Lagrangian Multiform Structures, Discrete Systems and Quantisation

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Submitted in accordance with the requirements for the degree of Doctor of Philosophy

The University of Leeds
School of Mathematics

August 2017

The candidate confirms that the work submitted is his own, except where work which has formed part of jointly authored publications has been included. The contribution of the candidate and other authors to this work has been explicitly indicated below. The candidate confirms that appropriate credit has been given where reference has been made to the work of others.

Chapters 2 and 3 are based on the work in [56], *Quantum Variational Principle and quantum multiform structure: the case of quadratic Lagrangians*, SD King and FW Nijhoff, arXiv preprint arXiv:1702.08709 (2017). SDK and FWN discussed ideas in conjunction. SDK carried out the majority of the computations. SDK wrote the majority of the paper and was corresponding author, FWN wrote the introduction and conclusion sections and offered corrections.

Acknowledgements

This thesis would not exist without the hard work, insights and suggestions of Frank Nijhoff, I am very grateful for many fruitful and interesting discussions. Thanks are also due to Simon Ruijsenaars for his comments at various points.

I would particularly like to thank Chuck and Shelly Parker, who encouraged me that it is possible to study mathematics to the glory of God, and set me off down this path. Also Adriana Pesci and Kerry Maunder, who willingly provided very kind references for a student who reappeared after several years. My parents were very supportive of my decision to return to academic study and I am grateful for their encouragement then and throughout.

An incredible debt of thanks is due to Louisa, who has been unwaveringly supportive, despite claiming not to understand what I do even slightly. Despite this, she gives a surprisingly good summary of the research topic. Her patience and selflessness especially now, as she cares for a baby whilst I finish a thesis, has been incredible. Thanks also go to Benjamin who, although he didn't know it, gave me a very good reason to get on and finish.

Lastly, the PhD journey would have been altogether more dreary without the company of fellow students. Thanks to Andy for unnumbered hundreds of cups of tea, and almost as many puns. Thanks are also due to Luke, Anja, Josh, Adam, Colleen and Raph, all of whom have made the journey more enjoyable. I've had a wonderful time.

Abstract

Lagrangian multiforms are an important recent development in the study of integrable variational problems. In this thesis, we develop two simple examples of the discrete Lagrangian one-form and two-form structures. These linear models still display all the features of the discrete Lagrangian multiform; in particular, the property of Lagrangian closure. That is, the sum of Lagrangians around a closed loop or surface, on solutions, is zero. We study the behaviour of these Lagrangian multiform structures under path integral quantisation and uncover a quantum analogue to the Lagrangian closure property. For the one-form, the quantum mechanical propagator in multiple times is found to be independent of the time-path, depending only on the endpoints. Similarly, for the two-form we define a propagator over a surface in discrete space-time and show that this is independent of the surface geometry, depending only on the boundary.

It is not yet clear how to extend these quantised Lagrangian multiforms to non-linear or continuous time models, but by examining two such examples, the generalised McMillan maps and the Degasperis-Ruijsenaars model, we are able to make some steps towards that goal. For the generalised McMillan maps we find a novel formulation of the r-matrix for the dual Lax pair as a normally ordered fraction in elementary shift matrices, which offers a new perspective on the structure. The dual Lax pair may ultimately lead to commuting flows and a one-form structure. We establish the relation between the Degasperis-Ruijsenaars model and the integrable Ruijsenaars-Schneider model, leading to a Lax pair and two particle Lagrangian, as well as finding the quantum mechanical propagator. The link between these results is still needed.

A quantum theory of Lagrangian multiforms offers a new paradigm for path integral quantisation of integrable systems; this thesis offers some first steps towards this theory.

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1 Introduction

Discrete systems occupy an important position in the theory of integrability. Many standard integrability properties have turned out to have discrete counterparts that in some sense are more fundamental than their continuous relatives, so that in recent years there has been a large growth in the study of such systems [15, 22, 51, 68, 75, 80]. These systems have surprisingly wide application in areas as diverse as discrete differential geometry, cluster algebras, Painlevé equations, random matrix theory and others [16]. Discrete integrable systems also give rise to interesting new mathematical theories through their underlying structures; as we allow these systems to speak for themselves, we uncover new principles that are suitable for describing their structures and integrability properties.

There are two parallel but equivalent approaches to classical mechanics: that is the Hamiltonian and Lagrangian formalisms. Traditionally, the Hamiltonian perspective has been dominant in the study of integrable systems: Liouville integrability requires the existence of a sufficient number of commuting invariants, leading to direct linearisations of integrable models. These invariants are interpreted as Hamiltonians generating multiple commuting time-flows [5, 22, 116, 117]. A recent development, however, has been the

Lagrangian multiform structure discovered behind many integrable models [17–19, 51, 62–65, 110, 112, 111, 123–126]. This is a variational framework able to capture the aspect of multiple equations holding simultaneously on the same set of variables, which appears to exist on a fundamental level for discrete models; the known continuous examples have been derived by continuum limit. In the case of evolutionary equations, these continuous Lagrangian one-forms are related to the multiple commuting Hamiltonians by Legendre transform. Although Hamiltonian and Lagrangian formulations are equivalent, as Dirac writes, "there are reasons for believing that the Lagrangian one is the more fundamental" [27].

A particularly interesting feature of these Lagrangian multiform structures is their accompanying variational principle. In the usual least action principle we extremise the action under variation of the *dependent* variables, producing the equations of motion. In the multiform case, however, the full set of simultaneous equations of motion arises from the variation of both independent *and* dependent variables: that is, the action must be stationary under a variation of the underlying geometry of the independent variables. This has been observed in the case of evolutionary equations, where variation is over the time-path through multiple time variables [125, 126], and for lattice field equations, where an underlying two dimensional, space-time surface is varied in a third (or higher) dimensions [63]. The outcome of this variational principle is that the resulting Euler-Lagrange equations become the defining equations for the Lagrangians themselves; only those Lagrangians with the so-called *closure property* yield actions which are stationary in the extended variational principle.

The preference for Hamiltonian descriptions of integrable systems is also reflected in a preference for *canonical* quantisation of such systems in the literature. Yet the alternative Feynman's path integral, or sum over histories, quantisation has been known for almost seventy years [36, 37]. Moreover, the sum over histories approach is known to be advantageous in many areas of physics, not least because (in contrast to the Hamiltonian approach) it can be written in a manifestly covariant, relativistic way [107, 114]. Up until recently, however, an integrable understanding of Lagrangian structures was missing from the literature. Without a good understanding of the necessary *classical* Lagrangian structures, the corresponding quantum question was unapproachable. The development of the Lagrangian multiform structure therefore offers a tantalising possibility: can a path integral quantisation for integrable systems, capturing the multiple flows that are a hallmark of integrability, now be developed? In this thesis we make some tentative first steps towards answering this question.

Discrete systems are particularly important in this context. From an integrability perspective, discrete systems represent a more fundamental set of models than their continuum limits. From a quantum perspective, the path integral for continuous systems can be problematic due to the difficulty of establishing the measure in an infinitesimal time slicing. For discrete systems, such difficulties are avoided by the finiteness of the time-steps: there is no infinitesimal time-slicing limit for a discrete system. Additionally, calculating the path integral in practice typically requires a discretisation of the Lagrangian, but the correct discretisation is not necessarily obvious since many different discrete models might lead to the same continuum limit. However, *integrable* discretisations are generally somewhat unique, potentially resolving this ambiguity. The interpolating continuous flows that belong to integrable discrete systems then suggest the possibility of establishing a path integral without any need for the time slicing limit at all.

In this introduction we review some helpful groundwork. In section 1.1 we consider integrable lattice models and their fundamental property of multi-dimensional consistency, which is the basis for the Lagrangian 2-form structure in the discrete case. In section 1.2 we look at discrete mappings and their integrable structures, essentially captured in multiple commuting Hamiltonians. We consider the discrete Ruijsenaars-Schneider model as an example of an integrable discrete mapping that exhibits commuting *discrete* flows, which give rise to a Lagrangian 1-form structure. We review some important ideas of quantum mechanics in section 1.3, and clarify what we understand by a quantum discrete system (of which many variations appear in the literature). In section 1.4 we consider possible links between this work and ideas in the field of quantum gravity. Integrable systems give us an insight into fundamental mathematical structures, and progress in the field of integrable systems may in turn guide the mathematics necessary in other areas of physics. Section 1.5 contains a brief overview of the thesis.

1.1 Classical lattice models

A key starting point in the study of integrable discrete-time systems is to consider partial difference equations on a space-time lattice - systems where time and space directions are

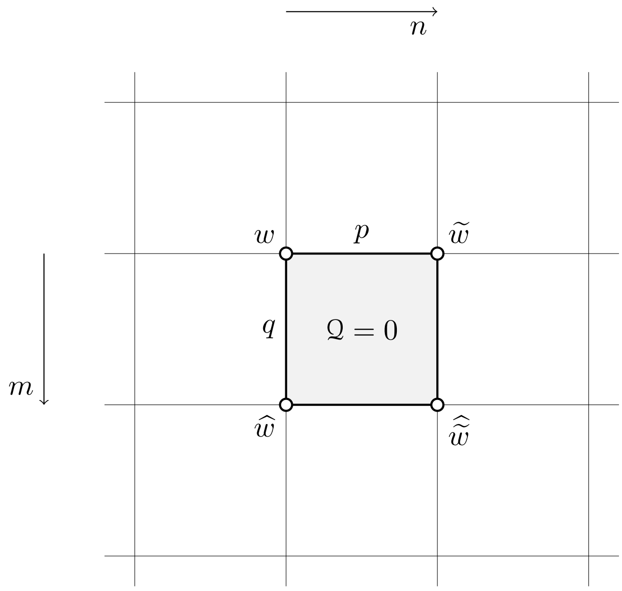

Figure 1.1: The quad equation $\mathcal{Q} = 0$ embedded within a square lattice.

both discrete and essentially on an equal footing. We consider field variables w at each lattice site on a two-dimensional square lattice, labelled by discrete independent variables n, m. We use "shift" notation, denoting movement in the n direction by a tilde and in the m direction by a hat. Under-accents represent backwards shifts, so that if we define the notation $w := w_{n,m}$, then

$$\widetilde{w} := w_{n+1,m} \ , \quad \underset{\sim}{w} := w_{n-1,m} \ , \quad \widehat{w} := w_{n,m+1} \ , \quad \underset{\wedge}{w} := w_{n,m-1} \ . \tag{1.1}$$

Equations determine the dynamics of the field variables across the lattice. In particular, we study quad equations - equations that link the four variables at the corners of each elementary plaquette on the lattice, and are then repeated across the entire lattice, shown in figure 1.1. Such an equation has the form

$$\mathcal{Q}(w, \widetilde{w}, \widehat{w}, \widehat{\widetilde{w}}; p, q) = 0 \ , \tag{1.2}$$

where p and q represent lattice parameters in the n and m directions.

The integrability of such quad equations is characterised by their *multi-dimensional consistency* [15, 83, 71]. The quad equation (1.2) and two-dimensional lattice are embedded within a three (or higher) dimensional square lattice, labelled by three discrete variables n, m, l. The third shift is labelled by a bar $^{-}$, with lattice parameter r, so that for field variable $w := w_{n,m,l}$ we have

$$\overline{w} := w_{n,m,l+1} \ , \quad \underline{w} := w_{n,m,l-1} \ . \tag{1.3}$$

The quad equation (1.2) produces companion equations,

$$\mathcal{Q}(w,\widehat{w},\overline{w},\widehat{\overline{w}};q,r)=0\ ,\quad \mathcal{Q}(w,\overline{w},\widetilde{w},\widetilde{\overline{w}};r,p)=0\ , \tag{1.4}$$

that hold across elementary plaquettes in the $(m-l)$ and $(l-n)$ lattice planes respectively.

An alternative notation labels the lattice directions $(1,2,3)$, introducing shifts

$$w_1 := w_{n+1,m,l}\ ,\quad w_2 := w_{n,m+1,l}\ ,\quad w_3 := w_{n,m,l+1}\ , \tag{1.5}$$

with lattice parameters p_i, $i=1,2,3$. Then the quad equations (1.2) and (1.4) can be summarised by the general form

$$\mathcal{Q}(w,w_i,w_j,w_{ij};p_i,p_j)=0\ . \tag{1.6}$$

Notice that implicitly this requires symmetry of the quad equation under the interchange of the lattice directions.

The question then remains: for a given quad equation $\mathcal{Q}=0$, can this embedding in a higher dimensional lattice be done consistently? Beginning with initial conditions $w,\widetilde{w},\widehat{w},\overline{w}$, it is clear from figure 1.2 that the opposite corner of the elementary cube, $\widehat{\widetilde{\overline{w}}}$, can be calculated via three different routes. Multi-dimensional consistency is captured in the property of *closure around the cube*: the multi-dimensional embedding (1.6) is consistent if the three possible values for $\widehat{\widetilde{\overline{w}}}$ coincide. This is the key integrability criterion for such lattice models [82, 83].

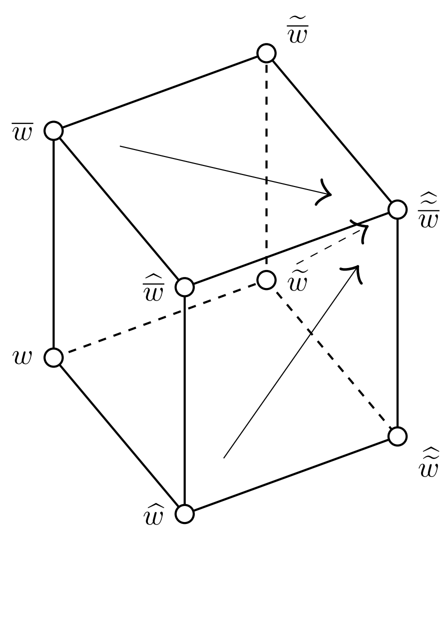

Figure 1.2: Closure around the cube: arrows show the three routes to calculate $\widehat{\widetilde{\overline{w}}}$.

Closure around the cube is the basis of the well-known classification of integrable quad equations in the ABS list [1, 2]. Demanding that the quad equation $\mathcal{Q} = 0$ (1.6) satisfy conditions of affine linearity, symmetry under D_4 on the square, and three-dimensional consistency, the authors classified all possible such equations in three lists, up to Möbius transformations on the variables and point transformations of the lattice parameters. These are labelled (Q1) - (Q4), (H1) - (H3) and (A1) - (A2). The affine-linearity property in particular is justified by restricting attention to those equations which can be uniquely solved for any argument: that is, those with a unique solution across the lattice for appropriate initial conditions.

An alternative approach to deriving integrable lattice equations is found in [80], based on a direct linearisation method. Such lattice equations are more general than the quad equations above, in that they involve not only the field variables around a square, but also field variables at the next-nearest and possibly further lattice sites. These equations form an integrable hierarchy: the *lattice Gelfand-Dikii hierarchy*, integrable due to the existence of an underlying Lax structure. The simplest member of the hierarchy is the so-called lattice Korteweg-de Vries (KdV) equation [75, 88, 120],

$$(p - q + \widehat{w} - \widetilde{w})(p + q - \widehat{\widetilde{w}} + w) = p^2 - q^2 \ . \tag{1.7}$$

This is an important example: in addition to sitting in the lattice Gelfand-Dikii hierarchy, the lattice KdV equation is also a quad equation (it links variables around an elementary square plaquette) and in fact is equivalent to (H1) of the ABS list under a simple transformation.

1.1.1 Key example: the Lattice KdV Equation

We consider the lattice KdV equation (1.7) as an illustrative example for the integrability of multi-dimensionally consistent lattice equations. Embedding the equation within a three-dimensional lattice, we write the multi-dimensional form (1.6)

$$(p_i - p_j + w_j - w_i)(p_i + p_j + w - w_{ij}) = p_i^2 - p_j^2 \ . \tag{1.8}$$

Beginning with initial values on four corners of a cube, $w, \widetilde{w}, \widehat{w}, \overline{w}$, as in figure 1.2, it is then possible to calculate the opposite corner $\widehat{\widetilde{\overline{w}}}$ in three different ways. For instance,

$$\widehat{\widetilde{\overline{w}}} = p + q + \overline{w} - \frac{p^2 - q^2}{p - q + \widehat{\overline{w}} - \widetilde{\overline{w}}} \ . \tag{1.9}$$

This is most easily expressed in terms of transformed variables w (indicated by Roman script) where a copy of the lattice parameter is absorbed in the shift [51],

$$\mathrm{w} := w\ ,\ \widetilde{\mathrm{w}} := \widetilde{w} - p\ ,\ \widehat{\mathrm{w}} := \widehat{w} - q\ ,\ \overline{\mathrm{w}} := \overline{w} - r\ ,\ \widehat{\widetilde{\mathrm{w}}} := \widehat{\widetilde{w}} - p - q\ ,\ \ldots, \tag{1.10}$$

(and similar). Using the quad equation (1.8) to write $\overline{\widehat{\widetilde{w}}}$ (1.9) in terms of the initial values, with the shifted variables w, we find

$$\overline{\widehat{\widetilde{\mathrm{w}}}} = -\frac{(p^2-q^2)\widetilde{\mathrm{w}}\widehat{\mathrm{w}} + (q^2-r^2)\widehat{\mathrm{w}}\overline{\mathrm{w}} + (r^2-p^2)\overline{\mathrm{w}}\widetilde{\mathrm{w}}}{(p^2-q^2)\overline{\mathrm{w}} + (q^2-r^2)\widetilde{\mathrm{w}} + (r^2-p^2)\widehat{\mathrm{w}}}\ . \tag{1.11}$$

This opposite corner $\overline{\widehat{\widetilde{w}}}$ only depends on $\widetilde{w}$, $\widehat{w}$ and $\overline{w}$, in a symmetric form. This is sufficient for closure around the cube, since the symmetry makes manifest that the alternative calculations will yield the same result. As commented in [1], in fact all known examples with the closure around the cube property have this "tetrahedron" form, linking $\overline{\widehat{\widetilde{w}}}$ with the three other points in the elementary cube $\widetilde{w}$, $\widehat{w}$ and $\overline{w}$. The closure around the cube proves the multi-dimensional consistency of the lattice equation (1.8).

This multi-dimensional consistency is the key integrability condition for lattice equations. For the lattice KdV equation, we explore this below to see that a Bäcklund transform and Lax representation, standard features of integrable systems, arise naturally from the multi-dimensional consistency. It is also possible to derive soliton solutions and continuum limits, but we omit these here.

Bäcklund transform and Lax pair

We can exploit the multi-dimensional consistency to derive a Bäcklund transform for the initial lattice KdV equation (1.7). The shift in the third lattice direction, $\overline{w}$, is interpreted as the introduction of a transformed variable v, with the lattice parameter representing a Bäcklund parameter k,

$$v := \overline{w}\ , \quad k := r\ . \tag{1.12}$$

The lattice KdV equation in the $(1-3)$ and $(2-3)$ directions (1.8) is given by

$$(p-k+v-\widetilde{w})(p+k-\widetilde{v}+w) = p^2-k^2\ , \tag{1.13a}$$

$$(q-k+v-\widehat{w})(q+k-\widehat{v}+w) = q^2-k^2\ , \tag{1.13b}$$

which give equations for v in terms of w. The multi-dimensional consistency (1.11) guarantees that the new variable v will also obey the lattice KdV equation (1.7),

$$(p-q+\widehat{v}-\widetilde{v})(p+q-\widehat{\widetilde{v}}+v) = p^2-q^2\ . \tag{1.14}$$

This represents an auto-Bäcklund transform for the system; the transform takes us from one solution of the equation w (1.7) to another solution of the same equation, v.

The Bäcklund transform (1.13) then leads to a Lax pair for the lattice KdV equation. Writing the fractional form

$$v - k =: \psi/\phi \ , \tag{1.15}$$

then the "tilde" equation (1.13a) can be rewritten as

$$\widetilde{\psi} = (p + w)\widetilde{\phi} - \frac{(p^2 - k^2)\phi\widetilde{\phi}}{(p - \widetilde{w})\phi + \psi} \ . \tag{1.16}$$

We make a choice for $\widetilde{\phi}$ so that this reduces to the linearised equations

$$(p - k)\widetilde{\phi} \ = \ (p - \widetilde{w})\phi + \psi \ , \tag{1.17a}$$

$$(p - k)\widetilde{\psi} \ = \ \big[(p + w)(p - \widetilde{w}) - p^2 + k^2\big]\phi + (p + w)\psi \ . \tag{1.17b}$$

Introducing the vector $\Phi := (\phi, \psi)^T$, (1.17) can be written in matrix form,

$$(p - k)\widetilde{\Phi} = \begin{pmatrix} p - \widetilde{w} & 1 \\ (p + w)(p - \widetilde{w}) - p^2 + k^2 & p + w \end{pmatrix} \Phi \ . \tag{1.18}$$

An entirely similar construction can be performed for the hat shift (1.13b), which together with the matrix form of the tilde equation (1.18) produces the Lax pair

$$(p - k)\widetilde{\Phi} = \mathscr{L}\Phi \ , \quad (q - k)\widehat{\Phi} = \mathscr{M}\Phi \ , \tag{1.19}$$

with matrices

$$\mathscr{L} \ = \ W\mathfrak{P}_k\widetilde{W}^{-1} := \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix} \begin{pmatrix} p & 1 \\ k^2 & p \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\widetilde{w} & 1 \end{pmatrix} \ , \tag{1.20a}$$

$$\mathscr{M} \ = \ W\mathfrak{Q}_k\widehat{W}^{-1} := \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix} \begin{pmatrix} q & 1 \\ k^2 & q \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\widehat{w} & 1 \end{pmatrix} \ . \tag{1.20b}$$

A key feature of this matrix Lax pair is the appearance of the field variable w in lower triangular matrices. Consistency of the spectral problem (1.19) gives rise to a zero-curvature condition,

$$\widehat{\mathscr{L}}\mathscr{M} = \widetilde{\mathscr{M}}\mathscr{L} \ . \tag{1.21}$$

Requiring this condition to hold produces the initial lattice KdV equation (1.7) as a compatibility condition. The existence of a Lax pair is a standard feature of integrable systems, which here is a direct consequence of the multi-dimensional consistency of the parent equation.

Variational Principle

The lattice KdV equation (1.7) can be derived from a variational or least action principle, by defining a discrete action across the two-dimensional lattice,

$$\mathcal{S} = \sum_{n,m\in\mathbb{Z}} \mathcal{L}(w_{n,m}, w_{n+1,m}, w_{n,m+1}; p, q) \ , \tag{1.22}$$

where $\mathcal{L}(w_{n,m}, w_{n+1,m}, w_{n,m+1}; p, q)$ is a *Lagrangian density* defined on individual plaquettes of the lattice. Note that for quad equations like lattice KdV, it is sufficient to consider 3-point Lagrangian densities with no dependence on $w_{n+1,m+1}$ [1, 62]. In general, there is sufficient freedom within the action to define the Lagrangian densities to depend only on three points of the elementary plaquette. As in the continuous least action principle, we demand the action be stationary under variation of the field, $\delta\mathcal{S}/\delta w = 0$, which applied to the action (1.22) leads to the *discrete Euler-Lagrange field equations*,

$$\frac{\partial}{\partial w}\left(\mathcal{L}(w,\widetilde{w},\widehat{w}) + \underset{\sim}{\mathcal{L}}(\underset{\sim}{w}, w, \underset{\sim}{\widehat{w}}) + \underset{\wedge}{\mathcal{L}}(\underset{\wedge}{w}, \underset{\wedge}{\widetilde{w}}, w)\right) = 0 \ . \tag{1.23}$$

The *three-leg form* of a quad equation is given by

$$f(w,\widetilde{w};p) - f(w,\widehat{w};q) + g(w,\widehat{\widetilde{w}};p,q) = 0 \ , \tag{1.24a}$$

for some functions of the field variables and lattice parameters f and g [1, 51]. By writing the lattice KdV equation (1.7) in this form,

$$(\widetilde{w} - p) - (\widehat{w} - q) + \frac{p^2 - q^2}{p + q + w - \widehat{\widetilde{w}}} = 0 \ , \tag{1.24b}$$

so that $f(w,\widetilde{w};p) = \widetilde{w} - p$ and $g(w,\widehat{\widetilde{w}};p,q) = (p^2-q^2)/(p+q+w-\widehat{\widetilde{w}})$, it can be seen to arise from the three point Lagrangian [24]

$$\mathcal{L}(w,\widetilde{w},\widehat{w};p,q) = -w(p - q - \widetilde{w} + \widehat{w}) + (p^2 - q^2)\log(p - q - \widetilde{w} + \widehat{w}) \ . \tag{1.25}$$

The discrete Euler-Lagrange equation (1.23) yields the equation of motion,

$$p + q + \underset{\sim}{w} - \widehat{w} - \frac{p^2 - q^2}{p - q - w + \underset{\sim}{\widehat{w}}} = p + q + \underset{\wedge}{w} - \widetilde{w} - \frac{p^2 - q^2}{p - q - \underset{\wedge}{\widetilde{w}} + w} \ . \tag{1.26}$$

This is not the quad equation (1.7), but a weaker version of it. Comparing with the three-leg form (1.24b), it is clear that two copies of the equation are produced. This is a general feature of quadrilateral equations under a variational principle [62, 69].

In chapter 2, we will use a simple example to discuss the *Lagrangian 2-form structure*: a Lagrangian structure and accompanying variational principle due to Lobb and Nijhoff [62–65]. The Lagrangian 2-form structure is built on a closure property for the Lagrangians $\mathcal{L}(w)$ (1.25) around the elementary cube of figure 1.2, first shown to hold for lattice equations of the ABS list, and the lattice KdV equation in particular. This extended Lagrangian structure results in not only the strong form of the quad equation arising directly from the variational principle, but also the full set of multidimensionally consistent equations (1.6): as such it is an *integrable* Lagrangian structure, capturing the multiple consistent equations of these lattice models.

1.2 Classical discrete mappings

Whilst the lattice equations of section 1.1 treat both space and time as discrete variables on an equal footing, in a discrete mapping it is time only that is the discrete variable. Lattice equations therefore resemble field theories or PDEs, whereas discrete mappings resemble evolution equations or ODEs. Position often plays the role of dependent, rather than independent, variable. Discrete mappings therefore emerge somewhat differently to lattice equations; in this section we discuss some of the basic theory and introduce the notion of an integrable mapping [22, 116, 117].

1.2.1 Integrable Symplectic Mappings

A symplectic map is defined in [22] on a differentiable manifold $\mathcal{M}$ of dimension $2N$. The manifold is equipped with a symplectic structure $\omega(u)$,

$$\omega(u) = \sum_{r,s} J_{rs}(u)\mathrm{d}u_r \wedge \mathrm{d}u_s \ , \quad r,s = 1,\ldots,2N \ , \tag{1.27}$$

with Jacobi conditions on the matrix J_{rs}. Equivalently this is expressed by the Poisson bracket,

$$\{\mathscr{F},\mathscr{G}\}_u := \sum_{r,s=1}^{2N} \frac{\partial \mathscr{F}}{\partial u_r}(J^{-1})_{rs}\frac{\partial \mathscr{G}}{\partial u_s} \ . \tag{1.28}$$

A well known result [5] is that locally the symplectic structure can be described in terms of *canonical co-ordinates*: position and momentum variables $x_i, X_i,\ i = 1,\ldots,N$, in terms of which the canonical structure can be written more simply,

$$\omega(x,X) = \sum_{j=1}^{N} \mathrm{d}x_j \wedge \mathrm{d}X_j \ , \tag{1.29a}$$

$$\{\mathscr{F},\mathscr{G}\}_{x,X} = \sum_{j=1}^{N} \left(\frac{\partial \mathscr{F}}{\partial x_j}\frac{\partial \mathscr{G}}{\partial X_j} - \frac{\partial \mathscr{F}}{\partial X_j}\frac{\partial \mathscr{G}}{\partial x_j} \right) \ . \tag{1.29b}$$

In practice, this canonical form will be the most useful for our purposes. We adopt the notation using lower and upper case pairs of letters to denote conjugate position and momentum variables, so x, y indicate positions, and X, Y their respective conjugate momenta.

A symplectic mapping is then a function from $\mathcal{M}$ to itself,

$$\begin{aligned} \Phi : \quad \mathcal{M} &\to \mathcal{M} \ , \\ (x_j, X_j) &\mapsto (\widehat{x}_j, \widehat{X}_j) \ , \end{aligned} \tag{1.30a}$$

defining transformed co-ordinates

$$\widehat{x}_j := f_j(x,X) \ , \quad \widehat{X}_j := g_j(x,X) \ , \tag{1.30b}$$

such that the symplectic structure (i.e. the Poisson bracket) (1.29b) is preserved:

$$\{f \circ \Phi, g \circ \Phi\}_{x,X} = \{f,g\}_{\widehat{x},\widehat{X}} \ , \tag{1.30c}$$

for any pair of functions f, g. A consequence is that

$$\{\widehat{x}_i, \widehat{x}_j\} = \{\widehat{X}_i, \widehat{X}_j\} = 0 \ , \quad \{\widehat{x}_i, \widehat{X}_j\} = \delta_{ij} \ , \tag{1.30d}$$

for the new co-ordinates. Such a map is also called a *canonical transform.*

A symplectic map defines a *discrete-time system* by iteration. Introducing a discrete time variable $m \in \mathbb{Z}$, we interpret the map as the evolution of a system under a single, discrete time step. The mapping equations (1.30b) are rewritten as

$$x_j(m+1) := f_j\big(x(m), X(m)\big) \ , \quad X_j(m+1) := g_j\big(x(m), X(m)\big) \ , \tag{1.31}$$

such that the Poisson bracket structure (1.29) is preserved under every step of the evolution. We use the "hat" notation to indicate a time-step evolution: if $x := x(m)$, then $\widehat{x} := x(m+1)$.

An important result is that symplecticity of a map is equivalent to the existence of a *generating function* [43]. Different forms of generating function can be written depending on the specific form of the mapping equations (1.30b). For example, so long as the Jacobian $|\partial g_i/\partial X_j| \neq 0$, we can write a generating function depending on the initial position and final momentum variables, $F(x_i, \widehat{X_i})$, such that the equations of the mapping are given by

$$\widehat{x}_j = \frac{\partial F}{\partial \widehat{X_j}}(x_i, \widehat{X_i}) \ , \quad X_j = \frac{\partial F}{\partial x_j}(x_i, \widehat{X_i}) \ . \tag{1.32}$$

Under appropriate conditions, these relations can be inverted to write the mapping in its canonical form (1.30b). The equations (1.32) and related forms are sometimes referred to as *discrete Hamilton's equations*, although this terminology must be used cautiously. The generating function $F(x_i, \widehat{X_i})$ is emphatically not a Hamiltonian as, unlike in the continuous case, discrete generating functions are in general not preserved under the mapping, $\widehat{F}(\widehat{x}_i, \widehat{\widehat{X_i}}) \neq F(x_i, \widehat{X_i})$.

We are most interested in the *Lagrangian* form of the generating function, $\mathcal{L}(x_i, \widehat{x}_i)$. For such a form, the mapping equations naturally arise from a *principle of least action*. Defining the action $\mathcal{S}$ as a functional of the *path* in discrete time $x_i(m)$,

$$\mathcal{S}[x_i(m)] = \sum_{m \in \mathbb{Z}} \mathcal{L}\big(x_i(m), x_i(m+1)\big) \ , \tag{1.33}$$

the mapping arises at the stationary point of the action $\mathcal{S}$:

$$\frac{\delta \mathcal{S}}{\delta x_i(m)} = 0 \ . \tag{1.34}$$

This least action principle yields the *discrete Euler-Lagrange equations*,

$$\frac{\partial}{\partial x_i(m)} \Big[\mathcal{L}\big(x_i(m-1), x_i(m)\big) + \mathcal{L}\big(x_i(m), x_i(m+1)\big) \Big] = 0 \ , \quad i = 1, \ldots, N \ , \tag{1.35}$$

or equivalently

$$\widehat{\frac{\partial \mathcal{L}}{\partial x_i}} + \frac{\partial \mathcal{L}}{\partial \widehat{x}_i} = 0 \ , \quad i = 1, \ldots, N \ . \tag{1.36}$$

In this view, the discrete Lagrangian $\mathcal{L}(x, \widehat{x})$ is the defining object for the symplectic map, with the canonical momenta and Poisson structure arising as a consequence of the generating function, via the equations

$$X_i = -\frac{\partial \mathcal{L}}{\partial x_i} \ , \quad \widehat{X_i} = \frac{\partial \mathcal{L}}{\partial \widehat{x}_i} \ , \tag{1.37}$$

so that we once more have the symplectic structure (1.29). Note that so long as the action (1.33) is unchanged, then the Euler-Lagrange equations (1.36) will also remain the same, but that the conjugate momenta (1.37) are affected by changes in the Lagrangian, even when the action stays the same.

In principle, the two generating functions $\mathcal{L}(x_i, \widehat{x}_i)$, $F(x_i, \widehat{X}_i)$ are related by a discrete Legendre transform [22, 43]. Notice the differentials from (1.32), (1.37),

$$\mathrm{d}F = \sum_i (X_i \mathrm{d}x_i + \widehat{x}_i \mathrm{d}\widehat{X}_i) \ , \tag{1.38a}$$

$$\mathrm{d}\mathcal{L} = \sum_i (-X_i \mathrm{d}x_i + \widehat{X}_i \mathrm{d}\widehat{x}_i) \ , \tag{1.38b}$$

so that we can write the Legendre transform

$$F(x_j, \widehat{X}_j) = \sum_i \widehat{X}_i \widehat{x}_i - \mathcal{L}(x_j, \widehat{x}_j) \ . \tag{1.39}$$

However, in practice, performing the transform depends on inverting the equations (1.37) to eliminate $\widehat{x}_i$ (respectively $\widehat{X}_i$)

$$\widehat{x}_i = \widehat{x}_i(x_j, \widehat{X}_j) \ . \tag{1.40}$$

This is not always possible, and so this Legendre transform is not a universal construction for any Lagrangian.

The symplectic structure allows a discrete analogue of Arnol'd-Liouville integrability. A $2N$-dimensional discrete mapping is said to be *completely integrable* if there exist N functionally independent invariants in involution [22], $I_j(x_i, X_i)$, $i = 1, \ldots, N$,

$$\widehat{I}_j(\widehat{x}_i, \widehat{X}_i) = I_j(x_i, X_i) \ , \quad \text{where } \{I_j, I_k\} = 0 \ , \quad j, k = 1, \ldots, N \ . \tag{1.41}$$

This integrability results from a canonical transform of the map into "action-angle" variables, new variables where the invariants I_j become the canonical momenta $Y_j(m)$, with some positions $y_j(m)$, $j = 1, \ldots, N$, so that

$$Y_j(m+1) = Y_j(m) = I_j \ , \tag{1.42a}$$

$$y_j(m+1) = G_j(y(m), Y(m)) \ . \tag{1.42b}$$

That is, the new momenta are constant in time. We must have preservation of the Poisson bracket,

$$\{y_i(m+1), Y_i(m+1)\} = \{y_i(m+1), Y_i(m)\} = \delta_{ij} \ , \tag{1.43}$$

which implies

$$y_i(m+1) = y_i(m) + \nu_i(Y) \ , \tag{1.44}$$

where the frequencies ν_i are some functions of the momenta Y_j, and therefore constant. In other words, such a transformation linearises the mapping so that it can easily be integrated. This is explored in more detail in [22]; integrability as sufficiently many invariants in involution is the basis of many known discrete integrable models in the literature [77, 78, 81, 88, 90], and is an important feature of the example discussed below.

1.2.2 Lagrangian One-form Structures

Continuous integrable models are characterised by a complete set of invariants in involution; these describe Hamiltonians that generate commuting time-flows. Recently, the Lagrangian one-form structure has been developed: this structure captures the multiple commuting flows of an integrable systems in an extended variational principle. In the discrete case, for some integrable models there have been found commuting maps analogously to the multi-dimensional consistency of the lattice models discussed in section 1.1, these commuting maps give rise to a discrete Lagrangian one-form structure.

A first example of the Lagrangian one-form was uncovered for the discrete Calogero-Moser (CM) model in [124, 125], and subsequently extended to the relativistic generalisation [126] and Toda-type systems [18], with the continuous theories arising in well chosen limits. Some additional exploration of the general theory has been done in [19, 110].

We consider the discrete Ruijsenaars-Schneider (relativistic Calogero-Moser) model of [126] as an illustrative example. The Ruijsenaars-Schneider (RS) model is an integrable, continuous-time, multi-particle model in one spatial dimension, initially found in [98] as a relativistic generalisation of the CM system [23, 85]. An integrable discrete-time model (discrete mapping) which produces the continuous-time model in a well chosen limit was discovered in [81], with a Lagrangian-type generating function of the form (1.33).

Following the treatment by Lagrangian one-form of the CM model [125], the commuting discrete flows of the RS model and corresponding Lagrangian one-form were uncovered in [126], also enabling the authors to expand the theoretical underpinnings of the discrete one-form structure. In the continuum limit, the authors discovered the Lagrangian one-form for the continuous-time model, expanding the Lagrangian description for the RS model given in [20].

We adopt similar notation to section 1.1. We consider vector functions of two discrete time-variables n, m, such that $x_i := x_i(n, m)$, using tilde and hat to indicate shifts in the

n and m directions, $\widetilde{x}_i = x_i(n+1,m)$, $\widehat{x}_i = x_i(n,m+1)$. Note that here n,m denote two distinct time variables, with two corresponding evolutions, rather than the discrete space-time of section 1.1.

The time-discrete RS model is derived from a Lax pair (inspired by the known Lax representation for the continuous-time model [21, 96]) and has discrete equations of motion

$$\frac{p}{\underset{\sim}{p}} \prod_{\substack{k=1\\k\neq i}}^{N} \frac{\sigma(x_i - x_k + \lambda)}{\sigma(x_i - x_k - \lambda)} = \prod_{k=1}^{N} \frac{\sigma(x_i - \widetilde{x}_k)\sigma(x_i - \underset{\sim}{x}_k + \lambda)}{\sigma(x_i - \underset{\sim}{x}_k)\sigma(x_i - \widetilde{x}_k - \lambda)} \ , \quad \text{for } i = 1,\dots,N \ . \tag{1.45}$$

The variables x_i, $i = 1,\dots,N$ are interpreted as the positions for N identical particles, evolving under discrete time n (x_i is a function of n). λ is a relativistic parameter, such that it is possible to regain the discrete CM model in a non-relativistic limit as $\lambda \to 0$. $\sigma(x)$ is the Weierstrass sigma function (with implicit primitive periods ω_1 and ω_2), and $p = p(n)$ is introduced as a free parameter that may depend on n: it relates to a free choice regarding the centre of mass motion, and we take it to be a constant (so $p/\underset{\sim}{p} = 1$). The Lax pair naturally implies an isospectrality which yields the invariants for the model and hence integrability.

Notice that the equations of motion (1.45) give the particle positions $\widetilde{x}_i$ at time $n+1$ *implicitly*, in terms of positions at earlier times x_i, $\underset{\sim}{x}_i$. In fact the map is multi-valued, but in a precise way. As a result of the integrability, it is possible to construct exact solutions as the eigenvalues of an $N \times N$ matrix; the multi-valuedness manifests itself as *indistinguishability* of the N identical particles (the positions can be exactly calculated, but in the discrete case there is no way of ascribing particular eigenvalues to particular particles). It is interesting that this typically quantum phenomenon occurs here in the classical, discrete case.

Inspired by the commuting flows of the related discrete-time CM case [78, 125], in [126] the authors posed a commuting, discrete flow for the time-discrete RS model. By introducing an alternative Darboux matrix into the Lax pair, they found a second set of discrete equations of motion in a second time variable m, so that $x_i = x_i(n,m)$, with shifts in m labelled by a hat,

$$\frac{q}{\underset{\wedge}{q}} \prod_{\substack{k=1\\k\neq i}}^{N} \frac{\sigma(x_i - x_k + \lambda)}{\sigma(x_i - x_k - \lambda)} = \prod_{k=1}^{N} \frac{\sigma(x_i - \widehat{x}_k)\sigma(x_i - \underset{\wedge}{x}_k + \lambda)}{\sigma(x_i - \underset{\wedge}{x}_k)\sigma(x_i - \widehat{x}_k - \lambda)} \ , \quad i = 1,\dots,N \ . \tag{1.46}$$

$q = q(m)$ is also a centre of mass parameter, similar to p, which is also chosen to be constant.

The compatibility of these two discrete time flows (1.45) and (1.46) is guaranteed on the level of the Lax matrices, and requires a further two *constraint equations*,

$$\frac{p}{q} = \prod_{j=1}^{N} \frac{\sigma(x_i - \widetilde{x}_j)\sigma(x_i - \widehat{x}_j - \lambda)}{\sigma(x_i - \widehat{x}_j)\sigma(x_i - \widetilde{x}_j - \lambda)} \;, \tag{1.47a}$$

$$\frac{p}{q} = \prod_{j=1}^{N} \frac{\sigma(x_i - \underset{\sim}{x}_j)\sigma(x_i - \underset{\wedge}{x}_j - \lambda)}{\sigma(x_i - \underset{\wedge}{x}_j)\sigma(x_i - \underset{\sim}{x}_j - \lambda)} \;. \tag{1.47b}$$

So there are now four equations of motion describing the evolution in two discrete times of the particle positions $x_i(n,m)$: (1.45), (1.46), (1.47). Remarkably, these four equations are consistent, with a joint solution $x_i(n,m)$ expressible once again in terms of the eigenvalues of an $N \times N$ matrix.

The evolution in discrete time (1.45) (respectively also (1.46)) is a symplectic mapping (section 1.2.1) and can be expressed through a variational principle on a generating function of Lagrangian form (1.33) [81]. We choose the specific form of the Lagrangian given in [126],

$$\begin{aligned}\mathcal{L}_{(n)}(x,\widetilde{x}) = \sum_{i,j=1}^{N} \Big(f(x_i - \widetilde{x}_j) - f(x_i - \widetilde{x}_j - \lambda) \Big) - \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \Big(f(x_i - x_j + \lambda) \\ + f(\widetilde{x}_i - \widetilde{x}_j + \lambda) \Big) + \log|p| \sum_{i=1}^{N} (\widetilde{x}_i - x_i) \;, \end{aligned} \tag{1.48a}$$

where

$$f(x) = \int^{x} \log|\sigma(\xi)| \; \mathrm{d}\xi \tag{1.48b}$$

is an *elliptic dilogarithm*, and the final term of (1.48a) represents the centre of mass motion. The Lagrangian $\mathcal{L}_{(n)}$ yields the equation of motion (1.45) for the tilde evolution under discrete Euler-Lagrange equations, as in (1.36). A similar Lagrangian, $\mathcal{L}_{(m)}(x,\widehat{x})$, yields the equation of the hat evolution (1.46).

The critical observation is the *closure relation* of the Lagrangians (1.48) [126]. That is, when we apply the equations of motion (1.45), (1.46) and the constraint equations (1.47), the equality holds,

$$\mathcal{L}_{(m)}\big(x,\widehat{x}\big) + \widehat{\mathcal{L}}_{(n)}\big(\widehat{x},\widehat{\widetilde{x}}\big) - \widetilde{\mathcal{L}}_{(m)}\big(\widetilde{x},\widehat{\widetilde{x}}\big) - \mathcal{L}_{(n)}\big(x,\widetilde{x}\big) = 0 \;. \tag{1.49}$$

Note that for a generic choice of $x_i(n,m)$, the sum of Lagrangians is non-zero, so that the closure relation holds only on solutions. For the discrete RS system, this result must be shown by direct calculation, which was done explicitly for the rational case in [126], and requires the specific choice of Lagrangians given in (1.48). The closure relation (1.49) can be understood as a closure around the elementary square in the lattice of time variables, shown in figure 1.3.

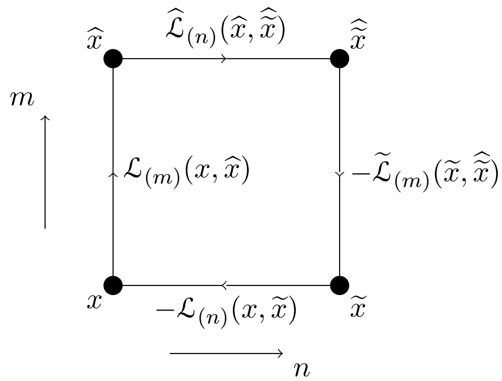

Figure 1.3: Oriented Lagrangians around a square in the time directions n, m.

This closure relation (1.49) is the key requirement for the discrete one-form structure. The essential observation is that all four equations of motion (1.45), (1.46), (1.47) (i.e. including the constraint equations) arise from Euler-Lagrange equations on the Lagrangians (1.48). In order to make a general statement we suppress the vector index i and mark shifts in the lattice by α, $\alpha = 1, 2$, so that $x_1 = \widetilde{x}$, $x_2 = \widehat{x}$, and label the Lagrangians $\mathcal{L}_n$ and $\mathcal{L}_m$ by $\mathcal{L}_\alpha$, $\alpha = 1, 2$, similarly. Then there are four *elementary configurations* of the action that yield the four possible Euler-Lagrange equations, shown in figure 1.4. In each case a pair of Lagrangians, with variation over the middle variable, yields an Euler-Lagrange equation of the form

$$\frac{\partial}{\partial x_\alpha}\Big(\mathcal{L}_\alpha(x, x_\alpha) + \mathcal{L}_\beta(x_\alpha, x_{\alpha\beta})\Big) = 0 \ . \tag{1.50}$$

As derived in [126], it is then straightforward to show that the elementary curves of figure 1.4 produce the equations of motion for the system. Clearly figures (i) and (ii) correlate with the known single time variable case, and yield the respective equations of motion (1.45) and (1.46). It can then also be shown that curves (iii) and (iv) yield the constraint equations (1.47a) and (1.47b) respectively, so that the complete set of equations of motion are described by this Lagrangian one-form. More detail of the Lagrangian one-form structure, and its accompanying variational principle, will be discussed via the simple example derived in chapter 2.

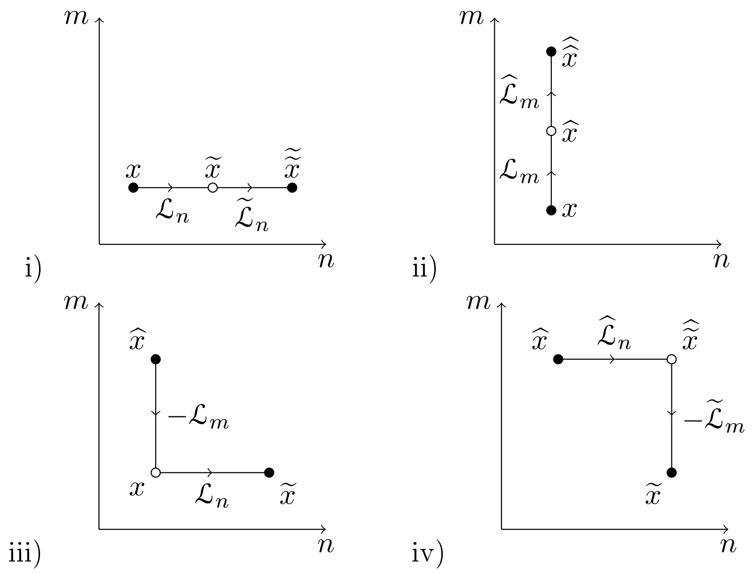

Figure 1.4: Elementary discrete curves for variables m and n.

The Lagrangian one-form structure therefore captures the multi-dimensional consistency of the system, encoding all the equations of motion and constraint equations within a single, extended, Lagrangian structure. The discrete RS system is essentially an iterated Bäcklund transform (as are many other integrable discrete time systems) [122] such that the Lagrangian one-form captures the commutativity of the Bäcklund transform under different choices of parameter. In the RS and CM cases, the discrete one-form structures have also been shown to lead, in well chosen continuum limits, to continuous one-form structures that capture the commuting flows of integrable Hamiltonian systems.

Note that the elliptic potential $\sigma(x)$ of the RS equations (1.45) can be simplified through limits on the primitive periods of the Weierstrass sigma function. There are three cases of interest,

$$\text{Hyperbolic:} \qquad \sigma(x) \rightarrow \sinh(x)\ , \tag{1.51a}$$

$$\text{Trigonometric:} \qquad \sigma(x) \rightarrow \sin(x)\ , \tag{1.51b}$$

$$\text{Rational:} \qquad \sigma(x) \rightarrow x\ . \tag{1.51c}$$

This limit can be performed directly on the sigma function $\sigma(x)$ in the results of this section to yield the appropriate equations of motion (1.45), (1.46), constraint equations (1.47), and Lagrangians (1.48) via the function $f(x)$ (1.48b). In particular, all the calculations for closure discussed in this section can be performed explicitly in the rational case [126].

1.3 Quantum Mechanics

There are two primary competing paradigms in quantum mechanics: canonical quantisation, a Hamiltonian approach, and path integral quantisation, whose fundamental object is the Lagrangian. In this section we review some essential notions of canonical and path integral quantisation, in particular developing how familiar quantum mechanical treatments extend naturally to discrete-time systems through the Heisenberg picture. We do not seek to provide a full introduction to quantum mechanics,[1] but only to clarify our particular perspective on quantum mappings, since many different views of "discrete quantum mechanics" exist in the literature.

In the context of integrable systems, Hamiltonian approaches to quantisation have received a lot of attention, whilst the path integral has been relatively neglected. The machinery of quantum inverse scattering and the Bethe ansatz is a key tool for understanding canonical quantum integrability, through the derivation of commuting quantum invariants in analogy to classical notions of integrability [58, 103]. Such invariants often require a *quantum correction* involving the Planck constant $\hbar$, relative to the classical case [50, 49]. Despite this, making a precise notion of "quantum integrability" is not straightforward due to the difficulties of establishing functional independence in the operator case. So far, there is no clear notion of quantum integrability for path integral quantisation, although an attempt was made by de Vega [25].

First, we consider systems evolving in continuous time t. In the Hamiltonian (canonical) approach, the canonical variables of position and momentum x_i, X_i become operators, acting on a (typically) infinite dimensional Hilbert space. Where it aids clarity, we use a bold type to indicate operators $\mathbf{x}_i, \mathbf{X}_i$. The Poisson bracket of the classical theory (1.29) becomes an operator commutator bracket between the position and momentum operators,

$$[\mathbf{x}_i, \mathbf{X}_j] = i\hbar\delta_{ij} \ . \tag{1.52}$$

Operators act on states in the Hilbert space, which are labelled using Dirac notation, $|\psi\rangle$. Eigenstates for the position and momentum operators $|x\rangle, |X\rangle$ are postulated such that

$$\mathbf{x}\,|x\rangle = x\,|x\rangle \ , \quad \mathbf{X}\,|X\rangle = X\,|X\rangle \ . \tag{1.53}$$

Such eigenstates then allow us to define position (or momentum) space wave-functions,

[1] of which many are available [29, 100]

$\psi(x,t) := \langle x|\psi\rangle$. In particular, the "free wave" is a position space representation of a momentum eigenstate, $\langle X|x\rangle = \exp(-ixX/\hbar)$.

The fundamental object in the continuous time theory is the Hamiltonian $\mathcal{H}(\mathbf{x},\mathbf{X})$, which generates the time-flow via the Schrödinger equation,

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = \mathcal{H}(\mathbf{x},\mathbf{X})|\psi\rangle \ . \tag{1.54}$$

This governs how states evolve in time in the *Schrödinger picture*, where time-dependence is assumed to sit in states, with operators time-independent. In this view of quantum mechanics, one often either considers stationary states, such that $\mathcal{H}(\mathbf{x},\mathbf{X})|\psi\rangle = E|\psi\rangle$, in which case the goal is typically to investigate the spectrum of the Hamiltonian operator. Or, one maintains an interest in the time dependence, in which case the usual goal involves calculating correlations functions to analyse some scattering process.

For Hamiltonians that do not explicitly depend on t, the consequence of the Schrödinger equation is that the time dependence of a state is easily expressed in terms of the Hamiltonian,

$$|\psi(t)\rangle = e^{-it\mathcal{H}/\hbar}|\psi(0)\rangle \ . \tag{1.55}$$

This leads to the obvious definition of the *time evolution operator*, a unitary operator for Hermitian Hamiltonians,

$$U(t) := e^{-it\mathcal{H}/\hbar} \ . \tag{1.56}$$

We transition to the *Heisenberg picture* by using the time evolution operator to relocate the time-dependence from states to operators. States become time-independent, whilst the time evolution of operators is expressed by conjugation

$$\mathbf{O}(t) = U^\dagger(t)\mathbf{O}(0)U(t) \ , \tag{1.57}$$

and eigenstates naturally also become time-dependent,

$$|x(t)\rangle = U^\dagger(t)|x(0)\rangle \ , \quad |X(t)\rangle = U^\dagger(t)|X(0)\rangle \ . \tag{1.58}$$

Wave functions (and all physical consequences of the theory) remain unchanged,

$$\psi(x,t) = \langle x(0)|\psi(t)\rangle = \langle x(0)|U(t)|\psi(0)\rangle = \langle x(t)|\psi(0)\rangle \ . \tag{1.59}$$

The Schrödinger equation (1.54) governs the time evolution of states, and so is no longer relevant in the Heisenberg picture. Instead, the time-evolution of operators is governed by the operator equation of motion,

$$\frac{\partial}{\partial t}\mathbf{O}(t) = \frac{i}{\hbar}\big[\mathcal{H}, \mathbf{O}\big] , \tag{1.60}$$

so that the Hamiltonian generates time-evolution of the operators by commutation: this directly parallels time evolution via Poisson bracket in the classical Hamiltonian formalism.

An alternative view of quantum mechanics due to Feynman [36] moves away from the Hamiltonian to the Lagrangian, and away from an operator view of quantum objects to considering particle trajectories. Although the origins of this view are in Dirac's work on the place of the Lagrangian in quantum mechanics [27, 28], Feynman's development initially faced some resistance, returning to particle trajectories at a time when many had rejected such a notion as unphysical. This theory of *path integrals* led to many new developments in quantum mechanics and wider physics (particularly statistical mechanics) and has been expounded since its inception in a number of texts [37, 47, 99, 102].

The primary object in this "sum over histories" formulation is the propagator, the matrix elements of the time evolution operator,

$$K(x_a, t_a; x_b, t_b) := \langle x_b(t_b)|x_a(t_a)\rangle = \langle x_b|U(t_b - t_a)|x_a\rangle . \tag{1.61}$$

This expresses the *probability amplitude* for a particle to travel from a position x_a at time t_a, to position x_b at some later time t_b (note that we have tacitly assumed time independence of the Hamiltonian). Completeness of the position eigenstates, $\int |x\rangle\langle x|\mathrm{d}x = 1$, gives the propagator the important group property,

$$K(x_a, t_a; x_b, t_b) = \int \mathrm{d}x_c \; K(x_a, t_a; x_c, t_c) \; K(x_c, t_c; x_b, t_b) , \tag{1.62}$$

for any intermediate time t_c.

Derivations of the sum over histories formalism (as early as Dirac's original paper) use the group property to begin with a "time-slicing" of the interval $[t_a, t_b]$ into segments,

$$K(x_a, t_a; x_b, t_b) = \int \prod_{i=1}^{N-1} \mathrm{d}x_i \prod_{j=0}^{N-1} \langle x_{j+1}|U(t_{j+1} - t_j)|x_j\rangle . \tag{1.63}$$

The derivation then depends on a small-time approximation: taking N to be large as the time becomes finely sliced, we allow $t_{j+1} - t_j =: \epsilon$, small, so that there is a *time-slicing limit* on the propagator,

$$K(x_a, t_a; x_b, t_b) = \lim_{\substack{N\to\infty \\ \epsilon\to 0}} \int \prod_{i=1}^{N-1} \mathrm{d}x_i \prod_{j=0}^{N-1} \langle x_{j+1}| \exp\big[-i\epsilon\mathcal{H}/\hbar\big]|x_j\rangle . \tag{1.64}$$

It is usually supposed that the Hamiltonian is written in a Newtonian, separable form,

$$\mathcal{H}(\mathbf{x},\mathbf{X}) = T(\mathbf{X}) + V(\mathbf{x}) = \frac{1}{2}\mathbf{X}^2 + V(\mathbf{x}) \ , \tag{1.65}$$

such that in the time slicing limit the evolution operator can be written in a separated form,

$$U(t_b - t_a) = \lim_{\substack{N\to\infty \\ \epsilon\to 0}} \Big[\exp\big[-i\epsilon\mathcal{H}/\hbar\big]\Big]^N \ , \tag{1.66a}$$

$$= \lim_{\substack{N\to\infty \\ \epsilon\to 0}} \Big[e^{-i\epsilon T(\mathbf{X})/\hbar} e^{-i\epsilon V(\mathbf{x})/\hbar}\Big]^N \ , \tag{1.66b}$$

where this equality holds only within the time-slicing limit [99].

Inserting a complete set of momentum eigenstates into each propagator segment of (1.64), and using the analytic continuation of the Gaussian integral formula,

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}iax^2+ibx}\mathrm{d}x = \sqrt{\frac{2\pi}{ia}}e^{ib^2/a} \ , \tag{1.67}$$

the infinitesimal piece of the propagator becomes

$$\lim_{\epsilon\to 0}\langle x_{j+1}|e^{-i\epsilon\mathcal{H}/\hbar}|x_j\rangle = \exp\left[\frac{i\epsilon}{\hbar}\left(\frac{1}{2}\left(\frac{x_{j+1}-x_j}{\epsilon}\right)^2 - V(x_j)\right)\right] \ , \tag{1.68a}$$

$$= \exp\left(\frac{i\epsilon}{\hbar}\ \mathcal{L}(x_j, x_{j+1})\right) \ . \tag{1.68b}$$

$\mathcal{L}(x_j, x_{j+1})$ is a *discretisation* of the Lagrangian that corresponds to the Legendre transform of the Hamiltonian $\mathcal{H}$ (1.65). The full propagator becomes

$$K(x_a, t_a; x_b, t_b) = \lim_{\substack{N\to\infty \\ \epsilon\to 0}} \int \prod_{i=1}^{N-1} \mathrm{d}x_i \exp\left(\frac{i\epsilon}{\hbar}\sum_{j=0}^{N-1}\mathcal{L}(x_j, x_{j+1})\right) \ , \tag{1.69a}$$

$$=: \int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}[x(t)] \exp\Big(i\mathcal{S}[x(t)]/\hbar\Big) \ . \tag{1.69b}$$

In this last equality, the *path integral* notation is defined. Notice especially the appearance of the action $\mathcal{S}[x(t)]$, a functional of the path $x(t)$. This has the well known physical interpretation as sum over all possible paths, or sum over histories, for the particle.

From a discrete-time perspective, there are a number of points of interest, especially for integrable systems. Many important integrable systems do not have a Hamiltonian of Newtonian form (1.65) (the Ruijsenaars-Schneider model of section 1.2.2 is one such case) but the derivation of the path integral relied upon such an assumption. For non-Newtonian models, it is not clear whether canonical and path integral quantisation are equivalent. If not, which is "correct"?

We noted also the appearance of a discretised Lagrangian in (1.68b). Many discrete integrable models can be described by Lagrangians, though these are often not of a Newtonian form, and may have non-trivial continuum limits (again, consider the Lagrangian for the discrete Ruijsenaars-Schneider model in (1.48a)). Additionally, in the discrete case the infinitesimal time-slicing is no longer needed, since we are concerned with finite time steps. This may offer a resolution to some of analytical difficulties that face the path integral as a result of the time slicing limit.

In studies of quantum gravity, the discretisation necessary for the path integral can be problematic as discretisations generally break the symmetries of the continuous model [10]. Additionally, the discretisation of the Lagrangian is non-unique for a given continuum model, yet choice of discretisation can change the result of the path integral calculation: for even the simple example of a Newtonian particle evolving under a vector potential, the "wrong choice" of discretisation leads to the wrong propagator [99]. Discrete integrable systems potentially resolve these problems; invariants of the continuum model are typically also preserved as commuting flows in the discrete case, and integrable models do not in general have such freedom in the choice of discretisation. Indeed, the discrete models themselves are perhaps the truly fundamental system, rather than their continuum limit.

1.3.1 Quantum mappings

Quantum mappings (or discrete-time systems) have a wide range of interpretations. The notion of a quantum mapping essentially began in [14], but has been advanced a great deal especially in the area of integrability, via canonical quantisation of known integrable maps [73, 74, 76, 91]. This has led naturally to the notion of a "quantum canonical transform" [3, 4], which is a transformation preserving commutation relations. For a discrete-time quantum system, we insist on formal unitarity of the time evolution operator, which is a stronger condition. We consider systems where the time variable t is replaced by the discrete variable n, which takes integer values; wave-functions become therefore $\psi(q,n) = \psi_n(q)$. We also discuss the quantisation of lattice equations, such as (1.8): in such cases both time and position become discrete, and we are essentially examining a discrete quantum field theory. Such integrable models have received some attention, e.g. in [33, 118, 119].

Clearly, in discrete time the Schrödinger equation (1.54) becomes redundant as a starting point, since time derivatives are no longer meaningful. Thus in the discrete case it is the operator equations of motion that take the centre stage. In general one takes discrete equations of motion from the classical case and transforms them into *operator equations of motion*, with commutator brackets defined from classical Poisson brackets, as in (1.52). This process is not necessarily unambiguous, as there may be issues with operator ordering. The emphasis on operator equations of motion means that the Heisenberg picture of quantum mechanics is often the more natural for discrete systems.

Canonical quantisation of discrete integrable systems is essentially linked to their Lax representations through the quantum inverse scattering method [58]; Lax representations encode the invariants of the system essential for integrability, as classically. In the quantum case, as operator ordering becomes non-trivial, quantum corrections are required to the invariants to account for the commutation relations. An *R-matrix structure* guarantees the commutation of the invariants, and the preservation of the symplectic structure under the mapping; this will be discussed at more length in chapter 4.

For a *canonical transform* (section 1.2.1), the operator equations of motion may have a form of the kind,

$$\mathbf{x}_{n+1} = f(\mathbf{x}_n, \mathbf{X}_n) \ , \quad \mathbf{X}_{n+1} = g(\mathbf{x}_n, \mathbf{X}_n) \ , \tag{1.70}$$

although such a form cannot always be written explicitly. Recalling the time evolution of operators by conjugation with the operator U (1.57), in the discrete case we seek an *elementary* time evolution operator, U, evolving states or operators by

$$|\psi\rangle_{n+1} = U|\psi\rangle_n \ , \quad \mathbf{O}_{n+1} = U^{-1}\mathbf{O}_n U \ . \tag{1.71}$$

Time evolution occurs by iteration of the operator U,

$$\mathbf{O}_n = U^{-n}\mathbf{O}_0 U^n \ . \tag{1.72}$$

We insist that U be formally unitary, so that $UU^\dagger = \mathbb{I}$.

Comparing (1.70) with (1.71), the functions f, g arise from the time evolution operator,

$$U^{-1}\mathbf{x}U = f(\mathbf{x}, \mathbf{X}) \ , \quad U^{-1}\mathbf{X}U = g(\mathbf{x}, \mathbf{X}) \ . \tag{1.73}$$

For many quantum discrete mappings it is then possible to write the evolution operator in a separated form,

$$U = \exp(-iT(\mathbf{X})/\hbar)\exp(-iV(\mathbf{x})/\hbar) \ . \tag{1.74}$$

Recalling the commutation relations (1.52), the conjugations follow,

$$e^{iT(\mathbf{X})/\hbar}\mathbf{x}e^{-iT(\mathbf{X})/\hbar} = \mathbf{x} + T'(\mathbf{X}) \;, \tag{1.75a}$$

$$e^{iV(\mathbf{x})/\hbar}\mathbf{X}e^{-iV(\mathbf{x})/\hbar} = \mathbf{X} - V'(\mathbf{x}) \;. \tag{1.75b}$$

In the case of a Newtonian kinetic term $T(\mathbf{X}) = \mathbf{X}^2/2$, the operator equations of motion (1.70) become

$$\mathbf{x}_{n+1} - \mathbf{x}_n = \mathbf{X}_n - V'(\mathbf{x}_n) \;, \tag{1.76a}$$

$$\mathbf{X}_{n+1} - \mathbf{X}_n = -V'(\mathbf{x}_n) \;. \tag{1.76b}$$

Of course, many other forms of time evolution operator U and equations of motion are possible. The generalised McMillan maps [41, 74], for example, are known to have "cross terms" in the unitary operator of the form

$$\exp(-i\mathbf{x}_i\mathbf{X}_j/\hbar) \;, \quad \text{for } i \neq j \;, \tag{1.77}$$

which act on operators by conjugation as

$$e^{i\mathbf{x}_i\mathbf{X}_j/\hbar}\mathbf{x}_k e^{-i\mathbf{x}_i\mathbf{X}_j/\hbar} = \mathbf{x}_k + \delta_{jk}\mathbf{x}_i \;, \tag{1.78a}$$

$$e^{i\mathbf{x}_i\mathbf{X}_j/\hbar}\mathbf{X}_k e^{-i\mathbf{x}_i\mathbf{X}_j/\hbar} = \mathbf{X}_k - \delta_{ik}\mathbf{X}_j \;. \tag{1.78b}$$

As we have seen, separability of the time evolution operator is an important part of the path integral derivation (1.66b) - these ideas have been used in [38, 40] to make some first steps towards path integral quantisation of discrete systems, which remains an under explored area, especially considering the many recent advances in the classical theory of such systems. The separability of the evolution operator opens the possibility of performing the path integral without the need for an infinitesimal time limit.

Recalling the centrality of the action in the path integral (1.69b), it is natural to ask how the Lagrangian one-form structure should inform the quantisation of these systems. Traditionally, the variational principle of the classical action becomes a sum over histories in quantum mechanics. For integrable systems and the one-form structure, the correct variational principle includes a variation over underlying geometries of the independent variables: how should this translate to quantum mechanics?

1.4 Reparametrisation invariance and quantum gravity

We mention some recent ideas in the field of quantum loop gravity that, although holding a different perspective, share some striking similarities with our work. Quantum gravity naturally poses two questions: what is the appropriate way to consider time in quantum mechanics, when looking towards relativistic concerns? And, what is an appropriate way to view the smallest length scales, as we approach the Planck length? These questions may be related to the questions we are posing in the context of integrable systems: what is a correct view of the independent variable (time) in a Lagrangian formulation? And, how do systems whose time (and perhaps length) scales are inherently discrete behave?

In [94, 95], Rovelli considers a *reparametrisation invariant* form of the Harmonic oscillator, and its discretisation. He poses a toy model where the independent variable can be freely reparametrised, without changing the physical observables of the model: so called *Diff-invariance*. The key conceptual step is the reclassification of the time variable t, such that rather than a system defined by the variable $x(t)$, we instead have the two variables $x(\tau)$, $t(\tau)$ in terms of an evolution parameter or real-time τ. Physical configurations of the system are then defined in terms of the *pair* $(x(\tau), t(\tau))$, meaning that there is a very large gauge invariance: any reparametrisation of τ leads to the same physical output. This is a view of time also investigated in [11] in the context of Machian schemes, and similar steps were made in a different context in [10].

The action governing the system is given by

$$\mathcal{S} = \frac{m}{2} \int \mathrm{d}\tau \left(\frac{\dot{x}^2}{\dot{t}} - \omega^2 \dot{t} x^2 \right) \,, \tag{1.79}$$

which is easily seen to be the same as the usual action for the harmonic oscillator. In the two variables, the action (1.79) yields two equations of motion,

$$\frac{d}{d\tau} \frac{\dot{x}}{\dot{t}} = -\omega^2 \dot{t} x \quad , \quad \frac{d}{d\tau} \left(\frac{\dot{x}^2}{\dot{t}^2} + \omega^2 x^2 \right) = 0 \,. \tag{1.80}$$

These are simply equivalent to the usual harmonic oscillator equation of motion $d^2x/dt^2 = -\omega^2 x$, and the corresponding conservation of energy. So far, so unremarkable! The Diff-invariance of the system is in the very large gauge invariance: the model is physically invariant under arbitrary reparametrisations of τ.

Rovelli's observation is that this reparametrisation invariance yields some interesting properties under discretisation. As in the continuum case, time is no longer considered the independent variable to be discretised. Rather, we discretise τ so that $\tau_n = na$ for some fixed step size a, yielding discrete position and times $x_n = x(\tau_n)$, $t_n = t(\tau_n)$. In other words, the time-step $t_{n+1} - t_n$ is no longer fixed (as is standard) but is allowed to vary. The discretised action from (1.79) yields,

$$\mathcal{S}_N = \frac{m}{2} \sum_{n=0}^{N-1} \left(\frac{(x_{n+1} - x_n)^2}{t_{n+1} - t_n} - \omega^2 (t_{n+1} - t_n) x_n^2 \right) , \tag{1.81}$$

which has the critical property that it is *independent of the step variable* a - this is the discrete realisation of reparametrisation invariance. A continuum limit to the original model (1.79) is now achieved in the limit $N \to \infty$ with no need to tune the parameter $a \to 0$; the shrinking of the step size is an automatic consequence of the $N \to \infty$ limit.

The elevation of the time t_n to a dependent variable means that variation of the action (1.81) yields two independent equations of motion,

$$v_{n+1} = v_n - (t_{n+1} - t_n)\omega^2 x_n , \quad \tfrac{1}{2} v_{n+1}^2 + \tfrac{1}{2}\omega^2 x_n^2 = \tfrac{1}{2} v_n^2 + \tfrac{1}{2}\omega^2 x_{n-1}^2 , \tag{1.82}$$

where we have introduced the discrete velocity $v_{n+1} := (x_{n+1} - x_n)/(t_{n+1} - t_n)$, and the second equation has the form of an *energy conservation*, $E_{n+1} = E_n$. Here is a striking feature: energy is not conserved under generic discretisations, but in this case the time parametrisation has been performed in such a way as to preserve an energy. In essence, the gauge freedom of (1.80) has been sacrificed in order to fix the energy; the time steps are chosen in such a way that energy is held constant. But, this is a feature of discrete integrable systems! As discussed in section 1.2.1, an integrable discrete flow is precisely characterised by the preservation of a sufficient number of invariants, with multiple invariants at the continuum level leading to a Lagrangian form structure by Legendre transform.

Rovelli comments further on the resulting quantum structure through a path integral. The correct, physical object to consider is a propagator (1.61), since objects depending on the parameter τ are physically meaningless. So in the discretisation we have the propagator

$$K_N(x_a, t_a; x_b, t_b) = \mathscr{N} \int \mathrm{d}\mu(x_n, t_n) \; e^{i\mathcal{S}_N(x_n, t_n)/\hbar} , \tag{1.83}$$

with a normalisation $\mathscr{N}$ and some integration measure $\mathrm{d}\mu$ over the variables x_n, t_n. Unlike in the non-parametrised case, t_n itself becomes an integration variable, under the obvious restrictions

$$t_i < t_n < t_{n+1} < t_f . \tag{1.84}$$

As N becomes large, the equations of motion (1.82) approximate to $v_{n+1}^2/2 \approx v_n^2/2$, which is the free particle. Since Rovelli is interested in the continuum limit as $N \to \infty$, he fixes the integration measure by considering this free particle case. This is the regime Rovelli calls "Ditt-invariance", that is, an *almost* Diff-invariance recaptured in the large N limit for the discrete case. The approximation justifies the choices

$$\mathscr{N} = \frac{N!}{(t_f - t_i)^N \sqrt{\omega}} \ , \quad \mathrm{d}\mu(x_n, t_n) = \left(\frac{m}{\hbar}\right)^{N/2} \frac{\prod_{n=1}^{N} \mathrm{d}x_n \mathrm{d}t_n}{\prod_{n=0}^{N} \sqrt{2\pi(t_{n+1} - t_n)}} \ , \tag{1.85}$$

so that (1.83) yields the expected harmonic oscillator propagator in the $N \to \infty$ limit.

There are some interesting parallels between these ideas and the quantisation of Lagrangian form structures. As discussed in [63] (and explored in chapter 2), a variational principle for Lagrangian forms extends beyond variation of the dependent variables to variation of the underlying geometry belonging to the independent variables. In his talk [72], Nijhoff proposed an extended path integral quantisation for a (continuous) Lagrangian 1-form structure in multiple times $\boldsymbol{t}$,

$$K(\boldsymbol{x}_a, \boldsymbol{t}_a, s_a; \boldsymbol{x}_b, \boldsymbol{t}_b, s_b) = \int_{\boldsymbol{t}(s_a)=\boldsymbol{t}_a}^{\boldsymbol{t}(s_b)=\boldsymbol{t}_b} \mathcal{D}[\boldsymbol{t}(s)] \int_{\boldsymbol{x}(\boldsymbol{t}_a)=\boldsymbol{x}_a}^{\boldsymbol{x}(\boldsymbol{t}_b)=\boldsymbol{x}_b} \mathcal{D}[\boldsymbol{x}(\boldsymbol{t})] e^{i\mathcal{S}(\boldsymbol{x})/\hbar} \ . \tag{1.86}$$

In other words, a path in multiple time-variables is parametrised by some variable s. The propagator results (in line with the variation in the classical theory) from a sum over histories, including a sum over geometries of the time path. Such a system would clearly be expected to satisfy Diff-invariance: that is, invariance under arbitrary reparametrisation of s. Under some appropriate discretisation, this idea is not very dissimilar to Rovelli's discretised propagator (1.83), and indeed a "sum over geometries" has appeared elsewhere in the study of quantum gravity [92, 93]. Clearly the journey from a discretised one-form propagator to a continuous expression such as (1.86) may not be straightforward, but in this thesis we take some tentative first steps to solving this problem.

1.5 Organisation of the Thesis

In **chapter 2**, we introduce the linearised integrable lattice model, and discuss its Lagrangian 2-form structure. Linear mappings are derived using a periodic staircase initial value problem, and we exploit the multi-dimensional consistency to derive commuting flows, and hence a "simplest possible" discrete Lagrangian 1-form. Commuting continuous flows with an interchange of continuous and discrete parameters and variables are observed. The next member of the family is also considered as a generalisation. These examples are used to demonstrate the extended variational principle for Lagrangian multiform structures.

In **chapter 3** we consider the quantisation of the linear models of chapter 2 as a first study of the quantisation of Lagrangian form structures. For the linear discrete mapping, it is found that the propagator for the one-form structure has a time-path independence, which depends on the correct initial choice of Lagrangian generating function. Similarly, for the lattice model, the Lagrangian 2-form structure can be path integral quantised, and the propagator is surface independent in the multi-dimensional structure. In other words, there is a quantum analogue to the classical Lagrangian closure and variational principle for forms.

Chapter 4 considers a discrete non-linear model: the generalised McMillan maps, derived from the lattice KdV equation. We make some investigations into possible commuting flows following the method of chapter 2, although these are predictably more complex in the non-linear case and do not so far yield a Lagrangian form structure. We consider quantum aspects in the McMillan case, writing some simple propagators for this non-linear example which may, ultimately, feed into a full path integral quantisation. We demonstrate that there are potentially consistent ways of viewing the Hilbert space, despite singularities of the model, and highlight a possible quantisation of the mapping parameter. The main result is a novel formulation of the r-matrix for the dual Lax pair of the generalised family of maps. The r-matrix can be written as a normally ordered fraction in elementary shift matrices, leading to new insights on the nature of the structure. We replicate some known results with this new formulation, and propose a possible quantum structure for the dual Lax pair.

In **chapter 5** we study a simple, but non-trivial, continuous model, related to the Ruijsenaars-Schneider model; here called the Degasperis-Ruijsenaars (DR) model. This has a non-Newtonian Hamiltonian that nonetheless yields the harmonic oscillator as its equation of motion. We are able to derive a Lagrangian for the model by embedding it within a 2-particle system, following known results for the Ruijsenaars-Schneider case. We also derive the precise link between the RS and DR models, and hence a Lax pair for the DR system. Nonetheless, the precise nature of discrete integrable systems leaves an integrable discretisation for the model out of reach. Consideration of the known quantum solution for the model yields an expression for the propagator, but the path integral quantisation that would link the propagator to the Lagrangian description remains elusive. However, its exact solution and integrability make this a promising model for path integral quantisation of a non-Newtonian system.

Chapter 6 contains a brief summary of the results and some discussion of future outlook and outstanding research questions.

2

Multiform Structures for Linear Models

"The career of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing levels of abstraction."

– Sidney Coleman

The Lagrangian multiform has been introduced in a number of recent works, the two-form for integrable lattice models [62–65] and the one-form for continuous and discrete evolutionary equations [17–19, 72, 110, 123–126]. As illustrated particularly in [63, 126], the Lagrangian multiform gives rise to a novel variational principle for systems with commuting, compatible flows. The system sits at a critical point in an extended variational principle, varying over not only the dependent variables but also the underlying geometry of the independent variables.

We apply these ideas to simplest models in the lattice and evolutionary cases: a linear lattice equation, and the discrete harmonic oscillator. These systems are simpler than previous examples of the Lagrangian multiform, indeed such systems are generally considered too simple to have any meaningful integrable structures. Nonetheless, in the lattice case it emerges that multi-dimensional consistency is sufficient to produce a two-form structure even in the linear case, and we use this simple example to illustrate the general theory in section 2.1.

By applying a periodic initial value problem to the linear lattice equation, we derive the equation of motion for the discrete harmonic oscillator in section 2.2. This unusual starting point, however, endows the model with the multi-dimensional consistency of its parent, and we show that it is therefore possible to find a commuting discrete flow for the equation. In the same way as for the discrete Ruijsenaars-Schneider model, these commuting flows are captured by a Lagrangian one-form structure (1.48). This is surprising for such a simple model.

In section 2.3 we consider a higher dimensional reduction of the lattice equation by lengthening the periodic initial condition, and find that the one-form structure continues to hold. However, the general case remains out of reach for the moment, as the invariants of the system cannot currently be captured in a Lax pair.

2.1 Linearised Lattice KdV Equation

Recall the lattice KdV equation of section 1.1.1, (1.7),

$$(p - q + \widehat{w} - \widetilde{w})(p + q - \widehat{\widetilde{w}} + w) = p^2 - q^2 \ . \tag{2.1}$$

This holds across a 2 dimensional quadrilateral lattice, with a field variable $w(n, m)$ at each lattice point, and lattice parameters p and q associated to the n and m directions on the lattice.

We are interested in the *linearisation* of the lattice KdV equation (2.1). Expanding about a fixed point, for a small parameter η,

$$w_{n,m} = w_0 + \eta u_{n,m} + O(\eta^2), \tag{2.2}$$

leads to the linearised lattice equation at first order in η,

$$(p + q)(\widetilde{u} - \widehat{u}) = (p - q)(u - \widehat{\widetilde{u}}) \ . \tag{2.3}$$

This equation is supposed to hold on every elementary plaquette across a two dimensional lattice; the elementary plaquette is shown in figure 2.1. The linear lattice equation (2.3)

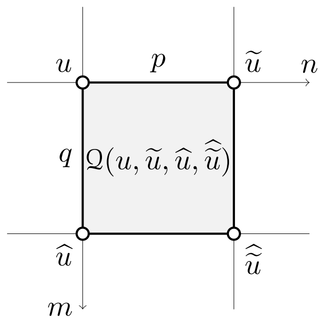

Figure 2.1: An elementary plaquette in the lattice

can be written as a quad equation,

$$\mathcal{Q}(u, \widetilde{u}, \widehat{u}, \widehat{\widetilde{u}}; p, q) = 0 \ , \tag{2.4}$$

comparing with (1.2). Although we have derived (2.3) here as a linearisation from the lattice KdV equation, it is the natural linearisation for nearly all the quad equations of the ABS list [1].

The linear lattice equation (2.3) can be derived from a variational principle on the three-point Lagrangian

$$\mathcal{L}(u, \widetilde{u}, \widehat{u}) = u(\widetilde{u} - \widehat{u}) - \frac{1}{2}\frac{p+q}{p-q}(\widetilde{u} - \widehat{u})^2 \ , \tag{2.5}$$

where, for the action, we sum across every plaquette in the lattice,

$$\mathcal{S} = \sum_{n,m \in \mathbb{Z}} \mathcal{L}(u_{n,m}, u_{n+1,m}, u_{n,m+1}; p, q) \ , \tag{2.6}$$

as in (1.22). The Lagrangian (2.5) is also the natural linearisation of the Lagrangian (1.25) of the lattice KdV equation, at $O(\eta^2)$. Lower order terms in η are either constants or total differences, which are absorbed in the action. The Lagrangian (2.5) gives rise to the lattice equation (2.3) via the Euler-Lagrange equations (1.23), which yield

$$(p - q)(\underset{\sim}{u} - \widehat{u}) - (p + q)(u - \widehat{\underset{\sim}{u}}) = (p - q)(\underset{\wedge}{u} - \widetilde{u}) - (p + q)(\widetilde{\underset{\wedge}{u}} - u) \ . \tag{2.7}$$

In the same way as the lattice KdV equation (section 1.1.1), this is a weaker version of the lattice equation (2.3): two copies are produced. We will see that this problem is remedied by the Lagrangian form structure associated with the multi-dimensional consistency of the model.

2.1.1 Multi-dimensional Consistency

The linear lattice equation (2.3) can be embedded into a *multi-dimensional* lattice, with directions labelled by subscripts i, j, k. Across an elementary plaquette in the $i-j$ plane, the linear lattice equation (2.3) takes the form

$$(p_i + p_j)(u_i - u_j) = (p_i - p_j)(u - u_{ij}) \ , \tag{2.8}$$

where u_i indicates u shifted once in the i direction on the lattice, and p_i is the lattice parameter associated to the i direction. Notice the symmetry of (2.8) under interchange of the lattice directions $i,\ j$.

This embedding can be performed *consistently* if (2.8) exhibits closure around the cube [83]. Considering an elementary cube within the multi-dimensional lattice, and initial conditions u, $\widetilde{u} \equiv u_1$, $\widehat{u} \equiv u_2$, $\overline{u} \equiv u_3$, there are three routes to calculate the variable $\widehat{\widetilde{\overline{u}}}$; for the equation to be multidimensionally consistent, all three possibilities must yield the same result (see figure 1.2). Around the cube, there are three elementary quad equations (2.8) and their shifts,

$$\begin{aligned}
(p+q)(\widetilde{u} - \widehat{u}) &= (p-q)(u - \widehat{\widetilde{u}}) \ , && (2.9a)\\
(q+r)(\widehat{u} - \overline{u}) &= (q-r)(u - \widehat{\overline{u}}) \ , && (2.9b)\\
(r+p)(\overline{u} - \widetilde{u}) &= (r-p)(u - \widetilde{\overline{u}}) \ , && (2.9c)\\
(p+q)(\widetilde{\overline{u}} - \widehat{\overline{u}}) &= (p-q)(\overline{u} - \widehat{\widetilde{\overline{u}}}) \ , && (2.9d)\\
(q+r)(\widehat{\widetilde{u}} - \widetilde{\overline{u}}) &= (q-r)(\widetilde{u} - \widehat{\widetilde{\overline{u}}}) \ , && (2.9e)\\
(r+p)(\widehat{\overline{u}} - \widehat{\widetilde{u}}) &= (r-p)(\widehat{u} - \widehat{\widetilde{\overline{u}}}) \ . && (2.9f)
\end{aligned}$$

Beginning from (2.9d) and substituting (2.9a,2.9b,2.9c) we deduce an expression for $\widehat{\widetilde{\overline{u}}}$ in terms of the initial values,

$$\widehat{\widetilde{\overline{u}}} = -\frac{p+q}{p-q}\frac{r+p}{r-p}\ \widetilde{u} - \frac{q+r}{q-r}\frac{p+q}{p-q}\ \widehat{u} - \frac{r+p}{r-p}\frac{q+r}{q-r}\ \overline{u} \ . \tag{2.10}$$

The symmetry of this expression is sufficient to guarantee the closure around the cube.

Note that this derivation required the critical partial fraction expression

$$\frac{p+q}{p-q}.\frac{q+r}{q-r} + \frac{q+r}{q-r}.\frac{r+p}{r-p} + \frac{r+p}{r-p}.\frac{p+q}{p-q} + 1 = 0 \ . \tag{2.11}$$

This fraction relation turns out to be a key combinatorial property for many of the results in this chapter. As in the lattice KdV case, the multi-dimensional consistency is the key integrability property of the linear lattice equation (2.3), leading to a Bäcklund transform, Lax representation, and ultimately Lagrangian two-form structure.

Plane wave solution

The multi-dimensional consistency of the system allows us to derive a Bäcklund transformation - and this in turn can be used to generate solutions. Beginning from equations (2.9b), (2.9c), we take $v := \overline{u}$ and let $\lambda := r$. We then have the transformation $u \to v$ with Bäcklund parameter λ,

$$\begin{aligned} (p+\lambda)(v-\widetilde{u}) &= (p-\lambda)(\widetilde{v}-u) \ , \\ (q+\lambda)(v-\widehat{u}) &= (q-\lambda)(\widehat{v}-u) \ . \end{aligned} \tag{2.12}$$

These equations imply that

$$\big[(p+q)(\widehat{v}-\widetilde{v})-(p-q)(\widehat{\widetilde{v}}-v)\big]+\big[(p+q)(\widehat{u}-\widetilde{u})-(p-q)(\widehat{\widetilde{u}}-u)\big]=0 \ , \tag{2.13}$$

or, in other words, if (2.3) holds for u, then it also must hold for v. The shift $u \to v$ is therefore an auto-Bäcklund transform.

Taking a trivial seed solution of (2.3) $u(n,m) = 0$, we apply the Bäcklund transform to gain the solution

$$v(n,m) = \left(\frac{p+\lambda}{p-\lambda}\right)^n \left(\frac{q+\lambda}{q-\lambda}\right)^m v(0,0) \ . \tag{2.14}$$

This is the discrete analogue of a plane wave, producing an exponential plane-wave factor in a continuum limit. By linear superposition we therefore have the general plane wave solution,

$$u(n,m) = \int_\Gamma \left(\frac{p+\lambda}{p-\lambda}\right)^n \left(\frac{q+\lambda}{q-\lambda}\right)^m w(\lambda)\mathrm{d}\lambda \ , \tag{2.15}$$

with an appropriate weight function and integration contour.

Lax representations

In a similar way to the lattice KdV equation, the Bäcklund transform also gives rise to a Lax pair for (2.3). To construct a linear spectral problem for an already linear system is perhaps a curious thing to do, but it is a feature the model shares with its non-linear relatives. Taking the Bäcklund transform (2.12) and writing the spectral variables $\phi := v$, $k := \lambda$, we find the linear inhomogeneous spectral problem

$$\widetilde{\phi} = u + \frac{p+k}{p-k}(\phi-\widetilde{u}) \ , \quad \widehat{\phi} = u + \frac{q+k}{q-k}(\phi-\widehat{u}) \ . \tag{2.16}$$

The lattice equation (2.3) arises from the compatibility condition $\widehat{\widetilde{\phi}} = \widetilde{\widehat{\phi}}$.

Introducing the vector $\Phi := (\phi, 1)^T$, we can pose (2.16) in a matrix form,

$$\widetilde{\Phi} = L\Phi \ , \quad \widehat{\Phi} = M\Phi \ , \tag{2.17a}$$

with matrices

$$L = \begin{pmatrix} \frac{p+k}{p-k} & u - \frac{p+k}{p-k}\widetilde{u} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{p+k}{p-k} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -\widetilde{u} \\ 0 & 1 \end{pmatrix} , \tag{2.17b}$$

$$M = \begin{pmatrix} \frac{q+k}{q-k} & u - \frac{q+k}{q-k}\widehat{u} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{q+k}{q-k} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -\widehat{u} \\ 0 & 1 \end{pmatrix} . \tag{2.17c}$$

These factorisations define the matrices $L =: U\mathbb{P}_k\widetilde{U}^{-1}$, and $M =: U\mathbb{Q}_k\widehat{U}^{-1}$. The lattice equation (2.3) now arises from the compatibility of the matrix spectral problem (2.17a), which is the zero-curvature condition

$$\widehat{L}M = \widetilde{M}L \ . \tag{2.18}$$

The lattice equation (2.3) appears as the coefficient at $\mathcal{O}(k)$ in the $(1,2)$ entry of the matrix; all other entries are trivially satisfied.

An alternative Lax formulation can be derived by exploiting the origin of the lattice equation (2.3) as a linearisation of the lattice KdV equation (2.1). Beginning with the spectral problem for the lattice KdV equation (1.19), we introduce the linearisation (2.2) and expand,

$$\Phi = \Phi_0 + \eta\Phi_1 \ , \quad \mathscr{L} = L_0 + \eta L_1 \ , \quad \mathscr{M} = M_0 + \eta M_1 \ , \tag{2.19}$$

where Φ_0, L_0, M_0 are fixed points. Expanding the form of the matrices (1.20) in η yields

$$L_0 = W_0\mathfrak{P}_k W_0^{-1} = \begin{pmatrix} p - w_0 & 1 \\ k^2 - w_0^2 & p + w_0 \end{pmatrix} , \tag{2.20a}$$

$$M_0 = W_0\mathfrak{Q}_k W_0^{-1} = \begin{pmatrix} q - w_0 & 1 \\ k^2 - w_0^2 & q + w_0 \end{pmatrix} , \tag{2.20b}$$

$$L_1 = \begin{pmatrix} -\widetilde{u} & 0 \\ (p - w_0)u - (p + w_0)\widetilde{u} & u \end{pmatrix} , \tag{2.20c}$$

$$M_1 = \begin{pmatrix} -\widehat{u} & 0 \\ (q - w_0)u - (q + w_0)\widehat{u} & u \end{pmatrix} , \tag{2.20d}$$

so that L_0 and M_0 are clearly non-dynamical, and commuting. To lowest order in η, the spectral problem (1.19) gives

$$(p-k)\Phi_0 = L_0\Phi_0 \ , \quad (q-k)\Phi_0 = M_0\Phi_0 \ . \tag{2.21}$$

So, Φ_0 is a joint eigenvector of L_0 and M_0; we can take $\Phi_0^T = (1, w_0 - k)^T\phi_0$. The interesting part of the spectral problem is at first order in η, where we have

$$(p-k)\widetilde{\Phi}_1 = L_0\Phi_1 + L_1\Phi_0 \ , \quad (q-k)\widehat{\Phi}_1 = M_0\Phi_1 + M_1\Phi_0 \ . \tag{2.22}$$

Seeking a zero-curvature condition for this spectral problem leads to a condition on the coefficients of Φ_0,

$$\widehat{L}_1 M_0 + L_0 M_1 = \widetilde{M}_1 L_0 + M_0 L_1 \ . \tag{2.23}$$

(The Φ_1 coefficients are automatically satisfied due to the commuting of L_0 and M_0.) This linearised zero-curvature condition then yields the linear lattice equation (2.3), as desired. This represents an entirely alternative Lax description to (2.18), and is a somewhat unusual Lax pair compared to those normally associated with lattice equations.

The challenge in exploiting this Lax pair for the integrability of the model is that, in the linearising limit from lattice KdV, the invariants appear at *quadratic* order in η, whereas the equations of motion and Lax pair appear at first order. It is unclear whether the invariants can be recovered from this linearised Lax pair.

2.1.2 Lagrangian Two-form Structure

The multi-dimensional consistency of quad equations is the key to their integrability, leading to Bäcklund transforms, soliton solutions, and Lax pairs. Such multi-dimensionally consistent quad equations can be seen as a set of compatible equations that all hold simultaneously on the same set of variables $u_{n,m,l}$, as in (2.8).[1] The consistency of these equations is guaranteed by the closure around the cube (2.10). Quad equations (and specifically the linear lattice equation) also arise from a Lagrangian, but the usual variational principle produces only the basic lattice equation, and not the full multi-dimensional family. How, then, can these multiple, consistent equations be recovered from an extended variational principle?

[1] The multi-dimensional consistency can be freely extended into an arbitrary number of dimensions: there is no need to stop at three. Equally, three dimensions is sufficient to illustrate everything we need.

For such lattice systems, Lobb and Nijhoff recently introduced a multi-form variational principle that captures the multiplicity of equations of motion within a single Lagrangian 2-form and variational principle [62, 63]. Generalising the Lagrangian (2.5) to the multi-dimensional case,

$$\mathcal{L}_{ij}(u) := \mathcal{L}(u, u_i, u_j; p_i, p_j) = u(u_i - u_j) - \tfrac{1}{2}s_{ij}(u_i - u_j)^2 \ , \tag{2.24a}$$

$$\text{where } s_{ij} = \frac{p_i + p_j}{p_i - p_j} \ , \tag{2.24b}$$

we consider an action over a 2-dimensional surface σ, embedded within the multi-dimensional lattice. σ is composed of oriented, elementary plaquettes σ_{ij}, shown in figure 2.2. To each oriented plaquette σ_{ij} is associated a Lagrangian $\mathcal{L}_{ij}(u)$, so that the action is the sum of Lagrangians over the surface σ,

$$\mathcal{S}_\sigma = \sum_{\sigma_{ij} \in \sigma} \mathcal{L}(u, u_i, u_j; p_i, p_j) \ . \tag{2.25}$$

This is a natural generalisation of the action (2.6). Note the antisymmetry of the Lagrangians with respect to the orientation of the plaquette, $\mathcal{L}_{ji}(u) = -\mathcal{L}_{ij}(u)$.

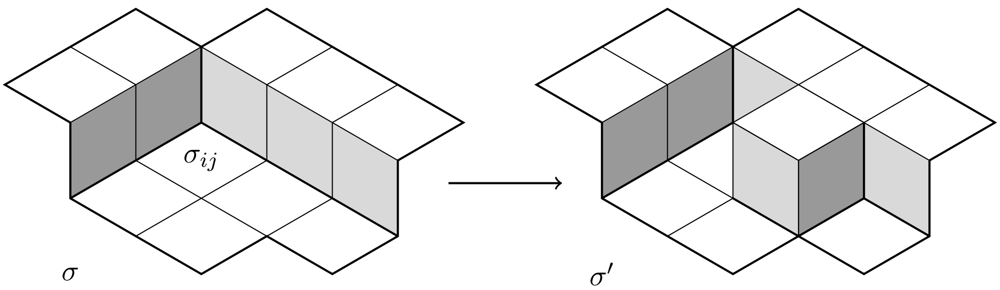

Figure 2.2: The multi-dimensional surface σ, deformed by an elementary move to the surface σ'. The elementary plaquette is σ_{ij}.

The crucial observation is the *closure* property of the Lagrangians $\mathcal{L}_{ij}(u)$ (2.24). Considering the combination of oriented Lagrangians on the faces of a cube, *on the equations of motion*, the Lagrangians sum to zero. Introducing the notation of a difference operator Δ_i in the direction i, so that $\Delta_i u := u_i - u$, we have the sum of oriented Lagrangians around the cube,

$$\Delta_1\mathcal{L}_{23}(u) + \Delta_2\mathcal{L}_{31}(u) + \Delta_3\mathcal{L}_{12}(u)$$

$$:= \mathcal{L}_{23}(u_1) - \mathcal{L}_{23}(u) + \mathcal{L}_{31}(u_2) - \mathcal{L}_{31}(u) + \mathcal{L}_{12}(u_3) - \mathcal{L}_{12}(u) \ , \tag{2.26a}$$

$$= 0 \ , \tag{2.26b}$$

where the final equality (2.26b) holds only when we apply the lattice equation (2.8). The implication of Lagrangian closure (2.26) is that a local move deforming the surface $\sigma \to \sigma'$ will leave the action $\mathcal{S}_\sigma$ (2.25) on the equations of motion unchanged (see figure 2.2). In other words, the action $\mathcal{S}_\sigma$ is invariant under deformations of the surface σ. Note that Lagrangian closure for the linear case (2.24) is a property that it inherits from its parent models of the ABS list, which were shown to have the Lagrangian closure property in [62].

This simple observation leads to a much deeper idea: the multi-form variational principle. The traditional variational principle holds the surface σ fixed and demands the action be stationary under variation of the dependent variables u. This leads to the equations of motion arising as Euler-Lagrange equations, although in the case of quadrilateral equations we find a weaker version of the equation (1.26), (2.7). Lobb and Nijhoff extended this variational principle by demanding that the action be stationary not only under variations of the dependent variable, but also under variation of the *surface itself*; that is, under variation of the *independent* variables. This leads to not a single Euler-Lagrange equation, but a system of Euler-Lagrange equations, corresponding to different configurations of the surface.

In [63], the authors derive three *elementary configurations* of the surface, that yield all the fundamental Euler-Lagrange equations of the model, shown in figure 2.3. The three configurations arise from considering the Euler-Lagrange equations around a cube - the simplest possible closed surface. The usual variational principle on this surface produces the three Euler-Lagrange equations,

$$\frac{\partial}{\partial u}\Big(\mathcal{L}_{ij}(u) + \mathcal{L}_{jk}(u) + \mathcal{L}_{ki}(u)\Big) = 0\ , \tag{2.27a}$$

$$\frac{\partial}{\partial u}\Big(\mathcal{L}_{ij}(u_{-i}) - \mathcal{L}_{jk}(u) + \mathcal{L}_{ki}(u_{-i})\Big) = 0\ , \tag{2.27b}$$

$$\frac{\partial}{\partial u}\Big(\mathcal{L}_{ij}(u_{-j}) + \mathcal{L}_{ki}(u_{-k})\Big) = 0\ , \tag{2.27c}$$

recalling antisymmetry of the Lagrangians under the change of orientation. These Euler-Lagrange equations are *elementary* in the sense that, by varying the surface, we demand that the equations should hold everywhere in the lattice; all possible Euler-Lagrange equations are then constructed from these three elementary choices. In each case, the action is constructed from a sum of neighbouring Lagrangians, indicated by the shaded triangles in figure 2.3 - the triangle indicates the three variables appearing in each Lagrangian, consisting of a base field variable u and two evolved variables u_i, u_j. The white circle indicates the field variable over which we perform the usual variational principle, with black circles indicating other field variables appearing in the equations.

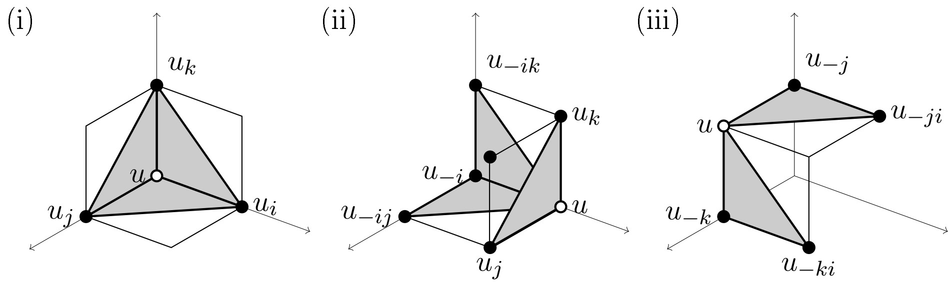

Figure 2.3: Three elementary configurations of the lattice.

Lobb and Nijhoff [63] then found that, in order to be consistent, such a system must be described by a Lagrangian of the form

$$\mathcal{L}(u, u_i, u_j; p_i, p_j) = A(u, u_i; p_i) - A(u, u_j; p_j) + C(u_i, u_j; p_i, p_j) \ , \tag{2.28}$$

where C_{ij} must be antisymmetric under interchange of i and j. Notice that the Lagrangian for the linear lattice equation (2.24) is already in this form. We must have

$$A(u, u_i; p_i) - A(u, u_j; p_j) = u(u_i - u_j) \ , \tag{2.29}$$

so that

$$A(u, u_i; p_i) = uu_i + \lambda u + \mu \ , \tag{2.30a}$$

$$C(u_i, u_j; p_i, p_j) = -\frac{1}{2}\frac{p_i + p_j}{p_i - p_j}(u_i - u_j)^2 \ , \tag{2.30b}$$

for arbitrary constants λ and μ. The Euler-Lagrange equations (2.27) then yield the equation on a single plaquette [63]

$$\frac{\partial}{\partial u_i}\Big(A(u, u_i; p_i) - A(u_i, u_{ij}; p_j) + C(u_i, u_j; p_i, p_j)\Big) = 0 \ , \tag{2.31}$$

which produces the lattice equation

$$(p_i + p_j)(u_i - u_j) = (p_i - p_j)(u - u_{ij} - \lambda) \ , \tag{2.32}$$

for all pairs of lattice directions i, j. Such an equation for any λ must, by construction, be multidimensionally consistent; taking $\lambda = 0$ we recover precisely the linear lattice equation in its multi-dimensional form (2.8).

The Lagrangian 2-form therefore captures the full set of multi-dimensionally consistent lattice equations via an extended variational principle. By varying the underlying surface geometry σ, the full set of lattice equations can be forced to hold simultaneously at the critical point for the action. This structure for the linear lattice equation is inherited entirely from its non-linear parents in the ABS list, for which equations the 2-form has been shown explicitly (including the case of the lattice KdV equation discussed in section 1.1.1). It is nonetheless interesting that the structure should continue to hold even for this simple, linear model.

2.1.3 Uniqueness

We have shown that the linear lattice Lagrangian (2.5) has a Lagrangian 2-form structure. In fact, it is the almost unique quadratic Lagrangian 2-form (i.e. that exhibits the closure property). The general form for a three-point Lagrangian 2-form is given in (2.28), with the lattice equation arising from the 2-form Euler-Lagrange equation (2.31). If we restrict our attention to quadratic Lagrangians, we therefore have the general form

$$\mathcal{L}_{ij}(u,u_i,u_j;p_i,p_j) = \left(\tfrac{1}{2}a_i u^2 + c_i u u_i\right) - \left(\tfrac{1}{2}a_j u^2 + c_j u u_j\right) + \left(\tfrac{1}{2}b_{ij}u_i^2 - \tfrac{1}{2}b_{ji}u_j^2 + \delta_{ij}u_i u_j\right) \ , \tag{2.33}$$

where we require $\delta_{ji} = -\delta_{ij}$ in order to have antisymmetry. Here, subscripts on coefficients indicate dependence on the lattice parameters p_i and p_j.

This Lagrangian 2-form (2.33) yields the equation of motion

$$c_i u - c_j u_{ij} = (a_j - b_{ij})u_i - \delta_{ij}u_j \ . \tag{2.34}$$

This is a quad equation, and as such it is required to be symmetric under the interchange of i and j. This leads to the conditions

$$c_i = c_j = c \ , \ \text{constant}, \tag{2.35a}$$

$$a_j - b_{ij} = \delta_{ij} \ . \tag{2.35b}$$

Noting that the Lagrangian (2.33) already obeys the closure relation (2.26) on the equations of motion (2.34), we use the freedom to multiply by an overall constant to let $c = 1$, and hence the general quadratic Lagrangian 2-form is given by

$$\mathcal{L}_{ij}(u,u_i,u_j) = u(u_i - u_j) - \tfrac{1}{2}\delta_{ij}(u_i - u_j)^2 + \tfrac{1}{2}a_i(u^2 - u_j^2) - \tfrac{1}{2}a_j(u^2 - u_i^2) \ . \tag{2.36}$$

This has the same form as the linear lattice Lagrangian (2.5), but with a more general dynamical, anti-symmetric parameter δ_{ij}, and the free parameter a_i that does not effect the equations of motion and is absorbed in the action.

The extended variational principle for 2-forms therefore does more than give the multi-dimensional lattice equations. It also restricts the class of permissible Lagrangians to those obeying a closure relation; only such Lagrangians give an action that can be stationary with respect to variations of the surface σ. In some sense, then, the extended variational principle also selects appropriate Lagrangians describing the model. The inverse problem of Lagrangian mechanics is, given an equation, to find a Lagrangian description, of which there may be many possible choices. The multi-form variational principle perhaps offers some resolution to this problem.

2.2 One Dimensional Reduction: The Discrete Harmonic Oscillator

2.2.1 Periodic Reduction

A common procedure in the literature is the construction of integrable symplectic mappings as reductions of lattice equations, by the application of some boundary conditions [24, 75, 80, 88]. Considering the linearisation of the lattice KdV equation, we follow the same reduction procedure as has been previously studied for non-linear quad equations.

The linear lattice equation (2.3) is reduced to a difference equation in one dimension by a periodic initial value problem. The evolution of the data progresses through the lattice according to a dynamical map, constructed via the lattice equation. We begin with initial data a_0, a_1 and a_2, and let $\widehat{a}_2 = a_0$, according to figure 2.4. This unit is then repeated periodically across an infinite staircase in the lattice. This is the simplest meaningful reduction we can perform on the lattice equation. The lattice variable m becomes a discrete time, labelling iteration of the mapping.

Applying the linear lattice equation (2.3) to each plaquette, we can write equations for

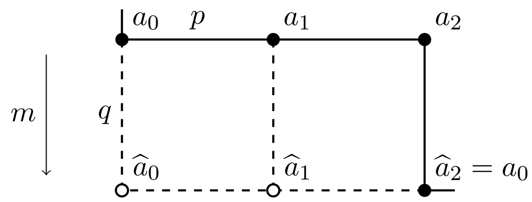

Figure 2.4: The simplest periodic initial value problem on the lattice equation.

the dynamical mapping $(a_0, a_1, a_2) \to (\widehat{a}_0, \widehat{a}_1, \widehat{a}_2)$, as

$$\begin{aligned} \widehat{a}_0 &= a_1 + s(\widehat{a}_1 - \widehat{a}_2) \ , \\ \widehat{a}_1 &= a_2 + s(a_0 - a_1) \ , \qquad \text{where } s := \frac{p-q}{p+q} \ , \\ \widehat{a}_2 &= a_0 \ . \end{aligned} \tag{2.37}$$

This is a finite-dimensional discrete system; we introduce the reduced variables

$$x := a_1 - a_0 \ , \qquad y := a_2 - a_1 \ . \tag{2.38}$$

In terms of these, the equations of the map become

$$\widehat{x} = y - sx + s\widehat{y} \ , \quad \widehat{y} = -x - y + sx \ , \tag{2.39}$$

and, by eliminating y, we write the second order difference equation

$$\widehat{x} + 2bx + \underset{\wedge}{x} = 0 \ , \quad b := 1 + 2s - s^2 \ . \tag{2.40}$$

This equation can be expressed by a Lagrangian-type generating function

$$\mathcal{L}(x, \widehat{x}) = -x\widehat{x} - bx^2 \ , \tag{2.41}$$

and so is symplectic, $\mathrm{d}\widehat{x} \wedge \mathrm{d}\widehat{y} = \mathrm{d}x \wedge \mathrm{d}y$. The map also possesses an exact invariant,

$$I_b(x, \widehat{x}) = x^2 + \widehat{x}^2 + 2bx\widehat{x} \ . \tag{2.42}$$

The difference equation (2.40) is a discrete harmonic oscillator. It is not difficult to see that the general solution to (2.40) is

$$x_m = c_1 \sin \mu m + c_2 \cos \mu m \ , \tag{2.43}$$

where $\cos \mu = -b$ and m is the discrete variable. This has an obvious relation to the solution for the continuous time harmonic oscillator.

Continuous Flow

This solution (2.43) can alternatively be written as

$$x_m = A\lambda^m + B\lambda^{-m} \ , \quad \lambda = -b + \sqrt{b^2 - 1} \ . \tag{2.44}$$

By considering derivatives with respect to the *parameter* b, we write

$$\frac{\partial x_m}{\partial b} = \frac{-m}{\sqrt{b^2-1}}(A\lambda^m - B\lambda^{-m}) \ . \tag{2.45}$$

Writing $x := x_m$, $\widehat{x} := x_{m+1}$ and $\underset{\wedge}{x} := x_{m-1}$ allows us to derive the semi-discrete equations

$$\frac{dx}{db} = \frac{m}{1-b^2}(bx + \widehat{x}) \ , \quad \frac{dx}{db} = -\frac{m}{1-b^2}(bx + \underset{\wedge}{x}) \ , \tag{2.46}$$

and eliminating $\widehat{x}$, $\underset{\wedge}{x}$ yields the second order differential equation in b

$$(1-b^2)\frac{d^2x}{db^2} - b\frac{dx}{db} + m^2 x = 0 \ . \tag{2.47}$$

A remarkable exchange has taken place: the parameter and independent variable of the discrete case, b and m, have exchanged roles to become the independent variable and parameter of a continuous time model. Note that the differential equation (2.47) can be simplified by taking $\mu := \cos^{-1}(-b)$ as the "time" variable, so that

$$\frac{d^2x}{d\mu^2} + m^2 x = 0 \ . \tag{2.48}$$

This is the equation for the harmonic oscillator, with a quantised frequency $\omega = m$, formerly the discrete time variable. Note also that the solution (2.43) guarantees the compatibility of the discrete and continuous flows.

2.2.2 Commuting Discrete Flow

Recalling that the linear lattice equation (2.3) is multi-dimensionally consistent, we can introduce a third direction to the reduction, with parameter r, and the shifted variables $\overline{a}_i$, as shown in figure 2.5.

To derive the mapping, we now use the lattice equations (compare (2.8))

$$(q+r)(\widehat{u} - \overline{u}) = (q-r)(u - \widehat{\overline{u}}) \ , \tag{2.49a}$$

$$(r+p)(\overline{u} - \widetilde{u}) = (r-p)(u - \widetilde{\overline{u}}) \ . \tag{2.49b}$$

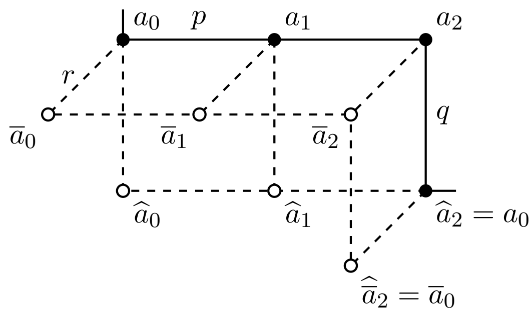

Figure 2.5: Commuting flow: the variables $\overline{a}_i$ extend from the plane in a third direction.

In terms of the a_i, these equations give

$$\begin{aligned} \overline{a}_0 &= a_1 + t(\overline{a}_1 - a_0) \ , \\ \overline{a}_1 &= a_2 + t(\overline{a}_2 - a_1) \ , \quad \text{where} \quad \begin{aligned} t &:= \tfrac{p-r}{p+r} \ , \\ t' &:= \tfrac{q-r}{q+r} \ . \end{aligned} \\ \overline{a}_2 &= a_0 + t'(\overline{a}_0 - a_2) \ . \end{aligned} \tag{2.50}$$

Again, we are interested in the reduction to (x, y) variables (2.38) which yields the map $(x, y) \to (\overline{x}, \overline{y})$, given by

$$\overline{x} = y + t(\overline{y} - x) \ , \tag{2.51a}$$

$$\overline{y} = -y - \frac{1-t}{1-tt'}(x + t'\overline{x}) \ . \tag{2.51b}$$

This map can be written in a matrix form,

$$\begin{pmatrix} 1 & -t \\ \frac{(1-t)t'}{1-tt'} & 1 \end{pmatrix} \begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} = \begin{pmatrix} -t & 1 \\ -\frac{1-t}{1-tt'} & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \ , \tag{2.52}$$

from which it can be shown to be area preserving, $\mathrm{d}\overline{x} \wedge \mathrm{d}\overline{y} = \mathrm{d}x \wedge \mathrm{d}y$.

As before, we eliminate y to produce a second order difference equation in x,

$$\overline{x} + 2ax + \underline{x} = 0 \ , \quad \text{with} \quad 2a := \frac{(2t + 1 - t^2) - t'(2t - 1 + t^2)}{1 - t^2 t'} \ . \tag{2.53}$$

This equation has the same form as (2.40), that of a discrete harmonic oscillator, along with invariant

$$I_a(x, \overline{x}) = x^2 + \overline{x}^2 + 2ax\overline{x} \ . \tag{2.54}$$

Using the reduced forms of the two mappings (equations (2.39) and (2.52)) we can write both maps $(x, y) \to (\widehat{x}, \widehat{y})$ and $(x, y) \to (\overline{x}, \overline{y})$ in matrix form,

$$\widehat{\mathbf{x}} = \mathbf{S}\,\mathbf{x} \ , \quad \overline{\mathbf{x}} = \mathbf{T}\,\mathbf{x} \ , \quad \text{for } \mathbf{x} := \begin{pmatrix} x \\ y \end{pmatrix} \ , \tag{2.55}$$

with matrices

$$\mathbf{S} = \begin{pmatrix} s^2-2s & 1-s \\ s-1 & -1 \end{pmatrix} \ , \quad \mathbf{T} = \frac{1}{\Delta}\begin{pmatrix} t\frac{tt'+t-2}{1-tt'} & 1-t \\ t-1 & -\frac{1+t'-2tt'}{1-tt'} \end{pmatrix} \ , \tag{2.56}$$

where $\Delta := (1-t^2t')/(1-tt')$. It is then clear that the two maps commute, $(\widehat{\overline{x}}, \widehat{\overline{y}}) = (\overline{\widehat{x}}, \overline{\widehat{y}})$, since we have $[\mathbf{S}, \mathbf{T}] = 0$. This last relation relies on the parameter identity

$$stt' = s - t + t' \ , \tag{2.57}$$

which is a reformulation of the partial fraction identity (2.11) and is easily shown using the definitions for s, t and t'.

Our equations are slightly simplified by introducing the parameters

$$P := p^2 + pq \ , \quad Q := q^2 \ , \quad R := r^2 \ , \tag{2.58}$$

in terms of which $a = (P-R)/(P+R)$ and $b = (P-Q)/(P+Q)$. By returning to the reduced forms of the mappings (2.39), (2.51) and eliminating y in a different manner, we can derive "corner equations" for the evolution. These link x, $\widehat{x}$ and $\overline{x}$, or $\widehat{x}$, $\overline{x}$ and $\widehat{\overline{x}}$, respectively. Thus,

$$\left(\frac{P-Q}{q} - \frac{P-R}{r}\right) x = \frac{P+R}{r}\overline{x} - \frac{P+Q}{q}\widehat{x} \ , \tag{2.59a}$$

$$\left(\frac{P-Q}{q} - \frac{P-R}{r}\right) \widehat{\overline{x}} = \frac{P+R}{r}\widehat{x} - \frac{P+Q}{q}\overline{x} \ . \tag{2.59b}$$

We therefore have multiple, consistent equations of motion (2.40), (2.53), (2.59) all holding simultaneously on the same variable x. Such a set of multiply consistent equations is precisely the scenario best described by a Lagrangian one-form.

Joint Solutions

The compatibility of the two discrete evolutions and their corner equations allows a joint solution to the equations, $x_{m,n}$. We allow m to label the hat evolution, and n to label the bar evolution, so that $x = x_{m,n}$, $\widehat{x} = x_{m+1,n}$, $\overline{x} = x_{m,n+1}$, and so on. We have the equations of motion (2.40), (2.53)

$$\widehat{x} + 2bx + \underset{\wedge}{x} = 0 \ , \quad \overline{x} + 2ax + \underline{x} = 0 \ , \tag{2.60}$$

which have solutions (2.43)

$$x_m = \beta_1 \sin(\mu m) + \beta_2 \cos(\mu m) \ , \quad \text{with} \quad \cos\mu = -b \ , \tag{2.61a}$$

$$x_n = \alpha_1 \sin(\nu n) + \alpha_2 \cos(\nu n) \ , \quad \text{with} \quad \cos\nu = -a \ . \tag{2.61b}$$

Now β_1, β_2 in the full case can depend on n, so that

$$x_{m,n} = \alpha \sin(\mu m)\sin(\nu n) + \beta \sin(\mu m)\cos(\nu n) + \gamma \cos(\mu m)\sin(\nu n) + \delta \cos(\mu m)\cos(\nu n) \ . \qquad (2.62)$$

Requiring $x_{m,n}$ to also obey the corner equations (2.59) imposes conditions on the constants; the application of some basic trigonometric identities leads to the general solution

$$x_{m,n} = a_1 \sin(\mu m + \nu n) + a_2 \cos(\mu m + \nu n) \ . \qquad (2.63)$$

Note the comparison with the interpolating continuous time solution,

$$x(t) = c_1 \sin \omega T + c_2 \cos \omega T \ . \qquad (2.64)$$

In a standard continuous limit, it is clear that the commuting discrete flows must degenerate to a single continuous-time harmonic oscillator flow. However, in section 2.2.5 we consider an alternative continuous flow where this may not be true, whilst in section 2.3 we examine flows of higher dimension where this degeneracy may be avoidable.

2.2.3 Lagrangian one-form structure

Recall the discrete Ruijsenaars-Schneider model discussed in section 1.2.2. There we had a set of variables x_i with compatible evolutions in two distinct discrete-time directions (1.45), (1.46), supplemented by constraint equations that govern the compatibility of the flows (1.47). This structure of two commuting, discrete flows can be described by a Lagrangian one-form structure. But this is the same scenario for the commuting, discrete harmonic oscillators: the Lagrangians generating the flows $x \to \widehat{x}$ and $x \to \overline{x}$ (2.40), (2.53) should form the components of a *difference 1-form*, each associated with an oriented direction on a 2D lattice.

The action functional is defined as a sum of elementary Lagrangian elements over an arbitrary discrete curve Γ in the two time variables m, n, shown in figure 2.6,

$$\mathcal{S}[x(\mathbf{n});\Gamma] = \sum_{\gamma(\mathbf{n})\in\Gamma} \mathcal{L}_i(x(\mathbf{n}), x(\mathbf{n}+\mathbf{e}_i)) \ . \qquad (2.65)$$

$\gamma(\boldsymbol{n})$ are the unit elements that compose the discrete curve Γ, such that $\gamma(\boldsymbol{n})$ corresponds to the single time-step evolution of the system from time $\boldsymbol{n} = (m, n)$ to time $\boldsymbol{n} + \mathbf{e}_i$ (where

$\mathbf{e}_i$ is the unit vector in time co-ordinate i). The usual variational principle demands that, on the equations of motion, the action $\mathcal{S}_\Gamma$ be stationary under the variation of the dynamical variables x. In the variational principle for forms, we also demand that $\mathcal{S}_\Gamma$ be stationary under variations of the curve Γ itself. This principle will lead to the compatibility of equations of motion and corner equations, under the condition of *closure* of the Lagrangians. That is, on the equations of the motion, the action should be invariant under local deformations to the curve, $\Gamma \to \Gamma'$ as shown in figure 2.6. This requires

$$\Box\mathcal{L} \;:=\; \mathcal{L}_b(x,\widehat{x}) + \mathcal{L}_a(\widehat{x},\widehat{\overline{x}}) - \mathcal{L}_b(\overline{x},\widehat{\overline{x}}) - \mathcal{L}_a(x,\overline{x}) \;=\; 0\;, \tag{2.66}$$

where this last equality holds only on the equations of motion. (Recall that Lagrangian closure is a closure around the elementary square, shown in figure 1.3.)

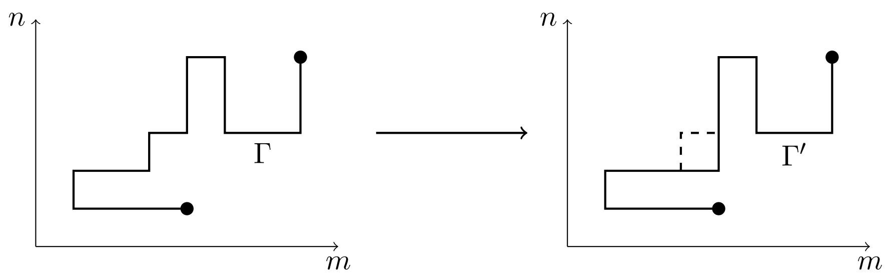

Figure 2.6: A curve Γ in the discrete variables, transformed to Γ' by an elementary move.

In the model we are considering, we already have compatible flows with consistent corner equations, and so it is natural for us to seek a Lagrangian one-form exhibiting closure. However, if we naively seek to satisfy the closure relation (2.66) with simple Lagrangians of the form (2.41), we will find that these do not suffice - the closure requires a specific form for the Lagrangians. We can write a family of quadratic Lagrangians which generate the correct equations of motion (2.40), (2.53),

$$\mathcal{L}_a \;=\; \alpha\big(-x\overline{x} - (a-a_0)x^2 - a_0\overline{x}^2\big)\;, \tag{2.67a}$$

$$\mathcal{L}_b \;=\; \beta\big(-x\widehat{x} - (b-b_0)x^2 - b_0\widehat{x}^2\big)\;, \tag{2.67b}$$

and apply the closure $\Box\mathcal{L} = 0$ to these as a condition.

Recall that closure holds only on the solutions to the equations of motion, so that we apply the corner equations (2.59) to $\Box\mathcal{L}$, eliminating x and $\widehat{\overline{x}}$ to give $\Box\mathcal{L}$ in terms only of $\widehat{x}$ and $\overline{x}$. Comparing coefficients of the remaining terms and demanding that α, a_0 and

β, b_0 be independent of Q and R respectively, we find the conditions on the coefficients

$$\alpha = \frac{P+R}{r}\gamma \,, \qquad a_0 = \frac{r}{P+R}f(P) + \frac{1}{2}a \,, \tag{2.68a}$$

$$\beta = \frac{P+Q}{q}\gamma \,, \qquad b_0 = \frac{q}{P+Q}f(P) + \frac{1}{2}b \,, \tag{2.68b}$$

where γ is some overall constant, and $f(P)$ is a free function of P. f does not make any further contribution to what follows, and so we ignore it; let $f \equiv 0$, so $a_0 = a/2$ and $b_0 = b/2$.

This yields the Lagrangians

$$\mathcal{L}_a(x,\overline{x}) = \frac{1}{r}\Big(-(P+R)\,x\overline{x} - \frac{1}{2}(P-R)\,(x^2+\overline{x}^2)\Big) \,, \tag{2.69a}$$

$$\mathcal{L}_b(x,\widehat{x}) = \frac{1}{q}\Big(-(P+Q)\,x\widehat{x} - \frac{1}{2}(P-Q)\,(x^2+\widehat{x}^2)\Big) \,. \tag{2.69b}$$

By construction, these obey the condition $\Box\mathcal{L} = 0$ on the equations of motion, and also yield the equations of motion (2.40) and (2.53) by the usual variational principle. This eliminates a great deal of the usual freedom in choosing our Lagrangian: requiring the closure condition mandates a specific form for the Lagrangians. As we saw for the two-form case, the extended variational principle for forms restricts the class of admissible Lagrangians, so that the Lagrangian itself is in some sense a solution to the least action principle.

The equations of motion (2.40) and (2.53) arise from a variational principle on this action by construction, but the extended variational principle on the action $\mathcal{S}_\Gamma$ also yields the corner equations (2.59). As we allow the curve Γ to vary (leaving the action unchanged) there are four elementary curves in the space of two discrete variables, shown in figure 2.7. Across each curve, we can define an action, and then a variation with respect to the middle point, which leads to an equation of motion.

The action for elementary curve 2.7(i) is

$$S = \mathcal{L}_a(x,\overline{x}) + \mathcal{L}_b(\overline{x},\widehat{\overline{x}}) \,, \tag{2.70}$$

with Euler-Lagrange equation

$$\frac{\partial S}{\partial \overline{x}} = 2\left[-\left(\tfrac{P-R}{r} + \tfrac{P-Q}{q}\right)\overline{x} - \tfrac{P+R}{r}x - \tfrac{P+Q}{q}\widehat{\overline{x}}\right] = 0 \,, \tag{2.71}$$

which is compatible with the corner equations (2.59).

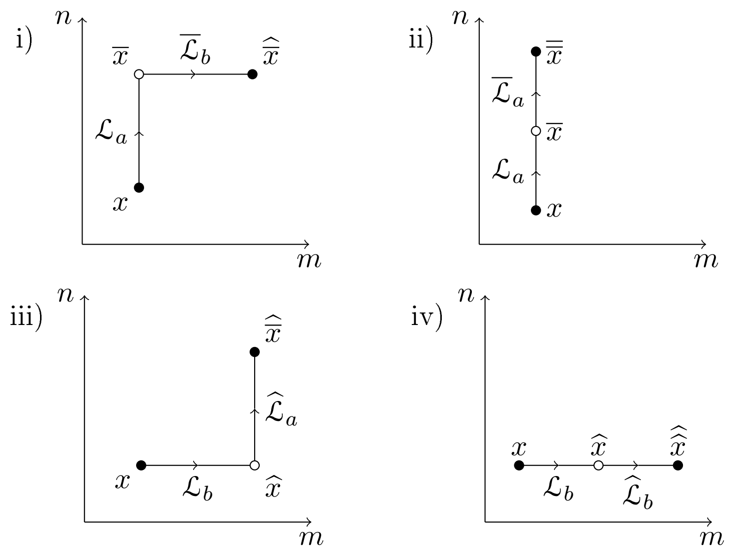

Figure 2.7: Simple discrete curves for variables m and n.

The action for elementary curve 2.7(ii) is

$$S = \mathcal{L}_a(x, \overline{x}) + \mathcal{L}_a(\overline{x}, \overline{\overline{x}}) \ , \tag{2.72}$$

with Euler-Lagrange equation

$$\frac{\partial S}{\partial \overline{x}} = 2\Big[-2\tfrac{P-R}{r}\overline{x} - \tfrac{P+R}{r}\left(x + \overline{\overline{x}}\right) \Big] = 0 \ , \tag{2.73}$$

which is equation (2.53). (i.e. this is a standard Euler-Lagrange equation).

Curves 2.7(iii) and (iv) yield similarly the equation of motion (2.40) and the other part of the corner equations (2.59). We therefore have, *for the specific choice of Lagrangians described*, a consistent 1-form structure, yielding the equations of motion and corner equations, and obeying a Lagrangian closure relation. The discrete harmonic oscillator, despite its simplicity, nonetheless has an underlying structure of a Lagrangian one-form expressing commuting flows: this is the simplest example yet discovered of such a structure.

2.2.4 Invariants

Recall that the discrete evolutions (2.40), (2.53) possess the invariants (2.42) and (2.54) respectively. It is straightforward to show using the equations of motion that both invariants are preserved under both evolutions,

$$\widehat{I}_b = \overline{I}_b = I_b \ , \quad \overline{I}_a = \widehat{I}_a = I_a . \tag{2.74}$$

It is not clear, however, that these invariants are necessarily equal: I_b has an apparent dependence on Q, and I_a on R, that must be resolved.

Taking the special choice of Lagrangians for the one-form (2.69), we can define canonical momenta (as discussed for (1.37)) and rewrite the invariants in those terms. Writing X_a as the momentum conjugate to x in $\mathcal{L}_a$, and X_b similarly for $\mathcal{L}_b$, we find

$$X_a = -\frac{\partial \mathcal{L}_a}{\partial x} = \frac{P+R}{r}\,\overline{x} + \frac{P-R}{r}\,x\ , \tag{2.75a}$$
$$X_b = -\frac{\partial \mathcal{L}_b}{\partial x} = \frac{P+Q}{q}\,\widehat{x} + \frac{P-Q}{q}\,x\ . \tag{2.75b}$$

Then, it follows as a direct consequence of the corner equation (2.59) that $X_a = X_b =: X$. In other words, we can define a common conjugate momentum for both evolutions. Writing the invariants in terms of x and X we find after multiplication by a constant (which clearly does not change the nature of the invariants) that

$$I_a = I_b = \frac{1}{2}X^2 + 2Px^2\ . \tag{2.76}$$

Note that this in entirely Q and R independent, and that it is nothing other than the Hamiltonian for the continuous harmonic oscillator, with angular frequency $\omega = 2\sqrt{P}$.

Notice that this is dependent on the choice of Lagrangians. A different choice, such as (2.41), yields different conjugate momenta that are no longer equal, and where also the equivalence of the invariants is no longer apparent. Requiring equality of the invariants turns out to be an equivalent condition to demanding Lagrangian closure [110].

To illustrate this, suppose that we again consider the more general quadratic Lagrangians of (2.67). These are associated to the conjugate momenta

$$X_a = -\frac{\partial \mathcal{L}_a}{\partial x} = \alpha\big(\overline{x} + 2(a-a_0)x\big)\ , \tag{2.77a}$$
$$X_b = -\frac{\partial \mathcal{L}_b}{\partial x} = \beta\big(\widehat{x} + 2(b-b_0)x\big)\ . \tag{2.77b}$$

Demanding equality of the momenta $X_a = X_b$, and comparing with the corner equation (2.59a), yields the conditions

$$\alpha = \frac{P+R}{r}\gamma\ , \qquad \beta = \frac{P+Q}{q}\gamma\ , \tag{2.78a}$$
$$2\beta(b-b_0) - 2\alpha(a-a_0) = \left(\frac{P-Q}{q} - \frac{P-R}{r}\right)\gamma\ , \tag{2.78b}$$

for some constant multiplier γ. Supposing we again require $\mathcal{L}_a$ and $\mathcal{L}_b$ to be Q and R independent, respectively, the condition (2.78b) reduces to the requirement

$$a_0 = \frac{1}{2}a + \frac{1}{2}\frac{r}{P+R}f(P) \; , \tag{2.78c}$$

$$b_0 = \frac{1}{2}b + \frac{1}{2}\frac{q}{P+Q}f(P) \; , \tag{2.78d}$$

for f a function of P only. But these are precisely the conditions on α, β, a_0 and b_0 that were required for the Lagrangian closure (2.68). Requiring the canonical momenta to coincide is an equivalent condition to the Lagrangian closure. Note that we could alternatively have demanded equality of the invariants I_a and I_b (up to an overall constant), which yields the same conditions on the coefficients in the Lagrangian.

2.2.5 Commuting Continuous Flows

In the same way as the parameter b generates a continuous flow compatible with the discrete evolution (2.47), so we can find a continuous flow in the parameter a. By manipulating the solution to the bar evolution (2.53) (analogously to (2.46)) we derive the semi-discrete equations

$$\frac{dx}{da} = \frac{n}{1-a^2}(ax+\overline{x}) \; , \quad \frac{dx}{da} = -\frac{n}{1-a^2}(ax+\underline{x}) \; , \tag{2.79}$$

leading to a differential equation for x as a function of a,

$$(1-a^2)\frac{d^2x}{da^2} - a\frac{dx}{da} + n^2 x = 0 \; . \tag{2.80}$$

The joint solution (2.63) guarantees the compatibility of the a and b flows with the commuting discrete evolutions. Recall that in terms of the lattice parameters a is given by $a = (P-R)/(P+R)$, such that the singularities of (2.80) at $a = \pm 1$ correspond to the limits on the lattice variable $R \to 0$ and $R \to \infty$ - that is, the lattice either collapses or is stretched to infinity.

The compatibility of the continuous flows can be further verified by checking the relation

$$\frac{\partial}{\partial a}\frac{\partial x}{\partial b} = \frac{\partial}{\partial b}\frac{\partial x}{\partial a} \; , \tag{2.81}$$

using (2.46) and (2.79). Adopting the shorthand for velocities $x_b := \partial x/\partial b$, the continuous time-flows are generated by the usual Euler-Lagrange equations on continuous time

Lagrangians of the form

$$\mathcal{L}_b(x, x_b) = \frac{1}{2m}\sqrt{1-b^2}\; x_b^2 - \frac{m}{2\sqrt{1-b^2}}x^2 \,, \tag{2.82a}$$

$$\mathcal{L}_a(x, x_a) = \frac{1}{2n}\sqrt{1-a^2}\; x_a^2 - \frac{n}{2\sqrt{1-a^2}}x^2 \,. \tag{2.82b}$$

Using the corner equations (2.59) these Lagrangians exhibit *continuous* multiform compatibility, obeying the relations

$$\frac{\partial \mathcal{L}_a}{\partial x_a} = \frac{\partial \mathcal{L}_b}{\partial x_b} \,, \quad \frac{\partial}{\partial a}\left(\frac{\partial \mathcal{L}_b}{\partial x}\right) = \frac{\partial}{\partial b}\left(\frac{\partial \mathcal{L}_a}{\partial x}\right) \,. \tag{2.83}$$

So, by considering the discrete parameters a, b now as continuous *variables*, we find a continuous-time 1-form structure [125, 126].

As in [26], the harmonic oscillator continues to display surprising new features. On the discrete level, we discover compatible flows that can be expressed through the structure of a Lagrangian form, even for this very simple case. This deeper structure then extends beyond the discrete case also into compatible continuous flows and we have an interplay between these discrete and continuous one-form structures.

2.3 Longer periods

The periodic reduction defined in section 2.2.1 is part of a more general family of periodic staircase initial value problems [24, 80, 88]. In general, we define $2P$ initial conditions, $a_0, a_1, \ldots, a_{2P-1}$ such that $a_0 = \widehat{a}_{2P-1}$, along a staircase as shown in figure 2.8. The linear lattice equation (2.3) defines a dynamical map $(a_0, a_1, \ldots, a_{2P-1}) \to (\widehat{a}_0, \widehat{a}_1, \ldots, \widehat{a}_{2P-1})$. As before, we introduce reduced variables $x_1, \ldots, x_{P-1}$, $y_1, \ldots, y_{P-1}$ and can eliminate the y_i to give a $P-1$ dimensional system of second order difference equations in terms of the x_i variables.

The $P = 2$ case yields a one dimensional mapping that is entirely equivalent to the case we have considered in section 2.2.1, except the lattice parameters combine in a slightly different way to give the coefficient of the harmonic oscillator.

The $P = 3$ case is the next case of interest, as here we find a system of coupled harmonic oscillators in x_1 and x_2, with two commuting invariants, and a similar commuting flow structure.

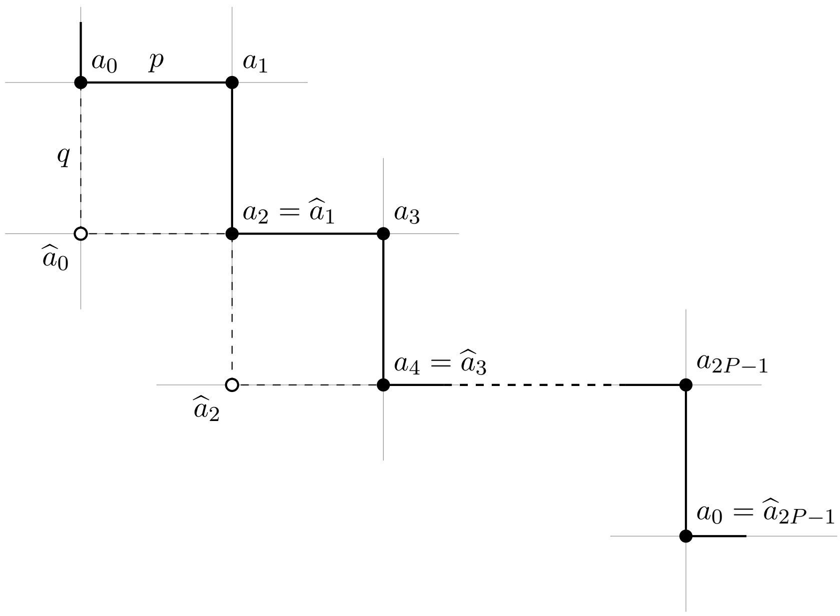

Figure 2.8: The periodic staircase for period P.

2.3.1 $P = 3$ Discrete Flow

As in section 2.2.1, we can derive equations for a discrete flow. We begin with the initial value problem on the staircase for lattice variables $a_0, \ldots, a_5$, as in figure 2.8. Applying the lattice equation (2.3) we have the mapping $\{a_i\} \to \{\widehat{a}_i\}$,

$$\widehat{a}_{2i-1} = a_{2i} \ , \quad \widehat{a}_{2i} = a_{2i+1} + s(a_{2i+2} - a_{2i}) \ . \tag{2.84}$$

(s is given in (2.37)). We can then define reduced variables, as before, but due to the increased degrees of freedom of the system we require four reduction variables,

$$\begin{aligned} x_1 &= a_2 - a_0 \quad , \quad y_1 = a_3 - a_1 \ , \\ x_2 &= a_4 - a_2 \quad , \quad y_2 = a_5 - a_3 \ , \end{aligned} \tag{2.85}$$

in terms of which the mapping equations (2.84) become

$$\begin{aligned} \widehat{y}_1 &= x_2 \quad &, \quad \widehat{x}_1 &= y_1 - s(x_1 - x_2) \ , \\ \widehat{y}_2 &= -(x_1 + x_2) \quad &, \quad \widehat{x}_2 &= y_2 - s(x_1 + 2x_2) \ . \end{aligned} \tag{2.86}$$

By eliminating y_i, we derive paired equations for a discrete flow in variables x_1 and x_2,

$$\widehat{x}_1 + \widehat{x}_2 + \underset{\wedge}{x}_1 + s(2x_1 + x_2) = 0 \ , \tag{2.87a}$$

$$\widehat{x}_2 + \underset{\wedge}{x}_1 + \underset{\wedge}{x}_2 + s(x_1 + 2x_2) = 0 \ . \tag{2.87b}$$

These are an entangled pair of discrete harmonic oscillators.

Now, the equations (2.87) can be generated as discrete Euler-Lagrange equations from a Lagrangian

$$\mathcal{L}(x,\widehat{x}) = -x_1(\widehat{x}_1+\widehat{x}_2) - x_2\widehat{x}_2 - s(x_1^2+x_1x_2+x_2^2) \ , \tag{2.88}$$

which is sufficient to prove this is a symplectic map. Note that the Lagrangian above produces canonically conjugate momenta,

$$X_1 = y_1+y_2 \ , \quad X_2 = y_2 \ . \tag{2.89}$$

In particular, y_i is not canonically conjugate to x_i. We have the symplectic structure $d\widehat{x}_i \wedge d\widehat{X_i} = dx_i \wedge dX_i$.

Writing the vector $\boldsymbol{X} = (x_1,x_2,y_1,y_2)^T$, we can pose the map (2.86) in a matrix form,

$$\widehat{\boldsymbol{X}} = A\boldsymbol{X} \ , \quad A = \begin{pmatrix} -s & s & 1 & 0 \\ -s & -2s & 0 & 1 \\ 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{pmatrix} . \tag{2.90}$$

As the map is linear, we seek quadratic invariants. These have the form $I = \boldsymbol{X}^T J\boldsymbol{X}$ for some matrix J. The condition for invariance is then

$$\begin{aligned} \widehat{I} &= I \ , \\ \Rightarrow \quad \widehat{\boldsymbol{X}}^T J\widehat{\boldsymbol{X}} &= \boldsymbol{X}^T J\boldsymbol{X} \ , \\ \Rightarrow \quad A^T JA &= J \ . \end{aligned} \tag{2.91}$$

By writing the matrices A and J in 2×2 block matrix form,

$$A = \left(\begin{array}{c|c} S & I \\ \hline E & 0 \end{array}\right) \ , \quad J = \left(\begin{array}{c|c} J_1 & J_2 \\ \hline J_3 & J_4 \end{array}\right) \ , \tag{2.92}$$

and using the condition (2.91) we are able to find two independent matrices J_1, J_2 satisfying the conditions, and hence two independent invariants of the mapping. After some simplification, these have the form

$$I_1 = x_1y_1 - x_1y_2 + 2x_2y_1 + x_2y_2 \ , \tag{2.93a}$$

$$I_2 = x_1^2+x_2^2+y_1^2+y_2^2+x_1x_2+y_1y_2 - s(2x_1y_1+x_1y_2+x_2y_1+2x_2y_2) \ . \tag{2.93b}$$

It is straightforward to check using the mapping equations (2.86) that $\widehat{I}_1 = I_1$ and $\widehat{I}_2 = I_2$.

Additionally, the conjugate momenta (2.89) define Poisson bracket relations which are preserved under the mapping,

$$\begin{aligned} \{x_1, y_1\} &= -1 \quad , \qquad \{x_1, y_2\} = 0 \; , \\ \{x_2, y_1\} &= 1 \quad , \qquad \{x_2, y_2\} = -1 \; . \end{aligned} \tag{2.94}$$

With respect to these the two invariants are in involution,

$$\{I_1, I_2\} = 0 \; . \tag{2.95}$$

So this map satisfies the standard criteria for an integrable map: it has sufficient invariants in involution. This symplectic structure is also preserved under the mapping.

Rearranging the mapping equations (2.87), we can write

$$\begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ \widehat{x}_1 \end{pmatrix} = \begin{pmatrix} -x_2 - 2s\widehat{x}_2 - \widehat{\widehat{x}}_2 \\ \underset{\wedge}{x}_2 + s x_2 \end{pmatrix} \; . \tag{2.96}$$

This allows us to eliminate x_1 from the equations of motion and derive a fourth order ordinary difference equation for x_2,

$$(\widehat{\widehat{x}}_2 + \underset{\wedge\wedge}{x}_2) + 3s(\widehat{x}_2 + \underset{\wedge}{x}_2) + (1 + 3s^2)x_2 = 0 \; . \tag{2.97}$$

Seeking solutions of the form $x_2 = \lambda^m$, we find that the auxiliary equation is solvable, giving four solutions $\lambda_{\pm}^{\pm 1}$, where

$$\lambda_{\pm} = -\frac{3}{4}s \pm \frac{1}{2}\sqrt{1 - \tfrac{3}{4}s^2} \;+\; \left[\left(-\frac{3}{4}s \pm \frac{1}{2}\sqrt{1 - \tfrac{3}{4}s^2}\right)^2 - 1\right]^{1/2} \; . \tag{2.98}$$

Using (2.96) to write x_1 in terms of x_2, we can express solutions to both in terms of $\lambda_{\pm}$.

$$x_2(m) = A\lambda_+^m + B\lambda_+^{-m} + C\lambda_-^m + D\lambda_-^{-m} \; , \tag{2.99}$$

$$\begin{aligned} x_1(m) = \frac{-1}{1-s^2}\Big(& A\big(s\lambda_+^{-1} + 1 + s^2 + 2s\lambda_+ + \lambda_+^2\big)\lambda_+^m \\ & + B\big(s\lambda_+ + 1 + s^2 + 2s\lambda_+^{-1} + \lambda_+^{-2}\big)\lambda_+^{-m} \\ & + C\big(s\lambda_-^{-1} + 1 + s^2 + 2s\lambda_- + \lambda_-^2\big)\lambda_-^m \\ & + D\big(s\lambda_- + 1 + s^2 + 2s\lambda_-^{-1} + \lambda_-^{-2}\big)\lambda_-^{-m}\Big) \; . \end{aligned} \tag{2.100}$$

Thus an explicit solution to the discrete system exists.

An area for further investigation would be to consider whether these solutions exhibit compatible continuous flows by differentiation with respect to the parameter s, following the continuous parameter flows for the simpler model of section 2.2.5.

Commuting Discrete Flow

As in section 2.2.2, we can also derive a commuting discrete flow for the evolution. Beginning with the a_i, and extending in a third, "bar" direction with lattice parameter r (similarly to figure 2.5) application of the multi-dimensional property of the linear lattice equation (2.8) gives

$$\begin{aligned} a_{2i+1} - \overline{a}_{2i} &= t(a_{2i} - \overline{a}_{2i+1}) \ , \\ a_{2i} - \overline{a}_{2i+1} &= t'(a_{2i+1} - \overline{a}_{2i}) \ , \end{aligned} \quad \text{where} \quad \begin{aligned} t &:= \tfrac{p-r}{p+r} \ , \\ t' &:= \tfrac{q-r}{q+r} \ . \end{aligned} \tag{2.101}$$

In terms of the reduced variables (2.85) these produce the equations

$$\overline{x}_1 - (y_1 - y_2) = -t(x_1 - (\overline{y}_1 - \overline{y}_2)) \ , \tag{2.102a}$$

$$\overline{x}_2 - y_2) = -t(x_2 - \overline{y}_2) \ , \tag{2.102b}$$

$$\overline{y}_1 + x_1 = -t'(y_1 + \overline{x}_1) \ , \tag{2.102c}$$

$$\overline{y}_2 + x_1 + x_2 = -t'(y_2 + \overline{x}_1 + \overline{x}_2) \ . \tag{2.102d}$$

Finally, by eliminating y_i we derive paired second order difference equations for the x_i variables,

$$\begin{aligned} (1+tt')(\overline{x}_1 + \underline{x}_1) + \overline{x}_2 + tt'\underline{x}_2 + (t+t')(2x_1 + x_2) &= 0 \ , \\ (1+tt')(\overline{x}_2 + \underline{x}_2) + tt'\overline{x}_1 + \underline{x}_1 + (t+t')(x_1 + 2x_2) &= 0 \ . \end{aligned} \tag{2.103}$$

Comparing these with the "hat" equations of motion (2.87), we are then naturally interested in their compatibility. This "bar" evolution (2.103) technically represents a Bäcklund transform of the discrete system (2.87) with parameter r.

Note that the equations for the commuting flow (2.103) arise as Euler-Lagrange equations from a discrete variational principle on the Lagrangian

$$\mathcal{L}(x, \overline{x}) = -(1+tt')(x_1\overline{x}_1 + +x_2\overline{x}_2) - x_1\overline{x}_2 - tt'x_2\overline{x}_1 - (t+t')(x_1^2 + x_1x_2 + x_2^2) \ . \tag{2.104}$$

This guarantees that the map is symplectic.

Recalling the vector $\boldsymbol{X}$ (2.90), we can write the mapping equations (2.102) in a matrix form

$$\left(\begin{array}{cc|cc} 1 & 0 & -t & t \\ 0 & 1 & 0 & -t \\ \hline t' & 0 & 1 & 0 \\ t' & t' & 0 & 1 \end{array}\right) \overline{\boldsymbol{X}} = \left(\begin{array}{cc|cc} -t & 0 & 1 & -1 \\ 0 & -t & 0 & 1 \\ \hline -1 & 0 & -t' & 0 \\ -1 & -1 & 0 & -t' \end{array}\right) \boldsymbol{X} \ , \tag{2.105a}$$

$$\Rightarrow \qquad B\overline{\boldsymbol{X}} = C\boldsymbol{X} \ . \tag{2.105b}$$

Comparing with the matrix A encoding the map $\{\boldsymbol{X}\} \to \{\widehat{\boldsymbol{X}}\}$ (2.90), we find that the matrices commute, $[B^{-1}C, A] = 0$. Since the matrices generating the maps commute, so do the maps themselves (as in (2.55)).

Additionally, considering the matrix form for the invariants in (2.91) we can describe the evolution of I_1 (2.93a) and I_2 (2.93b) under the commuting flow by

$$\overline{I} = \overline{\boldsymbol{X}}^T J \overline{\boldsymbol{X}} = \boldsymbol{X}(B^{-1}C)^T J B^{-1} C \boldsymbol{X} \ . \tag{2.106}$$

But a calculation shows that for both choices of J we have $(B^{-1}C)^T J B^{-1} C = J$, and hence the invariants are also preserved under the commuting flow, $\overline{I}_1 = I_1$, $\overline{I}_2 = I_2$. Since the invariants of the "hat" evolution are also invariant under the "bar" evolution, we again have two invariants and integrability for this second evolution.[2]

One-form structure

By eliminating the y_i from the reduced equations (2.86) and (2.102) in a different combination, we can derive corner equations for the x_i variables under the hat and bar evolutions,

$$3t'x_1 = (1-tt')(2\widehat{x}_1 + \widehat{x}_2 - \overline{x}_2) - (2+tt')\overline{x}_1 \ , \tag{2.107a}$$

$$3t'x_2 = (1-tt')(-\widehat{x}_1 + \widehat{x}_2 + \overline{x}_1) - (1+2tt')\overline{x}_2 \ , \tag{2.107b}$$

$$3t'\widehat{\overline{x}}_1 = (1-tt')(\widehat{x}_2 + \overline{x}_1 - \overline{x}_2) - (1+2tt')\widehat{x}_1 \ , \tag{2.107c}$$

$$3t'\widehat{\overline{x}}_2 = (1-tt')(-\widehat{x}_1 + \overline{x}_1 + 2\overline{x}_2) - (2+tt')\widehat{x}_2 \ . \tag{2.107d}$$

As in the one dimensional case of section 2.2.3, the existence of compatible equations of motion (2.87), (2.103), (2.107) on the variables x_i suggests a Lagrangian 1-form structure.

Seeking a closed form of the Lagrangians (2.88) and (2.104), we take the choice

$$\begin{aligned}\mathcal{L}_1(x,\widehat{x}) &= -x_1(\widehat{x}_1 + \widehat{x}_2) - x_2\widehat{x}_2 \\ &\quad -\tfrac{1}{2}s\big(x_1^2 + x_1x_2 + x_2^2 + \widehat{x}_1^2 + \widehat{x}_1\widehat{x}_2 + \widehat{x}_2^2\big) \ ,\end{aligned} \tag{2.108a}$$

$$\begin{aligned}\mathcal{L}_2(x,\overline{x}) &= -\frac{1+tt'}{1-tt'}(x_1\overline{x}_1 + x_2\overline{x}_2) - \frac{1}{1-tt'}(x_1\overline{x}_2 + tt'x_2\overline{x}_1) \\ &\quad -\frac{1}{2}\frac{t+t'}{1-tt'}\big(x_1^2 + x_1x_2 + x_2^2 + \overline{x}_1^2 + \overline{x}_1\overline{x}_2 + \overline{x}_2^2\big) \ ,\end{aligned} \tag{2.108b}$$

[2]Technically, we must also show involutivity of I_1, I_2 with respect to the canonical structure of the *bar* evolution. However, this will follow from the one-form structure.

such that for *these* Lagrangians the closure property holds on the equations of motion (2.87), (2.103), (2.107),

$$\square\mathcal{L} \;:=\; \mathcal{L}_1(x,\widehat{x}) + \mathcal{L}_2(\widehat{x},\widehat{\overline{x}}) - \mathcal{L}_1(\overline{x},\widehat{\overline{x}}) - \mathcal{L}_2(x,\overline{x}) \;=\; 0 \;. \tag{2.109}$$

Note that this closure depends on the parameter identity (2.57). The Lagrangian 1-form structure then has a multiform variational principle which produces all the equations of motion, as described for the one-dimensional case in section 2.2.3.

Notice the symmetrical form of the potential terms in the closed form Lagrangians (2.108) which appears to be a feature of such discrete Lagrangians (compare, for example, (1.48a), (2.69) and [125]). Also, note that the closure determines the relative scaling between the two Lagrangians (2.108a) and (2.108b), fixing much of the usual freedom in choosing a Lagrangian.

The Lagrangians (2.108a), (2.108b) define the momenta conjugate to x_1, x_2 by the usual formula $X_i = -\partial\mathcal{L}_1/\partial x_i$, so that

$$X_1 = \widehat{x}_1 + \widehat{x}_2 + \tfrac{1}{2}s(2x_1 + x_2) \;, \quad X_2 = \widehat{x}_2 + \tfrac{1}{2}s(x_1 + 2x_2) \;, \tag{2.110}$$

with respect to which we have the invariant Poisson structure $\{x_i, X_j\} = \delta_{ij}$, preserved under the mappings. We could also write expressions for the momenta X_i using the second Lagrangian $\mathcal{L}_2$; equality of these expressions is given by the corner equations (2.107). The invariants can then be rewritten in terms of the momenta X_i (2.110),

$$I_1 \;=\; x_1X_1 - 2x_1X_2 + 2x_2X_1 - x_2X_2 \;, \tag{2.111a}$$

$$I_2 \;=\; X_1^2 - X_1X_2 + X_2^2 + \left(1 - \tfrac{3}{4}s^2\right)\left(x_1^2 + x_1x_2 + x_2^2\right) . \tag{2.111b}$$

The canonical structure yields involutivity of the invariants, $\{I_1, I_2\} = 0$. The invariance and involutivity of these can be shown by direct calculation, also guaranteeing integrability for both maps.

In the simple, one-dimensional periodic reduction, we faced the problem that there were no longer meaningful commuting flows in a continuum limit, due to the insufficient number of degrees of freedom: in a continuum limit the commuting flows must ultimately coincide. For this period 3, two dimensional example, however, we expect this limitation will no longer be a problem. We have two commuting, independent invariants (2.111) that are generating Hamiltonians for two commuting, continuous time-flows. An outstanding problem is to derive an appropriate continuous Lagrangian one-form structure for these flows, perhaps in a continuum limit from the discrete structure.

2.3.2 Lax Pair

It seems probable that the one-form structure can be generalised for a staircase of arbitrary length $2P$ (figure 2.8). However, an important aspect of the integrability of the maps is the presence of a sufficient number of invariants: a staircase of length $2P$ produces a $P-1$ dimensional map, and hence integrability requires the existence of $P-1$ independent invariants in involution.

A Lax representation for the staircase reduction can be derived from that of the linear lattice equation (2.17), (2.18) (compare with previous work on staircase reductions from the lattice KdV equation [24, 88, 91]). Recall the staircase configuration of figure 2.8, and define the general reduction variables by

$$x_i = a_{2i} - a_{2i+2} \ , \quad y_i = a_{2i-1} - a_{2i+1} \ . \tag{2.112}$$

The linear lattice equation causes the a_i to evolve according to the mapping (2.84), which leads to the equations of motion

$$\widehat{x}_j = y_{j+1} + s(x_{j+1} - x_j) \ , \quad \widehat{y}_j = x_j \ . \tag{2.113}$$

It is these equations that we wish to capture in a Lax pair.

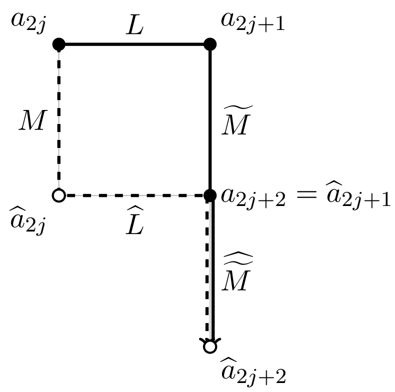

Figure 2.9: Two alternative routes through the lattice. The bold line and the dashed line must be equivalent.

Consider the two alternative routes along the staircase shown in figure 2.9, and the linear lattice Lax pair of section 2.1.1. There are two alternative combinations of the Lax matrices, according to the bold or dashed routes through the lattice, yielding the equality

$$\widehat{\widetilde{M}}\widetilde{M}L = \widehat{\widetilde{M}}\widehat{L}M \ . \tag{2.114}$$

This follows as a straightforward consequence from the zero-curvature condition (2.18). Using the factorisations of the Lax matrices (2.17), and recalling the definitions of x_i, y_i (2.112), we therefore have

$$\mathbb{Q}_k \widehat{X}_j \mathbb{P}_k \widehat{Y}_j \mathbb{Q} \widehat{A}_{2j}^{-1} A_{2j-1} = \mathbb{Q} \widehat{A}_{2j+2}^{-1} A_{2j+1} \mathbb{Q}_k X_j \mathbb{P}_k Y_j \ , \tag{2.115}$$

in which we have introduced notation for the upper triangular matrices,

$$X_j := \begin{pmatrix} 1 & x_j \\ 0 & 1 \end{pmatrix} \ , \quad Y_j := \begin{pmatrix} 1 & y_j \\ 0 & 1 \end{pmatrix} \ , \quad A_j := \begin{pmatrix} 1 & a_j \\ 0 & 1 \end{pmatrix} \ . \tag{2.116}$$

The equation (2.115) simplifies significantly as we identify the Lax pair for the mapping reduction,

$$L_j := \mathbb{Q}_k X_j \mathbb{P}_k Y_j \ , \quad M_j := \mathbb{Q}_k \widehat{A}_{2j}^{-1} A_{2j-1} = \mathbb{Q}_k \begin{pmatrix} 1 & y_j + s x_j \\ 0 & 1 \end{pmatrix} \ , \tag{2.117a}$$

(where we have used (2.113) in the expression for M_j) in terms of which (2.115) becomes

$$\widehat{L}_j M_j = M_{j+1} L_j \ . \tag{2.117b}$$

This Zakharov-Shabat condition generates the equations of motion for the mapping as a compatibility condition.

In general for an integrable system, it is then possible to extract invariants from such a Lax pair by the construction of a monodromy matrix,

$$T(k^2) = L_P L_{P-1} \dots L_2 L_1 \ . \tag{2.118}$$

The Zakharov-Shabat condition (2.117) guarantees the preservation of the spectral data of $T(k^2)$ under the map. However, for the linear reduction mapping, these spectral data are trivial: no invariants are encoded. Indeed, this Lax representation is closely related to that of the reduction from the lattice KdV equation (which will be considered in chapter 4), which encodes not only the invariants, but also has an r-matrix structure which is sufficient to demonstrate their involutivity; but in the linearisation this r-matrix structure is also lost.

Recall the alternative Lax formulation for the lattice equation as the linear limit from the Lax pair of the lattice KdV equation (2.23). On this level, the reduction equations and Lax representation appear at first order in the linearising parameter η. However, the invariants of the reduction are *quadratic*: they appear at quadratic order in η as the limits of the non-linear invariants, and similarly at quadratic order in the Lax pair. This may explain why the invariants are not easily encoded in the Lax structure for the linear model: in the limit they appear at a different order. Nonetheless, the invariants *do* exist, and it is possible that they retain the independence and involutivity of their parents in the non-linear case, but this remains to be shown. It is possible that the one-form structure may be a viable alternative for exploring the integrability of this model.

2.4 Summary

By considering the linearised lattice equation that arises naturally from the ABS list, its multi-dimensional consistency and Lagrangian structure, we have found a "simplest example" of a Lagrangian 2-form structure. There is a choice of Lagrangian for the lattice equation that possesses the closure property on the level of the multi-dimensional equations, leading naturally to the multi-form variational principle for lattice equations. The family of multi-dimensionally consistent lattice equations arise from an action that must be stationary both under variation of the dependent variables *and* under variation of the surface itself. Furthermore, we have seen that the quadratic Lagrangian 2-form for lattice equations is almost unique; the class of admissible Lagrangians is restricted by the closure requirement.

The multi-dimensional consistency of the linear lattice equation led to a novel perspective on the mappings that arise from periodic reductions of the lattice. Applying a periodic boundary condition led to a discrete harmonic oscillator which, by exploiting the multi-dimensional consistency of the parent lattice equation, was found to have a commuting, discrete flow. These commuting flows can be expressed in a Lagrangian 1-form structure, which is equivalent to the maps possessing a shared conjugate momentum. The Lagrangian 1-form captures the full set of equations of motion through a least action principle, varying both dependent variables and the *time curve* of the evolution. Due to their joint solutions, these commuting, discrete flows also have compatible continuous flows in a regime where discrete variables and continuous parameters exchange roles.

The reduction can also be extended to higher dimensional cases by lengthening the periodic initial value problem on the lattice. We examined the next case (in two dimensions) as an example, deriving the Lagrangian 1-form. Although this family of reductions have a local Lax pair inherited from the multi-dimensional consistency of the lattice equation, an outstanding problem is to encode the invariants of such mappings on the level of the Lax pair and hence relate the Lax problem to the Lagrangian form structure.

3

Quantum Multiform Structures for Linear Models

The development of the multiform variational principle is an important step forwards in the study of integrable systems from a Lagrangian perspective, both for evolutionary systems in the one-form case [18, 110, 125, 126] and for two-dimensional (particularly lattice) models in the two-form case [62–65]. These multiform structures capture the aspect of multi-dimensional consistency for integrable systems, parallel to the commuting Hamiltonian flows that have long been understood.

So far, the Lagrangian multiform structure has been used to describe the classical mechanics of integrable systems, but it is natural to speculate about the quantum analogue of these constructions. Canonical quantisation of integrable systems depends on the classical invariants becoming commuting Hamiltonian operators, but a Lagrangian approach to quantisation means a path integral, or sum over histories, approach [27, 28, 36, 37] - and as early as in [27] Dirac claimed that the Lagrangian formulation is the more fundamental approach. The classical principle of least action becomes a sum over all paths, with the action dictating the phase. For Lagrangian multiforms, the least action principle involves a variation over the underlying geometries of the independent variables;

we might wonder how the Lagrangian closure and variation of surfaces, which manifest the multi-dimensional consistency of integrable models, will appear at the quantum level.

In chapter 2 we explored discrete, linear models with a Lagrangian multiform structure: the linear lattice equation and the harmonic oscillator. But, the harmonic oscillator is the essential toy model for quantum mechanics; the path integral can be calculated explicitly for quadratic Lagrangians using straightforward methods. For the discrete harmonic oscillator we can also avoid the problematic infinite time-slicing that is inherent to path integral methods. In this chapter we make use of the toy models of chapter 2 to consider how the multi-dimensional properties of the Lagrangian multiform manifest in a quantum setting. For the discrete harmonic oscillator, we find in section 3.1 that the natural propagator for the Lagrangian one-form is independent from the path taken in the discrete time variables. That is, it depends only on the end-points. This is a quantum analogue to the Lagrangian closure condition. We also find that this is uniquely true (in the quadratic case) for the Lagrangian one-form structure of chapter 2. In section 3.2 we extend these ideas to the lattice two-form case and find similar results. We define a propagator over a surface on the lattice that, for the linear two-form, is independent of variations of the surface, depending only on the boundary. This can also be viewed as a quantum analogy for Lagrangian closure and, similarly to the one-form case, this holds uniquely for the linear Lagrangian two-form of chapter 2.

3.1 The Quantum Reduction

From the earliest works on creating a Lagrangian approach to quantum mechanics, the harmonic oscillator has provided the key example [27, 36]. In chapter 2 we found that, by exploiting lattice multi-dimensional consistency, we were able to construct commuting discrete flows for the harmonic oscillator and therefore to endow the model with a discrete Lagrangian one-form structure. The discrete harmonic oscillator is therefore an important toy model for exploring the quantisation of the Lagrangian one-form.

Rovelli in [94, 95] also tackles path integral quantisation of a discrete harmonic oscillator; although he considers reparametrisation invariant discretisations, whereas we proceed from what is essentially a Bäcklund transform. Nonetheless, there are important similarities, such as the preservation of the energy of the continuous model even in the discretisation.

3.1.1 Feynman Propagators

Consider the simple discrete harmonic oscillator model of section 2.2. The commuting flows of this model were described by a Lagrangian-type generating function (2.69) which has a one-form structure,

$$\mathcal{L}_b(x,\widehat{x}) = -\frac{P+Q}{q}x\widehat{x} - \frac{P-Q}{2q}(x^2+\widehat{x}^2)\ . \tag{3.1}$$

Recall that, of the possible quadratic Lagrangians generating the discrete model, the Lagrangian one-form is almost unique. The Lagrangian defines conjugate momenta

$$X \;=\; -\tfrac{\partial \mathcal{L}_b}{\partial x} \;=\; \frac{P+Q}{q}\widehat{x} + \frac{P-Q}{q}x\ , \tag{3.2a}$$

$$\widehat{X} \;=\; \tfrac{\partial \mathcal{L}_b}{\partial \widehat{x}} \;=\; -\frac{P+Q}{q}x - \frac{P-Q}{q}\widehat{x}\ , \tag{3.2b}$$

which are equivalent to discrete Hamilton's equations.

In canonical quantisation, position x and momentum X become *operators* $\mathbf{x}$ and $\mathbf{X}$, with equal time commutation relations $[\mathbf{x},\mathbf{X}] = i\hbar$. The momentum equations (3.2) become operator equations of motion,

$$\widehat{\mathbf{x}} - \mathbf{x} \;=\; -\frac{2P}{P-Q}\mathbf{x} - \frac{q}{P-Q}\widehat{\mathbf{X}}\ , \tag{3.3a}$$

$$\widehat{\mathbf{X}} - \mathbf{X} \;=\; \frac{4Pq}{P-Q}\mathbf{x} + \frac{2P}{P-Q}\widehat{\mathbf{X}}\ . \tag{3.3b}$$

In continuous time quantum mechanics the principle object of interest is the Schrödinger equation, with the operator equations of motion playing a more secondary role. In the discrete regime, however, the Schrödinger equation is no longer relevant, and so it is these operator equations of motion that become the primary objects of study. We make some assumptions on the Hilbert space, such that these operators have complete sets of orthogonal eigenfunctions; since the system is in essence the harmonic oscillator these are not difficult assumptions.

To understand the discrete time evolution, we express the Hamilton's equations (3.3) in terms of a time-evolution operator U_b. As in (1.71), we require

$$\mathbf{x} \to \widehat{\mathbf{x}} = U_b^{-1}\mathbf{x}U_b\ , \qquad \mathbf{X} \to \widehat{\mathbf{X}} = U_b^{-1}\mathbf{X}U_b\ . \tag{3.4}$$

This is a "canonical" approach to discrete quantisation, see for example [74]. Comparing equations (3.3) with (3.4), and using the conjugations (1.75), it is not hard to see that an appropriate choice of U_b is given by

$$\begin{aligned} U_b &= e^{-iV(\mathbf{x})/2\hbar} e^{-iT(\mathbf{X})/\hbar} e^{-iV(\mathbf{x})/2\hbar} , && (3.5a) \\ &= \exp\left(\frac{-iP\mathbf{x}^2}{\hbar q}\right) \exp\left(\frac{-iq\mathbf{X}^2}{2\hbar(P+Q)}\right) \exp\left(\frac{-iP\mathbf{x}^2}{\hbar q}\right) . && (3.5b) \end{aligned}$$

In other words, a separated form for U_b exists, but it is required to have three terms. This contrasts with previous one-dimensional examples in the literature, where two terms are normally considered sufficient (see (1.74)) [74]. Note that (3.5) is not a unique form for U_b; it can alternatively be written

$$\begin{aligned} U_b &= e^{-i\overline{T}(\mathbf{X})/2\hbar} e^{-i\overline{V}(\mathbf{x})/\hbar} e^{-i\overline{T}(\mathbf{X})/2\hbar} , && (3.6a) \\ &= \exp\left(\frac{-i\mathbf{X}^2}{4\hbar q}\right) \exp\left(\frac{-2iPq\mathbf{x}^2}{\hbar(P+Q)}\right) \exp\left(\frac{-i\mathbf{X}^2}{4\hbar q}\right) , && (3.6b) \end{aligned}$$

but when working in position space we will find (3.5) a more helpful form.

We use bra/ket notation to write $|x\rangle$ as the eigenstate for the position operator $\mathbf{x}$, so that $\mathbf{x}|x\rangle = x|x\rangle$. A subscript $|x\rangle_m$ indicates Heisenberg picture states with a dependence on the discrete time variable m. The propagator for a single, discrete time-step is

$$K_b(x, m; \widehat{x}, m+1) = {}_{m+1}\langle\widehat{x}|x\rangle_m = \langle\widehat{x}|U_b|x\rangle , \qquad (3.7a)$$

where we have moved in the second equality from time-dependent, Heisenberg picture eigenstates to time-independent, Schrödinger picture eigenstates.

Since we have an explicit form for U_b (3.5), we can calculate the propagator,

$$\begin{aligned} \langle\widehat{x}|U_b|x\rangle &= \langle\widehat{x}|e^{-iV(\mathbf{x})/2\hbar} e^{-iT(\mathbf{X})/\hbar} e^{-iV(\mathbf{x})/2\hbar}|x\rangle , && (3.7b) \\ &= e^{-iV(\widehat{x})/2\hbar}\langle\widehat{x}|e^{-iT(\mathbf{X})/\hbar}|x\rangle e^{-iV(x)/2\hbar} . && (3.7c) \end{aligned}$$

Making some assumptions on the Hilbert space we insert a complete set of momentum eigenstates,

$$\begin{aligned} \langle\widehat{x}|U_b|x\rangle &= \int \mathrm{d}X \langle\widehat{x}|X\rangle\langle X|x\rangle \exp\left[\frac{i}{\hbar}\left(-T(X) - \tfrac{1}{2}(V(\widehat{x}) + V(x))\right)\right] , && (3.7d) \\ &= \frac{1}{2\pi\hbar}\int \mathrm{d}X \exp\left[\frac{i}{\hbar}\left(\frac{-qX^2}{2(P+Q)} - X(\widehat{x}-x) - \frac{P}{q}\left(x^2+\widehat{x}^2\right)\right)\right] , && (3.7e) \\ &= \left(\frac{P+Q}{2\pi i\hbar q}\right)^{1/2} \exp\left[\frac{i}{\hbar}\left(-\frac{P+Q}{q}x\widehat{x} - \frac{P-Q}{2q}(x^2+\widehat{x}^2)\right)\right] , && (3.7f) \end{aligned}$$

where the last equality results from a Gaussian integral. The linearity of the model justifies taking the integration region over the whole real line. But, it is clear to see by comparison with the Lagrangian (3.1) that this is simply

$$\langle\widehat{x}|U_b|x\rangle = \left(\frac{P+Q}{2\pi i\hbar q}\right)^{1/2} \exp\left[\frac{i}{\hbar}\mathcal{L}_b(x,\widehat{x})\right] . \tag{3.8}$$

This is what might be expected for a “one-step" path integral (such as in [38, 40]) noting that this approach has also specified the normalisation constant. As early as Dirac's paper [27] it was shown that the Lagrangian appears in such a way for quantum mechanical systems, but here for the discrete evolution we have recovered precisely the chosen Lagrangian one-form.

This is sufficient to define the discrete-time path integral. By iterating the one-step propagator (3.8) over M steps, we can write the propagator for the discrete system,

$$K_b(x_0,0;x_M,M) = \left(\frac{P+Q}{2\pi i\hbar q}\right)^{M/2} \int_{-\infty}^{\infty}\prod_{m=1}^{M-1}\mathrm{d}x_m\ e^{i\mathcal{S}[x_m]/\hbar} , \tag{3.9}$$

with action

$$\mathcal{S}[x_m] = \sum_{m=0}^{M-1}\mathcal{L}_b(x_m,x_{m+1}) . \tag{3.10}$$

In this discrete case, equation (3.9) gives a precise definition to the path integral notation

$$K_b(x_0,0;x_M,M) = \int_{x(0)=x_0}^{x(M)=x_M}[\mathcal{D}x(m)]\ e^{i\mathcal{S}[x(m)]/\hbar} . \tag{3.11}$$

Notice in particular that the normalisation associated to the measure is unambiguous. We can deduce a similar expression for the bar evolution with parameter a.

Calculating the Discrete Propagator

We define the discrete-time path integral propagator in equations (3.9) and (3.10). In the quadratic regime, we can now calculate this explicitly.

Since we are working with a quadratic Lagrangian, we begin the evaluation (as in the continuous case [37]) by expanding x_m around the classical solution,

$$x_m = x_m^{cl} + y_m , \tag{3.12}$$

where x_m^{cl} is the classical solution, and y_m is the *quantum deviation* from the classical trajectory, with the endpoints fixed so that $y_0 = y_M = 0$. The action is expanded in a

Taylor expansion, which separates for quadratic Lagrangians,

$$\mathcal{S}[x] \;=\; \mathcal{S}[x^{cl}+y] \;=\; \mathcal{S}[x^{cl}_m]+\mathcal{S}[y_m]\;. \tag{3.13}$$

As in the continuous case, the path integral separates into a part depending on the action of the classical path, and a path integral with no dependence on the endpoints,

$$K_b(x_0,0;x_M,M)=\exp\left[\frac{i}{\hbar}\mathcal{S}[x^{cl}_m]\right]\int_{y(0)=0}^{y(M)=0}\mathcal{D}[y_m]\;e^{i\mathcal{S}[y_m]/\hbar}\;. \tag{3.14}$$

These two parts can be evaluated separately.

First, we evaluate the *classical* action for the path beginning at x_0, and reaching x_M after M time steps. Recalling the discrete equation of motion (2.40) and classical solution (2.43), and applying the boundary value problem, we rewrite the classical trajectory as

$$x_m=\frac{1}{\sin\mu M}\left(x_M\sin\mu M-x_0\sin\mu(m-M)\right)\;, \tag{3.15}$$

where $b=-\cos\mu$. Substituting (3.15) into the action gives

$$\begin{aligned}\mathcal{S}_{cl} \;&=\; \sum_{m=0}^{M-1}\left(-\frac{P+Q}{q}x_m x_{m+1}-\frac{P-Q}{2q}(x_m^2+x_{m+1}^2)\right)\;, && (3.16a)\\ &=\; \frac{\sqrt{P}}{\sin\mu M}\left[(x_0^2+x_M^2)\cos\mu M-2x_0x_M\right]\;, && (3.16b)\end{aligned}$$

where we have used the identities

$$\cos\mu=-b=-\frac{P-Q}{P+Q}\;,\qquad \sin\mu=\frac{2q\sqrt{P}}{P+Q}\;. \tag{3.17}$$

and made extensive use of trigonometric formulae to reach the second, simplified, expression (3.16). Notice two things about this result. First, there is no explicit Q dependence in $\mathcal{S}_{cl}$; all Q dependence is contained within the parameter μ, which only appears as μM. Second, the structure is identical to the classical action in the continuous case, in the limit $\mu M\to\omega T$.

It is left for us to evaluate the discrete path integral for the quantum deviation,

$$\tilde{K}_M(0,0):=\int_{y(0)=0}^{y(M)=0}\mathcal{D}[y_m]\;e^{i\mathcal{S}[y_m]/\hbar}\;. \tag{3.18}$$

In the discrete case, we can consider this via a time slicing procedure without needing to worry about the problematic shrinking to zero. Following the propagator definition (3.9), we have

$$\begin{aligned}\tilde{K}_M(0,0)=\mathscr{N}^M\int \mathrm{d}y_1\ldots\int \mathrm{d}y_{M-1}&\\ \times\exp\left\{\frac{i}{\hbar q}\sum_{m=0}^{M-1}\left(-(P+Q)y_my_{m+1}-\frac{1}{2}(P-Q)(y_m^2+y_{m+1}^2)\right)\right\}\;,&\quad (3.19a)\end{aligned}$$

where $\mathscr{N} := [i(P+Q)/2\pi\hbar q]^{1/2}$ is the normalising factor appearing in (3.9), and $y_0 = y_M = 0$. This expression is quadratic in all y_m variables, and so can be evaluated as a set of $M-1$ Gaussian integrals. This is most easily achieved by writing the equation in a matrix form (as in [99], for example). We define $\mathbf{y}^T = (y_1, \ldots, y_{M-1})$, in order to write

$$\tilde{K}_M(0,0) = \mathscr{N}^M \int \mathrm{d}^{M-1}\mathbf{y}\ \exp(-\mathbf{y}^T\sigma\mathbf{y}) \ , \tag{3.19b}$$

for σ a symmetric, tri-diagonal matrix,

$$\sigma = \frac{i(P+Q)}{\hbar q}\begin{pmatrix} \frac{P-Q}{P+Q} & 1/2 & & \\ 1/2 & \frac{P-Q}{P+Q} & \ddots & \\ & \ddots & \ddots & 1/2 \\ & & 1/2 & \frac{P-Q}{P+Q} \end{pmatrix} . \tag{3.19c}$$

As σ is symmetric, it can be diagonalised by a unitary matrix V, so that $\sigma = V^\dagger \sigma_D V$, and the vector becomes $\xi = V\mathbf{y}$. The propagator is then

$$\begin{aligned} \tilde{K}_M(0,0) &= \mathscr{N}^M \int \mathrm{d}^{M-1}\xi\ \exp(-\xi^T\sigma_D\xi) \ , && (3.19\text{d}) \\ &= \mathscr{N}^M \prod_{\alpha=1}^{M-1}\sqrt{\frac{\pi}{\sigma_\alpha}} \ , \quad \text{where } \sigma_\alpha \text{ are the entries of } \sigma_D \ , && (3.19\text{e}) \\ &= \mathscr{N}^M . \frac{\pi^{(M-1)/2}}{\sqrt{\det\sigma}} \ , && (3.19\text{f}) \end{aligned}$$

and hence it remains to calculate $\det\sigma$.

σ is a tri-diagonal matrix. Let X_n be a determinant for a general tri-diagonal matrix,

$$X_n = \begin{vmatrix} A & B & & \\ B & A & \ddots & \\ & \ddots & \ddots & B \\ & & B & A \end{vmatrix} \quad \text{of size } n \times n \ . \tag{3.20a}$$

The determinant is found by forming a recursion relation on the size of the matrix. Performing the cofactor expansion and solving the resulting discrete system, we find the determinant

$$\begin{aligned} X_n = \frac{1}{2^n}\Bigg[&\left(A + \frac{A^2-2B^2}{\sqrt{A^2-4B^2}}\right)\left(A+\sqrt{A^2-4B^2}\right)^{n-1} \\ &+ \left(A - \frac{A^2-2B^2}{\sqrt{A^2-4B^2}}\right)\left(A-\sqrt{A^2-4B^2}\right)^{n-1}\Bigg] . \end{aligned} \tag{3.20b}$$

For the matrix σ (3.19c), we have $A = (P-Q)/(P+Q) = \cos\mu$ and $B = 1/2$, so that $\sqrt{A^2 - 4B^2} = i\sin\mu$. This leads to significant simplifications of the above expression, so that

$$\det\sigma = \left(\frac{P+Q}{2i\hbar q}\right)^{M-1} \frac{\sin\mu M}{\sin\mu} . \tag{3.20c}$$

Putting this together with the normalisation constant $\mathscr{N}$ (3.8), we have the propagator for the deviation

$$\tilde{K}_M(0,0) = \sqrt{\frac{P+Q}{2\pi i\hbar q}.\frac{\sin\mu}{\sin\mu M}} . \tag{3.21}$$

With the classical action (3.16) we therefore find the propagator for the discrete harmonic oscillator,

$$K_b(x_0, 0; x_M, M) = \left(\frac{\sqrt{P}}{\pi i\hbar\sin\mu M}\right)^{1/2} \times \exp\left[\frac{i\sqrt{P}}{\hbar\sin\mu M}\left((x_0^2 + x_M^2)\cos\mu M - 2x_0 x_M\right)\right] . \tag{3.22}$$

Note that this has the same form as the propagator for the continuous time harmonic oscillator. Dependence on the parameter b is evident through $\cos\mu = -b$. We note, then, that the propagator is common to the discrete evolution and its interpolating continuous time flow.

3.1.2 Quantum Invariants

Using the operator equations of motion (3.3) it is easy to see that the discrete harmonic oscillator has an operator invariant given by

$$\mathbf{I}_b = \frac{1}{2}\mathbf{X}^2 + 2P\mathbf{x}^2 = \frac{1}{2}\left(-\hbar^2\frac{\partial^2}{\partial x^2} + 4Px^2\right) , \tag{3.23}$$

where we have taken the standard position space representation for the momentum $\mathbf{X} = -i\hbar\,\partial/\partial x$. This invariant is, of course, simply the operator version of the classical invariant (2.42), and is precisely the Hamiltonian for the continuous time harmonic oscillator, where $4P = \omega^2$.

The invariant can also be considered from the perspective of path integrals and the time evolution operator following the method of [40], where the authors investigated quantum systems possessing invariants under a one time-step path integral evolution. We begin by considering the evolution generated by Lagrangian $\mathcal{L}_b(x, \widehat{x})$ (3.1). We know this has a "one time-step" propagator given in (3.8). A wave-function $\psi_m(x)$ evolves in discrete time under this transformation according to

$$\psi_{m+1}(\widehat{x}) = \mathscr{N} \int \mathrm{d}x \exp\left(\frac{i}{\hbar}\mathcal{L}_b(x,\widehat{x})\right) \psi_m(x) \ , \tag{3.24}$$

with $\mathscr{N}$ the normalising constant. To look for an invariant we desire ψ_m and ψ_{m+1} to be solutions of the same eigenvalue problem, with the same eigenvalue:

$$M_x\psi_m(x) = E\psi_m(x) \quad \Rightarrow \quad M_{\widehat{x}}\psi_{m+1}(\widehat{x}) = E\psi_{m+1}(\widehat{x}) \ . \tag{3.25}$$

M_x is a differential operator, and we restrict to considering the second order case,

$$M_x = p_0(x)\frac{\partial^2}{\partial x^2} + p_1(x)\frac{\partial}{\partial x} + p_2(x) \ , \tag{3.26}$$

for $p_i(x)$ some functions of x which are to be determined. The quadratic restriction is justified as we are seeking an invariant of the form (3.23). Now,

$$\begin{aligned} E\psi_{m+1}(\widehat{x}) &= \mathscr{N}\int \exp\left(\frac{i}{\hbar}\mathcal{L}_b(x,\widehat{x})\right) E\psi_m(x)\mathrm{d}x \ , & \text{(3.27a)}\\ &= \mathscr{N}\int \exp\left(\frac{i}{\hbar}\mathcal{L}_b(x,\widehat{x})\right) (M_x\psi_m(x))\,\mathrm{d}x \ , & \text{(3.27b)}\\ &= \mathscr{N}\int \left(\overline{M}_x \exp\left(\frac{i}{\hbar}\mathcal{L}_b(x,\widehat{x})\right)\right) \psi_m(x)\mathrm{d}x \ + S \ , & \text{(3.27c)} \end{aligned}$$

where $\overline{M}_x$ is an adjoint to M_x constructed under integrations by parts, and S is the resulting surface term,

$$\begin{aligned} \overline{M}_x &= \frac{\partial^2}{\partial x^2}\circ p_0(x) - \frac{\partial}{\partial x}\circ p_1(x) + p_2(x) \ , & \text{(3.28a)}\\ S &= \left[p_0 e^{i\mathcal{L}/\hbar}\psi'_m - (p_0 e^{i\mathcal{L}/\hbar})'\psi_m + p_1 e^{i\mathcal{L}/\hbar}\psi_m\right]_{-\infty}^{\infty} \ . & \text{(3.28b)} \end{aligned}$$

If we assume ψ_m and ψ'_m vanish at infinity (a reasonable physical assumption) then the surface term S vanishes.

We can also write,

$$\begin{aligned} E\psi_{m+1}(\widehat{x}) &= M_{\widehat{x}}\psi_{m+1}(\widehat{x}) \ , & \text{(3.29a)}\\ &= \mathscr{N}\int_C \left(M_{\widehat{x}} \exp\left(\frac{i}{\hbar}\mathcal{L}_b(x,\widehat{x})\right)\right) \psi_m(x)\ \mathrm{d}x \ . & \text{(3.29b)} \end{aligned}$$

So, comparing (3.27c) with (3.29b), we require

$$\overline{M}_x \exp\left(\frac{i}{\hbar}\mathcal{L}_b(x,\widehat{x})\right) = M_{\widehat{x}} \exp\left(\frac{i}{\hbar}\mathcal{L}_b(x,\widehat{x})\right) \ . \tag{3.30}$$

Following the analysis in [40], and using the given Lagrangian (3.1), we find this can only hold under the restrictions

$$p_0(x) = -\hbar^2 C_0 \ , \quad p_1(x) \equiv 0 \ , \quad p_2(x) = 4PC_0 x^2 + C_2 \ , \tag{3.31}$$

so that

$$M_x = C_0 \left(-\hbar^2 \frac{\partial^2}{\partial x^2} + 4Px^2 \right) + C_2 \ . \tag{3.32}$$

This is precisely the Hamiltonian for the harmonic oscillator. In terms of the operator invariant (3.23),

$$M_x = 2C_0 \mathbf{I}_b + C_2 \ , \tag{3.33}$$

or in other words we have precisely the invariant $\mathbf{I}_b$, which is in accord with the invariants found in [40].

The invariant $\mathbf{I}_b$ (3.23) is further related to the evolution operator U_b (3.5) in principle by a Campbell-Baker-Hausdorff expansion [87, 115]; an explicit form is given by algebraic manipulation [56],

$$U_b = \exp\left[\frac{1}{\hbar\sqrt{P}} \operatorname{arctanh}\left(\frac{i\sqrt{P}}{q} \right) \mathbf{I}_b \right] \ . \tag{3.34}$$

So it can be clearly seen how the discrete quantum evolution relates to an interpolating continuous time-flow via these alternative expressions of the unitary time evolution operator.

3.1.3 Path independence of the propagator

In equation (3.22) we established the propagator for an evolution in one discrete time variable. But recall that in the classical case there are two compatible discrete flows: the bar and hat evolutions. These can be viewed as two discrete time evolutions. In the same way as the hat evolution, the bar evolution is characterised by a Lagrangian $\mathcal{L}_a(x, \overline{x})$ (2.69),

$$\mathcal{L}_a = -\frac{P+R}{r} x\overline{x} - \frac{P-R}{2r}(x^2 + \overline{x}^2) \ , \tag{3.35}$$

with time evolution generated by an operator U_a as in (3.5). This leads to a one time-step propagator similarly to (3.8),

$$K_a(x, \overline{x}; 1) = \left(\frac{P+R}{2\pi i\hbar r} \right)^{1/2} \exp\left[\frac{i}{\hbar} \mathcal{L}_a(x, \overline{x}) \right] \ . \tag{3.36}$$

Noting that the invariant $\mathbf{I}_b$ (3.23) is independent of q, it is clear that the bar evolution shares the same invariant, $\overline{\mathbf{I}}_b = \mathbf{I}_b$. We remark that, as we have here a second time direction, we might plausibly introduce a second $\hbar$ parameter. We ignore such considerations for the time being and allow $\hbar$ to be the same in both time directions.

In general, if we begin at a time co-ordinate $(0,0)$ and evolve along integer time co-ordinates to a new time (M,N), the propagator could depend not only on the endpoints, but also on the path Γ taken through the time variables (see figure 2.6). We associate to the path an action $\mathcal{S}_\Gamma := \mathcal{S}[x(\boldsymbol{n});\Gamma]$ (2.65),

$$\mathcal{S}_\Gamma = \sum_{\gamma(\boldsymbol{n})\in\Gamma} \mathcal{L}_i(x(\boldsymbol{n}), x(\boldsymbol{n}+\boldsymbol{e}_i)) \ , \tag{3.37}$$

where the summation takes place over unit elements $\gamma(\boldsymbol{n})$ of the discrete time curve Γ, each of which is associated to a single Lagrangian $\mathcal{L}_i(x(\boldsymbol{n}), x(\boldsymbol{n}+\boldsymbol{e}_i))$. We can then define a propagator for the evolution along the time-path Γ, made up of the one-step elements (3.8), (3.36)

$$K_\Gamma\big(x_a,(0,0);x_b,(M,N)\big) := \mathscr{N}_\Gamma \int \prod_{(m,n)\in\Gamma} \mathrm{d}x_{m,n} \ \exp\left[\frac{i}{\hbar}\mathcal{S}_\Gamma[x(\boldsymbol{n})]\right] \ , \tag{3.38}$$

where the integration is over all internal points $x_{m,n}$ on the curve Γ. Here $\mathscr{N}_\Gamma$ represents the product of normalisation factors from the relevant elements of (3.8), (3.36); the selection of the constant $\mathscr{N}_\Gamma$ is explained in appendix A.

Consider the simple case of an evolution of one step in each direction. There are two routes to achieve this, shown in figure 3.1. Either we evolve first in the hat direction, followed by an evolution in the bar direction, or vice versa.

In path (i), we evolve first according to the hat evolution $\mathcal{L}_b$, and then according to the bar evolution $\mathcal{L}_a$. The propagator is

$$K_\lrcorner(x,\widehat{\overline{x}}) = \left(\frac{(P+Q)(P+R)}{(2\pi i\hbar)^2 qr}\right)^{1/2} \int_{-\infty}^{\infty} \mathrm{d}\widehat{x} \ \exp\left\{\frac{i}{\hbar}\left(\mathcal{L}_b(x,\widehat{x}) + \mathcal{L}_a(\widehat{x},\widehat{\overline{x}})\right)\right\} \ , \tag{3.39a}$$

$$= \left(\frac{(P+Q)(P+R)}{2\pi i\hbar(P-qr)(q+r)}\right)^{1/2} \exp\left\{\frac{i}{\hbar}\left[\frac{(P+Q)(P+R)}{(P-qr)(q+r)}x\widehat{\overline{x}} + \frac{1}{2}\left(\frac{P(q+r)}{P-qr} - \frac{P-qr}{q+r}\right)(x^2+\widehat{\overline{x}}^2)\right]\right\} \ , \tag{3.39b}$$

where we have substituted the Lagrangians (3.1), (3.35) and made use of the Gaussian integral. Notice that $K_\lrcorner$ is totally symmetric under interchange of the parameters q and r. This symmetry makes is very straightforward to evaluate the propagator for the alternative path, (ii).

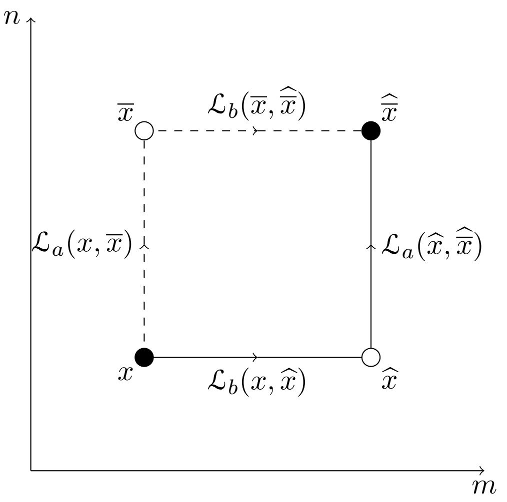

Figure 3.1: The solid line shows path (i) for $K_{\lrcorner}$, and the dashed line path (ii) for $K_{\ulcorner}$. The white circles represent variables that are integrated over.

In path (ii), we evolve first by the bar evolution $\mathcal{L}_a$, and then the hat evolution $\mathcal{L}_b$, so we have the propagator

$$K_{\ulcorner}(x, \widehat{\overline{x}}) = \left(\frac{(P+Q)(P+R)}{(2\pi i\hbar)^2 qr} \right)^{1/2} \int_{-\infty}^{\infty} \mathrm{d}\overline{x} \, \exp \left\{ \frac{i}{\hbar} \left(\mathcal{L}_a(x, \overline{x}) + \mathcal{L}_b(\overline{x}, \widehat{\overline{x}}) \right) \right\} \, . \tag{3.40}$$

But, this expression is identical to (3.39a) with the parameters q and r exchanged and a relabelling of the integration variable. We know that $K_{\lrcorner}$ is symmetric under the interchange of q and r, and so

$$K_{\ulcorner}(x, \widehat{\overline{x}}) = K_{\lrcorner}(x, \widehat{\overline{x}}) \, . \tag{3.41}$$

It is an obvious corollary of this result that, so long as we take only forward steps in time, the propagator $K_{M,N}(x_a, x_b)$ (3.38) is independent of the path taken in the time variables.

Backwards time-steps

The discrete nature of the time evolution suggests the possibility of including backwards time steps in the path Γ, via inverse canonical transforms. As in the classical case, we can construct an action for such a trajectory, using an appropriate orientation for the Lagrangians. In the quantum case we perform a path integral over this action, integrating over all the intermediate points. As the unitary operator U_b generates a time-step in the b direction (section 3.1.1), its inverse U_b^{-1} generates the backwards evolution.

Considering once more the simplest case, we imagine a trajectory around three sides of a square, with action

$$\mathcal{S}_{\sqcap}[x(m,n)] = \mathcal{L}_a(x,\overline{x}) + \mathcal{L}_b(\overline{x},\widehat{\overline{x}}) - \mathcal{L}_a(\widehat{x},\widehat{\overline{x}}) \ . \tag{3.42}$$

This path is shown in figure 3.2.

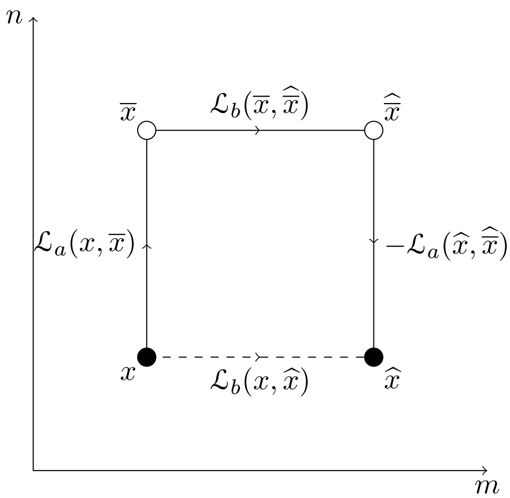

Figure 3.2: The path for action $\mathcal{S}_{\sqcap}$. In the propagator, we integrate over the variables at the white circles. Note the minus sign on the backwards step, $\mathcal{L}_a(\widehat{x},\widehat{\overline{x}})$.

Including the normalisation factors from (3.8) we therefore have the propagator for this three-step path

$$K_{\sqcap}(x,\widehat{x}) = \frac{(P+Q)^{1/2}(P+R)}{(2\pi\hbar)^{3/2}(iq)^{1/2}r}\int_{-\infty}^{\infty}\mathrm{d}\overline{x}\int_{-\infty}^{\infty}\mathrm{d}\widehat{\overline{x}}\ e^{i\mathcal{S}_{\sqcap}[x_{n,m}]/\hbar} \ . \tag{3.43}$$

This is easily calculated using $K_{\ulcorner}$ which we have already found, and another Gaussian integral, and yields

$$\begin{aligned} K_{\sqcap}(x,\widehat{x}) &= \left(\frac{P+Q}{2\pi i\hbar q}\right)^{1/2}\exp\left[\frac{i}{\hbar}\left(-\frac{P-Q}{2q}(x^2+\widehat{x}^2)-\frac{P+Q}{q}x\widehat{x}\right)\right] , &(3.44\text{a})\\ &= \left(\frac{P+Q}{2\pi i\hbar q}\right)^{1/2}\exp\left(\frac{i}{\hbar}\mathcal{L}_b(x,\widehat{x})\right) = K_b(x,\widehat{x};1) \ . &(3.44\text{b})\end{aligned}$$

So we regain exactly the one step propagator from (3.8). Remarkably we again achieve Lagrangian closure, but now on the quantum level. Recall that classically Lagrangian closure held only on the equations of motion: here we have left the equations of motion behind, and yet this key result still holds.

Time Loops

We can also consider the possibility of a *loop* in the discrete variables, illustrated in figure 3.3(i). We imagine some unspecified incoming and outgoing actions $\mathcal{S}_{in}(x_a, x_1)$ and $\mathcal{S}_{out}(x_5, x_b)$, a simple loop in discrete steps, and five integration variables $x_1, \ldots, x_5$. Note that we assign two integration variables x_1 and x_5 to the same vertex, as it is visited twice by the path: the following calculation justifies this choice.

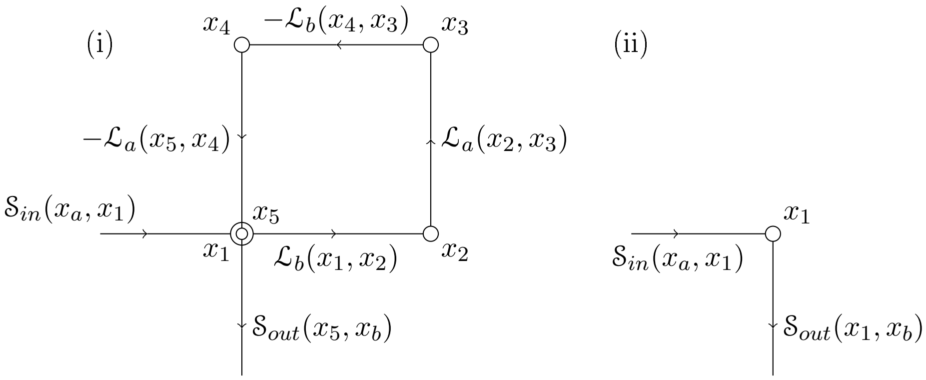

Figure 3.3: (i) shows the loop in discrete variables. (ii) is what remains after collapse of the loop.

Consider the action for the loop,

$$\mathcal{S}_{loop} = \mathcal{L}_b(x_1, x_2) + \mathcal{L}_a(x_2, x_3) - \mathcal{L}_b(x_4, x_3) - \mathcal{L}_a(x_5, x_4) \ , \tag{3.45}$$

noting the orientations on the Lagrangians (3.1) that correspond to backwards time-steps. With normalising factors from (3.8), (3.36) (including complex conjugates for the backwards steps), the propagator is then

$$K_{loop}(x_a, x_b) = \frac{P+Q}{2\pi\hbar q}\frac{P+R}{2\pi\hbar r}\int \mathrm{d}x_1 \ldots \int \mathrm{d}x_5 \ \exp\left\{\frac{i}{\hbar}(\mathcal{S}_{in} + \mathcal{S}_{loop} + \mathcal{S}_{out})\right\} \ . \tag{3.46}$$

The x_2 and x_4 integrals are evaluated as in (3.39a) yielding,

$$\begin{aligned} K_{loop}(x_a, x_b) = & \frac{(P+Q)(P+R)}{2\pi\hbar(P-qr)(q+r)} \iiint \mathrm{d}x_1 \ \mathrm{d}x_3 \ \mathrm{d}x_5 \\ & \times \exp\left\{\frac{i}{\hbar}\left[\mathcal{S}_{in}(x_a, x_1) + \frac{(P+Q)(P+R)}{(P-qr)(q+r)}(x_1 - x_5)x_3 \right.\right. \\ & \left.\left. - \frac{1}{2}\left(\frac{P-qr}{q+r} - \frac{P(q+r)}{P-qr}\right)(x_1^2 - x_5^2) + \mathcal{S}_{out}(x_5, x_b)\right]\right\} \ . \end{aligned} \tag{3.47}$$

Critically, the quadratic term in x_3 has disappeared, and so the integral over x_3 produces a Dirac delta function, $\delta(x_1 - x_5)$. Combined with the integral over x_5 this forces $x_5 = x_1$ (as expected) and we finally conclude,

$$K_{loop}(x_a, x_b) = \int \mathrm{d}x_1 \ \exp\left[\frac{i}{\hbar}\Big(\mathcal{S}_{in}(x_a, x_1) + \mathcal{S}_{out}(x_1, x_b)\Big)\right] \ . \tag{3.48}$$

So, the path integral over the loop action and the normalising factors have cancelled out. Diagrammatically, this is equivalent to the disappearance of the loop, shown in figure 3.3(ii). Loops in the discrete variables therefore "close" and do not effect the overall propagator.

The sum of these results leads to the proposition:

Proposition 1 *For the special choice of Lagrangians (3.1), (3.35) the propagator K_Γ along the time path Γ (3.38) is independent of the choice of Γ, depending only on the end points.*

Proof

Equations (3.41), (3.44) and (3.48) together show that the propagator is unchanged under elementary deformations of the curve Γ. Since we have a simple topology, a curve Γ_1 can be deformed into any other curve Γ_2 (with the same endpoints) by a series of elementary deformations. The proposition follows. □

This proposition amounts to a quantum analogue for the classical closure condition on the Lagrangians. In the classical case, the action is invariant under variations of the curve Γ, on the equations of motion. In the quantum regime, the equivalent condition is invariance of the propagator under changes in Γ. This quantum Lagrangian closure holds despite the redundancy of the equations of motion in the quantum regime.

The multi-time propagator

In proposition 1 we established the independence of propagator K_Γ (3.38) from the time path Γ. This now allows us to calculate the general propagator for M steps in the hat direction and N steps in the bar direction. We denote such a propagator from position x_a to x_b by $K_{M,N}(x_a, x_b)$. As a consequence of the path independence, it is then clear that we can calculate this as

$$K_{M,N}(x_a, x_b) = \int \mathrm{d}x \ K_{M,0}(x_a, x)K_{0,N}(x, x_b) \ . \tag{3.49}$$

In other words, we can make all the hat steps first, followed by all the bar steps.

Taking the discrete propagator in one time direction from (3.22), we can calculate the two-time propagator (3.49) via another Gaussian integral; but in fact the result follows immediately from the group property of the propagator, using its shared form with the continuous time case. So,

$$K_{M,N}(x_a,x_b) = \left(\frac{\sqrt{P}}{\pi i\hbar\sin(\mu M+\eta N)}\right)^{1/2} \times \exp\left[\frac{i\sqrt{P}}{\hbar\sin(\mu M+\eta N)}\Big((x_a^2+x_b^2)\cos(\mu M+\eta N)-2x_a x_b\Big)\right] . \quad (3.50)$$

As in the one-time case (3.22), this has a clear relation to the continuous time propagator for the discrete harmonic oscillator. As we noted classically, in a continuum limit these commuting discrete flows will degenerate to a single continuous-time flow - inevitably given the single, shared invariant. To study commuting flows at the continuous level will require a model with more degrees of freedom.

3.1.4 Uniqueness

For the special choice of Lagrangians (3.1), (3.35) the propagator K_Γ is independent of the path in the time variables, Γ (proposition 1). This is a special property of the choice of Lagrangians that does not hold in general. As in the classical case of section 2.2.3, we consider the generalised, quadratic, oscillator Lagrangians (2.67),

$$\mathcal{L}_a = \alpha\big(-x\overline{x}-(a-a_0)x^2-a_0\overline{x}^2\big) , \quad (3.51a)$$

$$\mathcal{L}_b = \beta\big(-x\widehat{x}-(b-b_0)x^2-b_0\widehat{x}^2\big) . \quad (3.51b)$$

We view the time-path independence of the propagator as the quantum analogue of the classical closure condition on the Lagrangians, and seek conditions on the Lagrangians (3.51) such that the propagator exhibits path independence.

We define propagators around two corners of a square as in equations (3.39a) and (3.40) (see figure 3.1),

$$K_{\lrcorner}(x,\widehat{\overline{x}}) = \mathcal{N}_{\lrcorner}\int_{-\infty}^{\infty} \mathrm{d}\widehat{x}\ \exp\left\{\frac{i}{\hbar}\Big(\mathcal{L}_b(x,\widehat{x})+\mathcal{L}_a(\widehat{x},\widehat{\overline{x}})\Big)\right\} , \quad (3.52a)$$

$$K_{\ulcorner}(x,\widehat{\overline{x}}) = \mathcal{N}_{\ulcorner}\int_{-\infty}^{\infty} \mathrm{d}\overline{x}\ \exp\left\{\frac{i}{\hbar}\Big(\mathcal{L}_a(x,\overline{x})+\mathcal{L}_b(\overline{x},\widehat{\overline{x}})\Big)\right\} . \quad (3.52b)$$

Note that we have generalised the normalisation constants, $\mathscr{N}_{\lrcorner}$ and $\mathscr{N}_{\ulcorner}$, and that a and b are free oscillator parameters. The analogue of closure is equality of these expressions,

$$K_{\lrcorner}(x,\widehat{\overline{x}}) = K_{\ulcorner}(x,\widehat{\overline{x}}) \ , \tag{3.53}$$

up to a possible normalisation.

Calculating the propagators via a Gaussian integral yields

$$\begin{aligned} K_{\lrcorner}(x,\widehat{\overline{x}}) = \mathscr{N}_{\lrcorner} & \left(\frac{\pi\hbar}{i(\beta b_0 + \alpha(a - a_0))} \right)^{1/2} \\ & \times \exp \left\{ \frac{i}{\hbar} \left[\left(\frac{\beta^2}{4(\beta b_0 + \alpha(a - a_0))} - \beta(b - b_0) \right) x^2 \right. \right. \\ & \left. \left. + \left(\frac{\alpha^2}{4(\beta b_0 + \alpha(a - a_0))} - \alpha a_0 \right) \widehat{\overline{x}}^2 + \frac{\alpha\beta}{2(\beta b_0 + \alpha(a - a_0))} x\widehat{\overline{x}} \right] \right\} \ , \end{aligned} \tag{3.54a}$$

$$\begin{aligned} K_{\ulcorner}(x,\widehat{\overline{x}}) = \mathscr{N}_{\ulcorner} & \left(\frac{\pi\hbar}{i(\alpha a_0 + \beta(b - b_0))} \right)^{1/2} \\ & \times \exp \left\{ \frac{i}{\hbar} \left[\left(\frac{\alpha^2}{4(\alpha a_0 + \beta(b - b_0))} - \alpha(a - a_0) \right) x^2 \right. \right. \\ & \left. \left. + \left(\frac{\beta^2}{4(\alpha a_0 + \beta(b - b_0))} - \beta b_0 \right) \widehat{\overline{x}}^2 + \frac{\alpha\beta}{2(\alpha a_0 + \beta(b - b_0))} x\widehat{\overline{x}} \right] \right\} \ . \end{aligned} \tag{3.54b}$$

By comparing the coefficients of x^2, $\widehat{\overline{x}}^2$ and $x\widehat{\overline{x}}$ in the exponent, we derive conditions for time-path independence on the coefficients,

$$\frac{\beta^2}{4(\beta b_0 + \alpha(a - a_0))} - \beta(b - b_0) = \frac{\alpha^2}{4(\alpha a_0 + \beta(b - b_0))} - \alpha(a - a_0) \ , \tag{3.55a}$$

$$\frac{\alpha^2}{4(\beta b_0 + \alpha(a - a_0))} - \alpha a_0 = \frac{\beta^2}{4(\alpha a_0 + \beta(b - b_0))} - \beta b_0 \ , \tag{3.55b}$$

$$\frac{\alpha\beta}{2(\beta b_0 + \alpha(a - a_0))} = \frac{\alpha\beta}{2(\alpha a_0 + \beta(b - b_0))} \ . \tag{3.55c}$$

Note that an immediate consequence of (3.55c) is that the multiplicative factors in (3.54a) and (3.54b) are the same: we can allow $\mathscr{N}_{\lrcorner} = \mathscr{N}_{\ulcorner}$. Analysis of the three conditions (3.55) leads to the necessary and sufficient conditions on the coefficients:

$$a_0 = \frac{1}{2}a + \frac{f}{2\alpha} \ , \qquad b_0 = \frac{1}{2}b + \frac{f}{2\beta} \ , \tag{3.56a}$$

$$\alpha = \frac{\gamma}{\sqrt{a^2 - 1}} \ , \qquad \beta = \frac{\gamma}{\sqrt{b^2 - 1}} \ . \tag{3.56b}$$

Here, f must be independent of the oscillator parameters a, b, and γ is an overall multiplier.

Now, these are the same conditions (2.68) that arose in the classical case by demanding that the Lagrangians should obey the classical closure condition. As in that case, the constant f makes no contribution and we ignore it. The general Lagrangians (3.51) are therefore restricted to a symmetric form, with a specified overall constant given by the oscillator parameters a, b,

$$\mathcal{L}_a = \frac{\gamma}{\sqrt{a^2-1}}\big(-x\overline{x} - \tfrac{1}{2}a(x^2+\overline{x}^2)\big) , \tag{3.57a}$$

$$\mathcal{L}_b = \frac{\gamma}{\sqrt{b^2-1}}\big(-x\widehat{x} - \tfrac{1}{2}b(x^2+\widehat{x}^2)\big) . \tag{3.57b}$$

Note that taking $a = (P-R)/(P+R)$, $b = (P-Q)/(P+Q)$ leads exactly to the conditions of (2.68) and the Lagrangians (3.1) and (3.35).

In conclusion,

Proposition 2 *For given oscillator parameters a and b, the Lagrangians (3.57) are the unique quadratic Lagrangians, up to constants γ and f (3.56), such that the multi-time propagator K_Γ is independent of the time path Γ.*

In other words, demanding time-path independence of the propagator is the natural quantum analogue of the closure relation on the Lagrangian.

We note that a general problem in the theory of path integrals is how to choose the correct Lagrangian. For example, in the case of a Newtonian system with a vector potential, the result of the path integral is sensitive to the choice of discretisation for the Lagrangian [47]. Here, for the Lagrangian one-form, we now have a more definite choice where the discretisation is precisely specified in order to preserve the time-path independence.

3.1.5 Quantum Variational Principle: One dimension

Consider a quantum mechanical evolution from an initial time $(0,0)$ to a new time (M,N), along a time-path Γ (shown in figure 2.6). We associate an action to the path, $\mathcal{S}_\Gamma$ (3.37), and consider the propagator for the evolution $K_\Gamma(x_a;x_b)$ defined in (3.38). We have shown that, in the special case of quadratic Lagrangian one-forms (3.57), the propagator K_Γ is independent of the path Γ (it depends only on the endpoints) but that this is not true in general. For a generic Lagrangian, K_Γ will depend on the time-path chosen, as shown in section 3.1.4.

Classically, a Lagrangian one-form defines a system as the critical point in a variational principle over both the dependent *and* independent variables. That is, the one-form system is a critical point with respect to variations of the time-path. This not only yields all the compatible equations of motion for the system (as in figure 2.7) but also selects certain "permissible" Lagrangians which obey a closure relation (2.66). This results in a system of extended Euler-Lagrange equations where the solution is not only the equations of motion, but also in some sense the Lagrangian itself.

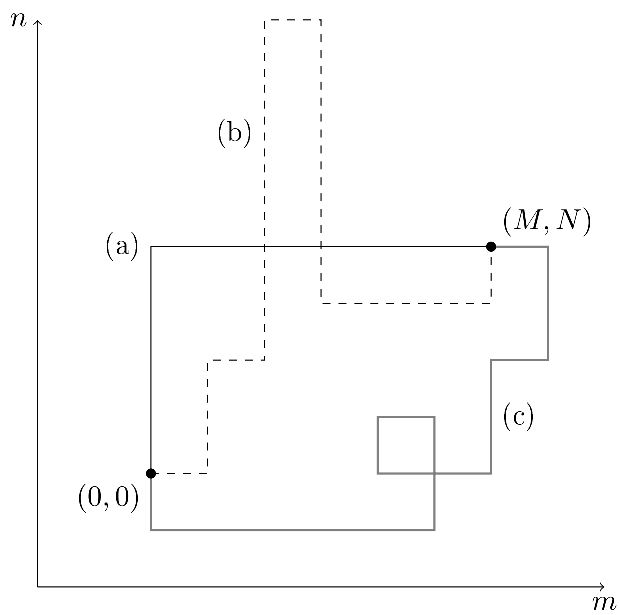

Figure 3.4: Three possible paths in the time-variables. Path (a) is a direct path. Path (b) extends for some distance in the m direction before returning. Path (c) includes a loop in the time variables.

In the quantum case, consider the dependence of the propagator on the discrete time-path Γ between fixed initial and final times. In general, there are an infinite number of possible time paths from $(0, 0)$ to (M, N), including shortest time-paths as well as those with long diversions, or loops, as illustrated in figure 3.4. For a generic Lagrangian, as we vary the time path, each Γ yields a different propagator (3.38) viewed as a functional of the path. In the special case of the Lagrangian one-form (3.57), however, the propagator K_Γ is *independent* of the path taken through the time variables, and so remains unchanged across the variation of the time-path Γ. This path independence property is the natural quantum analogue of the Lagrangian closure condition (2.66).

We might then speculate about the quantum analogue of the one-form variational principle. Let us view the propagator K_Γ as a functional of not only the path $x(\boldsymbol{n})$ and the time-path Γ, but also of the Lagrangian. The Lagrangian one-form (3.57) can then be thought of as representing a critical point (in a properly chosen function space of Lagrange functions) for the path-dependent propagator, with regard to variations of the time-path. That is, it is uniquely the choice of Lagrangian such that the propagator remains fixed over variations of Γ. Suppose that the time-path Γ can be varied in such a way that the critical point analysis *selects* the Lagrangian one-form from the space of possible Lagrangians. This was the point of view put forward in [63] in the classical case. This principle could perhaps be represented in a path integral by a "sum over all time-paths," by posing a new quantum object of the type first proposed in the continuous time-case in [72], (see (1.86))

$$K\big(x_b,(M,N);x_a,(0,0)\big) = \sum_{\Gamma\in\mathfrak{P}} \mathscr{N}_\Gamma \; K_\Gamma\big(x_b,(M,N);x_a,(0,0)\big)\ . \tag{3.58}$$

Here, $\mathfrak{P}$ the set of all possible time-paths from $(0,0)$ to (M,N), and $\mathscr{N}_\Gamma$ some normalisation factor to be specified. Although this formula is currently meaningless (its meaning is yet to be defined) we include (3.58) as an illustration of the concept. As a functional of the Lagrangian such an object would have a singular point for those Lagrangians with the quantum closure condition, where the path-independent propagators K_Γ would all contribute the same amount. However, controlling and regularising the singular behaviour of such an object is presently beyond our understanding. We note, however, that there are countably many time-paths in the set $\mathfrak{P}$, so that some reasonable renormalisation of the sum over time paths may be achievable.

In the continuous case, a "sum over time-paths" might require some parametrisation of the (multi-)time in terms of a parameter s, so that $\boldsymbol{t}(s)$. This might bear some relation to the reparametrisation invariant path integrals of Rovelli [94, 95] discussed in section 1.4. For a single time variable, Rovelli implemented a parametrisation, treating the time as a dynamical variable and integrating over the possible parametrisations. These ideas may help to extend the Lagrangian one-form into the continuous time, quantum case. However, this would require a model with a non-trivial one-form structure in the continuum case; one such possibility is discussed below.

3.1.6 Quantisation of the $P = 3$ case

A natural extension to the ideas of section 3.1 is to apply them to the two-dimensional evolution of the higher period staircase reduction, described in section 2.3. There are again two commuting, discrete flows, $\{x_1, x_2\} \to \{\widehat{x}_1, \widehat{x}_2\}$ and $\{x_1, x_2\} \to \{\overline{x}_1, \overline{x}_2\}$, but now with two position variables, and correspondingly two commuting invariants. The discrete harmonic oscillator model of section 3.1 has the disadvantage that it does not seem to retain the one-form structure in a continuous limit, due to degeneration of the flows; a higher order system could potentially avoid this pitfall and offer a richer insight into the continuous structures.

Beginning with the "hat" flow generated by the Lagrangian (2.108a) (which obeys the closure property)

$$\mathcal{L}_1(x, \widehat{x}) = -x_1(\widehat{x}_1 + \widehat{x}_2) - x_2\widehat{x}_2 - \tfrac{1}{2}s\left(x_1^2 + x_1x_2 + x_2^2 + \widehat{x}_1^2 + \widehat{x}_1\widehat{x}_2 + \widehat{x}_2^2\right) , \qquad (3.59)$$

we generate operator equations of motion

$$\begin{aligned}
\widehat{\mathbf{x}}_1 &= \mathbf{X}_1 - \mathbf{X}_2 - \tfrac{1}{2}s(\mathbf{x}_1 - \mathbf{x}_2) , & (3.60a)\\
\widehat{\mathbf{x}}_2 &= \mathbf{X}_2 - \tfrac{1}{2}s(\mathbf{x}_1 + 2\mathbf{x}_2) , & (3.60b)\\
\widehat{\mathbf{X}}_1 &= -s\mathbf{X}_1 + \tfrac{1}{2}s\mathbf{X}_2 - (1 - \tfrac{3}{4}s^2)\mathbf{x}_1 , & (3.60c)\\
\widehat{\mathbf{X}}_2 &= -\tfrac{1}{2}s\mathbf{X}_1 - \tfrac{1}{2}s\mathbf{X}_2 - (1 - \tfrac{3}{4}s^2)(\mathbf{x}_1 + \mathbf{x}_2) . & (3.60d)
\end{aligned}$$

The system has equal time canonical commutation relations,

$$[\mathbf{x}_i, \mathbf{x}_j] = [\mathbf{X}_i, \mathbf{X}_j] = 0 , \quad [\mathbf{x}_i, \mathbf{X}_j] = i\hbar\delta_{ij} , \qquad (3.61)$$

and is essentially a pair of harmonic oscillators.

We wish to express this mapping by the action of a time evolution operator U_1 (3.4), but the appropriate form for such an operator is not immediately apparent. However, we are inspired by the "cross terms" for such evolution operators in [74] (see equation (1.77)) and the three term time-evolution operator found in (3.5). We pose an ansatz for the evolution operator,

$$U_1 = \exp\left(-\tfrac{i}{\hbar}V(\mathbf{x})\right) \exp\left(-\tfrac{i}{\hbar}T(\mathbf{X})\right) \exp\left(-\tfrac{i}{\hbar}\mathbf{x}_1\mathbf{X}_2\right) \exp\left(-\tfrac{i}{\hbar}\overline{V}(\mathbf{x})\right) , \qquad (3.62)$$

and working through a lengthy calculation leads eventually to the expressions for the potentials,

$$T(\mathbf{X}) = \tfrac{1}{2}(\mathbf{X}_1^2 + \mathbf{X}_2^2) \ , \tag{3.63a}$$

$$V(\mathbf{x}) = \tfrac{1}{2}(1+s)(\mathbf{x}_1^2 + \mathbf{x}_2^2) + \tfrac{1}{2}s\mathbf{x}_1\mathbf{x}_2 \ , \tag{3.63b}$$

$$\overline{V}(\mathbf{x}) = \tfrac{1}{2}(2+s)(\mathbf{x}_1^2 + \mathbf{x}_1\mathbf{x}_2) + \tfrac{1}{2}(1+s)\mathbf{x}_2^2 \ . \tag{3.63c}$$

It is straightforward to verify that conjugation by U_1 generates the operator equations of motion (3.60).

In the same way as the one-dimensional case, it is then possible to calculate the one time-step propagator by using the joint position eigenstates,

$$K_1(x_1, x_2, m; \widehat{x}_1, \widehat{x}_2, m+1) = \langle \widehat{x}_1, \widehat{x}_2 | U_1 | x_1, x_2 \rangle \ . \tag{3.64}$$

Noting in particular that the commutation relations permit the evaluation with position and momentum eigenstates of the cross term,

$$\langle X_1, X_2 | e^{-i\mathbf{x}_1\mathbf{X}_2/\hbar} | x_1, x_2 \rangle = e^{-ix_1X_2/\hbar} \langle X_1, X_2 | x_1, x_2 \rangle \ , \tag{3.65}$$

we find the propagator in the same way as (3.8)

$$\langle \widehat{x} | U_1 | x \rangle = \frac{1}{2\pi i\hbar} \exp\left(\frac{i}{\hbar} \mathcal{L}_1(x, \widehat{x}) \right) \ . \tag{3.66}$$

As expected, we recover the exact form of the Lagrangian from (3.59).

More work is needed to delve into this system further. In particular, we expect a link between the time evolution operator U_1 and the known invariants of the model (2.111). The commuting flow should lead to a second time-evolution operator U_2, such that $[U_1, U_2] = 0$, we would hope then to see how the interplay of the commuting invariants relates to these commuting time flows. Perhaps the most important avenue to pursue is the continuum limit for this model: how do the commuting discrete flows and one-form structure go over into a continuous one-form model that can be examined on the quantum level?

3.2 Quantisation of the Lattice Equation

In section 2.1 we introduced the linearised lattice KdV equation (2.8). Having considered the quantisation of its finite dimensional reduction, we now turn to quantisation of the lattice equation itself. Quantisation of lattice models has been previously considered from a canonical (quantum inverse scattering method) perspective [13, 33, 118, 119], but here we will consider a Lagrangian, path integral perspective.

We have the linear lattice equation,

$$(p_i + p_j)(u_i - u_j) = (p_i - p_j)(u - u_{ij}) \ , \tag{3.67}$$

where p_i is the lattice parameter in the i direction, and u_i indicates a shift of the field variable u in the i direction. Classically, we suppose this equation holds on all plaquettes in the multi-dimensional lattice at the same time, so that there are multiple consistent equations holding on the same lattice variable u. The equation is generated by the oriented Lagrangian (2.24)

$$\mathcal{L}_{ij}(u, u_i, u_j; p_i, p_j) = u(u_i - u_j) - \frac{1}{2}s_{ij}(u_i - u_j)^2 \ , \quad \text{where } s_{ij} = \frac{p_i + p_j}{p_i - p_j} \ . \tag{3.68}$$

The Lagrangian itself is a critical point of the classical variational principle over surfaces: it obeys the closure property on the classical equations of motion, such that the surface can be allowed to freely vary under local moves. Indeed, classically it is also fairly unique, as seen in section 2.1.3.

How might we proceed to quantise such a system? A canonical approach is to transform (3.67) into an operator equation of motion, but we are concerned here with a Lagrangian approach. The clear analogy is to quantum field theory: we have a discretised space-time and a Lagrangian in two dimensions over field variables $u(\boldsymbol{n})$ indexed by a discrete vector $\boldsymbol{n}$. Field theoretic equations such as the Klein-Gordon equation can be quantised using two plane-wave factors to form creation and annihilation operators [107, 114], but the linear lattice equation (3.68) is an essentially first-order equation with only a single plane wave solution (2.14).

We imagine some space-time boundary $\partial\sigma$ enclosing a surface σ made up of elementary plaquettes σ_{ij} (such as in figure 2.2). We can then construct an action by summing the directed Lagrangians over the surface, as in the classical case,

$$\mathcal{S}_\sigma = \sum_{\sigma_{ij} \in \sigma} \mathcal{L}_{ij}(u) \ , \tag{3.69}$$

where we define the shorthand $\mathcal{L}_{ij}(u) := \mathcal{L}(u, u_i, u_j; p_i, p_j)$. We then consider the propagator over the surface σ

$$K_\sigma(\partial\sigma) = \int [\mathcal{D}u_{n,m}]_\sigma \; e^{i\mathcal{S}_\sigma[u_{n,m}]/\hbar} \,, \tag{3.70a}$$

$$= \mathscr{N}_\sigma \int \prod_{\boldsymbol{n}\in\sigma} \mathrm{d}\mu(u(\boldsymbol{n})) \; e^{i\mathcal{S}_\sigma[u(\boldsymbol{n})]/\hbar} \,. \tag{3.70b}$$

The integration is over all interior field variables, with some measure $\mathrm{d}\mu(u(\boldsymbol{n}))$. The propagator depends, in principle, on the surface σ and is a function of the field variables on the boundary $\partial\sigma$, which form some boundary value problem (see a similar point made in [95]). We will see as we go on that this object is subject to infra-red divergences, as particular surface configurations produce integrations yielding volume factors. Since our main statements involve only the combinatorics of the exponential factors, involving the action and arising through Gaussian integrals, we tacitly assume that the propagator K_σ can be renormalised by an appropriate choice of normalisation factor $\mathscr{N}_\sigma$.

$K_\sigma(\partial\sigma)$ describes a propagator in the sense of a surface gluing procedure. Two propagators K_{σ_1} and K_{σ_2} are combined to form a new propagator by multiplication and integration over all interior variables on the shared boundary $\partial\sigma_1 \cap \partial\sigma_2$, leaving a new propagator $K_{\sigma_1\cup\sigma_2}$ dependent only on field variables that lie on the new boundary, $\partial(\sigma_1 \cup \sigma_2)$. This surface gluing is illustrated in figure 3.5, where white circles indicate the interior field variables on the shared boundary which are integrated over. The black circles show the two field variables that lie both on the shared boundary but also on the boundary of the new surface $\sigma_1 \cup \sigma_2$, and hence are not integration variables. The product $*$ in (3.71a) indicates this procedure. Thus, the one-step surface gluing can be written symbolically as,

$$K_{\sigma_1\cup\sigma_2} = \int_{\partial\sigma_1\cap\partial\sigma_2} K_{\sigma_1} * K_{\sigma_2} \,, \tag{3.71a}$$

$$:= \mathscr{N}_{\partial\sigma_1\cap\partial\sigma_2} \left[\int \prod_{\boldsymbol{n}\in\partial\sigma_1\cap\partial\sigma_2} \mathrm{d}u(\boldsymbol{n}) \right] K_{\sigma_1}(\partial\sigma_1).K_{\sigma_2}(\partial\sigma_2) \,, \tag{3.71b}$$

where the integral is over appropriately chosen coordinates of the joined boundary. Iterating the gluing formula is tantamount to setting up a "surface-slicing" procedure for the path integral.

3.2.1 Motivation: The pop-up cube

The classical variational principle for two-forms includes a variation of the surface σ, so that the Lagrangian and equations of motion sit at a critical point. The action is

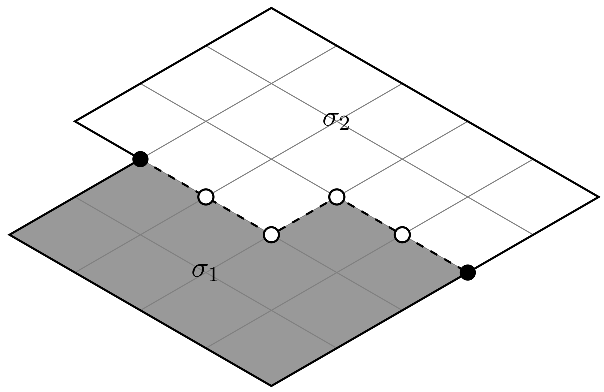

Figure 3.5: The gluing of surface σ_1 to σ_2, forming a new surface $\sigma_1 \cup \sigma_2$. White circles indicate interior, integration variables in the gluing procedure, whilst the black circles remain boundary variables.

stationary under variation of not only the dependent variables u, but also the variation of the surface itself. As we move to the quantum regime, we naturally ask: what happens to the propagator $K_\sigma(\partial\sigma)$ (3.70) under variation of the surface σ? In particular, we will be interested in variation of the surface under *local moves*, that is changes in the surface on the level of a single cube in the multi-dimensional lattice, such that any variation of the surface can be achieved through a series of local moves. We consider the effect of a simple variation of the surface: a local move from a flat surface to a popped-up cube, see figure 3.6.

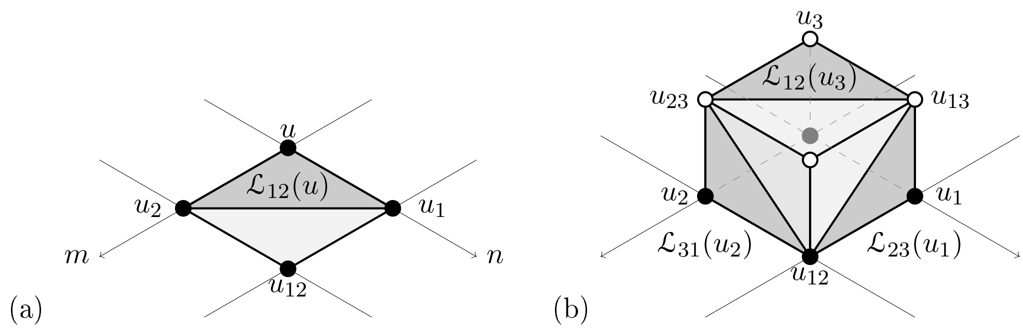

Figure 3.6: A flat surface in (a), compared to a pop-up cube shown in (b).

The following conventions are used for such local moves, as in figure 3.6. Black circles indicate variables on the boundary of the surface (in the move) that are not integrated over - the contribution to the overall propagator depends on these field variables. White circles indicated interior variables on the surface that are integrated over to give the contribution to the propagator from that surface configuration. As in figure 2.3, shaded triangles indicate the three variables on which each oriented Lagrangian depends. Lagrangians are oriented in local moves according to the classical closure relation (2.26), such that when two sides of a local move are compared, the signs on the Lagrangians must match the signs on the corresponding classical closure. For example, the closure relation (2.26) is rearranged for the local move of figure 3.6 as

$$\mathcal{L}_{12}(u) = \mathcal{L}_{23}(u_1) + \mathcal{L}_{31}(u_2) + \mathcal{L}_{12}(u_3) - \mathcal{L}_{23}(u) - \mathcal{L}_{31}(u) \ , \tag{3.72}$$

which will be used below. The closure relation can be rearranged as appropriate for the local move under consideration.

The contribution to the action given by surface (a) is therefore a single Lagrangian, $\mathcal{L}_{12}(u)$. On surface (b) we have five plaquettes, with a contribution to the action given by the sum of oriented Lagrangians

$$\mathcal{S}_{pop}[u_{n,m}] = \mathcal{L}_{23}(u_1) + \mathcal{L}_{31}(u_2) + \mathcal{L}_{12}(u_3) - \mathcal{L}_{23}(u) - \mathcal{L}_{31}(u) \ . \tag{3.73}$$

Note that the orientations lead to the negative contributions, shown in (3.72). In the path integral perspective (3.70), to calculate the propagator we must also integrate over the interior, "popped-up", variables u_3, u_{23}, u_{31} and u_{123}, shown by white circles in figure 3.6(b). The boundary variables on which the contributions depend are u, u_1, u_2 and u_{12}. So altogether the contribution from the pop-up cube to the propagator is

$$K_{pop} = \mathcal{N}_{pop} \iiiint \mathrm{d}u_3 \, \mathrm{d}u_{31} \, \mathrm{d}u_{23} \, \mathrm{d}u_{123} \ \exp\left(\frac{i}{\hbar}\mathcal{S}_{pop}[u_{n,m}]\right) \ . \tag{3.74}$$

$\mathcal{N}_{pop}$ is a normalising constant. The action $\mathcal{S}_{pop}[u_{n,m}]$ does not depend on u_{123}, so the integral $\int \mathrm{d}u_{123}$ produces a volume factor V.

Equation (3.74) can then be written in a matrix form,

$$\begin{aligned} K_{pop} = \ & V\mathcal{N}_{pop} \int \mathrm{d}^3\boldsymbol{u} \ \exp\frac{i}{\hbar}\left(\frac{1}{2}\boldsymbol{u}^T A\boldsymbol{u} + \boldsymbol{B}^t\boldsymbol{u}\right) \\ & \times \exp\left\{\frac{i}{2\hbar}\left[s_{31}(u_1^2 - u_{12}^2) + s_{23}(u_2^2 - u_{12}^2) + (u+u_{12})(u_1-u_2)\right]\right\} \ , \end{aligned} \tag{3.75a}$$

where

$$\boldsymbol{u}^T = (u_3, u_{31}, u_{23}) \ , \tag{3.75b}$$

$$A = \begin{pmatrix} s_{23}+s_{31} & 1 & -1 \\ 1 & -(s_{12}+s_{23}) & s_{12} \\ -1 & s_{12} & -(s_{12}+s_{31}) \end{pmatrix} , \tag{3.75c}$$

$$\boldsymbol{B}^T = \big(-s_{31}u_1 - s_{23}u_2, \ -u_1 + s_{23}u_{12}, \ u_2 + s_{31}u_{12}\big) \ . \tag{3.75d}$$

In principle, equation (3.75a) could be solved as a set of three Gaussian integrals, but the matrix A is singular. We calculate

$$\det A = (s_{23} + s_{31})(s_{12}s_{23} + s_{23}s_{31} + s_{31}s_{12} + 1) \ , \tag{3.76}$$

and recall the identity for the parameters s_{ij} (3.68)

$$s_{12}s_{23} + s_{23}s_{31} + s_{31}s_{12} + 1 = 0 \ , \tag{3.77}$$

such that $\det A = 0$. We therefore resolve (3.75a) by carrying out only two Gaussian integrals. The third integration variable must leave an exponent that is at most linear.

Performing the Gaussian integrations with respect to u_3 and u_{31}, we therefore have

$$\begin{aligned} K_{pop} &= V\mathscr{N}_{pop}\frac{2\pi\hbar}{s_{23}}\int \mathrm{d}u_{23} \exp\frac{i}{\hbar}\left(u(u_1-u_2) - \tfrac{1}{2}s_{12}(u_1-u_2)^2\right) \ , &(3.78a)\\ &= V^2\mathscr{N}_{pop}\frac{2\pi\hbar}{s_{23}}\exp\left(\frac{i}{\hbar}\mathcal{L}_{12}(u,u_1,u_2)\right) \ , &(3.78b)\end{aligned}$$

where in the first equality we note that all terms containing u_{23} have vanished entirely. This is now *exactly* the exponent expected from the diagram (a) in figure 3.6! So, whilst it is clear that there are non-trivial issues to resolve with respect to volume factors and normalisation factors in (3.78),[1] in the critical issue of the contribution to the *action* in the exponent between diagrams 3.6(a) and 3.6(b), the two pictures make the *same* contribution. In other words, there is some sense in which the action is unchanged by the local move that transforms the surface σ by the pop-up cube. Inspired by this discovery, we consider a more general situation.

3.2.2 Surface Independence of the Propagator

For classical lattice two-forms, Lobb and Nijhoff [63] (also [19]) identified three *elementary configurations* of the surface in three dimensions that yield three elementary Euler-Lagrange equations (see figure 2.3). The Euler-Lagrange equations for any configuration of the lattice can be derived from these three elementary configurations (2.27).

In the quantum case we no longer have Euler-Lagrange equations, but alterations to the propagator (3.70) under deformations of the surface σ. Deformations can be considered to take place one cube at a time in a series of local moves. The three elementary configurations

[1] The asymmetrical factor of s_{23} in the pre-factor is an indicator that all is not as it should be, and the multiple infinities due to volume factors are an obvious concern.

of the classical case yield three *elementary moves* that form the basis for deformations of the surface σ. Combined with the pop-up cube of figure 3.6 these give a full set of four elementary moves on the surface such that any local move on σ can then be recreated by a series of elementary moves. The three additional elementary moves are shown in figures 3.7, 3.8 and 3.9.

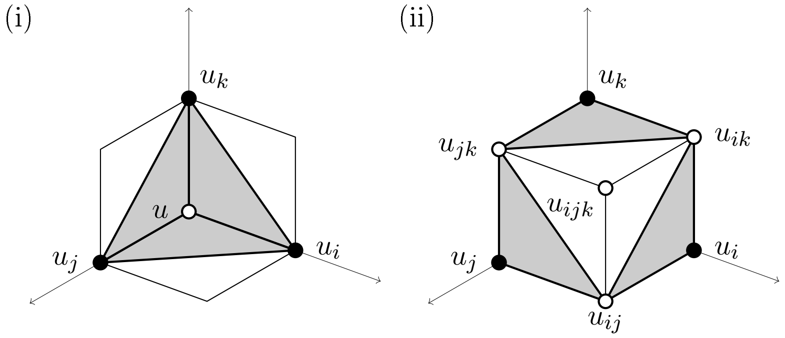

Figure 3.7: Elementary move (a). We pass between (i) and (ii); white circles indicate variables to be integrated over in the move.

The first move is shown in figure 3.7. We denote the two sides of the elementary move with the subscripts + and −, so that figure 3.7(i) is denoted by the subscript $a+$ and figure 3.7(ii) by $a-$. The action for figure 3.7(i) is given by

$$\mathcal{S}_{a+} = \mathcal{L}_{ij}(u) + \mathcal{L}_{jk}(u) + \mathcal{L}_{ki}(u) \ , \tag{3.79a}$$

$$= -\tfrac{1}{2}s_{ij}(u_i - u_j)^2 - \tfrac{1}{2}s_{jk}(u_j - u_k)^2 - \tfrac{1}{2}s_{ki}(u_k - u_i)^2 \ . \tag{3.79b}$$

As described in (3.70) we integrate over *interior* field variables on the surface, marked by white circles on the figure, so that the contribution to the propagator (3.70) is given by the integral over u,

$$K_{a+} = \mathscr{N}_{a+} \int \mathrm{d}u \exp\left[i\mathcal{S}_{a+}/\hbar\right] \ , \tag{3.80a}$$

$$= V\mathscr{N}_{a+} \exp\left\{\frac{-i}{2\hbar}\left(s_{ij}(u_i - u_j)^2 + s_{jk}(u_j - u_k)^2 + s_{ki}(u_k - u_i)^2\right)\right\} \ . \tag{3.80b}$$

$\mathscr{N}_{a+}$ indicates a normalisation factor. We note that all the u terms in the exponent $\mathcal{S}_{a+}$ cancel out, so the integral reduces to a volume factor, V.

In contrast, the action for figure 3.7 (ii) is given by

$$\mathcal{S}_{a-} = \mathcal{L}_{ij}(u_k) + \mathcal{L}_{jk}(u_i) + \mathcal{L}_{ki}(u_j) \ . \tag{3.81}$$

The contribution to the propagator is then

$$K_{a-} = \mathscr{N}_{a-} \iiiint \mathrm{d}u_{ij}\mathrm{d}u_{jk}\mathrm{d}u_{ki}\mathrm{d}u_{ijk} \exp\left[i\mathcal{S}_{a-}/\hbar\right] , \tag{3.82a}$$

$$= V\mathscr{N}_{a-} \int \mathrm{d}^3\boldsymbol{u} \ \exp\frac{i}{\hbar}\left(-\frac{1}{2}\boldsymbol{u}^T A\boldsymbol{u} + \boldsymbol{B}^t\boldsymbol{u}\right) , \tag{3.82b}$$

where the integral over u_{ijk} has produced a volume factor, and

$$\boldsymbol{u}^T = (u_{ij}, u_{jk}, u_{ki}) , \tag{3.83a}$$

$$A = \begin{pmatrix} s_{jk}+s_{ki} & -s_{ki} & -s_{jk} \\ -s_{ki} & s_{ki}+s_{ij} & -s_{ij} \\ -s_{jk} & -s_{ij} & s_{ij}+s_{jk}) \end{pmatrix} , \tag{3.83b}$$

$$\boldsymbol{B}^T = (u_i - u_j, u_j - u_k, u_k - u_i) . \tag{3.83c}$$

Critically, $\det A = 0$, so again this is a singular integral. Carrying out two integrals in turn, so that the third integration produces a volume factor, we therefore have

$$K_{a-} = V^2\mathscr{N}_{a-}2\pi\hbar \ \exp\left\{\frac{-i}{2\hbar}\left(s_{ij}(u_i-u_j)^2 + s_{jk}(u_j-u_k)^2 + s_{ki}(u_k-u_i)^2\right)\right\} . \tag{3.84}$$

Thus, the *exponents* in K_{a+} and K_{a-} (3.80), (3.84) are the same. With the correct choice of normalisations $\mathscr{N}_{a+}$ and $\mathscr{N}_{a-}$, the two configurations make identical contributions to the propagator.

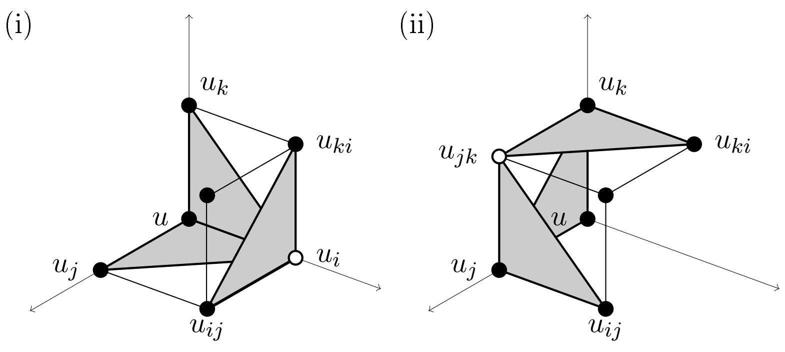

Figure 3.8: Elementary move (b). White circles indicate integration variables.

Considering elementary move (b), shown in figure 3.8, the action for figure 3.8(i) is given by

$$\mathcal{S}_{b+} = \mathcal{L}_{ij}(u) + \mathcal{L}_{ki}(u) - \mathcal{L}_{jk}(u_i) . \tag{3.85}$$

The contribution to the propagator is

$$K_{b+} = \mathscr{N}_{b+} \int \mathrm{d}u_i \exp\left[i\mathcal{S}_{b+}/\hbar\right] , \tag{3.86a}$$

$$= \mathscr{N}_{b+} \left(\frac{2\pi\hbar}{i(s_{ij}+s_{ki})}\right)^{1/2} \exp\left\{\frac{i}{\hbar(s_{ij}+s_{ki})}\left[\tfrac{1}{2}(u_j-u_k)^2 - \tfrac{1}{2}s_{ij}s_{ki}(u_{ij}-u_{ki})^2 - (s_{ij}u_j+s_{ki}u_k)(u_{ij}-u_{ki})\right] - \frac{i}{\hbar}\mathcal{L}_{jk}(u)\right\} . \tag{3.86b}$$

On the other side of the elementary move, for figure 3.8(ii) we have

$$\mathcal{S}_{b-} = \mathcal{L}_{ij}(u_k) + \mathcal{L}_{ki}(u_j) - \mathcal{L}_{jk}(u) , \tag{3.87}$$

with contribution to the propagator

$$K_{b-} = \mathscr{N}_{b-} \int \mathrm{d}u_{jk} \exp\left[i\mathcal{S}_{b-}/\hbar\right] , \tag{3.88a}$$

$$= \mathscr{N}_{b-} \left(\frac{2\pi\hbar}{i(s_{ij}+s_{ki})}\right)^{1/2} \exp\left\{\frac{i}{\hbar(s_{ij}+s_{ki})}\left[\tfrac{1}{2}(u_j-u_k)^2 - \tfrac{1}{2}s_{ij}s_{ki}(u_{ij}-u_{ki})^2 - (s_{ij}u_j+s_{ki}u_k)(u_{ij}-u_{ki})\right]\right\} \times \exp\left[-\frac{i}{\hbar}\mathcal{L}_{jk}(u)\right] , \tag{3.88b}$$

$$= K_{b+} , \tag{3.88c}$$

once we allow $\mathscr{N}_{b+} = \mathscr{N}_{b-}$. So here the contributions to the propagator are easily seen to be identical, without any volume factor concerns.

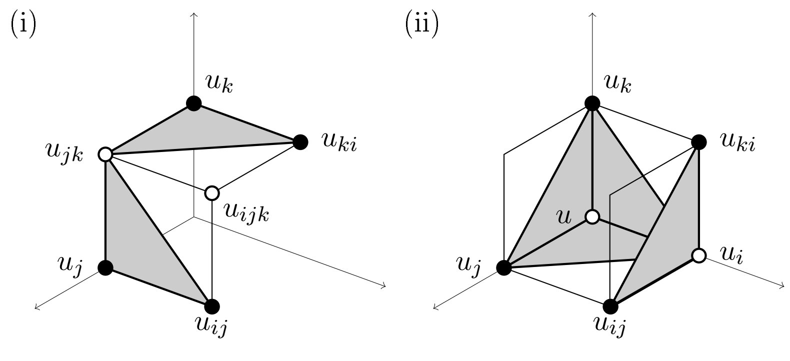

Figure 3.9: Lattice elementary move (c).

Lastly, consider elementary move (c) shown in figure 3.9. This move has a clear relation to figure 3.8: the element $\mathcal{L}_{jk}(u)$ has been shifted from one diagram to the other, inducing also a slight change in the integration variables. 3.9(i) is easily calculated from 3.8(ii):

$$\mathcal{S}_{c+} = \mathcal{L}_{ij}(u_k) + \mathcal{L}_{ki}(u_j) \ , \tag{3.89}$$

$$K_{c+} = \mathcal{N}_{c+} \iint \mathrm{d}u_{jk} \mathrm{d}u_{ijk} \exp\left[i\mathcal{S}_{c+}/\hbar\right] \ , \tag{3.90a}$$

$$= V\mathcal{N}_{c+} \left(\frac{2\pi\hbar}{i(s_{ij}+s_{ki})}\right)^{1/2} \exp\left\{\frac{i}{\hbar(s_{ij}+s_{ki})}\left[\tfrac{1}{2}(u_j-u_k)^2 - \tfrac{1}{2}s_{ij}s_{ki}(u_{ij}-u_{ki})^2 - (s_{ij}u_j + s_{ki}u_k)(u_{ij}-u_{ki})\right]\right\} . \tag{3.90b}$$

Similarly, the other side of the move 3.9(ii) is derived from 3.8(i) with an additional integral over u. The action is

$$\mathcal{S}_{c-} = \mathcal{L}_{ij}(u) + \mathcal{L}_{jk}(u) + \mathcal{L}_{ki}(u) - \mathcal{L}_{jk}(u_i) \ , \tag{3.91}$$

and the contribution to the propagator

$$K_{c-} = \mathcal{N}_{c-} \iint \mathrm{d}u \ \mathrm{d}u_i \ \exp\left[i\mathcal{S}_{c-}/\hbar\right] \ , \tag{3.92a}$$

$$= V\mathcal{N}_{c-} \left(\frac{2\pi\hbar}{i(s_{ij}+s_{ki})}\right)^{1/2} \exp\left\{\frac{i}{\hbar(s_{ij}+s_{ki})}\left[\tfrac{1}{2}(u_j-u_k)^2 - \tfrac{1}{2}s_{ij}s_{ki}(u_{ij}-u_{ki})^2 - (s_{ij}u_j + s_{ki}u_k)(u_{ij}-u_{ki})\right]\right\} , \tag{3.92b}$$

$$= K_{c+} \ . \tag{3.92c}$$

We have once more allowed the normalisations to be the same, $\mathcal{N}_{c+} = \mathcal{N}_{c-}$, yielding identical contributions to the propagator.

These results lead to the following proposition:

Proposition 3 *For the Lagrangian two-form (3.68), the surface propagator $K_\sigma(\partial\sigma)$ (3.70), correctly renormalised, is independent of the surface configuration σ, depending only on the boundary $\partial\sigma$.*

Proof

The combination of elementary moves above (3.80), (3.84), (3.88), (3.92), combined with the pop-up of figure 3.6, allows us to deform any surface σ to another topologically equivalent surface σ' by a series of elementary moves, without changing the exponent in the propagator. This free deformation gives us independence from the surface. □

The propagator (3.70) therefore depends only on the surface boundary $\partial\sigma$, and the field variables specified there - i.e. it is a function only of the boundary value problem. Note that since different topologies are specified by changes of the boundary, we have not considered these explicitly. This represents a quantum analogy to the Lagrangian closure property of the classical case. As in the classical case we have invariance of the action under deformations of the surface σ, so in the quantum case this carries over to invariance of the *propagator* under such deformations.

3.2.3 Uniqueness of the Surface Independent Lagrangian

The Lagrangian two-form (3.68) produces a propagator (3.70) which is independent of variations of the surface σ (proposition 3). In fact, it turns out that (3.68) is the unique, quadratic Lagrangian two-form such that this holds. Consider a general, 3-point, quadratic Lagrangian, imposing antisymmetry under interchange of i and j (required for the form structure)

$$\mathcal{L}_{ij}(u,u_i,u_j) = \tfrac{1}{2}a_{ij}u^2 + \tfrac{1}{2}b_{ij}u_i^2 - \tfrac{1}{2}b_{ji}u_j^2 + c_{ij}uu_i - c_{ji}uu_j + d_{ij}u_iu_j \ , \tag{3.93}$$

(compare (2.33). On the coefficients a_{ij}, b_{ij}, c_{ij}, d_{ij} subscripts i,j indicate dependence on the lattice parameters p_i, p_j, with the ordering of subscripts important (e.g. $a_{ij} := a(p_i,p_j) \neq a(p_j,p_i)$). The two-form structure requires that a_{ij} and d_{ij} are anti-symmetric under interchange of the parameters, $a_{ji} = -a_{ij}$, $d_{ji} = -d_{ij}$. Our interest is in the subset of Lagrangians that display the surface independence property for the propagator. We therefore look for conditions on the Lagrangian such that elementary moves will leave the contribution to the action (i.e. the exponent in the propagator) unchanged. We assume that external factors and even volume factors can be resolved by renormalisation, so that we only consider that part of the propagator in the exponent.

Consider the propagator for the general, quadratic Lagrangian (3.93) under elementary move (a), shown in figure 3.7. The contributions to the propagator, K_{a+} and K_{a-}, are calculated according to (3.80a) and (3.82a).

The integrations involved in these calculations fall into three types: they may appear in the exponent as quadratic, linear, or zero - leading to Gaussian, Dirac delta function, or volume factor integrals, respectively.

$$\int \mathrm{d}x\ e^{-iax^2} = \sqrt{\frac{\pi}{ia}}\ , \quad \int \mathrm{d}x\ e^{iax} = \delta(a)\ , \quad \int \mathrm{d}x = V\ . \tag{3.94}$$

However, a Dirac delta function would force linear dependence of field variables at different lattice points. This is an undesirable outcome and so we exclude this possibility: we will apply conditions to prevent such integrals arising.

In configuration (i) (figure 3.7(i)) the contribution to the propagator is

$$\begin{aligned} K_{a+} =& \mathscr{N}_{a+} \int \mathrm{d}u\ \exp\left\{ \frac{i}{\hbar} \left[\tfrac{1}{2}(a_{ij} + a_{jk} + a_{ki})u^2 \right.\right. \\ &+ \Big((c_{ij} - c_{ik})u_i + (c_{jk} - c_{ji})u_j + (c_{ki} - c_{kj})u_k \Big) u \\ &\left.\left. + \tfrac{1}{2}(b_{ij} - b_{ik})u_i^2 + cyclic + d_{ij}u_iu_j + cyclic \right] \right\} . \end{aligned} \tag{3.95}$$

The integration takes place over u, so that the cases for this integral divide on the totally antisymmetric parameter,

$$\mathfrak{a}_{ijk} := a_{ij} + a_{jk} + a_{ki}\ . \tag{3.96}$$

When $\mathfrak{a}_{ijk} \neq 0$, (3.95) is resolved by a Gaussian integral, such that

$$\begin{aligned} K_{a+,G} =& \mathscr{N}_{a+} \left(\frac{2\pi i\hbar}{\mathfrak{a}_{ijk}} \right)^{1/2} \exp\left\{ \frac{i}{\hbar} \left[\frac{1}{2}\Big(b_{ij} - b_{ik} - \frac{1}{\mathfrak{a}_{ijk}}(c_{ij} - c_{ik})^2 \Big) u_i^2 + cyclic \right.\right. \\ &\left.\left. + \Big(d_{ij} - \frac{1}{\mathfrak{a}_{ijk}}(c_{ij} - c_{ik})(c_{jk} - c_{ji}) \Big) u_iu_j + cyclic \right] \right\} . \end{aligned} \tag{3.97}$$

The subscript G indicates that this is the Gaussian case. The alternative case is that $\mathfrak{a}_{ijk} = 0$. In this case we require the integral to reduce to a volume factor - terms linear in u in the exponent of (3.95) must disappear - to prevent a Dirac delta function in the propagator. This requires the conditions

$$\begin{aligned} \mathfrak{a}_{ijk} &= 0 \quad , \quad & c_{ij} - c_{ik} &= 0\ , \\ \Rightarrow \quad a_{ij} &= a_i - a_j \quad , \quad & c_{ij} &= c_i\ . \end{aligned} \tag{3.98}$$

That is, a_{ij} must separate into p_i and p_j dependent parts, and c_{ij} must be a function of p_i only. Applying these conditions to (3.95) yields the contribution to the propagator

$$K_{a+,V} = V\mathscr{N}_{a+}\ \exp\left\{ \frac{i}{\hbar} \left[\frac{1}{2}(b_{ij} - b_{ik})u_i^2 + cyclic + d_{ij}u_iu_j + cyclic \right] \right\}\ , \tag{3.99}$$

where the subscript V indicates that this is the volume factor case. So for configuration (i), the propagator contribution K_a divides into two cases: (3.97) when $\mathfrak{a}_{ijk} \neq 0$ and (3.99) when $\mathfrak{a}_{ijk} = 0$.

Notice that under the conditions (3.98) for the volume factor case (when $\mathfrak{a}_{ijk} = 0$) we can write the Lagrangian as

$$\begin{aligned}\mathcal{L}_{ij}(u, u_i, u_j) &= \left(\tfrac{1}{2}a_i u^2 + c_i u u_i\right) - \left(\tfrac{1}{2}a_j u^2 + c_j u u_j\right) \\ &\quad + \tfrac{1}{2}(b_{ij}u_i^2 - b_{ji}u_j^2) + d_{ij}u_i u_j \ , \\ &= A_i(u, u_i) - A_j(u, u_j) + C_{ij}(u_i, u_j) \ , \end{aligned} \tag{3.100}$$

with $C_{ij}(u_i, u_j)$ antisymmetric under interchange of i and j. This is the general, quadratic, lattice Lagrangian two-form (2.28), (2.33) as found in [63].

Conversely, for configuration (ii) (figure 3.7(ii), equation (3.82a)) we have the propagator contribution

$$K_{a-} = V\mathscr{N}_{a-} \iiint \mathrm{d}^3\boldsymbol{u} \ \exp\left\{\frac{i}{\hbar}\left(\frac{1}{2}\boldsymbol{u}^T A\boldsymbol{u} + \boldsymbol{B}^t\boldsymbol{u} + \frac{1}{2}(a_{jk}u_i^2 + cyclic)\right)\right\} \ , \tag{3.101a}$$

where

$$\boldsymbol{u}^T = (u_{ij}, u_{jk}, u_{ki}) \ , \tag{3.101b}$$

$$A = \begin{pmatrix} b_{jk} - b_{ik} & d_{ki} & d_{jk} \\ d_{ki} & b_{ki} - b_{ji} & d_{ij} \\ d_{jk} & d_{ij} & b_{ij} - b_{kj} \end{pmatrix} \ , \tag{3.101c}$$

$$\boldsymbol{B}^T = (c_{jk}u_i - c_{ik}u_j, \mathrm{perm}\ (ijk), \mathrm{perm}\ (kji)) \ . \tag{3.101d}$$

This also separates into two distinct cases, resting on the critical point of $\det A$. For $\det A \neq 0$, the propagator contribution (3.101a) can be evaluated using the matrix form of Gaussian integration,

$$K_{a-,G} = V\mathscr{N}_{a-}\sqrt{\frac{(2\pi i\hbar)^3}{\det A}} \exp\left\{\frac{i}{\hbar}\left(-\tfrac{1}{2}\boldsymbol{B}^T A^{-1}\boldsymbol{B} + \left(\tfrac{1}{2}a_{jk}u_i^2 + cyclic\right)\right)\right\} \ . \tag{3.102}$$

This yields a very complicated expression! Alternatively, when $\det A = 0$, there are no longer three Gaussian integrations in (3.101a), and one of them must therefore reduce to a volume factor. $\det A = 0$ if

$$b_{ij} = -d_{ij} \ , \tag{3.103a}$$

so that b_{ij} is also anti-symmetric. Carrying out the integrations of (3.101a) in turn, we also find the condition for preventing a delta function integral,

$$c_{ij} - c_{ji} = 0 \qquad \forall i, j \ . \tag{3.103b}$$

I.e. c_{ij} must be symmetric. The $\det A = 0$ case then yields the propagator contribution

$$K_{a-,V} = V^2 \frac{2\pi\hbar}{(1-\Lambda_{ijk})^{1/2}} \mathscr{N}_{a-} \ \exp\left\{ \frac{i}{\hbar}\left[\tfrac{1}{2} a_{jk} u_i^2 + cyclic \right.\right.$$
$$\left.\left. - \frac{1}{2}\frac{d_{ij}}{1-\Lambda_{ijk}}(c_{jk}u_i - c_{ki}u_j)^2 + cyclic \right]\right\} , \tag{3.104}$$

where we have introduced the totally symmetric parameter

$$\Lambda_{ijk} := d_{ij}d_{jk} + d_{jk}d_{ki} + d_{ki}d_{ij} + 1 \ . \tag{3.105}$$

Configuration (ii) therefore also yields two cases for K_{a-}: (3.102) when $\det A \neq 0$ and (3.104) when $\det A = 0$.

We now compare the two configurations of the elementary move, and demand that the exponents from each configuration be the same, i.e. both K_{a+} (3.95) and K_{a-} (3.101a) make the same contribution to the propagator.

In the generic case ($\mathfrak{a}_{ijk} \neq 0$, $\det A \neq 0$) we compare equation (3.97) with (3.102). Comparing coefficients of u_i^2 and $u_i u_j$ in the exponent produces the functional equations on the coefficients of the Lagrangian,

$$b_{ij} - b_{ik} - \frac{1}{\mathfrak{a}_{ijk}}(c_{ij}-c_{ik})^2$$
$$= a_{jk} + \frac{1}{\det A}\Big\{ \big(d_{ij}^2 - (b_{ki}-b_{ji})(b_{ij}-b_{kj})\big)c_{jk}^2$$
$$+ \big(d_{ki}^2 - (b_{jk}-b_{ik})(b_{ki}-b_{ji})\big)c_{kj}^2 + 2\big(d_{ij}d_{ki} - d_{jk}(b_{ki}-b_{ji})\big)c_{jk}c_{kj} \Big\} , \tag{3.106a}$$

$$d_{ij} - \frac{1}{\mathfrak{a}_{ijk}}(c_{ij}-c_{ik})(c_{jk}-c_{ji})$$
$$= \frac{1}{\det A}\Big[\big((b_{ki}-b_{ji})(b_{ij}-b_{kj}) - d_{ij}^2\big)c_{jk}c_{ik} - \big(d_{ij}d_{jk} - d_{ki}(b_{ij}-b_{kj})\big)c_{jk}c_{ki}$$
$$+ \big(d_{jk}d_{ki} - d_{ij}(b_{jk}-b_{ik})\big)c_{ki}c_{kj} - \big(d_{ij}d_{ki} - d_{jk}(b_{ki}-b_{ji})\big)c_{ik}c_{kj} \Big] , \tag{3.106b}$$

along with cyclic permutations of these. It is not at all obvious that a solution to these equations, under the constraints on $\mathfrak{a}_{ijk}$ and $\det A$, exists.

However, there *is* a solution to the problem that exists at the critical point of the system, when $\mathfrak{a}_{ijk} = 0$ and $\det A = 0$. Considering the propagator contributions for these special cases (3.99) and (3.104), and comparing coefficients for u_i^2 and $u_i u_j$ in the exponents, we derive the conditions

$$b_{ij} - b_{ik} = a_{jk} - \frac{1}{1-\Lambda_{ijk}}(d_{ij} + d_{ki})c_{jk}^2 \ , \tag{3.107a}$$

$$d_{ij} = \frac{d_{ij}}{1-\Lambda_{ijk}} c_{jk}c_{ki} \ . \tag{3.107b}$$

Recall that we also have conditions preventing Dirac delta functions (3.98) and (3.103), which together imply

$$c_{ij} = c \ , \text{ constant.} \tag{3.107c}$$

The conditions (3.107a) and (3.107b) reduce to

$$\Lambda_{ijk} = 1 - c^2 \ , \quad a_{ij} = 0 \ . \tag{3.107d}$$

Finally, since the Lagrangian is defined only up to an overall multiple, we let $c = 1$ without loss of generality, so that $\Lambda_{ijk} = 0$. We therefore find the unique quadratic Lagrangian satisfying the requirements,

$$\mathcal{L}_{ij}(u, u_i, u_j) = u(u_i - u_j) - \tfrac{1}{2} d_{ij}(u_i - u_j)^2 \ , \tag{3.108}$$

with the condition on d_{ij} that $\Lambda_{ijk} = 0$. Comparing the definition of $\Lambda_{ijk} = 0$ (3.105) with the parameter identity on the s_{ij} (3.77), we see that this is precisely the condition

$$d_{ij} = s_{ij} \ . \tag{3.109}$$

So the resulting Lagrangian (3.108) is uniquely the Lagrangian two-form (3.68). We already know from section 3.2.2 that this Lagrangian also exhibits surface independence for the other elementary moves.

We summarise this result in the proposition:

Proposition 4 *We assume a valid renormalisation, and demand independence of field variables (i.e. excluding delta function integrals). Then, provided that we discount the generic case (3.106), the linear lattice Lagrangian two-form (3.68) is the unique quadratic lattice Lagrangian yielding a surface independent propagator (3.70).*

Proof

(3.108), with the restriction $\Lambda_{ijk} = 0$ (3.105), is the unique Lagrangian exhibiting surface independence for elementary move (a), and is identical to the Lagrangian (3.68). We also have from proposition 3 that Lagrangian (3.68) has surface independence under all other elementary moves. □

The principle of surface independence is therefore sufficient to determine the Lagrangian uniquely: even more so than in the classical case (section 2.1.3).

3.2.4 Quantum Variational Principle: Two dimensions

This result suggests a quantum variational principle in analogy to the one dimensional case of section 3.1.5. We consider the propagator over a discrete surface σ defined in (3.70),

$$K_\sigma(\partial\sigma) = \mathscr{N}_\sigma \int \prod_{\boldsymbol{n}\in\sigma} \mathrm{d}\mu(u(\boldsymbol{n}))\ e^{i\mathcal{S}_\sigma[u(\boldsymbol{n})]/\hbar}\ . \tag{3.110}$$

We have shown that, for the special choice of the Lagrangian two-form (3.68), the propagator $K_\sigma(\partial\sigma)$ is *independent* of the surface σ. It depends only on the variables sitting on the boundary, $\partial\sigma$. Additionally, this is a very unique choice of Lagrangian; for a generic Lagrangian, $K_\sigma(\partial\sigma)$ will depend also on the geometry of the surface σ itself.

Recall that, classically, the Lagrangian two-form structure arises from a variational principle *over surfaces* as in [63]. An extended set of Euler-Lagrange equations arise as we vary not only the dependent field variables $u(\boldsymbol{n})$, but also the surface σ. This restricts the class of admissible Lagrangians to those obeying the closure property (2.26): it is only for such Lagrangians and equations of motion that the classical action remains stationary under variations of the surface.

As we move to the quantisation, we consider the variation over all possible surfaces σ with a fixed boundary $\partial\sigma$, parallel to the argument in the one-form case. For a generic Lagrangian, as we vary the surface σ the propagator $K_\sigma(\partial\sigma)$ (3.70) changes. However, for the special "integrable" choice of Lagrangian two-form (3.68) the propagator $K_\sigma(\partial\sigma)$ remains unchanged as we vary the surface. The integrable choice of Lagrangian therefore represents a critical, indeed singular, point for some new quantum object in a space of possible Lagrangian functions: a "sum over surfaces" of all possible surface dependent propagators,

$$K(\partial\sigma) = \sum_{\sigma\in\mathfrak{S}} \mathscr{N}_\sigma K_\sigma(\partial\sigma)\ , \tag{3.111}$$

where $\mathfrak{S}$ is the set of all surfaces with boundary $\partial\sigma$, and $\mathscr{N}_\sigma$ signifies some normalisation and regularisation. As in the one-form case, this formula does not yet have a meaning -

but at this stage merely illustrates the concept. The key idea is that this sum over surfaces somehow forces the integrable, two-form choice of Lagrangian through the dominance of the singular point. As in the one-form case, how to control the singular behaviour, or write a concise mathematical definition, of such an object is currently unknown. Nonetheless, such an object may be the natural quantum analogue of the classical Lagrangian variational principle for two-forms, and could perhaps form an ingredient for the quantisation of lattice models that are integrable in the sense of multi-dimensional consistency.

The idea of a quantisation by a sum over surfaces is not a new one, see [92, 93], but these are concerned with a sum over topologically inequivalent surfaces. By demanding a fixed boundary, we restrict to surfaces that are topologically equivalent, and seek a Lagrangian with the special property of allowing a resolution to the sum over surfaces. However, the authors note in [93] that the contribution of a surface for their models depends only on its topology: for the special case of the Lagrangian two-form we have the same result.

3.3 Summary

By quantising the linear models of chapter 2 we have begun to form a picture of the quantum analogue to Lagrangian multiform structures. The mapping equations of the discrete harmonic oscillator were reinterpreted as operator equations of motion for canonical quantum mechanics, leading to a time evolution operator. Examination of the propagator then showed that this is equivalent for these cases to the Lagrangian (path integral) approach. The commuting discrete flows of the classical model led to multiple time evolutions in the quantum case. We discovered that the propagator for such multiple-time evolutions is independent of the path taken through the time variables: that is, it depends only on the endpoints of the path; this is a quantum analogue to the Lagrangian closure condition. Remarkably, although the classical Lagrangian closure holds only on the equations of motion, the quantum equivalent holds over the whole sum over histories (i.e. despite the redundancy of the equations of motion in the quantum regime). Further, we then found that in the case of quadratic Lagrangians, this is uniquely true for the Lagrangian one-form structure of chapter 2.

We have begun to extend some of this work into the 2 dimensional case (paired discrete harmonic oscillators), where the time evolution operator must be written with an interaction term $\exp(-i\mathbf{x}_1\mathbf{X}_2/\hbar)$. There is more to consider in this case, in particular how the time evolution operators of the commuting discrete flows relate to the commuting invariants of the model. The multiple invariants also makes this system a candidate for investigating the behaviour of the Lagrangian one-form in a continuum limit.

To quantise the two-form case, we considered propagators over a surface within the multi-dimensional lattice, where a path integral quantisation is enacted by integrating over the interior field variables. Motivated by the discovery that "popping up" a cube onto a flat surface does not change the propagator, we discovered that for the Lagrangian two-form of chapter 2 the surface propagator is independent of the geometry of the surface: it depends only on the boundary, and the field variables found there. This result directly corresponds with the time-path independence of the one-form case, and is also a quantum analogy to the Lagrangian closure property. Classically the Lagrangian two-form is closed only on the equations of motion, but in the quantum case this closure holds within the path integral, despite leaving the equations of motion behind. Further, we found that the Lagrangian two-form for the linear lattice equations is precisely the unique quadratic three-point lattice Lagrangian such that this holds, including specifying the necessary parameter.

These path- and surface-independence results lead us to propose a quantum multiform variational principle: by integrating over all possible time paths, or all possible surfaces, such an object could perhaps select exactly the integrable Lagrangian form structures. Understanding, and taming, such an object is beyond the scope of this thesis.

4

Generalised McMillan Maps: Commuting flows and r-matrix structure

In chapters 2 and 3, we extended the Lagrangian multiform structure in a new way to discrete linear models, which yielded insights into the possible operation of such structures in a quantum regime. The important and unresolved question is how to understand a quantum Lagrangian multiform in the non-linear (integrable) case. This turns out to be a complicated problem, which we are not able to solve in this thesis. Two distinct types of model suggest themselves as the most feasible candidates for study: on the one hand, models of Calogero-Moser type have a known integrable discrete structure and Lagrangian one-form, which are the essential ingredients of interest [78, 125]. On the other hand, we have discrete mappings arising from integrable lattice equations, the non-linear relative of the mappings considered in chapters 2 and 3. In this chapter we consider the latter option. Although the one-form structure (even classically) remains elusive for this model, we find commuting discrete flows by following the method of chapter 2. We investigate the dual Lax pair as a possible route to understanding these commuting flows, and find a novel realisation of the r-matrix structure.

The so-called generalised McMillan maps arise as reductions from the lattice KdV equation (1.7), and are an integrable family of maps of dimension $2N$ [24, 79, 80, 91]. The linear maps of chapter 2 arise essentially as their linearisation. As the multi-dimensional consistency of the linear lattice equation resulted in commuting discrete flows for the linear map, we might hope that the multi-dimensional consistency of the lattice KdV equation [1, 75, 80] will ultimately lead to commuting discrete flows for the generalised McMillan maps. The lattice KdV equation has a known Lagrangian two-form structure [62, 64], which makes its reductions a very natural place to extend the work of chapter 2 and to seek a Lagrangian one-form.

A significant advantage of these maps is that, although non-linear, they have a *Newtonian form*: they are generated by a Lagrangian of the type $\mathcal{L} = T - V$. As a consequence, in the quantum regime the time evolution operator U (1.71) can be written in a separated form (1.74) (compare (3.5)), which leads naturally to the construction of a discrete path integral, beginning from the canonical perspective. In contrast, such a form is not known for maps of Calogero-Moser type.

As integrable maps, the generalised McMillan maps have a family of commuting invariants. These also suggest the possibility of a compatible commuting flow, although unlike in the Calogero-Moser case the invariant flows are not known to arise from a continuum limit on the mapping. There is a well known Lax representation with a corresponding integrable quantum structure for these maps [46, 74, 76]; the maps retain integrability on the quantum level, which perhaps can be uncovered in the path integral regime. Furthermore, it is not currently known how the Lax structure relates to a variational formulation for most models, and in particular how this might be manifest in a path integral quantisation: the generalised McMillan maps offer a possible avenue for this research. Consideration of the spectral curves and Sklyanin variables [104] for these maps also leads to a semi-linearisation, even on the quantum level, in the *Dubrovin equations* for the map [39, 70], through the study of a quantum determinant. Perhaps such approaches might offer further insight into a path integral quantisation?

In this chapter we consider some issues around the generalised McMillan map, working towards a deeper understanding of the Lagrangian one-form and path integral structures. In section 4.1 we briefly overview some known classical and quantum results for these models that will set the scene for the rest of the chapter. In section 4.2 we consider commuting flows to the map, following chapter 2. For this non-linear case, we find that such commuting flows are significantly more complicated, so that it is not straightforward to express these flows via a generating function. In section 4.3 we examine some early steps to the path integral quantisation of the simplest member of the family, the McMillan map. Although a quantum one-form structure is not yet clear, it is possible to make some observations which point towards how the Hilbert space for such a model might look, and the relevance of the choice of Lagrangian. Finally, we examine the dual (or big) Lax pair for the generalised McMillan maps and its r-matrix structure, uncovering some new details that may lead to better insights into the structure of the model. The big Lax pair offers another way of considering the time-evolution of the map, whose connection to the Lagrangian perspective is not currently known.

4.1 The Generalised McMillan Maps

4.1.1 Staircase reductions from the Lattice KdV Equation

In chapter 2, we described a linearisation of the lattice KdV equation, a multi-dimensionally consistent, quadrilateral equation on a square lattice (2.3). By applying a periodic initial value problem along a staircase in the lattice (figure 2.8) the system of quad equations was reduced to a linear, finite-dimensional mapping. In other words, a discrete-time system. Although in chapter 2 we were primarily interested in the linearised system, such reductions were first studied for the parent equation, and other integrable lattice equations [24, 80]. Recall the lattice KdV equation (1.7), which holds on elementary plaquettes in the lattice on variables $w_{n,m}$,

$$(\delta + w_{n,m+1} - w_{n+1,m})(\epsilon - w_{n+1,m+1} + w_{n,m}) = \epsilon\delta \ . \tag{4.1}$$

In terms of the lattice parameters p and q, we use the notation $\delta = p - q, \ \epsilon = p + q$.

As in figure 2.8, we apply a periodic initial value problem along a staircase of length $2P$, introducing initial values

$$w_{j,j} =: a_{2j} \ , \quad w_{j+1,j} =: a_{2j+1} \ , \quad \text{for } j = 1, \ldots, P \ , \quad a_{i+2P} = a_i \ , \tag{4.2}$$

noting the periodic condition (compare [41], where non-periodic boundary conditions are used for the same problem). The lattice KdV equation (4.1) then describes on each elementary quadrilateral the evolution of the variables a_j. Evolution in the m (or hat) direction is taken to correspond to the mapping as a discrete time evolution (note that we could equivalently consider the evolution of n in the tilde direction; these two possibilities were compared in [70]). We therefore have evolution equations for the a_j,

$$\widehat{a}_{2j} = a_{2j+1} - \delta + \frac{\epsilon\delta}{\epsilon - a_{2j+2} + a_{2j}} \ , \quad \widehat{a}_{2j+1} = a_{2j+2} \ . \tag{4.3}$$

We introduce "reduced variables" v_i such that

$$v_i := \epsilon + a_i - a_{i+2} \ , \quad i = 1, \ldots, 2P \ , \tag{4.4}$$

so that the evolution of the a_j (4.3) yields the equations of motion

$$\widehat{v}_{2j-1} = v_{2j} \ , \qquad \widehat{v}_{2j} = v_{2j-1} + \frac{\epsilon\delta}{v_{2j}} - \frac{\epsilon\delta}{v_{2j+2}} \ . \tag{4.5}$$

It is easy to see that the map has two Casimirs,

$$\sum_{i=1}^{P} v_{2i} = \sum_{i=1}^{P} v_{2i-1} = P\epsilon \ , \tag{4.6}$$

so that this staircase initial value problem produces a $2P-2$ dimensional map.

This is an alternative choice of reduction variables to the $\{x_i, y_j\}$ used in (2.112), which have a *non-ultralocal* Poisson bracket structure,

$$\{v_i, v_j\} = \delta_{i+1,j} - \delta_{i,j+1} \ . \tag{4.7}$$

In other words, only nearest neighbours of the v_i have non-trivial Poisson brackets. Note that is is possible to eliminate the odd (or even) labelled v_i from the map, and hence write the mapping equations (4.5) as a canonical map in a Lagrangian form.

Recall from section 1.1.1 that the multidimensional consistency of the lattice KdV equation (4.1) leads to a Lax representation (1.20), (1.21). The staircase reduction (4.2) can be applied on the level of this Lax pair, producing a local Lax pair for the mapping (4.5). The mapping equations are produced by the Zakharov-Shabat condition,

$$\widehat{L}_n(\lambda) \,.\, M_n(\lambda) = M_{n+1}(\lambda) \,.\, L_n(\lambda) \ , \quad n = 1, \ldots, N \ , \tag{4.8}$$

for matrices,

$$L_j(\lambda) = V_{2j} V_{2j-1} \ , \text{ where } V_j = \begin{pmatrix} v_j & 1 \\ l_j & 0 \end{pmatrix} \ , \tag{4.9a}$$

$$M_j(\lambda) = \begin{pmatrix} v_{2j-1} - \epsilon\delta v_{2j}^{-1} & 1 \\ \lambda & 0 \end{pmatrix} \ , \tag{4.9b}$$

where $l_{2j} := \lambda$ and $l_{2j-1} := \lambda - \epsilon\delta$, with λ the spectral parameter. The invariants of the mapping arise through the monodromy matrix

$$T(\lambda) = L_N(\lambda) \dots L_1(\lambda) \ , \tag{4.10}$$

where the Zakharov-Shabat condition (4.8) ensures that the trace of $T(\lambda)$ will be invariant under the map (4.5). "Sufficiently many" (i.e. $P-1$) invariants arise by expanding $\mathrm{tr}T(\lambda)$ in powers of λ. We can equivalently consider the spectral curve,

$$\det(\eta\mathbb{I} - T(\lambda)) = \eta^2 - \mathrm{tr}T(\lambda)\eta + \det T(\lambda) = 0 \ , \tag{4.11}$$

which yields the invariants in the same way.

4.1.2 Classical r-matrix structures

The generalised McMillan mappings (4.5) have sufficiently many independent invariants for integrability, guaranteed by the existence of the monodromy matrix $T(\lambda)$ (4.10). However, for the mappings to be integrable, these invariants must be in involution: this is shown by constructing an r-matrix structure for the Lax pair [79].

We introduce tensor product spaces for the local Lax matrices of (4.9). The space of 2×2 matrices $GL(2)$ is embedded in a product space, $GL(2) \times GL(2)$. Index notation indicates which space a particular matrix sits in,

$$\overset{1}{A} \equiv A_1 := A \otimes \mathbb{I} \ , \quad \overset{2}{A} \equiv A_2 := \mathbb{I} \otimes A \ . \tag{4.12}$$

Matrices labelled with multiple indices, such as r_{12}, sit across both tensor spaces, and are represented as 4×4 matrices. Occasionally we will also need to use a third (or higher) matrix product space, but the notation generalises in the obvious way. Note that typically a different spectral parameter is associated with each element in the product space, for example

$$L_{n,1} := L_n(\lambda_1) \otimes \mathbb{I} \ , \quad L_{n,2} := \mathbb{I} \otimes L_n(\lambda_2) \ , \tag{4.13}$$

where λ_1 and λ_2 are distinct spectral parameters. In the notation, this dependence is often left implicit.

Using this notation, the Poisson bracket structure of the v_i (4.7) can be encoded on the Lax matrices (4.9) with the use of matrices $r_{12}^{\pm}$ and $s_{12}^{\pm}$, that live in the tensor product space [74, 79],

$$\{L_{n,1}, L_{m,2}\} = -\delta_{n,m+1} L_{n,1} s_{12}^{+} L_{m,2} + \delta_{n+1,m} L_{m,2} s_{12}^{-} L_{n,1} + \delta_{n,m}\big(r_{12}^{+} L_{n,1} L_{m,2} - L_{n,1} L_{m,2} r_{12}^{-}\big) \ . \quad (4.14)$$

The matrices $r_{12}^{\pm}$ and $s_{12}^{\pm}$ are given in [74]. The requirements on the Poisson bracket for skew-symmetry and the Jacobi identity yield conditions on $r_{12}^{\pm}$ and $s_{12}^{\pm}$ that must hold, most notably the classical Yang-Baxter equation,

$$[r_{12}^{\pm}, r_{13}^{\pm}] + [r_{12}^{\pm}, r_{23}^{\pm}] + [r_{13}^{\pm}, r_{23}^{\pm}] = 0 \ , \quad (4.15)$$

which is a classical limit of the famous Yang-Baxter equation [53, 101]. The usual difficulty in finding such r-matrix structures is in identifying solutions to the Yang-Baxter equation.

A consequence of the r-matrix structure (4.14) is that traces of the monodromy matrix $\mathrm{tr}T(\lambda)$ (4.10) are in involution,

$$\{\mathrm{tr}T(\lambda_1), \mathrm{tr}T(\lambda_2)\} = 0 \ . \quad (4.16)$$

But therefore the invariants of the system are in involution, and hence the generalised McMillan maps are integrable in the Arnol'd-Liouville sense [22].

4.1.3 The Quantum Map

In the quantum regime, the reduced variables v_i (4.4) become *operators* $\mathbf{v}_i$, with the Poisson bracket (4.7) replaced by a commutator bracket,

$$\big[\mathbf{v}_i, \mathbf{v}_j\big] = i\hbar(\delta_{i+1,j} - \delta_{i,j+1}) \ . \quad (4.17)$$

The mapping equations of motion (4.5) become operator equations of motion, under the assumption that we can create the inverse operator $\mathbf{v}_i^{-1}$ [46, 74, 76],

$$\widehat{\mathbf{v}}_{2j-1} = \mathbf{v}_{2j} \ , \qquad \widehat{\mathbf{v}}_{2j} = \mathbf{v}_{2j-1} + \epsilon\delta\mathbf{v}_{2j}^{-1} - \epsilon\delta\mathbf{v}_{2j+2}^{-1} \ . \quad (4.18)$$

We will not consider the analysis of this assumption at length.

Although the equations of motion transfer to the quantum regime in an obvious way, the integrability of the quantum map is more difficult. The Lax pair (4.9) is also a Lax pair for the quantum mapping, so long as care is taken with operator ordering. But, the identification of invariants is more difficult. Due to the non-commuting operators, the trace of the monodromy matrix tr$T(\lambda)$ (4.10) is no longer preserved under the mapping: a more subtle approach is needed.

Integrability follows from a quantum R-matrix structure which encodes the commutation relations on the level of the Lax matrices (4.9), using the same tensor notation as previously. We have matrices $R^{\pm}_{12}$, S^{+}_{12} such that

$$\begin{aligned} R^{+}_{12} L_{n,1} \cdot L_{n,2} &= L_{n,2} \cdot L_{n,1} R^{-}_{12} \ , & \text{(4.19a)} \\ L_{n+1,1} \cdot S^{+}_{12} L_{n,2} &= L_{n,2} L_{n+1,1} \ , & \text{(4.19b)} \\ L_{n,1} L_{m,2} &= L_{m,2} L_{n,1} \ , \quad \text{for } |n-m| \geq 2 & \text{(4.19c)} \end{aligned}$$

recalling the periodicity in $n = 1, \ldots, P$. The matrices $R^{\pm}_{12}$, S_{12} are given in [74]. The same structure also holds for the other part of the Lax pair, M_j (4.9b). These equations describe the commutation of the operators within the Lax matrices.

Proving the existence and commutativity of the invariants in the quantum case depends on the object

$$\tau(\lambda) = \mathrm{tr}\big(T(\lambda)K(\lambda)\big) \ , \quad \text{where } K(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 1 + i\hbar/\lambda \end{pmatrix} \ . \tag{4.20}$$

The R-matrix relations (4.19) guarantee that $\tau(\lambda)$ is preserved under the mapping, $\widehat{\tau}(\lambda) = \tau(\lambda)$, and that as operators $\tau(\lambda)$ commute for different choices of the spectral parameter,

$$\big[\tau(\lambda_1), \tau(\lambda_2)\big] = 0 \ . \tag{4.21}$$

In other words, $\tau(\lambda)$ produces the quantum invariants of the model, and they are commuting. The proof of these results is by no means trivial [74]! The effect of the matrix $K(\lambda)$ (4.20) is that these invariants have a "quantum correction" from the classical invariants, such that they reduce to the classical invariants in the limit $\hbar \to 0$. Notice that in the quantum case the R-matrix structure is needed not only to prove commutativity of the invariants, but also to derive the invariants themselves [103].

4.2 Commuting Discrete Flows

In section 2.2 we discovered commuting discrete flows for the staircase reductions of the linearised lattice KdV equation. The multi-dimensional consistency of the parent lattice equation allowed us to extend the staircase reduction into a third direction in the lattice, producing a second mapping. The map was consistent with the initial mapping as a consequence of the closure-around-the-cube property. The generalised McMillan maps of this chapter are staircase reductions from the lattice KdV equation (section 4.1.1) which has the same key property of multi-dimensional consistency. Although the non-linearity means that the commuting flow equations will be more complex, we can search for compatible maps in the same manner as the linear case. This is potentially a fruitful avenue of research as, unlike in the linear case, the generalised McMillan maps possess a known meaningful Lax representation encoding the invariants.

To simplify the calculations, we first consider the simplest possible reduction: the "$P = 1.5$" mapping shown in figure 2.4. We begin with initial values a_0, a_1, a_2 and apply the periodic boundary condition $\widehat{a}_2 = a_0$. The lattice equation (4.1) then yields equations for the mapping

$$(p - q + \widehat{a}_0 - a_1)(p + q + a_0 - \widehat{a}_1) = p^2 - q^2 \ , \tag{4.22a}$$

$$(p - q + \widehat{a}_1 - a_2)(p + q + a_1 - a_0) = p^2 - q^2 \ , \tag{4.22b}$$

$$\widehat{a}_2 = a_0 \ . \tag{4.22c}$$

As in the linear case, we introduce the reduction variables x and y,

$$x := a_1 - a_0 \quad , \quad y := a_2 - a_1 \ , \tag{4.23}$$

in terms of which the mapping equations (4.22) become

$$\widehat{x} = y + \frac{p^2 - q^2}{p + q + x} - \frac{p^2 - q^2}{p + q + \widehat{y}} \ , \tag{4.24a}$$

$$\widehat{y} = p - q - x - y + \frac{p^2 - q^2}{p + q + x} \ , \tag{4.24b}$$

so that it is clear $\{\widehat{x}, \widehat{y}\}$ can be calculated from $\{x, y\}$.

This map is most naturally expressed in Hamiltonian form, in terms of the variables x and $\widehat{y}$, as

$$y = p - q - x - \widehat{y} + \frac{p^2 - q^2}{p + q + x} \ , \tag{4.25a}$$

$$\widehat{x} = p - q - x - \widehat{y} - \frac{p^2 - q^2}{p + q + \widehat{y}} \ , \tag{4.25b}$$

which can be derived from the Hamiltonian-type generating function

$$F(x, \widehat{y}) = (p - q)(x + \widehat{y}) - \tfrac{1}{2}(x + \widehat{y})^2 + (p^2 - q^2) \log\left(\frac{p + q + x}{p + q + \widehat{y}}\right) \ . \tag{4.26}$$

This guarantees the symplectic structure of the map, such that $\mathrm{d}x \wedge \mathrm{d}y = \mathrm{d}\widehat{x} \wedge \mathrm{d}\widehat{y}$. Notice that this form is somewhat more complicated than the usual maps of generalised McMillan type (4.5).

Following section 2.2.2 we extend the mapping into a third lattice direction, illustrated in figure 2.5. The lattice equation (4.1) is first embedded within the multi-dimensional lattice,

$$(p_i - p_j + w_j - w_i)(p_i + p_j + w - w_{ij}) = p_i^2 - p_j^2 \ , \tag{4.27}$$

where p_i indicates the lattice parameter associated to the lattice direction i, and w_i indicates a shift of the lattice variable w in the i direction. This yields the set of equations for the staircase variables a_i,

$$(p - r + \overline{a}_0 - a_1)(p + r + a_0 - \overline{a}_1) = p^2 - r^2 \ , \tag{4.28a}$$

$$(p - r + \overline{a}_1 - a_2)(p + r + a_1 - \overline{a}_2) = p^2 - r^2 \ , \tag{4.28b}$$

$$(q - r + \overline{a}_2 - a_0)(q + r + a_2 - \overline{a}_0) = q^2 - r^2 \ . \tag{4.28c}$$

We wish to write this map in terms of the reduced variables x and y (4.23), but now the non-linearity of the mapping equations (4.28) makes this rather more difficult than in the linear case.

By eliminating $\overline{a}_1$ and $\overline{a}_2$, it is possible to write a quadratic for $\overline{a}_0$ in terms of the initial conditions,

$$\begin{aligned} A\overline{a}_0^2 - \Big[(2r + q - p + a_1 + a_2)A + (r^2 - q^2)(2p - x - y) + (p^2 - r^2)(p + q + x)\Big]\overline{a}_0 \\ - (p - r - a_1)\big[(r + q + a_2)A + (r^2 - q^2)(2p - x - y)\big] \\ + (p^2 - r^2)\big[(p + q + x)(r + q + a_2) + r^2 - q^2\big] = 0 \ , \end{aligned} \tag{4.29a}$$

where

$$A := (2p - x - y)(p + q + x) - p^2 + r^2 \ . \tag{4.29b}$$

This yields the double-valued expression for $\overline{a}_0$,

$$\overline{a}_0 = \frac{1}{2A}\left((2r + q - p + a_1 + a_2)A + C \pm \sqrt{B}\right) \ , \tag{4.30a}$$

where we have added the shorthand

$$\begin{aligned} B := \Big[(p + q + y)A + (r^2 - q^2)(2p - x - y) \\ -(p^2 - r^2)(p + q + x)\Big]^2 + 4(p^2 - r^2)^2(r^2 - q^2) \ , \end{aligned} \tag{4.30b}$$

$$C := (r^2 - q^2)(2p - x - y) + (p^2 - r^2)(p + q + x) \ . \tag{4.30c}$$

The mapping equations (4.28) then allow the derivation of similar expressions for $\overline{a}_1$ and $\overline{a}_2$,

$$\overline{a}_1 = p + r + a_0 - \frac{2(p^2 - r^2)A}{(p+q+y)A + C \pm \sqrt{B}} , \tag{4.31a}$$

$$\overline{a}_2 = r - q + a_0 - \frac{2(r^2 - q^2)A}{(p+q+y)A - C \mp \sqrt{B}} , \tag{4.31b}$$

noting that the multi-valuedness of these expressions is determined by the single choice in $\overline{a}_0$.

These expressions lead finally to expressions for the commuting map in terms of the reduction variables, for $\overline{x}, \overline{y}$ in terms of x, y,

$$\begin{aligned} \overline{x} &= \overline{a}_1 - \overline{a}_0 , \\ &= \frac{(3p - q - 2x - y)A - C \mp \sqrt{B}}{2A} - \frac{2(p^2 - r^2)A}{(p+q+y)A + C \pm \sqrt{B}} , \end{aligned} \tag{4.32a}$$

$$\begin{aligned} \overline{y} &= \overline{a}_2 - \overline{a}_1 , \\ &= -(p+q) + \frac{2(p^2 - r^2)A}{(p+q+y)A + C \pm \sqrt{B}} - \frac{2(r^2 - q^2)A}{(p+q+y)A - C \mp \sqrt{B}} . \end{aligned} \tag{4.32b}$$

The new variables $\overline{x}, \overline{y}$ can be calculated from these expressions, but are *dual-valued*: there is a choice of sign for the square root. In this form, the map appears to be double valued.

The compatibility of the map $\{x, y\} \rightarrow \{\overline{x}, \overline{y}\}$ with the map $\{x, y\} \rightarrow \{\widehat{x}, \widehat{y}\}$ is guaranteed by the multi-dimensional consistency of the parent equation (4.1), but other pertinent questions such as whether the map is symplectic, or preserves invariants of any kind, are harder to answer at present. We would hope to express the mapping via a generating function, but the form of the mapping equations (4.32) doesn't seem amenable to this; however, it is possible that a better form may exist. The lattice KdV equation from which the mapping originates possesses a Lax pair, with a matrix associated to every lattice direction - discussed in section 1.1.1. Indeed, the Local Lax representation for the generalised McMillan maps (4.9) is derived from this lattice Lax. In deriving the commuting flows above, we have not yet exploited this Lax representation; it is possible that this may resolve some of the present difficulties. More research is needed into these commuting maps if multi-form structures analogous to the linear case are to be uncovered.

4.3 Quantum McMillan Map

For the generalised McMillan maps there is a well established canonical quantum structure, discussed in section 4.1.3, via the quantum inverse scattering method. We have found that, classically, establishing commuting flows for these non-linear maps is more difficult than in the linear case. Approaching a path integral quantisation for these non-linear maps is also more challenging: in the linear case (chapter 3) we were able to make use of repeated Gaussian integrals, but such a technique no longer works in this non-linear case.

In the section below, we consider the simplest member of the family of mapping reductions: the McMillan map. Considering this simple, two dimensional map allows us to explore questions regarding the path integral quantisation for a non-linear mapping, examining forms for the time evolution operator and the resulting propagator, and considering the behaviour of the operator invariant. This is essential groundwork to prepare the way for studying Lagrangian one-form path integrals.

4.3.1 The McMillan map: $P = 2$

Recall that the maps we have been studying in this chapter arise from staircase reductions of the lattice KdV equation, as in section 4.1.1. The simplest non-trivial such mapping arises when $P = 2$: this yields the famous "McMillan map" [66]. This map can be expressed in terms of the variables v_1, v_2 (4.4), but it will be more convenient in this section to describe the map in terms of the alternative reduction variable q_n, where

$$q_n = \frac{a_2 - a_0}{\epsilon} \ . \tag{4.33}$$

The subscript n labels the discrete time evolution, so that $\widehat{q}_n = q_{n+1}$, and so forth. The evolution of the a_i (4.3) yields the equation of motion

$$q_{n+1} + q_{n-1} = \frac{2\alpha q_n}{1 - q_n^2} \ , \quad \text{where } \alpha := -\delta/\epsilon \ , \tag{4.34}$$

which is a well known form for the McMillan map.

The McMillan map (4.34) is symplectic, since it arises from a variational principle on a discrete Lagrangian,

$$\mathcal{L}(q, \widehat{q}) = -q\widehat{q} - \alpha \ln(1 - q^2) \ , \tag{4.35}$$

where the equation of motion results from the discrete-time Euler-Lagrange equations (1.36) (we have suppressed the subscript n for ease of notation). It is easy to check that the map has an invariant,

$$I_n(q) = (1 - q_n^2)(1 - q_{n-1}^2) + 2\alpha q_n q_{n-1} \ . \tag{4.36}$$

As such, the McMillan map is an integrable mapping of standard type, corresponding to a specific parameter choice of the type (i) mapping described by Suris [109].

Using the functional relation

$$\mathrm{sn}(a+b) = \frac{\mathrm{sn}a\,\mathrm{cn}b\,\mathrm{dn}b + \mathrm{sn}b\,\mathrm{cn}a\,\mathrm{dn}a}{1 - k^2\,\mathrm{sn}^2 a\,\mathrm{sn}^2 b} \ , \tag{4.37}$$

the McMillan map (4.34) can be explicitly solved in terms of the elliptic sn function,

$$q_n = A\,\mathrm{sn}(\zeta n + \xi_0) \ , \tag{4.38a}$$

where

$$\alpha = \mathrm{cn}\zeta\,\mathrm{dn}\zeta \ , \quad A^2 = k^2\,\mathrm{sn}^2\zeta \ . \tag{4.38b}$$

k is the modulus of the elliptic function, and ζ represents the discrete time step. From the perspective of a discrete time evolution, this solves the mapping for all time. Comparing this to the classical solution for the discrete harmonic oscillator (2.43), $x(t) = A\sin(\delta t + \theta)$, the McMillan solution (4.38a) is in some sense an elliptic generalisation of the linear case.

4.3.2 Quantisation of the map

In section 4.4.4, we consider the canonical quantisation of the generalised McMillan maps using an R-matrix. Following chapter 3 we would like to investigate a sum-over-histories quantisation, to extend the quantum Lagrangian one-form ideas into a non-linear example. Very little has been written on the path integral quantisation of integrable maps, although some early work exists in [38, 40] which we will build on in this section.

We consider the simple case of the McMillan map given in (4.34). In the quantum case, this as a map of quantum operators [76],

$$(\mathbf{q}_{n+1} + \mathbf{q}_{n-1})(1 - \mathbf{q}_n^2) = 2\alpha\mathbf{q}_n \ . \tag{4.39}$$

Bold symbols here denote operators. This quantum McMillan map is an operator equation of motion in the Heisenberg picture. However, making sense of such an operator equation

(4.39) is problematic. The Hilbert space on which these operators act is, a priori, unknown. In order to understand this quantum mechanical system, we need to understand the Hilbert space.

To understand the quantum McMillan equation (4.39) requires commutation relations between the position operators $\mathbf{q}_n$ at different times n. These commutation relations are defined by the conjugate momenta, and so the interpretation depends in an essential way on the choice of discrete Lagrangian. A different Lagrangian yields different conjugate momenta, hence different commutation relations. The question remains: what is the *correct* choice of Lagrangian?

A natural choice of Lagrangian that expresses a relation to standard approaches in continuous time[1] is given by [74]

$$\mathcal{L}(q_n, q_{n+1}) = \frac{1}{2}(q_{n+1} - q_n)^2 - q_n^2 - \alpha \log(1 - q_n^2)\ . \tag{4.40}$$

This Lagrangian has a kinetic and potential part,

$$\mathcal{L} = T(q_{n+1} - q_n) - V(q_n)\ , \tag{4.41}$$

in a Newtonian form. It is straightforward to check that discrete Euler-Lagrange equations yield the mapping equation (4.39). The Lagrangian (4.40) yields canonical momenta

$$p_{n+1} \ := \ \frac{\partial \mathcal{L}}{\partial q_{n+1}} \ = \ q_{n+1} - q_n\ , \tag{4.42a}$$
$$p_n \ := \ -\frac{\partial \mathcal{L}}{\partial q_n} \ = \ q_{n+1} + q_n - \frac{2\alpha q_n}{1 - q_n^2}\ , \tag{4.42b}$$

which endow the model with canonical commutation relations,

$$[\mathbf{q}_n, \mathbf{p}_n] = i\hbar\ . \tag{4.43}$$

The momentum equations (4.42) then define commutations between the position operators $\mathbf{q}_n$ at different times, so that we have as a consequence of (4.43),

$$[\mathbf{q}_{n-1}, \mathbf{q}_n] = i\hbar\ . \tag{4.44}$$

We can then show that the two factors of the mapping equation (4.39) commute,

$$(\mathbf{q}_{n+1} + \mathbf{q}_{n-1})(1 - \mathbf{q}_n^2) = (1 - \mathbf{q}_n^2)(\mathbf{q}_{n+1} + \mathbf{q}_{n-1})\ , \tag{4.45}$$

[1] This Lagrangian is of Newtonian form, and has a kinetic term that goes to $\dot{q}^2/2$ in a simple continuum limit.

which is essential for the consistency of the equation.

Having established commutation relations for the $\mathbf{q}_n$, we can also find the quantum correction to the classical invariant (4.36), such that $\mathbf{I}_{n+1} = \mathbf{I}_n$ is the *quantum* invariant,

$$\mathbf{I}_n = (1-\mathbf{q}_n^2)(1-\mathbf{q}_{n-1}^2) + 2(\alpha + i\hbar)\mathbf{q}_n\mathbf{q}_{n-1} \ . \tag{4.46}$$

The $\hbar$ correction addresses the operator ordering issues; but defining the invariant is therefore dependent on establishing commutation relations (4.44), or equivalently on choosing a generating Lagrangian. Such invariants (including in the McMillan case) were also considered in the paper [40], from a single-step path integral perspective, which we have included for the linear case in section 3.1.2.

4.3.3 Unitary operator and One-step Propagator

In [74], the authors used the Lagrangian (4.40) and the resulting canonical momenta (4.42) to describe the quantum McMillan map in terms of a time-evolution operator U, such that

$$\mathbf{p}_n \ \mapsto \ \mathbf{p}_{n+1} \ = \ U^{-1}\mathbf{p}_n U \ , \tag{4.47a}$$

$$\mathbf{q}_n \ \mapsto \ \mathbf{q}_{n+1} \ = \ U^{-1}\mathbf{q}_n U \ . \tag{4.47b}$$

The Newtonian form of the Lagrangian (4.41) means that it is straightforward to write U in the separated form,

$$U = e^{-iT(\mathbf{p})/\hbar}e^{-iV(\mathbf{q})/\hbar} \ , \tag{4.48a}$$

where

$$T(\mathbf{p}) = \tfrac{1}{2}\mathbf{p}^2 \ , \quad V(\mathbf{q}) = \mathbf{q}^2 + \alpha\log(1-\mathbf{q}^2) \ , \tag{4.48b}$$

are kinetic and potential terms. Writing the momentum equations (4.42) in the form

$$\begin{aligned} \mathbf{q}_{n+1} &= \mathbf{q}_n + \mathbf{p}_n - 2\mathbf{q}_n + \tfrac{2\alpha q_n}{1-q_n^2} &&= \mathbf{q}_n - T'(\mathbf{p}_n) + V'(\mathbf{q}_n) \ , \\ \mathbf{p}_{n+1} &= \mathbf{p}_n - 2\mathbf{q}_n + \tfrac{2\alpha q_n}{1-q_n^2} &&= \mathbf{p}_n + V'(\mathbf{q}_n) \ , \end{aligned} \tag{4.49}$$

and recalling the conjugations (1.75) it is easy to see that the time evolution operator U (4.48) leads to the correct operator equations of motion.

With the McMillan map expressed in terms of a time-evolution operator U, we can consider the one time-step propagator (as in chapter 3)

$$\begin{aligned}
{}_{n+1}\langle \widehat{q}|q\rangle_n &= \langle \widehat{q}|U|q\rangle \ , && (4.50a)\\
&= \int \mathrm{d}p \ \exp\left[\frac{i}{\hbar}\Big(-\tfrac{1}{2}p^2 + p(\widehat{q}-q) + V(q)\Big)\right] \ , && (4.50b)\\
&= \sqrt{\frac{2\pi\hbar}{i}}\exp\left[\frac{i}{\hbar}\left(\frac{1}{2}(\widehat{q}-q)^2 - q^2 - \alpha\log(1-q^2)\right)\right] \ , && (4.50c)\\
&= \sqrt{\frac{2\pi\hbar}{i}}\exp\left[\frac{i}{\hbar}\mathcal{L}(q,\widehat{q})\right] \ , && (4.50d)
\end{aligned}$$

so that the Lagrangian (4.40) reappears. We assume that we can introduce a complete set of momentum eigenstates; the integral over p is interpreted as a Gaussian integral. This assumption is to some extent validated by the resurfacing of the Lagrangian at the final step. Although this is not a new result [74], it is important to note the way that the propagator depends essentially on the initial choice of Lagrangian. We highlight this below by considering an alternative, simpler, choice of Lagrangian.

As given in (4.35) we can also choose the Lagrangian

$$\mathcal{L}(q_n, q_{n+1}) = -q_n q_{n+1} - \alpha\log(1-q_n^2) \ , \tag{4.51}$$

this represents a simplest possible choice of Lagrangian generating the McMillan equation (4.39), and is clearly equivalent to the Lagrangian (4.40) in the action. This alternative choice of Lagrangian defines different conjugate momenta, or Hamilton's equations,

$$\mathbf{p}_{n+1} = -\mathbf{q}_n \ , \quad \mathbf{p}_n = \mathbf{q}_{n+1} - \frac{2\alpha\mathbf{q}_n}{1-\mathbf{q}_n^2} \ . \tag{4.52}$$

Compare (4.42). These also yield the commutation relations (4.44).

The momentum equations (4.52) define a subtly different evolution to the previous choice (4.42), and correspondingly this is described by a different choice of time evolution operator U. We rewrite the momentum equations (4.52) in the form

$$\begin{aligned}
\mathbf{q}_{n+1} &= \mathbf{q}_n + \mathbf{p}_n - \mathbf{q}_n + \frac{2\alpha\mathbf{q}_n}{1-\mathbf{q}_n^2} \ = \ \mathbf{q}_n + T'(\mathbf{p}_n) - V'(\mathbf{q}_n) \ , && (4.53a)\\
\mathbf{p}_{n+1} &= \mathbf{p}_n - \mathbf{q}_n - \mathbf{p}_n \ = \ \mathbf{p}_n - \overline{V}'(\mathbf{q}_n) - T'(\mathbf{p}_n) \ . && (4.53b)
\end{aligned}$$

Inspired by the three-part time evolution operators found in chapter 3 (3.5), we write the time-evolution operator for this Lagrangian

$$U = e^{-i\overline{V}(\mathbf{q})/\hbar}e^{-iT(\mathbf{p})/\hbar}e^{-iV(\mathbf{q})/\hbar} \ , \tag{4.54}$$

where

$$\overline{V}(\mathbf{q}) = \tfrac{1}{2}\mathbf{q}^2 \ , \quad T(\mathbf{p}) = \tfrac{1}{2}\mathbf{p}^2 \ , \quad V(\mathbf{q}) = \tfrac{1}{2}\mathbf{q}^2 + \alpha \log(1-\mathbf{q}^2) \ . \tag{4.55}$$

This generates the operator equations of motion by the conjugation (4.47), as before.

Once again, as in (4.50), it is possible to write the one time-step propagator,

$$\begin{aligned} {}_{n+1}\langle \widehat{q} | q \rangle_n &= \langle \widehat{q} | e^{-i\overline{V}(\mathbf{q})/\hbar} e^{-iT(\mathbf{p})/\hbar} e^{-iV(\mathbf{q})/\hbar} | q \rangle \ , & (4.56a) \\ &= \sqrt{2\pi i\hbar}\ \exp\left[\frac{i}{\hbar}\left(-q\widehat{q} - \alpha\log(1-q^2)\right)\right] \ , & (4.56b) \end{aligned}$$

so that we recover the specific form of the chosen Lagrangian, given by (4.51). Note the sensitivity of the one-step propagator to the choice of Lagrangian; a key question remains whether a one-form structure that might fix the Lagrangian exists for this integrable model.

4.3.4 Quantum Mechanical Propagators

Following the one time-step propagator (4.56), a path integral quantisation of the McMillan map requires the extension of the propagator into multiple time-steps. The Hilbert space for the operator equation of motion (4.39) is not known; perhaps the path integral approach can ultimately offer new insights into the system.

One approach by Field and Nijhoff [38] begins by postulating Heisenberg picture position eigenstates at time n, $|q\rangle_n$, and proceeds from the operator equation of motion (4.39). The equation of motion is sandwiched between the quantum states ${}_{n+1}\langle \widehat{q}|$ and $|q\rangle_n$ to yield an equation for the "one time-step" propagator,

$${}_{n+1}\langle \widehat{q} | (\mathbf{q}_{n+1} + \mathbf{q}_{n-1})(1-\mathbf{q}_n^2) | q \rangle_n = 2\alpha\, {}_{n+1}\langle \widehat{q} | \mathbf{q}_n | q \rangle_n \ . \tag{4.57}$$

Evaluation of this operator equation depends once more on our choice of Lagrangian and conjugate momenta. Taking the Lagrangian (4.51) with operator momentum equations (4.52), we choose a standard representation of the operators $\mathbf{q}_n$ and $\mathbf{p}_n$ as differential operators,

$$\begin{aligned} {}_n\langle q | \mathbf{q}_n | \psi \rangle &= q\, {}_n\langle q | \psi \rangle \ , & (4.58a) \\ {}_n\langle q | \mathbf{p}_n | \psi \rangle &= -i\hbar \frac{\partial}{\partial q}\, {}_n\langle q | \psi \rangle \ . & (4.58b) \end{aligned}$$

This leads to the differential equation from (4.57)

$$(1-q^2)\left(\widehat{q} - i\hbar\frac{\partial}{\partial q}\right) K_1(q,\widehat{q}) = 2\alpha q K_1(q,\widehat{q}) \ , \tag{4.59}$$

where we have introduced the one time-step propagator

$$K_1(q,\widehat{q}) := {}_{n+1}\langle\widehat{q}|q\rangle_n \ . \tag{4.60}$$

It is possible to solve the differential equation (4.59) for the propagator, so that

$$K_1(q,\widehat{q}) = \mathscr{N}(\widehat{q}) \ \exp\left[\frac{i}{\hbar}\left(-q\widehat{q} - \alpha\log(1-q^2)\right)\right] \ . \tag{4.61}$$

The $\widehat{q}$ dependence of the integration factor is fixed by comparing different evaluations of ${}_{n+1}\langle\widehat{q}|\mathbf{q}_n|q\rangle_n$, using the equation $\mathbf{q}_n = -\mathbf{p}_{n+1}$ (4.52),

$${}_{n+1}\langle\widehat{q}|\mathbf{q}_n|q\rangle_n = qK_1(q,\widehat{q}) = i\hbar\frac{\partial}{\partial\widehat{q}}K_1(q,\widehat{q}) \ , \tag{4.62}$$

which yields that $\mathscr{N}$ = constant. We therefore have the result

$$\begin{aligned} K_1(q,\widehat{q}) &= \mathscr{N}\exp\left[\frac{i}{\hbar}\left(-q\widehat{q} - \alpha\log(1-q^2)\right)\right] \ , && (4.63a)\\ &= \mathscr{N}\exp\left[\frac{i}{\hbar}\mathcal{L}(q,\widehat{q})\right] \ , && (4.63b)\end{aligned}$$

where $\mathcal{L}(q,\widehat{q})$ is the discrete Lagrangian generating the map, given by (4.51). Note the consistency of this result with the propagator derived through the time evolution operator U (4.56). This equation of motion approach could similarly have used conjugate momenta defined from the alternative Lagrangian (4.40), leading to the alternative propagator expression (4.50).

The two time-step propagator

The equation of motion method of section 4.3.4 can be extended to derive an equation for the two-step propagator,

$$K_2(\widehat{q},\underset{\wedge}{q}) := {}_{n+1}\langle\widehat{q}|\underset{\wedge}{q}\rangle_{n-1} \ . \tag{4.64}$$

This method is again due to [38]: here we explore the full propagator expression in more detail, and relate the result to the one-step propagator of (4.63).

Sandwiching the operator equation of motion (4.39) with the Heisenberg position eigenstates ${}_{n+1}\langle\widehat{q}|$ and $|\underset{\wedge}{q}\rangle_{n-1}$ we can write

$${}_{n+1}\langle\widehat{q}|(\mathbf{q}_{n+1}+\mathbf{q}_{n-1})(1-\mathbf{q}_n^2)|\underset{\wedge}{q}\rangle_{n-1} = 2\alpha\,{}_{n+1}\langle\widehat{q}|\mathbf{q}_n|\underset{\wedge}{q}\rangle_{n-1} \ . \tag{4.65}$$

Again using the operator equations of motion (4.52), the commutation relation (4.43) and the representation (4.58), this yields the second order differential equation in terms of $\widehat{q}$,

$$\left[(\widehat{q}+\underset{\wedge}{q})\left(1+\hbar^2\frac{\partial^2}{\partial\widehat{q}^2}\right)-2i\hbar(\alpha+i\hbar)\frac{\partial}{\partial\widehat{q}}\right]\circ K_2(\widehat{q},\underset{\wedge}{q})=0\;. \tag{4.66}$$

This differential equation can be solved in terms of Bessel functions. Writing

$$z:=\widehat{q}+\underset{\wedge}{q}\;,\quad w(z):=K_2(\widehat{q},\underset{\wedge}{q})\;, \tag{4.67}$$

where $\underset{\wedge}{q}$ is treated as a parameter, the differential equation (4.66) can be rewritten as

$$w''-\frac{2i(\alpha+i\hbar)}{\hbar z}w'+\frac{1}{\hbar^2}w=0\;. \tag{4.68}$$

This has the form of equation (10.13.4) from [30], which is solved in terms of a cylindrical function $\mathscr{C}_\nu$,

$$w(z)=z^{\pm\nu}\mathscr{C}_\nu\left(\frac{z}{\hbar}\right)\;,\quad \nu=\pm\left(\frac{1}{2}-\frac{i\alpha}{\hbar}\right)\;. \tag{4.69}$$

This solution can be elaborated by seeking a series solution to the differential equation (4.68) around $z=0$. There are two such solutions, given by

$$w_+(z)=\sum_{n=0}^{\infty}(-1)^n\frac{1}{n!(\nu+1)_n}\left(\frac{z}{2\hbar}\right)^{2n}\;, \tag{4.70a}$$

$$w_-(z)=z^{-2\nu}\sum_{n=0}^{\infty}(-1)^n\frac{1}{n!(-\nu+1)_n}\left(\frac{z}{2\hbar}\right)^{2n}\;, \tag{4.70b}$$

$$\text{where}\quad \nu=1/2-i\alpha/\hbar\;. \tag{4.70c}$$

The notation $(a)_n$ represents the Pochhammer symbol of a,

$$(a)_n=a(a+1)\dots(a+n-1)\;. \tag{4.71}$$

The Bessel function is defined by

$$J_\nu(z)=(z/2)^\nu\sum_{k=0}^{\infty}(-1)^k\frac{(z^2/4)^k}{k!\;\Gamma(\nu+k+1)}\;, \tag{4.72}$$

[30, 121] so that the solutions for the two time-step propagator (4.64) are

$$K_2(\widehat{q},\underset{\wedge}{q})=a_+(\underset{\wedge}{q})\;w_+(\widehat{q}+\underset{\wedge}{q})+a_-(\underset{\wedge}{q})\;w_-(\widehat{q}+\underset{\wedge}{q})\;, \tag{4.73a}$$

$$\text{where}\quad w_\pm(z)=\left(\widehat{q}+\underset{\wedge}{q}\right)^{-\nu}J_{\pm\nu}\Big((\widehat{q}+\underset{\wedge}{q})/\hbar\Big)\;, \tag{4.73b}$$

with ν given in (4.70c), and $a_\pm(\underset{\wedge}{q})$ undetermined integration factors.

It remains to resolve the $\underset{\wedge}{q}$ dependence of the integration factors $a_\pm$, but this can be approached in a similar manner to the one time-step case (4.62). Observe that from the momentum equations (4.52) we have

$$\mathbf{q}_n \;=\; -\mathbf{p}_{n+1} \;=\; \mathbf{p}_{n-1} + \frac{2\alpha \mathbf{q}_{n-1}}{1-\mathbf{q}_{n-1}^2}\ , \tag{4.74}$$

so that ${}_{n+1}\langle \widehat{q}|\mathbf{q}_n|\underset{\wedge}{q}\rangle_{n-1}$ can be evaluated in two different ways,

$$\begin{aligned} {}_{n+1}\langle \widehat{q}|(-\mathbf{p}_{n+1})|\underset{\wedge}{q}\rangle_{n-1} &= {}_{n+1}\langle \widehat{q}| \left[\mathbf{p}_{n-1} + \frac{2\alpha \mathbf{q}_{n-1}}{1-\mathbf{q}_{n-1}^2}\right] |\underset{\wedge}{q}\rangle_{n-1}\ , && (4.75a)\\ \Rightarrow\quad i\hbar\frac{\partial}{\partial \widehat{q}} K_2(\widehat{q},\underset{\wedge}{q}) &= \left(i\hbar\frac{\partial}{\partial \underset{\wedge}{q}} + \frac{2\alpha \underset{\wedge}{q}}{1-\underset{\wedge}{q}^2}\right) K_2(\widehat{q},\underset{\wedge}{q})\ . && (4.75b)\end{aligned}$$

Now, noting that

$$\frac{\partial w_i(z)}{\partial \widehat{q}} = \frac{\partial w_i(z)}{\partial \underset{\wedge}{q}}\ , \tag{4.76}$$

and substituting (4.73) into (4.75), the condition becomes

$$0 = i\hbar\Big(a_+'(\underset{\wedge}{q})w_+(z) + a_-'(\underset{\wedge}{q})w_-(z)\Big) + \frac{2\alpha \underset{\wedge}{q}}{1-\underset{\wedge}{q}^2}\Big(a_+(\underset{\wedge}{q})w_+(z) + a_-(\underset{\wedge}{q})w_-(z)\Big)\ . \tag{4.77}$$

This is most easily solved by resolving the $a_\pm(\underset{\wedge}{q})$ parts separately as coefficients of $w_\pm(z)$ respectively, yielding

$$a_\pm(\underset{\wedge}{q}) = a_\pm\ \exp\left[\frac{i}{\hbar}\Big(-\alpha\log(1-\underset{\wedge}{q}^2)\Big)\right]\ , \tag{4.78}$$

for constants $a_\pm$. Hence the solution to the differential equation (4.66) is given by

$$\begin{aligned} K_2(\widehat{q},\underset{\wedge}{q}) = \exp&\left[\frac{i}{\hbar}\Big(-\alpha\log(1-\underset{\wedge}{q}^2)\Big)\right](\widehat{q}+\underset{\wedge}{q})^{-\nu} \\ &\times\left[a_+\ J_\nu\Big((\widehat{q}+\underset{\wedge}{q})/\hbar\Big) + a_-\ J_{-\nu}\Big((\widehat{q}+\underset{\wedge}{q})/\hbar\Big)\right]\ . \end{aligned} \tag{4.79}$$

This approach does not specify the constants $a_\pm$, but we will see that they follow from the group property of the propagator.

A natural question is: how does the two time-step propagator (4.79) link to the one time-step propagator (4.56), (4.63)? An expected property of quantum mechanical propagators is the group property, which is a composition rule

$$K(x,y;T+T') = \int \mathrm{d}\mu(z)\, K(x,z;T)K(z,y;T')\ . \tag{4.80}$$

In the discrete time case, the two step propagator has a natural time slicing into one step propagators, but this requires the introduction of a complete set of position eigenstates, with an appropriate integration measure. Without knowing the form of the Hilbert space, it is unclear what this should look like.

We postulate the complete set of position eigenstates

$$\int \mathrm{d}\mu(q)\ |q\rangle\langle q| = 1\ , \tag{4.81}$$

where the measure $\mathrm{d}\mu(q)$ is to be specified. Using the time evolution operator U (4.54), we time-slice the two step propagator into one step pieces,

$$\begin{aligned} K_2(\widehat{q}, \underset{\wedge}{q}) &= \langle\widehat{q}|U^2|\underset{\wedge}{q}\rangle = \int \mathrm{d}\mu(q)\ \langle\widehat{q}|U|q\rangle\langle q|U|\underset{\wedge}{q}\rangle\ , && \text{(4.82a)}\\ &= \int \mathrm{d}\mu(q)\ K_1(\widehat{q}, q)K_1(q, \underset{\wedge}{q})\ . && \text{(4.82b)} \end{aligned}$$

So this is a simplest possible manifestation of the composition rule (4.80) and time-slicing.

Substituting in the expression for K_1 (4.63) we find the integral for the two step propagator

$$\begin{aligned} K_2(\widehat{q}, \underset{\wedge}{q}) = \mathscr{N}^2 \exp\left[\frac{i}{\hbar}\Big(- \alpha\log(1 - \underset{\wedge}{q}^2)\Big)\right]& \\ \times \int \mathrm{d}\mu(q)\ \exp\left[\frac{i}{\hbar}\Big(- q(\widehat{q} + \underset{\wedge}{q}) - \alpha\log(1 - q^2)\Big)\right]\ .& \quad (4.83) \end{aligned}$$

Now, consider the following integral representations for the Bessel function [30, 121],

$$\begin{aligned} J_\nu(z) &= \frac{(z/2)^\nu}{\Gamma(\nu + 1/2)\sqrt{\pi}} \int_{-1}^{1} \mathrm{d}t\, e^{izt}(1 - t^2)^{\nu - 1/2} \quad , \quad \Re\nu > \frac{1}{2}\ , && \text{(4.84a)}\\ J_\nu(x) &= \frac{2(x/2)^{-\nu}}{\Gamma(1/2 - \nu)\sqrt{\pi}} \int_{1}^{\infty} \mathrm{d}t\ \frac{\sin(xt)}{(t^2 - 1)^{\nu + 1/2}} \quad , \quad |\Re\nu| < \frac{1}{2}\ . && \text{(4.84b)} \end{aligned}$$

In fact, we will need to assert the analytic continuation of these identities to the region $\Re\nu = \pm 1/2$. Suppose we make the identification,

$$\int \mathrm{d}\mu(q) = \int_{-\infty}^{\infty} \mathrm{d}q\ , \tag{4.85}$$

which is a standard assumption for a physical Hilbert space. It is then possible to rearrange the integral (4.83) into two parts,

$$\begin{aligned} K_2(\widehat{q}, \underset{\wedge}{q}) = 2\pi\hbar \exp\left[\frac{i}{\hbar}\Big(- \alpha\log(1 - \underset{\wedge}{q}^2)\Big)\right]& \\ \times \left(\int_{-1}^{1} \mathrm{d}q\ e^{ixq}(1 - q^2)^{-i\alpha/\hbar} + 2i \int_{1}^{\infty} \mathrm{d}q \sin(xq)(q^2 - 1)^{-i\alpha/\hbar} \right)\ .& \quad (4.86) \end{aligned}$$

But these integrals are solvable with Bessel function representations (4.84), so that the propagator is given by

$$K_2(\widehat{q}, \underset{\wedge}{q}) = a_0 \left(1 - \underset{\wedge}{q}^2\right)^{-i\alpha/\hbar} (\widehat{q} + \underset{\wedge}{q})^{-\nu} \Big(J_\nu\big((\widehat{q} + \underset{\wedge}{q})/\hbar\big) + iJ_{-\nu}\big((\widehat{q} + \underset{\wedge}{q})/\hbar\big) \Big) , \tag{4.87}$$

with $\nu = 1/2 - i\alpha/\hbar$ as before (4.70c), and the identification

$$a_0 = (2\hbar)^{\nu+1}\pi^{3/2}\ \Gamma(\nu + 1/2) . \tag{4.88}$$

But this expression (4.87) is precisely compatible with that found through the differential equation (4.79), where the relative values of $a_\pm$ are now fixed. Only an overall normalisation constant remains. This composition rule approach yields a compatible answer with the operator equation of motion. A key outstanding difficulty, however, are the singularities at $q = \pm 1$. How to resolve singularities in discrete-time quantum mechanics is not in general currently clear, but the favourable properties of integrable systems suggest it may ultimately be possible to understand the behaviour of solutions around these singularities in the quantum regime. Perhaps integrable systems may point the way for the broader theory.

4.3.5 Operator Invariant

Recall the invariant for the quantum McMillan map (4.46). In terms of the conjugate momenta (4.52), this can be written

$$\mathbf{I}_n = (1 - \mathbf{q}_n^2)(1 - \mathbf{p}_n^2) + 2(\alpha + i\hbar)\mathbf{q}_n\mathbf{p}_n . \tag{4.89}$$

We can consider eigenstates of the operator invariant, such eigenstates will have a fixed eigenvalue under the time evolution,

$$\mathbf{I}_{n+1}|\psi\rangle \ = \ \mathbf{I}_n|\psi\rangle \ = \ E|\psi\rangle . \tag{4.90}$$

These represent stationary states for the quantum mapping. In the discrete-time case, there is not a Schrödinger equation in the same way as for continuous time systems, but the invariant can be thought of as a Hamiltonian generating a continuous-time flow that is compatible with the map, with stationary Schrödinger equation given by the eigenvalue problem (4.90).

Taking a standard position space representation by conjugation with the position eigenstates $|q\rangle_n$, the eigenvalue problem (4.90) leads to the differential equation in position space

$$(1-q^2)\psi_n''(q) - 2\frac{i}{\hbar}(\alpha + i\hbar)q\psi_n'(q) + \frac{1}{\hbar^2}(1-q^2)\psi_n(q) = \frac{E}{\hbar^2}\psi_n(q) \ , \tag{4.91}$$

where $\psi_n(q)$ is the wave function at discrete time n. It is observed in [40] that this is an equation of confluent Heun class. By substituting $\psi_n(q) = G(q)v(q)$, (4.91) can be written in the form

$$\frac{d}{dq}(1-q^2)\frac{dv}{dq} + \left[b^2(1-q^2) - \eta + \mu(\mu+1) - \frac{\mu^2}{1-q^2}\right]v(q) = 0 \ , \tag{4.92a}$$

where

$$\mu = i\alpha/\hbar - 2 \ , \tag{4.92b}$$

$$b^2 = 1/\hbar^2 \ , \tag{4.92c}$$

$$\eta = E/\hbar^2 \ , \tag{4.92d}$$

$$G(q) = \exp\left[-\frac{\mu}{2}\log(1-q^2)\right] \ . \tag{4.92e}$$

Equation (4.92a) is the *spheroidal wave equation* for $v(q)$, see equation (30.2.1) of [30]. The Sturm-Liouville form of (4.92a) is sufficient for orthogonality of eigenfunctions either on the range $[-1, 1]$ or over the whole real line with vanishing boundary conditions, with weight function $\sigma(q) = 1$. Such boundary conditions would need to account in some other manner for the singularities occurring at $q = \pm 1$. However, we can gain some further insight from the existing literature on the spheroidal wave equation.

The spheroidal wave equation (4.92a) is in general multi-valued across the complex plane; literature on solutions is generally focused on the case where a single-valued function can be found [6, 12, 34, 67, 106]. For this single-valuedness, it is required that the parameter μ is an integer (4.92b). In that case there is a set of eigenvalues,

$$\lambda := \mu(\mu+1) - \eta = \lambda_\nu^\mu(b^2) \ , \quad \text{numbered by } \nu = \mu, \ \mu+1, \ \mu+2, \dots \ , \tag{4.93}$$

where $\lambda_\nu^\mu(b^2) < \lambda_{\nu+1}^\mu(b^2)$. These eigenvalues correspond to eigenfunctions $\mathrm{Ps}_\nu^\mu(q, b^2)$, called spheroidal wave functions, which are bounded, complete and orthogonal on $(-1, 1)$. They have the form $(1-q^2)^{\mu/2}g(q)$, for $g(q)$ an entire function. In other words, if $v(q)$ takes this form, then the quantum wave function $\psi_n(q)$ (4.91) is an entire function.

One must then consider the physical requirements on the solution $\psi_n(q)$. The results above in the literature for the spheroidal wave function are suggestive that a physically meaningful solution to the equation (4.92a) requires that μ (4.92b) be integer, so that we make the parameter choice

$$\alpha = -(\mu + 2)i\hbar \ , \tag{4.94}$$

for integer μ. In other words, physical restrictions on the wave function $\psi_n(q)$ could lead to a *quantisation of the mapping parameter itself.* Under such a quantisation, the wave functions would then be

$$\psi_n(q) = (1 - q^2)^{-\mu/2}\,\mathrm{Ps}_\nu^\mu(q) \ , \tag{4.95}$$

for spheroidal wave functions Ps_ν^μ, with eigenvalues given by λ_ν^μ, and ν enumerating the energy levels for the model. These functions are naturally bounded (hence normalisable), orthogonal and complete over $(-1, 1)$. Perhaps a suitable basis for a Hilbert space theory of the model can be found in these functions.

Note that a second special case of (4.92a) corresponds to the choice $\mu = \frac{1}{2}$. In that case, the equation becomes the Mathieu equation, equation (28.2.1) of [30].

4.4 Dual Lax and r-matrix Structure

There are a number of outstanding challenges in the path integral quantisation of the McMillan map. As we saw in section 4.2, it remains unclear whether we can establish a one-form structure, due to the difficulties in describing the commuting flow. Similarly, in section 4.3 the early inroads into the path integral itself so far only hold for the simplest case, and only as far as the two time-step propagator. But, the McMillan maps have extensive integrability structures that we have not yet leveraged to their full extent, particularly the Lax pair.

In the section below, we consider the Lax pair in more depth. It is not yet clear how the Lax pair relates to the Lagrangian structure, but rather it is a parallel approach. The local Lax structure of the generalised McMillan maps (discussed in section 4.1) has a related *dual* structure of large Lax matrices [80]. In the following section we discuss the r-matrix structure belonging to this dual Lax pair, identifying some novel features, by expressing the r-matrix as a normal-ordered matrix fraction. It emerges that this dual Lax pair may have a simpler quantisation than the one already known for the local Lax case, although this is not yet clear in the general case. In the discrete Calogero-Moser and Ruijsenaars-Schneider models (section 1.2.2), the Lagrangian one-form structures were associated to commuting discrete flows - these commuting flows arose through the introduction of additional elements into the large Lax pair. Further, for the CM case, the one-form action actually arises directly from the Lax pair as a log-determinant of the ordered sequence of Darboux matrices $\mathcal{M}$ generating the time evolution. By elaborating the large Lax structure for the McMillan maps, it is possible that we will be able to pursue a Lagrangian one-form structure through this avenue.

4.4.1 Dual Lax Matrix

The generalised McMillan map in variables v_i (4.5) has a well studied "local" Lax pair (4.9), but this is not the unique Lax description for the model. It is possible to move to an alternative, "dual" Lax matrix formulation [24], by considering eigenvectors θ_1 of the monodromy matrix $T(\lambda)$ (4.10),

$$T(\lambda)\,\theta_1 = h\theta_1 \ . \tag{4.96}$$

Recalling that $T(\lambda) = V_{2N}\dots V_1$, the local matrices V_j (4.9a) define a sequence of vectors θ_j,

$$\theta_{j+1} := V_j\theta_j \quad \Rightarrow \quad \theta_{2N+1} = h\theta_1 \ , \tag{4.97a}$$

such that

$$\theta_j = \begin{pmatrix} \phi_j \\ \psi_j \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} \phi_{j+1} \\ \psi_{j+1} \end{pmatrix} = \begin{pmatrix} v_j\phi_j + \psi_j \\ \lambda_j\phi_j \end{pmatrix} . \tag{4.97b}$$

Eliminating ψ_j from these equations gives a system of equations for ϕ_j,

$$\phi_{j+1} = v_j\phi_j + \lambda_{j-1}\phi_{j-1} \ , \quad j = 2,\dots,2N-1 \ , \tag{4.98a}$$

$$h\phi_1 = v_{2N}\phi_{2N} + \lambda_{2N-1}\phi_{2N-1} \ , \tag{4.98b}$$

$$h\phi_2 = hv_1\phi_1 + \lambda_{2N}\phi_{2N} \ . \tag{4.98c}$$

Forming the new vector $\Phi = (\phi_1,\dots,\phi_{2N})^T$, we can cast these equations (4.98) into the matrix form

$$\mathcal{L}(h)\Phi = \lambda\Phi \ . \tag{4.99}$$

This is a dual spectral form to the local Lax matrix (4.9), with the dual $(2N \times 2N)$ Lax matrix given by

$$\mathcal{L}(h) = \begin{pmatrix} \epsilon\delta & -v_2 & 1 & & & & \\ & 0 & -v_3 & 1 & & & \\ & & \epsilon\delta & -v_4 & \ddots & & \\ & & & 0 & \ddots & 1 & \\ & & & & \ddots & -v_{2N-1} & 1 \\ h & & & & & \epsilon\delta & -v_{2N} \\ -hv_1 & h & & & & & 0 \end{pmatrix} . \tag{4.100}$$

Note that the roles of h and λ as eigenvalue and spectral parameter are reversed from the local Lax case (4.96).

The Lax matrix $\mathcal{L}(h)$ is more easily expressed in terms of the $(2N \times 2N)$ shift matrix Σ_h,

$$\Sigma_h := \sum_{j=1}^{2N-1} E_{j,j+1} + hE_{2N,1} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ h & & & & 0 \end{pmatrix} , \tag{4.101}$$

which acts on elementary $2N \times 2N$ matrices $E_{i,j}$ by

$$\Sigma_h E_{i,j} = E_{i-1,j} , \qquad \Sigma_h E_{1,j} = hE_{2N,j} , \tag{4.102a}$$

$$E_{i,j}\Sigma_h = E_{i,j+1} , \qquad E_{i,2N}\Sigma_h = hE_{i,1} . \tag{4.102b}$$

Σ_h has the additional periodic property

$${\Sigma_h}^{2N} = h\mathbb{I} . \tag{4.103}$$

In terms of Σ_h, the dual Lax matrix $\mathcal{L}(h)$ can be written as

$$\mathcal{L}(h) = \Sigma_h^2 - \Sigma_h V + \Lambda , \tag{4.104a}$$

$$:= \Sigma_h^2 - \Sigma_h \left(\sum_{j=1}^{2N} v_j E_{j,j} \right) + \epsilon\delta \sum_{j=1}^{N} E_{2j-1,2j-1} , \tag{4.104b}$$

where we have also introduced the diagonal matrices V and Λ. Hence $\mathcal{L}(h)$ is expressible in terms of a combination of Σ_h and diagonal matrices [80].

Since Σ_h allows us to express the Lax pair in terms of diagonal matrices, we introduce the simplifying notation D_j (with a single subscript) to indicate the $(2N \times 2N)$ elementary diagonal matrix with a 1 in the (j,j) entry, and zero everywhere else, $D_j := E_{j,j}$. This reduces some unnecessary subscripts in the calculations to follow. Σ_h also commutes with these diagonal matrices according to the simple rule

$$D_j \Sigma_h = \Sigma_h D_{j+1} \ , \tag{4.105}$$

which is easily shown from the definition (4.101).

Having written the Lax matrix as a quadratic in Σ_h (4.104), it is easy to identify factorisations of $\mathcal{L}(h)$, of which there are two possibilities. Either

$$\mathcal{L}(h) = \mathcal{N}(h)\mathcal{M}(h) \ , \tag{4.106a}$$

with

$$\begin{aligned} \mathcal{M}(h) &= \Sigma_h - \epsilon\delta \sum_j v_{2j}^{-1} D_{2j-1} \ , & (4.106b) \\ \mathcal{N}(h) &= \Sigma_h - \sum_j v_{j+1} D_j + \epsilon\delta \sum_j v_{2j+2}^{-1} D_{2j} \ . & (4.106c) \end{aligned}$$

Or the alternative decomposition,

$$\mathcal{L}(h) = \mathcal{M}'(h)\mathcal{N}'(h) \ , \tag{4.107a}$$

with

$$\begin{aligned} \mathcal{M}'(h) &= \Sigma_h - \epsilon\delta \sum_j v_{2j-1}^{-1} D_{2j-1} \ , & (4.107b) \\ \mathcal{N}'(h) &= \Sigma_h - \sum_j v_j D_j + \epsilon\delta \sum_j v_{2j-1}^{-1} D_{2j} \ . & (4.107c) \end{aligned}$$

The equations of motion for the map (4.5) then arise from a conjugation with the matrix $\mathcal{M}(h)$,

$$\widehat{\mathcal{L}}(h)\,\mathcal{M}(h) = \mathcal{M}(h)\mathcal{L}(h) \qquad \Rightarrow \qquad \widehat{\mathcal{L}}(h) = \mathcal{M}(h)\mathcal{N}(h) \ . \tag{4.108}$$

In other words, evolving the map by one time step corresponds to interchanging the factors of the Lax matrix $\mathcal{L}(h)$ [80]. Note that the Darboux matrix $\mathcal{M}(h)$ could also be derived from the local Lax matrices $L_i(\lambda)$, $M_j(\lambda)$ (4.9a), (4.9b) in the same way as the Lax matrix $\mathcal{L}(h)$. Although these interchanges generate the dynamics, the possible link to a variational formulation, or indeed to a path integral quantisation, is not currently known. For the discrete Calogero-Moser map, the generating Lagrangian arises as the log determinant of the Darboux matrix, $\mathcal{M}$, but such a connection has not yet been discovered for other discrete-time mappings. Interchange of the alternative factors $\mathcal{M}'$ and $\mathcal{N}'$ (4.107) yields equations of motion corresponding to the backwards time evolution.

The evolution of the Lax matrix $\mathcal{L}(h)$ (4.108) additionally guarantees the invariance of the spectral curve,

$$\det\big(\lambda\mathbb{I} - \mathcal{L}(h)\big) = 0\ . \tag{4.109}$$

This yields invariants for the model as the minors of $\mathcal{L}(h)$. Note that the spectral curve (4.109) is closely related to the spectral curve of the local Lax pair (4.11), such that the invariants for the dual Lax matrix are the same as those yielded by the monodromy matrix $T(\lambda)$ (4.10) [24].

4.4.2 r-matrix structure

As described in (4.14), the Poisson bracket structure of the local Lax matrices $L_i(\lambda)$ (4.9a) can be encoded in a classical r-matrix structure. We would expect there to be a similar structure for the dual Lax matrix $\mathcal{L}(h)$ (4.100), but such a structure for the generalised McMillan case surprisingly does not exist in the literature; although it has been found for a selection of other models with similar Lax structures [104, 105]. However, in [60] the authors consider a similar dual Lax structure for the Dimer-Self-Trapping model, and the r-matrix structure in that case turns out also to be the correct one for the McMillan map.

We begin by choosing the r-matrix from [60] which lives in the matrix tensor product space,

$$r_{12}(h,h') = \frac{1}{h-h'}\left(h'\sum_{j\geq i} + h\sum_{j<i}\right) E_{ij}\otimes E_{ji}\ . \tag{4.110}$$

The authors do nothing more with r_{12} except to comment on its non-unitarity, that is $r_{21} \neq -r_{12}$. They mention that in some cases it is possible to gauge transform to an alternative, unitary form of the r-matrix, although this is not done for the specific case.

The r-matrix (4.110) however, can alternatively be written in terms of the shift matrix Σ_h (4.101), which reveals some interesting properties. We suppress the spectral parameter to write

$$\Sigma_1 := \Sigma_h \otimes \mathbb{I} = \overset{1}{\Sigma}_h\ , \quad \Sigma_2 := \mathbb{I}\otimes\Sigma_{h'} = \overset{2}{\Sigma}_{h'}\ . \tag{4.111}$$

Recalling the periodicity property of Σ_h (4.103), we write the r-matrix as

$$r_{12}(h,h') = \frac{1}{h-h'}\left(\sum_{j<i}\Sigma_1^{2N}\overset{1}{E}_{ij}\overset{2}{E}_{ji}+\sum_{j\geq i}\overset{1}{E}_{ij}\overset{2}{E}_{ji}\,\Sigma_2^{2N}\right), \tag{4.112a}$$

$$= \frac{1}{h-h'}\left(\sum_{j<i}\Sigma_1^{2N-i+j}\overset{1}{E}_{jj}\overset{2}{E}_{jj}\,\Sigma_2^{i-j} + \sum_{j\geq i}\Sigma_1^{j-i}\overset{1}{E}_{jj}\overset{2}{E}_{jj}\,\Sigma_2^{2N-j+i}\right), \tag{4.112b}$$

$$= \frac{1}{h-h'}\sum_{k=1}^{2N}\Sigma_1^{2N-k}\mathcal{E}_{12}\Sigma_2^{k}\,, \tag{4.112c}$$

where we have introduced the sparse diagonal matrix across the tensor product space,

$$\mathcal{E}_{12} = \sum_{j=1}^{2N} E_{jj}\otimes E_{jj}\ . \tag{4.113}$$

The expression (4.112c) then gives a form for r_{12} in terms only of the shift matrix Σ_h and $\mathcal{E}_{12}$ - a form inherently compatible with the dual Lax matrix $\mathcal{L}(h)$ (4.104).

We can take the summation expression for the r-matrix (4.112c) further by noting a relation to a fractional expression of the Σ_h matrices. Notice that

$$(\Sigma_1-\Sigma_2)^{-1} = \frac{1}{h-h'}\sum_{k=1}^{2N}\Sigma_1^{2N-k}\Sigma_2^{k-1}\,, \tag{4.114}$$

which is easily checked by multiplication by $(\Sigma_1-\Sigma_2)$ and use of the periodicity property (4.103). Comparing with the expression for the r-matrix (4.112c), we can write the normal ordered expression

$$r_{12}(h,h') = \frac{1}{h-h'}\left[\sum_{k=1}^{2N}\Sigma_1^{2N-k}\mathcal{E}_{12}\Sigma_2^{k-1}\right]\Sigma_2\,, \tag{4.115a}$$

$$= :\frac{\mathcal{E}_{12}}{\Sigma_1-\Sigma_2}:\Sigma_2\ . \tag{4.115b}$$

The notation $:\quad:$ indicates a normal ordering on the factors Σ_1 and Σ_2. In a series expansion of $:f(\Sigma_i,\Sigma_j):$ the normal ordering indicates that factors of Σ_i should be placed to the left, and factors of Σ_j to the right. Note that the ordering of factors is indicated by the order that the Σ_h factors are written inside the function, so that for example

$$:\frac{\mathcal{E}_{12}}{\Sigma_1-\Sigma_2}: := \frac{1}{h-h'}\sum_{k=1}^{2N}\Sigma_1^{2N-k}\mathcal{E}_{12}\Sigma_2^{k-1}$$

$$\neq \frac{1}{h-h'}\sum_{k=1}^{2N}\Sigma_2^{2N-k}\mathcal{E}_{12}\Sigma_1^{k-1} = -:\frac{\mathcal{E}_{12}}{\Sigma_2-\Sigma_1}:\ . \tag{4.116}$$

In the first normal-ordered fraction, Σ_1 factors are placed to the front and Σ_2 factors to the back, whereas in the second expression this ordering choice is reversed, with the two choices not equivalent. This yields a very satisfying form for r_{12} as a *normally ordered fraction of* Σ_h *matrices.* In practice, the expansion (4.112c) remains the more useful for calculations.

Now, recalling the non-ultralocal Poisson bracket structure for the v_i (4.7) we can calculate the Poisson bracket for the dual Lax matrix $\mathcal{L}(h)$ (4.104),

$$\begin{aligned} \{\mathcal{L}_1, \mathcal{L}_2\} &= \Sigma_1 \sum_{i=1}^{2N} \left(\overset{1}{D}_{i+2} - \overset{1}{D}_i \right) \overset{2}{D}_i \, \Sigma_2 \ , && (4.117a) \\ &= \Sigma_2 \mathcal{E}_{12} \Sigma_1 - \Sigma_1 \mathcal{E}_{12} \Sigma_2 \ , && (4.117b) \end{aligned}$$

so that this is expressed in terms of Σ_h and $\mathcal{E}_{12}$. Returning to the expression for r_{12} (4.112c) it is straightforward to show that the Poisson bracket (4.117) can be expressed in linear r-matrix form as

$$\{\mathcal{L}_1(h), \mathcal{L}_2(h')\} = [r_{12}, \mathcal{L}_1] - [r_{21}, \mathcal{L}_2] \ . \tag{4.118}$$

Such r-matrix forms have been found previously [9, 104] in particular for the Dimer-Self-Trapping case that we have been following [60], but this is a new result for the generalised McMillan maps. Notice that in comparison to the local Lax case (section 4.1.2) this is a *linear* r-matrix structure, rather than the typically quadratic form of (4.14).

A number of properties for the r-matrix arise from its normal ordered fraction form (4.115). Respecting the normal ordering, we can write the inversion of the fraction, such that

$$\Sigma_1 \left(: \frac{\mathcal{E}_{12}}{\Sigma_1 - \Sigma_2} : \right) - \left(: \frac{\mathcal{E}_{12}}{\Sigma_1 - \Sigma_2} : \right) \Sigma_2 = \mathcal{E}_{12} \ . \tag{4.119}$$

This is easy to verify by expanding the fraction and using the periodicity property of Σ_h: there is a natural telescoping of the terms created. But, this is equivalent to a classical pseudo-skew-symmetry for the r-matrix,

$$r_{12} + r_{21} = -\mathcal{E}_{12} \ . \tag{4.120}$$

Notice that in order for the r-matrix structure of the Lax matrix (4.118) to define a proper Poisson bracket, it must be skew-symmetric. But this is automatic in this case. Hence the non skew-symmetry of the r-matrix, $r_{12} \neq -r_{21}$, is not problematic.

In many cases, the classical r-matrix *is* skew-symmetric, and has often emerged as the classical limit of some unitary solution to the Yang-Baxter equation [53]. In [60], the authors mention another case (the Toda system) where a gauge transform puts a non skew-symmetric r-matrix (of the kind we are considering) into a skew-symmetric form. However, they do not suggest what an appropriate transform might be in the DST case, and it is similarly unclear how to make such a transformation for the generalised McMillan case. The r-matrix of (4.112c) is, however, sufficiently interesting to merit further study despite its non skew-symmetry.

To define a proper Poisson bracket, the r-matrix structure (4.118) must also have the Jacobi property,

$$\{\mathcal{L}_1, \{\mathcal{L}_2, \mathcal{L}_3\}\} + \{\mathcal{L}_2, \{\mathcal{L}_3, \mathcal{L}_1\}\} + \{\mathcal{L}_3, \{\mathcal{L}_1, \mathcal{L}_2\}\} = 0 \ . \tag{4.121}$$

To hold, this requires that r_{12} obeys a classical Yang-Baxter condition [32, 53],

$$\Big[\mathcal{L}_1 \, , \, [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{32}, r_{13}]\Big] \;=\; 0 \ , \tag{4.122}$$

which is clearly satisfied if the sum of r-matrix commutators is zero. Using the summation form of the r-matrix (4.112c) we can show that the three-term relation holds,

$$r_{12}r_{13} = r_{13}r_{32} + r_{23}r_{12} \ , \tag{4.123}$$

from which it is easy to show that the classical Yang-Baxter equation (4.122) follows, and hence the Poisson bracket (4.118) has the Jacobi property.

This three term relation appears somewhat mysterious, but is related to the normal-ordered fraction form of the r-matrix (4.115). The three term relation (4.123) is written using the fractional form as

$$\begin{aligned} &: \frac{\mathcal{E}_{12}}{\Sigma_1 - \Sigma_2}\Sigma_2 :: \frac{\mathcal{E}_{13}}{\Sigma_1 - \Sigma_3}\Sigma_3 : \\ &\qquad = : \frac{\mathcal{E}_{13}}{\Sigma_1 - \Sigma_3}\Sigma_3 :: \frac{\mathcal{E}_{32}}{\Sigma_3 - \Sigma_2}\Sigma_2 : \ + \ : \frac{\mathcal{E}_{23}}{\Sigma_2 - \Sigma_3}\Sigma_3 :: \frac{\mathcal{E}_{12}}{\Sigma_1 - \Sigma_2}\Sigma_2 : \ . \end{aligned} \tag{4.124}$$

We note two properties. First, since Σ_h and $\mathcal{E}_{ij}$ are numerical matrices, matrices in different tensor product spaces can commute through each other freely. Second, using the commutation rule between the D_j and Σ_h (4.105), it is possible to reverse the factors in the normal ordering,

$$: \frac{\mathcal{E}_{ij}}{\Sigma_i - \Sigma_j}\Sigma_j : \;=\; \frac{1}{h_i - h_j}\sum_{k=1}^{2N} \Sigma_i^{2N-k}\mathcal{E}_{ij}\Sigma_j^k \ , \tag{4.125a}$$

$$= \frac{1}{h_i - h_j}\sum_{k=1}^{2N} \Sigma_j^k\mathcal{E}_{ij}\Sigma_j^{2N-k} \;=\; - : \Sigma_j \frac{\mathcal{E}_{ji}}{\Sigma_j - \Sigma_i} : \ . \tag{4.125b}$$

Note that the exchange only works in this simple form because there are precisely $2N$ of the Σ matrices in each term of the summation. Exploiting these two properties, the extra factors of Σ_2 and Σ_3 in (4.124) can be cancelled out, leaving a relation in terms of normal-ordered fractions only,

$$: \frac{\mathcal{E}_{12}}{\Sigma_1 - \Sigma_2} :: \frac{\mathcal{E}_{13}}{\Sigma_1 - \Sigma_3} : = : \frac{\mathcal{E}_{23}}{\Sigma_2 - \Sigma_3} :: \frac{\mathcal{E}_{12}}{\Sigma_1 - \Sigma_2} : - : \frac{\mathcal{E}_{13}}{\Sigma_1 - \Sigma_3} :: \frac{\mathcal{E}_{23}}{\Sigma_2 - \Sigma_3} : \ . \tag{4.126}$$

But this has the form of a *partial fraction identity* for the normally ordered matrix fractions. Note that the ordering of the factors in the identity is non-trivial. Indeed, showing that this identity holds directly seems to be quite tedious. Nonetheless, it hints at possible deeper elements to these normally ordered fractions which merit further exploration.

4.4.3 Structure under the mapping

Considering the wider Lax structure of (4.106), we find that the r-matrix structure (4.112c), (4.118) can also be extended to the Darboux matrix $\mathcal{M}(h)$ (the matrix generating the time evolution). Since $\mathcal{M}(h)$ depends only on the even numbered v_i, the non ultra-local Poisson structure means it has trivial Poisson bracket,

$$\{\mathcal{M}_1, \mathcal{M}_2\} = 0 \ . \tag{4.127}$$

Considering the r-matrix, it is also straightforward to show that the sum of the commutators is zero,

$$[r_{12}, \mathcal{M}_1] - [r_{21}, \mathcal{M}_2] = 0 \ . \tag{4.128}$$

But the sum of these results is that $\mathcal{M}(h)$ has the same r-matrix structure as $\mathcal{L}(h)$,

$$\{\mathcal{M}_1, \mathcal{M}_2\} = [r_{12}, \mathcal{M}_1] - [r_{21}, \mathcal{M}_2] = 0 \ , \tag{4.129}$$

albeit the structure holds in a "trivial" sense.

In order to describe the full r-matrix structure for the mapping, we also need a description of the interaction between $\mathcal{L}(h)$ and its factor $\mathcal{M}(h)$ that generates the time evolution. It is easily possible to calculate the relevant Poisson bracket,

$$\{\mathcal{L}_1, \mathcal{M}_2\} = \epsilon\delta \ \Sigma_1 \sum_j v_{2j}^{-2} \left(\overset{1}{D}_{2j+1} - \overset{1}{D}_{2j-1} \right) \overset{2}{D}_{2j-1} \ . \tag{4.130}$$

The "missing piece" of the r-matrix structure is to express this Poisson bracket in terms of the r-matrix (or possibly an "s-matrix" as in the local case (4.14)). The Poisson bracket $\{\mathcal{L}_1, \mathcal{M}_2\}$ describes the relations for the discrete time-evolution, and is needed for a complete description particularly in the quantum case, where establishing invariants is more difficult.

To be useful for the discrete time-evolution, the r-matrix structure (4.118) must be preserved under the mapping; in other words we need

$$\{\widehat{\mathcal{L}}_1, \widehat{\mathcal{L}}_2\} = [r_{12}, \widehat{\mathcal{L}}_1] - [r_{21}, \widehat{\mathcal{L}}_2] \ . \tag{4.131}$$

The evolved matrix $\widehat{\mathcal{L}}(h)$ is given in (4.108). To calculate the r-matrix structure we would ideally use the bracket $\{\mathcal{L}_1, \mathcal{M}_2\}$, but this part of the r-matrix structure remains unknown. However, using the expressions for $\mathcal{M}(h)$ and $\mathcal{N}(h)$ (4.106) we can calculate $\widehat{\mathcal{L}}(h)$ directly. In terms of the v_i, we find

$$\widehat{\mathcal{L}}(h) = \Sigma_h^2 - \Sigma_h \left(\sum_i v_{i+1} D_i + \epsilon\delta \sum_j (v_{2j}^{-1} - v_{2j+2}^{-1}) D_{2j} \right) + \epsilon\delta \sum_j D_{2j-1} \ . \tag{4.132}$$

From this, and using r_{12} (4.112c) and the Poisson bracket (4.7), it is possible (though tedious) to show directly that (4.131) holds, and

$$\{\widehat{\mathcal{L}}_1, \widehat{\mathcal{L}}_2\} \ = \ [r_{12}, \widehat{\mathcal{L}}_1] - [r_{21}, \widehat{\mathcal{L}}_2] \ = \ \Sigma_2 \mathcal{E}_{12} \Sigma_1 - \Sigma_1 \mathcal{E}_{12} \Sigma_2 \ . \tag{4.133}$$

This means that the r-matrix structure (4.118) is indeed preserved under the mapping. Recalling (4.117), the last equality of (4.133) then also gives us that

$$\{\widehat{\mathcal{L}}_1, \widehat{\mathcal{L}}_2\} = \{\mathcal{L}_1, \mathcal{L}_2\} \ , \tag{4.134}$$

so that the Poisson structure of the v_i can be clearly seen to be preserved by the mapping $\{v_i\} \to \{\widehat{v}_i\}$.

The r-matrix structure is then useful to prove that the system possesses sufficiently many invariants *in involution*, with respect to the Poisson bracket (4.7). The time evolution of the Lax matrix is described by a zero-curvature condition (4.108), which ensures that the trace of powers of $\mathcal{L}(h)$ are preserved under the mapping,

$$\mathrm{tr}\Big(\widehat{\mathcal{L}}^n(h)\Big) = \mathrm{tr}\Big(\mathcal{M}(h)\mathcal{L}^n(h)\mathcal{M}^{-1}(h)\Big) = \mathrm{tr}\Big(\mathcal{L}^n(h)\Big) \ . \tag{4.135}$$

Varying the power n yields sufficient invariants $(P-1)$ for the mapping. Note the link to the invariants derived from the local monodromy matrix $T(\lambda)$ (4.10) through the relation of the spectral curves (4.109).

Involutivity of these invariants is proved by considering $\mathrm{tr}_{12}\{\mathcal{L}_1^n(h), \mathcal{L}_2^m(h')\}$ (that is, the trace across both parts of the tensor product space) [9]. On the one hand,

$$\mathrm{tr}_{12}\{\mathcal{L}_1^n(h), \mathcal{L}_2^m(h')\} = \{\mathrm{tr}\mathcal{L}^n(h), \mathrm{tr}\mathcal{L}^m(h')\} \ . \tag{4.136}$$

On the other hand, using the r-matrix relation (4.118),

$$\begin{aligned}
\mathrm{tr}_{12}\{\mathcal{L}_1^n(h), \mathcal{L}_2^m(h')\} &= \mathrm{tr}_{12} \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \mathcal{L}_1^k \mathcal{L}_2^l \{\mathcal{L}_1, \mathcal{L}_2\} \mathcal{L}_1^{n-1-k} \mathcal{L}_2^{m-1-k} \ , && (4.137a)\\
&= \mathrm{tr}_{12} \sum_{k,l} \mathcal{L}_1^k \mathcal{L}_2^l \big([r_{12}, \mathcal{L}_1] - [r_{21}, \mathcal{L}_2] \big) \mathcal{L}_1^{n-1-k} \mathcal{L}_2^{m-1-k} \ , && (4.137b)\\
&= \mathrm{tr}_{12} \left(\sum_l \mathcal{L}_2^l [r_{12}, \mathcal{L}_1^n] \mathcal{L}_2^{m-1-l} - \sum_k \mathcal{L}_1^k [r_{21}, \mathcal{L}_2^m] \mathcal{L}_1^{n-1-k} \right) \ , && (4.137c)\\
&= m \ \mathrm{tr}_2 \Big(\mathrm{tr}_1 \big([r_{12}, \mathcal{L}_1^n] \big) \ \mathcal{L}_2^{m-1} \Big) - n \ \mathrm{tr}_1 \Big(\mathrm{tr}_2 \big([r_{21}, \mathcal{L}_2^m] \big) \ \mathcal{L}_1^{n-1} \Big) \ , && (4.137d)\\
&= 0 \ . && (4.137e)
\end{aligned}$$

Hence the invariants are in involution,

$$\{\mathrm{tr}\mathcal{L}^n(h), \mathrm{tr}\mathcal{L}^m(h')\} = 0 \ , \tag{4.138}$$

and therefore we have classical integrability of the map.

Much of the literature on classical r-matrices focuses on the skew-symmetric case [104], where the linear Poisson bracket algebra (4.118) can be recast into a quadratic form,

$$\{\mathcal{L}_1(h), \mathcal{L}_2(h')\} = [r_{12}(h - h'), \mathcal{L}_1(h)\mathcal{L}_2(h')] \ . \tag{4.139}$$

Such quadratic Poisson algebras then have a known relation to quantum R-matrix structures. The desire for skew-symmetric r-matrices is sufficiently strong that non skew-symmetric r-matrices have tended to receive less attention, even when they are the natural structure for a particular model [60]. We have uncovered here a novel formulation of the non skew-symmetric r-matrix as a normally ordered partial fraction, with interesting properties of its own that seem to be inherited from the fractional form. Moreover, this is the natural r-matrix for the McMillan map, with no apparent transformation to a unitary form. The outstanding problem is to relate the evolution as derived from the Lax pair to a variational formulation from the map, and hence (perhaps) to the possibility of a commuting flow and Lagrangian one-form structure.

4.4.4 The Quantum Case

The Local Lax matrices of the generalised McMillan map permit a quantisation via the quantum inverse scattering method - a quantum R-matrix structure (4.19). This structure allows the derivation of commuting quantum invariants, which is less straightforward than in the classical case, requiring the construction of the object $\tau(\lambda)$ (4.20). The classical Yang-Baxter structure of section 4.1.2 and (4.14) then arises naturally in the small $\hbar$ limit of this quantum structure. The dual r-matrix structure of (4.118) is a linear and non skew-symmetric structure, and it is less clear how this might arise from a wider quantum structure. However, there is some literature on the quantisation of non skew-symmetric r-matrices [42], and it is known that such linear structures *can* be transferred easily into the quantum regime via a quantum $\mathfrak{r}$-matrix [104]. We make some further remarks about the quantisation here.

In the quantum regime, the variables v_i become operators $\mathbf{v}_i$ with the Poisson bracket replaced by a commutator (4.17). Denoting the dual Lax matrix (4.104) in the operator case by $\underline{\mathcal{L}}(h)$, the linear nature of the structure yields a commutator that is a direct translation of the classical Poisson bracket (4.118) [104],

$$[\underline{\mathcal{L}}_1, \underline{\mathcal{L}}_2] = i\hbar\big([r_{12}, \underline{\mathcal{L}}_1] - [r_{21}, \underline{\mathcal{L}}_2]\big) \ . \tag{4.140}$$

This leads us to introduce the quantum $\mathfrak{r}$-matrix,

$$\mathfrak{r}_{12} = i\hbar r_{12} \ . \tag{4.141}$$

The linearity also means that the classical calculation of preservation of the Poisson bracket structure (4.134) carries over directly in the quantum regime to give

$$[\widehat{\underline{\mathcal{L}}}_1, \widehat{\underline{\mathcal{L}}}_2] \ = \ [\underline{\mathcal{L}}_1, \underline{\mathcal{L}}_2] \ = \ [\mathfrak{r}_{12}, \underline{\mathcal{L}}_1] - [\mathfrak{r}_{21}, \underline{\mathcal{L}}_2] \ . \tag{4.142}$$

So, the commutator bracket structure is also preserved under the mapping.

Recall the pseudo-skew-symmetry of r_{12} (4.120). The r-matrix remarkably also exhibits a quantum-type unitarity. Consider the product,

$$\begin{aligned} r_{12}(h,h') \ . \ r_{21}(h',h) &= \frac{1}{h-h'}\sum_{k=1}^{2N} \Sigma_1^{2N-k}\mathcal{E}_{12}\Sigma_2^k \ . \ \frac{1}{h'-h}\sum_{l=1}^{2N} \Sigma_2^{2N-l}\mathcal{E}_{21}\Sigma_1^l \ , \\ &= \frac{-1}{(h-h')^2}\sum_{k,l,i,j=1}^{2N} \Sigma_1^{2N-k+l} \ \overset{1}{D}_{i+l}\overset{2}{D}_i\overset{1}{D}_j\overset{2}{D}_{j-k} \ \Sigma_2^{2N-l+k} \ , \\ &= \frac{-hh'}{(h-h')^2}\mathbb{I}\otimes\mathbb{I} \ . \end{aligned} \tag{4.143}$$

So, r_{21} is a scaled inverse for r_{12}: the r-matrices are essentially unitary. In terms of the quantum $\mathfrak{r}$-matrix, we can write the unitarity as

$$\frac{h-h'}{h}\mathfrak{r}_{12}(h,h') \;.\; \frac{h'-h}{h'}\mathfrak{r}_{21}(h',h) = (i\hbar)^2 \;. \tag{4.144}$$

This is unusual: r_{12} is a *classical* r-matrix, exhibiting a quantum property. This property allows for a rearrangement of the commutator bracket (4.140) as

$$\Big[\underline{\mathcal{L}}_1 + \mathfrak{r}_{21} \,,\, \underline{\mathcal{L}}_2 + \mathfrak{r}_{12}\Big] = 0 \;. \tag{4.145}$$

This represents the quantum commutation relations for the dual Lax matrix. Compatibility of the bracket (4.145) under a Jacobi-type relation leads to the classical Yang-Baxter equation (4.122) as a condition, which we already know is satisfied by $\mathfrak{r}_{12}$, due to the three term relation (4.123). Notably, the Yang-Baxter equation does not appear as a requirement, as in many standard solutions for quantum inverse scattering.

This quantum commutation relation (4.145) can be extended into a more general case. Consider additional $\mathfrak{r}$-matrices in the product,

$$\begin{aligned}(\underline{\mathcal{L}}_1 + \mathfrak{r}_{21} + \mathfrak{r}_{31})\,(\underline{\mathcal{L}}_2 + \mathfrak{r}_{12} + \mathfrak{r}_{32}) \\ = (\underline{\mathcal{L}}_1 + \mathfrak{r}_{21})\,(\underline{\mathcal{L}}_2 + \mathfrak{r}_{12}) + \mathfrak{r}_{31}\,(\underline{\mathcal{L}}_2 + \mathfrak{r}_{12}) + (\underline{\mathcal{L}}_1 + \mathfrak{r}_{21})\,\mathfrak{r}_{32} + \mathfrak{r}_{31}\mathfrak{r}_{32} \;. \end{aligned} \tag{4.146a}$$

Now, using the commutator (4.145), and identifying those matrices which commute, almost all terms can be commuted,

$$= (\underline{\mathcal{L}}_2 + \mathfrak{r}_{12})\,(\underline{\mathcal{L}}_1 + \mathfrak{r}_{21}) + \underline{\mathcal{L}}_2\mathfrak{r}_{31} + \mathfrak{r}_{32}\underline{\mathcal{L}}_1 + \mathfrak{r}_{31}\mathfrak{r}_{12} + \mathfrak{r}_{21}\mathfrak{r}_{32} + \mathfrak{r}_{31}\mathfrak{r}_{32} \;. \tag{4.146b}$$

Finally, the paired $\mathfrak{r}$-matrices can be reversed by exploiting the three-term relation (4.123), so that these entire expressions commute,

$$(\underline{\mathcal{L}}_1 + \mathfrak{r}_{21} + \mathfrak{r}_{31})\,(\underline{\mathcal{L}}_2 + \mathfrak{r}_{12} + \mathfrak{r}_{32}) = (\underline{\mathcal{L}}_2 + \mathfrak{r}_{12} + \mathfrak{r}_{32})\,(\underline{\mathcal{L}}_1 + \mathfrak{r}_{21} + \mathfrak{r}_{31}) \;. \tag{4.146c}$$

This proof extends inductively to the addition of an arbitrary number of $\mathfrak{r}$-matrix terms, so that

$$\Big[\underline{\mathcal{L}}_1 + i\hbar(r_{21} + \ldots + r_{n1})\,,\, \underline{\mathcal{L}}_2 + i\hbar(r_{12} + r_{32} + \ldots + r_{n2})\Big] = 0 \;. \tag{4.147}$$

It follows that n such mutually commuting terms can be created,

$$\left[\Big(\underline{\mathcal{L}}_j + i\hbar\sum_{\substack{i=1\\ i\neq j}}^{n} r_{ij}\Big)\,,\,\Big(\underline{\mathcal{L}}_k + i\hbar\sum_{\substack{i=1\\ i\neq k}}^{n} r_{ik}\Big)\right] = 0\,, \quad \forall j,k = 1,\ldots,n \;. \tag{4.148}$$

It remains unclear, however, how such a structure could be used to derive the quantum invariants of this dual Lax matrix.

Mysteriously, despite the lack of a quadratic R-matrix structure, $\mathfrak{r}_{12}$ *is* a solution to the Yang-Baxter equation. Proceeding from the expression for r_{12} (4.112c), we can write

$$\begin{aligned} r_{12}\, r_{13}\, r_{23} &= \frac{1}{H_{123}} \sum_{k,l,m=1}^{2N} \left(\Sigma_1^{2N-k}\mathcal{E}_{12}\Sigma_2^k\right)\left(\Sigma_1^{2N-l}\mathcal{E}_{13}\Sigma_3^l\right)\left(\Sigma_2^{2N-m}\mathcal{E}_{23}\Sigma_3^m\right) && (4.149a)\\ &= \frac{1}{H_{123}} \sum_{k,l,m=1}^{2N} \Sigma_1^{4N-k-l}\Sigma_2^{2N+k-m}\Sigma_3^{l+m} \\ &\quad \times \sum_{i,j,s=1}^{2N} \overset{1}{D}_{i-l}\overset{2}{D}_{i+k-m}\overset{1}{D}_j\overset{3}{D}_{j+l+m}\overset{2}{D}_s\overset{3}{D}_{s+m} \ , && (4.149b)\\ &= \frac{h'}{H_{123}} \sum_{k,l=1}^{2N} \Sigma_1^{4N-k-l}\left(\sum_{i=1}^{2N}\overset{1}{D}_i\overset{2}{D}_{i+k}\overset{3}{D}_i\right)\Sigma_3^{k+l} \ . && (4.149c) \end{aligned}$$

In the final equality we have used the product for elementary diagonal matrices $D_iD_j = \delta_{ij}D_i$, and we have used the shorthand for the spectral parameters $H_{123} = (h-h')(h-h'')(h'-h'')$. Then, the remarkable result is that, with careful relabelling of the parameters, we acquire the same result for the product $r_{23}\, r_{13}\, r_{12}$, so that we have the Yang-Baxter equation for the quantum $\mathfrak{r}$-matrix,

$$\mathfrak{r}_{12}\, \mathfrak{r}_{13}\, \mathfrak{r}_{23} = \mathfrak{r}_{23}\, \mathfrak{r}_{13}\, \mathfrak{r}_{12} \ . \qquad (4.150)$$

However, so far it is unclear where, or whether, a requirement arises in the structure for $\mathfrak{r}_{12}$ to obey the Yang-Baxter equation: in other words, it is so far unclear how this result might be useful.

As discussed in [74, 103] uncovering the invariants in the quantum case is more complicated than classically, since we must deal with the issue of operator ordering. Whilst classically it is straightforward to find invariants using the trace (4.135), in the quantum case this is no longer sufficient: non-trivial commutation relations mean that the cyclic property of the trace does not hold in general. In order to prove both invariance and commutativity, some quantum corrected object may be required, as in the local Lax case (4.20). What is missing to prove these properties, however, is an $\mathfrak{r}$-matrix structure encompassing the time evolution element $\mathcal{M}(h)$ - exactly as in the classical case (4.130). An alternative approach for some models has been the creation of central objects in the operator algebra called quantum determinants [39, 103, 104], but these have generally been applied to 2×2 Lax matrices.

However, we can make some progress in this example by considering the simplest case, when $2P = 4$: the McMillan map. Then the quantum dual Lax matrix (4.100) is given by

$$\underline{\mathcal{L}}(h) = \begin{pmatrix} \epsilon\delta & -\mathbf{v}_2 & 1 & 0 \\ 0 & 0 & -\mathbf{v}_3 & 1 \\ h & 0 & \epsilon\delta & -\mathbf{v}_4 \\ -h\mathbf{v}_1 & h & 0 & 0 \end{pmatrix} . \tag{4.151}$$

The classical invariant for the map is given by $\mathrm{tr}(\mathcal{L}^4)$ (or equivalently $\det(\lambda - \mathcal{L}(h))$),

$$I = (v_4 v_3 - \epsilon\delta)(v_2 v_1 - \epsilon\delta) . \tag{4.152}$$

In the quantum case there is a correction to the McMillan invariant, so that it is given by

$$\mathbf{I} = (\mathbf{v}_4\mathbf{v}_3 - \epsilon\delta)(\mathbf{v}_2\mathbf{v}_1 - \epsilon\delta) + i\hbar\mathbf{v}_3\mathbf{v}_2 . \tag{4.153}$$

(See section 4.3.) In the local Lax case this is derived from $\tau(\lambda)$ (4.20), but can also be verified with the equations of motion (4.5) and commutator bracket (4.17) [74].

For the quantum Lax matrix $\underline{\mathcal{L}}(h)$, we then consider the trace $\mathrm{tr}(\underline{\mathcal{L}}^4(h))$. We evaluate this trace with consideration for the operator ordering in the matrix product, and recall the algebra Casimirs (4.6), so that

$$\begin{aligned} \mathrm{tr}(\underline{\mathcal{L}}^4) &= 2\big((\epsilon\delta)^2 + h\big)^2 + 2h^2 + 16h\epsilon^3\delta + h\Big[\mathbf{v}_3\mathbf{v}_4\mathbf{v}_1\mathbf{v}_2 + \mathbf{v}_1\mathbf{v}_2\mathbf{v}_3\mathbf{v}_4 \\ &\qquad + (2\epsilon\delta + \mathbf{v}_4\mathbf{v}_1)(2\epsilon\delta + \mathbf{v}_2\mathbf{v}_3) + (2\epsilon\delta + \mathbf{v}_2\mathbf{v}_3)(2\epsilon\delta + \mathbf{v}_4\mathbf{v}_1)\Big] , && \text{(4.154a)} \\ &= 2\big((\epsilon\delta)^2 + h\big)^2 + 2h^2 + 16h\epsilon^3\delta + 4h\Big[(\mathbf{v}_4\mathbf{v}_3 - \epsilon\delta)(\mathbf{v}_2\mathbf{v}_1 - \epsilon\delta) \\ &\qquad + i\hbar\mathbf{v}_3\mathbf{v}_2 + (\epsilon\delta)^2 + 4\epsilon^3\delta - 3i\hbar\epsilon^2 + \tfrac{1}{2}\hbar^2\Big] , && \text{(4.154b)} \\ &= 4h\mathbf{I} + 2\big((\epsilon\delta)^2 + h\big)^2 + 2h^2 + 16h\epsilon^3\delta \\ &\qquad + 4h\Big[(\epsilon\delta)^2 + 4\epsilon^3\delta - 3i\hbar\epsilon^2 + \tfrac{1}{2}\hbar^2\Big] . && \text{(4.154c)} \end{aligned}$$

So, in the McMillan case, $\mathrm{tr}(\underline{\mathcal{L}}^4(h))$ yields *exactly* the correct quantum invariant $\mathbf{I}$ (4.153). The natural ordering of the factors in the matrix product turns out to yield the correct operator ordering. In this simplest case, then, the traces of the powers of $\underline{\mathcal{L}}$ are indeed conserved in the quantum regime: this may be a crucial hint to uncovering the general structure.

4.5 Summary

By exploiting the multi-dimensional consistency of the lattice KdV equation, we investigated discrete flows that commute with the McMillan maps. We have written explicit equations for such a commuting map for the simplest possible reduction. However, due to the difficulties of the non-linearity, we have not been able to write this map in a generating function form. It remains to be shown that these maps must be canonical, and that they preserve the invariants of the initial maps, so that uncovering the desired Lagrangian one-form structure has not so far been possible.

For the simplest member of the mapping family, the McMillan map, we have shown that the choice of initial Lagrangian leads to the time evolution operator and one-step propagator. Approaching the propagator via the time evolution operator, derived from the Hamilton's equations, appears to be equivalent to deriving the propagator though a sandwiching of the Euler-Lagrange equations between position eigenstates. The key element is allowing the choice of Lagrangian to determine the commutation relations. There are possible insights to be gained regarding the Hilbert space from considering the group property of the propagators, and also from considering stationary states of the operator invariant. However, the key outstanding question is how to resolve the two singularities occurring at $q = \pm 1$. The resolution of singularities in quantum mechanics is an important question in the study of integrable systems, for which this model may be a useful test case.

Considering an alternative approach, we examined the dual Lax matrix structure for the generalised McMillan maps, and uncovered a novel expression for the r-matrix as a normal-ordered fraction of shift matrices Σ_h in the matrix tensor product space. This structure leads to interesting formulations of the pseudo-skew-symmetry and a three-term relation that is a stronger form of the classical Yang-Baxter equation, and can also be used to prove preservation and involutivity of the invariants. The r-matrix structure is also preserved under the mapping, although the full structure (in particular, relating to the Darboux matrix $\mathcal{M}(h)$) is not known completely. This dual r-matrix structure leads to a linear, quantum $\mathfrak{r}$-matrix structure that encodes the quantum commutation relations, and where the matrix $\mathfrak{r}_{12}$ is remarkably a solution of the Yang-Baxter equation. It appears possible that the quantum invariants may be encoded simply in the trace of powers of the Lax matrix $\mathcal{L}(h)$, but the full structure required to show this in the general case remains unknown.

The McMillan map remains, from the perspective of the Lagrangian one-form, unsolved. Although one-form structures have been found to describe a range of integrable maps, it is not yet clear how this structure might apply in cases like this one. A key missing element is the commuting discrete flow that is typically described by the one-form structure. However, by deepening our understanding of the Lax structures, in particular for the dual Lax matrix, perhaps commuting flows can be uncovered in a similar way to the discrete time models of Calogero-Moser type [77, 81, 125, 126].

5

The Degasperis-Ruijsenaars Model: Propagator and Lagrangian Formalism

The Degasperis-Ruijsenaars (DR) model was proposed in [26], where it is studied from the perspective of *Newton equivalence*. The DR model has the classical equation of motion for a harmonic oscillator, but derived from a non-Newtonian Hamiltonian which is multiplicatively separable. This Hamiltonian is "Newton equivalent" to the standard harmonic oscillator, since the classical Newton equations of motion are the same. The DR Hamiltonian has a *relativistic* form and is closely related to the integrable Ruijsenaars-Schneider[1] (RS) model [96, 98]. Additionally to its classical properties, there is an exact canonical quantisation of the DR model by creation and annihilation operators, in analogy to a standard approach to the quantum harmonic oscillator. However, this quantisation is *not* unitary equivalent to the standard harmonic oscillator, and differs by a shift in the energy levels. In other words, although it is classically equivalent to the harmonic oscillator, it is different on the quantum level. As for the one-form structure investigated

[1] or relativistic Calogero-Moser

in chapters 2 and 3, the DR model reveals hidden depths to the humble harmonic oscillator.

As discussed in section 1.2.2, the Ruijsenaars-Schneider model has a known integrable discretisation [81] and, recently, a Lagrangian one-form structure [126]. Conversely, the Degasperis-Ruijsenaars model is a harmonic oscillator: it may be a suitable example for a path integral quantisation of a non-Newtonian model, allowing a comparison with the established canonical quantisation. Through its relation to the RS model this may provide crucial insights into the path integral quantisation of integrable models with a Lagrangian one-form structure and to the discrete counterpart. Although the DR model is a model in continuous time, path integral quantisation naturally raises the question of discretisation in the time-slicing approach. For integrable models, is the correct time-slicing one that utilises the integrable discretisation?

After introducing the Degasperis-Ruijsenaars model, we develop a novel Lagrangian description of the model by embedding it as a centre of mass system in a two-particle setting. We see that this again yields the harmonic oscillator equation of motion (i.e. is a Newton equivalent description) although the separation of variables is not manifest on the level of the Lagrangian. By making explicit the connection to the Ruijsenaars-Schneider model, we are also able to write a Lax pair for the DR system. Finally, considering the quantisation of the model, we exploit the known results of the canonical quantisation to derive an expression for the propagator, and consider some aspects of the Lagrangian structure that may lead towards a possible path integral quantisation.

5.1 The Degasperis-Ruijsenaars Model

In [26] a new system was derived from the classical Newtonian equation of motion for the harmonic oscillator, with the Hamiltonian

$$H_\beta = \frac{1}{\beta^2 m} \cosh\left(\beta p\right) (1 + \lambda^2 x^2)^{1/2} \;, \tag{5.1}$$

with position x and momentum p. Hamilton's equations yield

$$\dot{x} = \frac{1}{\beta m} \sinh(\beta p)\, (1 + \lambda^2 x^2)^{1/2} \;, \tag{5.2a}$$

$$\dot{p} = -\frac{\lambda^2}{\beta^2 m} \cosh(\beta p)\, x(1 + \lambda^2 x^2)^{1/2} \;, \tag{5.2b}$$

so that, when the momentum p is eliminated, the equation of motion is

$$\ddot{x} = -\left(\frac{\lambda}{\beta m}\right)^2 x \ . \tag{5.3}$$

In other words, the Hamiltonian H_β (5.1) generates the harmonic oscillator, with angular frequency $\omega = \lambda/\beta m$.

This makes this a very interesting model. The harmonic oscillator (also discussed in chapters 2 and 3) is the standard example for understanding path integral quantisation, due to the advantages gained from its quadratic Hamiltonian [37, 47, 99]. But, here is an alternative, non-Newtonian Hamiltonian yielding the same classical equation of motion. As the authors go on to show to [26], this Hamiltonian can also be canonically quantised. Can such a model, despite losing the benefits of the quadratic Hamiltonian, also be path integral quantised?

The Hamiltonian (5.1) is interpreted as a *relativistic harmonic oscillator*, with the parameter β playing the role of the inverse speed of light [26]. This is justified by embedding the Hamiltonian into the two particle model,

$$H_R = \frac{1}{\beta^2 m}\Big[\cosh(\beta p_1) + \cosh(\beta p_2)\Big]\Big(1 + \lambda^2 (x_1 - x_2)^2\Big)^{1/2} \ . \tag{5.4}$$

By changing to centre of mass co-ordinates,

$$x = x_1 - x_2 \ , \qquad X = \frac{1}{2}(x_1 + x_2) \ , \tag{5.5a}$$

$$p = \frac{1}{2}(p_1 - p_2) \ , \qquad P = p_1 + p_2 \ , \tag{5.5b}$$

the Hamiltonian H_R can be written in the form

$$H_R = 2\cosh\left(\frac{\beta P}{2}\right) H_\beta(x,p) \ , \tag{5.6}$$

with H_β (5.1) appearing as the centre of mass Hamiltonian. The two particle Hamiltonian H_R separates *multiplicatively* into an (X, P) and an (x, p) component. Note that separation of variables usually occurs *additively*; nonetheless we will see that this separation is effective due to the total momentum P being an integral of the motion.

The relativistic aspect of the model then arises by considering the Lie algebra of symmetries for H_R (5.4) (the invariants (5.9)) which represent the Lorentz group. Additionally, considering the non-relativistic limit on H_R as $\beta \to 0$, recalling that

$\lambda = \beta m\omega$,

$$H_R = \frac{1}{\beta^2}\left(\frac{2}{m} + \beta^2\Big(\tfrac{1}{2m}p_1^2 + \tfrac{1}{2m}p_2^2 + \tfrac{1}{2}m\omega^2x^2\Big) + O(\beta^4)\right) , \tag{5.7a}$$

$$= \frac{2}{m\beta^2} + H_{HO} + O(\beta^2) . \tag{5.7b}$$

The standard Newtonian Hamiltonian for the harmonic oscillator, H_{HO}, reappears in the non-relativistic limit, with a constant shift to the energy $-2/m\beta^2$.

Considering the dynamics of the two particle Hamiltonian H_R (5.4), Hamilton's equations yield immediately that

$$\dot{P} = 0 , \tag{5.8a}$$

so that the total momentum P is an invariant of the motion. We also find equations for the positions,

$$\ddot{x}_1 = -\ddot{x}_2 = -\frac{2\lambda^2}{\beta^2m^2}\left(\cosh\left(\frac{\beta P}{2}\right)\right)^2 x , \tag{5.8b}$$

so that in terms of the centre of mass variables,

$$\ddot{X} = 0 , \quad \ddot{x} = -\left(\frac{2\lambda}{\beta m}\cosh\left(\frac{\beta P}{2}\right)\right)^2 x . \tag{5.8c}$$

So the total velocity $\dot{X}$ is constant, and the centre of mass variable x obeys the harmonic oscillator equation (5.3), with angular frequency $\omega = 2\lambda\cosh(\beta P/2)/\beta m$, as in the one particle model (5.1) - compare to the one-particle equation of motion (5.3). Notice that we also have two integrals of the motion, P (5.8a) and $\dot{X}$ (5.8c), so that

$$I_1 = P = p_1 + p_2 , \tag{5.9a}$$

$$I_2 = \dot{X} = \frac{1}{2\beta m}\big(\sinh(\beta p_1) + \sinh(\beta p_2)\big)(1 + \lambda^2x^2)^{1/2} . \tag{5.9b}$$

These are independent and in involution, $\{I_1, I_2\} = 0$, and hence we have Liouville integrability of the two particle model.

Other "relativistic oscillators" have been considered in the literature in a number of places; Degasperis and Ruijsenaars highlight [7, 8] as describing a relativistic oscillator Hamiltonian that is directly related to the DR model by a unitary similarity transform.

5.2 Lagrangian Description

In the section below we derive and investigate a Lagrangian description for the Degasperis-Ruijsenaars model of [26]. We search for a Lagrangian that is Newton equivalent to the harmonic oscillator; that is, a Lagrangian that produces the harmonic oscillator as an equation of motion, but does not have the standard Newtonian form for a harmonic oscillator Lagrangian. This Lagrangian perspective has been previously considered in [108] where the authors derived a relativistic form of Newton-equivalent Lagrangian, but our perspective differs in that we seek a Lagrangian explicitly related to the DR model by Legendre transform.

5.2.1 Legendre Transform

The Hamiltonian perspective on the Degasperis-Ruijsenaars oscillator is well understood [26], but for our purposes we are interested in the companion variational approach. In general, the Lagrangian is found via a Legendre transform from the Hamiltonian H_β (5.1), but attempting such a Legendre transform for the DR model produces an undesirably complicated Lagrangian that does not seem amenable for further study. However, the two particle Hamiltonian H_R (5.4) [26] turns out to be an appropriate setting for a Legendre transform and a variational formulation of the model. This is unsurprising: the two particle form of the DR oscillator is closely related to the Ruijsenaars-Schneider model, which has a known Lagrangian form [20]. This relation will be established in more detail in section 5.2.3.

Establishing the Lagrangian for the two particle Hamiltonian H_R (5.4) rests on the integrals of the motion. For Lagrangian one-form structures, Legendre transforms relate the hierarchy of commuting integrals to the components of the one-form structure [110, 126]. Although in the two particle case we expect only a single Lagrangian, the invariants maintain their importance. Considering the integrals of motion I_1, I_2 (5.9), it is easy to see that we equivalently have commuting integrals

$$S_1 = e^P = e^{p_1+p_2} , \tag{5.10a}$$

$$S_2 = 2\beta^2 m e^{\beta P/2} H_\beta = \left(e^{\beta p_1} + e^{\beta p_2}\right)(1+\lambda^2 x^2)^{1/2} , \tag{5.10b}$$

with H_β the one particle Hamiltonian given in (5.1). Performing a Legendre transform on the two particle Hamiltonian H_R (5.4) is also difficult. However, if we consider the time flow generated by the invariant S_2 (5.10b), then we are able to perform a Legendre transform. In other words, we take S_2 to be the two particle Hamiltonian.

We generate Hamilton's equations from S_2,

$$\dot{x}_i = \beta e^{\beta p_i}(1+\lambda^2 x^2)^{1/2}, \tag{5.11}$$

and Legendre transform the invariant S_2 in the usual way. This yields the Lagrangian

$$\mathcal{L}_\beta(x,\dot{x}) = \frac{1}{\beta}\left[\dot{x}_1 \ln \dot{x}_1 + \dot{x}_2 \ln \dot{x}_2 - \frac{1}{2}(\dot{x}_1+\dot{x}_2)\ln(1+\lambda^2 x^2) - (\dot{x}_1+\dot{x}_2)\right]. \tag{5.12}$$

Removing the overall multiplier $1/\beta$ and a total derivative term, this is equivalent to the Lagrangian

$$\mathcal{L}_{DR}(x,\dot{x}) = \dot{x}_1 \ln \dot{x}_1 + \dot{x}_2 \ln \dot{x}_2 - \frac{1}{2}(\dot{x}_1+\dot{x}_2)\ln(1+\lambda^2 x^2)\ . \tag{5.13}$$

As for the two particle Hamiltonian (5.4), this closely resembles the known Lagrangian for the Ruijsenaars-Schneider model [20]. Indeed such $\dot{x}\ln\dot{x}$ kinetic terms are characteristic of these relativistic models [18, 20, 97, 126]. Note that the relativistic parameter β has disappeared from the Lagrangian, amounting to a fixing of the gauge, but is easily reintroduced.

5.2.2 Lagrangian dynamics

Is the Lagrangian $\mathcal{L}_{DR}$ (5.13) really equivalent to the Degasperis-Ruijsenaars Hamiltonian H_R (5.4)? We saw in section 5.2.1 that they are not related by a Legendre transform. However, it is easy to see from the dynamics of $\mathcal{L}_{DR}$ that we recover the essential harmonic oscillator motion on the level of the centre of mass motion that characterises the DR oscillator.

The Lagrangian $\mathcal{L}_{DR}$ (5.13) yields the Euler-Lagrange equations

$$\ddot{x}_1 \ = \ -2\lambda^2 x\frac{\dot{x}_1\dot{x}_2}{1+\lambda^2x^2}\ , \qquad \ddot{x}_2 \ = \ 2\lambda^2 x\frac{\dot{x}_1\dot{x}_2}{1+\lambda^2x^2}\ . \tag{5.14}$$

In terms of the centre of mass variables $x = x_1 - x_2$ and $X = (x_1+x_2)/2$ (5.5), the equations of motion are

$$\ddot{x} \ = \ -4\lambda^2 x\frac{\dot{x}_1\dot{x}_2}{1+\lambda^2x^2}\ , \qquad \ddot{X} \ = \ 0\ , \tag{5.15}$$

so that there is a constant centre of mass motion, and $\dot{X}$ is an integral of the motion.

Now, considering the conjugate momenta arising from the Lagrangian $\mathcal{L}_{DR}$,

$$p_i \ = \ \frac{\partial\mathcal{L}}{\partial\dot{x}_i} \ = \ \ln\dot{x}_i + 1 - \tfrac{1}{2}\ln(1+\lambda^2x^2)\ , \tag{5.16}$$

the Euler-Lagrange equations (5.14) yield the conservation of the total momentum,

$$\frac{d}{dt}P = \frac{\partial \mathcal{L}_{DR}}{\partial x_1} + \frac{\partial \mathcal{L}_{DR}}{\partial x_2} = 0 \; . \tag{5.17}$$

Hence expressing the conjugate momenta in terms of x, $\dot{x}$ (5.16) we can write the integral of the motion

$$I_1 \;=\; \frac{\dot{x}_1 \dot{x}_2}{1+\lambda^2 x^2} \;=\; e^{p_1+p_2-2} \; . \tag{5.18}$$

The equation of motion in the centre of mass variables (5.15) therefore reduces to

$$\ddot{x} = -4 I_1 \lambda^2 x^2 \; , \tag{5.19}$$

which is the equation of motion for a one-dimensional harmonic oscillator, as in the Hamiltonian description (5.3) and (5.8c), with angular frequency $\omega^2 = 4I_1\lambda^2$. In other words, the Lagrangian $\mathcal{L}_{DR}$ (5.13) once more results in the harmonic oscillator equation of motion; in the terminology of [26] it is “Newton equivalent” to the usual harmonic oscillator Lagrangian. Additionally, the equations of motion describe the same dynamics as the two particle DR Hamiltonian (5.8c), and so the Lagrangian $\mathcal{L}_{DR}$ and Hamiltonian H_R are equivalent, up to a gauge choice of momentum variables.

In addition to the total momentum, yielding the invariant I_1 (5.18), the centre of mass velocity $\dot{X}$ is also an invariant of the motion (5.15),

$$I_2 \;=\; \frac{1}{2}(\dot{x}_1 + \dot{x}_2) \;=\; \frac{1}{2e}(e^{p_1} + e^{p_2})(1+\lambda^2 x^2)^{1/2} \; , \tag{5.20}$$

which is essentially the generating Hamiltonian S_2 (5.10b).

5.2.3 Comparison to two particle Ruijsenaars-Schneider Model

Writing q_i, p_i for position and momentum, the two particle Ruijsenaars-Schneider model has a Hamiltonian of the form [96]

$$H_{RS}(q,p) \;=\; mc^2(\cosh p_1 + \cosh p_2)\left(\wp(\nu) - \wp(q_1 - q_2)\right)^{1/2} \; , \tag{5.21a}$$

$$\;=\; 2mc^2 \cosh P \cosh p \left(\wp(\nu) - \wp(q)\right)^{1/2} \; , \tag{5.21b}$$

for centre of mass variables q, p, P (compare (5.5)), parameter ν, and where $\wp(q)$ is the Weierstrass $\wp$ function. This has invariants,

$$S_1 = e^{p_1+p_2} \; , \qquad S_2 = (e^{p_1} + e^{p_2})(\wp(\nu) - \wp(q))^{1/2} \; . \tag{5.22}$$

So, notice that the Hamiltonian and invariants here have the same forms as the Degasperis-Ruijsenaars case (5.4), (5.10a) and (5.10b), with a different potential term.

The invariant S_2 undergoes a Legendre transform to

$$\mathcal{L}_{RS} = \dot{q}_1 \ln \dot{q}_1 + \dot{q}_2 \ln \dot{q}_2 - \frac{1}{2}(\dot{q}_1 + \dot{q}_2) \ln \left(\wp(\nu) - \wp(q)\right) \ , \tag{5.23}$$

in the same way as the DR Lagrangian $\mathcal{L}_{DR}$ (5.13). Similarly, the equations of motion reduce in the centre of mass variable q to

$$\ddot{q} \ = \ 2\dot{q}_1\dot{q}_2 \frac{\wp'(q)}{\wp(\nu) - \wp(q)} \ = \ 2e^{P-2}\wp'(q) \ . \tag{5.24}$$

This has the same form as the centre of mass equations of motion for the DR model (5.15), with an altered potential.

In fact it is possible to make the connection between the Ruijsenaars-Schneider and Degasperis-Ruijsenaars models explicit. The DR model arises as a specific linearisation from the *trigonometric* case of the RS model.[2] The trigonometric case arises from a reduction on the periods of the elliptic $\wp$ function to produce a trigonometric potential, $\wp(q) \to \operatorname{cosec}^2 q$ (compare the limits of (1.51)), so that the Lagrangian (5.23) and equations of motion (5.24) become

$$\mathcal{L}_{Trig}(q, \dot{q}) \ = \ \dot{q}_1 \ln \dot{q}_1 + \dot{q}_2 \ln \dot{q}_2 - \frac{1}{2}(\dot{q}_1 + \dot{q}_2) \ln \left(\operatorname{cosec}^2\nu - \operatorname{cosec}^2 q\right) \ , \tag{5.25a}$$

$$\ddot{q} \ = \ -4\dot{q}_1\dot{q}_2 \frac{\operatorname{cosec}^3 q \cos q}{\operatorname{cosec}^2\nu - \operatorname{cosec}^2 q} \ . \tag{5.25b}$$

To reduce to the equations for the DR model, we introduce the small parameter ϵ, and expand with an angular shift,

$$q = \frac{\pi}{2} + \epsilon x \ , \quad \nu = \frac{\pi}{2} + \epsilon \mu \ . \tag{5.26}$$

Applied to the trigonometric Lagrangian and equation of motion (5.25), these yield to highest order in ϵ

$$\mathcal{L}(x, \dot{x}) \ = \ \dot{x}_1 \ln \dot{x}_1 + \dot{x}_2 \ln \dot{x}_2 - \frac{1}{2}(\dot{x}_1 + \dot{x}_2) \ln(\mu^2 - x^2) \ , \tag{5.27a}$$

$$\ddot{x} \ = \ \frac{4\dot{x}_1\dot{x}_2 x}{\mu^2 - x^2} \ . \tag{5.27b}$$

But, these are exactly the equations for the Degasperis-Ruijsenaars model under the parameter relabelling $\mu^2 = -1/\lambda^2$: the replacement yields precisely the DR Lagrangian $\mathcal{L}_{DR}$ (5.13) and equations of motion (5.15).

[2] I am grateful to S. Ruijsenaars for this hint.

We exploit this direct limit from the trigonometric Ruijsenaars-Schneider model to derive two additional results below: an alternative Lagrangian and a Lax pair for the Degasperis-Ruijsenaars model.

Alternative Choice of Lagrangian

In [126], the authors derived a discrete Lagrangian one-form structure for the integrable discretisation of the Ruijsenaars-Schneider model, discussed in section 1.2.2. A continuum limit led to an alternative Lagrangian to $\mathcal{L}_{RS}$ (5.23) for the *rational* RS model, in the continuous Lagrangian one-form. In the rational limit, the elliptic function reduces to a rational function, $\wp(q) \to 1/q^2$, and the lowest member of the Lagrangian one-form hierarchy (for the two particle model) is given by

$$\mathcal{L}_{rat}(q, \dot{q}) = \dot{q}_1 \ln \dot{q}_1 + \dot{q}_2 \ln \dot{q}_2 - \dot{q}_1 \big(\ln(q + \nu) - \ln q \big) - \dot{q}_2 \big(\ln(q - \nu) - \ln q \big) \ . \qquad (5.28)$$

Compared to the Lagrangian $\mathcal{L}_{RS}$ the potential term has been factorised and rearranged. Guided by this Lagrangian, we seek an alternative Lagrangian to $\mathcal{L}_{DR}$ (5.13) for the Degasperis-Ruijsenaars model. The key observation is the factorisation of the potential term,

$$1 + \lambda^2 x^2 = (1 + i\lambda x)(1 - i\lambda x) \ , \qquad (5.29)$$

as used in the canonical quantisation of the model [26] (see section 5.3.1). A little investigation reveals the alternative Lagrangian for the DR model,

$$\overline{\mathcal{L}}_{DR}(x, \dot{x}) = \dot{x}_1 \ln \dot{x}_1 + \dot{x}_2 \ln \dot{x}_2 - \dot{x}_1 \ln(1 + i\lambda x) - \dot{x}_2 \ln(1 - i\lambda x) \ . \qquad (5.30)$$

Note that although the Lagrangian $\overline{\mathcal{L}}_{DR}$ (5.30) does not Legendre transform to the same Hamiltonian as $\mathcal{L}_{DR}$ (5.13), it nonetheless has the same classical dynamics. Essentially, it differs in a gauge choice of momentum variables. The simplicity of this form of Lagrangian is particularly appealing, especially in seeking a suitable path integral quantisation.

Lax Pair

The close relation between the Degasperis-Ruijsenaars model and the two particle Ruijsenaars-Schneider model allows us to find a Lax pair encoding the DR system and capturing the integrals of the motion, by modifying the known Lax pair for RS. Recalling that the systems are linked via the trigonometric RS model (5.25), which has the Lax pair [59, 96]

$$L_{trig}(\kappa) = \begin{pmatrix} \dot{q}_1 \frac{\sin(\kappa+\nu)}{\sin\kappa\sin\nu} & (\dot{q}_1\dot{q}_2)^{1/2}\frac{\sin(\kappa+\nu+q)}{\sin\kappa\sin(\nu+q)} \\ (\dot{q}_1\dot{q}_2)^{1/2}\frac{\sin(\kappa+\nu-q)}{\sin\kappa\sin(\nu-q)} & \dot{q}_2 \frac{\sin(\kappa+\nu)}{\sin\kappa\sin\nu} \end{pmatrix}, \tag{5.31a}$$

$$M_{trig}(\kappa) = \begin{pmatrix} \begin{array}{l} \dot{q}_1(\cot\kappa+\cot\nu) \\ +\frac{1}{2}\dot{q}_2\big(\cot(\nu+q) \\ \qquad +\cot(\nu-q)\big) \end{array} & (\dot{q}_1\dot{q}_2)^{1/2}\frac{\sin(\kappa+\nu)}{\sin\kappa\sin\nu} \\ (\dot{q}_1\dot{q}_2)^{1/2}\frac{\sin(\kappa+\nu)}{\sin\kappa\sin\nu} & \begin{array}{l} \dot{q}_2(\cot\kappa+\cot\nu) \\ +\frac{1}{2}\dot{q}_1\big(\cot(\nu-q) \\ \qquad +\cot(\nu+q)\big) \end{array} \end{pmatrix}. \tag{5.31b}$$

The Lax pair encodes the equations of motion by the relation

$$\dot{L}_{trig} = \left[M_{trig}, L_{trig}\right] . \tag{5.32}$$

We note that, in the matrices above, the spectral variable κ is separable and therefore redundant to the dynamics.

Applying the linearising limit (5.26) to the RS Lax matrices therefore leads to a Lax pair for the DR model,

$$L_{DR}(\kappa) = \begin{pmatrix} \dot{x}_1\left(\frac{1}{\mu}+\frac{1}{\kappa}\right) & (\dot{x}_1\dot{x}_2)^{1/2}\left(\frac{1}{\mu}+\frac{1}{\kappa}-\frac{1}{\mu-x}\right) \\ (\dot{x}_1\dot{x}_2)^{1/2}\left(\frac{1}{\mu}+\frac{1}{\kappa}-\frac{1}{\mu+x}\right) & \dot{x}_2\left(\frac{1}{\mu}+\frac{1}{\kappa}\right) \end{pmatrix}, \tag{5.33a}$$

$$M_{DR}(\kappa) = \begin{pmatrix} \begin{array}{l} \dot{x}_1\left(\frac{1}{\mu}+\frac{1}{\kappa}\right) \\ -\frac{1}{2}\dot{x}_2\left(\frac{1}{\mu+x}+\frac{1}{\mu-x}\right) \end{array} & (\dot{x}_1\dot{x}_2)^{1/2}\left(\frac{1}{\mu}+\frac{1}{\kappa}\right) \\ (\dot{x}_1\dot{x}_2)^{1/2}\left(\frac{1}{\mu}+\frac{1}{\kappa}\right) & \begin{array}{l} \dot{x}_2\left(\frac{1}{\mu}+\frac{1}{\kappa}\right) \\ -\frac{1}{2}\dot{x}_1\left(\frac{1}{\mu+x}+\frac{1}{\mu-x}\right) \end{array} \end{pmatrix}, \tag{5.33b}$$

where a spectral parameter κ has been used to mimic the form of the RS Lax pair. Recall that the parameter μ is related to the oscillator parameter λ by $\mu^2 = -1/\lambda^2$. The DR equations of motion (5.14) arise from the commutator,

$$\dot{L}_{DR}(\kappa) = \left[M_{DR}(\kappa), L_{DR}(\kappa)\right] , \tag{5.34}$$

with the invariants (5.20) encoded by the spectral curve,

$$\det\left(L_{DR}(\kappa) - \eta\mathbb{I}\right) = 0 . \tag{5.35}$$

The form of the Lax pair $L_{DR}(\kappa)$, $M_{DR}(\kappa)$ might encourage us to think we could lift the model to a discrete time version, since it is closely related to the RS Lax pair for which an integrable discretisation is known [81]. However, in the RS case this depends upon the Lagrange interpolation formula, but in the DR model some critical factors in the application of this formula are missing, so that an analogous derivation does not seem to be possible. This is perhaps unsurprising, as the nature of the integrable discretisation is very precise. However, it is possible that a correct application of the linearisation (5.26) to the discrete *trigonometric* RS model might yield some integrable discretisation for DR.

5.3 The Quantum System

We consider the quantisation of the Degasperis-Ruijsenaars oscillator. The quantum DR oscillator is not unitary equivalent to the harmonic oscillator, but it shares many of its nice properties. These allow a canonical quantisation, in particular the construction of a creation and annihilation operator algebra [26]. This leads us to question whether a path integral quantisation is possible, despite the non-Newtonian Lagrangian. Indeed, its relation to the integrable Ruijsenaars-Schneider model and discrete counterpart, together with their Lagrangian one-form structures, make this an interesting and potentially fruitful avenue for exploring the nature of path integral quantisations in such integrable cases. As commented in [26], the move from a classical equation of motion to quantum mechanics depends essentially on choosing either a Lagrangian or Hamiltonian function as a starting point: in the non-Newtonian case it is not necessarily clear that these two approaches need be equivalent.

5.3.1 Canonical Quantisation

A remarkable feature of the Degasperis-Ruijsenaars Hamiltonian H_β (5.1) is that it can be quantised [26]. Position and momentum become operators $\mathbf{x}$, $\mathbf{p}$, and a particular ordering prescription is chosen for H_β to ensure the Hamiltonian is self-adjoint and parity invariant as a quantum operator,

$$\hat{H}_\beta = \frac{1}{\beta^2 m}\left[(1+i\lambda\mathbf{x})^{1/2}e^{\beta\mathbf{p}}(1-i\lambda\mathbf{x})^{1/2} + (1-i\lambda\mathbf{x})^{1/2}e^{-\beta\mathbf{p}}(1+i\lambda\mathbf{x})^{1/2}\right] \, . \tag{5.36}$$

Under a standard operator representation $\mathbf{p} = -i\hbar\partial_x$, the exponentiated momentum operators act as *analytical difference operators*,

$$e^{\beta\mathbf{p}} f(x) = e^{-i\hbar\beta\partial_x} f(x) = f(x - i\hbar\beta) \; . \tag{5.37}$$

The Hilbert space theory for such models is by no means straightforward [26, 96]. The authors proceed by establishing a set of complete, orthogonal eigenstates for the model, and then treating $\hat{H}_\beta$ as a proper Hamiltonian operator restricted to the Hilbert space formed by the eigenstates.

Following [26] we use dimensionless variables, making the replacements $\sqrt{(m\omega/\hbar)}\, x \to x$, and $\sqrt{\hbar\beta\lambda} \to \lambda$. We replace $\hat{H}_\beta$ with the Hamiltonian in terms of the dimensionless quantities,

$$\hat{H}_\lambda = \frac{1}{2\lambda^2}\left[(1+i\lambda\mathbf{x})^{1/2} e^{-i\lambda\partial_x}(1-i\lambda\mathbf{x})^{1/2} + (1-i\lambda\mathbf{x})^{1/2} e^{i\lambda\partial_x}(1+i\lambda\mathbf{x})^{1/2}\right] \; . \tag{5.38}$$

By establishing an algebra of creation and annihilation operators similarly to the standard treatment of the quantum harmonic oscillator, the authors find explicitly a complete set of eigenstates and energy levels for the system. In particular, this is therefore an exactly solvable quantum system in 1 dimension, with a non-Newtonian Hamiltonian.

The Hamiltonian $\hat{H}_\lambda$ (5.38) possesses a complete set of normalised, orthogonal, energy eigenstates $\{\hat{\psi}_n^{(\lambda)}(x) \,|\, n = 0, 1, 2, \ldots\}$, so that

$$\hat{H}_\lambda \psi_n^{(\lambda)} = E_n \psi_n^{(\lambda)} \; , \quad \text{with } E_n = \lambda^{-2} + n \; . \tag{5.39}$$

Notice that the model has quantised energy eigenstates. We use $\psi_n^{(\lambda)}$ to denote the unnormalised state, whereas the addition of a hat $\hat{\psi}_n^{(\lambda)}$ indicates the normalised eigenstate. The un-normalised eigenstates are given by

$$\psi_n^{(\lambda)}(x) = \psi_0^{(\lambda)}(x) p_n^{(\lambda)}(x) \; , \tag{5.40a}$$

with a ground state

$$\psi_0^{(\lambda)}(x) = \left[\Gamma(\lambda^{-2} + i\lambda^{-1}x)\Gamma(\lambda^{-2} - i\lambda^{-1}x)\right]^{1/2} \; , \tag{5.40b}$$

and excited states given by the polynomials

$$p_n^{(\lambda)}(x) = n! \left(\frac{\lambda}{2}\right)^n P_n^{(\lambda^{-2})}(x/\lambda; \pi/2) \; , \tag{5.40c}$$

where $P_n^{(\alpha)}(x;\theta)$ are the Meixner-Pollaczek polynomials [30, 57], a family of orthogonal polynomials. The normalisations follow,

$$(\psi_n^{(\lambda)}, \psi_n^{(\lambda)}) = n! \left(\frac{\lambda}{2}\right)^{2n} \frac{\Gamma(n+2\lambda^{-2})}{\Gamma(2\lambda^{-2})} (\psi_0^{(\lambda)}, \psi_0^{(\lambda)}) \, , \tag{5.41a}$$

$$(\psi_0^{(\lambda)}, \psi_0^{(\lambda)}) = 2^{1-2\lambda^{-2}} \pi \lambda \Gamma(2\lambda^{-2}) \, . \tag{5.41b}$$

The authors note additionally that some alternative ordering choices can be made in $\hat{H}_\lambda$, but the analytical requirements mean that these alternative choices are not necessarily straightforward. Notice that these results apply to the reduced (centre-of-mass) model, rather than the full two particle model. Some additional treatment is required for the centre of mass, which may not be trivial as the separation of variables in the model is not of the standard additive form.

In quantum mechanics, the Hamiltonian or Lagrangian play a much more fundamental role than the classical case. Here, the Hamiltonian has been chosen as the fundamental object, leading to a canonical quantisation. Alternatively, a Lagrangian for the model (5.13), (5.30) can be chosen as the fundamental object. This leads to a quantisation procedure via the path integral. For non-Newtonian Lagrangians (as we consider here) it is far from obvious that the path integral and canonical quantisations need even be equivalent: standard derivations of the path integral assume Hamiltonians of Newtonian type [37, 47, 99]. The explicit canonical quantisation of this model therefore makes it an interesting study for the development of non-Newtonian path integral quantisation.

5.3.2 The Propagator

Using known results for the Meixner-Pollaczek polynomials, it is possible to derive a formula for the propagator for the Degasperis-Ruijsenaars model, beginning from the Hamiltonian $\hat{H}_\lambda$ (5.38). Here we apply a *Mehler formula* for the Meixner-Pollaczek polynomials to perform the sum over eigenfunctions required for the propagator, producing an expression in terms of a hypergeometric function ${}_2F_1$.

Recall that for the simple harmonic oscillator, the canonical quantisation leads to energy eigenfunctions expressed in terms of the Hermite polynomials,

$$\psi_n^{HO}(x) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) \, . \tag{5.42}$$

The eigenfunctions lead to the quantum mechanical propagator (1.61) through the Mehler formula for the Hermite polynomials,

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\left(\tfrac{1}{2}t\right)^n = (1-t^2)^{-1/2}\exp\left[\frac{2xyt-(x^2+y^2)t^2}{1-t^2}\right] . \tag{5.43}$$

This leads to the well known propagator for the harmonic oscillator (3.22) discussed in chapter 3.

The Meixner-Pollaczek polynomials of the DR eigenfunctions (5.40) also have a Mehler formula [52],

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{n!}{(2\alpha)_n} t^n P_n^{(\alpha)}\left(\xi;\frac{\pi}{2}\right) P_n^{(\alpha)}\left(\eta;\frac{\pi}{2}\right)& \\ = (1-t)^{-\alpha-i\eta}(1-t)^{-\alpha-i\xi}(1+t)^{i\xi+i\eta}\ {}_2F_1&\left(\begin{matrix}\alpha+i\xi,\alpha+i\eta \\ 2\alpha\end{matrix};\frac{-4t}{(1-t)^2}\right) .\end{aligned} \tag{5.44}$$

${}_2F_1$ is the hypergeometric function,

$${}_2F_1\left(\begin{matrix}a,b \\ c\end{matrix};z\right) := \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{s=0}^{\infty}\frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)s!}z^s , \tag{5.45}$$

(defined for $|z| \geq 1$ by analytic continuation) and $(a)_k$ indicates the Pochhammer symbol for integer k,

$$(a)_k = a(a+1)(a+2)\dots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)} . \tag{5.46}$$

This Mehler formula allows us to evaluate the sum over eigenfunctions that appears in the propagator calculation.

The Hamiltonian $\hat{H}_\lambda$ (5.38) is time independent, and so the propagator (1.61) for the DR model is given by

$$\begin{aligned} K(x,y;T) &= \langle y|e^{-i\tau\hat{H}_\lambda}|x\rangle\,\Theta(\tau) , && (5.47a) \\ &= \sum_{n=0}^{\infty}\hat{\psi}_n^{(\lambda)}(x)\hat{\psi}_n^{(\lambda)}(y)e^{-iE_n\tau}\Theta(\tau) , && (5.47b)\end{aligned}$$

by inserting a complete set of energy eigenstates $\sum_n |\psi_n\rangle\langle\psi_n| = \mathbb{I}$. We use the shorthand $\tau = T/\hbar$, and $\Theta(\tau)$ is the Heavyside step function. Then, using the known results for the energy eigenfunctions (5.39), (5.40a) and (5.41), we can write the propagator

$$\begin{aligned} K(x,y;T) = \frac{1}{(\psi_0^{(\lambda)},\psi_0^{(\lambda)})}\,\psi_0^{(\lambda)}(x)\psi_0^{(\lambda)}(y)\,e^{-i\lambda^{-2}\tau}\Theta(\tau)& \\ \times\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{2}{\lambda}\right)^{2n}\frac{\Gamma(2\lambda^{-2})}{\Gamma(2\lambda^{-2}+n)}p_n^{(\lambda)}(x)p_n^{(\lambda)}(y)e^{-in\tau}& .\end{aligned} \tag{5.47c}$$

But we can apply the Meixner-Pollaczek Mehler formula (5.44) to this summation, with $t = e^{-i\tau}$. Using the expressions for the energy eigenfunctions (5.40b) and (5.41) yields the propagator,

$$K(x,y;T) = \frac{1}{2\pi\lambda\Gamma(2\lambda^{-2})}\Theta(\tau)\left(i\sin\frac{\tau}{2}\right)^{-2\lambda^{-2}}\left(\frac{\cos\frac{\tau}{2}}{i\sin\frac{\tau}{2}}\right)^{\frac{i}{\lambda}(x+y)}$$
$$\times\left[\Gamma\left(\lambda^{-2}+\frac{i}{\lambda}x\right)\Gamma\left(\lambda^{-2}-\frac{i}{\lambda}x\right)\Gamma\left(\lambda^{-2}+\frac{i}{\lambda}y\right)\Gamma\left(\lambda^{-2}-\frac{i}{\lambda}y\right)\right]^{1/2}$$
$$\times\, {}_2F_1\left(\begin{matrix}\lambda^{-2}+\frac{i}{\lambda}x, \lambda^{-2}+\frac{i}{\lambda}y \\ 2\lambda^{-2}\end{matrix}; \left(\sin\frac{\tau}{2}\right)^{-2}\right). \qquad (5.48)$$

This is a new expression for the DR propagator.

Clearly the propagator (5.48) is a complicated expression, and there are aspects of its expected behaviour that are worthy of further study. Showing explicitly that $K(x,y;T)$ is a solution to the time dependent Schrödinger equation and examining the small time limit for the propagator are not straightforward. Quantum mechanical propagators also obey the group structure composition rule, which in this case will manifest as an integral identity for hypergeometric functions. The resulting identity is in some sense a generalisation of the known orthogonality for Meixner-Pollaczek polynomials, with the weight function (5.40b) appearing; the Meixner-Pollaczek polynomials themselves are a specialisation of the hypergeometric function [30, 57].

However, we are chiefly interested in an alternative question: how does the propagator derived from the canonical quantisation compare with a path integral approach? In particular, the canonical approach dealt carefully with ordering ambiguities and the Hilbert space problems arising from the analytic difference operators appearing in the Hamiltonian. In the path integral approach we face alternative difficulties, including an ambiguous choice of Lagrangian, but more particularly how to carry out the time-slicing procedure in a non-Newtonian case, if indeed this is an appropriate approach.

5.3.3 Path integral quantisation

The path integral quantisation begins with the Lagrangian as its fundamental object, posing the propagator

$$K(x',t';x'',t'') = \int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}[x(t)]\,\exp\left(i\mathcal{S}[x(t)]\right)\,, \qquad (5.49)$$

with the action appearing in the exponent. $\mathcal{D}[x(t)]$ indicates the path integral measure, which is usually understood by a discrete time-slicing procedure beginning from the Hamiltonian. In a non-Newtonian case such as the Degasperis-Ruijsenaars model, however, it is not clear that the canonical and path integral quantisations are equivalent; nor can we say definitively which is the more fundamental. However, investigating path integrals of non-Newtonian models is in general not straightforward.

We consider in this section some possible hints that may lead to a path integral quantisation for an integrable case like the DR model. Other work on path integrals for similar models does exist. A path integral for some three body cases of Calogero-Moser has been evaluated explicitly [35, 44, 54, 55]. In [45] a path integral for the Ruijsenaars-Schneider model appears as an interpretation of the theory considered, but this does not include an evaluation of the propagator for RS. In a different vein, a new approach to path integrals with a stochastic viewpoint is being developed by Hallnas and O'Connell [48, 84]; it is not currently clear how this relates to our point of view, but this approach could prove a fruitful avenue of research in the future.

Recall the Lagrangians for the DR model, $\mathcal{L}_{DR}$ (5.13) and $\overline{\mathcal{L}}_{DR}$ (5.30). The exponentiated action for these Lagrangians contains kinetic terms of the form $\exp(\frac{i}{\hbar}\dot{x}_j \ln \dot{x}_j)$. Now, using the freedom to add total derivative terms into the Lagrangian, these can be rewritten,

$$\exp\left(\tfrac{i}{\hbar}\dot{x}_j \ln(i\dot{x}_j/\hbar)\right) \ . \tag{5.50}$$

But to factors of this form, we can apply Stirling's formula [30],

$$(\alpha x)^{\alpha x} \sim \frac{1}{\sqrt{2\pi}}\,\Gamma(\alpha x)\,[\alpha x]^{1/2} e^{\alpha x} \quad \text{as } \alpha \to \infty \ . \tag{5.51}$$

In the path integral, a time slicing approach is typically taken where the positions x are discretised, and the approximation taken for the velocity $\dot{x}(t) \sim (x_{n+1} - x_n)/\epsilon$. The time slicing limit entails the shrinking $\epsilon \to 0$. So in such a limit (or equivalently, understanding that the velocity in the path integral is typically large) the kinetic factors (5.50) can be rewritten using Stirling's formula (5.51),

$$\exp\left(\tfrac{i}{\hbar}\dot{x}_j \ln(i\dot{x}_j/\hbar)\right) \sim \frac{1}{\sqrt{2\pi}}\,\Gamma\left(\frac{i\dot{x}_j}{\hbar}\right) \exp\left[\frac{i\dot{x}_j}{\hbar} + \frac{1}{2}\ln\left(\frac{i\dot{x}_j}{\hbar}\right)\right] \ , \tag{5.52a}$$

$$\sim \frac{1}{\sqrt{2\pi}}\,\Gamma\left(\frac{i\dot{x}_j}{\hbar}\right) \exp\left(\tfrac{1}{2}\ln \dot{x}_j\right) \ . \tag{5.52b}$$

In the last line we have again used the freedom to add total derivatives and constant terms to the Lagrangian. So this suggests a way of rewriting the kinetic terms in the path integral, perhaps in some time-slicing limit, to produce gamma functions. Note that such "small ϵ" limits for time-sliced path integrals have been used previously in other cases [54].

A pertinent question for integrable systems is, since in many cases one can find *integrable* discretisations of the Lagrangian, what is the correct way to discretise the Lagrangian in the path integral? Kinetic terms of the form $\dot{x}\ln\dot{x}$ are characteristic of "relativistic" integrable systems of this family, and indeed appear in known integrable discretisations. The observation of the Stirling's formula approximation for such terms (5.52) may then have wider application in the path integral quantisation of these relativistic, integrable models.

A second observation applies to the exponentiated potential term of the Lagrangian $\overline{\mathcal{L}}_{DR}$ (5.30),

$$\exp\left[\frac{i}{\hbar}\left(-\dot{x}_1\ln(1+i\lambda x)-\dot{x}_2\ln(1-i\lambda x)\right)\right] . \tag{5.53}$$

By writing $\dot{x}_1$ and $\dot{x}_2$ in terms of the centre of mass variables x, X we observe that this has the form of a *generating function for the Meixner-Pollaczek polynomials* [57, 30],

$$\exp\left[\left(-\frac{i\dot{X}}{\hbar}+\frac{i\dot{x}}{2\hbar}\right)\ln(1-i\lambda x)+\left(-\frac{i\dot{X}}{\hbar}-\frac{i\dot{x}}{2\hbar}\right)\ln(1+i\lambda x)\right] = \sum_{n=0}^{\infty} P_n^{(i\dot{X}/\hbar)}\left(\frac{\dot{x}}{2\hbar};\frac{\pi}{2}\right)(\lambda x)^n . \tag{5.54}$$

$P_n^{(\alpha)}(x;\theta)$ are the Meixner-Pollaczek polynomials that appeared in the excited energy eigenstates for the canonical quantisation of DR (5.40): it is interesting that they should also appear here.

Combining the observation of the Stirling formula approximation (5.52) with the (5.54) leads to a tantalising final possibility. Beginning with the Lagrangian $\overline{\mathcal{L}}_{DR}$ (5.30), we have, in terms of the centre of mass variables,

$$e^{i\overline{\mathcal{L}}_{DR}/\hbar} \sim \frac{1}{2\pi}\Gamma\left(\frac{i\dot{X}}{\hbar}+\frac{i\dot{x}}{2\hbar}\right)\Gamma\left(\frac{i\dot{X}}{\hbar}-\frac{i\dot{x}}{2\hbar}\right) \times\left(\dot{X}^2-\tfrac{1}{4}\dot{x}^2\right)^{1/2}\sum_{n=0}^{\infty} P_n^{(i\dot{X}/\hbar)}\left(\frac{\dot{x}}{2\hbar};\frac{\pi}{2}\right)(\lambda x)^n . \tag{5.55}$$

But, recalling the eigenfunctions (5.40), these gamma functions are precisely the appropriate weight function for the Meixner-Pollaczek polynomials appearing in the expansion. Although this is not yet sufficient for the DR propagator, perhaps with the necessary further insights a full path integral quantisation may be possible.

5.4 Summary

By placing the Degasperis-Ruijsenaars model into a two particle setting, we were able to derive a Lagrangian for the model. In the same way as the DR Hamiltonian, this Lagrangian is Newton equivalent to the standard harmonic oscillator on the level of the centre-of-mass variables. Two invariants arise in the equations of motion which guarantee integrability, and produce the simple harmonic oscillator. We have also made explicit the connection between DR and the Ruijsenaars-Schneider model, which is as a well chosen linearisation from the trigonometric case of RS. The Hamiltonian, Lagrangian and equations of motion all arise naturally from the RS case in this reduction. Additionally, we derived an alternative form of Lagrangian from the one-form structure of the RS model, and also a Lax pair as a limit from the RS case.

In the quantum regime, we have derived a propagator for the DR model using a Mehler formula for the Meixner-Pollaczek polynomials, which give the excitations of the energy eigenfunctions. The unresolved problem is to contrast this with a propagator derived from a Lagrangian (path integral) approach. However, we have made some observations on the exponentiated DR Lagrangian that may lead to an eventual solution. Further research is needed in this area, in particular regarding possible discretisations of the DR Lagrangian in order to carry out a time-slicing of the path integral. Given the Meixner-Pollaczek generating function arising in the exponentiated Lagrangian, an appropriate discretisation may also lead to the correct one time-step stationary states following the method of [40], discussed for a simple case in section 3.1.2.

6 Conclusion

6.1 Summary

The Lagrangian multiform structure captures the integrability of both discrete and continuous systems in a new way, expressing the feature of commuting flows through a variational framework. Through studying simple, linear examples we have extended the one-form and two-form structures to simplest possible discrete examples. These have enabled us to consider how such structures could be quantised. Naturally for Lagrangian structures this takes the form of a Feynman path integral, but with novel features associated to the Lagrangian multiform structure. Extending the multiform path integral for non-linear examples is a more challenging prospect, but we have considered both the generalised McMillan maps as an example of a non-linear discrete system, and the Degasperis-Ruijsenaars model as a non-Newtonian system in continuous time which captures the harmonic oscillator in an unusual way. Although we have not yet come to a quantised non-linear Lagrangian one-form, we have uncovered new aspects of these models.

In chapter 2 we derived Lagrangian one- and two-form structures for simpler models than have been previously considered. Beginning from a linearised lattice equation, we showed that the equation can be described by a discrete Lagrangian two-form, in the same way as the non-linear, multi-dimensionally consistent quad equations from which it is derived. The correct choice of Lagrangian for the linear lattice equation has the closure property, so that the action is stationary on solutions under deformations of the underlying surface geometry. Moreover, this linear lattice two-form is fairly unique. By imposing a periodic staircase initial value problem, we reduced the lattice equation to a linear, discrete mapping, and exploited the multi-dimensional consistency of the parent lattice equation to derive discrete commuting flows. These commuting flows can be described by a Lagrangian one-form structure, where the action along a time-path Γ remains invariant under deformations of the path - another manifestation of Lagrangian closure. The Lagrangian one-form was also shown to be uniquely determined by the choice of oscillator parameters.

The simplicity of the linear models explored in chapter 2 makes them helpful examples of the discrete multiform structure. As models with quadratic Lagrangians, they are useful toy models for exploring the quantisation of the Lagrangian multiform in chapter 3, since the path integral can be explicitly calculated for quadratic Lagrangians. In the one-form case, we constructed a propagator in multiple times, capturing the commuting flows of the discrete system. For the Lagrangian one-form, this propagator is *independent of the path taken in the time variables*. That is, the time-path can be freely deformed without changing the propagator, so that the propagator depends only on the endpoints. This is the quantum analogue of the classical Lagrangian closure condition. Moreover, from all possible choices of quadratic Lagrangian, this property holds uniquely for the one-form. For the lattice equation, we defined a quantum propagator over a space-time surface, depending on boundary values. When this propagator is evaluated with the Lagrangian two-form it has the remarkable property of *surface independence*: we showed that the surface can be freely deformed whilst leaving the propagator unchanged, so that the propagator is independent of the surface geometry, depending only on the boundary. As in the one-form case, this is a quantum analogue to the Lagrangian closure condition. Additionally, this surface independence was also shown to hold uniquely (in the linear case) for the linear Lagrangian two-form. Classically, the Lagrangian closure depended on the equations of motion, leading to invariance of the action under deformations of the time-path or surface. In the quantum analogue, the propagators continue to display time-path or surface independence, despite the redundancy of the equations of motion.

In chapter 4 we considered a non-linear model derived from the lattice KdV equation, the parent equation of the linear lattice model. A staircase reduction on the lattice KdV equation gives rise to the generalised McMillan maps. Exploiting the multi-dimensional consistency of the lattice KdV equation, we derived a commuting flow for a simple member of the mapping family, but the non-linearity of the parent lattice equation resulted in complicated expressions for the commuting flow that have so far not been written in a canonical, generating function form. It remains an outstanding problem to establish a Lagrangian one-form for these mappings. We have made first steps for the McMillan map towards performing the discrete path integral, by establishing the two-step propagator using Bessel functions, demonstrating the consistency of the propagator group property with a differential equation established via the operator equation of motion. This two-step propagator may be sufficient to establish whether path independence for the propagator could hold in this case, if a classical one-form structure can be uncovered.

Additionally, investigating the complementary canonical quantisation led to new insights for the dual (or large) form of the Lax pair. For the discrete Calogero-Moser model, the large Lax pair leads to commuting flows and hence to the Lagrangian one-form structure, from which the one-form for the continuous model is derived in a continuum limit. Moreover, there is a tantalising relation between the Darboux matrix $\mathcal{M}$ of CM and the Lagrangian form - the action is precisely the log determinant of the ordered product of Darboux matrices generating the time evolution. We have some hope that the dual Lax pair for McMillan may also lead to helpful insights in that case. We encoded the Poisson bracket structure of this Lax matrix with a classical r-matrix, which is expressible as a normally ordered partial fraction in terms of elementary shift matrices. This new formulation of the r-matrix reveals some interesting results: the r-matrix has a pseudo-skew-symmetry which is equivalent to an inverse fraction relation, and it obeys a three-term relation that is a strong version of the classical Yang-Baxter equation and is equivalent to a normally-ordered partial fraction expansion. The r-matrix structure can be used to prove the involutivity of the classical invariants and the preservation of the Poisson bracket structure under the mapping. Additionally, the linear r-matrix structure gives rise to a quantum $\mathfrak{r}$-matrix structure, where the $\mathfrak{r}$-matrix has a unitarity and is a solution to the Yang-Baxter equation. On the quantum level, it is not yet clear how the quantum invariants arise from the dual Lax structure, or how to establish their commutativity, since the time-evolution part of the $\mathfrak{r}$-matrix structure remains unknown. But, the simplest case of the McMillan map offers a hint that the trace of powers of the Lax matrix may be sufficient to resolve the operator ordering ambiguity; in other words, that the natural ordering in the matrix product may be the correct one.

In chapter 5, we studied the Degasperis-Ruijsenaars model, a non-Newtonian Hamiltonian system that produces the harmonic oscillator equation of motion. By embedding the DR model as a centre-of-mass motion in a two particle system, we have shown that it is a linearisation of the integrable trigonometric Ruijsenaars-Schneider system; this relation gives rise to a Lagrangian and Lax pair for the DR model. The RS model has a known discrete counterpart with commuting flows and an established Lagrangian one-form structure, giving rise to a continuous Lagrangian one-form structure in a well-chosen limit. The semi-linearity of the DR model and its known canonical quantisation then makes this an interesting model for investigating the continuum quantisation of one-form structures. By exploiting the known quantum eigenfunctions (established through a canonical quantisation) we were able to write the propagator explicitly for the DR model. The outstanding problem is to establish a link from the Lagrangian structure to the known propagator: we have uncovered some starting hints towards this calculation, but a crucial aspect remains the correct time-slicing of the Lagrangian. Perhaps a solution may be possible through further elaboration of the links to the discrete RS model and its corresponding one-form structure.

By investigating one-form and two-form structures for discrete, linear models, we have uncovered a quantum analogue of the Lagrangian closure condition that holds for these linear cases. However, extending the quantisation of Lagrangian multiform structures for non-linear examples is a challenging problem, due to the difficulty of carrying out the required time-slicings and integrals. Nonetheless, investigating the McMillan maps and Degasperis-Ruijsenaars model we have made some steps towards that goal. The dual Lax matrix structure of the McMillan maps suggests a possible avenue for the derivation of commuting flows and hence a one-form structure in that case; whilst the propagator and Lagrangian description for the DR model may lead to a connection between these two results, and hence insights into the quantised continuous Lagrangian structures.

6.2 Outlook

The discrete, linear one-form and two-form structures investigated in chapter 2 have a natural path integral quantisation (found in chapter 3) that reveals a quantum analogue to the Lagrangian closure condition. That is, the one-form and two-form propagators are time-path and space-time surface independent, respectively. This quantum Lagrangian multiform theory is in a very early stage of investigation, and so a number of avenues require further study; in particular there is a need for more examples of the quantum multiform structure, both in continuous time and for non-linear models. The results of the linear, discrete theory may guide the necessary calculations. There is also a great deal that remains unknown about the multiform structures themselves, some areas of which we discuss below.

The linear discrete mapping has a commuting flow whose compatibility with the initial flow is guaranteed by the multi-dimensional consistency of the underlying lattice equation (found in chapter 2). Many multi-dimensionally consistent lattice equations are known, so we would expect compatible, commuting mappings to arise in these non-linear cases. However, as found in chapter 4, the non-linearity makes this not necessarily straightforward. There is an open question about how to capture these commuting flows in the general, non-linear case. In particular, many of these lattice models are captured by Lax pairs, whose link to the commuting flows and one-form structures is not yet clear. Further research is also needed in understanding the relation between the two-form structures of the lattice and the one-form structures of the mapping reductions.

Path integral quantisation of the discrete Lagrangian one-form and two-form of chapter 2 revealed a surface independence of the discrete quantum propagators. For other examples in the literature, discrete one-form structures undergo well chosen continuum limits to Lagrangian one-forms describing compatible continuous flows [123, 125, 126]. The commuting invariants of the discrete mappings become generating Hamiltonians for these compatible continuous flows. The continuous Lagrangian one-forms also have a Lagrangian closure relation, a differential relation representing local independence of the action under variation of the continuous time-path. A suitable quantum analogue of this *continuous* Lagrangian closure is not yet clear. The higher dimensional linear reductions of section 2.3 may be suitable candidates for investigating these continuous structures, as they have a known discrete one-form and also commuting invariants such that it may be possible to avoid degeneration of the flows in a continuum limit. The discrete and continuous flows are also described by quadratic Lagrangians, so that the path integral can be carried out.

For discrete linear models, the path integral can be resolved by repeated Gaussian integration. However, many integrable models with Lagrangian multiform structures are non-linear, indeed in some cases even non-Newtonian: it is not known in general how to path integral quantise these models. Creating a general theory for the quantisation of Lagrangian multiforms clearly requires exploration of non-linear examples. There is, however, a clear relation between one time-step propagators for integrable discrete models and the notion of a Baxter Q-operator [60, 61, 89, 105]. In fact, it is known that there is a semi-classical relation between integral kernels for the Baxter Q-operator $Q_\lambda(x, \widehat{x})$ and generating functions for classical Bäcklund transforms $F_\lambda(x, \widehat{x})$ [61],

$$Q_\lambda(x, \widehat{x}) \sim \exp\left(-\frac{i}{\hbar} F_\lambda(x, \widehat{x})\right) , \quad \text{as } \hbar \to 0 . \tag{6.1}$$

But discrete-time integrable systems are nothing other than iterated Bäcklund transforms, where the Lagrangians are the generating functions. Perhaps the technology of the Baxter Q-operator may be precisely what is required to solve the multiform path integral in the non-linear cases. For non-linear models, resolution of singularities also becomes a significant issue, as is revealed even for the relatively simple case of McMillan. Further study of the Degasperis-Ruijsenaars model may also yield a crucial connection to an integrable discrete time model, which would suggest a way to approach the path integral time-slicing for this non-Newtonian model. If the Lagrangian one-form can be quantised for a greater number of models, it will then be possible to see to what extent the time-path independence of the propagator holds in the general case.

Studying the generalised McMillan maps, we uncovered a novel formulation of the r-matrix structure as a normally-ordered fraction of elementary shift matrices. The generalised McMillan maps arose as periodic reductions of the lattice KdV equation, and their Lax pairs as a consequence of the underlying Lax structure of the lattice equation. Now, the lattice KdV equation itself is the first member of a hierarchy: the lattice Gelfand-Dikii hierarchy [80]. A significant question is whether the r-matrix structure of the generalised McMillan maps is a *universal* structure; a natural setting to answer this question is the next equation in the hierarchy, the lattice Boussinesq equation,

$$\begin{aligned}\frac{p^3-q^3}{p-q+\widehat{\widetilde{w}}-\widetilde{\widetilde{w}}}-\frac{p^3-q^3}{p-q+\widehat{\widehat{w}}-\widehat{\widetilde{w}}}-\widehat{w}\widehat{\widehat{\widetilde{w}}}+\widetilde{w}\widehat{\widetilde{\widetilde{w}}}&\\ +\widehat{\widehat{\widetilde{\widetilde{w}}}}\Big(p-q+\widehat{\widehat{\widetilde{w}}}-\widehat{\widetilde{\widetilde{w}}}\Big)+w\big(p-q+\widehat{w}-\widetilde{w}\big)&\\ =(2p-q)\Big(\widetilde{w}+\widehat{\widehat{\widetilde{w}}}\Big)-(p+2q)\Big(\widehat{w}+\widehat{\widetilde{\widetilde{w}}}\Big)\,.&\qquad(6.2)\end{aligned}$$

In contrast to the four field variables of the lattice KdV equation (1.7), the lattice Boussinesq equation depends on nine points in the lattice. Dual Lax matrices on the reduction will be cubic in the shift matrix Σ_h, and thus the equation may offer a deeper insight into the nature of the r-matrix. Elements of the quantum structure have been found in [73], and a two-form structure established in [64] - this model may be useful to explore the mysterious relation between Lax pairs and Lagrangian multiforms. There is a known link for the discrete time Calogero-Moser model between the Darboux matrix $\mathcal{M}$ generating the time evolution of the Lax pair and the Lagrangian one-form structure, but such links are not known for any other models [125]. Perhaps the rich structures of the lattice Gelfand-Dikii hierarchy may offer further insights.

In sections 3.1.5 and 3.2.4, we propose a quantum variational principle for the one-form and two-form cases. Classically, the Lagrangian closure leads to a wider variational principle: the action should be stationary under variation of both the dependent *and* independent variables. In the path integral, the classical variation of the dependent variables become a *sum* over all possible configurations. The suggestion is that the variation of the independent variables for the multiform - i.e. the variation over time paths - should become a sum over all possible time-paths in the quantum regime (respectively, a sum over all surfaces for the two-form). As discussed in chapter 3, how to calculate such a sum is currently unknown, but the suggested principle is that in the unique case of the one-form all terms of the sum will converge to the same, path independent, value. With the correct renormalisation, this unique value will yield the desired propagator and also the desired Lagrangian multiform. Taming the unusual behaviour of such an object is a subject for future research.

6.3 Final Remarks

In this thesis we have made an early investigation into the quantisation of Lagrangian multiform structures on the discrete level: many important questions remain, but perhaps these findings will be a useful starting point. The Lagrangian multiform theory is a new way of understanding integrability from a variational perspective, even suggesting new ways of thinking about Lagrangians themselves as solutions to a variational problem. In order to quantise the range of integrable systems where a Lagrangian multiform structure has been discovered, we require a new approach to path integral quantisation that can capture the integrability of these models.

In ideas discussed by Rovelli, a parametrisation is introduced for time so that both time and space become integration variables in the path integral, creating a reparametrisation invariant system [94, 95] (section 1.4). Rovelli is far from alone in suggesting the need for new ways to think about space and time in quantum mechanics - Barbour [11] considers a Machian view, where he dispenses of an independent time variable, t'Hooft [113] suggests the long discarded view that it may be possible to view quantum mechanics as a hidden variable problem, using cellular automata, even Einstein himself [31] suggested that quantum mechanics must be incomplete without a proper theory of discrete functions. Rovelli's reparametrisation invariance is not so dissimilar from the time-path independence of the linear mapping in chapter 3; if such a time-path independence were to hold for Lagrangian one-forms on the continuous level, a description of such systems would require parametrisation of the time path by some "real time" s. The suggestion by Nijhoff [72] of a path integral over time-paths then looks somewhat similar to Rovelli's ideas. The suggested sum over all time-paths of chapter 3 (3.58) is an attempt to create a concrete realisation of this idea for a simple case: if such an object can be tamed and understood it will offer an entirely new way of thinking about the quantisation of integrable systems.

Appendix

A Normalisation constants for the one-form path integral

The discrete path integral for one forms is defined along a time-path Γ (3.38),

$$K_\Gamma\big(x_a, (0,0); x_b, (M,N)\big) := \mathscr{N}_\Gamma \int \prod_{(m,n)\in\Gamma} \mathrm{d}x_{m,n} \ \exp\left[\frac{i}{\hbar}\mathcal{S}_\Gamma[x(\boldsymbol{n})]\right] . \tag{A.1}$$

This is made up of discrete elements for time-steps in m and n directions. A single time-step in the m direction is given by (3.8),

$$K_m(x,\widehat{x};1) = \left(\frac{P+Q}{2\pi i\hbar q}\right)^{1/2} \exp\left[\frac{i}{\hbar}\mathcal{L}_b(x,\widehat{x})\right] , \tag{A.2}$$

and in the n direction by (3.36),

$$K_n(x,\overline{x};1) = \left(\frac{P+R}{2\pi i\hbar r}\right)^{1/2} \exp\left[\frac{i}{\hbar}\mathcal{L}_a(x,\overline{x})\right] . \tag{A.3}$$

Backwards time steps are given by the complex conjugates on K_m and K_n.

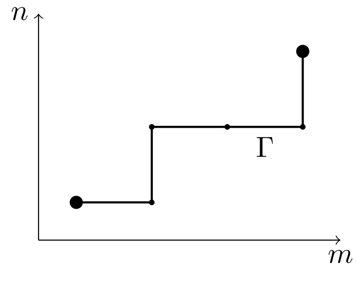

Figure A.1: A simple time-path Γ in the discrete variables.

The normalisation constant $\mathscr{N}_\Gamma$ (A.1) is simply a product of the constants for the individual time steps. Consider the simple curve Γ illustrated in figure A.1. This time-path is made up of three steps in the m direction, and 2 in the n direction, hence the normalisation constant results from the product

$$\mathscr{N}_\Gamma = \left(\sqrt{\frac{P+Q}{2\pi i\hbar q}}\right)^3 \left(\sqrt{\frac{P+R}{2\pi i\hbar r}}\right)^2 \quad . \tag{A.4}$$

A general time-path Γ from time co-ordinate $(0,0)$ to (M,N) is made up of $M+k$ forward steps and k backwards steps in the m direction, and $N+l$ forward steps and l backwards steps in the n direction, so that the normalisation constant in the general case is given by

$$\mathscr{N}_\Gamma = \left(\frac{P+Q}{2\pi i\hbar q}\right)^{(M+k)/2} \left(\frac{P+Q}{-2\pi i\hbar q}\right)^{k/2} \left(\frac{P+R}{2\pi i\hbar r}\right)^{(N+l)/2} \left(\frac{P+R}{-2\pi i\hbar r}\right)^{l/2} \quad . \tag{A.5}$$

The ordering of factors is unimportant. This normalisation is unambiguous for any given Γ.

Bibliography

[1] VE Adler, AI Bobenko, and Yu B Suris, *Classification of integrable equations on quad-graphs. The consistency approach*, Communications in Mathematical Physics **233** (2003), no. 3, 513–543.

[2] VE Adler, AI Bobenko, and Yu B Suris, *Discrete nonlinear hyperbolic equations. Classification of integrable cases*, Functional Analysis and its Applications **43** (2009), no. 1, 3–17.

[3] A Anderson, *Quantum canonical transformations and integrability. Beyond unitary transformations*, Physics Letters B **319** (1993), no. 1-3, 157–162.

[4] A Anderson, *Canonical transformations in quantum mechanics*, Annals of Physics **232** (1994), no. 2, 292–331.

[5] V I Arnol'd, *Mathematical methods of classical mechanics*, Springer, 1978.

[6] FM Arscott, *Periodic differential equations*, Pergamon, 1964.

[7] NM Atakishiyev, RM Mir-Kasimov, and Sh M Nagiev, *Quasipotential models of a relativistic oscillator*, Theoretical and Mathematical Physics **44** (1980), no. 1, 592–603.

[8] NM Atakishiyev and KB Wolf, *Generalized coherent states for a relativistic model of the linear oscillator in a homogeneous external field*, Reports on Mathematical Physics **27** (1989), no. 3, 305–311.

[9] O Babelon and CM Viallet, *Hamiltonian structures and Lax equations*, Physics Letters B **237** (1990), no. 3-4, 411–416.

[10] B Bahr, B Dittrich, and S Steinhaus, *Perfect discretization of reparametrization invariant path integrals*, Physical Review D **83** (2011), no. 10, 105026.

[11] JB Barbour and B Bertotti, *Mach's principle and the structure of dynamical theories*, Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences **382** (1982), no. 1783, 295–306.

[12] H Bateman, A Erdélyi, W Magnus, F Oberhettinger, and FG Tricomi, *Higher transcendental functions*, vol. 2, McGraw-Hill, New York, 1955.

[13] VV Bazhanov, VV Mangazeev, and SM Sergeev, *Exact solution of the Faddeev–Volkov model*, Physics Letters A **372** (2008), no. 10, 1547–1550.

[14] MV Berry, NL Balazs, M Tabor, and A Voros, *Quantum maps*, Annals of Physics **122** (1979), no. 1, 26–63.

[15] AI Bobenko and Yu B Suris, *Integrable systems on quad-graphs*, International Mathematics Research Notices (2002), no. 11, 573–611.

[16] AI Bobenko and Yu B Suris, *Discrete differential geometry*, Graduate Studies in Mathematics, vol. 98, American Mathematical Society, 2008.

[17] AI Bobenko and Yu B Suris, *Discrete pluriharmonic functions as solutions of linear pluri-Lagrangian systems*, Communications in Mathematical Physics **336** (2015), no. 1, 199–215.

[18] R Boll, M Petrera, and Yu B Suris, *Multi-time Lagrangian 1-forms for families of Bäcklund transformations: Toda-type systems*, Journal of Physics A: Mathematical and Theoretical **46** (2013), no. 27, 275204.

[19] R Boll, M Petrera, and Yu B Suris, *What is integrability of discrete variational systems?*, Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences **470** (2013), no. 2162.

[20] HW Braden and R Sasaki, *The Ruijsenaars-Schneider model*, Progress of theoretical physics **97** (1997), no. 6, 1003–1017.

[21] M Bruschi and F Calogero, *The Lax representation for an integrable class of relativistic dynamical systems*, Communications in Mathematical Physics **109** (1987), no. 3, 481–492.

[22] M Bruschi, O Ragnisco, PM Santini, and GZ Tu, *Integrable symplectic maps*, Physica D: Nonlinear Phenomena **49** (1991), no. 3, 273–294.

[23] F Calogero, *Solution of a three-body problem in one dimension*, Journal of Mathematical Physics **10** (1969), no. 12, 2191–2196.

[24] HW Capel, FW Nijhoff, and VG Papageorgiou, *Complete integrability of Lagrangian mappings and lattices of KdV type*, Physics Letters A **155** (1991), no. 6, 377–387.

[25] HJ de Vega, *Functional integration through inverse scattering variables*, Physical Review D **21** (1980), no. 2, 395.

[26] A Degasperis and SNM Ruijsenaars, *Newton-equivalent Hamiltonians for the harmonic oscillator*, Annals of Physics **293** (2001), no. 1, 92–109.

[27] PAM Dirac, *The Lagrangian in quantum mechanics*, Physikalische Zeitschrift der Sowjetunion **3** (1933), no. 1, 64–72.

[28] PAM Dirac, *On the analogy between classical and quantum mechanics*, Rev. Mod. Phys. **17** (1945), 195–199.

[29] PAM Dirac, *The principles of quantum mechanics*, no. 27, Oxford University Press, 1981.

[30] *NIST Digital Library of Mathematical Functions*, http://dlmf.nist.gov/, Release 1.0.10 of 2015-08-07, Online companion to [86].

[31] A Einstein, B Podolsky, and N Rosen, *Can quantum-mechanical description of physical reality be considered complete?*, Physical Review **47** (1935), no. 10, 777.

[32] LD Faddeev and LA Takhtajan, *Hamiltonian methods in the theory of solitons*, Springer-Verlag, Berlin, 1987.

[33] LD Faddeev and A Yu Volkov, *Algebraic quantization of integrable models in discrete space-time*, Oxford Lecture Series in Mathematics and its applications **16** (1999), 301–320.

[34] Peter E Falloon, PC Abbott, and JB Wang, *Theory and computation of spheroidal wavefunctions*, Journal of Physics A: Mathematical and General **36** (2003), no. 20, 5477.

[35] G Fano, G Turchetti, F Ortolani, and AG Teolis, *Functional integration and the S energy levels of a three-body system*, Il Nuovo Cimento B Series 11 **40** (1977), no. 1, 51–66.

[36] RP Feynman, *Space-time approach to non-relativistic quantum mechanics*, Rev. Mod. Phys. **20** (1948), 367–387.

[37] RP Feynman and AR Hibbs, *Quantum mechanics and path integrals*, McGraw-Hill, 1965.

[38] CM Field, *On the quantization of integrable discrete-time systems*, Ph.D. thesis, University of Leeds, 2005.

[39] CM Field and FW Nijhoff, *Quantum discrete Dubrovin equations*, Journal of Physics A: Mathematical and General **37** (2004), no. 33, 8065.

[40] CM Field and FW Nijhoff, *Time-sliced path integrals with stationary states*, Journal of Physics A: Mathematical and General **39** (2006), no. 20, L309.

[41] CM Field, FW Nijhoff, and HW Capel, *Exact solutions of quantum mappings from the lattice KdV as multi-dimensional operator difference equations*, Journal of Physics A: Mathematical and General **38** (2005), no. 43, 9503.

[42] L Freidel and JM Maillet, *Quadratic algebras and integrable systems*, Physics Letters B **262** (1991), no. 2-3, 278–284.

[43] H Goldstein, C Poole, and J Safko, *Classical mechanics*, AAPT, 2002.

[44] MJ Goovaerts, *Path-integral evaluation of a nonstationary Calogero model*, Journal of Mathematical Physics **16** (1975), 720–723.

[45] A Gorsky and N Nekrasov, *Relativistic Calogero-Moser model as gauged WZW theory*, Nuclear Physics B **436** (1995), no. 3, 582–608.

[46] B Grammaticos, A Ramani, VG Papageorgiou, and FW Nijhoff, *Quantization and integrability of discrete systems*, Journal of Physics A: Mathematical and General **25** (1992), no. 23, 6419.

[47] C Grosche and F Steiner, *Handbook of Feynman path integrals*, vol. 1, Springer, 1998.

[48] M Hallnäs and S Ruijsenaars, *Kernel functions and Bäcklund transformations for relativistic Calogero-Moser and Toda systems*, Journal of Mathematical Physics **53** (2012), no. 12, 123512.

[49] J Hietarinta, *Classical versus quantum integrability*, Journal of Mathematical Physics **25** (1984), no. 6, 1833–1840.

[50] J Hietarinta and B Grammaticos, *On the $\hbar^2$ correction terms in quantum integrability*, Journal of Physics A: Mathematical and General **22** (1989), no. 9, 1315.

[51] J Hietarinta, N Joshi, and FW Nijhoff, *Discrete systems and integrability*, Cambridge Texts in Applied Mathematics, vol. 54, Cambridge University Press, 2016.

[52] MEH Ismail and D Stanton, *Classical orthogonal polynomials as moments*, Canadian Journal of Mathematics **49** (1997), no. 3, 520–542.

[53] M Jimbo, *Introduction to the Yang-Baxter equation*, International Journal of Modern Physics A **4** (1989), no. 15, 3759–3777.

[54] DC Khandekar and SV Lawande, *Path integral for a three body problem*, Journal of Physics A: General Physics **5** (1972), no. 6, 812.

[55] DC Khandekar and SV Lawande, *Path integration of a three body problem*, Journal of Physics A: General Physics **5** (1972), no. 7, L57.

[56] SD King and FW Nijhoff, *Quantum variational principle and quantum multiform structure: the case of quadratic lagrangians*, arXiv preprint arXiv:1702.08709 (2017).

[57] R Koekoek, PA Lesky, and RF Swarttouw, *Hypergeometric orthogonal polynomials and their q-analogues*, Springer Science & Business Media, 2010.

[58] VE Korepin, NM Bogoliubov, and AG Izergin, *Quantum inverse scattering method and correlation functions*, Cambridge University Press, 1993.

[59] VB Kuznetsov, FW Nijhoff, and EK Sklyanin, *Separation of variables for the Ruijsenaars system*, Communications in Mathematical Physics **189** (1997), no. 3, 855–877.

[60] VB Kuznetsov, M Salerno, and EK Sklyanin, *Quantum Bäcklund transformation for the integrable DST model*, Journal of Physics A: Mathematical and General **33** (2000), no. 1, 171.

[61] VB Kuznetsov and EK Sklyanin, *On Bäcklund transformations for many-body systems*, Journal of Physics A: Mathematical and General **31** (1998), no. 9, 2241.

[62] SB Lobb and F Nijhoff, *Lagrangian multiforms and multidimensional consistency*, Journal of Physics A: Mathematical and Theoretical **42** (2009), no. 45, 454013.

[63] SB Lobb and F Nijhoff, *A variational principle for discrete integrable systems*, arXiv preprint arXiv:1312.1440 (2013).

[64] SB Lobb and FW Nijhoff, *Lagrangian multiform structure for the lattice Gel'fand–Dikii hierarchy*, Journal of Physics A: Mathematical and Theoretical **43** (2010), no. 7, 072003.

[65] SB Lobb, FW Nijhoff, and GRW Quispel, *Lagrangian multiform structure for the lattice KP system*, Journal of Physics A: Mathematical and Theoretical **42** (2009), no. 47, 472002.

[66] EM McMillan, *A problem in the stability of periodic systems*, Topics in modern physics, a tribute to EV Condon, 1971, pp. 219–244.

[67] J Meixner, FW Schäfke, and G Wolf, *Mathieu functions and spheroidal functions and their mathematical foundations: Further studies*, vol. 837, Springer, 1980.

[68] J Moser and AP Veselov, *Discrete versions of some classical integrable systems and factorization of matrix polynomials*, Communications in Mathematical Physics **139** (1991), no. 2, 217–243.

[69] M Nieszporski, PM Santini, and A Doliwa, *Darboux transformations for 5-point and 7-point self-adjoint schemes and an integrable discretization of the 2d Schrödinger operator*, Physics Letters A **323** (2004), no. 3, 241–250.

[70] FW Nijhoff, *Discrete Dubrovin equations and separation of variables for discrete systems*, Chaos, Solitons & Fractals **11** (2000), no. 1, 19–28.

[71] FW Nijhoff, *Lax pair for the Adler (lattice Krichever–Novikov) system*, Physics Letters A **297** (2002), no. 1, 49–58.

[72] FW Nijhoff, *Lagrangian multiform theory and variational principle for integrable systems* (2013), https://www.newton.ac.uk/seminar/20130710093010001.

[73] FW Nijhoff and HW Capel, *Integrability and fusion algebra for quantum mappings*, Journal of Physics A: Mathematical and General **26** (1993), no. 22, 6385.

[74] FW Nijhoff and HW Capel, *Quantization of integrable mappings*, Geometric and quantum aspects of integrable systems, Springer, 1993, pp. 187–211.

[75] FW Nijhoff and HW Capel, *The discrete Korteweg-de Vries equation*, Acta Applicandae Mathematica **39** (1995), no. 1-3, 133–158.

[76] FW Nijhoff, HW Capel, and VG Papageorgiou, *Integrable quantum mappings*, Physical Review A **46** (1992), 2155–2158.

[77] FW Nijhoff and GD Pang, *A time-discretized version of the Calogero-Moser model*, Physics Letters A **191** (1994), no. 1, 101–107.

[78] FW Nijhoff and GD Pang, *Discrete-time Calogero-Moser model and lattice KP equations*, CRM Proceedings and Lecture Notes **9** (1996).

[79] FW Nijhoff, VG Papageorgiou, and HW Capel, *Integrable time-discrete systems: lattices and mappings*, Quantum groups (1992), 312–325.

[80] FW Nijhoff, VG Papageorgiou, HW Capel, and GRW Quispel, *The lattice Gel'fand-Dikii hierarchy*, Inverse Problems **8** (1992), no. 4, 597.

[81] FW Nijhoff, O Ragnisco, and VB Kuznetsov, *Integrable time-discretisation of the Ruijsenaars-Schneider model*, Communications in Mathematical Physics **176** (1996), no. 3, 681–700.

[82] FW Nijhoff, A Ramani, B Grammaticos, and Y Ohta, *On discrete Painlevé equations associated with the lattice KdV systems and the Painlevé vi equation*, Studies in Applied Mathematics **106** (2001), no. 3, 261–314.

[83] FW Nijhoff and AJ Walker, *The discrete and continuous Painlevé vi hierarchy and the Garnier systems*, Glasgow Mathematical Journal **43** (2001), no. A, 109–123.

[84] N O'Connell, *Stochastic Bäcklund transformations*, In Memoriam Marc Yor-Séminaire de Probabilités XLVII, Springer, 2015, pp. 467–496.

[85] MA Olshanetsky and AM Perelomov, *Classical integrable finite-dimensional systems related to Lie algebras*, Physics Reports **71** (1981), no. 5, 313–400.

[86] FWJ Olver, DW Lozier, RF Boisvert, and CW Clark (eds.), *NIST Handbook of Mathematical Functions*, Cambridge University Press, New York, NY, 2010, Print companion to [30].

[87] JA Oteo, *The Baker–Campbell–Hausdorff formula and nested commutator identities*, Journal of mathematical physics **32** (1991), no. 2, 419–424.

[88] VG Papageorgiou, FW Nijhoff, and HW Capel, *Integrable mappings and nonlinear integrable lattice equations*, Physics Letters A **147** (1990), no. 2-3, 106–114.

[89] V Pasquier and M Gaudin, *The periodic Toda chain and a matrix generalization of the Bessel function recursion relations*, Journal of Physics A: Mathematical and General **25** (1992), no. 20, 5243.

[90] GRW Quispel, HW Capel, VG Papageorgiou, and FW Nijhoff, *Integrable mappings derived from soliton equations*, Physica A: Statistical Mechanics and its Applications **173** (1991), no. 1-2, 243–266.

[91] GRW Quispel and FW Nijhoff, *Integrable two-dimensional quantum mappings*, Physics Letters A **161** (1992), no. 5, 419–422.

[92] MP Reisenberger, *A left-handed simplicial action for Euclidean general relativity*, Classical and Quantum Gravity **14** (1997), no. 7, 1753.

[93] MP Reisenberger and C Rovelli, *"Sum over surfaces" form of loop quantum gravity*, Physical Review D **56** (1997), no. 6, 3490.

[94] C Rovelli, *Discretizing parametrized systems: the magic of Ditt-invariance*, arXiv preprint arXiv:1107.2310 (2011).

[95] C Rovelli, *On the structure of a background independent quantum theory: Hamilton function, transition amplitudes, classical limit and continuous limit*, arXiv preprint arXiv:1108.0832 (2011).

[96] SNM Ruijsenaars, *Complete integrability of relativistic Calogero-Moser systems and elliptic function identities*, Communications in Mathematical Physics **110** (1987), no. 2, 191–213.

[97] SNM Ruijsenaars, *Relativistic Toda systems*, Communications in mathematical physics **133** (1990), no. 2, 217–247.

[98] SNM Ruijsenaars and H Schneider, *A new class of integrable systems and its relation to solitons*, Annals of Physics **170** (1986), no. 2, 370–405.

[99] LS Schulman, *Techniques and applications of path integration*, vol. 140, Dover, 2005.

[100] J Schwinger, *Quantum mechanics: symbolism of atomic measurements*, AAPT, 2003.

[101] MA Semenov-Tian-Shansky, *What is a classical r-matrix?*, Funktsional'nyi Analiz i ego Prilozheniya **17** (1983), no. 4, 17–33.

[102] B Simon, *Functional integration and quantum physics*, vol. 86, Academic press, 1979.

[103] EK Sklyanin, *Quantum inverse scattering method. Selected topics*, arXiv preprint hep-th/9211111 (1992).

[104] EK Sklyanin, *Separation of variables: New trends*, Progress of Theoretical Physics Supplement **118** (1995), 35–60.

[105] EK Sklyanin, *Bäcklund transformations and Baxter's Q-operator*, Integrable Systems: from Classical to Quantum (1999, Montreal), CRM Proc. Lecture Notes, vol. 26, 2000, pp. 227–250.

[106] S Yu Slavyanov and W Lay, *Special functions: a unified theory based on singularities*, Oxford University Press, 2000.

[107] M Srednicki, *Quantum field theory*, Cambridge University Press, 2007.

[108] K Surawuttinack, S Yoo-Kong, and M Tanasittikosol, *The relativistic-like Lagrangian: Multiplicative form*, (2015).

[109] Yu B Suris, *Integrable mappings of the standard type*, Functional Analysis and Its Applications **23** (1989), no. 1, 74–76.

[110] Yu B Suris, *Variational formulation of commuting Hamiltonian flows: Multi-time Lagrangian 1-forms.*, Journal of Geometric Mechanics **5** (2013), no. 3.

[111] Yu B Suris, *Variational symmetries and pluri-lagrangian systems*, Dynamical Systems, Number Theory and Applications: A Festschrift in Honor of Armin Leutbecher's 80th Birthday, World Scientific, 2016, pp. 255–266.

[112] Yu B Suris and M Vermeeren, *On the lagrangian structure of integrable hierarchies*, Advances in Discrete Differential Geometry, Springer, 2016, pp. 347–378.

[113] G t'Hooft, *The cellular automaton interpretation of quantum mechanics. a view on the quantum nature of our universe, compulsory or impossible?*, arXiv preprint arXiv:1405.1548 (2014).

[114] D Tong, *Lectures on quantum field theory*, http://www.damtp.cam.ac.uk/user/tong/qft.html, 2007.

[115] VS Varadarajan, *Lie groups, Lie algebras, and their representations*, vol. 102, Springer Science & Business Media, 2013.

[116] AP Veselov, *Integrable maps*, Russian Mathematical Surveys **46** (1991), no. 5, 1–51.

[117] AP Veselov, *What is an integrable mapping?*, What is integrability?, Springer, 1991, pp. 251–272.

[118] A Yu Volkov, *Quantum lattice KdV equation*, Letters in Mathematical Physics **39** (1997), no. 4, 313–329.

[119] A Yu Volkov and LD Faddeev, *Quantum inverse scattering method on a spacetime lattice*, Theoretical and Mathematical Physics **92** (1992), no. 2, 837–842.

[120] HD Wahlquist and FB Estabrook, *Bäcklund transformation for solutions of the Korteweg-de Vries equation*, Physical review letters **31** (1973), no. 23, 1386.

[121] GN Watson, *A treatise on the theory of Bessel functions*, Cambridge University Press, 1922.

[122] S Wojciechowski, *The analogue of the Bäcklund transformation for integrable many-body systems*, Journal of Physics A: Mathematical and General **15** (1982), no. 12, L653.

[123] P Xenitidis, FW Nijhoff, and SB Lobb, *On the Lagrangian formulation of multidimensionally consistent systems*, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences **467** (2011), no. 2135, 3295–3317.

[124] S Yoo-Kong, *Calogero-Moser type systems, associated KP systems, and Lagrangian structures*, Ph.D. thesis, University of Leeds, 2011.

[125] S Yoo-Kong, SB Lobb, and FW Nijhoff, *Discrete-time Calogero–Moser system and Lagrangian 1-form structure*, Journal of Physics A: Mathematical and Theoretical **44** (2011), no. 36, 365203.

[126] S Yoo-Kong and FW Nijhoff, *Discrete-time Ruijsenaars-Schneider system and Lagrangian 1-form structure*, arXiv preprint arXiv:1112.4576 (2011).