identical logic in the last argument also apply to here. For equation (4.42) and equation (4.43), a larger difference between AiH and AiL means a larger difference between the reciprocals of AiH and AiL due to the monotonic feature of the reciprocal function. Furthermore, $(1/AiH)-(1/AiL)$ is negative. Hence, a larger $AiH-AiL$ means a smaller $(1/AiH)-(1/AiL)$. Repeatedly, sufficiently large $AiH-AiL$ can yield the result that

$$\left[\frac{1}{AiH}-p_L\left(\frac{1}{AiH}-\frac{1}{AiL}\right)\right]^{-1}<\left[\frac{1}{AiL}+p_H\left(\frac{1}{AiH}-\frac{1}{AiL}\right)\right]^{-1} \tag{4.45}$$

Once more, if this effect overcomes the effect of $\phi'(i_L)>\phi'(i_H)$, the model ends up with $\left[r_t^f\middle|s_H\right]>\left[r_t^f\middle|s_L\right]$.

Q.E.D. ■

In addition to these possibilities, the extreme case also helps to illustrate the argument. When $p_L\to 1$, $A_L\to 0$ and $i_L\to 0$, we have $\left[r_t^f\middle|s_t=s_L\right]\to 0$. The economic growth approaches the lowest level $\phi(0)+1-\delta$. In this case, there is no incentive for firms to invest. Therefore the technology dose not have enough development to pull the economy out of the low state BGP. Investors have low expectations in consumption growth. The risk-free rate approaches to the zero lower bound.

For the counter-cyclicality of the risk premium, the model is also indeterminated. It depends on the interactions between the marginal efficiency of the investment $\phi'(i)$, the probabilities p and $AiH-AiL$. In other words, the model should have enough degrees of freedom to generate (1) pro-cyclical expected risk return $E[R_{t+1}|s_H]>E[R_{t+1}|s_L]$, (2) pro-cyclical risk-free rate $\left[r_t^f\middle|s_H\right]>\left[r_t^f\middle|s_L\right]$ and (3) counter-cyclical risk premium

$$E[R_{t+1}|s_H]-\left[r_t^f\middle|s_H\right]<E[R_{t+1}|s_L]-\left[r_t^f\middle|s_L\right] \tag{4.46}$$

simultaneously.

To summarise, firstly, expectations are crucial to the BGP level investment, technology and the long-run growth in the model. Moreover, when pessimism dominates, the economy tend to be trapped in the stagnated state with low technology, low investment, low growth and risk-free rate bounded by 0.

Nevertheless, proposition 4.5 only shows the theoretical potentialities of mimicking the patter in the data. In next section, I calibrate the model and match the model predictions with data moments.

4.7 Calibration

This section aims to match the theoretical predicted moments to historical data moments. Firstly, I consider two alternative functional forms for the efficiency function $\phi(i)$ of the investment to capital ratio i, which are borrowed from Eberly and Wang (2009) and Gourio (2012). The former is in a log form given by

$$\phi(i) = \alpha + \Gamma \log\left(1 + \frac{i}{\theta}\right) \tag{4.47}$$

The latter is the frequently used quadratic form

$$\phi(i) = i - \frac{\Gamma(i-\theta)^2}{2} \tag{4.48}$$

Now the model is really for calibration. Given A_L, A_H, p_L, p_H, β, δ and parameterisations in the efficiency function $\phi(i)$, the closed form solutions for the investment-GDP ratio $I/Y = i/A$, the growth rate of the economy $\phi(i) + 1 - \delta$, the risk-free rate r^f and the expected risky asset return $E(R)$ in both states are given by

$$\left[\frac{I_t}{K_t}\middle| s_t = s_L\right] = i_L, \quad \left[\frac{I_t}{K_t}\middle| s_t = s_H\right] = i_H \tag{4.49}$$

$$\left[\frac{I_t}{Y_t}\middle| s_t = s_L\right] = \frac{i_L}{A_L}, \quad \left[\frac{I_t}{Y_t}\middle| s_t = s_H\right] = \frac{i_H}{A_H} \tag{4.50}$$

$$\left[\frac{Y_{t+1}}{Y_t}\middle| s_t = s_L\right] = \phi(i_L) + 1 - \delta \tag{4.51}$$

$$\left[\frac{Y_{t+1}}{Y_t}\middle| s_t = s_H\right] = \phi(i_H) + 1 - \delta \tag{4.52}$$

$$\left[r_t^f \middle| s_t = s_L\right] = \frac{\phi'(i_L)}{1-\beta}\left[\frac{1}{AiH} - p_L\left(\frac{1}{AiH} - \frac{1}{AiL}\right)\right]^{-1} \tag{4.53}$$

$$\left[r_t^f \middle| s_t = s_H\right] = \frac{\phi'(i_H)}{1-\beta}\left[\frac{1}{AiL} + p_H\left(\frac{1}{AiH} - \frac{1}{AiL}\right)\right]^{-1} \tag{4.54}$$

$$E\left[R_{t+1} \middle| s_t = s_L\right] = \frac{\phi'(i_L)}{1-\beta}\left[AiH - p_L(AiH - AiL)\right] \tag{4.55}$$

$$E\left[R_{t+1} \middle| s_t = s_H\right] = \frac{\phi'(i_H)}{1-\beta}\left[AiL + p_H(AiH - AiL)\right] \tag{4.56}$$

Additionally, the i_L and i_H are determined within the model. The Euler equation (4.13) in general equilibrium splits into two equations in the economy with sunspots.

$$1 = p_L\left[\beta\frac{\left(A_L - i_L + \frac{\phi(i_L)+1-\delta}{\phi'(i_L)}\right)\phi'(i_L)}{\frac{A_L-i_L}{A_L-i_L}(\phi(i_L)+1-\delta)}\right] + \left(1-p_L\right)\left[\beta\frac{\left(A_H - i_H + \frac{\phi(i_H)+1-\delta}{\phi'(i_H)}\right)\phi'(i_L)}{\frac{A_H-i_H}{A_L-i_L}(\phi(i_L)+1-\delta)}\right] \tag{4.57}$$

$$1 = p_H\left[\beta\frac{\left(A_H - i_H + \frac{\phi(i_H)+1-\delta}{\phi'(i_H)}\right)\phi'(i_H)}{\frac{A_H-i_H}{A_H-i_H}(\phi(i_H)+1-\delta)}\right] + \left(1-p_H\right)\left[\beta\frac{\left(A_L - i_L + \frac{\phi(i_L)+1-\delta}{\phi'(i_L)}\right)\phi'(i_H)}{\frac{A_L-i_L}{A_H-i_H}(\phi(i_H)+1-\delta)}\right] \tag{4.58}$$

Ideally, the calibration procedure should firstly solve this equation system for i_L and i_H. Then, we substitute them back into the equations (4.49) to (4.56) to obtain their values. Finally, by varying the parameters, the model obtains different predictions

	α	Γ	θ	δ	β	A_L	A_H	p_L	p_H
Log	0.1	0.015	0.035	0.13	0.98	0.171	2.05	0.9935	0.998
Qud	–	1.7	0.025	0.13	0.98	0.11	0.14	0.9983	0.998

Table 4.1 Parameterisation

for the moments. In fact, with the condition $p_H \in (0, 1)$ and $p_L \in (0, 1)$, the above equation system collapses to

$$\frac{\phi(i_H) + 1 - \delta}{\phi'(i_H)(A_H - i_H)} = \frac{\beta}{1-\beta} \tag{4.59}$$

$$\frac{\phi(i_L) + 1 - \delta}{\phi'(i_L)(A_L - i_L)} = \frac{\beta}{1-\beta} \tag{4.60}$$

The calculation is in appendix 4.9.9. Basically, this means that, with or without the information of the transitional matrix, the investor makes same decision on the level of BGP.

In terms of the parameterisations, the benchmark of the parameters in the efficiency function $\phi(i)$ is based on the paper by Eberly and Wang (2009) and Gourio (2012). The others are adjusted to match the model predictions to data moments. The table 4.1 reports the parameterisations chosen in end.

The moments generated by the model and calculated from the data are in table 4.2. I use the data of the US including investment-capital ratio, investment-GDP ratio, per capita GDP growth rate, treasury bond rate, and equity risk premium. All data are in real terms. Data description is in appendix 4.9.10. Particularly, I use the period between 1992 to 2001 as the representation of high-growth state with a 2.28% per capita GDP growth. For the low-growth state, I use 2005 to 2014. In this period, on average, the growth rate of per capita GDP is 0.65%.

As shown in table 4.2, overall, the investment adjustment function in the log form slightly performs better than the the quadratic from. Figures in bold are those moments cannot match the data moments even in a rough sense. In general, the log-form model can match the moments of GDP growth rate and counter-cyclical risk premium in both states. Especially, it captures the property that in the low-

%	Calibration - Log		Calibration - Qud		Data of US	
	High	Low	High	Low	High	Low
$i = I/K$	**84**	5.3	17.5	12.7	11.5	10.4
I/Y	40.9	31.0	**86.1**	**83.8**	34.8	32
Growth Rate	1.8	**-1.6**	2.5	**-1.1**	2.28	0.65
risk-free Rate	**2.0**	0.9	4.6	0.85	3.58	1.1
Risk Premium	**1.7**	5.5	**0.00**	**0.00**	2.4	5.6

Table 4.2 Calibration and Data

growth state risk-free rate drop dramatically to 0.9%. Meanwhile, it yields a high risk premium of 5.5%. However, there is a trade-off in the calibration. The log-form can not obtain proper investment-capital ratio. The 84% of i_H shows that the model needs a very high $A_H = 2.05$ to generate high growth. In terms of the quadratic form, it obtains the proper values for investment-capital ratio at around 10% and well performed risk-free rates of $r_H^f = 4.6\%$ and $r_L^f = 0.85\%$. Nonetheless, it cannot offer an appropriate distance between technology A and i to generate the investment-GDP ratio I/Y. In addition, although it generates counter-cyclical risk premium, they are negligible.

Generally, the calibration shows the model can capture some patterns shown in the data. However, there are trade-offs. Those parameters that can generate correct macro-fundamental moments cannot generate reasonable moments of financial variables and vice versa. To overcome this problem, chapter 5 introduces Epstein and Zin utility to call for more parameters and study influence of the intertemporal elasticity of substitution (IES). There are many studies show that the recursive utility can provide some theoretical explanations of the behaviour of the risk-free rate. Introducing this preference might also improve the calibration of this model.

4.8 Concluding Remarks

This paper explores secular stagnation and its implications on asset pricing. I develop an AK model. With the log utility and the AK production function, the baseline model reveals a linkage between technology and investment. The productivity determines the firms' willingness to investment. Besides, proposition 4.3 shows that a significant increase in technology A does not necessarily mean a strong improvement in economic growth. Additionally, with exogenous i.i.d technology, the baseline model shows counter-cyclical risk premia.

Furthermore, I endogenise the technology A. By doing that, the model exhibits multiple equilibria with different economic growth rates, risk-free rates, risk returns and so forth. I introduce Markovian regime switching sunspots, which serve as a selecting device. The sunspots represent the beliefs and activate the switch between different BGPs in the economy. In general, the model produces arbitrarily long period of low growth accompanied by decreases in risk-free rate and expansion in risk premia.

The model has a simple structure and closed form solutions for the moments of related variables. In the calibration, the model captures the growth rate and risk premium relatively well but has a trade-off when we consider the investment to capital ratio. A further work of the paper in chapter 5 modifies the log utility to the Epstein and Zin framework to capture more information of the risk-free rate.

4.9 Appendix

4.9.1 Convexity of the Adjustment Costs in Investment

Here I show that our restrictions on the capital accumulation function are consistent with the convex restriction on the adjustment costs of investments in the literature. Our capital accumulation condition (4.2) can be re-expressed as

$$K_{t+1} = I_t + (1-\delta) K_t - [i_t - \phi(i_t)] K_t \tag{4.61}$$

If the third term is zero, the current period investment is directly transformed into tomorrow's capital input without cost. Accordingly, I can treat the term in the bracket as adjustment costs of investment. Denote it as $Cst(i)$. Then our restrictions $1 > \phi'(i) > 0$ and $\phi''(i) < 0$ immediately lead to,

$$Cst'(i) = 1 - \phi'(i) > 0 \tag{4.62}$$

$$Cst''(i) = -\phi''(i) > 0 \tag{4.63}$$

i.e. our restrictions on $\phi(i)$ indicate adjustment costs is convex.

4.9.2 FOCs and Euler Equation of the Firm's Problem

This subsection offers the derivation of the firm's problem in the baseline model.
Proof. The firms' problem can be written in recursive form. The Bellman equation is

$$V_t = \max_{I_t} \quad A_t K_t - I_t + E_t \left[\beta \frac{\Lambda_{t+1}}{\Lambda_t} V_{t+1} \right] \tag{4.64}$$

The first order condition with respect to investment I is simply calculated by taking derivatives with respect to I and equalling it to 0.

$$E_t\left[\beta\frac{\Lambda_{t+1}}{\Lambda_t}\frac{\partial V_{t+1}}{\partial K_{t+1}}\right]=\frac{1}{\phi'\left(i_t\right)} \tag{4.65}$$

For the FOC for capital K, one can either use the envelop theorem or directly apply the Hayashi (1982) proposition in this model. For a problem like this, I have a condition derived by Hayashi (1982) stating that “marginal q” $(\partial V/\partial K)$ equals to the “average Q” (V/K). A rigorous proof is in the appendix 4.9.3. In fact, this condition simplifies the calculation. Therefore, by allowing of the Hayashi proposition, I divide both side of the Bellman equation (4.64) by K_t to obtain the first order condition for capital as

$$\frac{V_t}{K_t}=A_t-i_t+E_t\left[\beta\frac{\Lambda_{t+1}}{\Lambda_t}\frac{V_{t+1}}{K_{t+1}}\frac{K_{t+1}}{K_t}\right] \tag{4.66}$$

$$\frac{V_t}{K_t}=A_t-i_t+\frac{\phi\left(i_t\right)+1-\delta}{\phi'\left(i_t\right)} \tag{4.67}$$

The second equality is obtain by plugging in the FOC of investment I and capital accumulation condition. Finally, I forward the expression (4.67) for one period and substitute it back into equation (4.65) to achieve the Euler equation.

$$E_t\left\{\beta\frac{\Lambda_{t+1}}{\Lambda_t}\left[A_{t+1}-i_{t+1}+\frac{\phi\left(i_{t+1}\right)+1-\delta}{\phi'\left(i_{t+1}\right)}\right]\right\}=\frac{1}{\phi'\left(i_t\right)} \tag{4.68}$$

Q.E.D. ■

4.9.3 Proof of Hayashi Proposition in the Baseline Model

To simplify the notion, I use $V_{K,t}\equiv\partial V_t/\partial K_t$. Essentially I need to prove, in this model with the assumed functional form, $V_{K,t}=V_t/K_t$.

Proof. I start from the FOC for I_t,

$$E_t\left[\beta\left(\frac{\Lambda_{t+1}}{\Lambda_t}\right)V_{K_{t+1}}\right]=\frac{1}{\phi'(i_t)} \tag{4.69}$$

I use the envelope theorem to derive the FOC of capital K_t,

$$V_{Kt}=A_t+E_t\left[\beta\left(\frac{\Lambda_{t+1}}{\Lambda_t}\right)V_{Kt+1}\right]\left[1-\delta+\phi(i_t)-i\phi'(i_t)\right] \tag{4.70}$$

I multiply both sides by K_t, yield

$$V_{Kt}K_t=A_tK_t+E_t\left[\beta\left(\frac{\Lambda_{t+1}}{\Lambda_t}\right)V_{Kt+1}\right]\left[K_t(1-\delta+\phi(i_t))-i_t\phi'(i_t)K_t\right] \tag{4.71}$$

$$=A_tK_t+E_t\left[\beta\left(\frac{\Lambda_{t+1}}{\Lambda_t}\right)V_{Kt+1}K_{t+1}\right]-E_t\left[\beta\left(\frac{\Lambda_{t+1}}{\Lambda_t}\right)V_{Kt+1}\right]\phi'(i_t)I_t \tag{4.72}$$

$$=A_tK_t-I_t+E_t\left[\beta\left(\frac{\Lambda_{t+1}}{\Lambda_t}\right)V_{Kt+1}K_{t+1}\right] \tag{4.73}$$

The third equality I use the FOC for I_t in (4.69). Equation (4.73) can be forward and iterated to obtain,

$$V_{K,t}K_t=E_t\left\{\sum_{\tau=t}^{\infty}\beta^{\tau-t}\left(\frac{\Lambda_\tau}{\Lambda_t}\right)(A_\tau K_\tau-I_\tau)\right\} \tag{4.74}$$

$$=V_t \tag{4.75}$$

Q.E.D. ■

4.9.4 Consumer's Problem

The representative consumer faces a standard infinite horizon utility maximization problem given by,

$$J_t=\underset{C}{Max}\quad E_t\left[\sum_{\tau=t}^{\infty}\beta^{\tau-t}U(C_t)\right] \tag{4.76}$$

subject to budget constrain

$$S_{t+1}P_t = S_t\left(P_t + D_t\right) - C_t \tag{4.77}$$

$$C_t > 0, \quad S_t > 0 \tag{4.78}$$

$$P_t > 0, \quad D_t > 0 \tag{4.79}$$

The Bellman equation can be written as

$$J_t\left(S_t\right) = \underset{C}{Max} \quad U\left(C_t\right) + \beta E_t\left(J_{t+1}\left(S_{t+1}\right)\right) \tag{4.80}$$

FOC of consumption C_t gives,

$$U'\left(C_t\right) = \beta E_t\left(\frac{J_{St+1}}{P_t}\right) \tag{4.81}$$

Envelope theorem offers

$$J_{St} = \beta E_t\left[J_{St+1}\left(\frac{P_t + D_t}{P_t}\right)\right] \tag{4.82}$$

Combine the two, yield

$$J_{St} = U'\left(C_t\right)\left(P_t + D_t\right) \tag{4.83}$$

Further, I forward 1 period and substitute back into (4.82). I obtain

$$U'\left(C_t\right)\left(P_t + D_t\right) = \beta E_t\left[U'\left(C_{t+1}\right)\left(P_{t+1} + D_{t+1}\right)\left(\frac{P_t + D_t}{P_t}\right)\right] \tag{4.84}$$

$$U'\left(C_t\right)P_t = \beta E_t\left[U'\left(C_{t+1}\right)\left(P_{t+1} + D_{t+1}\right)\right] \tag{4.85}$$

which is the Euler equation.

4.9.5 Transversality Condition of the Baseline Model

In the firms problem, the transversality condition is

$$\lim_{\tau\to\infty}\beta^{\tau}\frac{\Lambda_{\tau}}{\Lambda_t}\frac{\partial D_{\tau}}{\partial K_{\tau+1}}K_{\tau+1}=0 \tag{4.86}$$

$$\lim_{\tau\to\infty}\beta^{\tau}\frac{\Lambda_{\tau}}{\Lambda_t}\phi'\left(i_{\tau}\right)^{-1}K_{\tau+1}=0 \tag{4.87}$$

$$\frac{1}{\Lambda_t}\lim_{\tau\to\infty}\beta^{\tau}\frac{K_{\tau+1}}{A_{\tau}K_{\tau}-I_{\tau}}\phi'\left(i_{\tau}\right)^{-1}=0 \tag{4.88}$$

$$\frac{1}{\Lambda_t}\lim_{\tau\to\infty}\beta^{\tau}\frac{\phi\left(i_{\tau}\right)+1-\delta}{\left(A-i_{\tau}\right)\phi'\left(i_{\tau}\right)}=0 \tag{4.89}$$

Since i is a constant in the equilibrium, the condition is satisfied.

The consumer's problem has a transversality condition given by

$$\lim_{t\to\infty}\beta^{t}U'\left(C_t\right)P_t=0$$

In the equilibrium, consumption constantly grow at rate $\phi\left(i\right)+1-\delta$, given C_t

$$\lim_{t\to\infty}\beta^{t}U'\left(C_t\right)P_t=\lim_{t\to\infty}\beta^{t}\frac{P_t}{D_t} \tag{4.90}$$

$$=\lim_{t\to\infty}\beta^{t}\frac{V_t-D_t}{D_t} \tag{4.91}$$

$$=\lim_{t\to\infty}\beta^{t}\left(\frac{V_t/K_t}{A_t-i_t}-1\right) \tag{4.92}$$

$$=\lim_{t\to\infty}\beta^{t}\left(\frac{A_t-i_t+\frac{\phi(i_t)+1-\delta}{\phi'(i_t)}}{A_t-i_t}-1\right) \tag{4.93}$$

$$=\lim_{t\to\infty}\beta^{t}\left(\frac{\phi\left(i_t\right)+1-\delta}{\left(A_t-i_t\right)\phi'\left(i_t\right)}\right) \tag{4.94}$$

In order to follow the typical notation of Euler equation, our asset price P in the consumer's problem is slightly different from the stock price V in the firm's problem. In fact, I have $P+D=V$, which is used in the second equality.

4.9.6 Root of Equation (4.17) in the Baseline Model

Firstly, I consider the number of roots to this equation.

$$\frac{\phi(i)+1-\delta}{\phi'(i)(A-i)}=\frac{\beta}{1-\beta} \tag{4.95}$$

With the restriction of $A>i$, $\phi(i)>0$, $\phi'(i)>0$ and $\phi''(i)<0$, I can write

$$(1-\beta)[\phi(i)+1-\delta]-\beta\phi'(i)(A-i)=0 \tag{4.96}$$

If I define the left hand side as $f(i)$, in the defined range $A>i>0$, we have

$$\frac{\partial f}{\partial i}=(1-\beta)\phi'(i)-\beta\left[\phi''(i)(A-i)-\phi'(i)\right] \tag{4.97}$$

$$\frac{\partial f}{\partial i}>0 \tag{4.98}$$

Additionally,

$$f(0)=(1-\beta)[\phi(0)+1-\delta]-\beta\phi'(0)A \tag{4.99}$$

$$f(A)=(1-\beta)[\phi(A)+1-\delta] \tag{4.100}$$

Thus the sufficient condition guarantees the uniqueness of the root is

$$A>\frac{(1-\beta)[\phi(0)+1-\delta]}{\beta\phi'(0)} \tag{4.101}$$

which guarantees $f(A)>0>f(0)$ and is identical to the condition in proposition 4.2.

Secondly, I derive the relation between A and i. Implicit function theorem offers

$$\frac{\partial i}{\partial A}=\frac{\beta\phi'(i)}{\phi'(i)-\beta(A-i)\phi''(i)} \tag{4.102}$$

$$\frac{\partial^2 i}{\partial A^2}=\frac{\beta^2\phi'(i)\phi''(i)}{[\phi'(i)-\beta(A-i)\phi''(i)]^2} \tag{4.103}$$

In the first derivative, I have that $\beta\phi'(i) < \phi'(i)$ and $-\beta(A-i)\phi''(i) > 0$. With the previous conditions, I have $1 > \partial i/\partial A > 0$ and $\partial^2 i/\partial A^2 < 0$.

Third, accordingly,

$$\frac{\partial(A-i)}{\partial A} = 1 - \frac{\partial i}{\partial A} > 0 \tag{4.104}$$

With $1 > \partial i/\partial A > 0$, naturally, $A_H - i[A_H] > A_L - i[A_L]$.

4.9.7 Expected Asset Return and Risk-free Rate

This section in the Appendix I derive the return for the risky asset R_{t+1} the risk-free rate r_t^f. I use the following conditions derived in the baseline model.

$$K_{t+1} = (1-\delta)K_t + \phi(i_t)K_t \tag{4.105}$$

$$\frac{V(K_t)}{K_t} = A_t - i_t + \frac{\phi(i_t)+1-\delta}{\phi'(i_t)} \tag{4.106}$$

$$\frac{\phi(i_t)+1-\delta}{\phi'(i_t)(A_t-i_t)} = \frac{\beta}{1-\beta} \tag{4.107}$$

I start from the included dividends asset return,

$$R_{t+1}^{IND} = \frac{V_{t+1}}{V_t} \tag{4.108}$$

$$= \frac{V_{t+1}/K_{t+1}}{V_t/K_t}\frac{K_{t+1}}{K_t} \tag{4.109}$$

$$= \frac{A_{t+1} - i_{t+1} + \frac{\phi(i_{t+1})+1-\delta}{\phi'(i_{t+1})}}{A_t - i_t + \frac{\phi(i_t)+1-\delta}{\phi'(i_t)}}\left(\phi\left(i_t\right)+1-\delta\right) \tag{4.110}$$

$$= \frac{A_{t+1} - i_{t+1} + \left(A_{t+1} - i_{t+1}\right)\frac{\phi(i_{t+1})+1-\delta}{\phi'(i_{t+1})(A_{t+1}-i_{t+1})}}{A_t - i_t + \left(A_t - i_t\right)\frac{\phi(i_t)+1-\delta}{\phi'(i_t)(A_t-i_t)}}\left(\phi\left(i_t\right)+1-\delta\right) \tag{4.111}$$

$$= \frac{\left(A_{t+1} - i_{t+1}\right)}{\left(A_t - i_t\right)}\left[\phi\left(i_t\right)+1-\delta\right] \tag{4.112}$$

$$= \left(A_{t+1} - i_{t+1}\right)\phi'\left(i_t\right)\frac{\left[\phi\left(i_t\right)+1-\delta\right]}{\phi'\left(i_t\right)\left(A_t - i_t\right)} \tag{4.113}$$

$$= \left(A_{t+1} - i_{t+1}\right)\phi'\left(i_t\right)\frac{\beta}{1-\beta} \tag{4.114}$$

Since V_t is the included dividends price I derive the ex-dividends return as

$$R_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t} \tag{4.115}$$

$$= \frac{V_{t+1}}{V_t - D_t} \tag{4.116}$$

$$= \left[\frac{1}{R_{t+1}^{IND}} - \frac{D_t}{V_{t+1}} \right]^{-1} \tag{4.117}$$

$$= \left[\frac{1}{R_{t+1}^{IND}} - \frac{D_t/K_t}{V_{t+1}/K_{t+1}} \frac{K_t}{K_{t+1}} \right]^{-1} \tag{4.118}$$

$$= \left[\frac{1}{R_{t+1}^{IND}} - \frac{A_t - i_t}{A_{t+1} - i_{t+1} + \frac{\phi(i_{t+1})+1-\delta}{\phi'(i_{t+1})}} \frac{1}{\phi(i_t) + 1 - \delta} \right]^{-1} \tag{4.119}$$

$$= \left[\frac{1}{R_{t+1}^{IND}} - \frac{1}{A_{t+1} - i_{t+1} + (A_{t+1} - i_{t+1}) \frac{\beta}{1-\beta}} \frac{1}{\left(\frac{\beta}{1-\beta}\right) \phi'(i_t)} \right]^{-1} \tag{4.120}$$

$$= \left[\frac{1}{R_{t+1}^{IND}} - \frac{1}{(A_{t+1} - i_{t+1}) \left(\frac{1}{1-\beta}\right)} \frac{1}{\left(\frac{\beta}{1-\beta}\right) \phi'(i_t)} \right]^{-1} \tag{4.121}$$

$$= \left[\frac{1}{(A_{t+1} - i_{t+1}) \phi'(i_t) \frac{\beta}{1-\beta}} - \frac{1}{(A_{t+1} - i_{t+1}) \left(\frac{1}{1-\beta}\right)} \frac{1}{\left(\frac{\beta}{1-\beta}\right) \phi'(i_t)} \right]^{-1} \tag{4.122}$$

$$= \left\{ \frac{1}{(A_{t+1} - i_{t+1}) \phi'(i_t) \frac{\beta}{1-\beta}} \left[1 - \frac{1}{\left(\frac{1}{1-\beta}\right)} \right] \right\}^{-1} \tag{4.123}$$

$$= (A_{t+1} - i_{t+1}) \frac{\phi'(i_t)}{1-\beta} \tag{4.124}$$

The capital accumulation condition (4.105) and the solution for marginal q (4.106) are used in equality (4.110). The rest of the equality I repeatedly use the solution condition (4.107) for BGP level of i.

One can check the relation indicated by the Euler equation,

$$1 = E_t \left[\beta \frac{\Lambda_{t+1}}{\Lambda_t} \cdot R_{t+1} \right] \tag{4.125}$$

For the risk-free rate, by definition it is the inverse of expectation of SDF as,

$$
\begin{aligned}
r_t^f &= \frac{1}{E\left(\beta \frac{\Lambda_{t+1}}{\Lambda_t}\right)} && (4.126)\\
&= \left[E\left(\beta \cdot \frac{A_t - i_t}{A_{t+1} - i_{t+1}} \frac{K_t}{K_{t+1}}\right)\right]^{-1} && (4.127)\\
&= \left[E\left(\frac{1}{A_{t+1} - i_{t+1}}\right)\beta\left(\frac{A_t - i_t}{\phi(i_t) + 1 - \delta}\right)\right]^{-1} && (4.128)\\
&= \left[E\left(\frac{1}{A_{t+1} - i_{t+1}}\right)\beta\left(\frac{1-\beta}{\beta\phi'(i_t)}\right)\right]^{-1} && (4.129)\\
&= \left[E\left(\frac{1}{A_{t+1} - i_{t+1}}\right)\right]^{-1}\frac{\phi'(i_t)}{1-\beta} && (4.130)
\end{aligned}
$$

The capital accumulation condition (4.105) is used in the third equality and the solution condition(4.107) for of i is used in the forth equality.

4.9.8 Asset Return and Risk-Free Rate under Sunspots

I recall that under proposition 4.4 and assumption 4.1, I have the expressions for asset return and risk-free rate as

$$
\begin{aligned}
E_t\left(R_{t+1} | i_t = i_L\right) &= \frac{\phi'(i_L)}{1-\beta} E_t\left[A_{t+1} - i\left[A_{t+1}\right] | i_t = i_L\right] && (4.131)\\
E_t\left(R_{t+1} | i_t = i_H\right) &= \frac{\phi'(i_H)}{1-\beta} E_t\left[A_{t+1} - i\left[A_{t+1}\right] | i_t = i_H\right] && (4.132)\\
\left[r_t^f \middle| i_t = i_L\right] &= \frac{\phi'(i_L)}{1-\beta}\left[E\left(\frac{1}{A\left(s^{t+1}\right) - i\left[A\left(s^{t+1}\right)\right]} \middle| i_t = i_L\right)\right]^{-1} && (4.133)\\
\left[r_t^f \middle| i_t = i_H\right] &= \frac{\phi'(i_H)}{1-\beta}\left[E\left(\frac{1}{A\left(s^{t+1}\right) - i\left[A\left(s^{t+1}\right)\right]} \middle| i_t = i_H\right)\right]^{-1} && (4.134)
\end{aligned}
$$

With the sunspots assumption 4.2, the expectation is calculated simply by adding two realisations with their corresponding possibilities. For example,

$$E_t(R_{t+1}|\,i_t = i_L) = \frac{\phi'(i_L)}{1-\beta} E_t[A_{t+1} - i[A_{t+1}]|\,i_t = i_L] \tag{4.135}$$

$$= \frac{\phi'(i_L)}{1-\beta}[(1-p_L)AiH + p_L AiL] \tag{4.136}$$

$$= \frac{\phi'(i_L)}{1-\beta}[AiH - p_L(AiH - AiL)] \tag{4.137}$$

where $AiL \equiv A_L - i_L$ and $AiH \equiv A_H - i_H$. The rest is obtained by identical methods.

4.9.9 Calculation of the Equation System in the Calibration

Basically, I have the equation system

$$1 = p_L\left[\beta\left(\frac{A_L - i_L}{A_L - i_L}(\phi(i_L) + 1 - \delta)\right)^{-1}\left(A_L - i_L + \frac{\phi(i_L) + 1 - \delta}{\phi'(i_L)}\right)\phi'(i_L)\right] + \tag{4.138}$$

$$(1-p_L)\left[\beta\left(\frac{A_H - i_H}{A_L - i_L}(\phi(i_L) + 1 - \delta)\right)^{-1}\left(A_H - i_H + \frac{\phi(i_H) + 1 - \delta}{\phi'(i_H)}\right)\phi'(i_L)\right]$$

$$1 = p_H\left[\beta\left(\frac{A_H - i_H}{A_H - i_H}(\phi(i_H) + 1 - \delta)\right)^{-1}\left(A_L - i_L + \frac{\phi(i_H) + 1 - \delta}{\phi'(i_H)}\right)\phi'(i_H)\right] + \tag{4.139}$$

$$(1-p_H)\left[\beta\left(\frac{A_L - i_L}{A_H - i_H}(\phi(i_H) + 1 - \delta)\right)^{-1}\left(A_L - i_L + \frac{\phi(i_L) + 1 - \delta}{\phi'(i_L)}\right)\phi'(i_H)\right]$$

To simplify the notation, I rearrange it into

$$p_L\beta\left(\frac{1}{F_L} + 1\right) + (1-p_L)\frac{\beta}{F_L}(1 + F_H) = 1 \tag{4.140}$$

$$p_H\beta\left(\frac{1}{F_H} + 1\right) + (1-p_H)\frac{\beta}{F_H}(1 + F_L) = 1 \tag{4.141}$$

where the notation of F follows the proposition 4.2.

$$F_H = \frac{\phi(i_H) + 1 - \delta}{\phi'(i_H)(A_H - i_H)} \tag{4.142}$$

$$F_L = \frac{\phi(i_L) + 1 - \delta}{\phi'(i_L)(A_L - i_L)} \tag{4.143}$$

Further, after some basic algebra, with the conditions that $\beta \neq 0$, $p_L \in (0, 1)$ and $p_H \in (0, 1)$, the equation system consisting of equation (4.140) and (4.140) has and only has one solution,

$$F_H = \frac{\beta}{1 - \beta} \tag{4.144}$$

$$F_L = \frac{\beta}{1 - \beta} \tag{4.145}$$

Since I have proofed the one to one relation between i and F in proposition 4.2, the conclusion follows.

4.9.10 Data

Data come from Fred Economic Data[8] and Quandl[9]. All data are quarterly data and adjusted by CPI growth rate. GDP growths are in per capita form. The Risk premium is calculated by the real return on stock index S&P 500 neglecting the real 10-year treasury constant maturity rate. The chapter uses the 1992 - 2001 as the high growth state and 2005 to 2014 as the low growth state.

[8]research.stlouisfed.org

[9]www.quandl.com/data/MULTPL/SP500_INFLADJ_MONTH

Chapter 5

A Regime Switching Model of the Prolonged Slump

5.1 Abstract

This chapter constructs an endogenous growth model to study the oscillation between the robust growth and stagnation in the economy. The model features constant return to scale in capital and extrinsic randomness, which imply multiple perfect foresight balanced growth paths. The presence of Epstein and Zin utility expands the degree of freedom in the model. Further, it enables the model to match the historical data of macroeconomic quantities and asset prices. The calibration suggests that the historical data moments can be accommodated by the model with persistent regimes and a high level of intertemporal elasticity of substitution.

5.2 Introduction

Though macroeconomic literature has not typically focused on medium-run evolution, there are discussions about the oscillation between healthy growth and relative stagnation of many developed countries. In an expanding amount of literature, Blanchard et al. (1997) and Comin and Gertler (2006) in particular refer to those oscillations that are longer than the business cycles in the traditional view as medium-term business cycles. These two papers share the point that the economy leaves many unexplained fluctuations if we simply treat the middle run as a period of transition from business cycle to steady growth. There are medium-term oscillations that do not fit in the business cycle theory since they happen with a relatively low frequency. Additionally, these oscillations are often accompanied by large volatilises in financial variables like the risk-free rate and asset returns.

This paper builds a model which incorporates the medium-term shifts of economic growth and the fluctuations in the risk-free rate and risk premium. It is a general equilibrium model in the production economy with complete markets. Additionally, it possesses (1) constant return to scale in product function, (2) convex cost on investment of firms (3) Epstein and Zin (1989) (EZ) preference of households, (4) the assumption that investment can feed back to productivity and (5) sunspots. Basically, it extends the model of the last chapter to a model with EZ preference. The recursive preference offers a degree of freedom on the intertemporal elasticity of substitution when I attempt to calibrate the model to match the predicted moments to the historical data. In addition, the regime switching property introduced by the sunspots also assists the calibration. Empirically, the calibrated regime switching model accommodates the data's persistent downturn in growth and investment, the long-term decrease in the risk-free rate and the counter-cyclicality in risk premium.

The constant return to scale in capital ensure endogenous growth in the equilibrium. Due to the adjustment cost of investment, investment is less efficient when it is at a high level. This inefficiency of investment introduces a trade-off in a firm's investment plan. Hence, firms need to identify an optimal level of investment according to the exogenous productivity parameter. Furthermore, I endogenise the technology. The productivity parameter is assumed to be in a one-to-one relation with the investment capital ratio. Since the equilibrium growth path is determined by the technology level and its corresponding optimal investment capital ratio, this assumption generates multiplicity. The economy can be trapped because, in some equilibria, its investment cannot sustain a relatively high level of productivity and this in return discourages the firm from investing. In fact, the idea can be dated back to literature on the self-reinforcement of economic growth. Among many others, Azariadis and Stachurski (2005) and Mookherjee and Ray (2001) survey the growth models dealing with "poverty traps" in the economic development.

Admittedly, this chapter inherits the main structure of the endogenous growth model from chapter 4. Nevertheless, this chapter replaces the log utility by the EZ framework to enrich the decision-making mechanism of the consumers. Since the utilisation of the EZ framework complicates the setups, the solutions are mathematically less elegant. Fortunately, most of the theoretical findings still remain. More importantly, two chapters have different aims. The main purpose of this chapter is to match the model predictions with the historical data in terms of the asset returns and the macroeconomic quantities. In other words, this chapter attempts to improve the calibration in the last chapter. The calibration in the last chapter introduced a trade-off between the macro-fundamental variables and the financial variables. The parameterisations that can generate proper macro-fundamental moments cannot offer reasonable values of financial variables and vice versa. In fact, half of the contents in this chapter focuses on the calibration, which also shapes the main contribution of this chapter. The EZ preference has a good reputation in explaining behaviours of the risk-free rate and the risk premium. With the help of minimisation of the loss function, the regime switching model under the EZ preference can match all of the 10 historical data moments with reasonable parameter values. The calibration suggests that the model needs parameters, which makes the regime very persistent, and a high level of intertemporal elasticity of substitution to generate the collected data moments.

There are some literature which attempt to provide theoretical foundations to the medium-term fluctuations. A closely related paper is Bambi et al. (2014). They add implementation delays to a standard endogenous growth model, which means there is time lag between the technology innovation and the adoption of this innovation in the production processes. By adding this assumption, economy features permanent endogenous fluctuations in macro-variables such as consumption and output. However, the mechanisms to achieve medium-term cycles between their paper and this one are different. This paper rely on multiplicity and regime-switching self-fulfilling sunspots. Additionally, this paper also consider fluctuations in risk-free rate and risk-premium.

The chapter is constructed as follows. Section 5.3 sets the model and briefly reviews the EZ framework. Section 5.4 establishes the optimal conditions and multiple equilibria. Section 5.5 calibrates the model in two different scenarios. Section 5.6 concludes.

5.3 Baseline Model Setups

This section builds the general equilibrium structures of the baseline model. Some proportions of the setups are inherited from the chapter 4. Nonetheless, for the independence of this chapter, I recall the key structures.

Time t discretely runs from 0 to infinity. There are a large number of identical firms and consumers in the economy. The production function is linear and given by $Y_t = A_t K_t$. Y, K and A denote the output, capital stock and exogenous technology scale factor respectively. Firm uses operation profit to pay the dividends $D_t \equiv A_t K_t - I_t$, where I is the investment. Same as in the last chapter, I restrict dividends to be positive, namely $D(s^t) > 0$.

The firm maximises its stock value represented by the discounted cash flow,

$$V_t = \underset{I}{Max} \quad \sum_{\tau=t}^{\infty} E_t \left[\beta^{\tau-t} \frac{\Lambda_\tau}{\Lambda_t} D_\tau \right] \tag{5.1}$$

and is subject to the constrains

$$\frac{K_{t+1}}{K_t} = g\left(i_t\right) \tag{5.2}$$

$$A_t > 0, \quad K_t > 0, \quad \Lambda_t > 0 \tag{5.3}$$

$$V_t > 0, \quad I_t > 0 \tag{5.4}$$

$$K_0 \; is \; given. \tag{5.5}$$

Where $\beta \in (0,\ 1)$ is the time preference parameter and $i \equiv I/K$. β and Λ together constitute the discount factor. The firms are price takers. They observe the discount factor and act accordingly. The equation (5.2) is the capital accumulation condition. The $g(\cdot)$ function describes the growth rate of the capital accumulation. On one hand, it captures the effectiveness in converting investment to capital inputs. On the other hand, one can take it as the production function of the capital goods. I use the name efficiency function or the production function for capital goods interchangeably. In most parts of this paper, the efficiency function is subject to restrictions as $1 > g'(i) > 0$ and $g''(i) \leq 0$. This assumption incorporates the concepts of investment adjustment cost and is consistent with the assumption that adjustment cost function is convex in the literature. However, later, the calibration section of this paper has a detailed discussion of this assumption. Lastly, I omit the subscript t when there is no ambiguity.

For the consumers, I briefly recall the origins of the recursive utility. In the standard utility time-separable preference, the consumers have the objective function as

$$J_t = E_t\left[\sum_{\tau=t}^{\infty} \beta^{\tau-t} U\left(C_\tau\right)\right] \tag{5.6}$$

With the constant relative risk aversion (CRRA) preference, it has a recursive equivalence given by

$$J_t = \frac{C_t^{1-\alpha} - 1}{1-\alpha} + \beta E_t\left(J_{t+1}\right) \tag{5.7}$$

With Epstein and Zin framework, the linear time-separable preference is generalised into a non-linear function given by

$$J_t = \left\{ C_t^{1-\rho} + \beta \left[E_t \left(J_{t+1}^{1-\gamma} \right) \right]^{\frac{1-\rho}{1-\gamma}} \right\}^{\frac{1}{1-\rho}} \tag{5.8}$$

where ρ is the reciprocal of the intertemporal elasticity of substitution (IES) of deterministic variations. γ is the risk aversion coefficient. If $\rho = \gamma$, it reduces to the power utility. Basically, the EZ framework separates ρ from γ.

The discount factor in this case is well established as shown in Weil (1989) and Cochrane (2005). Derivation is in the appendix 5.7.1.

$$DF_{t+1} = \left[\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\rho} \right]^{\theta} \left(\frac{1}{R_{t+1}} \right)^{1-\theta} \tag{5.9}$$

where R represents the return on the wealth portfolio and $\theta \equiv (1-\gamma)/(1-\rho)$. Similar to those general equilibrium models in production economy, the discount factor plays a vital role in the optimal condition of the equilibrium in next section.

5.4 Model Solution

The Bellman equation for the firm's problem is written as

$$V_t = \underset{I}{Max} \quad E_t \left(D_t + \beta \frac{\Lambda_{t+1}}{\Lambda_t} V_{t+1} \right) \tag{5.10}$$

The first order condition (FOC) with respect to investment to capital ratio i is given by,

$$E_t\left[\beta\frac{\Lambda_{t+1}}{\Lambda_t}\frac{\partial V_{t+1}}{\partial K_{t+1}}g'(i_t)\right]=1 \tag{5.11}$$

I follow the tradition to name the marginal price of capital $\partial V/\partial K$ marginal q. The $g(I/K)$ follows the assumption of Hayashi (1982) that it is homogeneous of degree one in I and K. Accordingly, the proposition assures that, under our conditions, marginal q $(\partial V/\partial K)$ equals the average Q (V/K). As a result, I can rewrite the bellman equation (5.10) into

$$\frac{V_t}{K_t}=\underset{I}{Max}\quad E_t\left[A_t-i_t+\beta\frac{\Lambda_{t+1}}{\Lambda_t}\frac{V_{t+1}}{K_{t+1}}g(i_t)\right] \tag{5.12}$$

The firm's Euler equation comes from the combination of the FOC (5.11) and equation (5.12).

$$E_t\left[\beta\frac{\Lambda_{t+1}}{\Lambda_t}\left(A_{t+1}-i_{t+1}+\frac{g(i_{t+1})}{g'(i_{t+1})}\right)g'(i_t)\right]=1 \tag{5.13}$$

In the standard asset pricing representation, it can be reformed into the form of Euler equation,

$$E_t\left[SDF_{t+1}R_{t+1}\right]=1 \tag{5.14}$$

where SDF stands for the stochastic discount factor. The expectation of the product of unique discount factor and any asset return should be unity. With the identity of the asset return as $R_{t+1}=V_{t+1}/(V_t-D_t)$, I combine the results from both two sides to construct the general equilibrium condition in proposition 5.1.

Proposition 5.1 *In the modelled economy, I join the first order condition from both firm's with consumer's side to obtain the following optimal stochastic difference* *equation.*

$$E_t\left\{\left[\beta\left(\frac{A_{t+1}-i_{t+1}}{A_t-i_t}\cdot g\left(i_t\right)\right)^{-\rho}\left(A_{t+1}-i_{t+1}+\frac{g\left(i_{t+1}\right)}{g'\left(i_{t+1}\right)}\right)g'\left(i_t\right)\right]^{\theta}\right\}=1 \quad (5.15)$$

Secondly, the expressions for stochastic discount factor SDF_{t+1}, risk free rate r_t^f and the expected risky asset return $E_t\left(R_{t+1}\right)$ are given by

$$SDF_{t+1}=\beta^{\theta}\left(\frac{A_t-i_t}{g\left(i_t\right)\left(A_{t+1}-i_{t+1}\right)}\right)^{\theta\rho}\left[\left(A_{t+1}-i_{t+1}+\frac{g\left(i_{t+1}\right)}{g'\left(i_{t+1}\right)}\right)g'\left(i_t\right)\right]^{\theta-1} \quad (5.16)$$

$$r_t^f=\frac{1}{E_t\left(SDF_{t+1}\right)} \quad (5.17)$$

$$R_{t+1}=g'\left(i_t\right)\left(A_{t+1}-i_{t+1}+\frac{g\left(i_{t+1}\right)}{g'\left(i_{t+1}\right)}\right) \quad (5.18)$$

For later reference, here I define a functional representation of the variables as $SDF_{t+1}=SDF\left(i_t,\, i_{t+1}\right)$, $r_t^f=r^f\left(i_t,\, i_{t+1}\right)$ and $R_{t+1}=R\left(i_t,\, i_{t+1}\right)$.

Proof. See appendix 5.7.2.

Unfortunately, the stochastic difference equation 5.15 dose not have a general solution as we had in the last chapter. However, solution for the deterministic BGP is still available.

5.4.1 Deterministic BGPs

In this subsection, I examine the deterministic growth path. Ignoring the stochastic components and fixing the productivity parameter A, proposition 5.2 offers the condition, which allows for a BGP in the economy.

Proposition 5.2 *According to the stochastic difference equation* (5.15)*, the deterministic BGP should be at the level of $i=\bar{i}$ which solves the following equation*

$$\frac{(A-i)\,g'(i)+g(i)}{g(i)^{\rho}}=\frac{1}{\beta} \quad (5.19)$$

In this BGP, the transversality condition holds. Appendix 5.7.4 presents a calculation. Additionally, the marginal effect of the technology A to investment to capital ratio is derived by the implicit function theorem.

$$\frac{\partial i}{\partial A} = \left[(A-i)\left(\rho\frac{g'(i)}{g(i)} - \frac{g''(i)}{g'(i)}\right) + \rho\right]^{-1} > 0 \tag{5.20}$$

$$\frac{\partial^2 i}{\partial A^2} = -\left(\rho\frac{g'(i)}{g(i)} - \frac{g''(i)}{g'(i)}\right)\left[(A-i)\left(\rho\frac{g'(i)}{g(i)} - \frac{g''(i)}{g'(i)}\right) + \rho\right]^{-2} < 0 \tag{5.21}$$

Proof. In the deterministic case where the productivity parameter A is a constant, according to the difference equation (5.15), the economy obtains the balanced growth when

$$\left[\beta\left(\frac{A-i}{A-i}\cdot g(i)\right)^{-\rho}\left(A-i+\frac{g(i)}{g'(i)}\right)g'(i)\right]^{\theta} = 1 \tag{5.22}$$

Basic algebra offers,

$$\frac{(A-i)\,g'(i)+g(i)}{g(i)^{\rho}} = \frac{1}{\beta} \tag{5.23}$$

I directly apply the single variable implicit function theorem to obtain the derivatives. The signs of the derivatives are implied by the assumptions of $g\prime(i) > 0$ and $g\prime\prime(i) < 0$

Q.E.D. ■

The logics here are similar to those in the last chapter. Static comparison shows that firm raises its investment to catch up to the productivity. This is optimal since it increases future output and the present value of dividend cash flow. However, there are trade-offs involved. The first trade off comes from the adjustment cost of the investment. As investment being raised, the investment itself becomes less and less efficient, which harms the potential growth of the further output. The second trade off is the typical story of the asset pricing mechanism. The substitution effect conflicts with the income effects. Hence, a high dividend growth alone dose not guarantee the high price of the asset since it means low marginal utility in the future.

Again, it is necessary to develop proposition 5.3 to guarantee that the equation (5.19) has the unique root.

Proposition 5.3 *The following condition ensures that equation* (5.19) *has the unique BGP.*

$$\frac{Ag'(0)+g(0)}{g(0)^{\rho}} > \frac{1}{\beta} > g(A)^{1-\rho} \tag{5.24}$$

Proof. The left hand side of the equation (5.19) has the derivative with respect to i as

$$g'(i)\,g(i)^{\rho}\left[(A-i)\left(\frac{g''(i)}{g'(i)}-\rho\frac{g'(i)}{g(i)}\right)-\rho\right]<0 \tag{5.25}$$

Since in the model setups, the dividends payments are assumed to be positive $D>0$, the investment-capital ratio i chosen by the firm will not exceed technology scale A. Hence, the above expression can be draw under the condition of $A-i>0$, $g(i)>0$, $g'(i)>0$ and $g''(i)<0$.

As a result, the left hand side of the equation (5.19) is monotonically decreasing with the investment to capital ration i in the range of $(0,A)$. Since it is a continuous function, we only need to specify that the starting and ending points are at opposite sides of $1/\beta$. Therefore, the inequality in the proposition follows.

Q.E.D. ■

Accordingly, the expressions of the other variables on this BGP are derived in the following proposition.

Proposition 5.4 *Due to the solution of the investment to capital ratio $\bar{i}$ to equation* (5.19)*, the following economic variables keep constant at this BGP.*

$$\frac{C_{t+1}}{C_t} = g(\bar{i}) \tag{5.26}$$

$$r^f_{t+1} = \frac{1}{DF_{t+1}} = \frac{g(\bar{i})^{\rho}}{\beta} \tag{5.27}$$

where r^f is the risk free rate.

Proof. See Appendix 5.7.3.

Furthermore, naturally, the stability or local determinacy of the growth path is of interest. Hence I present proposition 5.5 to show that, similar to the situation in the last section, the BGP defined in equation (5.19) is locally unstable.

Proposition 5.5 *According to the Euler difference equation* (5.15)*, I conclude that the BGP defined at proposition 5.2 is locally unstable. If we initialise the economy in the local area of the BGP, the economy diverges.*

Proof. I apply the implicit function theorem on the difference equation (5.15) and yield

$$\begin{aligned}\frac{\partial i_{t+1}}{\partial i} = & \frac{g(i_{t+1}) + (A_{t+1} - i_{t+1})\, g'(i_{t+1})}{g(i_t)} \times \\ & \frac{A_{t+1} - i_{t+1}}{A_t - i_t} \times \frac{g'(i_{t+1})}{g'(i_t)} \times \\ & \frac{(A_t - i_t)\,\rho g'(i_t)^2 + g(i_t)\left(\rho g'(i_t) - (A_t - i_t)\, g''(i_t)\right)}{(A_{t+1} - i_{t+1})\,\rho g'(i_{t+1})^2 + g(i_{t+1})\left(\rho g'(i_{t+1}) - (A_{t+1} - i_{t+1})\, g''(i_{t+1})\right)}\end{aligned} \tag{5.28}$$

Thus,

$$\left.\frac{\partial i_{t+1}}{\partial i_t}\right|_{\substack{i_{t+1} \to i_t \to \bar{i} \\ A_{t+1} \to A_t \to A}} = 1 + \frac{(A - \bar{i})\, g'(\bar{i})}{g(\bar{i})} > 1 \tag{5.29}$$

Therefore in the local area around the BGP, the paths are unstable.

Q.E.D. ■

Due to the non-linear structure of the difference equation (5.15), it is rather difficult to derive the analytical solution and to examine all the paths. Since the aim is to study BGPs, this paper ignores the unstable equilibrium paths. With the all these properties in the BGP, the model is ready to accommodate multiplicities.

5.4.2 Threshold Assumption and Multiplicity

Similar to the assumption in chapter 4, this section offers the threshold assumption between the productivity parameter A and investment to capital ratio i.

Assumption 5.1 *Productivity parameter A is a threshold function of the investment to capital ratio i.*

$$A(i_t) = \begin{cases} A_H; & if\ i_t \geq i^* \\ A_L; & if\ i_t < i^* \end{cases} \tag{5.30}$$

I define i_H and i_L to be the corresponding solution to the equation (5.19) *with A_H and A_L respectively.*

The assumption describes a threshold relation. The investment-capital ratio should be maintained above a certain level, i^*, to assure a high performance of technology level A_H. Otherwise, the technology is trapped in a relatively low level A_L. Further, the discontinuous function of technology A ensures two solutions of equation (5.19). There are two equilibria such as $\{A_H,\ i_H\}$ and $\{A_L,\ i_L\}$ for the firms to select.

Now the baseline model is fully developed in the Epstein and Zin framework. The theoretical model generally re-exhibits the properties shown in the log utility. However, the recursive structure and the separation of IES $1/\rho$ from risk aversion γ make the solutions less elegant and less intuitive. In the last chapter, I show that the model in the log utility has trade-offs in the calibration. It cannot captures the macro-fundamental variables and the financial variables simultaneously. In the next

Variables	$i = I/K$	I/Y	Growth	r^f	$E(R) - r^f$
1992 - 2001 (High Growth)	11.5%	34.8%	2.28%	3.58%	2.4%
2005 - 2014 (Low Growth)	10.4%	32%	0.65%	1.1%	5.6%

Table 5.1 Data Moments

section, the model in the EZ framework is calibrated in two alternative ways to overcome the problem appeared in the last chapter.

5.5 Historical Data Moments and Calibration

In this section, the model is calibrated to compare the predictions with the data moments in the US. Table 5.1 shows the 10 years mean of indicated variables in the US. Data are collected from 2 different decades. From 1992 to 2001, the US had a relative fast growing decade with average real GDP growth at 2.28%. For the slow growing decade, between 2005 and 2014, the US' average real GDP growth slowed down to only 0.65%. Data used in here are the same as used in chapter 4. Different from the method used in the last chapter, the strategy in here is taking the moments presented in table 5.1 back to the model and see if the model can find reasonable values for the parameters which coordinates with the data moments.

The model predicts a branch of analytical solution for the above moments indifferent balanced growth paths. In fact, on one hand, the model yields the theoretical moments for

$$\frac{I}{Y} = \frac{I/K}{AK/K} = \frac{i}{A} \tag{5.31}$$

$$\frac{Y_{t+1}}{Y_t} = \frac{K_{t+1}}{K_t} = g(\bar{i}) \tag{5.32}$$

$$r^f_{t+1} = \frac{1}{DF_{t+1}} = \frac{g(\bar{i})^{\rho}}{\beta} \tag{5.33}$$

On the other hand, I substitute the collected data moments into the model and assume

$$i_L = 0.104, \qquad i_H = 0.115 \tag{5.34}$$

$$\frac{i_L}{A_L} = 0.32, \qquad \frac{i_H}{A_H} = 0.348 \tag{5.35}$$

$$g(i_L) = 1.0065, \quad g(i_H) = 1.0228 \tag{5.36}$$

$$r_L^f = 1.011, \qquad r_H^f = 1.0358 \tag{5.37}$$

$$E(R_L) = 1.067, \qquad E(R_H) = 1.0698 \tag{5.38}$$

Additionally, for the convenience of later calculation, the first two rows simply offers that

$$A_L = 0.325, \qquad A_H = 0.3305 \tag{5.39}$$

The experiments are in two different situation. Start from the simple one, the model is calibrated in two separate BGPs. Then, I construct the regime switching model and examine the corresponding parameters yield by that model.

5.5.1 Two Separate BGPs

Proposition 5.4 offers the solution for the risk-free rate. For two separate BGPs, we can obtain the value parameters for ρ and β from the solution of the following equation system.

$$\frac{g(i_L)^\rho}{\beta} = r_L^f \tag{5.40}$$

$$\frac{g(i_H)^\rho}{\beta} = r_H^f \tag{5.41}$$

This equation system can be solved for $\rho = 1.51$ and $\beta = 0.998$. Secondly, the equation (5.19) system is rearranged to

$$(A_L - i_L)\, g'(i_L) + g(i_L) = r_L^f \tag{5.42}$$

$$(A_H - i_H)\, g'(i_H) + g(i_H) = r_H^f \tag{5.43}$$

The equation system receives the values from data moments expect for $g'(i_L)$ and $g'(i_H)$. Hence, it can be solved for $g'(i_L) = 0.02$ and $g'(i_H) = 0.06$.

Since the data show that $\bar{i}_L = 0.104$ and $\bar{i}_H = 0.115$, the calibration results actually against our assumption in the theoretically model that $g''(i) < 0$. I save the discussion of this in next subsection. In fact, it is indeed a bit unreasonable to assume the data moments are generated by two unrelated separate BGPs. However, the exercises in this subsection offers anchors of the parameter values of β, ρ, $g'(i_L)$ and $g'(i_H)$.

5.5.2 Regime Switches

This subsection develops a situation which offers shift between two BGPs. The framework follows the setups of regime switching equilibrium in the last chapter. Here, I offer assumption 5.2.

Assumption 5.2 *s_t is an extrinsic random variable which governs the system and acts like a selection device in the model. It can represent the symbol of beliefs of the agents in the economy. $\{s_t\}_0^\infty$ is a Markov chain with state space $\{s_L, s_H\}$ and transitional matrix $\mathbf{P_s}$ given by*

$$\begin{pmatrix} p_L & 1-p_L \\ 1-p_H & p_H \end{pmatrix} \tag{5.44}$$

The scenario is straight forward to explain. There are two exogenous level of the productivity parameter A, namely A_L and A_H. Agents in the economy know the distribution of the Markovian process $\{s_t\}_0^\infty$. When they observe the symbol, they select the equilibrium and act optimally according to the Euler equation. More importantly, the purpose is to examine if the model can jointly generate all the moments in table 5.1.

Again, the calibration starts from key stochastic difference Euler equation (5.15).

$$E_t\left\{\left[\beta\left(\frac{A_{t+1}-i_{t+1}}{A_t-i_t}\cdot g\left(i_t\right)\right)^{-\rho}\left(A_{t+1}-i_{t+1}+\frac{g\left(i_{t+1}\right)}{g'\left(i_{t+1}\right)}\right)g'\left(i_t\right)\right]^{\theta}\right\}=1 \tag{5.45}$$

In the functional expression suggested in proposition 5.1, it has the reduced form given by

$$E_t\left[SDF\left(A_t,\, A_{t+1},\, i_t,\, i_{t+1}\right)R\left(A_{t+1},\, i_t,\, i_{t+1}\right)\right]=1 \tag{5.46}$$

Once again, I substitute all the historical data moments back into the model and see if the model can have reasonable parameters to coordinate with them. According to the probability distribution of the regime switching framework, the economy faces the following difference equation system.

$$\begin{aligned}1 &= p_L SDF\left(A_L,\, A_L,\, i_L,\, i_L\right)R\left(A_L,\, i_L,\, i_L\right)+\\ &\quad\left(1-p_L\right)SDF\left(A_L,\, A_H,\, i_L,\, i_H\right)R\left(A_H,\, i_L,\, i_H\right) \qquad (5.47)\\ 1 &= p_H SDF\left(A_H,\, A_H,\, i_H,\, i_H\right)R\left(A_H,\, i_H,\, i_H\right)+\\ &\quad\left(1-p_H\right)SDF\left(A_H,\, A_L,\, i_H,\, i_L\right)R\left(A_L,\, i_H,\, i_L\right) \qquad (5.48)\end{aligned}$$

To explain, these two equations represent the current state being at slow growth state or high growth state respectively. In total, there are seven unknowns in the equations, which are β, ρ, γ, p_L, p_H, $g'(i_L)$ and $g'(i_H)$.

Additionally, I decompose the Euler equation to find places for the risk-free rate r^f and expected risk assets returns $E(R)$ of the data moments. In fact, the expectation of product between two terms has the feature that

$$1 = E_t\left[SDF\left(A_t,\, A_{t+1},\, i_t,\, i_{t+1}\right)R\left(A_{t+1},\, i_t,\, i_{t+1}\right)\right] \tag{5.49}$$

$$= E_t\left[SDF\left(A_t,\, A_{t+1},\, i_t,\, i_{t+1}\right)\right]E_t\left[R\left(A_{t+1},\, i_t,\, i_{t+1}\right)\right]+ Cov\left[SDF\left(A_t,\, A_{t+1},\, i_t,\, i_{t+1}\right),\, R\left(A_{t+1},\, i_t,\, i_{t+1}\right)\right] \tag{5.50}$$

Therefore, the first two expectation terms in the equation (5.50) are the places to accommodate r_L^f, r_H^f, $E\left(R_L\right)$ and $E\left(R_H\right)$. For example, with the definition of the risk-free rate in proposition 5.1 and the the definition for covariance, the model in the low growth state should follow

$$1 = \frac{E\left(R_L\right)}{r_L^f} + E_t\left\{\left[SDF\left(A_t,\, A_{t+1},\, i_t,\, i_{t+1}\right) - \frac{1}{r_L^f}\right]\left[R\left(A_{t+1},\, i_t,\, i_{t+1}\right) - E\left(R_L\right)\right]\right\} \tag{5.51}$$

Furthermore, the expectation in the second term can be calculated by the probability p_L and p_H. In all, the decomposition of the Euler difference equation system offers

$$1 = \frac{E\left(R_L\right)}{r_L^f} + p_L\left[SDF\left(A_L,\, A_L,\, i_L,\, i_L\right) - \frac{1}{r_L^f}\right]\left[R\left(A_L,\, i_L,\, i_L\right) - E\left(R_L\right)\right] + \left(1-p_L\right)\left[SDF\left(A_L,\, A_H,\, i_L,\, i_H\right) - \frac{1}{r_L^f}\right]\left[R\left(A_H,\, i_L,\, i_H\right) - E\left(R_L\right)\right] \tag{5.52}$$

$$1 = \frac{E\left(R_H\right)}{r_H^f} + p_H\left[SDF\left(A_H,\, A_H,\, i_H,\, i_H\right) - \frac{1}{r_H^f}\right]\left[R\left(A_H,\, i_H,\, i_H\right) - E\left(R_H\right)\right] + \left(1-p_H\right)\left[SDF\left(A_H,\, A_L,\, i_H,\, i_L\right) - \frac{1}{r_H^f}\right]\left[R\left(A_L,\, i_H,\, i_L\right) - E\left(R_H\right)\right] \tag{5.53}$$

Once again, this is a equation system for unknowns of β, ρ, γ, p_L, p_H, $g'\left(i_L\right)$ and $g'\left(i_H\right)$. Together with the previous two equations (equation (5.47) and (5.48)), we have a equation system with 4 equations and 7 unknowns. Ideally, if we calibrate three of them, say β, ρ and γ, the rest of the variables can be solved.

Unfortunately, the equation system is highly non-linear and has no closed form solution. In fact, numerical methods provided by computational software also have difficulties to offer a numerical solution. Therefore, I resort to the minimisation of loss function. The loss function is simply set as the sum of the square of the difference in 4 equations. To simplify of the notation, Υ_1, Υ_2, Υ_3 and Υ_4 are used to denote the right hand sides of the equation (5.47), (5.48), (5.52) and (5.53) respectively in the following loss function.

$$LF(\eta) = \sum_{n=1}^{4} \left[1 - \Upsilon_n(\eta)\right]^2 \tag{5.54}$$

where η is the vector of unknowns, namely $(\beta, \gamma, \rho, p_L, p_H, g'(i_L), g'(i_H))$.

To save the computational power, I calibrate the risk aversion parameter $\gamma = 7$ following the suggestion in Bansal and Yaron (2004). Standard numerical minimisation method in the computational software then offers that when $\beta = 0.979$, $\rho = 0.11$, $p_L = 0.952$, $p_H = 0.999$, $g'(i_L) = 0.034$ and $g'(i_H) = 0.016$, the loss function LF is minimised to 2.7×10^{-11}. Meanwhile, we have $\Upsilon_1 = 0.9994$, $\Upsilon_2 = 0.9990$, $\Upsilon_3 = 1.00004$ and $\Upsilon_4 = 1.00169$. Since the purpose of this experiments is to conduct a calibration, I conclude the values above are fairly close and reasonable. With these parameterisations, the model matches the predictions with the data moments well.

When all of the data moments are taken into consideration, the first feature of the regime switching model is that it needs considerably persistent parameters of $p_L = 0.952$ and $p_H = 0.999$ to match the data moments. With the probabilities close to 1, the economy tend to stay in the current state. If the current state is the slow growth state, the economy tend to be trapped for a long period.

Second, the derivatives of the efficiency function $g'(i_L) = 0.034$ and $g'(i_H) = 0.016$ are different from the values obtained in the calibrations of two separate BGPs, namely $g'(i_L) = 0.02$ and $g'(i_H) = 0.06$. In fact, the former still follows the assumption of $g''(i) < 0$. In fact, literature has no consensus on the proper curvature of the production function $g(i)$ of the capital goods in reality. Many researchers assume that the adjustment cost for investment is convex so that production function of the capital goods should be concave. However, one also can justify the convex production function of the capital goods by complementarities or externalities. Though the model in this paper is not capable of judge these two, it is clear that the second case is empirically more plausible since it can simultaneously capture all 10 moments from the data.

Third, the parameter of 1/IES, $\rho = 0.11$, is significantly different from the value $\rho = 1.51$ in the model of two separated model. In fact, in the literature, the value for IES is controversial. Estimations from Hall (1988) suggest low values for IES around 0.5 which makes ρ around 2. However, research such as Bansal and Yaron (2004) and van Binsbergen et al. (2012) update the value to a level which is larger than 1. In Bansal and Yaron (2004), they suggest the estimation of IES should be modified to around 1.5 when takes the effects of time-varying consumption volatility into consideration. Hence, the parameter ρ is around 0.66. Given the complex structures in the calibrations, the model is not able to justify the values in the calibration. However, according to the literature, I believe that the value obtained is not entirely ridiculous.

5.6 Conclusion

This chapter inherits the key setups from chapter 4 and explores the empirical feasibility of the model to account for the stagnated growth, the long period of lowered risk-free rate and the counter-cyclicality in risk premium.

However, I make an important modification in the preference. This chapter replace the log utility used in the last chapter by the recursive utility which is the EZ preference. This framework gives the degree of freedom on the intertemporal elasticity of substitution. Therefore, the model exhibits more explanatory power on the data.

In contrast to the calibration in chapter 4, whether or not the agents are aware of the regime switches and the corresponding probability distribution is important. The calibration of the regime switching model is empirically more plausible than the calibration of the two separated BGPs. With reasonable values of parameters, the former is capable of accommodating the data moments not only from the macro-fundamentals as growth rate, investment to capital ratio, investment GDP ratio, but also from the asset pricing side namely risk free rate and risk premium from two periods.

Admittedly, this paper is a preliminary attempt for bring the theoretical framework of the endogenous growing regime-switching model in chapter 4 into the data. A further work could focus on some more serious empirical studies. For example, although the non-linear structure of the Euler difference equation puts some difficulties on the estimation we still have many methods in the field of Bayesian estimation to deal with non-linear dynamic stochastic general equilibrium models.

5.7 Appendix

5.7.1 Derivation of the Discount Factor in the Epstein and Zin Framework

I start with the recursive value function for the consumer,

$$J_t = \left\{ C_t^{1-\rho} + \beta \left[E_t \left(J_{t+1}^{1-\gamma} \right) \right]^{\frac{1-\rho}{1-\gamma}} \right\}^{\frac{1}{1-\rho}} \tag{5.55}$$

Since the value function J_t is homogeneous of degree one, I can rewrite the function according to Euler's Homogeneous Function Theorem

$$J_t = \frac{\partial J_t}{\partial C_t} \cdot C_t + E_t \left(\frac{\partial J_t}{\partial J_{t+1}} \cdot J_{t+1} \right) \tag{5.56}$$

The partial derivatives can be derived as

$$\frac{\partial J_t}{\partial C_t} = J_t^{\rho} C_t^{-\rho} \tag{5.57}$$

$$\frac{\partial J_t}{\partial \left(\left[E_t \left(J_{t+1}^{1-\gamma} \right) \right]^{\frac{1}{1-\gamma}} \right)} = J_t^{\rho} \beta \left(\left[E_t \left(J_{t+1}^{1-\gamma} \right) \right]^{\frac{1}{1-\gamma}} \right)^{-\rho} \tag{5.58}$$

$$\frac{\partial \left(\left[E_t \left(J_{t+1}^{1-\gamma} \right) \right]^{\frac{1}{1-\gamma}} \right)}{\partial J_{t+1}} = \left(\left[E_t \left(J_{t+1}^{1-\gamma} \right) \right]^{\frac{\gamma}{1-\gamma}} \right) J_{t+1}^{-\gamma} \tag{5.59}$$

$$\frac{\partial J_t}{\partial J_{t+1}} = \frac{\partial J_t}{\partial \left(\left[E_t \left(J_{t+1}^{1-\gamma} \right) \right]^{\frac{1}{1-\gamma}} \right)} \frac{\partial \left(\left[E_t \left(J_{t+1}^{1-\gamma} \right) \right]^{\frac{1}{1-\gamma}} \right)}{\partial J_{t+1}} \tag{5.60}$$

$$= J_t^{\rho} \beta \left(\left[E_t \left(J_{t+1}^{1-\gamma} \right) \right]^{\frac{1}{1-\gamma}} \right)^{\gamma-\rho} J_{t+1}^{-\gamma} \tag{5.61}$$

Hence, the intertemporal marginal rate of substitution is given by

$$IMRS_{t+1} = \frac{\frac{\partial J_t}{\partial J_{t+1}} \frac{\partial J_{t+1}}{\partial C_{t+1}}}{\frac{\partial J_t}{\partial C_t}} \tag{5.62}$$

$$= \frac{J_t^{\rho} \beta \left(\left[E_t \left(J_{t+1}^{1-\gamma} \right) \right]^{\frac{1}{1-\gamma}} \right)^{\gamma-\rho} J_{t+1}^{-\gamma} J_{t+1}^{\rho} C_{t+1}^{-\rho}}{J_t^{\rho} C_t^{-\rho}} \tag{5.63}$$

$$= \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\rho} \left(\frac{J_{t+1}}{\left[E_t \left(J_{t+1}^{1-\gamma} \right) \right]^{\frac{1}{1-\gamma}}} \right)^{\rho-\gamma} \tag{5.64}$$

Define the total wealth W_t and the cum-dividend return $R_{w,t+1}$ on wealth to be

$$W_t = C_t + E_t IMRS_{t+1} W_{t+1} \tag{5.65}$$

$$R_{w,t+1} = \frac{W_{t+1}}{W_t - C_t} \tag{5.66}$$

Recall that I have

$$J_t = \frac{\partial J_t}{\partial C_t} \cdot C_t + E_t \left(\frac{\partial J_t}{\partial V_{t+1}} \cdot J_{t+1} \right) \tag{5.67}$$

Therefore,

$$W_t = \frac{J_t}{\partial J_t / \partial C_t} \tag{5.68}$$

Further, the relation between wealth return and $IMRS_{t+1}$ is established as

$$R_{w,t+1} = \frac{\frac{J_{t+1}}{\partial J_{t+1}/\partial C_{t+1}}}{\frac{J_t}{\partial J_t/\partial C_t} - C_t} \tag{5.69}$$

$$= \left(\frac{C_{t+1}}{C_t}\right)^{\rho} \frac{J_{t+1}^{1-\rho}}{J_t^{1-\rho} - C_t^{1-\rho}} \tag{5.70}$$

$$= \frac{1}{\beta}\left(\frac{C_{t+1}}{C_t}\right)^{\rho} \left(\frac{J_{t+1}}{\left[E_t\left(J_{t+1}^{1-\gamma}\right)\right]^{\frac{1}{1-\gamma}}}\right)^{1-\rho} \tag{5.71}$$

$$\frac{J_{t+1}}{\left[E_t\left(J_{t+1}^{1-\gamma}\right)\right]^{\frac{1}{1-\gamma}}} = \left[R_{w,t+1}\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\rho}\right]^{\frac{1}{1-\rho}} \tag{5.72}$$

$$IMRS_{t+1} = \beta\left(\frac{C_{t+1}}{C_t}\right)^{-\rho}\left(\frac{J_{t+1}}{\left[E_t\left(J_{t+1}^{1-\gamma}\right)\right]^{\frac{1}{1-\gamma}}}\right)^{\rho-\gamma} \tag{5.73}$$

$$= \beta\left(\frac{C_{t+1}}{C_t}\right)^{-\rho}\left[R_{w,t+1}\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\rho}\right]^{\frac{\rho-\gamma}{1-\rho}} \tag{5.74}$$

$$= \beta^{1+\frac{\rho-\gamma}{1-\rho}}\left(\frac{C_{t+1}}{C_t}\right)^{-\rho\left(1+\frac{\rho-\gamma}{1-\rho}\right)} R_{w,t+1}^{\frac{\rho-\gamma}{1-\rho}} \tag{5.75}$$

$$= \left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\rho}\right]^{\theta}\left(\frac{1}{R_{w,t+1}}\right)^{1-\theta} \tag{5.76}$$

Where $\theta \equiv (1-\gamma)/(1-\rho)$. In the third equality, I apply the fact that $J_t^{1-\rho} = C_t^{1-\rho} + \beta\left[E_t\left(J_{t+1}^{1-\gamma}\right)\right]^{\frac{1-\rho}{1-\gamma}}$.

5.7.2 Proof of Proposition 5.1

Proof. We have the optimal condition for the firm's problem as

$$E_t\left[\beta\frac{\Lambda_{t+1}}{\Lambda_t}\left(A_{t+1} - i_{t+1} + \frac{g(i_{t+1})}{g'(i_{t+1})}\right)g'(i_t)\right] = 1 \tag{5.77}$$

In the standard asset pricing representation, it can be reformed into the form of Euler equation,

$$E_t\left[DF_{t+1}R_{t+1}\right] = 1 \tag{5.78}$$

It is obvious to identify the asset return in the model to be

$$R_{t+1} = \left(A_{t+1} - i_{t+1} + \frac{g\left(i_{t+1}\right)}{g'\left(i_{t+1}\right)}\right) g'\left(i_t\right) \tag{5.79}$$

Recall the discount factor derived from consumer's problem as

$$DF_{t+1} = \left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\rho}\right]^{\theta}\left(\frac{1}{R_{t+1}}\right)^{1-\theta} \tag{5.80}$$

With all these preparation and the market clearing condition $C = D$, our Euler equation can be rewritten as

$$E_t\left\{\left[\beta\left(\frac{\left(A_{t+1} - i_{t+1}\right)K_{t+1}}{\left(A_t - i_t\right)K_t}\right)^{-\rho}\left(A_{t+1} - i_{t+1} + \frac{g\left(i_{t+1}\right)}{g'\left(i_{t+1}\right)}\right) g'\left(i_t\right)\right]^{\theta}\right\} = 1 \tag{5.81}$$

$$E_t\left\{\left[\beta\left(\frac{A_{t+1} - i_{t+1}}{A_t - i_t}\cdot g\left(i_t\right)\right)^{-\rho}\left(A_{t+1} - i_{t+1} + \frac{g\left(i_{t+1}\right)}{g'\left(i_{t+1}\right)}\right) g'\left(i_t\right)\right]^{\theta}\right\} = 1 \tag{5.82}$$

By definition, the risk free rate is

$$r_t^f = \frac{1}{E_t\left[DF_{t+1}\right]} \tag{5.83}$$

$$= \left[E_t\left\{\beta^{\theta}\left(\frac{A_{t+1} - i_{t+1}}{A_t - i_t}\cdot g\left(i_t\right)\right)^{-\theta\rho} \times \left[\left(A_{t+1} - i_{t+1} + \frac{g\left(i_{t+1}\right)}{g'\left(i_{t+1}\right)}\right) g'\left(i_t\right)\right]^{\theta-1}\right\}\right]^{-1} \tag{5.84}$$

For the price dividend ratio and decomposition of the asset return, I firstly derive the average price of capital V_t/K_t as a corner stone. I have FOC for the firm's problem as

$$E_t\left[\beta\frac{\Lambda_{t+1}}{\Lambda_t}\frac{\partial V_{t+1}}{\partial K_{t+1}}g'(i_t)\right]=1 \tag{5.85}$$

Hayashi (1982) has the proposition which enables us to rewrite the bellman equation (5.10)

$$\frac{V_t}{K_t}=\underset{I}{Max}\quad E_t\left[A_t-i_t+\beta\frac{\Lambda_{t+1}}{\Lambda_t}\frac{V_{t+1}}{K_{t+1}}g(i_t)\right] \tag{5.86}$$

Together, I can write

$$\frac{V_t}{K_t}=A_t-i_t+\frac{g(i_t)}{g'(i_t)} \tag{5.87}$$

Hence, the price dividend ratio is

$$\frac{V_t-D_t}{D_t}=\frac{V_t/K_t}{D_t/K_t}-1 \tag{5.88}$$

$$=\frac{A_t-i_t+\frac{g(i_t)}{g'(i_t)}}{A_t-i_t}-1 \tag{5.89}$$

$$=\frac{g(i_t)}{g'(i_t)(A_t-i_t)} \tag{5.90}$$

Further, the decomposition of the asset return can be derived,

$$E_t\left(\frac{D_{t+1}}{V_t-D_t}\right)=E_t\left(\frac{D_{t+1}/K_{t+1}}{V_t/K_t-D_t/K_t}\cdot g\left(i_t\right)\right) \tag{5.91}$$

$$=E_t\left(\frac{A_{t+1}-i_{t+1}}{A_t-i_t+\frac{g(i_t)}{g'(i_t)}-A_t+i_t}\cdot g\left(i_t\right)\right) \tag{5.92}$$

$$=g'\left(i_t\right)E_t\left(A_{t+1}-i_{t+1}\right) \tag{5.93}$$

$$\tag{5.94}$$

$$E_t\left(\frac{V_{t+1}-D_{t+1}}{V_t-D_t}\right)=E_t\left(\frac{V_{t+1}/K_{t+1}-D_{t+1}/K_{t+1}}{V_t/K_t-D_t/K_t}\cdot g\left(i_t\right)\right) \tag{5.95}$$

$$=E_t\left(\frac{A_{t+1}-i_{t+1}+\frac{g(i_{t+1})}{g'(i_{t+1})}-A_{t+1}+i_{t+1}}{A_t-i_t+\frac{g(i_t)}{g'(i_t)}-A_t+i_t}\cdot g\left(i_t\right)\right) \tag{5.96}$$

$$=g'\left(i_t\right)E_t\left(\frac{g\left(i_{t+1}\right)}{g'\left(i_{t+1}\right)}\right) \tag{5.97}$$

Q.E.D. ■

5.7.3 Proof of Proposition 5.4

Proof. For growth rate of consumption and output, I have

$$\frac{Y_{t+1}}{Y_t}=\frac{AK_{t+1}}{AK_t} \tag{5.98}$$

$$=g\left(\bar{i}\right) \tag{5.99}$$

$$\frac{C_{t+1}}{C_t}=\frac{\left(A-\bar{i}\right)K_{t+1}}{\left(A-\bar{i}\right)K_t}=g\left(\bar{i}\right) \tag{5.100}$$

$$DF_{t+1} = \left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\rho}\right]^{\theta}\left(\frac{1}{R_{t+1}}\right)^{1-\theta} \tag{5.101}$$

$$= \left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\rho}\right]^{\theta}\left[\left(A_{t+1} - i_{t+1} + \frac{g(i_{t+1})}{g'(i_{t+1})}\right)g'(i_t)\right]^{\theta-1} \tag{5.102}$$

$$= \left[\beta\left(g(i_t)\right)^{-\rho}\right]^{\theta}\left[(A_t - i_t)g'(i_t) + g(i_t)\right]^{\theta-1} \tag{5.103}$$

$$1/DF_{t+1} = \left[\beta\left(g(i_t)\right)^{-\rho}\right]^{-\theta}\left[(A_t - i_t)g'(i_t) + g(i_t)\right]^{1-\theta} \tag{5.104}$$

$$= \frac{1}{\beta^{\theta}}\left(g(i)\right)^{\theta\rho}\left[(A - i)g'(i) + g(i)\right]^{1-\theta} \tag{5.105}$$

$$= \frac{1}{\beta^{\theta}}\left(g(i)\right)^{\theta\rho}\left[\frac{g(i)^{\rho}}{\beta}\right]^{1-\theta} \tag{5.106}$$

$$= \frac{1}{\beta^{\theta}}\left(g(i)\right)^{\theta\rho}\frac{g(i)^{\rho-\theta\rho}}{\beta^{1-\theta}} \tag{5.107}$$

$$= \frac{1}{\beta}g(i)^{\rho} \tag{5.108}$$

Q.E.D. ■

5.7.4 Transversality Condition of the Baseline Model

Jointly, in the equilibrium, the growth rate of the firm's market capitalization has to be capped by the household's discounting behaviour. If I represent the Euler equation in the balanced growth path as

$$DF \times R = 1 \tag{5.109}$$

Then, in the firms problem, the transversality condition is

$$\lim_{\tau\to\infty} DF^{\tau}\frac{\partial D_{\tau}}{\partial K_{\tau+1}}K_{\tau+1}=0 \tag{5.110}$$

$$\lim_{\tau\to\infty} DF^{\tau}R^{\tau}R^{-\tau}\frac{\partial D_{\tau}}{\partial K_{\tau+1}}K_{\tau+1}=0 \tag{5.111}$$

$$\lim_{\tau\to\infty} R^{-\tau}g'(i_{\tau})^{-1}K_{\tau+1}=0 \tag{5.112}$$

$$\lim_{\tau\to\infty} R^{-\tau}g'(i_{\tau})^{-1}K_{0}g(\bar{i})^{\tau+1}=0 \tag{5.113}$$

$$\lim_{\tau\to\infty} \left[(A-\bar{i})g'(\bar{i})+g(\bar{i})\right]^{-\tau}g'(i_{\tau})^{-1}K_{0}g(\bar{i})^{\tau+1}=0 \tag{5.114}$$

$$\lim_{\tau\to\infty} \left[\frac{g(\bar{i})}{(A-\bar{i})g'(\bar{i})+g(\bar{i})}\right]^{\tau}\frac{g(\bar{i})}{g'(\bar{i})}K_{0}=0 \tag{5.115}$$

I use the expression for R derived in proposition 5.1 in the equilibrium in equality (5.114). Since i is a constant and $(A-i)g'(i)>0$ in the equilibrium, the condition is satisfied.

Chapter 6

Sectoral Growth and Asset Pricing in a Two-Sector Production Economy

6.1 Abstract

This chapter explores the spread of firm-level idiosyncratic shocks in a general equilibrium framework. I build an endogenous growth model with two parallel sectors. With a general equilibrium structure, the model shows strong co-movement in the growths and asset returns between two sectors. Two channels make the spillover of the idiosyncratic shocks possible. The first is the unified stochastic discount factor. The other is the endogenised spillover of technology. Different patterns of the cross-sectional co-movements of a firms' growth, investment ratio and asset prices are examined in this theoretical framework.

6.2 Introduction

Conventionally, macroeconomic models often assume a representative firm and neglect the idiosyncratic shocks. However, heterogeneity is not neglectable in reality. In recent years, there is increasing theoretical and empirical research studying the asymmetric cross-sectional distribution of firms. Gabaix (2011) discusses the distribution of firms' sizes and argues that the idiosyncratic shocks are not ignorable. Bachmann and Bayer (2014) examine the reasons for pro-cyclicality of the cross-sectional dispersion of firm-level investment rates. In addition, asset price also reflects the heterogeneity. Although we see strong co-movement in stock returns across different sections, there are significant cross-sectional differences. Fama and French (1993) show that factors such as the book-market ratio can explain the differences in cross-section returns. In this paper, I build a endogenous growth model to study cross-sectional growth, investment behaviour and other macroeconomic fundamentals.

The model is a general equilibrium model in the production economy. The model assumes (1) two parallel productive sectors, (2) constant return to scale in production function and (3) spillovers and complementarities in productivity. The "AK" framework assures the endogenous growth of the firms. In addition, I assume there is a shared component in the technology scales of two sectors. This common part is determined by the investment level of two sectors. By this means, I endogenise the technology scale and build a channel linking the dynamics of two sectors together. In fact, the model allows two channels for cross-sectoral interactions. The other is the unique stochastic discount factor (SDF) formed by the investor's consumption choice across states of natures. Since there are two parallel sectors, the SDF is constructed on the base of the dividend growth rate from two sectors. The channels between two sectors enables the propagation of the idiosyncratic shocks and co-movements in terms of cross-sectoral investments, growth and asset returns.

The general equilibrium structure is built on a production economy with complete market. The firms are owned by consumers and take the stochastic discount factor (SDF) as given. The firm faces an adjustment cost of the investment. The household's problem follows the consumption-based capital asset pricing model (CCAPM). As usual, the household problem offers the Euler equation to determine the SDF.

With proper setups in parameters and functional forms, the model generates varies patterns of propagation of sector specific shocks. The channel formed by SDF enables an opposite responses to an idiosyncratic shock in the economy between two sectors. Basically, the substitution effect in asset pricing transits the shock from one sector to the other through the pricing kernel. By this means, the shock affects the investment decision of both sectors.

The spillovers in productivity make the second channel. A specific technology shock to one sector stimulates the aggregate technology A, which is the common part in the productivity in all sectors. In fact, all parallel sectors in the economy adjust their behaviour according to this idiosyncratic shock. These synchronizations also have a impact on the asset returns of individual sectors.

There are models dealing with multi-sectors in the literature. Cochrane et al. (2008) and Martin (2013) develop the Lucas (1978) tree model into a multi-assets framework. These models are in the exchange economy, assuming exogenous supply of assets. They show that an dividend shock to a specific asset propagates even though assets have independent cash flow. Since the SDF is constructed by marginal rates of substitution in aggregate consumption, a shock to dividend alters the consumer's behaviour on pricing the asset. My model investigates a similar mechanism. It provides the linkages between cross-sectional firms regarding the investment decision, the dividends growth and the stock prices.

Other related literatures include Kogan (2001, 2004) and Eberly and Wang (2009). The former mainly focuses on the effect of irreversible investment. The author assume one sector with irreversible investment and one with reversible investment and studies the impacts on stock returns, whereas my model mainly studies the interactions between two sectors with only the adjustment cost of investment. The latter has a similar parallel sectoral structure as in this paper. Nonetheless, it focuses on capital reallocation between sectors. The restructuring is costly and leads to a delayed economic growth. In the methodology, their economy does not have a balanced growth path (BGP). The two sectors eventually converge to one. The model in this paper extends their framework to incorporate stable growth in both sectors. This allows my model to generate implications about how heterogeneity between sectors interactively influences sectoral and systematic growth.

The next section introduces the baseline for the model to study equilibrium and the first channel. Section 6.4 extends the framework into the second channel with the productivity externalities and examines the propagation of the shocks. Section 6.5 calibrates the BGPs in the deterministic model. Section 6.6 extends the calibration to the stochastic model and shows the impulse responses functions to idiosyncratic shocks. Section 6.7 concludes the paper.

6.3 The Baseline Model

The model in this chapter follows the basic setups in the chapter 4. It still describes a production economy with an endogenous growth framework. Most of the first order conditions are similar. However, for the independence of this chapter and later reference, this section has some repetition. Yet I omit some of the intermediate derivations in the next subsection and directly jump to the optimal conditions.

Firstly, I consider a deterministic production economy with two productive sectors as the baseline model. Two sectors are denoted by $n = 1$ and 2 respectively. Time t discretely runs from 0 to infinity. The production function is linear $Y_{nt} = A_{nt}K_{nt}$. Y, K and A denote the output, the capital stock and the constant exogenous technology scale factor respectively. Firm uses operation profit to pay the dividends as $D_{nt} \equiv A_{nt}K_{nt} - I_{nt}$, where I is the investment. Capital accumulates at rate $K_{n,t+1}/K_{n,t} = g_n(i_{nt})$ with $i \equiv I/K$ being the investment to capital ratio. The $g(\cdot)$ function on one hand describes the growth rate of the capital accumulation. On the other hand, it captures the effectiveness in converting investment to capital inputs. Nonetheless, the restrictions in previous chapters are still assumed binding, namely $g(i) > 0$, $1 > g'(i) > 0$ and $g''(i) \leq 0$. I omit the subscript t when there is no ambiguity.

The representative firm maximises its stock value represented by the discounted cash flow,

$$V_{nt} = \underset{I}{Max} \quad \sum_{\tau=t}^{\infty} \beta^{\tau-t} \frac{\Lambda_\tau}{\Lambda_t} D_{nt} \tag{6.1}$$

subject to the constrains

$$\frac{K_{n,t+1}}{K_{n,t}} = g_n\left(i_{nt}\right) \tag{6.2}$$

$$A_{nt} > 0, \quad K_{nt} > 0, \quad \Lambda_t > 0 \tag{6.3}$$

$$V_t > 0, \quad I_t > 0 \tag{6.4}$$

$$K_{n0} \textit{ are given.} \tag{6.5}$$

In this firm-specific problem, both sector take the discount factor $\beta^{\tau-t}\Lambda_\tau/\Lambda_t$ as given.

The representative consumer accumulates assets and consumes the dividends. He or she maximises the life time utility that

$$J_t = \underset{C}{Max} \quad \sum_{\tau=t}^{\infty} \beta^{\tau-t} U\left(C_\tau\right) \tag{6.6}$$

subject to budget constrain

$$S_{t+1}P_t = S_t\left(P_t + D_t\right) - C_t \tag{6.7}$$

where S is the stock shares holding by the consumers. P is the asset portfolio price. D is the asset portfolio's dividends. C is the consumption.

Finally, consumption goods are not storable. Market is cleared by the condition that produced goods can only be either consumed or invested.

$$A_1K_1 - I_1 + A_2K_2 - I_2 = C \tag{6.8}$$

or equivalently,

$$D_1 + D_2 = C \tag{6.9}$$

To summarise, in a sense, this model incorporates the two-asset model in the exchange economy in chapter 3 and the endogenous growth model in the production economy in chapter 4 and 5.

6.3.1 General Equilibrium and BGP

The firms problem is handled by dynamic programming method. For the individual sector, the problem of the representative firm are identical to the case solved in chapter 4 and 5. Here, I directly present the Euler equation for the firms in both two sectors.

$$\beta \frac{\Lambda_{t+1}}{\Lambda_t} \left[A_{nt+1} - i_{nt+1} + \frac{g_n\left(i_{nt+1}\right)}{g_n'\left(i_{nt+1}\right)} \right] g_n'\left(i_{nt}\right) = 1 \tag{6.10}$$

For now the sectors are independent just like the two assets in the chapter 3. Taking the discount factor as given, this difference equation governs the investment behaviour in both sectors. The investment path is the result of balancing intertemporal marginal firm value of investment according to the Euler equation (6.10).

In terms of the consumer's problem, the Euler equation of the consumer's problem offers the unique discount factor in the market. In the model we have, it comes from the Euler equation of the standard consumption-based capital asset pricing model (CCAPM). Explicitly, it is determined by the intertemporal marginal substitution in consumption.

$$DF_{t+1} = \beta \frac{U'\left(C_{t+1}\right)}{U'\left(C_t\right)} \tag{6.11}$$

For simplicity, I consider the log-utility preference. With market clearing condition, it is equivalent to

$$DF_{t+1} = \beta \left(\frac{D_{1,t+1} + D_{2,t+1}}{D_{1t} + D_{2t}} \right)^{-1} \tag{6.12}$$

With the ingredients from both sides, the general equilibrium condition is consist of the Euler equation (6.10) of firm, the discount factor (6.12) provided by the consumer and the market clearing condition (6.8).

$$DF_{t+1} \left(A_{1t+1} - i_{1t+1} + \frac{g_1\left(i_{1t+1}\right)}{g_1'\left(i_{1t+1}\right)} \right) g_1'\left(i_{1t}\right) = 1 \tag{6.13}$$

$$DF_{t+1} \left(A_{2t+1} - i_{2t+1} + \frac{g_2\left(i_{2t+1}\right)}{g_2'\left(i_{2t+1}\right)} \right) g_2'\left(i_{2t}\right) = 1 \tag{6.14}$$

where

$$DF_{t+1} = \beta \left(\frac{\left(A_{1t+1} - i_{1t+1}\right) g_1\left(i_{1t}\right) K_{1t} + \left(A_{2,t+1} - i_{2,t+1}\right) g_2\left(i_{2t}\right) K_{2t}}{\left(A_{1t} - i_{1t}\right) K_{1t} + \left(A_{2t} - i_{2t}\right) K_{2t}} \right)^{-1} \tag{6.15}$$

The difference equation system involves the time series of the predetermined capital stock K_{nt}, the exogenous technology scale A_{nt} and the choice variable investment to capital ratio i_{nt}. One can directly notice that there is *no* balanced growth path (BGP) in this economy. By definition of the BGP, if the two constant exogenous variables are fixed at $A_{nt+1} = A_{nt} = A_n$ we should be able to find two constant investment capital that $i_{nt+1} = i_{nt} = i_n$. However, this could not be the case in here. Even in the deterministic case when $A_{nt+1} = A_{nt} = A_n$, since the capital K_{nt} is accumulating over time, it is impossible to maintain the investment capital i_n at a constant level. The only scenario with constant capital K is $g\left(i\right) = 1$ for all t. This economy is of little interest since it has no growth at all. This issue is the main reason why the model in Eberly and Wang (2009) dose not incorporate steady growth. However, the BGP is important to the model because the paper is interested in the dynamics of growths and the relation between growth and asset returns of two sectors. Hence, I make following modification in the model to assure that there is a BGP.

Assumption 6.1 *I adjust the log utility from* $U(C) = \log(C)$ *to*

$$U(C_t) = \frac{D_{1t} + D_{2t}}{2\sqrt{D_{1t}D_{2t}}} \times \log(C) \tag{6.16}$$

Accordingly,

$$U'(C_t) = \frac{1}{2\sqrt{D_{1t}D_{2t}}} \tag{6.17}$$

After the adjustment, the discount factor shown in (6.12) *is replaced by*

$$\beta \frac{U'(D_{1,t+1} + D_{2,t+1})}{U'(D_{1t} + D_{2t})} = \beta \sqrt{\frac{D_{1t}D_{2t}}{D_{1t+1}D_{2t+1}}} \tag{6.18}$$

This is a reverse engineering. The purpose of this assumption is to ensure the existence of two respective BGPs for two sectors and to revisit the issue of sectoral co-movements in line with Eberly and Wang (2009) in the economy with BGPs. The extra component in the utility function (6.16) is the ratio of arithmetic average and geometric average of dividends payments by two sectors, namely

$$\frac{D_{1t} + D_{2t}}{2} \bigg/ \sqrt{D_{1t}D_{2t}} \tag{6.19}$$

This ratio achieves high value as the difference between D_1 and D_2 becomes large. On the other hand, when $D_1 \approx D_2$ the ratio becomes 1. In later section, we shall see that, with this assumption 6.1, two sectors respectively grow at 2 constant rates in the BGPs. Inevitably, over time, one will overwhelmingly outgo the other. Indeed, the micro-foundation for the argument that an imbalanced sectoral structure enhances consumer's utility is thin. Therefore, the following model which adapts this assumption is more suitable to describe those periods of economy with two similarly sized sectors.

However, even in absence of this assumption, the existence of the connections between two sectors preserve as shown in Eberly and Wang (2009). The point is that we cannot analytically and numerically explore those predictions in the context of steady growth of economy. The purpose is to establish the two distinguish steady growth paths for two sectors.

Methodologically, it is not the first model that adjusts the preference. One can take the unconventional component in equation (6.16) as a habit stock. The studies of habit formation in the utility function are originated back to Abel (1990). For example, in Carroll et al. (2000), they introduce a utility function as

$$U(H,C) = H^{-\gamma(1-\sigma)} \times \frac{C^{1-\sigma}}{1-\sigma} \tag{6.20}$$

where H is the habit stock. It originally stands for the proxy of the past consumption or a lagged average of the standard of living. Here, I borrow the structure.

Further, with assumption 6.1, proposition 6.1 establishes the BGP in general equilibrium.

Proposition 6.1 *With assumption 6.1, the solution to the following equation system indicates the BGP in the economy described by the deterministic baseline model.*

$$\beta^2 \left[(A_1 - i_1) g_1'(i_1) + g_1(i_1)\right]^2 - g_1(i_1) g_2(i_2) = 0 \tag{6.21}$$

$$\beta^2 \left[(A_2 - i_2) g_2'(i_2) + g_2(i_2)\right]^2 - g_1(i_1) g_2(i_2) = 0 \tag{6.22}$$

Proof. After the adjustment, the difference equation system (6.13) which governs the equilibrium is replaced by,

$$DF_{t+1}\left(A_{1t+1} - i_{1t+1} + \frac{g_1(i_{1t+1})}{g_1'(i_{1t+1})}\right) g_1'(i_{1t}) = 1 \tag{6.23}$$

$$DF_{t+1}\left(A_{2t+1} - i_{2t+1} + \frac{g_2(i_{2t+1})}{g_2'(i_{2t+1})}\right) g_2'(i_{2t}) = 1 \tag{6.24}$$

where

$$DF_{t+1} = \beta \left(\frac{(A_{1t+1} - i_{1t+1})\, g_1(i_{1t}) \times (A_{2,t+1} - i_{2,t+1})\, g_2(i_{2t})}{(A_{1t} - i_{1t}) \times (A_{2t} - i_{2t})} \right)^{-\frac{1}{2}} \tag{6.25}$$

The assumption 6.1 successfully eliminates the impact of the dynamics of capital stock K_{nt}. Therefore, if we fix the dynamics of A_{nt} as $A_{nt+1} = A_{nt} = A_n$, the optimal level of i_{nt} is solved by the equation system given by

$$\beta \Big(g_1(i_1)\, g_2(i_2) \Big)^{-\frac{1}{2}} \left(A_1 - i_1 + \frac{g_1(i_1)}{g_1'(i_1)} \right) g_1'(i_1) = 1 \tag{6.26}$$

$$\beta \Big(g_1(i_1)\, g_2(i_2) \Big)^{-\frac{1}{2}} \left(A_2 - i_2 + \frac{g_2(i_2)}{g_2'(i_2)} \right) g_2'(i_2) = 1 \tag{6.27}$$

The proposition immediately follows.

Q.E.D. ■

This deterministic equilibrium follows a two-variables-two-equation system. Ideally, the system can be solved for $i_1 = i_1(A_1, A_2)$ and $i_2 = i_2(A_1, A_2)$. Accordingly, we can analyse the impact of each technology scale A_1 and A_2 on the investment decision of the firm in each sector. However, the non-linear structure of the efficiency function $g(i)$ and the equation system itself makes it difficult to solve for the closed form solution. Hence, with the multi-variable implicit function theorem, I derive proposition 6.2.

Proposition 6.2 *Applying implicit function theorem to the equation system in* (6.26)*, we have the following partial derivatives*

$$\frac{\partial i_1(A_1, A_2)}{\partial A_1} > 0, \quad \frac{\partial i_2(A_1, A_2)}{\partial A_2} > 0 \tag{6.28}$$

$$\frac{\partial i_1(A_1, A_2)}{\partial A_2} < 0, \quad \frac{\partial i_2(A_1, A_2)}{\partial A_1} < 0 \tag{6.29}$$

and obviously,

$$\frac{\partial i_1(A_1,A_2)}{\partial A_1}\frac{\partial i_2(A_1,A_2)}{\partial A_1}<0 \tag{6.30}$$

$$\frac{\partial i_1(A_1,A_2)}{\partial A_2}\frac{\partial i_2(A_1,A_2)}{\partial A_2}<0 \tag{6.31}$$

Proof. See proof in appendix 6.8.1

In Proposition 6.2, changes in technology scale A of a specific sector have opposing effects on investments of two sectors. To explain, suppose that sector one has a technological innovation. The firm in this sector raises the investment to utilise the improved productivity and to optimise the stock value. However, a higher growth and further a high risk-free rate wears down the effect since high growth makes the asset less attractive due to the substitution effect in asset pricing. On the other hand, since discount factor is lowered by the investment in the first sector, the best move for the second sector is to downgrade its investment to offset the negative affect brought by the first sector. Therefore, the technological idiosyncratic shock in A_n is corresponded differently in 2 sectors.

Hence, the model already has the first channel between two sectors which allows the idiosyncratic shock to spread. In fact, it is a minor extension to the multi-assets pricing models in the exchange economy which are represented by Cochrane et al. (2008) and Martin (2013). These papers demonstrate that the pricing mechanism of a specific asset is affect by the idiosyncratic shocks of other assets even though they have independent cash flows. The model in this paper establish similar relation between sectors. This relation not only affects the pricing mechanism but also influences the sectoral investment behaviour. In next section, I build another channel for the spillover.

6.4 The Second Channel: Externalities

The last section constructs a bridge between two sectors. However, the micro-structure of the propagation of the business cycle is more complicated than predictions of the first channel. Like stated in the Kogan and Papanikolaou (2012), there is evidence of strong comovement in the cross-section of stock returns. This section establishes another channel to account for this comovement. The idea follows the intuition in the previous chapters. Based on equation system in (6.21), I propose an assumption to partly endogenise the technology and its spillover effects.

Assumption 6.2 *The technology scales A_1 and A_2 in equation system* (6.21) *are assumed to be*

$$A_1 = A_{tr1}\left[A_1\left(i_1\right)\right]^{\alpha_1}\left[A_2\left(i_2\right)\right]^{1-\alpha_1} \tag{6.32}$$

$$A_2 = A_{tr2}\left[A_1\left(i_1\right)\right]^{1-\alpha_2}\left[A_2\left(i_2\right)\right]^{\alpha_2} \tag{6.33}$$

Where A_{tr1} and A_{tr2} are constant.

The technology functions $A_1\left(i_1\right)$ and $A_2\left(i_2\right)$ are a logistic functions given by

$$A_1\left(i_1\right) = \frac{\eta_1}{1+\exp\left[-\xi_1\left(i_1-i_1^*\right)\right]} + A_{L,1} \tag{6.34}$$

$$A_2\left(i_2\right) = \frac{\eta_2}{1+\exp\left[-\xi_2\left(i_2-i_2^*\right)\right]} + A_{L,2} \tag{6.35}$$

Figure 6.1 is an example of the logistic function. It is a sigmoid shape function. The parameters A_L and $A_L+\eta$ pin down the lower and upper bound respectively. The i^* indicates the position with highest steepness.

I would like to emphasis two points in this assumption. Firstly, a component of the technology scale A is endogenously driven by the investment to capital ratio. It is another version of the threshold assumption in the previous chapters. The concern here is still for methodology. The discontinuity of threshold function is inconvenient for later analytical exploration based on the differencing of this function.

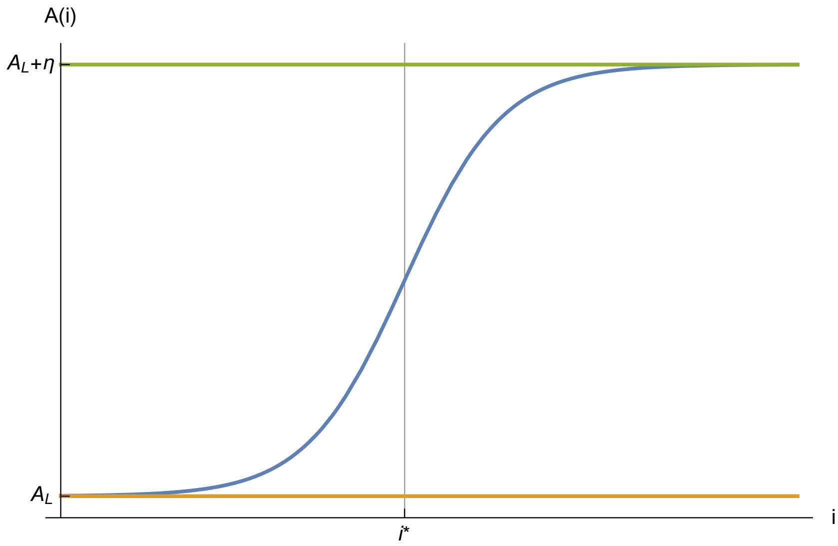


Fig. 6.1 Logistic Function

Secondly, the shared component in technology scale is in a Cobb-Douglas form and consists of the productivity spillovers from two sectors. To elaborate, productivity which comes from $A_t r$ and $A(i)$ are private factor. Specifically, $A(i)$ is controlled by each sector and $A_t r$ are idiosyncratic and exogenous. However, in the aggregate level or social level, the productivity for a sector is affected by the spillovers coming from the other sector. Parameter α controls the scale of the spillovers. This becomes the second channel for the spillover between sectors.

Accordingly, in equilibrium, the equation system (6.21) is adjusted into

$$\beta\sqrt{\frac{1}{g_1(i_1)g_2(i_2)}}\left(A_{tr1}\left[A_1(i_1)\right]^{\alpha_1}\left[A_2(i_2)\right]^{1-\alpha_1}-i_1+\frac{g_1(i_1)}{g_1'(i_1)}\right)g_1'(i_1)=1 \quad (6.36)$$

$$\beta\sqrt{\frac{1}{g_1(i_1)g_2(i_2)}}\left(A_{tr2}\left[A_1(i_1)\right]^{1-\alpha_2}\left[A_2(i_2)\right]^{\alpha_2}-i_2+\frac{g_2(i_2)}{g_2'(i_2)}\right)g_2'(i_2)=1 \quad (6.37)$$

The equation system remains a two-variable-two-equation system yet it becomes more difficult to solve both analytically and numerically. However, the marginal effect of productivity parameter A's on investment-capital ratios are still of interest. In fact, proposition 6.3 shows that the effect is ambiguous and subject to parameterisations.

Proposition 6.3 *For an economy which has the BGP governed by equation system* (6.36)*, the sign of interactions of the sectoral investments*

$$\frac{\partial i_1\left(A_{tr1}, A_{tr2}\right)}{\partial A_{tr1}} \times \frac{\partial i_2\left(A_{tr1}, A_{tr2}\right)}{\partial A_{tr1}} \tag{6.38}$$

is ambiguous in general. In other words, it depends on specific parameterisations.

Proof. See appendix 6.8.2.

With assumption 6.2, the spread of the idiosyncratic shocks has two pathways. A positive transitory shock to sector one urges the sector one to invest. Since the technology is endogenised, it raises the externalities and the aggregate productivity in the society. sector two also responds to this by raising its investment. However, this affect interacts with the first channel in the last section. The two channels intertwine with each other. In next section, I attempt to numerically calibrate the effects from two different channels.

6.5 Calibration of the BGPs

This section calibrates the economy BGPs. Firstly, assumption 6.3 offers the quadratic functional form for the efficiency function $g(i)$.

Assumption 6.3 *The efficiency function $g(i)$ follows the quadratic function given by Gourio (2012),*

$$\frac{K_{nt+1}}{K_{nt}} = g_n\left(i_n\right) \tag{6.39}$$

$$= i_n - \frac{\Gamma_n\left(i_n - \theta_n\right)^2}{2} + 1 - \delta_n \tag{6.40}$$

where $n = 1$ or 2, $\theta_n > 0$, $\Gamma_n > 0$ and $\delta_n \in (0, 1)$. Due to the assumptions on $g(i)$, the investment to capital ratio is restricted to the left half of the parabola, namely $i < \theta + 1/\Gamma$.

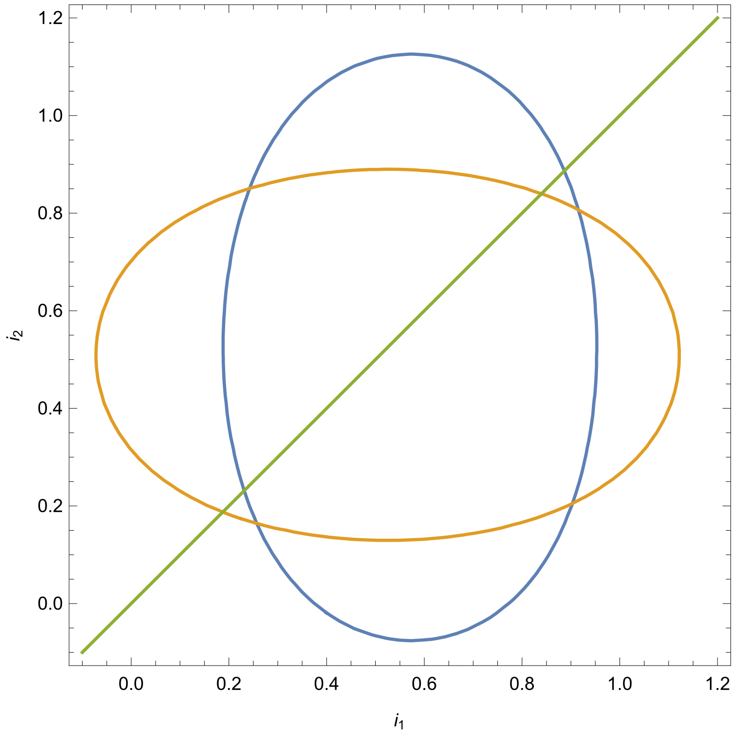


Fig. 6.2 BGP in Equation System (6.21)
Parameterisation: $\beta = 0.83$, $A_1 = 0.6$, $A_2 = 0.5$, $\Gamma_1 = \Gamma_2 = 2$,
$\theta_1 = \theta_2 = 0.025$, $\delta_1 = \delta_2 = 0.13$.

The parameters Γ and θ capture the shape of the efficiency function. δ can be understood as the capital depreciation rate. Now the model is ready for the calibrations.

6.5.1 The First Channel

Firstly, I consider the equation system (6.21) in proposition 6.1 with the functional form given by assumption 6.3. The BGP exists on the solution of the equation system. Hence, after substituting the parameterisation, I plot the two equations separately and look for the intersection. Figure 6.2 indicates the BGPs in the economy. Parameterisations are indicated below the figure. These parameters follow the calibration in Gourio (2012) and Eberly and Wang (2009).

Figure 6.2 shows the plot of i_1 against i_2 according to the equation system (6.21). The diagonal line is $i_1 = i_2$. I deliberately break the symmetry by assigning $A_1 \neq A_2$.

At the first glimpse, there are 4 roots at 4 intersections on the figure. However, due to restrictions such as $A - i > 0$ and $i < \theta + 1/\Gamma$, the only feasible BGP is the one at the left-down side with $i_1 = 0.26$ and $i_2 = 0.16$. Accordingly, on this BGP, the first sector grows at rate $g_1(i_1) = 1.07$ and the second sector grows at rate $g_2(i_2) = 1.01$. With relatively heavy discount factor $\beta = 0.83$ and capital depreciation $\delta = 0.13$, the model manage to generate a high growth sector with 7% growth and a low growth sector with 1% growth.

On the other hand, the model shows the possibilities of multiplicity. Ideally, with proper assigned functional form for the efficiency function $g(i)$ and parameters, it is possible to form multiple BGPs under our restrictions. Nonetheless, exploring multiplicity and its implications are not the priority in this paper, so I leave it aside and focus on the feasible solution.

6.5.2 Two Channels Together

Analogue to the last section, figure 6.3 is plotted according to the equation system (6.36) with the logistic function $A(i)$ and the quadratic efficiency function $g(i)$.

With the parameterisations of figure 6.3, sector one has more productivity than sector two as $\bar{A}_{tr1} > \bar{A}_{tr2}$. Besides, sector one has a heavier weight in the aggregate productivity of its own sector, namely $\alpha_1 > \alpha_2$. The rest of the parameters are the same in both sectors, hence I omit the subscripts.

Figure 6.3 shows possible roots to the equation system (6.36). The equation system is highly non-linear and the plot severely varies with the changes in the parameterisations. I choose these values because they generate relatively good calibration of the deterministic BGP in here and the stochastic version of the model in next section.

Again, there are 4 possible roots in the graph. The one at the left-down corner is the only one fulfilling our conditions. This root consists $i_1 = 0.13$ and $i_2 = 0.11$. The respective growth of each sector are 1.022 and 1.000. The former is growing at 2.2% per period and the latter is barely growing. Compare to the previous case with

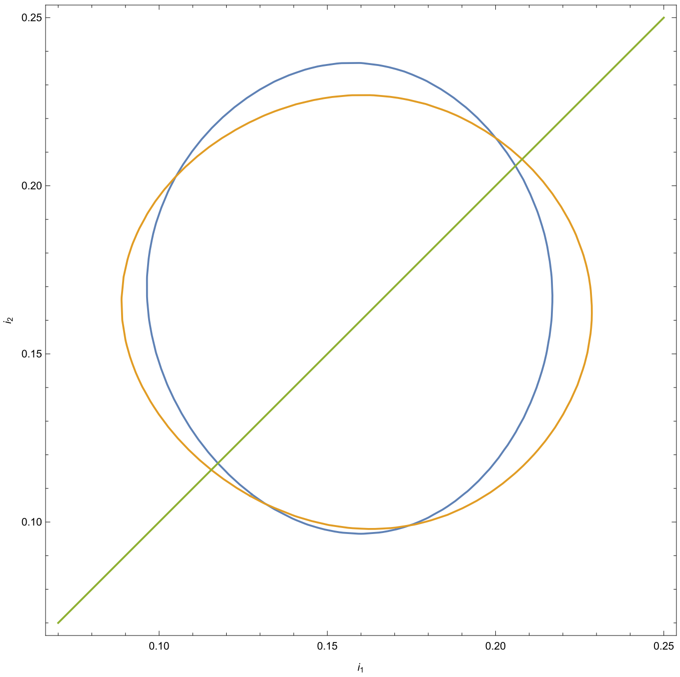


Fig. 6.3 BGP in Equation System (6.36)
Parameterisation: $\bar{A}_{tr1} = 1.3$, $\bar{A}_{tr2} = 1.29$, $\alpha_1 = 0.6$, $\alpha_2 = 0.55$, $\beta = 0.95$, $\Gamma = 1.7$, $\theta = 0.025$, $\delta = 0.09$, $A_L = 0.12$,$\eta = 1$, $\xi = 20$, $i^* = 0.315$.

$A_1 = 0.6$ and $A_2 = 0.5$, the total factor productivity in this case is given by,

$$A_{tr1}\left[A_1\left(i_1\right)\right]^{\alpha_1}\left[A_2\left(i_2\right)\right]^{1-\alpha_1} = 0.183 \tag{6.41}$$

$$A_{tr2}\left[A_1\left(i_1\right)\right]^{1-\alpha_2}\left[A_2\left(i_2\right)\right]^{\alpha_2} = 0.179 \tag{6.42}$$

In the last subsection, the high growth sector has a 7% growth rate. Here, due to the suppressed productivity yet higher time preference β, the growth in our economy becomes relatively stagnated. However, the model still reflects a relatively healthy sector and a sluggish sector even though the above total factor productivities are close to each other.

6.6 Calibration of the Stochastic Model - Impulse Response Functions

Furthermore, besides the deterministic models, the specific asset returns are particularly of interest. This section explores the stochastic version of the model and the idiosyncratic shocks. The stochastic model is similar to the models used in the previous sections. Naturally, the Euler equation system is the a stochastic difference equation system.

$$E_t\left[SDF_{t+1}\left(A_{1t+1}-i_{1t+1}+\frac{g_1\left(i_{1t+1}\right)}{g_1'\left(i_{1t+1}\right)}\right)g_1'\left(i_{1t}\right)\right]=1 \tag{6.43}$$

$$E_t\left[SDF_{t+1}\left(A_{2t+1}-i_{2t+1}+\frac{g_2\left(i_{2t+1}\right)}{g_2'\left(i_{2t+1}\right)}\right)g_2'\left(i_{2t}\right)\right]=1 \tag{6.44}$$

where

$$SDF_{t+1}=\beta\left[\frac{\left(A_{1t+1}-i_{1t+1}\right)g_1\left(i_{1t}\right)\left(A_{2,t+1}-i_{2,t+1}\right)g_2\left(i_{2t}\right)}{\left(A_{1t}-i_{1t}\right)\left(A_{2t}-i_{2t}\right)}\right]^{-\frac{1}{2}} \tag{6.45}$$

It also takes the reduced form given by,

$$E_t\left(SDF_{t+1}\times R_{1t+1}\right)=1 \tag{6.46}$$

$$E_t\left(SDF_{t+1}\times R_{2t+1}\right)=1 \tag{6.47}$$

where SDF_{t+1} is the stochastic discount factor corresponding to the DF_{t+1} in the previous sections. R_{nt} stand for the asset returns. Further, I follow the standard definition and define the risk-free rate as

$$r_t^f=E_t\left(\frac{1}{SDF_{t+1}}\right) \tag{6.48}$$

Basically, computational software solves the stochastic difference equation system by the perturbation method. In terms of the shock, I study the orthogonalised shocks with independent AR(1) process

$$\log\left(A_{nt+1}\right) = \rho \log\left(A_{nt}\right) + \varepsilon_{nt+1} \tag{6.49}$$

where ε_{nt}'s follow independent and identical normal distribution. In the practise of the calibration, I specify the standard deviation of the idiosyncratic shock to be 0.25%.

6.6.1 The First Channel

Again, firstly, this subsection verifies the opposing responds in two sectors predicted in proposition 6.2. With the uncertainty, the equation system (6.21) becomes

$$E_t\left[SDF_{t+1}\left(A_{1t+1} - i_{1t+1} + \frac{g_1\left(i_{1t+1}\right)}{g_1'\left(i_{1t+1}\right)}\right)g_1'\left(i_{1t}\right)\right] = 1 \tag{6.50}$$

$$E_t\left[SDF_{t+1}\left(A_{2t+1} - i_{2t+1} + \frac{g_2\left(i_{2t+1}\right)}{g_2'\left(i_{2t+1}\right)}\right)g_2'\left(i_{2t}\right)\right] = 1 \tag{6.51}$$

where

$$SDF_{t+1} = \beta\left(\frac{\left(A_{1t+1} - i_{1t+1}\right)g_1\left(i_{1t}\right)\left(A_{2,t+1} - i_{2,t+1}\right)g_2\left(i_{2t}\right)}{\left(A_{1t} - i_{1t}\right)\left(A_{2t} - i_{2t}\right)}\right)^{-\frac{1}{2}} \tag{6.52}$$

The risk-free rate follow the definition in (6.48) and the expected assets returns take the form

$$E_t\left[R_{1t+1}\right] = E_t\left[\left(A_{1t+1} - i_{1t+1} + \frac{g_1\left(i_{1t+1}\right)}{g_1'\left(i_{1t+1}\right)}\right)g_1'\left(i_{1t}\right)\right] \tag{6.53}$$

$$E_t\left[R_{2t+1}\right] = E_t\left[\left(A_{2t+1} - i_{2t+1} + \frac{g_2\left(i_{2t+1}\right)}{g_2'\left(i_{2t+1}\right)}\right)g_2'\left(i_{2t}\right)\right] \tag{6.54}$$

With the quadratic efficiency function $g\left(i\right)$, the processes of shocks in equation (6.49) and the parameterisations in figure 6.3, the model obtains the impulse responds functions in figure 6.4 and 6.5. The first and second panel represents the

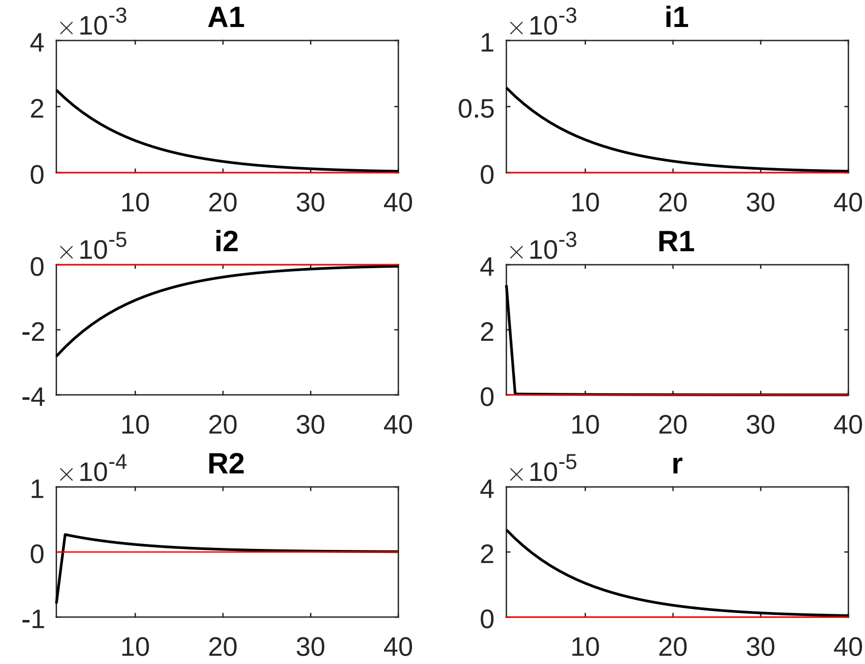


Fig. 6.4 Orthogonalised Shock to A_1

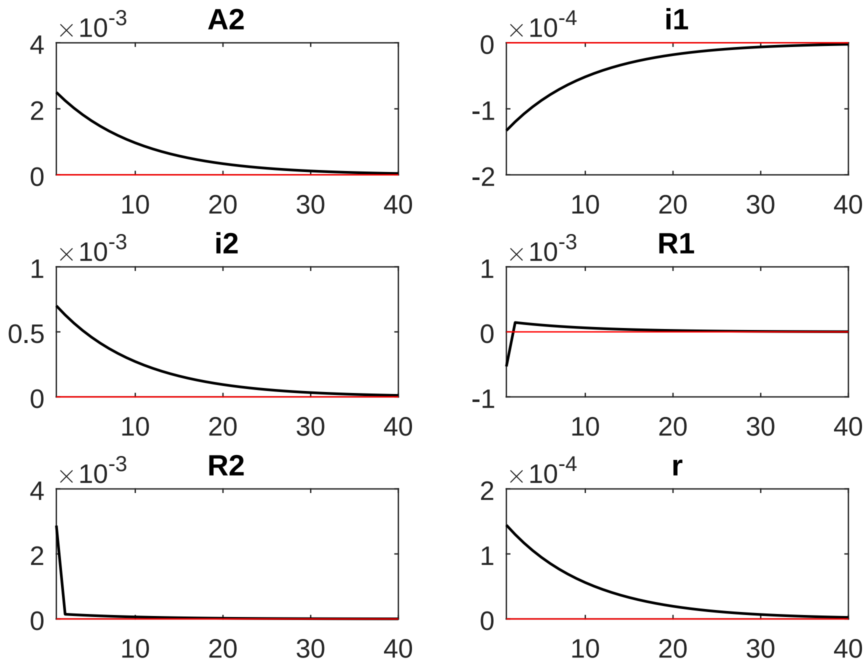


Fig. 6.5 Orthogonalised Shock to A_2

responds of variables to an orthogonalised shock to technology scale A_1 and A_2, respectively.

The first picture in both panels shows how persistence is the shock in technology scale A. It is controlled by the parameter ρ.[1] The rest of the pictures are the responses of the system to the unique orthogonalised shock. The figure 6.4 and 6.5 confirm the theoretical prediction of the asynchronous responds of two different sectors. In both panels, the investment to capital ratios i_1 and i_2 move to the opposing directions when the economy is hit by an idiosyncratic shock. Moreover, R represents the expected return of the risky asset. The expected returns show interesting pattern. Firstly, the expected return reacts more violently to the shock to its own sector. Secondly, asset return shows an “over-shoot” effect. For example, the picture of R_2 in the upper panel shows the reaction of the expected asset return of the sector two to the shock hitting sector one. Clearly, positive shock decreases the discount factor, the non-shocked sector reacts. However, the calibration shows it over-adjust in the first a few periods and then back to the BGP asymptotically. For the risk-free rate, it mirrors the discount factor which is lowered by the boosted economic growth. In terms of the scale, take the upper panel for example, the investment to capital ratio i_1 and the asset return R_1 respond on the same magnitudes, namely 0.1%. The impact on the variables of the other sector such as i_2 and R_2 is weaker. The magnitude is 0.01%.

6.6.2 Two Channels Together

For the model incorporating two channels, the stochastic difference Euler equation system analogous to the last section expect for the replacement of A_1 and A_2.

$$A_1 = A_{tr1}\left[A_1\left(i_1\right)\right]^{\alpha_1}\left[A_2\left(i_2\right)\right]^{1-\alpha_1} \tag{6.55}$$

$$A_2 = A_{tr2}\left[A_1\left(i_1\right)\right]^{1-\alpha_2}\left[A_2\left(i_2\right)\right]^{\alpha_2} \tag{6.56}$$

[1] I set $\rho = 0.9$.

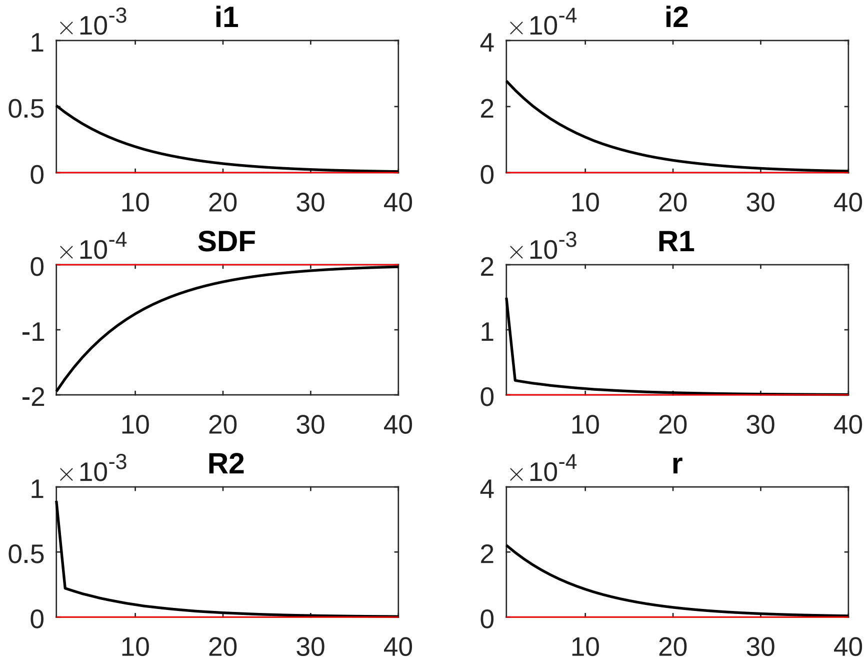


Fig. 6.6 Orthogonalised Shock to A1

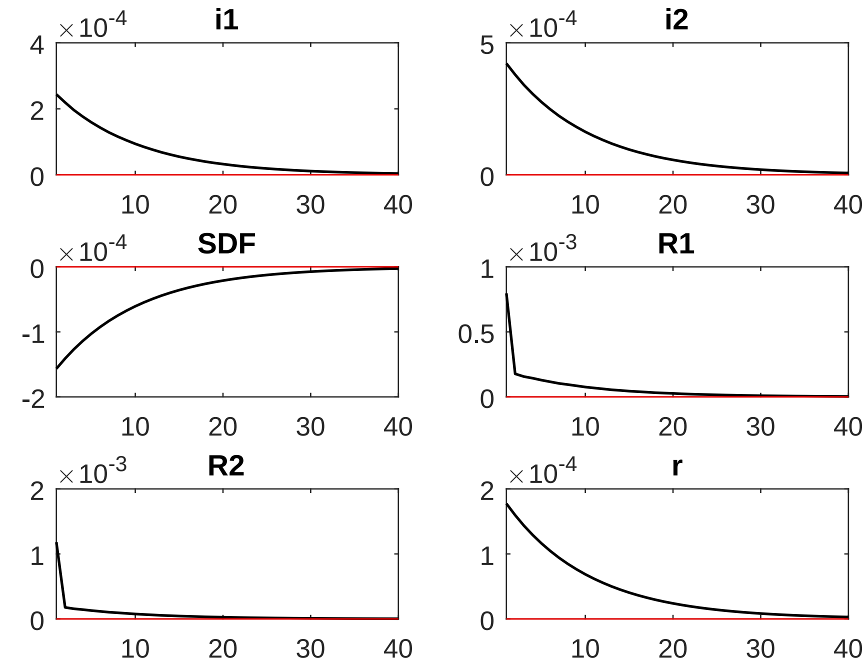


Fig. 6.7 Orthogonalised Shock to A2

‱	IRF	Figures 6.4 & 6.5	Figures 6.6 & 6.7
Shock to sector one	i_2	-0.03	0.3
	R_2	-0.08	1
Shock to sector two	i_1	-0.14	0.22
	R_1	-0.5	0.75

Table 6.1 Comparison of Two Models

Compare to the last section, a slight difference in the reported impulse response functions is that the plot for A is replaced by the stochastic discount factor (SDF) since the dynamics of A's are unchanged. In both panels, variables in two sectors respond in the same direction. It verifies that, at least in some situations, a shock to one specific sector is responded similarly in both sectors. Therefore, we see sectoral synchronization. To elaborate, figure 6.6 and 6.7 shows that there is comovement in the investment capital ratios and expected risky assets return between two sectors when the system is hit by an idiosyncratic shock. Two channels are twisted together. This situation affects the firms investment choice in a fundamental way.

For the magnitudes, the shock's effect on its own sector is still stronger. More importantly, if we compare figure 6.4 and 6.5 with figure 6.6 and 6.7, the non-shocked sector responds more heavily in the model with two channels. To elaborate, I report the magnitudes of the first period responses of investment-capital ratios and assets returns in both figures in table 6.1. In general, if we compare column 3 to column 4 the magnitudes shown in column 4 is larger. It shows that the model with two channels represent a closer connection between two sectors. Hence, the responses of the variables in one sector to the idiosyncratic shocks in the opposing sector are more severer in the model with two channels.

6.7 Concluding Remarks

This paper explores a two-sector general equilibrium model in a production economy. I am interested in the sectoral relation of the growth and the asset returns. The model is in"AK" framework and possesses a BGP. Additionally, to assure the stable BGP in the general equilibrium, habit formation are used to adjust the discount factor which equally balances the two sectors.

To enable the comovements of sectoral variables and spillovers of the idiosyncratic shocks, the model incorporates two channels between sectors. The first is the unique stochastic discount factor in the market. As shown in proposition 6.2, this bridge enables opposing responses to an idiosyncratic shock in two sectors. Generally, the sector with positive technological shock wants to raise the investment since its asset price benefits more from the income effect in asset pricing. However, the sector without the technology shock decreases the investment to offset the impact form the SDF which is originated from the first sector. This channel builds a extension from the Lucas tree exchange economy to a production economy with an endogenous growth.

The second channel is established on the productive externalities. I endogenise the technology. Assumption 6.2 defines that the aggregate level of productivity consist of two technology scales from two sectors. This channel reinforces the interlink between two sectors. With this assumption, a positive technology shock to sector one supports the productivity in the aggregate level, hence sector two also benefits from it. Therefore, the model manages to mimic comovements in terms of investments, growths and assets returns in the impulse responds function in figure 6.6 and 6.7.

The shortage of the model is in the adjustment of utility function and the discount factor. The further studies can focus on building a more appropriate model which includes the stable sectoral growth and the fact that high growth sector gradually takes over the sluggish sector. Admittedly, the highly non-linear structure of the BGP makes it difficult to examine the equilibrium conditions empirically. However, future studies can be carried by simplifying or linearising the optimal conditions and then examining the model empirically.

6.8 Appendix

6.8.1 Proof of Proposition 6.2

Proof. With a slight change in (6.21) and the assumption of functional form for the solution as $i_1 = i_1(A_1, A_2)$ and $i_2 = i_2(A_1, A_2)$, I have

$$\beta\sqrt{\frac{1}{g_1(i_1)\,g_2(i_2)}}\left[(A_1 - i_1)\,g_1'(i_1) + g_1(i_1)\right] = 1 \tag{6.57}$$

$$\beta\sqrt{\frac{1}{g_1(i_1)\,g_2(i_2)}}\left[(A_2 - i_2)\,g_2'(i_2) + g_2(i_2)\right] = 1 \tag{6.58}$$

$$\beta^2\left[(A_1 - i_1)\,g_1'(i_1) + g_1(i_1)\right]^2 - g_1(i_1)\,g_2(i_2(A_1, A_2)) = 0 \tag{6.59}$$

$$\beta^2\left[(A_2 - i_2)\,g_2'(i_2) + g_2(i_2)\right]^2 - g_1(i_1(A_1, A_2))\,g_2(i_2) = 0 \tag{6.60}$$

Implicit function theorem offers the following equation system,

$$\frac{\partial i_1(A_1, A_2)}{\partial A_1} = -\frac{2\beta^2 g_1'(i_1)\left[g_1(i_1) + (A_1 - i_1)\,g_1'(i_1)\right] - g_1(i_1)\,g_2'(i_2)\,\frac{\partial i_2(A_1, A_2)}{\partial A_1}}{2(A_1 - i_1)\,\beta^2\left[g_1(i_1) + (A_1 - i_1)\,g_1'(i_1)\right]g_1''(i_1) - g_2(i_2)\,g_1'(i_1)} \tag{6.61}$$

$$\frac{\partial i_1(A_1, A_2)}{\partial A_2} = \frac{g_1(i_1)\,g_2'(i_2)\,\frac{\partial i_2(A_1, A_2)}{\partial A_2}}{2(A_1 - i_1)\,\beta^2\left[g_1(i_1) + (A_1 - i_1)\,g_1'(i_1)\right]g_1''(i_1) - g_2(i_2)\,g_1'(i_1)} \tag{6.62}$$

$$\frac{\partial i_2(A_1, A_2)}{\partial A_2} = -\frac{2\beta^2 g_2'(i_2)\left[g_2(i_2) + (A_2 - i_2)\,g_2'(i_2)\right] - g_2(i_2)\,g_1'(i_1)\,\frac{\partial i_1(A_1, A_2)}{\partial A_2}}{2(A_2 - i_2)\,\beta^2\left[g_2(i_2) + (A_2 - i_2)\,g_2'(i_2)\right]g_2''(i_2) - g_1(i_1)\,g_2'(i_2)} \tag{6.63}$$

$$\frac{\partial i_2(A_1, A_2)}{\partial A_1} = \frac{g_2(i_2)\,g_1'(i_1)\,\frac{\partial i_1(A_1, A_2)}{\partial A_1}}{2(A_2 - i_2)\,\beta^2\left[g_2(i_2) + (A_2 - i_2)\,g_2'(i_2)\right]g_2''(i_2) - g_1(i_1)\,g_2'(i_2)} \tag{6.64}$$

For simplicity, I apply the following donation,

$$\frac{\partial i_1(A_1,A_2)}{\partial A_1} = -\frac{X_L - X_S\frac{\partial i_2(A_1,A_2)}{\partial A_1}}{X_1}, \quad \frac{\partial i_1(A_1,A_2)}{\partial A_2} = \frac{X_S\frac{\partial i_2(A_1,A_2)}{\partial A_2}}{X_1} \tag{6.65}$$

$$\frac{\partial i_2(A_1,A_2)}{\partial A_2} = -\frac{Y_L - Y_S\frac{\partial i_1(A_1,A_2)}{\partial A_2}}{Y_1}, \quad \frac{\partial i_2(A_1,A_2)}{\partial A_1} = \frac{Y_S\frac{\partial i_1(A_1,A_2)}{\partial A_1}}{Y_1} \tag{6.66}$$

where

$$X_S = g_1(i_1)\, g_2'(i_2) \tag{6.67}$$

$$X_L = 2\beta^2 g_1'(i_1)\left[g_1(i_1) + (A_1 - i_1)\, g_1'(i_1)\right] \tag{6.68}$$

$$X_1 = 2(A_1 - i_1)\beta^2\left[g_1(i_1) + (A_1 - i_1)\, g_1'(i_1)\right] g_1''(i_1) - g_2(i_2)\, g_1'(i_1) \tag{6.69}$$

$$Y_S = g_2(i_2)\, g_1'(i_1) \tag{6.70}$$

$$Y_L = 2\beta^2 g_2'(i_2)\left[g_2(i_2) + (A_2 - i_2)\, g_2'(i_2)\right] \tag{6.71}$$

$$Y_1 = 2(A_2 - i_2)\beta^2\left[g_2(i_2) + (A_2 - i_2)\, g_2'(i_2)\right] g_2''(i_2) - g_1(i_1)\, g_2'(i_2) \tag{6.72}$$

It can be deduced from the assumptions that,

$$X_S > 0, \quad X_L > 0, \quad X_1 < 0 \tag{6.73}$$

$$Y_S > 0, \quad Y_L > 0, \quad Y_1 < 0 \tag{6.74}$$

$$X_1 Y_1 - X_S Y_S > 0 \tag{6.75}$$

The last inequality is obtained by expanding the terms. Then, it is not difficult to solve the 4 variables 4 equation system for $\partial i_1/\partial A_1$, $\partial i_1/\partial A_2$, $\partial i_2/\partial A_2$ and

$\partial i_2/\partial A_1$ as

$$\frac{\partial i_1(A_1,A_2)}{\partial A_1} = -\frac{X_L Y_1}{X_1 Y_1 - X_S Y_S} > 0 \tag{6.76}$$
$$\frac{\partial i_2(A_1,A_2)}{\partial A_2} = -\frac{Y_L X_1}{X_1 Y_1 - X_S Y_S} > 0 \tag{6.77}$$
$$\frac{\partial i_1(A_1,A_2)}{\partial A_2} = -\frac{X_S Y_L}{X_1 Y_1 - X_S Y_S} < 0 \tag{6.78}$$
$$\frac{\partial i_2(A_1,A_2)}{\partial A_1} = -\frac{X_L Y_S}{X_1 Y_1 - X_S Y_S} < 0 \tag{6.79}$$

For the signs, I have assumptions for efficiency function $g(i)$ as $g(i) > 0$, $1 > g'(i) > 0$ and $g''(i) \leq 0$. Additionally, I have $A - i > 0$ since $AK - I \equiv C > 0$. Accordingly, the signs for numerators are straight forward. For denominators, if I expand the first term into polynomials, the first term cancels out $g_1'(i_1)\,g_2'(i_2)$ and 3 terms left are all positive. Hence, I have proposition 6.2.

Q.E.D. ■

6.8.2 Proof of Proposition 6.3

Proof. I follow similar procedure as the last proposition. Firstly I substitute $i_1 = i_1(A_{tr1}, A_{tr2})$ and $i_2 = i_2(A_{tr1}, A_{tr2})$ into the equation system (6.36). After some rearrangement, I obtain

$$\beta^2\left[\left(A_{tr1}\left[A_1(i_1)\right]^{\alpha_1}\left[A_2(i_2(A_{tr1},A_{tr2}))\right]^{1-\alpha_1} - i_1\right)g_1'(i_1) + g_1(i_1)\right]^2 - g_1(i_1)\,g_2(i_2(A_{tr1},A_{tr2})) = 0 \tag{6.80}$$

$$\beta^2\left[\left(A_{tr2}\left[A_1(i_1(A_{tr1},A_{tr2}))\right]^{1-\alpha_2}\left[A_2(i_2)\right]^{\alpha_2} - i_2\right)g_2'(i_2) + g_2(i_2)\right]^2 - g_1(i_1(A_{tr1},A_{tr2}))\,g_2(i_2) = 0 \tag{6.81}$$

I define

$$F_1(i_1, A_{tr1}, A_{tr2}) = \beta^2 \Bigg[\Big(A_{tr1} [A_1(i_1)]^{\alpha_1} [A_2(i_2(A_{tr1}, A_{tr2}))]^{1-\alpha_1} - i_1 \Big) g_1'(i_1) + g_1(i_1) \Bigg]^2 - g_1(i_1) g_2(i_2(A_{tr1}, A_{tr2})) \tag{6.82}$$

$$F_2(i_2, A_{tr1}, A_{tr2}) = \beta^2 \Bigg[\Big(A_{tr2} [A_1(i_1(A_{tr1}, A_{tr2}))]^{1-\alpha_2} [A_2(i_2)]^{\alpha_2} - i_2 \Big) g_2'(i_2) + g_2(i_2) \Bigg]^2 - g_1(i_1(A_{tr1}, A_{tr2})) g_2(i_2) \tag{6.83}$$

for the convenience of later reference.

Implicit function theorem offers,

$$\frac{\partial i_1(A_{tr1}, A_{tr2})}{\partial A_{tr1}} = -\frac{\frac{\partial F_1(i_1, A_{tr1}, A_{tr2})}{\partial A_{tr1}}}{\frac{\partial F_1(i_1, A_{tr1}, A_{tr2})}{\partial i_1}} \tag{6.84}$$

$$\frac{\partial i_1(A_{tr1}, A_{tr2})}{\partial A_{tr2}} = -\frac{\frac{\partial F_1(i_1, A_{tr1}, A_{tr2})}{\partial A_{tr2}}}{\frac{\partial F_1(i_1, A_{tr1}, A_{tr2})}{\partial i_1}} \tag{6.85}$$

$$\frac{\partial i_2(A_{tr1}, A_{tr2})}{\partial A_{tr2}} = -\frac{\frac{\partial F_2(i_2, A_{tr1}, A_{tr2})}{\partial A_{tr2}}}{\frac{\partial F_2(i_2, A_{tr1}, A_{tr2})}{\partial i_2}} \tag{6.86}$$

$$\frac{\partial i_2(A_{tr1}, A_{tr2})}{\partial A_{tr1}} = -\frac{\frac{\partial F_2(i_2, A_{tr1}, A_{tr2})}{\partial A_{tr1}}}{\frac{\partial F_2(i_2, A_{tr1}, A_{tr2})}{\partial i_2}} \tag{6.87}$$

This 4 variables 4 equations linear equation system for $\partial i_1(A_{tr1}, A_{tr2}) / \partial A_{tr1}$, $\partial i_1(A_{tr1}, A_{tr2}) / \partial A_{tr2}$, $\partial i_2(A_{tr1}, A_{tr2}) / \partial A_{tr2}$ and $\partial i_2(A_{tr1}, A_{tr2}) / \partial A_{tr1}$ can be solved for

$$\frac{\partial i_1(A_{tr1}, A_{tr2})}{\partial A_{tr1}} = -\frac{Y_1 X_B X_C}{X_1 Y_1 - (X_A - X_D)(Y_A - Y_D)} \tag{6.88}$$

$$\frac{\partial i_2(A_{tr1}, A_{tr2})}{\partial A_{tr1}} = -\frac{(Y_A - Y_D) X_B X_C}{X_1 Y_1 - (X_A - X_D)(Y_A - Y_D)} \tag{6.89}$$

$$\frac{\partial i_1(A_{tr1}, A_{tr2})}{\partial A_{tr2}} = -\frac{(X_A - X_D) Y_B Y_C}{X_1 Y_1 - (X_A - X_D)(Y_A - Y_D)} \tag{6.90}$$

$$\frac{\partial i_2(A_{tr1}, A_{tr2})}{\partial A_{tr2}} = -\frac{X_1 Y_B Y_C}{X_1 Y_1 - (X_A - X_D)(Y_A - Y_D)} \tag{6.91}$$

Use the equation system (6.36) itself to simply the expression, I have following mapping,

$$X_A = g_1(i_1)\, g_2'(i_2) \tag{6.92}$$

$$X_B = 2\beta g_1'(i_1)\sqrt{g_1(i_1)\, g_2(i_2)} \tag{6.93}$$

$$X_C = [A_1(i_1)]^{\alpha_1}[A_2(i_2)]^{1-\alpha_1} \tag{6.94}$$

$$X_D = 2A_{tr1}(1-\alpha_1)\beta [A_1(i_1)]^{\alpha_1}[A_2(i_2)]^{-\alpha_1} A_2'(i_2)\, g_1'(i_1)\sqrt{g_1(i_1)\, g_2(i_2)} \tag{6.95}$$

$$Y_A = g_1'(i_1)\, g_2(i_2) \tag{6.96}$$

$$Y_B = 2\beta g_2'(i_2)\sqrt{g_1(i_1)\, g_2(i_2)} \tag{6.97}$$

$$Y_C = [A_1(i_1)]^{1-\alpha_2}[A_2(i_2)]^{\alpha_2} \tag{6.98}$$

$$Y_D = 2A_{tr2}(1-\alpha_2)\beta [A_2(i_2)]^{\alpha_2}[A_1(i_1)]^{-\alpha_2} A_1'(i_1)\, g_2'(i_2)\sqrt{g_1(i_1)\, g_2(i_2)} \tag{6.99}$$

$$X_1 = -Y_A + 2\beta\sqrt{g_1(i_1)\,g_2(i_2)}\times \left\{A_{tr1}\alpha_1\left[\frac{A_2(i_2)}{A_1(i_1)}\right]^{1-\alpha_1} A_1'(i_1)\,g_1'(i_1) + g_1''(i_1)\left(A_{tr1}[A_1(i_1)]^{\alpha_1}[A_2(i_2)]^{1-\alpha_1} - i_1\right)\right\} \tag{6.100}$$

$$Y_1 = -X_A + 2\beta\sqrt{g_1(i_1)\,g_2(i_2)}\times \left\{A_{tr2}\alpha_2\left[\frac{A_1(i_1)}{A_2(i_2)}\right]^{1-\alpha_2} A_2'(i_2)\,g_2'(i_2) + g_2''(i_2)\left(A_{tr2}[A_2(i_2)]^{\alpha_2}[A_1(i_1)]^{1-\alpha_2} - i_2\right)\right\} \tag{6.101}$$

I use the equation system (6.36) to simply the expression. Further, I have

$$\frac{\partial i_1(A_{tr1}, A_{tr2})}{\partial A_{tr1}}\frac{\partial i_2(A_{tr1}, A_{tr2})}{\partial A_{tr1}} = \frac{(X_B X_C)^2\,Y_1\,(Y_A - Y_D)}{[X_1 Y_1 - (X_A - X_D)(Y_A - Y_D)]^2} \tag{6.102}$$

The sign of this term obviously depends on $Y_1(Y_A - Y_D)$, where

$$Y_1 = -g_1(i_1)\,g_2'(i_2) + 2\beta\sqrt{g_1(i_1)\,g_2(i_2)}\times \left\{A_{tr2}\alpha_2\left[\frac{A_1(i_1)}{A_2(i_2)}\right]^{1-\alpha_2} A_2'(i_2)\,g_2'(i_2) + g_2''(i_2)\left(A_{tr2}[A_2(i_2)]^{\alpha_2}[A_1(i_1)]^{1-\alpha_2} - i_2\right)\right\} \tag{6.103}$$

$$Y_A - Y_D = g_1'(i_1)\,g_2(i_2) - 2A_{tr2}(1-\alpha_2)\,\beta\,[A_2(i_2)]^{\alpha_2}[A_1(i_1)]^{-\alpha_2}A_1'(i_1)\,g_2'(i_2)\sqrt{g_1(i_1)\,g_2(i_2)} \tag{6.104}$$

It is ambiguous according to our assumptions.

Q.E.D. ■

Chapter 7

Conclusions, Limitations and Future Work

In this thesis, I attempt to explore the relations between investments, growth and asset prices with four models. Chapter 3 starts from a model with multiple assets in the exchange economy. It studies the co-movements of two assets when their cash flows are independent. The model shows that the proportions that one asset takes in the aggregate consumption is crucial to its pricing mechanism. Chapters 4 and 5 turn to the general equilibrium model in the production economy. Together, they establish a theoretical framework in which beliefs play important roles. In a pessimistic mood, the economy in this framework generates long-run stagnant growth accompanied by a downturn in investment, weak productivity, a low risk-free rate and an expansive risk premium, which correspond to the predictions of the secular stagnation hypothesis. In a preliminary way, they are calibrated to match the data moments collected from two periods in the US. With the recursive utility, the calibration in chapter 5 performs relatively well in matching all 10 data moments. Chapter 6 returns to the exploration of cross-sectional co-movements. It extends the model studied chapter 3 to the production economy. With the externalities and complementarities of the production processes, the model generates various kinds of co-movement patterns in cross-sectional investments, growth and asset returns.

Nevertheless, I am aware that the models in this thesis are, in a sense, over-simplified and subject to many limitations.

First, as suggested by Martin (2013), the models with multiple assets are subject to the curse of dimensionality and become incredibly difficult to solve both theoretically and numerically when the number of assets increases. The models with this framework in the production economy also suffer from this issue. Since the dynamics of in individual firm among massive firms are of great interest, a future study could focus on joining the studies of the distribution of firms and this asset pricing framework. For example, Gabaix (2011) studies the relation between aggregate macro-fundamental variables and the distribution of firms under Zipf's Law. Future research can introduce this distribution into the existing framework.

Second, the core assumptions such as threshold assumption and Markovian sunspots are over-simplified. These assumptions, on the one hand, are ad hoc and lack micro-foundations. Future studies can enrich the model by adding the mechanisms that explain the source of the externalities and offer clearer rationales for the sunspots. On the other hand, the discontinuous function and the two-stage Markov chain in this thesis are preliminary. By expanding the functional form and Markovian processes, the model should have a better performance in the calibration. In addition, with the help of Bayesian estimation methods in non-linear models, future studies should be able to estimate the model with different data sets from various countries.

Finally, although the habit assumption 6.1 in chapter 6 enables the model to have BGPs, it also eliminates the dynamics in the sectors' sizes and growth rates. A more sophisticated model should capture the sectoral interactions as well as the relations between individual sector and the aggregate economy. Moreover, in this framework, future research can also explore more about the relation between cross-sectoral assets returns and the risk-free rate or the risk premium.

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