# Klein-Gordon Solutions on Non-Globally Hyperbolic Standard Static Spacetimes

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## Abstract

We construct a class of solutions to the Cauchy problem of the Klein-Gordon equation on any standard static spacetime. Specifically, we have constructed solutions to the Cauchy problem based on any self-adjoint extension (satisfying a technical condition: "acceptability") of (some variant of) the Laplace-Beltrami operator defined on test functions in a $L^2$ space of the static hypersurface. The proof of the existence of this construction completes and extends work originally done by Wald. Further results include the uniqueness of these solutions, their support properties, the construction of the space of solutions and the energy and symplectic form on this space and an analysis of certain symmetries on the space of solutions and of various examples of this method, including the construction of a non-bounded below acceptable self-adjoint extension generating the dynamics.

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# Acknowledgements

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# Author's Declaration

I declare that all the following is my own work unless referenced to the contrary.

# 1 Introduction

The first purpose of this thesis is to construct a class of solutions to the Cauchy problem of the Klein-Gordon equation on any (not necessarily globally hyperbolic) standard static spacetime $(M, g) = (\mathbb{R} \times \Sigma, V^2 dt^2 - h)$, where $(\Sigma, h)$ is a Riemannian manifold and $V$ is a smooth positive function on $\Sigma$ (Sanchez [30]). A class of solutions was originally constructed by Wald [37]. His solutions were given in terms of some fixed positive self-adjoint extension (s.a.e.) of a particular symmetric linear operator on $L^2(\Sigma, V^{-1} d\text{vol}_h)$. Our treatment of the existence of solutions differs from that of Wald in the following aspects:

1. Wald considered only positive s.a.e.s and so the linear operators $C(t, A_E)$ and $S(t, A_E)$ (defined in Section 3) used to construct solutions were bounded. In this thesis however we also consider "acceptable" s.a.e.s (Definition 3.2). Incidentally, all bounded below s.a.e.s are acceptable. Under these conditions $C(t, A_E)$ and $S(t, A_E)$ may be unbounded linear operators so care is required with the domains.

2. We point out that a more recent result on the extendibility of subsets of the spacetime to smooth spacelike Cauchy surfaces in globally hyperbolic spacetimes by Bernal and Sanchez [6] is needed to complete the proof on the existence of Wald solutions.

The remainder of the thesis deals with proving various properties satisfied by the solutions and analysing some examples. Since many already known results are quoted in this work for completeness, we shall for clarity list the other main new results of this thesis:

1. We show in detail the properties satisfied by the solutions only implicit in the paper by Wald and Ishibashi [38]. In that paper, they assumed

certain conditions on the dynamics (e.g. constraints on the support of solutions, how they are transformed under time translation and reflection in time and the existence of an energy norm) and then proved that it must be generated by a particular s.a.e. $A_E$. In this thesis we answer the natural question: "to what extent are these conditions on the dynamics necessary?", that is, does the dynamics generated by a particular choice of acceptable s.a.e. $A_E$ satisfy these conditions? We shall answer mostly to the affirmative. However we note that Assumption 1 in Wald and Ishibashi [38] (the support of solutions/"the causality assumption": see below) is not always true of dynamics generated by an arbitrary acceptable s.a.e..

To amplify this point, Assumption 1 on the dynamics in Wald and Ishibashi [38] states that the support of the solution to the Klein-Gordon equation corresponding to Cauchy data always lies within the union of the causal future and past of the support of that data. In Section 12.5 we give a simple example of a standard static spacetime and a choice of s.a.e. $A_E$ such that the dynamics generated satisfies: $\operatorname{supp}\phi \nsubseteq J(K)$ for some initial data $(\phi_0, \dot{\phi}_0)$, where $K = \operatorname{supp}\phi_0 \cup \operatorname{supp}\dot{\phi}_0$. We show in Section 8 however that, in general, $\operatorname{supp}\phi$ is contained in $J(K)$ up until the time at which the data can "hit" any edge in the spacetime. We prove that this weaker form of Assumption 1 is true of all dynamics constructed in this thesis, using the previous results on the uniqueness of solutions in Section 6 and results on the causal structure of the spacetime Section 7.

2. An important property satisfied by the "Wald solutions" is that the value of the standard symplectic form evaluated at any pair of solutions

is independent of the static hypersurface on which it is calculated, so the space of solutions has a natural symplectic space structure. Since it is this structure which allows the quantisation of the theory, by the construction of the Weyl algebra (Bär et al. [3]), it is an important result in Section 10 that even after extending Wald's method to the case of only acceptable s.a.e.s, we retain the conservation of the symplectic form even in the cases where the positive definiteness of the energy form (Section 9) is lost.

3. In Section 11 we prove how the solutions are transformed under time translation and reflection. We show that the previously constructed energy form is invariant under both time translation and reflection of its arguments whereas the symplectic form is time-translation invariant but acquires a minus sign under reflection of its arguments in time. (These properties correspond to assumptions 2(i), 2(ii), 3(i) and 3(ii) of Wald and Ishibashi [38].)

4. In Sections 12.1-12.3 and Appendices F,G and H we consider three simple one-dimensional Riemannian manifolds ($S^1$, $(0,\infty)$ and $(0,a)$) with their usual differential structures and Riemannian metrics), each of which will then generate a standard static spacetime with $V = 1$. In order to classify the dynamics generated on the latter spacetimes by the construction of this thesis, we have analysed in great detail the s.a.e.s of minus the Laplacian on $S^1$, $(0,\infty)$ and $(0,a)$. In particular we have classified the s.a.e.s, determined their spectra and resolvents. The proofs of these statements are to be found in Appendices F,G and H. Specifically, it is shown there that the functions given in Sections 12.1, 12.2 and 12.3 do indeed generate the resolvents for all the self-adjoint

extensions of minus the Laplacian on the manifolds $S^1$, $(0,\infty)$ and $(0,a)$ respectively. It should be mentioned that, while it may be surprising, various parts of this do not seem to be readily accessible in the literature. The parts which can already be found in the literature are as follows: although it is the simplest of the examples considered here, the Green's function for the first case was not found in the literature. The Green's function for the second case is stated in Stakgold [32] and the expressions for the domains of the extensions of the third case can be found in Posilicano [26]. The form of the Green's functions for the third case can with some effort be reached from the methods in Posilicano. However our approach, proving directly that the stated expressions give the resolvents of the extensions, we have not found in the literature.

5. In Section 12.6 we construct an acceptable non-bounded below s.a.e. $A_E$ of minus the Laplacian on $\Sigma = \mathbb{Z}\times(0,\infty)$. This example then shows that the extension of theory of Wald [37] from bounded-below s.a.e.s to acceptable s.a.e.s carried out in this thesis is non-trivial (Wald's paper only deals with positive s.a.e.s).

The structure of the thesis is as follows:

In Section 2 we recall the definitions of static and standard static spacetimes found in for example Sanchez [30]. In Section 3 we define the Klein-Gordon equation, describe the functional analytic method employed in its solution and define the notion of an acceptable s.a.e.. In Section 4 we discuss the causal structure of the spacetime, and, in particular, characterise the causal future $J^+(K)$ of a compact subset $K \subseteq \Sigma_0 = \{0\} \times \Sigma$ of $M$ and find a simple expression for the Cauchy development (also called the domain of dependence) of $\Sigma_0$. These facts are especially useful for proving the support property of the solutions (Section 8).

In Section 5 we prove the existence of Wald solutions with respect to any acceptable s.a.e. $A_E$.

In Section 6 we prove a uniqueness theorem concerning Wald solutions and define the space of solutions. This uniqueness theorem, together with Section 7, which contains more results on the causal structure of the spacetime, allows us to prove the required support properties of solutions in Section 8.

In Sections 9 and 10 we show that on the space of solutions we can define an energy form and a symplectic form. In Section 11 we show how solutions are transformed under time-translation and time-reversal. It is shown that the energy form is invariant under both transformations and that the symplectic form is invariant under time translation but picks up a minus sign under time-reversal.

In Sections 12.1 to 12.3 we give some simple examples of Riemannian manifolds $(\Sigma, h)$ (that is: $S^1$, $(0,\infty)$ and $(0,a)$ mentioned above) and give all the self-adjoint extensions of minus the Laplacian $A$ ($V = 1, m = 0$) on $L^2(\Sigma, d\text{vol}_h)$. Since all the s.a.e.s $A_E$ of $A$ are bounded-below, they all generate a solution of the Laplace-Beltrami equation on $(\mathbb{R} \times \Sigma, dt^2 - h)$ by the construction in this thesis. For completeness we also state the resolvents of the s.a.e.s though we leave the proofs thereof to Appendices F,G and H. In Section 12.4 we discuss the effect on the s.a.e.s and their corresponding resolvents of adding a non-zero mass to the linear operator $A$. In Section 12.5 we give a simple example of a standard static spacetime and a choice of s.a.e. $A_E$ such that the dynamics generated satisfies: $\operatorname{supp}\phi \notin J(K)$ for some initial data (this corresponds to 1. of the second list on p.6). In Section 12.6 we construct an acceptable non-bounded below s.a.e. $A_E$ of minus the Laplacian on a particular (disconnected) Riemannian manifold (specifically: $\Sigma = \mathbb{Z} \times (0, \infty)$ with the Riemannian metric induced from that of $\mathbb{R}^2$). This example then shows that the extension of theory of Wald [37] from bounded-below s.a.e.s to acceptable s.a.e.s carried out in this thesis is non-trivial (Wald considered only positive s.a.e.s).

The appendix contains much varied material. Parts are results which have been postponed so as to improve readability of the main body of the thesis. Appendices A and B fall under this category. The former shows when the linear operators $C(t, A_E)$ and $S(t, A_E)$, introduced in Section 3, are bounded and also constructs a subspace invariant with respect to both linear operators and on which both linear operators are strongly differentiable with respect to $t$. These results were quoted in the earlier Section 3. Appendix B concerns the well-posedness of the Klein-Gordon equation on globally hyperbolic spacetimes with respect to arbitrary smooth initial data specified on a Cauchy surface. While the result is probably well known (Corollary 5, Section 3.5.3 in Ginoux's contribution in Bär and Fredenhagen (Eds.) [4]), it is included for completeness.

Appendices C, D and E do not contain new results but are included for completeness. Appendix C is a reminder of some elements of metric space theory that are needed in Section 4. Appendix D.1 constructs measure on manifolds form densities. Appendix D.2 is an introduction to Partial Differential Operators based on Chapter 10 of Nicolaescu [22]. Appendix D.3 defines the $L^p$ spaces, distributions and Sobolev spaces on manifolds, which are required constructions for much of this thesis. Appendix E deals primarily with the derivation of the expressions for the energy and symplectic forms used in Sections 9 and 10. In this appendix we also prove various propositions which although well known are not easily found in the standard texts on Lorentzian Geometry. The remaining Appendices F, G and H deal with proving that the functions given in Sections 12.1, 12.2 and 12.3 do indeed generate the resolvents for all the self-adjoint extensions of minus the Laplacian on the manifolds $S^1$, $(0, \infty)$ and $(0, a)$ respectively.

# 2 Static versus Standard Static Spacetimes

We will shortly give the definition of the class of spacetimes of interest but we first introduce some necessary concepts from differential geometry. (See Sachs and Wu [29], Sanchez [30] and O'Neill [23].)

**Definition 2.1.** *Given a smooth n-dimensional manifold $M$ and $m$ an integer, $0 \leq m \leq n$, an (m-dimensional, smooth)* ***distribution*** *$W$ maps each $p \in M$ to an m-dimensional subspace of $T_p(M)$, $W : p \mapsto W(p) \subseteq T_p(M)$ such that for all $p \in M$ there exists an open neighbourhood $U$ of $p$ and $m$ smooth vector fields $(X_i)_{i=1...m}$ on $U$ s.t. $\forall q \in U : \langle\{X_1(q), ..., X_m(q)\}\rangle = W(q)$, where $\langle S\rangle$ is the linear span of a subset $S$ of a vector space. Such a collection of locally defined smooth vector fields is called a (smooth)* ***local basis*** *for the smooth distribution $W$.*

A **local section** $X$ of a (smooth) distribution $W$ is a smooth vector field defined on an open set $U$ s.t. $X_p \in W(p)$ $\forall p \in U$. A smooth distribution is called **involutive** if for all local sections $X, Y$ of $W$, $[X, Y]$ is a local section of $W$. A necessary and sufficient condition for a smooth distribution to be involutive is that for each $p \in M$ there exists an open neighbourhood $U$ of $p$ and a local basis $(X_i)_{i=1...m}$ of $W$ on $U$ such that $[X_i, X_j]_q \in W(q)$ $\forall q \in U$. Note that if one local basis has this property then all do. Clearly every 1-dimensional smooth distribution is involutive.

We have the following four examples of smooth distributions:

1. Given a non-vanishing smooth vector field $X$ on a manifold $M$ a simple example of a 1-dimensional smooth distribution is given by: $W(p) = \langle\{X_p\}\rangle \subseteq T_p(M)$. Clearly in this example the local basis is given by $X$ and its domain of definition $U = M$.

2. Similarly, given a collection of smooth vector fields $(X_i)_{i=1..m}$ on $M$ which are linearly independent at every point, then we have the m-dimensional smooth distribution: $W(p) = \langle \{(X_1)_p, \ldots, (X_m)_p\} \rangle$.

3. $W(p) = T_p(M)$ is an n-dimensional smooth distribution on $M$. It is smooth as a local basis is given by the $n$ smooth vector fields $\{\frac{\partial}{\partial x^1}, .., \frac{\partial}{\partial x^n}\}$ defined on $U$ where $(U, \phi)$ is a chart and $x^i$ are the components of $\phi$.

4. Let $(M, g)$ be a n-dimensional Lorentzian manifold (in this thesis, always of signature $+ - \ldots -$) and $X$ be a smooth timelike vector field on M, define the $(n-1)$-dimensional distribution $W(p) = \{X_p\}^\perp = \{Y_p \in T_p(M)$ s.t. $g_p(X_p, Y_p) = 0\}$. Note that the subspaces $W(p)$ will consist of only spacelike vectors and the zero-vector. It is shown in Proposition E.6 that $W$ so defined is a smooth distribution.

Note: with our choice of signature on a Lorentzian manifold, a vector $X_p \in T_p(M) \backslash \{0\}$ is **timelike** if $g_p(X_p, X_p) > 0$; **null** if $g_p(X_p, X_p) = 0$; **causal** if $g_p(X_p, X_p) \geq 0$; spacelike if $g_p(X_p, X_p) < 0$. A smooth vector field $X$ is as usual timelike, null, causal or spacelike if $X_p$ is timelike, null, causal or spacelike at each $p \in M$.

**Definition 2.2.** *A smooth timelike vector field on a Lorentzian manifold is called* ***irrotational*** *if the smooth distribution, which it defines according to Example 4 above, is involutive.*

**Definition 2.3.** *Define the following:*

1. *A Lorentzian manifold $(M, g)$ is* ***time-orientable*** *if there exists on $M$ a smooth timelike vector field and time-oriented if one such is fixed.*

2. *A* ***spacetime*** *is a time-oriented Lorentzian manifold.*

3. *A spacetime is **stationary** if there exists on $(M,g)$ a smooth timelike Killing vector field on $M$.*

4. *A spacetime is **static** if stationary and there exists a smooth timelike Killing vector field that is irrotational. (Such a vector field is called a **static vector field**.)*

Note that, unlike some authors, we are not assuming orientability in the definition of a spacetime. The following is a very important example of a static spacetime. It is only this class of spacetimes with which this thesis is concerned.

**Definition 2.4.** *(Sanchez [30]) A **standard static spacetime** is defined by:*

$$(M,g) = (\mathbb{R} \times \Sigma, V^2 dt^2 - h),$$

*where $(\Sigma, h)$ is a smooth Riemannian manifold; $M$ is given the usual product topology and differential structure; $dt^2$ is the Euclidean metric on $\mathbb{R}$; $V \in C^\infty(\Sigma)$ with $V > 0$. The time-orientation is that given by the timelike vector field $X = \frac{\partial}{\partial t}$.*

*Remark.* In the above expression for the metric $g = V^2 dt^2 - h$, we are using a slightly sloppy notation for conciseness. Denoting by $\pi_1 : \mathbb{R} \times \Sigma \to \mathbb{R}$ and $\pi_1 : \mathbb{R} \times \Sigma \to \Sigma$ the two projection (bundle) maps, then more precisely: $g = \pi_2^*(V^2)\pi_1^*(dt^2) - \pi_2^*(h)$, where $\pi_i^*$ is the pull-back applied here to metrics and functions and $dt^2 = dt \otimes dt$ is the standard Riemannian metric on $\mathbb{R}$.

Let $(M,g) = (\mathbb{R} \times \Sigma, V^2 dt^2 - h)$ be a standard static spacetime. As $\Sigma$ is a smooth manifold then for each $t \in \mathbb{R}$ the map $\pi_t : \Sigma \to M$ given by $x \to (t,x)$ is a smooth embedding from $\Sigma$ to $M$ and for each embedded submanifold $\Sigma_t := \{t\} \times \Sigma = \pi_t(\Sigma) \subset M$ there exists a unique unit future-pointing smooth timelike vector field $n_t = V^{-1}\frac{\partial}{\partial t}$ normal to each tangent space of $\Sigma_t$. Note that we have not assumed that the manifold $\Sigma$ is orientable.

A standard static spacetime is static since a static vector field is given by $X = \frac{\partial}{\partial t}$, which we shall check shortly. (This vector field also defines the time-orientation of $M$.) Note that this is a globally defined smooth vector field since the map $(t,x) \to t$ is one of the coordinates of each of the charts in an atlas of $M$. It is easy to check that this Killing vector field $X$ is complete and so defines a global group of isometries. Also, every point in $M$ lies in the orbit of a unique point in $\{0\} \times \Sigma$ under this isometry (equivalently in the integral curve of a unique point in $\{0\} \times \Sigma$ under the vector field $X$).

**Proposition 2.5.** *Every standard static spacetime is static.*

*Proof.* We show first that the vector field $X = \frac{\partial}{\partial t}$ is irrotational. Consider the smooth distribution $W$ defined as orthogonal to $X$. Clearly $W(t,p) = T_p\Sigma \subseteq T_{(t,p)}M$. A simple local basis of $W$ is as follows: Given $(t,p) \in M$ then let $(U,\phi)$ be a chart in $\Sigma$ containing $p$. This then induces $n$ smooth vector fields on $U$: $\frac{\partial}{\partial x^1} \cdots \frac{\partial}{\partial x^n}$, where $x^i$ are the components of the chart map. Then define the local basis $\{X_i\}_{i\in\{1,..n\}}$ on the domain $\mathbb{R}\times U$ as: $X_i(s,q) = \frac{\partial}{\partial x^i}|_{(s,q)}$. Viewed as smooth vector fields on $U \subseteq \Sigma$, $[X_i, X_j]$ is a smooth vector field on $U$ by definition of the Lie bracket. Viewed as smooth vector fields on $\mathbb{R}\times U$, $[X_i, X_j]_{(t,p)} \in T_p(\Sigma) = W(t,p)$ for each $i,j$ and so $W$ is involutive and $X$ is irrotational.

We now show that $X$ is a Killing vector field. We make use of a well-known fact (see e.g. p.650 of Misner et al. [21]) that given a spacetime $(M,g)$ of dimension $n$ and a fixed integer $j$ $(1 \leq j \leq n)$, then if there exists an atlas $(U_\alpha, \phi_\alpha)$ on $M$ s.t. for each $\alpha$ the induced metric components do not depend on $x_\alpha^j$ (where $x_\alpha^i$ is, as usual, short-hand for the $i$-th component of the map $\phi_\alpha$) and if $x_\alpha^j$ and $x_{\alpha'}^j$ differ by a constant on $U_\alpha \cap U_{\alpha'}$ then $\frac{\partial}{\partial x^j}$ is a smooth Killing vector field for $(M, g)$. Applying this to our case, then given an atlas $(U_\alpha, \phi_\alpha)$ for $\Sigma$, then $(\mathbb{R} \times U_\alpha, \Psi_\alpha)$ is an atlas for $M$, where $\Psi_\alpha(t, p) = (t, \phi_\alpha(p))$. In the coordinates of one of these charts, the metric has components: $g_{00} = V^2$, $g_{0i} = g_{i0} = 0$ and $g_{ij} = -h_{ij}$, all of which are independent of $t$. We have already pointed out that $X = \frac{\partial}{\partial t}$ is a smooth vector field and thus is a smooth Killing vector field.

Thus $X$ is a smooth timelike irrotational Killing vector field and so is a static vector field by definition. □

*Remark.* Note that any open subset of a standard static spacetime with the induced spacetime structure is static though need not be standard static. See Sanchez [30] for more discussion and sufficient conditions guaranteeing that a static spacetime is standard static.

In the next section, we define the Klein-Gordon equation and propose a solution to a similar problem in functional analysis which shall prove to be the first step in the solution of the Klein-Gordon equation.

# 3 The Cauchy problem of the Klein-Gordon Equation on Standard Static Spacetimes: The construction of candidate solutions as vector-valued functions

We wish to solve the Klein-Gordon equation on an arbitrary standard static spacetime. For an arbitrary spacetime and mass $m \geq 0$ the Klein-Gordon equation reads:

$$(\Box_g + m^2)\phi = 0, \tag{3.1}$$

where $\Box_g = \text{div}_g \circ \text{grad}_g$ is the Laplace-Beltrami operator (see Appendix D.2), sometimes locally given by: $\nabla^\mu \nabla_\mu$ where $\nabla_\mu$ is the covariant derivative defined by the metric.

Alternatively, the Klein-Gordon equation can be expressed in local coordinates (see p.86 and p.213 of O'Neill [23]):

$$\Box_g = \frac{1}{\sqrt{|g|}} \partial_\mu g^{\mu\nu} \sqrt{|g|}\, \partial_\nu$$

where $g := \det(g_{\mu\nu})$.

*Remark.* Note also that we shall demand that $\phi \in C^\infty(M)$, where $C^\infty(M)$ is defined as the space of all smooth $\mathbb{K}$-valued functions on $M$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We are removing the dependence of the field of scalars from our notation for $C^\infty(M)$ purely for brevity. We shall find that the results of this thesis apply equally well to solving the Klein-Gordon equation for real-valued functions as for complex-valued functions. In the sequel we shall take all function spaces, Hilbert spaces etc. to be either over $\mathbb{R}$ or $\mathbb{C}$ as required. We shall on occasion in this thesis mention where we may have to treat the two cases separately. For instance Sections 12.1-12.3 only apply to the complex case as will be discussed there.

Our spacetime of interest is: $M = \mathbb{R} \times \Sigma$ with $g = V^2 dt^2 - h$, where $\Sigma$ is a smooth manifold with smooth Riemannian metric $h$ and $V \in C^\infty(\Sigma)$, $V > 0$.

For this spacetime, we define a solution to the Cauchy problem for the Klein Gordon equation to be a linear map:

$$\Psi : C_0^\infty(\Sigma_0) \times C_0^\infty(\Sigma_0) \to C^\infty(M)$$
$$(\phi_0, \dot{\phi}_0) \mapsto \phi$$

such that: $\forall \phi_0, \dot{\phi}_0 \in C_0^\infty(\Sigma_0)$, if $\Psi(\phi_0, \dot{\phi}_0) = \phi$ then:

1. $(\Box_g + m^2)\phi = 0$
2. $\phi|_{\Sigma_0} = \phi_0$
3. $\partial_t \phi|_{\Sigma_0} = \dot{\phi}_0$

In this thesis we shall construct solutions to the Cauchy problem. (We shall in fact find a solution to an extension of this problem, that is, extend the space of test functions $C_0^\infty(\Sigma_0)$ to a certain subspace $\chi_E$ of $C^\infty(\Sigma_0)$.) We start by expressing the Klein-Gordon equation in a simpler form. Given an atlas $(U_\alpha, \phi_\alpha)$ for $\Sigma$ we have the following atlas for $M$: $(\mathbb{R} \times U_\alpha, t \times \phi_\alpha)$ and in these coordinates the metric components are: $g_{00} = V^2$, $g_{0i} = g_{i0} = 0$ and $g_{ij} = -h_{ij}$.

The components $g^{\mu\nu}$ of the (2,0) tensor field are defined as inverse to those of the (0,2) tensor field $g_{\mu\nu}$. Similarly, the components $h^{ij}$ are defined as inverse to those of $h_{ij}$. Thus $g^{00} = V^{-2}$, $g^{0i} = g^{i0} = 0$, $g^{ij} = -h^{ij}$ and $\sqrt{|g|} = V\sqrt{h}$. In the last expression we denote by $g$ the determinant of the components $g_{\mu\nu}$ of the metric tensor $g$, similarly for $h$. The use of the symbol $g$ to denote the determinant shall be restricted only to such expressions and so is not to be confused with the metric itself.

So, in these local coordinates the Laplace-Beltrami operator reads:

$$\begin{aligned}\Box_g &= \frac{1}{V\sqrt{h}}\partial_t V^{-2}.V\sqrt{h}\partial_t - \frac{1}{V\sqrt{h}}\partial_i h^{ij}.V\sqrt{h}\partial_j \\ &= V^{-2}\partial_t^2 - \frac{1}{V\sqrt{h}}\partial_i h^{ij}.V\sqrt{h}\partial_j \\ &= V^{-2}\partial_t^2 - V^{-1}D^i V D_i,\end{aligned}$$

where $D_i$ is the covariant derivative on $\Sigma$ induced by $h$. Thus equation (3.1) reads:

$$(V^{-2}\partial_t^2 - V^{-1}D^i V D_i + m^2)\phi = 0$$

which is true iff:

$$(\partial_t^2 - VD^i V D_i + m^2 V^2)\phi = 0$$

iff:

$$\partial_t^2 \phi = -A\phi \tag{3.2}$$

where $A = -VD^iVD_i + m^2V^2$. Note that in coordinate free notation: $A = -V\mathrm{div}_h V\mathrm{grad}_h + m^2V^2$. See Section D.2 for more details.

We solve this form of the Klein-Gordon equation with the methods of functional analysis on the (real or complex) Hilbert space $L^2(\Sigma, V^{-1}d\mathrm{vol}_h)$. (This space is defined in Section D.3. The following definitions are taken from Chapter VIII Reed and Simon [27].)

In the next few pages we shall introduce some necessary concepts from functional analysis, fixing our notation. Then in equation (3.7) on p.25 we shall propose a solution to the related problem (equation (3.6) on p.25) to the new form of the Klein-Gordon equation (equation (3.2) above). It is this functional analytic solution that will provide the first step in solving the Klein Gordon equation.

In this thesis, by a **linear operator** $A$ on a real or complex Hilbert space $H$ we shall mean a linear map $A$: $D(A) \to H$, where $D(A)$ is a subspace of $H$, called the domain of the linear operator $A$. Some authors denote linear operators by $(A, D(A))$. For the sake of brevity we shall denote such a linear operator by $A$, however its domain $D(A)$ is always to be given.

- $A$ is called **densely defined** if $D(A)$ is dense in $H$.

Given a densely defined linear operator $A$ we define its **adjoint** $A^*$ as follows:

$$D(A^*) = \{\phi \in H \text{ s.t. } \exists \chi \in H \text{ s.t. } \langle \phi, A\theta \rangle = \langle \chi, \theta \rangle \text{ for all } \theta \in D(A)\}$$
$$A^*\phi = \chi \text{ for } \phi \in D(A^*), \text{ where } \chi \text{ is as in the previous line.}$$

Note that as $A$ is densely defined then $A^*$ is well defined. A partial order $\leq$ is defined on the set of linear operators on a Hilbert space $H$ as follows. Given linear operators $A, B$ then $A \leq B$ iff $D(A) \subseteq D(B)$ and $B|_{D(A)} = A$.

- A linear operator $A$ is called **symmetric** if it is densely defined and $A \leq A^*$. In other words, if it is densely defined and:

$$\langle \phi, A\theta \rangle = \langle A\phi, \theta \rangle \text{ for all } \phi, \theta \in D(A).$$

- A linear operator $A$ is called **self-adjoint** if it is densely defined and $A = A^*$, that is, $A$ is symmetric and the following is true:

  For all $\phi \in H$, if there exists $\chi \in H$ such that

$$\langle \phi, A\theta \rangle = \langle \chi, \theta \rangle \text{ for all } \theta \in D(A),$$

  then $\phi \in D(A)$.

- A linear operator $A$ is **positive** if $\langle Ax|x\rangle \geq 0$ for all $x \in D(A)$.

- A linear operator $A$ is **bounded-below** if there exists $M \in \mathbb{R}$ s.t. $\langle Ax|x\rangle \geq -M\|x\|^2$ for all $x \in D(A)$.

  *Remark.* Note that both positive and bounded-below linear operators satisfy: $\langle Ax|x\rangle \in \mathbb{R}$ for all $x \in D(A)$. If $H$ is a complex Hilbert space, then via the polarisation identity, it follows that such an operator $A$ is symmetric if also densely defined. If $A$ is a self-adjoint linear operator then it can be shown by the spectral theorem that

  $$\langle Ax|x\rangle \geq -M\|x\|^2 \text{ for all } x \in D(A) \text{ iff } \sigma(A) \subseteq [-M, \infty).$$

- $A$ is **closable** if, given $x_n \in D(A)$ and $y \in H$ satisfying $x_n \to 0$ and $Ax_n \to y$, then $y = 0$.

  *Remark.* Note that denoting $\Gamma(A) := \{(x, Ax) \in H \oplus H : x \in D(A)\}$ the graph of the linear operator $A$, then the definition of closable is a necessary and sufficient condition for $\overline{\Gamma(A)}$ to be the graph of a linear operator, denoted $\overline{A}$. It is shown in Reed and Simon [27] (Theorem VIII.1) that a densely-defined linear operator $A$ is closable iff its adjoint $A^*$ is densely defined.

- A closable linear operator $A$ is called closed if $A = \overline{A}$.

- A symmetric linear operator $A$ is called **essentially self-adjoint** (e.s.a.) if $A^* \leq A^{**}$, which is true iff $\overline{A}$ is self-adjoint.

On the (real or complex) Hilbert space $L^2(\Sigma, V^{-1}d\text{vol}_h)$, we have the following linear operator $A$:

$$D(A) = [C_0^\infty(\Sigma)] \tag{3.3}$$

$$A[\phi] = [(-VD^iVD_i + m^2V^2)\phi], \tag{3.4}$$

where $D_i$ is the covariant derivative on $(\Sigma, h)$ and $\phi \in C_0^\infty(\Sigma)$. That $A$ is symmetric and positive is proven in Proposition D.10 in the appendix.

Note that the adjoint is a well-defined linear operator since $A$ is densely defined in $L^2(\Sigma, V^{-1}d\text{vol}_h)$. The adjoint $A^*$ is given by :

$$\begin{aligned} D(A^*) &= \{\phi \in L^2(\Sigma, V^{-1}d\text{vol}_h) \text{ s.t. } A\phi \in L^2(\Sigma, V^{-1}d\text{vol}_h)\} \\ A^*\phi &= A\phi, \end{aligned}$$

where in both lines $A\phi$ is meant distributionally and a priori $\phi, A\phi \in D'(\Sigma)$. Here, functions are interpreted as distributions by use of the smooth measure $V^{-1}d\text{vol}_h$ on $\Sigma$.

*Remark.* Note that since $[C_0^\infty(\Sigma)] \subseteq D(A^*)$ then $D(A^*)$ is densely defined and so $A$ is closable. Also, be aware that the reason for appearance of the partial differential operator $A$ instead of its formal adjoint $A^*$ in the above definition of the linear operator $A^*$ is that $A$ is formally self-adjoint with respect to the smooth measure $V^{-1}d\text{vol}_h$. See Section sec:pdos for definitions of these terms.

The domain of the closure $\overline{A}$ of $A$ is given by the closure of $[C_0^\infty(\Sigma)]$ in the Hilbert space $D(A^*)$ with the inner product $\langle\cdot,\cdot\rangle_{A^*}$:

$$\langle\phi,\theta\rangle_{A^*} = \langle\phi,\theta\rangle_{L^2(\Sigma,V^{-1}d\text{vol}_h)} + \langle A^*\phi, A^*\theta\rangle_{L^2(\Sigma,V^{-1}d\text{vol}_h)},$$

It is important to note that $A$ is not necessarily essentially self-adjoint (e.s.a.). The following theorem gives a case where $A$ is e.s.a..

**Theorem 3.1** (Essential Self-Adjointness of minus the Laplacian on Complete Riemannian Manifolds)**.** *Let $(\Sigma, h)$ be a complete Riemannian manifold. Then letting $V = 1$ and $m = 0$, we have $A = -div_h grad_h = -\Delta_h$, minus the Laplacian corresponding to the metric $h$. Then if $D(A) = [C_0^\infty(\Sigma)]$ in the Hilbert space $H = L^2(\Sigma, dvol_h)$, then $A$ is essentially self-adjoint.*

*Proof.* See e.g. Taylor [35] Proposition 8.2.4. □

As pointed out by Wald [37], since $A$ is a symmetric positive linear operator then at least one positive self-adjoint extension exists. We do not restrict ourselves however to using a single extension, but we are forced to only consider a certain class of s.a.e.s of $A$, which we define shortly. We wish to first make a remark concerning the choice of the field of scalars.

*Remark.* Note that if we define $H_{\mathbb{K}} = L^2(\Sigma, \mathbb{K}, V^{-1}d\mathrm{vol}_h)$ as the space of equivalence classes of $\mathbb{K}$-valued square-integrable Borel-measurable functions, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $f \sim g$ iff $f = g$ a.e., then we can view $A$ as a symmetric linear operator on either the real Hilbert space $H_{\mathbb{R}}$ or the complex Hilbert space $H_{\mathbb{C}}$. The set of self-adjoint extensions of these operators are related. To see how, take the general situation of a real Hilbert space $H$ and its complexification $H_{\mathbb{C}}$. Now, on $H_{\mathbb{C}}$ can be defined a natural complex conjugation operator $C$. It is shown in Section 2 of Seggev [31] that the self-adjoint extensions of a symmetric linear operator $A$ on $H$ are are in bijection with the self-adjoint extensions of the symmetric linear operator $A_{\mathbb{C}}$ on $H_{\mathbb{C}}$, which commute with $C$, where $A_{\mathbb{C}}$ is the complexification of $A$.

We now introduce our new notion of an acceptable s.a.e.:

**Definition 3.2.** *A s.a.e. $A_E$ of $A$ is called **acceptable** if it satisfies:*

$$[C_0^\infty(\Sigma)] \subseteq \bigcap_{t>0} D(\exp(A_E^-)^{1/2}t), \tag{3.5}$$

*where $A_E^- := x^-(A_E)$ is the positive self-adjoint operator defined via continuous functional calculus using the function $x^- : \mathbb{R} \to [0, \infty)$ defined by:*

$$x^-(y) := \begin{cases} -y, & y \leq 0 \\ 0, & \textit{otherwise.} \end{cases}$$

*The operator $A_E^-$ is called the negative part of the operator $A_E$ and it's bounded iff $A_E$ is bounded-below.*

*Remark.* In the paper [37] by Wald, he considered only positive s.a.e.s of $A$. Clearly, a positive linear operator is bounded-below. If $A_E$ is a bounded-below s.a.e. then $A_E^-$ is a bounded linear operator, as is $(A_E^-)^{1/2}$. Then $\exp(A_E^-)^{1/2}t$ is also a bounded linear operator for all $t$ and so:

$$[C_0^\infty(\Sigma)] \subseteq L^2(\Sigma, V^{-1}d\text{vol}_h) = \bigcap_{t>0} D(\exp(A_E^-)^{1/2}t).$$

Thus every bounded-below s.a.e $A_E$ is also acceptable. Thus we are extending the method of Wald to more s.a.e.s of $A$.

The approach (taken from Wald [37]) is to find a map $\mathbb{R} \to D(A_E) \subseteq H$ where $H = L^2(\Sigma, V^{-1}d\text{vol}_h)$. $t \to \phi_t$, for each pair of data $\phi_0, \dot{\phi}_0 \in C_0^\infty(\Sigma)$. We demand that the map $t \to \phi(t)$ is twice differentiable as a vector-valued function with double-derivative:

$$\frac{d^2\phi_t}{dt^2} = -A_E\phi_t \tag{3.6}$$

Our intended solution to this problem is given in terms of any acceptable s.a.e. $A_E$ of $A$:

$$[\phi_t] = \cos(A_E^{1/2}t)[\phi_0] + A_E^{-1/2}\sin(A_E^{1/2}t)[\dot{\phi}_0] \tag{3.7}$$

Our immediate problem is to show that this expression makes sense. If $A_E$ was positive self-adjoint then, following Wald [37], we can take the square root to form a positive self-adjoint unbounded linear operator $A_E^{1/2}$ and then construct the two bounded linear operators $\cos(A_E^{1/2}t)$ and $A_E^{-1/2}\sin(A_E^{1/2}t)$ by applying the multiplication operator form of the Spectral Theorem, as in Reed and Simon [27]. If $A_E$ was not positive but merely bounded-below, then we shall show that this method still works and $\cos(A_E^{1/2}t)$ and $A_E^{-1/2}\sin(A_E^{1/2}t)$ are still well-defined bounded linear operators despite the non-existence of the square root. If, however $A_E$ is not bounded-below then these linear operators will be unbounded and we must concern ourselves with their (dense) domains. We shall show that even when $A_E$ is not bounded-below, but is acceptable (Definition 3.2), then we can solve the Cauchy problem with respect to smooth initial data of compact support.

In this thesis, in order to avoid expressions involving square roots of non-positive self-adjoint linear operators we introduce an alternative representation of equation (3.7).

Define the functions $C, S : \mathbb{R}^2 \to \mathbb{R}$:

$$C(t,x) = \cos(x^{1/2}t) = \begin{cases} \cos(x^{1/2}t) & \text{for } x \geq 0 \\ \cosh((-x)^{1/2}t) & \text{for } x < 0 \end{cases}$$

$$S(t,x) = t\frac{\sin(x^{1/2}t)}{x^{1/2}t} = \begin{cases} x^{-1/2}\sin(x^{1/2}t) & \text{for } x \geq 0 \\ (-x)^{-1/2}\sinh((-x)^{1/2}t) & \text{for } x < 0 \end{cases}$$

where:

$$\frac{\sin z}{z} := \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n+1)!} \text{ and } \frac{\sinh z}{z} := \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n+1)!}$$

are analytic functions on $\mathbb{C}$, both being invariant under $z \to -z$ (the same is true of course of the functions $\cos z$ and $\cosh z$). This makes the definitions of $C(t,x)$ and $S(t,x)$ independent of the choice of square root.

The functions $C$ and $S$ are well-defined as if a different root of $x$ is taken then the value of $C(t,x)$ is unchanged as cos is an even function. Similarly for $S(t,x)$ as long as the same root of $x$ is used for the numerator as for the denominator. Since $C(t,\cdot)$ and $S(t,\cdot)$ are (unbounded) real-valued measurable functions for each fixed $t$, then by functional calculus we can construct the (possibly unbounded) self-adjoint linear operators $C(t,A_E)$ and $S(t,A_E)$, for any s.a.e. $A_E$ of $A$. It is shown in the remarks following Propositions A.7 and A.8 that for $t \geq 0$:

$$D(\exp(A_E^-)^{1/2}t) = D(C(t,A_E)) \subseteq D(S(t,A_E)).$$

Thus:

$$[C_0^\infty(\Sigma)] \subseteq \bigcap_{t>0} D(\exp(A_E^-)^{1/2}t) = \bigcap_{t>0} D(C(t, A_E))$$

and the condition on $A_E$ that it is acceptable is precisely what is required for $C(t, A_E)$ (and so $S(t, A_E)$) to be defined on equivalence classes of test functions.

Given an acceptable s.a.e. $A_E$ of $A$, let our proposed solution to Equation (3.6) for arbitrary $\phi_0, \dot{\phi}_0 \in C_0^\infty(\Sigma)$ define:

$$[\phi_t] = C(t, A_E)[\phi_0] + S(t, A_E)[\dot{\phi}_0]. \tag{3.8}$$

If $A_E$ is bounded-below, then $C(t, A_E)$ and $S(t, A_E)$ are bounded linear operators for all $t$ (proven in Appendix A) and $[\phi_t]$ is a well-defined element of $L^2(\Sigma, V^{-1}d\mathrm{vol}_h)$. If not, then the condition on $A_E$ in Definition 3.2 is precisely what is required for the RHS to make sense. We wish to show that in fact the map $t \to [\phi_t]$ is infinitely differentiable and that $[\phi_t] \in [C^\infty(\Sigma)] \cap L^2(\Sigma, V^{-1}d\mathrm{vol}_h)$ for all $t \in \mathbb{R}$.

The following proposition is vital for this thesis. It is an application of Sobolev theory. It is taken from Wald [37] and reproduced here for completeness. (For the definitions of $L^p$ spaces, distributions and Sobolev spaces $W^{k,p}(M, \mu)$ on a Riemannian manifold $M$ with smooth measure $\mu$, see Appendix D.3.)

**Theorem 3.3.** *Any s.a.e. $A_E$ of $A$ satisfies: $D(A_E^\infty) \subseteq [C^\infty(\Sigma)]$.*

*Proof.* (Wald [37]) We know that $[C_0^\infty(\Sigma)] = D(A) \subseteq D(A_E^\infty)$. Take $\phi \in D(A_E^\infty)$. Since $L^2(\Sigma, V^{-1}d\mathrm{vol}_h) \subseteq L^1_{loc}(\Sigma, V^{-1}d\mathrm{vol}_h) \subseteq D'(\Sigma)$, the space of distributions on the manifold $\Sigma$, then for all $f \in C_0^\infty(\Sigma)$:

$$\phi(A^n f) = \langle \phi, [A^n f] \rangle = \langle \phi, A^n[f] \rangle = \langle \phi, A_E^n[f] \rangle = \langle A_E^n \phi, [f] \rangle = (A_E^n \phi)(f)$$

Thus $A^n\phi \in D(A_E^\infty) \subseteq L^2(\Sigma, V^{-1}d\text{vol}_h)$, where $A^n\phi$ is interpreted in the sense of distributions (since $A$ is a formally self-adjoint partial differential operator of second order w.r.t. $V^{-1}d\text{vol}_h$, see Appendix D.2).

Take an open set $\Omega \subseteq \Sigma$, which is precompact in the domain of a chart on $\Sigma$. Letting $N := \dim\Sigma$, then denote the resulting chart map $\Psi : \Omega \to \mathbb{R}^N$. Restricting $\phi$ to $\Omega$, we have

$$A^n\phi \in L^2(\Omega, V^{-1}d\text{vol}_h) = W^{0,2}(\Omega, V^{-1}d\text{vol}_h).$$

As $V^{-1}$ and $|\det(h_{ij})|$ are bounded by below on $\Omega$, then $A^n\phi \in W^{0,2}(\Psi(\Omega)) \subseteq W^{0,2}_{loc}(\Psi(\Omega))$, where we are now viewing $\phi$ as a function and $A$ as a p.d.o. on $\Psi(\Omega) \subseteq \mathbb{R}^N$. As $A^n$ is an elliptic p.d.o. of order $2n$, then, by an elliptic regularity theorem (Theorem D.12), $\phi \in W^{2n,2}_{loc}(\Psi(\Omega))$ for all $n$. And by Sobolev's lemma (Theorem D.13), we have (after possibly changing $\phi$ on a null set) that $\phi \in C^l(\Psi(\Omega))$ for any non-negative integer $l < 2n - \frac{N}{2}$. Since $n$ and $\Omega$ are arbitrary, then $\phi \in C^\infty(\Sigma)$. □

We begin by defining what we mean by strongly differentiable:

**Definition 3.4.** *If $A(t), B$ are densely defined linear operators for every $t \in \mathbb{R}$ and $D \subseteq H$ is a dense subspace satisfying: $D \subseteq D(B) \cap D(A(t))$ $\forall t \in (t_0 - \epsilon, t_0 + \epsilon)$ and:*

$$\text{for all } x \in D: \quad \left\| \left( \frac{A(t_0 + h) - A(t_0)}{h} - B \right) x \right\| \to 0 \text{ as } h \to 0,$$

*then we write $\frac{d}{dt}A(t)|_{t_0} = B$ on $D$ and say that $A(t)$ is **strongly differentiable** at $t_0$ on $D$ with strong derivative $B$.*

Similarly, if $A(t)$ is strongly differentiable at all times $t \in \mathbb{R}$ on $D_t$ with derivative $B(t)$ we write $\frac{d}{dt}A(t) = B(t)$ on $D_t$. Note then, by definition, we must have that for each $t$ there exists $\epsilon > 0$ s.t. $D_t \subseteq D(A(t')) \cap D(B(t))$ for all $t' \in (t-\epsilon, t+\epsilon)$.

Using Theorem 3.3, we define a space of smooth functions $\chi_E$ which contains all compactly supported smooth functions (as $A_E$ is acceptable). We shall show in later sections that we can solve the Klein-Gordon equation with respect to data in the space $\chi_E$.

**Proposition 3.5.** *Given an acceptable s.a.e. $A_E$ of $A$, define:*

$$\chi_E := \{f \in C^\infty(\Sigma) \ s.t. \ [f] \in D(A_E^\infty) \cap \bigcap_{t>0} D(\exp((A_E^-)^{1/2} t))\}$$

*Then the linear operators $C(t, A_E)$ and $S(t, A_E)$ satisfy the following:*

$$C(t, A_E),\ S(t, A_E) : [\chi_E] \to [\chi_E]$$

*Also, the maps $t \to C(t, A_E)$ and $t \to S(t, A_E)$ are infinitely often strongly differentiable on $[\chi_E]$, where for $n \in \mathbb{N} \cup \{\infty\}$:*

$$D(A_E^n) = \{x \in D(A_E) :\ A_E^m x \in D(A_E) \ for\ all\ m = 1, ..., n-1\}$$

*In fact, for $n \in \mathbb{N}$ the following strong derivatives hold on the dense subspace $[\chi_E]$ of $L^2(\Sigma, V^{-1} dvol_h)$:*

$$\begin{aligned}
\frac{d^{2n}}{dt^{2n}} C(t, A_E) &= (-1)^n A_E^n C(t, A_E) \\
\frac{d^{2n-1}}{dt^{2n-1}} C(t, A_E) &= (-1)^n A_E^n S(t, A_E) \\
\frac{d^{2n}}{dt^{2n}} S(t, A_E) &= (-1)^n A_E^n S(t, A_E) \\
\frac{d^{2n+1}}{dt^{2n+1}} S(t, A_E) &= (-1)^n A_E^n C(t, A_E)
\end{aligned}$$

*Proof.* See Appendix A. □

**Lemma 3.6.** *Thus just as in equation* (3.8)*, given initial data* $\phi_0, \dot{\phi}_0 \in \chi_E$ *and letting* $[\phi_t] = C(t, A_E)[\phi_0] + S(t, A_E)[\dot{\phi}_0]$*, then* $[\phi_t]$ *is differentiable as a vector valued function to arbitrary order and to even order:*

$$\frac{d^{2n}}{dt^{2n}}[\phi_t] = (-1)^n A_E^n[\phi_t]$$

*Thus in particular for* $n = 1$ *we have reproduced equation* (3.6) *and:*

$$\begin{aligned}[\phi_0] &= [\phi_t]|_{t=0} \\ [\dot{\phi}_0] &= \frac{d}{dt}[\phi_t]\bigg|_{t=0}\end{aligned}$$

Since $[\chi_E]$ is an invariant subspace of $L^2(\Sigma, V^{-1}d\text{vol}_h)$ w.r.t. the linear operators $C(t, A_E)$ and $S(t, A_E)$, so for all initial data $\phi_0, \dot{\phi}_0 \in \chi_E$, the solution given in Proposition 3.6, satisfies:

$$[\phi_t] \in D(A_E^\infty) \subseteq [C^\infty(\Sigma_t)] \;\; \forall t \in \mathbb{R}.$$

Thus we have solved the Hilbert space version of the Klein-Gordon Equation (equation (3.6)). We shall use this in Section 5 to construct solutions of the Klein-Gordon equation itself (equation (3.2)).

# 4 Causal Structure of Standard Static Spacetimes (i)

Before we construct solutions to the Klein-Gordon equation in Section 5, we shall find it useful to introduce some concepts from geometry, namely we shall define the causality relations and define the causal future and causal past of a set. After some preliminaries concerning Riemannian manifolds we shall then analyse the causal structure of an arbitrary standard static spacetime. Later in the section, since we shall need to quote results concerning the well-posedness of the Klein-Gordon equation on globally-hyperbolic spacetimes when we construct our solutions in Section 5, so we define the terms globally hyperbolic, Cauchy surfaces and Cauchy developments. Subsequently, returning to standard static spacetimes, we shall then in Proposition 4.22 re-express the Cauchy development $D(\Sigma_0)$ of the hypersurface $\Sigma_0$. Lastly, in Theorem 4.25 we shall quote the well-known result concerning the well-posedness of the Klein-Gordon equation on globally-hyperbolic spacetimes.

We first define the causality relations, the causal and chronological future of a point $p$ and the future and past Cauchy developments of a set in an arbitrary spacetime $M$.

**Definition 4.1.** *Given $p, q \in M$ then:*

1. *$p \ll q$ iff there is a future-pointing smooth timelike curve from $p$ to $q$.*

2. *$p < q$ iff there is a future-pointing smooth causal curve from $p$ to $q$.*

By a smooth curve from $p$ to $q$ we mean a smooth map $\gamma: [a,b] \to M$, where $a, b \in \mathbb{R}$, $a < b$, $[a,b]$ is viewed as a smooth manifold with boundary and $M$ is a smooth manifold. This is equivalent to there existing a smooth extension of $\gamma$ to $\gamma': (a-\epsilon, b+\epsilon) \to M$ for some $\epsilon > 0$.

We note that the relations would have been unchanged had we only used piecewise smooth curves. We prove this using a result from Penrose ([24] p.15):

**Proposition 4.2.** *The following statements are true:*

1. *$p \ll q$ iff there is a piecewise smooth future-pointing timelike geodesic from $p$ to $q$.*

2. *$p < q$ iff there is a piecewise smooth future-pointing causal geodesic from $p$ to $q$.*

**Corollary 4.3.** *The following statements are true:*

1. *$p \ll q$ iff there is a piecewise smooth future-pointing timelike curve from $p$ to $q$.*

2. *$p < q$ iff there is a piecewise smooth future-pointing causal curve from $p$ to $q$.*

*Proof.* Given a piecewise smooth future-pointing timelike curve $\gamma$ from $p$ to $q$, then each of its segments can be replaced by a piecewise smooth future-pointing timelike geodesics. Add these curves together, $\gamma$ can similarly be replaced and (again via Proposition 4.2) $p$ and $q$ can be connected by a smooth future-pointing timelike curve. An identical argument works for causal curves too. □

Note that it's the piecewise smooth formulation of the causal relations that allows one to most easily see that the relations are transitive, i.e. that $p \ll q \ll r \Rightarrow p \ll r$ and similarly for the relation $<$.

Using these relations, given $A \subseteq M$ we define its chronological and causal future ($I^+(A), J^+(A)$ respectively) as follows:

Given $A \subseteq M$, let:

$$I^+(A) = \{q \in M : \exists p \in A \text{ s.t. } p \ll q\}$$

$$J^+(A) = \{q \in M : \exists p \in A \text{ s.t. } p \leq q\}$$

(As usual $p \leq q$ means that either $p < q$ or $p = q$). The chronological and causal past are defined similarly:

$$I^-(A) = \{q \in M : \exists p \in A \text{ s.t. } q \ll p\}$$

$$J^-(A) = \{q \in M : \exists p \in A \text{ s.t. } q \leq p\}$$

Additionally, define: $J(A) = J^+(A) \cup J^-(A)$, $I(A) = I^+(A) \cup I^-(A)$.

We shall now analyse the causal structure of a standard static spacetime:

Before we begin, we shall find it useful to discuss metrics on Riemannian manifolds (here we use the term "metric" as in "metric space" rather than as in "metric tensor"!). It is well known that a Riemannian manifold $(\Sigma, h)$ is naturally metrisable. A metric $d : \Sigma \times \Sigma \to [0, \infty)$ is given by:

$$d(p,q) = \inf \left\{ \begin{array}{ll} \int_a^b |\dot{\sigma}(t)| dt \text{ s.t.} & \sigma : [a,b] \to \Sigma \text{ is a piecewise smooth} \\ & \text{curve in } \Sigma \text{ with } \sigma(a) = p,\ \sigma(b) = q. \end{array} \right\},$$

where $|\dot{\sigma}(t)| := [h_{\sigma(t)}(\dot{\sigma}(t), \dot{\sigma}(t))]^{1/2}$.

**Theorem 4.4.** *Given a Riemannian manifold* $(\Sigma, h)$*, then the metric* $d$ *given above induces the topology on* $\Sigma$.

*Proof.* See for example Lee [19], Lemma 6.2. □

For a choice of standard static spacetime $(M, g) = (\mathbb{R} \times \Sigma, V^2 dt^2 - h)$, we shall always choose the metric on $\Sigma$ induced by the Riemannian metric $V^{-2}h$ on $\Sigma$. The importance of choosing a metric on $\Sigma$ dependent on $V$ shall be seen in Proposition 4.12.

**Proposition 4.5.** *Consider a standard static spacetime* $(M,g) = (\mathbb{R}\times\Sigma, dt^2 - h)$. *Given a smooth curve* $\sigma: [0,t] \to \Sigma$ *(smooth in the sense of* $[0,t]$ *being a smooth manifold with boundary) satisfying:*

$$\sigma(0) = x$$
$$\sigma(t) = y$$
$$|\dot{\sigma}(s)| \leq 1 \quad \forall s \in [0,t]$$

*then define the smooth curve* $\gamma: [0,t] \to M$, *by:* $\gamma(s) = (s, \sigma(s))$. *Then* $\gamma$ *is a smooth future-pointing causal curve from* $(0,x)$ *to* $(t,y)$ *and thus* $(t,y) \in J^+((0,x))$ *and* $\gamma(s) \in \Sigma_s \ \ \forall s \in [0,t]$.

*Proof.* Clearly $\gamma$ is smooth. It is also causal since

$$g_{\gamma(s)}(\dot{\gamma}(s), \dot{\gamma}(s)) = 1 - h_{\sigma(s)}(\dot{\sigma}(s), \dot{\sigma}(s)) \geq 0.$$

From $g_{\gamma(s)}\left(\dot{\gamma}(s), \frac{\partial}{\partial t}\Big|_{\gamma(s)}\right) = 1 > 0$ it follows that $\gamma$ is future-pointing. □

In fact all future-pointing causal curves from $(0,x)$ to $(t,y)$ are of this form, or are reparametrisations thereof as the next proposition shows.

**Proposition 4.6.** $(t,y) \in J^+(0,x)$ *in the spacetime* $(M,g) = (\mathbb{R}\times\Sigma, dt^2 - h)$ *iff* $\exists$ *a smooth curve* $\sigma: [0,t] \to \Sigma$ *s.t.:*

$$\sigma(0) = x$$
$$\sigma(t) = y$$
$$|\dot{\sigma}(s)| \leq 1 \quad \forall s \in [0,t]$$

*Proof.* We have already proven that this condition is sufficient for $(t,y) \in J^+(0,x)$.

Conversely, if $(t,y) \in J^+(0,x)$ then there exists $\gamma: [a,b] \to \mathbb{R}\times\Sigma$ which is smooth future-pointing and causal s.t. $\gamma(a) = (0,x)$ and $\gamma(b) = (t,y)$.

Let $\gamma(s) = (\gamma_1(s), \gamma_2(s))$, where $\gamma_1 : [a, b] \to \mathbb{R}$ and $\gamma_2 : [a, b] \to \Sigma$ are both smooth curves defined using the smooth projection maps. So:

$$g_{\gamma(s)}(\dot{\gamma}(s), \dot{\gamma}(s)) = |\dot{\gamma}_1(s)|^2 - h_{\gamma_2(s)}(\dot{\gamma}_2(s), \dot{\gamma}_2(s)) \geq 0$$

The condition of $\gamma$ being future-pointing gives us: $g_{\gamma(s)}(\dot{\gamma}(s), \frac{\partial}{\partial t}\big|_{\gamma(s)}) = \dot{\gamma}_1(s) > 0$. We wish to reparametrise this curve and show that it is of the form of the previous proposition. For this purpose, let $\Phi : [a, b] \to \mathbb{R}$ be given by:

$$\Phi(s) = \int_a^s \dot{\gamma}_1(u) du$$

Since $\dot{\Phi}(s) = \dot{\gamma}_1(s) > 0$ then by the Inverse Function Theorem there exists a smooth inverse $\Phi^{-1} : [0, c] \to [a, b]$, where $\Phi(a) = 0$, $\Phi(b) = c$.

Define the reparametrisation: $\gamma'(s) = \gamma(\Phi^{-1}(s))$. $\gamma' : [0, c] \to \mathbb{R} \times \Sigma$ is then a smooth curve, satisfying:

$$\dot{\gamma}_1'(s) = \dot{\Phi}^{-1}(s)\dot{\gamma}_1(\Phi^{-1}(s)) = \frac{\dot{\gamma}_1(\Phi^{-1}(s))}{\dot{\gamma}_1(\Phi^{-1}(s))} = 1$$

Thus $\gamma_1'(s) = s \ \forall s \in [0, c]$ and let $\sigma = \gamma_2'$ so that $\gamma(s) = (s, \sigma(s))$ and:

$$(0, \sigma(0)) = \gamma'(0) = \gamma(a) = (0, x)$$
$$(c, \sigma(c)) = \gamma'(c) = \gamma(b) = (t, y)$$

Thus $c = t$, $\sigma(0) = x$, $\sigma(t) = y$ and as $\gamma'$ is still causal then $|\dot{\sigma}(s)| \leq 1$ for every $s \in [0, t]$. □

This easily adapts to an alternative description of the causal future of a subset $K$ of $\Sigma_0$.

**Proposition 4.7.** *Given the standard static spacetime $(M, g) = (\mathbb{R} \times \Sigma, dt^2 - h)$ and $K \subseteq \Sigma_0$, then $(t, y) \in J^+(K)$ iff there exists a smooth curve $\sigma : [0, t] \to \Sigma$*

*s.t.:*

$$\sigma(0) = x \in \pi(K)$$
$$\sigma(t) = y$$
$$|\dot{\gamma}(s)| \leq 1 \quad \forall s \in [0,t]$$

We recall a property of Riemannian manifolds which will be very useful to us. Clearly it is false for Lorentzian manifolds.

**Proposition 4.8** (Mean Value Theorem)**.** *Let $(\Sigma, h)$ be a Riemannian manifold. Then, for any piecewise smooth curve $\sigma: [a,b] \to \Sigma$, (where $a, b \in \mathbb{R}$, $a < b$):*

$$d(\sigma(a), \sigma(b)) \leq L(\sigma) = \int_a^b |\dot{\sigma}(s)| ds \leq (b-a) \sup_{s\in[a,b]} \{|\dot{\sigma}(s)|\}$$

Note that this implies that for any such curve:

$$d(\sigma(t), \sigma(t')) \leq |t'-t| \sup_{s\in[a,b]} \{|\dot{\sigma}(s)|\}$$

for any $t, t' \in [a,b]$. Also, as the speed of $\sigma$ is bounded over $[a,b]$ (a compact set), then $\sigma$ is uniformly continuous on $[a,b]$. Similarly, if $\sigma: (a,b) \to \Sigma$ is a smooth curve such that $|\dot{\sigma}|$ is bounded on $(a,b)$, then $\sigma$ is uniformly continuous on $(a,b)$.

**Definition 4.9** (Extendibilty of Curves)**.** *Given $I$, an open interval of $\mathbb{R}$ and a smooth manifold $M$, a smooth curve $\gamma: I \to M$ is (continuously)* ***extendible*** *if, denoting $I = (A,B)$, $A \in \mathbb{R} \cup \{-\infty\}$, $B \in \mathbb{R} \cup \{\infty\}$, then $\gamma(t)$ converges either as $t \to A$ or $t \to B$. A smooth curve is called* ***inextendible*** *if it is not extendible. Note that a curve is extendible iff one of its reparametrisations is extendible. If $\gamma$ is a future-pointing smooth or piecewise smooth causal curve in a spacetime $M$, we say $\gamma$ is* ***future-extendible*** *if $\gamma(t)$ converges as $t \to B$*

*and **past-extendible** if $\gamma(t)$ converges either as $t \to A$. Again $\gamma$ is called **future(past)-inextendible** if it is not future(past)-extendible. A geodesic $\gamma: I \to M$ (where $I \subseteq \mathbb{R}$ is open) is called **geodesically extendible** if we can extend it to a geodesic $\gamma': I' \to M$ defined on a strictly larger open domain $I' \supsetneq I$. If $\gamma$ is not defined on an open interval, e.g. $[a,b)$ then the notions of extendibility shall refer to the open end-point, in this case $b$. For instance, the geodesic $\gamma: [a,b) \to M$ is geodesically extendible if it can be extended to a geodesic $\gamma': [a, b+\epsilon) \to M$ for some $\epsilon > 0$.*

We shall shortly need the following theorem from O'Neill [23] (Lemma 5.8) on the extendibility of geodesics. We quote it here for the reader's convenience:

**Theorem 4.10.** *Given $b < \infty$ then a geodesic $\gamma: [a,b) \to M$ in a Lorentzian or Riemannian manifold $M$ is geodesically extendible iff it is (continuously) extendible.*

**Lemma 4.11.** *Let $\gamma: [a,b) \to M$ be a geodesic in a Riemannian manifold, then it is geodesically extendible iff there exists a compact set $C \subseteq M$ s.t. $[\gamma] := \gamma([a,b)) \subseteq C$.*

*Proof.* If $\gamma$ is geodesically extendible then its extension $\gamma': [a,b] \to M$ is continuous so $C = \gamma'([a,b])$ is compact. Conversely, if $\gamma([a,b)) \subseteq C$ then since it is a geodesic it has constant speed and so uniformly continuous. As $C$ is compact it is complete as a metric space. Thus we can extend $\gamma$ continuously to $[a,b]$ by basic functional analysis. So $\gamma$ is continuously extendible and so geodesically extendible by Theorem 4.10. □

Note that this lemma is also true in the case of Lorentzian manifolds. The proof can be reached by applying Lemma 1.56, Proposition 3.38 and Lemma 5.8 of O'Neill [23]. In this thesis we only need the result in its current form.

**Proposition 4.12.** *Consider the spacetime* $(M, g) = (\mathbb{R} \times \Sigma, dt^2 - h)$. *Let* $t \geq 0$ *and* $K \subseteq \Sigma_0$ *be compact. Then the following statements are equivalent:*

1. $C(K,t)$ *is compact,*
2. $\overline{B_t(0)} \subseteq \epsilon_p$ *for all* $p \in K$,
3. $J(K) \cap \Sigma_t$ *is compact.*

*If Statements 1-3 are true, then* $C(K,t) = \bigcup_{p\in K} \exp_p[\overline{B_t(0)}] = J(K) \cap \Sigma_t$.

*Remark.* In order to make sense of this proposition, take note of the following definitions: Given a metric space $(X, d)$ and $K \subseteq X$, then for $t \geq 0$ define:

$$C(K,t) := \{p \in X \text{ such that } d(p,K) \leq t\},$$

where $d(p,K) := \inf_{q\in K}\{d(p,q)\}$ (see Appendix C). We are implicitly using the metric $d$ on $\Sigma_0$ induced by the Riemannian metric $h$ via Theorem 4.4. We define $\epsilon_p \subseteq T_pM$ to be the domain of the exponential map $\exp_p$ at $p$, induced by the Riemannian metric $h$. The set $B_t(0) \subseteq T_pM$ is the open ball of radius $t$ centered on $0 \in T_pM$ with respect to the norm induced by $h_p$. Note that, since $C(K,t)$ is given in terms of the metric $d$ induced by the Riemannian metric $h$, then all three expressions $C(K,t), \overline{B_t(0)}$ and $J(K) \cap \Sigma_t$ depend on $h$. Indeed, if a different equivalent metric $d$ was chosen, then this proposition would be, in general, false.

*Proof of Proposition 4.12.*
*(1⇒2)*
$C(K,t)$ is compact $\Rightarrow$ $C(p,t)$ is compact for all $p \in K$.
Take $p \in K$ and let $X_p \in \overline{B_t(0)}$, so $|X_p| \leq t$. Let $\sigma$ be the maximal geodesic in $\Sigma$ through $p$ s.t.: $\dot{\sigma}(0) = X_p$, $\sigma\colon [0,b) \to \Sigma$ and so $|\dot{\sigma}(s)| = |X_p| \leq t$ $\forall s \in [0,b)$.

If $b \le 1$, then $L(\sigma|_0^{t'}) = \int_0^{t'} |\dot{\sigma}(s)| ds \le t't \le bt \le t$ and so $d(p, \sigma(s)) \le L(\sigma|_0^{t'}) \le t \;\; \forall s \in [0, b)$.

And so $\sigma(s) \in C(p,t) \;\forall s \in [0,b)$ but by Theorem 4.10, then $\sigma$ can be extended to a geodesic defined on $[0, b+\epsilon)$, contradiction. Thus $b > 1$ and $X_p \in \epsilon_p \; \forall p \in K$.

*(3⇒2)*

As in the proof of (1⇒ 2), choose $p \in K$, $X_p \in \overline{B_t(0)}$ and $\sigma : [0,b) \to \Sigma$ be the maximal geodesic through $p$ with speed $X_p$ at $p$. If $b \le 1$, for $c \in [0,b)$, let $\sigma' : [0,t] \to \Sigma$, $\sigma'(s) = \sigma(s\frac{c}{t})$ and $|\dot{\sigma}'(s)| = \frac{c}{t}|\dot{\sigma}(s\frac{c}{t})| \le c < b \le 1$.

Thus $(t, \sigma(s)) \subseteq J(p) \cap \Sigma_t \subseteq J(K) \cap \Sigma_t$ (compact by assumption) $\forall s \in [0,b)$. Again by Theorem 4.10, then $\sigma$ can be extended to a geodesic defined on $[0, b+\epsilon)$, contradiction. Thus $b > 1$ and $X_p \in \epsilon_p$ for all $p \in K$ such that $|X_p| \le t$.

*(2⇒1)*

$\overline{B_t(0)} \subseteq \epsilon_p \;\; \forall p \in K$ implies $C(p,t) = \exp_p[\overline{B_t(0)}]$, which is Corollary 5.6.4 in Petersen [25]. Note that since the exponential map is certainly continuous then it follows that $C(p,t)$ is compact $\forall p \in K$. It follows from this that $C(K,t) = \bigcup_{p \in K} C(p,t)$ is compact, as is shown in Proposition C.7 in the appendix.

*(1⇒3)*

Given $q_n \in J(K) \cap \Sigma_t \subseteq C(K,t)$, then by compactness there exists a subsequence $q_{n_k} \to q \in C(K,t) = \bigcup_{p \in K} C(p,t)$. Thus $q \in C(p,t) = \exp_p[\overline{B_t(0)}]$ for some $p \in K$.

So there exists a geodesic $\sigma : [0,1] \to \Sigma$ s.t. $\sigma(0) = p, \sigma(1) = q, |\dot{\sigma}(s)| = |\dot{\sigma}(0)| \le t$. So define $\sigma' : [0,t] \to \Sigma$, $\sigma'(s) = \sigma(s/t)$, $|\dot{\sigma}'(s)| = \frac{1}{t}|\dot{\sigma}(s/t)| \le 1$,

$\sigma'(0) = p$, $\sigma'(1) = q$ and $(t, q) \in J(K) \cap \Sigma_t$ by Proposition 4.7, so $J(K) \cap \Sigma_t$ is compact.

(The final statement)
When Statements *1-3* are true then, fixing $p \in K$, we have $C(p,t) = \exp_p[\overline{B_t(0)}]$ from Corollary 5.6.4 in Petersen [25]. But we have $J(p) \cap \Sigma_t \subseteq C(p,t) = \exp_p[\overline{B_t(0)}]$ $\forall t \in \mathbb{R}$. Furthermore, the argument in the proof of (1⇒ 3) shows: $\exp_p[\overline{B_t(0)}] \subseteq J(p) \cap \Sigma_t$. So $C(p,t) = \exp_p[\overline{B_t(0)}] = J(p) \cap \Sigma_t$ $\forall p \in K$. Thus $C(K,t) = \bigcup_{p \in K} C(p,t) = \bigcup_{p \in K} \exp_p[\overline{B_t(0)}] = \bigcup_{p \in K} J(p) \cap \Sigma_t = J(K) \cap \Sigma_t$. □

In particular, the content of this proposition is true when $K = \{p\}$ is any point in $\Sigma_0$. The usefulness of this proposition arises from the fact that $C(K,t)$ is easier to visualise than $J(K) \cap \Sigma_t$ as Section 7 utilises.

We recall the notion of a globally hyperbolic spacetime:

**Definition 4.13.** *A spacetime $(M, g)$ is **globally hyperbolic** if:*

1. *It obeys the causality condition: there exist no closed causal curves.*

2. *$J^+(p) \bigcap J^-(q)$ is compact $\forall p, q \in M$.*

(A curve $\gamma : [a,b] \to M$ is called closed if $\gamma(a) = \gamma(b)$.) Note, it is shown in Bernal and Sanchez [7] that condition *1* may be equivalently replaced by the "strong causality condition".

**Definition 4.14.** *Given a spacetime $(M, g)$, a **Cauchy surface** (of $(M, g)$) is a subset $S$ of $M$ that is met exactly once by every inextendible smooth timelike curve in $M$.*

To explain the name, we note that every such set is an achronal closed topological embedded hypersurface in $M$ (see Lemma 14.29 in O'Neill [23]).

It is an important fact that global hyperbolicity is equivalent to the existence of a Cauchy surface in the spacetime as the following theorem states, which we include for completeness and future reference. Before we state it, we first define the concept of an acausal set in a spacetime, since this notion shall be used in the following theorem.

**Definition 4.15** (Acausal Set). *A subset $S$ of a spacetime $M$ is called **acausal** if it is met at most once by any causal curve in $M$.*

An **achronal** set is defined similarly with "any causal curve" replaced by "any timelike curve".

**Theorem 4.16.** *A spacetime $(M, g)$ is globally hyperbolic iff it possesses a Cauchy surface. If so, then it also possesses a smooth spacelike Cauchy surface. Additionally, if $H$ is a smooth spacelike acausal compact $m$-dimensional embedded submanifold with boundary in $M$, then there exists a smooth spacelike Cauchy surface $S$ in $M$ that contains $H$.*

*Proof.* If $S \subseteq M$ is Cauchy surface then $M$ is globally hyperbolic by Corollary 14.39 in O'Neill [23]. That $M$ is globally hyperbolic implies that it possesses a smooth spacelike Cauchy surface is proved in Theorem 1 in Bernal and Sanchez [8]. For the last statement see Theorem 1.1 of Bernal and Sanchez [6]. □

Note that if $M$ is $n$-dimensional, then in the above theorem: $m \in \{0, ..., n-1\}$.

In order to aid the understanding of this theorem, we shall shortly give an example to illustrate why we cannot remove the condition of compactness. In order to state this example, we first introduce the concept of Cauchy deveploment of a set, a notion that will be frequently used later.

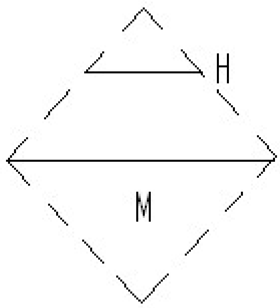


Figure 1: A smooth spacelike acausal embedded submanifold of a globally hyperbolic spacetime need not be extendible to a Cauchy surface.

**Definition 4.17** (The Past and Future Cauchy Developments)**.** *Given a subset $S$ of a spacetime $M$, then the* ***future Cauchy development*** $D^+(S) \subseteq M$ *is defined as:*

$$D^+(S) = \left\{ p \in M \colon \begin{array}{l} \text{Every past-inextendible future-pointing smooth} \\ \text{causal curve through } p \text{ intersects } S. \end{array} \right\}$$

*The* ***past Cauchy development*** $D^-(S)$ *of $S$ is defined similarly with "past-inextendible" replaced by "future-inextendible". The* ***Cauchy development*** $D(S)$ *of $S$ is then defined:* $D(S) = D^+(S) \cup D^-(S)$.

Using this definition, let $M = D(\{0\} \times (0,1))$ be an open subset of 2-dimensional Minkowski space and fix $0 < |t| < 1/2$. Then $H = \{t\} \times (0,1) \cap M$ is a 1-dimensional smooth spacelike, acausal embedded submanifold and so also a submanifold with boundary (just with empty boundary!). However, it is non-compact in $M$ and is contained in no Cauchy surface (see Figure 1).

The following proposition gives a necessary and sufficient condition for a standard static spacetime to be globally hyperbolic. The content of this proposition is already known, however we give an alternative proof. See Lemma A.5.14 in Bär, Ginoux and Pfäffle [3] or Theorem 3.67 in Beem, Ehrlich and Easley [5] for other proofs.

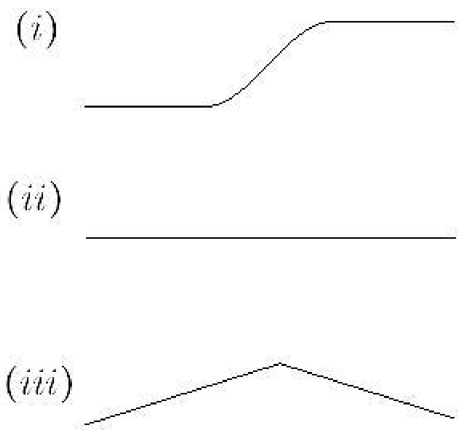


Figure 2: Examples of Cauchy surfaces in the globally hyperbolic spacetime $\mathbb{M}^{1+1}$ which are (i) smooth hypersurfaces but which contain both null and spacelike tangent vectors, (ii) smooth and contain only spacelike tangent vectors and (iii) not $C^1$.

**Proposition 4.18.** *Given the Riemannian manifold* $(\Sigma, h)$*, then the standard static spacetime* $(M, g) = (\mathbb{R} \times \Sigma, dt^2 - h)$ *is globally hyperbolic iff* $(\Sigma, h)$ *is a complete Riemannian manifold.*

*Proof.* Let $(\Sigma, h)$ be complete. We wish to give two proofs that $(M, g)$ is globally hyperbolic, namely that it possesses a smooth spacelike Cauchy surface and that it satisfies the definition of global hyperbolicity (equivalent by Theorem 4.16).

We wish to show that $\Sigma_0 = \{0\} \times \Sigma$ is a smooth spacelike Cauchy surface. Let $\gamma: I \to M$ be a smooth inextendible causal curve w.l.o.g. given by $\gamma(t) = (t, \sigma(t))$, where $\sigma: I \to \Sigma$ is a smooth inextendible curve in $\Sigma$ with speed bounded by 1. Then if $I \neq \mathbb{R}$ then as $\sigma$ is uniformly continuous and $\Sigma$ is complete then $\sigma$ can be continuously extended to the closure $\overline{I}$ of $I$ in $\mathbb{R}$, where $I \neq I'$, contradicting the inextendibility of $\sigma$. Thus $I = \mathbb{R}$, $\gamma(0) \in \Sigma_0$, and so any inextendible smooth causal curve passes $\Sigma_0$. Note also the parametrisation: $\gamma(t) = (t, \sigma(t))$ also shows that it must pass $\Sigma_0$ once and only once.

We now show that $M$ satisfies the definition of a globally hyperbolic spacetime. That it is causal follows from the previous argument. We must now show that $J^+(x) \cap J^-(y)$ is compact for all $x, y \in M$. Let $x = (t, p)$ and $y = (t', q)$. By Proposition 4.12, since $(\Sigma, h)$ is complete we know that $J^+(x) \cap \Sigma_s = \{s\} \times C(p, s-t)$ for all $s \geq t$. Thus:

$$
\begin{aligned}
J^+(x) \cap J^-(y) &= \left[ \bigcup_{s \geq t} \{s\} \times C(p, s-t) \right] \cap \left[ \bigcup_{s' \leq t'} \{s'\} \times C(q, t'-s') \right] \\
&= \bigcup_{s \geq t, s' \leq t'} \{s\} \times C(p, s-t) \cap \{s'\} \times C(q, t'-s') \\
&= \bigcup_{t \leq s \leq t'} \{s\} \times [C(p, s-t) \cap C(q, t'-s)]
\end{aligned}
$$

Note that $C(p, s-t) \cap C(q, t'-s)$ is compact since complete Riemannian manifolds obey the Heine-Borel property (Theorem 16 of Petersen [25]). Let $z_n = (s_n, r_n) \in J^+(x) \cap J^-(y)$, so $s_n \in [t, t']$ and $r_n \in [C(p, s_n - t) \cap C(q, t' - s_n)]$. By taking successive subsequences we have that $s_{n_k} \to s \in [t, t']$ and $r_{n_k} \to r \in C(p, s-t) \cap C(q, t'-s)$. So $z_{n_k} \to z = (s, r) \in J^+(x) \cap J^-(y)$ and the latter is compact.

Now for the converse: If $(\Sigma, h)$ is not complete, then $\epsilon_p \neq T_p\Sigma$ for some $p \in \Sigma$. So $\exists X_p \in T_p\Sigma$ s.t. $|X_p| = R$, $B(0, R) \subseteq \epsilon_p$ and $X_p \notin \epsilon_p$. Consequently, there exists a geodesic $\sigma : [0, t) \to \Sigma$ that is (continuously) inextendible by Theorem 4.10 and has unit speed. Let $x = (0, p)$, $y = (2t, p)$ so $(t, \sigma(s)) \in J^+(x) \cap J^-(y)$ $\forall s \in [0, t)$. If $(M, g)$ is globally hyperbolic, then $J^+(x) \cap J^-(y)$ is compact and for all $s \in [0, t)$: $\sigma(s) \in \pi_t^{-1}[\Sigma_t \cap J^+(x) \cap J^-(y)]$, where the RHS is compact in $\Sigma$. So, $\sigma$ is extendible by Lemma 4.11, which is a contradiction. $\square$

Note that since global hyperbolicity is preserved under conformal transformations then we also have the following result:

**Lemma 4.19.** *Given the Riemannian manifold $(\Sigma, h)$ and the smooth function $V \in C^\infty(\Sigma)$, $V > 0$ then the standard static spacetime $(M, g) = (\mathbb{R} \times$*

$\Sigma, V^2dt^2 - h)$ *is globally hyperbolic iff* $(\Sigma, V^{-2}h)$ *is a complete Riemannian manifold.*

We shll now analyse the Cauchy development $D(\Sigma_0)$ of the set $\Sigma_0$ in a standard static spacetime. Note that the Cauchy development of a set may be open (in the case of $D(\{0\} \times \mathbb{R}) \subseteq \mathbb{M}^{1+1}$) or closed (as in the case of $D(\{0\} \times [a, b]) \subseteq \mathbb{M}^{1+1}$). The following proposition states that in a standard static spacetime the Cauchy development of $\Sigma_0$ is open. Thus it is a smooth embedded submanifold and the metric $g$ gives it the structure of a spacetime. Note that this spacetime is also static but not in general standard static (e.g. let $M = \mathbb{R} \times (0, 1)$ be a strip in Minkowski space with the induced Lorentzian metric. Then $D(\Sigma_0)$ is an open diamond.) In fact, this spacetime $D(\Sigma_0)$ is also globally hyperbolic.

**Proposition 4.20.** *Let* $(M, g) = (\mathbb{R} \times \Sigma, V^2dt^2 - h)$ *be the standard static spacetime in Definition 2.4 then* $\Sigma_0$ *is an acausal smooth embedded spacelike hypersurface in* $M$*. Also,* $M$ *satisfies the causality condition.*

*Proof.* We have shown in Proposition 4.6 that if $\gamma : I \to M$ is a causal curve meeting $\Sigma_0$ (where $I$ is an open interval of $\mathbb{R}$) then, after taking a reparametrisation, we may let $\gamma(t) = (t, \sigma(t))$, where $\sigma : I \to \Sigma$ is a smooth curve with speed bounded by 1 and $0 \in I$ and clearly $\gamma$ only passes $\Sigma_0$ once. A similar argument also works for the second statement. □

**Proposition 4.21.** *Given any acausal topological hypersurface* $S$ *in a spacetime* $(M, g)$*, then* $D(S)$ *is open in* $M$ *and* $(D(S), g)$ *is a globally hyperbolic spacetime. In fact* $S$ *is a Cauchy surface for* $(D(S), g)$*.*

*Proof.* See Propositions 14.38 and 14.43 of O'Neill [23]. □

Thus a consequence of the previous two propositions is that given a standard static spacetime $(M,g) = (\mathbb{R}\times\Sigma, V^2dt^2-h)$ then $(D(\Sigma_0),g)$ is a globally hyperbolic spacetime.

We give here an explicit form of $D(\Sigma_0)$ and an alternative proof that it is an open set in $M$.

**Proposition 4.22.** *Given a standard static spacetime* $(M,g) = (\mathbb{R}\times\Sigma, V^2dt^2-h)$ *then the following statements are true:*

1. $D^+(\Sigma_0) = \{(t,p) \in M \colon C(p,t)$ *is compact in* $\Sigma, t \geq 0\}$.

2. $D^-(\Sigma_0) = \{(-t,p) \in M \colon C(p,t)$ *is compact in* $\Sigma, t \geq 0\} = TD^+(\Sigma_0)$.

3. $D(\Sigma_0) = D^+(\Sigma_0) \cup D^-(\Sigma_0) = \{(t,p) \in M \colon C(p,|t|)$ *is compact in* $\Sigma\}$.

4. $D(\Sigma_0)$ *is open in* $M$,

where $C(p,t)$ is the closed ball centered on $p$ of radius $t$ in the metric on $\Sigma$ induced by the Riemannian metric $V^{-2}h$ (see Theorem 4.4), $T : M \to M$ is the smooth map: $T(t,p) = (-t,p)$ and $\Sigma_0 = \{0\} \times \Sigma$.

Before we prove Proposition 4.22, we prove the following very useful result:

**Proposition 4.23.** *With the definitions of the previous proposition, let* $K \subseteq \Sigma$ *and* $C(K,t)$ *be compact in* $\Sigma$*, where* $t \geq 0$*, then* $\{t\} \times K \subseteq D^+(\Sigma_0)$.

*Proof.* As usual we let w.l.o.g. $V = 1$ for simplicity. Let $p \in K$ and $\gamma\colon I \to \mathbb{R}\times\Sigma$ be an inextendible future-pointing smooth causal curve through $(t,p)$, where $I$ is an open interval of $\mathbb{R}$. By Proposition 4.7 w.l.o.g. we can set $\gamma(s) = (s,\sigma(s))\ \forall s \in I$, where $\sigma\colon I \to \Sigma$ is a smooth curve with $t \in I$, $\sigma(t) = p$ and $|\dot{\sigma}(s)| \leq 1\ \forall s \in I$.

Let $I = (a, b)$ where $b \in \mathbb{R} \cup \{\infty\}$. If $a \geq 0$ then for $s \in (a, t)$:

$$d(p, \sigma(s)) \leq L(\sigma|_s^t) = \int_s^t |\dot{\sigma}(s')| ds' \leq t - s \leq t$$

and so $\sigma(s) \in C(K, t)$ $\forall s \in (a, t)$. But since $\sigma$ is a smooth curve in $\Sigma$ with speed bounded by 1 it is uniformly continuous by the Mean Value Theorem (Theorem 4.8) and since it is contained in the compact (and thus complete) set $C(p, t)$, then it can be continuously extended, contradiction. Thus $a < 0$ and $\gamma$ passes $\Sigma_0$. □

**Corollary 4.24.** *Again, with the definitions of the previous propositions, if $C(K, t)$ is compact in $\Sigma$ then $\{s\} \times C(K, t - s) \subseteq D^+(\Sigma_0)$ for all $s \in [0, t]$.*

*Proof.* So (by Proposition C.4) $C(K, t) = C(C(K, t - s), s)$ is compact. By the previous proposition then $\{s\} \times C(K, t - s) \subseteq D^+(\Sigma_0)$ for all $s \in [0, t]$. □

*Proof of Proposition 4.22.* Again, for simplicity and w.l.o.g. assume $V = 1$. We start by proving Statement *1*:

$$D^+(\Sigma_0) = \{(t, p) \in M \colon C(p, t) \text{ is compact in } \Sigma, t \geq 0\}$$

That the RHS is contained in the LHS follows from Corollary 4.24 with $K = \{p\}$ and $s = t$. For the converse, let $(t, p) \in D^+(\Sigma_0)$. So $t \geq 0$ and any past-inextendible future-pointing inextendible smooth causal curve through $(t, p)$ passes $\Sigma_0$. Thus, using the symmetry of the spacetime, any future-pointing future-inextendible smooth causal curve through $(t, p)$ passes $\Sigma_{2t}$. Take for instance the curve $\gamma \colon I \to M$, where $0 \in I$, $\gamma(s) = (t + s, \sigma(s))$ and $\sigma$ is an inextendible geodesic with $\sigma(0) = p$ and $\dot{\sigma}(0) = X_p$, with $|X_p| \leq 1$. Let $I = (a, b)$. The curve $\gamma$ is thus causal and inextendible and so passes $\Sigma_{2t}$ and so $b > t$. Alternatively if $|X_p| \leq t$ then (by Lemma 5.8 (Rescaling Lemma) in Lee [19]) $b > 1$ and so $\overline{B(0, t)} \subseteq \epsilon_p$ or, by Proposition 4.12, $C(p, t)$ is compact in $\Sigma$.

Statements *2* and *3* follow. Now for the Statement *4* that $D(\Sigma_0)$ is open:

Let $(t,p) \in D(\Sigma_0)$ w.l.o.g. $t \geq 0$. So $C(p,t)$ is compact in $\Sigma$ and from Corollary C.6, there exists $\epsilon > 0$ s.t. $C(p, t+\epsilon)$ is also compact. We propose that:

$$\left(-\left(t+\frac{\epsilon}{2}\right), t+\frac{\epsilon}{2}\right) \times B\left(p, \frac{\epsilon}{2}\right) \subseteq D(\Sigma_0).$$

This follows by showing that if $(s,q) \in (-(t+\frac{\epsilon}{2}), t+\frac{\epsilon}{2}) \times B(p, \frac{\epsilon}{2})$ then $C(q,s)$ is compact (the result then follows from the description of $D(\Sigma_0)$ just proven). Firstly, we can set w.l.o.g. $s \in [0, t+\frac{\epsilon}{2})$, $d(p,q) < \frac{\epsilon}{2}$. But $r \in C(q,s) \Rightarrow d(r,q) \leq s$ and so:

$$d(r,p) \leq d(r,q) + d(q,p) < s + \frac{\epsilon}{2} < t + \frac{\epsilon}{2} + \frac{\epsilon}{2} = t + \epsilon$$

So $C(q,s) \subseteq C(p, t+\epsilon)$ and as the RHS is compact then so is $C(q,s)$ □

It is well known that given a globally hyperbolic spacetime and smooth initial data of compact support defined on a smooth spacelike Cauchy surface then the Klein-Gordon equation can be solved uniquely with respect to this data:

**Theorem 4.25** (Existence and Uniqueness of Classical Solutions on Globally Hyperbolic Spacetimes with respect to compactly supported initial data)**.** *(Bär et al. [3] Theorem 3.2.11) Let $(M,g)$ be a globally hyperbolic spacetime with smooth, spacelike Cauchy surface $S$. Then the Klein-Gordon equation has a well-posed initial value formulation, that is, given data $\phi_0, \dot{\phi}_0 \in C_0^\infty(S)$ then there exists a unique solution $\psi \in C^\infty(M)$ to:*

$$(\Box_g + m^2)\psi = 0$$

$$\psi|_S = \phi_0$$

$$\nabla_n \psi|_S = \dot{\phi}_0,$$

*where $n$ is the unique unit smooth future-pointing timelike vector field along $S$ normal to $S$. Moreover:*

$$supp\,\psi \subseteq J(K)$$

*where $K = supp\,\phi_0 \bigcup supp\,\dot{\phi}_0$.*

Note that there exists along any smooth spacelike surface $S$ in a spacetime $M$ such a smooth vector field $n$ along $S$ normal to $S$ (the smooth vector field $n$ is not to be confused with the dimension of the spacetime). For completeness, this is proven in Proposition E.1. Note that the orientability of $M$ or $S$ is not assumed.

We shall use a modification of this theorem in the next section, that is, we can drop the condition on the data of being of compact support. This is shown in the Appendix (Theorem B.1).

# 5 The Existence of Wald solutions

In this section we show how to construct our solution to the Klein-Gordon equation from the vector-valued function $t \to [\phi_t]$. This section is based on the paper by Wald [37], but is extended in the following aspects. The more recent result by Bernal and Sanchez [6] on the extendibility of subsets of the spacetime to smooth spacelike Cauchy surfaces in globally hyperbolic spacetimes (the second half of Theorem 4.16) is needed to complete the proof on the existence of Wald solutions. We also extend Wald's proof to the case of acceptable s.a.e.s. The reference for the results on globally hyperbolic spacetimes is, as usual, Bär et al. [3]. This section is of great importance to us as it proves that the construction of Section 3 defines a smooth solution to the Klein-Gordon equation. We answer in the next section the question of its uniqueness.

Now we finally come to the statement concerning the agreement between our solution (Equation (3.8), p.27) to the Hilbert space version of the Klein-Gordon equation (Equation 3.6, p.25) and that arising from an application of Theorem B.1.:

**Theorem 5.1.** *Given initial data $\phi_0, \dot{\phi}_0 \in \chi_E$, where $A_E$ is an acceptable s.a.e. of $A$, choose $\phi_t \in \chi_E$ s.t. $[\phi_t] = C(t, A_E)[\phi_0] + S(t, A_E)[\dot{\phi}_0]$. If we define the function $\phi$ on $M$ by: $\phi(t,x) = \phi_t(x)$, and let $\psi$ be the unique smooth solution in $D(\Sigma_0)$ satisfying this smooth Cauchy data according to Theorem B.1 then $\phi = \psi$ in $D(\Sigma_0)$ and, in particular, $\phi|_{D(\Sigma_0)}$ is smooth and solves the Klein-Gordon equation there.*

Note that if $A_E$ is bounded-below then $A_E^-$ is bounded and $\chi_E = \{f \in C^\infty(\Sigma) \text{ s.t. } [f] \in D(A_E^\infty)\}$.

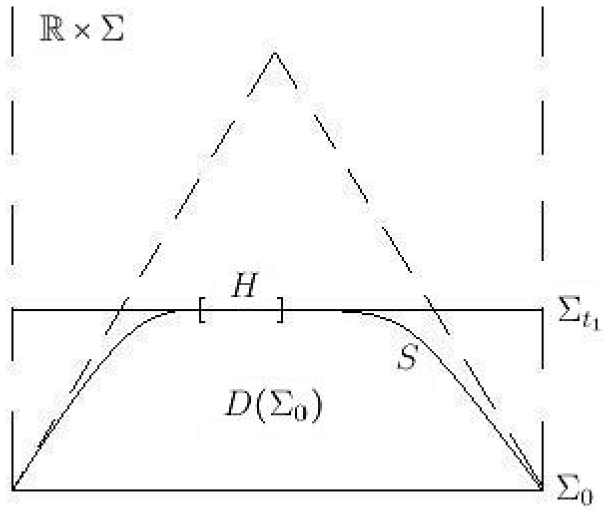


Figure 3: $\phi = \psi$ in $D(\Sigma_0)$, where $(M,g) = (\mathbb{R}\times(0,1), dt^2 - dx^2)$

We will prove this theorem by contradiction. The proof is based on Wald [37] but completed (with a more recent result of Bernal and Sanchez [6] on the existence of smooth spacelike Cauchy surfaces) and extended.

**Proposition 5.2.** *If there exists $t_1$ such that $\phi \neq \psi$ everywhere in a non-null set in $\Sigma_{t_1} \cap D(\Sigma_0)$, then there exists a compact set $H$ in $\Sigma_{t_1} \cap D(\Sigma_0)$ and a smooth spacelike Cauchy surface $S$ for $D(\Sigma_0)$ s.t. $H \subseteq S$ and $vol_h\{(t_1,x) \in H\colon \psi(t_1,x) \neq \phi(t_1,x)\} > 0$.*

*Proof.* So, by assumption there exists $t_1 \in \mathbb{R}$ such that

$$\mathrm{vol}_h\{(t_1,x) \in \Sigma_{t_1} \cap D(\Sigma_0)\colon \psi(t_1,x) \neq \phi(t_1,x)\} > 0. \tag{5.1}$$

Now we construct a smooth compact embedded submanifold with boundary $H$ of $\Sigma_{t_1} \cap D(\Sigma_0)$ s.t.

$$\mathrm{vol}_h\{(t_1,x) \in H\colon \psi(t_1,x) \neq \phi(t_1,x)\} > 0.$$

Firstly, let $U = \{(t_1, x) \in \Sigma_{t_1} \cap D(\Sigma_0)\colon \psi(t_1, x) \neq \phi(t_1, x)\}$, so $\mathrm{vol}_h(U) > 0$. Since any manifold has a countable atlas (see e.g. [39] Lemma 1.9) then there exists such an atlas $(V_n, \phi_n)_{n\geq 0}$ of $\Sigma_{t_1} \cap D(\Sigma_0)$ with $U = \bigcup_{n\geq 0} U \cap V_n$ and $\mathrm{vol}_h(U) \leq \sum_{n\geq 0} \mathrm{vol}_h(U \cap V_n)$ and so there must be one chart $(V_n, \phi_n)$ s.t. $\mathrm{vol}_h(U \cap V_n) > 0$. Let $(V, \phi) = (V_n, \phi_n)$.

Secondly, by a similar argument, as $\phi(V)$ is a open subset of $\mathbb{R}^4$ and any open subset of $\mathbb{R}^n$ can be covered by a countable number of open balls then there exists an open ball $B = \{x \in \mathbb{R}^4 \text{ s.t. } \|x\| < r\}$ (w.l.o.g. centered at 0) s.t. $B \subseteq \phi(V)$ and $\mathrm{vol}_h(U \cap \phi^{-1}(B)) > 0$.

Lastly, since $B = \{x \in \mathbb{R}^4 \text{ s.t. } \|x\| < r\}$ is covered by the countable collection of closed balls $C_n = \{x \in \mathbb{R}^4 \text{ s.t. } \|x\| \leq r_n\}$ where $(r_n)_{n\geq 1}$ is any sequence of positive reals s.t. $r_n \nearrow r$ and as before there must exist $n \geq 1$ s.t. $\mathrm{vol}_h(U \cap \phi^{-1}(C_n)) > 0$. Let $H = \phi^{-1}(C_n)$ be the desired smooth compact submanifold with boundary of $\Sigma_{t_1} \cap D(\Sigma_0)$. Since $H \subseteq \Sigma_{t_1} \cap D(\Sigma_0)$ and the latter is a smooth spacelike acausal embedded submanifold then $H$ is a smooth compact acausal spacelike embedded submanifold with boundary of the spacetime $(D(\Sigma_0), g)$. The reason for this construction is that it allows us to apply Theorem 4.16. Thus there exists a smooth spacelike Cauchy surface $S$ of $(D(\Sigma_0), g)$ which contains $H$. □

Now let $\dot{f}_{t_1}$ be a smooth compactly supported function on $S$ with support in $S \cap \Sigma_{t_1}$ such that $\dot{f}_{t_1} \geq 0$ and $\dot{f}_{t_1} = 1$ on $H$. Thus:

$$\int_{S\cap\Sigma_{t_1}} \dot{f}_{t_1}(\psi - \phi)V^{-1} d\mathrm{vol}_h \neq 0$$

and define $f$ to be the unique smooth solution to the Klein-Gordon equation on $D(\Sigma_0)$ with Cauchy data $(0, \dot{f}_1)$ on $S$, according to Theorem 4.25.

We define $F\colon [0,t_1]\times\Sigma \to \mathbb{R}$ as:

$$F(p) = \begin{cases} f(p), & p \in [0,t_1]\times\Sigma \cap D(\Sigma_0). \\ 0, & \text{otherwise.} \end{cases}$$

**Proposition 5.3.** *F satisfies the following:*

1. *supp* $F$ *is compact in* $[0,t_1]\times\Sigma\cap D(\Sigma_0)$.
2. *It is compactly supported on each* $\Sigma_t \cap D(\Sigma_0)$ *for* $0 \le t \le t_1$.
3. $F \in C^\infty([0,t_1]\times\Sigma)$ *(as a smooth manifold with boundary).*
4. $(\Box_g + m^2)F = 0$ *(as an element of* $C^\infty([0,t_1]\times\Sigma)$*).*
5. *supp* $(\partial_t F)\cap\Sigma_{t_1}$ = *supp* $\dot{f}_{t_1} \subseteq S\cap\Sigma_{t_1}$ *and* $\partial_t F(p) = \dot{f}_{t_1}(p)$ *for* $p \in S\cap\Sigma_{t_1}$.
6. $F|_{\Sigma_{t_1}} = 0$.

*Proof.* By construction supp $\dot{f}_{t_1}$ compact in $S$ and contained in $D(\Sigma_0)\cap\Sigma_{t_1}$. As all hypersurfaces concerned are embedded, then all have their topologies induced from that of $M$ and thus supp $\dot{f}_{t_1}$ compact in $D(\Sigma_0)$.

But, the causal past of a compact set intersected with the causal future of a Cauchy surface $S$ (in a globally hyperbolic spacetime) is always compact (see Corollary A.5.4 of Bär et al. [3]). Thus:

$$J^-_{D(\Sigma_0)}(\operatorname{supp}\dot{f}_1)\cap J^+_{D(\Sigma_0)}(\Sigma_0) \text{ is compact in } D(\Sigma_0).$$

So,

$$J^-_{D(\Sigma_0)}(\operatorname{supp}\dot{f}_1)\cap[0,t_1]\times\Sigma \text{ is compact in } D(\Sigma_0),$$

and so also in $[0,t_1]\times\Sigma\cap D(\Sigma_0)$.

Thus, $\operatorname{supp} F \subseteq \operatorname{supp} f \cap [0,t_1]\times\Sigma$ is compact in $[0,t_1]\times\Sigma\cap D(\Sigma_0)$ and Statement *1* is proved.

Statements *2* and *3* follow directly from *1*. Now, since by definition $F$ is locally equal to either $f$ or 0, where both are smooth solutions to the Klein-Gordon equation, then Statement *4* follows. Statements *5* and *6* result straight from the definitions of $F$ and $f$. □

**Theorem 5.4.** *The functions $\phi$ and $\psi$ are equal on $D(\Sigma_0)$.*

*Proof.* If there exists $t_1$ such that $\phi \neq \psi$ everywhere in a non-null set in $\Sigma_{t_1} \cap D(\Sigma_0)$, construct $H, S$ and $F$ as above. Now define:

$$c(t) = \int_{\Sigma_t} V^{-1} \left[ F \left( \frac{\partial \psi}{\partial t} - \frac{d\phi_t}{dt} \right) - \frac{\partial F}{\partial t} (\psi - \phi_t) \right] d\mathrm{vol}_h \tag{5.2}$$

Clearly since $\phi_t$ and $\frac{d\phi_t}{dt}$ are only defined a.e. in $\Sigma$ we should point out that any other choices in the same respective equivalence classes would yield an identical value of $c$. As both functions are in $\mathcal{L}^2(\Sigma)$, then multiplying by the smooth functions of compact support, $F$ and $\frac{\partial F}{\partial t}$, we obtain an element of $\mathcal{L}^1(\Sigma)$.

The smooth function $\psi$ is only defined in $D(\Sigma_0)$ and so on each hypersurface $\Sigma_t \cap D(\Sigma_0)$, $\psi$ and $\frac{\partial \psi}{\partial t}$ are smooth functions but as $F$ and $\frac{\partial F}{\partial t}$ are compactly supported smooth functions on each $\Sigma_t \cap D(\Sigma_0)$, then $f \frac{\partial \psi}{\partial t}$ and $\frac{\partial F}{\partial t} \psi$ are easily definable and smooth on each $\Sigma_t$, $t \in [0, t_1]$. Indeed they are of compact support also so they are integrable on $\Sigma_t$ (since we are dealing with a Radon measure). Thus:

$$
\begin{aligned}
\frac{dc}{dt} &= \int_{\Sigma_t} V^{-1}\left[\frac{\partial F}{\partial t}\left(\frac{\partial \psi}{\partial t} - \frac{d\phi_t}{dt}\right) + F\left(\frac{\partial^2 \psi}{\partial t^2} - \frac{d^2\phi_t}{dt^2}\right) - \frac{\partial^2 F}{\partial t^2}(\psi - \phi_t) - \frac{\partial F}{\partial t}\left(\frac{\partial \psi}{\partial t} - \frac{d\phi_t}{dt}\right)\right] d\mathrm{vol}_h \\
&= \int_{\Sigma_t} V^{-1}\left[F\frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 F}{\partial t^2}\psi\right] d\mathrm{vol}_h - \int_{\Sigma_t} V^{-1}\left[F\frac{d^2\phi_t}{dt^2} - \frac{\partial^2 F}{\partial t^2}\phi_t\right] d\mathrm{vol}_h \\
&= \int_{\Sigma_t} V^{-1}\left[FVD^i(VD_i\psi) - VD^i(VD_iF)\psi\right] d\mathrm{vol}_h - \int_{\Sigma_t} V^{-1}\left[F\frac{d^2\phi_t}{dt^2} - \frac{\partial^2 F}{\partial t^2}\phi_t\right] d\mathrm{vol}_h \\
&= \int_{\Sigma_t} \left[FD^i(VD_i\psi) - D^i(VD_if)\psi\right] d\mathrm{vol}_h - \int_{\Sigma_t} V^{-1}\left[F\frac{d^2\phi_t}{dt^2} - \frac{\partial^2 F}{\partial t^2}\phi_t\right] d\mathrm{vol}_h \\
&= \int_{\Sigma_t} \left[-(D^iF)(VD_i\psi) + (VD_iF)(D^i\psi)\right] d\mathrm{vol}_h - \int_{\Sigma_t} V^{-1}\left[F\frac{d^2\phi_t}{dt^2} - \frac{\partial^2 F}{\partial t^2}\phi_t\right] d\mathrm{vol}_h \\
&= \int_{\Sigma_t} V^{-1}\left[-F\frac{d^2\phi_t}{dt^2} + \frac{\partial^2 F}{\partial t^2}\phi_t\right] d\mathrm{vol}_h \\
&= \langle F, A_E\phi_t\rangle - \langle A_E F, \phi_t\rangle \\
&= 0.
\end{aligned}
$$

But, $\psi|_{\Sigma_0} = \phi_0$ and $\frac{\partial\psi}{\partial t}|_{\Sigma_0} = \dot{\phi}_0$, so $c(0) = 0$ and since $F|_{t_1} = 0$ by definition, we have:

$$
\begin{aligned}
c(t_1) &= -\int_{\Sigma_t} V^{-1}\dot{F}_{t_1}(\psi - \phi_t) d\mathrm{vol}_h \\
&\neq 0.
\end{aligned}
$$

However $c \in C^1[0, t_1]$ and so this last statement contradicts the Intermediate Value Theorem, yielding that $\phi = \psi$ a.e. in $D(\Sigma_0) \cap \Sigma_t$ for all $t$. Since $\phi$ and $\psi$ are continuous, then $\phi = \psi$ in $D(\Sigma_0) \cap \Sigma_t$ for all $t$ and so $\phi = \psi$ in $D(\Sigma_0)$. □

Thus we have proven Theorem 5.1. We shall now show that $\phi$ solves the Klein-Gordon equation everywhere in $M$.

**Theorem 5.5** (Existence of Wald Solutions)**.** *Let $A_E$ be an acceptable s.a.e. of $A$. Given any pair of functions $\phi_0, \dot{\phi}_0 \in \chi_E$, for each $t \in \mathbb{R}$ define $\phi_t \in \chi_E$ uniquely by: $[\phi_t] = C(t, A_E)[\phi_0] + S(t, A_E)[\dot{\phi}_0]$ and define the function $\phi$ on $M$ as $\phi(t,x) = \phi_t(x)$, where $\phi_t \in C^\infty(\Sigma)$. This function is smooth, solves* *the Klein-Gordon equation and satisfies the Cauchy data* $(\phi_0, \dot{\phi}_0)$*, that is* $\phi|_{\Sigma_0} = \phi_0$*,* $\partial_t \phi|_{\Sigma_0} = \dot{\phi}_0$.

*Proof.* Given $p = (t_1, x) \in M$, we wish to find an open neighbourhood of $p$ in $M$ in which $\phi$ is smooth and satisfies the Klein-Gordon equation. We begin by reformulating our vector-valued solution. We propose that:

$$\begin{aligned}
[\phi_t] &= C(t, A_E)[\phi_0] + S(t, A_E)[\dot{\phi}_0] \\
&= C(t - t_1 + t_1, A_E)[\phi_0] + S(t - t_1 + t_1, A_E)[\dot{\phi}_0] \\
&= [C(t - t_1, A_E)C(t_1, A_E) - A_E S(t - t_1, A_E)S(t_1, A_E)][\phi_0] \\
&\quad + [S(t - t_1, A_E)C(t_1, A_E) + C(t - t_1, A_E)S(t_1, A_E)][\dot{\phi}_0] \\
&= C(t - t_1)[C(t_1, A_E)[\phi_1] + S(t_1, A_E)[\dot{\phi}_0]] \\
&\quad + S(t - t_1, A_E)[-A_E S(t_1, A_E)[\phi_0] + C(t_1, A_E)[\dot{\phi}_0]] \\
&= C(t - t_1, A_E)[\phi_{t_1}] + S(t - t_1, A_E)[\dot{\phi}_{t_1}].
\end{aligned}$$

Here, we have used the identities:

$$\begin{aligned}
C(t_1 + t_2, A_E) &= C(t_1, A_E)C(t_2, A_E) - A_E S(t_1, A_E)S(t_2, A_E) \\
S(t_1 + t_2, A_E) &= S(t_1, A_E)C(t_2, A_E) + C(t_1, A_E)S(t_2, A_E)
\end{aligned}$$

on $D(A_E)$. But $\phi_{t_1}, \dot{\phi}_{t_1} \in \chi_E$ and Theorem (5.1) can be applied to this data to show that $\phi$ is smooth in the open neighbourhood $D(\Sigma_{t_1})$ of $p$ and satisfies the Klein-Gordon equation there. □

# 6 Uniqueness of Wald Solutions

We so far have concerned ourselves with constructing a class of solutions to the Klein-Gordon equation on standard static spacetimes. Our set of prescriptions is parametrised by acceptable s.a.e.s $A_E$ of the linear operator $A$ on the (real or complex) Hilbert space $L^2(\Sigma, V^{-1}d\text{vol}_h)$. For each such linear operator $A_E$ we show that the solution to the Klein-Gordon equation w.r.t. chosen Cauchy data it generates is unique up to some conditions yet to be stated. We will use this result to define a vector space of solutions, corresponding to each acceptable s.a.e. $A_E$.

**Theorem 6.1** (Uniqueness of Solutions (i)). *Let $A$ be the symmetric linear operator on the (real or complex) Hilbert space $L^2(\Sigma, V^{-1}dvol_h)$, defined by: $D(A) = [C_0^\infty(\Sigma)]$, $A([\phi]) = [(-VD^iVD_i + m^2V^2)\phi]$ for $\phi \in C_0^\infty(\Sigma)$. Let $A_E$ be an acceptable s.a.e. of $A$ and if $\Psi \in C^2(M)$ satisfies:*

$$(\Box_g + m^2)\Psi = 0$$

$$\Psi|_{\Sigma_0} = \partial_t \Psi|_{\Sigma_0} = 0$$

$$[\pi_t^*(\Psi|_t)] \in D(A_E)$$

$$[\pi_t^*(\partial_t\Psi|_t)] \in L^2(\Sigma, V^{-1}dvol_h)$$

*(where $\pi_t^*$ is the pull-back of the map $\pi_t : \Sigma \to \Sigma_t$), then $\Psi = 0$.*

We start with a proposition, which has its roots in distribution theory on arbitrary Riemannian manifolds.

**Proposition 6.2.** *Take $A$ and $A_E$ as above. If $\phi \in C^2(\Sigma)$ such that $[\phi] \in D(A_E)$, then $A_E[\phi] = [(-VD^iVD_i + m^2V^2)\phi]$.*

*Proof.* We know (already stated on p.23), that the adjoint $A^*$ of the linear operator $A$ is given by:

$$D(A^*) = \{\phi \in L^2(\Sigma, V^{-1}d\text{vol}_h) \text{ s.t. } A\phi \in L^2(\Sigma, V^{-1}d\text{vol}_h)\},$$

since $A$ is formally self-adjoint with respect to the smooth measure $V^{-1}d\mathrm{vol}_h$, which is proven in Proposition D.10. We can strengthen Proposition D.10 to the following case:

$$\int_\Sigma (A\phi)\theta V^{-1}d\mathrm{vol}_h = \int_\Sigma \phi(A\theta)V^{-1}d\mathrm{vol}_h,$$

for all $\phi \in C^2(\Sigma)$ and $\theta \in C_0^\infty(\Sigma)$, since $A$ is of second order and commutes with complex conjugation. The proof is similar. Then, if $\phi \in C^2(\Sigma)$ and $[\phi] \in D(A^*)$, we have:

$$A^*[\phi](\theta) = \int_\Sigma \phi(A\theta)V^{-1}d\mathrm{vol}_h = \int_\Sigma (A\phi)\theta V^{-1}d\mathrm{vol}_h = [A\phi](\theta),$$

where $A^*[\phi]$ is meant distributionally. Therefore $A^*[\phi] = [A\phi]$. Lastly, since $A_E$ is a s.a.e. of $A$, then $A \leq A_E$ and we have: $A_E \leq A^*$. So, $A_E$ is the restriction of $A^*$ to space $D(A_E)$. Therefore, if $\phi \in C^2(\Sigma)$ and $[\phi] \in D(A_E)$, then $A_E[\phi] = A^*[\phi] = [A\phi]$. □

*Proof of Theorem 6.1.* We use a proof by contradiction.
Firstly, we point out that if $(\Box_g + m^2)\Psi = 0$ then $\partial_t^2\Psi = -A\Psi$.
But as $\pi_t^*(\Psi|_t) \in D(A_E)$, by the previous proposition:

$$A(\pi_t^*(\Psi|_t)) = A_E(\pi_t^*(\Psi|_t)) \in L^2(\Sigma, V^{-1}d\mathrm{vol}_h)$$

and thus $\pi_t^*(\partial_t^2\Psi|_t) \in L^2(\Sigma, V^{-1}d\mathrm{vol}_h)$ also.

If $\Psi \neq 0$ then $\exists t_1 \in \mathbb{R}$ s.t. $\Psi|_{\Sigma_{t_1}} \neq 0$.
Let $\dot{f}_{t_1} \in C_0^\infty(\Sigma_{t_1})$ s.t. $\int_{\Sigma_{t_1}} \dot{f}_{t_1}\Psi V^{-1}d\mathrm{vol}_h \neq 0$
Let $f_t = S(t - t_1, A_E)(\pi_{t_1}^*\dot{f}_{t_1})$ be the vector-valued function. According to Theorem 5.5 on the existence of smooth Wald solutions, this function can be represented by the smooth solution $f \in C^\infty(M)$ to the Cauchy problem with smooth initial data $(0, \dot{f}_{t_1})$, of compact support on $\Sigma_{t_1}$.

We now evaluate the symplectic form at our two solutions $\Psi$ and $f$:

$$
\begin{aligned}
c(t) &= \int_{\Sigma_t} [\partial_t f \Psi - f \partial_t \Psi] V^{-1} d\text{vol}_h \\
&= \int_{\Sigma} \pi_t^*(\partial_t f|_t) \pi_t^*(\Psi_t) - \pi_t^*(f|_t) \pi_t^*(\partial_t \Psi_t) V^{-1} d\text{vol}_h.
\end{aligned}
$$

Then, the following are true:

1. $\pi_t^*(f|_t) \in L^2(\Sigma, V^{-1} d\text{vol}_h) \cap [C^\infty(\Sigma)]$.

2. $\pi_t^*(\partial_t f|_t) \in L^2(\Sigma, V^{-1} d\text{vol}_h) \cap [C^\infty(\Sigma)]$.

3. $\pi_t^*(\partial_t^2 f|_t) \in L^2(\Sigma, V^{-1} d\text{vol}_h) \cap [C^\infty(\Sigma)]$.

4. $\pi_t^*(\Psi|_t) \in D(A_E) \cap C^2(\Sigma) \subseteq L^2(\Sigma, V^{-1} d\text{vol}_h) \cap [C^2(\Sigma)]$.

5. $\pi_t^*(\partial_t \Psi|_t) \in L^2(\Sigma, V^{-1} d\text{vol}_h) \cap [C^1(\Sigma)]$.

6. $\pi_t^*(\partial_t^2 \Psi|_t) \in L^2(\Sigma, V^{-1} d\text{vol}_h) \cap [C(\Sigma)]$.

Then clearly $c(t_1) \neq 0$ and $c(0) = 0$ but:

$$
\begin{aligned}
\frac{dc(t)}{dt} &= \int_{\Sigma_t} [\partial_t^2 f \Psi - f \partial_t^2 \Psi] \\
&= -\langle Af, \Psi \rangle + \langle f, A\Psi \rangle \\
&= -\langle A_E f, \Psi \rangle + \langle f, A_E \Psi \rangle \\
&= 0,
\end{aligned}
$$

which is a contradiction. □

**Lemma 6.3** (Uniqueness of Solutions (ii))**.** *Let $A_E$ be an acceptable s.a.e. of $A$. Given two solutions $\Psi_1, \Psi_2 \in C^2(M)$ of the Klein-Gordon equation: $(\Box_g + m^2)\Psi_i = 0$ for $i \in \{1, 2\}$, corresponding to Cauchy data $\phi_0 \in C^2(\Sigma_0)$ s.t.*

$[\phi_0] \in D(A_E)$ *and* $\dot{\phi}_0 \in C^1(\Sigma) \cap \mathcal{L}^2(\Sigma, V^{-1}dvol_h)$ *such that* $\forall i \in \{1,2\}$*:*

$$\Psi_i|_{\Sigma_0} = \phi_0$$
$$\partial_t \Psi_i|_{\Sigma_0} = \dot{\phi}_0$$
$$[\pi_t^*(\Psi_i|_t)] \in D(A_E)$$
$$[\pi_t^*(\partial_t \Psi_i|_t)] \in L^2(\Sigma, V^{-1}dvol_h)$$

*then* $\Psi_1 = \Psi_2$

*Proof.* Let $\Psi = \Psi_1 - \Psi_2$, then $\Psi \in C^2(M)$ and satisfies the conditions of the previous proposition since all operations concerned are linear and $D(A_E)$ and $L^2(\Sigma, V^{-1}d\text{vol}_h)$ are vector spaces. Thus $\Psi = 0$. □

We note here the following trivial generalisation, the proof of which is similar to those previous. It will be this result that will be of use in Section 8 in describing the support of the Wald solution $\phi$.

**Lemma 6.4** (Uniqueness of Solutions (iii))**.** *Let $A_E$ be an acceptable s.a.e. of $A$. Given two solutions*

$$\Psi_1, \Psi_2 \in C^2([t_1, t_2) \times \Sigma)$$

*of the Klein-Gordon equation (regarding $[t_1, t_2) \times \Sigma$) as a smooth manifold with boundary):* $(\Box_g + m^2)\Psi_i = 0$ *for* $i \in \{1,2\}$*, corresponding to Cauchy data* $\phi_{t_1} \in C^2(\Sigma_{t_1})$ *s.t.* $[\phi_{t_1}] \in D(A_E)$ *and* $\dot{\phi}_{t_1} \in C^1(\Sigma) \cap \mathcal{L}^2(\Sigma, V^{-1}dvol_h)$ *such that* $\forall i \in \{1,2\}$ *and* $\forall t \in [t_1, t_2)$*:*

$$\Psi_i|_{\Sigma_{t_1}} = \phi_{t_1}$$
$$\partial_t \Psi_i|_{\Sigma_{t_1}} = \dot{\phi}_{t_1}$$
$$[\pi_t^*(\Psi_i|_t)] \in D(A_E)$$
$$[\pi_t^*(\partial_t \Psi_i|_t)] \in L^2(\Sigma, V^{-1}dvol_h)$$

*then* $\Psi_1 = \Psi_2$

Using Theorems 5.5 and 6.1 on the existence and uniqueness of solutions to the Klein-Gordon equation, we will find it useful to define a vector space of solutions, for each acceptable s.a.e. $A_E$ of $A$. We show that it can be given a natural symplectic structure in Section 10. It's this structure that is required for the construction of the Weyl-algebra, however we will not be concerned with quantisation in this thesis.

**Definition 6.5** (Space of Solutions)**.** *Given an acceptable s.a.e. $A_E$ of $A$, define the **space of solutions**, $S_E$ to be:*

$$S_E = \{\phi \in C^\infty(M) \colon (\Box_g + m^2)\phi = 0, \pi_t^{-1}(\phi_t), \pi_t^{-1}(\dot{\phi}_t) \in \chi_E \textit{ for all } t\}$$

**Proposition 6.6.** *We have the linear isomorphism: $\Psi \colon \chi_E \times \chi_E \to S_E$, defined by $\Psi(\phi_0, \dot{\phi}_0) = \phi$, where $\phi$ is constructed using Theorem 5.5 on the existence of Wald solutions.*

*Proof.* Clearly $\Psi$ is linear. Surjectivity follows since, if $\psi \in S_E$, then $\psi_0, \dot{\psi}_0 \in \chi_E$. Let $\phi$ be the Wald solution, satisfying the Cauchy data $(\psi_0, \dot{\psi}_0)$. Then $\psi$ and $\phi$ satisfy all the conditions of Theorem 6.1 on uniqueness and so $\psi = \phi$. □

# 7 Causal Structure of Standard Static Spacetimes (ii)

We shall in Section 8 further analyse some of the properties of our constructed solutions to the Klein-Gordon equation. However, we must first prove some basic properties of the causal structure of standard static spacetimes. One apparently simple result of this section is that if $K$ is a compact subset of $\Sigma_0$ then for all sufficiently small $t$, $J^+(K) \cap \Sigma_t$ is compact in $\Sigma_t$. It will be this result and the adapted uniqueness result of Lemma 6.4 which will prove useful in the next section. We shall also need to prove more properties of $J^+(K)$ to be used in Section 8.

For all the results of this section, let $(M, g) = (\mathbb{R} \times \Sigma, V^2 dt^2 - h)$ be a standard static spacetime as in Definition 2.4, however in all the statements we can set w.l.o.g $V = 1$, since both the Cauchy development and causal future of a set in a spacetime are invariant under conformal transformations of the spacetime to itself.

**Proposition 7.1.** *Let $K \subseteq \Sigma_0$ be a compact set. If $J(K) \cap \Sigma_t$ is compact then $J(K) \cap \Sigma_{t'}$ is compact for all $|t'| \leq |t|$. Define:*

$$t^\infty(K) := \sup\{t \geq 0 : J^+(K) \cap \Sigma_t \text{ is compact in } \Sigma_t\}.$$

*Then $t^\infty(K) \in (0, \infty]$. Furthermore, the following are true:*

1. *$J(K) \cap \Sigma_t$ is compact for all $|t| < t^\infty(K)$.*
2. *If $t^\infty(K) < \infty$ then $J(K) \cap \Sigma_t$ is not compact for all $|t| \geq t^\infty(K)$.*
3. *If $\Sigma$ is complete, then $C(K, t)$ is compact for all $t$ and $t^\infty(K) = \infty$.*
4. *If $t^\infty(K) = \infty$ for any non-empty compact set $K$, then $\Sigma$ is complete.*

Note that $\Sigma$ is complete as a metric space iff geodesically complete by the Hopf-Rinow Theorem (see e.g. Theorem 6.13 Lee [19]). If so, then $\Sigma$ obeys the Heine-Borel property, that is $K \subseteq \Sigma$ is compact iff $K$ is closed and bounded (see e.g. Theorem 16 in Petersen [25]).

*Proof.* Let $t \geq 0$. If $J(K) \cap \Sigma_t$ is compact, then, by Proposition 4.12, $J(K) \cap \Sigma_t = C(K,t)$. But as $C(K,t)$ is compact, it easily follows that $C(K,t')$ is compact for all $|t'| \leq |t|$ and similarly for $J(K) \cap \Sigma_{t'}$. That $t^\infty(K) > 0$ is proven as follows. As $K$ is compact, then, by Proposition C.5, $C(K,t)$ is compact for some $t > 0$ and so $J^+(K) \cap \Sigma_t$ is compact by Proposition 4.12. It then follows that $t^\infty(K) > 0$ and also that Statement *1* is true. If $t^\infty(K) < \infty$ and $J(K) \cap \Sigma_{t^\infty(K)}$ is compact then $C(K, t^\infty(K))$ is compact, as is $C(K, t^\infty(K) + \epsilon)$ for some $\epsilon > 0$ (by Proposition C.6), and so also $J(K) \cap \Sigma_{t^\infty(K)+\epsilon}$ which contradicts the definition of $t^\infty(K)$. This proves Statement *2*. If $\Sigma$ is complete, then, for all $t$, as $C(K,t)$ is closed and bounded, so it's also compact by the Heine-Borel property. Statement *3* then follows from Proposition 4.12. If $p_n$ is a Cauchy sequence, then it is bounded and so contained in the compact set $C(K,t)$ for some t and so $p_n$ converges, which proves Statement *4*. □

**Proposition 7.2.** *Let $C(K,t)$ be compact in $\Sigma$, where $K$ is a compact subset of $\Sigma$ and $t \geq 0$, then $\{\frac{t}{2}\} \times C(K, \frac{t}{2}) \subseteq D(\Sigma_0)$.*

*Proof.* This follows from Corollary 4.24 with $s = \frac{t}{2}$. □

**Corollary 7.3.** *If $J(K) \cap \Sigma_t$ is compact, then $J(K) \cap \Sigma_{\frac{t}{2}} \subseteq D(\Sigma_0)$.*

*Proof.* It follows easily from Proposition 7.2 and repeated use of Proposition 4.12. □

**Proposition 7.4.** $\forall 0 \leq t_1 \leq t_2$*:*

*1.* $\pi(J(p)\cap\Sigma_{t_1})\subseteq\pi(J(p)\cap\Sigma_{t_2})$

*2.* $\pi(D(\Sigma_0)\cap\Sigma_{t_2})\subseteq\pi(D(\Sigma_0)\cap\Sigma_{t_1})$,

*where* $\pi{:}\,\mathbb{R}\times\Sigma\to\Sigma$ *is the map:* $\pi(t,x)=x$.

*Proof.* We can set w.l.o.g $V=1$ since otherwise:

$$\begin{aligned}\pi(J(p)_{V^2dt^2-h}\cap\Sigma_{t_1})=\pi(J(p)_{dt^2-V^{-2}h}\cap\Sigma_{t_1})&\subseteq\pi(J(p)_{dt^2-V^{-2}h}\cap\Sigma_{t_2})\\&=\pi(J(p)_{V^2dt^2-h}\cap\Sigma_{t_2}),\end{aligned}$$

where our subscript notation highlights the dependence of $J(p)$ on the metric.

To prove *1*: If $q\in LHS$, then $\exists\gamma{:}\,[0,t_1]\to\mathbb{R}\times\Sigma,\gamma(t)=(t,\sigma(t))$, $|\dot\sigma(t)|\le1$, $\sigma(0)=p$, $\sigma(t_1)=q$.

Let $\gamma'{:}\,[0,t_2]\to\mathbb{R}\times\Sigma$, $\gamma'(t)=(t,\sigma(t\frac{t_1}{t_2}))$, $|\dot\gamma'|^2=1-(\frac{t_1}{t_2})^2|\dot\sigma(t\frac{t_1}{t_2})|^2\le0$, $\gamma'(0)=(0,p)$, $\gamma'(t_2)=(t_2,q)$, so that $q\in RHS$.

Statement *2* follows from Proposition 4.22. □

**Corollary 7.5.** $J(p)\cap\Sigma_{t_1}\nsubseteq D(\Sigma_0)\Rightarrow J(p)\cap\Sigma_{t_2}\nsubseteq D(\Sigma_0)\ \forall 0\le t_1\le t_2$.

**Proposition 7.6.** *If* $t^\infty(K)<\infty$*, then* $J(K)\cap\Sigma_{t^\infty(K)/2}\nsubseteq D(\Sigma_0)$.

*Proof.* Again w.l.o.g let $V=1$. We know via Propositions 7.1 and 4.12, that: $\overline{B(p,t)}\subseteq\epsilon_p$ for all $t<t^\infty(K)$ and $p\in K$; $B(p,t^\infty(K))\subseteq\epsilon_p$ for all $p\in K$, and that there exists $p\in K$ such that $\overline{B(p,t^\infty(K))}\nsubseteq\epsilon_p$.

Thus there exists $X_p\in T_p\Sigma\backslash\epsilon_p$ with $|X_p|=t^\infty(K)$. We hold that there must then exist a geodesic $\sigma{:}\,[0,1)\to\Sigma$ inextendible to 1 such that $\dot\sigma(0)=X_p$.

To show this is true, let $\sigma:[0,a)\to\Sigma$ the maximal geodesic, starting at $p$ with $\dot\sigma(0)=X_p$. If $a>1$, then $X_p\in\epsilon_p$ by definition, which is however a contradiction. If $a < 1$, then $\gamma$ be the geodesic through $p$ with $\dot{\gamma}(0) = aX_p$. By the rescaling Lemma, $\sigma$ being inextendible to $a$ implies that $\gamma$ is inextendible to 1. So, by definition, $aX_p \notin \epsilon_p$. But $|aX_p| < |X_p| = t^\infty(K)$, which is a contradiction.

Now that the existence of the geodesic $\sigma$ is proven, define $\sigma' : [0, t^\infty(K)) \to \Sigma$ via: $\sigma'(s) = \sigma(\frac{s}{t^\infty(K)})$. It is satisfies: $|\dot{\sigma}'(s)| = \frac{1}{t^\infty(K)} |\dot{\sigma}(\frac{s}{t^\infty(K)})| = 1$ and $\sigma'(0) = p$. So, from Proposition 4.7, $x = (\frac{t^\infty(K)}{2}, \sigma'(\frac{t^\infty(K)}{2})) \in J(K) \cap \Sigma_{t^\infty(K)/2}$.

Now define $\alpha : (0, t^\infty(K)/2] \to \mathbb{R} \times \Sigma$, $\alpha(s) = (s, \sigma'(t^\infty(K) - s))$. Since $\sigma$ is inextendible to 1 then $\sigma'$ is inextendible to $t^\infty(K)$ and so $\alpha$ is past-inextendible to 0. Clearly, $\alpha$ does not pass $\Sigma_0$ although $\alpha(t^\infty(K)/2) = (t^\infty(K)/2, \sigma'(t^\infty(K)/2)) = x$. Since $\alpha$ is a future-pointing past-inextendible smooth causal curve passing $x$ but not $\Sigma_0$, then $x \notin D(\Sigma_0)$. Thus $x \in J(K) \cap \Sigma_{t^\infty(K)/2} \backslash D(\Sigma_0)$. □

**Corollary 7.7.** *The following statements are true:*

1. $J(K) \cap \Sigma_t \subseteq D(\Sigma_0)$ $\forall 0 \le t < t^\infty(K)/2$.

2. $t^\infty(K) < \infty \Rightarrow J(K) \cap \Sigma_t \nsubseteq D(\Sigma_0)$ $\forall t \ge t^\infty(K)/2$.

3. $t_1(K) := \sup\{t : J^+(K) \cap \Sigma_t \subseteq D(\Sigma_0)\} = t^\infty(K)/2$.

For the purposes of the following section, we continue these arguments to define an increasing sequence:

$$t_{n+1}(K) := \sup\{t : J^+(K) \cap \Sigma_t \subseteq D(\Sigma_{t_n(K)})\},$$

where $t_0(K) = 0$ and the resulting definition of $t_1(K)$ agrees with that used above. We are led to the following corollary:

**Corollary 7.8.** *The following statements are true:*

1. $J(K) \cap \Sigma_t \subseteq D(\Sigma_{t_n(K)})$ $\forall t_n(K) \leq t < (1 - \frac{1}{2^n})t^\infty(K)$.

2. $t^\infty(K) < \infty \Rightarrow J(K) \cap \Sigma_t \nsubseteq D(\Sigma_{t_n(K)})$ $\forall t \geq (1 - \frac{1}{2^n})t^\infty(K)$.

3. $t_n(K) = (1 - \frac{1}{2^n})t^\infty(K) \nearrow t^\infty(K)$ *as* $n \to \infty$.

# 8 Support of Wald Solutions

We now prove a result concerning the support of our "Wald solutions". It is in fact not true that given Cauchy data consisting of two test functions $(\phi_0, \dot{\phi}_0)$ then the support of the corresponding solution $\phi$ (w.r.t. some acceptable s.a.e. $A_E$ of $A$) as constructed in Theorem 5.5, is necessarily contained in $J(K)$, where $K = \operatorname{supp}\phi_0 \cup \operatorname{supp}\dot{\phi}_0$ and $J(K)$ is as usual the union of the causal future and past of $K$: $J(K) = J^+(K) \cup J^-(K)$. A counterexample is given in Section 12.5.

It would however be natural to guess that up until a time at which data can pass to a possible edge, the support of $\phi$ is contained in $J(K)$. More precisely, if we define:

$$t^\infty(K) = \sup\{t \geq 0 \colon J^+(K) \cap \Sigma_t \text{ is compact in } \Sigma_t\} \in (0, \infty],$$

then we propose that:

$$\operatorname{supp}\phi \cap [-t^\infty, t^\infty] \times \Sigma \subseteq J(K).$$

It was proven in Proposition 7.1 that $t^\infty(K) > 0$, so this is a non-trivial statement. At first sight it might appear that this result is trivial. Since $D(\Sigma_0)$ is a globally hyperbolic spacetime we know that $\operatorname{supp}\phi \cap D(\Sigma_0) \subseteq J(K)$ however this does not show that $\phi$ is zero in the shaded triangular region in Figure 4. Thus this does not even prove that $\phi$ is compactly supported on $\Sigma_t$ for small $t$.

The proof we give shortly uses the uniqueness result of Lemma 6.4 and the sequence $t_n(K)$ constructed in the previous section. We shall define $\Psi \colon (-t_1(K), t_1(K)) \times \Sigma \to \mathbb{R}$ to be equal to $\phi$ inside $J(K)$ and zero outside it. We shall show that $\Psi$ so defined is smooth, compactly supported on $\Sigma_t$ for

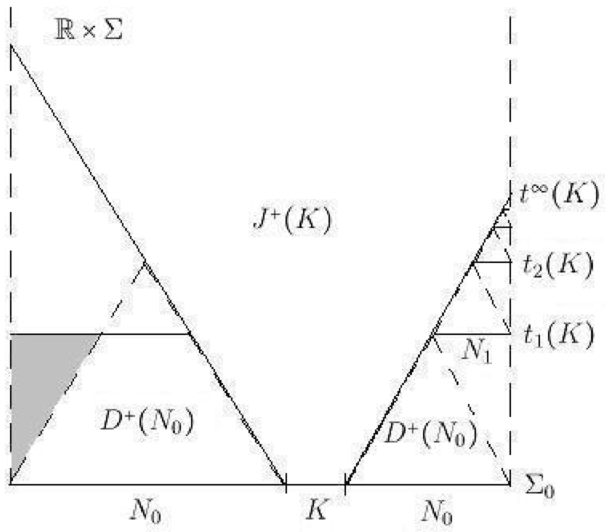


Figure 4: The construction of $t^\infty(K)$ and $t_n(K)$ in for example $(M,g) = (\mathbb{R}\times(0,1), dt^2 - dx^2)$.

$t \in (-t_1(K), t_1(K))$ and satisfies the Klein-Gordon equation in its domain. Thus $[\Psi|_{\Sigma_t}] \in D(A) \subseteq D(A_E)$ and so $\Psi = \phi$ in the domain of $\Psi$ by Lemma 6.4. By induction and the fact that $t_n(K) \nearrow t_\infty(K)$ the result then follows.

**Proposition 8.1.** *Given $\phi_0, \dot{\phi}_0 \in C_0^\infty(\Sigma)$ let $K = supp\,\phi_0 \cup supp\,\dot{\phi}_0$. Define $t^\infty(K)$ as earlier. Let $\phi$ be the solution to the Klein-Gordon equation generated by some acceptable s.a.e. $A_E$ of $A$ and data $(\phi_0, \dot{\phi}_0)$ via Theorem 5.5. Then:*

1. *If $t^\infty(K) = \infty$ then: $supp\,\phi \subseteq J(K)$*

2. *If $t^\infty(K) < \infty$ then: $supp\,\phi \cap [-t^\infty(K), t^\infty(K)] \times \Sigma \subseteq J(K)$*

*Proof.* If $t^\infty(K) = \infty$ then by Proposition 7.1 $(\Sigma, V^{-2}h)$ is a complete Riemannian manifold and so $M$ is globally hyperbolic by Lemma 4.19 and $\operatorname{supp}\phi \subseteq J(K)$ follows from Theorem 4.25.

If $t^\infty(K) < \infty$, construct a strictly increasing sequence $(t_n(K))_{n\geq 0}$ inductively as follows. Let $t_0(K) = 0$ and:

$$t_{n+1}(K) := \sup\{t : J^+(K) \cap \Sigma_t \subseteq D(\Sigma_{t_n(K)})\}.$$

From Corollary 7.8 we know that $t_n(K) \nearrow t^\infty(K)$. Define $\Psi : (-t_1(K), t_1(K)) \times \Sigma \to \mathbb{R}$ as:

$$\Psi(x) = \begin{cases} \phi(x), & \text{for } x \in (-t_1(K), t_1(K)) \times \Sigma \cap J(K) \\ 0, & \text{otherwise.} \end{cases}$$

The first problem is to show that the function $\Psi$ so defined is smooth. We do this by finding for each $x \in (-t_1(K), t_1(K)) \times \Sigma$ an open neighbourhood $U$ s.t. $\Psi$ either equals $\phi$ on $U$ or is zero on $U$.

If $x \in (-t_1(K), t_1(K)) \times \Sigma \cap D(\Sigma_0) = U$, an open neighbourhood (since $D(\Sigma_0)$ is open by Proposition 4.21), then $\Psi = \phi$ on $U$. This is because if $y \in U$ then either $y \in J(K)$ and so $\Psi(y) = \phi(y)$ by definition, or $y \in D(\Sigma_0)\backslash J(K) = D(\Sigma_0\backslash K)$ and $\Psi(y) = 0 = \phi(y)$ (by the uniqueness of solutions to the Klein-Gordon equation on the globally hyperbolic spacetime $D(\Sigma_0\backslash K)$ (Theorem 4.25), where $\Sigma_0\backslash K$ is an acausal topological hypersurface and so $D(\Sigma_0\backslash K)$ is an open set in $M$ and a globally hyperbolic spacetime by Proposition 4.21).

If $x \in (-t_1(K), t_1(K)) \times \Sigma\backslash D(\Sigma_0) \subseteq (-t_1(K), t_1(K)) \times \Sigma\backslash J(K) =: U$ (from Corollary 7.7, Statement *1*), then $\Psi = 0$ on $U$ by definition.

Thus $\Psi \in C^\infty((-t_1(K), t_1(K)) \times \Sigma)$ and

$$[\Psi_t], [\partial_t \Psi_t] \in [C_0^\infty(\Sigma)] = D(A) \subseteq D(A_E),$$

for all $t \in [0, t_1(K))$. We also have $\Psi|_{\Sigma_0} = \phi_0$ and $\partial_t \Psi|_{\Sigma_0} = \dot{\phi}_0$. Since $\Psi$ is locally either equal to $\phi$, or zero, both being solutions of the Klein-Gordon equation, then so is $\Psi$, that is $(\Box_g + m^2)\Psi = 0$ on $(-t_1(K), t_1(K)) \times \Sigma$. Moreover, by definition $\Psi = 0$ on $[0, t_1(K)) \times \Sigma \backslash J(K)$.

By uniqueness of the Wald solution (Lemma 6.4), then:

$$\phi = \Psi \text{ in } [0, t_1(K)) \times \Sigma.$$

Therefore $\phi = 0$ on $[0, t_1(K)) \times \Sigma \backslash J(K)$. But since $\phi$ is smooth, then also $\partial_t \phi = 0$ on $[0, t_1) \times \Sigma \backslash J(K)$. In particular then, $\phi = \partial_t \phi = 0$ on $\Sigma_{t_1(K)} \backslash J(K) = N_1$.

Using the constructed sequence $(t_n(K))_{n \geq 0}$, we prove the proposition by induction. Our inductive hypothesis $P(n)$ is as follows:

$$P(n)\text{: } \operatorname{supp} \phi \cap [0, t_n(K)] \times \Sigma \subseteq J(K)$$

We have already proven the statement for $n = 1$. If $P(n)$ is true, by smoothness $\phi, \partial_t \phi$ are zero on $N_n = \Sigma_{t_n(K)} \backslash J(K)$. Now, as before, define: $\Psi\colon [t_n(K), t_{n+1}(K)) \times \Sigma \to \mathbb{R}$ as:

$$\Psi(x) = \begin{cases} \phi(x), & \text{for } x \in [t_n(K), t_{n+1}(K)) \times \Sigma \cap J(K) \\ 0, & \text{otherwise.} \end{cases}$$

Similarly to the previous argument $\Psi \in C^\infty([t_n(K), t_{n+1}(K)) \times \Sigma)$ as a manifold with boundary. Also:

$$[\Psi_{\Sigma_t}] \in [C_0^\infty(\Sigma)] = D(A) \subseteq D(A_E) \;\; \forall t \in [t_n(K), t_{n+1}(K)),$$

$$\Psi|_{\Sigma_{t_n(K)}} = \phi_{t_n(K)} \text{ and } \partial_t \Psi|_{\Sigma_0} = \dot{\phi}_{t_n(K)}.$$

By the uniqueness theorem (Lemma 6.4), $\phi = \Psi$ in $[t_n(K), t_{n+1}(K)) \times \Sigma$. Thus $\phi = 0$ on $[t_n(K), t_{n+1}(K)) \times \Sigma \backslash J(K)$. But since $\phi$ is smooth, then also $\partial_t \phi = 0$ on $[t_n(K), t_{n+1}(K)) \times \Sigma \backslash J(K)$. In particular then, $\phi = 0$ on $\Sigma_{t_{n+1}(K)} \backslash J(K) = N_1$ and $P(n+1)$ is proven.

Hence $\operatorname{supp}\phi \cap [0, t_n(K)] \times \Sigma \subseteq J(K)$ for all $n$. But as $t_n(K) \nearrow t^\infty(K)$, then $\operatorname{supp}\phi \cap [0, t^\infty(K)) \times \Sigma \subseteq J(K)$, and by continuity:

$$\operatorname{supp}\phi \cap [0, t^\infty(K)] \times \Sigma \subseteq J(K).$$

Finally, since the spacetime is symmetric around $\Sigma_0$, we have:

$$\operatorname{supp}\phi \cap [-t^\infty(K), t^\infty(K)] \times \Sigma \subseteq J(K).$$

□

# 9 Energy form on the Space of Solutions

In Sections 9 to 11, we shall prove the existence of certain structures on the space of solutions $S_E$ (Definition 6.5), corresponding to a particular acceptable s.a.e. $A_E$. Specifically, we shall show the existence of an energy form, a symplectic form and certain symmetries: time translation and time-reversal. These were all conditions placed on the dynamics in the paper by Wald and Ishibashi [38]. It is important for us to show that these conditions are in fact necessary, even in our extended case of dynamics generated by an acceptable s.a.e. $A_E$. In this section we show that there is a natural bilinear symmetric form $E$ on our constructed space of solutions $S_E$ to the Klein-Gordon equation. In general, it is not a norm. However, if our choice of acceptable self-adjoint extension $A_E$ is positive and zero is not an eigenvalue, then $E$ is a norm on $S_E$.

Given two pairs of smooth Cauchy data: $(\phi_0, \dot{\phi}_0), (\phi'_0, \dot{\phi}'_0) \in \chi_E^2 \subseteq C^\infty(\Sigma)^2$ then we have by the existence of Wald solutions (Theorem 5.5) two corresponding solutions $\phi, \phi'$ to the Klein Gordon equation on our spacetime. For each time $t \in \mathbb{R}$ we define the energy at time $t$ to be:

$$E(\phi, \phi')(t) = \langle \dot{\phi}_t, \dot{\phi}'_t \rangle_{\Sigma_t} + \langle \phi_t, A_E \phi'_t \rangle_{\Sigma_t}$$

Our task is to show that $E(\phi, \phi')$ is in fact independent of time. Remember that:

$$\phi_t = C(t, A_E)\phi_0 + S(t, A_E)\dot{\phi}_0$$
$$\dot{\phi}_t = -A_E S(t, A_E)\phi_0 + C(t, A_E)\dot{\phi}_0$$

Thus for all $t \in \mathbb{R}$:

$$
\begin{aligned}
E(\phi,\phi')(t) &= \langle -A_E S(t,A_E)\phi_0 + C(t,A_E)\dot{\phi}_0, -A_E S(t,A_E)\phi_0' + C(t,A_E)\dot{\phi}_0' \rangle \\
&\qquad + \langle C(t,A_E)\phi_0 + S(t,A_E)\dot{\phi}_0, A_E C(t,A_E)\phi_0' + A_E S(t,A_E)\dot{\phi}_0' \rangle \\
&= \langle A_E S(t,A_E)\phi_0, A_E S(t,A_E)\phi_0' \rangle - \langle A_E S(t,A_E)\phi_0, C(t,A_E)\dot{\phi}_0' \rangle \\
&\qquad - \langle C(t,A_E)\dot{\phi}_0, A_E S(t,A_E)\phi_0' \rangle + \langle C(t,A_E)\dot{\phi}_0, C(t,A_E)\dot{\phi}_0' \rangle \\
&\qquad + \langle C(t,A_E)\phi_0, A_E C(t,A_E)\phi_0' \rangle + \langle C(t,A_E)\phi_0, A_E S(t,A_E)\dot{\phi}_0' \rangle \\
&\qquad + \langle S(t,A_E)\dot{\phi}_0, A_E C(t,A_E)\phi_0' \rangle + \langle S(t,A_E)\dot{\phi}_0, A_E S(t,A_E)\dot{\phi}_0' \rangle \\
&= \langle \phi_0, A_E(A_E S(t,A_E)^2 + C(t,A_E)^2)\phi_0' \rangle \\
&\qquad + \langle \dot{\phi}_0, (A_E S(t,A_E)^2 + C(t,A_E)^2)\dot{\phi}_0' \rangle \\
&= \langle \phi_0, A_E \phi_0' \rangle + \langle \dot{\phi}_0, \dot{\phi}_0' \rangle \\
&= E(\phi,\phi')(0)
\end{aligned}
$$

where we have used the following identity: $A_E S(t,A_E)^2 + C(t,A_E)^2 = \mathbb{I}$ on $[\chi_E]$. Hence $E(t)$ has the same value at all times.

Using the linear isomorphism $\Psi\colon \chi_E \times \chi_E \to S_E$ between $\chi_E^2$ and the space of solutions $S_E$ defined in Proposition 6.5 then $E$ defined above is a bilinear symmetric form on $S_E$ (the symmetry of $E$ follows easily since as $A_E$ is self-adjoint it is certainly symmetric) and is called the **energy form**.

# 10 The Symplectic Form on the Space of Solutions

Similarly to the previous section, we show that there exists a natural symplectic form on the real vector space of our space of solutions $S_E$.

Given two pairs of smooth Cauchy data: $(\phi_0, \dot{\phi}_0), (\phi'_0, \dot{\phi}'_0) \in \chi_E^2 \subseteq C^\infty(\Sigma)^2$ then we have by the existence of Wald solutions (Theorem 5.5) two corresponding solutions $\phi, \phi'$ to the Klein Gordon equation on our spacetime. For each time $t \in \mathbb{R}$ we define the **symplectic form** at time $t$ to be:

$$\sigma_E(\phi, \phi')(t) = \langle \phi_t, \dot{\phi}'_t \rangle - \langle \dot{\phi}_t, \phi'_t \rangle$$

We show again that this form is independent of time. For all $t \in \mathbb{R}$:

$$\begin{aligned}
\sigma_E(\phi, \phi')(t) &= \langle C(t, A_E)\phi_0 + S(t, A_E)\dot{\phi}_0, -A_E S(t, A_E)\phi'_0 + C(t, A_E)\dot{\phi}'_0 \rangle \\
&\quad + \langle A_E S(t, A_E)\phi_0 - C(t, A_E)\dot{\phi}_0, C(t, A_E)\phi'_0 + S(t, A_E)\dot{\phi}'_0 \rangle \\
&= -\langle C(t, A_E)\phi_0, A_E S(t, A_E)\phi'_0 \rangle + \langle C(t, A_E)\phi_0, C(t, A_E)\dot{\phi}'_0 \rangle \\
&\quad - \langle S(t, A_E)\dot{\phi}_0, A_E S(t, A_E)\phi'_0 \rangle + \langle S(t, A_E)\dot{\phi}_0, C(t, A_E)\dot{\phi}'_0 \rangle \\
&\quad + \langle A_E S(t, A_E)\phi_0, C(t, A_E)\phi'_0 \rangle + \langle A_E S(t, A_E)\phi_0, S(t, A_E)\dot{\phi}'_0 \rangle \\
&\quad - \langle C(t, A_E)\dot{\phi}_0, C(t, A_E)\phi'_0 \rangle - \langle C(t, A_E)\dot{\phi}_0, S(t, A_E)\dot{\phi}'_0 \rangle \\
&= \langle \phi_0, (A_E S(t, A_E)^2 + C(t, A_E)^2)\dot{\phi}'_0 \rangle \\
&\quad - \langle \dot{\phi}_0, (A_E S(t, A_E)^2 + C(t, A_E)^2)\phi'_0 \rangle \\
&= \langle \phi_0, \dot{\phi}'_0 \rangle - \langle \dot{\phi}_0, \phi'_0 \rangle \\
&= \sigma_E(\phi, \phi')(0)
\end{aligned}$$

Here, we have again made use of the identity $A_E S(t, A_E)^2 + C(t, A_E)^2 = \mathbb{I}$ on $[\chi_E]$. Thus we have a map $\sigma_E: S_E \times S_E \to \mathbb{R}$, where $S_E$ is the real vector space of solutions. It is clearly bilinear, antisymmetric and also weakly nondegenerate, since if $\phi \in S_E$ is non-zero then (by uniqueness) $(\phi_0, \dot{\phi}_0) \neq (0,0) \in \chi_E \times \chi_E$. Consequently, let $\phi' = \Psi(-\dot{\phi}_0, \phi_0)$. Then, $\sigma_E(\phi, \phi') = ||\phi_0||^2 + ||\dot{\phi}_0||^2 > 0$ as either $\phi_0$ or $\dot{\phi}_0$ is non-zero and so has non-zero norm (as both are continuous). Thus, $(S_E, \sigma_E)$ is a real symplectic space.

# 11 Symmetries

In this section, we derive some symmetries satisfied by the linear isomorphism $\Psi : \chi_E \times \chi_E \to S_E$ defined in Proposition 6.6. Consider the maps $T_t, P : C^\infty(M) \to C^\infty(M)$ given by:

$$(T_t F)(s,x) = F(s-t,x)$$
$$(PF)(s,x) = F(-s,x)$$

**Proposition 11.1.** *Given a standard static spacetime and the linear operator $A$ defined as usual on the Hilbert space $L^2(\Sigma, dvol_h)$ then for any acceptable s.a.e. $A_E$ of $A$. The maps $T_t$ and $P$ satisfy: $T_t, P\colon S_E \to S_E$. Then letting $\phi_t = \Psi(\phi_0, \dot{\phi}_0)|_{\Sigma_t}$ and $\dot{\phi}_t = \partial_t \Psi(\phi_0, \dot{\phi}_0)|_{\Sigma_t}$ we have:*

$$\Psi(\phi_t, \dot{\phi}_t) = T_{-t}[\Psi(\phi_0, \dot{\phi}_0)]$$
$$\Psi(\dot{\phi}_0, -A_E\phi_0) = \frac{\partial}{\partial t}[\Psi(\phi_0, \dot{\phi}_0)]$$
$$\Psi(\phi_0, -\dot{\phi}_0) = P[\Psi(\phi_0, \dot{\phi}_0)]$$

*In particular this also proves that $\frac{\partial}{\partial t}\colon \chi_E \to \chi_E$. Additionally, for all $\Psi_1, \Psi_2 \in S_E$:*

$$E(T_t\Psi_1, T_t\Psi_2) = E(\Psi_1, \Psi_2)$$
$$E(P\Psi_1, P\Psi_2) = E(\Psi_1, \Psi_2)$$
$$\sigma_E(T_t\Psi_1, T_t\Psi_2) = \sigma_E(\Psi_1, \Psi_2)$$
$$\sigma_E(P\Psi_1, P\Psi_2) = -\sigma_E(\Psi_1, \Psi_2)$$

(Note that the first five properties correspond to Assumptions 2(i), 2(ii), 3(i) and 3(ii) in Wald and Ishibashi [38].)

*Proof.*

$$\Psi(\phi_t, \dot{\phi}_t)(s, x) = \begin{bmatrix} C(s, A_E)(C(t, A_E)\phi_0 + S(t, A_E)\dot{\phi}_0) \\ +S(s, A_E)(-A_E S(t, A_E)\phi_0 + C(t, A_E)\dot{\phi}_0) \end{bmatrix}(x)$$
$$= [C(s + t, A_E)\phi_0 + S(s + t, A_E)\dot{\phi}_0](x)$$
$$= \Psi(\phi_0, \dot{\phi}_0)(t + s, x)$$
$$= T_{-t}(\Psi(\phi_0, \dot{\phi}_0))(s, x)$$

$$\Psi(\dot{\phi}_0, -A_E\phi_0)(t, s) = \left[C(t, A_E)\dot{\phi}_0 + S(t, A_E)(-\phi_0))\right](x)$$
$$= \dot{\phi}_t(x)$$
$$= \frac{\partial}{\partial t}[\Psi(\phi_0, \dot{\phi}_0)](t, x)$$

$$\Psi(\phi_0, -\dot{\phi}_0)(t, x) = [C(t, A_E)\phi_0 + S(t, A_E)(-\dot{\phi}_0))](x)$$
$$= [C(-t, A_E)\phi_0 + S(-t, A_E)\dot{\phi}_0)](x)$$
$$= \Psi(\phi_0, \dot{\phi}_0)(-t, x)$$
$$= P(\Psi(\phi_0, \dot{\phi}_0))(t, x)$$

The remaining properties are easily proven from the time independence of $E(\Psi_1, \Psi_2)(t)$ and $\sigma(\Psi_1, \Psi_2)(t)$ (Sections 9 and 10). □

# 12 Examples

In this section we shall discuss a few simple examples of standard static spacetimes. In all the examples we examine we shall let $V = 1$ for simplicity. Thus the spacetime $(M, g) = (\mathbb{R} \times \Sigma, dt^2 - h)$ and the solutions to the Cauchy problem of the Klein-Gordon equation constructed in this thesis for each of these spacetimes will be indexed by the acceptable s.a.e.s $A_E$ of the symmetric linear operator $A$ on $L^2(\Sigma, d\mathrm{vol}_h)$ generated by the partial differential operator (also labelled by) $A = -\mathrm{div}_h \mathrm{grad}_h$, minus the Laplace-Beltrami operator, and $D(A) = [C_0^\infty(\Sigma)]$.

We shall also only consider the case of the solving the Klein-Gordon case for complex-valued data and so we only consider complex Hilbert spaces. Note that it's only on complex Hilbert spaces that we can define the deficiency spaces $H^\pm$ of a densely defined operator $A$ as $H^\pm := \ker(A^* \mp i)$. We note the following theorem (see Theorems 83.1 and 85.1 in Akhiezer and Glazman [2]):

**Theorem 12.1.** *Let $A$ be a positive symmetric linear operator with equal and finite deficiency indices, that is, denoting $n^\pm := \dim\ker(A^* \mp i)$, we have $n^+ = n^- = n < \infty$. Then every s.a.e. $A_E$ of $A$ is bounded-below. Furthermore, every s.a.e. $A_E$ has the same continuous spectrum as $A$, each of the s.a.e.s has only a finite number of negative eigenvalues and the sum of the multiplicities of the negative eigenvalues of any particular s.a.e. $A_E$ is not greater than $n$.*

Thus if $A$ has finite deficiency indices then in particular every s.a.e. $A_E$ of $A$ is acceptable. In all the following examples the deficiency indices are finite and are equal to 0, 1 or 2.

## 12.1 Self-Adjoint Extensions of minus the Laplacian on $S^1$

In our first example we let $\Sigma = S^1$. We equip $S^1$ with its (unique) differential structure, the Riemannian metric induced from that on $\mathbb{R}^2$ and the induced smooth measure from this metric. Since $S^1$ is compact in its topology induced from $\mathbb{R}^2$, then it is also compact in its topology induced from the Riemannian metric $h$ (Theorem 4.4), so it is also complete in this metric and so complete as a Riemannian manifold by the Hopf-Rinow Theorem (See e.g. Theorem 6.13 Lee [19]). Thus the linear operator $A$ given by:

$$D(A) = [C_0^\infty(S^1)] = [C^\infty(S^1)]$$
$$A([\phi]) = -[\phi''] \text{ for } \phi \in C^\infty(S^1)$$

is essentially self-adjoint by Theorem 3.1. Thus $\overline{A} = A^*$ is the unique s.a.e. of $A$ and

$$D(\overline{A}) = W^{2,2}(S^1) = \{\phi \in L^2(S^1) \text{ s.t. } \phi',\ \phi'' \in L^2(S^1)\}.$$

Note the Sobolev space $W^{2,2}(S^1)$ is defined in Appendix D.3, where we are implicitly adopting the standard Riemannian metric on $S^1$ as on all the manifolds in Section 12.

The spectrum of $\overline{A}$ is shown in the Appendix to be:

$$\sigma(\overline{A}) = \sigma_{disc}(\overline{A}) = \{n^2 \colon n \in \mathbb{N}_0\}$$

If we identify $S^1\backslash\{1\}$ with $(0, 2\pi)$ by the chart: $\phi\colon U = S^1\backslash\{1\} \to (0, 2\pi)$, $\phi^{-1}(\theta) = \exp i\theta$, then define the function $g\colon U \times U \times \mathbb{C}\backslash\{n^2\colon n \in \mathbb{N}\} \to \mathbb{C}$ by:

$$g(\theta, \phi; \lambda) = \frac{i}{2\sqrt{\lambda}}\left[\exp i\sqrt{\lambda}|\theta - \phi| + \frac{2\cos\sqrt{\lambda}(\theta - \phi)}{\exp(-2\pi i\sqrt{\lambda}) - 1}\right].$$

As $\{1\} \subseteq S^1$ is clearly null, $h$ generates a well-defined integral kernel.

The Green's function for $\lambda \in \rho(\overline{A}) = \mathbb{C} \backslash \{n^2 \colon n \in \mathbb{N}_0\}$ is given by $g(\cdot,\cdot,\lambda)$, which does not depend on the choice of square root of $\lambda$ used to define it.

## 12.2 Self-Adjoint Extensions of minus the Laplacian on $(0,\infty)$

In the remaining cases the domains $D(A_E)$ of the s.a.e.s of $A$ shall be given by conditions placed on the domain of the adjoint of $A$, that is $D(A^*)$. Since $A \leq A_E \leq A^*$ then all the s.a.e.s of $A$ are restrictions of $A^*$ to their domain $D(A_E)$. The conditions placed on the domain will be in terms of "trace maps". The derivation of these maps is to be found in Lions and Magenes [20].

**Theorem 12.2.** *Let $\Omega$ be an open interval of $\mathbb{R}$. Consider the linear maps:*

$$\rho \colon C_0^\infty(\overline{\Omega}) \to \mathbb{C}^{|\partial\Omega|}, \;\; \phi \mapsto \phi|_{\partial\Omega}$$
$$\tau \colon C_0^\infty(\overline{\Omega}) \to \mathbb{C}^{|\partial\Omega|}, \;\; \phi \mapsto \phi'|_{\partial\Omega}.$$

*These maps extend by continuity to a unique continuous maps*

$$\rho, \tau \colon W^{2,2}(\Omega) \to \mathbb{C}^{|\partial\Omega|}.$$

*Letting $\Phi = (\rho,\tau)$, then $\Psi$ is linear and surjective. Additionally: $W_0^{2,2}(\Omega) = ker\Psi = \{\phi \in W^{2,2}(\Omega) \colon \rho(\phi) = \tau(\phi) = 0\}$.*

Note that $C_0^\infty(\overline{\Omega})$ is defined as the space of smooth functions on the smooth manifold with boundary $\overline{\Omega}$ which are of compact support. If $\overline{\Omega}$ is compact then clearly $C_0^\infty(\overline{\Omega}) = C^\infty(\overline{\Omega})$. Note also that we define:

$$W_0^{2,2}(\Omega) := \overline{[C_0^\infty(\Omega)]}^{W^{2,2}(\Omega)},$$

the closure of $[C_0^\infty(\Omega)]$ in the Sobolev norm on $W^{2,2}(\Omega)$.

If $\Sigma = (0,\infty)$ then the s.a.e.s of $A$ are indexed by $\alpha \in (-\pi/2, \pi/2]$, denoted $A_\alpha$. Their domains are given by:

$$D(A_\alpha) = \{\phi \in W^{2,2}(0,\infty) \text{ s.t. } \cos\alpha\, \rho(\phi) = \sin\alpha\, \tau(\phi)\}$$

The spectra of these s.a.e.s are given by the following:

$$\sigma(A_\alpha) = \begin{cases} [0,\infty) & \text{for } \alpha \in [0,\pi/2] \\ [0,\infty) \cup \{-\cot^2\alpha\} & \text{for } \alpha \in (-\pi/2, 0) \end{cases}$$

The pure point spectrum $\sigma_{pp}(A_\alpha)$, continuous spectrum $\sigma_{cont}(A_\alpha)$ and discrete spectrum $\sigma_{disc}(A_\alpha)$ of the operator $A_\alpha$ are given by the following statements:

1. $\sigma_{cont}(A_\alpha) = [0,\infty)$ for all $\alpha$.
2. If $\alpha \in [0, \frac{\pi}{2}]$: $\sigma(A_\alpha) = \sigma_{cont}(A_\alpha) = [0,\infty)$.
3. If $\alpha \in (-\frac{\pi}{2}, 0)$: $\sigma_{pp}(A_\alpha) = \sigma_{disc}(A_\alpha) = \{-\cot^2\alpha\}$.

Thus for $\alpha \in [0, \frac{\pi}{2}]$, $A_\alpha$ is a positive s.a.e. of $A$ and for $\alpha \in (-\frac{\pi}{2}, 0)$, $A_\alpha$ is not positive but it is bounded-below. Thus all the s.a.e.s of $A$ are acceptable according to Definition 3.2.

For completeness we also now give the Green's function for each s.a.e. $A_\alpha$, that is $g\colon (0,\infty) \times (0,\infty) \times \rho(A_\alpha) \to \mathbb{C}$ given by:

$$g(x,\xi,\lambda) = A[\cos\alpha \sin(\sqrt{\lambda}x_<) + \sqrt{\lambda}\sin\alpha\cos(\sqrt{\lambda}x_<)]\exp(i\sqrt{\lambda}x_>),$$

where $A = [\sqrt{\lambda}(\cos\alpha - i\sqrt{\lambda}\sin\alpha)]^{-1}$, $x_> = \max\{x,\xi\}$, $x_< = \min\{x,\xi\}$ and $\sqrt{\lambda} = a + bi, b > 0$ is defined as the unique square root of $\lambda$ in the upper-half plane, possible since $\lambda \notin [0,\infty)$. These statements are proven in Appendix G.

## 12.3 Self-Adjoint Extensions of minus the Laplacian on $(0, a)$

Now consider the case where $\Sigma = (0, a)$. Again, the domains of the s.a.e.s of $A$ are given in terms of the elements of $W^{2,2}(0, a)$ satisfying conditions placed on them via the trace map. The set of all s.a.e.s of $A$ are given by the Dirichlet extension and two groups of extensions which we shall describe presently. The first group shall also contain the Neumann extension.

We shall give a very brief explanation of the origin of this classification of the self-adjoint extensions of the $A$. We refer the reader to Posilicano [26]. The set of s.a.e.s of $A$ is indexed by pairs:

$$\left\{ (\Pi, \Theta) \colon \begin{array}{l} \Pi \text{ is an orthogonal projection operator on the Hilbert space } \mathbb{C}^2 \\ \Theta \text{ is a bounded s.a. linear operator on the Hilbert space } Im(\Pi) \end{array} \right\}.$$

Given the pair $(\Pi, \Theta)$ the s.a.e. $A_{\Pi,\Theta}$ is then given by:

$$D(A_{\Pi,\Theta}) = \left\{ \phi \in W^{2,2}(0, a) \colon \rho\phi \in Im(\Pi),\ \Pi\tau\phi = \Theta\rho\phi \right\},$$

where:

$$\rho \colon W^{2,2}(0, a) \to \mathbb{C}^2,\ \ \rho(\phi) = \begin{pmatrix} \phi(0) \\ \phi(a) \end{pmatrix}$$

$$\tau \colon W^{2,2}(0, a) \to \mathbb{C}^2,\ \ \tau(\phi) = \begin{pmatrix} \phi'(0) \\ -\phi'(a) \end{pmatrix}.$$

Here we are implicitly using Theorem 12.2. Note that $\tau$ evaluates the inward-pointing derivative at the boundary, hence the sign.

There are three natural collections of s.a.e.s of $A$ according as $rank(\Pi) = 0, 1$ or $2$.

Picking $rank(\Pi) = 0$, then we have $\Pi = 0$ and $\Theta = 0$. This is the Dirichlet s.a.e. $A_D$ (which we may also call the **s.a.e. of the zeroth kind**), defined by the following domain:

$$D(A_D) = \{\phi \in W^{2,2}(0,a) \text{ s.t. } \phi(0) = \phi(1) = 0\}.$$

Picking $rank(\Pi) = 2$, we obtain the next collection of s.a.e.s, which we shall call the **s.a.e.s of the first kind**, in agreement with the language of Posilicano [26]. They are obtained by setting $\Pi = \mathbb{I}$. Then $Im\Pi = \mathbb{C}^2$ and let $\Theta$ be defined by a self-adjoint complex $2 \times 2$ matrix

$$\theta = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \overline{\theta}_{12} & \theta_{22} \end{pmatrix},$$

where $\theta_{11}, \theta_{22} \in \mathbb{R}, \theta_{12} \in \mathbb{C}$. The domain of the extension $D(A_\theta)$ is defined as those elements $\phi \in W^{2,2}(0,a)$ such that $\Pi\tau\phi = \Theta\rho\phi$, that is:

$$\begin{pmatrix} \phi'(0) \\ -\phi'(a) \end{pmatrix} = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \overline{\theta}_{12} & \theta_{22} \end{pmatrix} \begin{pmatrix} \phi(0) \\ \phi(a) \end{pmatrix},$$

or:

$$D(A_\theta) = \left\{ \phi \in W^{2,2}(0,a) \text{ s.t.:} \quad \begin{matrix} \theta_{11}\phi(0) - \phi'(0) + \theta_{12}\phi(a) = 0 \\ \overline{\theta_{12}}\phi(0) + \theta_{22}\phi(a) + \phi'(a) = 0 \end{matrix} \right\}.$$

Note that letting $\theta_{11} = \theta_{22} = \theta_{12} = 0$ we obtain the Neumann extension $A_N = A_0$:

$$D(A_N) = \{\phi \in W^{2,2}(0,a) \text{ s.t. } \phi'(0) = \phi'(a) = 0\}.$$

Picking $rank(\Pi) = 1$, we obtain the last collection of s.a.e.s, which we shall call the **s.a.e.s of the second kind**. They are obtained by setting

$$\Pi = w \otimes w = \begin{pmatrix} |w_1|^2 & w_1\overline{w_2} \\ \overline{w_1}w_2 & |w_2|^2 \end{pmatrix},$$

where $w = (w_1, w_2) \in \mathbb{C}^2$ is a unit vector, spanning a one-dimensional subspace in $\mathbb{C}^2$ and defining $\Pi$ by orthogonal projection onto this subspace. We set $\Theta$ to be defined as multiplication by $\theta \in \mathbb{R}$. These s.a.e.s are then indexed by triples: $\{(w_1, w_2, \theta)$: $w_1, w_2 \in \mathbb{C}$ s.t. $|w_1|^2 + |w_2|^2 = 1$ and $\theta \in \mathbb{R}\}$. The domain of the extension $D(A_{w_1 w_2 \theta})$ is then those elements $\phi \in W^{2,2}(0,a)$ such that:

$$\rho(\phi) = \begin{pmatrix} \phi(0) \\ \phi(a) \end{pmatrix} \in Im\Pi = \left\langle \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right\rangle$$

$$\begin{aligned} \begin{pmatrix} \theta\phi(0) \\ \theta\phi(a) \end{pmatrix} = \theta\rho\phi = \Pi\tau\phi \\ &= \Pi \begin{pmatrix} \phi'(0) \\ -\phi'(a) \end{pmatrix} \\ &= \begin{pmatrix} |w_1|^2 & w_1\overline{w_2} \\ \overline{w_1}w_2 & |w_2|^2 \end{pmatrix} \begin{pmatrix} \phi'(0) \\ -\phi'(a) \end{pmatrix} \end{aligned}$$

Then, from the first condition: $w_2\phi(0) = w_1\phi(a)$. And from the second:

$$\begin{aligned} \overline{w_1}\theta\phi(0) + \overline{w_2}\theta\phi(a) &= \overline{w_1}|w_1|^2\phi'(0) - \overline{w_2}|w_1|^2\phi'(a) + \overline{w_1}|w_2|^2\phi'(0) - \overline{w_2}|w_2|^2\phi'(a) \\ &= \overline{w_1}\phi'(0) - \overline{w_2}\phi'(a) \end{aligned}$$

Thus,

$$D(A_{w_1 w_2 \theta}) = \left\{ \phi \in W^{2,2}(0,a) \text{ s.t.: } \begin{array}{l} w_2\phi(0) - w_1\phi(a) = 0 \\ \overline{w_1}(\theta\phi(0) - \phi'(0)) + \overline{w_2}(\theta\phi(a) + \phi'(a)) = 0 \end{array} \right\}$$

*Remark.* Note that replacing $w_i$ with $w_i e^{i\phi}$ for $i = 1, 2$ then we obtain identical boundary conditions and so the same s.a.e. of $A$. Clearly this is because both choices yield the same 1-dimensional subspace in $\mathbb{C}^2$ and so the same orthogonal projection operator $\Pi$.

We shall now give the spectra of all these self-adjoint extensions and the their corresponding Green's functions. By analysing the spectra or by the finiteness of the deficiency indices and using Theorem 12.1, all s.a.e.s are bounded-below. These results can either be reached via the approach of Posilicano [26] (Example 5.1) or the methods of Stakgold [32]. The proofs of all but the case of the Dirichlet extension are found in Section H. The case of the Dirichlet extension itself is simpler, along similar lines and is to be found in Stakgold [32]. The numbering below corresponds to the three groupings of s.a.e.s previously introduced: s.a.e.s of zeroth, first and second kinds.

0. Denoting $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, we have:
$$\sigma(A_D) = \left\{ \left( \frac{n\pi}{a} \right)^2, \ n \in \mathbb{N} \right\}.$$

1. If $\lambda \neq 0$ then $\lambda \in \sigma(A_\theta)$ iff
$$\begin{aligned} 0 &= \theta_{11}\sqrt{\lambda}\cos\sqrt{\lambda}a + \theta_{22}\sqrt{\lambda}\cos\sqrt{\lambda}a - \lambda\sin\sqrt{\lambda}a \\ &\quad + \theta_{11}\theta_{22}\sin\sqrt{\lambda}a - |\theta_{12}|^2\sin\sqrt{\lambda}a + 2\mathfrak{R}(\theta_{12})\sqrt{\lambda} \end{aligned}$$
(note that the validity of this condition is independent of which square root of $\lambda$ we take) and:
$$0 \in \sigma(A_\theta) \text{ iff } a|\theta_{12}|^2 - \theta_{11} - a\theta_{11}\theta_{22} - \theta_{22} - 2\mathfrak{R}(\theta_{12}) = 0.$$
For instance, letting $\theta_{11} = \theta_{22} = \theta_{12} = 0$, we have that:
$$\sigma(A_N) = \left\{ \left( \frac{n\pi}{a} \right)^2, \ n \in \mathbb{N}_0 \right\}.$$

2. If $\lambda \neq 0$ then $\lambda \in \sigma(A_{w_1 w_2 \theta})$ iff
$$-\sqrt{\lambda}\cos\sqrt{\lambda}a + 2\mathfrak{R}(w_1\overline{w_2})\sqrt{\lambda} - \theta\sin\sqrt{\lambda}a = 0$$
and $0 \in \sigma(A_{w_1 w_2 \theta})$ iff $a\theta - 2\mathfrak{R}(w_1\overline{w_2}) + 1 = 0$.

The Green's functions for each of these cases is treated in the following:

0. The Green's function for the Dirichlet extension $A_D$ is given in terms of the kernel $g(x,y;\lambda)$.

   For $\lambda \in \mathbb{C}\backslash\{(\frac{n\pi}{a})^2, n \in \mathbb{N}_0\}$, define:

$$g(x,y;\lambda) = \frac{\sin\sqrt{\lambda}(a-x_>)\sin\sqrt{\lambda}x_<}{\sqrt{\lambda}\sin a\sqrt{\lambda}},$$

   where $x_< := \min\{x,y\}$, $x_> := \max\{x,y\}$. And for $\lambda = 0$, let:

$$g(x,y;0) = \frac{(a-x_>)x_<}{a}.$$

1. The Green's function for the s.a.e. of the first kind is given by the following.

   For $\lambda \in \rho(A_\theta)\backslash\{0\}$:

$$g(x,y;\lambda) = A\left[\begin{array}{l} \lambda\cos\sqrt{\lambda}(a-x_>)\cos\sqrt{\lambda}x_< + \theta_{22}\sqrt{\lambda}\sin\sqrt{\lambda}(a-x_>)\cos\sqrt{\lambda}x_< \\ +\theta_{11}\sqrt{\lambda}\cos\sqrt{\lambda}(a-x_>)\sin\sqrt{\lambda}x_< + \theta_{11}\theta_{22}\sin\sqrt{\lambda}(a-x_>)\sin\sqrt{\lambda}x_< \\ +|\theta_{12}|^2\sin\sqrt{\lambda}(x_>-a)\sin\sqrt{\lambda}x_< + C(x,y)(\theta_{12})\sqrt{\lambda}\sin\sqrt{\lambda}(x_<-x_>) \end{array}\right],$$

   where

$$A^{-1} = \sqrt{\lambda}\left[\begin{array}{l} \theta_{11}\sqrt{\lambda}\cos\sqrt{\lambda}a + \theta_{22}\sqrt{\lambda}\cos\sqrt{\lambda}a - \lambda\sin\sqrt{\lambda}a \\ +\theta_{11}\theta_{22}\sin\sqrt{\lambda}a - |\theta_{12}|^2\sin\sqrt{\lambda}a + 2\Re(\theta_{12})\sqrt{\lambda} \end{array}\right],$$

   and for $k \in \mathbb{C}$,

$$C(x,y)(k) = \begin{cases} k \text{ if } x < y. \\ \overline{k} \text{ if } x \geq y. \end{cases}$$

   If $0 \in \rho(A_\theta)$, then

$$g(x,y,0) = A\left[\begin{array}{l} (a-x_>)x_<|\theta_{12}|^2 - \theta_{11}x_< + (x_>-a)x_<\theta_{11}\theta_{22} \\ +(x_>-a)\theta_{22} - 1 + C(x,y)(\theta_{12})(x_>-x_<) \end{array}\right],$$

where

$$A^{-1} = a|\theta_{12}|^2 - \theta_{11} - a\theta_{11}\theta_{22} - \theta_{22} - 2\Re(\theta_{12}).$$

In particular, the Green's function for the Neumann extension $A_N$ is given as follows:

For $\lambda \in \mathbb{C}\backslash\{(\frac{n\pi}{a})^2, n \in \mathbb{N}_0\}$, define:

$$g(x,y;\lambda) = -\frac{\cos\sqrt{\lambda}(a - x_>)\cos\sqrt{\lambda}x_<}{\sqrt{\lambda}\sin a\sqrt{\lambda}}.$$

2. The Green's function for the s.a.e. of the second kind is given as follows.

   For $\lambda \in \rho(A_{w_1 w_2 \theta})\backslash\{0\}$,

$$g(x,y;\lambda) = A\left[\begin{array}{l} |w_1|^2\sqrt{\lambda}\sin\sqrt{\lambda}(x_> - a)\cos\sqrt{\lambda}x_< + \sqrt{\lambda}C(x,y)(w_1\overline{w_2})\sin\sqrt{\lambda}(x_< - x_>) \\ +\theta\sin\sqrt{\lambda}(x_> - a)\sin\sqrt{\lambda}x_< - |w_2|^2\sqrt{\lambda}\cos\sqrt{\lambda}(x_> - a)\sin\sqrt{\lambda}x_< \end{array}\right],$$

   where

$$A^{-1} = \sqrt{\lambda}\left[-\sqrt{\lambda}\cos\sqrt{\lambda}a + 2\Re(w_1\overline{w_2})\sqrt{\lambda} - \theta\sin\sqrt{\lambda}a\right].$$

   If $0 \in \rho(A_{w_1 w_2 \theta})$, then

$$g(x,y;0) = A[\theta(a - x_>)x_< + C(x,y)(w_1\overline{w_2})(x_> - x_<) + |w_1|^2(a - x_>) + |w_2|^2 x_<],$$

   where

$$A^{-1} = a\theta - 2\Re(w_1\overline{w_2}) + 1.$$

## 12.4 Self-adjoint Extensions of minus the Laplacian plus mass

We shall show here that the s.a.e.s of the operator $A = -\text{div}_h \circ \text{grad}_h + m^2$ is easily given in terms of the s.a.e.s of $-\text{div}_h \circ \text{grad}_h$. The corresponding Green's functions are then easily constructible.

This situation is covered by the following more general problem:

**Proposition 12.3.** *Let $A$ be a linear operator on a (real or complex) Hilbert space $H$. For $\mu \in \mathbb{R}$ define the linear operator $A+\mu$ via the domain $D(A+\mu) = D(A)$, $(A+\mu)\phi = A\phi + \mu\phi$. Then the following are true:*

1. *$A$ is closable $\Leftrightarrow$ $A+\mu$ is closable.*

2. *If $A$ is closable then: $\overline{A+\mu} = \overline{A} + \mu$.*

3. *$A$ is self-adjoint $\Leftrightarrow$ $A+\mu$ self-adjoint.*

4. *$A$ is e.s.a. iff $A+\mu$ is e.s.a..*

5. *If $A$ is a symmetric linear operator and $\{A_\gamma \colon \gamma \in \Gamma\}$ are all the s.a.e.s of $A$ ($\Gamma = \phi$ is possible). Then the s.a.e.s of $A+\mu$ are precisely $\{A_\gamma + \mu \colon \gamma \in \Gamma\}$. Additionally, $\sigma(A_\gamma + \mu) = \sigma(A_\gamma) + \mu$ and if $G_\lambda$ is the resolvent of $A_\gamma$ at $\lambda \in \rho(A_\gamma)$ then $G_{\lambda+\mu}$ is the resolvent of $A_\gamma + \mu$ for $\lambda + \mu \in \rho(A_\gamma + \mu)$.*

*Proof.* The proposition is easily proven directly by the definitions of closability, self-adjointness etc. □

We now apply this proposition to the problem of finding the s.a.e.s of the Klein-Gordon operator on a Riemannian manifold, for which we already know all the s.a.e.s of minus the Laplacian. For instance let $\Sigma = S^1$. This was treated in Section 12.1. Let $H = L^2(S^1)$ (the Borel measure on $S^1$ being induced by the Riemannian metric on $S^1$). Define the linear operator $A$ on $H$:

$$D(A) = [C_0^\infty(S^1)] = [C^\infty(S^1)]$$
$$A([\phi]) = -[\phi''] \text{ for } \phi \in C^\infty(S^1).$$

Consider $B = A + m^2$. We are using the previous notation. So $D(B) = D(A)$. Then, according to the previous proposition, as $A$ is e.s.a. so also is $B$ and:

$$D(\overline{B}) = D(\overline{A}) = W^{2,2}(S^1) = \{\phi \in L^2(S^1) \text{ s.t. } \phi',\ \phi'' \in L^2(S^1)\}.$$

Since $\sigma(\overline{A}) = \sigma_{disc}(\overline{A}) = \{n^2\colon n \in \mathbb{N}_0\}$, then the spectrum of $\overline{B}$ is:

$$\sigma(\overline{B}) = \sigma_{disc}(\overline{B}) = \{n^2 + m^2\colon n \in \mathbb{N}_0\}.$$

Using the chart: $\phi\colon U = S^1\backslash\{1\} \to (0, 2\pi)$, $\phi^{-1}(\theta) = \exp i\theta$, define the function $g\colon U \times U \times \mathbb{C}\backslash\{n^2 + m^2\colon n \in \mathbb{N}_0\} \to \mathbb{C}$ by:

$$g(\theta, \phi; \lambda) = \frac{i}{2\sqrt{\lambda - m^2}} \left[ \exp i\sqrt{\lambda - m^2}|\theta - \phi| + \frac{2\cos\sqrt{\lambda - m^2}(\theta - \phi)}{\exp(-2\pi i\sqrt{\lambda - m^2}) - 1} \right].$$

Clearly as before, this expression for $g$ does not depend on the choice of square root of $\lambda - m^2$ taken. It follows from Proposition 12.3 that $g$ so defined is the Green's function for $\overline{B}$, that is, it generates its resolvent.

## 12.5 Example of Wald Dynamics satisfying $\operatorname{supp}\phi \notin J(K)$

We shall show here by means of the examples just given that there exist simple standard static spacetimes and Wald dynamics generated by a s.a.e. $A_E$ such that $\operatorname{supp}\phi \notin J(K)$ for some initial data $(\phi_0, \dot{\phi}_0)$, where $K = \operatorname{supp}\phi_0 \cup \operatorname{supp}\dot{\phi}_0$.

Consider the example considered in Section 12.3, that is $\Sigma = (0, a)$, so $M = \mathbb{R} \times (0, a)$, $g = dt^2 - dx^2$. (See Figure 5). Take for instance

$$\theta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

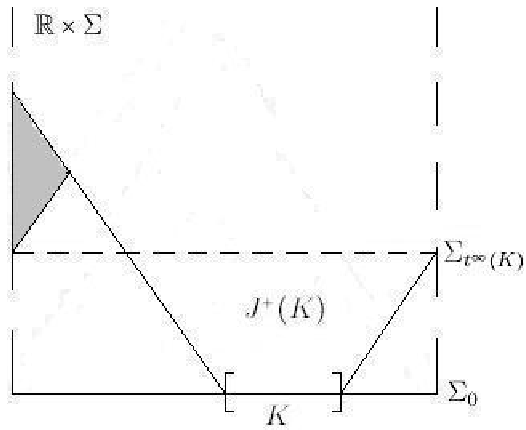


Figure 5: Wald dynamics satisfying: $\text{supp}\,\phi \nsubseteq J(K)$, where $(M, g) = (\mathbb{R} \times (0,1), dt^2 - dx^2)$. For some s.a.e.s $A_E$ there exist points in the shaded area at which $\phi$ is non-zero though they are clearly not contained in $J^+(K)$.

and pick the s.a.e. $A_\theta$ of $A$. Thus its domain is given by:

$$D(A_\theta) = \{\phi \in W^{2,2}(0,a) \colon \phi'(0) = \phi(a),\ \phi(0) = -\phi'(a)\}.$$

From this we can see that if $\phi'(t,a) \neq 0$ then so is $\phi(t,0)$ and hence $\phi$ is non-zero in a neighbourhood of $(0,t)$ and so also non-zero at points outside $J(K)$.

## 12.6 Example of non-bounded below acceptable self-adjoint extensions of minus the Laplacian

We shall, in this section, construct examples of non-bounded below acceptable s.a.e.s of minus the Laplacian on certain choices of simple Riemannian manifolds (though our example shall be on a disconnected manifold). In particular, this shows that the class of solutions to the Cauchy problem of the

Klein-Gordon equation on non-globally hyperbolic spacetimes constructed in this thesis a nontrivial extension of the application of theory of Wald [37] from bounded-below s.a.e.s to acceptable s.a.e.s (Wald considered only those that were positive).

Before we begin the construction, we shall briefly describe some necessary background. It concerns the the direct sum of linear operators on Hilbert spaces. Given a sequence $H_n$ of (real or complex) Hilbert spaces, then the direct sum is defined as usual as:

$$H = \bigoplus_{n \in \mathbb{N}} H_n := \left\{ (\phi_n)_n \in \mathbb{N} \text{ such that } \sum_{n \in \mathbb{N}} \|\phi_n\|_n^2 < \infty \right\},$$

where $\|\cdot\|_n$ is the norm in the Hilbert space $H_n$.

**Definition 12.4.** *For each $n \in \mathbb{N}$, let $A_n$ be a linear operator on the Hilbert space $H_n$. Then define the linear operator $A$ on $H$ as the direct sum of the linear operators $A_n$ as follows:*

$$D(A) = \left\{ \phi = (\phi_n)_{n \in \mathbb{N}} \text{ such that } \phi_n \in D(A_n) \text{ and } \sum_n \|A_n \phi_n\|_n^2 < \infty \right\}$$

$$(A\phi)_n = A_n \phi_n.$$

*We then define the countable direct sum $\bigoplus_{n \in \mathbb{N}} A_n$ of the operators $A_n$ to be the operator $A$.*

**Proposition 12.5.** *The following are true:*

1. *If all the linear operators $A_n$ are densely defined (closed, symmetric, self-adjoint), then $A$ is densely-defined (closed, symmetric, self-adjoint) respectively.*

2. *If all the linear operators $A_n$ are bounded, then the sequence $(\|A_n\|)$ is bounded iff $A = \bigoplus_{n \in \mathbb{N}} A_n$ is bounded.*

3. *The spectrum $\sigma(A)$ is obtained from the spectra $\sigma(A_n)$ by the relation:* $\sigma(A) = \bigcup_n \sigma(A_n)$.

4. *If each operator $A_n$ is a orthogonal projection operator on $H_n$, then $A$ is an orthogonal projection operator on $H$.*

5. *If all the operators $P_n$ are projection-valued measures (p.v.m.s) on $H_n$, then $P$ is a p.v.m. on $H$ defined by:*

$$P_\Omega = \bigoplus_{n\in\mathbb{N}} (P_n)_\Omega \text{ for each } \Omega \subseteq \mathbb{R} \text{ Borel.}$$

   *We shall denote this p.v.m. $P$ by $\bigoplus_{n\in\mathbb{N}} P_n$.*

6. *Let all the operators $A_n$ be self-adjoint. If $P_n$ is the projection-valued measure (p.v.m.) on $H_n$ associated to $A_n$ via the spectral theorem and $P$ is the p.v.m. on $H$ associated to $A$, then:*

$$P = \bigoplus_{n\in\mathbb{N}} P_n.$$

7. *If all the operators $A_n$ are self-adjoint and $f : \mathbb{R} \to \mathbb{R}$ is Borel measurable, then:*

$$f(\bigoplus_{n\in\mathbb{N}} A_n) = \bigoplus_{n\in\mathbb{N}} f(A_n),$$

   *where we are using the spectral theorem to define the self-adjoint operators $f(\bigoplus_{n\in\mathbb{N}} A_n)$ and $f(A_n)$.*

The proof of this proposition is an exercise in functional analysis (see e.g. Reed and Simon [27] or Birman and Solomjak [9]). The proof is omitted here for brevity.

Using this notation, we construct such extensions as follows: Given a fixed Riemannian manifold $(\Sigma, h)$, we shall first consider the case of constructing a s.a.e. $A$ of minus the Laplacian on $(\Sigma', h) = (\mathbb{Z} \times \Sigma, h)$ from s.a.e.s $(A_n)_{n\in\mathbb{Z}}$

of minus the Laplacian on $(\Sigma, h)$. We then show that if all the s.a.e.s $A_n$ are acceptable, then so is $A$. We then give necessary and sufficient conditions for $A$ to be non-bounded below before giving a concrete example. We state our results in the form of the following proposition.

**Proposition 12.6.** *Fix a Riemannian manifold $(\Sigma, h)$ and define $(\Sigma', h) = (\mathbb{Z}\times\Sigma, h)$. Considering the Hilbert spaces $L^2(\Sigma, dvol_h)$ and $L^2(\Sigma', dvol_h)$ then we have the following isomorphism:*

$$L^2(\Sigma', dvol_h) \cong \bigoplus_{n\in\mathbb{Z}} L^2(\Sigma, dvol_h).$$

*Define the following linear operators $A$ and $A'$ on the Hilbert spaces $L^2(\Sigma, dvol_h)$ and $L^2(\Sigma', dvol_h)$ as follows: $D(A) = [C_0^\infty(\Sigma)]$, $A[\phi] = [-\Box_h\phi]$ for $\phi \in C_0^\infty(\Sigma)$ and similarly for $A'$. For simplicity, we shall treat the aforementioned isomorphism as an identification. Then we have the following relationship between $D(A)$ and $D(A')$:*

$$D(A') = \left\{\phi \in \bigoplus_{n\in\mathbb{Z}} D(A) \text{ such that } \phi_n \neq 0 \text{ for at most finitely many } n\right\}.$$

*Now, for each $n \in \mathbb{Z}$, let $A_{E,n}$ be a s.a.e. of $A$ and define the operator $A'_E = \bigoplus_{n\in\mathbb{Z}} A_{E,n}$. Then:*

1. *$A'_E$ is a s.a.e. of $A'$.*
2. *If for all $n$, $A_{E,n}$ is an acceptable s.a.e. of $A$, then $A'_E$ is an acceptable s.a.e. of $A'$.*
3. *$\sigma(A'_E) = \bigcup_{n\in\mathbb{Z}} \sigma(A_{E,n})$.*

*Therefore, if for all $n$, $A_{E,n}$ is an acceptable s.a.e. of $A$ and if $\bigcup_{n\in\mathbb{Z}} \sigma(A_{E,n})$ has no lower bound in $\mathbb{R}$, then $A'_E$ is a non-bounded below acceptable s.a.e. of $A'$.*

*Remark.* In the equation relating $D(A)$ with $D(A')$ we are adopting the following notation: If $H = \bigoplus_{n\in\mathbb{Z}} H_n$ is the countable direct sum of Hilbert spaces, then if for each $n$, $V_n \leq H_n$ is a (not necessarily closed) subspace, then we define:

$$\bigoplus_{n\in\mathbb{Z}} V_n := \left\{\phi \in \bigoplus_{n\in\mathbb{Z}} H_n \text{ such that } \phi \in V_n \text{ for each } n\right\}.$$

With this notation, note that in general: $D(\bigoplus_{n\in\mathbb{Z}} A_n) \neq \bigoplus_{n\in\mathbb{Z}} D(A_n)$ but rather:

$$D(\bigoplus_{n\in\mathbb{Z}} A_n) = \left\{\phi \in \bigoplus_{n\in\mathbb{Z}} D(A_n) \text{ such that } \sum_{n\in\mathbb{Z}} \|A_n\phi_n\|^2 < \infty\right\}.$$

*Proof of Proposition 12.6.* It follows from the previous proposition, that $A_E$ is a self-adjoint operator on $L^2(\Sigma', d\text{vol}_h)$. We must first show that it is in fact a self-adjoint extension of $A'$.

By definition of $A'_E$ we have:

$$\begin{aligned} D(A') &= \left\{\phi \in \bigoplus_{n\in\mathbb{Z}} D(A) \text{ such that } \phi_n \neq 0 \text{ for at most finitely many } n\right\} \\ &\subseteq \left\{\phi \in \bigoplus_{n\in\mathbb{Z}} D(A_{E,n}) \text{ such that } \phi_n \neq 0 \text{ for at most finitely many } n\right\} \\ &\subseteq \left\{\phi \in \bigoplus_{n\in\mathbb{Z}} D(A_{E,n}) \text{ such that } \sum_{n} \|A_{E,n}\phi_n\|_n^2 < \infty\right\} \\ &= D(A'_E) \end{aligned}$$

If $\phi \in D(A') = \left\{\phi \in \bigoplus_{n\in\mathbb{Z}} D(A) \text{ such that } \phi_n \neq 0 \text{ for at most finitely many } n\right\}$, then $(A_E\phi)_n = A_{E,n}\phi_n = A\phi_n = (A'\phi)_n$. This proves Statement *1*.

Let $A_{E,n}$ be an acceptable s.a.e. of $A$ for each $n$. Then, for all $t > 0$:

$$
\begin{aligned}
[C_0^\infty(\Sigma')] &= D(A') \\
&= \left\{ \phi \in \bigoplus_{n\in\mathbb{Z}} D(A) \text{ such that } \phi_n \neq 0 \text{ for at most finitely many } n \right\} \\
&\subseteq \left\{ \begin{array}{l} \phi \in \bigoplus_{n\in\mathbb{Z}} D(\exp((A_{E,n}^-)^{1/2}t)) \\ \text{and } \sum_{n\in\mathbb{Z}} \|\exp((A_{E,n}^-)^{1/2}t)\phi_n\|^2 < \infty \end{array} \right\} \\
&= D(\exp((A_E'^-)^{1/2}t)),
\end{aligned}
$$

where the last equality follows from Statement *7* of Proposition 12.5, which gives: $\exp((A_E'^-)^{1/2}t) = \bigoplus_{n\in\mathbb{Z}} \exp((A_{E,n}^-)^{1/2}t)$. Therefore, $A'_E$ is also an acceptable s.a.e. of $A'$.

The last statement follows from Statement *3* of the previous proposition. □

**Lemma 12.7.** *Let $\Sigma = (0,\infty)$ and pick a sequence $\alpha_n \in (-\frac{\pi}{2}, \frac{\pi}{2}]$ indexed by $n \in \mathbb{Z}$ such that for all $0 < \epsilon < \frac{\pi}{2}$ there exists $n$ with $\alpha_n \in (-\epsilon, 0)$. Define $A_{E,n} = A_{\alpha_n}$. Then, the corresponding operator $A'_E = \bigoplus_{n\in\mathbb{Z}} A_{\alpha_n}$ is a non-bounded below acceptable s.a.e. of minus the Laplacian.*

*Proof.* Note that the s.a.e.s $A_\alpha$ were defined in Section 12.2 as:

$$D(A_\alpha) = \{\phi \in W^{2,2}(0,\infty) \text{ s.t. } \cos\alpha\, \rho(\phi) = \sin\alpha\, \tau(\phi)\}.$$

The spectra of these s.a.e.s were given by the following:

$$\sigma(A_\alpha) = \begin{cases} [0,\infty) & \text{for } \alpha \in [0,\pi/2] \\ [0,\infty) \cup \{-\cot^2\alpha\} & \text{for } \alpha \in (-\pi/2, 0) \end{cases}.$$

So, since $\lim_{x\to 0} \cot^2 x = \infty$ then, by Statement *3* of Proposition 12.6, $\sigma(A'_E) = \bigcup_{n\in\mathbb{Z}} \sigma(A_{\alpha_n})$ has no lower bound in $\mathbb{R}$, i.e. $\inf \sigma(A'_E) = -\infty$ and $A'_E$ is not bounded below. That $A'_E$ is still an acceptable s.a.e. of minus the Laplacian follows from the previous proposition. □

# 13 Summary

We have shown the existence and uniqueness properties of solutions of the Klein-Gordon equation on arbitrary standard static spacetimes based on "acceptable" self-adjoint extensions $A_E$ of the symmetric linear operator $A$, as defined in equation (3.3). The proof of the existence (Section 5) was based on work by Wald [37], though differs in the following: Our treatment utilised the more recent result of Bernal and Sanchez [6]. Also, we have shown that the construction of solutions is valid also when the self-adjoint extension is merely acceptable (Definition 3.2).

Separate to the work of Wald, we proved in this thesis a result concerning the uniqueness of the Wald solutions and used this to prove a result on their support. The stronger statement: $\operatorname{supp}\phi \subseteq J(K)$ for $K = \operatorname{supp}(\phi_0) \cup \operatorname{supp}(\dot{\phi}_0)$, which was a condition on the dynamics in the paper by Wald and Ishibashi on this topic [38], was seen to be false in general. In Section 12.5 we gave a simple example where $\operatorname{supp}\phi \nsubseteq J(K)$.

Also, using the uniqueness result, we defined the space of solutions in Definition 6.5, constructed both the "energy form" and the "symplectic form" on the space of solutions (Sections 9 and 10 respectively) and analysed some symmetries of the space of solutions (Section 11).

In Sections 12.1 to 12.3 we considered three simple one-dimensional Riemannian manifolds ($S^1$, $(0,\infty)$ and $(0,a)$ with their usual Riemannian metrics), determined all the self-adjoint extensions of minus the Laplacian on each of these spaces, found their spectra and proved the form of their resolvents as integral operators. This then specifies the dynamics as constructed in this thesis as generated by each of these s.a.e.s on the standard static spacetimes $(M, g) = (\mathbb{R} \times \Sigma, dt^2 - h)$, as $\Sigma = S^1$, $(0, \infty)$ or $(0, a)$. In Section 12.5 we give a simple example of a standard static spacetime and a choice of s.a.e. $A_E$ such that the dynamics generated satisfies: $\operatorname{supp} \phi \notin J(K)$ for some initial data (this corresponds to 1. of the second list on p.6). In Section 12.6 we constructed an acceptable non-bounded below s.a.e. $A_E$ of minus the Laplacian on a particular (disconnected) Riemannian manifold (specifically: $\Sigma = \mathbb{Z} \times (0, \infty)$ with the Riemannian metric induced from that of $\mathbb{R}^2$). This example then shows that the extension of theory of Wald [37] from bounded-below s.a.e.s to acceptable s.a.e.s carried out in this thesis is non-trivial (Wald considered only positive s.a.e.s).

We shall now discuss avenues of further work on the subject of this thesis. We list them as follows, some of which are related:

1. The well-posedness of the Cauchy problem for the Klein-Gordon equation often has a stronger meaning than that used in this thesis. The stronger sense includes continuity of the map

$$
\begin{aligned}
C_0^\infty(\Sigma) \times C_0^\infty(\Sigma) &\to C^\infty(M) \\
(\phi_0, \dot{\phi}_0) &\to \phi.
\end{aligned}
$$

   A problem unanswered in this thesis is whether our solution to the Cauchy problem generated by an acceptable s.a.e. is well-posed in this sense.

2. We constructed in Section 12.6 an example of an acceptable non-bounded below s.a.e. on a disconnected Riemannian manifold. It would be of interest to construct examples on connected ones.

3. Once the answer to Statement 1 is known, a natural question in line with the paper by Wald and Ishibashi [38] is whether there are necessary

and sufficient conditions on a solution to the Cauchy problem to be generated by an acceptable s.a.e. via this thesis. Since their paper dealt with the case of sufficient conditions for the Cauchy problem to be generated by a positive s.a.e. then this would be an extension of their work to the present case.

4. An important question, connected with Statement 3, is whether or not there exists dynamics conserving the symplectic form constructed in Section 10 (but possibly not conserving an energy form), that is **not** generated by a s.a.e. via the construction in this thesis. This question posed by Kay and Studer [17] (Appendix A.2) is still unanswered.

# Appendices

## A The Linear Operators $C(t, A_E)$ and $S(t, A_E)$

In the first part of this appendix we show that if the s.a.e. $A_E$ of $A$ is bounded below then the linear operators $C(t, A_E)$ and $S(t, A_E)$ are both bounded on $L^2(\Sigma, V^{-1}d\text{vol}_h)$. In the second part of the appendix we prove Proposition 3.5 concerning the strong derivatives of these linear operators.

We begin my defining a multiplication operator and stating our required form of the Spectral Theorem. In the following, denote by $\mathcal{L}^2(M, \Omega, \mu)$ the space of (real or complex valued) square-integrable measurable functions (by measurability here we mean with respect to the $\sigma$-algebra $\Omega$ on $M$ and the $\sigma$-algebra of Borel sets on $\mathbb{R}$ or $\mathbb{C}$ as required). Denote by $L^2(M, \Omega, \mu)$ the (real or complex) Hilbert space consisting of equivalence classes of elements in $\mathcal{L}^2(M, \Omega, \mu)$.

**Definition A.1** (Multiplication Operators (see e.g. Reed and Simon [27])). *Let $(M, \Omega, \mu)$ be a measure space (a triple consisting of a set $M$, a $\sigma$-algebra $\Omega$ of subsets of $M$ and measure $\mu$ on $\Omega$) and $f: M \to \mathbb{R}$ be a measurable function. Then define the linear operator $T_f$ on $L^2(M, \Omega, \mu)$ by:*

$$D(T_f) = \{[\phi] \in L^2(M, \Omega, \mu) \;\; s.t\; f(\cdot)\phi(\cdot) \in \mathcal{L}^2(M, \Omega, \mu)\},$$
$$T([\phi]) = [f(\cdot)\phi(\cdot)] \;\; on\; D(T_f).$$

The following are true of the linear operator $T_f$ (see Reed and Simon [27]):

- It is a self-adjoint linear operator.
- It is bounded iff $f \in \mathcal{L}^\infty(M, \Omega, \mu)$.

- It is bounded-below iff $f \geq -M$ a.e. for some $M \in \mathbb{R}$ i.e. iff $f^- \in \mathcal{L}^\infty(M, \Omega, \mu)$.

**Theorem A.2** (Multiplication Operator Version of the The Spectral Theorem (See e.g. Reed and Simon [27]))**.** *Let $H$ be a complex separable Hilbert space and $A$ be a self-adjoint linear operator on $H$. Then there exists a measure space $(M, \Omega, \mu)$, real-valued measurable function $f$ on $M$ and a unitary operator $U: H \to L^2(M, \Omega, \mu)$ s.t. $A$ and $T_f$ are unitarily equivalent via $U$, that is:*

$$U(D(A)) = D(T_f)$$
$$UAU^{-1}[\phi] = [f(\cdot)\phi(\cdot)], \text{ for all } [\phi] \in D(T_f).$$

In the following propositions, using the Spectral Theorem in the form stated above, we shall show that if $A_E$ is a bounded-below self-adjoint operator, then $C(t, A_E)$ and $S(t, A_E)$ are bounded linear operators. We shall then consider their strong derivatives.

**Proposition A.3.** *Using the above definitions, let $f \geq -M$ a.e. where $M \geq 0$. Then for all $t \in \mathbb{R}$: $C(t, f(\cdot)) \in L^\infty(M, \Omega, \mu)$ and so $T_{C(t,f(\cdot))}$ is a bounded self-adjoint operator by the previous proposition. In fact,*

$$||T_{C(t,f(\cdot))}|| \leq 1 + \cosh(M^{1/2}t).$$

*Proof.* $|C(t, f(m))| = |\cos(f(m)^{1/2}t)| \leq 1$ for $f(m) \geq 0$
$|C(t, f(m))| = |\cosh((-f(m))^{1/2}t)| \leq \cosh(M^{1/2}t)$ for $f(m) < 0$ as cosh is a monotonic decreasing function on the negative reals.
Thus $|C(t, f(m))| \leq 1 + \cosh(M^{1/2}t)$ for all $m$. □

Using the last theorem we can state the properties of $C(t, A_E)$ for an arbitrary bounded-below self-adjoint operator $A_E$:

**Lemma A.4.** *Let $A_E$ be a bounded-below self-adjoint operator. Defining $C(t,x)$ as above then $C(t,A_E)$ is a bounded self-adjoint operator satisfying*

$$\|C(t,A_E)\| \leq 1 + \cosh(M^{1/2}t) \text{ for all } t \in \mathbb{R}.$$

**Proposition A.5.** *Using the above definitions, let $f \geq -M$ a.e. where $M \geq 0$. Then $S(t,f(\cdot)) \in L^\infty(M,\Omega,\mu)$ and so $T_{S(t,f(\cdot))}$ is a bounded self-adjoint operator by the previous proposition. In fact $\|T_{S(t,f(\cdot))}\| \leq t(1+M^{-1/2}\sinh(M^{1/2}t))$*

*Proof.* $|S(t,f(m))| = |f(m)^{-1/2}\sin(f(m)^{1/2}t)| \leq t$ for $f(m) \geq 0$
$|S(t,f(m))| = |(-f(m))^{-1/2}\sinh((-f(m))^{1/2}t)| \leq tM^{-1/2}\sinh(M^{1/2}t)$ for $f(m) < 0$ as $x^{-1}\sinh x$ monotonic decreasing function on the negative reals.
Thus $|S(t,f(m))| \leq t(1+M^{-1/2}\sinh(M^{1/2}t))$ for all $m$.
The result follows. □

**Lemma A.6.** *Let $A_E$ be a bounded-below self-adjoint operator. Defining $S(t,x)$ as above then $S(t,A_E)$ is a bounded self-adjoint operator satisfying $\|S(t,A_E)\| \leq t(1+M^{-1/2}\sinh(M^{1/2}t))$ for all $t \in \mathbb{R}$.*

Before we further consider the linear operators $C(t,A_E)$ and $S(t,A_E)$ we make a simple proposition which allows us to quickly compare the domains of multiplication operators, and hence also functions of self-adjoint operators in any Hilbert space by the above version of the Spectral Theorem.

**Proposition A.7.** *Let $(M,\Omega,\mu)$ be a measure space and $f: M \to \mathbb{R}$ be a Borel measurable function. If $g,h:\mathbb{R}\to\mathbb{R}$ are continuous functions satisfying:*

$$g(x) = O(h(x)) \text{ as } |x| \to \infty$$

*(that is $|g(x)| \leq A|h(x)|$ for all $|x| > K$ and some $A \geq 0$), then $D(T_{h\circ f}) \subseteq D(T_{g\circ f})$.*

*Proof.* Firstly let $|g(x)| \le A|h(x)|$ for all $|x| > K$. If $[\phi] \in D(T_{h\circ f})$ then:

$$\begin{aligned}
&\int_M |g\circ f(m)|^2|\phi(m)|^2 d\mu(m) \\
&\quad = \int_{|f|\le K} |g\circ f(m)|^2|\phi(m)|^2 d\mu(m) + \int_{|f|>K} |g\circ f(m)|^2|\phi(m)|^2 d\mu(m) \\
&\quad \le (\sup_{|x|\le K} |g(x)|)^2 \int_{|f|\le K} |\phi(m)|^2 d\mu(m) + A\int_{|f|>K} |h\circ f(m)|^2|\phi(m)|^2 d\mu(m) \\
&\quad < \infty,
\end{aligned}$$

since the first term is finite as $g$ is continuous and $[\phi] \in L^2(M,\Omega,\mu)$ and the second term is finite since $[\phi] \in D(T_{h\circ f})$. □

*Remark.* The result is also true (by a similar proof) if $g(x) = O(1)$ as $x \to \infty$ and $g(x) = O(h(x))$ as $x \to -\infty$. Using this result we have for instance that $D(T_{C(t,f)}) \subseteq D(T_{S(t,f)})$ and so $D(C(t,A_E)) \subseteq D(S(t,A_E))$ for all $t$.

**Lemma A.8.** *Let $(M,\Omega,\mu)$ and $f$ be as in the last proposition. If $g,h : \mathbb{R} \to (0,\infty)$ are continuous functions satisfying:*

$$\left|\frac{g(x)}{h(x)}\right| \to c > 0 \text{ as } |x| \to \infty,$$

*then $D(T_{g\circ f}) = D(T_{h\circ f})$.*

This lemma follows easily from the last proposition. Similarly to the previous remark it would be sufficient if $g$ and $h$ were bounded on $[0,\infty)$ and $\left|\frac{g(x)}{h(x)}\right| \to c > 0$ as $x \to -\infty$. Using this result with $g(x) = C(t,x)$ and $h(x) = \exp t(x^-)^{1/2}$ for some $t > 0$ results in: $D(T_{C(t,f)}) = D(T_{\exp t(x^-)^{1/2}})$ and hence $D(C(t,A_E)) = D(\exp t(A_E^-)^{1/2})$.

We now find the strong derivatives of the linear operators $C(t,A_E)$ and $S(t,A_E)$ (Proposition 3.5) and show that $[\chi_E]$ is an invariant space with respect to both linear operators.

**Proposition A.9.** *Given an acceptable s.a.e.* $A_E$ *of* $A$*, define:*

$$\chi_E := \{f \in C^\infty(\Sigma) \text{ s.t. } [f] \in D(A_E^\infty) \cap \bigcap_{t>0} D(\exp((A_E^-)^{1/2}t))\}$$

*then the linear operators* $C(t, A_E)$ *and* $S(t, A_E)$ *satisfy the following:*

$$C(t, A_E),\ S(t, A_E) : [\chi_E] \to [\chi_E].$$

*Also, the maps* $t \to C(t, A_E)$ *and* $t \to S(t, A_E)$ *are infinitely often strongly differentiable on* $[\chi_E]$*, where for* $n \in \mathbb{N} \cup \{\infty\}$*:*

$$D(A_E^n) = \{x \in D(A_E) :\ A_E^m x \in D(A_E) \text{ for all } m = 1, ..., n-1\}.$$

*In fact, for* $n \in \mathbb{N}$ *the following strong derivatives hold on the dense subspace* $[\chi_E]$ *of* $L^2(\Sigma, V^{-1}dvol_h)$*:*

$$\frac{d^{2n}}{dt^{2n}} C(t, A_E) = (-1)^n A_E^n C(t, A_E),$$
$$\frac{d^{2n-1}}{dt^{2n-1}} C(t, A_E) = (-1)^n A_E^n S(t, A_E),$$
$$\frac{d^{2n}}{dt^{2n}} S(t, A_E) = (-1)^n A_E^n S(t, A_E),$$
$$\frac{d^{2n+1}}{dt^{2n+1}} S(t, A_E) = (-1)^n A_E^n C(t, A_E).$$

*Proof.* It suffices to prove these facts for the unitarily equivalent case of $T_f$, where $f : M \to \mathbb{R}$ is measurable.

We show first that

$$T_{C(t,f)} : \bigcap_{t>0} D(T_{\exp((f^-)^{1/2}t)}) \to \bigcap_{t>0} D(T_{\exp((f^-)^{1/2}t)})$$

and then that:

$$T_{C(t,f)} : D(T_{f^n}) \cap \bigcap_{t>0} D(T_{\exp((f^-)^{1/2}t)}) \to D(T_{f^n}).$$

Indeed, if $\phi \in \bigcap_{t>0} D(T_{\exp((f^-)^{1/2}t)})$ then:

$$
\begin{aligned}
\int_M & |C(t,f(m)|^2|\exp f^-(m)^{1/2}t'|^2|\phi(m)|^2 d\mu(m) \\
&\leq \int_{f\leq 0} |C(t,f(m)|^2 \exp 2f^-(m)^{1/2}t'|\phi(m)|^2 d\mu(m) \\
&\qquad + \int_{f>0} \exp 2f^-(m)^{1/2}t'|\phi(m)|^2 d\mu(m) \\
&= \int_{f\leq 0} \cosh^2 tf^-(m)^{1/2} \exp 2f^-(m)^{1/2}t'|\phi(m)|^2 d\mu(m) + C \\
&= \frac{1}{4}\int_{f\leq 0} \exp 2(t+t')f^-(m)^{1/2}|\phi(m)|^2 d\mu(m) \\
&\qquad + \frac{1}{2}\int_{f\leq 0} \exp 2f^-(m)^{1/2}t'|\phi(m)|^2 d\mu(m) \\
&\qquad + \frac{1}{4}\int_{f\leq 0} \exp -2(t+t')f^-(m)^{1/2}|\phi(m)|^2 d\mu(m) + C \\
&< \infty.
\end{aligned}
$$

Secondly,

$$
\begin{aligned}
\int_M & |f(m)|^{2n}|C(t,f(m))|^2|\phi(m)|^2 d\mu(m) \\
&= \int_{f<0} |f(m)|^{2n} \cosh^2(tf^-(m)^{1/2})|\phi(m)|^2 d\mu(m) \\
&\qquad + \int_{f\geq 0} |f(m)|^{2n}|\cos(tf(m)^{1/2})|^2|\phi(m)|^2 d\mu(m) \\
&\leq \frac{1}{4}\int_{f<0} |f(m)|^{2n} \exp 2tf^-(m)^{1/2}t)|\phi(m)|^2 d\mu(m) \\
&\qquad + \frac{1}{2}\int_{f<0} |f(m)|^{2n}|\phi(m)|^2 d\mu(m) \\
&\qquad + \frac{1}{4}\int_{f<0} |f(m)|^{2n} \exp -2tf^-(m)^{1/2}t)|\phi(m)|^2 d\mu(m) + C \\
&< \infty.
\end{aligned}
$$

Thus $T_{C(t,f)}$ maps the subspace $D(T_f^\infty) \cap \bigcap_{t>0} D(T_{\exp((f^-)^{1/2}t)})$ to itself. Again, arguing by the Spectral Theorem, we have $C(t,A_E) : [\chi_E] \to [\chi_E]$ for any s.a.e. $A_E$. The proof for the linear operator $S(t,A_E)$ is similar. We prove the last properties by a sequence of lemmas. $\square$

**Lemma A.10.** *Let $f$ be measurable. Then the map $t \to T_{\cos tf}$ is strongly differentiable at $t$ on $D(T_{f^2})$ with strong derivative $T_{-f \sin tf}$.*

*Proof.*

$$\lim_{h\to 0}\left\|\frac{1}{h}(T_{\cos(t+h)f} - T_{\cos tf})\phi - T_{-f\sin tf}\phi\right\|^2$$
$$= \lim_{h\to 0}\int_M \left[\begin{array}{l} \frac{1}{h}\left(\cos(t+h)f(m) - \cos tf(m)\right) \\ +f(m)\sin tf(m) \end{array}\right]^2 |\phi(m)|^2 d\mu(m) \qquad \text{(A.1)}$$

As $t$ is fixed, the integrand approaches zero pointwise in $M$ as $h \to 0$. In order to show that the integral converges to zero, we use the Dominated Convergence theorem (D.C.T.). By Taylor's theorem, for all $h$ there exists $k$ such that $|k| < |h|$ and

$\cos(t+h)f(m) = \cos tf(m) - hf(m)\sin tf(m) - \dfrac{h^2}{2}f(m)^2\cos(t+k)f(m)$, so

$$\left|\frac{1}{h}\left(\cos(t+h)f(m) - \cos tf(m)\right) + f(m)\sin tf(m)\right| \le \frac{|h|}{2}|f(m)|^2.$$

We define $g(m) = \frac{1}{4}|f(m)|^4|\phi(m)|^2$.

So for $|h| < 1$, $\left[\frac{1}{h}\left(\cos(t+h)f(m) - \cos tf(m)\right) + f(m)\sin tf(m)\right]^2 |\phi(m)|^2 \le g(m) \in \mathcal{L}^1(M,\Omega,\mu)$ as $\phi \in D(T_{f^2})$.

Thus by the D.C.T., the RHS of (A.1) is zero. □

**Lemma A.11.** *Letting $f$ be measurable, then the map $t \to T_{\cosh tf}$ is strongly differentiable at $t > 0$ on $D(T_{f^2 \cosh t'f})$ (for any $t' > t$) with strong derivative $T_{f \sinh tf}$.*

*Proof.*

$$\lim_{h\to 0}\left\|\frac{1}{h}(T_{\cosh(t+h)f} - T_{\cosh tf})\phi - T_{f\sinh tf}\phi\right\|^2$$
$$= \lim_{h\to 0}\int_M \left[\begin{array}{l} \frac{1}{h}\left(\cosh(t+h)f(m) - \cosh tf(m)\right) \\ -f(m)\sinh tf(m) \end{array}\right]^2 |\phi(m)|^2 d\mu(m) \qquad \text{(A.2)}$$

For $|h| < t' - t$ there exists $k: |k| < |h|$ such that:

$$\cosh(t+h)f(m) = \cosh tf(m) + hf(m)\sinh tf(m) - \frac{h^2}{2}f(m)^2\cosh(t+k)f(m).$$

So

$$\left|\frac{1}{h}(\cosh(t+h)f(m) - \cosh tf(m)) - f(m)\sinh tf(m)\right| \\ \leq \frac{|h|}{2}|f(m)|^2\cosh(t+k)f(m).$$

So we define $g(m) = \frac{1}{4}|f(m)|^4|\phi(m)|^2\cosh^2 t'f(m)$.
So for $|h| < 1$, $\left[\frac{1}{h}(\cosh(t+h)f(m) - \cosh tf(m)) - f(m)\sinh tf(m)\right]^2|\phi(m)|^2 \leq g(m) \in \mathcal{L}^1(M,\Omega,\mu)$ as $\phi \in D(T_{f^2\cosh t'f})$.

Thus by the D.C.T., the RHS of (A.2) is zero. $\square$

**Lemma A.12.** *The map* $t \to T_{C(t,f)}$ *is strongly differentiable at* $t > 0$ *on* $D(T_{f^+}) \cap D(T_{f^-\cosh t'(f^-)^{1/2}})$ *(for any* $t' > t$*) with strong derivative* $T_{-fS(t,f)}$.

*Proof.*

$$
\begin{aligned}
&\lim_{h\to 0}\left\|\frac{1}{h}(T_{C(t+h,f)}-T_{C(t,f)})\phi+T_{fS(t,f)}\phi\right\|^2 \\
&=\lim_{h\to 0}\int_M\left[\begin{array}{l}\frac{1}{h}(C(t+h,f(m))-C(t,f(m))) \\ +f(m)S(t,f(m))\end{array}\right]^2|\phi(m)|^2d\mu(m) \\
&=\lim_{h\to 0}\int_{f<0}\left[\begin{array}{l}\frac{1}{h}(\cosh(t+h)(-f(m))^{1/2}-\cosh t(-f(m))^{1/2}) \\ +f(m)(-f(m))^{-1/2}\sinh t(-f(m))^{1/2}\end{array}\right]^2|\phi(m)|^2d\mu(m) \\
&\quad+\lim_{h\to 0}\int_{f\geq 0}\left[\begin{array}{l}\frac{1}{h}(\cos(t+h)f(m)^{1/2}-\cos tf(m)^{1/2}) \\ +f(m)(f(m))^{-1/2}\sin t(f(m))^{1/2}\end{array}\right]^2|\phi(m)|^2d\mu(m) \\
&=\lim_{h\to 0}\int_M\left[\begin{array}{l}\frac{1}{h}(\cosh(t+h)f^-(m)^{1/2}-\cosh t(f^-(m))^{1/2}) \\ -f^-(m)^{1/2}\sinh tf^-(m)^{1/2}\end{array}\right]^2|\phi(m)|^2d\mu(m) \\
&\quad+\lim_{h\to 0}\int_M\left[\begin{array}{l}\frac{1}{h}(\cos(t+h)f^+(m)^{1/2}-\cos tf^+(m)^{1/2}) \\ +f^+(m)^{1/2}\sin tf^+(m)^{1/2}\end{array}\right]^2|\phi(m)|^2d\mu(m) \\
&\to 0,
\end{aligned}
$$

since: $\phi\in D(T_{f^+})\cap D(T_{f^-\cosh t'(f^-)^{1/2}})$. □

Note that:

$$
\begin{aligned}
\bigcap_{n\geq 0,t>0} D(T_{f^n})\cap D(T_{C(t,f)}) &\subseteq D(T_f)\cap D(T_{fC(t',f)}) \\
&\subseteq D(T_{f^+})\cap D(T_{f^-\cosh t'(f^-)^{1/2}})
\end{aligned}
$$

(All inclusions between domains of multiplication operators follow from either Proposition A.7 or the remark thereafter.) So $t\to T_{C(t,f)}$ is also strongly differentiable on

$$\bigcap_{n\geq 0,t>0} D(T_{f^n})\cap D(T_{C(t,f)}).$$

Thus if $A_E$ is an acceptable s.a.e. of $A$ then $t \to C(t, A_E)$ is strongly differentiable on

$$D(A_E^\infty) \cap \bigcap_{t>0} D(C(t, A_E)) =: [\chi_E] \subseteq D(A_E^\infty) \subseteq [C^\infty(\Sigma)],$$

where the last inclusion is the content of Theorem 3.3.

The following statements follow by similar arguments.

**Lemma A.13.** *Again with $f$ measurable, then the following are true:*

1. *The map $t \to T_{\sin tf}$ is strongly differentiable at $t$ on $D(T_{f^2})$ with strong derivative $T_{f \cos tf}$.*
2. *The map $t \to T_{\sinh tf}$ is strongly differentiable at $t > 0$ on $D(T_{f^2 \sinh t'f})$ for $t' > t$ with strong derivative $T_{f \cosh tf}$.*
3. *The map $t \to T_{S(t,f)}$ is strongly differentiable at $t > 0$ on*

$$D(T_{(f^-)^{1/2} \sinh t'(f^-)^{1/2}}) \cap D(T_{(f^+)^{1/2}})$$

*for $t' > t$ with strong derivative $T_{C(t,f)}$.*

Using the last two lemmas we obtain the required last properties given in Proposition A.9.

# B Klein-Gordon Solutions on Globally Hyperbolic Spacetimes not of Compact Support on any Cauchy surface

This section is devoted to proving an extension of Theorem 4.25. It is included here for completeness. (See also Corollary 5, Section 3.5.3 in Ginoux's contribution in Bär and Fredenhagen (Eds.) [4].)

**Theorem B.1** (Existence and Uniqueness of Classical Solutions on Globally Hyperbolic Spacetimes with respect to smooth initial data:)**.** *Let $(M, g)$ be a globally hyperbolic spacetime with smooth, spacelike Cauchy surface $S$. Then the Klein-Gordon equation has a well-posed initial value formulation, that is, given data $\phi_0, \dot{\phi}_0 \in C^\infty(S)$ then there exists a unique solution $\phi \in C^\infty(M)$ to:*

$$(\Box_g + m^2)\psi = 0$$
$$\psi|_S = \phi_0$$
$$\nabla_n \psi|_S = \dot{\phi}_0,$$

*where $n$ is the unique unit smooth future-pointing timelike vector field on $S$ normal to the smooth spacelike Cauchy surface $S$. Moreover:*

$$supp\, \psi \subseteq J(K),$$

*where $K = supp\, \phi_0 \bigcup supp\, \dot{\phi}_0$.*

To show this, we begin by proving some basic lemmas. Throughout, $(M, g)$ is a globally hyperbolic spacetime and $S$ is some smooth spacelike Cauchy hypersurface of $M$. We refer the reader to O'Neill [23] for a thorough introduction to causality theory.

**Lemma B.2.** $D(W)^C = J(S \backslash W) \;\; \forall W \subseteq S$

*Proof.* $x \in$ LHS

iff $\exists$ an inextendible causal curve that does not pass through $W$.

iff $\exists$ an inextendible causal curve that passes $S\backslash W$.

iff $x \in$ RHS $\square$

**Lemma B.3.** $W \subseteq D(J(W) \cap S) \ \ \forall W \subseteq M$

*Proof.* If $x \in W$, take $\gamma$ to be an inextendible causal curve through p. As $S$ is a smooth Cauchy surface, $\gamma$ passes through $S$ at some $y \in S$. So $y \in J(W) \cap \Sigma$ and $x \in D(J(W) \cap S)$. $\square$

**Lemma B.4.** *$K \subseteq S$ is closed $\Rightarrow$ $J(K)$ is closed in $M$.*

*Proof.* If $K$ is closed then $S\backslash K$ is open in $S$. Thus $S\backslash K$ is an acausal topological hypersurface in $M$ and, according to Theorem 4.21, $D(S\backslash K)$ is open in $M$. But $D(S\backslash K) = J(K)^C$ due to Lemma B.2 and so $J(K)$ is closed. $\square$

*(Proof of Theorem B.1).* We start by proving existence of our solution. Given any $p \in M$, then $J(p) \cap S$ is compact in $S$. Let $f \in C_0^\infty(S)$ s.t. $f = 1$ on an open neighbourhood $U$ of $J(p) \cap S$. Since $\phi_0 f, \dot{\phi}_0 f \in C_0^\infty(S)$, then define $\phi_{p,f} \in C^\infty(M)$ as the solution to the Cauchy problem w.r.t compactly supported smooth data $(\phi_0 f, \dot{\phi}_0 f)$, via Theorem 4.25. Now, define the function $\phi: M \to \mathbb{K}$ as $\phi(p) = \phi_{p,f}(p)$. We shall first show that the value $\phi(p)$ is independent of which function $f$ we take. Let $f, g \in C_0^\infty(M)$ such that $f = g = 1$ on an open neighbourhood $U$ of $J(p) \cap S$. So, $f - g = 0$ on $U$ and:

$$\phi_0(f-g), \dot{\phi}_0(f-g) = 0 \text{ on } U \supseteq J(p) \cap S.$$

Therefore, $\phi_{p,f-g} = \phi_{p,f} - \phi_{p,g} = 0$ on $D(U)$. But $p \in D(J(p) \cap S) \subseteq D(U)$ by Lemma B.4 and so $\phi_{p,f}(p) = \phi_{p,g}(p)$. Thus $\phi$ is a well-defined function on $M$.

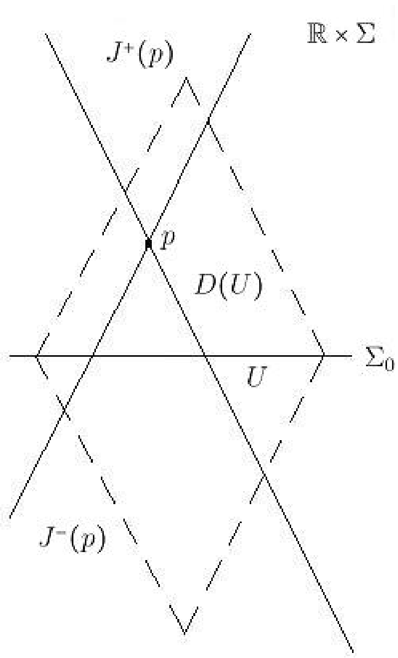


Figure 6: The definition of the solution of the Cauchy problem to arbitrary smooth data.

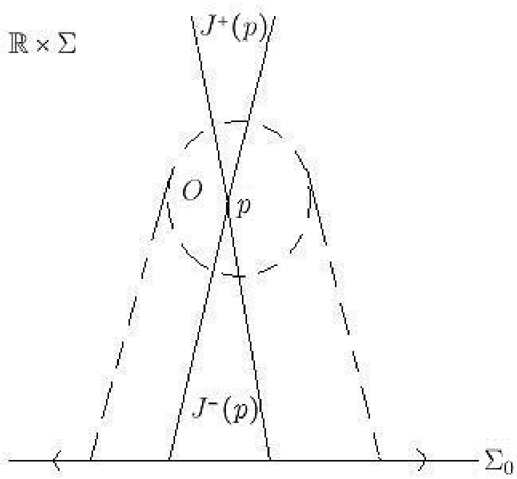


Figure 7: A reformulation of the defintion of the solution of the Cauchy problem to arbitrary smooth data.

We must now show that it is smooth, solves the K-G equation with respect to smooth inital data $(\phi_0, \dot{\phi}_0)$. This is proven by an alternative characterisation of the function $\phi$. For any precompact open subset $O$, let $g \in C_0^\infty(S)$ such that $g = 1$ on an open neighbourhood $V$ of $J(\overline{O}) \cap S$. Let $\phi_{\overline{O},g} \in C^\infty(M)$ be the solution to the Klein-Gordon equation corresponding to data $(\phi g, \dot{\phi}_0 g)$ according to Theorem 4.25. It follows, from the previous argument, that for all $p \in O$: $\phi(p) = \phi_{\overline{O},g}(p)$, where $\phi$ was defined previously, since $J(p) \subseteq J(O)$ and if $g = 1$ on an open neighbourhood of $J(O) \cap S$, then so also on an open neighbourhood of $J(p) \cap S$.

As $\phi_{\overline{O},g}$ is smooth and solves the Klein-Gordon equation, then so does $\phi$. If $p \in S$, then $\phi$ satisfies:

$$\phi(p) = \Psi(p) = \phi_0(p) f(p) = \phi_0(p)$$
$$\nabla_n \phi(p) = \nabla_n \Psi(p) = \dot{\phi}_0(p) f(p) = \dot{\phi}_0(p).$$

So, $\phi|_S = \phi_0$ and $\nabla_n \phi|_S = \dot{\phi}_0$.

We now check the property concerning the support of $\phi$. If $J(K) = M$ then the statement is trivially true. If $p \notin J(K)$, then as $K$ is closed and $J(p) \cap S$ compact in $S$, there exists an open neighbourhood $U$ of $J(p) \cap S$ and $f \in C_0^\infty(S)$ s.t. $f = 1$ on $U$ and $f = 0$ on $K$. So, $\phi_0, \dot{\phi}_0 = 0$ on $U$ and $U \cap K = \phi$ then $\phi_0 f = \dot{\phi}_0 f = 0$ on $S$. Therefore, $\phi(p) = \Psi(p) = 0$ and so $p \notin J(K) \Rightarrow \phi(p) = 0$ or $\{\phi \neq 0\} \subseteq J(K)$ and

$$\operatorname{supp} \phi = \overline{\{\phi \neq 0\}} \subseteq \overline{J(K)} = J(K),$$

by Lemma B.4.

The proof of the uniqueness of such a solution is given by the following short argument. Consider the following statements:

1. We have uniqueness of the Cauchy problem w.r.t. arbitrary smooth data.

2. If $\phi \in C^\infty(M)$ satisfies $(\Box_g + m^2)\phi = 0$, $\phi|_S = 0$ and $\nabla_n \phi|_S = 0$ then $\phi = 0$.

3. We have uniqueness of the Cauchy problem w.r.t. arbitrary smooth data of compact support.

The following equivalences are easy to show, by the linearity of the Klein-Gordon operator, the covariant derivative and the restrictions onto the surface $S$:

$$1. \Leftrightarrow 2. \Leftrightarrow 3.$$

Thus, since Statement 3. is true by Theorem 4.25, then Statement 1. is true. □

# C Basic Metric Space Theory

In this appendix we deal with the some relatively basic results in metric space theory needed in the proofs of Propositions 4.22 and 4.12. We include them for completeness.

Let $(X, d)$ be a metric space. Given $K \subseteq X$, then we define:

$$d(q, K) := \inf_{p \in K} \{d(p, q)\}.$$

The closed ball around $K$ of radius $t$ is then defined as:

$$C(K, t) := \{q \colon d(q, K) \leq t\}.$$

Note its alternative characterisation:

$$q \in C(K, t) \text{ iff for all } \epsilon > 0 \text{ there exists } p \in K \text{ such that } d(p, q) < t + \epsilon.$$

**Proposition C.1.** *Let $(X, d)$ be a metric space and $K \subseteq X$ a compact subset. Then:*

$$\begin{aligned} C(K, t) &:= \{q \colon d(q, K) \leq t\} \\ &= \left\{q \colon \inf_{p \in K} \{d(p, q)\} \leq t\right\} \\ &= \bigcup_{p \in K} C(p, t). \end{aligned}$$

Note that in the last step we have used the compactness of $K$ and the continuity of the metric. We now check that the closed ball $C(K, t)$ is indeed closed.

**Proposition C.2.** *Let $(X, d)$ be a metric space and $K \subseteq X$. Then, $C(K, t)$ is closed for all $t \geq 0$.*

*Proof.* Let $q_n \in C(K,t)$ and $q_n \to q \in X$. We want to show that

$$\inf_{p \in K} \{d(p,q)\} \leq t.$$

If this were not true, then there would exist $\epsilon > 0$ such that $d(p,q) > t + \epsilon$ for all $p \in K$. But since $q_n \to q$, then $d(p,q_n) > t + \epsilon/2$ for all $p \in K$ and all $n > N$ for some $N$. So $d(q_n, K) = \inf_{p \in K}\{d(p,q_n)\} \geq t + \epsilon/2$, which is a contradiction. □

We shall now prove some more simple properties of the closed ball:

**Proposition C.3.** *Letting $(X,d)$ be a metric space and $A, B \subseteq X$, then:*

1. *If $A \subseteq B$, then $C(A,t) \subseteq C(B,t)$.*
2. *$C(A \cup B, t) \subseteq C(A,t) \cup C(B,t)$.*
3. *$C(C(A,t),s) \subseteq C(A, t+s)$.*

*Proof.* To prove Statement *1*, note that:

$$\{d(p,q) : q \in A\} \subseteq \{d(p,q) : q \in B\}.$$

So, if $p \in C(A,t)$ then:

$$t \geq d(A,p) = \inf\{d(p,q) : q \in A\} \geq \inf\{d(p,q) : q \in B\} = d(B,p)$$

and so $p \in C(B,t)$.

To prove Statement *2*:

$$\{d(p,q) : q \in A \cup B\} = \{d(p,q) : q \in A\} \cup \{d(p,q) : q \in B\}$$

and so

$$d(p, A \cup B) = \inf\{d(p,q) : q \in A \cup B\} = \min\{d(p,A), d(p,B)\}.$$

So if $p \in C(A \cup B, t)$ then $d(p, A \cup B) \leq t$ and either $d(p, A) < t$ or $d(p, B) < t$, that is, $p \in C(A, t)$ or $p \in C(B, t)$.

To prove Statement *3*: If $q \in C(C(A, t), s)$ and $\epsilon > 0$, then $\exists r \in C(A, t)$ such that $d(q, r) < s + \epsilon$. Additionally, $\exists p \in A$ such that $d(p, r) < t + \epsilon$. Thus,

$$d(p, q) \leq d(p, r) + d(q, r) < s + t + 2\epsilon$$

and so $q \in C(A, t + s)$. □

*Remark.* Note that equality in Statement *3* is in fact false for general metric spaces. However, it is true in the case of interest in this thesis: when $(\Sigma, h)$ is a Riemannian manifold and $(\Sigma, d)$ is the induced metric space, via Theorem 4.4. This is shown in the following proposition, together with the result that the closure of the open ball is the closed ball, another statement which is false for general metric spaces.

**Proposition C.4.** *Let $(\Sigma, h)$ be a Riemannian manifold, $(\Sigma, d)$ the induced metric space, via Theorem 4.4, and $A \subseteq \Sigma$. Then:*

1. $C(C(A, t), s) = C(A, s + t)$ *for all* $s, t \geq 0$.
2. $\overline{B(p, t)} = C(p, t)$ *for all* $p \in \Sigma$ *and* $t \geq 0$..

*Proof.* We start by proving Statement *1*: The inclusion $\subseteq$ was already proven in Proposition C.3. For the converse: Let $q \in C(A, s + t)$, so $d(q, A) \leq s + t$. If $d(q, A) \leq t$ then $q \in LHS$. So, let w.l.o.g. $d(q, A) > t$. Pick $\gamma_n : [0, 1] \to \Sigma$ piecewise smooth such that $\gamma_n(0) = p_n \in A$, $\gamma_n(1) = q$ for all $n$ and

$$L(\gamma_n) \searrow d(q, A) \leq s + t.$$

We must have $L(\gamma_n) > t$ for all $n$. For each $n$, let $s_n \in [0, 1]$ such that $L(\gamma_n|_{[0, s_n]}) = t$ and let $q_n = \gamma_n(s_n) \in C(A, t)$. Then:

$$d(q_n, q) \leq L(\gamma_n|_{[s_n, 1]}) = L(\gamma_n) - L(\gamma_n|_{[0, s_n]}) = L(\gamma_n) - t \searrow d(q, A) - t \leq s + t - t = s.$$

Thus $d(C(A,t),q) \le s$ and so $q \in C(C(A,t),s)$.

To prove Statement *2*: The inclusion $\subseteq$ is clear. Conversely, let $q \in C(p,t)$. If $d(p,q) < t$ then we are done. So, let $d(p,q) = t$. Let $\gamma_n : [0,1] \to \Sigma$ piecewise smooth such that $\gamma_n(0) = p$, $\gamma_n(1) = q$ and:

$$L(\gamma_n) \searrow d(p,q) = t.$$

Pick $s_n \in [0,1]$ such that $L(\gamma_n|_{[0,s_n]}) = t(1 - \frac{1}{n})$ and set $q_n = \gamma_n(s_n)$. Thus $d(p,q_n) \le L(\gamma_n|_{[0,s_n]}) = t(1-\frac{1}{n}) < t$ and so $q_n \in B(p,t)$. Finally:

$$d(q,q_n) \le L(\gamma_n|_{[s_n,1]}) = L(\gamma_n) - L(\gamma_n|_{[0,s_n]}) = L(\gamma_n) - t + t/n \searrow 0.$$

So $q_n \to q$ and $q \in \overline{B(p,t)}$. □

**Proposition C.5.** *Let $X$ be a locally compact metric space. Given $K \subseteq X$ compact, then there exists $\epsilon > 0$ s.t. $C(K,\epsilon)$ is compact.*

*Proof.* As $X$ is locally compact, for each $p \in X$ choose $\epsilon_p > 0$ s.t. $C(p,\epsilon_p)$ is compact. Then since $\bigcup_{p\in K} B(p,\frac{\epsilon_p}{2})$ is an open cover of $K$. By compactness of $K$ there exists a finite collection of points $(p_n)_{1\le n\le N}$ in $K$ s.t. $\bigcup_{1\le n\le N} B(p_n,\frac{\epsilon_{p_n}}{2})$ is an open cover of $K$. Thus

$$K \subseteq \bigcup_{1\le n\le N} B\left(p_n,\frac{\epsilon_{p_n}}{2}\right) \subseteq \bigcup_{1\le n\le N} C\left(p_n,\frac{\epsilon_{p_n}}{2}\right).$$

Let $\epsilon = \min\{\epsilon_{p_n} \colon 1 \le n \le N\}$ and so

$$\begin{aligned} C\left(K,\frac{\epsilon}{2}\right) &\subseteq C\left(\bigcup_{1\le n\le N} C(p_n,\frac{\epsilon_{p_n}}{2}),\frac{\epsilon}{2}\right) \\ &\subseteq \bigcup_{1\le n\le N} C\left(C(p_n,\frac{\epsilon_{p_n}}{2}),\frac{\epsilon}{2}\right) \\ &\subseteq \bigcup_{1\le n\le N} C\left(C(p_n,\frac{\epsilon_{p_n}}{2}),\frac{\epsilon_{p_n}}{2}\right) \\ &\subseteq \bigcup_{1\le n\le N} C\left(p_n,\epsilon_{p_n}\right), \end{aligned}$$

where we have used all the properties proven in Proposition C.3. The RHS is compact as a finite union of compact sets and so the LHS is compact. □

Note that the previous proposition applies to the case of $(\Sigma, d)$ in Proposition C.4, as this is locally compact, as is any (finite dimensional) smooth manifold. We would like to prove that, if $K \subseteq \Sigma$ and $C(K,t)$ is compact then $C(K, t+\epsilon)$ is compact for small $\epsilon$. However this is false in general, even for locally compact metric spaces: for example let $X = \mathbb{R}\backslash(0,1]$ with the Euclidean metric and $K = \{0\}$. Then, $C(0,1)$ is compact but $C(0, 1+\epsilon)$ is never compact for any $\epsilon > 0$. However, let the (locally compact) metric space $(\Sigma, d)$ be induced by a Riemannian manifold $(\Sigma, h)$. Then, by the previous proposition, if $C(K,t)$ is compact then $C(C(K,t), \epsilon)$ is compact for small $\epsilon$. Then, by Proposition C.4, $C(K, t+\epsilon) = C(C(K,t), \epsilon)$ is compact, which we state in the following corollary:

**Corollary C.6.** *Let $(\Sigma, h)$ be a Riemannian manifold and $(\Sigma, d)$ the induced (locally compact) metric space. If $K \subseteq \Sigma$ and $C(K,t)$ is compact for some $t \geq 0$, then $C(K, t+\epsilon)$ is compact for sufficiently small $\epsilon > 0$.*

We apply this in the following useful proposition:

**Proposition C.7.** *Let $(\Sigma, h)$ be a Riemannian manifold and $(\Sigma, d)$ the induced metric space. Let $K$ be a compact set in $\Sigma$. If $C(p,t)$ is compact for all $p \in K$, then $C(K,t) = \bigcup_{p \in K} C(p,t)$ is compact in $\Sigma$.*

*Proof.* Let $q_n \in C(K,t)$. So, there exists a sequence $p_n \in K$ s.t. $d(p_n, q_n) \leq t$. By compactness, there exists a subsequence $p_{n_k} \to p \in K$. We can take w.l.o.g. $p_n \to p$. Then, for all $\epsilon > 0$ there exists $N |\ d(p, q_n) \leq t + \epsilon\ \forall n > N$. But $C(p,t)$ is compact implies that $C(p, t+\epsilon)$ is compact for some $\epsilon$, by the previous corollary. So $q_n \in C(p, t+\epsilon)$ for all $n > N$. Again, by compactness there exists a convergent subsequence $q_{n_k}$ s.t. $q_{n_k} \to q \in \Sigma$. By the closure of $C(K,t)$ proved in Proposition C.2, we have $q \in C(K,t)$. □

# D Partial Differential Operators on Manifolds

In this section we shall mainly deal with the topic of partial differential operators on manifolds, but we shall find it useful to first introduce smooth measures on manifolds as these will be needed in the later parts of this appendix. In Section D.2 we discuss the concept of partial differential operators (p.d.o.s) on manifolds and in particular define such terms as elliptic and formally self-adjoint with the view to checking that these properties do in fact hold for our p.d.o. $A$ first introduced on p.20 in Section 3. We will then define the $L^p$ spaces, distributions and Sobolev spaces in Section D.3, which are used in much of this thesis. We have tried to give a fuller account of this material than that found in the literature. In particular we have tried to use only global notation and definitions, motivated by the approach of Bär et al. [3] and Nicolaescu [22]. We shall not be including all proofs of the theory in this section for brevity, especially those already to be found in Nicolaescu [22] and Treves [36], but give full references.

## D.1 Smooth Measures on Smooth Manifolds

In thus section we wish to give an account of the construction of smooth measures on manifolds from volume elements (special types of densities), in particular from pseudo-Riemannian metrics. This is well known and covered in many books (e.g. Nicolaescu [22]), however one usually constructs a functional on the space of test-functions and then invokes Riesz Representation Theorem (Theorem 12.31 Driver [12]) to construct the measure if desired. However, for completeness we would like to give here a more self-contained and direct approach to constructing a measure.

We start by defining densities, not via giving the transition maps and invoking the Existence Theorem of Vector Bundles (Steenrod [33]), but by giving its explicit construction (p.107 Nicolaescu [22]). We shall then define volume elements, as certain types of densities, show how pseudo-Riemannian metrics generate volume elements and subsequently show how volume elements generate smooth measures.

We first define densities on an n-dimensional manifold. Given a smooth manifold $M$, let:

$$|\Lambda|_p = \left\{ \begin{array}{l} V_p : \Lambda^n(T_pM) \to \mathbb{R} \\ \text{s.t. } V_p(\lambda e_p) = |\lambda| V_p(e_p) \text{ for all } e_p \in \Lambda^n(T_pM) \end{array} \right\},$$

where $\Lambda^n(T_pM)$ is the n-th exterior product of the vector space $T_pM$. Thus $|\Lambda|_p$ is a 1-dimensional vector space. Now define:

$$|\Lambda| = \bigcup_{p \in M} (p, |\Lambda|_p).$$

A **density** is then a map

$$V : M \to |\Lambda|$$

$$p \mapsto V_p$$

$$\text{s.t. } V_p \in |\Lambda|_p \ \forall p \in M.$$

Note that if $\omega_p \in \Lambda^n T_p^* M$ then define $|\omega|_p \in |\Lambda|_p$ by: $|\omega_p|(e_p) = |\omega_p(e_p)|$, which makes sense since $\omega_p \in \Lambda^n T_p^* M \cong (\Lambda^n(T_pM))^*$.

If $(U_\alpha, \phi_\alpha)$ is an atlas then we can write $V$ locally as: $V = V_\alpha |dx_\alpha^1 \wedge ... \wedge dx_\alpha^n|$, where $x_\alpha^1, ..., x_\alpha^n$ are the coordinates of $\phi_\alpha$ and $V_\alpha : U_\alpha \to \mathbb{R}$. We then define a density to be smooth if $V_\alpha \in C^\infty(U_\alpha)$ for all $\alpha$.

Using these definitions we show here the transformation law for the coordinates of a density. Given local coordinates $x^1, ..., x^n$ and $y^1, ..., y^n$, then:

$$V = v|dx^1 \wedge ... \wedge dx^n| = v'|dy^1 \wedge ... \wedge dy^n|.$$

But as $dy^i = \frac{\partial y^i}{\partial x^j} dx^j$, then:

$$\begin{aligned} V &= v' \left| \frac{\partial y^1}{\partial x^{j_1}} dx^{j_1} \wedge \ldots \wedge \frac{\partial y^n}{\partial x^{j_n}} dx^{j_n} \right| \\ &= v' \left| \det \left( \frac{\partial y^i}{\partial x^j} \right) dx^1 \wedge \ldots \wedge dx^n \right| \\ &= v' \left| \det \left( \frac{\partial y^i}{\partial x^j} \right) \right| |dx^1 \wedge \ldots \wedge dx^n|, \end{aligned}$$

and so $v = \left| \det \left( \frac{\partial y^i}{\partial x^j} \right) \right| v'$ and $v' = \left| \det \left( \frac{\partial x^i}{\partial y^j} \right) \right| v$.

**Definition D.1.** *A* ***volume element*** *on a smooth manifold $M$ is a smooth density such that given an atlas $(U_\alpha, \phi_\alpha)$, then $v_\alpha > 0$ for all $\alpha$. Equivalently, a smooth density $V$ is a volume element if for all $p \in M$, $V_p : \Lambda^n(T_pM) \to (0, \infty)$.*

Note that a pseudo-Riemannian metric (and a symplectic form) generate a volume element. To see this note that the components of a metric transform as: $g'_{ij} = \frac{\partial x^l}{\partial y^i} \frac{\partial x^m}{\partial y^j} g_{lm}$. Thus:

$$\begin{aligned} \det g'_{ij} &= \det \left( \frac{\partial x^l}{\partial y^i} \frac{\partial x^m}{\partial y^j} g_{lm} \right) \\ &= \det \left( \frac{\partial x^i}{\partial y^j} \right)^2 \det(g_{ij}). \end{aligned}$$

So: $|\det g'_{ij}|^{1/2} = \left| \det \left( \frac{\partial x^i}{\partial y^j} \right) \right| |\det(g_{ij})|^{1/2}$ and $|\det(g_{ij})|^{1/2}$ transforms as the component of a density. Thus $V_g := |\det(g_{ij})|^{1/2} |dx^1 \wedge \ldots \wedge dx^n|$ is a (global) volume element on $M$. Since on every manifold can be defined a Riemannian metric then on every manifold there exists a volume element.

Before we state a theorem on the existence of a Borel measure w.r.t. any volume element $V$, we prove a statement regarding Borel sets on the manifold. Remember, that on any topological space $(M, \tau)$, the Borel $\sigma$-algebra $\sigma(\tau)$ is defined as the smallest $\sigma$-algebra containing all open sets in the topology $\tau$.

**Proposition D.2.** *Let $(M,\tau)$ be a topological space and $U_n$ be a countable open cover of $M$. Given $U \subseteq M$, then:*

1. *For all $n$, $\sigma(\tau_{U_n}) = \{U \cap U_n,\ U \in \sigma(\tau)\}$*
2. *$U \in \sigma(\tau)$ iff $U \cap U_n \in \sigma(\tau_{U_n})$ for all $n$,*

*where $\tau_{U_n}$ is the induced topology of $\tau$ on $U_n$.*

*Proof.* To prove *1*: By definition of the induced topology, we know that if $U \subseteq M$ then:

$$U \in \tau \text{ iff } U \cap U_n \in \tau_{U_n} \text{ for all } n.$$

We now prove that $U \in \sigma(\tau) \Rightarrow U \cap U_n \in \sigma(\tau_{U_n})$. This then proves $RHS \subseteq LHS$ of Statement *1* and $LHS \Rightarrow RHS$ of Statement *2*. To do this consider:

$$\Pi := \{U \subseteq M \text{ s.t. } U \cap U_n \in \sigma(\tau_{U_n}) \text{ for all } n\}.$$

We check that $\Pi$ is in fact a $\sigma$-algebra on $M$ that clearly contains $\tau$:

(i) $\phi, M \in \Pi$.

(ii) $U \in M \Rightarrow U \cap U_n \in \sigma(\tau_{U_n})$ and so $U^c \cap U_n = U_n \backslash (U \cap U_n) \in \sigma(\tau_{U_n})$ and so $U^c \in \Pi$.

(iii) $U_m \in \Pi \Rightarrow U_m \cap U_n \in \sigma(\tau_{U_n})$ for all $m, n$. Thus $(\bigcup_m U_m) \cap U_n = \bigcup_m (U_m \cap U_n) \in \sigma(\tau_{U_n})$ and so $\bigcup_m U_m \in \Pi$.

Thus, if $U \in \tau$ then $U \cap U_n \in \tau_{U_n} \subseteq \sigma(\tau_{U_n})$ for all $n$ and $U \in \Pi$. So we have $\tau \subseteq \Pi$, which implies $\sigma(\tau) \subseteq \Pi$, i.e.

$$U \in \sigma(\tau) \Rightarrow U \cap U_n \in \sigma(\tau_{U_n}) \text{ for all } n.$$

Now, let $\Pi_n := \{U \cap U_n :\ U \in \sigma(\tau)\}$. We shall check that for all $n$, $\Pi_n$ is a $\sigma$-algebra on $U_n$, which contains the topology $\tau_{U_n}$.

(i) $\pi, U_n \in \Pi_n$

(ii) $U \in \sigma(\tau) \Rightarrow U_n \backslash (U \cap U_n) = U^c \cap U_n$ where $U^c \in \sigma(\tau)$. So: $U_n \backslash (U \cap U_n) \in \Pi_n$.

(iii) $U_m \in \sigma(\tau) \Rightarrow (\bigcup_m U_m) \cap U_n = \bigcup_m (U_m \cap U_n) \in \Pi_n$.

In particular we have that $\tau_{U_n} \subseteq \Pi_n$ and so $\sigma(\tau_{U_n}) \subseteq \Pi_n$ for all $n$. Thus the remaining direction of *1* is proven. Now to prove that $LHS \Leftarrow RHS$ of Statement *2*, let $U \subseteq M$ and $U \cap U_n \in \sigma(\tau_{U_n})$ for all $n$. Then by *1* we have for each $n$: $U \cap U_n = V_n \cap U_n$ for $V_n \in \sigma(\tau)$. Then:

$$\begin{aligned} U &= \bigcup_{n \geq 1} (U \cap U_n) \\ &= \bigcup_{n \geq 1} (V_n \cap U_n) \\ &\in \sigma(\tau) \text{ (since } U_n \in \tau_U \subseteq \sigma(\tau_U)). \end{aligned}$$

□

The following definition is taken from p.332 of Folland [14], the definition of a smooth measure on a manifold.

**Definition D.3.** *A* ***smooth measure*** *on a manifold $M$ is a Borel measure $\mu$ such that on any chart $(U, \phi)$ on $M$, $d\mu = f \phi_*^{-1}(d\lambda)$, where $f \in C^\infty(U)$, $f > 0$ and $\phi_*^{-1}(d\lambda)$ is the push-forward of the Lebesgue measure $d\lambda$ to $U$ along $\phi^{-1}$.*

**Theorem D.4.** *Given a smooth manifold $M$ with a volume element $V$, then there exists a unique smooth measure $\mu$ on $M$ such that if $(U_n, \phi_n)$ is a countable atlas of $M$ then for all $n \in \mathbb{N}$:*

$$\mu|_{U_n} = v_n (\phi_n^{-1})_* (d\lambda),$$

*where $v_n \in C^\infty(U_n)$ is the component of $V$ in the local basis of densities induced by the chart $(U_n, \phi_n)$.*

*Furthermore, the following are true of $\mu$:*

1. *Given any locally finite atlas* $(U_n, \phi_n)$ *and any partition of unity* $\{f_n : f_n \in C^\infty(M)\}$ *subordinate to* $(U_n)$*, then for* $U \in \sigma(\tau)$*,* $\mu(U)$ *is given by:*

$$\mu(U) = \sum_n \int_{\phi_n(U \cap U_n)} f_n \circ \phi_n^{-1}.v_n \circ \phi_n^{-1} d\lambda$$

2. *The measure* $\mu$ *is Radon and regular.*

3. *The null sets w.r.t.* $\mu$ *are independent of the volume element* $V$*. Explicitly,* $N \subseteq M$ *is null iff there exists a countable atlas* $(U_n, \phi_n)$ *s.t. for each* $n$*:* $\phi_n(N \cap U_n)$ *is null w.r.t. the Lebesgue measure.*

4. *Every smooth measure on* $M$ *arises in this way from some volume element.*

Note that according to Statement *3*, the set of Lebesgue sets in $M$ w.r.t. $\mu$ (the completion of $\sigma(\tau)$ w.r.t. $\mu$) is independent of the volume element used to define $\mu$.

We begin with a Lemma:

**Lemma D.5.** *Given volume element* $V$*, charts* $(U_n, \phi_n)$ *and* $(U_m, \phi_m)$*,* $f \in C_0^\infty(M)$ *of compact support in* $U_n \cap U_m$ *and* $U \subseteq U_n \cap U_m$ *Borel, then:*

$$\int_{\phi_n(U)} f \circ \phi_n^{-1}.v_n \circ \phi_n^{-1} d\lambda = \int_{\phi_m(U)} f \circ \phi_m^{-1}.v_m \circ \phi_m^{-1} d\lambda$$

*Proof.*

$$\begin{aligned}
&\int_{\phi_n(U)} f \circ \phi_n^{-1}.v_n \circ \phi_n^{-1} d\lambda \\
&\qquad = \int_{\phi_n(U)} f \circ \phi_n^{-1}.v_n \circ \phi_n^{-1}.|\det D(\phi_m \circ \phi_n^{-1})|^{-1} |\det D(\phi_m \circ \phi_n^{-1})| d\lambda \\
&\qquad = \int_{\phi_n(U)} f \circ \phi_n^{-1}.v_m \circ \phi_n^{-1}.|\det D(\phi_m \circ \phi_n^{-1})| d\lambda \\
&\qquad = \int_{\phi_m(U)} f \circ \phi_m^{-1}.v_m \circ \phi_m^{-1} d\lambda
\end{aligned}$$

□

*Proof of Theorem D.4.* We first prove uniqueness. As $(U_n)$ is a countable cover of $M$ then define $V_n = \bigcup_{n=1}^{n} U_n$ and $W_1 = V_1$ and $W_n = V_n \backslash V_{n-1}$ for $n \geq 2$. So $(W_n)$ is a countable cover of $M$ by pairwise disjoint measurable sets. As $W_n \subseteq U_n$ so $\mu$ is uniquely determined on each $W_n$ and so also on $M$.

In order to prove the remaining properties we shall first find an alternative expression for our proposed Borel measure $\mu$. If $\mu$ is a Borel measure on $M$ satisfying the condition, then if $U \subseteq M$ is Borel, $(U_n, \phi_n)$ is a locally finite atlas and $\{f_n : \ f_n \in C^\infty(M)\}$ is a partition of unity subordinate to $(U_n)$, then:

$$\begin{aligned}
\mu(U) &= \int_M \mathbb{1}_U d\mu \\
&= \int_M (\sum_n f_n).\mathbb{1}_U d\mu \\
&= \sum_n \int_M f_n.\mathbb{1}_U d\mu \\
&= \sum_n \int_{U_n} f_n.\mathbb{1}_U d\mu \\
&= \sum_n \int_{U_n} f_n.\mathbb{1}_U v_n (\phi_n^{-1})_*(d\lambda) \\
&= \sum_n \int_{\phi_n(U_n)} f_n \circ \phi_n^{-1}.\mathbb{1}_U \circ \phi_n^{-1}.v_n \circ \phi_n^{-1} d\lambda \\
&= \sum_n \int_{\phi_n(U_n) \cap \phi_n(U)} f_n \circ \phi_n^{-1}.v_n \circ \phi_n^{-1} d\lambda \\
&= \sum_n \int_{\phi_n(U_n \cap U)} f_n \circ \phi_n^{-1}.v_n \circ \phi_n^{-1} d\lambda
\end{aligned}$$

We must show that if we take this final expression to define $\mu$, then it is indeed a measure. So we are defining, for $U \in \sigma(\tau)$:

$$\mu(U) = \sum_n \int_{\phi_n(U_n \cap U)} f_n \circ \phi_n^{-1}.v_n \circ \phi_n^{-1} d\lambda$$

If $U = \bigcup_m U_m$ is a disjoint union of Borel sets in $M$, then:

$$\begin{aligned}
\mu(U) &= \sum_n \int_{\phi_n(U_n \cap (\bigcup_m U_m))} f_n \circ \phi_n^{-1}.v_n \circ \phi_n^{-1} d\lambda \\
&= \sum_n \int_{\phi_n(\bigcup_m (U_n \cap U_m))} f_n \circ \phi_n^{-1}.v_n \circ \phi_n^{-1} d\lambda \\
&= \sum_n \int_{\bigcup_m (\phi_n(U_n \cap U_m))} f_n \circ \phi_n^{-1}.v_n \circ \phi_n^{-1} d\lambda \\
&= \sum_n \sum_m \int_{\phi_n(U_n \cap U_m)} f_n \circ \phi_n^{-1}.v_n \circ \phi_n^{-1} d\lambda \\
&= \sum_m \sum_n \int_{\phi_n(U_n \cap U_m)} f_n \circ \phi_n^{-1}.v_n \circ \phi_n^{-1} d\lambda \\
&= \sum_m \mu(U_m)
\end{aligned}$$

So $\mu$ is countably additive. Clearly we also have: $\mu(\phi) = \phi$. Thus $\mu$ is a Borel measure on $M$. If $U \subseteq U_m$ then:

$$\begin{aligned}
\mu(U) &= \sum_n \int_{\phi_n(U_n \cap U)} f_n \circ \phi_n^{-1}.v_n \circ \phi_n^{-1} d\lambda \\
&= \sum_n \int_{\phi_m(U_n \cap U)} f_n \circ \phi_m^{-1}.v_m \circ \phi_m^{-1} d\lambda \text{ (using the Lemma)} \\
&= \sum_n \int_{\phi_m(U)} f_n \circ \phi_m^{-1}.v_m \circ \phi_m^{-1} d\lambda \\
&= \int_{\phi_m(U)} \sum_n f_n \circ \phi_m^{-1}.v_m \circ \phi_m^{-1} d\lambda \\
&= \int_{\phi_m(U)} v_m \circ \phi_m^{-1} d\lambda
\end{aligned}$$

Thus the main statement of the theorem and Statement *1* are proven. Now to prove the remaining properties of $\mu$.

To prove Statement *2*, take a compact subset $K$ of $M$. We will show that $\mu(K) < \infty$. Firstly, given any countable locally finite atlas $(U_n, \phi_n)$ then as $K$ is compact then $K$ is covered by $U_{n_1}, ..., U_{n_N}$. Pick any partition of unity $\{f_k : k = 1...N\}$ of the submanifold $\bigcup_k U_{n_k}$ subordinate to $(U_{n_k})$.

So $f_k \in C^\infty(M)$ with $\operatorname{supp} f_k \subseteq U_{n_k}$. Similarly to the previous parts of the theorem, then $\mu(K)$ is given by:

$$\begin{aligned}\mu(K) &= \sum_{k=1}^{N} \int_{\phi_{n_k}(U_{n_k} \cap K)} f_k \circ \phi_{n_k}^{-1}.V_{n_k} \circ \phi_{n_k}^{-1} d\lambda \\ &= \sum_{k=1}^{N} \int_{\phi_{n_k}(U_{n_k}) \cap \phi_{n_k}(K \cap \operatorname{supp}(f_{n_k}))} f_k \circ \phi_{n_k}^{-1}.V_{n_k} \circ \phi_{n_k}^{-1} d\lambda \\ &< \infty\end{aligned}$$

as each term is finite since $\phi_{n_k}(K \cap \operatorname{supp}(f_{n_k}))$ is compact in $\phi_{n_k}(U_{n_k})$ and $f_k \circ \phi_{n_k}^{-1}.V_{n_k} \circ \phi_{n_k}^{-1} d\lambda$ is a Radon measure since $d\lambda$ is and all functions are smooth. Thus $\mu$ is Radon. That all Radon measures on a locally compact, Hausdorff, second countable topological space are regular is a consequence of Theorem 12.32 in Driver [12].

Property *3* regarding null sets follows easily from the above description of $\mu$.

Property *4* follows from the Change of Variables Theorem (Theorem 21.1 of Driver [12]).

□

Thus according to the previous theorem, the set of smooth measures on a manifold $M$ can be identified with the set of volume elements on $M$. As an aside, note that since the set of densities is the set of sections of a vector bundle, so it can be given the structure of a Fréchet space. The set of volume elements is then given the induced topology and the set of smooth measures on a manifold will thus inherit a topology from the set of volume elements. We conclude this section by defining some notation which is used in much of the thesis.

**Definition D.6.** *If $g$ is a pseudo-Riemannian metric on a manifold $M$, we know that $V_g = |\det(g_{ij})|^{1/2}|dx^1 \wedge ... \wedge dx^n|$ is a volume element on $M$. Using Theorem D.4, define* $\mathbf{d}\boldsymbol{vol}_{\mathbf{g}}$ *as the Borel measure on $M$ generated by $V_g$.*

Note that most authors do not distinguish in notation between volume elements and smooth measures. We do so here for clarity. In the next section we shall define partial differential operators (p.d.o.s) and analyse the p.d.o. $A$ used throughout this thesis.

## D.2 Partial Differential Operators and an Analysis of $A$

In this appendix we introduce the concept of a partial differential operator (p.d.o.) on a smooth manifold $M$, following Chapter 10 of Nicolaescu [22]. Given two vector bundles $E$ and $F$ over $M$, define

$$\mathbf{Op}(\mathbf{E}, \mathbf{F}) := L(\Gamma(E), \Gamma(F)),$$

that is, the space of linear maps from the vector space of smooth sections of the vector bundle $E$ to that of the vector bundle $F$.

We shall now define the set of partial differential operators of order at most n, $\mathbf{PDO}^{(\mathbf{n})}(\mathbf{E}, \mathbf{F})$. Firstly, define $PDO^0(E, F) = \Gamma(Hom(E, F))$, where $Hom(E, F)$ is the homomorphism bundle between $E$ and $F$. Remember that $Hom(E, F)$ is a vector bundle over $M$ whose fibre at $x \in M$ is: $Hom(E, F)_x = L(E_x, F_x)$. Note that if $T \in \Gamma(Hom(E, F))$ and $f \in \Gamma(E)$ then define $Tf \in \Gamma(F)$ by: $(Tf)(x) = T(x)f(x)$. Thus we view $T \in L(\Gamma(E), \Gamma(F)) = Op(E, F)$ and $\Gamma(Hom(E, F)) \subseteq Op(E, F)$.

Given $f \in C^\infty(M)$, define $ad(f)$: $Op(E, F) \to Op(E, F)$ by $ad(f)(T) =$

$T \circ f - f \circ T$ and then set:

$PDO^{(n)}(E,F)$

$= ker\ ad^{n+1}$

$= \{T \in Op(E,F)\colon ad(f_1)ad(f_2)...ad(f_{n+1})T = 0 \text{ for all } f_1,...f_{n+1} \in C^\infty(M)\}$

We define

$$\mathbf{PDO^n(E,F)} := PDO^{(n)}(E,F) \backslash PDO^{(n-1)}(E,F),$$

the set of partial differential operators between $E$ and $F$ of order $n$.

In order to define the concept of an elliptic p.d.o. we first introduce the principal symbol of a partial differential operator. It is covered in full detail on p.430 of Nicolaescu [22]. If $P \in PDO^{(n)}(E,F)$ then for $f_1,...,f_n \in C^\infty(M)$ by definition $ad(f_1)ad(f_2)...ad(f_n)P \in PDO^0(E,F) = \Gamma(Hom(E,F))$. It can be shown that for $x_0 \in M$, the linear map $ad(f_1)ad(f_2)...ad(f_n)P|_{x_0} \in L(E_{x_0},F_{x_0})$ depends only on the values $df_i(x_0)$ of the functions $f_i$. Using the fact that $[ad(f),ad(g)] = 0$ for all $f,g \in C^\infty(M)$, then we have a symmetric multilinear map:

$$\sigma(P)(x_0)\colon T^*_{x_0}M \times ... \times T^*_{x_0}M \to L(E_{x_0},F_{x_0})$$
$$(\xi_1,...,\xi_n) \mapsto \frac{1}{n!} ad(f_1)ad(f_2)...ad(f_n)P|_{x_0},$$

where $df_i(x_0) = \xi_i$. As shown on p.312 of Nicolaescu [22], as for any symmetric multilinear map, $\sigma(P)(x_0)$ is completely determined by its values on $\{(\xi,...,\xi)\colon \xi \in T^*_{x_0}M\}$.

Define $\sigma_n(P)(x_0,\xi) = \sigma(P)(x_0)(\xi,...,\xi)$. Then for each $(x_0,\xi) \in T^*M$ and $\lambda \in \mathbb{R}$:

$$\sigma_n(P)(x_0,\xi) \in L(E_{x_0},F_{x_0})$$
$$\sigma_n(P)(x_0,\lambda\xi) = \lambda^n \sigma_n(P)(x_0,\xi)$$

In fact,

$$\sigma_n(P) \in \Gamma(\pi^* Hom(E,F)) \cong \Gamma(Hom(\pi^* E, \pi^* F)) = PDO^0(\pi^* E, \pi^* F),$$

where $\pi^*(E)$ of a vector bundle $E$ over $M$ is the pullback bundle (also called the induced bundle) along the bundle map $\pi : T^*M \to M$, that is, it is a vector bundle with base space $T^*M$ and its fibre at $(x_0, \xi) \in T^*M$ is $E_{x_0}$. Additionally "$\cong$" denotes a vector space isomorphism between the respective spaces of sections between isomorphic bundles (the induced bundle is constructed on p.47 of Steenrod [33]).

It is important to note that if $E, F, G$ are all vector bundles over the same manifold $M$ and $P \in PDO^{(m)}(F,G)$ and $Q \in PDO^{(n)}(E,F)$ then $P \circ Q \in PDO^{(n+m)}(E,G)$ (see p.428 of Nicolaescu [22]). With this notation then we also have: $\sigma_{m+n}(P \circ Q) = \sigma_m(P) \circ \sigma_n(Q)$.

We shall now consider an alternative definition of $PDO^n(E,F)$ and define an elliptic p.d.o..

Recalling that $P \in PDO^{(n)}(E,F)$ then $P \in PDO^n(E,F)$ iff $\exists f_1, .., f_n \in C^\infty(M)$ s.t. $ad(f_1)..ad(f_n)P \neq 0$. Since $\sigma(P)$ is reconstructible from $\sigma_n(P)$ then if one is zero then so is the other. Thus $P \in PDO^n(E,F)$ iff $\sigma_n(P) \neq 0 \in \Gamma(Hom(\pi^* E, \pi^* F))$.

**Definition D.7.** *The operator $P \in PDO^n(E,F)$ is said to be **elliptic** if $\sigma_n(P)(x_0,\xi)\colon E_{x_0} \to F_{x_0}$ is a linear isomorphism for all $x_0 \in M$ and $\xi \in T_{x_0}M \backslash \{0\}$.*

*Remark.* Clearly if $P \in PDO(E,F)$ is an elliptic operator then $rank(E) = rank(F)$.

We shall now introduce some common partial differential operators between certain bundles, consider their principal symbols, recall the partial differential operator $A$ of interest to us in this thesis and show that it is of second order and elliptic.

Given a pseudo-Riemannian manifold $(M, g)$ (called semi-Riemannian by O'Neill [23]), we define the partial differential operator $\mathrm{div}_g \in PDO^1(TM, M)$. Here we denote by $M$ the trivial bundle and by $TM$ the tangent bundle. For the vector field $X$ on $M$:

$$\mathrm{div}_g(X) := trace(\nabla X).$$

Note that $\nabla$ is the covariant derivative induced by $g$ and so $\nabla X$ is a rank $(1,1)$ tensor. In fact $\mathrm{div}_g$ is of order 1. Thus:

$$\mathrm{div}_g\colon \Gamma(TM) \to C^\infty(M).$$

We will show that $\mathrm{div}_g \in PDO^1(TM, \mathbb{R} \times M)$. It is easily shown that $\mathrm{div}_g(fX) = f\mathrm{div}_g X + X(f)$. From this it follows:

$$\mathrm{div}_g(f_1 f_2 X) = f_1 \mathrm{div}_g(f_2 X) + f_2 \mathrm{div}_g(f_1 X) - f_1 f_2 \mathrm{div}_g X$$

and hence $ad(f_1)ad(f_2)(\mathrm{div}_g) = 0$ for all $f_1, f_2 \in C^\infty(M)$.
Thus $\mathrm{div}_g \in PDO^{(1)}(TM, \mathbb{R} \times M)$.

To calculate the principal symbol of $\mathrm{div}_g$ let $f \in C^\infty$ with $df(x_0) = \xi \neq 0$. Then:

$$\begin{aligned}
\sigma_1(\mathrm{div}_g)(x_0, \xi) &= [ad(f)\mathrm{div}_g]|_{x_0} \\
&= [\mathrm{div}_g \circ f - f \circ \mathrm{div}_g]|_{x_0} \\
&= df|_{x_0} \\
&= \xi\colon T_{x_0} \to \mathbb{R} \\
&\neq 0
\end{aligned}$$

Thus $\mathrm{div}_g \in PDO^1(TM, \mathbb{R} \times M)$.

We also define the p.d.o. $\mathrm{grad}_g \in PDO^1(\mathbb{R} \times M, TM)$, also of order 1. Given $f \in C^\infty(M)$:

$$\mathrm{grad}_g f = df|^\#,$$

where $^\#\colon \Gamma(T^*M) \to \Gamma(TM)$ is the "index-raising" map induced by the metric $g$. Thus:

$$\mathrm{grad}_g\colon C^\infty(M) \to \Gamma(TM)$$

i.e. $\mathrm{grad}_g \in Op(\mathbb{R}\times M, TM)$. We now show that $\mathrm{grad}_g \in PDO^{(1)}(\mathbb{R}\times M, TM)$. It is easy to show that $\mathrm{grad}_g(f_1 f_2) = f_1 \mathrm{grad}_g(f_2) + f_2 \mathrm{grad}_g(f_1)$, from which it follows that

$$f_1 f_2 \mathrm{grad}_g f_3 - f_1 \mathrm{grad}_g f_2 f_3 - f_2 \mathrm{grad}_g f_1 f_3 + \mathrm{grad}_g f_1 f_2 f_3 = 0$$

for all $f_3 \in C^\infty(M)$. Thus $\mathrm{grad}_g$ is a p.d.o. of order at most 1. The principal symbol of $\mathrm{grad}_g$ is:

$$\begin{aligned}
\sigma_1(\mathrm{grad}_g)(x_0, \xi) &= [ad(f)\mathrm{grad}_g]|_{x_0} \\
&= [\mathrm{grad}_g \circ f - f \mathrm{grad}_g]|_{x_0} \\
&= \mathrm{grad}_g f(x_0) \\
&= \xi^\# \\
&\neq 0,
\end{aligned}$$

where $f \in C^\infty(M)$ satisfies $df(x_0) = \xi \neq 0$. Thus $\mathrm{grad}_g \in PDO^1(\mathbb{R} \times M, TM)$.

Recall that the Laplace-Beltrami operator is the composition of these two operators:

$$\Box_g = \mathrm{div}_g \circ \mathrm{grad}_g\colon C^\infty(M) \to C^\infty(M).$$

We have $\Box_g \in PDO^{(2)}(\mathbb{R} \times M, \mathbb{R} \times M)$, where $\mathbb{R} \times M$ is the trivial real vector bundle over $M$ of rank 1.

*Remark.* Note that, for the sake of definiteness, one can let $C^\infty(M) := C^\infty(M, \mathbb{R})$, the space of real-valued smooth functions on $M$. We can also define $\Box_g$ to act on $C^\infty(M, \mathbb{C})$, instead. One can either define it to act on real and imaginary parts separately, or, in line with the above treatment, one can first define $\operatorname{grad}_g \in PDO^1(\mathbb{C} \times M, TM_{\mathbb{C}})$ and $\operatorname{div}_g \in PDO^1(TM_{\mathbb{C}}, \mathbb{C} \times M$, where $\mathbb{C} \times M$ is the trivial complex vector bundle over $M$ and $TM_{\mathbb{C}}$ is the complexified tangent bundle. Then $\Box_g = \operatorname{div}_g \circ \operatorname{grad}_g : C^\infty(M, \mathbb{C}) \to C^\infty(M, \mathbb{C})$.

In this thesis, of fundamental importance is the partial differential operator:

$$A = -V \operatorname{div}_h V \operatorname{grad}_h + m^2 V^2 \colon C^\infty(\Sigma) \to C^\infty(\Sigma),$$

where $(\Sigma, h)$ is a Riemannian manifold and $V \in C^\infty(\Sigma, V > 0$.

Here we check that the p.d.o. $A = -V \operatorname{div}_h V \operatorname{grad}_h + m^2 V^2$ on a Riemannian manifold $(\Sigma, h)$ used in this thesis is of second order, elliptic and formally self-adjoint w.r.t. $V^{-1} d\mathrm{vol}_h$. The consequences of this last property is that the linear operator $A$ generated by this p.d.o. with the domain $D(A) = [C_0^\infty(\Sigma)]$ on the Hilbert space $L^2(\Sigma, V^{-1} d\mathrm{vol}_h)$ is symmetric.

First, by composition of p.d.o.s, again we know that $A \in PDO^{(2)}(\Sigma, \Sigma)$. To show that $A$ is in fact of second order, we calculate its principal symbol $\sigma_2(A)$. Given $f \in C^\infty(\Sigma)$ with $df(x_0) = \xi \neq 0$, then:

$$\begin{aligned}
\sigma_2(A)(x_0, \xi) &= -\sigma_0(V)(x_0, \xi) \circ \sigma_1(\operatorname{div}_h)(x_0, \xi) \circ \sigma_0(V)(x_0, \xi) \circ \sigma_1(\operatorname{grad}_h)(x_0, \xi) \\
&= -V(x_0) \xi V(x_0) \xi^\# \\
&= -V(x_0)^2 \xi(\xi^\#) \\
&= -V(x_0)^2 \|\xi\|_h^2 \colon \mathbb{R} \to \mathbb{R} \\
&\neq 0,
\end{aligned}$$

where $|\cdot|_h$ is the inner product induced on each cotangent space $T_{x_0}\Sigma$ by the Riemannian metric $h$. Since $V > 0$ and as $h$ is Riemannian then $\xi \neq 0 \Rightarrow \|\xi\|_h \neq 0$. Thus $A$ is an elliptic partial differential operator of order 2.

We shall now show that the p.d.o. $A$ is also formally self-adjoint with respect to the smooth measure $V^{-1}d\text{vol}_h$. We shall define here what we mean by this (Nicolaescu [22]) but first give the definition of a (Riemannian or Hermitian) metric on a (real or complex) vector bundle.

**Definition D.8.** *(see e.g. p.167 of Bär et al. [3]) Given a real vector bundle $E$ over $M$, a* ***Riemannian metric*** *on $E$ is a smooth choice of inner product $\langle\cdot,\cdot\rangle_x$ (on a real vector space this is a positive definite symmetric bilinear form) on each fibre $E_x$ for $x \in M$. The choice is smooth if for all smooth sections $u, v \in \Gamma(E)$, the map $x \mapsto \langle u(x), v(x)\rangle_x$ is smooth. If $E$ is a complex vector bundle then a* ***Hermitian metric*** *on $E$ is a smooth choice of inner product $\langle\cdot,\cdot\rangle_x$ on $E_x$ for all $x$ (on a complex vector space this means a positive definite conjugate-symmetric sesquilinear form). The condition of smoothness is as before.*

Given a partial differential operator $P \in PDO(E,F)$, between vector bundles $E$ and $F$ (both with Riemannian metrics if real and Hermitian metrics if complex vector bundles) over a manifold $M$ with smooth measure $\mu$, then the **formal adjoint** $P^*$ of $P$ w.r.t. $\mu$ is a partial differential operator $P^* \in PDO(F,E)$, defined uniquely by:

$$\int_M \langle\psi, P\phi\rangle_F d\mu = \int_M \langle P^*\psi, \phi\rangle_E d\mu$$

for all $\phi \in \Gamma(E)$ and $\psi \in \Gamma(F)$ s.t. $\operatorname{supp}\phi \cap \operatorname{supp}\psi$ is compact in $M$.

Any p.d.o. $P \in PDO(E,F)$ between vector bundles $E$ and $F$ over $M$ (with Hermitian or Riemannian metrics) has, with respect to any smooth measure

$\mu$ on $M$, a unique formal adjoint $P^* \in PDO(F,E)$. The proof of this for the case when $\mu = d\text{vol}_g$ and $M$ is oriented is the content of Proposition 10.1.30 in Nicolaescu [22]. His proof generalises to the case of nonorientable manifolds. Lastly, fixing a Riemannian metric $g$, any smooth measure $\mu$ can be written $\mu = f d\text{vol}_g$ for some smooth function $f > 0$. Denoting $P_g^*$ the formal adjoint of $P$ w.r.t. $g$ then it is shown: $P^* = f^{-1} P_g^* \circ f$. Note that the formal adjoint $P^*$ does depend on the choice of smooth measure $\mu$ although for brevity we are omitting it from the notation.

If $P$ is a p.d.o. from the vector bundle $E$ (with Riemannian or Hermitian metric) to itself (i.e. $P \in PDO(E,E)$), then $P$ is called formally self-adjoint w.r.t. the smooth measure $\mu$ on $M$ if $E = E^*$. In order to show that the p.d.o. $A$ is in fact formally self-adjoint w.r.t. $V^{-1} d\text{vol}_h$ we first need the following:

**Proposition D.9.** *Let $(\Sigma, h)$ be a (not necessarily orientable) Riemannian manifold. Denote by $dvol_h$ the smooth measure on $\Sigma$ generated by the metric $h$ (Section D.1). Then the following holds:*

$$\int_\Sigma df(X) dvol_h = - \int_\Sigma f\, div_h(X) dvol_h$$

*for all $f \in C_0^\infty(\Sigma)$ and all $X \in \Gamma_0(TM)$, the space of compactly supported smooth vector fields on $M$.*

*Proof.* This results from an application of Gauss Theorem (Theorem E.4) to the Riemannian case (here the manifold has no boundary: $\partial M = \phi$) and $df(X) = \text{div}_h(fX) - f\text{div}_h X$. □

The following proposition is therefore sufficient to show that $A$ is formally self-adjoint w.r.t. $V^{-1} d\text{vol}_h$.

**Proposition D.10.** *Let* $A = -V\,div_h V\,grad_h + m^2V^2$*, where* $(\Sigma, h)$ *is a Riemannian manifold and* $V \in C^\infty(\Sigma)$*,* $V > 0$*. Then:*

$$\int_\Sigma (Af) g V^{-1} dvol_h = \int_\Sigma f(Ag) V^{-1} dvol_h$$

*for all* $f, g \in C_0^\infty(\Sigma)$*.*

*Proof.*

$$\begin{aligned}
\int_\Sigma (Af) g V^{-1} d\text{vol}_h &= \int_\Sigma -V \text{div}_h(V \text{grad}_h f) g V^{-1} d\text{vol}_h + \int_\Sigma m^2V^2 fgV^{-1} d\text{vol}_h \\
&= \int_\Sigma dg(V \text{grad}_h f) d\text{vol}_h + \int_\Sigma m^2 V fg d\text{vol}_h \\
&= \int_\Sigma V dg(\text{grad}_h f) d\text{vol}_h + \int_\Sigma m^2 V fg d\text{vol}_h \\
&= \int_\Sigma V g(df, dg) d\text{vol}_h + \int_\Sigma m^2 V fg d\text{vol}_h \\
&= \int_\Sigma V df(\text{grad}_h g) d\text{vol}_h + \int_\Sigma m^2 V fg d\text{vol}_h \\
&= \int_\Sigma df(V \text{grad}_h g) d\text{vol}_h + \int_\Sigma m^2 V fg d\text{vol}_h \\
&= \int_\Sigma -f \text{div}_h(V \text{grad}_h g) d\text{vol}_h + \int_\Sigma m^2 V fg d\text{vol}_h \\
&= \int_\Sigma f(Ag) V^{-1} d\text{vol}_h.
\end{aligned}$$

□

Thus the linear operator $A$ on the real Hilbert space $L^2(\Sigma, V^{-1}d\text{vol}_h)$ as defined in Section 3 is symmetric, that is:

$$\begin{aligned}
D(A) &= [C_0^\infty(\Sigma)] \\
A[\phi] &= [(-VD^iVD_i + m^2V^2)\phi].
\end{aligned}$$

Additionally, consider now the case of the linear operator $A$ on the complex Hilbert space $L^2(\Sigma, V^{-1}d\text{vol}_h)$ (the space of equivalence classes of complex-valued square-integrable measurable functions). Since the partial differential operator $A$ commutes with complex conjugation, that is: $A\overline{f} = \overline{Af}$ for $f \in C^\infty(\Sigma)$ smooth and complex valued, then:

$$\int_\Sigma \overline{(Af)} g V^{-1} d\mathrm{vol}_h = \int_\Sigma \overline{f}(Ag) V^{-1} d\mathrm{vol}_h$$

for all $f, g \in C_0^\infty(\Sigma)$ complex-valued. Therefore, the linear operator $A$ now defined on the complex Hilbert space is also symmetric.

## D.3 Definitions of $L^p$ spaces, Distributions and Sobolev spaces on Manifolds

In this section, given any manifold $M$ with smooth measure $\mu$ and a vector bundle $\pi : E \to M$ (with Riemannian metric if $E$ is real and a Hermitian metric if a complex vector bundle), we define the spaces $L^p(M, E, \mu)$ and the Sobolev spaces $W^{k,p}(M, \mu)$. In order to give our preferred definitions we refer to various constructions found in Nicolaescu [22], Bär et el. [3] and Treves [36]. We adopt the approach of [3] in defining the spaces of distributions, and apply this method to define the Sobolev spaces, globally and not via taking the abstract completion of a normed vector space (c.f. Hebey [16]). Thus we aim here to give a more intrinsic definition of the Sobolev spaces than that found in Hebey [16]. Our treatment differs from Nicolaescu [22], who in Section 10.2.4 defines Sobolev spaces on oriented Riemannian manifolds. The manifolds we shall consider here may be nonorientable and have a both a smooth measure (Section D.1) and Riemannian metric defined on them (they are not necessary related).

We remind the reader that any pseudo-Riemannian metric $g$ defines a smooth measure $\mathrm{vol}_g$. This in turn defines a regular Borel measure on the manifold $M$, also often denoted $\mathrm{vol}_g$ (see Section D.1). Before we define the $L^p$ spaces we define the space of Borel measurable sections of the vector bundle $E$:

$$\Gamma_{Bor}(E) = \{u : M \to E \text{ Borel measurable s.t. } u(x) \in E_x \text{ a.e.}\},$$

where $E_x = \pi^{-1}(x)$ is the fibre of $x \in M$ and $\pi : E \to M$ is the bundle map. Note that here "a.e." refers to any smooth measure on $M$ but is independent of which smooth measure we take. Given a manifold $M$ with a smooth measure $\mu$ and a vector bundle $\pi : E \to M$ with a Riemannian metric (or with Hermitian metric), then define the following spaces:

$$L^p(M, E, \mu) = \left\{ u \in \Gamma_{Bor}(E) \text{ s.t. } \int_M \|u(x)\|^p_{E_x} d\mu(x) < \infty \right\} \Big/ \sim$$

$$L^\infty(M, E) = \{u \in \Gamma_{Bor}(E) \text{ s.t. } \exists C \in \mathbb{R} \text{ with } \|u(x)\|_{E_x} < C \text{ a.e. } \} / \sim$$

$$L^p_{loc}(M, E) = \left\{ u \in \Gamma_{Bor}(E) \text{ s.t. } \forall K \subseteq_c M\text{: } \int_K \|u(x)\|^p_{E_x} d\mu(x) < \infty \right\} \Big/ \sim$$

$$L^\infty_{loc}(M, E) = \left\{ \begin{array}{l} u \in \Gamma_{Bor}(E) \text{ s.t. } \forall K \subseteq_c M \ \exists C \in \mathbb{R} \text{ s.t.:} \\ \|u(x)\|_{E_x} < C \text{ a.e. in } K \end{array} \right\} \Big/ \sim,$$

Note the following:

1. The expression $K \subseteq_c M$ denotes that $K$ is a compact subset of $M$.
2. $\|\cdot\|_{E_x}$ denotes the norm on the fibre $E_x$ defined by the Riemannian metric on the vector bundle $E$.
3. In each case, $\sim$ denotes the equivalence relation on $\Gamma_{Bor}(E)$: $u \sim v$ iff $u = v$ a.e.. Thus each of the spaces just defined consists of equivalence classes of Borel measurable sections of $E$.
4. If we set $E$ to be the trivial bundle $\mathbb{R} \times M$ then the above defines $L^p(M, \mu)$ etc..
5. Of the above spaces only $L^p(M, E, \mu)$ depends on the choice of smooth measure $\mu$ on $M$ as the notation might suggest. This follows from the

following lemma. In spite of this, we shall sometimes include a smooth measure in the expressions for the remaining spaces, e.g. $L^\infty_{loc}(M, E, \mu)$, when we are dealing with a particular choice of smooth measure $\mu$ on $M$.

6. For $p \in [1, \infty]$, $L^p(M, E, \mu)$ is a Banach space while $L^p_{loc}(M, E)$ is a Fréchet space.

**Lemma D.11.** *Let $\mu$ and $\lambda$ be smooth measures on a smooth manifold $M$. If $K \subseteq_c M$, then there exists $A, B > 0$ s.t. $A\mu \leq \lambda \leq B\mu$ on $K$.*

*Proof.* Let $h$ be a Riemannian metric on $M$. Let $\mu = f\ d\text{vol}_h$ and $\lambda = g\ d\text{vol}_h$. So $f, g > 0$ are smooth functions on $M$. Let $A = \min_K \frac{g}{f}$ and $B = \max_K \frac{g}{f}$. So,

$$A\mu = \min_K \left(\frac{g}{f}\right) f\ d\text{vol}_h = \min_K (g)\ d\text{vol}_h \leq g\ d\text{vol}_h = \lambda$$

and similarly for the remaining inequality. □

Before we define the Sobolev spaces, we first describe some necessary constructions from Bär et al. ([3] Section 1.1.2). We start with a different but related definition of the formal adjoint of a p.d.o. from that given in Section D.2. Given a partial differential operator $P \in PDO(E, F)$ between vector bundles $E$ and $F$ over a manifold $M$ with smooth measure $\mu$, then the **formal adjoint** $P^*$ of $P$ w.r.t. $\mu$ is a partial differential operator $P^* \in PDO(F^*, E^*)$, defined uniquely by:

$$\int_M \psi[P\phi] d\mu = \int_M (P^*\psi)[\phi] d\mu$$

for all $\phi \in \Gamma_0(E)$ and $\psi \in \Gamma_0(F^*)$.

The proof of existence is based on the proof of the corresponding construction in Section D.2. Equip the vector bundles $E$ and $F$ with Hermitian metrics and denote $I_E : \Gamma(E) \to \Gamma(E^*)$ and $I_F : \Gamma(F) \to \Gamma(F^*)$ be the induced isomorphisms. Note $I_E \in PDO^0(E, E^*)$ and $I_F \in PDO^0(F, F^*)$. It's then shown that, denoting $P^*_H$ the formal adjoint of $P$ with respect to the metrics just defined using the definition of Section D.2, $P^* = I_E \circ P^*_H \circ I_F^{-1}$ is a formal adjoint of $P$. Uniqueness follows then by showing that if $\theta \in \Gamma_0(E^*)$ and $\int_M \theta[\phi] d\mu = 0$ for all $\phi \in \Gamma_0(E)$ then $\theta = 0$, which is proven by assuming the contrary, picking a local frame of $E$, and corresponding coordinates of $\theta$, choosing an appropriate $\phi$ using a bump function and reaching a contradiction.

We also need to define the spaces of distributions over a vector bundle $E$. As ever, the analogy should always be with the construction of distributions on $\mathbb{R}$, where they are defined as the continuous dual to the space of test functions. Thus we must first define the space of compactly supported smooth sections of the bundle $E$ denoted:

$$\Gamma_0(E) = \{u \in \Gamma(E) \text{ s.t. } \operatorname{supp} u \subseteq_c M\},$$

where $\operatorname{supp} u = \overline{\{x \in M : u(x) \neq 0 \in E_x\}}$. We define a Fréchet topology on $\Gamma(E)$ (see Section 1.1.1 Bär et al. [3]). $\Gamma_0(E)$, the space of smooth sections of $E$ of compact support is given a LF topology (see Chapter 13 in Treves [36]). The dual to a LF space is a locally convex topological vector space, when given either the strong or the weak topology.

We then define the **space of distributions in** $E$, denoted $D'(M, E)$, as the continuous dual to $\Gamma_0(E^*)$:

$$D'(M, E) := \Gamma_0(E^*)'$$

The reason for the perhaps surprising occurrence of the dual bundle $E^*$ in the definition is to ensure that if we pick out a smooth measure $\mu$ on $M$, then we have the continuous linear injection $i : \Gamma(E) \to D'(M,E)$, given by $(i\phi)(\theta) = \int_M \theta(\phi)d\mu$ for $\phi \in \Gamma(E)$ and $\theta \in \Gamma_0(E^*)$.

The definitions of a p.d.o. guarantee that a p.d.o. is "local" (see Lemma 10.1.3 in Nicolaescu [22]) and the map $P : \Gamma(E) \to \Gamma(F)$ is linear and continuous, where $\Gamma(E)$ and $\Gamma(F)$ are given the Fréchet topologies previously described. Restricted to compact sections we have: $P : \Gamma_0(E) \to \Gamma_0(F)$ is then also continuous, where this time both spaces are LF spaces.

Using this we extend (using the Corollary following Proposition 19.5 in Treves [36]) this map to the continuous linear operator $P: D'(M,E) \to D'(M,F)$ by:

$$P(\Psi)(\phi) = \Psi(P^*\phi),$$

for all $\Psi \in D'(M,E)$ and $\phi \in \Gamma_0(F)$. In other words, we extend a p.d.o. $P$ by taking the transpose (in the sense of the transpose of a continuous linear map between Fréchet spaces, e.g. Treves [36]) of the p.d.o. $P^* \in PDO(F^*,E^*)$ (the formal adjoint p.d.o. of $P$). We apply this construction to the k-th covariant derivative $\nabla^k \in PDO^k(\mathbb{K} \times M, T^*M^{\otimes k})$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. (See Definition 3.17 in O'Neill [23] and Example 10.1.19 in Nicolaescu [22] for the covariant derivative and then compose it repeatedly with itself. Also see p.457 in [22].) We will then find the extension $\nabla^k : D'(M) \to D'(M, T^*M^{\otimes k})$.

We first note that for any vector bundle $E$ (with Riemannian or Hermitian metric) over the Riemannian manifold $M$, we have the following continuous linear injections for all $k \in \mathbb{N}_0$:

$$L^p(M,E,\mu) \overset{i_1}{\to} L^1_{loc}(M,E,\mu) \overset{i_2}{\to} D'(M,E)$$

given by: $i_1(u) = u$ and $i_2(v)(\phi) = \int_M \phi(x)(v(x))d\mu(x)$ for $u \in L^p(M,E,\mu)$, $v \in L^1_{loc}(M,E,\mu)$ and $\phi \in \Gamma_0(E^*)$. Since $\mu$ is a smooth measure then according to Section D.1 it is also Radon and so the map $i_1$ is well-defined. Note that both maps $i_1$ and $i_2$ are continuous linear injections though are not necessarily embeddings.

Now we finally define for any Riemannian manifold $M$, equipped with a smooth measure $\mu$, and for every $p \in [1,\infty]$ and $k \in \mathbb{N}_0$ the **Sobolev space**

$$W^{k,p}(M,\mu) = \left\{F \in L^p(M,\mu) \text{ s.t. } \forall 1 \leq j \leq k \text{ integer: } \nabla^j F \in L^p(M, T^*M^{\otimes j}, \mu)\right\}.$$

Note that if we had instead started with the k-th covariant derivative $\nabla^k \in PDO^k((\mathbb{R} \times M) \otimes E, T^*M^{\otimes k} \otimes E)$, where $E$ is a Riemannian vector bundle over $M$, then we would have obtained the generalised Sobolev spaces

$$W^{k,p}(M,E,\mu) = \left\{ \begin{array}{l} F \in L^p(M,E,\mu) \text{ s.t. } \forall 0 \leq j \leq k \text{ integer:} \\ \nabla^j F \in L^p(M, T^*M^{\otimes j} \otimes E, \mu) \end{array} \right\}.$$

However, in this thesis we shall only need the usual Sobolev spaces $W^{k,p}(M,\mu)$.

Note that as usual, the Sobolev spaces are Banach Spaces with the norm:

$$||F|| = \sum_{j=0}^{k} ||\nabla^j F||_{L^p(M,T^*M^{\otimes j},\mu)}$$

for $F \in W^{k,p}(M,\mu)$. Letting $p = 2$, one usually defines $H^k(M,\mu) := W^{k,2}(M,\mu)$, which is a Hilbert space with the (equivalent) norm:

$$||F|| = \left[\sum_{j=0}^{k} ||\nabla^j F||^2_{L^2(M,T^*M^{\otimes j},\mu)}\right]^{1/2}$$

and the corresponding inner-product. Note that for all $p \in [1,\infty]$ and $k \in \mathbb{N}_0$, we have the inclusion $[C_0^\infty(M)] \subseteq W^{k,p}(M,\mu)$ (where $[C_0^\infty(M)]$ is the set of equivalence classes of test functions) and so we can define:

$$W_0^{k,p}(M,\mu) := \overline{[C_0^\infty(M)]},$$

where we are closing $[C_0^\infty(M)]$ in the topology of $W^{k,p}(M,\mu)$ defined by the norm $\|\cdot\|$ just given. Thus, as a closed subspace of a Banach space, $W_0^{k,p}(M,\mu)$ is itself a Banach space.

We shall also need to define the local Sobolev spaces $W_{loc}^{k,p}(M,\mu)$. They are defined very similarly to Sobolev spaces $W^{k,p}(M,\mu)$ treated above. Fix a Riemannian manifold $M$ equipped with a smooth measure $\mu$. Then, for every $p \in [1,\infty]$ and $k \in \mathbb{N}_0$, the **local Sobolev space** is defined as :

$$W_{loc}^{k,p}(M,\mu) = \left\{ \begin{array}{l} F \in L_{loc}^p(M,\mu) \text{ s.t. } \forall 1 \leq j \leq k \text{ integer:} \\ \nabla^j F \in L_{loc}^p(M, T^*M^{\otimes j}, \mu) \end{array} \right\}.$$

We could then, similarly to above, define the generalised local Sobolev space, but again, this is not needed in this thesis. Actually, we shall only need the case of $M = \Omega \subseteq \mathbb{R}^N$ (an open subset) with the usual metric and Lebesgue measure. For simplicity we shall denote the corresponding Sobolev space by $W^{k,p}(\Omega)$ and the local Sobolev space by $W_{loc}^{k,p}(\Omega)$.

*Remark.* The spaces $W^{k,p}(\Omega)$ and $W_{loc}^{k,p}(\Omega)$ are also defined in p.50-51 in Reed and Simon [28]. The defintions there agree with those here but not the notation. The space $W^{m,2}(\mathbb{R}^n)$ corresponds to the expression $W_m$ used there. Our term $W_{loc}^{m,2}(\Omega)$ corresponds to their expression $W_m(\Omega)$. (In their notation be aware that $W_m(\mathbb{R}^n) \neq W_m$ since the first expression is a local Sobolev space and the second is not). In addition, note that the definitions there extend the cases of $W^{k,p}(\Omega)$ and $W_{loc}^{k,p}(\Omega)$ from integer $k$ to arbitary real $k$. For definitions more similar to our approach see Definition 10.2.33 in Nicolaescu [22].

We wish to quote for completeness two theorems on Sobolev theory, which were needed in the proof of Theorem 3.3.

The first is an elliptic regularity theorem. It is Theorem 10.3.6 in Nicolaescu [22].

**Theorem D.12** (Elliptic Regularity Theorem)**.** *Let* $p \in (0, \infty)$ *and let* $P \in PDO^m(E, E)$ *be an elliptic p.d.o. from the vector bundle* $E$ *(with Riemannian or Hermitian metric) to itself, where* $E$ *is over the Riemannian manifold* $M$*. Fix a smooth measure* $\mu$ *on* $M$ *(for instance* $\mu = dvol_g$*). Remember that* $P$ *can be extended to act:* $P : D'(M, E) \to D'(M, E)$*. If* $u \in L^p_{loc}(M, E, \mu)$ *and* $Pu = v \in W^p_{loc}(M, E, \mu)$*, then* $u \in W^{m,p}_{loc}(M, E, \mu)$*.*

The second theorem to be quoted is Theorem IX.24 in Reed and Simon [28].

**Theorem D.13** (Sobolev's Lemma)**.** *If* $\Omega \subseteq \mathbb{R}^N$ *is open and let* $T \in W^{k,2}_{loc}(\Omega)$ *for non-negative integer* $k > N/2$*. Then, if* $l$ *is a non-negative integer satisfying* $l < k - N/2$*, then* $T$ *is equal to a* $C^l$ *function.*

# E Construction of Energy/Symplectic Forms

In this section we shall introduce the energy-momentum tensor associated with a smooth function on a spacetime satisfying the Klein-Gordon equation. We shall then specialise this to a standard static spacetime. The expression obtained shall be shown to agree with the energy-form introduced in Section 9. Indeed this is why this form was chosen by Wald.

Given a spacetime $(M, g)$, then a smooth function $\phi \in C^\infty(M)$ satisfies the Klein-Gordon equation iff

$$(\Box_g + m^2)\phi = 0.$$

As can be found using Noether's theorem, the associated Energy-Momentum Tensor $T[\phi]$ is a rank (0,2) tensor, $T[\phi] \in \Gamma(T^*M \otimes T^*M)$, given in components by:

$$T_{\mu\nu}[\phi] = \frac{1}{2}\nabla_\mu\phi^*\nabla_\nu\phi + \frac{1}{2}\nabla_\nu\phi^*\nabla_\mu\phi - \frac{1}{2}g_{\mu\nu}(\nabla^\sigma\phi^*\nabla_\sigma\phi - m^2\phi^*\phi)$$

In coordinate independent form this is:

$$T[\phi] = \frac{1}{2}\nabla\phi^* \otimes \nabla\phi + \frac{1}{2}\nabla\phi \otimes \nabla\phi^* - \frac{1}{2}g\left[\nabla\phi((\nabla\phi^*)^\#) - m^2\phi^*\phi\right],$$

where $\nabla\colon C^\infty(M) \to \Gamma(T^*M)$ is the covariant derivative generated by the Levi-Civita connection of the metric $g$ and $^\#\colon \Gamma(T^*M) \to \Gamma(TM)$ is the "index-raising" map induced by the metric.

It is known from Noether's theorem that the Energy-Momentum Tensor $T[\phi]$ satisfies: $\nabla_\mu T^{\mu\nu}[\phi] = 0$. For sake of completeness we now show this directly, in coordinates.

$$T^{\mu\nu}[\phi] = \frac{1}{2}\nabla^\mu\phi^*\nabla^\nu\phi + \frac{1}{2}\nabla^\nu\phi^*\nabla^\mu\phi - \frac{1}{2}g^{\mu\nu}(\nabla^\sigma\phi^*\nabla_\sigma\phi - m^2\phi^*\phi)$$

and

$$
\begin{aligned}
\nabla_\mu T^{\mu\nu}[\phi] &= \frac{1}{2}\nabla_\mu\nabla^\mu\phi^*\nabla^\nu\phi + \frac{1}{2}\nabla^\mu\phi^*\nabla_\mu\nabla^\nu\phi + \frac{1}{2}\nabla_\mu\nabla^\nu\phi^*\nabla^\mu\phi \\
&\quad + \frac{1}{2}\nabla^\nu\phi^*\nabla_\mu\nabla^\mu\phi - \frac{1}{2}g^{\mu\nu}\nabla_\mu(\nabla^\sigma\phi^*\nabla_\sigma\phi - m^2\phi^*\phi) \\
&= -\frac{1}{2}m^2\phi^*\nabla^\nu\phi + \frac{1}{2}\nabla^\mu\phi^*\nabla_\mu\nabla^\nu\phi + \frac{1}{2}\nabla_\mu\nabla^\nu\phi^*\nabla^\mu\phi \\
&\quad - \frac{1}{2}m^2\nabla^\nu\phi^*\phi - \frac{1}{2}g^{\mu\nu}\nabla_\mu\nabla^\sigma\phi^*\nabla_\sigma\phi - \frac{1}{2}g^{\mu\nu}\nabla^\sigma\phi^*\nabla_\mu\nabla_\sigma\phi \\
&\quad - \frac{1}{2}g^{\mu\nu}(-m^2\nabla_\mu\phi^*\phi - m^2\phi^*\nabla_\mu\phi) \\
&= \frac{1}{2}\nabla^\mu\phi^*\nabla_\mu\nabla^\nu\phi + \frac{1}{2}\nabla_\mu\nabla^\nu\phi^*\nabla^\mu\phi \\
&\quad - \frac{1}{2}g^{\mu\nu}\nabla_\mu\nabla^\sigma\phi^*\nabla_\sigma\phi - \frac{1}{2}g^{\mu\nu}\nabla^\sigma\phi^*\nabla_\mu\nabla_\sigma\phi \\
&= \frac{1}{2}\nabla^\mu\phi^*\nabla_\mu\nabla^\nu\phi + \frac{1}{2}\nabla_\mu\nabla^\nu\phi^*\nabla^\mu\phi \\
&\quad - \frac{1}{2}\nabla^\nu\nabla^\mu\phi^*\nabla_\mu\phi - \frac{1}{2}\nabla^\mu\phi^*\nabla^\nu\nabla_\mu\phi \\
&= \frac{1}{2}\nabla^\mu\phi^*\nabla_\mu\nabla^\nu\phi + \frac{1}{2}\nabla_\mu\nabla^\nu\phi^*\nabla^\mu\phi \\
&\quad - \frac{1}{2}\nabla^\nu\nabla_\mu\phi^*\nabla^\mu\phi - \frac{1}{2}\nabla^\mu\phi^*\nabla^\nu\nabla_\mu\phi \\
&= 0
\end{aligned}
$$

Note that in the first equality we used the fact that we are dealing with the covariant derivative defined by a metric connection and so $\nabla g = 0$ and also $\nabla_\mu g^{\mu\nu} = 0$. In the last equality we have used that $\nabla^\nu\nabla_\mu\phi = \nabla_\mu\nabla^\nu\phi = 0$ for all smooth functions $\phi$. This itself follows from the symmetry of the covariant Hessian (Lemma 3.49 in O'Neill [23]) for the covariant derivative of a symmetric connection and a second use of the fact $\nabla g = 0$. In particular this is true for the covariant derivative defined by the Levi-Civita connection of our metric.

The Energy-Momentum tensor is of most use when there exists on the spacetime a Killing vector field $\xi$, since the field given in components as $\xi^\nu T^\mu_\nu$

will then be a divergence-free smooth vector field as:

$$\begin{aligned}\nabla_\mu(\xi_\nu T^{\mu\nu}) &= (\nabla_\mu \xi_\nu)T^{\mu\nu} + \xi_\nu(\nabla_\mu T^{\mu\nu}) \\ &= (\nabla_{(\mu}\xi_{\nu)})T^{\mu\nu} + \xi_\nu(\nabla_\mu T^{\mu\nu}) \text{ (as } T^{\mu\nu} = T^{\nu\mu}) \\ &= 0,\end{aligned}$$

since the first term $\nabla_{(\mu}\xi_{\nu)}$ is zero as a consequence of $\xi$ being a Killing vector field and $\nabla_\mu T^{\mu\nu} = 0$ as shown above.

We shall next use Gauss' Divergence Theorem. Before we state it we mention here a well-known result that on the boundary $\partial M$ of a smooth pseudo-Riemannian manifold with boundary $M$ such that $g|_{\partial M}$ is nondegenerate there exists a unique outward-pointing unit normal vector field $N$ along $\partial M$. Gauss' divergence theorem is expressed in terms of this vector field. (In this section we shall only require the Lorentzian case.)

**Proposition E.1** (Existence of Normal to Boundary of a pseudo-Riemannian Manifold with Boundary)**.** *Given a smooth pseudo-Riemannian manifold with boundary* $(M, g)$ *such that* $g|_{\partial M}$ *is nondegenerate at all points in* $\partial M$*, then there exists a unique outward-pointing smooth unit vector field* $N$ *along* $\partial M$ *such that* $N_p$ *is normal to* $T_p(\partial M)$ *for all* $p \in \partial M$*.*

*Remark.* Thus $N$ is a smooth map $N: \partial M \to TM$ (between smooth manifolds with boundary) such that $\pi \circ N = id$ where $\pi: TM \to M$ is the smooth bundle map. By a unit vector $X_p$ we mean that $|g_p(X_p, X_p)| = 1$ and similarly for a unit vector field.

Note that even if $M$ is connected $\partial M$ need not be. Note that the metric $g$ has the same signature on each connected component of $\partial M$ (by the continuity of the signature which follows from Lemma E.3 and the fact that $g|_{\partial M}$ is nondegenerate).

The content of the proposition may at first sight be surprising since we haven't assumed the orientability of $M$ or its boundary $\partial M$. However it is precisely the concept of outward-pointing vectors on the boundary which gives the preferred direction with which we define the normal vector field $N$. Although this result is well known, for completeness sake and since it cannot be found easily in the literature, we give a proof similar to that of the proof of the existence of local pseudo-orthonormal frames in O'Neill [23] (see after Corollary 3.46). It should be noted that the Riemannian version of this proposition is given in Proposition 10.39 of Lee [18].

We begin by stating a proposition.

**Proposition E.2.** *Let $P$ be an $m$-dimensional smoothly embedded submanifold of a pseudo-Riemannian manifold $M$ (with or without boundary) such that $g|_P$ is nondegenerate ($\dim M = n$). Then for all $p \in P$ there exists an open neighbourhood $U$ of $p$ in $P$ and $n$ smooth pseudo-orthonormal vector fields $\{X_1, ..., X_n\}$ in $M$ along $U$ such that $X_1, ..., X_m$ are tangent to $P$.*

*Proof.* Given $p \in P$, choose a pseudo-orthonormal set of vectors

$$\{(X_1)_p, ..., (X_n)_p\}$$

in $T_p(M)$ such that

$$(X_1)_p, ..., (X_m)_p \in T_p(P).$$

(This is possible by for example Bishop and Goldberg [10] (Theorem 2.21.1).) Pick a normal neighbourhood $U \subseteq P$ with respect to the metric $g|_P$. Define the vector fields $\{X_1, ..., X_m\}$ on $U$ by parallel transporting in $P$ along the unique minimal geodesics in $U$ between $p$ and the other points in $U$. Define $\{X_{m+1}, ..., X_n\}$ by parallel transporting via the normal connection (see e.g. Lemma 4.40 p.119 in O'Neill [23]). All the resulting vector fields are smooth by the same argument as in O'Neill ([23] after Corallary 3.46). The vector fields are pseudo-orthonormal as parallel transporting always preserves the metric. □

*Proof of Proposition E.1.* We first show uniqueness of such a vector field $N$. From the following proof of the local existence of $n$ will then follow its global existence.

Since the boundary $\partial M$ of $M$ is a hypersurface (i.e. $\dim \partial M = \dim M - 1$) then for every point $p \in \partial M$, $T_p(\partial M)^\perp$ is one-dimensional. As $g|_{\partial M}$ is nondegenerate, then $g|_{T_p(\partial M)^\perp}$ is non-zero and we can pick $N_p$ such that $g_p(N_p, N_p) = \pm 1$. Demanding that $N_p$ is also outward-pointing determines it uniquely.

To show the local existence, apply the previous proposition to the embedded hypersurface $\partial M$ in $M$. Since w.l.o.g. $U$ is connected then the smooth vector field $X_n$ is either inward-pointing or outward pointing. Switching the sign of $X_n$ if necessary we then set $N = X_n$. □

**Lemma E.3** (The signature of a pseudo-Riemannian metric is locally constant)**.** *Let $M$ be a smooth manifold. Let $g$ be a smooth symmetric nondegenerate (0,2) tensor field (also called a pseudo-Riemannian metric). Define the following function $f$ on $M$. For $p \in M$ let $f(p)$ be the number of diagonal elements that are $+1$ in the matrix representation of $g$ in any pseudo-orthonormal basis of $T_p(M)$. Then $f$ is locally constant. The same result is true if $f$ gives instead the number of $-1$ terms. In particular the signature of $g$ is also locally constant and constant if $M$ is connected.*

*Proof (O'Neill [23] (see after Corollary 3.46)).* Note first that $f$ is well-defined as the number of $+1$ terms is independent of the pseudo-orthonormal basis. See for example Bishop and Goldberg [10] (Theorem 2.21.1)). Given $p \in M$ construct a local pseudo-orthonormal frame as follows: Let $U \subseteq M$ be a normal neighbourhood of $p$ and pick a pseudo-orthonormal basis $\{(X_1)_p, ...(X_n)_p\}$ of $T_p(M)$. Extend these vectors to smooth pseudo-orthonormal vector fields $\{X_1, ...X_n\}$ by parallel transporting along the unique minimal geodesics in $U$ to points in $U$. If $\{Y_1, ...Y_n\}$ is another pseudo-orthonormal frame on $U$ then the number of fields $Y_i$ such that $g(Y_i, Y_i) = 1$ equals the number of fields $X_i$ with $g(X_i, X_i) = 1$ since the both equal the number of +1 terms in the matrix representation of $g_p$ for any point $p \in U$. Thus $f$ is constant on $U$. □

*Remark.* Note that the content of this lemma is false if we relax the condition of non-degeneracy of $g$. For instance multiplying any pseudo-Riemannian metric by a bump function $f$ in $M$ yields a somewhere degenerate symmetric (0,2) tensor field whose signature is not constant.

We shall need here and elsewhere in this thesis the following important and well-known theorem. The oriented Riemannian case is probably the most common (e.g. Lee [19]), though as we shall need both the nonorientable Riemannian and Lorentzian case we shall quote here the yet more general case for a nonorientable pseudo-Riemannian manifold. The statement of the Lorentzian case is found in Theorem 1.3.16 of Bär et al. [3]. The general nonorientable pseudo-Riemannian case is reached by application of Theorem 7.2.15 in Abraham et al. [1] (the nonorientable Stokes' Theorem) and a mimicking of Theorem 7.2.9 and Corollary 7.2.10 in Abraham et al..

**Theorem E.4** (Gauss' Divergence Theorem)**.** *Given a smooth pseudo-Riemannian manifold with boundary $(M, g)$ such that $g|_{\partial M}$ is nondegenerate at all points in $\partial M$, then for all compactly supported smooth vector fields*

*$X$ in $M$:*

$$\int_M div_g(X)\, dvol_g = \int_{\partial M} \epsilon_N g(X,N)\, dvol_{g|_{\partial M}},$$

*where $dvol_g$ and $dvol_{g|_{\partial M}}$ are respectively the induced measures on $M$ and $\partial M$ induced from the metrics $g$ and $g|_{\partial M}$. $N$ is the unique outward-pointing smooth unit vector field $N$ along $\partial M$ orthogonal to $\partial M$ given in Proposition E.1 and $\epsilon_N = g(N,N) \in C^\infty(\partial M)$ is locally constant with $|\epsilon_N| = 1$.*

Now returning to the construction of the Energy form we know that $\xi^\nu T^\mu_\nu$ are the components of smooth vector field in $M$ where $\xi$ is a particular Killing vector field on $M$. Specialising now to the class of spacetimes of interest here, standard static spacetimes $(M,g) = \mathbb{R}\times\Sigma, V^2dt^2 - h)$, then $\xi = \frac{\partial}{\partial t}$ is the static vector field and so in particular Killing. We shall consider the case where the scalar field $\phi$ obeying the Klein-Gordon equation in $M$ is compactly supported. Thus the corresponding Energy-Momentum tensor $T[\phi]$ also has compact support as also the vector field with components $\xi^\nu T^\mu_\nu$. We shall now apply the Divergence Theorem to this vector field and the smooth embedded manifold with boundary $[t_1, t_2] \times \Sigma$. With the induced metric, this forms a Lorentzian manifold with boundary and the metric is clearly nondegenerate on the boundary as it is negative-definite there. The boundary clearly has two connected components: $\{t_1\} \times \Sigma$ and $\{t_2\} \times \Sigma$. The outward pointing unit normal $N$ is $N_1 = V^{-1}\frac{\partial}{\partial t}$ on the former and $N_2 = -V^{-1}\frac{\partial}{\partial t}$ on the latter connected component of the boundary. Both vector fields are timelike so $\epsilon_{N_1} = \epsilon_{N_2} = 1$.

Inserting all this into the Divergence Theorem, we obtain:

$$0 = \int_{\{t_1\}\times\Sigma} g(X,N_1)\mathrm{dvol}_h + \int_{\{t_2\}\times\Sigma} g(X,N_2)\mathrm{dvol}_h,$$

or:

$$\int_{\{t_1\}\times\Sigma} g(X,\frac{\partial}{\partial t})V^{-1}\mathrm{dvol}_h = \int_{\{t_2\}\times\Sigma} g(X,\frac{\partial}{\partial t})V^{-1}\mathrm{dvol}_h.$$

In components:

$$g(X, \frac{\partial}{\partial t}) = (\frac{\partial}{\partial t})^\mu \xi^\nu T_{\mu\nu}[\phi] = T_{00}[\phi]$$
$$= \frac{1}{2}\partial_t\phi^*\partial_t\phi + \frac{1}{2}V^2 D^i\phi^* D_i\phi + \frac{1}{2}m^2V^2\phi^*\phi,$$

and thus:

$$\int_{\{t_1\}\times\Sigma}[\frac{1}{2}\partial_t\phi^*\partial_t\phi + \frac{1}{2}V^2D^i\phi^*D_i\phi + \frac{1}{2}m^2V^2\phi^*\phi]V^{-1}d\mathrm{vol}_h$$
$$= \int_{\{t_2\}\times\Sigma}[\frac{1}{2}\partial_t\phi^*\partial_t\phi + \frac{1}{2}V^2D^i\phi^*D_i\phi + \frac{1}{2}m^2V^2\phi^*\phi]V^{-1}d\mathrm{vol}_h.$$

Thus if we define for each $t \in \mathbb{R}$: $E(\phi,\phi)(t) = \langle\dot{\phi}_t, \dot{\phi}_t\rangle_{\Sigma_t} + \langle\phi_t, A\phi_t\rangle_{\Sigma_t}$ where $A = -VD^iVD_i + m^2V^2$ and we are working in $L^2(\Sigma_t, V^{-1}d\mathrm{vol}_h)$ for each $t$, then we have shown that $E(\phi,\phi)(t)$ is independent of time when $\phi$ is a smooth compactly supported solution of the Klein-Gordon equation.

It follows however by the polarization identity that defining $E(\phi,\phi')(t) = \langle\dot{\phi}_t, \dot{\phi}'_t\rangle_{\Sigma_t} + \langle\phi_t, A\phi'_t\rangle_{\Sigma_t}$ then $E(\phi,\phi')(t)$ is also independent of time. Replacing $A$ by $A_E$ gives us the energy form as given in Section 9.

We shall now justify the expression of the symplectic form as given in Section 10. Given two compactly supported smooth solutions $\phi, \phi'$ to the Klein-Gordon equation define the smooth vector field $X$ given by components: $X^\mu = \phi\nabla^\mu\phi' - \phi'\nabla^\mu\phi$. In coordinate free notation this is: $X = \phi(\nabla\phi')^\# - \phi'(\nabla\phi)^\#$. Then:

$$\nabla_\mu X^\mu = \nabla_\mu\phi\nabla^\mu\phi' + \phi\nabla_\mu\nabla^\mu\phi' - \nabla_\mu\phi'\nabla^\mu\phi - \phi'\nabla_\mu\nabla^\mu\phi$$
$$= \phi\nabla_\mu\nabla^\mu\phi' - \phi'\nabla_\mu\nabla^\mu\phi$$
$$= -m^2\phi\phi' + m^2\phi\phi'$$
$$= 0.$$

Since $X$ is also compactly supported we can again then apply the Divergence Theorem to the submanifold with boundary $[t_1, t_2] \times \Sigma$: Again using:

$$\int_{\{t_1\}\times\Sigma} g(X, \frac{\partial}{\partial t}) V^{-1} d\mathrm{vol}_h = \int_{\{t_2\}\times\Sigma} g(X, \frac{\partial}{\partial t}) V^{-1} d\mathrm{vol}_h,$$

we have

$$\int_{\{t_1\}\times\Sigma} [\phi\dot{\phi}' - \phi'\dot{\phi}] V^{-1} d\mathrm{vol}_h = \int_{\{t_2\}\times\Sigma} [\phi\dot{\phi}' - \phi'\dot{\phi}] V^{-1} d\mathrm{vol}_h$$

and so

$$\begin{aligned} \sigma(\phi, \phi')(t) &= \int_{\{t\}\times\Sigma} [\phi\dot{\phi}' - \phi'\dot{\phi}] V^{-1} d\mathrm{vol}_h \\ &= \langle \phi_t, \dot{\phi}'_t \rangle_{\Sigma_t} - \langle \dot{\phi}_t, \phi'_t \rangle_{\Sigma_t} \end{aligned}$$

is independent of time (where the brackets refer to the inner-product in $L^2(\Sigma_t, V^{-1} d\mathrm{vol}_h)$). This is the form of the symplectic map as used in Section 10 and also in Theorem 5.4, the important step in proving the existence of the Wald solutions.

**Proposition E.5.** *Let $(M, g)$ be a spacetime and let $X$ be a smooth timelike vector field on $M$ defining the time-orientation. If $S \subseteq M$ is a smooth spacelike hypersurface in $M$ then there exists a unique smooth timelike f.p. unit normal vector field $n$ along $S$.*

*Remark.* Again, we haven't the orientability of either $M$ or $S$. Here, it is the fact that $(M, g)$ is time-oriented that allows us to construct $n$.

*Proof.* We first show the uniqueness of a tangent vector $n_p$ for $p \in S$ timelike f.p. unit vector orthogonal to $T_p(S)$. Uniqueness of the vector field $n$ will then follow. If $p \in S$ then $T_p(S)^{\perp}$ is one-dimensional in $T_p(M)$. The conditions of being unit and future-pointing determine a unique vector in $T_p(S)^{\perp}$.

Existence is next to be proven. Given $p \in S$ let $\{Z_1, .., Z_{n-1}\}$ be a pseudo-orthonormal frame of $(S, g|_S)$ defined on $U \subseteq S$ open, with $p \in U$. Thus $g(Z_i, Z_j) = -\delta_{ij}$ as smooth functions on $U$. (Such a frame is easily definable from any coordinate frame using the Gram-Schmidt orthogonalisation procedure since $g|_S$ is negative definite as $S$ spacelike.)

Define $n$ on $U$ by: $n = X + \sum_{i=1}^{n-1} g(Z_i, X)Z_i$. So $n$ is a smooth vector field along $U$, such that $n_p$ is orthogonal to $T_p(s)$ for each $p \in U$ since:

$$\begin{aligned} g(n, Z_j) &= g(X, Z_j) + \sum_i g(Z_i, X)g(Z_i, Z_j) \\ &= g(X, Z_j) - \sum_i g(Z_i, X)\delta_{ij} \\ &= 0. \end{aligned}$$

$n$ is a timelike vector field because:

$$\begin{aligned} g(n, n) &= g(X, X) + \sum_{i,j} g(Z_i, X)g(Z_j, X)g(Z_i, Z_j) + 2\sum_i g(Z_i, X)g(Z_i, X) \\ &= g(X, X) - \sum_i g(Z_i, X)g(Z_i, X) + 2\sum_i g(Z_i, X)g(Z_i, X) \\ &= g(X, X) + \sum_i g(Z_i, X)g(Z_i, X) \\ &\geq g(X, X) \\ &> 0, \end{aligned}$$

and $n$ is also future-pointing as:

$$g(n, X) = g(X, X) + \sum_{i=1}^{n-1} g(Z_i, X)g(Z_i, X) \geq g(X, X) > 0.$$

After normalising $n$, then it has all the desired properties. Thus we have proven local existence. As usual, this together with uniqueness at every $p \in S$ proves the global existence.

Alternatively, if $\{Y_1, ..Y_{n-1}\}$ is another pseudo-orthonormal frame defined on $V \subseteq S$, then $g(Y_i, Y_j) = -\delta_{ij}$ and $Z_i = A_{ij}Y_j$ where $A_{ij}A_{ik} = A_{ji}A_{ki} = \delta_{jk}$. The two frames produce the vector fields $n_Z$, $n_Y$ along $U \cap V$. Then

$$\begin{aligned} n_Z &= X + g(Z_i, X)Z_i \\ &= X + g(A_{ij}Y_j, X)A_{ik}Y_k \\ &= X + g(Y_j, X)A_{ij}A_{ik}Y_k \\ &= X + g(Y_j, X)\delta_{jk}Y_k \\ &= X + g(Y_j, X)Y_j \\ &= n_Y. \end{aligned}$$

Thus $n$ is well-defined on all $S$. □

**Proposition E.6.** *Let $(M, g)$ be a Lorentzian manifold and $X$ a smooth timelike vector field on $M$. Then $W(p) = \{X_p\}^\perp$ defines a smooth distribution in $M$ and around any point $p$ there exists a pseudo-orthonormal basis of the distribution.*

*Proof.* Firstly, w.l.o.g. let $g(X, X) = 1$. Given $p \in M$, let $U \subseteq M$ be open and $\{X_1, ...X_n\}$ be a pseudo-orthonormal basis on $U$. So $g(X_1, X_1) = 1$, $g(X_i, X_j) = -\delta_{ij}$, $g(X_1, X_i) = 0$ for $i, j \geq 2$.

As $X$ and $X_1$ are both timelike then $g(X, X_1)$ never vanishes in $U$. So $X = \sum_i f_i X_i$ where $f_i$ are smooth functions on $U$ with $f_1$ never vanishing.

Thus, since

$$\begin{aligned} X_1 &= f_1^{-1}X - f_1^{-1}\sum_{i\geq 2} f_i X_i \\ X_i &= (X_i - g(X, X_i)X) + g(X, X_i)X \quad \text{for } i \geq 2, \end{aligned}$$

each element in one of the following sets is in the span of the other set:

$$\{X_1, X_2, ..., X_n\}$$
$$\{X, X_2 - g(X, X_2)X, ..., X_n - g(X, X_n)X\}.$$

As the former set is a local basis of $M$ then so is the latter set of vector fields. Define $Y_i = X_i - g(X, X_i)X \;\; i \geq 2$. These vector fields satisfy: $g(X, Y_i) = g(X, X_i) - g(X, X_i)g(X, X) = 0$. Thus $\{Y_2, ..., Y_n\}$ is a local basis of the distribution $W$. As also $g$ is negative definite on $W(p) = \{X(p)\}^{\perp}$ then we can apply the Gram-Schmidt orthonormalisation procedure to $\{Y_2, ..., Y_n\}$ to form the vector fields $\{Z_2, ..., Z_n\}$ with $g(Z_i, Z_j) = -\delta_{ij}$. □

# F Green's Function for the Closure of minus the Laplacian on $S^1$

We showed in Section 12.1 that in the case $\Sigma = S^1$, $V = 1$, $m = 0$, $A$ is e.s.a. and

$$D(\overline{A}) = W^{2,2}(S^1) = \{\phi \in L^2(S^1) \text{ s.t. } \phi',\ \phi'' \in L^2(S^1)\},$$

$\overline{A}([\phi]) = -[\phi'']$ for $[\phi] \in W^{2,2}(S^1)$. We wish to directly determine the spectrum of $\overline{A}$ by stating its eigenvalues, eigenvectors and determining its resolvents.

**Proposition F.1.** *The spectrum of $\overline{A}$ is given by:*

$$\sigma(\overline{A}) = \sigma_{disc}(\overline{A}) = \{n^2 : n \in \mathbb{N}_0\}$$

*We are using the chart: $\phi: U = S^1 \backslash \{1\} \to (0, 2\pi)$, $\phi^{-1}(\theta) = \exp i\theta$, define the function $h: U \times U \times \mathbb{C}\backslash\mathbb{Z} \to \mathbb{C}$ by:*

$$h(\theta, \phi; z) = \frac{i}{2z}\left[\exp iz|\theta - \phi| + \frac{2\cos z(\theta - \phi)}{\exp(-2\pi i z) - 1}\right]$$

*As $\{1\} \subseteq S^1$ is clearly null, $h$ generates a well-defined integral kernel.*

*Then $h(\theta, \phi; z) = h(\theta, \phi; -z)$ and $g(\theta, \phi; \lambda) = h(\theta, \phi; \sqrt{\lambda})$ is the Green's function for $\lambda \in \rho(\overline{A}) = \mathbb{C}\backslash\{n^2 : n \in \mathbb{N}_0\}$ and $g$ does not depend on the choice of square root of $\lambda$ used to define it.*

*The eigenspace corresponding to the eigenvalue $n^2$, ($n \in \mathbb{N}_0$) is spanned by the vectors $[\phi_n^{\pm}]$, where $\phi_n^{\pm}(\theta) = \exp(\pm in\theta)$. So $\dim E_0 = 1$ and $\dim E_{n^2} = 2$ for $n \in \mathbb{N}$.*

*Proof.* Firstly check that

$$\frac{1}{\exp(-2\pi i z) - 1} + \frac{1}{\exp(2\pi i z) - 1} = -1,$$

for all $z \in \mathbb{C}\backslash\mathbb{Z}$. Then:

$$\begin{aligned}0 &= \exp iz(\theta-\phi) + \exp -iz(\theta-\phi)\\ &\qquad + 2\cos z(\theta-\phi)\left[\frac{1}{\exp(-2\pi iz)-1} + \frac{1}{\exp(2\pi iz)-1}\right]\\ &= \exp iz|\theta-\phi| + \exp -iz|\theta-\phi|\\ &\qquad + 2\cos z(\theta-\phi)\left[\frac{1}{\exp(-2\pi iz)-1} + \frac{1}{\exp(2\pi iz)-1}\right],\end{aligned}$$

so:

$$\begin{aligned}\exp iz|\theta-\phi| + \frac{2\cos z(\theta-\phi)}{\exp(-2\pi iz)-1} &= -\left[\exp -iz|\theta-\phi| + \frac{2\cos z(\theta-\phi)}{\exp(2\pi iz)-1}\right]\\ \frac{i}{2z}\left[\exp iz|\theta-\phi| + \frac{2\cos z(\theta-\phi)}{\exp(-2\pi iz)-1}\right] &= \frac{i}{-2z}\left[\exp -iz|\theta-\phi| + \frac{2\cos z(\theta-\phi)}{\exp(2\pi iz)-1}\right],\end{aligned}$$

and

$$h(\theta,\phi;z) = h(\theta,\phi;-z).$$

Thus, we are setting:

$$g(\theta,\phi;\lambda) = \frac{i}{2\sqrt{\lambda}}\left[\exp i\sqrt{\lambda}|\theta-\phi| + \frac{2\cos\sqrt{\lambda}(\theta-\phi)}{\exp(-2\pi i\sqrt{\lambda})-1}\right]$$

Since for each fixed $\lambda \in \mathbb{C}\backslash\{n^2 : n \in \mathbb{N}\}$, $g$, viewed as a function on $(0,2\pi)\times(0,2\pi)$, has a continuous extension to $[0,2\pi]\times[0,2\pi]$ then, for each $\lambda$, $g(\cdot,\cdot;\lambda)$ is bounded on $U \times U$. Also, since $S^1$ is of finite measure, then for each fixed $\lambda$, $g(\cdot,\cdot;\lambda)$ is a Hilbert-Schmidt integral kernel. (It can be shown that in fact $g(\cdot,\cdot;\lambda)$ is a continuous function of $S^1 \times S^1$ for each $\lambda \in \mathbb{C}\backslash\{n^2 : n \in \mathbb{N}\}$ and is even smooth on $S^1 \times S^1\backslash\{(p,p) : p \in S^1\}$.) In particular $G_\lambda$ defines a bounded linear map (even compact) from $L^2(S^1)$ to itself as follows:

$$\begin{aligned}G_\lambda &: L^2(S^1) \to L^2(S^1)\\ G_\lambda(f)(\theta) &= \int_0^{2\pi} g(\theta,\phi;\lambda)f(\phi)d\phi.\end{aligned}$$

(Clearly we are being sloppy here. We actually mean:

$$G_\lambda([f]) = [\int_0^{2\pi} g(\theta,\phi;\lambda) f(\phi)d\phi].$$

We shall however adopt this same notation for all the examples here in the appendix for the sake of brevity.)

We must now show that this is indeed the resolvent for $\overline{A}$. So:

$$\begin{aligned}-2i\sqrt{\lambda}G_\lambda(f)(\theta) &= \int_0^\theta \exp i\sqrt{\lambda}(\theta-\phi)f(\phi)d\phi + \int_\theta^{2\pi} \exp i\sqrt{\lambda}(\phi-\theta)f(\phi)d\phi \\ &+ \frac{2}{\exp -2\pi i\sqrt{\lambda}-1}\int_0^{2\pi}\cos\sqrt{\lambda}(\theta-\phi)f(\phi)d\phi.\end{aligned}$$

It not at first clear that we can differentiate this expression. We can however evaluate its distributional derivative. We shall follow this procedure in all the other examples too without further mention. We have

$$\begin{aligned}-2i\sqrt{\lambda}G_\lambda(f)'(\theta) &= \int_0^\theta (i\sqrt{\lambda})\exp i\sqrt{\lambda}(\theta-\phi)f(\phi)d\phi + \cancel{f(\theta)} \\ &+ \int_\theta^{2\pi}(-i\sqrt{\lambda})\exp i\sqrt{\lambda}(\phi-\theta)f(\phi)d\phi - \cancel{f(\theta)} \\ &- \frac{2}{\exp -2\pi i\sqrt{\lambda}-1}(\sqrt{\lambda})\int_0^{2\pi}\sin\sqrt{\lambda}(\theta-\phi)f(\phi)d\phi\end{aligned}$$

and

$$\begin{aligned}-2i\sqrt{\lambda}G_\lambda(f)''(\theta) &= \int_0^\theta -\lambda\exp i\sqrt{\lambda}(\theta-\phi)f(\phi)d\phi + i\sqrt{\lambda}f(\theta) \\ &+ \int_\theta^{2\pi} -\lambda\exp i\sqrt{\lambda}(\phi-\theta)f(\phi)d\phi + i\sqrt{\lambda}f(\theta) \\ &- \frac{2\lambda}{\exp -2\pi i\sqrt{\lambda}-1}\int_0^{2\pi}\cos\sqrt{\lambda}(\theta-\phi)f(\phi)d\phi,\end{aligned}$$

so

$$G_\lambda(f)''(\theta) = (-\lambda G_\lambda(f) + 2iA\sqrt{\lambda}f)(\theta).$$

Note that we have shown that $G_\lambda(f), G'_\lambda(f)$ and $G''_\lambda(f)$ (viewed a priori as distributions) are elements of $L^2(S^1)$. And so $G_\lambda\colon L^2(S^1) \to W^{2,2}(S^1) =$

$D(\overline{A})$. Additionally:

$$
\begin{aligned}
-G_\lambda(f)'' - \lambda G_\lambda(f) &= f \\
(\overline{A} - \lambda)(G_\lambda(f)) &= f \\
(\overline{A} - \lambda) \circ G_\lambda &= id \text{ on } L^2(S^1) \text{ for all } \lambda \in \mathbb{C} \backslash \{n^2 : n \in \mathbb{N}_0\}
\end{aligned}
$$

Therefore $G_\lambda$ is a right-inverse of $\overline{A} - \lambda$ for $\lambda \in \mathbb{C} \backslash \{n^2 : n \in \mathbb{N}_0\}$. Now, to check that it is also a left-inverse:

Let $f \in D(\overline{A}) = W^{2,2}(S^1)$, then:

$$
\begin{aligned}
&- 2i\sqrt{\lambda} \int_0^{2\pi} g(\theta, \phi; \lambda) f''(\theta) d\theta \\
&= \int_0^\theta \exp i\sqrt{\lambda}(\theta - \phi) f''(\phi) d\phi + \int_\theta^{2\pi} \exp i\sqrt{\lambda}(\phi - \theta) f''(\phi) d\phi \\
&\quad + \frac{2}{\exp -2\pi i\sqrt{\lambda} - 1} \int_0^{2\pi} \cos \sqrt{\lambda}(\theta - \phi) f''(\phi) d\phi \\
&= \int_0^\theta -\lambda \exp i\sqrt{\lambda}(\theta - \phi) f(\phi) d\phi \\
&\quad + [\exp i\sqrt{\lambda}(\theta - \phi) f'(\phi) + i\sqrt{\lambda} \exp i\sqrt{\lambda}(\theta - \phi) f(\phi)]_0^\theta \\
&\quad + \int_\theta^{2\pi} -\lambda \exp i\sqrt{\lambda}(\phi - \theta) f(\phi) d\phi \\
&\quad + [\exp i\sqrt{\lambda}(\phi - \theta) f'(\phi) - i\sqrt{\lambda} \exp i\sqrt{\lambda}(\phi - \theta) f(\phi)]_\theta^{2\pi} \\
&\quad + \frac{2}{\exp -2\pi i\sqrt{\lambda} - 1} \int_0^{2\pi} -\lambda \cos \sqrt{\lambda}(\theta - \phi) f(\phi) d\phi \\
&\quad + \frac{2}{\exp -2\pi i\sqrt{\lambda} - 1} [\cos \sqrt{\lambda}(\theta - \phi) f'(\phi) - \sqrt{\lambda} \sin \sqrt{\lambda}(\theta - \phi) f(\phi)]_0^{2\pi}.
\end{aligned}
$$

Thus:

$$
\begin{aligned}
&-2i\sqrt{\lambda}\int_0^{2\pi} g(\theta,\phi;\lambda)f''(\phi)d\phi \\
&= 2i\lambda\sqrt{\lambda}\int_0^{2\pi} g(\theta,\phi;\lambda)f(\phi)d\phi \\
&\quad + f'(\theta) + i\sqrt{\lambda}f(\theta) - \exp i\sqrt{\lambda}\theta f'(0) - i\sqrt{\lambda}\exp i\sqrt{\lambda}\theta f(0) \\
&\quad + \exp i\sqrt{\lambda}(2\pi-\theta)f'(2\pi) - i\sqrt{\lambda}\exp i\sqrt{\lambda}(2\pi-\theta)f(2\pi) \\
&\quad - f'(\theta) + i\sqrt{\lambda}f(\theta) \\
&\quad + \frac{2}{\exp -2\pi i\sqrt{\lambda}-1}\left[\begin{array}{l} \cos\sqrt{\lambda}(\theta-2\pi)f'(2\pi) - \sqrt{\lambda}\sin\sqrt{\lambda}(\theta-2\pi)f(2\pi) \\ -\cos\sqrt{\lambda}\theta f'(0) + \sqrt{\lambda}\sin\sqrt{\lambda}\theta f(0) \end{array}\right].
\end{aligned}
$$

The boundary terms can be collected together and using the fact that $f(0) = f(2\pi)$ and $f'(0) = f'(2\pi)$.

The coefficient of the $f(0)$ term is proportional to:

$$
\begin{aligned}
&(\exp(-2\pi i\sqrt{\lambda}) - 1)(-i\sqrt{\lambda}\exp i\sqrt{\lambda}\theta - i\sqrt{\lambda}\exp i\sqrt{\lambda}(2\pi-\theta) \\
&\quad + 2(-i\sqrt{\lambda}\sin\sqrt{\lambda}(\theta-2\pi) + \sqrt{\lambda}\sin\sqrt{\lambda}\theta) \\
&= -\exp i\sqrt{\lambda}(\theta-2\pi) - \exp(-i\sqrt{\lambda}\theta) + \exp i\sqrt{\lambda}\theta \\
&\quad + \exp i\sqrt{\lambda}(2\pi-\theta) + 2i\sin\sqrt{\lambda}(\theta-2\pi) - 2i\sin\sqrt{\lambda}\theta \\
&= 0.
\end{aligned}
$$

The coefficient of the $f'(0)$ term is proportional to:

$$
\begin{aligned}
&(\exp(-2\pi i\sqrt{\lambda}) - 1)(-\exp i\sqrt{\lambda}\theta + \exp i\sqrt{\lambda}(2\pi-\theta) \\
&\quad + 2(\cos\sqrt{\lambda}(\theta-2\pi) - \cos\sqrt{\lambda}\theta) \\
&= -\exp i\sqrt{\lambda}(\theta-2\pi) - \exp(-i\sqrt{\lambda}\theta) + \exp i\sqrt{\lambda}\theta \\
&\quad - \exp i\sqrt{\lambda}(2\pi-\theta) + 2\cos\sqrt{\lambda}(\theta-2\pi) - 2\cos\sqrt{\lambda}\theta \\
&= 0
\end{aligned}
$$

Thus:

$$\begin{aligned}
&\int_0^{2\pi} g(\theta,\phi;\lambda) f''(\phi) d\phi = -\lambda \int_0^{2\pi} g(\theta,\phi;\lambda) f(\phi) d\phi - f(\theta) \\
&\int_0^{2\pi} g(\theta,\phi;\lambda)((\overline{A}-\lambda) f)(\phi) d\phi = f(\theta) \\
&G_\lambda \circ (\overline{A}-\lambda)(f) = f \text{ for } f \in D(\overline{A}) \\
&G_\lambda \circ (\overline{A}-\lambda) = id \text{ on } D(\overline{A})
\end{aligned}$$

. □

# G The Self-Adjoint Extensions of minus the Laplacian on $(0, \infty)$ and Spectral Analysis

Consider the linear operator given by minus the Laplacian on the Riemannian manifold $(0, \infty)$. More precisely, let:

$$D(A) = [C_0^\infty(0, \infty)] \subseteq L^2(0, \infty)$$
$$A([\phi]) = -[\phi''] \ \ \forall \phi \in C_0^\infty(0, \infty)$$

This linear operator is symmetric on the Hilbert space $L^2(0, \infty)$, that is: $\langle \phi, A\theta \rangle = \langle A\phi, \theta \rangle \ \ \forall \phi, \theta \in D(A)$. We also have that

1. The adjoint of $A$ is: $D(A^*) = W^{2,2}(0, \infty)$
2. The closure of $A$ is: $D(\overline{A}) = W_0^{2,2}(0, \infty) := \overline{[C_0^\infty(0, \infty)]}^{W^{2,2}(0,\infty)}$, that is, the closure of the domain of $A$ in the norm of $W^{2,2}(0, \infty)$.

These last two statements are not trivial from the definitions of the closure and adjoint of linear operators. It does follow from the definitions that: $D(A^*) = \{\phi \in L^2(0, \infty) \text{ s.t. } \phi'' \in L^2(0, \infty)\}$, where the derivative is as usual in the sense of distributions. However, $W^{2,2}(0, \infty) = \{\phi \in L^2(0, \infty) \text{ s.t. } \phi', \phi'' \in L^2(0, \infty)\}$. That $\phi, \phi'' \in L^2(0, \infty)$ implies that $\phi' \in L^2(0, \infty)$ also follows from a more general theorem in Lions and Magenes [20]. Indeed, more is true: $\phi \mapsto \phi'$ is continuous as a linear map from $D(A^*)$ to $L^2(0, \infty)$ (where $D(A^*)$ is given the graph norm: $||\phi||_{A^*}^2 = ||\phi||^2 + ||A^*\phi||^2 = ||\phi||^2 + ||\phi''||^2$). Thus in fact $||\phi||_{A^*}$ and $\sqrt{||\phi||^2 + ||\phi'||^2 + ||\phi''||^2}$ are equivalent norms on $D(A^*)$.

Statement (2) follows from the following. If $A$ is a symmetric linear operator on a Hilbert space, then $D(\overline{A}) = \overline{D(A)}^{D(A^*)}$ (the last expression denotes the closure of $D(A)$ in the graph norm on $D(A^*)$). The result then follows from the previous discussion of equivalent norms on $D(A^*)$.

We now follow the method of Reed and Simon [28] to construct all the self-adjoint extensions of this linear operator. We start by constructing explicitly the form of the deficiency spaces $H^{\pm} = \ker(A^{*} \mp I) \subseteq D(A^{*})$.

By elementary analysis, $H^{\pm} = \langle\{[\phi^{\pm}]\}\rangle$ where $\phi^{\pm}(x) = 2^{\frac{1}{4}}\exp(\frac{(-1\pm i)x}{\sqrt{2}})$.
Now, since $n^{+} = dimH^{+} = dimH^{-} = n^{-}$, then the self-adjoint extensions are labelled by the group $U(1) = \{u \in \mathbb{C}\colon |u| = 1\}$.

Given a unitary map $U\colon H^{+} \to H^{-}$ we define the self-adjoint extension $A_U$ by the domain:

$$D(A_U) = D(\overline{A}) + (I + U)H^{+}.$$

Given $u \in \mathbb{C}, |u| = 1$, let $U(u)(\lambda\phi^{+}) = u\lambda\phi^{-}$ and thus let $A_u := A_{U(u)}$. So:

$$D(A_u) = \{\phi_0 + \lambda\phi^{+} + \lambda u\phi^{-}\colon\ \phi_0 \in D(\overline{A}), \lambda \in \mathbb{C}\}$$

$$A_u(\phi_0 + \lambda\phi^{+} + \lambda u\phi^{-}) = \overline{A}\phi + i\lambda\phi^{+} - i\lambda u\phi^{-}.$$

Now we have an expression for the domains of our extensions we wish to re-express them more familiarly in terms of their boundary behaviour. This is the content of the following proposition:

**Proposition G.1.** *Define the bijection* $\{u \in \mathbb{C}\colon |u| = 1\} \to (-\pi/2, \pi/2]$ *given by* $u \mapsto \alpha(u)$ *where:*

$$\alpha(u) = \begin{cases} \cot^{-1}[2^{-1/2}\frac{(1-u)i-1-u}{u+1}] & \text{for } u \neq -1 \\ 0 & \text{for } u = -1 \end{cases}$$

*Then, in terms of this map:*

$$D(A_{U(u)}) = \{\phi \in W^{2,2}(0,\infty)\colon \cos\alpha(u)\ \phi(0) = \sin\alpha(u)\ \phi'(0)\}.$$

*Proof.* Letting $\phi = \phi_0 + \lambda\phi^+ + \lambda u\phi^- \in D(A_u)$, then:

$$\phi(0) = \lambda 2^{\frac{1}{4}} + u\lambda 2^{\frac{1}{4}} = 2^{\frac{1}{4}}\lambda(u+1),$$
$$\phi'(0) = \frac{\lambda}{2^{1/4}}((1-u)i - 1 - u).$$

Eliminating $\lambda$ we get:

$$2^{1/4}\phi'(0)(u+1) = \phi(0)2^{-1/4}((1-u)i - 1 - u)$$

and so

$$\sqrt{2}\phi'(0)(u+1) = \phi(0)((1-u)i - 1 - u).$$

If $u \neq -1$ then

$$\phi'(0) = 2^{-1/2}\phi(0)\frac{(1-u)i - 1 - u}{u+1} = \gamma\phi(0),$$

where $\gamma = 2^{-1/2}\frac{(1-u)i-1-u}{u+1} = \gamma^*$. Using the fact that $u^* = u^{-1}$ it is easy to show that $\gamma \in \mathbb{R}$. Reformulating this condition we have:

$$\frac{\phi'(0)}{(1+\gamma^2)^{1/2}} = \frac{\gamma\phi(0)}{(1+\gamma^2)^{1/2}},$$

which can then be put into the form

$$\cos\alpha\ \phi(0) = \sin\alpha\ \phi'(0),$$

where $\alpha \in (-\pi/2, \pi/2]$, $\cot\alpha = \gamma = 2^{-1/2}\frac{(1-u)i-1-u}{u+1}$. If $u = -1$ then $0 = \phi(0)(2i)$ so $\phi(0) = 0$ and $\cos\alpha\ \phi(0) = \sin\alpha\ \phi'(0)$ is satisfied as $\alpha(-1) = 0$.

We have shown one direction of the inclusion. For the other: if $u = -1$ and $\phi \in RHS$, then let $\lambda = -i2^{-3/4}\phi'(0)$ and $\theta = \phi - (I+U)(\lambda\phi^+)$. An easy computation shows that $\theta(0) = \theta'(0) = 0$ and so by Theorem 12.2, then $\theta \in W^{2,2}(0,\infty) = D(\overline{A})$ and $\phi \in LHS$.

If $u \neq -1$ and $\phi \in RHS$ then let $\lambda = 2^{-1/4}(1+u)^{-1}\phi(0)$ and $\theta = \phi - (I + U)(\lambda\phi^+)$. Again it is seen that $\theta \in W^{2,2}(0,\infty) = D(A^*)$ and $\theta(0) = \theta'(0) = 0$. Thus by Theorem 12.2 $\theta \in D(\overline{A})$ and $\phi \in LHS$. □

Conclusion: the self-adjoint extensions of the linear operator $A$ are indexed by $U(1)$. Given $\alpha \in (-\pi/2, \pi/2]$:

$$D(A_\alpha) = \{\phi \in W^{2,2}(0,\infty) \colon \cos\alpha\, \phi(0) = \sin\alpha\, \phi'(0)\}.$$

Now, given the precise form of the self-adjoint extensions $A_\alpha$ we determine their spectra and resolvents.

Define $\rho(A_\alpha) = \mathbb{C} \backslash \sigma(A_\alpha)$, where:

$$\sigma(A_\alpha) := \begin{cases} [0,\infty), & \alpha \in [0,\pi/2]. \\ [0,\ \infty) \cup \{-\cot^2\alpha\}, & \alpha \in (-\pi/2, 0). \end{cases}$$

We define $\sigma(A_\alpha)$ and $\rho(A_\alpha)$ in this way for brevity. We show in the following that they are the spectrum and resolvent set of $A_\alpha$ respectively. For fixed $\alpha \in (-\pi/2, \pi/2]$, define the Green's function, $g\colon (0,\infty) \times (0,\infty) \times \rho(A_\alpha) \to \mathbb{C}$ by:

$$g(x,\xi,\lambda) = A[\cos\alpha \sin(\sqrt{\lambda} x_<) + \sqrt{\lambda}\sin\alpha\cos(\sqrt{\lambda}x_<)]\exp(i\sqrt{\lambda}x_>),$$

where $A = [\sqrt{\lambda}(\cos\alpha - i\sqrt{\lambda}\sin\alpha)]^{-1}$, $x_> = \max\{x,\xi\}$, $x_< = \min\{x,\xi\}$ and $\sqrt{\lambda} = a + bi$, $b > 0$ is the unique square root of $\lambda$ in the upper-half plane (possible since $\lambda \notin [0,\infty)$). This function is given on p.487 of Stakgold [32]. Alternatively to the methods applied there we shall check directly that this defines the resolvent of $A_\alpha$.

Fix $\alpha \in (-\pi/2, \pi/2]$ and $\lambda \in \rho(A_\alpha)$. Given a function $f \in \mathcal{L}(0,\infty)$, define $G_\lambda(f)\colon (0,\infty) \to \mathbb{C}$ by:

$$G_\lambda(f)(x) = \int_0^\infty g(x,\xi,\lambda) f(\xi) d\xi.$$

We will show that this function $G_\lambda(f)$ is the resolvent for the linear operator $A_\alpha$. However first we discuss some relevant functional analysis, that is, the definition of a Holmgren kernel, after which we show that the function $g$ defined above is such a kernel.

**Definition G.2.** *(See Stakgold [32] p.324) Given a measure space* $(M, \Omega, \mu)$ *then a* ***Holmgren kernel*** *is a measurable function* $k: M \times M \to \mathbb{C}$ *s.t.:* $\exists C \in \mathbb{R}$ *s.t. for all* $\eta \in M$:

$$\int_M d\xi \int_M dx |k(x,\xi)||k(x,\eta)| < C$$

**Proposition G.3.** *A Holmgren kernel defines a bounded linear map* $K$: $\mathcal{L}^2(M) \to \mathcal{L}^2(M)$ *according to the prescription:*

$$K(u) = \int_M k(x,\xi)u(\xi)d\xi.$$

*It generates a bounded linear map* $K'$: $L^2(M) \to L^2(M)$ *via* $K'([u]) = [K(u)]$.

*Proof.* We first must show that $Ku$: $M \to \mathbb{C}$ is a measurable function and that if $u_1, u_2$: $M \to \mathbb{C}$ are measurable functions almost everywhere equal then $K(u_1) = K(u_2)$ almost everywhere. In fact it will be sufficient to show that if $u = 0$ *a.e.* then $K(u) = 0$ everywhere, which itself follows from standard properties of Lebesgue integration.

We now show that $K$ maps $\mathcal{L}^2$ functions to $\mathcal{L}^2$ functions and in fact is a bounded linear map of the semi-normed space $\mathcal{L}^2(M)$ to itself:

$$\begin{aligned}
\int_M dx|(Ku)(x)|^2 &= \int dx \left| \int_M d\xi \; k(x,\xi)u(\xi) \right|^2 \\
&\leq \int_M dx \left[ \int_M d\xi \; |k(x,\xi)|^{1/2} |k(x,\xi)|^{1/2} u(\xi) \right]^2 \\
&\leq \int_M dx \int_M d\xi \; |k(x,\xi)| \int_M d\eta \; |k(x,\eta)| |u(\eta)|^2 d\eta \\
&= \int_M d\eta \; |u(\eta)|^2 \int_M d\xi \int_M dx \; |k(x,\xi)||k(x,\eta)| \\
&\leq C \int_M d\eta \; |u(\eta)|^2 \\
&= C||\eta||^2
\end{aligned}$$

Thus $K$ generates a bounded linear map from $L^2(M)$ to itself. □

**Lemma G.4.** *A sufficient condition that a measurable function* $k$: $M \times M \to \mathbb{C}$ *is a Holmgren kernel is that it satisfies:*

$$\begin{aligned}
&\int_M d\xi \; |k(x,\xi)| < C \textit{ for all } x \in M \\
&\int_M dx \; |k(x,\xi)| < C \textit{ for all } \xi \in M.
\end{aligned}$$

*Proof.*

$$\begin{aligned}
\int_M d\xi \int_M dx \; |k(x,\xi)||k(x,\eta)| &= \int_M dx \; |k(x,\eta)| \int_M d\xi \; |k(x,\xi)| \\
&\leq C \int_M dx \; |k(x,\eta)| \\
&\leq C^2
\end{aligned}$$

□

Thus for instance if $k$ satisfies $\int_M d\xi \; |k(x,\xi)| < C \;\; \forall x \in M$ and $k(x,\xi) = k(\xi,x) \;\; \forall x,\xi$ then $k$ is a Holmgren kernel. It is this case that will be of use to us in determining the spectra of the self-adjoint extensions $A_\alpha$ of $A$.

**Proposition G.5** (The Green's function is a Holmgren kernel)**.** *Recalling the Green's function:*

$$g(x,\xi;\lambda) = A[\cos\alpha\sin(\sqrt{\lambda}x_<) + \sqrt{\lambda}\sin\alpha\cos(\sqrt{\lambda}x_<)]\exp(i\sqrt{\lambda}x_>).$$

*Then $g$ is a Holmgren kernel.*

*Proof.* We have already seen that $g(x,\xi;\lambda) = g(\xi,x,;\lambda))\ \ \forall x,\xi \in (0,\infty)$, thus we only have one integral to evaluate:

$$\begin{aligned}
&\int_0^\infty |g(x,\xi;\lambda)|d\xi \\
&= |A|\int_0^\infty |[\cos\alpha\sin(\sqrt{\lambda}x_<) + \sqrt{\lambda}\sin\alpha\cos(\sqrt{\lambda}x_<)]\exp(i\sqrt{\lambda}x_>)|d\xi \\
&= |A|\int_0^x |[\cos\alpha\sin(\sqrt{\lambda}\xi) + \sqrt{\lambda}\sin\alpha\cos(\sqrt{\lambda}\xi)]\exp(i\sqrt{\lambda}x)|d\xi \\
&\quad + |A|\int_x^\infty |[\cos\alpha\sin(\sqrt{\lambda}x) + \sqrt{\lambda}\sin\alpha\cos(\sqrt{\lambda}x)]\exp(i\sqrt{\lambda}\xi)|d\xi \\
&= |A|\int_0^x |[\cos\alpha\sin((a+ib)\xi) + (a+ib)\sin\alpha\cos((a+ib)\xi)]\exp(i(a+ib)x)|d\xi \\
&\quad + |A|\int_x^\infty |[\cos\alpha\sin((a+ib)x) + (a+ib)\sin\alpha\cos((a+ib)x)]\exp(i(a+ib)\xi)|d\xi \\
&\leq |A|e^{-bx}|\cos\alpha|\int_0^x |\sin((a+ib)\xi)|d\xi + |A|e^{-bx}|(a+ib)\sin\alpha|\int_0^x |\cos((a+ib)\xi)|d\xi \\
&\quad + |A||\cos\alpha\sin(a+ib)x|\int_0^\infty \exp(-b\xi)d\xi \\
&\quad + |A||(a+ib)\sin\alpha\cos(a+ib)x|\int_x^\infty \exp(-b\xi)d\xi \\
&= |A|e^{-bx}|\cos\alpha|\int_0^x |\sin((a+ib)\xi)|d\xi + |A|e^{-bx}|(a+ib)\sin\alpha|\int_0^x |\cos((a+ib)\xi)|d\xi \\
&\quad + |A||\cos\alpha\sin(a+ib)x|\frac{1}{b}\exp(-bx) + |A||(a+ib)\sin\alpha\cos(a+ib)x|\frac{1}{b}\exp(-bx) \\
&\leq |A||\cos\alpha|\frac{\sqrt{2}}{be} + |A||(a+ib)\sin\alpha|\frac{\sqrt{2}}{be} + |A||\cos\alpha|\frac{1}{b} + |A||(a+ib)\sin\alpha|\frac{1}{b} \\
&\leq \frac{|A|}{b}(\frac{\sqrt{2}}{e} + |a+ib|\frac{\sqrt{2}}{e} + 1 + |a+ib|) \\
&= \frac{|A|}{b}(1+\frac{\sqrt{2}}{e})(1+|a+ib|) \\
&= \frac{|A|}{b}(1+\frac{\sqrt{2}}{e})(1+|\lambda|^{1/2}),
\end{aligned}$$

where we have used the four inequalities of the following lemma. □

**Lemma G.6.** *The following are true for all $x, b > 0$:*

1. $e^{-\beta x}|\sin((\alpha + i\beta)x)| \leq 1$.
2. $e^{-\beta x}|\cos((\alpha + i\beta)x)| \leq 1$.
3. $e^{-\beta x}\int_0^x |\sin[(\alpha + i\beta)\xi]|d\xi \leq \frac{\sqrt{2}}{\beta e}$.
4. $e^{-\beta x}\int_0^x |\cos[(\alpha + i\beta)\xi]|d\xi \leq \frac{\sqrt{2}}{\beta e}$.

*Proof.* For the first inequality, we have

$$\begin{aligned}
e^{-2\beta x}|\sin((\alpha + i\beta)x)|^2 &= e^{-2\beta x}|\sin(\alpha x)\cosh(\beta x) + i\cos(\alpha x)\sinh(\beta x)|^2 \\
&= e^{-2\beta x}[\sin^2(\alpha x)\cosh^2(\beta x) + \cos^2(\alpha x)\sinh^2(\beta x)] \\
&= e^{-2\beta x}\begin{bmatrix} (1 - \cos^2(\alpha x))(1 + \sinh^2(\beta x)) \\ + \cos^2(\alpha x)\sinh^2(\beta x) \end{bmatrix} \\
&= e^{-2\beta x}[1 + \sinh^2(\beta x) - \cos^2(\alpha x)] \\
&\leq e^{-2\beta x}[1 + \frac{1}{4}e^{2\beta x} + \frac{1}{4}e^{-2\beta x} - \frac{1}{2}] \\
&= \frac{1}{4} + \frac{1}{2}e^{-2\beta x} + \frac{1}{4}e^{-4\beta x} \\
&\leq 1.
\end{aligned}$$

Similarly for the second inequality. Now, for the third inequality:

$$\begin{aligned}
e^{-\beta x}\int_0^x |\sin[(\alpha+i\beta)\xi]|d\xi &= e^{-\beta x}\int_0^x (1+\sinh^2(\beta\xi)-\cos^2(\alpha\xi))^{1/2}\\
&\le e^{-\beta x}\int_0^x (2+\sinh^2(\beta\xi))^{1/2}\\
&\le e^{-\beta x}\int_0^x (\sqrt{2}+\sinh(\beta\xi))\\
&= e^{-\beta x}[\sqrt{2}\xi+\frac{1}{2\beta}(e^{\beta\xi}+e^{-\beta\xi})]_0^x\\
&= e^{-\beta x}[\sqrt{2}x+\frac{1}{2\beta}e^{\beta x}+\frac{1}{2\beta}e^{-\beta x}-\frac{1}{\beta}]\\
&= \sqrt{2}xe^{-\beta x}+\frac{1}{2\beta}+\frac{1}{2\beta}e^{-2\beta x}-\frac{1}{\beta}e^{-\beta x}\\
&\le \frac{\sqrt{2}}{\beta e}+\frac{1}{2\beta}+\frac{1}{2\beta}-\frac{1}{\beta}\\
&= \frac{\sqrt{2}}{\beta e}.
\end{aligned}$$

Where we have used the simple inequality $(a^2+b^2)^{1/2}\le a+b \quad \forall a,b\ge 0$. The fourth inequality follows similarly. □

We now show that:

**Proposition G.7.** *For each $\alpha\in(-\pi/2,\pi/2]$, the Green's function for $A_\alpha$ is the function $g\colon(0,\infty)\times(0,\infty)\times\rho(A_\alpha)\to\mathbb{C}$ defined above as:*

$$g(x,\xi,\lambda)=A[\cos\alpha\sin(\sqrt{\lambda}x_<)+\sqrt{\lambda}\sin\alpha\cos(\sqrt{\lambda}x_<)]\exp(i\sqrt{\lambda}x_>),$$

*where $A=[\sqrt{\lambda}(\cos\alpha-i\sqrt{\lambda}\sin\alpha)]^{-1}$, $x_>=\max\{x,\xi\}$, $x_<=\min\{x,\xi\}$ and $\sqrt{\lambda}=a+bi$, $b>0$ is the unique square root of $\lambda$ in the upper-half plane (possible since $\lambda\notin[0,\infty)$).*

*Proof.* Recalling,

$$G_\lambda(f)(x)=A\int_0^\infty[\cos\alpha\sin(\sqrt{\lambda}x_<)+\sqrt{\lambda}\sin\alpha\cos(\sqrt{\lambda}x_<)]\exp(i\sqrt{\lambda}x_>)f(\xi)d\xi$$

we show here that $(A_\alpha-\lambda I)\circ G_\lambda=I_{L^2(0,\infty)}$.

Note first that for fixed $x$, the term in the square brackets is a bounded function of $\xi$ and the remaining two terms are square integrable. Thus the integral converges absolutely for all $x \in (0, \infty)$.

$G_\lambda(f)(0) = A\sqrt{\lambda}\sin\alpha \int_0^\infty \exp(i\sqrt{\lambda}\xi) f(\xi) d\xi$

$$\begin{aligned} G_\lambda(f)(x) = A &\int_0^x \left[\cos\alpha \sin(\sqrt{\lambda}\xi) + \sqrt{\lambda}\sin\alpha\cos(\sqrt{\lambda}\xi)\right] \exp(i\sqrt{\lambda}x) f(\xi) d\xi \\ &+ A\int_x^\infty \left[\cos\alpha\sin(\sqrt{\lambda}x) + \sqrt{\lambda}\sin\alpha\cos(\sqrt{\lambda}x)\right]\exp(i\sqrt{\lambda}\xi) f(\xi) d\xi \end{aligned}$$

$$\begin{aligned} &A^{-1}G_\lambda(f)'(x) \\ &= i\sqrt{\lambda}\int_0^x \left[\cos\alpha\sin(\sqrt{\lambda}\xi) + \sqrt{\lambda}\sin\alpha\cos(\sqrt{\lambda}\xi)\right]\exp(i\sqrt{\lambda}x) f(\xi) d\xi \\ &\quad + \left[\cos\alpha\sin(\sqrt{\lambda}x) + \sqrt{\lambda}\sin\alpha\cos(\sqrt{\lambda}x)\right]\exp(i\sqrt{\lambda}x) f(x) \\ &\quad + \left[\sqrt{\lambda}\cos\alpha\cos(\sqrt{\lambda}x) - \lambda\sin\alpha\sin(\sqrt{\lambda}x)\right]\int_x^\infty \exp(i\sqrt{\lambda}\xi) f(\xi) d\xi \\ &\quad - \left[\cos\alpha\sin\sqrt{\lambda}x) + \sqrt{\lambda}\sin\alpha\cos(\sqrt{\lambda}x)\right]\exp(i\sqrt{\lambda}x) f(x) \\ &= i\sqrt{\lambda}\int_0^x \left[\cos\alpha\sin(\sqrt{\lambda}\xi) + \sqrt{\lambda}\sin\alpha\cos(\sqrt{\lambda}\xi)\right]\exp(i\sqrt{\lambda}x) f(\xi) d\xi \\ &\quad + \left[\sqrt{\lambda}\cos\alpha\cos(\sqrt{\lambda}x) - \lambda\sin\alpha\sin(\sqrt{\lambda}x)\right]\int_x^\infty \exp(i\sqrt{\lambda}\xi) f(\xi) d\xi \end{aligned}$$

Thus in particular: $G_\lambda(f)'(0) = A\sqrt{\lambda}\cos\alpha \int_0^\infty \exp(i\sqrt{\lambda}\xi) f(\xi) d\xi$ and so

$$\cos\alpha G_\lambda(f)(0) = \sin\alpha G'_\lambda(f)(0).$$

Thus $G_\lambda$: $L^2(0,\infty) \to D(A_\alpha)$ for $\lambda \in \rho(A_\alpha)$. We now show that $G_\lambda$ is actually the Green's function of $A_\alpha$. Differentiating again we have:

$$\begin{aligned} A^{-1}G_\lambda(f)''(x) = -\lambda &\int_0^x \left[\cos\alpha\sin(\sqrt{\lambda}\xi) + \sqrt{\lambda}\sin\alpha\cos(\sqrt{\lambda}\xi)\right]\exp(i\sqrt{\lambda}x) f(\xi) d\xi \\ &+ i\sqrt{\lambda}\left[\cos\alpha\sin(\sqrt{\lambda}x) + \sqrt{\lambda}\sin\alpha\cos(\sqrt{\lambda}x)\right]\exp(i\sqrt{\lambda}x) f(x) \\ &+ \left[-\lambda\cos\alpha\sin(\sqrt{\lambda}x) - \lambda\sqrt{\lambda}\sin\alpha\cos(\sqrt{\lambda}x)\right]\int_x^\infty \exp(i\sqrt{\lambda}\xi) f(\xi) d\xi \\ &- \left[\sqrt{\lambda}\cos\alpha\cos(\sqrt{\lambda}x) - \lambda\sin\alpha\sin(\sqrt{\lambda}x)\right]\exp(i\sqrt{\lambda}x) f(x). \end{aligned}$$

Thus:

$$
\begin{aligned}
&-G_\lambda(f)''(x) - \lambda G_\lambda(f)(x) \\
&= -A\begin{bmatrix} i\sqrt{\lambda}(\cos\alpha\sin(\sqrt{\lambda}x) + \sqrt{\lambda}\sin\alpha\cos(\sqrt{\lambda}x)) \\ -\sqrt{\lambda}\cos\alpha\cos(\sqrt{\lambda}x) + \lambda\sin\alpha\sin(\sqrt{\lambda}x) \end{bmatrix}\exp(i\sqrt{\lambda}x)f(x) \\
&= -A\begin{bmatrix} \sqrt{\lambda}\cos\alpha(-\cos(\sqrt{\lambda}x) + i\sin(\sqrt{\lambda}x)) \\ +\lambda\sin\alpha(i\cos(\sqrt{\lambda}x) + \sin(\sqrt{\lambda}x) \end{bmatrix}\exp(i\sqrt{\lambda}x)f(x) \\
&= -A\left[-\sqrt{\lambda}\cos\alpha\exp(-i\sqrt{\lambda}x) + \lambda\sin\alpha i\exp(-i\sqrt{\lambda}x)\right]\exp(i\sqrt{\lambda}x)f(x) \\
&= -A\left[-\sqrt{\lambda}\cos\alpha + \lambda i\sin\alpha\right]f(x) \\
&= A\sqrt{\lambda}\left[\cos\alpha - i\sqrt{\lambda}\sin\alpha\right]f(x) \\
&= f(x),
\end{aligned}
$$

and so $(A_\alpha - \lambda I)(G_\lambda(f)) = -G_\lambda(f)''(x) - \lambda G_\lambda(f) = f \;\; \forall f \in L^2(0,\infty)$. Now, to prove that $G_\lambda \circ (A_\alpha - \lambda I) = I_{D(A_\alpha)}$. Let $\phi \in D(A_\alpha)$, then

$$
\begin{aligned}
&G_\lambda(A_\alpha\phi)(x) \\
&\quad = A\int_0^\infty g(x,\xi,\lambda)(A_\alpha\phi)(\xi)d\xi \\
&\quad = -A\int_0^\infty g(x,\xi,\lambda)\phi''(\xi)d\xi \\
&\quad = -A\int_0^\infty \left[\cos\alpha\sin(\sqrt{\lambda}x_<) + \sqrt{\lambda}\sin\alpha\cos(\sqrt{\lambda}x_<)\right]\exp(i\sqrt{\lambda}x_>)\phi''(\xi)d\xi
\end{aligned}
$$

And:

$$
\begin{aligned}
&-A^{-1}G_\lambda(A_\alpha\phi)(x) \\
&\qquad = \int_0^x \left[\cos\alpha\sin(\sqrt{\lambda}\xi) + \sqrt{\lambda}\sin\alpha\cos(\sqrt{\lambda}\xi)\right]\exp(i\sqrt{\lambda}x)\phi''(\xi)d\xi \\
&\qquad + \int_x^\infty \left[\cos\alpha\sin(\sqrt{\lambda}x) + \sqrt{\lambda}\sin\alpha\cos(\sqrt{\lambda}x)\right]\exp(i\sqrt{\lambda}\xi)\phi''(\xi)d\xi
\end{aligned}
$$

We shall proceed by integrating by parts so as to convert the $\phi''$ term into $\phi$. Doing so will produce various 'boundary terms', some of which will cancel, whereas others will cancel upon imposition of the boundary conditions. We shall consider the two terms in the right-hand-side of the last expression separately at first for simplicity.

Thus the first term is proportional to:

$$\begin{aligned}
&\int_0^x \left[\cos\alpha\sin(\sqrt{\lambda}\xi) + \sqrt{\lambda}\sin\alpha\cos(\sqrt{\lambda}\xi)\right]\phi''(\xi)d\xi \\
&\quad = \left[(\cos\alpha\sin(\sqrt{\lambda}\xi) + \sqrt{\lambda}\sin\alpha\cos(\sqrt{\lambda}\xi)\phi'(\xi)\right]_0^x \\
&\qquad - \int_0^\infty \left[\sqrt{\lambda}\cos\alpha\cos(\sqrt{\lambda}\xi) - \lambda\sin\alpha\sin(\sqrt{\lambda}\xi)\right]\phi'(\xi)d\xi \\
&\quad = (\cos\alpha\sin(\sqrt{\lambda}x) + \sqrt{\lambda}\sin\alpha\cos(\sqrt{\lambda}x)\phi'(x) - \sqrt{\lambda}\sin\alpha\phi'(0) \\
&\qquad - \left[(\sqrt{\lambda}\cos\alpha\cos(\sqrt{\lambda}\xi) - \lambda\sin\alpha\sin(\sqrt{\lambda}\xi)\phi(\xi)\right]_0^x \\
&\qquad + \int_0^x \left[-\lambda\cos\alpha\sin(\sqrt{\lambda}\xi) - \lambda^{3/2}\sin\alpha\cos(\sqrt{\lambda}\xi)\right]\phi(\xi)d\xi \\
&\quad = (\cos\alpha\sin(\sqrt{\lambda}x) + \sqrt{\lambda}\sin\alpha\cos(\sqrt{\lambda}x)\phi'(x) - \sqrt{\lambda}\sin\alpha\phi'(0) \\
&\qquad - \sqrt{\lambda}\cos\alpha\cos(\sqrt{\lambda}x) + \lambda\sin\alpha\sin(\sqrt{\lambda}x)\phi(x) + \sqrt{\lambda}\cos\alpha\phi(0) \\
&\qquad - \lambda\int_0^x \left[\cos\alpha\sin(\sqrt{\lambda}\xi) + \sqrt{\lambda}\sin\alpha\cos(\sqrt{\lambda}\xi)\right]\phi(\xi)d\xi
\end{aligned}$$

The second term is proportional to:

$$\begin{aligned}
&\int_x^\infty \exp(i\sqrt{\lambda}\xi)\phi''(\xi)d\xi \\
&\quad = \left[\exp(i\sqrt{\lambda}\xi)\phi'(\xi)\right]_x^\infty - i\sqrt{\lambda}\int_x^\infty \exp(i\sqrt{\lambda}\xi)\phi'(\xi)d\xi \\
&\quad = -\exp(i\sqrt{\lambda}x)\phi'(x) - i\sqrt{\lambda}\left[\exp(i\sqrt{\lambda}\xi)\phi(\xi)\right]_0^\infty - \lambda\int_x^\infty \exp(i\sqrt{\lambda}\xi)\phi(\xi)d\xi \\
&\quad = -\exp(i\sqrt{\lambda}x)\phi'(x) + i\sqrt{\lambda}\exp(i\sqrt{\lambda}x)\phi(x) - \lambda\int_x^\infty \exp(i\sqrt{\lambda}\xi)\phi(\xi)d\xi
\end{aligned}$$

Combining these terms we find:

$$
\begin{aligned}
&-A^{-1}G_\lambda(A_\alpha\phi)(x)\\
&=\exp(i\sqrt{\lambda}x)\begin{bmatrix} (\cos\alpha\sin(\sqrt{\lambda}x)+\sqrt{\lambda}\sin\alpha\cos(\sqrt{\lambda}x))\phi'(x)-\sqrt{\lambda}\sin\alpha\phi'(0)\\ -\sqrt{\lambda}\cos\alpha\cos(\sqrt{\lambda}x)\phi(x)+\lambda\sin\alpha\sin(\sqrt{\lambda}x)\phi(x)+\sqrt{\lambda}\cos\alpha\phi(0)\\ -\lambda\int_0^x\left[\cos\alpha\sin(\sqrt{\lambda}\xi)+\sqrt{\lambda}\sin\alpha\cos(\sqrt{\lambda}\xi)\right]\phi(\xi)d\xi \end{bmatrix}\\
&+(\cos\alpha\sin(\sqrt{\lambda}x)+\sqrt{\lambda}\sin\alpha\cos(\sqrt{\lambda}x))\times\\
&\left[-\exp(i\sqrt{\lambda}x)\phi'(x)+i\sqrt{\lambda}\exp(i\sqrt{\lambda}x)\phi(x)-\lambda\int_x^\infty\exp(i\sqrt{\lambda}\xi)\phi(\xi)d\xi\right]
\end{aligned}
$$

Cancelling, we therefore get that

$$
\begin{aligned}
&-A^{-1}G_\lambda(A_\alpha\phi)(x)=\\
&\exp(i\sqrt{\lambda}x)\begin{bmatrix} \cancel{\cos\alpha\sin(\sqrt{\lambda}x)\phi'(x)}+\cancel{\sqrt{\lambda}\sin\alpha\cos(\sqrt{\lambda}x)\phi'(x)}\\ \cancel{-\cos\alpha\sin(\sqrt{\lambda}x)\phi'(x)}-\cancel{\sqrt{\lambda}\sin\alpha\cos(\sqrt{\lambda}x)\phi'(x)}\\ -\sqrt{\lambda}\cos\alpha\cos(\sqrt{\lambda}x)\phi(x)+\lambda\sin\alpha\sin(\sqrt{\lambda}x)\phi(x)+\sqrt{\lambda}\cos\alpha\phi(0)\\ +i\sqrt{\lambda}\cos\alpha\sin(\sqrt{\lambda}x)\phi(x)+i\sqrt{\lambda}\sin\alpha\cos(\sqrt{\lambda}x)\phi(x)\\ \cancel{-\sqrt{\lambda}\sin\alpha\phi'(0)+\sqrt{\lambda}\cos\alpha\phi(0)} \end{bmatrix}\\
&-\lambda\int_0^x\left[\cos\alpha\sin(\sqrt{\lambda}\xi)+\sqrt{\lambda}\sin\alpha\cos(\sqrt{\lambda}\xi)\right]\exp(i\sqrt{\lambda}x)\phi''(\xi)d\xi\\
&-\lambda\int_x^\infty\left[\cos\alpha\sin(\sqrt{\lambda}x)+\sqrt{\lambda}\sin\alpha\cos(\sqrt{\lambda}x)\right]\exp(i\sqrt{\lambda}\xi)\phi''(\xi)d\xi
\end{aligned}
$$

$$
\begin{aligned}
&=\exp(i\sqrt{\lambda}x)\begin{bmatrix} -\sqrt{\lambda}\cos\alpha(\cos(\sqrt{\lambda}x)-i\sin(\sqrt{\lambda}x))\phi(x)\\ +i\lambda\sin\alpha(\cos(\sqrt{\lambda}x)-i\sin(\sqrt{\lambda}x))\phi(x) \end{bmatrix}\\
&\qquad-\lambda\int_0^\infty\left[\cos\alpha\sin(\sqrt{\lambda}x_<)+\sqrt{\lambda}\sin\alpha\cos(\sqrt{\lambda}x_<)\right]\exp(i\sqrt{\lambda}x_>)\phi''(\xi)d\xi\\
&=\left[-\sqrt{\lambda}\cos\alpha\phi(x)+i\lambda\sin\alpha\phi(x)\right]-\lambda A^{-1}G_\lambda(\phi)(x)\\
&=-A^{-1}\phi(x)-\lambda A^{-1}G_\lambda(\phi)(x)
\end{aligned}
$$

Consequently, $G_\lambda(A_\alpha\phi)(x)=\phi(x)+\lambda(G_\lambda(\phi))(x)$ and so

$$G_\lambda((A_\alpha-\lambda)\phi)=G_\lambda(A_\alpha\phi-\lambda\phi)=G_\lambda(A_\alpha\phi)-\lambda G_\lambda(\phi)=\phi\quad\forall\phi\in D(A_\alpha).$$

Therefore $G_\lambda$ is the inverse of $A_\alpha - \lambda$ as required. It follows from Proposition G.5 and Proposition G.3 that $G_\lambda$ is a bounded linear operator. □

Having proven that $G_\lambda$ is the Green's function for $A_\alpha$ at $\lambda \in \rho(A_\alpha)$, we shall show that $\rho(A_\alpha)$, defined earlier, is contained in the resolvent set for $A_\alpha$. We shall then proceed to prove that $[0,\infty) \subseteq \sigma_{cont}(A_\alpha)$, the continuous spectrum of $A_\alpha$, for all $\alpha \in (-\pi/2, \pi/2]$ and that for $\alpha \in (-\pi/2, 0)$, we have $\{-\cot^2\alpha\} \subseteq \sigma_{pp}(A_\alpha)$. Combining all these together yields:

$$\sigma(A_\alpha) = \begin{cases} [0,\infty), & \alpha \in [0,\pi/2]. \\ [0,\infty) \cup \{-\cot^2\alpha\}\,, & \alpha \in (-\pi/2, 0). \end{cases}$$

and

1. $\sigma_{cont}(A_\alpha) = [0,\infty)$ for all $\alpha$.
2. If $\alpha \in [0, \frac{\pi}{2}]$: $\sigma(A_\alpha) = \sigma_{cont}(A_\alpha) = [0,\infty)$.
3. If $\alpha \in (-\frac{\pi}{2}, 0)$: $\sigma_{pp}(A_\alpha) = \sigma_{disc}(A_\alpha) = \{-\cot^2\alpha\}$.

We first remind the reader of the decomposition of the spectrum being used, before proving these statements. Let $A$ be a closed linear operator $A$ on a Hilbert space $H$ and $\lambda \in \mathbb{C}$. Then we define

$\lambda \in \sigma(A)$ iff $A - \lambda$ is not invertible (to a bounded linear operator $H \to H$).

$\lambda \in \sigma_{pp}(A)$ iff $A - \lambda$ is not injective: $\exists \phi \in D(A)\backslash\{0\}$: $A\phi = \lambda\phi$.

$\lambda \in \sigma_{cont}(A)$ iff $A - \lambda$ is injective and $Im(A - \lambda)$ is a proper dense subspace of $H$.

$\lambda \in \sigma_{res}(A)$ iff $A - \lambda$ injective and $Im(A - \lambda)$ is not dense in $H$.

$\sigma(A)$, $\sigma_{pp}(A)$, $\sigma_{cont}(A)$ and $\sigma_{res}(A)$ are respectively called the spectrum, pure point spectrum, continuous spectrum and residual spectrum of $A$. Note:

$$\sigma(A) = \sigma_{pp}(A) \cup \sigma_{cont}(A) \cup \sigma_{res}(A)$$

A necessary and sufficient condition for $\lambda \in \sigma_{cont}(A)$ is that $A - \lambda$ is injective and $\nexists c > 0$ s.t. $\|(A-\lambda)x\| \geq c\|x\| \quad \forall x \in D(A)$. This is clear as the continuity of the inverse would contradict this statement. Thus if $A - \lambda$ is injective, then $\lambda \in \sigma_{cont}(A)$ iff there exists a sequence $(u_n)_{n\geq 1}$ in $D(A)\backslash\{0\}$ with:

$$\frac{\|(A-\lambda)u_n\|}{\|u_n\|} \to 0 \text{ as } n \to \infty$$

**Proposition G.8.** $[0,\infty) \subseteq \sigma_{cont}(A_\alpha)$ *for all* $\alpha \in (-\pi/2, \pi/2]$.

*Proof.* (We shall be using the methods of Stakgold [32]). Given $\lambda > 0$, we define $u_n$ as follows. Pick $F \in C^2[0,1]$ s.t. $F(0) = \sqrt{\lambda}\sin\,\alpha$, $F'(0) = \sqrt{\lambda}\cos\,\alpha$ and $F(1) = F'(1) = 0$, where $\sqrt{\lambda} > 0$. Define the sequence of functions

$$u_n(x) = \begin{cases} \cos\,\alpha \sin\,\sqrt{\lambda}x + \sqrt{\lambda}\sin\,\alpha\cos\,\sqrt{\lambda}x, & 0 \leq x \leq l_n \\ F(x-l_n), & l_n \leq x \leq l_n+1 \\ 0, & x \geq l_n + 1 \end{cases}$$

It is readily seen that $u_n \in D(A_\alpha)$ for all $n$ and:

$$A_\alpha u_n(x) = \begin{cases} \lambda u_n, & 0 \leq x \leq l_n \\ -F''(x-l_n), & l_n \leq x \leq l_n+1 \\ 0, & x \geq l_n + 1. \end{cases}$$

Thus:

$$\begin{aligned} \|u_n\|^2 &= \int_0^{l_n} (\cos\,\alpha\sin\,\sqrt{\lambda}x + \sqrt{\lambda}\sin\,\alpha\cos\,\sqrt{\lambda}x)2dx + \int_{l_n}^{l_n+1} F(x-l_n)^2 dx \\ &= \frac{1}{2}\cos^2\alpha l_n + \frac{1}{2}\lambda\sin^2\alpha l_n + \int_0^1 F^2 dx \end{aligned}$$

and:

$$\begin{aligned} \|(A_\alpha - \lambda)u_n\|^2 &= \int_{l_n}^{l_n+1} (-F''(x-l_n) - \lambda F(x-l_n))^2 dx \\ &= \int_0^1 (F''(x) + \lambda F(x))^2 dx \\ &=: C \end{aligned}$$

Thus

$$\frac{||(A_\alpha - \lambda)u_n||^2}{||u_n||^2} = \frac{C}{\frac{1}{2}(\cos^2\alpha + \lambda\sin^2\alpha)l_n + \int_0^1 F^2 dx} \to 0$$

as $n \to \infty$.

We show similarly that $0 \in \sigma_{cont}(A_\alpha)$ for all $\alpha$:

Let $F, G \in C^2[0,1]$ satisfy: $F(0) = F'(0) = G(0) = 1$, $F(1) = F'(1) = G'(0) = G(1) = G'(1) = 0$. Then define

$$u_n(x) = \begin{cases} x\cos\ \alpha + \sin\ \alpha, & 0 \le x \le n \\ n\cos\ \alpha F(\frac{x-n}{n}) + \sin\ \alpha G(\frac{x-n}{n}), & n \le x \le 2n \\ 0, & x \ge 2n \end{cases}$$

It follows that $u_n \in D(A_\alpha)$ for all $n$ and:

$$||u_n||^2 = \left[ \begin{array}{l} n^3\cos^2\ \alpha\left(\frac{1}{3} + \int_0^1 F^2 dx\right) + n^2\sin 2\alpha\left(\int_0^1 FG dx + 1/2\right) \\ +n\sin^2\alpha\left(1 + \int_0^1 G^2 dx\right) \end{array} \right]$$

and:

$$||A_\alpha u_n||^2 = \frac{1}{n}\int_0^1 [n^2\cos^2\alpha F'' + \sin^2\alpha G'' + n\sin 2\alpha F''G'']dx.$$

$$\frac{||A_\alpha u_n||^2}{||u_n||^2}$$

$$= \frac{n^2\cos^2\alpha\int_0^1 F''dx + \sin^2\alpha\int_0^1 G''dx + n\sin 2\alpha\int_0^1 F''G''dx}{n^4\cos^2\ \alpha(\frac{1}{3} + \int_0^1 F^2 dx) + n^3\sin 2\alpha(\int_0^1 FGdx + 1/2) + n^2\sin^2\alpha(1 + \int_0^1 G^2 dx)}$$

$\to 0$ as $n \to \infty$.

Note that we can check the last statement in the two cases: $\alpha \neq \pi/2$ and $\alpha = \pi/2$. □

The previous statement, regarding the pure point spectrum of $A_\alpha$ for $\alpha \in (-\pi/2, 0)$ is easily proven as follows:

**Lemma G.9.** *For $\alpha \in (-\pi/2, 0)$, we have $-\cot^2\alpha \in \sigma_{pp}(A_\alpha)$.*

*Proof.* Fixing $\alpha \in (-\pi/2, 0)$, let $\phi_\alpha(x) = \exp(x\cot\alpha)$. Then as $\cot\alpha < 0$, we know $[\phi_\alpha] \in L^2(0,\infty)$. Clearly $[\phi_\alpha] \in W^{2,2}(0,\infty)$ and $\cos\alpha\,\phi_\alpha(0) = \cos\alpha = \sin\alpha\cot\alpha = \sin\alpha\,\phi'_\alpha(0)$. So, $[\phi_\alpha] \in D(A_\alpha)$ and $A_\alpha[\phi_\alpha] = -[\phi''_\alpha] = -\cot^2\alpha\,[\phi_\alpha]$. □

# H The Self-Adjoint Extensions of minus the Laplacian on $(0, a)$ and Spectral Analysis

Given $\theta_{11}, \theta_{22} \in \mathbb{R}, \theta_{12} \in \mathbb{C}$ denote the self-adjoint $2 \times 2$ complex matrix

$$\theta = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \overline{\theta}_{12} & \theta_{22} \end{pmatrix}.$$

The domain of the extension $D(A_\theta)$ is defined as those elements $\phi \in W^{2,2}(0, a)$ such that:

$$\theta_{11}\phi(0) - \phi'(0) + \theta_{12}\phi(a) = 0$$
$$\overline{\theta_{12}}\phi(0) + \theta_{22}\phi(a) + \phi'(a) = 0$$

**Proposition H.1.** *The Green's function for the s.a.e. of the first kind for $\lambda \in \rho(A_\theta)\backslash\{0\}$ is given by the following.*

$$g(x, y; \lambda) = A \left[ \begin{array}{l} \lambda \cos\sqrt{\lambda}(a - x_>) \cos\sqrt{\lambda}x_< + \theta_{22}\sqrt{\lambda}\sin\sqrt{\lambda}(a - x_>)\cos\sqrt{\lambda}x_< \\ +\theta_{11}\sqrt{\lambda}\cos\sqrt{\lambda}(a - x_>)\sin\sqrt{\lambda}x_< + \theta_{11}\theta_{22}\sin\sqrt{\lambda}(a - x_>)\sin\sqrt{\lambda}x_< \\ +|\theta_{12}|^2\sin\sqrt{\lambda}(x_> - a)\sin\sqrt{\lambda}x_< + C(x, y)(\theta_{12})\sqrt{\lambda}\sin\sqrt{\lambda}(x_< - x_>) \end{array} \right]$$

*where:*

$$A^{-1} = \sqrt{\lambda} \left[ \begin{array}{l} \theta_{11}\sqrt{\lambda}\cos\sqrt{\lambda}a + \theta_{22}\sqrt{\lambda}\cos\sqrt{\lambda}a - \lambda\sin\sqrt{\lambda}a \\ +\theta_{11}\theta_{22}\sin\sqrt{\lambda}a - |\theta_{12}|^2\sin\sqrt{\lambda}a + 2\Re(\theta_{12})\sqrt{\lambda} \end{array} \right],$$

*and for $k \in \mathbb{C}$:*

$$C(x, y)(k) = \begin{cases} k & \text{if } x < y. \\ \overline{k} & \text{if } x > y. \end{cases}$$

*Remark.* We have not specified which square root $\sqrt{\lambda}$ to take as it can be seen that the expression for $g$ is invariant under replacing $\sqrt{\lambda}$ with $-\sqrt{\lambda}$.

*Proof.* We first show that, for each $\theta$ and $\lambda \in \rho(A_\theta)$, $G_\lambda$: $L^2(0,a) \to D(A_\theta)$. By definition:

$$G_\lambda(f)(x) = \int_0^a g(x,y;\lambda)f(y)dy,$$

and so

$$\begin{aligned} &G_\lambda(f)(0) \\ &\quad = \int_0^a g(0,y;\lambda)f(y)dy \\ &\quad = A\int_0^a \left[\lambda\cos\sqrt{\lambda}(a-y) + \theta_{22}\sqrt{\lambda}\sin\sqrt{\lambda}(a-y) - \theta_{12}\sqrt{\lambda}\sin\sqrt{\lambda}y\right] f(y)dy. \end{aligned}$$

Similarly:

$$\begin{aligned} G_\lambda(f)(a) &= \int_0^a g(a,y;\lambda)f(y)dy \\ &= A\int_0^a \left[\lambda\cos\sqrt{\lambda}y + \theta_{11}\sqrt{\lambda}\sin\sqrt{\lambda}y + \overline{\theta_{12}}\sqrt{\lambda}\sin\sqrt{\lambda}(y-a)\right] f(y)dy \end{aligned}$$

We have the following "boundary values":

$$G_\lambda(f)'(0) = A\int_0^a \left[\begin{array}{l} \theta_{11}\lambda\cos\sqrt{\lambda}(a-y) + \theta_{11}\theta_{22}\sqrt{\lambda}\sin\sqrt{\lambda}(a-y) \\ +|\theta_{12}|^2\sqrt{\lambda}\sin\sqrt{\lambda}(y-a) + \theta_{12}\lambda\cos\sqrt{\lambda}y \end{array}\right] f(y)dy$$

$$G_\lambda(f)'(a) = A\int_0^a \left[\begin{array}{l} -\theta_{22}\lambda\cos\sqrt{\lambda}y - \theta_{11}\theta_{22}\sqrt{\lambda}\sin\sqrt{\lambda}y \\ +|\theta_{12}|^2\sqrt{\lambda}\sin\sqrt{\lambda}y - \overline{\theta_{12}}\lambda\cos\sqrt{\lambda}(y-a) \end{array}\right] f(y)dy$$

Using this we check that $G_\lambda(f) \in D(A_\theta)$:

$$\begin{aligned} &\theta_{11}G_\lambda(f)(0) + \theta_{12}G_\lambda(f)(a) \\ &= A\int_0^a \left[\begin{array}{l} \theta_{11}\lambda\cos\sqrt{\lambda}(a-y) + \theta_{11}\theta_{22}\sqrt{\lambda}\sin\sqrt{\lambda}(a-y) \\ -\theta_{11}\theta_{12}\sqrt{\lambda}\sin\sqrt{\lambda}y + \theta_{12}\lambda\cos\sqrt{\lambda}y \\ +\theta_{11}\theta_{12}\sqrt{\lambda}\sin\sqrt{\lambda}y + |\theta_{12}|^2\sqrt{\lambda}\sin\sqrt{\lambda}(y-a) \end{array}\right] f(y)dy \\ &= G_\lambda(f)'(0) \end{aligned}$$

and

$$
\begin{aligned}
&\overline{\theta_{12}} G_\lambda(f)(0) + \theta_{22} G_\lambda(f)(a) \\
&= A \int_0^a \left[ \begin{array}{l} \overline{\theta_{12}} \lambda \cos\sqrt{\lambda}(a-y) + \overline{\theta_{12}}\theta_{22}\sqrt{\lambda}\sin\sqrt{\lambda}(a-y) \\ -|\theta_{12}|^2\sqrt{\lambda}\sin\sqrt{\lambda}y + \theta_{22}\lambda\cos\sqrt{\lambda}y \\ +\theta_{11}\theta_{22}\sqrt{\lambda}\sin\sqrt{\lambda}y + \overline{\theta_{12}}\theta_{22}\sqrt{\lambda}\sin\sqrt{\lambda}(y-a) \end{array} \right] f(y)dy \\
&= -G_\lambda(f)'(a).
\end{aligned}
$$

In order to show that $g$ is the Green's function and to greatly simplify the calculations we analyse the effect of each of the terms in $g$ at first separately. Consider the following

$g \to [-G_\lambda(f)'' - \lambda G_\lambda(f)](x)$

We summarise here this map for the following integral kernels:

$$
\begin{aligned}
\cos\sqrt{\lambda}(a-x_>)\cos\sqrt{\lambda}x_< &\to -\sqrt{\lambda}\sin\sqrt{\lambda}af(x) \\
\sin\sqrt{\lambda}(a-x_>)\cos\sqrt{\lambda}x_< &\to \sqrt{\lambda}\cos\sqrt{\lambda}af(x) \\
\cos\sqrt{\lambda}(a-x_>)\sin\sqrt{\lambda}x_< &\to \sqrt{\lambda}\cos\sqrt{\lambda}af(x) \\
\sin\sqrt{\lambda}(a-x_>)\sin\sqrt{\lambda}x_< &\to \sqrt{\lambda}\sin\sqrt{\lambda}af(x) \\
C(x,y)(\theta_{12})\sin\sqrt{\lambda}(x_< - x_>) &\to 2\sqrt{\lambda}\mathfrak{R}(\theta_{12})f(x).
\end{aligned}
$$

Adding all these terms together gives the function $[-G_\lambda(f)'' - \lambda G_\lambda(f)](x)$ for the candidate Green's function:

$$
\begin{aligned}
&[-G_\lambda(f)'' - \lambda G_\lambda(f)](x) \\
&= A\sqrt{\lambda} \left[ \begin{array}{l} \theta_{11}\sqrt{\lambda}\cos\sqrt{\lambda}a + \theta_{22}\sqrt{\lambda}\cos\sqrt{\lambda}a - \lambda\sin\sqrt{\lambda}a \\ +\theta_{11}\theta_{22}\sin\sqrt{\lambda}a - |\theta_{12}|^2\sin\sqrt{\lambda}a + 2\mathfrak{R}(\theta_{12})\sqrt{\lambda} \end{array} \right] f(x) \\
&= f(x)
\end{aligned}
$$

Thus $(A_\theta - \lambda)(G_\lambda(f)) = -G_\lambda(f)'' - \lambda G_\lambda(f) = f$ and $(A_\theta - \lambda) \circ G_\lambda$ on $D(A_\theta)$ for $\lambda \in \rho(A_\theta)\backslash\{0\}$. So $G_\lambda$ is the left-inverse of $A_\theta - \lambda$. We now show that it is also the right-inverse:

Denoting $g \to [-G_\lambda(f'') - \lambda G_\lambda(f)](x)$, we summarise the action of this map acting on the following integral kernels:

$$\begin{aligned}
&\cos\sqrt{\lambda}(a - x_>)\cos\sqrt{\lambda}x_< \to \\
&\qquad -\sqrt{\lambda}\sin\sqrt{\lambda}af(x) + \cos\sqrt{\lambda}(a-x)f'(0) - \cos\sqrt{\lambda}xf'(a) \\
&\sin\sqrt{\lambda}(a - x_>)\cos\sqrt{\lambda}x_< \to \\
&\qquad \sqrt{\lambda}\cos\sqrt{\lambda}af(x) + \sin\sqrt{\lambda}(a-x)f'(0) - \sqrt{\lambda}\cos\sqrt{\lambda}xf(a) \\
&\cos\sqrt{\lambda}(a - x_>)\sin\sqrt{\lambda}x_< \to \\
&\qquad \sqrt{\lambda}\cos\sqrt{\lambda}af(x) - \sqrt{\lambda}\cos\sqrt{\lambda}(a-x)f(0) - \sin\sqrt{\lambda}xf'(a) \\
&\sin\sqrt{\lambda}(a - x_>)\sin\sqrt{\lambda}x_< \to \\
&\qquad \sqrt{\lambda}\sin\sqrt{\lambda}af(x) - \sqrt{\lambda}\sin\sqrt{\lambda}(a-x)f(0) - \sqrt{\lambda}\sin\sqrt{\lambda}xf(a) \\
&C(x,y)(\theta_{12})\sin\sqrt{\lambda}(x_< - x_>) \to \\
&\qquad 2\sqrt{\lambda}\Re(\theta_{12})f(x) - \overline{\theta}_{12}\sin\sqrt{\lambda}xf'(0) - \overline{\theta}_{12}\sqrt{\lambda}\cos\sqrt{\lambda}xf(0) \\
&\qquad - \theta_{12}\sin\sqrt{\lambda}(x-a)f'(a) - \theta_{12}\sqrt{\lambda}\cos\sqrt{\lambda}(x-a)f(a).
\end{aligned}$$

Adding all these terms together gives the function $[-G_\lambda(f'') - \lambda G_\lambda(f)](x)$

for the candidate Green's function:

$$
\begin{aligned}
&[-G_\lambda(f'') - \lambda G_\lambda(f)](x) \\
&= f(x) + A\left[\begin{array}{l}
\lambda\cos\sqrt{\lambda}(a-x)f'(0) - \lambda\cos\sqrt{\lambda}xf'(a) \\
+\theta_{22}\sqrt{\lambda}\sin\sqrt{\lambda}(a-x)f'(0) - \theta_{22}\lambda\cos\sqrt{\lambda}xf(a) \\
-\theta_{11}\lambda\cos\sqrt{\lambda}(a-x)f(0) - \theta_{11}\sqrt{\lambda}\sin\sqrt{\lambda}xf'(a) \\
-\theta_{11}\theta_{22}\sqrt{\lambda}\sin\sqrt{\lambda}(a-x)f(0) - \theta_{11}\theta_{22}\sqrt{\lambda}\sin\sqrt{\lambda}xf(a) \\
+|\theta_{12}|^2\sqrt{\lambda}\sin\sqrt{\lambda}(a-x)f(0) + |\theta_{12}|^2\sqrt{\lambda}\sin\sqrt{\lambda}xf(a) \\
-\overline{\theta}_{12}\sqrt{\lambda}\sin\sqrt{\lambda}xf'(0) - \overline{\theta}_{12}\lambda\cos\sqrt{\lambda}xf(0) \\
-\theta_{12}\sqrt{\lambda}\sin\sqrt{\lambda}(x-a)f'(a) - \theta_{12}\lambda\cos\sqrt{\lambda}(x-a)f(a)
\end{array}\right] \\
&= f(x) + A\left[\begin{array}{l}
\lambda\cos\sqrt{\lambda}(a-x)[f'(0) - \theta_{11}f(0) - \theta_{12}f(a)] \\
\sqrt{\lambda}\sin\sqrt{\lambda}(a-x)[-\theta_{22}f'(0) - \theta_{11}\theta_{22}f(0) + |\theta_{12}|^2f(0) + \theta_{12}f'(a)] \\
-\lambda\cos\sqrt{\lambda}x[f'(a) + \theta_{22}f(a) + \overline{\theta}_{12}f(0)] \\
-\sqrt{\lambda}\sin\sqrt{\lambda}x[\theta_{11}f'(a) + \theta_{11}\theta_{22}f(a) - |\theta_{12}|^2f(a) + \overline{\theta}_{12}f'(0)]
\end{array}\right] \\
&= f(x),
\end{aligned}
$$

where the cancellations all follow from the condition that $f \in D(A_\theta)$.
Thus $G_\lambda((A_\theta - \lambda)(f)) = -G_\lambda(f'') - \lambda G_\lambda(f) = f$
and $G_\lambda \circ (A_\theta - \lambda) = id$ on $D(A_\theta)$. □

**Proposition H.2.** *If $0 \in \rho(A_\theta)$, the Green's function for the s.a.e. of the first kind at $\lambda = 0$ is given by the following integral kernel:*

$$
g(x,y,0) = A\left[\begin{array}{c}
(a - x_>)x_<|\theta_{12}|^2 - \theta_{11}x_< + (x_> - a)x_<\theta_{11}\theta_{22} \\
+(x_> - a)\theta_{22} - 1 + C(x,y)(\theta_{12})(x_> - x_<)
\end{array}\right],
$$

*where*

$$
A^{-1} = a|\theta_{12}|^2 - \theta_{11} - a\theta_{11}\theta_{22} - \theta_{22} - 2\Re(\theta_{12})
$$

*Proof.* Again, we first show that $G_0$: $L^2(0,a) \to D(A_\theta)$. From the Green's function, we can find the following boundary values of $G_0(f)$:

$$\begin{aligned} G_0(f)(0) &= A\int_0^a \left[(y-a)\theta_{22} - 1 + \theta_{12}y\right] f(y)dy \\ G_0(f)(a) &= A\int_0^a \left[-\theta_{11}y - 1 + \overline{\theta_{12}}(a-y)\right] f(y)dy \\ G_0(f)'(0) &= A\int_0^a \left[(a-y)|\theta_{12}|^2 - \theta_{11} + (y-a)\theta_{11}\theta_{22} - \theta_{12}\right] f(y)dy \\ G_0(f)'(a) &= A\int_0^a \left[-y|\theta_{12}|^2 + y\theta_{11}\theta_{22} + \theta_{22} + \overline{\theta_{12}}\right] f(y)dy. \end{aligned}$$

Thus to check that $G_0(f) \in D(A_\theta)$ we evaluate:

$$\begin{aligned} &\theta_{11}G_0(f)(0) + \theta_{12}G_0(f)(a) \\ &\quad = A\int_0^a \begin{bmatrix} (y-a)\theta_{11}\theta_{22} - \theta_{11} + \cancel{\theta_{11}\theta_{12}y} \\ \cancel{-\theta_{11}\theta_{12}y} - \theta_{12} + |\theta_{12}|^2(a-y) \end{bmatrix} f(y)dy \\ &\quad = G_0(f)'(0) \end{aligned}$$

and

$$\begin{aligned} &\overline{\theta_{12}}G_0(f)(0) + \theta_{22}G_0(f)(a) \\ &\quad = A\int_0^a \begin{bmatrix} \cancel{(y-a)\overline{\theta_{12}}\theta_{22}} - \overline{\theta_{12}} + |\theta_{12}|^2 y \\ -\theta_{11}\theta_{22}y - \theta_{12} + \cancel{\overline{\theta_{12}}\theta_{22}(a-y)} \end{bmatrix} f(y)dy \\ &\quad = -G_0(f)'(a). \end{aligned}$$

Now we show that $G_0$ is indeed the Green's function for $A_\theta$.

We first list the effect of the map $g \to -G_0(f)''(x)$ on the following integral kernels:

$$\begin{array}{lll} (a - x_>) \to 1 & x_< \to 1 & (x_> - a)x_< \to -a \\ (x_> - a) \to -1 & 1 \to 0 & C(x,y)(\theta_{12})(x_> - x_<) \to -\Re(\theta_{12}). \end{array}$$

Summing these terms yields: $-G_0(f)''(x) = A[a|\theta_{12}|^2 - \theta_{11} - a\theta_{11}\theta_{22} - \theta_{22} - 2\Re(\theta_{12})]f(x) = f(x)$.

Thus $A_\theta \circ G_0 = id$ on $L^2(0,a)$ and $G_0$ is the right-inverse of $A_\theta$. Now to show that it is also the left-inverse:

Again we list the effect of the map $g \to -G_0(f'')(x)$ on each of the terms:

$$(a - x_>) \to f(x) + (a-x)f'(0) - f(a)$$

$$x_< \to f(x) - f(0) - xf(a)$$

$$(x_> - a)x_< \to -af(x) + (a-x)f(0) + xf(a)$$

$$(x_> - a) \to -f(x) + (x-a)f'(0) + f(a)$$

$$1 \to f'(0) - f'(a)$$

$$C(x,y)(\theta_{12})(x_> - x_<) \to$$
$$- 2\Re(\theta_{12})f(x) + \overline{\theta}_{12}xf'(0) - \overline{\theta}_{12}f(0) - \theta_{12}(a-x)f'(a) + \theta_{12}f(a)$$

Adding these terms together after multiplying by a constant as it appears in our proposed Green's function we obtain:

$$
\begin{aligned}
-G_0(f'')(x) &= Af(x)[a|\theta_{12}|^2 - \theta_{11} - a\theta_{11}\theta_{22} - \theta_{22} - 2\mathfrak{R}(\theta_{12})] \\
&+ A\left[\begin{array}{l}
-|\theta_{12}|^2(a-x)f(0) - |\theta_{12}|^2 x f(a) \\
+\theta_{11}f(0) + \theta_{11}xf'(a) \\
+\theta_{11}\theta_{22}(a-x)f(0) - \theta_{11}\theta_{22}xf(a) \\
+\theta_{22}(x-a)f'(0) + \theta_{22}f(a) \\
-f'(0) + f'(a) \\
\overline{\theta}_{12}xf'(0) + \overline{\theta}_{12}f(0) \\
-\theta_{12}(a-x)f'(a) + \theta_{12}f(a)
\end{array}\right] \\
&= f(x) + A\left[\begin{array}{l}
[\theta_{11}f(0) - f'(0) + \theta_{12}f(a)] \\
+[\theta_{22}f(a) + f'(a) + \overline{\theta}_{12}f(0)] \\
+(a-x)[-|\theta_{12}|^2 f(0) + \theta_{11}\theta_{22}f(0) - \theta_{22}f'(0) - \theta_{12}f'(a)] \\
+x[-|\theta_{12}|^2 f(a) + \theta_{11}f'(a) + \theta_{11}\theta_{22}f(a) + \overline{\theta}_{12}f'(0)]
\end{array}\right] \\
&= f(x)
\end{aligned}
$$

Thus $G_0(A_\theta(f)) = -G_0(f'') = f$ and $G_0 \circ A_\theta = id$ on $D(A_\theta)$. □

Now, we do the same for the s.a.e.s of the second kind. Remembering that the domain of this extension is defined as:

$$
D(A_{w_1 w_2 \theta}) = \left\{ \phi \in W^{2,2}(0,a) \text{ s.t.: } \begin{array}{l} w_2\phi(0) - w_1\phi(a) = 0 \\ \overline{w_1}(\theta\phi(0) - \phi'(0)) + \overline{w_2}(\theta\phi(a) + \phi'(a)) = 0 \end{array} \right\},
$$

where $w_1, w_2 \in \mathbb{C}$, $|w_1|^2 + |w_2|^2 = 1$ and $\theta \in \mathbb{R}$. We shall prove the following:

**Proposition H.3.** *The Green's function for the s.a.e. of the second kind for* $\lambda \in \rho(A_{w_1 w_2 \theta})\backslash\{0\}$ *is given by:*

*For* $\lambda \in \rho(A_{w_1 w_2 \theta})\backslash\{0\}$*:*

$$g(x,y;\lambda) = A\left[\begin{array}{l} |w_1|^2\sqrt{\lambda}\sin\sqrt{\lambda}(x_> - a)\cos\sqrt{\lambda}x_< + \sqrt{\lambda}C(x,y)(w_1\overline{w_2})\sin\sqrt{\lambda}(x_< - x_>) \\ +\theta\sin\sqrt{\lambda}(x_> - a)\sin\sqrt{\lambda}x_< - |w_2|^2\sqrt{\lambda}\cos\sqrt{\lambda}(x_> - a)\sin\sqrt{\lambda}x_< \end{array}\right],$$

*where*

$$A^{-1} = \sqrt{\lambda}\left[-\sqrt{\lambda}\cos\sqrt{\lambda}a + 2\Re(w_1\overline{w_2})\sqrt{\lambda} - \theta\sin\sqrt{\lambda}a\right]$$

*Remark.* We have not specified which square root $\sqrt{\lambda}$ to take as it can be seen that the expression for $g$ is invariant under replacing $\sqrt{\lambda}$ with $-\sqrt{\lambda}$.

*Proof.* Again, we first check that $G_\lambda$: $L^2(0,a) \to D(A_{w_1 w_2 \theta})$. Following the previous method we can find the following boundary values of $G_\lambda(f)$:

$$G_\lambda(f)(0) = A\int_0^a \left[|w_1|^2\sqrt{\lambda}\sin\sqrt{\lambda}(y-a) - \sqrt{\lambda}w_1\overline{w_2}\sin\sqrt{\lambda}y\right]f(y)dy$$

$$G_\lambda(f)(0) = A\int_0^a \left[\overline{w_1}w_2\sqrt{\lambda}\sin\sqrt{\lambda}(y-a) - \sqrt{\lambda}|w_2|^2\sin\sqrt{\lambda}y\right]f(y)dy$$

$$G_\lambda(f)'(0) = A\int_0^a \left[\begin{array}{l} \lambda w_1\overline{w_2}\sqrt{\lambda}\cos\sqrt{\lambda}y + \theta\sqrt{\lambda}\sin\sqrt{\lambda}(y-a) \\ -|w_2|^2\lambda\cos\sqrt{\lambda}(y-a) \end{array}\right]f(y)dy$$

$$G_\lambda(f)'(a) = A\int_0^a \left[\begin{array}{l} \lambda|w_1|^2\sqrt{\lambda}\cos\sqrt{\lambda}y - \lambda\overline{w_1}w_2\cos\sqrt{\lambda}(y-a) \\ +\theta\sqrt{\lambda}\sin\sqrt{\lambda}y \end{array}\right]f(y)dy$$

We can see from inspection that $w_2 G_\lambda(f)(0) = w_1 G_\lambda(f)(a)$. Furthermore:

$$\begin{aligned}&\overline{w_1}(\theta G_\lambda(f)(0) - G_\lambda(f)'(0)) \\ &= A\int_0^a \left[\begin{array}{l} \theta\overline{w_1}|w_1|^2\sqrt{\lambda}\sin\sqrt{\lambda}(y-a) - \theta\sqrt{\lambda}|w_1|^2\overline{w_2}\sin\sqrt{\lambda}y \\ -\lambda|w_1|^2\overline{w_2}\cos\sqrt{\lambda}y - \theta\overline{w_1}\sqrt{\lambda}\sin\sqrt{\lambda}(y-a) \\ +\overline{w_1}|w_2|^2\lambda\cos\sqrt{\lambda}(y-a) \end{array}\right]f(y)dy\end{aligned}$$

and

$$\overline{w_2}(\theta G_\lambda(f)(a) - G_\lambda(f)'(a))$$

$$= A\int_0^a \begin{bmatrix} \theta\overline{w_1}|w_2|^2\sqrt{\lambda}\sin\sqrt{\lambda}(y-a) - \theta\sqrt{\lambda}|w_2|^2\overline{w_2}\sin\sqrt{\lambda}y \\ +\lambda|w_1|^2\overline{w_2}\cos\sqrt{\lambda}y \\ -\overline{w_1}|w_2|^2\lambda\cos\sqrt{\lambda}(y-a) + \theta\overline{w_2}\sqrt{\lambda}\sin\sqrt{\lambda}y \end{bmatrix} f(y)dy,$$

from which we see that

$$\overline{w_1}(\theta G_\lambda(f)(0) - G_\lambda(f)'(0)) + \overline{w_2}(\theta G_\lambda(f)(a) - G_\lambda(f)'(a)) = 0$$

and so $G_\lambda(f) \in D(A_{w_1 w_2 \theta})$. We now show that $G_\lambda$ is indeed the resolvent of $A_{w_1 w_2 \theta}$. Using the first table of integral kernels in Proposition H.1, by the same argument we have:

$$\begin{aligned} &[-G_\lambda(f)'' - \lambda G_\lambda(f)](x) \\ &= Af(x)\begin{bmatrix} -|w_1|^2\lambda\cos\sqrt{\lambda}a + 2\lambda\Re(w_1\overline{w_2}) \\ -\theta\sqrt{\lambda}\sin\sqrt{\lambda}a - |w_2|^2\lambda\cos\sqrt{\lambda}a \end{bmatrix} \\ &= Af(x)[-\lambda\cos\sqrt{\lambda}a + 2\lambda\Re(w_1\overline{w_2}) - \theta\sqrt{\lambda}\sin\sqrt{\lambda}a] \\ &= f(x) \end{aligned}$$

Thus $A_{w_1 w_2 \theta} \circ G_\lambda = id$ on $L^2(0,a)$ for $\lambda \in \rho(A_{w_1 w_2 \theta})$.

Using the second table of integral kernels in Proposition H.1, we have:

$$
\begin{aligned}
&[-G_\lambda(f'') - \lambda G_\lambda(f)](x) \\
&= f(x) + A\begin{bmatrix}
-|w_1|^2\sqrt{\lambda}\sin\sqrt{\lambda}(a-x)f'(0) + |w_1|^2\lambda\cos\sqrt{\lambda}xf(a) \\
-\sqrt{\lambda}\overline{w_1}w_2\sin\sqrt{\lambda}xf'(0) - \lambda\overline{w_1}w_2\cos\sqrt{\lambda}xf(0) \\
-\sqrt{\lambda}w_1\overline{w_2}\sin\sqrt{\lambda}(x-a)f'(a) - \lambda w_1\overline{w_2}\cos\sqrt{\lambda}(x-a)f(a) \\
+\theta\sqrt{\lambda}\sin\sqrt{\lambda}(a-x)f(0) + \theta\sqrt{\lambda}\sin\sqrt{\lambda}xf(a) \\
+|w_2|^2\lambda\cos\sqrt{\lambda}(a-x)f(0) + |w_2|^2\sqrt{\lambda}\sin\sqrt{\lambda}xf'(a)
\end{bmatrix} \\
&= f(x) + A\begin{bmatrix}
\sqrt{\lambda}\sin\sqrt{\lambda}(a-x)[-|w_1|^2f'(0) + w_1\overline{w_2}f'(a) + \theta f(0)] \\
\lambda\cos\sqrt{\lambda}x[|w_1|^2f(a) - \overline{w_1}w_2f(0)] \\
+\sqrt{\lambda}\sin\sqrt{\lambda}x[-\overline{w_1}w_2f'(0) + \theta f(a) + |w_2|^2f'(a)] \\
+\lambda\cos\sqrt{\lambda}(a-x)[-w_1\overline{w_2}f(a) + |w_2|^2f(0)]
\end{bmatrix} \\
&= f(x)
\end{aligned}
$$

Thus $G_\lambda((A_{w_1w_2\theta} - \lambda)(f)) = -G_\lambda(f'') - \lambda G_\lambda(f) = f$
and $G_\lambda \circ (A_{w_1w_2\theta} - \lambda) = id$ on $D(A_{w_1w_2\theta})$. □

**Proposition H.4.** *If $0 \in \rho(A_{w_1w_2\theta})$ then the Green's function for the s.a.e. of the second kind at $\lambda = 0$ is:*

$$g(x,y;0) = A[\theta(a-x_>)x_< + C(x,y)(w_1\overline{w_2})(x_> - x_<) + |w_1|^2(a-x_>) + |w_2|^2x_<],$$

*where*

$$A^{-1} = a\theta - 2\Re(w_1\overline{w_2}) + 1$$

*Proof.* As before, we first check that $G_0$: $L^2(0,a) \to D(A_{w_1w_2\theta})$. The following boundary values can be calculated:

$$G_0(f)(0) = A\int_0^a \left[w_1\overline{w_2}y + |w_1|^2(a-y)\right] f(y)dy$$
$$G_0(f)(a) = A\int_0^a \left[\overline{w_1}w_2(a-y) + |w_2|^2 y\right] f(y)dy$$
$$G_0(f)'(0) = A\int_0^a \left[\theta(a-y) - w_1\overline{w_2} + |w_2|^2\right] f(y)dy$$
$$G_0(f)'(a) = A\int_0^a \left[-\theta y + \overline{w_1}w_2 - |w_1|^2\right] f(y)dy$$

It can be seen that we already have: $w_2 G_0(f)(0) = w_1 G_0(f)(a)$. Additionally:

$$\begin{aligned}&\overline{w_1}(\theta G_0(f)(0) - G_0(f)'(0))\\ &\quad = A\int_0^a \left[\begin{array}{l}\theta|w_1|^2\overline{w_2}y + \theta\overline{w_1}|w_1|^2(a-y)\\ -\theta\overline{w_1}(a-y) + |w_1|^2\overline{w_2} - \overline{w_1}|w_2|^2\end{array}\right] f(y)dy\end{aligned}$$

and:

$$\begin{aligned}&\overline{w_2}(\theta G_0(f)(a) - G_0(f)'(a))\\ &\quad = A\int_0^a \left[\begin{array}{l}\theta|w_2|^2\overline{w_1}(a-y) + \theta\overline{w_2}|w_2|^2 y\\ -\theta\overline{w_2}y + |w_2|^2\overline{w_1} - \overline{w_2}|w_1|^2\end{array}\right] f(y)dy,\end{aligned}$$

whence:

$$\begin{aligned}&\overline{w_1}(\theta G_0(f)(0) - G_0(f)'(0)) + \overline{w_2}(\theta G_0(f)(a) - G_0(f)'(a))\\ &\quad = A\int_0^a \left[\begin{array}{l}\theta|w_1|^2\overline{w_2}y + \theta\overline{w_1}(a-y)\\ -\theta\overline{w_1}(a-y) + \theta\overline{w_2}|w_2|^2 y\\ -\theta\overline{w_2}y\end{array}\right] f(y)dy\\ &\quad = 0.\end{aligned}$$

Thus $G_0(f) \in D(A_{w_1 w_2 \theta})$. We now show that $G_0$ is indeed the resolvent of

$A_{w_1 w_2 \theta}$. Using the first table of integral kernels in Proposition H.2:

$$\begin{aligned} -G_0(f)''(x) &= A[a\theta - 2\Re(w_1\overline{w_2}) + |w_1|^2 + |w_2|^2]f(x) \\ &= A[a\theta - 2\Re(w_1\overline{w_2}) + 1]f(x) \\ &= f(x). \end{aligned}$$

Thus $A_{w_1 w_2 \theta} \circ G_0 = id$ on $L^2(0,a)$.

Now we prove that $G_0$ is also the left-inverse of $A_{w_1 w_2 \theta}$. Using the second table of integral kernels in Proposition H.2, we have:

$$\begin{aligned} &- G_0(f'')(x) \\ &= f(x) + A\left[\begin{array}{l} \theta(x-a)f(0) - \theta x f(a) \\ +\overline{\theta}_{12} x f'(0) + \overline{\theta}_{12} f(0) - \theta_{12}(a-x)f'(a) + \theta_{12} f(a) \\ -|w_1|^2 f(a) + |w_1|^2 (a-x) f'(0) \\ -|w_2|^2 f(0) - |w_2|^2 x f'(a) \end{array}\right] \\ &= f(x) + A\left[\begin{array}{l} \overline{\theta}_{12} f(0) + \theta_{12} f(a) - |w_1|^2 f(a) - |w_2|^2 f(0) \\ +(x-a)[\theta f(0) + \theta_{12} f'(a) - |w_1|^2 f'(0)] \\ +x[-\theta f(a) + \overline{\theta}_{12} f'(0) - |w_2|^2 f'(a)] \end{array}\right] \\ &= f(x) \end{aligned}$$

Thus $G_0(A_{w_1 w_2 \theta}(f)) = -G_0(f'') = f$
and $G_0 \circ A_{w_1 w_2 \theta} = id$ on $D(A_{w_1 w_2 \theta})$. □

# References

[1] R. Abraham, J.E. Marsden and T. Ratiu, Manifolds, Tensor Analysis, and Applications (2001) Springer

[2] N.I. Akhiezer and I.M. Glazman, Theory of Linear Operators in Hilbert Space (1993) Dover

[3] C. Bär, N. Ginoux and F. Pfäffle, Wave Equations on Lorentzian Manifolds and Quantisation (2007) European Mathematical Society

[4] C. Bär and K. Fredenhagen (Eds.), Quantum Field Theory on Curved Spacetimes: Concepts and Mathematical Foundations (2009) Springer

[5] J.K. Beem, P.E. Ehrlich and K.L. Easley, Global Lorentzian Geometry (1996) CRC Press

[6] A.N. Bernal and M. Sanchez, Further results on the smoothability of Cauchy hypersurfaces and Cauchy time functions, 2007, arXiv:gr-qc/0512095v1

[7] A.N. Bernal and M. Sanchez, Globally hyperbolic spacetimes can be defined as "causal" instead of "strongly causal", 2006, arXiv:gr-qc/0611138/v1

[8] A.N. Bernal and M. Sanchez, On Smooth Cauchy Hypersurfaces and Geroch's Splitting Theorem, Communications in Mathematical Physics, (2003) 243, 3, 461-470

[9] M.S. Birman and M.Z. Solomjak, Spectral Theory of Self-Adjoint Operators in Hilbert Space (1987) D.Reidel Publishing Company

[10] R.L. Bishop and S.I. Goldberg, Tensor Analysis on Manifolds (1980) Dover

[11] H.O. Cordes, Self-Adjointness of Powers of Elliptic Operators, Mathematische Annalen 195 (1972) 257-272

[12] B. Driver, Analysis Tools with Applications, `http://www.math.ucsd.edu/~bdriver/231-02-03/lecture_notes.htm`

[13] N. Dunford and J.T. Schwartz, Linear Operators Part I: General Theory (1958) Interscience Publishers

[14] G.B Folland, Real Analysis, Modern Techniques and Their Applications (1984) Wiley

[15] G. Grubb, A characterisation of the non local boundary value problems associated with an elliptic operator, Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3,22,3 (1968) 425-513

[16] E. Hebey, Sobolev Spaces on Riemannian Manifolds (1991) Springer

[17] B.S. Kay and U.M. Studer, Boundary Conditions for Quantum Mechanics on Cones and Fields Around Cosmic Strings, Communications in Mathematical Physics 139, 103-139 (1991)

[18] J.M. Lee, Introduction to Smooth Manifolds (2003) Springer

[19] J.M. Lee, Riemannian Manifolds: An Introduction to Curvature (1991) Springer

[20] J.L Lions and E. Magenes, Non-Homogeneous Boundary Value Problems and Applications (1972) Springer

[21] C.W. Misner, K.S. Thorne and J.A. Wheeler, Gravitation (1973) W.H. Freeman and Company

[22] L.I. Nicolaescu, Lectures on the Geometry of Manifolds (2007) World Scientific Publishing

[23] B. O'Neill, Semi-Riemannian Geometry (1983) Academic Press

[24] R. Penrose, Techniques of Differential Topology in Relativity (1972) Society for Industrial and Applied Mathematics

[25] P. Petersen, Riemannian Geometry (2006) Springer

[26] A. Posilicano, Self-adjoint Extensions of Restrictions, 2008, arXiv:math-ph/0703078

[27] M. Reed and B. Simon, Methods of Modern Mathematical Physics, I: Functional Analysis (1980) Academic Press

[28] M. Reed and B. Simon, Methods of Modern Mathematical Physics, II: Fourier Analysis, Self-Adjointness (1975) Academic Press

[29] R.K. Sachs and H. Wu, General Relativity for Mathematicians (1983) Springer

[30] M. Sanchez, On the Geometry of Static Spacetimes, 2004, arXiv:math/0406332v2

[31] I. Seggev, Dynamics in stationary, non-globally hyperbolic spacetimes, Classical Quantum Gravity 21 (2004) 2651-2668

[32] I. Stakgold, Green's Functions and Boundary Value Problems (1998) Wiley

[33] N. Steenrod, The Topology of Fibre Bundles (1951) Princeton University Press

[34] M.E. Taylor, Partial Differential Equations I (1996) Springer

[35] M.E. Taylor, Partial Differential Equations II (1996) Springer

[36] F. Treves, Topological Vector Spaces, Distributions and Kernels (1967) Academic Press

[37] R.M. Wald, Dynamics in nonglobally hyperbolic, static space-times, 1980, Journal of Mathematical Physics 21(12) 2802

[38] R.M. Wald and A. Ishibashi, Dynamics in non-globally-hyperbolic static spacetimes: II. General analysis of prescriptions for dynamics, Class. Quantum Grav. 20 (2003) 3815-3826

[39] F.W. Warner, Foundations of Differentiable Manifolds and Lie Groups (1983) Springer