Derived Categories of Surfaces and Group Actions

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To Debbie.

Abstract

This thesis focuses on two distinct projects on the bounded derived category of coherent sheaves of surfaces and group actions from different directions.

The first project studies bielliptic surfaces, which arise as quotients of products of elliptic curves by a finite group acting freely. We prove a structure theorem describing the group of exact autoequivalences of the bounded derived category of coherent sheaves on a bielliptic surface over $\mathbb{C}$. We also list the generators of the group in some cases.

The second project studies semi-orthogonal decompositions of the bounded equivariant derived category of a surface S with an effective action of a finite abelian group G. These semi-orthogonal decompositions are constructed by studying the geometry of the quotient stack $[S/G]$. We produce new examples of semi-orthogonal decompositions of the equivariant derived category of surfaces with a finite abelian group action. We give a new proof of the Derived McKay correspondence in dimension 2. Using this, we construct semi-orthogonal decompositions of the equivariant derived category of $\mathbb{C}^2$ with an effective action of the Dihedral group D_{2n}. Moreover, we show that these semi-orthogonal decompositions satisfy a conjecture of Polishchuk and Van den Bergh.

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Chapter 1

Introduction

1.1 The Derived Category

In homological algebra, we often define properties using resolutions. The derived category allows us to consider objects and their different resolutions as "the same" in a precise way. A consequence of this idea is how it allows us to define derived functors as functors between derived categories.

Although appearing abstract at first, the bounded derived category of coherent sheaves $D(X) = D^b \operatorname{Coh}(X)$ of a variety X contains a great deal of geometric information about the projective variety. Suppose the variety is smooth and projective over an algebraically closed field of characteristic zero and the (anti)-canonical bundle is ample. Then the variety is determined uniquely up to isomorphism by its derived category. Moreover, the derived category of a variety contains information about the connectedness of the variety, properties of the canonical bundle, and the Cox ring. If two varieties X and Y have equivalent derived categories they have the same dimension, the same Kodaira dimension, and the canonical bundle ω_X is ample or nef if and only if ω_Y is ample or nef.

The derived category is a powerful tool which allows us to understand different relationships between varieties. For example, two $K3$ surfaces which have equivalent derived categories can be expressed as moduli spaces of sheaves on each other, generalizing the Torelli Theorem. This interaction has allowed people to prove results on moduli spaces of sheaves which do not mention derived categories using derived techniques.

This thesis is the culmination of two distinct projects. The first studies the group of symmetries of the derived category for bielliptic surfaces - a surprisingly difficult problem. The second studies decompositions of the equivariant derived category with respect to a finite group acting effectively on a smooth projective variety. This allows us to describe new semi-orthogonal decompositions of equivariant derived categories for a minimal surface of general type, give a new proof of the derived McKay correspondence in dimension 2, and prove a conjecture of Polishchuk and Van den Bergh for an action of the dihedral group D_{2n} on $\mathbb{C}^2$.

1.2 Autoequivalences of the Derived Category

Let X be a smooth projective variety over the complex numbers. An important question in the study of the derived category $D(X)$ is to describe its group of symmetries: the group $\operatorname{Aut} D(X)$ of exact $\mathbb{C}$-linear autoequivalences of $D(X)$ considered up to isomorphism as functors. We think of these autoequivalences as "higher" symmetries of the variety.

Several autoequivalences of $D(X)$ arise naturally forming the subgroup

$$\operatorname{Aut}_{st} D(X) = (\operatorname{Aut} X \ltimes \operatorname{Pic} X) \times \mathbb{Z}$$

of *standard autoequivalences* of $\operatorname{Aut} D(X)$. This subgroup is generated by pulling back along automorphisms of X, tensoring by line bundles and by powers of the shift functor. These autoequivalences always exist. The central question becomes: are there any non-standard autoequivalences? Can we classify them?

When the (anti-)canonical bundle of X is ample, Bondal and Orlov [13, Theorem 3.1] showed that $\operatorname{Aut} D(X) = \operatorname{Aut}_{st} D(X)$, i.e. there are no non-standard autoequivalences of $D(X)$. The first example of a non-standard autoequivalence was observed by Mukai [55] for principally polarized abelian varieties. Many have studied non-standard autoequivalences of the derived category but the full group $\operatorname{Aut} D(X)$ is only understood in a small number of cases. The only complete description in all dimensions of $\operatorname{Aut} D(X)$ for varieties X with neither ω_X ample or ω_X^{-1} ample is given by Orlov [60] for Abelian varieties.

Together with Bondal and Orlov's result, this classifies the group of autoequivalences of the derived category of smooth projective curves.

Theorem 1.2.1 (Bondal-Orlov, Orlov)**.** *Let X be a smooth projective curve of genus g over an algebraically closed field of characteristic zero.*

- *If $g = 0$ or $g \geq 2$, then*

$$\operatorname{Aut} D(X) = \operatorname{Aut}_{st} D(X) = (\operatorname{Aut} X \ltimes \operatorname{Pic} X) \times \mathbb{Z}.$$

- *If $g = 1$, there is a short exact sequence of groups*

$$1 \longrightarrow \mathbb{Z} \times (\operatorname{Aut}(X) \ltimes \operatorname{Pic}^0(X)) \longrightarrow \operatorname{Aut} D(X) \longrightarrow \operatorname{SL}(2, \mathbb{Z}) \longrightarrow 1.$$

Substantial progress has been made for surfaces. Broomhead and Ploog [19] computed the group for many rational surfaces (including most toric surfaces). Bayer and Bridgeland [5] described the group for K3 surfaces of Picard rank 1 using the theory of stability conditions. Uehara [74] conjectured a description of the group for smooth projective elliptic surfaces of non-zero Kodaira dimension and proved the conjecture when each reducible fibre is a cycle of (-2)-curves. Furthermore, he describes the group for elliptic ruled surfaces [75]. Ishii and Uehara [41] computed the group for smooth projective surfaces (not necessarily minimal) of general type whose canonical model has at worst A_n singularities. However, these are the only examples that are completely understood at this time for surfaces. We describe the group $\operatorname{Aut} D(S)$ when S is a bielliptic surface.

Bielliptic surfaces are minimal projective surfaces S of Kodaira dimension zero with geometric genus $p_g = \dim_{\mathbb{C}} H^2(S, \mathcal{O}_S) = 0$ and irregularity $q = \dim_{\mathbb{C}} H^1(S, \mathcal{O}_S) = 1$. They were classified by Bagnera and de Franchis as quotients of products of elliptic curves $A \times B$ by a finite group acting freely. They have torsion canonical bundle of order $n = 2, 3, 4, 6$. Using the torsion canonical bundle we can construct an abelian surface $\widetilde{S}$, the canonical cover of S, realizing S as the quotient of $\widetilde{S}$ by a free action of a cyclic group of order $2, 3, 4$ or 6 respectively.

Bielliptic surfaces come equipped with two elliptic fibrations $p_A \colon S \to A/G$ and $p_B \colon S \to B/G$ induced by the projections from the product $A \times B$ onto each factor. The first is smooth with fibres isomorphic to B, the second has smooth fibres isomorphic to A and multiple fibres over the fixed points of the action of G on B.

We study the group of autoequivalences of the bielliptic surface S by studying the action of $\operatorname{Aut} D(S)$ on the numerical Grothendieck group $N(S)$ of S, which is a quotient of the Grothendieck group $K(S)$. To any complex $E^\bullet \in D(S)$, we associate its class $[E^\bullet] = \sum_i (-1)^i [\mathcal{H}^i(E^\bullet)]$ in $N(S)$ as the alternating sum of its cohomology sheaves. This gives a natural action of $\operatorname{Aut} D(S)$ on $N(S)$ by

$$\rho \colon \operatorname{Aut} D(S) \to \operatorname{Aut} N(S)$$

where $\rho(\Phi)([E^\bullet]) = [\Phi(E^\bullet)]$. As autoequivalences preserve Hom sets, their image under ρ preserves the Euler form on $N(S)$. So $\rho(\Phi)$ is an isometry of $N(S)$. Moreover, $\rho(\Phi)$ preserves the subgroup

$$\Delta = \left\{ [E] \in N(S) \middle| [E] = \pi_!([\widetilde{E}]) \text{ for some } [\widetilde{E}] \in N(\widetilde{S}) \right\} \subset N(S)$$

where $\pi_! \colon N(\widetilde{S}) \to N(S)$ is induced by the pushforward on K-theory. Denote by $O_\Delta(N(S))$ the subgroup of isometries of $N(S)$ which preserve Δ. The main Theorem of Chapter 3 is the following:

Theorem 1.2.2. *There is an exact sequence*

$$1 \longrightarrow (\operatorname{Aut} S \ltimes \operatorname{Pic}^0 S) \times \mathbb{Z} \longrightarrow \operatorname{Aut} D(S) \xrightarrow{\rho} O_\Delta(N(S))$$

where $\mathbb{Z}$ is generated by the second shift [2]. The map ρ is induced by the natural action of $\operatorname{Aut} D(S)$ on $N(S)$ given by $\rho(\Phi)[E] = [\Phi(E)]$. Furthermore, the image of ρ is a subgroup of $O_\Delta(N(S))$ of index 4 if S of type A2 or B2 and index 2 otherwise (see

Table 3.1).

Bridgeland in [14] describes a family of autoequivalences associated to an elliptic fibration called *relative Fourier-Mukai Transforms.* As a bielliptic surface has two elliptic fibrations we get two families of autoequivalences. When the canonical cover of S is a product of elliptic curves (we call such S *cyclic*) we describe the generators of $\operatorname{Aut} D(S)$.

Theorem 1.2.3. *Suppose S is a cyclic bielliptic surface. Then $\operatorname{Aut} D(S)$ is generated by standard autoequivalences and relative Fourier-Mukai transforms along the two elliptic fibrations.*

We expect Theorem 1.2.3 to extend to all bielliptic surfaces.

1.3 Semi-orthogonal Decompositions of Equivariant Derived Categories

We now introduce the second project which studies decompositions of the equivariant derived category with respect to finite group actions. First, we review the McKay correspondence which focuses on the local case before explaining the approach we will take to studying the global case using the language of Deligne-Mumford stacks.

1.3.1 The McKay Correspondence

The McKay correspondence, and its derivatives, originated from an observation by John McKay in [52] of a bijection between non-trivial irreducible representations of finite subgroups $G \subset \mathrm{SL}(2, \mathbb{C})$ and rational curves in the exceptional locus of the minimal resolution $Y \to \mathbb{C}^2/G$ of the quotient singularity. Precisely, McKay gave an argument that links affine Dynkin diagrams arising from the representation theory (the *McKay* graph) of a finite group $G \subset \mathrm{SL}_2(\mathbb{C})$ with the dual intersection graph of irreducible exceptional curves on the resolution of the singularity $\mathbb{C}^2/G$.

This bijection was realized geometrically by Gonzalez-Springberg and Verdier [34] using vector bundles $\mathcal{L}_\rho$ called *tautological bundles* on the minimal resolution, which are constructed from non-trivial irreducible representations ρ of G. Moreover, this bijection gives an isomorphism between the Grothendieck group $K^G(\mathbb{C}^2)$ of G-equivariant coherent sheaves on $\mathbb{C}^2$ and $K(Y)$ the Grothendieck group of the minimal resolution Y of $\mathbb{C}^2/G$.

The bounded derived category of coherent sheaves on a smooth projective variety can be thought of as a "categorification" of the Grothendieck group. We would expect the isomorphism

$$K^G(\mathbb{C}^2) \cong K(Y)$$

to lift to an equivalence of derived categories. Kapranov and Vasserot [44] proved that it does.

Theorem 1.3.1. *Let X be a surface equipped with a holomorphic symplectic form ω and suppose that the G-action on X preserves ω. Then*

$$D^b(Y) \cong D^G(X)$$

where $Y \to X/G$ is the minimal resolution of X/G and $D^G(X) = D^b(\mathrm{Coh}^G(X))$ is the bounded derived category of G-equivariant coherent sheaves on X.

As a corollary, we have the following version of the McKay Correspondence often referred to as the derived McKay Correspondence for subgroups of $\mathrm{SL}_2(\mathbb{C})$.

Corollary 1.3.2. *Let $G \subset \mathrm{SL}(2, \mathbb{C})$ be a finite subgroup and $Y \to \mathbb{C}^2/G$ the minimal resolution of $\mathbb{C}^2/G$. Then there is an equivalence*

$$D^b(Y) \cong D^G(\mathbb{C}^2).$$

This equivalence was extended by Bridgeland, King, and Reid [16] to 3-folds.

The philosophy behind the McKay Correspondence is, as stated by Reid [65], that

> any question about the G-equivariant geometry of $\mathbb{C}^n$ should have an answer related to the geometry of a crepant resolution $Y \to \mathbb{C}^n/G$.

Further work on the McKay Correspondence has diverged in two different directions:

1. Studying the higher dimensional case where we consider finite subgroups $G \subset \mathrm{SL}(n, \mathbb{C})$ and crepant resolutions (see [65] for a survey) with the aim of relating the representation theory of G to the geometry of a crepant resolution (when one exists) of $\mathbb{C}^n/G$.
2. Considering more general groups $G \subset \mathrm{GL}(2, \mathbb{C})$ and try to relate the representation theory of G to the geometry of the minimal resolution $Y \to \mathbb{C}^2/G$.

We will follow the second case.

1.3.2 The Special McKay Correspondence

Finite subgroups of $G \subset \mathrm{GL}_n(\mathbb{C})$ may contain elements which fixed a codimension 1 hyperplane in $\mathbb{C}^n$, which we call *pseudo-reflections*. A subgroup which contains no pseudo-reflections is called *small*.

If we are only interested in properties of the singularity we can reduce to the study of small subgroups of $\mathrm{GL}_2(\mathbb{C})$. Let $N \subset G$ be the subgroup generated by pseudo-reflections. Then by the Chevalley-Shephard-Todd Theorem [69] $\mathbb{C}^n/N \cong \mathbb{C}^n$, so

$$\mathbb{C}^n/G \cong (\mathbb{C}^n/N)/(G/N).$$

Thus if we only were interested in the singularity and bijections arising from resolving the singularity we are reduced to studying small subgroups G of $\mathrm{GL}_n(\mathbb{C})$.

We now consider small finite subgroups of $\mathrm{GL}_2(\mathbb{C})$. Unlike in the $\mathrm{SL}_2(\mathbb{C})$ case there is no bijection between irreducible exceptional curves and non-trivial irreducible exceptional curves - the representation theory of G can be strictly larger.

Wunram [78] and Riemenschneider [66] re-established a bijection by considering a subset of *special* representations of G corresponding to reflexive modules on the quotient $\mathbb{C}^2/G$ which lift to full sheaves supported on irreducible components of the exceptional locus. This bijection is referred to as the *special McKay correspondence.*

The non-special representations of G measure the failure of the minimal resolution to capture the equivariant geometry of G. On the level of the derived category, this measure of failure will be expressed using a semi-orthogonal decomposition.

1.3.3 Semi-orthogonal Decompositions

The derived category is a complicated object. One way to simplify it is to decompose the derived category into simpler pieces. A semi-orthogonal decomposition does this by filtering objects.

A semi-orthogonal decompositions of a triangulated category $\mathcal{D}$ is a pair of strict full triangulated subcategories $\mathcal{A}, \mathcal{B}$ such that:

1. For all $A \in \mathcal{A}$ and $B \in \mathcal{B}$, $\mathrm{Hom}_{\mathcal{D}}(B, A) = 0$.
2. The triangulated category $\mathcal{D}$ is generated by $\mathcal{A}$ and $\mathcal{B}$ by taking shifts, cone of morphisms and direct sums from objects. Equivalently, any object $D \in \mathcal{D}$ has a decomposition
$$D_{\mathcal{A}} \longrightarrow D \longrightarrow D_{\mathcal{B}} \longrightarrow T(D_{\mathcal{A}})$$
where T is the shift functor encoded in the triangulated structure on $\mathcal{D}$, $D_{\mathcal{A}} \in \mathcal{A}$ and $D_{\mathcal{B}} \in \mathcal{B}$.

We write $\mathcal{D} = \langle \mathcal{A}, \mathcal{B} \rangle$ for such a semi-orthogonal decomposition. Using induction we can define a semi-orthogonal decomposition with more than two pieces. A semi-orthogonal decomposition $\mathcal{D} = \langle \mathcal{A}, \mathcal{B} \rangle$ is *orthogonal* if additionally for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$, $\mathrm{Hom}_{\mathcal{D}}(A, B) = 0$.

The derived category of a connected noetherian scheme has no orthogonal decompositions by Bridgeland [15, Example 3.2]. However, many connected varieties have semi-orthogonal decompositions. The most famous example was given by Beilinson [8].

Theorem 1.3.3 (Beilinson)**.** *There is a semi-orthogonal decompositions*

$$D^b(\mathbb{P}^n) = \langle \mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), \dots, \mathcal{O}_{\mathbb{P}^n}(n-1) \rangle$$

where $\mathcal{O}_{\mathbb{P}^n}(i)$ denotes the full triangulated subcategory generated by $\mathcal{O}_{\mathbb{P}^n}(i)$.

Let G be a finite group acting faithfully on a curve X over an algebraically closed field of characteristic zero. Denote by $D_1, \ldots, D_n$ the special fibres of $\pi\colon X \to X/G$ with the non-reduced scheme structure. Denote by $m_1, \ldots, m_n$ the multiplicities of the special fibres. Then we have the following due to Polishchuk [63].

Theorem 1.3.4 ([63, Theorem 1.2]). *For each $i = 1, \ldots, n$, denote the full triangulated subcategory of $D^G(X)$ generated by $\mathcal{O}_{kD_i}$ for $1 \leq k \leq m_i - 1$ by*

$$\mathcal{B}_i = \langle \mathcal{O}_{(m_i-1)D_i}, \ldots, \mathcal{O}_{2D_i}, \mathcal{O}_{D_i} \rangle.$$

Note that the subcategories $\mathcal{B}_i$ and $\mathcal{B}_j$ are mutually orthogonal for $i \neq j$. There is a semi-orthogonal decomposition

$$D^G(X) = \left\langle \pi^* D^b(X/G), \mathcal{B}_1, \ldots, \mathcal{B}_n \right\rangle.$$

Ishii and Ueda [40] interpreted the special McKay correspondence in terms of the derived category using semi-orthogonal decompositions in the following way.

Theorem 1.3.5 ([40, Theorem 1.2]). *Let G be a finite small subgroup of $\mathrm{GL}(2, \mathbb{C})$ and let $Y \to \mathbb{C}^2/G$ be the minimal resolution of the quotient. Then there is a semi-orthogonal decomposition*

$$D^G(\mathbb{C}^2) = \left\langle \Phi_Y D^b(Y), E_1, \ldots, E_n \right\rangle$$

where E_i are exceptional objects and n is the number of non-special representations of G.

Kawamata extended this to general $G \subset \mathrm{GL}_2(\mathbb{C})$ in [45] and $G \subset \mathrm{GL}_3(\mathbb{C})$ in [46] by understanding how the Toric Minimal Model program affects the derived category of smooth Deligne-Mumford stacks associated to pairs (X, B).

1.3.4 Stacks and the McKay Correspondence

We are interested in global versions of the McKay correspondence when X is a smooth projective surface over a field k and G an arbitrary finite group acting effectively on X. It is easy to construct examples where G acts via $\mathrm{SL}_2(\mathbb{C})$ on an affine chart but via $\mathrm{GL}_2(\mathbb{C})$ on another (consider the action $(x:y:z) \mapsto (-x:-y:z)$ of $\mathbb{Z}_2$ on $\mathbb{P}^2$).

In this thesis, we will study G-equivariant sheaves on X by studying sheaves on the quotient stack $[X/G]$ as we have the following equivalence of categories

$$\mathrm{Coh}^G(X) \cong \mathrm{Coh}([X/G]).$$

In Chapter 5 we construct semi-orthogonal decompositions of $D([X/G]) = D^b(\mathrm{Coh}([X/G])$ by studying the geometry of the quotient stack $[X/G]$.

This uses previous work by Satriano and Geraschenko who give a structure theorem for smooth separated tame Deligne-Mumford stacks $\mathcal{X}$ in terms of their coarse moduli space X. They use two constructions in their theorem: the canonical stack construction and the root stack construction. The former contains information about "stackiness" in codimension greater than one and the later about codimension one "stackiness". The reduction in the McKay correspondence to studying small groups amounts to reducing to the canonical stack.

Both of these constructions were studied by Ishii and Ueda in [40] and recently in further generality by Bergh, Lunts, and Schnürer [9]. They prove the following which we state in more generality below.

Theorem 1.3.6. *Let $\mathcal{X}$ be a smooth separated tame Deligne-Mumford stack with trivial generic stabilizer. Then we have a decomposition*

$$\begin{tikzcd} \mathcal{X} \arrow[r, "f"] \arrow[rr, bend left, "\pi"] & X^{can} \arrow[r, "\epsilon"] & X \end{tikzcd}$$

of the coarse moduli space map. Assume:

1. *That the morphism $\pi\colon \mathcal{X} \to X$ is an isomorphism outside a simple normal crossing divisor $D = \sum_{i=1}^{n} D_i$. Denote by $\mathcal{D} = \sum_{i=1}^{n} \mathcal{D}_i$ the pullback of D to X^{can}.*
2. *The pull back $f^*(\mathcal{D}_i)$ is a multiple of a prime divisor of order r_i.*

Then there exists a semi-orthogonal decomposition of $D(\mathcal{X})$ with one piece given by the derived categories of X^{can} and the rest by derived categories of intersections of the divisors D_i.

We derive the immediate Corollary below for a quotient stack $[X/G]$ when G is an abelian group.

Corollary 1.3.7. *Let X be a smooth quasi-projective variety over k and G a finite abelian group whose order is coprime to the characteristic of k. Let $D = \sum_{i=1}^{n} D_i$ on X/G be the branch divisor. Denote by $\mathcal{D}$ the pullback of D to the canonical stack $(X/G)^{can}$.*

Then there is a semi-orthogonal decomposition of $D^G(X) \cong D([X/G])$ with pieces given by:

- *The derived category $D((X/G)^{can})$ of the canonical stack $(X/G)^{can}$.*
- *The derived category $D(\mathcal{D}_i)$ of the irreducible components of the branch divisor.*
- *The derived category of the intersections of branch divisors.*

More generally, for any non-abelian group smooth quotient stack $[X/G]$ (or smooth separated Deligne-Mumford stacks $\mathcal{X}$) we have the following theorem.

Theorem 1.3.8. *Let $\mathcal{X}$ be a smooth separated Deligne-Mumford stack with trivial generic stabilizer over a field k of characteristic zero with coarse moduli space X. Denote the canonical stack associated to X by X^{can} and let $f\colon \mathcal{X} \to X^{can}$ be the unique map given by the universal property of X^{can}. Then the functor*

$$f^*\colon D(X^{can}) \to D(\mathcal{X})$$

is fully faithful.

1.3.5 Applications

Using the theory developed in Chapter 5 we give several applications in Chapter 6.

1. We describes new semi-orthogonal decompositions of equivariant derived categories of minimal surfaces of general type with actions of finite groups in several examples. We also discuss the case for smooth abelian Galois covers of smooth projective varieties in sections 6.2 and 6.3.

2. We give a new proof of the derived McKay correspondence in dimension 2 in Section 6.4:

   Theorem 1.3.9. *Let $G \subset \mathrm{GL}(2,\mathbb{C})$ be a finite subgroup acting faithfully on $\mathbb{C}^2$. Then there is a semi-orthogonal decomposition of the equivariant derived category*

   $$D^G(\mathbb{C}^2) = \left\langle E_1, \dots, E_n, \Phi_{\widetilde{D_1}} D(\widetilde{D}_1), \dots, \Phi_{\widetilde{D_n}} D(\widetilde{D}_m), \Phi_{\tilde{Y}} D(\widetilde{Y}) \right\rangle$$

   where $\widetilde{Y}$ is the minimal resolution of $\mathbb{C}^2/G$, $\widetilde{D}_i$ are the normalizations of the irreducible components of the branch divisor $D = \sum_{i=1}^m D_i$ and $E_1, \dots, E_n$ are exceptional objects.

3. Using our new proof of the derived McKay correspondence in dimension two we compute semi-orthogonal decompositions for the action of the Dihedral group

   $$D_{2n} = \left\{\tau, \sigma \middle| \tau^n = \sigma^2 = e,\, \tau\sigma\tau = \sigma\right\}.$$

   acting effectively on $\mathbb{C}^2$ by $\rho\colon D_{2n} \to \mathrm{GL}(2,\mathbb{C})$, given by

   $$\rho(\tau) = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}, \quad \rho(\sigma) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

   where $\xi^n = 1$ is an n-th root of unity. Denote by $D \subset \mathbb{C}^2/D_{2n}$ the branch divisor.

   Theorem 1.3.10. *Let D_{2n} act on $\mathbb{C}^2$ as above. Then we have two cases:*

   Odd n: *There is a semi-orthogonal decomposition*

   $$D^{D_{2n}}(\mathbb{C}^2) = \left\langle \pi^* D(\mathbb{C}^2), \Phi_{\widetilde{D}}(D(\widetilde{D})), E_1, \dots, E_{\frac{n-1}{2}} \right\rangle$$

where $\widetilde{D}$ is the normalization of D.

Even n: *There is a semi-orthogonal decomposition*

$$D^{D_{2n}}(\mathbb{C}^2) = \left\langle \pi^* D(\mathbb{C}^2), \Phi_{\widetilde{D}_1}(D(\widetilde{D}_1)), \Phi_{\widetilde{D}_2}(D(\widetilde{D}_2)), E_1, \ldots, E_{\frac{n}{2}} \right\rangle$$

where $D = D_1 \cup D_2$ is reducible and $\widetilde{D}_i$ are the normalization of D_i.

Furthermore, we verify that these semi-orthogonal decompositions agree with the motivic decomposition conjecture of Polishchuk and Van den Bergh [64].

Conjecture 1.3.11 (Motivic Decomposition)**.** *Assume that a finite group G acts effectively on a smooth quasi-projective variety X over an algebraically closed field and that all the quotients $X^g/C(g)$ are smooth for $g \in G$. Then there exists a semi-orthogonal decomposition of the derived category $D^G(X)$ of G-equivariant sheaves on X such that the pieces $\mathcal{C}_{[g]}$ of this decomposition are in bijection with the conjugacy classes of g in G and $\mathcal{C}_{[g]} \cong D(X^g/C(g))$.*

We expect that the theory of developed in Chapter 5 will allow us to prove Conjecture 1.3.11 for all abelian groups.

1.4 Structure of this Thesis

This thesis is structured as follows:

In Chapter 2 we review the necessary background on derived categories and derived functors before introducing properties of autoequivalences and semi-orthogonal decompositions.

In Chapter 3 we prove the main theorems in section 1.2 on the group of autoequivalences of the derived category of a bielliptic surface.

In Chapter 4 we review the background on Deligne-Mumford stacks that will be used in Chapters 5 and 6. We also introduce the derived category of a stack and derived functors between them.

In Chapter 5 we review the theorem of Ishii and Ueda and the structure theorem for smooth separated Deligne-Mumford stacks by Geraschenko and Satriano. Using their description we describe semi-orthogonal decompositions of the derived categories for quotient stacks $[X/G]$ when G is abelian. We also prove that for a general smooth Deligne-Mumford stack $\mathcal{X}$ with coarse moduli space X, the derived category of the canonical stack X^{can} associated to X embeds fully faithfully into $D(\mathcal{X})$.

In Chapter 6 we give several application of the results in Chapter 5. In particular, we construct new examples of semi-orthogonal decompositions for abelian groups acting on smooth projective surfaces. These include explicit examples for surfaces of general type, Godeaux surfaces, and Burniat surfaces. We give a new proof of the derived

McKay Correspondence in dimension 2. As a consequence of this, we describe new semi-orthogonal decompositions for Dihedral groups D_{2n} acting on $\mathbb{C}^2$ and prove Polishchuk and Van den Bergh's Motivic decompositions conjecture for them.

1.4.1 Notation and Conventions

We denote the category of schemes over S by Sch/S.

We will consider all schemes and stacks over a base scheme S. All stacks in this thesis are Deligne-Mumford stacks over a base scheme S.

We will denote the cyclic group $\mathbb{Z}/n\mathbb{Z}$ by $\mathbb{Z}_n$.

For an abelian category $\mathcal{A}$ we denote the unbounded derived category by $D(\mathcal{A})$ and by $D^*(\mathcal{A})$ where $* = +, -, b$ the bounded below, bounded above and bounded derived categories of $\mathcal{A}$.

We will denote the bounded category of coherent sheaves on a scheme X by $D(X) = D^b(\mathrm{Coh}(X))$ compared to $D(\mathrm{Coh}(X))$ which denotes the *unbounded* derived category of coherent sheaves on X.

Chapter 2

Background on Derived Categories

In this chapter, we review background material on the derived category of an abelian category before focusing on the derived category of coherent sheaves on a scheme.

First, we recall the basic construction of the derived category of an abelian category and properties of the derived category in section 2.1. We then recall some basic properties of the derived categories of (quasi)-coherent sheaves on a noetherian scheme and derived functors between them in section 2.2. Next, we review the theory of Fourier-Mukai transforms and autoequivalences in section 2.3. Finally, we review semi-orthogonal decompositions of triangulated categories in section 2.4 and give some examples.

2.1 Constriction and Properties of Derived categories

We give an overview of the construction of the derived category and properties it has following chapters III and IV in [29]. The derived category was first constructed by Grothendieck and studied by Verdier in his thesis [76] to generalize Serre duality and put the theory of derived functors on a more conceptual level.

2.1.1 Basic Construction

Let $\mathcal{A}$ be an abelian category. Denote by $\mathrm{Ch}(\mathcal{A})$ the category of chain complexes over $\mathcal{A}$ which has objects chain complexes denoted by $A^\bullet$. Throughout this thesis we will use ascending degree notation, i.e. the i-th differential increases degree $d_i\colon A^i \to A^{i+1}$.

Recall that a morphism of chain complexes $f\colon E^\bullet \to F^\bullet$ is a *quasi-isomorphism* if the induced maps $f_*\colon H^i(E^\bullet) \to H^i(F^\bullet)$ are isomorphisms for all $i \in \mathbb{Z}$. The derived category can be constructed by localizing the category $\mathrm{Ch}(\mathcal{A})$ of chain complexes by quasi-isomorphisms.

Definition 2.1.1. *Let $\mathcal{A}$ be an abelian category and $\mathrm{Ch}(\mathcal{A})$ the category of chain complexes over $\mathcal{A}$. The derived category of $\mathcal{A}$ is a category $D(\mathcal{A})$ and a functor $Q\colon \mathrm{Ch}(\mathcal{A}) \to D(\mathcal{A})$ which satisfies the following properties:*

(i) For any quasi-isomorphism f, $Q(f)$ is an isomorphism,

(ii) The pair $(Q, D(\mathcal{A}))$ is universal in the following way: given any other functor $F\colon \mathrm{Ch}(A) \to \mathcal{D}$ such that for an quasi-isomorphism f, $F(f)$ is an isomorphism there exists a unique functor $G\colon D(\mathcal{A}) \to \mathcal{D}$ such that $F = G \circ Q$.

$$\begin{tikzcd} \mathrm{Ch}(\mathcal{A}) \arrow[r, "Q"] \arrow[dr, "F"'] & D(\mathcal{A}) \arrow[d, "\exists ! G"] \\ & \mathcal{D} \end{tikzcd}$$

We call the category $D(\mathcal{A})$ the derived category *of $\mathcal{A}$.*

The above definition asserts, if it exists, that the derived category is unique up to unique equivalence of categories. However, it does not guarantee that it does exists. An elementary proof of existence can be found in [29, III §2.2] which constructs $D(\mathcal{A})$ formally by adjoining inverses to quasi-isomorphisms. This does not, however, give a concrete description of the morphisms between any two objects. To get a better grasp of the morphism we construct $D(\mathcal{A})$ by localization.

Let $K(\mathcal{A})$ denote the homotopy category of $\mathrm{Ch}(A)$ whose objects are chain complexes over $\mathcal{A}$ and morphisms are homotopy classes of morphisms between chain complexes (see [29, III §4]). We often impose the following finiteness conditions on complexes. Denote by $K^+(\mathcal{A})$ the subcategory of $K(\mathcal{A})$ with objects with

$$E^i = 0 \text{ for } i \geq i_0(E^\bullet) \text{ for some } i_0(E^\bullet) \in \mathbb{Z}$$

and $K^-(\mathcal{A})$ the subcategory of $K(\mathcal{A})$ with objects with

$$E^i = 0 \text{ for } i \leq i_0(E^\bullet) \text{ for some } i_0(E^\bullet) \in \mathbb{Z}\,.$$

Let $K^b(\mathcal{A}) = K^+(\mathcal{A}) \cap K^-(\mathcal{A})$ which has objects with $E^i = 0$ for $|i| > i_0(E^\bullet) \in \mathbb{Z}$.

We construct $D(\mathcal{A})$ by localizing $K(\mathcal{A})$ by quasi-isomorphisms using a generalization of localization for non-commutative rings using the Ore conditions (see [29, III §2.6-2.10]).

Proposition 2.1.2 ([29, III §4 Proposition 2])**.** *The localization of $K(\mathcal{A})$ by quasi-isomorphisms is canonically isomorphic to the derived category $D(\mathcal{A})$. The same holds for $K^*(\mathcal{A})$ and $D^*(\mathcal{A})$ with $* = +, -, b$.*

The objects of $D(\mathcal{A})$ are the same as objects of $K(\mathcal{A})$ and $\mathrm{Ch}(\mathcal{A})$. A morphism between two chain complexes $E^\bullet$ and $F^\bullet$ in $D(\mathcal{A})$ is an equivalence class of diagrams called a *roof*.

$$\begin{tikzcd} & Z^\bullet \arrow[dl, "s"'] \arrow[dr, "f"] & \\ E^\bullet & & F^\bullet \end{tikzcd}$$

where f and s are morphisms in $K(\mathcal{A})$ and s is a quasi-isomorphism. Two diagrams are equivalent if there is a further roof that makes everything commute.

2.1.2 Properties of the Derived Category

The derived category (and $K(\mathcal{A})$) are not usually abelian. They do, however, possess a triangulated structure.

Definition 2.1.3. *Let $\mathcal{D}$ be an additive category. A triangulated structure on $\mathcal{D}$ is specified by the data:*

a) An additive endomorphism $T\colon \mathcal{D} \to \mathcal{D}$.

b) A class of distinguished triangles

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X).$$

A morphism of distinguished triangles is given by a diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & T(X) \arrow[d, "T(f)"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & T(X'). \end{tikzcd}$$

We require that this data satisfies the following axioms:

1. *For any $X \in \mathcal{D}$,*
$$X \xrightarrow{id} X \longrightarrow 0 \longrightarrow T(X)$$
is a distinguished triangle.

2. *The set of distinguished triangles is closed under isomorphism.*

3. *Any morphism $u\colon X \to Y$ can be extended to a distinguished triangle*
$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X).$$

4. *Any triangles*
$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X).$$
is distinguished if and only if
$$Y \xrightarrow{u} Z \xrightarrow{v} T(X) \xrightarrow{-T(u)} T(Y).$$
is distinguished.

5. *Given a diagram*

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] & T(X) \arrow[d, "T(f)"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & T(X').
\end{tikzcd}$$

Then the diagram can be completed to a morphism of distinguished triangles by a morphism (not necessarily unique) $h\colon Z \to Z'$.

6. *(The Octahedral Axiom) Given a commutative diagram*

$$\begin{tikzcd}
X_1 \arrow[r, "u_3"] \arrow[d, "id_{X_1}"] & X_2 \arrow[r, "v_3"] \arrow[d, "u_1"] & Z_3 \arrow[r, "w_3"] \arrow[d, "m_1"] & T(X_1) \arrow[d, "id_{T(X_1)}"] \\
X_1 \arrow[r, "u_1 \circ u_3"] \arrow[d, "u_3"] & X_3 \arrow[r, "v_2"] \arrow[d, "id_{X_3}"] & Z_2 \arrow[r, "w_2"] \arrow[d, "m_1"] & T(X_1) \arrow[d, "T(u_3)"] \\
X_2 \arrow[r, "u_1"] \arrow[d, "v_3"] & X_3 \arrow[r, "v_1"] \arrow[d, "v_2"] & Z_1 \arrow[r, "w_1"] \arrow[d, "id_{Z_1}"] & T(X_2) \arrow[d, "T(v_3)"] \\
Z_3 \arrow[r, "m_1"] & Z_2 \arrow[r, "m_3"] & Z_1 \arrow[r, "T(v_3)\circ w_1"] & T(Z_3)
\end{tikzcd}$$

such that the top three rows are distinguished triangles and the maps induce maps of distinguished triangles. Then the bottom row is a distinguished triangle.

The triangulated structure on $K(\mathcal{A})$ is given as follows:

1. The additive endomorphism is given by the shift functor $[1]\colon K(\mathcal{A}) \to K(\mathcal{A})$ where $E^\bullet[1]$ is defined by

$$(E^\bullet[1])^i = E^{i+1}, \quad d^i_{E^\bullet[1]} = -d^{i+1}_{E^\bullet}\colon E^{i+1} \to E^{i+2}.$$

2. The set of distinguished triangles are given by the cone construction. Let $f\colon E^\bullet \to F^\bullet$ be a morphism of chain complexes. Then define $C(f)$, the *cone of* f, by

$$C(f)^i = E^{i+1} \oplus F^i, \quad d^i_{C(f)} = \begin{pmatrix} -d^{i+1}_{E^\bullet} & 0 \\ f[1] & d_{F^\bullet} \end{pmatrix}.$$

A distinguished triangle in $K(\mathcal{A})$ is any diagram isomorphic to

$$E^\bullet \xrightarrow{u} F^\bullet \xrightarrow{v} C(u) \xrightarrow{w} E^\bullet[1].$$

This triangulated structure of $K(\mathcal{A})$ induces triangulated structure on $K^*(\mathcal{A})$ for $* = +, -, b$. Because the triangulated structure is compatible with quasi-isomorphisms, the derived category inherits a triangulated structure from $K(\mathcal{A})$ with the shift functor as the additive endomorphism and the image of distinguished triangles under $Q\colon K(\mathcal{A}) \to D(\mathcal{A})$ defining distinguished triangles in $D(\mathcal{A})$.

There is a natural way to view $\mathcal{A}$ sitting inside $D(\mathcal{A})$ by considering an object $E \in \mathcal{A}$ as a complex concentrated in degree 0.

Proposition 2.1.4 ([29, III §5.2]). *Denote by $F\colon \mathcal{A} \hookrightarrow D^*(\mathcal{A})$ the inclusion defined by*

$$F(A) = \cdots \to 0 \to A \to 0 \to \cdots .$$

Then F is fully faithful and the essential image of F is the full subcategory given by

$$\left\{E^\bullet \in D(\mathcal{A}) \middle| H^i(E^\bullet) = 0 \text{ for all } i \neq 0\right\}.$$

Remark 2.1.5. *Using this we define for $E, F \in \mathcal{A}$,*

$$\mathrm{Ext}^i(E, F) = \mathrm{Hom}_{D(\mathcal{A})}(E, F[i]).$$

One can show that this definition of Ext^i is equivalent to the definition using derived functors if $\mathcal{A}$ admits enough injectives.

2.1.3 Derived Functors

We now define derived functors associated to left (resp. right) exact functors between abelian categories. In this section, we follow [29, III §6].

First note that exact functors between abelian categories induce exact functors between derived categories.

Proposition 2.1.6 ([29, III §6.2]). *Assume that $F\colon \mathcal{A} \to \mathcal{B}$ is exact.*

1. *Then the induced functor*

$$K^*(F)\colon K^*(\mathcal{A}) \to K^*(\mathcal{B})$$

defined by $K^*(F)(E^\bullet)^i = F(E^i)$ sends quasi-isomorphisms to quasi-isomorphisms and induces a functor

$$D^*(F)\colon D^*(\mathcal{A}) \to D^*(\mathcal{B}).$$

2. *The functor $D^*(F)$ is an exact functor, i.e. it sends distinguished triangles to distinguished triangles.*

For left (resp. right) exact functors we define right (resp. left) derived functors as follows.

Definition 2.1.7. *The* derived functor *of an additive left exact functor $F\colon \mathcal{A} \to \mathcal{B}$ is a pair consisting of an exact functor $RF\colon D^+(\mathcal{A}) \to D^+(\mathcal{B})$ and a natural transformation (morphism of functors) $\epsilon_F\colon Q_\mathcal{B} \circ K^+(F) \to RF \circ Q_\mathcal{A}$ where $Q_\mathcal{A}$ and $Q_\mathcal{B}$ are the* *localization functors and $K^+(F)\colon K^+(\mathcal{A}) \to K^+(\mathcal{B})$ is the induced functor.*

$$\begin{tikzcd} & D^+(\mathcal{A}) \arrow[dr, "RF"] & \\ K^+(\mathcal{A}) \arrow[ur, "Q_{\mathcal{A}}"] \arrow[dr, "K^+(F)"'] & & D^+(\mathcal{B}) \\ & K^+(\mathcal{B}) \arrow[ur, "Q_{\mathcal{A}}"'] & \end{tikzcd}$$

This pair satisfies the following universal property: for any exact functor $G\colon D^+(\mathcal{A}) \to D^+(\mathcal{B})$ and any morphism of functors $\epsilon\colon Q_{\mathcal{B}} \circ K^+(F) \to G \circ Q_{\mathcal{A}}$, there exists a unique morphism of functors $\eta\colon RF \to G$ such that

$$\begin{tikzcd} & Q_{\mathcal{B}} \circ K^+(F) \arrow[dr, "\epsilon"] & \\ RF \circ Q_{\mathcal{A}} \arrow[ur, "\epsilon_F"] \arrow[rr, "\eta \circ Q_{\mathcal{A}}"] & & G \circ Q_{\mathcal{A}} \end{tikzcd}$$

commutes.

Similarly, the left derived functor *of a right exact functor $F\colon \mathcal{A} \to \mathcal{B}$ is a pair consisting of an exact functor $LF\colon D^-(\mathcal{A}) \to D^-(\mathcal{B})$ and a natural transformation $\epsilon_F\colon LF \circ Q_{\mathcal{A}} \to Q_{\mathcal{B}} \circ K^-(F)$ satisfying a universal property similar to above but with a morphism $\eta\colon G \to LF$.*

Remark 2.1.8. *By a standard categorical argument the right (resp. left) derived functor of an additive left (resp. right) exact functor is unique up to unique isomorphism.*

We now explain how to construct the right (resp. left) derived functor of a left (resp. right) exact functor using adaptive classes of objects.

Definition 2.1.9. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left (right) exact functor. A class of object $\mathcal{R} \subset Ob(\mathcal{A})$ is said to be* adapted *to F if it is stable under finite direct sums and satisfies the following two conditions:*

a) A left (right) exact functor F maps any acyclic complex from $Ch^+(\mathcal{R})$ $(\mathrm{Ch}^-(\mathcal{R}))$ into an acyclic complex.

b) For a left (right) exact functor F, any object of $\mathcal{A}$ is a sub-object (quotient) of an object from $\mathcal{R}$.

Proposition 2.1.10 ([29, III §5.4 and §5.8])**.** *Let $\mathcal{R}$ be a class of objects adapted to a left exact functor $F\colon \mathcal{A} \to \mathcal{B}$ and $S_{\mathcal{R}}$ be a class of quasi-isomorphisms in $K^+(\mathcal{R})$. Then $S_{\mathcal{R}}$ is a localizing class of morphisms in $K^+(\mathcal{R})$ and the canonical functor*

$$K^+(\mathcal{R})[S_{\mathcal{R}}^{-1}] \to D^+(\mathcal{A})$$

is an equivalence of categories. A similar statement holds for right exact functors.

Following [29, III §5.5] we construct the right derived functor RF of a left exact functor F as follows. First, we define $\bar{F}\colon K^+(\mathcal{R})[S_{\mathcal{R}}^{-1}] \to D^+(\mathcal{B})$ by

$$\bar{F}(E^\bullet)^i = F(E^i)$$

for $E^\bullet \in K^+(\mathcal{R})$. Using Proposition 2.1.10 we choose an equivalence $\Phi\colon D^+(\mathcal{A}) \to K^+(\mathcal{R})[S_{\mathcal{R}}^{-1}]$. Using this, we define $RF\colon D^+(\mathcal{A}) \to D^+(\mathcal{B})$ by

$$RF(E^\bullet) = \bar{F}(\Phi(E^\bullet)).$$

There is a similar construction for the left derived functor of a right exact functor. By [29, III §5.8] the functor RF defined above is the right derived functor of F.

For applications, we will need to produce an adaptive class of objects. Two classes of adaptive objects are given by injective and projective objects of $\mathcal{A}$ if we have enough of them.

Definition 2.1.11. *We say an abelian category $\mathcal{A}$ has* enough injectives *(resp. enough projectives) if for every object $A \in Ob(\mathcal{A})$ is a sub-object (resp. quotient object) of an injective (resp. projective) object.*

Theorem 2.1.12 ([29, III §6.12])**.** *If $\mathcal{A}$ contains enough injective (resp. projective) objects, then the class $\mathcal{I}$ (resp. $\mathcal{P}$) of injective (resp. projective) objects is adapted to any left (resp. right) exact functor $F\colon \mathcal{A} \to \mathcal{B}$.*

Remark 2.1.13. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor and $RF\colon D^+(\mathcal{A}) \to D^+(\mathcal{B})$ its right derived functor. Then we can define the* classical i-th derived functor of F *by $R^iF = H^0(RF[i]) = H^i(RF)$. A similar statement holds for left derived functors.*

Example 2.1.14. *Let $\mathcal{A}$ be an abelian category with enough injectives. Fix an object $X \in \mathcal{A}$ and consider $\operatorname{Hom}_{\mathcal{A}}(X,-)\colon \mathcal{A} \to \mathrm{Ab}$. This functor is left exact. Then we have*

$$\operatorname{Ext}^i_{\mathcal{A}}(X,-) = R^i \operatorname{Hom}_{\mathcal{A}}(X,-).$$

We will use the following criteria to see when a derived functor descends to a derived functor between bounded derived category.

Proposition 2.1.15 ([38, Corollary 2.68])**.** *Suppose that $F\colon K^+(\mathcal{A}) \to K^+(\mathcal{B})$ is an exact functor that admits a right derived functor $RF\colon D^+(\mathcal{A}) \to D^+(\mathcal{B})$.*

If $RF(A) \in D^b(\mathcal{B})$ for any object $A \in \mathcal{A}$, then $RF(E^\bullet) \in D^b(\mathcal{B})$ for any complex $E^\bullet \in D^b(\mathcal{A})$, i.e. RF descends to an exact functor

$$RF\colon D^b(\mathcal{A}) \to D^b(\mathcal{B}).$$

2.1.4 Serre Functors

We now introduce the notion of a Serre functor on a triangulated category. This abstracts the notion of Serre Duality for sheaves to arbitrary triangulated categories. One use of Serre functors is to construct adjoints. We follow [38, §1.1 and §1.2].

Definition 2.1.16. *A k-linear category is an additive category $\mathcal{A}$ such that the group* $\mathrm{Hom}_{\mathcal{A}}(A, B)$ *are k-vector spaces and all compositions are k-bilinear.*

All additive functors $F\colon \mathcal{A} \to \mathcal{B}$ between two k-linear categories over a common base field k will be assumed to be k-linear, i.e. for any two objects $A, B \in \mathcal{A}$ the induced map $\mathrm{Hom}_{\mathcal{A}}(A, B) \to \mathrm{Hom}_{\mathcal{B}}(F(A), F(B))$ *is k-linear.*

Definition 2.1.17. *Let $\mathcal{A}$ be a k-linear category. A* Serre functor *is a k-linear equivalence $S\colon \mathcal{A} \to \mathcal{A}$ such that for any two objects $A, B \in \mathcal{A}$ there exists an isomorphism*

$$\eta_{A,B}\colon \mathrm{Hom}_{\mathcal{A}}(A, B) \cong \mathrm{Hom}_{\mathcal{A}}(B, S(A))^*$$

which is functorial in A and B.

One use for Serre functors is to construct adjoints using the remark below.

Remark 2.1.18 ([38, Remark 1.31])**.** *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor between k-linear categories endowed with Serre functors $S_{\mathcal{A}}$ and $S_{\mathcal{B}}$ respectively. Also, assume that all* Hom *sets are finite dimensional. Then*

$$G \dashv F \Rightarrow F \dashv S_{\mathcal{A}} \circ G \circ S_{\mathcal{B}}^{-1}.$$

A similar argument holds for the construction of a left adjoint given a right adjoint. Thus for functors between categories with Serre functors the existence of the left or right adjoint guarantees the existence of the other.

2.2 The Derived Category of a Scheme

We now focus on the abelian category of quasi-coherent and coherent sheaves on a scheme X. We follow [38, §3].

Definition 2.2.1. *Let X be a scheme. Its derived category $D(X)$ is the bounded derived category of the abelian category* $\mathrm{Coh}(X)$*, i.e.*

$$D(X) := D^b(\mathrm{Coh}(X)).$$

Definition 2.2.2. *Two schemes over a field k are called* derived equivalent *if there is a k-linear exact equivalence $D(X) \cong D(Y)$. We say that Y is a* Fourier-Mukai partner *of X if X and Y are derived equivalent.*

Proposition 2.2.3 ([38, Proposition 3.3]). *Suppose X is a noetherian scheme. Then any quasi-coherent sheaf F admits a resolution*

$$0 \to F \to \mathcal{I}^0 \to \mathcal{I}^1 \to \cdots$$

by quasi-coherent sheaves $\mathcal{I}^i$ which are injective as $\mathcal{O}_X$-modules, i.e. $\mathrm{QCoh}(X)$ *has enough injectives.*

Proposition 2.2.4 ([38, Proposition 3.5]). *Let X be a noetherian scheme. Then the natural functor*

$$D(X) \to D^b(\mathrm{QCoh}(X))$$

defines an equivalence between the bounded derived category $D(X)$ and the full triangulated subcategory $D^b_{coh}(\mathrm{QCoh}(X))$ of bounded complexes of quasi-coherent sheaves with coherent cohomology.

Remark 2.2.5. *Let X be a noetherian scheme of finite type over a field k. Then the derived categories $D^*(\mathrm{QCoh}(X))$ and $D^*(\mathrm{Coh}(X))$ are k-linear categories.*

When X is a smooth projective variety over a field, Serre Duality endows $D(X)$ with a Serre functor.

Theorem 2.2.6 ([38, Theorem 3.12]). *Let X be a smooth projective variety of dimension n over a field k. Define the exact functor*

$$S_X \colon D(X) \to D(X)$$

by $S_X(E^\bullet) = E^\bullet \otimes \omega_X[n]$. Then their exists functorial isomorphisms

$$\eta_{E,F} \colon \mathrm{Hom}_{D(X)}(E^\bullet, F^\bullet) \cong \mathrm{Hom}_{D(X)}(F^\bullet, S_X(E^\bullet))^* = \mathrm{Hom}_{D(X)}(F^\bullet, E^\bullet \otimes \omega_X[n])^*$$

where $\mathrm{Hom}_{D(X)}(F^\bullet, S_X(E^\bullet))^*$ is the dual vectorspace to $\mathrm{Hom}_{D(X)}(F^\bullet, S(E^\bullet))$, i.e. S_X is a Serre functor for $D(X)$.

The above theorem can be used to prove the following Proposition.

Proposition 2.2.7 ([38, Proposition 3.13]). *Suppose F and G are coherent sheaves on a smooth projective variety of dimension n. Then*

$$\mathrm{Ext}^i_X(F, G) = 0 \text{ for } i > n.$$

A consequence of the above Proposition is the following characterization of the derived category of a curve.

Corollary 2.2.8 ([38, Corollary 3.15]). *Let C be a smooth projective curve. Then any object $E^\bullet$ of $D(C)$ is isomorphic to a direct sum $\bigoplus_i E_i$ where E_i are coherent sheaves on C.*

2.2.1 Derived Functors and Schemes

We now derived the left and right exact functors between categories of quasi-coherent and coherent sheaves such as the direct image functor, Hom functor, tensor product functor $-\otimes-$ and pullback functor. Throughout we will assume that X is noetherian.

Direct Image

As $\mathrm{QCoh}(X)$ contains enough injectives, we can derive the direct image functor on the level of $\mathrm{QCoh}(X)$.

Let $f\colon X\to Y$ denote a quasi-compact quasi-separated morphism of schemes. Then the direct image functor maps quasi-coherent sheaves to quasi-coherent sheaves and

$$f_*\colon\ \mathrm{QCoh}(X)\to\mathrm{QCoh}(Y)$$

is left exact. As $\mathrm{QCoh}(X)$ contains enough injectives, there is a right derived functor

$$Rf_*\colon D^+(\mathrm{QCoh}(X))\to D^+(\mathrm{QCoh}(Y)).$$

Remark 2.2.9. *If X is a scheme over a field k, the global section functor $\Gamma\colon\ \mathrm{QCoh}(X)\to\mathrm{Vec}_k$ is a special case of the direct image under the structure morphism $f\colon X\to\operatorname{Spec}k$.*

Theorem 2.2.10. *For any quasi-coherent sheaf F on X, and a morphism $f\colon X\to Y$ of noetherian schemes, the classical higher direct image sheaves R^if_* are trivial for $i>\dim(X)$.*

Thus using Theorem 2.2.10 and Proposition 2.1.15, Rf_* induces an exact functor

$$Rf_*\colon D^b(\mathrm{QCoh}(X))\to D^b(\mathrm{QCoh}(Y)).$$

To descend to the coherent level we need the following Theorem

Theorem 2.2.11. *If $f\colon X\to Y$ is a proper morphism of noetherian schemes, then the higher direct images $R^if_*(F)$ of a coherent sheaf F are again coherent.*

Thus for any proper morphism between noetherian schemes, we obtain a right derived functor

$$Rf_*\colon D(X)\to D(Y).$$

The Hom Functor

Let $F\in\mathrm{QCoh}(X)$. Then

$$\mathcal{H}om_X(F,-)\colon\ \mathrm{QCoh}(X)\to\mathrm{QCoh}(X)$$

is a left exact functor. Note if $F \in \mathrm{Coh}(X)$ $\mathcal{H}om$ descends to

$$\mathcal{H}om_X(F, -)\colon \mathrm{Coh}(X) \to \mathrm{Coh}(X).$$

As X is noetherian, $\mathrm{QCoh}(X)$ contains enough injectives. Thus the derived functors

$$R\mathcal{H}om_X(F, -)\colon D^+(\mathrm{QCoh}(X)) \to D^+(\mathrm{QCoh}(X))$$

exists. We define

$$\mathcal{E}xt^i_X(F, E) = R^i\mathcal{H}om_X(F, E)$$

for any quasi-coherent sheaves E, F.

If F is coherent we have the following description of the stalk of $\mathcal{E}xt^i_X(F, E)$ at $x \in X$

$$\mathcal{E}xt^i_X(F, E)_x = \mathrm{Ext}^i_{\mathcal{O}_{X,x}}(F_x, E_x).$$

Note that $\mathcal{E}xt^i_X(F, E)$ is coherent if F and E are.

If additionally, we assume that X is regular, then $\mathcal{H}om$ descends to the level of the bounded derived category for $F \in \mathrm{Coh}(X)$

$$\mathcal{H}om_X(F, -)\colon D(X) \to D(X).$$

To prove this we use the following

Proposition 2.2.12. *If X is regular, then $F^\bullet \in D(X)$ is isomorphic to a bounded complex of locally free sheaves $G^\bullet \in D(X)$.*

Remark 2.2.13. *The above proposition can also be used to replace F by a complex of locally free sheaves and compute $R\mathcal{H}om(F^\bullet, -)$ using $\mathcal{H}om(G^\bullet, -)$.*

Tensor Product

As X is noetherian, any coherent sheaf F admits a resolution by locally free sheaves, i.e. there exists a surjection

$$F^0 \twoheadrightarrow F$$

with F^0 locally free. If E is an acyclic bounded complex with all E^i locally free, then $F \otimes E$ is still acyclic. Thus the class of locally free sheaves in $\mathrm{Coh}(X)$ is adapted to the right exact functor $F \otimes -$. Thus the left derived functor

$$F \otimes^L -\colon D^-(\mathrm{Coh}(X)) \to D^-(Coh(X))$$

exists (c.f. [38, pp.78–79]). By definition

$$\mathcal{T}or_i(F, E) := \mathcal{H}^{-i}(F \otimes^L E).$$

When X is regular, $F \otimes^L -$ restricts to

$$F \otimes^L -\colon D(X) \to D(X)$$

because any coherent sheaf E admits a locally free resolution of length n, so $\mathcal{T}or_i(F, E) = 0$ for $i > n$.

Pullback

Let $f\colon X \to Y$ be a morphism of schemes. Then the pullback functor

$$f^*\colon \mathcal{O}_Y - \mathrm{Mod} \to \mathcal{O}_X - \mathrm{Mod}$$

is the composite of the exact functor

$$f^{-1}\colon \mathcal{O}_Y - \mathrm{Mod} \to \mathcal{O}_{f^{-1}\mathcal{O}_Y} - \mathrm{Mod}$$

and the right exact functor

$$\mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_Y)} -\colon \mathcal{O}_{f^{-1}\mathcal{O}_Y} - \mathrm{Mod} \to \mathcal{O}_X - \mathrm{Mod}\,.$$

Then f^* is right exact and if $\mathcal{O}_X \otimes^L_{f^{-1}\mathcal{O}_Y} (-)$ if the left derived functor of $\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} (-)$ then

$$Lf^*\colon = \left(\mathcal{O}_X \otimes^L_{f^{-1}\mathcal{O}_Y} -\right) \circ f^{-1}\colon D^-(Y) \to D^-(X)$$

is the left derived functor of f^*.

Remark 2.2.14. *Note that the previous discussion deriving the tensor product functor does not strictly apply but can be adapted to this more general situation.*

Remark 2.2.15. *Often f will be flat, so f^* is exact and we will not need to derive f.*

Projection Formula

We will use the following compatibility relation frequently. Let $f\colon X \to Y$ be a proper morphism of projective schemes over a field k. For any $F^\bullet \in D(X)$ and $E^\bullet \in D(Y)$ there exists a natural isomorphism

$$Rf_*(F^\bullet) \otimes^L E^\bullet \xrightarrow{\sim} Rf_*(F^\bullet \otimes^L Lf^* E^\bullet).$$

This is a consequence of the classical projection formula $f_*(F) \otimes E \cong f_*(F \otimes f^* E)$ for a locally free sheaf E and arbitary sheaf F.

Grothendieck-Verdier Duality

Let $f\colon X \to Y$ be a morphism of smooth proper schemes over a field k of relative dimension $\dim(f) = \dim(X) - \dim(Y)$. Then the relative dualizing bundle is

$$\omega_f := \omega_X \otimes f^*\omega_Y.$$

Consider the functor

$$\begin{aligned} f^!\colon D(Y) &\to D(X) \\ E^{bullet} &\mapsto Lf^*(E^\bullet) \otimes \omega_f[\dim(f)] \end{aligned}$$

Then Grothendieck-Verdier duality states that $f^!$ is right adjoint to f_*

Theorem 2.2.16. *For any $F^\bullet \in D(X)$ and $E^\bullet \in D(Y)$ there exists a functorial isomorphism*

$$Rf_*R\mathcal{H}om_X(F^\bullet, f^!(E^\bullet)) \cong R\mathcal{H}om(Rf_*(F^\bullet), E^\bullet).$$

Moreover, $f^!$ is right adjoint to Rf_*. Thus we have

$$Lf^* \dashv Rf_* \dashv f^!.$$

2.2.2 Support of a Complex

Recall that the support of a coherent sheaf E on X is the closed subset

$$\mathrm{supp}(E) = \{x \in X | E_x \neq 0\}.$$

Definition 2.2.17. *The support of a complex $E^\bullet \in D(X)$ is the union of the support its cohomology sheaves. Explicitly, it is the closed subset*

$$\mathrm{supp}(E^\bullet) := \bigcup \mathrm{supp}(H^i(E^\bullet)).$$

Lemma 2.2.18 ([38, Lemma 3.9])**.** *Suppose $E^\bullet \in D(X)$ and $\mathrm{supp}(E^\bullet) = Z_1 \coprod Z_2$ where $Z_1, Z_2 \subset X$ are disjoint closed subsets. Then $E^\bullet \cong E_1^\bullet \oplus E_2^\bullet$ with $\mathrm{supp}(E_i^\bullet) \subset Z_i$ for $i = 1, 2$.*

A consequence of this lemma is the following result due to Bridgeland.

Proposition 2.2.19 ([38, Proposition 3.10])**.** *Let X be a noetherian scheme. Then $D(X)$ is an indecomposable triangulated category if and only if X is connected.*

We will frequently use the following

Proposition 2.2.20. *Let E and F be coherent sheaves on X such that $\mathrm{supp}(E) \cap \mathrm{supp}(F) = \emptyset$. Then*

$$\mathrm{Ext}^i_X(E, F) = 0 \text{ for all } i.$$

Proof. Consider the following spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^q(E,F)) \Rightarrow \mathrm{Ext}^{p+q}(E,F).$$

Then $\mathcal{E}xt^i(E,F)_x = \mathrm{Ext}^i_{\mathcal{O}_{X,x}}(E_x, F_x)$ is zero for all $x \in X$ as one of E_x or F_x is zero because E and F have disjoint support. Hence $E_2^{p,q} = 0$ for all p and q. Hence $\mathrm{Ext}^i(E,F) = 0$ for all i. $\square$

Notation 2.2.21. *From now on we will write $f_*, f^*, \otimes, \mathcal{H}om$ for the derived functors $Rf_*, Lf^*, \otimes^L, R\mathcal{H}om$ between derived categories.*

2.3 Autoequivalences and Fourier-Mukai Transforms

We now review the theory of Fourier-Mukai transforms and autoequivalences of the bounded derived category of a smooth projective variety X over a field k. In this section, we study the group $\mathrm{Aut}\, D(X)$ of k-linear exact autoequivalences of $D(X)$. We follow [38, §5]. All functors between derived categories will be derived appropriately.

First, we give some examples of autoequivalences of $D(X)$ which arise naturally

Example 2.3.1.

1. *Let $f\colon X \to X$ be an automorphism of X. Then $f_*\colon D(X) \to D(X)$ is an autoequivalence of $D(X)$ with inverse f^*.*
2. *Let $L \in \mathrm{Pic}(X)$ be a line bundle. Then the functor $- \otimes L\colon D(X) \to D(X)$ is an autoequivalence with inverse $- \otimes L^*$.*
3. *Let $n \in \mathbb{Z}$. The shift functor $[n]\colon D(X) \to D(X)$ is an autoequivalence of $D(X)$ with inverse $[-n]$.*

These autoequivalence form the subgroup of standard autoequivalences

$$\mathrm{Aut}_{stand}\, D(X) = \mathbb{Z} \times (\mathrm{Aut}(X) \ltimes \mathrm{Pic}(X))$$

of $\mathrm{Aut}\, D(X)$.

When the (anti)-canonical bundle of X is ample, the following result of Bondal and Orlov tells us there are no other autoequivalences.

Theorem 2.3.2 ([13, Theorem 2.5])**.** *Let X be a smooth projective variety with ample (anti-)canonical bundle. Then the group of autoequivalences is just the group* $\mathrm{Aut}_{stand}\, D(X)$ *of standard autoequivalences.*

We now recall the notion of a Fourier-Mukai transform (or integral transform) between derived categories following [38, §5]

Definition 2.3.3. *Let X and Y be smooth projective varieties and $\mathcal{P}^\bullet \in D(X \times Y)$. Denote the two projections by*

$$\begin{tikzcd} & X \times Y \arrow[dl, "q"'] \arrow[dr, "p"] & \\ X & & Y. \end{tikzcd}$$

The Fourier-Mukai transform with kernel $\mathcal{P}^\bullet$ is the functors

$$\Phi_{\mathcal{P}^\bullet} \colon D(X) \to D(Y)$$

defined by $\Phi_{\mathcal{P}^\bullet}(-) = p_*(q^*(-) \otimes \mathcal{P}^\bullet)$. Note that p_*, q^* and $\otimes$ denote the derived functors between derived categories. We have the usual pullback functor q^* because q is flat. Note that $q^*(-) \otimes \mathcal{P}^\bullet$ is the usual tensor product if $\mathcal{P}^\bullet$ is a complex of locally free sheaves. As p_*, q^* and $\otimes$ are all exact, so is $\Phi_{\mathcal{P}}$.

Example 2.3.4. *Let $f \colon X \to Y$ be a morphism. Then*

$$f_* = \Phi_{\mathcal{O}_{\Gamma_f}} \colon D(X) \to D(Y)$$

where $\Gamma_f \subset X \times Y$ is the graph of f. This is because the following string of equivalences

$$\begin{aligned} q\Phi_{\mathcal{O}_{\Gamma_f}}(E^\bullet) =& p_*(q^*(E^\bullet) \otimes \mathcal{O}_{\Gamma_f}) = p_*(q^*(E^\bullet \otimes (id, f)_*\mathcal{O}_X)) \\ \cong& p_* \circ (id, f)_*((id, f)^* q^*(E^\bullet) \otimes \mathcal{O}_X) \quad \textit{(Projection Formula)} \\ \cong& (p \circ (id, f))_*((q \circ (id, f))^*(E^\bullet)) \\ \cong& f_*(id^*(E^\bullet) = f_*(E^\bullet). \end{aligned}$$

using the commutativity of the diagram

$$\begin{tikzcd} & X \arrow[d, "{(id,f)}"] \arrow[ddl, bend right, "id"'] \arrow[ddr, bend left, "f"] & \\ & X \times Y \arrow[dl, "q"'] \arrow[dr, "p"] & \\ X & & Y \end{tikzcd}$$

and $\mathcal{O}_{\Gamma_f} = (id, f)_*(\mathcal{O}_X)$.

We have the following properties of Fourier-Mukai transforms.

Facts 2.3.5.

1. *Fourier-Mukai Transforms are exact because they are the composition of exact functors.*

2. *The composite of Fourier-Mukai transforms is a Fourier-Mukai transform [38, Proposition 5.10].*

3. *A Fourier-Mukai transform $\Phi_{\mathcal{P}^\bullet}$ admits left and right adjoints $\Phi_{\mathcal{P}_L}$ and $\Phi_{\mathcal{P}_R}$ respectively where*

$$\mathcal{P}_L^\bullet = (\mathcal{P}^\bullet)^* \otimes p^*\omega_Y[\dim(Y)], \quad \mathcal{P}_R^\bullet = (\mathcal{P}^\bullet)^* \otimes q^*\omega_X[\dim(X)].$$

The following theorem gives a criterion for when a functor between derived categories is a Fourier-Mukai transform whose proof we omit (see [38, Theorem 5.14] for more details).

Theorem 2.3.6 (Orlov). *Let X and Y be two smooth projective varieties and let*

$$F\colon D(X) \to D(Y)$$

be a fully faithful exact functor. If F admits left and right adjoints, then their exists an object $\mathcal{P}^\bullet \in D(X \times Y)$ unique up to isomorphism such that F is isomorphic to $\Phi_{\mathcal{P}^\bullet}$.

Remark 2.3.7. *Theorem 2.3.6 is usually applied to functors which are equivalences [38, Corollary 5.17].*

Remark 2.3.8. *Rizzardo and Van den Bergh [67] have shown that the result is false if we remove the fully faithfulness assumption.*

We can use Theorem 2.3.6 to give a criterion for when an autoequivalence is standard using the following.

Corollary 2.3.9 ([38, Corollary 5.23]). *Suppose $\Phi\colon D(X) \to D(Y)$ is an equivalence such that for any closed point $x \in X$ there exists a closed point $f(x) \in Y$ with*

$$\Phi(\mathcal{O}_x) \cong \mathcal{O}_{f(x)}.$$

Then $f\colon X \to Y$ defines an isomorphism and Φ is the composite of f_* with a twist by some line bundle $M \in \operatorname{Pic}(Y)$, i.e.

$$\Phi \cong f_*(M \otimes (-)).$$

Example 2.3.10. *Let $E = \mathbb{C}/\Gamma$ be an elliptic curve defined by a lattice $\Gamma \subset \mathbb{C}$. Denote by $\mathcal{P}$ the Poincaré line bundle on $E \times E$. Note that $\mathcal{P}$ is the universal family for the moduli functor parameterizing degree 0 line bundles on E. Then the Fourier-Mukai transform*

$$\Phi_{\mathcal{P}}\colon D(E) \to D(E)$$

with kernel $\mathcal{P}$ is an autoequivalence of $D(E)$.

Moreover, for any closed point $x \in E$, $\Phi_{\mathcal{P}}(\mathcal{O}_x)$ is the degree zero line bundle $\mathcal{O}_E([0]-x)$ where $[0]$ is the image of $0 \in \mathbb{C}$ is E. This shows that $\Phi_{\mathcal{P}}$ is not standard.

2.4 Semi-orthogonal Decompositions

The derived category of a projective variety is a complicated object and we might want to decompose the derived category into simpler pieces. As long as the variety is connected there are no direct sum decompositions of the derived category. So we search for weaker decompositions called semi-orthogonal decomposition. We follow [38, §1.4].

Definition 2.4.1. *Let $\mathcal{D}$ be a triangulated category. A* semi-orthogonal decomposition *of $\mathcal{D}$ is a pair of strictly full triangulated subcategories $\mathcal{A}, \mathcal{B}$ of $\mathcal{D}$ such that:*

1. *For any $B \in \mathcal{B}$ and $A \in \mathcal{A}$, $\operatorname{Hom}(B, A) = 0$.*
2. *The largest triangulated category generated by $\mathcal{A}$ and $\mathcal{B}$ by taking cones, shifts and direct sums is $\mathcal{D}$. I.e. for all $D \in \mathcal{D}$, there is a distinguished triangle*

$$D_{\mathcal{B}} \longrightarrow D \longrightarrow D_{\mathcal{A}} \longrightarrow D_{\mathcal{B}}[1]$$

with $D_{\mathcal{A}} \in \mathcal{A}$ and $D_{\mathcal{B}} \in \mathcal{B}$.

We call the distinguished triangle

$$D_{\mathcal{B}} \longrightarrow D \longrightarrow D_{\mathcal{A}} \longrightarrow D_{\mathcal{B}}[1]$$

the *decomposition triangle for D*. Moreover, this decomposition is functorial in D, i.e. the projections

$$D \to D_{\mathcal{A}}$$
$$D \to D_{\mathcal{B}}$$

are functors.

We can generalize this definition to a semi-orthogonal decomposition of more than two strictly full triangulated subcategories of $\mathcal{D}$ as follows.

Definition 2.4.2. *A semiorthogonal decomposition of $\mathcal{D}$ with n components is a collection $\mathcal{A}_1, \dots, \mathcal{A}_n$ of strictly full triangulated subcategories in $\mathcal{D}$ such that*

1. *For any $A_i \in \mathcal{A}_i$ and $A_j \in \mathcal{A}_j$, $\operatorname{Hom}(A_i, A_j) = 0$ for $i > j$.*
2. *For all $T \in \mathcal{D}$ we have a filtration*

$$0 = D_n \longrightarrow D_{n-1} \longrightarrow \cdots \longrightarrow D_1 \longrightarrow D_0 = D$$

such that $Cone(D_i \to D_{i-1}) \in \mathcal{A}_i$.

For $n = 2$ we can see this definition is equivalent to the previous one as we have a filtration

$$0 \longrightarrow D_{\mathcal{B}} \longrightarrow D$$

and we have that

$$C(0 \to D_{\mathcal{B}}) = D_{\mathcal{B}} \in \mathcal{B}$$

and

$$C(D_{\mathcal{B}} \to D) = D_{\mathcal{A}} \in \mathcal{A}$$

If we have a semi-orthogonal decomposition of $\mathcal{D}$ by $\mathcal{A}_1, \ldots, \mathcal{A}_n$ we write

$$\mathcal{D} = \langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle$$

Now assume that $\alpha \colon \mathcal{A} \to \mathcal{D}$ is a full embedding of a triangulated subcategory of $\mathcal{D}$.

Definition 2.4.3. *We call $\mathcal{A}$ a* right (resp. left) admissible *subcategory of $\mathcal{D}$ if there is a right (resp. left) adjoint $\alpha^! \colon \mathcal{D} \to \mathcal{A}$ (resp. $\alpha^* \colon \mathcal{D} \to \mathcal{A}$). We call a subcategory* admissible *if it is both right and left admissible.*

Right and left admissible subcategory are the foundation of constructing semi-orthogonal decompositions due to the following.

Proposition 2.4.4. *Suppose that $\mathcal{A}$ is a right (resp. left) admissible subcategory of $\mathcal{D}$. Then one has a semi-orthogonal decomposition*

$$\mathcal{D} = \langle \mathcal{A}^{\perp}, \mathcal{A} \rangle$$

(resp. $\mathcal{D} = \langle \mathcal{A}, {}^{\perp}\mathcal{A} \rangle$) where

$$\mathcal{A}^{\perp} = \{ D \in \mathcal{D} | \operatorname{Hom}(D, \mathcal{A}) = 0 \}$$

and

$${}^{\perp}\mathcal{A} = \{ D \in \mathcal{D} | \operatorname{Hom}(\mathcal{A}, D) = 0 \} .$$

This is proved using the following general argument. Suppose that $\alpha \colon \mathcal{A} \to \mathcal{D}$ is a right admissible subcategory of $\mathcal{D}$ and let $\alpha^! \colon \mathcal{D} \to \mathcal{A}$ denote the right adjoint to α. Then the required semi-orthogonal decomposition is given by

$$\mathcal{D} = \langle \ker(\alpha !), \operatorname{im}(\alpha) \rangle$$

where $\ker(\alpha^!) = \left\{ D \in \mathcal{D} | \alpha^!(D) = 0 \right\}$ and $\operatorname{im}(\alpha)$ is the essential image of α.

If $\mathcal{D}$ admits a Serre functor and the Hom-spaces of $\mathcal{D}$ are finite dimensional, then any left admissible subcategory is right admissible and vice versa by Remark 2.1.18.

2.4.1 Exceptional Collections

We now give the simplest collection of examples of semi-orthogonal decompositions.We follow [38, §1.4 and §8.3].

Definition 2.4.5. *An object E of a triangulated k-linear category $\mathcal{D}$ is called* exceptional *if*

$$\dim_k \operatorname{Hom}^i_{\mathcal{D}}(E,E) = \operatorname{Hom}_{\mathcal{D}}(E,E[i]) = \begin{cases} 1 & \text{if } i=0, \\ 0 & \text{otherwise} \end{cases}$$

Now assume that $\mathcal{D}$ has finite dimensional Hom sets over k and

$$\bigoplus_{i\in\mathbb{Z}} \dim_k \operatorname{Hom}_{\mathcal{D}}(A,B[i]) < \infty$$

for any pair $A,B \in D$. Denote for $A,B\in\mathcal{D}$

$$\operatorname{Hom}^{\bullet}_{\mathcal{D}}(A,B) = \bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}^i_{\mathcal{D}}(A,B) = \bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{D}}(A,B[i]).$$

which is a finite dimensional vectorspace over k.

Let Vec_{fd} denotes the abelian category of finite dimensional vector spaces over k and $E\in\mathcal{D}$. Consider the functor

$$\alpha_E \colon D(\mathrm{Vec}_{fd}) \to \mathcal{D}$$

given by $V^\bullet \mapsto V^\bullet \otimes_k E$. This admits a right adjoint

$$\alpha_E^! \colon \mathcal{D} \to D(\mathrm{Vec}_{fd})$$

given by $\alpha_E^!(D) = \operatorname{Hom}^\bullet(E,D)$. Then α_E is fully faithful (i.e. $\alpha_E^! \circ \alpha_E = id_{D(k)}$) if and only if E is exceptional. Thus when E is exceptional we get a semi-orthogonal decomposition

$$\mathcal{D} = \langle \ker(\alpha_E^!), \alpha_E(D(k))\rangle = \langle E^\perp, E\rangle.$$

Example 2.4.6. *Let X be a smooth projective variety over k and suppose that $h^{i,0}(X) = 0$ for $i>0$ (e.g. X Fano). Then any line bundle L on X is exceptional and we have a semi-orthogonal decomposition*

$$D(X) = \langle L^\perp, L\rangle.$$

Definition 2.4.7. *An* exceptional collection *is a collection of objects $E_1,\dots,E_n$ such that*

1. *Each E_i is exceptional for $i=1,\dots,n$.*
2. *For $i>j$, the vector space $\operatorname{Hom}^\bullet(E_i,E_j)=0$ (i.e. there are no maps from right to left).*

We call $E_1, \dots, E_n$ a full *exceptional collection if $E_1, \dots, E_n$ is an exceptional collection and they generate $\mathcal{D}$, i.e. $\mathcal{D} = \langle E_1 \dots, E_m \rangle$.*

Any exceptional collection gives rise to a semi-orthogonal decomposition of the derived category

$$\mathcal{D} = \langle E_1^\perp \cap E_2^\perp \cap \dots \cap E_n^\perp, E_1, E_2, \dots, E_n \rangle.$$

Note that $E_1, \dots, E_n$ is a full exceptional collection if and only if $E_1^\perp \cap E_2^\perp \cap \dots \cap E_n^\perp = 0$.

Example 2.4.8. *Suppose that X is a Fano variety of Picard rank 1. Then $-K_X = \mathcal{O}_X(r.H)$ for some generator H of the Picard group. Here r is the* Fano index *of X. Then*

$$\mathcal{O}_X, \mathcal{O}_X(H), \dots, \mathcal{O}_X((r-1)H)$$

is an exceptional collection because

$$\mathrm{Ext}^\bullet(\mathcal{O}_X(iH), \mathcal{O}_X(jH)) = H^p(X, \mathcal{O}_X((j-i)H) = 0 \quad (\textit{for } i > j).$$

So

$$D(X) = \langle \mathcal{A}, \mathcal{O}_X, \mathcal{O}_X(H), \dots, \mathcal{O}_X((r-1)H) \rangle.$$

Understanding the derived category in this way using semi-orthogonal decompositions often is reduced to understanding the orthogonal component $\mathcal{A}$.

Example 2.4.9. *Let $X = \mathbb{P}^n$. Then $-K_X = \mathcal{O}(n+1)$ and*

$$D^b(X) = \langle \mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n) \rangle$$

is a full exceptional collection due to Beilinson.

It is not too difficult to extend this to the relative setting

$$\pi \colon \mathbb{P}(\mathcal{N}) \to X$$

where $\mathcal{N}$ is a vector bundle on X and $\mathbb{P}(\mathcal{N})$ is the projectivization of $\mathcal{N}$.

Proposition 2.4.10 ([38, Corollary 8.36])**.** *Let $\mathcal{N}$ be a vector bundle of rank r. Then for any $a \in \mathbb{Z}$ the sequence if full subcategories*

$$\pi^* D(X) X \otimes \mathcal{O}(a), \dots, \pi^* D(X) \otimes \mathcal{O}(a + r - 1) \subset D(\mathbb{P}(\mathcal{N}))$$

gives a semi-orthogonal decomposition of $D(\mathbb{P}(\mathcal{N}))$.

2.4.2 Orlov's Blow Up Formula

We now discuss Orlov's famous blow-up formula for a smooth variety blown up in a smooth centre of codimension ≥ 2. The semi-orthogonal decomposition of the blow up contains terms corresponding to the blown up variety and several copies of the centre. We follow [38, §11.2]. The original paper is [59].

Proposition 2.4.11 ([38, Proposition 11.13])**.** *Suppose $f\colon S \to T$ is a projective morphism of smooth projective varieties such that $f_*\mathcal{O}_S \cong \mathcal{O}_T$ in $D(T)$. Then*

$$f^*\colon D(T) \to D(S)$$

is fully faithful. Thus f^* realizes $D(T)$ as an admissible subcategory of $D(S)$.

Proof. This follows from the adjunction of f^* and f_* and the projection formula to show that $id \cong f_* f^*$. The second statement follows from f^* admitting a right adjoint. □

Example 2.4.12. *Suppose $q\colon \widetilde{X} \to X$ is the blow up of $Y \subset X$ with X and Y smooth. As the fibres are projective spaces $q_*\mathcal{O}_{\widetilde{X}} \cong \mathcal{O}_X$. So using $q^*\colon D(X) \to D(\widetilde{X})$ we can view $D(X)$ as an admissible subcategory of $D(\widetilde{X})$.*

We now describe the orthogonal pieces to $q^*D(X)$ in $D(\widetilde{X})$. We consider the following setup: let X be a smooth projective variety over k and $Y \subset X$ a smooth projective subvariety of codimension $c \geq 2$ and $\widetilde{X}$ the blow up of X in Y. Denote by $j\colon E \to \widetilde{X}$ the inclusion of the exceptional divisor and $\pi\colon E \to Y$ the projection so we have the diagram

$$\begin{tikzcd} E \arrow[r, "j"] \arrow[d, "\pi"'] & \widetilde{X} \arrow[d, "q"] \\ Y \arrow[r, "\bar{j}"] & X. \end{tikzcd}$$

Proposition 2.4.13. *Suppose $Y \subset X$ has codimension $c \geq 2$. Then the functor*

$$\Phi_k\colon = j_*(\mathcal{O}_E(kE) \otimes \pi^*(-))\colon D(Y) \to D(\widetilde{X})$$

is fully faithful for any k. Moreover, Φ_k admits a right adjoint functor.

To prove Proposition 2.4.13 we will need the following results. First, we will use the following criteria for when a functor is fully faithful due to Bondal and Orlov.

Proposition 2.4.14 ([38, Proposition 7.1])**.** *Let $\Phi_{\mathcal{P}}\colon D(X) \to D(Y)$ be a Fourier-Mukai transform with kernel $\mathcal{P}$. Then $\Phi_{\mathcal{P}}$ is fully faithful if and only if for any two closed points $x, y \in X$ one has*

$$\mathrm{Hom}_{D(X)}(\Phi_{\mathcal{P}}(\mathcal{O}_x), \Phi_{\mathcal{P}}(\mathcal{O}_y)) = \begin{cases} k & \text{if } x = y \text{ and } i = 0 \\ 0 & \text{if } x \neq y \text{ or } i < 0 \text{ or } i > \dim(X). \end{cases}$$

We will also need the following description of self Ext groups of the push forward of the structure sheaf along an arbitrarily closed embedding.

Proposition 2.4.15 ([38, Proposition 11.8]). *Let $j\colon Y \hookrightarrow X$ be an arbitrarily closed embedding of smooth varieties. Then there exist isomorphisms*

$$\mathcal{H}^i(j^* j_* \mathcal{O}_Y) \cong \bigwedge^{-i} \mathcal{N}^*_{Y/X}$$
$$\mathcal{E}xt^i_X(j_*\mathcal{O}_Y, j_*\mathcal{O}_Y) \cong \bigwedge^{i} \mathcal{N}_{Y/X}$$

where $\mathcal{N}_{Y/X}$ is the normal bundle of Y in X.

Proof of Proposition 2.4.13. Note that Φ_k is a Fourier-Mukai transform with kernel $\mathcal{O}_E(kE)$ considered as an object of $D(Y \times \widetilde{X})$. We will use Proposition 2.4.14 to prove that Φ_k is fully faithful.

First, let us show that $\mathrm{Hom}_{D(Y)}(\Phi_k(\mathcal{O}_x), \Phi_k(\mathcal{O}_y)[i]) = 0$ for all i and $x \neq y$. If $x \neq y$, then $\Phi_k(\mathcal{O}_x) = j_*\mathcal{O}_{F_x}(-k)$ and $\Phi_k(\mathcal{O}_y) = j_*\mathcal{O}_{F_y}(-k)$ where F_x and F_y are the fibres of π over x and y respectively. This is because they have disjoint support so there are no non-trivial maps between them.

Suppose $x = y$. Then we show that

$$\mathrm{Ext}^i_{\widetilde{X}}(j_*\mathcal{O}_{F_x}(-k), j_*\mathcal{O}_{F_x}(-k)) \cong \mathrm{Ext}^i_{\widetilde{X}}(j_*\mathcal{O}_{F_x}, j_*\mathcal{O}_{F_x})$$

vanishes for i out side the interval $[0, d]$ (where $d = \dim X$) and has dimension 1 for $i = 0$. We do this using the spectral sequence

$$E_2^{p,q} = H^p(\widetilde{X}, \mathcal{E}xt^q_{\widetilde{X}}(j_*\mathcal{O}_{F_x}, j_*\mathcal{O}_{F_x})) \Rightarrow \mathrm{Ext}^{p+q}_{\widetilde{X}}(j_*\mathcal{O}_{F_x}, j_*\mathcal{O}_{F_x}).$$

By Proposition 2.4.15 we have

$$\bigwedge^{i} \mathcal{N}_{F_x/\widetilde{X}} \cong \mathcal{E}xt^i_{\widetilde{X}}(j_*\mathcal{O}_{F_x}, j_*\mathcal{O}_{F_x}).$$

so the spectral sequence becomes

$$E_2^{p,q} = H^p(\widetilde{X}, \bigwedge^{q} \mathcal{N}_{F_x/\widetilde{X}}) \Rightarrow \mathrm{Ext}^{p+q}_{\widetilde{X}}(j_*\mathcal{O}_{F_x}, j_*\mathcal{O}_{F_x}).$$

We need to understand $\mathcal{N}_{F_x/\widetilde{X}}$. Consider the short exact sequence

$$0 \longrightarrow \mathcal{N}_{F_x/E} \longrightarrow \mathcal{N}_{F_x/\widetilde{X}} \longrightarrow \mathcal{N}_{E/\widetilde{X}}|_{F_x} \longrightarrow 0.$$

As $\mathcal{N}_{E/\widetilde{X}} \cong \mathcal{O}_E(E) = \mathcal{O}_{\widetilde{X}}(E)|_E$ and $\mathcal{N}_{F_X/E} \cong \mathcal{O}_{F_x}^{\oplus d}$, we see that $\mathcal{N}_{F_x/\widetilde{X}}$ is an extension of $\mathcal{O}_{F_x}(-1)$ by $\mathcal{O}_{F_X}^{\oplus d}$. As F_x is isomorphic to a projective space, there are no non-trivial extensions of $\mathcal{O}_X(-1)$. Hence $N_{F_x/\widetilde{X}} \cong \mathcal{O}_{F_x}^{\oplus d} \oplus \mathcal{O}_{F_x}(-1)$.

So $E_2^{p,q} = 0$ for all pairs p, q with $p > 0$ or $p = 0$ and $q > d$. Therefore,

$$\mathrm{Ext}^q_{\widetilde{X}}(j_*\mathcal{O}_{F_x}, j_*\mathcal{O}_{F_x}) = E^{(0,q)} = 0$$

for $q > d$ and

$$\mathrm{Ext}^0_{\widetilde{X}}(j_*\mathcal{O}_{F_x}, j_*\mathcal{O}_{F_x}) = E_2^{0,0} \cong k.$$

Since the negative Ext groups vanish for the usual reasons, Φ_k satisfies the conditions of Proposition 2.4.14. □

We now introduce some notation to describe the semi-orthogonal decomposition of the derived category of the blow up $\widetilde{X}$. For $k = -c+1, \dots, -1$ denote the essential images

$$\mathcal{D}_k = \mathrm{im}(\Phi_{-k} \colon D(Y) \to D(\widetilde{X})).$$

The full subcategory $q^*D(X)$ will be denoted $\mathcal{D}_0$.

Theorem 2.4.16 (Orlov,[38, Proposition 11.18])**.** *There is a semi-orthogonal decomposition*

$$D(\widetilde{X}) = \langle \mathcal{D}_{-c+1}, \dots, \mathcal{D}_{-1}, \mathcal{D}_0 \rangle.$$

Proof. We show semi-orthogonality, then we prove fullness.

First, we show that

$$\mathcal{D}_l \subset \mathcal{D}_k^{\perp} \text{ for } -c+1 \leq l < k < 0.$$

Let $E^\bullet, F^\bullet \in D(Y)$, then the adjunction between $j^* \dashv j_*$ gives

$$\begin{aligned}\mathrm{Hom}_{D(\widetilde{X})}&(j_*(\pi^*F^\bullet \otimes \mathcal{O}_E(-kE)), j_*(\pi^*E^\bullet \otimes \mathcal{O}_E(-lE))) \\ &\cong \mathrm{Hom}_{D(E)}(j^*j_*\pi^*F^\bullet, \pi^*E^\bullet \otimes \mathcal{O}_E((k-l)E)).\end{aligned}$$

By taking the cone of the unit morphism we have a distinguished triangles

$$\pi^*F^\bullet \otimes \mathcal{O}_E(-E)[1] \longrightarrow j^*j_*\pi^*F^\bullet \longrightarrow \pi^*F^\bullet \longrightarrow \pi^*F^\bullet \otimes \mathcal{O}_E(-E)[2].$$

This reduces the claim to showing the following vanishing

$$\begin{aligned}\mathrm{Hom}_{D(E)}&(\pi^*F^\bullet, \pi^*E^\bullet \otimes \mathcal{O}_E((k-l)E)) = 0 \\ &= \mathrm{Hom}_{D(E)}(\pi^*F^\bullet \otimes \mathcal{O}_E(-E), \pi^*E^\bullet \otimes \mathcal{O}_E((k-l)E))\end{aligned}$$

for all $E^\bullet, F^\bullet \in D(Y)$. These both follow from the adjunction $\pi^* \dashv \pi_*$, the projection formula and $\pi_*(\mathcal{O}_E((k-l)E)) = 0$ for $-c+1 \leq l-k < 0$ as the fibres of π are all projective spaces.

Next, we show

$$\mathcal{D}_l \subset \mathcal{D}_0^{\perp} \text{ for } -c+1 \leq l < 0.$$

Again, we use $\pi_*(\mathcal{O}_E(-lE)) = 0$ for $-c+1 \le l < 0$ to deduce for all $E^\bullet \in D(X)$ and $F^\bullet \in D(Y)$ that

$$\begin{aligned}\mathrm{Hom}_{D(\widetilde{X}}(q^*E^\bullet, j_*(\pi^*F^\bullet \otimes \mathcal{O}_E(-lE))) &\cong \mathrm{Hom}_{D(X)}(E^\bullet, q_*j_*(\pi^*F^\bullet \otimes \mathcal{O}_E(-lE))) \\ &\cong \mathrm{Hom}_{D(X)}(E^\bullet, \bar{j}_*\pi_*(\pi^*F^\bullet \otimes \mathcal{O}_E(-lE))) \\ &= 0.\end{aligned}$$

Finally, we prove fullness. Assume that $E^\bullet \in \mathcal{D}_l^\perp$ for all $-c+1 \le l < 0$. Then we will show there exists $G^\bullet \in D(Y)$ with $j^*E^\bullet \otimes \mathcal{O}_E((1-c)E) \cong \pi^*G^\bullet$.

By our assumption on $E^\bullet$ we have

$$\mathrm{Hom}_{D(\widetilde{X})}(j_*(\pi^*F^\bullet \otimes \mathcal{O}_E(-lE)), E^\bullet) = 0$$

for all $-c+1 \le l < 0$ and all $F^\bullet \in D(Y)$. Grothendieck-Verdier Duality and $j^!E^\bullet \cong j^*E^\bullet \otimes \mathcal{O}_E(E)[-1]$ show that

$$\mathrm{Hom}_{D(E)}(\pi^*F^\bullet \otimes \mathcal{O}_E(-lE), j^*E^\bullet) = 0$$

for all $-c+2 \le l < 1$ and $F^\bullet \in D(Y)$. Then by the semi-orthogonal decomposition of the projectivization $\mathbb{P}(\mathcal{N})$ of a locally free sheaf $\mathcal{N}$ we have that the pullback $j^*E^\bullet$ is contained in $\pi^*D(Y) \otimes \mathcal{O}_E((1-c)E)$ which is the semi-orthogonal complement of $\langle \pi^*D(Y)(k)\rangle_{k=-c+2,\dots,0}$ in $D(E)$.

Suppose that $E_0^\bullet \in D(\widetilde{X})$ such that $j^*E_0^\bullet \cong \pi^*G^\bullet$ for some $G^\bullet \in D(Y)$. If $G^\bullet \cong 0$, then $E_0^\bullet$ has support outside the exceptional divisor E and $E^\bullet \in \mathcal{D}_0$. Suppose $G^\bullet \not\cong 0$. Then for some closed point $x \in Y$ and $m \in \mathbb{Z}$, $\mathrm{Hom}_{D(\widetilde{X})}(E_0^\bullet, q^*\mathcal{O}_x[m]) \neq 0$. To see this consider the spectral sequence

$$E_2^{r,s} = \mathrm{Hom}_{D(\widetilde{X})}(E_0^\bullet, H^s(q^*\mathcal{O}_x)[r]) \Rightarrow \mathrm{Hom}_{D(\widetilde{X})}(E_0^\bullet, q^*\mathcal{O}_x[r+s]).$$

By applying [38, Proposition 11.12] to $Z = x \subset Y$ we have $H^s(q^*\mathcal{O}_x) \cong \Omega^s_{F_x}(-s)$. This and our assumption $j^*E_0^\bullet \cong \pi^*G^\bullet$ gives

$$\begin{aligned}E_2^{r,s} = \mathrm{Hom}_{D(\widetilde{X})}(E_0^\bullet, j_*(\Omega^s_{F_x}(s))[r]) &\cong \mathrm{Hom}_{D(E)}(j^*E_0^\bullet, \Omega^s_{F_x}([r])) \\ \cong \mathrm{Hom}_{D(E)}(\pi^*G^\bullet, \Omega^s_{F_x}(s)[r]) &\cong \mathrm{Hom}_{D(Y)}(G^\bullet, \pi_*\Omega^s_{F_x}(s)[r]) = 0\end{aligned}$$

except for $s = 0$. Hence

$$\mathrm{Hom}_{D(\widetilde{X})}(E_0^\bullet, q^*\mathcal{O}_x[m]) = E_2^{m,0} = \mathrm{Hom}_{D(Y)}(G^\bullet, \mathcal{O}_x[m]) \neq 0$$

for some $m \in \mathbb{Z}$ and $x \in Y$ as the closed points of Y span the derived category $D(Y)$.

By applying this to the complexes $E^\bullet$ and $E_0^\bullet \cong E^\bullet \otimes \mathcal{O}_{\widetilde{X}}(-(c-1)E)$ we get

$$\begin{aligned}
0 &\neq \mathrm{Hom}_{D(\widetilde{X})}(E^\bullet \otimes \mathcal{O}_{\widetilde{X}}(-(c-1)E), q^*\mathcal{O}_x[m]) \\
&\cong \mathrm{Hom}_{D(\widetilde{X})}(q^*\mathcal{O}_x, E^\bullet \otimes \mathcal{O}_{\widetilde{X}}(-(c-1)E) \otimes \omega_{\widetilde{X}}[\dim(X) - m])^* \\
&\cong \mathrm{Hom}_{D(\widetilde{X})}(q^*\mathcal{O}_x, E^\bullet[\dim(X) - m])^*.
\end{aligned}$$

Thus if $E^\bullet \in \mathcal{D}_l^\perp$ for all $-c+1 \leq l < 0$ we cannot have $E^\bullet \in \mathcal{D}_0^\perp$. So $\mathcal{D}_{-c+1}, \ldots, \mathcal{D}_{-1}, \mathcal{D}_0$ generate $D(\widetilde{X})$. □

Chapter 3

Derived Autoequivalences of Bielliptic Surfaces

In this chapter, we describe the group of autoequivalences of the bounded derived category of a bielliptic surface over the complex numbers. First we review some background on bielliptic surfaces in section 3.1, the numerical Grothendieck group of these surfaces in section 3.2 and their canonical cover in section 3.3. In section 3.4 we review some background on moduli space of sheaves.

In section 3.5 we review the construction of relative Fourier-Mukai transforms along an elliptic fibration and prove Theorem 1.2.3. In section 3.6 we sketch an argument to fix a gap in the proof of Theorem 3.6.1 concerning Fourier-Mukai partners of bielliptic surfaces. In section 3.7 we construct some non-standard autoequivalences for bielliptic surfaces using moduli spaces of sheaves. Finally, in section 3.8 we prove Theorem 1.2.2. Throughout this chapter, all varieties will be over the complex numbers.

3.1 Bielliptic Surfaces

Bielliptic surfaces are minimal surfaces which are to Abelian surfaces what Enriques surfaces are to $K3$ surfaces. Precisely, we define a bielliptic surface in the following way:

Definition 3.1.1. *A* bielliptic *(or* hyperelliptic*) surface S is a minimal projective surface of Kodaira dimension zero with $q = 1$ and $p_g = 0$.*

Bielliptic surfaces are constructed by taking the quotient of the product of two elliptic curves $A \times B$ by a finite subgroup G of A acting on A by translations and on B via automorphisms, which are not all translations. These surfaces are classified by Bagnera and De Franchis into seven families [3, §V.5] determined by the group G, the lattice Γ such that $B = \mathbb{C}/\Gamma$, and the action of G on B (see Table 3.1).

Type	Γ	G	Action of G on B
A1	Arbitrary	$\mathbb{Z}_2$	$b \mapsto -b$
A2	Arbitrary	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$b \mapsto -b$,
			$b \mapsto b + \beta$, where $2\beta = 0$
B1	$\mathbb{Z} \oplus \mathbb{Z}\omega$	$\mathbb{Z}_3$	$b \mapsto \omega b$
B2	$\mathbb{Z} \oplus \mathbb{Z}\omega$	$\mathbb{Z}_3 \oplus \mathbb{Z}_3$	$b \mapsto \omega b$,
			$b \mapsto b + \beta$, where $\omega\beta = \beta$
C1	$\mathbb{Z} \oplus \mathbb{Z}i$	$\mathbb{Z}_4$	$b \mapsto ib$
C2	$\mathbb{Z} \oplus \mathbb{Z}i$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2$	$b \mapsto ib$,
			$b \mapsto b + \beta$, where $i\beta = \beta$
D	$\mathbb{Z} \oplus \mathbb{Z}\omega$	$\mathbb{Z}_6$	$b \mapsto -\omega b$

Table 3.1: ($\omega^3 = 1$ and $i^4 = 1$ are complex roots of unity.)

Definition 3.1.2. *We call a bielliptic surface* cyclic *if it is of type $A1$,$B1$,$C1$, or D and* non-cyclic *otherwise (see Table 3.1).*

Remark 3.1.3. *By construction bielliptic surfaces have torsion canonical bundle of order $2, 3, 4$ and 6 for bielliptic surfaces of type A, B, C and D respectively.*

Remark 3.1.4. *Associated with a bielliptic surface S are two elliptic fibrations:*

$$p_A \colon S \to A/G$$
$$p_B \colon S \to B/G$$

with A/G an elliptic curve and $B/G \cong \mathbb{P}^1$.

The projection $A \to A/G$ is étale, so all the fibres of p_A are smooth. The fibre of p_B over a point $P \in B/G$ is a multiple of a smooth elliptic curve. The multiplicity of the fibre of p_B at P is the same as the multiplicity of the projection $B \to B/G \cong \mathbb{P}^1$. As all smooth fibres of p_A (respectively p_B) are isomorphic to B (respectively A) we will denote the class of the smooth fibre of p_A and p_B in $H^2(S, \mathbb{Q})$ by B and A respectively.

3.2 The Numerical Grothendieck Group

We will study the group of autoequivalences by studying how it acts on the numerical Grothendieck group of the surface.

The *Grothendieck group* $K(X)$ of a smooth projective variety X is the free abelian group generated by isomorphism classes of objects in $D(X)$ modulo an equivalence relation given by distinguished triangles [38, §5]. There is a natural bilinear form on this group, the Euler form, given by

$$\chi([E], [F]) = \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{C}} \operatorname{Hom}^i_{D(X)}(E, F).$$

Consider the left radical of the Euler form

$$^{\perp}\chi = \{v \in K(X) | \chi(v,w) = 0 \text{ for all } w \in K(X)\}.$$

Serre duality implies that $\chi(v,w) = 0$ for all w if and only if $\chi(w,v) = 0$ for all w. Thus when we take the quotient $N(X) = K(X)/{}^{\perp}\chi$, the Euler form descends to a non-degenerate bilinear form on $N(X)$. We call $N(X)$ the *numerical Grothendieck group* of X. Recall that $\mathrm{Num}(X)$ is the (free abelian) group of divisors on X modulo numerical equivalence $\equiv$.

Proposition 3.2.1. *Let S be a bielliptic surface. Then the Chern character*

$$\mathrm{ch}\colon K(S) \to H^{2*}(S,\mathbb{Q})$$

induces an isomorphism between $N(S)$ and the group

$$H^0(S,\mathbb{Z}) \oplus \mathrm{Num}(S) \oplus H^4(S,\mathbb{Z}) \cong \mathbb{Z} \oplus \mathrm{Num}(S) \oplus \mathbb{Z}.$$

Under this identification, for $\mathrm{ch}(E) = (r, D, s)$ and $\mathrm{ch}(F) = (r', D', s')$ the Euler form becomes $\chi(E,F) = rs' + r's - D \cdot D'$.

Proof. For $v = (v_0, v_2, v_4) \in H^{2*}(S,\mathbb{Q})$ define $v^{\vee} = (v_0, -v_2, v_4) \in H^{2*}(S,\mathbb{Q})$. Recall that the Mukai pairing on $H^{2*}(S,\mathbb{Q})$ is defined by

$$\langle v, v' \rangle = \int_X v^{\vee} \cdot v'$$

where the product in the integral is the cup product of cohomology classes. The Todd classes $td(X)$ of abelian and bielliptic surfaces X are $(1,0,0)$ because $\chi(\mathcal{O}_X) = 0$ and K_X is trivial in cohomology. Then by Hirzebruch-Riemann-Roch for $[E],[F] \in K(S)$

$$\chi([E],[F]) = \langle \mathrm{ch}(E), \mathrm{ch}(F) \rangle.$$

Thus the Euler form for $\mathrm{ch}(E) = (r, D, s)$ and $\mathrm{ch}(F) = (r', D', s')$ can be written as

$$\chi([E],[F]) = \langle (r,D,s), (r',D',s') \rangle = rs' + r's - D \cdot D'.$$

A class lies in the radical of the Euler form if and only if it lies in the radical of the Mukai pairing. As the Mukai pairing is non-degenerate an element of $K(S)$ lies in the radical of the Euler form if and only if it has zero Chern Character. Hence $\ker(\mathrm{ch}) =^{\perp} \chi$ and $\mathrm{im}(\mathrm{ch}) \cong N(S)$.

Using this alternative description of the Euler form, we see that the class of a numerically trivial divisor D, $[\mathcal{O}_S(D)]$ is equivalent to $[\mathcal{O}_S]$. Therefore, the image of the Chern character restricted to the group $H^2(S,\mathbb{Q})$ is the group $\mathrm{Num}(S)$. Furthermore, by Hirzebruch-Riemann-Roch we have $\mathrm{ch}_2(E) = \chi(E) \in \mathbb{Z}$ for all E. Thus we have an isomorphism

$$N(S) \cong H^0(X,\mathbb{Z}) \oplus \mathrm{Num}(S) \oplus H^4(X,\mathbb{Z}) \cong \mathbb{Z} \oplus \mathrm{Num}(S) \oplus \mathbb{Z}\,.$$

□

Remark 3.2.2. *These isomorphisms generalize to other surfaces using the Mukai vector and Mukai lattice. The Mukai vector of a sheaf E on X is defined by*

$$v(E) = \mathrm{ch}(E)\sqrt{td(X)}$$

where $td(X)$ is the Todd class of the surface. For bielliptic and abelian surfaces we have $td(X) = \sqrt{td(X)} = 1$, so the Mukai vector coincides with the Chern character.

Remark 3.2.3. *We will study the group $\mathrm{Aut}\, D(S)$ by studying its action on the numerical Grothendieck group given by the homomorphism*

$$\rho\colon\ \mathrm{Aut}\, D(S) \to \mathrm{Aut}(N(S))$$

defined by $\rho(\Phi)([E]) = [\Phi(E)]$. Autoequivalences of $D(S)$ preserve the Hom^i_S groups, thus the Euler form. Hence the image of ρ is contained in the group of isometries $O(N(S))$ of $N(S)$.

3.3 Canonical Covers of Bielliptic Surfaces

To any bielliptic surface S we can associate an étale cover $\widetilde{S}$ which has trivial canonical bundle. This cover is called the *canonical cover* of S.

Proposition 3.3.1 ([17, §2], [38, §7.3],[4, §7.2])**.** *Let X be a smooth projective variety whose canonical bundle ω_X has finite order, i.e. there exists n such that $\omega_X^{\otimes n} \cong \mathcal{O}_X$. Then there exists a smooth projective variety $\widetilde{X}$ with trivial canonical bundle, and an étale cover $\pi\colon \widetilde{X} \to X$ of degree n such that*

$$\pi_*(\mathcal{O}_{\widetilde{X}}) \cong \bigoplus_{i=0}^{n-1} \omega_X^{\otimes i}.$$

Furthermore, $\widetilde{X}$ is uniquely defined up to isomorphism, and there is a free action of the cyclic group $\widetilde{G} = \mathbb{Z}_n$ on $\widetilde{X}$ such that $\pi\colon \widetilde{X} \to X = \widetilde{X}/\widetilde{G}$ is the quotient morphism.

The canonical cover of a bielliptic surface will play an important role in determining the group of autoequivalences. We list the following facts about the canonical cover of a bielliptic surface.

Proposition 3.3.2. *Let S be a bielliptic surface which is realized as a quotient of $A \times B$ be a finite group G of order l as in Table 3.1. Then there exists an abelian surface $\widetilde{S}$ which is the canonical cover of S. Moreover,*

- *If S is cyclic, then $\widetilde{S} \cong A \times B$.*
- *If S is non-cyclic, then $\widetilde{S}$ is a quotient of $A \times B$ by a cyclic subgroup $H \subset G$ of order k acting on $A \times B$ purely by translations. We have $G \cong \mathbb{Z}_n \oplus \mathbb{Z}_k$.*

Remark 3.3.3. *The canonical cover $\widetilde{S}$ has two fibrations*

$$\widetilde{p}_A \colon \widetilde{S} \to A/H$$
$$\widetilde{p}_B \colon \widetilde{S} \to B/H.$$

Both $\widetilde{p}_A$ and $\widetilde{p}_B$ are smooth fibrations with fibres isomorphic to B and A respectively. We will denote the class of these fibres by $\widetilde{B}$ and $\widetilde{A}$ in $\mathrm{Num}(\widetilde{S})$ respectively. The degree of the intersection $\widetilde{B} \cdot \widetilde{A} = k = |H|$.

Serrano [68, §1] described the structure of $\mathrm{Num}(S)$ in the following way.

Lemma 3.3.4. *Let S be a bielliptic surface constructed as a quotient of $A \times B$ by a finite abelian group G where A and B are elliptic curves.*

Recall that S admits a canonical cover $\pi \colon \widetilde{S} \to S$ where $\widetilde{S}$ is an abelian surface. The canonical cover $\widetilde{S}$ is constructed as a quotient of $A \times B$ by a cyclic group of order $1, 2$ or 3 with quotient map $\widetilde{\pi} \colon (A \times B) \to \widetilde{S} = (A \times B)/H$. Denote $\deg \pi = n$ and $\deg \widetilde{\pi} = k$.

Recall that S has two elliptic fibrations and $p_A \colon A/G$ and $p_B \colon S \to B/G$ whose smooth fibres are isomorphic to B and A respectively. We will write B and A to denote the classes of these fibres in $H^2(S, \mathbb{Q})$.

The pairing on $H^2(S, \mathbb{Q})$ is the intersection pairing.

Then:

1. *The second rational cohomology group $H^2(S, \mathbb{Q})$ is generated by A and B.*
2. *The second integral cohomology group $H^2(S, \mathbb{Z})$ is generated by $\frac{1}{n}A$ and $\frac{1}{k}B$.*

3.3.1 Canonical Covers and the Derived Category

Consider the category $\mathrm{Sp}\text{-}\mathrm{Coh}(S)$ of coherent $\pi_*(\mathcal{O}_S)$-modules on S. A sheaf E lies in the essential image of the forgetful map $\mathrm{Sp}\text{-}\mathrm{Coh}(S) \to \mathrm{Coh}(S)$ if and only if $E \otimes \omega_S \cong E$. We call such sheaves *special.*

Denote by $\mathrm{Coh}^{\widetilde{G}}(\widetilde{S})$ the category of $\widetilde{G}$-equivariant sheaves on $\widetilde{S}$. An object of $\mathrm{Coh}^{\widetilde{G}}(\widetilde{S})$ is a pair $(E, \{\lambda_{\tilde{g}}\}_{\tilde{g} \in \widetilde{G}})$ which satisfies some axioms (see [16] for more details - later we will see that $\mathrm{Coh}^{\widetilde{G}}(\widetilde{S}) \cong \mathrm{Coh}([\widetilde{S}/\widetilde{G}]$ where $[\widetilde{S}/\widetilde{G}]$ is the quotient stack). As $\widetilde{G}$ is cyclic, an object of $\mathrm{Coh}^{\widetilde{G}}(\widetilde{S})$ is given by a pair $(E, \lambda_{\tilde{g}})$ where $\lambda_{\tilde{g}} \colon E \cong \tilde{g}^* E$ where $\tilde{g}$ is generator of $\widetilde{G}$.

The following results relate these categories to the category of coherent sheaves on $\widetilde{S}$ and S respectively.

Lemma 3.3.5 ([17, Lemma 2.4]). *The functors*

$$\begin{aligned} \pi_*\colon \ \mathrm{Coh}(\widetilde{S}) &\to \text{Sp-Coh}(S) \\ \pi^*\colon \ \mathrm{Coh}(S) &\to \mathrm{Coh}^{\widetilde{G}}(\widetilde{S}) \end{aligned}$$

are equivalences.

On the level of derived categories, we have

Proposition 3.3.6 ([17, Proposition 2.5]). *Let E be an object of $D(S)$. Then there is an object $\widetilde{E}$ of $D(\widetilde{S})$ such that $\mathbf{R}\pi_*(\widetilde{E}) \cong E$ if and only if $E \otimes \omega_S \cong E$.*

Remark 3.3.7. *Recall $\pi_!\colon N(\widetilde{S}) \to N(S)$ is defined by ([38, §5.2])*

$$\pi_![E] = \sum_{i\in\mathbb{Z}} (-1)^i [R^i\pi_*(E)].$$

After taking Chern characters, $\pi_!$ coincides with the pushforward π_* on cohomology by Grothendieck-Riemann-Roch. This is due to the Todd classes of $\widetilde{S}$ and S being $(1, 0, 0)$.

First note that the composite $\pi_!\colon K(\widetilde{S}) \to K(S) \to N(S)$ descends to a map $\pi_!\colon N(\widetilde{S}) \to N(S)$ because for $v \in {}^\perp\chi$, $\pi_!(v) = 0$ because for any $w \in N(S)$

$$\chi(\pi_!(v), w) = \chi(v, \pi^* w) = 0$$

by adjunction. As χ is non-degenerate on $N(S)$, $\pi_!(v) = 0$.

On the level of the numerical Grothendieck group $N(S)$ we are interested in the subgroup Δ of *special classes* defined by

$$\Delta = \mathrm{im}(\pi_!) = \left\{ [E] \in N(S) \middle| [E] = \pi_!([\widetilde{E}]) \text{ for some } [\widetilde{E}] \in N(\widetilde{S}) \right\}.$$

Remark 3.3.8. *Note that the class $[E]$ of a special object $E \in D(S)$ lies in Δ by Proposition 3.3.6 as there exists $\widetilde{E} \in D(\widetilde{S})$ such that $[E] = [\pi_*(\widetilde{E})] = \pi_![\widetilde{E}]$.*

The subgroup Δ is important because the image of $\operatorname{Aut} D(S)$ under ρ preserves Δ. We recall the following results on functors between derived categories of smooth projective varieties with torsion canonical bundles and functors between the derived categories of the canonical cover.

Definition 3.3.9 ([38, Definition 7.15][17, Definition 4.2]). *Suppose X and Y are smooth projective varieties whose canonical bundles are torsion of order n and $\widetilde{X}$ and $\widetilde{Y}$ are their canonical covers respectively. Then a lift of a functor $\Phi\colon D(X) \to D(Y)$ is a functor $\widetilde{\Phi}\colon D(\widetilde{X}) \to D(\widetilde{Y})$ such that the following diagram commutes:*

$$\begin{tikzcd} D(\widetilde{X}) \arrow[r, "\widetilde{\Phi}"] \arrow[d, "\pi_{X,*}"'] & D(\widetilde{Y}) \arrow[d, "\pi_{Y,*}"] \\ D(X) \arrow[r, "\Phi"] & D(Y), \end{tikzcd} \qquad \begin{tikzcd} D(\widetilde{X}) \arrow[r, "\widetilde{\Phi}"] & D(\widetilde{Y}) \\ D(X) \arrow[r, "\Phi"] \arrow[u, "\pi_X^*"] & D(Y). \arrow[u, "\pi_Y^*"'] \end{tikzcd}$$

Theorem 3.3.10 ([38, Proposition 7.18] [17, Theorem 4.5])**.** *Suppose X and Y are smooth projective varieties whose canonical bundles are torsion of order n with canonical covers $\widetilde{X}$ and $\widetilde{Y}$ respectively. Then for any equivalence $\Phi\colon D(X) \to D(Y)$ there is a lift $\widetilde{\Phi}\colon D(\widetilde{X}) \to D(\widetilde{Y})$. Moreover, $\widetilde{\Phi}$ is an equivalence of categories and equivariant in the following way: there is an automorphism τ of G such that*

$$g_* \circ \widetilde{\Phi} = \widetilde{\Phi} \circ \tau(g)_*$$

for every $g \in G$.

Proposition 3.3.11. *Let $\Phi \in \operatorname{Aut} D(S)$. Then $\rho(\Phi)$ preserves Δ.*

Proof. Any autoequivalence $\Phi \in \operatorname{Aut} D(S)$ lifts to an equivariant autoequivalences $\widetilde{\Phi} \in \operatorname{Aut} D(\widetilde{S})$ by Theorem 3.3.10 such that

$$R\pi_* \circ \widetilde{\Phi} \cong \Phi \circ R\pi_*.$$

Consider $v \in \Delta$ and $\omega \in N(\widetilde{S})$ such that $v = \pi_!(w)$. Then

$$\rho(\Phi)(v) = \rho(\Phi)(\pi_!(w)) = \pi_!(\rho(\widetilde{\Phi})(w)) \in \Delta.$$

Therefore $\rho(\Phi)(\Delta) \subset \Delta$. $\square$

3.3.2 Autoequivalences which act trivially on $N(S)$

We now show for any bielliptic surface S that any autoequivalence Φ of the derived category $D(S)$ is a sheaf transform, i.e. $\Phi(E)$ is a shift of a sheaf for any sheaf E.

First, recall that any autoequivalence of belian surfaces is a sheaf transform.

Lemma 3.3.12 ([18, Corollary 2.10])**.** *Let $\widetilde{S}$ be an abelian surface and Y any surface. Then any equivalence $\widetilde{\Phi}\colon D(Y) \to D(\widetilde{S})$ is a sheaf transform.*

Lemma 3.3.13. *Let S be a bielliptic surface, Y any surface and $\Phi\colon D(Y) \to D(S)$ an equivalence. Then Φ is a sheaf transform.*

Proof. We proceed by contradiction. Let $E \in D(S)$ be a sheaf such that $\Phi(E)$ is not a shift of a sheaf. As Y is derived equivalent S, it admits a canonical cover $\widetilde{Y}$ which is derived equivalent to $\widetilde{S}$.

Consider the commutative diagram

$$\begin{array}{ccc} D(\widetilde{Y}) & \xrightarrow{\widetilde{\Phi}} & D(\widetilde{S}) \\ {\scriptstyle \pi^*}\uparrow & & {\scriptstyle \pi^*}\uparrow \\ D(S) & \xrightarrow{\Phi} & D(S) \end{array}$$

where $\widetilde{S}$ is the canonical cover of S and $\widetilde{\Phi}$ is a lift of Φ.

One way around the diagram gives $\widetilde{\Phi}(\pi_S^*(E))$, which is a sheaf by Lemma 3.3.12 because π is flat. The other way gives $\pi^*(\Phi(E))$, which is a complex. This is a contradiction.

Hence Φ is a sheaf transform. □

A corollary of this is a description of those autoequivalences in the kernel of ρ.

Corollary 3.3.14. *Let* $\rho\colon \operatorname{Aut} D(S) \to N(S)$ *be the natural representation of* $\operatorname{Aut} D(S)$ *given by* $\rho(\Phi)([E]) = [\Phi(E)]$. *Then*

$$\ker \rho = (\operatorname{Aut} S \ltimes \operatorname{Pic}^0 S) \times \mathbb{Z}[2].$$

Proof. Let Φ be a autoequivalences that act trivially on $N(S)$. Then $ch(\Phi(\mathcal{O}_s)) = (0,0,1)$. By Lemma 3.3.13, $\Phi(\mathcal{O}_s)$ is an even shift of a sheaf. Thus $\Phi(\mathcal{O}_s)[-2k] \cong \mathcal{O}_{s'}$ for some $s' \in S$ and $k \in \mathbb{Z}$. By Corollary 2.3.9 $\Phi = f_*(L \otimes -)[2k]$ where $k \in \mathbb{Z}$, L is a line bundle, and $f\colon S \to S$ is an automorphism.

As Φ acts trivially on $N(S)$, n is even. Tensoring by a line bundle L act trivially on $N(S)$ if and only if L has degree zero. Thus $L \in \operatorname{Pic}^0(S)$.

As automorphisms of S preserve effective divisors, they cannot exchange the fibres of the two different elliptic fibrations. This is because one has multiple fibres and the other does not. Hence f can be any automorphism of S. □

3.3.3 Structure of Δ

To describe the group of autoequivalences which preserve Δ we need the following results which describe the structure of Δ.

Lemma 3.3.15. *A class* $(r, D, s) \in \Delta$ *if and only if* $n \mid r$ *and* $(0, D, 0) \in \Delta$. *Thus*

$$\Delta = n\,\mathbb{Z} \oplus \pi_*(\operatorname{Num}(\widetilde{S})) \oplus \mathbb{Z} \subset \mathbb{Z} \oplus \operatorname{Num}(S) \oplus \mathbb{Z} \cong N(S).$$

Proof. Suppose $n \mid r$ and $(0, D, 0) \in \Delta$. Then $r = \tilde{r}n$ and there exists $\widetilde{D} \in \operatorname{Num}(\widetilde{S})$ such that $\pi_!(0, \widetilde{D}, 0) = (0, \pi_*(\widetilde{D}), 0) = (0, D, 0)$. Then

$$\pi_!(\tilde{r}, \widetilde{D}, s) = \pi_!(\tilde{r}, 0, 0) + \pi_!(0, \widetilde{D}, 0) + \pi_!(0, 0, s) = (r, D, s)$$

as $\pi_!(0, 0, 1) = (0, 0, 1)$.

Suppose that $(r, D, s) \in \Delta$. Then there exists $[E] \in N(\widetilde{S})$ such that $\pi_!([\widetilde{E}]) = (r, D, s)$. Note that $\pi^*(0, 0, 1) = (0, 0, n)$ as π is étale of degree n and $\pi^*(1, 0, 0) = (1, 0, 0)$. Also, $\pi_*(1, 0, 0) = (n, 0, 0)$ by construction of the canonical cover and $\pi_*(0, 0, 1) = (0, 0, 1)$ as π is étale.

Using the adjunction between π_* and π^* and by computing the Mukai pairing of (r, D, s) with the classes $(1, 0, 0)$ and $(0, 0, 1)$ we see that $\operatorname{ch}_2([E]) = s$ and $r = n\operatorname{rk}(\widetilde{E})$. So

$(r, 0, 0), (0, 0, s) \in \Delta$ as $\pi_!(\mathrm{rk}(E)[\mathcal{O}_{\widetilde{S}}]) = (r, 0, 0)$ and $\pi_!(s[\mathcal{O}_{\tilde{s}}]) = (0, 0, s)$. Then

$$(r, D, s) - (0, 0, s) - (r, 0, 0) = (0, D, 0) \in \Delta.$$

□

We now describe some elements of $\Delta \cap \mathrm{Num}(S)$. We will write $D \in \Delta$ for $D \in \mathrm{Num}(S)$ if $(0, D, 0) \in \Delta$. Recall that $\mathrm{Num}(S)$ is generated by $\frac{1}{n}A$ and $\frac{1}{k}B$.

Lemma 3.3.16.

1. *The classes* $A, B \in \Delta$.

2. *The classes* $\frac{m}{k}A$ *never lies in* Δ *for* $m \not\equiv 0 \pmod k$.

3. *If* S *is non-cyclic, then* $\frac{m}{k}B$ *never lies in* Δ *for* $m \not\equiv 0 \pmod k$.

Proof. 1. The classes $A, B \in \Delta$ as $\pi_*(\tilde{A}) = A$ and $\pi_*(\tilde{B}) = B$.

2. To show that $\frac{m}{n}A \notin \Delta$ for $m \not\equiv \pmod k$ it is enough to show that $\frac{1}{n}A \notin \Delta$. We proceed by contradiction.

Suppose that $\frac{1}{n}A \in \Delta$. Then there exist $0 \not\equiv \widetilde{D} \in \mathrm{Num}(\widetilde{S})$ such that $\pi_*(\widetilde{D}) = \frac{1}{n}A$. As $\widetilde{D} \cdot \widetilde{D} = n(\pi_* D, \pi_* D) = n(\frac{1}{n}A, \frac{1}{n}A) = 0$, by [43, Proposition 2.3], $\widetilde{D} \equiv mE$ for some $0 \neq m \in \mathbb{Z}$ and E an elliptic curve. Then by the push-pull formula we have

$$0 = A \cdot \pi_*(mE) = \pi_*(\pi^* A \cdot mE).$$

As the pushforward of points is injective on cohomology, we have

$$0 = \pi^* A \cdot mE = n\widetilde{A} \cdot mE = nm(\widetilde{A} \cdot E).$$

So $\widetilde{A} \cdot E = 0$. As E and $\widetilde{A}$ are irreducible curves, by [43, Proposition 2.1] $E = T_{\tilde{s}}(\widetilde{A})$, so $E \equiv \widetilde{A}$. But $\pi_*(mE) = \pi_*(m\widetilde{A}) = mA \not\equiv \frac{1}{k}A$, which is a contradiction. Hence $\frac{1}{k}A \notin \Delta$.

3. A similar argument holds for $\frac{m}{k}B$ when S is a non-cyclic bielliptic by replacing $\widetilde{A}$ by $\widetilde{B}$. □

Remark 3.3.17. *Note that the only non-zero isotropic elements* $(0, D, 0) \in \Delta$ *have* $D = aA, bB$ *with* $a, b \in \mathbb{Z}, a, b \neq 0$ *by Lemma 3.3.4 and Lemma 3.3.16.*

Remark 3.3.18. *Note that we prove nothing about classes of the form* $\frac{m}{k}(A + B)$.

A consequence of the above Lemmas is the following description of Δ when S is cyclic.

Corollary 3.3.19. *Suppose that* S *is a cyclic bielliptic surface. Then* Δ *is generated by the classes* $(n, 0, 0), (0, A, 0), (0, B, 0), (0, 0, 1)$.

3.4 Moduli Spaces of Sheaves

In general, the moduli space of coherent sheaves on a variety will form a stack. In order to produce a moduli space of sheaves which is a scheme, we need to impose extra conditions on our sheaves. We introduce the notions of Gieseker and slope stability which allow us to define schemes which parameterize stable coherent sheaves on X.

Moduli spaces of stable sheaves play an important role in understanding equivalences between two objects. Mukai first explored this for abelian varieties and their dual using a universal family of sheaves as a kernel for a Fourier-Mukai transform. This was extended to $K3$ surfaces by Mukai and Orlov who showed that for any derived equivalent $K3$ surfaces X and Y, we can express one as a moduli space of stable sheaves on the other.

3.4.1 Stability of Sheaves

We recall the notions of Gieseker and slope stability as well as simple facts about stable and semistable sheaves with respect to these two different notions of stability.

Definition 3.4.1 (Gieseker stability). *Fix an ample divisor H. Define the normalized Hilbert polynomial of a torsion-free coherent sheaf E with respect to H by*

$$p_E = p_{H,E}(m) = \frac{\chi(E \otimes \mathcal{O}(mH))}{\operatorname{rank} E}.$$

A torsion-free coherent sheaf E is stable *(resp.* semistable*) if $p_{H,F}(m) < p_{H,E}(m)$ (resp. if $p_{H,F}(m) \leq p_{H,E}(m)$) for $m \gg 0$ and all proper sub-sheaves $F \subset E$.*

A semistable sheaf is called polystable *if all its direct summands are stable sheaves.*

Definition 3.4.2 (Slope stability). *Fix an ample divisor H on X. Define the slope of a torsion-free coherent sheaf E with respect to H by*

$$\mu(E) = \frac{c_1(E) \cdot H}{\operatorname{rank} E}.$$

*A torsion-free coherent sheaf E is μ-*stable *(resp. μ-*semistable *if $\mu(F) < \mu(E)$ (resp. $\mu(F) \leq \mu(E)$) for all non-trivial sub-sheaves $F \subset E$ with $0 < \operatorname{rank} F < \operatorname{rank} E$.*

A μ-semistable sheaf is called polystable *if all its direct summands are μ-stable sheaves.*

Remark 3.4.3. *By Hirzebruch-Riemann-Roch we can write*

$$\chi(E \otimes \mathcal{O}(mH)) = \int_X \operatorname{ch}(E \otimes \mathcal{O}(mH)) \cdot td(X).$$

If X is a surface with $td(X) = (1, 0, 0)$ (i.e. X is bielliptic or abelian) then we have

$$\begin{aligned}\chi(E \otimes \mathcal{O}(mH)) &= \int_X \mathrm{ch}(E) \cdot exp(mH) \\ &= \int_X (\mathrm{rank}(E), c_1(E), \mathrm{ch}_2(E)) \cdot (1, mH, \frac{1}{2}m^2H^2) \\ &= \mathrm{ch}_2(E) + (c_1(E) \cdot H)m + \frac{\mathrm{rank}(E)H^2}{2}m^2.\end{aligned}$$

So

$$p_E = \frac{\mathrm{ch}_2(E)}{\mathrm{rank}(E)} + \mu(E)m + \frac{H^2}{2}m^2.$$

These notions of stability are related in the following ways.

Lemma 3.4.4 ([39, Lemma 1.2.13]). *We have the following implications*

$$E \text{ is } \mu\text{-stable } \Rightarrow E \text{ is stable} \Rightarrow E \text{ is semi-stable} \Rightarrow E \text{ is } \mu\text{-semistable.}$$

Proposition 3.4.5 ([39, Proposition 1.2.7]). *Let F and G be semi-stable torsion free coherent sheaves.*

(i) If $p(F) < p(G)$, then $\mathrm{Hom}_X(F, G) = 0$. If $p(F) = p(G)$ and $f\colon F \to G$ is non-trivial then f is injective if F is stable and surjective if G is stable.

(ii) If $p(F) = p(G)$ and $\mathrm{rank}(F) = \mathrm{rank}(G)$ then any non-trivial homomorphism $f\colon F \to G$ is an isomorphism provided F or G is stable.

Recall that a sheaf E on X is *simple* if $\mathrm{Hom}_X(E, E) \cong \mathbb{C}$.

Proposition 3.4.6. *Stable sheaves are simple. Moreover, any simple polystable sheaf is stable.*

Proof. The first statement follows from Proposition 3.4.5 part (ii) and that any finite dimensional division algebra over an algebraically closed field is trivial [39, Corollary 1.2.8].

Suppose E be a simple polystable sheaf. Then $E = \oplus_i E_i$ where E_i are stable. Then we have

$$\mathrm{Hom}(E, E) = \bigoplus_{i,j} \mathrm{Hom}(E_i, E_j).$$

As E is simple, all except one of the factors on the right hand side must be zero. Hence $E \cong E_i$ for some i, thus stable. □

3.4.2 Moduli Spaces of Sheaves and Universal Families

By considering families of Gieseker semi-stable sheaves we can construct moduli spaces which are schemes. This was first done by Gieseker in [32] and a modern treatment can be found in [39, §4.3-4.4]. This is achieved using the theory of geometric invariant theory which we will not discuss. We will denote the moduli space of H-semistable sheaves on X by M_H and the open subset of H-stable sheaves by $M_H^s \subset M_H$.

Recall that a family of sheaves on X parameterized by S (an S-family) is a coherent $\mathcal{O}_{X\times S}$-module F flat over S. Let $s \in S$ be a closed point and denote F_s the restriction of F to the fibre X_s over s.

Definition 3.4.7 ([39, Definition 4.6.1]). *A flat family $\mathcal{E}$ of stable sheaves on X parameterized by $M_H^s(v)$ is called* quasi-universal *if the following holds: if F is an S-flat family of stable sheaves on X with Hilbert polynomial P and $\phi_F \colon S \to M_H^s$ the morphism induced by F, which on closed points takes a point $s \in S$ to the sheaf $F_s \in M_H^s$. Then there is a locally free sheaf W of finite rank on S such that $F \otimes p^*W \cong \phi_F^*(\mathcal{E})$. A quasi-universal family is* universal *if W is a line bundle.*

There always exists a (not necessarily unique) quasi-universal family on $X \times M^s$ by [39, Proposition 4.6.2]. However, universal families exist if and only if M_H^s is a *fine moduli space* of stable sheaves. In our situation, we have the following sufficient criteria for the existence of a universal family.

Corollary 3.4.8 ([38, Lemma 10.22 and Corollary 10.23]). *Let X be a smooth surface and $v = (r, D, s)$. Suppose their exists v' such that $\langle v, v'\rangle = 1$. Then there exists an ample class H such that $gcd(r, D \cdot H, s) = 1$ and $M_H^s(v)$ is fine moduli space, i.e. there exists is a universal family on $M_H^s(v) \times X$.*

3.4.3 Properties of the Moduli Space of Sheaves

We now describe some properties of elements $v \in H^{2*}(X, \mathbb{Z})$ which give nice properties of the moduli space of (semi)stable sheaves of class v.

If we assume some generality conditions on our ample divisor H then we can say more.

Definition 3.4.9. *Let $v \in H^{2*}(X, \mathbb{Z})$. We say H is* general *with respect to v (or does not lie on a wall with respect to v) if for every μ-semistable sheaf E with $v(E) = v$ and every $0 \neq F \subset E$ which satisfies $\mu(F) = \mu(E)$ then*

$$\frac{c_1(F)}{\operatorname{rank} F} = \frac{c_1(E)}{\operatorname{rank} E}.$$

Remark 3.4.10. *The notion of H being general can be defined by defining open subsets in the ample cone which are complementary to codimension one subspaces called walls.*

Recall the following notions for an element $v \in H^{2*}(X, \mathbb{Z})$.

Definition 3.4.11. *Let $v = (r, D, s) \in H^{2*}(X, \mathbb{Z})$ with $D \in NS(X)$.*

1. *A class v is* primitive *if v is indivisible. I.e. if $v = dv_0$ with $d \in \mathbb{Z}$ then $d = \pm 1$.*

2. *A vector v is* isotropic *if $\langle v, v\rangle = D^2 - 2rs = 0$.*

The following theorem guarantees non-emptiness of moduli spaces for abelian surfaces.

Theorem 3.4.12 ([79, Lemma 1.2]). *Let X be an abelian surface and H an ample divisor. Assume that $v = (r, D, s) \in H^{2*}(X, \mathbb{Z})$ with $r > 0$ is primitive and isotropic. Then the moduli space $M_H(v)$ is non-empty and consists of μ-stable locally free sheaves.*

Remark 3.4.13. *By [79, Remark 1.1] $M_H(v)$ does not depend on H.*

3.4.4 Smoothness

We can understand smoothness of M_H at a point $[F]$, where F is a stable sheaf on a projective scheme X, by studying the self Ext groups of F. Through understanding the deformation theory of F we have the following characterization of the tangent space $T_{[F]}M_H$ and smoothness at $[F]$.

Corollary 3.4.14 ([39, Corollary 4.5.2]). *Let F be a stable sheaf on a projective scheme X represented by a point $[F] \in M_H$. Then the Zariski tangent space to M_H at F is given by*

$$T_{[F]}M_H \cong \operatorname{Ext}^1_X(F, F).$$

If $\operatorname{Ext}^2_X(F, F) = 0$, then M_H is smooth at $[F]$.

If we assume X is smooth, then we can improve upon the Corollary above. Let E be a locally free sheaf on X, then the trace map $tr\colon \mathcal{E}nd(E) \to \mathcal{O}_X$ induces maps

$$tr^i\colon\ \operatorname{Ext}^i_X(E, E) \to H^i(\mathcal{E}nd_X(E)) \to H^i(\mathcal{O}_X).$$

We can construct these trace maps even when F is not locally free by taking resolutions. These homomorphisms are surjective if the rank of F is non-zero. Denote by $\operatorname{Ext}^i_X(E, E)_0$ the kernel of tr^i.

Theorem 3.4.15 ([39, Theorem 4.5.4]). *Let X be a smooth projective variety and let F be a stable $\mathcal{O}_X$-module of rank $r > 0$. If $\operatorname{Ext}^2_X(F, F)_0 = 0$, then M_H is smooth at $[F]$.*

3.5 Relative Fourier-Mukai Transforms and Bielliptic Surfaces

Given any elliptic fibration $X \to C$ of a smooth projective surface we can consider sheaves supported on a smooth fibre of the fibration. When this moduli space is representable, certain sheaves on the product gives rise to equivalences between the derived category of the moduli space and of the surface. This was used to great effect by Bridgeland and Maciocia [18] to determine the Fourier-Mukai partners of surfaces with Kodaira dimension 0 and 1.

Bielliptic surfaces come with two elliptic fibrations. Thus we expect to get derived equivalences between certain moduli spaces of sheaves supported on the smooth fibres and the original surface. By Proposition 3.6.1 these induce autoequivalences of the derived category. Finally, we prove Theorem 1.2.3 which describes the generators of the group of autoequivalences for cyclic bielliptic surfaces.

3.5.1 Relative Fourier-Mukai Transforms

Recall that a relatively minimal elliptic surface is a projective surface X together with a fibration $\pi\colon X \to C$ whose generic fibre is isomorphic to an elliptic curve and there are no (-1)-curves in the fibres. We will only consider relatively minimal elliptic surfaces.

For an elliptic surface, $\pi\colon X \to C$ define λ_π to be the smallest positive integer such that π has a holomorphic λ_π-multisection. This is equivalent to

$$\lambda_\pi = \min\{f \cdot D > 0 | D \in \operatorname{Num}(X)\},$$

where f is the class of a smooth fibre of π. We call the D such that $D \cdot f = \lambda_\pi$ a λ multi-section for π.

Suppose $a > 0, b \in \mathbb{Z}$ with $\gcd(a\lambda_\pi, b) = 1$. Then we can construct the moduli space $J_X(a, b)$ of pure dimension 1 stable sheaves of class (a, b) supported on fibres of π. By [18, Lemma 4.2] we see that $J_X(a, b) \cong J_X(1, b) =: J_X(b)$ for all a. Bridgeland constructed equivalences between the derived category of X and the derived category of $J_X(b)$ [14]. We call these equivalences *relative Fourier-Mukai transforms.*

Theorem 3.5.1. *[14, Theorem 5.3] Let $\pi\colon X \to C$ be an elliptic surface and take an element*

$$\begin{pmatrix} c & a \\ d & b \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

such that λ_π divides d and $a > 0$. Then there exists a derived equivalence $\Phi\colon D(J_X(b)) \to D(X)$ such that for any closed point $y \in J_X(b)$, $\Phi(\mathcal{O}_y)$ has Chern character $(0, af, b)$, where f is the class of a fibre. Moreover, the functor satisfies

$$\begin{pmatrix} r(\Phi(E)) \\ d(\Phi(E)) \end{pmatrix} = \begin{pmatrix} c & a \\ d & b \end{pmatrix} \begin{pmatrix} r(E) \\ d(E) \end{pmatrix}$$

for all objects E of $D(J_X(b))$.

For a bielliptic surface S, relative Fourier-Mukai transforms with respect to either elliptic fibration p_A or p_B give rise to autoequivalences of $D(S)$ in the following way. The following argument is due to Bridgeland.

Proposition 3.5.2. *Let S be a bielliptic surface and $p_A\colon S \to A/G$ and $p_B\colon S \to B/G$ its two relatively minimal elliptic fibrations. Then a relative Fourier-Mukai transform with respect to either fibration induces an autoequivalence on $D(S)$ which is non-standard.*

Proof. Denote by λ the relative fibre degree of one of the elliptic fibration. Then we need to show that all relative Jacobians $J_S(b)$ are isomorphic to S for either elliptic fibration. By [18, Lemma 4.2] we can reduce to the case where b is coprime to λ. After tensoring by the line bundle corresponding to the multi-section we need only consider b modulo λ by [18, Remark 4.5]. So we are interested in invertible elements of $\mathbb{Z}_\lambda$. As $\lambda = 1, 2, 3, 4$ or 6, the only invertible elements in $\mathbb{Z}_\lambda$ are ± 1. As $J_S(1) \cong J_s(-1) \cong S$ by [18, Remark 4.5] we are done.

Let $\Phi\colon D(J_S(b)) \to D(S)$ be a relative Fourier-Mukai transform induced by one of the two fibrations. By the above argument $J_s(b)$ is isomorphic to S. After choosing an isomorphism $g\colon J_S(b) \to S$, the composite $\Psi = \Phi_{Rel} \circ g^*$ is an autoequivalence of $D(S)$. It is non-standard because $\mathrm{ch}(\Psi(\mathcal{O}_s)) = (0, af, b)$ where f is the fibre of the elliptic fibration. $\square$

To prove Theorem 1.2.3 we will need the following two autoequivalences induced by relative Fourier-Mukai transforms:

Example 3.5.3. *Note that for either fibration p_A or p_B of S we have an autoequivalence corresponding to the matrix*

$$P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

given by Theorem 3.5.1. We have an autoequivalence Ψ_B, constructed by composing the relative Fourier-Mukai transform along p_A associated to P and tensoring by a suitable line bundle, which acts on $N(S)$ by

$$\begin{aligned}
(1, 0, 0) &\mapsto (1, 0, 0)\\
(0, 0, 1) &\mapsto (0, B, 1)\\
(0, \frac{1}{k}B, 0) &\mapsto (0, \frac{1}{k}B, 0)\\
(0, \frac{1}{n}A, 0) &\mapsto (\lambda_{p_A}, \frac{1}{n}A, 0).
\end{aligned}$$

Note Ψ_B sends $(0, A, 0)$ to $(n, A, 0)$.

Suppose that S is cyclic. Then the fibration $p_A\colon S \to A/G$ admits a section, i.e. $\lambda_{p_A} = 1$. Then there is a relative Fourier-Mukai functor $\hat{\Psi}$ that corresponds to the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

given by Theorem 3.5.1 which acts on $N(S)$ *by*

$$\begin{aligned}(1,0,0) &\mapsto (0,(-1/n)A,0)\\(0,0,1) &\mapsto (0,B,0)\\(0,B,0) &\mapsto (0,0,1)\\(0,(1/n)A,0) &\mapsto (1,0,0).\end{aligned}$$

3.5.2 Proof of Theorem 1.2.3

We now prove Theorem 1.2.3.

Theorem 3.5.4 (Theorem 1.2.3)**.** *Suppose* S *is a cyclic bielliptic surface. Then* $\operatorname{Aut} D(S)$ *is generated by standard autoequivalences and relative Fourier-Mukai transforms along the two elliptic fibrations.*

Proof. As S is cyclic, $k=1$ and $|G|=n=\deg\pi$ and $\widetilde{S}\cong A\times B$. Let $\Phi\in\operatorname{Aut} D(S)$. Consider $v=\rho(\Phi)(0,0,1)$. Then $v\in\Delta$, $v^2=0$ and there exists $v'=\rho(\Phi)(1,0,0)$ such that $\langle v,v'\rangle=1$.

We will construct an autoequivalence $\Psi\in\operatorname{Aut} D(S)$ which is the composite of standard autoequivalences and relative Fourier-Mukai transforms along p_A and p_B such that $\rho(\Psi)(0,0,1)=v$.

We separate the argument into three cases:

1. Suppose that $v=\pm(0,0,1)$. Then $\Psi=id$ or $[1]$.

2. Suppose that $v=(0,D,s)$. As $\langle v,v\rangle=0$, $D=aA$ or bB for $a,b\in\mathbb{Z}$, $a,b\neq 0$. Suppose that $D=aA$. As there exists $v'=\varphi(1,0,0)=(r',(a'/n)A+b'B,s')$ such that $\langle v,v'\rangle=1$, we have
$$a(B\cdot A)b'-sr'=1.$$
As $\lambda_{p_B}=B\cdot A$, $\gcd(a\lambda_{p_B},s)=1$. Therefore there exists a relative Fourier-Mukai transform, $\hat{\Phi}$, along p_B such that $\rho(\hat{\Phi})$ sends $(0,0,1)$ to $v=(0,aA,s)$. Then set $\Psi=\hat{\Phi}$. A similar argument for $D=bB$ will work to construct a relative Fourier-Mukai transform along p_A which sends $(0,0,1)$ to $(0,bB,s)$.

3. Suppose that $v=(r,aA+bB,s)$ with $r\neq 0$. We can assume that $r>0$ after applying $\rho([1])$. Then $r=nc$ with $c\in\mathbb{N}$, as $v\in\Delta$. As $v^2=0$ we have
$$v=(nc,aA+bB,ab/c)\,.$$
Note one of a,b is non zero as otherwise v would be divisible.

Suppose $a = 0$, so $v = (nc, bB, 0)$. Then we can apply the relative Fourier-Mukai transform $\hat{\Psi}$ which sends

$$(nc, bB, 0) \mapsto (0, -cA, b)$$

and reduce to case (2).

Suppose that $a \neq 0$. After tensoring by A we can assume $a > 0$. Let $gcd(c, a) = d$ for some $d \in \mathbb{N}$. We can write $c = dc'$ and $a = da'$ with $gcd(a', c') = 1$. Thus v has the form

$$v = (ndc', da'A + bB, a'b/c').$$

We have two operations given by $\rho(- \otimes (-1/n)A)$ and $\rho(\Psi_B^{-1})$ which act on ndc' and da' in the following way:

$$\begin{aligned} \rho(- \otimes (-1/n)A) :&(ndc', da') \mapsto (ndc', d(a' - c')) \\ \rho(\Psi_B^{-1}) :&(ndc', da') \mapsto (nd(c' - a'), da'). \end{aligned}$$

This is just the Euclidean algorithm on c' and a'. Thus we can reduce a' to 1 and c' to 0 and reduce to case (2).

Consider the autoequivalence $\Psi^{-1} \circ \Phi$ whose image under ρ sends $(0, 0, 1)$ to $(0, 0, 1)$. So $\Psi^{-1} \circ \Phi$ is a standard autoequivalence by Corollary 3.3.14. Thus we can express Φ as a composite of standard autoequivalences and relative Fourier-Mukai transforms. □

3.6 Fourier-Mukai Partners for Bielliptic Surfaces

The derived category of a bielliptic surface S is a strong invariant of the surface due to the following result of Bridgeland and Maciocia.

Proposition 3.6.1 ([18, Proposition 6.2]). *Let S be a bielliptic surface and S' be a smooth projective minimal surface derived equivalent to S. Then S is isomorphic to S'.*

The proof of the above result only holds when the canonical cover $\widetilde{S}$ of S is the product of elliptic curves, i.e. S is cyclic. We sketch an argument due to Bridgeland (private correspondence) below for the non-cyclic case.

Assume that S is non-cyclic. Without loss of generality, assume S is of type A2. A similar argument should hold for bielliptic surfaces of type B2 and C2. Let $\Phi \colon D(Y) \to D(S)$ be an equivalence of derived categories where Y is a smooth projective surface which is derived equivalent to S. Consider

$$ch(\Phi(\mathcal{O}_y)) = v = (r, aA + bB, s)$$

where $r \geq 0$, a, b are either integers or 1/2-integers and $A \cdot B = 4$.

As $v^2 = 0$ we see that $rs = 4ab$. As Φ lifts to a equivariant equivalence $\widetilde{\Phi}\colon D(\widetilde{Y}) \to D(\widetilde{S})$, $v \in \Delta$ so $2|r$. Therefore a and b cannot be both $1/2$ integers, since then 2 would not divide $4ab = rs$. Hence as $aA + bB \in \Delta$ by Lemma 3.3.16, $a, b \in \mathbb{Z}$. But then $(aA + bB) \cdot C$ is divisible by 2 for any class $C \in \mathrm{Num}(S)$ since $\frac{1}{2}A$ and $\frac{1}{2}B$ generate $\mathrm{Num}(S)$. By primitivity of v, s is not divisible by 2.

Now consider the elliptic fibration p_B which admits a 2-multisection. Sheaves of class v restrict to the general fibre to give sheaves of rank r and degree $d = 4b$. Let $h = \gcd(r, d)$. By the relation $rs = ad$ and as $\gcd(2, s) = 1$, the rank r contains as big a factor of 2 as d, i.e. 2^k divides d implies 2^k divides r. Then h is the greatest common divisor of $2r$ and d. Thus we can find x and y with $yd - 2xr = h$. Consider the matrix

$$\begin{pmatrix} d/h & -r/h \\ 2x & y \end{pmatrix}$$

which has determinant 1. It maps a column vector (r, d) to $(0, -h)$. Then this matrix induces an autoequivalence of $D(S)$ by Proposition 3.5.2.

By composing with the relative Fourier-Mukai transform we get an equivalence $\Phi'\colon D(Y) \to D(S)$ which sends $(0, 0, 1)$ to $v' = (0, -hA, s)$. By primitivity of v' we can compose with a another relative Fourier-Mukai transform to get an equivalence $\Phi''\colon D(Y) \to D(S)$ which sends $(0, 0, 1)$ to $(0, 0, 1)$. By Lemma 3.3.13, Φ'' sends a skyscraper sheaf to the shift of a skyscraper sheaf and so induces an isomorphism $f\colon Y \to S$ by Corollary 2.3.9.

3.7 Moduli Spaces of Sheaves and Equivalences of Derived Categories

Mukai first observed [55] that the Poincaré line bundle on the product $A \times \hat{A}$ of an abelian variety and its dual can be used as the kernel of an integral transform to give an equivalence of derived categories $D(A) \cong D(\hat{A})$. Since then there has been an intimate relationship between moduli space of sheaves M on X and functors between the derived categories $D(M)$ and $D(X)$ given by integral transforms whose kernel is the universal family of the moduli space.

The following Proposition due to Bridgeland gives sufficient criteria on the moduli space of sheaves for the integral transform to be an equivalence.

Recall that a sheaf E on a variety X is *special* if $E \otimes \omega_X \cong E$.

Proposition 3.7.1 ([18, Corollary 2.8]). *Let X be a smooth projective surface with a fixed polarization, and let Y be a smooth, fine, complete, two-dimensional moduli space of special, stable sheaves on X. Then there is a universal sheaf $\mathcal{P}$ on $Y \times X$ and the functor $\Phi^{\mathcal{P}}_{Y \to X}\colon D(Y) \to D(X)$ is an equivalence.*

To prove Theorem 1.2.2 we will construct autoequivalences using certain moduli spaces of sheaves of our bielliptic surface.

Proposition 3.7.2. *Let S be a bielliptic surface and $\pi\colon \widetilde{S} \to S$ the canonical cover of S. Let $v = (r, D, s) \in \Delta$, $r > 0$, which is isotropic and $\langle v, v' \rangle = 1$ for some $v' \in N(S)$. Choose an ample line bundle H general with respect to v. Then there exists a two dimensional, projective, smooth, fine moduli space M of stable, special sheaves on S of class v.*

Moreover, the universal sheaf on $M \times S$ induces an autoequivalence Φ of $D(S)$ such that $[\Phi(\mathcal{O}_s)] = v$ for any closed point $s \in S$.

Proof. We first show that M is non-empty. As $v \in \Delta$, there exists $w = (\tilde{r}, \tilde{D}, \tilde{s}) \in N(\widetilde{S})$ such that $\pi_*(w) = v$. As $r > 0$, then $\tilde{r} > 0$ as $\pi_*(\tilde{r}) = \deg \pi \cdot \tilde{r} = r$.

As v is isotropic, so is w because $0 = \langle v, v \rangle = \langle \pi_* w, \pi_* w \rangle = n \langle w, w \rangle$. As v is primitive, we can see that w is primitive by applying adjunction and $1 = \langle \pi_* w, v' \rangle = \langle w, \pi^* v' \rangle$.

As w is isotropic and primitive with $\tilde{r} > 0$, the moduli space of $\pi^* H$-semistable sheaves of class w on the abelian surface $\widetilde{S}$ is non-empty and consists of $\mu_{\pi^* H}$-stable locally free sheaves of class w by Theorem 3.4.12.

Let F be a $\mu_{\pi^* H}$-stable locally free sheaf of class w. By [72, Proposition 1.7] $\pi_*(F)$ is μ_H-polystable. We now show that $\pi_* F$ is simple, therefore μ_H-stable.

Note that F is not the pullback of any sheaf on S because if so with $F \cong \pi^* E'$,

$$1 = \langle \pi_* F, v' \rangle = \langle \pi_* \pi^* E', v' \rangle = n \langle E', v' \rangle$$

as $n > 1$ we get a contradiction.

As $\widetilde{G}$ is cyclic, choose a generator $\tilde{g}$ of $\widetilde{G}$. Then

$$\begin{aligned} \operatorname{Hom}_S(\pi_* F, \pi_* F) \cong \operatorname{Hom}_{\widetilde{S}}(\pi^* \pi_*(F), F) &\cong \operatorname{Hom}_{\widetilde{S}} \left(\bigoplus_{i=0}^{n-1} (\tilde{g}^*)^i(F), F \right) \\ &\cong \bigoplus_{i=0}^{n-1} \operatorname{Hom}_{\widetilde{S}}((\tilde{g}^*)^i(F), F). \end{aligned}$$

As F does not lie in the essential image of

$$\pi^* \colon \operatorname{Coh}(S) \to \operatorname{Coh}^{\widetilde{G}}(\widetilde{S}) \to \operatorname{Coh}(\widetilde{S})$$

$F \ncong (g^*)(F)$. Therefore $F \ncong (g^*)^i(F)$ for any i.

As F is $\mu_{\pi^* H}$-stable, so is $(g^*)^i(F)$ with the same slope. As they are not isomorphic, $\operatorname{Hom}_{\widetilde{S}}((g^*)^i(F), F) = 0$ for all $i \neq 0$. Hence $\dim_{\mathbb{C}} \operatorname{Hom}_S(\pi_* F, \pi_* F) = 1$. Thus $\pi_* F$ is simple, hence μ-stable. By construction, $\operatorname{ch}(\pi_* F) = \pi_*(w) = v$.

Therefore, the moduli space M_H of stable sheaves of class v is non-empty. As H is general with respect to v, all H-semistable sheaves are stable, so the moduli space

$M_H(v) = \bar{M}_H(v)$ is projective. By [39, Proposition 4.6] there exists a quasi-universal family on $M_H \times S$. This family can be chosen to be universal as there exists v' such that $\langle v, v' \rangle = 1$ by Corollary 3.4.8.

Let E be a H-stable sheaf of class v corresponding to a point of M_H^v. As $v = [E]$ is isotropic and E is stable, $\dim_{\mathbb{C}} \operatorname{Hom}_S(E, E) = 1$ and

$$\dim_{\mathbb{C}} \operatorname{Ext}^1_S(E, E) = 1 + \dim_{\mathbb{C}} \operatorname{Ext}^2_S(E, E).$$

By Serre Duality, $\dim_{\mathbb{C}} \operatorname{Ext}^2_S(E, E) \cong \dim_{\mathbb{C}} \operatorname{Hom}_S(E, E \otimes \omega_S)$. As $\operatorname{ch}(E) = \operatorname{ch}(E \otimes \omega_S) \in H^*(S, \mathbb{Q})$ as K_S is numerically trivial, so $p(E) = p(E \otimes \omega_s)$ and $\operatorname{rk}(E) = \operatorname{rk}(E \otimes \omega_S)$. As E is stable, by Proposition 3.4.5, $\dim_{\mathbb{C}} \operatorname{Hom}_S(E, E \otimes \omega_S) = 0$ or 1. Hence $\dim_{\mathbb{C}} \operatorname{Ext}^1_S(E, E) \leq 2$.

By construction, M_H contains at least one closed point corresponding to a sheaf F which is a μ-stable sheaf which is the pushforward of a $\mu_{\pi^* H}$-stable sheaf on the canonical cover. Thus F is special by Proposition 3.3.6, so $F \otimes \omega_S \cong F$ and $\dim_{\mathbb{C}} \operatorname{Ext}^2_S(F, F) = 1$. Hence $\dim_{\mathbb{C}} \operatorname{Ext}^1_S(F, F) = 2$. By Serre Duality and [39, §4.5] M_H is smooth at F because the trace map on $\operatorname{Ext}^2_S(F, F)$ has zero kernel due to F being special.

As M is smooth at F, $\dim M'_H = \dim_{\mathbb{C}} \operatorname{Ext}^1_S(F, F) = 2$ for some connected irreducible M'_H of M_H. Hence $\dim_{\mathbb{C}} \operatorname{Ext}^1_S(E, E) \geq 2$ for all sheaves E corresponding to points of M'_H. So $\dim_{\mathbb{C}} \operatorname{Ext}^1_S(E, E) = 2$ for all such E. Thus M'_H is smooth of dimension 2. Set $M = M'^v_H$. Note that E is special as $\dim_{\mathbb{C}} \operatorname{Hom}_S(E, E \otimes \omega_S) = \dim_{\mathbb{C}} \operatorname{Ext}^2_S(E, E) = 1$ and as E is H-stable, $E \cong E \otimes \omega_S$.

Thus M is a two-dimensional, projective, smooth, fine moduli space of special stable sheaves on S of class v.

By [18, Corollary 2.8] the universal sheaf $\mathcal{P}$ on $M \times S$ induces an equivalence

$$\Phi_{\mathcal{P}} \colon D(M) \to D(S).$$

By Proposition 3.6.1, M is isomorphic to S. Thus the equivalence $\Phi_{\mathcal{P}}$ induces an autoequivalence Φ of $D(S)$ after choosing an isomorphism $M \cong S$. By construction $[\Phi(\mathcal{O}_s)] = [\mathcal{P}_s] = v$. □

3.8 Proof of Theorem 1.2.2

We now prove Theorem 1.2.2.

Theorem 3.8.1 (Theorem 1.2.2)**.** *There is an exact sequence*

$$1 \longrightarrow (\operatorname{Aut} S \ltimes \operatorname{Pic}^0 S) \times \mathbb{Z} \longrightarrow \operatorname{Aut} D(S) \xrightarrow{\rho} O_\Delta(N(S))$$

where $\mathbb{Z}$ is generated by the second shift [2]*. The map ρ is induced by the natural action of* $\operatorname{Aut} D(S)$ *on $N(S)$ given by $\rho(\Phi)[E] = [\Phi(E)]$. Furthermore, the image of ρ is a* *subgroup of* $O_\Delta(N(S))$ *of index* 4 *if* S *of type* $A2$ *or* $B2$ *and index* 2 *otherwise.*

Proof. To prove Theorem 1.2.2 we will compute the kernel and image of

$$\rho\colon \operatorname{Aut} D(S) \to O(N(S))$$

given by $\rho(\Phi)([E]) = [\Phi(E)]$.

The description of the kernel is given in Corollary 3.3.14.

We now characterize the image of ρ. Let $\varphi \in O_\Delta(N(S))$ and consider $v = \varphi(0,0,1) \in \Delta$. Then $v \in \Delta$, $v^2 = 0$ and there exists $v' = \varphi(1,0,0)$ such that $\langle v, v'\rangle = 1$. We will construct an autoequivalence Ψ such that $\rho(\Psi))$ sends v to $(0,0,1)$. We treat three separate cases:

1. Suppose $v = \pm(0,0,1)$. Then we can apply $\rho([1])$ to make $v = (0,0,1)$ if needed.
2. Suppose that $v = (0,D,s)$. As $\langle v, v\rangle = 0$ and $v \in \Delta$, $D = aA$ or bB for $a,b \in \mathbb{Z}$, $a,b \neq 0$. Suppose that $D = aA$. As there exists $v' = \varphi(1,0,0) = (r', (a'/n)A + b'B, s')$ such that $\langle v, v'\rangle = 1$, we have
$$a(B\cdot A)b' - sr' = 1.$$
As $\lambda_{p_B} = B\cdot A$, $\gcd(a\lambda_{p_B}, s) = 1$. Therefore there exists a relative Fourier-Mukai transform, $\hat{\Phi}$, along p_B such that $\rho(\hat{\Phi})$ sends $(0,0,1)$ to $v = (0,aA,s)$. Then set $\Psi = \hat{\Phi}^{-1}$. A similar argument for $D = bB$ will work to construct a relative Fourier-Mukai transform along p_A which sends $(0,0,1)$ to $(0,bB,s)$.
3. Suppose that $v = \pm(r,D,s)$. After applying $\rho([1])$ we can assume that $r > 0$. Hence by Proposition 3.7.2 there $\Phi \in \operatorname{Aut} D(S)$ such that $\rho(\Phi)(0,0,1) = v$. Set $\Psi = \Phi^{-1}$ or $\Psi = \Phi^{-1}\circ[1]$.

Consider the isometry
$$\varphi' = (\rho(\Psi))\circ\varphi\,.$$
Then $\varphi'(0,0,1) = (0,0,1)$. As $\varphi'(1,0,0) = (1,D,s)$ is isotropic, $D^2 = 2s$. Thus $s = D^2/2$ and $\varphi'(1,0,0) = (1,D,D^2/2)$ is the class of a line bundle L with $c_1(L) = D$. Consider the isometry
$$\varphi'' = \rho(L^*\otimes(-))\circ\varphi'\,.$$
Notice that φ'' acts by
$$id_{H^0}\oplus\psi\oplus id_{H^4}$$
on $N(S)$ where ψ is an isometry of $\operatorname{Num}(S)$. Note that φ'' respects the grading and is an element of $O_\Delta(N(S))$ as it is a composite of elements of $O_\Delta(N(S))$.

The group $\mathrm{Num}(S)$ is isomorphic as a lattice to a single hyperbolic plane U with underlying group $\mathbb{Z}^2$ [68, §1]. The group of isometries $O(U)$ is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$. It is generated by the involutions ι, which acts by $-id$ on U, and σ which exchanges the two copies of $\mathbb{Z}$. Both of these give rise to isometries of $N(S)$ by acting by the identity on $H^0(S, \mathbb{Z})$ and $H^4(S, \mathbb{Z})$ which we will denote by ι and σ by an abuse of notation.

Suppose the isometry ι is induced by an autoequivalence. As ι fixes the class of a point and acts non-trivially on $N(S)$, ι is induced by a standard autoequivalence which acts non-trivially on $N(S)$. But standard autoequivalences which act non-trivially on $N(S)$ act by tensoring by $\pm(1, D, D^2/2)$ for some line bundle L with $c_1(L) = D \neq 0$. However, ι does not acts on $N(S)$ in this way as $\iota(1, 0, 0) = (1, 0, 0)$. Hence ι is not induced by an autoequivalence. Similarly, σ and $\iota \circ \sigma$ are not induced by autoequivalences. Thus the image of ρ intersected with $O(\mathrm{Num}(S))$ is trivial.

Note that ι preserves Δ. However, σ may not preserve Δ. The index of the image of ρ will 2 or 4 in $O_\Delta(N(S))$ depending on whether σ preserves Δ. As σ acts trivially on the two copies of $\mathbb{Z}$ in $N(S)$ it is sufficient to study the action on $\mathrm{Num}(S)$ by the following Lemma.

If $(r, D, s) \in \Delta$ then $\sigma(r, D, s) = (r, \sigma(D), s) \in \Delta$ if and only if $(0, \sigma(D), 0) \in \Delta$. To determine whether σ preserves Δ we reduce to studying classes of the form $(0, D, 0)$. By abuse of notation, we will denote the class $(0, D, 0) \in N(S)$ by D and we write $D \in \Delta$ for $(0, D, 0) \in \Delta$.

Note that σ interchanges the generators of $\mathrm{Num}(S)$. We will consider separate cases to determine the index of the image of ρ.

We will use the following repeatedly: A class $D \in \Delta$ if and only if $D' = D + (aA + bB) \in \Delta$ with $a, b \in \mathbb{Z}$. Clearly if $D \in \Delta$ then $D' \in \Delta$. Conversely, if $D' \in \Delta$, then $D = D' - (aA + bB) \in \Delta$ as Δ is a subgroup.

Cyclic Bielliptic Suppose that S is cyclic. Then σ interchanges $\frac{1}{n}A$ and B. But by Lemma 3.3.16 $\frac{1}{n}A \notin \Delta$ but $B \in \Delta$, so σ does not preserve Δ. Hence the index is 2.

Bielliptic of type A2 By Lemma 3.3.16 we have $\frac{1}{2}A, \frac{1}{2}B \notin \Delta$ and $A, B \in \Delta$. Consider $D = \frac{a}{2}A + \frac{b}{2}B$ with $a, b \in \mathbb{Z}$. Then $\sigma(D) = \frac{b}{2}A + \frac{a}{2}B$. By adding or subtracting multiples of A and B we can reduce to the cases when $a, b \in \{0, 1\}$. We have 3 cases:

1. If $a = b = 0$ then $D \in \Delta$ and $\sigma(D) \in \Delta$.
2. Suppose $a = 0$ and $b = 1$. Then $\sigma(D) = \frac{1}{2}A \notin \Delta$ and $D = \frac{1}{2}B \notin \Delta$. A similar argument show that $D, \sigma(D) \notin \Delta$ for $a = 1$ and $b = 0$.
3. Suppose that $a = b = 1$. Then $D = \frac{1}{2}A + \frac{1}{2}B = \sigma(D)$. Hence $D \in \Delta$ if and only if $\sigma(D) \in \Delta$.

Thus σ preserves Δ and the index is 4.

Bielliptic of type B2 By Lemma 3.3.16 we have $\frac{1}{3}A, \frac{1}{3}B \notin \Delta$ and $A, B \in \Delta$. Consider $D = \frac{a}{3}A + \frac{b}{3}B$ with $a, b \in \mathbb{Z}$ and $\sigma(D) = \frac{b}{3}A + \frac{a}{3}B$. By adding or subtracting multiples of A and B we can reduce to the cases when $a, b \in \{0, 1, -1\}$. We have 4 cases:

1. If $a = b = 0$. Then $D \in \Delta$ and $\sigma(D) \in \Delta$.
2. Suppose that $a = b = 1$ Then $\sigma(D) = \frac{1}{3}A + \frac{1}{3}B = D$. Hence $D \in \Delta$ if and only if $\sigma(D) \in \Delta$. A similar argument works for $a = b = -1$.
3. Suppose that $m = a$ and $b = 1$. Then $D = \frac{1}{3}B \notin \Delta$ and $\sigma(D) = \frac{1}{3}A \notin \Delta$. Similarly for $a = 0, b = -1$ and $a = 1, -1, b = 0$ we have $D \notin \Delta$ and $\sigma(D) \notin \Delta$.
4. Suppose that $a = 1$ and $b = -1$. Then $\sigma(D) = -\frac{1}{3}A + \frac{1}{3}B = -D$. As Δ is a subgroup $-D \in \Delta$ if and only if $D \in \Delta$. Hence $D \in \Delta$ if and only if $\sigma(D) \in \Delta$. A similar argument works for $a = -1$ and $b = 1$.

Thus σ preserves Δ and the index is 4.

Bielliptic of type C2 Note that $\frac{1}{2}A \notin \Delta$ by a similar argument to Lemma 3.3.16. Then as σ interchanges $\frac{1}{2}A$ and $2(\frac{1}{2}B) = B$, σ does not preserve Δ. Hence the index is 2.

$\square$

Chapter 4

Background on Stacks

In this Chapter we review the background material on stacks required for Chapters 5 and 6. In section 4.1 review the definition of a Deligne-Mumford stack and properties of them. In section 4.2 we discuss presentations of Deligne-Mumford stacks which will be useful for performing calculations. In section 4.3 we define the category of (quasi-)coherent sheaves on a Deligne-Mumford stack and construct the associated derived category and derived functors.

4.1 Deligne-Mumford Stacks

Throughout this thesis, we will only consider Deligne-Mumford stacks. In this section, we summarize the basic definitions and properties of these stacks giving references for further details. These definitions can primarily be found in the Appendix of [77] and in [58]. We will give specific references in each section. We will fix a base scheme S. In Chapters 5 and 6 we will assume that $S = \operatorname{Spec} k$ where k is a field of arbitrary characteristic and not necessarily algebraically closed.

4.1.1 Étale Topology

In this section we introduce the étale topology following [36, §1] (see also [70, Tag 02GH]).

Let A be a local ring and denote by m_A its maximal ideal and $k(A)$ its residue field. Recall that a morphism of local rings $f\colon A \to B$ is a ring homomorphism such that $f(m_A) \subset m_B$. Recall that a field extension L over K is *separable* if for every element $\alpha \in L$, its minimal polynomial μ_α is separable, i.e. its formal derivative μ'_α is non-zero.

Definition 4.1.1.

1. *A morphism $f\colon A \to B$ of local rings is* unramified *if $f(m_A)B = m_B$ and $k(B)$ is a finite separable extension of $k(A)$.*

2. *A morphism of finite type $f\colon X \to Y$ of schemes is* unramified at $x \in X$ *if the associated morphism $f^{\#}\colon \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ of local rings at x is unramified. The morphism $f\colon X \to Y$ is* unramified *if it is unramified at every point of x.*

Definition 4.1.2.

1. *A morphism $f\colon A \to B$ of local rings is* étale *if f is flat and unramified.*

2. *A morphism of finite type of schemes $f\colon X \to Y$ is* étale at $x \in X$ *if the induced map of local rings at x is étale. A morphism is étale if it is étale at every point.*

Example 4.1.3. *Suppose $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$. Then f induces the map on rings $f^{\#}\colon A \to B$. Note f is étale if and only if $f^{\#}$ is. Then f is étale if*

1. *B is a finitely generated A-algebra*

2. *B is a flat A-algebra*

3. *For all maximal ideals m of B, B_m/mB_m is a finite separable extension of A_p/pAp where $p = (f^{\#})^{-1}(m)$.*

Example 4.1.4.

- *Let $f\colon U \to X$ be an open immersion. Then f is étale.*

- *Let G be a finite group acting freely on a quasi-projective variety X over an algebraically closed field. Then the quotient map $\pi\colon X \to X/G$ is étale.*

- *If $i\colon Z \hookrightarrow X$ is a closed immersion, then i is unramified but not flat, hence i is not étale.*

Remark 4.1.5. *Note that étale maps are open as they are flat. Moreover, étale morphisms are stable under composition and base change (c.f. [70, Tag 02GH]).*

Remark 4.1.6. *Suppose X and Y are smooth projective varieties over $\mathbb{C}$. Then a morphism between X and Y is étale if it is a local isomorphism in the analytic topology.*

The étale topology on Sch/S will be an example of a Grothendieck topology on Sch/S which specifies a collection of coverings.

Definition 4.1.7. *Let $\mathcal{C}$ be a category. A* Grothendieck topology *on $\mathcal{C}$ consists of a set* $\operatorname{Cov}(X)$ *of collections of morphisms $\{X_i \to X\}_{i\in I}$ for every object $X \in \mathcal{C}$ such that*

1. *If $V \to X$ is an isomorphism, then $\{V \to X\} \in \operatorname{Cov}(X)$.*

2. *If $\{X_i \to X\}_{i\in I} \in \operatorname{Cov}(X)$ and $Y \to X$ is any morphism in $\mathcal{C}$, then the fibre products $X_i \times_X Y$ exist and the collection of compositions*

$$\{X_1 \times_X Y \to Y\}_{i\in I}$$

is in $\operatorname{Cov}(Y)$.

3. *If* $\{X_i \to X\}_{i \in I} \in \mathrm{Cov}(X)$ *and for every* $i \in I$ *we are given* $\{V_{ij} \to X_i\}_{j \in J_i} \in \mathrm{Cov}(X_i)$*, then the collection of compositions*

$$\{V_{ij} \to X_i \to X\}_{i \in I, j \in J_i}$$

is in $\mathrm{Cov}(X)$.

A category with a Grothendieck topology is called a site.

Example 4.1.8 (Small Classical Site). *Let* X *be a topological space and consider the subcategory* $\mathrm{top}(X)$ *of* $\mathrm{Top}\,/X$ *whose objects are topological spaces* U *and an open imbedding* $U \to X$ *and morphisms are continuous maps* $f\colon U \to V$ *such that*

$$\begin{tikzcd} U \arrow[rr, "f"] \arrow[dr] & & V \arrow[dl] \\ & X & \end{tikzcd}$$

commutes. Then for $U \to X$ *we define* $\mathrm{Cov}(U)$ *to be the collection of morphisms* $\{U_i \to U\}_{i \in I}$ *in* $\mathrm{top}(X)$ *for which* $U = \cup_{i \in I} U_i$*. This defines a Grothendieck topology on* $\mathrm{top}(X)$ *called the* small classical site *on* X.

Example 4.1.9 (Big Classical Site). *Let* $\mathrm{Top}\,/X$ *be the category of topological spaces with a continuous morphism to* X*, with morphisms continuous maps* $f\colon U \to V$ *such that*

$$\begin{tikzcd} U \arrow[rr, "f"] \arrow[dr] & & V \arrow[dl] \\ & X & \end{tikzcd}$$

commutes.

For a topological space U *define* $\mathrm{Cov}(U)$ *to be the collection of morphisms* $\{U_i \to U\}_{i \in I}$ *over* Y *for which each* $U_i \to U$ *is an open imbedding and* $U = \cup_{i \in I} U_i$*. Note than only the covering maps are open imbeddings. Then* $\mathrm{Top}\,/X$ *equipped with this topology is the* big classical site *of* X.

Example 4.1.10 (Small Étale Site). *Let* S *be a scheme and define* $\acute{e}t(S)$ *to be the full subcategory of the category* $\mathrm{Sch}\,/S$ *of schemes over* S *whose objects are étale morphisms* $X \to S$ *and morphism are morphisms* $f\colon X \to Y$ *such that*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

commutes. A collection of morphisms $\{X_i \to X\}_{i \in I}$ *is in* $\mathrm{Cov}(X)$ *if the map*

$$\coprod_{i \in I} X_i \to X$$

is surjective. Note that all the morphisms in $\acute{e}t(S)$ *are étale as the composite of étale* *morphisms is étale. We call* $\acute{e}t(S)$ *the* small étale site of X.

Example 4.1.11 (Big Étale Site). *Let* S *be a scheme and let* Sch/S *be category of schemes over* S. *For* $X \in \mathrm{Sch}/S$ *define* $\mathrm{Cov}(X)$ *to be the collections* $\{X_i \to X\}_{i\in I}$ *of morphisms in* Sch/S *for which each morphism* $X_i \to X$ *is étale and the map*

$$\coprod_{i\in I} X_i \to X$$

is surjective. Note that only the covering maps are étale. We will write $\acute{E}t(S)$ *for the category* Sch/S *with this topology. We call* $\acute{E}t(S)$ *the* big étale site of S.

From now one we will only consider the big étale site of S (and write étale topology for the big étale topology).

We now define a sheaf on Sch/S equipped with the étale or classical Zariski topology.

Definition 4.1.12 ([70, Tag 00VL]). *Consider* Sch/S *with the étale or classical topology. A presheaf* F *on* Sch/S *is a functor*

$$F\colon \mathrm{Sch}/\mathcal{S}^{op} \to \mathrm{Set}\,.$$

We say that F *is a sheaf if for any covering* $\{U_i \to U\}_{i\in I}$ *the sequence*

$$F(U) \longrightarrow \textstyle\coprod_{i\in I} F(U_i) \underset{pr_j^*}{\overset{pr_i^*}{\rightrightarrows}} \coprod_{i,j\in I} F(U_i \times_U U_j)$$

is exact, i.e. the first arrow is the equalizer of pr_i^* *and* pr_j^*.

4.1.2 Categories Fibred in Groupoids

For this section, we follow the appendix in [77, §7]. A more general discussion can be found in [58, §3].

Definition 4.1.13. *A category fibred in groupoids over a scheme* S *is a category* $\mathcal{F}$ *and a functor* $p\colon \mathcal{F} \to \mathrm{Sch}/S$ *such that*

(1) If $f\colon X \to Y$ *is a morphism of* S*-schemes and* y *is an object of* $\mathcal{F}$ *such that* $p(y) = Y$, *then there exists a morphism* $\phi\colon x \to y$ *in* $\mathcal{F}$ *such that* $p(\phi) = f$. *Diagrammatically,*

$$\begin{tikzcd} x \arrow[r, dotted, "\exists"] \arrow[d, dash] & y \arrow[d, dash] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

where the vertical dashes denote $p(x) = X$ *and* $p(y) = p(y)$.

(2) If $\phi\colon x \to y$ *and* $\psi\colon z \to y$ *are morphisms in* $\mathcal{F}$ *and there exists* $h\colon p(x) \to p(z)$ *in* Sch/S *such that* $p(\psi) \circ h = p(\phi)$. *Then there exists a unique arrow* $\rho\colon x \to z$ *such*

that $\psi \circ \rho = \phi$ and $p(\rho) = h$. Diagrammatically,

$$\begin{tikzcd}
x \arrow[rr, "\phi"] \arrow[dr, dashed, "\exists!\rho"'] \arrow[dd, dash] & & y \arrow[dd, dash] \\
& z \arrow[ur, "\psi"] \arrow[dd, dash] & \\
X \arrow[rr, "p(\phi)"] \arrow[dr, "h"'] & & Y \\
& Z. \arrow[ur, "p(\psi)"'] &
\end{tikzcd}$$

Remark 4.1.14. *Note that* (2) *guarantees that the object in* (1) *is unique up to canonical isomorphism. We think of the object x as the pullback of y along f and write $x = f^*y$.*

Definition 4.1.15. *Let $p\colon \mathcal{F} \to \mathrm{Sch}/S$ be a category fibred in groupoids over S. Denote by $\mathcal{F}(X)$ the category whose objects are objects x of $\mathcal{F}$ such that $p(x) = X$ and morphisms are morphisms ϕ in $\mathcal{F}$ such that $p(\phi) = id_X$. The category $\mathcal{F}(X)$ is a groupoid by* (2)*.*

Example 4.1.16. *Suppose that $F\colon (\mathrm{Sch}/S)^{op} \to \mathrm{Set}$ is a functor. Then we can associate to F a category fibred in groupoids $\underline{F}$. An object of $\underline{F}$ is a pair (X, x) where $x \in F(X)$ and X is an S-scheme. A morphism $\phi\colon (X, x) \to (Y, y)$ is a morphism $\phi\colon X \to Y$ such that $F(\phi)(y) = x$. The functor $p\colon \underline{F} \to \mathrm{Sch}/S$ sends (X, x) to X.*

Let $Z \in \mathrm{Sch}/S$ and consider the functor $F = \mathrm{Hom}(-, Z)$. The associated category fibred in groupoids is $\underline{F} = \mathrm{Sch}/Z$ and the functor $p\colon \mathrm{Sch}/Z \to \mathrm{Sch}/S$ is given by composing with the structure map $Z \to S$. We will denote the category fibred in groupoids associated to the functor of points $\mathrm{Hom}_S(-, Z)$ of a scheme Z by $\underline{Z}$.

Definition 4.1.17. *A morphism of categories fibred in groupoids is a functor $\Phi\colon \mathcal{F} \to \mathcal{G}$ such that the following diagram*

$$\begin{tikzcd}
\mathcal{F} \arrow[rr, "\Phi"] \arrow[dr, "p_{\mathcal{F}}"'] & & \mathcal{G} \arrow[dl, "p_{\mathcal{G}}"] \\
& \mathrm{Sch}/S & .
\end{tikzcd}$$

commutes. Here $p_{\mathcal{G}} \circ \Phi = p_{\mathcal{F}}$ as functors.

Suppose $\Phi, \Psi\colon \mathcal{F} \to \mathcal{G}$ are morphisms of fibred categories, then a base preserving natural transformation *$\alpha\colon \Phi \to \Psi$ is a natural transformation of functors such that for every $u \in \mathcal{F}$ the morphism $\alpha_u\colon \Phi(u) \to \Psi(u)$ in $\mathcal{G}$ projects to the identity morphism in* Sch/S.

We denote by $HOM_{\mathrm{Sch}/S}(\mathcal{F}, \mathcal{G})$ the category whose objects are morphisms of fibred categories $\mathcal{F} \to \mathcal{G}$ and whose morphisms are base preserving natural transformations.

Lemma 4.1.18 (2-Yoneda lemma). *The functor*

$$\eta\colon HOM_{\mathrm{Sch}/S}(\underline{X}, \mathcal{F}) \to \mathcal{F}(X)$$

sending a morphism of fibred categories

$$\Phi\colon \underline{X} \to \mathcal{F}$$

to $\Phi(id_X)$ gives an equivalence of categories.

Proposition 4.1.19. *Consider the diagram*

$$\begin{tikzcd} & F_1 \arrow[d, "c"] \\ F_2 \arrow[r, "d"] & F_3 \end{tikzcd}$$

of categories fibred in groupoids over Sch/S. Then the fibred product $G = F_1 \times_{F_3} F_2$ exists and is unique up to unique isomorphism.

Proof. We only prove existence and refer [58, Proposition 3.4.13] for complete details. Let $p_i\colon F_i \to \mathrm{Sch}/S$ be the given functors to $\mathcal{C}$.

Define G to be the category of triples (x_1, x_2, σ) where $x_i \in F_i$ are objects such that $p_1(x_1) = p_2(x_2)$, and $\sigma\colon c(x_1) \to d(x_2)$ is an isomorphism in $F_3(p_1(x_1)) = F(p_2(x_2))$.

A morphism

$$(x_1, x_2, \sigma) \to (x_1', x_2', \sigma')$$

is a pair of morphisms $f_i\colon x_i' \to x_i$ in F_i $(i = 1, 2)$ such that $p_1(f_1) = p_2(f_2)$ and the diagram

$$\begin{tikzcd} c(x_1') \arrow[r, "c(f_1)"] \arrow[d, "\sigma"] & c(x_1) \arrow[d, "\sigma"] \\ d(x_2') \arrow[r, "d(f_2)"] & d(x_2) \end{tikzcd}$$

commutes.

Let $\alpha\colon G \to F_1$ be the functor sending (x_1, x_2, σ) to x_1 and $\beta\colon G \to F_2$ the functor sending (x_1, x_2, σ) to x_2. The isomorphisms σ define an isomorphism $\gamma\colon c \circ \alpha \to d \circ \beta$. $\square$

We now explain the main example of a category fibred in groupoids we will encounter.

Example 4.1.20. *Let $X \in \mathrm{Sch}/S$ and G be a finite group (or more generally a flat group scheme of finite type) acting on X on the right:*

$$a\colon X \times_S G \to X.$$

Define the category fibred in groupoids $[X/G]$ having objects objects triples (B, E, f) where

- *B is a scheme over S.*
- *E is a principal G-bundle over B which is locally trivial in the étale topology,*
- *$f : E \to X$ is a G-equivariant morphism.*

A morphism from $E' \to B'$ with equivariant morphism $f' : E' \to X$ to $E \to B$ is a commutative diagrams

$$\begin{tikzcd} E' \arrow[r, "g"] \arrow[d] & E \arrow[d] \\ B' \arrow[r] & B \end{tikzcd}$$

where $g : E' \to E$ is a G-equivariant morphism such that $gf = f'$.

There is a natural morphism $[X/G] \to \mathrm{Sch}/S$ forgetting everything except the base scheme B. There is also a morphism $q : X \to [X/G]$ given by the trivial bundle

$$\begin{tikzcd} X \times_S G \arrow[r, "a"] \arrow[d, "pr_1"] & X \\ X. & \end{tikzcd}$$

Thus we have the commutative diagram

$$\begin{tikzcd} X \times_S G \arrow[r, "a"] \arrow[d, "pr_1"] & X \arrow[d, "q"] \\ X \arrow[r, "q"] & {[X/G]}\,. \end{tikzcd}$$

Note that if $X = S$ we denote the category fibred in groupoids $[S/G]$ by $B_S G$. When $S = \operatorname{Spec} k$ where k is a field we recover the classifying space of G-torsors BG over k.

If G acts freely and the quotient X/G exists in the category of schemes (i.e. the orbit of every point of X is contained in an affine open subset of X [36, Expose V, Proposition 1.8]) then there is an equivalence of categories $\pi : [X/G] \to \underline{X/G}$.

4.1.3 Deligne-Mumford Stacks

We can now define a stack over S following sections [58, §4] on stacks and [58, §8.3] on Deligne-Mumford stacks.

Definition 4.1.21. *A category fibred in groupoids over S is a* stack *if:*

(i) For any $X \in \mathrm{Sch}/S$ and any two objects $x, y \in \mathcal{F}(X)$, the functor

$$\mathrm{Isom}_X(x, y) : \mathrm{Sch}/X \to \mathrm{Set}$$

which associates to a morphism $f\colon Y \to X$ *the set of isomorphisms in* $\mathcal{F}(Y)$ *between* f^*x *and* f^*y *is a sheaf in the étale topology.*

(ii) Let $\{X_i \to X\}$ *be a covering in the étale topology. Let* $x_i \in F(X_i)$ *and let*

$$\phi_{ij}\colon x_j|_{X_i\times_X X_j} \to x_i|_{X_i\times_X X_j}$$

be isomorphisms in $F(X_i \times_X X_j)$ *satisfying the cocycle relation. Then there is an* $x \in F(X)$ *with isomorphisms* $\psi_i\colon x|_{X_i} \to x_i$ *such that*

$$\phi_{ij} = \psi_i|_{X_i\times_X X_j} \circ (\psi_j|_{X_i\times_X X_j})^{-1}.$$

A morphism of stacks is a morphism of categories fibred in groupoids.

Example 4.1.22. *Let* $X \in \mathrm{Sch}/S$ *be an* S*-scheme and* $\underline{X}$ *the associated category fibred in groupoids. Then* $\underline{X}$ *is a stack.*

It is easy to see that $\underline{X}$ *satisfies condition (i) because if* $f, g\colon T \to X$ *are two elements of* $\underline{X}(T)$ *then* $Isom_T(f,g)(T')$ *is either empty or one point if* $f|_{T'} = g|_{T'}$*. Therefore* $Isom_X(f,g)$ *is either the constant or empty sheaf.*

Another way to see this is that for $f, g\colon T \to X$, $\mathrm{Isom}_X(f,g)$ *is the fibred product of categories fibred in groupoids*

$$\begin{array}{ccc} \mathrm{Isom}_X(f,g) & \longrightarrow & T \\ \big\downarrow & & \big\downarrow{\scriptstyle g} \\ T & \xrightarrow{\quad f \quad} & X. \end{array}$$

which is simply the fibred product $T \times_{f,X,g} T$*. Thus to see* $\mathrm{Isom}_x(f,g)$ *is a sheaf it suffices to show that* $T \times_{f,X,g} T$ *is a sheaf.*

Condition (ii) is non-trivial and follows from showing that $\mathrm{Hom}_S(-,X)$ *is a sheaf in the étale topology for any* $X \in \mathrm{Sch}/S$*. Condition (ii) is true in the Zariski topology and in the étale topology. It follows from the following theorem, originally due to Grothendieck.*

Theorem 4.1.23 ([70, Tag 02W4] and [70, Tag 023P])**.** *For any* S*-scheme* X*, the functor*

$$\mathrm{Hom}_S(-,X)\colon (\mathrm{Sch}/S)^{op} \to \mathrm{Set}$$

is a sheaf in the étale topology.

Example 4.1.24 ([25, Proposition 2.2])**.** *Recall that the category fibred in groupoids* $[X/G]$ *where* $X \in \mathrm{Sch}/S$ *and* G *a finite group (this holds more generally for any flat affine group scheme) acting on* X *on the right*

$$a\colon X \times_S G \to X.$$

Let $e, e'\colon B \to [X/G]$ *correspond to* G*-principal bundles* $E \to B$ *and* $E' \to B$ *with* G*-equivariant morphisms* $f\colon E \to X$ *and* $f'\colon E' \to X$*. Then* $\mathrm{Isom}_B(e,e')$ *is the étale sheaf* *which is the quotient of* $X \times_{X\times X} E \times_B E'$ *by the free product action of* G. *Moreover, this sheaf is a scheme.*

When $E = E'$ *and* $f = f'$ *these isomorphism correspond to elements of* G *which preserve* f.

Since any principal G*-bundle is locally trivial in the étale topology it determines descent data in the following way. Let* $\{B_i \to B\}$ *be an étale cover on which* $E \to B$ *is trivial. Then we have* G*-equivariant morphisms*

$$\phi_i \colon E \times_B B_i \to G \times B_i$$

If ϕ_{ij} *is the pullback of* ϕ_i *to* $B_i \times_B B_j$ *then the* ϕ_{ij} *satisfy the cocycle condition.*

Descent theory for principal G*-bundles gives the opposite direction. Given principal bundles (not necessarily trivial)* $E_i \to B_i$ *and isomorphisms* $E_i|_{B_i\times_B B_j} \to E_j|_{B_i\times_B B_j}$ *satisfying the cocycle condition, there exists a principal* G*-bundle* $E \to B$ *such that* $E_i \cong E \times_B B_i$. *Thus condition (ii) is satisfied.*

The definition of a stack is too general to *do* algebraic geometry. Thus we impose extra conditions which will allow us to define geometric properties of stacks that closely resemble properties of schemes.

Definition 4.1.25. *A morphism of stacks* $f \colon \mathcal{X} \to \mathcal{Y}$ *is representable by schemes if for every scheme* U *and morphism* $y \colon \underline{U} \to \mathcal{Y}$ *the fibre product*

$$\mathcal{X} \times_{\mathcal{Y}} \underline{U}$$

is isomorphic to $\underline{V}$ *for some scheme* V.

Remark 4.1.26. *The above definition means that we can pull back elements of* $\mathcal{Y}(U)$ *to elements of* $\mathcal{X}(\mathcal{X} \times_{\mathcal{Y}} U) = \mathcal{X}(V)$.

The following proposition motivates why we will ask for the diagonal to be representable.

Proposition 4.1.27. *Let* $\mathcal{X}/S$ *be a stack over* S. *Then the following two conditions are equivalent:*

1. *The diagonal map* $\Delta \colon \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ *is representable.*

2. *Every morphism* $\underline{U} \to \mathcal{X}$ *from a scheme* U *is representable.*

Proof. Suppose that Δ is representable and $f \colon \underline{X} \to \mathcal{X}$ and $g \colon \underline{Y} \to \mathcal{X}$ are morphisms with X and Y schemes. Then the fibred product obtained in the diagram

$$\begin{tikzcd}
\underline{X} \times_{\mathcal{X}} \underline{Y} \arrow[r] \arrow[d] & \underline{Y} \arrow[d, "g"] \\
\underline{X} \arrow[r, "f"] & \mathcal{X}
\end{tikzcd}$$

is equivalent to the fibred product in the diagram

$$\begin{tikzcd}
\mathcal{X} \times_{\mathcal{X}\times_S \mathcal{X}} (\underline{X \times_S Y}) \arrow[r] \arrow[d] & \underline{X \times_S Y} \arrow[d, "{(f,g)}"] \\
\mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times_S \mathcal{X}.
\end{tikzcd}$$

Hence $X \times_{\mathcal{X}} Y$ is a scheme.

Suppose that every morphism from a scheme to $\mathcal{X}$ is representable. Let $h\colon \underline{X} \to \mathcal{X} \times_S \mathcal{X}$ be a morphism with X a scheme given by a pair of maps $f\colon \underline{X} \to \mathcal{X}$ and $g\colon \underline{X} \to \mathcal{X}$. Then we have a tower of commutative squares

$$\begin{tikzcd}
\mathcal{X} \times_{\mathcal{X}\times_S \mathcal{X}} \underline{X} \arrow[r] \arrow[d] & \underline{X} \arrow[d, "\Delta_X"] \\
\underline{X} \times_{\mathcal{X}} \underline{X} \arrow[r] \arrow[d] & \underline{X \times_S X} \arrow[d, "{(f,g)}"] \\
\mathcal{X} \arrow[r, "\Delta_{\mathcal{X}}"] & \mathcal{X} \times_S \mathcal{X}
\end{tikzcd}$$

As $\underline{X} \times_{\mathcal{X}} \underline{X}$ is a scheme (as f and g are representable by our assumption), so $\mathcal{X} \times_{\mathcal{X}\times_S \mathcal{X}} \underline{X}$ is a scheme. Hence Δ is representable. $\square$

Now we can define a Deligne-Mumford stack following the definition in [58].

Definition 4.1.28. *A stack $\mathcal{X}/S$ is a* Deligne-Mumford stack *if the following holds:*

1. *The diagonal*
$$\Delta_{\mathcal{X}}\colon \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$$
is representable by schemes.

2. *There exists an étale surjective morphism $\pi\colon \underline{X} \to \mathcal{X}$ with X a scheme. That is, for any morphism from a scheme $\underline{T} \to \mathcal{X}$ the induced morphism of schemes*
$$X \times_{\mathcal{X}} \underline{T} \to T$$
is étale and surjective. Note that $X \times_{\mathcal{X}} \underline{T}$ is a scheme by (1)*. We call X an* atlas *for $\mathcal{X}$.*

A morphism of Deligne-Mumford stacks $f\colon \mathcal{X} \to \mathcal{Y}$ is a morphism of the underlying stacks.

Example 4.1.29. *It is easy to see that for any $X \in \mathrm{Sch}/S$, $\underline{X}$ is a Deligne-Mumford stack as every morphism from a scheme is representable, so the diagonal is representable and $\underline{id}\colon \underline{X} \to \underline{X}$ is a surjective étale cover.*

Another characterization of Deligne-Mumford stacks is using the notion of *formally unramified.* Recall that a morphism of schemes $g\colon V \to W$ is *formally unramified* if for any closed embedding of affine schemes $i\colon X_0 \hookrightarrow X$ defined by a square zero ideal, the natural map

$$\mathrm{Hom}_W(X, V) \to \mathrm{Hom}_W(X_0, V)$$

is injective.

Remark 4.1.30. *A morphism of schemes $g\colon V \to W$ is unramified as defined in Definition 4.1.1 if W is locally noetherian, g is formally unramified and locally of finite type (c.f. [70, Tag 024Q]).*

Proposition 4.1.31 ([77, Proposition 7.15] and [58, Theorem 8.3.3])**.** *Let $\mathcal{X}$ be a Deligne-Mumford stack over S. Then the diagonal*

$$\Delta_{\mathcal{X}}\colon \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$$

if formally unramified.

Example 4.1.32 ([25, Corollary 2.2])**.** *Let X/S be a noetherian scheme of finite type and G a finite group (more generally, a smooth affine group scheme of finite type over S) acting on X such that the stabilizers of geometric points are finite and reduced. Then $[X/G]$ is a Deligne-Mumford stack.*

The condition on the stabilizers ensure that $\mathrm{Isom}_B(E, E)$ *is formally unramified over E for any $\underline{B} \to [X/G]$. This implies that the diagonal is unramified. As* $\mathrm{Isom}_B(E, E')$ *is isomorphic to a scheme from Example 4.1.24 we have that the diagonal is representable. The atlas condition is satisfied by the morphism $q\colon \underline{X} \to [X/G]$.*

4.1.4 Properties of Stacks

We now define properties of Deligne-Mumford stacks and properties of morphisms following [58, §8.2] and [70, Tag 04X8].

We first define properties of Deligne-Mumford stacks using any atlas

Definition 4.1.33 ([70, Tag 0348])**.** *Let P be a property of schemes. We say that P is local in the étale topology if for any covering $\{U_i \to U\}_{i\in I}$ we have*

$$U \textit{ has } P \Leftrightarrow \textit{ each } U_i \textit{ has } P \textit{ for all } i.$$

Definition 4.1.34. *Let P be a property of schemes which is local with in the étale topology. We say that a Deligne-Mumford stack $\mathcal{X}$* has property P *if there exists a surjective étale morphism $X \to \mathcal{X}$ with X being a scheme having property P.*

Remark 4.1.35. *The following properties are local with respect to the étale topology: regular, locally noetherian, locally of finite type, quasi-compact, proper. Thus we can talk about Deligne-Mumford stacks of finite type over a field k (taking $S = \mathrm{Spec}\, k$) which are regular.*

4.1.5 Properties of Morphisms

Now we define properties of morphisms of Deligne-Mumford stacks following [70, Tag 04XB].

Definition 4.1.36 ([70, Tag 02KN]). *Let P be a property of schemes over a base S. We say P is* local on the target *if for any étale covering $\{Y_i \to Y\}_{i\in I}$ and any morphism of schemes $f\colon X \to Y$ over S we have*

$$f \text{ has } P \Leftrightarrow Y_i \times_Y X \to Y_i \text{ has } P \text{ for all } i.$$

We say P is local on the source *if for any étale covering $\{X_i \to X\}_{i\in I}$ and any morphism of schemes $f\colon X \to Y$ over S we have*

$$f \text{ has } P \Leftrightarrow \text{ each } X_i \to Y \text{ has } P.$$

Definition 4.1.37. *We say a property P of schemes is* stable with respect to the étale topology *if P is local on the target and preserved under arbitary base change.*

Definition 4.1.38. *Let P be a property of morphisms of schemes which is stable with respect to the étale topology. A representable morphism of algebraic stacks $f\colon \mathcal{X} \to \mathcal{Y}$ has property P if for every morphism $\underline{T} \to \mathcal{Y}$ with T a scheme, the morphism of schemes*

$$\mathcal{X} \times_{\mathcal{Y}} \underline{T} \to \underline{T}$$

has property P.

To define properties of arbitary morphisms we use the following notation following [58, §8.2.5]. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of Deligne-Mumford stacks over S. A *chart* for f by schemes is a diagram

$$\begin{tikzcd}
\underline{X} \arrow[r, "g"] \arrow[rr, bend left, "h"] \arrow[dr, "q"'] & \mathcal{X}' \arrow[r, "f'"] \arrow[d, "p'"] & \underline{Y} \arrow[d, "p"] \\
& \mathcal{X} \arrow[r, "f"] & \mathcal{Y}
\end{tikzcd}$$

where X and Y are schemes, the squares in the diagram are commutative, the right square is cartesian, and g and p are surjective and étale.

Definition 4.1.39 ([58, §8.2.6]). *Let P be a property of morphisms of schemes that is stable and local on the source with respect to the étale topology. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of Deligne-Mumford stacks. We say f* has property P *if there exists a chart for f such that h has property P.*

Remark 4.1.40. *The above definition allows us to define* flat *morphisms of Deligne-Mumford stacks.*

We now define the image of a morphism from a Deligne-Mumford stack to a scheme.

Definition 4.1.41 ([58, S 8.5]). *Let $\mathcal{X}/S$ be a Deligne-Mumford stack over S. For a morphism $f\colon \mathcal{X} \to Y$ to a scheme $Y \subseteq \mathrm{Sch}/S$, define the* image *of f to be the subset of Y which is the image of the composite*

$$U \xrightarrow{u} \mathcal{X} \xrightarrow{f} Y$$

where $u\colon U \to \mathcal{X}$ is an étale surjective morphism with U a scheme.

4.1.6 Open and Closed Substacks

We now define various substacks of Deligne-Mumford stacks.

Definition 4.1.42. *A morphism of Deligne-Mumford stacks $f\colon \mathcal{Z} \to \mathcal{X}$ is an* open (respectively closed) embedding *if it is representable and has property P in 4.1.38 where P is the property of being a closed (respectively open) embedding (also called immersion).*

An open substack *is a stack $\mathcal{U}$ and an open imbedding $\mathcal{U} \to \mathcal{X}$.*

A closed substack *of an algebraic stack $\mathcal{X}$ is defined by an equivalence class of closed imbeddings $\mathcal{Z} \to \mathcal{X}$ where two closed imbeddings $f_i\colon \mathcal{Z}_i \to \mathcal{X}$ $(i = 1, 2)$ are equivalent if there exists a pair (g, σ) with $g\colon \mathcal{Z}_1 \to \mathcal{Z}_2$ and $\sigma\colon f_2 \circ g \cong f_1$ an isomorphism.*

Example 4.1.43. *Let $\mathbb{Z}_2$ act on $\mathbb{A}^1_k$ by $x \mapsto -x$. Then the inclusion*

$$i_x\colon \operatorname{Spec} k \to [\mathbb{A}^1_k \,/\, \mathbb{Z}_2]$$

which corresponds to the trivial principal $\mathbb{Z}_2$-bundle over $\operatorname{Spec} k$ and the equivariant map $\mathbb{Z}_2 \to \mathbb{A}^1_k$ sending the identity element to $x \in \mathbb{A}^1_k$ and non-identity element to $-x$ is a closed immersion of stacks.

The morphism i_x is representable because $[\mathbb{A}^1_k \,/\, \mathbb{Z}_2]$ is a Deligne-Mumford stack and

$$\begin{array}{ccc} \mathbb{Z}_2 & \longrightarrow & \mathbb{A}^1_k \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \longrightarrow & [\mathbb{A}^1_k \,/\, \mathbb{Z}_2] \end{array}$$

is the fibre product $\operatorname{Spec} k \times_{[\mathbb{A}^1_k \,/\, \mathbb{Z}_2]} \operatorname{Spec} k[x]$ is isomorphic to $\mathbb{Z}\,/2$. The induced map $\mathbb{Z}_2 \to \mathbb{A}^1_k$ is the equivariant map which is a closed immbedding.

4.1.7 Separated and Proper Morphisms

Recall that a morphism $f\colon X \to Y$ of schemes is separated if the relative diagonal $\Delta_f\colon X \to X \times_Y X$ is a closed imbedding.

Recall that a morphism $f\colon X \to Y$ of schemes is *universally closed* if for any morphism $Z \to Y$ the induced morphism $X \times_Y Z \to Z$ is closed (i.e. the image of closed subsets are closed). Then a morphism $f\colon X \to Y$ of schemes is *proper* if it is separated, of finite type and universally closed. Two examples of proper morphisms of schemes are closed imbeddings and finite morphisms.

We now extend the definition of separated to Deligne-Mumford stacks.

Definition 4.1.44. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of Deligne-Mumford stacks over S and let*

$$\Delta_{\mathcal{X}/\mathcal{Y}}\colon \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$$

be the (relative) diagonal morphism.

- *We say f is* quasi-separated *if the diagonal Δ_f is quasi-compact and quasi-separated.*
- *We say f is* separated *if the diagonal $\Delta_{\mathcal{X}/\mathcal{Y}}$ is proper.*

If $\mathcal{Y} = S$ and f is the structure morphism, then we say that $\mathcal{X}$ is separated.

The following gives a way of characterizing whether a quotient stack is separated. Recall that a group action of G on X on the left is a morphism

$$a\colon G \times X \to X.$$

The action is *proper* if

$$(a, id_X)\colon G \times X \to X \times X$$

is proper.

Proposition 4.1.45 ([25, Corollary 2.2])**.** *Let X/S be a noetherian scheme of finite type over S and G a finite group (more generally a smooth affine group scheme of finite type over S) acting on X on the right such that the stabilizer groups of geometric points are finite and reduced. Then $[X/G]$ is separated if and only if the action is proper.*

Example 4.1.46 (Example of a seperated stack)**.** *Consider the quotient stack $\mathcal{X} = [\mathbb{A}^1 \,/\, \mathbb{Z}_2]$ where $\mathbb{Z}_2$ acts on $\mathbb{A}^1$ by $z \mapsto -z$. Then $\mathcal{X}$ is a seperated because the action is proper. We will give a different proof in Example 4.2.16 in Section 4.2.2 using groupoid presentations.*

Now we define proper morphisms for non-representable morphisms following [58, §10.1] and [70, Tag 0CL4].

Definition 4.1.47. *A morphism $f\colon \mathcal{X} \to \underline{Y}$ from a Deligne-Mumford stack $\mathcal{X}$ to a scheme Y is* closed *if for every closed substack $\mathcal{Z} \subset \mathcal{X}$ the image of $\mathcal{Z}$ in Y is closed.*

A morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of Deligne-Mumford stacks is universally closed *if for every morphism $\underline{Y} \to \mathcal{Y}$ where Y is a scheme, the morphism $\mathcal{X} \times_{\mathcal{Y}} Y \to Y$ is closed.*

A morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of Deligne-Mumford stacks is proper *if it is separated, of finite type, and universally closed.*

This new definition recovers our previous definition of proper morphism when the morphism is representable due to the following.

Proposition 4.1.48 ([58, Proposition 10.1.4]). *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a representable separated morphism of finite type. The f is universally closed if and only if f is proper in the previous sense.*

Example 4.1.49. *A closed imbedding of Deligne-Mumford stacks is proper.*

We now list some properties of proper morphisms of Deligne-Mumford stacks.

Proposition 4.1.50 ([58, Proposition 10.1.6]).

- *For a composite of morphisms of Deligne-Mumford stack*

  $$\mathcal{X} \xrightarrow{f} \mathcal{Y} \xrightarrow{g} \mathcal{Z}$$

  if f and g are proper, so if gf. If gf is proper and g is separated (e.g. proper) then f is proper.

- *Proper morphisms are closed under arbitary base change.*

4.1.8 Automorphism Groups of Points

We now define properties of Deligne-Mumford stacks which are dependent on properties of automorphism groups of points.

Definition 4.1.51. *Let $\mathcal{X}/S$ be a Deligne-Mumford stack over S and k be a field. For $x\colon \mathrm{Spec}(k) \to \mathcal{X}$ define the* automorphism group *of x to be the finite group scheme G_x defined as the fibred product of the diagram*

$$\begin{array}{ccc} G_x & \longrightarrow & \mathrm{Spec}(k) \\ \big\downarrow & & \big\downarrow{\scriptstyle x} \\ \mathrm{Spec}(k) & \xrightarrow{x} & \mathcal{X}. \end{array}$$

Definition 4.1.52. *Let $\mathcal{X}$ be a Deligne-Mumford stack separated and of finite type over S. We say that $\mathcal{X}$ is* tame *if for every geometric point (a morphism $\bar{x}\colon Spec(\bar{k}) \to \mathcal{X}$ where $\bar{k}$ is algebraically closed) the automorphism group $G_{\bar{x}}$ has order invertible in k.*

Remark 4.1.53. *If S is a scheme over a field of characteristic zero, then every separated Deligne-Mumford stack of finite type over S is tame.*

We also need the notion of trivial generic stabilizer which means that our stabilizer groups will be as small as possible.

Definition 4.1.54. *A Deligne-Mumford stack $\mathcal{X}$ over S has trivial generic stabilizer if for any atlas $U \to \mathcal{X}$ the automorphism group of the generic point of U in $\mathcal{X}$ is trivial.*

Example 4.1.55. *Let G be a finite group acting effectively on a quasi-projective scheme X over k. Then the quotient stack $[X/G]$ has trivial generic stabilizer.*

4.2 Presentations of stacks

In this section, we summarize the definitions and properties of algebraic groupoids and the stack associated to an algebraic groupoid. We give a dictionary between properties of a stack and properties of a groupoid presentation for that stack. More details (in greater generality) can be found in [58, §3.4], [70, Tag 04TJ] and [7].

4.2.1 Stack Associated to an Algebraic Groupoid

Recall that a groupoid is a small category in which every morphism has an inverse. It comprises of:

- a set of objects U
- a set of morphisms R,
- source and target maps $s, t\colon R \to U$,
- a composition map $m\colon R \times_{s,U,t} R \to R$,
- an inverse map $i\colon R \to R$
- a map giving identity map $\epsilon\colon U \to R$.

This can be abstracted in the following way where the sets of objects and morphisms are schemes.

Definition 4.2.1. *An* algebraic groupoid over S *is a collection of data*

$$(R, U, s, t, \epsilon, i, m)$$

with

1. *Objects R and U of* Sch/S.
2. *Morphisms over S*

$$s\colon R \to U, t\colon R \to U, \epsilon\colon U \to R,$$
$$i\colon R \to R, m\colon R \times_{s,U,t} R \to R.$$

This data is required to satisfy the "obvious" axioms of a groupoid where R denotes the morphisms, U the objects, s, t the source and target, ϵ is the identity map, i the inverse, and m describes how to compose morphisms. We will write $R \rightrightarrows U$ to denote the groupoid $(R, U, s, t, \epsilon, i, m)$.

Note that for any scheme T, the groupoid in sets $(U(T), R(T), s, t, \epsilon, i, m)$ is a groupoid in the usual sense.

Definition 4.2.2 ([70, Tag 0230])**.** *A morphism* $\Phi\colon (R \rightrightarrows U) \to (R' \rightrightarrows U')$ *of algebraic groupoids over* Sch/S *is a pair of morphisms of schemes* $\Phi\colon U \to U'$ *and* $\Phi\colon R \to R'$ *such that for any scheme* T *over* S *the map* Φ *defines a functor*

$$\Phi(T)\colon\ (R(T) \rightrightarrows U(T)) \to \big(R'(T) \rightrightarrows U'(T)\big)$$

between groupoid categories.

We now explain how to construct a category fibred in groupoids associated to an algebraic groupoid $R \rightrightarrows U$ following [70, Tag 04TJ]. For every $X \in \mathrm{Sch}/S$, consider the category $\{R \rightrightarrows U\}(X)$ whose objects are elements $x \in U(X) = \mathrm{Hom}_{\mathrm{Sch}/S}(X, U)$, and a morphism $x \to x'$ is an element $\xi \in R(X)$ for which $s(\xi) = x$ and $t(\xi) = x'$,

$$\begin{tikzcd} & R \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\ X \arrow[ur, "\xi"] \arrow[r, shift left, "x"] \arrow[r, shift right, "x'"'] & U. \end{tikzcd}$$

Given a composition

$$x'' \xrightarrow{\ \eta'\ } x' \xrightarrow{\ \eta\ } x$$

we define $\xi \circ \eta$ to be the image under m of the element

$$(\xi, \eta) \in R(X) \times_{s, U(X), t} R(X).$$

The axioms of a groupoid in Sch/S imply that $\{R \rightrightarrows U\}(X)$ is a category. In fact, it is a groupoid as the inverse of $\xi \in R(X)$ is given by $i(\xi)$.

To any morphism $f\colon X \to Y$ there is a functor

$$f^*\colon \{R \rightrightarrows U\}(Y) \to \{R \rightrightarrows U\}(X),$$

induced by the pullback maps

$$f^*\colon U(Y) \to U(X), \quad f^*\colon R(Y) \to R(X).$$

This allows us to define a fibred category

$$p\colon \{R \rightrightarrows U\} \to \mathrm{Sch}/S$$

with objects given by pairs (X, x) with $X \in \mathrm{Sch}/S$ and $x \in \{R \rightrightarrows U\}(X)$. A morphism

$$(X, x) \to (Y, y)$$

in $\{R \rightrightarrows U\}$ is a pair (f, α) where $f\colon X \to Y$ is a morphism in Sch/S and $\alpha\colon x \to f^*y$ is a isomorphism in $\{R \rightrightarrows U\}(X)$. The functor p sends a pair (X, x) to X and a morphism (f, α) to f.

To get a stack from $\{R \rightrightarrows U\} \to \mathrm{Sch}/S$ we *stackify* the category fibred in groupoids. To do this we construct the category $[R \rightrightarrows U] \to \mathrm{Sch}/S$ as category which has objects over $T \in \mathrm{Sch}/S$ the collection of data

$$(\{T_i \to T\}_{i \in I}, (t_i, \phi_{ij}))\,,$$

where $\{T_i \to T\}$ is an étale covering of $\mathcal{T}$ and (t_i, ϕ_{ij}) is an object of $\{R \rightrightarrows U\}(\{T_i \to T\})$.

A morphism

$$\left(\{T'_s \to T'\}, (\{t'_s\}, \psi_{st})\right) \to (\{T_i \to T\}, (\{t_i\}, \phi_{ij}))$$

is a pair (f, ρ) where $f\colon T' \to T$ is a morphism in Sch/S and $\rho\colon (\{t'_s\}, (t'_{st})) \to f^*(\{t_i\}, (\phi_{ij}))$ is a morphism between the induced objects of

$$\{R \rightrightarrows U\}(\{T'_s \times_{T'} T' \times_T T_i\}_{i,s} \to T').$$

More explicitly, an object of $[R \rightrightarrows U](T)$ is a tuple

$$(\{T_i \to T\}, (t_i, \phi_{ij}))$$

where $\{T_i \to T\}$ is a covering in Sch/S and $t_i\colon T_i \to U$ and $\phi_{ij}\colon T_i \times_{X_0} T_j \to R$ such that $s \circ \phi_{ij} = t_i$ and $t \circ \phi_{ij} = t_j$.

Remark 4.2.3. *There is a more general way to get a stack from a category fibred in groupoids [58, Theorem 4.6.5] which follows a similar procedure.*

Remark 4.2.4. *The stack $[R \rightrightarrows U]$ associated to an algebraic groupoid is not necessarily a Deligne-Mumford stack (c.f. [70, Tag 06PI]) as it need not admit an atlas.*

Definition 4.2.5. *An algebraic groupoid $R \rightrightarrows U$ is* étale *if the two maps $s\colon R \to U$ and $t\colon R \to U$ are étale.*

Theorem 4.2.6 ([70, Tag 04TJ])**.** *Let $R \rightrightarrows U$ be an étale groupoid over S. Then the associated stack $[R \rightrightarrows U]$ is a Deligne-Mumford stack over S with atlas U.*

Definition 4.2.7. *A presentation of a Deligne-Mumford stack $\mathcal{X}/S$ is an étale groupoid $R \rightrightarrows U$ such that $[R \rightrightarrows U] \cong \mathcal{X}$.*

Remark 4.2.8. *Note that any Deligne-Mumford stack $\mathcal{X}$ has a presentation $R \rightrightarrows U$ where U is an atlas for $\mathcal{X}$ and $R = U \times_{\mathcal{X}} U$ with s and t given by the projection maps. As any Deligne-Mumford stack has many atlases there are many different presentations. Thus an étale groupoid can be thought of as a Deligne-Mumford stack and a choice of an atlas.*

Example 4.2.9. *Let X be a quasi-projective variety and G a finite group acting effectively on the left on X with action map $a\colon G \times X \to X$. Then we can form the étale* *groupoid*

$$G \times X \overset{pr_2}{\underset{a}{\rightrightarrows}} X$$

and the Deligne-Mumford stack $[G \times X \rightrightarrows X]$ *is isomorphic to the quotient stack* $[X/G]$. *Thus we can interpret properties of* $[X/G]$ *in terms of the étale groupoid above.*

4.2.2 Properties of Stacks in Terms of Groupoids

Throughout this section let $\mathcal{X}$ be a Deligne-Mumford stack over a scheme S and $R \rightrightarrows U$ a presentation for $\mathcal{X}$ so $[R \rightrightarrows U] \cong \mathcal{X}$. Note that U is an atlas for $\mathcal{X}$ and $s, t \colon R \to U$ are étale.

First, we observe the following proposition which follows from Definition 4.1.34.

Proposition 4.2.10. *Let* P *be a property local in the étale topology. Then* $\mathcal{X}$ *has property* P *if and only if* U *does.*

We now show that any morphism of algebraic stacks induces a morphism of presentation.

Lemma 4.2.11 ([70, Tag 04Y6])**.** *Let* $f \colon \mathcal{X} \to \mathcal{Y}$ *be a morphism of Deligne-Mumford stacks over* S *represented by schemes. Let* $[R \rightrightarrows U]$ *be a presentation for* $\mathcal{Y}$. *Set* $U' = U \times_{\mathcal{Y}} \mathcal{X}$ *and* $R' = R \times_{\mathcal{Y}} \mathcal{X}$. *Then there is a groupoid of the form* $[R' \rightrightarrows U']$ *which is a presentation for* $\mathcal{X}$ *and a diagram*

$$\begin{array}{ccc} [R' \rightrightarrows U'] & \longrightarrow & \mathcal{X} \\ \big\downarrow{\scriptstyle pr} & & \big\downarrow{\scriptstyle f} \\ [R \rightrightarrows U] & \longrightarrow & \mathcal{Y} \end{array}$$

where pr *is induced by a morphism of groupoids* $(R' \rightrightarrows U) \to (R \rightrightarrows U)$.

We can also relate locally closed, open and closed substacks of $\mathcal{X}$ to invariant subspaces a presentation groupoid $[R \rightrightarrows U]$. We follow [70, Tag 04YK].

Definition 4.2.12 ([70, Tag 03LN])**.** *Let* $R \rightrightarrows U$ *be an étale groupoid over* S.

1. *A open subset* $W \subset U$ *is* R-invariant *if* $t(s^{-1}(W)) \subset W$.
2. *A closed subscheme* $Z \subset U$ *is* R-invariant *if* $s^{-1}(Z) = t^{-1}(Z)$ *where we take the scheme theoretic inverse image.*

If W *is an* R*-invariant open subscheme of* U, *the restriction of* R *to* W *is* $R_W = s^{-1}(W) = t^{-1}(W)$. *Similarly if* Z *is an* R*-invariant open subscheme of* U, *the restriction of* R *to* Z *is* $R_Z = s^{-1}(Z) = t^{-1}(Z)$

Lemma 4.2.13. *Let* $R \rightrightarrows U$ *be an étale groupoid over* S. *Let* $i : \mathcal{Z} \to [U/R]$ *be an immersion. Then there exists an* R*-invariant locally closed subspace* $Z \subset U$ *and a* *presentation* $[R_Z \rightrightarrows Z] \to \mathcal{Z}$ *where* R_Z *is the restriction of* R *to* Z *such that*

$$\begin{tikzcd}[R_Z \rightrightarrows Z] \arrow[rr] \arrow[dr] & & \mathcal{Z} \arrow[dl] \\ & [R \rightrightarrows U] & \end{tikzcd}$$

is 2*-commutative. If* i *is a closed (resp. open) immersion then* Z *is a closed (resp. open) subspace of* U*.*

Proposition 4.2.14. *Let* $[R \rightrightarrows U]$ *be a presentation of a Deligne-Mumford stack* $\mathcal{X}$ *over* S*. Then there is a canonical bijection*

$$\text{locally closed substacks } \mathcal{Z} \text{ of } \mathcal{X} \longleftrightarrow R\text{-invariant locally closed subspace } Z \text{ of } U$$

sending $\mathcal{Z}$ *to* $\mathcal{Z} \times_{\mathcal{X}} U$*. Similarly for closed and open substacks.*

We now relate properties of the diagonal $\Delta_{\mathcal{X}} \colon \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ to $j = (s,t) \colon R \to U \times U$.

Proposition 4.2.15 ([70, Tag 0DTX])**.** *Let* $\mathcal{X}$ *be a Deligne-Mumford stack over* S *and* $R \rightrightarrows U$ *a presentation for* $\mathcal{X}$*.*

Then

1. *If* $j \colon R \to U \times U$ *is separated, then* $\Delta_{\mathcal{X}}$ *is separated.*
2. *If* U *and* R *are separated, so is* $\Delta_{\mathcal{X}}$*.*
3. *If* $j \colon R \to U \times U$ *is proper, then* $\mathcal{X}$ *is separated.*
4. *If* $s, t \colon R \to U$ *are proper and* U *is separated, then* $\mathcal{X}$ *is separated.*

Example 4.2.16. *Consider the quotient stack* $\mathcal{X} = [\mathbb{A}^1 / \mathbb{Z}_2]$ *where* $\mathbb{Z}_2$ *acts on* $\mathbb{A}^1$ *by* $z \mapsto -z$*. We claimed earlier that this stack is separated. Consider the presentation* $\mathbb{A}^1 \times \mathbb{Z}_2 \rightrightarrows \mathbb{A}^1$ *of* $\mathcal{X}$ *where* s, t *are given by* pr_1 *and the action map* $a \colon \mathbb{A}^1 \times \mathbb{Z}_2 \to \mathbb{A}^1$*.*

By Proposition 4.2.15 (3) we see that $\mathcal{X}$ *is separated as* $\mathbb{A}^1$ *is separated,* $t = a$ *is proper as* $\mathbb{Z}_2$ *acts properly, and* $s = pr_1 \colon \mathbb{A}^1 \times \mathbb{Z}_2 \to \mathbb{A}^1$ *is proper.*

4.3 Sheaves on Stacks

In this section, we first define quasi-coherent and coherent sheaves on Deligne-Mumford stacks following [77, Appendix §7.18]. For more general algebraic stacks see [58, §9]. We then define the bounded derived category of coherent sheaves on a Deligne-Mumford stack using the construction from Section 2.1.1. Finally, we define derived push forward, pullback, tensor product, and Hom functors in the context of Deligne-Mumford stacks.

4.3.1 Coherent Sheaves on a Stack

Definition 4.3.1. *Let $\mathcal{X}/S$ be a Deligne-Mumford stack over S. A* quasi-coherent *sheaf F on $\mathcal{X}$ is the following collection of data:*

1. *For each atlas $U \to \mathcal{X}$ a* quasi-coherent *sheaf F_U on U.*
2. *For each commutative diagram*

$$\begin{tikzcd} U \arrow[rr, "f"] \arrow[dr] & & V \arrow[dl] \\ & \mathcal{X} & \end{tikzcd}$$

with U, V atlases an isomorphism $\alpha_f \colon F_U \to f^*F_V$.

These isomorphisms are required to satisfy the cocycle condition.

Suppose that $\mathcal{X}$ is locally noetherian. Then a quasi-coherent sheaf F on $\mathcal{X}$ is coherent *if $\mathcal{X}$ is locally noetherian (so every atlas $U \to \mathcal{X}$ is locally noetherian) and all sheaves F_U are coherent.*

If E and F are quasi-coherent sheaves on $\mathcal{X}$, a homomorphism $\phi \colon E \to F$ is a collection of homomorphisms $\phi_U \colon E_U \to F_U$ for any atlas U which is compatible with the α_f.

We will denote the categories of quasi-coherent (resp. coherent) sheaves on $\mathcal{X}$ by $\mathrm{QCoh}(\mathcal{X})$ *(resp.* $\mathrm{Coh}(\mathcal{X})$*).*

Example 4.3.2.

1. *The structure sheaf $\mathcal{O}_{\mathcal{X}}$ is defined by $(\mathcal{O}_{\mathcal{X}})_U = \mathcal{O}_U$ for any atlas $U \to \mathcal{X}$ [70, Tag 06TU].*
2. *The sheaf of differentials $\Omega_{\mathcal{X}/S}$ is defined by $(\Omega_{\mathcal{X}/S})_U = \Omega_{U/S}$. Since the map f has to be étale, there is a natural isomorphism $\Omega_{U/S} \cong f^*\Omega_{V/S}$.*

Remark 4.3.3. *The more general theory of quasi-coherent sheaves on an algebraic stack can be developed using the lisse-étale topology on* Sch/S *but this has difficulties defining the pullback of quasi-coherent sheaves (see [58, §9.3]). For Deligne-Mumford stacks the categories of quasi-coherent sheaves with respect to the lisse-étale topology and with respect to the étale topology are equivalent, allowing us to avoid these complications.*

Definition 4.3.4 ([70, Tag 06TN] and [70, Tag 06TI])**.** *Let $\Phi \colon \mathcal{X} \to \mathcal{Y}$ be a morphism of Deligne-Mumford stacks over S and F a quasi-coherent sheaf on $\mathcal{Y}$. Define the inverse image of F along Φ, Φ^*F by*

$$(\Phi^{-1}F)_U = F_{\Phi(U)}.$$

Just as for schemes, we can define the pullback Φ^*F of F by

$$(\Phi^*F)_U = F_{\Phi(U)} \otimes_{\Phi^{-1}\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{\mathcal{X}}.$$

Let $\Phi\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of Deligne-Mumford stacks over S and F a quasi-coherent sheaf on $\mathcal{X}$. Define the push forward of F, $\Phi_* F$ by

$$(\Phi_* F)_V = \lim_{\Phi(U)\to V} F_U.$$

By construction, these give an adjoint pair

$$\mathrm{Hom}_{\mathcal{X}}(\Phi^* G, F) \cong \mathrm{Hom}_{\mathcal{Y}}(G, \Phi_* F).$$

Remark 4.3.5. *If $\Phi\colon \mathcal{X} \to \mathcal{Y}$ is a representable morphism of Deligne-Mumford stacks over S, then we can compute the push forward of a sheaf F on $\mathcal{X}$ by Φ as follows. Consider the diagram*

$$\begin{tikzcd} \mathcal{X} \times_{\mathcal{Y}} V \arrow[r] \arrow[d] & V \arrow[d] \\ \mathcal{X} \arrow[r, "\Phi"] & \mathcal{Y}. \end{tikzcd}$$

where $V \to \mathcal{Y}$ is an atlas. Then $\mathcal{X} \times_{\mathcal{Y}} V \to \mathcal{X}$ is an atlas for $\mathcal{X}$ and $\Phi_*(F)_V = \Phi_*(F_{\mathcal{X}\times_{\mathcal{Y}} V})$.

4.3.2 Coarse Moduli Space

We now define the coarse moduli space of a Deligne-Mumford stack following [58, §11].

Definition 4.3.6. *Let $\mathcal{X}/S$ be a Deligne-Mumford stack over S. A* coarse moduli space *for $\mathcal{X}$ is a scheme X over S and a morphism $\pi\colon \mathcal{X} \to X$ such that:*

(i) π is initial for maps to a scheme over S.

(ii) For every algebraically closed field k the map $|\mathcal{X}(k)| \to X(k)$ is bijective where $|\mathcal{X}(k)|$ denotes the set of isomorphism classes in $\mathcal{X}(k)$.

The following theorem of Keel and Mori guarantee the existence of coarse moduli spaces for many Deligne-Mumford stacks.

Theorem 4.3.7 ([58, Theorem 11.1.2])**.** *Assume that S is locally noetherian and $\mathcal{X}$ a Deligne-Mumford stack of finite presentation over S with finite diagonal. Then there exists a coarse moduli space $\pi\colon \mathcal{X} \to X$. In addition:*

1. *X/S is locally of finite type, and if $\mathcal{X}/S$ is separated, so if X/S.*
2. *π is proper and $\mathcal{O}_X \to \pi_* \mathcal{O}_{\mathcal{X}}$ is an isomorphism.*
3. *If $X' \to X$ is a flat morphism, then $\pi'\colon \mathcal{X}' = \mathcal{X} \times_X X' \to X'$ is a coarse moduli space.*

Example 4.3.8. *Let X be a smooth quasi-projective variety over k and G a finite group acting on X. Then the quotient stack $[X/G]$ satisfies the assumptions above and has* *a coarse moduli space X/G given locally as the invariant ring of functions (c.f. [77, Proposition 2.11]).*

We will use the following Proposition to characterize morphism between Deligne-Mumford stacks with isomorphic coarse moduli spaces

Proposition 4.3.9. *Suppose $f\colon \mathcal{X} \to \mathcal{X}'$ is a morphism of Deligne-Mumford stacks and we have a commutative diagram*

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "f"] \arrow[dr, "\pi"'] & & \mathcal{X}' \arrow[dl, "\pi'"] \\ & X. & \end{tikzcd}$$

where X is the coarse moduli space for both $\mathcal{X}$ and $\mathcal{X}'$. Then f is proper.

Proof. The Proposition follows from the maps $\mathcal{X} \to X$ and $\mathcal{X}' \to X$ being proper and Proposition 4.1.50 as $\pi = \pi' \circ f$. □

We will use the following concerning the push forward of quasi-coherent sheaves from a tame Deligne-Mumford stack to its coarse moduli space.

Proposition 4.3.10 ([58, Proposition 11.3.4])**.** *Let $\mathcal{X}/S$ be a Deligne-Mumford stack locally of finite presentation over a locally noetherian scheme S with finite diagonal. Let $\pi\colon \mathcal{X} \to X$ be its coarse moduli space. If $\mathcal{X}$ is tame, then the functor*

$$\pi_*\colon \operatorname{QCoh}(\mathcal{X}) \to \operatorname{QCoh}(X)$$

is exact.

4.3.3 Effective Cartier Divisors

Just as for schemes we have a bijection between closed subschemes Z and ideal sheaves $\mathcal{I}_Z$ we have a similar bijection for closed substacks [50, Application 14.2.7]. We will denote by $\mathcal{I}_{\mathcal{Z}}$ the quasi-coherent ideal sheaf associated to a closed substack $\mathcal{Z}$.

Definition 4.3.11. *Let $\mathcal{X}$ be a Deligne-Mumford stack over S. An effective Cartier divisor on $\mathcal{X}$ is a closed substack $\mathcal{D} \subset \mathcal{X}$ whose ideal sheaf $\mathcal{I}_{\mathcal{D}}$ is a line bundle.*

Example 4.3.12. *Let $G = \mathbb{Z}/2\mathbb{Z}$ act on $\mathbb{C}^2$ by the matrix*

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Consider the quotient stack $[\mathbb{C}^2/G]$ Then the closed substack $\mathcal{D} = [D/G] \subset [\mathbb{C}^2/G]$ where $D = V(x) \subset \mathbb{C}^2$ is an effective Cartier divisor.

4.3.4 Coherent Sheaves on a Groupoid

We now relate sheaves on a Deligne-Mumford stack $\mathcal{X}$ to sheaves on a presentation $R \rightrightarrows U$. See [70, Tag 03LH] for more details.

Definition 4.3.13. *A* quasi-coherent sheaf *on an algebraic groupoid $R \rightrightarrows U$ is a quasi-coherent sheaf F on U with respect to the étale topology together with an isomorphism $\alpha\colon s^*F \cong t^*U$ which satisfies a cocycle condition given by associativity of the groupoid multiplication and that $\epsilon^*\alpha = id$. A quasi-coherent sheaf F on $R \rightrightarrows U$ is coherent if U is locally noetherian and F on U is coherent.*

A morphism of quasi-coherent sheaves $\phi\colon (F,\alpha) \to (G,\beta)$ of sheaves on $R \rightrightarrows U$ is a morphism of $\mathcal{O}_U$-modules $\phi\colon F \to G$ such that

$$\begin{array}{ccc} s^*F & \xrightarrow{\alpha} & t^*F \\ {\scriptstyle s^*\phi}\downarrow & & \downarrow{\scriptstyle t^*\phi} \\ s^*G & \xrightarrow{\beta} & t^*G \end{array}$$

commutes.

We will denote the category of quasi-coherent (resp. coherent) sheaves on $R \rightrightarrows U$ by $\mathrm{QCoh}(R \rightrightarrows U)$ *(resp.* $\mathrm{Coh}(R \rightrightarrows U)$*).*

Proposition 4.3.14 ([70, Tag 06WT])**.** *Let $R \rightrightarrows U$ be an étale groupoid over S and $\mathcal{X} = [R \rightrightarrows U]$ the associated algebraic stack. Then the category of quasi-coherent sheaves on $\mathcal{X}$ is equivalent to the category of quasi-coherent sheaves on the étale groupoid $R \rightrightarrows U$.*

Proof. Recall that an object $x = (T,u)$ is a scheme T and a map $u\colon T \to U$. A morphism $(T,u) \to (T',u')$ is given by a pair (f,r) where $f\colon T \to T'$ such that $u' \circ f = u$ and $r\colon T \to R$ such that $s \circ r = u$ and $t \circ r = u' \circ f$.

Let F be a quasi-coherent sheaf on $\mathcal{X}$. Then we obtain for every atlas $u\colon T \to \mathcal{X} \in [R(T) \rightrightarrows U(T)]$ a quasi-coherent sheaf $u^*F = F_T$ on T. Moreover, for any morphism $f\colon (T,u) \to (T',u')$ of atlases we have an isomorphism

$$\alpha_f\colon f^*F_{T'} \to F_T.$$

These isomorphisms are compatible with compositions. We construct a quasi-coherent sheaf on $R \rightrightarrows U$ in the following way: First the object $(U,id) \in [R \rightrightarrows U](U)$ corresponds to the quasi-coherent sheaf $F_{U,id}$ on U.

Recall that as $s,t\colon R \to U$ are surjective étale maps as they admit a section ϵ. Hence we have sheaves $F_{(R,s)}$ and $F_{(R,t)}$ on R corresponding to the elements $s,t\colon R \to U \in [R \rightrightarrows U](R)$.

The isomorphism $\alpha\colon t^*F_U \cong s^*F_U$ is obtained in the following way:

1. First, the element id_R gives an isomorphism between (R,s) and (R,t) in $\mathcal{X}(R)$ and so an isomorphism of sheaves $F_{(R,s)} \cong F_{(R,t)}$.

2. The morphism $(R, s) \to (U, id)$ gives an isomorphism $s^*F_{(U,id)} \cong F_{(R,s)}$.
3. The morphism $(R, t) \to (U, id)$ gives an isomorphism $t^*F_{(U,id)} \cong F_{R,t}$.

By composing these we obtain the necessary isomorphism α. This isomorphism satisfies the cocycle relation as the multiplication on $R \rightrightarrows U$ is associative.

Conversely, suppose that (F, α) is a quasi-coherent sheaf on $R \rightrightarrows U$. Then we define a presheaf $F_{\mathcal{X}}$ of $\mathcal{O}_{\mathcal{X}}$-modules on $\mathcal{X}$ by for any atlas $u\colon T \to \mathcal{X} \in [R \rightrightarrows U](T)$

$$F_{\mathcal{X}}(T, u) = \Gamma(T, u^*F).$$

Given a morphism $(f, r)\colon (T, u) \to (T', u')$ there is a map

$$\begin{aligned} F(T', u') =& \Gamma(T', (u')^*F) \\ =& \Gamma(T, f^*(u')^*F) = \Gamma(T, (u' \circ f)^*F) \\ =& \Gamma(T, (t \circ r)^*F) = \Gamma(T, r^*t^*F) \\ \cong& \Gamma(T, r^*s^*F) = \Gamma(T, (s \circ r)^*F) \\ =& \Gamma(T, u^*F) \\ =& F(T, u). \end{aligned}$$

The cocycle condition guarantees that this defined a presheaf of modules. Pulling $F_{\mathcal{X}}$ back to Sch/T shows that $F_{\mathcal{X}}$ is quasi-coherent. □

Example 4.3.15. *Let $\mathcal{X} = [X/G]$ where G is a finite group acting on a locally noetherian scheme X. Then $X \times G \rightrightarrows X$ is a presentation for $[X/G]$. Thus quasi-coherent sheaves on $[X/G]$ correspond to pairs (E, α) on X where $\alpha\colon pr_1^*E \to a^*E$ is an isomorphism which satisfies a cocycle condition. This is by definition a G-equivariant quasi-coherent sheaf on X. Thus we have equivalences of categories*

$$\mathrm{QCoh}([X/G]) \cong \mathrm{QCoh}^G(X), \quad \mathrm{Coh}([X/G]) \cong \mathrm{Coh}^G(X)$$

between (quasi-)coherent sheaves on X and G-equivariant sheaves on X.

Given a morphism of groupoids, we get two functors relating sheaves between the two groupoids.

Proposition 4.3.16. *Let $\Phi\colon (R \rightrightarrows U) \to (R' \rightrightarrows U)$ be a morphism of algebraic groupoids. Then Φ defines a functor*

$$\Phi^*\colon \mathrm{QCoh}(R' \rightrightarrows U') \to \mathrm{QCoh}(R \rightrightarrows U)$$

by $\Phi^*\colon (F', \alpha') \mapsto (\Phi^*F', \Phi^*\alpha')$.

Proposition 4.3.17. *Let $\Phi\colon (R \rightrightarrows U) \to (R' \rightrightarrows U)$ be a morphism of algebraic groupoids. Suppose that*

1. *$\Phi\colon U \to U'$ is quasi-compact and quasi-separated*

2. *The square*

$$\begin{array}{ccc} R & \xrightarrow{\Phi} & R' \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t'} \\ U & \xrightarrow{\Phi} & U' \end{array}$$

   commutes, and

3. *The morphisms s and t are flat.*

Then Φ gives a functor

$$\Phi_*\colon \ \mathrm{QCoh}(R \rightrightarrows U) \to \mathrm{QCoh}(R' \rightrightarrows U')$$

defined by $\Phi_*\colon (F, \alpha) \to (\Phi_* F, \Phi_* \alpha)$.

4.4 Derived Category of a Stack

We now construct the derived category (quasi)-coherent sheaves on a Deligne-Mumford stack and explain how to derive the usual functors between categories of (quasi)coherent sheaves.

Proposition 4.4.1 ([70, Tag 06WU])**.** *Let $\mathcal{X}$ be an Deligne-Mumford stack over S. Then the category* $\mathrm{QCoh}(\mathcal{X})$ *is abelian. Moreover, if $\mathcal{X}$ is locally noetherian then* $\mathrm{Coh}(\mathcal{X})$ *is an abelian subcategory of* $\mathrm{QCoh}(\mathcal{X})$.

Proof. By Proposition 4.3.14 we have an equivalence $\mathrm{QCoh}(\mathcal{X}) \cong \mathrm{QCoh}([R \rightrightarrows U])$. Thus it suffices to show that $\mathrm{QCoh}([R \rightrightarrows U]$ is abelian. This follows from [70, Tag 06VZ] and we sketch the argument below.

Recall that $R \rightrightarrows U$ is an étale groupoid, so s and t are both flat. Let $\phi\colon (F, \alpha) \to (G, \beta)$ be a morphism of quasi-coherent sheaves on $R \rightrightarrows U$. As s is flat the sequence

$$0 \longrightarrow s^* \ker \phi \longrightarrow s^* F \xrightarrow{s^*\phi} s^* G \longrightarrow s^* \operatorname{coker} \phi \longrightarrow 0$$

is exact. Moreover, we have a similar exact sequence for t^*. Then the isomorphisms $s^*\alpha$ and $s^*\beta$ induce isomorphisms $\kappa\colon s^* ker\phi \to t^* \ker \phi$ and $\lambda\colon s^* \operatorname{coker} \phi \to t^* \operatorname{coker} \phi$. The result then follows from showing $(\ker \phi, \kappa)$ and $(\operatorname{coker} \phi, \lambda)$ satisfy the universal property for kernels and cokernels using that $\mathrm{QCoh}(U)$ is abelian.

Suppose $\mathcal{X}$ is locally noetherian. Then U and R are also locally noetherian. Then s and t preserve coherent sheaves. Then $\mathrm{Coh}(R \rightrightarrows U)$ is an abelian subcategory we use the fact that s^* and t^* preserve coherent sheaves. ☐

Remark 4.4.2. *The above proposition allows us to apply the machinery in Section 2.1.1 to construct the following derived categories $D^*(\mathrm{QCoh}(\mathcal{X}))$ and $D^*(\mathrm{Coh}(\mathcal{X}))$ for $* = +, -, b$. Following Section 2.1.1 we will write $D(\mathcal{X}) = D^b(\mathrm{Coh}(\mathcal{X}))$.*

The following result generalizes a well-known result for noetherian schemes [38, Proposition 3.5] to noetherian Deligne-Mumford stacks.

Proposition 4.4.3 ([9, Proposition A.1])**.** *Let $\mathcal{X}$ be a noetherian Deligne-Mumford stack. Then the obvious functor defines an equivalence*

$$D^-(\mathrm{Coh}(\mathcal{X})) \cong D^-_{\mathrm{Coh}}(\mathrm{QCoh}(\mathcal{X})).$$

4.4.1 Derived Functors and Stacks

In this section we derive several of the common functors including $- \otimes -$ and for a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ the functors f^* and f_*, echoing Section 2.2.1 which treated the case of schemes.

Derived Tensor Product and Pullback

First, we treat the case of left derived functors for tensor product and pullback along a morphism of Deligne-Mumford stacks. On the level of bounded above complexes of (quasi)-coherent sheaves, we can derive the tensor product bi-functor.

Proposition 4.4.4 ([50, 13.2.6(i) and 15.6(i) and (ii)])**.** *Let $\mathcal{X}$ be a Deligne-Mumford stack over S.*

1. *If E and F are quasi-coherent sheaves on $\mathcal{X}$, then $E \otimes_{\mathcal{O}_\mathcal{X}} F$ is also a quasi-coherent sheaf. More generally, the functor $- \otimes^L_{\mathcal{O}_\mathcal{X}} -$ induces a functor*

$$- \otimes^L_{\mathcal{O}_\mathcal{X}} - \colon D^-(\mathrm{QCoh}(\mathcal{X})) \times D^-(\mathrm{QCoh}(\mathcal{X})) \to D^-(\mathrm{QCoh}(\mathcal{X})).$$

Moreover, if $\mathcal{X}$ is locally noetherian, then $- \otimes_{\mathcal{O}_\mathcal{X}} -$ induces a functor

$$- \otimes^L_{\mathcal{O}_\mathcal{X}} - \colon D^-(\mathrm{Coh}(\mathcal{X})) \times D^-(\mathrm{Coh}(\mathcal{X})) \to D^-(\mathrm{Coh}(\mathcal{X})).$$

For a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ we have a similar result on the level of bounded above complexes

Proposition 4.4.5 ([70, Tag 07BD])**.** *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of Deligne-Mumford stacks. Then the functor f^* induces a left derived functor*

$$Lf^*\colon D^-(\mathrm{QCoh}(\mathcal{Y})) \to D^-(\mathrm{QCoh}(\mathcal{X})).$$

To descend these functors to the bounded derived category we will need the language of perfect complexes.

Perfect Objects and the Bounded Derived Category

We refer to [70, Tag 08CL] and [9, Appendix A] for the following results.

Definition 4.4.6. *Let X be a scheme over S. An object $E \in D(\mathrm{QCoh}(X)$ is* perfect *if it is locally (in the étale topology) quasi-isomorphic to a bounded complex of finite free $\mathcal{O}_X$-modules.*

Let $\mathcal{X}$ be a Deligne-Mumford stack over S. An object $E \in D(\mathrm{QCoh}(\mathcal{X}))$ is perfect *if for any atlas $U \to \mathcal{X}$, E_U is perfect.*

We will denote the triangulated subcategory of $D(\mathrm{QCoh}(\mathcal{X}))$ of perfect objects in $D(\mathrm{QCoh}(\mathcal{X}))$ by $D_{pf}(\mathcal{X})$.

It is useful to talk about perfect complexes when considering the functors $- \otimes -$ and f^* for a morphism $f \colon \mathcal{X} \to \mathcal{Y}$ of stacks due to following results

Proposition 4.4.7 ([70, Tag 08CL])**.**

1. *Let $E^\bullet, F^\bullet \in D_{pf}(\mathcal{X})$. Then $E^\bullet \otimes^L F^\bullet \in D_{pf}(\mathcal{X})$. Thus $- \otimes -$ descends to a bi-functor*

$$- \otimes^L - \colon D_{pf}(\mathcal{X}) \times D_{pf}(\mathcal{X}) \to D_{pf}(\mathcal{X}).$$

2. *Let $f \colon \mathcal{X} \to \mathcal{Y}$ be a morphism of Deligne-Mumford stacks. Then if $E^\bullet \in D_{pf}(\mathcal{Y})$, $Lf^*(E^\bullet) \in D_{pf}(\mathcal{X})$. Thus we have a functors*

$$Lf^* \colon D_{pf}(\mathcal{Y}) \to D_{pf}(\mathcal{X}).$$

The category of perfect complexes is useful as it gives a way to descend to the bounded derived category using the following result

Proposition 4.4.8 ([9, Proposition A.2])**.** *Let $\mathcal{X}$ be a regular and quasi-compact. Then we have an equality*

$$D_{pf}(\mathcal{X}) = D^b(\mathrm{Coh}(\mathcal{X})).$$

Combining the previous two Propositions we have the following

Corollary 4.4.9. *Let $\mathcal{X}$ be a regular, noetherian Deligne-Mumford stack over S. Then there exists a derived bi-functor*

$$- \otimes^L - \colon D^b(\mathrm{Coh}(\mathcal{X})) \times D^b(\mathrm{Coh}(\mathcal{X})) \to D^b(\mathrm{Coh}(\mathcal{X}))$$

Let $f \colon \mathcal{X} \to \mathcal{Y}$ be a morphism of regular, noetherian Deligne-Mumford stacks over a scheme S. Then there exists a derived functor

$$Lf^* \colon D^b(\mathrm{Coh}(\mathcal{Y})) \to D^b(\mathrm{Coh}(\mathcal{X})).$$

Often the map $f \colon \mathcal{X} \to \mathcal{Y}$ will be flat. The the following result means that, as for schemes, we will not have to derive f^*.

Lemma 4.4.10 ([70, Tag 076W]). *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a flat morphism of Deligne-Mumford stacks. Then*

$$f^*\colon \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{X})$$

is exact.

Derived Hom Functor

As $\mathrm{QCoh}(\mathcal{X})$ contains enough injective we can derive $\mathcal{H}om_{\mathcal{X}}(F, -)$.

Proposition 4.4.11 ([57, Proposition 6.4]). *Let $\mathcal{X}$ be a locally noetherian Deligne-Mumford stack. Then the functor $R\mathcal{H}om_{\mathcal{X}}(-,-)$ induces functors*

$$R\mathcal{H}om_{\mathcal{X}}(-,-)\colon D^-(\mathrm{Coh}(\mathcal{X})) \times D^+(\mathrm{QCoh}(\mathcal{X})) \to D^+(\mathrm{QCoh}(\mathcal{X}))$$
$$R\mathcal{H}om_{\mathcal{X}}(-,-)\colon D^-(\mathrm{Coh}(\mathcal{X})) \times D^+(\mathrm{Coh}(\mathcal{X})) \to D^+(\mathrm{Coh}(\mathcal{X}))$$

To descend to the bounded level we have to assume that $\mathcal{X}$ is regular, just as for schemes.

Derived Pushforward Functor

In Section 2.2.1 we used for a scheme X that $\mathrm{QCoh}(X)$ has enough injectives. For a Deligne-Mumford stack $\mathcal{X}$ we also have that $\mathrm{QCoh}(\mathcal{X})$ enough injectives.

Proposition 4.4.12 ([70, Tag 06WU]). *Let $\mathcal{X}$ be a Deligne-Mumford stack over S. Then the category $\mathrm{QCoh}(\mathcal{X})$ has enough injectives.*

Thus on the level of $\mathrm{QCoh}(\mathcal{X})$ we can derive f_* assuming f is a quasi-compact morphism of quasi-compact quasi-separated Deligne-Mumford stacks.

Lemma 4.4.13 ([57, Lemma 6.5]). *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a quasi-compact morphism of quasi-compact quasi-separated Deligne-Mumford stacks. Then for any any quasi-coherent sheaf E on $\mathcal{X}$, the sheaf f_*E is a quasi-coherent sheaf on $\mathcal{Y}$.*

As $\mathrm{QCoh}(X)$ has enough injectives, by Section 2.1.3 there exists a derived functor

$$Rf_*\colon D^+(\mathrm{QCoh}(X)) \to D^+ \mathrm{QCoh}(Y).$$

Similarly, as for schemes, we have the following Theorem for on the level of coherent sheaves.

Theorem 4.4.14 ([57, Theorem 10.13]). *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a proper morphism between locally noetherian Deligne-Mumford stacks. Then for any coherent sheaf E on $\mathcal{X}$ and $i \geq 0$, the sheaves $R^i f_* E$ are coherent on $\mathcal{Y}$. More generally, we have a functor*

$$Rf_*\colon D^+(\mathrm{Coh}(\mathcal{X})) \to D^+(\mathrm{Coh}(\mathcal{Y})).$$

To descend to the bounded derived category we use the following theorem.

Theorem 4.4.15 ([58, Theorem 11.6.5])**.** *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a proper morphism of finite type Deligne-Mumford stacks over S and assume that S is quasi-compact. For a geometric point $\bar{x} \to \mathcal{X}$, let $G_{\bar{x}}$(resp. $H_{f(\bar{x})}$) denote the stabilizer group of $\bar{x}$ (resp. $f(\bar{x})$), and let $K_{\bar{x}}$ denote the kernel of the natural map $G_{\bar{x}} \to H_{f(\bar{x})}$. If for every geometric point $\bar{x}$ the order of the group $K_{\bar{x}}$ is invertible in the field $k(\bar{x})$, then there exists an integer n_0 such that for any quasi-coherent sheaf E on $\mathcal{X}$ we have $R^q f_* E = 0$ for $q > n_0$.*

Corollary 4.4.16. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a proper morphism of locally noetherian tame Deligne-Mumford stacks over a quasi-compact scheme S. Then Rf_* induces a functor*

$$Rf_*\colon D^b(\mathrm{Coh}(\mathcal{X})) \to D^b(\mathrm{Coh}(\mathcal{Y})).$$

Proof. This follows from apply Proposition 2.1.15 using Theorem 4.4.15. □

Remark 4.4.17. *Just as for schemes, the projection formula holds for Deligne-Mumford stacks as it holds for perfect objects [70, Tag 0943].*

Duality for Stacks

We now explain when Grothendieck-Verdier Duality lifts to stacks. For schemes Grothendieck Verdier Duality centers around constructing a right adjoint to f_*.

Theorem 4.4.18 ([56, Theorem 1.16])**.** *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a separated quasi-compact morphism of Deligne-Mumford stacks. Then the functor $Rf_*\colon D^+(\mathcal{X}) \to D^+(\mathcal{Y})$ has a right adjoint $f^!\colon D^+(\mathcal{Y}) \to D^+(\mathcal{X})$.*

For proper morphisms, we have the following description

Proposition 4.4.19 ([56, Corollary 2.10])**.** *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a proper morphism of Deligne-Mumford stacks and $F^\bullet \in D^+_c(\mathcal{X}), G^\bullet \in D^+(\mathcal{Y})$. Then the natural morphism*

$$Rf_* R\mathcal{H}om_{\mathcal{X}}(F^\bullet, f^! G^\bullet) \to R\mathcal{H}om_{\mathcal{Y}}(Rf_* F^\bullet, Rf_* f^! G^\bullet) \to R\mathcal{H}om_{\mathcal{Y}}(Rf_* F^\bullet, G^\bullet)$$

is an isomorphism.

For smooth proper Deligne-Mumford stacks we have Serre Duality just as for schemes.

Theorem 4.4.20. *Let $\mathcal{X}$ be a smooth proper Deligne-Mumford stack over k of dimension n. Then $\omega_{\mathcal{X}}[n]$ is a dualizing complex for $\mathcal{X}$. Hence $S_{\mathcal{X}} = - \otimes^L \omega_{\mathcal{X}}[n]$ is a Serre functor for $\mathcal{X}$.*

Chapter 5

Semi-orthogonal Decompositions for Deligne-Mumford Stacks

In this chapter, we recall the main tools used to construct semi-orthogonal decompositions of the bounded derived category of coherent sheaves on smooth separated tame Deligne-Mumford stacks over a field k.

In Sections 5.1 and 5.2 we describe the main constructions used to understand the geometry of these stacks: canonical stacks and root stacks. We then in Section 5.3 describe the geometry of these stacks using these constructions following [31].

In Section 5.4 we recall semi-orthogonal decompositions of root stacks and iterated root stacks constructed by Ishii and Ueda [40]. We then apply these results to describe semi-orthogonal decompositions of Deligne-Mumford quotient stacks. As far as the author knows, this perspective is new and not in the literature.

In Section 5.5 we prove that for any smooth separated tame Deligne-Mumford stack $\mathcal{X}$ over a field k with trivial generic stabilizer and coarse moduli space X, the derived category $D(X^{can})$ of the canonical stack X^{can} embeds fully faithfully into $D(\mathcal{X})$. Again, this result appears to be new.

The article [73] follows a similar approach from the perspective of Gromov-Witten theory which may be of interest to the reader.

Notation and Conventions

Throughout this chapter, a Deligne-Mumford stack will be a quasi-separated quasi-compact Deligne-Mumford stack of finite type over a field k, i.e $S = \operatorname{Spec} k$. We do not impose any additional assumptions on k. Throughout we will write X both the scheme X and the stack $\underline{X}$ associated to X.

We say a morphism between stacks is unique if it is unique up a unique 2-arrow. We denote by $\mathbb{G}_{\mathrm{m}}$ the sheaf of invertible sections in $\mathcal{O}_{\mathcal{X}}$.

5.1 Canonical Stacks

Canonical stacks were first studied by Vistoli in [77], as a way of associating a smooth stack to a scheme of finite type over k with tame quotient singularities (étale locally the quotient of a smooth variety by a finite group whose order is prime to the characteristic of k). In particular, the canonical stack is the first example of a *stacky resolution* of singularities. We hope to study schemes with tame quotient singularities by studying the associated canonical stack.

The notion of a smooth Deligne-Mumford stack being canonical corresponds to the subset of "stacky point" having codimension at least 2. We follow [26, §4]. Other references are [77] and [31].

Definition 5.1.1. *Let $\mathcal{X}$ be a smooth Deligne-Mumford stack with coarse moduli space X. We call $\mathcal{X}$* canonical *if the locus where the map $\pi\colon \mathcal{X} \to X$ is not an isomorphism has codimension ≥ 2 in X.*

Example 5.1.2. *Let $G \subset SL(n, \mathbb{C})$ be a finite subgroup. Then the quotient stack $[\mathbb{C}^n/G]$ is canonical. This follows from $\mathrm{Fix}(G) = \{0\} \in \mathbb{C}^n$. More generally, if $G \subset \mathrm{GL}(n, \mathbb{C})$ is* small *(contains no psuedoreflections) then $[\mathbb{C}^n/G]$ is canonical for similar reasons.*

We now recall some well known facts about canonical Deligne-Mumford stacks from [26, §4]

Remark 5.1.3. *Let $\mathcal{X}$ be a smooth canonical Deligne-Mumford stack with coarse moduli space X.*

- *The locus where the coarse map $\pi\colon \mathcal{X} \to X$ is an isomorphism is precisely $\pi^{-1}(X_{sm})$, where X_{sm} is the smooth locus of X.*
- *If X is smooth, π is an isomorphism and $\mathcal{X} \cong X$.*

Definition 5.1.4.

1. *A dominant morphism $f\colon V \to W$ of irreducible varieties is called* codimension preserving *if $\mathrm{codim}_V Z_V = \mathrm{codim}_W Z$ for any irreducible closed subset $Z \subset W$ and every irreducible component Z_V of $f^{-1}(Z)$.*
2. *A dominant morphism of Deligne-Mumford stacks with trivial generic stabilizers is* codimension preserving *if the induced map on every irreducible component of the coarse moduli space is codimension preserving.*

Remark 5.1.5. *Note that coarse moduli space map $\mathcal{X} \to X$ is codimension preserving because the induced map is the identity. Moreover, any flat morphism (therefore any étale and smooth) morphism is codimension preserving. A composite of codimension preserving maps is codimension preserving. Note that blowing up is* not *codimension preserving.*

We will now characterize canonical stacks using a universal property.

Theorem 5.1.6 ([26, Theorem 4.6]). *Let $\mathcal{X}$ be a canonical smooth Deligne-Mumford stack, $\pi\colon \mathcal{X} \to X$ the morphism to the coarse moduli space and $g\colon \mathcal{Y} \to X$ a dominant codimension preserving morphism with $\mathcal{Y}$ a smooth Deligne-Mumford stack with trivial generic stabilizer. Then there exists a unique morphism $f\colon \mathcal{Y} \to \mathcal{X}$ such that the diagram*

$$\begin{tikzcd} \mathcal{Y} \arrow[r, dotted, "\exists! f"] \arrow[dr, "g"'] & \mathcal{X} \arrow[d, "\epsilon"] \\ & X \end{tikzcd}$$

commutes.

The following corollary asserts the uniqueness of a canonical stack for the coarse moduli space X. This allows us to talk about *the* canonical stack with coarse moduli space X.

Corollary 5.1.7 ([26, Corollary 4.8]). *Let $\mathcal{X}, \mathcal{Y}$ be a canonical smooth Deligne-Mumford stacks with coarse moduli spaces X, Y respectively. Let $\bar{f}\colon X \to Y$ be an isomorphism. Then there is a unique isomorphism $f\colon \mathcal{X} \to \mathcal{Y}$ inducing $\bar{f}$.*

We now describe the unique canonical Deligne-Mumford stack with trivial generic stabilizer associated with a variety over a field k with (tame) quotient singularities. Recall that a variety X over a field k is said to have tame quotient singularities if it is étale locally the quotient of a smooth variety by a finite group whose order is prime to the characteristic of k.

Theorem 5.1.8 ([77, Proposition 2.8] and [26, Corollary 4.9]). *Let X be a variety over a field k with tame quotient singularities. Then there exists a smooth canonical Deligne-Mumford stack X^{can} over k with coarse moduli space X. Moreover, X^{can} is universal in the following way. Given any other smooth Deligne-Mumford stack $\mathcal{X}$ with coarse moduli space X there is a unique morphism $f\colon \mathcal{X} \to X^{can}$ making the following diagram commute*

$$\begin{tikzcd} \mathcal{X} \arrow[r, "f"] \arrow[dr, "\pi"'] & X^{can} \arrow[d, "\epsilon"] \\ & X. \end{tikzcd}$$

Proof. Note that by Theorem 5.1.6 the canonical stack has the required universal property and is unique up to unique isomorphism.

Now we construct X^{can}. Let $x \in X$ be a closed point. Then there is a smooth scheme V and a finite group G acting faithfully on V, with an étale morphism $V/G \to X$ whose image contains x. Let v be the inverse image of x. If G_v is the stabilizer of G at v, the morphism $V/G_v \to X$ is étale at v. By restricting V, we can assume that v is a fixed point of G. An element of G will be called a *pseudo-reflection at v* if it acts trivially on a divisor of V passing through v. By the Chevalley-Shephard-Todd Theorem [69], a subgroup $H \subset G$ is generated by psuedoreflections at v if and only if the quotient V/H

is smooth. By quotienting by the (normal) subgroup generated by psuedoreflections at v and restricting V we can assume the set of fixed points of any element of G lie in codimension at least 2. Thus the morphism $V \to X$ is étale in codimension 1. Thus there exists a finite set of schemes V_α and morphisms $V_\alpha \to X$ such that:

1. The V_α's are smooth,
2. The morphisms $V_\alpha \to X$ are étale in codimension 1,
3. For each α, there is a finite group G_α acting on V_α in such a way that $V_\alpha \to X$ is the composite of the projections $V_\alpha \to V_\alpha/G_\alpha$ with an étale morphism $V_\alpha/G_\alpha \to X$,
4. The union of the images of the V_α's cover X.

Denote by $V_{\alpha\beta}$ the normalization of $V_\alpha \times_X V_\beta$. Then the two projections from $V_{\alpha\beta}$ to V_α and V_β are étale in codimension 1. As V_α is smooth, the only ramification of the map $V_{\alpha\beta} \to V_\alpha$ is in codimension 1 by Zariski's Theorem on the purity of the branch locus [80]. As the maps $V_{\alpha\beta} \to V_\alpha$ are étale in codimension 1 $V_{\alpha\beta}$ are all smooth and the projections are étale. Thus we can form the étale algebraic groupoid

$$\coprod_{\alpha,\beta} V_{\alpha\beta} \rightrightarrows \coprod_{\alpha} V_\alpha.$$

The canonical stack X^{can} is the stackification of the fibred category associated to the above groupoid with atlas $\coprod_\alpha V_\alpha$. By construction it follows from [33, Proposition 9.2] that X is the coarse moduli space for X^{can}. $\square$

Example 5.1.9. *Let* $G = \frac{1}{4}(1,2)$ *be the cyclic group of order* 4 *acting on* $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec}\mathbb{C}[x,y]$. *The image of this group in* $\mathrm{GL}(2,\mathbb{C})$ *is generated by the matrix*

$$\begin{pmatrix} i & 0 \\ 0 & -1 \end{pmatrix}.$$

One can then compute the quotient $X = \mathbb{A}^2_{\mathbb{C}}/G$ *as*

$$X = \operatorname{Spec}\mathbb{C}[x,y]^G \cong \operatorname{Spec}\mathbb{C}[x^4, y^2, x^2y] \cong \operatorname{Spec}[u,v,w]/(uv - w^2)$$

which is the cone in $\mathbb{A}^3_{\mathbb{C}}$ *cut out by the equation* $uv - w^2$. *It is easy to compute that* $X \cong \mathbb{A}^2_{\mathbb{C}}/\mu_2$ *where* μ_2 *acts by the matrix*

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

on $\mathbb{A}^2_{\mathbb{C}}$. *Thus* $X^{can} \cong [\mathbb{A}^2_{\mathbb{C}}/\mu_2]$. *The map* $[\mathbb{A}^2_{\mathbb{C}}/G] \to [\mathbb{A}^2_{\mathbb{C}}/\mu_2]$ *is given by the quotient map* $G \to G/H \cong \mu_2$ *where* $H = < g^2 >$ *for a generator* g *of* G.

Example 5.1.10. *Let* G *acting effectively on* $\mathbb{A}^n_k$ *be generated by psuedoreflections.*

Then by Chevalley-Shepard-Todd [69] the quotient $\mathbb{A}^n_k/G \cong \mathbb{A}^n_k$ *is smooth. Thus*

$$(\mathbb{A}^n_k/G)^{can} \cong \mathbb{A}^n_k/G \cong \mathbb{A}^n_k.$$

Remark 5.1.11. *Note that by Proposition 4.1.50 the canonical morphism* $f\colon \mathcal{X} \to X^{can}$ *is proper.*

5.2 Root Stacks

The birational geometry of singular varieties often requires the treatment of $\mathbb{Q}$-Cartier divisors. This is equivalent to taking roots of line bundles. Whilst for schemes this is problematic, this can be achieved in the world of stacks using roots stacks. Root stacks were first constructed by Cadman in [22] and independently by Abramovich, Graber and Vistoli [1]. In this section, we define the notion of a root stack in several contexts: root stack of a line bundle, root stack of a line bundle with a section, and the iterated root stack.

5.2.1 The Root Stack of a Line Bundle

Let $\mathcal{X}$ be a Deligne-Mumford stack and $\mathcal{L}$ a line bundle on $\mathcal{X}$. We use the same notation as in [40, §5]. Let $r \in \mathbb{Z}$ be a postive integer. The the r-th root stack of $\mathcal{L}$, denoted $\sqrt[r]{\mathcal{L}/\mathcal{X}}$, is the fibred product

$$\begin{tikzcd} \sqrt[r]{\mathcal{L}/\mathcal{X}} \arrow[r] \arrow[d, "\pi"] & B\,\mathbb{G}_{\mathrm{m}} \arrow[d, "\tau_r"] \\ \mathcal{X} \arrow[r, "\mathcal{L}"] & B\,\mathbb{G}_{\mathrm{m}} \end{tikzcd}$$

where the morphism $\tau_r\colon B\,\mathbb{G}_{\mathrm{m}} \to B\,\mathbb{G}_{\mathrm{m}}$ is induced by the power map on $\mathbb{G}_{\mathrm{m}}$.

Explicitly, the objects over a scheme T is a triple (φ, M, ϕ) consisting of a morphism $\varphi\colon \underline{T} \to \mathcal{X}$ of stacks, a line bundle M on T and an isomorphism $\phi\colon M^{\otimes r} \cong \varphi^*\mathcal{L}$ of line bundles on T.

We will denote by $(\mathcal{M}, \Phi)$ the universal object on $\sqrt[r]{\mathcal{L}/\mathcal{X}}$ where $\mathcal{M}$ is a line bundle on $\sqrt[r]{\mathcal{L}/\mathcal{X}}$ and $\Phi\colon \mathcal{M}^{\otimes r} \cong \pi^*\mathcal{L}$.

5.2.2 The Root stack of a Line Bundle with a Section

In [22] the author defines the notion of a root of a line bundle and a global section. Let $(\mathcal{L}, s)$ be the pair of a line bundle $\mathcal{L}$ on $\mathcal{X}$ and a global section $s \in \Gamma(\mathcal{X}, \mathcal{L})$. Then we can form the root stack of $(\mathcal{L}, s)$ in the following way. Recall that $[\mathbb{A}^1_k/\mathbb{G}_{\mathrm{m}}]$ is the category of line bundles with a section.

Lemma 5.2.1 ([22, Lemma 2.1.1]). *Let $\mathcal{X}$ be a Deligne-Mumford stack. Then their is an equivalence of categories between the category of morphisms $\mathcal{X} \to [\mathbb{A}^1_k \,/\, \mathbb{G}_\mathrm{m}]$ and the category whose objects are pairs $(\mathcal{L}, s)$ where $\mathcal{L}$ is a line bundle on $\mathcal{X}$ and $s \in \Gamma(\mathcal{X}, \mathcal{L})$ and whose morphisms*

$$(\mathcal{L}, s) \to (\mathcal{L}', s')$$

are isomorphisms $\varphi\colon \mathcal{L} \to \mathcal{L}'$ such that $\varphi(s) = t$.

Denote by $\theta_r\colon [\mathbb{A}^1_k \,/\, \mathbb{G}_\mathrm{m}] \to [\mathbb{A}^1_k \,/\, \mathbb{G}_\mathrm{m}]$ the morphism induced by the power maps on $\mathbb{A}^1_k$ and $\mathbb{G}_\mathrm{m}$.

Definition 5.2.2. *Let $\mathcal{X}$ be a Deligne-Mumford stack, r be a positive integer and $(\mathcal{L}, s)$ a pair consisting of a line bundle $\mathcal{L}$ on $\mathcal{X}$ and a global section $s \in \Gamma(\mathcal{X}, \mathcal{L})$. Then define the r^{th} root stack $\sqrt[r]{(\mathcal{L}, s)/\mathcal{X}}$ of $(\mathcal{L}, s)$ on $\mathcal{X}$ as the fibred product*

$$\begin{tikzcd} \sqrt[r]{(\mathcal{L}, s)/\mathcal{X}} \arrow[r] \arrow[d, "\pi"] & {[\mathbb{A}^1 \,/\, \mathbb{G}_\mathrm{m}]} \arrow[d, "\theta_r"] \\ \mathcal{X} \arrow[r, "\mathcal{L}"] & {[\mathbb{A}^1 \,/\, \mathbb{G}_\mathrm{m}]}. \end{tikzcd}$$

The universal object is a pair $(\mathcal{M}, t)$ of a line bundle $\mathcal{M}$ on $\sqrt[r]{(\mathcal{L}, s)/\mathcal{X}}$ and a section $t \in \Gamma(\sqrt[r]{(\mathcal{L}, s)/\mathcal{X}}, \mathcal{M})$.

More explicitly, an object of $\sqrt[r]{\mathcal{D}/\mathcal{X}}$ over a scheme T is a quadruple (φ, M, ϕ, τ) consisting of an object (φ, M, ϕ) of $\sqrt[r]{\mathcal{D}/\mathcal{X}}$ over T and a section τ of $\mathcal{M}$ such that $\phi(\tau^{\otimes r}) = \varphi^* s$.

Let $\mathcal{D}$ be a Cartier divisor on $\mathcal{X}$ and denote by $1_\mathcal{D}$ the canonical section corresponding to the inclusion $\mathcal{O}_\mathcal{X}(\mathcal{D}) \to \mathcal{O}_\mathcal{X}$. We will denote the r-th root stack of $\mathcal{X}$ of $(\mathcal{O}_\mathcal{X}(\mathcal{D}), 1_\mathcal{D})$ by $\sqrt[r]{\mathcal{D}/\mathcal{X}}$.

Example 5.2.3. *Suppose that $X = \operatorname{Spec}(A)$ is an affine scheme and $0 \neq f \in A$ is a non-zero divisor and let $\mathcal{D} = V(f)$ be the associated effective Cartier divisor. Then the root stack $\sqrt[r]{\mathcal{D}/X}$ is isomorphic to the quotient stack $[\operatorname{Spec}(A[t]/(t^r - f)) \,/\, \mathbb{Z}_r]$. Note that this generalizes to any scheme X and $L = \mathcal{O}_X$ is the trivial line bundle and f a global section of $\mathcal{O}_X$.*

5.2.3 The Iterated Root Stack

The construction above can be iterated. Let $\mathbf{L} = (\mathcal{L}_1, \ldots, \mathcal{L}_n)$ be a collection on n line bundles on $\mathcal{X}$ and $\mathbf{s} = (s_1, \ldots, s_n)$ a collection of global sections with $s_i \in \Gamma(\mathcal{X}, \mathcal{L}_i)$ and $r = (r_1, \ldots, r_n)$ with $r_i \in \mathbb{Z}, r_i > 0$. Denote by $\Theta_r\colon [\mathbb{A}^n_k \,/\, {\mathbb{G}_\mathrm{m}}^n] \to [\mathbb{A}^n_k \,/\, {\mathbb{G}_\mathrm{m}}^n]$ the morphism induced by the power morphism $x \mapsto x^r$ and $t \mapsto t^r$ on $\mathbb{A}^n_k$ and ${\mathbb{G}_\mathrm{m}}^n$.

Definition 5.2.4. *Using the notation defined above, define the $\mathbf{r}$-th root stack of $(\mathbf{L}, \mathbf{s})$ on $\mathcal{X}$ as the fibred product*

$$\begin{tikzcd} \sqrt[\mathbf{r}]{(\mathbf{L},\mathbf{s})/\mathcal{X}} \arrow[r] \arrow[d] & {[\mathbb{A}^n_k\,/\,\mathbb{G}_{\mathrm{m}}{}^n]} \arrow[d, "\Theta_r"] \\ \mathcal{X} \arrow[r, "(\mathcal{L}_k, s_k)"] & {[\mathbb{A}^n\,/\,\mathbb{G}_{\mathrm{m}}{}^n]}. \end{tikzcd}$$

For a collection of divisors $\mathbf{D} = (\mathcal{D}_1, \dots, \mathcal{D}_n)$ we denote the $\mathbf{r}$-th root stack of $(\mathcal{O}_{\mathcal{X}}(\mathcal{D}_i), \mathbf{1}_{\mathcal{D}_i})_{i=1}^n$ by $\sqrt[\mathbf{r}]{\mathbf{D}/\mathcal{X}}$.

We have the following properties of iterated root stacks by [22, §2], [26, §1.3b] and [9, Proposition 3.3]

1. If $\mathcal{X}$ is a Deligne-Mumford stack then so is $\sqrt[\mathbf{r}]{\mathbf{D}/\mathcal{X}}$.
2. The fibre product of all $\sqrt[r_i]{\mathcal{D}_i/\mathcal{X}}$ over $\mathcal{X}$ for all i is isomorphic to $\sqrt[\mathbf{r}]{\mathbf{D}/\mathcal{X}}$.
3. The morphism $\sqrt[\mathbf{r}]{\mathbf{D}/\mathcal{X}} \to \mathcal{X}$ is an isomorphism over $\mathcal{X} \setminus \bigcup_i \mathcal{D}_i$.
4. If $\mathcal{X}$ is smooth, each $\mathcal{D}_i$ are smooth and $\mathcal{D}_i$ have simple normal crossings then $\sqrt[\mathbf{r}]{\mathbf{D}/\mathcal{X}}$ is smooth.
5. The morphism $\sqrt[r]{(\mathcal{L}, s)/\mathcal{X}} \to \mathcal{X}$ is proper, faithfully flat and birational.

Remark 5.2.5. *Let $\mathcal{D}_1, \mathcal{D}_2$ be two effective Cartier divisors on $\mathcal{X}$ which intersect. Then the root stacks $\sqrt[r]{(\mathcal{D}_1 \cup \mathcal{D}_2)/\mathcal{X}}$ and $\sqrt[(r,r)]{(\mathcal{D}_1, \mathcal{D}_2)/\mathcal{X}}$ are not isomorphic. Consider a point $x \in D_1 \cap D_2$ and it's preimage $\tilde{x}$ in $\sqrt[r]{\mathcal{D}_1 \cup \mathcal{D}_2/\mathcal{X}}$ and $\sqrt[(r,r)]{(\mathcal{D}_1, \mathcal{D}_2)/\mathcal{X}}$. In the former $\tilde{x}$ has stabilizer group $\mathbb{Z}_r$ while in latter it has $\mathbb{Z}_r \times \mathbb{Z}_r$.*

Remark 5.2.6. *Let $\mathcal{X} \to [\mathbb{A}^1_k\,/\,\mathbb{G}_{\mathrm{m}}]$ be induced by $\mathcal{D}_1$ and $\mathcal{X} \to [\mathbb{A}^{n-1}_k\,/\,\mathbb{G}_{\mathrm{m}}{}^{n-1}]$ be induced by the $n-1$ tuple $(\mathcal{D}_2, \dots, \mathcal{D}_n)$ and let $\mathbf{r} = (r_2, \dots, r_n)$. Then there is a canonical isomerism*

$$[\mathbb{A}^1_k\,/\,\mathbb{G}_{\mathrm{m}}] \times_{r_1, [\mathbb{A}^1_k\,/\,\mathbb{G}_{\mathrm{m}}]} \mathcal{X} \times_{[\mathbb{A}^{n-1}\,/\,\mathbb{G}_{\mathrm{m}}{}^{n-1}], \mathbf{r}} [\mathbb{A}^{n-1}_k\,/\,\mathbb{G}_{\mathrm{m}}{}^{n-1}] \cong \sqrt[\mathbf{r}]{\mathbf{D}/\mathcal{X}}$$

where $\mathbf{D} = (\mathcal{D}_1, \dots, \mathcal{D}_n)$ See [22, Remark 2.2.5] for more details.

5.2.4 Root stacks and Groupoid Presentations

If we restrict ourselves to Deligne-Mumford stacks of the form $[Z/G]$ where Z is a scheme and G a finite abelian group we can give a more concrete description of a root stack over $[Z/G]$ using groupoid presentations.

Let Z be a subvariety of $\mathbb{C}^n$ of codimension greater than or equal to two. Let G be a group acting on on Z such that $[Z/G]$ is a Deligne-Mumford stack. Then, as all line bundles on Z are trivial, a line bundle on $[Z/G]$ is $\mathcal{O}_Z$ and a representation $\chi\colon G \to \mathbb{C}^*$.

Lemma 5.2.7 ([26, Lemma 7.1]). *Let Z be a subvariety of $\mathbb{C}^n$ of codimension equal or higher than two and G an abelian finite group acting on Z such that $[Z/G]$ is a Deligne-Mumford stack.*

Let $(\mathbf{L}, \mathbf{s}) = ((\mathcal{L}_1, s_1), \ldots, (\mathcal{L}_n, s_n))$ *be a collection of* n *line bundles on* $[Z/G]$ *with a global sections* $s_i \in \Gamma(\mathcal{X}, \mathcal{L}_i)$*. Denote by* $\chi = (\chi_1, \ldots, \chi_n)$ *the representations associated to the line bundles* $\mathbf{L}_i$*. Let* $\mathbf{r} = (r_1, \ldots, r_n) \in \mathbb{Z}^n_{>0}$*.*

Then the root stack $\mathcal{X} = \sqrt[\mathbf{r}]{(\mathbf{L}, \mathbf{s})/[Z/G]}$ *is isomorphic to* $[\tilde{Z}/\tilde{G}]$ *where* $\tilde{Z}$ *and* $\tilde{G}$ *are defined by the fibred products:*

$$\begin{tikzcd} \tilde{Z} \arrow[r] \arrow[d] & \mathbb{A}^n \arrow[d, "\wedge \mathbf{d}"] \\ Z \arrow[r, "s"] & \mathbb{A}^n \end{tikzcd} \qquad \begin{tikzcd} \tilde{G} \arrow[r] \arrow[d] & \mathbb{G}_m^n \arrow[d, "\wedge \mathbf{d}"] \\ G \arrow[r, "\chi"] & \mathbb{G}_m^n \end{tikzcd}$$

The action of $\tilde{G}$ *on* $\tilde{Z}$ *is given by*

$$(g, (\lambda_1, \ldots, \lambda_n)) \cdot (z, (x_1, \ldots, x_n)) = (gz, (\lambda_1 x_1, \ldots, \lambda_n x_n))$$

for any $(g, \lambda_1, \ldots, \lambda_n)) \in \tilde{G}$ *and* $(z, (x_1, \ldots, x_n)) \in \tilde{Z}$*.*

Remark 5.2.8. *Note that Lemma 5.2.7 extends to any variety* Z *on which all line bundles are trivial (e.g.* $\mathbb{C}^n$*).*

We now use this to compute some examples.

Example 5.2.9. *Suppose* $G = \frac{1}{4}(1,2) \subset \mathrm{GL}(2, \mathbb{C})$ *acts on* $\mathbb{C}^2$*. The group* G *is generated by the matrix*

$$\begin{pmatrix} i & 0 \\ 0 & -1 \end{pmatrix}$$

Denote by $\pi \colon \mathcal{X} = [\mathbb{C}^2 / G] \to \mathbb{C}^2 / G = X$*. Note that* $\mathcal{X}$ *is not canonical and we have a factorization*

$$\begin{tikzcd} \mathcal{X} \arrow[r, "f"] \arrow[rr, bend left, "\pi"] & X^{can} \arrow[r, "\epsilon"] & X. \end{tikzcd}$$

The coarse moduli space is isomorphic to

$$\mathbb{C}^2 / G \cong \operatorname{Spec} \mathbb{C}[x, y]^G = \operatorname{Spec} \mathbb{C}[x^4, y^2, x^2 y] \cong V(uv - w^2) \subset \mathbb{C}^3$$

the A_1*-singularity. The canonical stack* X^{can} *is* $[\mathbb{C}^2 / \mathbb{Z}_2]$ *with* $\mathbb{Z}_2$ *generated by*

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The branch divisor D *lifts to the divisor* $\mathcal{D} = \epsilon^{-1}(D) \subset X^{can}$*. The branch divisor* $\mathcal{D}$ *is isomorphic to the quotient stack* $[V(a)/ \mathbb{Z} / 2 \mathbb{Z}]$ *where* $\mathbb{Z}_2$ *acts by* $b \mapsto -b$ *on* $V(a)$*.*

This divisor is an effective Cartier divisor and we construct the 2^{nd} *root stack of* X^{can} *along* $\mathcal{D}$*. By Lemma 5.2.7 we have have* $[\mathbb{C}^2 / G] \cong \sqrt[2]{\mathcal{D}/X^{can}}$*.*

Example 5.2.10. *Let $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ act on $\mathbb{C}^2$ by the matrices*

$$\sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then as G acts on $\mathbb{C}^2$ by psuedoreflections, in accordance with the theorem of Chevalley-Shephard-Todd [69],

$$X = \mathbb{C}^2/G = \operatorname{Spec}\mathbb{C}[x,y]^G = \operatorname{Spec}\mathbb{C}[x^2,y^2] \cong \operatorname{Spec}\mathbb{C}[a,b] \cong \mathbb{C}^2.$$

The branch divisor is the simple normal crossing divisor given by the coordinate axes on the quotient.

As X is smooth, $X^{can} \cong X$. Let $D = D_1 + D_2$ be the branch divisor with $D_1 = V(a)$ and $D_2 = V(b)$ the divisors corresponding to the coordinate axes on X.

We first form the 2^{nd} root stack $\sqrt[2]{D_1/X}$ of X along D_1. By Example 5.2.3 we have

$$\mathcal{X} = \sqrt[2]{D_1/X} \cong \left[\operatorname{Spec}(\mathbb{C}[a,b,t]/(a^2-t))/\mathbb{Z}_2\right] \cong [\operatorname{Spec}\mathbb{C}[b,t]/\mathbb{Z}_2]$$

with $\mathbb{Z}_2$ acting by $t \mapsto -t$ on $\mathbb{C}[a,b,t]/(a^2-t)$.

The pulled back divisor $\mathcal{D}_2$ of D_2 to $\mathcal{X}$ is Cartier. Like in Example 5.2.9, $\mathcal{D}$ is the quotient stack $\mathcal{D}_2 = [D_2/\mathbb{Z}_2] \subset \mathcal{X}$. By Lemma 5.2.7 we have

$$[\mathbb{C}^2/G] \cong \sqrt[2]{\mathcal{D}_2/\mathcal{X}}.$$

5.3 Structure Theorems for Smooth Deligne-Mumford Stacks

Much work has gone into understanding the geometric relationship between a smooth separated Deligne-Mumford stack $\mathcal{X}$ with trivial generic stabilizer and its coarse moduli space X. One might hope that there is a way to "bootstrap" a "stacky" structure to X to recover $\mathcal{X}$. This is, in fact, the case under certain conditions.

For Deligne-Mumford stacks of dimension 1 we have the following:

Theorem 5.3.1 ([6, Theorem 1.187])**.** *Let $\mathcal{X}$ be a smooth separated Deligne-Mumford stack with trivial generic stabilizer and of finite type over an algebraically closed field k with $char(k) = 0$. Suppose that the coarse moduli space X is an irreducible curve. Then there exists an effective divisor $D = (P_1, \dots, P_n)$ on X and $\boldsymbol{r} = (r_1, \dots, r_n) \in \mathbb{Z}^n_{>0}$ such that*

$$\mathcal{X} \cong \sqrt[\boldsymbol{r}]{D/X}.$$

The main result in [31] generalizes this idea to higher dimensions. A Deligne-Mumford stack $\mathcal{Y}$ has (tame) quotient singularities if there exist an atlas $U \to \mathcal{Y}$ where U is a scheme with (tame) quotient singularities. One can associate to a Deligne-Mumford stack $\mathcal{Y}$ with tame quotient singularities a canonical smooth Deligne-Mumford stack $\mathcal{Y}^{can}$ in a similar way to schemes with tame quotient singularities following [31, Background].

Theorem 5.3.2 ([31, Theorem 1]). *Let $\mathcal{X}/S$ be a smooth separated tame Deligne-Mumford stack with trivial generic stabilizer.*

Denote by X its coarse moduli space, $D \subset X$ the branch divisor of the coarse moduli map $\pi\colon \mathcal{X} \to X$ and $\mathcal{D} = \sum_{i=1}^{n} \mathcal{D}_i \subset X^{can}$ the pullback of D to X^{can}.

Let r_i be the ramification index of π over the irreducible components D_i of $D = \sum_{i=1}^{n} D_i$. Denote by $\sqrt[\mathbf{r}]{\mathbf{D}/X^{can}}$ the root stack along $\mathbf{D} = (\mathcal{D}_1, \dots, \mathcal{D}_n)$ of order $\mathbf{r} = (r_1, \dots, r_n)$.

Then $\sqrt[\mathbf{r}]{\mathcal{D}/X^{can}}$ has tame quotient singularities and π factors as

$$\mathcal{X} \cong \sqrt[\mathbf{r}]{\mathbf{D}/X^{can}}^{can} \to \sqrt[\mathbf{r}]{\mathbf{D}/X^{can}} \to X^{can} \to X.$$

Moreover, if D is Cartier, then $\sqrt{\mathbf{D}/X}$ has tame quotient singularities and π factors as

$$\mathcal{X} \cong \sqrt[\mathbf{r}]{\mathbf{D}/X}^{can} \to \sqrt[\mathbf{r}]{\mathbf{D}/X} \to X.$$

Remark 5.3.3. *In [31], the authors give a local description of this Theorem [31, Theorem 11] that the reader may find insightful. We give the statement below.*

Let V be a vectorspace over k and G an abstract finite group acting linearly and faithfully whose order is coprime to the characteristic of k. Let $H \subset G$ be the subgroup generated by psuedoreflections and $H' \subset H$ be its commutator subgroup ($H' = \{h' \in H : h'h = hh'\}$). Then the coarse moduli space map $\pi\colon \mathcal{X} = [V/G] \to V/G = X$ factors as

$$\begin{tikzcd}
\mathcal{X} \cong \sqrt[\mathbf{r}]{\mathbf{D}/X^{can}}^{can} \arrow[r] \arrow[d, equal] & \sqrt[\mathbf{r}]{\mathbf{D}/X^{can}} \arrow[r] \arrow[d, equal] & X^{can} \arrow[r] \arrow[d, equal] & X \arrow[d, equal] \\
{[V/G]} \arrow[r] & {[(V/H')/(G/H')]} \arrow[r] & {[(V/H)/(G/H)} \arrow[r] & V/G
\end{tikzcd}$$

A corollary of this theorem ([30]) is the following description for abelian global quotient stacks.

Corollary 5.3.4 ([30, Corollary 5.6]). *Suppose that X is a smooth quasi-projective variety over k and G a finite abelian group acting on X whose order is coprime to the characteristic of k. Then the induced map*

$$f\colon [X/G] \to (X/G)^{can}$$

to the canonical stack of X/G is a root stack morphism along a collection of smooth connected divisors with simple normal crossings, i.e. one can construct $[X/G]$ as an iterated root stack along a collection of smooth connected divisors $D = \sum D_i$ with simple normal crossings from $(X/G)^{can}$.

Proof. This follows from Theorem 5.3.2 and is a consequence of the root stack $\sqrt[r]{\mathbf{D}/X^{can}}$ being smooth so $[X/G] \cong \sqrt[r]{\mathbf{D}/(X/G)^{can}}$. □

5.4 Semi-orthogonal Decompositions for Root Stacks

Root stacks behave much like blow ups for schemes. The derived category of root stacks have been extensively studied, first by Ishii and Ueda in [40] and generalized by Bergh, Lunts and Schnürer in [9]. We shall only need the content of the theorem by Ishii and Ueda in this thesis and so refer to those. We describe the results below.

Theorem 5.4.1 ([40, Theorem 1.6])**.** *Let $\mathcal{D}$ be a smooth divisor on a smooth Deligne-Mumford stack $\mathcal{X}$ and let $\mathcal{Y} = \sqrt[r]{\mathcal{D}/\mathcal{X}}$ be the r-th root stack of $\mathcal{D}$ with $r > 1$. Then there are full and faithful functors*

$$\Phi_{\mathcal{X}} \colon D(\mathcal{X}) \to D(\mathcal{Y})$$
$$\Phi_{\mathcal{D}} \colon D(\mathcal{D}) \to D(\mathcal{Y})$$

embedding $D(\mathcal{X})$ and $D(\mathcal{D})$ as admissible subcategories of $D(\mathcal{Y})$. Moreover, there is a semi-orthogonal decomposition

$$D(\mathcal{Y}) = \left\langle \Phi_{\mathcal{D}}(D(\mathcal{D})) \otimes \mathcal{M}_{\mathcal{E}}^{\otimes(r-1)}, \dots, \Phi_{\mathcal{D}}(D(\mathcal{D})) \otimes \mathcal{M}_{\mathcal{E}}, \Phi_{\mathcal{X}}(D(\mathcal{X})) \right\rangle$$

where $\mathcal{M}_{\mathcal{E}}$ is the universal line bundle on $\mathcal{Y}$ corresponding to the universal object.

Proof. Consider the commutative diagram

$$\begin{tikzcd} \mathcal{E} = \sqrt[r]{\mathcal{D}/\mathcal{D}} \arrow[r, "j"] \arrow[d, "\pi_{\mathcal{D}}"'] & \mathcal{Y} \arrow[d, "\pi_{\mathcal{X}}"] \\ \mathcal{D} \arrow[r, "\bar{j}"] & \mathcal{X} \end{tikzcd}$$

where j sends a line bundle M over T to the same line bundle M over T with the zero section. We will denote by $\mathcal{E} = \sqrt[r]{\mathcal{D}/\mathcal{D}}$ the effective Cartier divisor on $\mathcal{Y}$.

First, we note that proof that $\Phi_{\mathcal{X}}$ is fully faithful is omitted in [40]. It does however follow from [9, Lemma, 4.4, Lemma 4.5 and Example 4.6].

Now we show that the functor

$$\Phi_{\mathcal{D}} = j_* \pi_{\mathcal{D}}^* \colon D(\mathcal{D}) \to D(\mathcal{Y})$$

is fully faithful. Let $E^\bullet, F^\bullet$ be objects of $D(\mathcal{D})$ and $q \in \mathbb{Z}$. We show that the natural morphism

$$\mathrm{Hom}^q_{D(\mathcal{D})}(E^\bullet, F^\bullet) \cong \mathrm{Hom}^q_{D(\mathcal{Y})}(j_*\pi_{\mathcal{D}}^* E^\bullet, j_*\pi_{\mathcal{D}}^* F^\bullet) \cong \mathrm{Hom}^q_{D(\mathcal{E})}(j^* j_* \pi_{\mathcal{E}}^* E^\bullet, \pi_{\mathcal{D}}^* F^\bullet)\ (*) \tag{5.1}$$

is an isomorphism. As $\mathcal{E}$ is a smooth divisor in $\mathcal{Y}$ we can use a stacky version of [12, Lemma 3.3] to obtain for $E^\bullet \in D(\mathcal{D})$ a distinguished triangle

$$\pi_{\mathcal{D}}^* E^\bullet \otimes \mathcal{O}_{\mathcal{E}}(-\mathcal{E})[1] \longrightarrow j^* j_* \pi_{\mathcal{D}}^* E^\bullet \longrightarrow \pi_{\mathcal{D}}^* E^\bullet \longrightarrow \pi_{\mathcal{D}}^* E^\bullet \otimes \mathcal{O}_{\mathcal{E}}(-\mathcal{E})[2]$$

The original proof uses a spectral sequence arguement but we feel using the above distinguished triangle is clearer.

Since $r > 1$, by [40, Theorem 1.5] the functor

$$\Phi\colon \operatorname{Coh}(\mathcal{D})^{\oplus r} \to \operatorname{Coh}(\mathcal{E})$$

defined by

$$\Phi(\bigoplus_{i=0}^{r-1} E_i) = \bigoplus_{i=1}^{r-1} \pi_{\mathcal{D}}^* E_i \otimes \mathcal{M}_{\mathcal{E}}^i$$

where $\mathcal{M}_{\mathcal{E}}$ is the universal line bundle on $\mathcal{E}$ is an equivalence.

Thus we see that

$$\operatorname{Hom}^q_{D(\mathcal{E})}(\pi_{\mathcal{D}}^* E \otimes \mathcal{O}_{\mathcal{E}}(-\mathcal{E}), \pi_{\mathcal{D}}^* F) = 0.$$

Also, as $\pi_{\mathcal{D}}^*$ is fully faithful

$$\operatorname{Hom}^q_{D(\mathcal{E})}(\pi_{\mathcal{D}}^* E, \pi_{\mathcal{D}}^* F) \cong \operatorname{Hom}^q(E, F)$$

for any q. By applying $\operatorname{Hom}_{D(\mathcal{E})}(-, \pi_{\mathcal{D}}^* F^\bullet)$ to the above triangle and using the above identities we see that $(*)$ is an isomorphism.

The essential images of $\Phi_{\mathcal{X}}$ and $\Phi_{\mathcal{D}}$ are admissible subcategories as $\pi_{\mathcal{X}}^*$ and $\Phi_{\mathcal{D}} = j_* \pi_{\mathcal{D}}^*$ admit left and right adjoints as j_* and $\pi_{\mathcal{D}}^*$ admit left right and left adjoints and the functor $(-) \otimes \mathcal{M}_{\mathcal{E}}^{\otimes i}$ is an equivalence.

We see that $\Phi_{\mathcal{D}} D^b(\mathcal{D}) \otimes \mathcal{M}_{\mathcal{E}}^{\otimes i}$ is right orthogonal to $\pi_{\mathcal{X}}^* D^b(\mathcal{X})$ for $1 \le i \le r-1$ by

$$\begin{aligned}
\operatorname{Hom}_{D(\mathcal{Y})}(\pi_{\mathcal{X}}^* E^\bullet, j_*(\pi_{\mathcal{D}}^* F^\bullet \otimes \mathcal{M}_{\mathcal{E}}^{\otimes i})) &\cong \operatorname{Hom}_{D(\mathcal{E})}(j^* \pi_{\mathcal{X}}^* E^\bullet, \pi_{\mathcal{D}}^* F^\bullet \otimes \mathcal{M}_{\mathcal{E}}^{\otimes i}) \\
&\cong \operatorname{Hom}_{D(\mathcal{E})}(\pi_{\mathcal{D}}^* \bar{j}^* E^\bullet, \pi_{\mathcal{D}}^* F^\bullet \otimes \mathcal{M}_{\mathcal{E}}^{\otimes i}) \\
&= 0.
\end{aligned}$$

Similarly, we have

$$\operatorname{Hom}_{D(\mathcal{Y})}(j_* \pi_{\mathcal{D}}^* E^\bullet \otimes \mathcal{M}_{\mathcal{E}}^{\otimes k}, j_* \pi_{\mathcal{D}}^* F^\bullet \otimes \mathcal{M}_{\mathcal{E}}^{\otimes l}) = 0$$

for $1 \le k \le l \le r-1$.

We now show fullness by showing that any object $E^\bullet$ of $D(\mathcal{Y})$ is obtained from an object of $j_* \pi_{\mathcal{D}}^* D(\mathcal{D}) \otimes \mathcal{M}^{\otimes i}$ for $1 \le i \le r-1$ and $\pi_{\mathcal{X}}^* D(\mathcal{X})$ by taking shifts and cones. Since $\pi_{\mathcal{X}}$ is an isomorphism outside of $\mathcal{D}$, the mapping cone in the triangle induced by the adjunction morphism

$$\pi_{\mathcal{X}}^* \pi_{\mathcal{X},*} E^\bullet \longrightarrow E^\bullet \longrightarrow F^\bullet \longrightarrow \pi_{\mathcal{X}}^* \pi_{\mathcal{X},*} E^\bullet[1]$$

has F supported on $\mathcal{E}$. Hence $E^\bullet$ can be obtained from $\pi_{\mathcal{X}}^* \pi_{\mathcal{X},*} E^\bullet$ and an object supported on $\mathcal{E}$ by taking cones.

By definition, any object E supported on $\mathcal{E}$ has cohomology sheaves supported on $\mathcal{E}$. By considering the standard filtration of E in terms of the cohomology sheaves of E, we see that E can be obtained from shifts of sheaves supported on $\mathcal{E}$ by taking cones. Thus any object supported on $\mathcal{E}$ is obtained from objects of $j_* D(\mathcal{E})$ by taking cones.

As $\Phi\colon \operatorname{Coh}(\mathcal{E}) \cong (\operatorname{Coh}(\mathcal{D})^{\oplus r}$ is an equivalence, objects of $j_* D(\mathcal{E})$ can be obtained from $j_* \pi_{\mathcal{D}}^* D^b(\mathcal{D}) \otimes \mathcal{M}_{\mathcal{E}}^{\otimes i}$ for $0 \leq i \leq r-1$ by taking cones.

Finally, we show that an object$F^\bullet \in j_* \pi_{\mathcal{D}}^* D(\mathcal{D})$ is obtained from objects of $\pi_{\mathcal{X}}^* D(\mathcal{X})$ and $j_* \pi_{\mathcal{D}}^* D(\mathcal{D}) \otimes \mathcal{M}_{\mathcal{E}}^{\otimes i}$ for $1 \leq i \leq r-1$. Then $\pi_{\mathcal{X}}^* \bar{j}_* F^\bullet$ has a filtration whose factors are $j_* \pi_{\mathcal{D}}^* F^\bullet \otimes \mathcal{M}_{\mathcal{E}}^{\otimes i}$ for $0 \leq i \leq r-1$ as $\operatorname{supp}(\pi_{\mathcal{X}}^* \bar{j}_* F^\bullet) \subset \mathcal{E}$. Thus $j_* \pi_{\mathcal{D}}^* F^\bullet$ is obtained from $\pi_{\mathcal{X}}^* \bar{j}_* F^\bullet$ and $j_* \pi_{\mathcal{D}}^* F^\bullet \otimes \mathcal{M}_{\mathcal{E}}^{\otimes i}$ for $1 \leq i \leq r-1$ by taking shifts and cones. □

By applying Theorem 5.4.1 iteratively we get a semi-orthogonal decomposition for iterated root stacks. Theorem 5.4.2 is an immediate generalization of [40, Proposition 7.2] whose proof is contained in the first part of the proof of [40, Proposition 7.2] which we give below (see [9, §4] for a more general version).

Theorem 5.4.2. *Let $\mathcal{X}$ be a smooth separated Deligne-Mumford stack with trivial generic stabilizer and coarse moduli space X. Assume:*

1. *The canonical morphism $\phi\colon \mathcal{X} \to X^{can}$ from $\mathcal{X}$ to the canonical stack of the coarse moduli space X is an isomorphism outside a simple normal crossing divisor $\sum_{i=1}^n \mathcal{D}_i$ on $\mathcal{X}^{can}$.*
2. *The pull back $\phi^*(\mathcal{D}_i) \equiv r_i E_i$ for some prime divisor E_i for $i = 1, \ldots, n$.*

Then there exists a semi-orthogonal decomposition of $D(\mathcal{X})$ with pieces given by the derived category of X^{can} and the derived categories of D_i and their intersections of irreducible components.

Proof. We proceed by induction. Consider

$$\mathcal{X}_1 = \sqrt[r_2]{D_2/\mathcal{X}^{can}} \times_{\mathcal{X}^{can}} \cdots \times_{\mathcal{X}^{can}} \sqrt[r_n]{D_n/\mathcal{X}^{can}}$$

and let $\mathcal{D}_1 \subset \mathcal{X}_1$ be the prime divisor corresponding to D_1. Then $\mathcal{X}$ is isomorphic to $\sqrt[r_1]{\mathcal{D}_1/\mathcal{X}_1}$ and we have a semi-orthogonal decomposition

$$D(\mathcal{X}) = \left\langle \Phi_{\mathcal{D}}(D(\mathcal{D})) \otimes \mathcal{M}^{\otimes(r_1-1)}, \ldots, \Phi_{\mathcal{D}}(D(\mathcal{D})) \otimes \mathcal{M}, \Phi_{\mathcal{X}_1}(D(\mathcal{X}_1)) \right\rangle$$

by Theorem 5.4.1. We obtain the required semi-orthogonal decomposition by induction. □

Example 5.4.3. *We use the semi-orthogonal decomposition for iterated root stacks to construct the semi-orthogonal decompositions*

$$D([\mathbb{C}^2/(\mathbb{Z}_2 \times \mathbb{Z}_2)] = \left\langle D(pt), D(D_1), D(D_2), D^b(\mathbb{C}^2) \right\rangle .$$

As the coarse moduli space of $[\mathbb{C}^2/(\mathbb{Z}_2 \times \mathbb{Z}_2)]$ *is smooth and as* $[\mathbb{C}^2/(\mathbb{Z}_2 \times \mathbb{Z}_2)]$ *is a iterated root stack over* (D_1, D_2) *we have the decomposition*

$$D([\mathbb{C}^2/(\mathbb{Z}_2 \times \mathbb{Z}_2)] = \left\langle D(\widetilde{D_1}), D(D_2), D(\mathbb{C}^2) \right\rangle .$$

where $\mathcal{D}_1$ *is the pullback of the divisor* D_1 *to* $\sqrt[2]{D_2/\mathbb{C}^2}$*. Then* $D(\mathcal{D}_1) = \langle D(pt), D(D_1) \rangle$ *as* $\mathcal{D}_1 = [D_1/\mathbb{Z}_2]$*. Hence we obtain the semi-orthogonal decomposition.*

Remark 5.4.4. *Note that if* G *is abelian then the branch divisor is a simple normal crossing divisor with smooth components by [30, Lemma 5.5].*

A corollary of these semi-orthogonal decompositions is the following new result for abelian groups acting on smooth quasi-projective varieties.

Corollary 5.4.5. *[Corollary 1.3.7] Let* X *be a smooth quasi-projective variety over* k *and* G *a finite abelian group whose order is coprime to the characteristic of* k*. Let* $D = \sum_{i=1}^n D_i$ *on* X/G *be the simple normal crossing branch divisor and* $\mathcal{D} = \sum_{i=1}^n \mathcal{D}_i$ *the pullback of the branch divisor to the canonical stack* $(X/G)^{can}$*.*

Then there is a semi-orthogonal decomposition of $D^G(X) = D^b([X/G])$ *with pieces given by*

- *The derived category* $D((X/G)^{can})$ *of the canonical stack* $(X/G)^{can}$*,*
- *The derived category* $D(\mathcal{D}_i)$ *of the irreducible components of the branch divisor* $\mathcal{D} = \sum \mathcal{D}_i$*,*
- *The derived category of the intersections of divisors.*

Proof. It follows from 5.3.4 that $[X/G]$ is an iterated root stack over the canonical stack along a simple normal crossing divisor. The result then follows from 5.4.2. □

Remark 5.4.6. *Note that when* G *is non-abelian, the irreducible components of the branch divisor need not be smooth. Consider the unique two-dimensional irreducible representation* $S_3 = D_6$*. Then the branch divisor is singular as it is the cubic cusp. In this case, the root stack will be singular. See [31] for a more detailed explanation.*

5.5 Semi-orthogonal Decompositions and the Canonical Stack

The above semi-orthogonal decompositions provide evidence that we should expect for any quotient stack $[X/G]$ with G a finite group acting faithfully on a smooth quasi-projective variety X (or more generally, any smooth, separated tame Deligne-Mumford stack $\mathcal{X}$ with trivial generic stabilizer) the derived category of $[X/G]$ to have a semi-orthogonal decomposition with one piece given by the canonical stack associated to the coarse moduli space X/G (respectively X).

By the universal property of the canonical stack, we have a decomposition of the coarse moduli map π

$$\begin{tikzcd} \mathcal{X} \arrow[r, "f"] \arrow[rr, bend right, "\pi"'] & X^{can} \arrow[r, "\epsilon"] & X. \end{tikzcd}$$

Theorem 5.5.1. *Let $\mathcal{X}$ be a smooth separated tame Deligne-Mumford stack with trivial generic stabilizer over an algebraically closed field k of characteristic zero with coarse moduli space X. Denote the canonical stack associated to X by X^{can} and let $f\colon \mathcal{X} \to X^{can}$ be the unique map in the decomposition above. Then the functor*

$$f^*\colon D^b(X^{can}) \to D^b(\mathcal{X})$$

is fully faithful.

Proof. By adjunction and the projection formula

$$\mathrm{Hom}_{D^b(\mathcal{X})}(f^*E, f^*F) \cong \mathrm{Hom}_{D^b(X^{can})}(E, f_*f^*F) \cong \mathrm{Hom}_{D^b(X^{can})}(E, f_*\mathcal{O}_{\mathcal{X}} \otimes F).$$

To prove fully faithfulness it suffices to show that $Rf_*\mathcal{O}_{\mathcal{X}} \cong \mathcal{O}_{X^{can}}$. The following argument from [71] generalizes the argument of [20, Theorem 3.1] to Deligne-Mumford stacks.

Let $g\colon Z \to X^{can}$ be an atlas for X^{can}. Then we have a diagram

$$\begin{tikzcd} \mathcal{X}' = \mathcal{X} \times_{X^{can}} Z \arrow[r, "f'"] \arrow[d, "g'"] & Z \arrow[d, "g"] \\ \mathcal{X} \arrow[r, "f"] & X^{can} \end{tikzcd}$$

As g is flat, by base change we have

$$Rf_*(g')^*\mathcal{O}_X \cong Rf'_*\mathcal{O}_{\mathcal{X}'} \cong g^*Rf_*\mathcal{O}_{\mathcal{X}}.$$

Thus to prove that $Rf_*\mathcal{O}_{\mathcal{X}} \cong \mathcal{O}_{X^{can}}$ it suffices to show that $f'_*\mathcal{O}_{\mathcal{X}'} \cong \mathcal{O}_Z$ as Z is an atlas for X^{can}.

Denote by $\pi'\colon \mathcal{X}' \to X'$ be the map from $\mathcal{X}'$ its coarse moduli space X'. By the universal property of the coarse moduli space, we have a factorization of f'

$$\begin{tikzcd} & X' \arrow[dr, "h"] & \\ \mathcal{X}' \arrow[ur, "\pi'"] \arrow[rr, "f'"] & & Z \end{tikzcd}$$

As $char(k) = 0$, $\mathcal{X}'$ is a tame Deligne-Mumford stack. Hence $\pi'_*\mathcal{O}_{\mathcal{X}'} \cong \mathcal{O}_{X'}$ and $R^q\pi'_*\mathcal{O}_{\mathcal{X}'} = 0$ for all $q > 0$ by Proposition 4.3.10 . Also, $h_*\mathcal{O}_{X'} \cong \mathcal{O}_Z$ and $R^q h_*\mathcal{O}_{X'} = 0$ for all $q > 0$ as f' is surjective and X' has rational singularities [48, Proposition 5.13] and [20, §3].

Hence $Rf'_*\mathcal{O}_{\mathcal{X}} \cong \mathcal{O}_Z$ so f^* is fully faithful. $\square$

Chapter 6

Semi-orthogonal Decompositions for Surfaces

In this chapter, we apply the theory developed in Chapter 5.

In section 6.1 we describe semi-orthogonal decompositions for abelian groups acting on smooth quasi-projective surfaces over a field of characteristic zero. In section 6.2 we give examples of semi-orthogonal decompositions related to abelian Galois covers. Then in section 6.3 we give explicit examples of semi-orthogonal decompositions for Godeaux surfaces with an action of $\mathbb{Z}_2$, and for Burniat surfaces with an action of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

In section 6.4 we give a new proof of the derived McKay Correspondence in dimension 2. Finally, using this new proof of the derived McKay Correspondence we describe semi-orthogonal decompositions for a natural action of D_{2n} on $\mathbb{C}^2$ and show that they satisfy a conjecture of Polishchuk and Van den Bergh.

Throughout this chapter, k will be a field of characteristic zero.

6.1 Semi-orthogonal Decompositions for Surfaces

Let X be a quasi-projective variety of dimension 2 over k. Then by Hironaka the minimal resolution of X/G exists and is unique. We can use this to give a finer semi-orthogonal decomposition of $D([X/G])$.

Following Ishii and Ueda we have the following description of the canonical stack associated with a surface over k with at worst quotient singularities.

Theorem 6.1.1 ([40, Theorem 1.6]). *Let X^{can} be the canonical stack associated with a surface X with at worst quotient singularities, and Y the minimal resolution of X. Then there is a fully faithful functor*

$$\Phi_Y \colon D(Y) \to D(X^{can})$$

and a semi-orthogonal decomposition

$$D(X^{can}) = \langle E_1, \ldots, E_n, \Phi_Y(D(Y))\rangle$$

where $E_1, \ldots, E_n$ are exceptional objects.

The following corollary follows from Corollary 5.4.5 and Theorem 6.1.1.

Corollary 6.1.2. *Suppose that X is a smooth quasi-projective surface over k and G a finite abelian group acting faithfully on X. Let $D = \sum_{i=1}^{n} D_i$ denote the branch divisor of $\pi\colon [X/G] \to X/G$. Let Y be the minimal resolution of X/G.*

Then there is a semi-orthogonal decomposition of $D^G(X)$ with pieces given by

1. *The derived category of the minimal resolution $D(Y)$*
2. *Multiple copies of the derived category of the irreducible components of the branch divisor $D(D_i)$ determined by the order of the stabilizer group of D_i.*
3. *Exceptional objects E_i arising from the intersection of the divisors D_i and D_j, where stabilizers jumps along a divisor at a point, and non-special representations of G acting by $GL_2(k)$ at an isolated fixed point.*

We give two examples of semi-orthogonal decompositions of surfaces with group actions.

Example 6.1.3. *Following on from Example 5.2.9, Corollary 6.1.2 we have a semi-orthogonal decompositions*

$$D^{\mathbb{Z}_4}(\mathbb{C}^2) \cong D([\mathbb{C}^2/(\mathbb{Z}_4)]) = \langle \Phi_{\mathcal{D}} D(\mathcal{D}), \Phi_Y D(Y)\rangle$$

where Y is the minimal resolution of $X = \mathbb{C}^2/(\mathbb{Z}_4)$ and $\mathcal{D}$ is the branch divisor in X^{can}. We have a further decomposition as $\mathcal{D} = [D'/(\mathbb{Z}_2)]$ so

$$D(\mathcal{D}) = \left\langle E, \pi_{D'}^{*} D(D'/\mathbb{Z}_2)\right\rangle$$

where $\pi_{D'}\colon D' \to D'/\mathbb{Z}_2$ is the quotient map. Notice that $D'/\mathbb{Z}_2 \cong D \subset X$. Thus we have a semi-orthogonal decomposition

$$D^{\mathbb{Z}_4}(\mathbb{C}^2) = \langle E, \Phi_D(D(D)), \Phi_Y D(Y)\rangle .$$

Example 6.1.4. *Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ act on $\mathbb{C}^2$ as in Example 5.2.10. Then we can express the quotient stack $[\mathbb{C}^2/G]$ as a root stack*

$$\left[\mathbb{C}^2/G\right] = \sqrt[2]{D_1/X} \times_X \sqrt[2]{D_2/X}$$

which can also be expressed as

$$\left[\mathbb{C}^2/G\right] = \sqrt[2]{\mathcal{D}/\sqrt[2]{D_1/X}}.$$

Thus we get a semi-orthogonal decomposition

$$D^G(\mathbb{C}^2) = D([\mathbb{C}^2/G]) = \left\langle \Phi_{\mathcal{D}} D(\mathcal{D}), \Phi_{D_1} D(D_1), \Phi_{\mathbb{C}^2} D(\mathbb{C}^2) \right\rangle.$$

As $\mathcal{D} = [D'/\mathbb{Z}_2]$ *where* $D' = V(b)$ *on* $\operatorname{Spec}\mathbb{C}[b,t]$*, by [63, Theorem 1.2] we have*

$$D(\mathcal{D}) = \left\langle E, \pi_{D'}^* D(D'/\mathbb{Z}_2) \right\rangle.$$

where $\pi_{D'} \colon D' \to D'/\mathbb{Z}_2$ *and* E *is a exceptional object. As* $D'/\mathbb{Z}_2 \cong D_2$ *we have a semi-orthogonal decomposition*

$$D([\mathbb{C}^2/G]) = \left\langle E, \Phi_{D_2} D(D_2), \Phi_{D_1} D(D_1), \Phi_{\mathbb{C}^2} D(\mathbb{C}^2) \right\rangle.$$

which is the semi-orthogonal decomposition as described in [64, §6.4].

6.2 Abelian Galois Covers

The theory of abelian Galois covers was first used by Catanese to produce surfaces of general type to prove that the moduli of surfaces of general type with fixed K^2 and χ is *not* equidimensional. This idea was expanded upon by Pardini [61] to describe a recipe for constructing such Galois covers. When the Galois cover is smooth, the associated quotient stack is smooth and we can describe semi-orthogonal decompositions of the derived category using Corollary 6.1.2. Moreover, these ideas provide a geometric realization of the root stack construction outlined in 5.2.

Throughout this section, we will assume that all varieties are defined over an algebraically closed field k.

6.2.1 Construction

Recall that a *Galois covering* is a finite surjective morphism of quasi-projective algebraic varieties $\pi \colon X \to Y$ where the function field $k(X)$ is a Galois extension of $k(Y)$ with $Gal(k(X)/k(Y)) = G$. If $\pi \colon X \to Y$ is a Galois cover then $Y = X/G$. A Galois covering is *abelian* if G is abelian. A Galois covering is *smooth* if X and Y are smooth.

Let $\pi \colon X \to Y$ be a smooth Galois cover. Denote by R and D the ramification and branch locus of π. We will characterize π in terms of two pieces of data: the algebra structure of $\pi_*(\mathcal{O}_X)$ and the action of the inertia groups on the irreducible components of the branch divisor.

Note that we have a decomposition

$$\pi_*\mathcal{O}_X = \bigoplus_{\chi \in G^*} L_\chi^{-1}.$$

where G acts on L_χ^{-1} by the character χ. The invariant summand is isomorphic to $\mathcal{O}_Y$.

Let R_i be a smooth irreducible component of R and define the inertia group of R_i by

$$H_i = \{h \in G | hx = x \text{ for all } x \in R_i\}$$

As R_i is smooth has codimension 1, H is cyclic and acts faithfully on the tangent space to T. Denote by χ_i a generator of H_i^*. For any component D_i of the branch locus D, all components of $\pi^{-1}(D_i)$ have the same inertia groups and isomorphic representations. Thus we can associated to each irreducible component D_i an inertia subgroup H_i and character $\chi_i \in H_i^*$.

Denote by $\mathfrak{C}$ the set of cyclic subgroups of G and for $H \in \mathfrak{C}$, the set of generators S_h of the group of characters of H^*. Thus we get a decomposition

$$D = \sum_{H \in \mathfrak{C}} \sum_{\phi \in S_H} D_{H.,\phi}.$$

We call the pair $\{D_{H,\phi}, L_\chi\}$ the *building data* of the abelian cover $\pi \colon X \to Y$.

The central theorem of [61] is the following:

Theorem 6.2.1. *Let G be an abelian group, Y a smooth variety and X a normal variety with $\pi \colon X \to Y$ an abelian cover with group G. The building data of π satisfies the following linear equivalences*

$$L_\chi + L_{\chi'} = L_{\chi\chi'} + \sum_{H \in \mathfrak{C}} \sum_{\phi \in S_H} \epsilon^{H,\phi}_{\chi,\chi'} D_{H,\phi}$$

where $\epsilon^{H,\phi}_{\chi,\chi'}$ are defined by

$$\epsilon^{H,\phi}_{\chi,\chi'} = \begin{cases} 0, & \text{if } i_\chi + i_{\chi'} < |H| \\ 1, & \text{otherwise} \end{cases}$$

where $\chi \mid_H = \phi^{i_\chi}$ and $\chi' \mid_H = \phi^{i_{\chi'}}$.

Conversely, to any data $\{L_\chi, D_{H,\phi}\}$ satisfying the above equivalences we can associated an abelian cover $\pi \colon X \to Y$ whose building data is given by $L_\chi, D_{\eta,\phi}$.

Moreover, if Y is proper, then π is determined uniquely up to isomorphism of Galois covers.

Remark 6.2.2. *Suppose that the abelian Galois cover $\pi \colon X \to Y$ is smooth. Then by Corollary 5.4.5 we get a semi-orthogonal decompositions of $D([X/G])$ in terms the derived categories of intersections of the divisors $D_{H,\phi}$ and $D(Y)$.*

6.2.2 Examples

We now focus on a few explicit examples of abelian covers and the induced semi-orthogonal decomposition.

Example 6.2.3 (Cyclic Covers). *Suppose that G is a cyclic group of order n and choose a generator $\chi \in G^*$. The building data for the abelian Galois cover $X \to Y = X/G$ consists of a line bundle $L = L_\chi$ and a collection of divisors $D_{H,\phi}$, possibly empty, for each (cyclic) subgroup $H \subset G$ such that the following relations are satisfied:*

$$nL = \sum_{H,\phi} n \frac{r_{H,\phi}}{m} D_{H,\phi}$$

where $\chi|_H = \phi^{r_{H,\phi}}$ with $0 \leq r_{H,\phi} < |H| = m$. Here ϕ is a root of unity of order at most m.

The quotient stack $[X/G]$ is the iterated root stack over Y of order m along the divisor $(D_{H,\phi})$ and we get the induced semi-orthogonal decomposition.

$$D([X/G]) = \left\langle D(Y), D(D_{H,\phi}), D(D_{H,\phi}) \otimes \chi, \ldots, D(D_{H,\phi}) \otimes \chi^{m-1}, \ldots \right\rangle .$$

This recovers results due to Lim [51] and Krug, Ploog and Sosna [49] when the inertia group for all divisors is G.

Example 6.2.4 ($(\mathbb{Z}/2\mathbb{Z})^s$-covers). *Suppose that $G = (\mathbb{Z}/2\mathbb{Z})^s$. Then G-covers are particular easy to describe. Let $\chi_1, \ldots, \chi_s$ be a basis for G^* and let $H_1, \ldots, H_r$ $r = 2^s - 1$ be the subgroups of order 2. Define $\epsilon_{ij} = 0$ if $\chi_j|_{H_i} = 1$ and $\epsilon_{ij} = 1$ otherwise. Then the building data consists of line bundles $L_1, \ldots, L_s$ and effective divisors $D_1, \ldots, D_r$ such that*

$$2L_j = \sum_i \epsilon_{ij} D_i, \quad j = 1, \ldots, s$$

So the quotient stack $[X/G]$ is the root stack $\sqrt[(2,\ldots,2)]{(D_i)/Y}$. Thus we get a semi-orthogonal decomposition

$$D([X/G]) = \langle D(Y), D(D_1), \ldots, D(D_r), \{E_k\} \rangle$$

where the number of exceptional objects is given by $\# \sum_{i,j,i\neq j} D_i \cap D_j$.

6.3 Semi-orthogonal Decompositions of Surfaces of General Type

We now describe semi-orthogonal decompositions of the equivariant derived categories of surfaces of general type with an abelian group action. Some of these equivariant derived categories with have full exceptional collections which are in contrast to the case for ordinary derived categories where Alexeev-Orlov, Gorchinskiy- Orlov, Boöhning-Graf von Bothmer-Katzarkov-Sosna, Boöhning-Graf von Bothmer-Sosna, Galkin-Shinder and Galkin-Katzarkov-Mellit-Shinder have discovered (quasi)-phantom categories (see [2], [35], [11], [10], [28], [27]). We will focus on two examples: numerical Godeaux surfaces with an involution and Burniat surfaces with an action of the Klein four group.

Throughout this section, we will assume that all varieties are over the complex numbers.

6.3.1 Godeaux Surfaces with an Involution

One of the first surfaces of general type with $p_g = 0$ was constructed by Godeaux in the 1931. Consider the Fermat quintic surface $X = V(x^5 + y^5 + z^5 + t^5) \subset \mathbb{P}^3$. Then the weighted diagonal action of $\mathbb{Z}_5$ acting by $(x : y : z : w) \mapsto (\xi x : \xi^2 y : \xi^3 z : \xi^4 w)$ acts freely on $\mathbb{P}^4$ where $\xi^5 = 1$ and preserves the quintic. Then $S = X/\mathbb{Z}_5$ is a surface of general type with $p_g = q = 0$ and $K^2 = 1$ [3, VII §11]. Now we call any minimal surface of general type with these numerical invariants a *numerical Godeaux surface.*

Definition 6.3.1. *A numerical Godeaux surface S is a smooth minimal surface of general type with $p_g = q = 0$ and $K_S^2 = 1$.*

Numerical Godeaux surfaces have been studied by several authors over the last 40 years. Many attempts have been made to classify such surfaces and understand their moduli space. An important invariant associated to a numerical Godeaux surface S is the torsion subgroup $\mathrm{Tors}(S) = Pic(S)_{tor}$ of the Picard group. Miyaoka [54, Lemma 11, Theorem 2'] proved that $\mathrm{Tors}(S)$ is cyclic of order at most 5.

When $\mathrm{Tors}(S) = \mathbb{Z}_5$, these surfaces fill up an irreducible component of the moduli space with expected dimension 8. This component consists of quotients of quintics in $\mathbb{P}^3$ by a $\mathbb{Z}_5$ action, recovering Godeaux's original example. Godeaux surfaces with smaller torsion subgroups have been constructed but no classification is known and their moduli spaces are still a mystery [23, §1]. However, many of these are equipped with an involution, an automorphism of order 2 of the surface.

Numerical Godeaux surfaces with an involution were first considered by Keum and Lee [47] and generalized by Calabri, Ciliberto and Mendes Lopes [23].

Theorem 6.3.2. *A numerical Godeaux surface S with an involution σ is birationally equivalent to one of the following:*

1. *A double plane of Campedelli type;*
2. *A double plane branched along a reduced curve which is the union of two distinct lines and a curve of degree* 12 *with specified singularities.*
3. *A double cover of an Enriques surface branched along a curve of arithmetic genus* 2*.*

In case (3)*,* $\mathrm{Tors}(S) = \mathbb{Z}/4\mathbb{Z}$ *and in cases* (1), (2) $\mathrm{Tors}(S) = \mathbb{Z}/2\mathbb{Z}$ *or* $\mathbb{Z}/4\mathbb{Z}$*.*

We will focus on the case (3) but a similar story holds for cases (1) and (2).

First start with some notation following [53]. Let S be a numerical Godeaux surface and $\sigma\colon S \to S$ an involution of S. Let $\pi\colon S \to \Sigma = S/\sigma$ be the quotient map. Then by

[23, Proposition 4.5] the fixed point set Fix(σ) consists of a smooth (possible reducible) curve R and 5 isolated fixed points $p_1, \ldots, p_5$. Set $q_i = \pi(p_i)$ and $B = \pi(R)$ the branch locus. We have a diagram

$$\begin{tikzcd} V \arrow[r, "\epsilon"] \arrow[d, "\tilde{\pi}"] & S \arrow[d, "\pi"] \\ W \arrow[r, "\eta"] & \Sigma \end{tikzcd}$$

where ϵ is the blow up of S at $p_1, \ldots, p_5$, $\eta\colon W \to \Sigma$ is the minimal resolution of Σ and $\tilde{\pi}$ is a flat double cover. The quotient Σ has 5 A_1 singularities at q_i and is smooth otherwise. Denote by $C_i \subset W$ the exceptional (-2)-curves over q_i for $1 \leq i \leq 5$.

By [23, Proposition 3.9 and Lemma 4.11] there exists a birational morphism $f\colon W \to Y$ with:

- Y a smooth Enriques surface
- the exceptional locus of f is disjoint from the C_i
- there is a flat double cover $p\colon X \to Y$ fitting into the diagram

$$\begin{tikzcd} X \arrow[d, "p"] & V \arrow[l, "g"'] \arrow[r, "\epsilon"] \arrow[d, "\tilde{\pi}"] & S \arrow[d, "\pi"] \\ Y & W \arrow[l, "f"'] \arrow[r, "\eta"] & \Sigma \end{tikzcd}$$

As Y is a Enriques, one can show that $p_a(B) = p_a(R) = 2$.

Thus we get the following Theorem.

Theorem 6.3.3. *Let S be a numerical Godeaux surface with an involution σ such that the quotient $\Sigma = S/\mathbb{Z}_2$ is birational to an Enriques surface. Then there is a semi-orthogonal decomposition*

$$D^{\mathbb{Z}_2}(S) = \langle D(Y), D(B), E_1, \ldots, E_k \rangle$$

where B is a curve of arithmetic genus 2, Y the minimal model of Σ and E_i exceptional objects resulting from the birational map $f\colon W \to Y$ with $k \leq 4$.

Proof. This follows from applying Corollary 6.1.2 to the above diagram. □

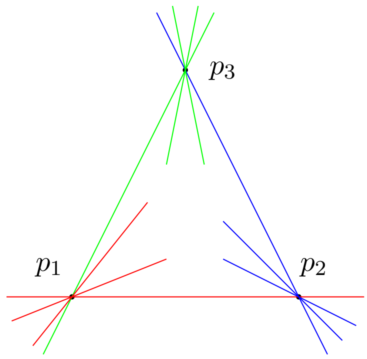
Remark 6.3.4. *There is a similar story for other Godeaux surfaces with an involution and numerical Campedelli surfaces with involutions ($p_g = 0$ and $K^2 = 2$) as outlined in [24] which will give similar semi-orthogonal decompositions. As the ramification divisor is a disjoint union of rational curves and the quotient is rational, the equivariant derived category will have an exceptional collection.*

6.3.2 Burniat Surfaces with a Klein Four Group Action

Burniat surfaces are minimal surfaces of general type constructed by Burniat in [21] with $p_g = q = 0$ and $K^2 = 2, 3, \dots, 6$. These surfaces can be constructed as a Klein four group Galois cover of a multiple blow ups of $\mathbb{P}^2$ branched over configurations of lines.

The case when $K^2 = 6$ was considered by Alexeev and Orlov [2] in which they show that the derived category contains an exceptional collection of length 6 which is not full. The orthogonal to this collection is an example of a "quasi"-phantom category (i.e. it has trivial Hochschild homology and torsion K-group).

We will construct a full exceptional collection of length 60 for the $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$-equivariant derived category of Burniat surfaces with $K^2 = 6$. We recall the construction of the Burniat surface with $K^2 = 6$ from [62]. Choose three points $\{p_1, p_2, p_3\}$ in $\mathbb{P}^2$, not colinear. Consider 3 reducible curves C_1, C_2, C_3 with each C_i consisting of 3 distinct lines passing through p_i and $p_{i+1} \in C_i$ but $p_{i+2} \notin C_i$ (indices are taken modulo 3). The curve C_1 corresponds to the red lines, C_2 to the blue lines and C_3 to the green lines in the diagram below.



Let $\sigma\colon P \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at the points p_1, p_2, p_3. Denote by $\widetilde{C}_i$ the strict transform of C_i and the exceptional divisors by E_i above the point p_i. Then

$$\widetilde{C}_i = \sigma^* C_i - 3E_i - E_{i+1} = 3H - 3E_i - E_{i+1}$$

Consider the curves $D_i = \widetilde{C}_i + E_{i+2}$. Then $D_i + D_j$ are 2-divisible. Set $D_i + D_j = 2F_k$ for i, j, k a cyclic permutation of $1, 2, 3$. Then the divisors $\{D_i\}_{i=1}^3$ and the line bundles $\{\mathcal{O}_P(-F_i)\}_{i=1}^3$ give the building data for a Galois $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ cover of P branched over $D = D_1 + D_2 + D_3$. Denote this cover by $Q \to P \to \mathbb{P}^2$. Then Q is the Burniat surface with $K^2 = 6$ [3, V §11] [62].

The quotient stack $[Q/G]$ is constructed as an iterated root stack over P along the divisors (D_1, D_2, D_3) of order $(2, 2, 2)$. As $D_i \cdot D_j = 10$ for $i \neq j \in \{1, 2, 3\}$ we have 30 exceptional objects E_{ij} arising from the points of intersection of D_i and D_j. We have also have components arising from the derived categories of D_i and of P. So we have a semi-orthogonal decomposition

$$D([Q/G]) = \langle \{E_{ij}\}, D(D_3), D(D_2), D(D_1), D(P) \rangle\,.$$

As P is the blow up of $\mathbb{P}^2$ at 3 points we see that $D(P)$ is generated by 6 exceptional objects by Orlov's blow up formula. Moreover, each D_i is a sum of rational curves and so $D(D_i)$ is generated by $4 \times 2 = 8$ exceptional objects. Thus we have that $D([Q/G])$ is generated by $30 + 8 \times 3 + 6 = 60$ exceptional objects.

6.4 Derived McKay Correspondence in Dimension 2

In this section, we give a new proof of the derived McKay Correspondence in dimension 2 for non-trivial finite subgroups of $\mathrm{GL}(2, \mathbb{C})$ compared with [45] which uses the McKay Correspondence for subgroups of $\mathrm{SL}(2, \mathbb{C})$ [44] and for cyclic subgroups of $\mathrm{GL}(2, \mathbb{C})$ [40].

Theorem 6.4.1. *Let $G \subset \mathrm{GL}(2, \mathbb{C})$ be a non-trivial finite subgroup acting on $\mathbb{C}^2$. Then there is a semi-orthogonal decomposition of the equivariant derived category*

$$D^G(\mathbb{C}^2) = \left\langle E_1, \ldots, E_n, \Phi_{\widetilde{D_1}} D(\widetilde{D}_1), \ldots, \Phi_{\widetilde{D_n}} D(\widetilde{D}_m), \Phi_{\widetilde{Y}} D(\widetilde{Y}) \right\rangle$$

where Y is the minimal resolution of $\mathbb{C}^2/G$, $\widetilde{D}_i$ are the normalizations of the irreducible components of the branch divisor $D = \sum_{i=1}^m D_i$ and $E_1, \ldots, E_n$ are exceptional objects.

Proof. Let $G \subset \mathrm{GL}(2, \mathbb{C})$ be a finite subgroup and set $H = \mathrm{SL}(2, \mathbb{C}) \cap G$. Then H is a normal subgroup of G and $A = G/H$ is a finite cyclic group of order r since $\det\colon \mathrm{GL}(2, \mathbb{C}) \to \mathbb{C}^*$ identifies A with a subgroup of $\mathbb{C}^*$. Let $Y = H - \mathrm{Hilb}(\mathbb{C}^2)$ be the minimal resolution of $\mathbb{C}^2/H$ by the McKay Correspondence for subgroups of $\mathrm{SL}(2, \mathbb{C})$.

There is a natural G action on Y where $g \in G$ sends a subscheme $Z \in Y = H - \mathrm{Hilb}(\mathbb{C}^2)$ to its image $g \cdot Z$ under the action $g\colon \mathbb{C}^2 \to \mathbb{C}^2$. Since Z is H-invariant (by definition of $Y = H - \mathrm{Hilb}(\mathbb{C}^2)$), the G action on Y descends to a $A = G/H$ action on Y.

Thus we have the following diagram

$$\begin{tikzcd}
\mathbb{C}^2 \arrow[dd] & & \\
& Y \arrow[dl, "f"'] \arrow[dd] & \\
\mathbb{C}^2/H \arrow[dd] & & M \arrow[dl] \arrow[dd] \\
& Y/A \arrow[dl, "\bar{f}"'] & \\
\mathbb{C}^2/G & & \widetilde{Y} \arrow[ll]
\end{tikzcd}$$

where M is the minimal resolution of Y/A and $\widetilde{Y}$ is the minimal resolution of $\mathbb{C}^2/G$. Note that $\bar{f}$ is a projective birational morphism (see [45, §7]). As Y/A is birational to $\mathbb{C}^2/G$, M is a resolution of $\mathbb{C}^2/G$. By contracting (-1)-curves in M we obtain the minimal resolution $\widetilde{Y}$.

We now follow the proof of [40, Theorem 4.1] to show that $D([Y/A]) \cong D([\mathbb{C}^2/G])$. Consider the diagram

$$\begin{tikzcd} & \mathcal{Z} \arrow[d, phantom, "\cap"] & \\ & Y \times \mathbb{C}^2 \arrow[dl, "p"'] \arrow[dr, "q"] & \\ Y & & \mathbb{C}^2 \end{tikzcd}$$

where $\mathcal{Z} \subset Y \times \mathbb{C}^2$ is the universal subscheme and p, q are the natural projections. As G acts diagonally on $Y \times \mathbb{C}^2$ and G preserves $\mathcal{Z}$, we can take the the quotient of the whole diagram with respect to the action of G. Thus we have a diagram

$$\begin{tikzcd} & {[\mathcal{Z}/G]} \arrow[d, phantom, "\cap"] & \\ & {[Y \times \mathbb{C}^2/G]} \arrow[dl, "\overline{p}"'] \arrow[dr, "\overline{q}"] & \\ {[Y/G]} & & {[\mathbb{C}^2/G]} \end{tikzcd}.$$

Consider the natural morphism

$$\varphi\colon [Y/G] \to [Y/A]$$

from the surjection $G \twoheadrightarrow A$. Then the pullback functor

$$\varphi^*\colon D([Y/A]) \to D([Y/G])$$

sends an A-equivariant coherent sheaf on Y to the same sheaf considered as a G-equivariant sheaf through the surjective homomorphism $G \twoheadrightarrow A$.

Then we can define the integral functor

$$\Phi\colon D([Y/A]) \to D([\mathbb{C}^2/G])$$

by

$$\Phi(E^\bullet) = \overline{q}_*(\mathcal{O}_{[\mathcal{Z}/G]} \otimes \overline{p}^*(\varphi^*(E^\bullet))).$$

This functor is an equivalence by [40, Theorem 4.1].

As A is abelian and Y is smooth, by Corollary 6.1.2 we have a semi-orthogonal decomposition

$$D([\mathbb{C}^2/G]) = D^G(\mathbb{C}^2) = \left\langle D(M), D(\widetilde{D}_1), \ldots, D(\widetilde{D}_m), E_1, \ldots, E_k \right\rangle$$

where E_i are exceptional objects and $\widetilde{D_i}$ are the irreducible components of the branch divisor on $(Y/A)^{can}$.

As M is a blow up of $\widetilde{Y}$, by Orlov's blow up formula we have the further semi-orthogonal decomposition

$$D^G(\mathbb{C}^2) = \left\langle D(\widetilde{Y}), D(\widetilde{D}_1), \ldots, D(\widetilde{D}_m), E_1, \ldots, E_l \right\rangle$$

where $\widetilde{Y}$ is the minimal resolution of $\mathbb{C}^2/G$.

As the diagram above commutes, the branch divisors of $Y \to Y/A$ are the strict transforms of the branch divisors of $\mathbb{C}^2/H \to \mathbb{C}^2/G$. As $\mathbb{C}^2 \to \mathbb{C}^2/H$ is only ramified in codimension 2, the branch divisor of $\mathbb{C}^2 \to \mathbb{C}^2/G$ is the same as $\mathbb{C}^2/H \to \mathbb{C}^2/G$. As $\widetilde{D}_i \to D_i$ is birational and $\widetilde{D}_i$ is normal, $\widetilde{D_i}$ is isomorphic to the normalization of D_i.

Thus we get the semi-orthogonal decomposition

$$D^G(\mathbb{C}^2) = \left\langle D(\widetilde{Y}), D(\widetilde{D}_1), \ldots, D(\widetilde{D}_m), E_1, \ldots, E_n \right\rangle .$$

□

Remark 6.4.2. *Note that when $G \subset \mathrm{SL}(2,\mathbb{C})$ then $G = H$ and $A = id$ and we recover the traditional McKay Correspondence.*

When $G \subset GL(2,\mathbb{C})$ is small (i.e. contains no psuedoreflections) the branch divisor on $\mathbb{C}^2/G$ is empty and so the category orthogonal to the minimal resolution is generated by an exceptional collection as described by Ishii and Ueda in [40]. This recovers the result of Ishii-Ueda but note their result for canonical stacks is central to the proof of the theorem.

6.5 Motivic Decomposition for Dihedral Groups

In [64] Polishchuk and Van den Bergh propose the following conjecture. Recall that for a group G the centralizer of $g \in G$ is

$$C(g) = \{h \in G | hg = gh\} .$$

Conjecture 6.5.1 ([64, Conjecture A])**.** *Assume that a finite group G acts effectively on a smooth quasi-projective variety X over an algebraically closed field and that all the quotients $X^g/C(g)$ are smooth for $g \in G$. Then there exists a semi-orthogonal decomposition of the derived category $D^G(X)$ of G-equivariant sheaves on X such that the pieces $\mathcal{C}_{[g]}$ of this decomposition are in bijection with the conjugacy classes of g in G and $\mathcal{C}_{[g]} \cong D(X^g/C(g))$.*

In this section, we describe a semi-orthogonal decomposition for a natural action of D_{2n} on $\mathbb{C}^2$ and prove that these semi-orthogonal decompositions satisfy Conjecture 6.5.1.

Recall that the dihedral group D_{2n} of order $2n$ for $n \geq 2$ has a presentation

$$D_{2n} = \left\{\tau, \sigma \middle| \tau^n = \sigma^2 = e,\, \tau\sigma\tau = \sigma\right\}.$$

Define the effective action of D_{2n} on $\mathbb{C}^2$ by $\rho\colon D_{2n} \to \mathrm{GL}(2, \mathbb{C})$ where

$$\rho(\tau) = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}, \quad \rho(\sigma) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with $\zeta^n = 1$ a complex n^{th} root of unity. Let $\pi\colon \mathbb{C}^2 \to \mathbb{C}^2 / D_{2n}$ denote the quotient map. As D_{2n} is generated by the reflections σ and $\sigma\tau$, the quotient $\mathbb{C}^2 / D_{2n} \cong \operatorname{Spec} \mathbb{C}[a, b]$ is smooth by the Chevalley-Shepard-Todd Theorem. The ramification divisor R is a collection of hyperplanes and the branch divisor $\pi(R) = D$ is given by the equation $V(a^2 - b^n)$, which is singular with an A_{n-1}-singularity at $(0, 0)$.

We now recall the following results on G-clusters and $G - \mathrm{Hilb}(\mathbb{C}^2)$ for cyclic groups from [42, §12]. Let $G = \mathbb{Z}_n$ be generated by τ and $n \geq 2$. Define the action of G on $\mathbb{C}^2$ by $\tau(x, y) = (\zeta x, \zeta^{-1} y)$ where $\zeta^{n+1} = 1$ is a complex n-th root of unity. Then $\mathbb{C}^2 / G$ is the simple singularity of type A_{n-1} and its minimal resolution $Y \to \mathbb{C}^2 / G$ is isomorphic to $G - \mathrm{Hilb}(\mathbb{C}^2)$. The following description of points of Y and affine charts covering Y will be useful.

Lemma 6.5.2 ([42, Lemma 12.2])**.** *Any $I \in G - \mathrm{Hilb}(\mathbb{C}^2)$ is one of the following ideals of colength n:*

$$I(\Sigma)\colon = \prod_{\mathfrak{p} \in \Sigma} \mathfrak{m}_{\mathfrak{p}} = (x^n - a^n, xy - ab, y^n - b^n), \tag{6.1}$$

where $\Sigma = G \cdot (a, b)$ is a G-orbit of $\mathbb{C}^2$ disjoint from the origin; or

$$I_i(p_i\colon q_i)\colon = (p_i x^i - q_i y^{n-i}, xy, x^{i+1}, y^{n+1-i}), \tag{6.2}$$

for some $1 \leq i \leq n-1$ and some $[p_i\colon q_i] \in \mathbb{P}^1$.

Theorem 6.5.3 ([42, Theorem 12.3])**.** *Let a, b be parameters of $\mathbb{C}^2$ on which the group G acts by $\tau(a, b) = (\zeta a, \zeta^{-1} b)$.*

Let $X = \mathbb{C}^2 / G\colon = \operatorname{Spec} \mathbb{C}[a^n, ab, b^n]$ and $Y \to X$ be its (toric) minimal resolution, with affine charts U_i defined by

$$U_i = \operatorname{Spec} \mathbb{C}[s_i, t_i] \text{ for } 1 \leq i \leq n,$$

where $s_i\colon = a^i / b^{n-i}$ and $t_i = b^{n+1-i} / a^{i-1}$. Then the isomorphism of Y to $G - \mathrm{Hilb}(\mathbb{C}^2)$ is given by (the morphism defined by the universal property of $\mathrm{Hilb}^n(\mathbb{C}^2)$ from) two dimensional flat families of subschemes defined by the G-invariant ideals of $\mathcal{O}_{\mathbb{C}^2}$

$$\mathcal{I}_i(s_i, t_i)\colon = (x^i - s_i y^{n-i}, xy - s_i t_i, y^{n+1-i} - t_i x^{i-1})$$

for $1 \leq i \leq n$.

Theorem 6.5.4. *Let D_{2n} act on $\mathbb{C}^2$ as above. Then we have two cases:*

Odd n**:** *There is a semi-orthogonal decomposition*

$$D^{D_{2n}}(\mathbb{C}^2) = \left\langle \pi^* D(\mathbb{C}^2), \Phi_{\widetilde{D}}(D(\widetilde{D})), E_1, \ldots, E_{\frac{n-1}{2}} \right\rangle$$

where $\widetilde{D}$ is the normalization of D.

Even n**:** *There is a semi-orthogonal decomposition*

$$D^{D_{2n}}(\mathbb{C}^2) = \left\langle \pi^* D(\mathbb{C}^2), \Phi_{\widetilde{D}_1}(D(\widetilde{D}_1)), \Phi_{\widetilde{D}_2}(D(\widetilde{D}_2)), E_1, \ldots, E_{\frac{n}{2}} \right\rangle$$

where $D = D_1 \cup D_2$ is reducible and $\widetilde{D}_i$ are the normalization of D_i.

Proof. As $\rho(D_{2n}) \cap \mathrm{SL}(2, \mathbb{C}) \cong \mathbb{Z}_n$ we have a diagram

$$\begin{tikzcd}
\mathbb{C}^2 \arrow[d] & \\
\mathbb{C}^2 / \mathbb{Z}_n \arrow[d] & Y \arrow[l] \arrow[d] \\
\mathbb{C}^2 / D_{2n} & Y/\mathbb{Z}_2 \arrow[l]
\end{tikzcd}$$

where Y is the minimal resolution of $\mathbb{C}^2 / \mathbb{Z}_n$ and $D = V(a^2 - b^n)$ is the branch divisor.

Recall from the proof of Theorem 6.4.1 that the $\mathbb{Z}_2$ action on Y is induced by the action of $\mathbb{Z}_2$ on $\mathbb{C}^2$. Let σ be a generator of $\mathbb{Z}_2$. Then $\sigma(a, b) = (b, a)$ for a point $(a, b) \in \mathbb{C}^2$.

Using the description of points of Y in terms of ideals in Lemma 6.5.2 we see that an ideal of the form in Equation (6.1) is fixed if and only if $a = b$, i.e. the cluster is supported on the fixed loci of $\mathbb{Z}_2$ acting on $\mathbb{C}^2$. These clusters form the fixed locus of $\mathbb{Z}_2$ acting on Y. By analyzing ideals of the form in Equation (6.2), we see that σ interchanges ideals $I_i(p_i : q_i)$ with ideals of the form $I_{n-i}(q_i : p_i)$.

We now consider the cases when n is odd or even.

Odd n: Suppose that n is odd. Then $n-1$ is even and Y contain $n-1$ (-2)-curves and the action of Z_2 interchanges each pair of (-2)-curves. The only fixed point occurs at the intersection of the (-2)-curves whose points correspond to ideals of the form $I_{\frac{n-1}{2}}(p_i : q_i)$ and $I_{\frac{n+1}{2}}(p_i : q_i)$. They meet at the point $I_{\frac{n-1}{2}}(1 : 1) = I_{\frac{n+1}{2}}(1 : 1)$.

By looking at the affine chart $U_{\frac{n+1}{2}} = \operatorname{Spec} \mathbb{C}[s, t]$ where $s = a^{\frac{n+1}{2}} / b^{\frac{n-1}{2}}$ and $t = b^{\frac{n+1}{2}} / a^{\frac{n-1}{2}}$ in Theorem 6.5.3, we see that $\mathbb{Z}_2$ acts by psuedoreflections at the only isolated fixed point of $\mathbb{Z}_2$. Thus $Y/\mathbb{Z}_2$ is smooth.

Hence $(Y/Z_2)^{can} \cong Y/Z_2$ and we have a semi-orthogonal decomposition

$$D([\mathbb{C}^2/D_{2n}] = \langle D(B), D(Y/Z_2)\rangle$$

where B is the branch divisor of $Y \to Y/\mathbb{Z}_2$.

Note that Y/Z_2 contains exactly $\frac{n-1}{2}$ irreducible curves, which are the image of the (-2)-curves on Y, that are contracted to a point by the birational morphism $f\colon Y/Z_2 \to \mathbb{C}^2/D_{2n}$. By [37, §V, Corollary 5.4] f can be factored as the composition of $\frac{n-1}{2}$ blow ups of $\mathbb{C}^2$. Hence by Theorem 6.4.1 we have a semi-orthogonal decomposition

$$D([\mathbb{C}^2/D_{2n}]) \cong D^{D_{2n}}(\mathbb{C}^2) = \left\langle \pi^* D(\mathbb{C}^2), \Phi_{\widetilde{D}}(D(B)), E_1, \dots, E_{\frac{n-1}{2}} \right\rangle.$$

As B is smooth and maps birationally to the branch divisor D, $B = \widetilde{D}$ is the normalization of D. As D is irreducible, so is $\widetilde{D}$. Hence we have the semi-orthogonal decomposition

$$D([\mathbb{C}^2/D_{2n}]) \cong D^{D_{2n}}(\mathbb{C}^2) = \left\langle \pi^* D(\mathbb{C}^2), \Phi_{\widetilde{D}}(D(\widetilde{D})), E_1, \dots, E_{\frac{n-1}{2}} \right\rangle.$$

Even n: Suppose that n is even, so $n-1$ is odd. Then the action of $\mathbb{Z}_2$ interchanges the (-2)-curves E_i and E_{n-i} on Y except when $i = n/2$. Then the points correspond to ideals of the form

$$I_{\frac{n}{2}}(p_i\colon q_i) = (p_i x^{n/2} - q_i y^{n/2}, xy, x^{n/2+1}, y^{n/2+1}).$$

Then $\mathbb{Z}_2$ acts freely on $E_{n/2}$ sending $(p_i\colon q_i)$ to $(-q_i\colon p_i)$. Hence $\mathbb{Z}_2$ acts without isolated fixed points and the quotient Y/Z_2 is smooth.

Note that Y/Z_2 contains exactly $\frac{n}{2}$ irreducible curves, which are the image of the (-2)-curves on Y, that are contracted to a point by the birational morphism $f\colon Y/Z_2 \to \mathbb{C}^2/D_{2n}$. By [37, §V, Corollary 5.4] f can be factored into the composition of $\frac{n-1}{2}$ blow ups of $\mathbb{C}^2$. Hence by Theorem 6.4.1 we have a semi-orthogonal decomposition

$$D([\mathbb{C}^2/D_{2n}]) \cong D^{D_{2n}}(\mathbb{C}^2) = \left\langle \pi^* D(\mathbb{C}^2), \Phi_{\widetilde{D}}(D(B)), E_1, \dots, E_{\frac{n}{2}} \right\rangle.$$

Note that the branch divisor $D = V(a^2 - b^n) = V\left((a^2 - b^{\frac{n}{2}})(a^2 + b^{\frac{n}{2}})\right)$ is reducible with $D = D_1 + D_2$. Therefore the normalization $\widetilde{D}$ is reducible and $\widetilde{D} = \widetilde{D}_1 + \widetilde{D}_2$ with $\widetilde{D}_i$ the normalization of D_i. As B maps birationally to the branch divisor D, $B = \widetilde{D}$ is the normalization of D. Hence $B = \widetilde{D} = \widetilde{D}_1 + \widetilde{D}_2$. By Theorem 6.4.1 we have a semi-orthogonal decomposition

$$D^{D_{2n}}(\mathbb{C}^2) = \left\langle \pi^* D(\mathbb{C}^2), \Phi_{\widetilde{D}_1}(D(\widetilde{D}_1)), \Phi_{\widetilde{D}_2}(D(\widetilde{D}_2)), E_1, \dots, E_{\frac{n}{2}} \right\rangle.$$

□

Corollary 6.5.5. *The semi-orthogonal decompositions described in Theorem 6.5.4 satisfy Conjecture 6.5.1.*

Proof. The bijection is given by

$$[g] \mapsto D(X^g/C(g))$$

which we describe explicitly for n odd and even.

Odd n: The conjugacy classes of D_{2n} are $[e], [b], [a^i]$ for $1 \leq i \leq (n-1)/2$. The bijection is given by

$$\begin{aligned} [e] &\longleftrightarrow D(\mathbb{C}^2/D_{2n}) \\ [b] &\longleftrightarrow D(\widetilde{D}) \\ [a^i] &\longleftrightarrow E_i \text{ for } 1 \leq i \leq \frac{n-1}{2}. \end{aligned}$$

Even n: The conjugacy classes of D_{2n} are $[e], [b], [ab], [a^{\frac{n}{2}}], [a^i]$ for $1 \leq i < (n-2)/2$. The bijection is given by

$$\begin{aligned} [e] &\longleftrightarrow D(\mathbb{C}^2/D_{2n}) \\ [b] &\longleftrightarrow D(\widetilde{D_1}) \\ [ab] &\longleftrightarrow D(\widetilde{D_2}) \\ [a^{n/2}] &\longleftrightarrow E_0 \\ [a^i] &\longleftrightarrow E_i \text{ for } 1 \leq i \leq \frac{n-2}{2}. \end{aligned}$$

□

Remark 6.5.6. *For $D_6 \cong S_3$ the semi-orthogonal decomposition described in [64] agrees with the one here.*

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