Combinatorial Questions for $S \wr_n \mathcal{T}_n$ for a semigroup S

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Abstract

We study combinatorial questions for the wreath product $S \wr_n \mathcal{T}_n$ (properly, $S \wr_{\underline{n}} \mathcal{T}_n$) and related semigroups, where S is a monoid and $\mathcal{T}_n$ is the full transformation monoid on $\underline{n} = \{1, 2, \ldots, n\}$. It is well known that $S \wr_n \mathcal{T}_n$ is isomorphic to the endomorphism monoid of a free S-act $F_n(S)$ on n generators and if S is a group, $F_n(S)$ is an example of an independence algebra. We determine the number of idempotents of $S \wr_n \mathcal{T}_n$, first in the more straightforward case where S is a group.

We investigate the monoid of partial endomorphisms $\mathcal{PT}_{\mathbf{A}}$ of an independence algebra $\mathbf{A}$, focussing on the special case where $\mathbf{A}$ is $\mathbf{F_n(G)}$. We determine Green's relations and Green's pre-orders on $\mathcal{PT}_{\mathbf{F_n(G)}}$. We also obtain formulae for the number of idempotents and the number of nilpotents in $\mathcal{PT}_{\mathbf{F_n(G)}}$.

We specialise Lavers' technique in order to construct a presentation for $M^n \rtimes \mathcal{T}_n$ from presentations of M^n and $\mathcal{T}_n$.

Finally, we find monoid presentations for some special subsemigroups of semidirect products. We suppose M and T are monoids such that M is a left T-act by endomorphisms and G and H are the groups of units of M and T, respectively. In addition, we suppose $N = M \setminus G$ and $S = T \setminus H$ are ideals of M and T, with N and G left S-acts, respectively. We then establish a monoid presentation for $C = (N^1 \rtimes \{1\}) \sqcup (M \rtimes S)$.

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Preface

Wreath products are particular kinds of semidirect products, and provide a powerful tool within both semigroup and group theory. For historical use of wreath products in groups see, for example, Loewy [29] and Polya [36]. The use of wreath products was extended to semigroups by B.H. Neumann [35] in 1960 and they are now widely used in this context.

Semigroups of transformations are fundamental mathematical objects in semigroup theory, since every semigroup S embeds into a full transformation semigroup, with this embedding being monoidal in the case S is a monoid. Among all transformation semigroups one can distinguish: the *symmetric group* $\mathcal{S}_X$ of all bijective transformations of a set X, the full *transformation semigroup* $\mathcal{T}_X$ of all transformations of a set X and the partial *transformation semigroup* $\mathcal{PT}_X$ of all partial transformations of a set X. Moreover, some of the most natural examples of monoids arise as the (partial) endomorphism monoids of various mathematical structures, including independence algebras. If X is a set with n elements, then it will be convenient to denote the set X as $\underline{n}$ or $X_n = \{1, 2, \cdots, n\}$. If $X = X_n$ we will write $\mathcal{S}_n$ for $\mathcal{S}_X$, $\mathcal{T}_n$ for $\mathcal{T}_X$ and $\mathcal{PT}_n$ for $\mathcal{PT}_X$. Note that the set $\mathrm{Sing}_n = \mathcal{T}_n \setminus \mathcal{S}_n$ is a subsemigroup (indeed, an ideal) of $\mathcal{T}_n$, and consists of all non-invertible (i.e., *singular*) transformations on X_n. Howie [24] first investigated Sing_n and his famous result states that Sing_n is generated by its non-invertible idempotents.

Let A be an alphabet, a *semigroup presentation* is an ordered pair $\langle A : R\rangle$, where $R \subseteq A^+ \times A^+$. For $R \subseteq A^+ \times A^+$, we denote by $R^\sharp$ the smallest congruence on A^+ generated by R. To say that a semigroup S has presentation $\langle A : R\rangle$ is to say that $S \cong A^+/R^\sharp$ or, equivalently, there is a semigroup epimorphism $\varphi : A^+ \to S$ with $\mathrm{Ker}\,\varphi = R^\sharp$. If such an epimorphism exists, then we say that S has presentation $\langle A : R\rangle$ via φ. Presentations have been obtained for certain singular semigroups of transformations and related structures [10, 11, 12, 13] and [31]. Specifically, a presentation for Sing_n is given in [11] in terms of the generating set consisting of all idempotents of rank $n-1$. Furthermore, a presentation for $M \wr_n \mathrm{Sing}_n$ is given in [14] in terms of a particularly natural idempotent generating set.

This thesis is organised as follows:

Chapter 1: We recall the basic definitions and results of semigroup theory, needed for full understanding of the subsequent work in this thesis. In particular we give some details of presentations in the context of semigroups, monoids and groups.

Chapter 2: We give an account of the notions of universal algebra that we require. We recall some facts concerning independence algebras (also known as v^*-algebras) and their endomorphism monoids.

Chapter 3: We recall some basic definitions concerning transformations semigroups $\mathcal{S}_X$, $\mathcal{T}_X$, $\mathcal{PT}_X$ and Sing_X. We recall that $\mathcal{PT}_n$ is isomorphic to $\overline{\mathcal{T}_{n,0}}$, which is a submonoid of the full transformation semigroup $\mathcal{T}_{n,0}$ on $X_{n,0} = \{0, 1, 2, \cdots, n\}$. Clearly, $\mathcal{T}_{n,0}$ is isomorphic to $\mathcal{T}_{n+1}$, the full transformation semigroup on $X_{n+1} = \{1, 2, \cdots, n+1\}$.

Chapter 4: The aim of this chapter is to study free (left) S-acts (where S is a monoid) and their endomorphism monoids. We recall that the endomorphism monoid of a free left S-act $F_n(S)$ on n generators is isomorphic to a wreath product $S \wr_n \mathcal{T}_n$. Our first new result is to count the number of idempotents in $\mathrm{End}\, F_n(S)$ where S is finite monoid.

Chapter 5: This chapter is devoted to the study of the partial endomorphism monoid $\mathcal{PT}_{\mathbf{A}}$, where $\mathbf{A}$ is an independence algebra. This extends the consideration of the *local automorphism monoid* $\mathcal{L}(\mathbf{A})$ by L. Lima [28] of all isomorphisms $\alpha : \mathbf{B} \to \mathbf{C}$, where $\mathbf{B}$ and $\mathbf{C}$ are subalgebras of $\mathbf{A}$. Indeed, $\mathcal{L}(\mathbf{A})$ is an inverse subsemigroup of $\mathcal{I}(A)$, the *symmetric inverse semigroup* on the set A (see, [28] Chapter 2), and

$\mathcal{PT}_{\mathbf{A}}$ is a *left restriction* monoid (see, [16]).

In particular, we consider the monoid $\mathcal{PT}_{\mathbf{F_n(G)}}$, which consists of all morphisms $\alpha : \mathbf{B} \to \mathbf{C}$, where $\mathbf{B}$ and $\mathbf{C}$ are subalgebras of $\mathbf{F_n(G)}$. If G is trivial clearly $\mathcal{PT}_{\mathbf{F_n(G)}}$ is isomorphic to $\mathcal{PT}_n$. If G is non-trivial, we prove that $\mathcal{PT}_{\mathbf{F_n(G)}}$ is isomorphic to $\operatorname{End} F_n(G)^0$, the endomorphism monoid of the left G-act given by $F_n(G)^0 = F_n(G)\dot{\cup}\{0\}$, where $\{0\}$ is a trivial left G-act. Also, we show that $\mathcal{PT}_{\mathbf{F_n(G)}}$ is embedded via φ in $G^0 \wr_{n+1} \mathcal{T}_{n,0}$, where G^0 is the group G with 0 adjoined and $\operatorname{Im}\varphi$ is the monoid $K_n(G)^0$, where

$$K_n(G)^0 = \{(0, g_1, \cdots, g_n, \alpha) : i\alpha = 0 \quad \text{if and only if} \quad g_i = 0 \\ \text{where} \quad 1 \leq i \leq n \quad \text{and} \quad \alpha \in \overline{\mathcal{T}_{n,0}}\}.$$

Furthermore, where G is a finite group we find formulae to count the number of idempotents and nilpotents in $\mathcal{PT}_{\mathbf{F_n(G)}}$.

Chapter 6: In this chapter we give presentations for certain subsemigroups of semidirect products. We find a monoid presentation for $M^n \rtimes \mathcal{T}_n$ from a presentation of M^n and $\mathcal{T}_n$ by using Lavers' technique [27]. We give a general presentation for a semidirect product $M \rtimes S$ which allows us to find a number of presentations for $M \wr_n \operatorname{Sing}_n$. In the case where $M \wr_n \operatorname{Sing}_n$ is idempotent generated, we give a presentation in terms of a particularly natural idempotent generating set: these results are taken from the joint paper [14], to which I contributed in small part. We find a monoid presentation for $(N^1 \wr_n \{1_n\}) \sqcup (M \wr_n \operatorname{Sing}_n)$ where M is a monoid, G is a group of units of M and $N = M \setminus G$ is an ideal of M; this is a minor adjustment of a known result [14]. Finally, we suppose M and T are monoids such that M is a left T-act by endomorphisms and G and H are the groups of units of M and T, respectively. In addition, we suppose $N = M \setminus G$ and $S = T \setminus H$ are ideals of M and T, respectively, with N and G are left S-acts. Then a monoid presentation for $(N^1 \rtimes \{1\}) \sqcup (M \rtimes S)$ is obtained.

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Author's Declaration

I declare that this thesis is a presentation of original work and I am the sole author. This work has not previously been presented for an award at this, or any other, University. All sources are acknowledged as References.

Chapter 4 studies free (left) S-acts $F_n(S)$ of finite rank n over a monoid S, and their endomorphism monoids. It is known that $\operatorname{End} F_n(S)$ is isomorphic to a wreath product $S \wr_n \mathcal{T}_n$. In the case where S is a group, $F_n(S)$ is an independence algebra. Green's relations on endomorphism monoids of independence algebras are known, but I give a direct account in this special case. This chapter also contains my first new results, giving the number of idempotents in $\operatorname{End} F_n(S)$ where S is a finite monoid: this formula may also be found in the joint paper [14]. Chapter 5 is devoted to the study of the monoid of partial endomorphisms $\mathcal{PT}_{\mathbf{A}}$, where $\mathbf{A}$ is an independence algebra. The results in this chapter are new, although whilst this thesis was under construction, the characterisation of Green's relations in $\mathcal{PT}_{\mathbf{F_n(G)}}$ presented here were extended to the case of $\mathcal{PT}_{\mathbf{A}}$ for an arbitrary independence algebra $\mathbf{A}$ and appeared in [43]. Finally, Chapter 6 finds presentations for certain subsemigroups of semidirect products. Some of the results in this chapter are taken from the joint paper [14], to which I contributed in small part. The results in Section 6.3, Subsection 6.3.1, are a very minor variation on a result of [14]. The results in Subsection 6.3.2 are new.

Chapter 1

Preliminaries I: Semigroup fundamentals

In this chapter we present the necessary background on semigroups for full understanding of the subsequent work. All of the definitions and results presented here are standard and can be found in [9], [25] and [33].

Throughout this thesis, mappings are written on the right of their arguments, which means composition of mappings is from left to right.

1.1 Semigroups, binary relations and equivalences

1.1.1 Semigroups

A *semigroup* is a pair (S, μ) where S is non-empty set and $\mu : S \times S \to S$ is a binary operation satisfying the associative law:

$$((x, y)\mu, z)\mu = (x, (y, z)\mu)\mu$$

for all $x, y, z \in S$. We usually write $(x, y)\mu$ as $x \cdot y$ or even more simply as xy, so the associative law can be expressed as $(xy)z = x(yz)$ for all $x, y, z \in S$. We refer to the binary operation μ as *multiplication* on S. Where μ is clear, we write simply

S rather than (S, μ) or $(S, \cdot)$.

A semigroup S is commutative if $ab = ba$ for all $a, b \in S$.

If a semigroup S contains an element 1 with property that for all $x \in S$

$$x\,1 \;=\; 1\,x \;=\; x,$$

we say that 1 is an *identity element* of S, and that S is a *semigroup with identity* or *monoid*.

Notice that every semigroup has at most one identity, since if $1^{'}$ also has property that $x\,1^{'} \;=\; 1^{'}\,x \;=\; x$ for all $x \in S$, then

$$\begin{aligned} 1^{'} &= 1\,1^{'} \quad (\text{as}\,1\,\text{is an identity}) \\ &= 1 \qquad (\text{as}\,1^{'}\,\text{is an identity}). \end{aligned}$$

Therefore, if S is a monoid we may refer to *the* identity of S.

If S has no identity element, then it is easy to adjoin an extra element 1 to S to form a monoid $S \cup \{1\}$ with $1\,s = s\,1 = s$ for all $s \in S$, and $1\,1 = 1$.

We define

$$S^1 = \begin{cases} S & \text{if } S \text{ has a identity element;} \\ S \cup \{1\} & \text{otherwise.} \end{cases}$$

We say that S^1 is the monoid obtained from S by *adjoining an identity if necessary.* Thus every semigroup S may be embedded into a monoid, S^1.

We call a semigroup S with at least two elements a *semigroup with zero* if there exists an element 0 in S such that for all $x \in S$, we have

$$0\,x = x\,0 = 0.$$

We say that 0 is a *zero element* of S. Note that every semigroup S has at most one zero element. For any semigroup S, we may adjoin an extra element 0 to S to give $S \cup \{0\}$ and define $0\,s = s\,0 = 0\,0 = 0$ for all s in S. We refer to S^0 as the semigroup with zero obtained from S *by adjoining a zero*. Thus every semigroup S can be embedded into a semigroup with zero, S^0.

If A and B are subsets of a semigroup S, then we define the *product* of A and B to be the set $\{ab : a \in A, b \in B\}$. It is easy to check the associativity survives to give $(AB)C = A(BC)$ for all $A, B, C \subseteq S$. Note that, $A^2 = \{a_1 a_2 : a_1, a_2 \in A\}$ rather than $\{a^2 : a \in A\}$. In the special case of singleton subsets $A = \{a\}$ or $B = \{b\}$, we write aB or Ab rather than $\{a\}B$ or $A\{b\}$.

An *idempotent* in a semigroup S is an element $e \in S$ such that $e^2 = e$. We denote by $E(S)$ the set of all idempotents in S and often use E if S is clear.

A non-empty subset T of a semigroup S is called a *subsemigroup* of S if it is closed with respect to the multiplication, which means that for all $x, y \in T$, we have $xy \in T$. Precisely, a non-empty subset T of S is a subsemigroup if and only if $T^2 \subseteq T$. If S is a monoid, then T is a *submonoid* of S if T is a subsemigroup and $1 \in T$.

Let S be a semigroup and $\{T_i : i \in I\}$ be an indexed set of subsemigroups of S. Then if the set $\bigcap_{i \in I} T_i$ is non-empty, it is a subsemigroup of S. Particularly, for any non-empty subset A of S the intersection of all the subsemigroups of S that contains A is non-empty and it is a subsemigroup of S. We use $\langle A \rangle$ to denote this subsemigroup, and note it consists of all elements of S that can be expressed as a finite products of elements in A. Further, if $\langle A \rangle = S$ we say that A is a *set of generators*, or a *generating set*, of S.

A non-empty subsemigroup T of a semigroup S is a *left* (*right*) *ideal* of S if and only if $ST \subseteq T$ ($TS \subseteq T$), and T is (*two-sided*) *ideal* if and only if $TS \subseteq T$ and $ST \subseteq T$. Notice that, if S is a monoid with identity 1, then an ideal T of S is equal to S if and only if $1 \in T$.

If a is an element of a semigroup S, then $Sa = \{sa : s \in S\}$, $aS = \{as : s \in S\}$ and $SaS = \{sas : s \in S\}$ are left, right and two sided ideals, respectively, but they might not contain a. Notice that

$$S^1 a = Sa \cup \{a\},$$

so $a \in S^1 a$. Also, $S^1 a$ is a subset of S (it does not contain any adjoined identity). In particular, $S^1 a$ is the smallest left ideal of S containing a, known as the *principal left ideal of S generated by a*. Dually, $aS^1 = aS \cup \{a\}$ is the smallest right ideal of S containing a, known as the *principal right ideal of S generated by a*. Moreover,

$$S^1 a S^1 = SaS \cup Sa \cup aS \cup \{a\},$$

is the smallest two sided ideal of S containing a, known as the *principal ideal generated by a*. If S is a monoid, then $S^1 a = Sa$, $aS^1 = aS$ and $S^1 aS^1 = SaS$. In fact, other conditions (such as a being idempotent) will also give $S^1 a = Sa, aS^1 = aS$ and $S^1 aS^1 = SaS$. Further conditions will be mentioned later.

A semigroup S has M_L if there are no infinite chains

$$S^1 a_1 \supset S^1 a_2 \supset S^1 a_3 \supset \cdots$$

of principal left ideals; M_L is the *descending chain condition* (d.c.c) on principal left ideals. The left-right dual of M_L is denoted by M_R.

Let S and T be semigroups. Then the map $\theta : S \to T$ is called a (*semigroup*) *morphism* if for all $x, y \in S$,

$$(xy)\theta = (x\theta)(y\theta).$$

If S and T are monoids, with 1_S and 1_T, respectively, then θ will be called a (*monoid*) *morphism* if θ is a semigroup morphism and $1_S\theta = 1_T$. On the other hand, if θ is one-one then we call it a *monomorphism*, and if θ is onto, then we say that θ an *epimorphism*. Finally, θ is called an *isomorphism* if it is bijective homomorphism.

A morphism θ from a semigroup S into itself is called an *endomorphism* of S, and an isomorphism from S onto S is called an *automorphism* of S. We denote by $\operatorname{End} S$ and $\operatorname{Aut} S$, respectively, the set of all endomorphisms of S and the set of all automorphisms of S. In fact, $\operatorname{End} S$ and $\operatorname{Aut} S$ are closed under composition. As the identity map $I_S : S \to S$ belongs to each of those sets, hence the set of all endomorphisms of S, $\operatorname{End} S$, under composition of maps, forms a monoid, which called the *endomorphism monoid* of S. The set of all automorphisms of S, $\operatorname{Aut} S$, under composition of maps forms a group called the *automorphism group* of S. Clearly, $\operatorname{Aut} S$ is the set of elements α of $\operatorname{End} S$ such that there is a $\beta \in \operatorname{End} S$ with $\alpha\beta = \beta\alpha = I_S$, that is, $\operatorname{Aut} S$ is the the group of units of $\operatorname{End} S$.

In fact, any algebraic structure $\mathbf{A}$ has a monoid of endomorphisms $\operatorname{End} \mathbf{A}$, with group of units $\operatorname{Aut} \mathbf{A}$, as we will explain in Chapter 5. Moreover, we also be considering $\mathcal{PT}_{\mathbf{F_n(A)}}$, the monoid of partial endomorphisms of $\mathbf{A}$.

1.1.2 Binary relations

A (*binary*) *relation* on a set X is a subset ρ of the cartesian product $X \times X$, i.e., a set of ordered pairs $(x, y) \in X \times X$, we say that x and y are ρ-related if $(x, y) \in \rho$.

For a binary relation $\rho \subseteq X \times X$, we usually prefer to write $x \,\rho\, y$ instead of $(x, y) \in \rho$. Notice that, every binary relation on X includes the empty subset $\emptyset$ of $X \times X$, and the whole set $\omega_X = X \times X$ includes every binary relation on X. The relation is called *universal* relation on X, in which $x \in X$ is related to every $y \in X$. The *equality* relation on a set X is defined as the set

$$\iota_X = \{(x, x) : x \in X\}$$

which is also known as the *diagonal* relation. Obviously, here two elements of X are related if and only if they are equal. Usually, $\mathcal{B}_X$ denotes the set of all binary relations on X. Define a multiplication $\circ$ on $\mathcal{B}_X$ by the rule that for all $\rho, \sigma \in \mathcal{B}_X$,

$$\rho \circ \sigma = \{(x, y) \in X \times X : (\exists z \in X)\,(x, z) \in \rho \text{ and } (z, y) \in \sigma\}.$$

As it is easy to check that the multiplication $\circ$ on $\mathcal{B}_X$ is associative, we have the following proposition.

Proposition 1.1.1. [25] *Let $\mathcal{B}_X$ be the set of all binary relations on a set X, then $(\mathcal{B}_X, \circ)$ is a semigroup.*

It is easy to show $\iota_X \circ \rho = \rho = \rho \circ \iota_X$ for all $\rho \in \mathcal{B}_X$, hence $\mathcal{B}_X$ is a monoid with identity ι_X.

For each $\rho \in \mathcal{B}_X$, we define the *domain* $\operatorname{Dom} \rho$ by

$$\operatorname{Dom} \rho = \{x \in X : (\exists y \in X)\,(x, y) \in \rho\}$$

and the *image* $\operatorname{Im} \rho$ by

$$\operatorname{Im} \rho = \{y \in X : (\exists x \in X)\,(x, y) \in \rho\}.$$

We remark that $\rho \subseteq \sigma$ implies $\operatorname{Dom} \rho \subseteq \operatorname{Dom} \sigma$ and $\operatorname{Im} \rho \subseteq \operatorname{Im} \sigma$, for all $\rho, \sigma \in \mathcal{B}_X$.

For each $\rho \in \mathcal{B}_X$, we define $\rho^{-1} \in \mathcal{B}_X$, the *converse* of ρ, by

$$\rho^{-1} = \{(x, y) \in X \times X : (y, x) \in \rho\}.$$

A binary relation α of $\mathcal{B}_X$ is called a *partial map* of X if $|x\alpha| = 1$ for all $x \in \operatorname{Dom} \alpha$. This means that for all $x, y_1, y_2 \in X$,

$$(x, y_1) \in \alpha \text{ and } (x, y_2) \in \alpha \Rightarrow y_1 = y_2.$$

The above condition is fulfilled by the empty relation $\emptyset$, which is therefore included among the partial maps.

Let $\alpha : U \to V$ be a map and $Z \subseteq U$. The restriction of α to the set Z is the map

$$\alpha|_Z : Z \to V, \ z \mapsto z\alpha \text{ for all } z \in Z.$$

Sometimes we treat the restriction $\alpha|_Z$ as a map with domain Z and codomain $Z\alpha$.

If α, β are partial maps of X, then α is a *restriction* of β if $\operatorname{Dom}\alpha \subseteq \operatorname{Dom}\beta$, and $\alpha = \beta|_{\operatorname{Dom}\alpha}$.

Let $\mathcal{PT}_X$ be the *set of all partial maps* of X.

Proposition 1.1.2. [25] *The subset $\mathcal{PT}_X$ of $\mathcal{B}_X$ consisting of all partial maps of X is a subsemigroup of $\mathcal{B}_X$.*

We remark that the converse α^{-1} of a partial map α need not be a partial map. For example, if $X = \{1, 3\}$, then $\alpha = \{(1,\ 1),\ (3,\ 1)\}$ is a partial map, but α^{-1} is not.

We will later discuss in more details the partial transformation semigroup $\mathcal{PT}_X$ in Chapter 3.

A binary relation $\leq$ on a set X is called a *partial order relation* if

(i) $\leq$ is *reflexive*, i.e., for all $x \in X$, $x \leq x$;

(ii) $\leq$ is *anti-symmetric*, i.e., for all $x, y \in X$, $x \leq y$, $y \leq x$ imply $x = y$;

(iii) $\leq$ is *transitive*, i.e., for all $x, y, z \in X$, $x \leq y$, $y \leq z$ imply $x \leq z$.

A binary relation $\leq$ on a set X is called a *total order relation* if $\leq$ is a partial order relation having an extra property, that if for all $x, y \in X$, $x \leq y$ or $y \leq x$. We call a pair $(X, \leq)$ with $\leq$ a partial order on X a *partially ordered set*, and with $\leq$ a total order on X a *totally ordered set.*

A pre-order (quasi-order) on a set X is defined to be a reflexive and transitive relation $\leq$ on X. Furthermore, there exist standard way of obtaining an equivalence relation $\equiv$ from a pre-order $\leq$, where we define $a \equiv b$ if and only if $a \leq b$ and $b \leq a$.

1.1.3 Equivalences

An *equivalence* (*relation*) on a set X is defined to be a binary relation ρ which is reflexive, symmetric and transitive. Traditionally, if ρ is an equivalence relation on a set X, we write $x\,\rho\,y$ or $x \equiv y\,(mod\,\rho)$ instead of $(x, y) \in \rho$, for all $x, y \in X$.

The set

$$x\rho = \{y \in X : (x, y) \in \rho\},$$

for all $x \in X$, called *ρ-classes*, or *equivalence classes* of x.

Let $\Omega(X) = \{B_i : i \in I\}$ be a family of non-empty subsets B_i of a set X, then $\Omega(X)$ is a *partition* of X, if for all $i \in I$ the subsets B_i are pairwise disjoint, i.e., $B_i \cap B_j = \emptyset$ for all $i \neq j$, and the union of all the subsets B_i in the partition is X, i.e., $X = \bigcup_{i \in I} B_i$. It is known that an equivalence relation ρ on a set X partitions X into equivalence classes. Conversely, corresponding to any partition of X, there exists an equivalence relation ρ on X where two elements of X are ρ-related if and only if they belong to the same block of partition, which means

$$x\,\rho\,y \Leftrightarrow x, y \in B_i,$$

for some $i \in I$ and $x, y \in X$.

We call a set

$$X/\rho = \{x\rho : x \in X\}$$

the quotient set of X by ρ.

The following proposition presents an important connection between maps and equivalences.

Proposition 1.1.3. [25] *If $\phi : X \to Y$ is a map, then $\phi \circ \phi^{-1}$ is an equivalence.*

We usually call the equivalence relation

$$\begin{aligned}\phi \circ \phi^{-1} &= \{(x,y) \in X \times X : (\exists z \in X)\,(x,z) \in \phi,\ (y,z) \in \phi\} \\ &= \{(x,y) \in X \times X, : x\phi = y\phi\},\end{aligned}$$

the *kernel* of ϕ, i.e., $\phi \circ \phi^{-1} = \operatorname{Ker}\phi$. Hence $\operatorname{Ker}\phi$ is an equivalence relation defined on X by the rule

$$x \operatorname{Ker}\phi\, y \Leftrightarrow x\phi = y\phi$$

for all $x, y \in X$.

It is important to emphasize that the $\operatorname{Ker}\phi$ classes partition X into disjoint subsets, and for any $x, y \in X$, x, y lie in the same class if and only if they have the same image under ϕ.

If $\phi : X \to Y$ is a map, then ϕ is one-one if and only if $\operatorname{Ker}\phi = \iota_X$, and constant if and only if $\operatorname{Ker}\phi = \omega_X$.

Let $\{\rho_i : i \in I\}$ be a non-empty family of equivalence relations on the set X. Then it is easy to check that $\bigcap_{i \in I} \rho_i$, the intersection of all ρ_i, $i \in I$, is also an equivalence relation on X. Moreover, for any given relation ρ on X, the family of all equivalence relations containing ρ is a non-empty set as we certainly have $\rho \subseteq X \times X$, hence the intersection of these equivalence relations is again an equivalence relation, which is the smallest equivalence relation containing ρ. We call it the equivalence relation *generated by* ρ, and denoted it by ρ^e.

However, this foregoing general description is not particularly useful, therefore, it is necessary to develop an alternative description to find the equivalence relation ρ^e generated by a given binary relation ρ on a set X.

Let ρ be an arbitrary reflexive relation on X. For any $m \in \mathbb{N}$, we define

$$\rho^m = \underbrace{\rho \circ \rho \cdots \circ \rho}_{m \text{ times}}.$$

Then we say that

$$\rho^\infty = \bigcup\{\rho^n : n \geq 1\}$$

is the *transitive closure* of the relation ρ. According to Howie [25], ρ^∞ is the smallest transitive relation on X containing ρ. Moreover, we have

Proposition 1.1.4. [25] *Let ρ be any fixed binary relation on X. Then the smallest equivalence relation on X containing ρ is given by*

$$\rho^e = \left(\rho \cup \rho^{-1} \cup \iota_X\right)^\infty.$$

We can rewrite the Proposition 1.1.4 as follows:-

Proposition 1.1.5. [25] *If ρ is a relation on a set X and ρ^e is the smallest equivalence on X containing ρ, then $(x, y) \in \rho^e$ if and only if either $x = y$ or, for some $n \in \mathbb{N}$, there is a sequence of transitions*

$$x = z_1 \to z_2 \to \cdots \to z_n = y$$

in which, for each $i \in \{1, 2, \cdots, n-1\}$, either $(z_i, z_{i+1}) \in \rho$ or $(z_{i+1}, z_i) \in \rho$.

1.2 Congruences

For a semigroup S, homomorphic images (monogenic acts) are not determined by subsemigroups, but rather by congruences (one-sided congruences).

Let ρ be a binary relation on a semigroup S. We say that ρ is *left compatible* (with the operation on S) if

$$(\forall s, t, a \in S)\,(s, t) \in \rho \Rightarrow (as, at) \in \rho$$

and *right compatible* if

$$(\forall s, t, a \in S)\,(s,t) \in \rho \Rightarrow (sa, ta) \in \rho$$

and ρ is said to be compatible if

$$(\forall s, t, s', t' \in S)\,[(s,t) \in \rho \text{ and } (s', t') \in \rho] \Rightarrow (ss', tt') \in \rho.$$

Remark: a left (right) compatible equivalence is called a *left (right) congruence*, and a compatible equivalence relation is called a *congruence*.

Proposition 1.2.1. [25] *A relation ρ on a semigroup S is a congruence if and only if it is both a left and right congruence.*

The following theorem is the Fundamental Theorem of Morphisms for Semigroups.

Theorem 1.2.2. [25] *Let ρ be a congruence on a semigroup S. Then the set*

$$S/\rho = \{a\rho :\ a \in S\}$$

together with the multiplication defined by the rule that $(a\rho)\,(b\rho) = (ab)\rho$ forms a semigroup, and the mapping $\rho^\natural$ defined by

$$\rho^\natural : S \longrightarrow S/\rho\ : a \mapsto a\rho$$

is morphism.

Now let ϕ be a morphism from S to T. Then the relation

$$\operatorname{Ker}\phi = \{(a,b) \in S \times S :\ a\phi = b\phi\}$$

is a congruence on S, $Im\,\phi$ is a subsemigroup of T, and $S/Ker\phi$ is isomorphic to $Im\,\phi$.

It is clear that the intersection of non-empty family of congruences on a semigroup S is a congruence on S and any relation ρ is contained in some congruence namely $X \times X$. Hence we can deduce that for every relation ρ on S there is a unique smallest congruence $\rho^\sharp$ on S containing ρ, namely the intersection of the family of *all* congruences on S containing ρ.

The following proposition is an analogous result for congruences to Proposition 1.1.4, and gives us a usable description of $\rho^\sharp$.

Proposition 1.2.3. [25] *For any fixed binary relation ρ on a semigroup S, the smallest congruence $\rho^\sharp$ containing ρ is defined by $\rho^\sharp = (\rho^c)^e$, where*

$$\rho^c = \{(xay, xby) : x, y \in S^1, (a, b) \in \rho\},$$

and ρ^c is the smallest left and right compatible relation containing ρ.

Now, if ρ is any relation on a semigroup S, and if $c, d \in S$ are such that

$$c = xay, \quad d = xby,$$

for some $x, y \in S^1$, where either (a, b) or (b, a) belongs to ρ, we say that c is connected to d by an *elementary ρ-transition.*

Then we have the following proposition as an analogous result for congruences of Proposition 1.1.5.

Proposition 1.2.4. [25] *Let ρ be a relation on a semigroup S, and let $a, b \in S$. Then $(a, b) \in \rho^\sharp$ if and only if either $a = b$ or, for some $n \in \mathbb{N}$, there is a sequence*

$$a = z_1 \to z_2 \to \cdots \to z_n = b$$

of elementary ρ-transitions connecting a to b.

The following lemma has another property for the congruence on a semigroup S:

Lemma 1.2.5. [25] *Let ρ, σ be congruences on a semigroup S, then*

$$(\rho \cup \sigma)^c = \rho^c \cup \sigma^c = \rho \cup \sigma.$$

Where ρ and σ are congruences on a semigroup S, we denote $(\rho \cup \sigma)^\sharp$ by $\rho \vee \sigma$. Hence from Proposition 1.2.3 and Lemma 1.2.5 we obtain

$$(\rho \cup \sigma)^\sharp = [(\rho \cup \sigma)^c]^e = (\rho \cup \sigma)^e = \rho \vee \sigma.$$

So $(\rho \cup \sigma)^e$ is the smallest congruence on S containing $\rho \cup \sigma$.

A useful result is provided by the following lemma:

Lemma 1.2.6. [25] *Let ρ, σ be congruences on a semigroup S such that $\rho \circ \sigma = \sigma \circ \rho$. Then $\rho \vee \sigma = \rho \circ \sigma$.*

1.3 Free semigroups, monoids and free groups

1.3.1 Free semigroups and monoids

An *alphabet* is a non-empty finite set A. A *letter* is an element of A and a *word* (or *string*) over A is a finite sequence $a_1 a_2 \cdots a_n$ of elements a_i, $1 \leq i \leq n$ of A.

If $a_1, a_2, \cdots, a_n, a'_1, \cdots, a'_m \in A$, then

$$a_1 a_2, \cdots a_n = a'_1 \cdots a'_m \Leftrightarrow n = m \text{ and } a_i = a'_i \text{ for } 1 \leq i \leq n.$$

Let $A^+ = \{a_1 a_2 \cdots a_n : n \in \mathbb{N}, a_i \in A, 1 \leq i \leq n\}$ be the set of all finite non-empty words over A. A binary operation is defined on A^+ by juxtaposition

$$(a_1 a_2 \cdots a_m)(b_1 b_2 \cdots b_n) = a_1 a_2 \cdots a_m b_1 b_2 \cdots b_n,$$

with respect to this operation, A^+ is a semigroup, called the *free semigroup* on A. It is clear that the set A is a generating set of A^+. As $A = A^+ \setminus (A^+)^2$, the set A is a unique minimum generating set for A^+.

Notice that $A^* = A^+ \cup \{\epsilon\}$, where ϵ is the *empty word* (containing no letter at all). So A^* with the following binary operation

$$(a_1 a_2 \cdots a_m)(b_1 b_2 \cdots b_n) = a_1 a_2 \cdots a_m b_1 b_2 \cdots b_n,$$
$$\epsilon\,\omega = \omega = \omega\,\epsilon \quad \text{for all} \quad \omega \in A^*$$

is a monoid called *free monoid* on A.

An abstract way to define a free semigroup on A can be given as follows: A semigroup F is called a *free semigroup* on a set A if

(F1) there is a map $\alpha : A \to F$;

(F2) for every semigroup S and every map $\phi : A \to S$ there exists a unique morphism $\psi : F \to S$ such that the diagram

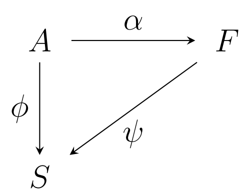


Figure 1.1: The commutative diagram of free semigroups.

commutes.

The uniqueness property guarantees that any two free semigroups on base sets of equal order are isomorphic.

It is easy to see the free semigroup A^+ as defined above does have the properties (F_1) and (F_2). We refer to the map $\alpha : A \to A^+$ that associated each $a \in A$ with the corresponding one-letter word in A^+ as the *standard embedding* of A in A^+. Now, where α is standard embedding map and $\phi : A \to S$ any given map with any semigroup S, we define $\psi : A^+ \to S$ by

$$(a_1 a_2 \cdots a_m)\psi = (a_1\phi)(a_2\phi)\cdots(a_m\phi)$$

for all $a_1 a_2 \cdots a_m \in A^+$. It is easy to see that ψ is a unique morphism and $\alpha\psi = \phi$.

If S is a semigroup and A is a generating set for S then the property (F_2) gives us a morphism ψ from A^+ onto S. Hence $S \simeq A^+/\operatorname{Ker}\psi$. As we can always find a generating set for S, even S itself we have the following lemma:

Lemma 1.3.1. [25] *Every semigroup may be expressed up to an isomorphism as a quotient of a free semigroup by a congruence.*

Corollary 1.3.2. *Every (finitely generated) semigroup S is a homomorphic image of a (finitely generated) free semigroup .*

By replacing A^+ by A^*, Corollary 1.3.2 becomes:

Corollary 1.3.3. *Every (finitely generated) monoid M is a homomorphic image of a (finitely generated) free monoid.*

1.3.2 Free groups

An abstract definition of a *free group* on a set X can be given as follows:

A group $FG(X)$ is said to be *free* on a non-empty set $X \subseteq FG(X)$ if for every group G and map $\theta : X \to G$, there exists a unique homomorphism $\psi : FG(X) \to G$ such that $x\theta = x\psi$, for all $x \in X$.

As $X \subseteq FG(X)$, let the map $\iota : X \longrightarrow FG(X)$ be the inclusion map, hence the following diagram commutes.

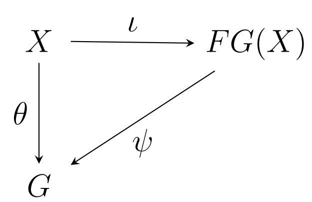


Figure 1.2: The commutative diagram of free groups.

To construct a concrete description of a free group on a set of generators, there are two approaches, which both begin in the same way by consideration of free monoids.

If X is the set of generators, let $X^{-1} = \{x^{-1} : x \in X\}$ be a set in one-one correspondence with X, then we consider the free monoid $(X \cup X^{-1})^*$. Notice that at this point the two approaches diverge.

One approach by letting ρ be the monoid congruence on $(X \cup X^{-1})^*$ generated by H, where

$$H = \{(xx^{-1}, \epsilon), (x^{-1}x, \epsilon) : x \in X\},$$

such that ϵ is the empty word in $(X \cup X^{-1})^*$. Then consider

$$G(X) = (X \cup X^{-1})^*/\rho = \{\omega\rho : \omega \in (X \cup X^{-1})^*\}.$$

It is easy to see that $G(X)$ is a group under the binary operation $(\omega\rho)(\omega'\rho) = (\omega\omega')\rho$, where $\omega, \omega' \in (X \cup X^{-1})^*$ with identity $\epsilon\rho$ and $(x_1^{\zeta_1} \cdots x_n^{\zeta_n})^{-1}\rho = (x_n^{-\zeta_n} x_{n-1}^{-\zeta_{n-1}} \cdots x_1^{-\zeta_1})\rho$, where $x_i \in X$, $\zeta_i \in \{1, -1\}$. We only need to check that $(x_n^{-\zeta_n} \cdots x_1^{-\zeta_1})\rho$ is the inverse of $(x_1^{\zeta_1} \cdots x_n^{\zeta_n})\rho$ as follows:

$$\begin{aligned}
(x_1^{\zeta_1} \cdots x_n^{\zeta_n})\rho(x_n^{-\zeta_n} \cdots x_1^{-\zeta_1})\rho =& \big((x_1^{\zeta_1} \cdots x_n^{\zeta_n})(x_n^{-\zeta_n} \cdots x_1^{-\zeta_1})\big)\rho \\
&= (x_1^{\zeta_1} \cdots x_n^{\zeta_n} x_n^{-\zeta_n} \cdots x_1^{-\zeta_1})\rho \\
&= (x_1^{\zeta_1} \cdots \epsilon \cdots x_1^{-\zeta_1})\rho \\
&\vdots \\
&= (x_1^{\zeta_1} x_1^{-\zeta_1})\rho \\
&= \epsilon\rho.
\end{aligned}$$

We claim that $G(X)$ is free group on a set X in the sense of the abstract definition of free group. We take the inclusion map $\iota : X \to G(X)$ such that $x \mapsto x\rho$, which represents the standard embedding of X into $G(X)$ associating each x in X

with the corresponding one-letter word in $G(X)$. Then for any given group G and an arbitrary map $\theta : X \to G$, we extend θ to have domain $X \cup X^{-1}$ by putting $x^{-1}\theta = (x\theta)^{-1}$. Now, as $(X \cup X^{-1})^*$ is free on $X \cup X^{-1}$ we have a well-define morphism $\psi : (X \cup X^{-1})^* \to G$ given by

$$(x_1^{\zeta_1} \cdots x_n^{\zeta_n})\psi = (x_1\theta)^{\zeta_1} \cdots (x_n\theta)^{\zeta_n},$$

where $x_i \in X$, $\zeta_i \in \{1, -1\}$.

Now, for any $(xx^{-1}, \epsilon) \in H$ we have $(xx^{-1})\psi = (x\theta)(x\theta)^{-1} = 1 = \epsilon\psi$ this implies $(xx^{-1}, \epsilon) \in \operatorname{Ker}\psi$. Similarly, $(x^{-1}x, \epsilon) \in \operatorname{Ker}\psi$, so $H \subseteq \operatorname{Ker}\psi$ and hence $\rho \subseteq \operatorname{Ker}\psi$. Therefore, there exists $\bar{\psi} : G(X) = (X \cup X^{-1})^*/\rho \to G$ given by $(\omega\rho)\bar{\psi} = \omega\psi$. Moreover, for any $x \in X$, $x\iota\bar{\psi} = (x\rho)\bar{\psi} = x\psi = x\theta$, so $\iota\bar{\psi} = \theta$.

Now, we check that $\bar{\psi}$ is unique morphism such that the following diagram commutes.

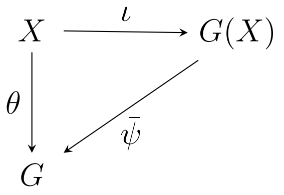


Figure 1.3: The commutative diagram of free group G(X).

Suppose that $\phi : G(X) = (X \cup X^{-1})^*/\rho \to G$ is such that $\iota\phi = \theta$ then for any $x\rho \in G(X)$ where $x \in X$ we have

$$(x\rho)\phi = x\iota\phi = x\theta = x\psi = (x\rho)\bar{\psi}.$$

Now, we are going to describe another approach to construct a free group.

If X is a subset of the free group on X, then different words of $(X \cup X^{-1})^*$ can give rise to the same element of the free group. For example, if $X = \{x, y\}$ then xy and $xx^{-1}xy$ must be equal in the free group. To overcome this, we consider the concept of a *reduced* word as follows:

A word $\omega = x_1^{\eta_1} \cdots x_n^{\eta_n} \in (X \cup X^{-1})^*$ where $x_i \in X$ and $\eta_i \in \{1, -1\}$, is *reduced* if it contains no subword xx^{-1} or $x^{-1}x$.

Examples of group-reduced words can easily given. Let $X = \{x, y\}$; then xy, $x^{-1}yxyx^{-1}$, $x^{-1}yyxy^{-1}$ are group-reduced words. However, $yxyxx^{-1}$ and $x^{-1}yxyy^{-1}$ are not.

Proposition 1.3.4. [4] *Let $\omega \in (X \cup X^{-1})^*$. Then $\omega \,\rho\, \omega^r$ for a unique reduced word ω^r.*

Now, define

$$R(X) = \{\omega \in (X \cup X^{-1})^* : \omega = \omega^r\}$$

with binary operation

$$\omega \cdot \nu = (\omega\nu)^r, \quad \text{for any} \quad \omega, \nu \in (X \cup X^{-1})^*.$$

Clearly, the identity element of the set $R(X)$ is the empty word $\epsilon \in (X \cup X^{-1})^*$, as $\epsilon = \epsilon^r$. If $\omega = x_1^{\zeta_1} \cdots x_n^{\zeta_n} \in R(X)$ then $x_n^{-\zeta_n} \cdots x_1^{-\zeta_1} \in R(X)$ and $x_1^{\zeta_1} \cdots x_n^{\zeta_n} \cdot x_n^{-\zeta_n} \cdots x_1^{-\zeta_1} = (x_1^{\zeta_1} \cdots x_n^{\zeta_n} \cdot x_n^{-\zeta_n} \cdots x_1^{-\zeta_1})^r = \epsilon^r = \epsilon$. Hence, to show the set $R(X)$ is group with the above binary operation we only check that the associativity holds for all $\omega \in R(X)$. To prove that, let $\omega, \nu, \kappa \in R(X)$, we need to show that $((\omega\nu)^r\kappa)^r \,\rho\, (\omega(\nu\kappa)^r)^r$. By Proposition 1.3.4, we have $((\omega\nu)^r\kappa)^r \,\rho\, (\omega\nu)^r\kappa \,\rho\, (\omega\nu)\kappa$, and as $(\omega\nu)\kappa = \omega(\nu\kappa)$, we obtain $\omega(\nu\kappa) \,\rho\, \omega(\nu\kappa)^r \,\rho\, (\omega(\nu\kappa)^r)^r$. As $((\omega\nu)^r\kappa)^r$ and $(\omega(\nu\kappa)^r)^r$ are both reduced words, then $((\omega\nu)^r\kappa)^r = (\omega(\nu\kappa)^r)^r$. Hence, $(\omega \cdot \nu) \cdot \kappa = \omega \cdot (\nu \cdot \kappa)$, so that $R(X)$ is a group.

Now, define a map $\phi : G(X) \to R(X)$ by $(\omega\rho)\phi = w^r$. It is easy to show that ϕ is an isomorphism map and then we deduce $G(X) \cong R(X)$. Hence, $R(X)$ is free group on a set of generators X.

1.4 Green's relations and regular semigroups

1.4.1 Green's relations

Green's relations are equivalence relations that characterize the elements of semigroup S in terms of the principal ideals they generate, and they were first introduced by J.A. Green in 1951 [20].

We define relations $\leq_{\mathcal{L}}$, $\leq_{\mathcal{R}}$ and $\leq_{\mathcal{J}}$ on a semigroup S by the rules:

$$\begin{aligned} a \leq_{\mathcal{L}} b &\iff S^1 a \subseteq S^1 b \\ a \leq_{\mathcal{R}} b &\iff a S^1 \subseteq b S^1 \\ a \leq_{\mathcal{J}} b &\iff S^1\, a\, S^1 \subseteq S^1\, b\, S^1. \end{aligned}$$

Remark that these relations are all quasi-orders; they are not, in general, partial orders.

If $e, f \in E(S)$, then we have

$$e \leq_{\mathcal{L}} f \iff ef = e \quad \text{and} \quad e \leq_{\mathcal{R}} f \iff fe = e.$$

Furthermore, if $a, b \in S$ then $ab \leq_{\mathcal{J}} a,\, b$. Thus the product always lies below its factors with respect to $\leq_{\mathcal{J}}$.

Note that $\leq_{\mathcal{L}}$ is right compatible, as if $a, b, c \in S$ and $a \leq_{\mathcal{L}} b$, then $S^1 a \subseteq S^1 b$, and so $S^1 ac \subseteq S^1 bc$, that is, $ac \leq_{\mathcal{L}} bc$. Dually, $\leq_{\mathcal{R}}$ is left compatible.

We now let $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{J}$ be the equivalence relations associated with $\leq_{\mathcal{L}}$, $\leq_{\mathcal{R}}$ and $\leq_{\mathcal{J}}$, respectively. Thus for any $a, b \in S$

$$\begin{aligned} a\, \mathcal{L}\, b &\iff S^1 a = S^1 b \\ a\, \mathcal{R}\, b &\iff a S^1 = b S^1 \\ a\, \mathcal{J}\, b &\iff S^1\, a\, S^1 = S^1\, b\, S^1. \end{aligned}$$

So, a and b are $\mathcal{L}$-related if they generate the same principal left ideal, a and b are $\mathcal{R}$-related if they generate the same principal right ideal, in addition a and b are

$\mathcal{J}$-related if they generate the same principal two-sided ideal.

The next proposition gives an alternative characterisation of the relations $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{J}$.

Proposition 1.4.1. [25] *Let S be a semigroup and $a, b \in S$. Then*

(i) $a \,\mathcal{L}\, b$ if and only if there exist $x, y \in S^1$ such that $xa = b$, $yb = a$;

(ii) $a \,\mathcal{R}\, b$ if and only if there exist $u, v \in S^1$ such that $au = b$, $bv = a$;

(iii) $a \,\mathcal{J}\, b$ if and only if there exist $x, y, u, v \in S^1$ such that $xay = b$, $ubv = a$.

Now, if $a \,\mathcal{L}\, b$ and $c \in S$, then $S^1a = S^1b$, so $S^1ac = S^1bc$ and hence $ac \,\mathcal{L}\, bc$, i.e., $\mathcal{L}$ is right compatible. Dually, $\mathcal{R}$ is a left compatible.

The following lemma gives an important property of $\mathcal{L}$ and $\mathcal{R}$ follows from the observation concerning the compatibility properties of $\leq_{\mathcal{R}}$ and $\leq_{\mathcal{L}}$.

Lemma 1.4.2. [25] *Let S be a semigroup. The relation $\mathcal{L}$ is a right congruence and $\mathcal{R}$ is a left congruence.*

As the intersection of two equivalences is again an equivalence, hence the intersection of the equivalence relations $\mathcal{L}$ and $\mathcal{R}$ on a semigroup S is again an equivalence relation on S, and we denote it by $\mathcal{H}$ such that $\mathcal{L} \cap \mathcal{R} = \mathcal{H}$.

We denote the equivalence relation $\mathcal{L} \vee \mathcal{R}$ by $\mathcal{D}$. We use L_a, R_a, H_a, J_a and D_a to be denote the $\mathcal{L}$-class, the $\mathcal{R}$-class, the $\mathcal{H}$-class, the $\mathcal{J}$-class and the $\mathcal{D}$-class of an element $a \in S$, respectively.

Proposition 1.4.3. [25] *The relations $\mathcal{L}$ and $\mathcal{R}$ commute, i.e., $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$.*

By Lemma 1.2.6, we have $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \vee \mathcal{R}$. It is clear $\mathcal{L} \subseteq \mathcal{J}$ and $\mathcal{R} \subseteq \mathcal{J}$, and as $\mathcal{D}$ is the smallest equivalence relation containing $\mathcal{L}$ and $\mathcal{R}$, we have $\mathcal{D} \subseteq \mathcal{J}$.

Observe that in certain classes of semigroups some of Green's relations coincide. For example, in a group G we have

$$G^1a = G = G^1b \text{ and } aG^1 = G = bG^1$$

for all $a, b \in G$. So $a\,\mathcal{L}\,b$ and $a\,\mathcal{R}\,b$ for all a, $b \in G$. Therefore, we have

$$\mathcal{L} = \mathcal{R} = \mathcal{H} = \mathcal{D} = \mathcal{J} = G \times G = \omega.$$

Another example, if S is a commutative semigroup then we have

$$\mathcal{L} = \mathcal{R} = \mathcal{H} = \mathcal{D} = \mathcal{J}.$$

It is worth to mention that in every finite semigroup S, we have $\mathcal{D} = \mathcal{J}$, as follows:

Proposition 1.4.4. [25] *If a semigroup S is a finite, then $\mathcal{D} = \mathcal{J}$.*

In the following theorem we will consider some semigroups for which $\mathcal{D} = \mathcal{J}$:

Theorem 1.4.5. [25] *let S be a semigroup. If S has M_L, M_R, then $\mathcal{D} = \mathcal{J}$.*

Observe that each $\mathcal{D}$-class in a semigroup S is a union of $\mathcal{L}$-class and also a union of $\mathcal{R}$-class. Further, the intersection of an $\mathcal{L}$-class and $\mathcal{R}$-class is either empty or is an $\mathcal{H}$-class. However, it follows from the definition of $\mathcal{D}$ and the fact $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$ that

$$a\,\mathcal{D}\,b \Leftrightarrow R_a \cap L_b \neq \emptyset \Leftrightarrow L_a \cap R_b \neq \emptyset.$$

It is often useful to visualize a $\mathcal{D}$-class of a semigroup S using a so called *egg-box diagram*, which is a grid depicted by the figure below, whose columns represent $\mathcal{L}$-class of $\mathcal{D}$, rows represent $\mathcal{R}$-class of $\mathcal{D}$, and the intersections of the columns and rows, that is the cells of the grid, represent the $\mathcal{H}$-class of $\mathcal{D}$.

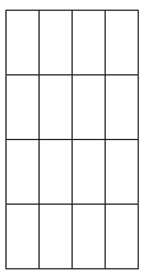


Figure 1.4: The egg-box of a typical $\mathcal{D}$-class.

The structure of $\mathcal{D}$-classes helps to determine the properties of a semigroups. The following important lemma, known as Green's Lemma, tells us that every $\mathcal{L}$-class and every $\mathcal{R}$-class in a $\mathcal{D}$-class has the same size.

We first explain the maps restricted to particular domains used in this lemma, known as *right translations* and *left translations.* A *right translation* of a semigroup is a map $\rho_s : S \longrightarrow S$ define by

$$x\rho_s = xs$$

for all $x \in S$. Dually, a *left translation* of a semigroup S is a map $\lambda_t : S \longrightarrow S$ such that

$$y\lambda_t = ty$$

for all $y \in S$.

Lemma 1.4.6. Green's Lemma: [25] *Let S be a semigroup and $a, b \in S$. Suppose $a\,\mathcal{D}\,b$*

(*i*) *If $a\,\mathcal{R}\,b$, let s, $s^{'} \in S^1$ such that*

$$as = b, \quad bs^{'} = a.$$

Then the right translations $\rho_s : L_a \longrightarrow L_b$ and $\rho_{s'} : L_b \longrightarrow L_a$ are mutually inverse $\mathcal{R}$-class preserving bijections from L_a onto L_b and L_b onto L_a, respectively.

(*ii*) *If $a\,\mathcal{L}\,b$, let t, $t^{'} \in S^1$ be such that*

$$ta = b, \quad t^{'}b = a.$$

Then the left translations $\lambda_t : R_a \longrightarrow R_b$ and $\lambda_{t'} : R_b \longrightarrow R_a$ are mutually inverse $\mathcal{L}$-class preserving bijections from R_a onto R_b and R_b onto R_a, respectively.

A consequence of Green's Lemma is that if a, b are any $\mathcal{D}$-equivalent elements in a semigroup S then there exists a bijection from H_a onto H_b, which means $|H_a| = |H_b|$. We have the following result, usually called Green's Theorem:

Theorem 1.4.7. Green's Theorem: [25] *If H is an $\mathcal{H}$-class in a semigroup S then either $H^2 \cap H = \emptyset$ or $H^2 = H$ and H is a subgroup of S.*

We now immediately deduce:

Corollary 1.4.8. [25] *If H is an $\mathcal{H}$-class in a semigroup S, then H is a subgroup of S if and only if H contains an idempotent of S. No $\mathcal{H}$-class in S can contain more than one idempotent.*

Given an idempotent $e \in E(S)$, and let G be a subgroup of a semigroup S containing e. For any $a \in G$, we know $a \,\mathcal{H}\, e$ in G, so $a \,\mathcal{H}\, e$ in S, and hence $G \subseteq H_e$, thus, the elements of G are all $\mathcal{H}$-related, where H_e is the $\mathcal{H}$-class of e in S. Therefore, we have H_e is a maximal subgroup of S containing e.

We end this subsection with the following result:

Proposition 1.4.9. [25] *Let a, b be elements in a $\mathcal{D}$-class of a semigroup S. Then $ab \in R_a \cap L_b$ if and only if $L_a \cap R_b$ contains an idempotent.*

1.4.2 Regular semigroups

An element a of a semigroup S is called *regular* if there exists x in S such that $axa = a$. The semigroup S is called *regular* if all its elements are regular.

If $b \in R_a$, there exist $u, v \in S^1$ such that $au = b$, $bv = a$, this gives

$$b = au = axau = axb = b(vx)b$$

means b is also regular. Similarly, if $c \in L_b$ so c is regular, and we have $a \,\mathcal{R}\, b \,\mathcal{L}\, c$ and this implies $a \,\mathcal{D}\, c$, so we have the following lemma:

Lemma 1.4.10. [25] *If a is a regular element of a semigroup S, then every element of D_a is regular.*

Consequently, if D is a $\mathcal{D}$-class of a semigroup S then either every element of D is regular or no element of D is regular. We call the $\mathcal{D}$-class *regular* if all it is elements are regular.

If S is a regular semigroup then for each $a \in S$, $a = axa \in aS$, Sa and SaS. Hence, Green's relations can be expressed in terms of S rather than S^1. Note also that if a is regular then $S^1a = Sa$, $aS^1 = aS$ and $S^1aS^1 = SaS$.

Notice that any idempotents e in a semigroup S are regular (as $e = eee$), it follows that every $\mathcal{D}$-class containing an idempotent is regular. Moreover, every regular $\mathcal{D}$-class must contain at least one idempotent, as if $a = axa$, then

$$(ax)^2 = (ax)(ax) = (axa)x = ax,$$

so $ax \in E(S)$ and dually, $xa \in E(S)$. Moreover,

$$\begin{aligned} a = axa, &\quad ax = ax \Longrightarrow a \,\mathcal{R}\, ax \\ a = axa, &\quad xa = xa \Longrightarrow a \,\mathcal{L}\, xa. \end{aligned}$$

Hence $ax \,\mathcal{R}\, a \,\mathcal{L}\, xa$, which has the following diagram

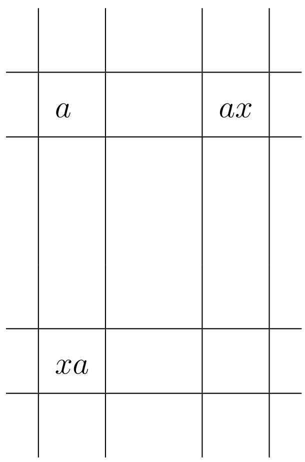


Figure 1.5: The egg-box diagram of D_a.

Lemma 1.4.11. [25] *In a regular $\mathcal{D}$-class, each $\mathcal{L}$-class and $\mathcal{R}$-class contains an idempotent.*

From the above lemma we deduce that a semigroup S is regular if and only if each $\mathcal{L}$-class and each $\mathcal{R}$-class contains an idempotent.

We remark that every idempotent e in a semigroup S is a left identity for R_e and a right identity for L_e.

1.5 Presentations

1.5.1 Semigroup presentations

Let A be an alphabet. A *semigroup presentation* is an ordered pair $\langle A : R\rangle$, where $R \subseteq A^+ \times A^+$. A relation $(u, v) \in R$ will usually written as an equation $u = v$. If $A = \{a_1, \cdots, a_m\}$ and $R = \{(u_1, v_1), \cdots, (u_n, v_n)\}$, we write $\langle a_1, \cdots, a_m : u_1 = v_1, \cdots, u_n = v_n\rangle$ instead of $\langle A : R\rangle$. The elements of A and R are called generator symbols and defining relations, respectively. For $R \subseteq A^+ \times A^+$, we denote by $R^\sharp$ the smallest congruence on A^+ generated by R. To say that a semigroup S has presentation $\langle A : R\rangle$ is to say that $S \cong A^+/R^\sharp$ or, equivalently there is a semigroup epimorphism $\varphi : A^+ \to S$ with $\operatorname{Ker}\varphi = R^\sharp$. If such an epimorphism exists, then S has presentation $\langle A : R\rangle$ via φ.

1.5.2 Monoid presentations

In the definition of the semigroup presentations if replace A^+ by A^*, we obtain a *monoid presentation* for a semigroup S.

The difference between these types of presentations is that monoid presentations may contain relations of the form $u = 1$.

If S is a semigroup and has a semigroup presentation $\langle A : R\rangle$ via $\phi : A^+ \to S$, then $S^1 = S \cup \{1\}$ has a monoid presentation $\langle A : R\rangle$ via $\phi^* : A^* \to S^1$. This means if S is not a monoid, then $\omega\phi^* = 1$ if and only if $\omega = \varepsilon$ (the empty word in A^*). If S is a monoid, then $\omega\phi^* = \omega\phi = 1$, for some $\omega \in A^+$ (as S has semigroup presentation $\langle A : R\rangle$ via ϕ). So, from every semigroup presentation of a semigroup S we may obtain a monoid presentation of the monoid S^1.

If M is a monoid and has a monoid presentation $\langle Y : Q\rangle$ via $\psi : Y^* \to M$, then we may not be able to regard $\langle Y : Q\rangle$ as a semigroup presentation, as it may contain relations of the form $u = 1$. Even if Q has no relations of this kind, $\psi|_{Y^+} : Y^+ \to M$ may not be onto. However, $\langle Z : P\rangle$ is a semigroup presentation for M via $\bar{\psi}$, where $Z = Y \cup \{e\}$ and

$$\begin{aligned} P = &\{u = v \,:\, u = v \in Q,\, u, v \in Y^+\} \cup \{u = e \,:\, u = 1 \in Q\} \\ &\cup \{e\,x = x,\, x\,e = x \,:\, x \in Z\}, \end{aligned}$$

and

$$y\bar{\psi} = \begin{cases} y\psi & \text{if} \quad y \in Y; \\ 1 & \text{if} \quad y = e. \end{cases}$$

1.5.3 Group presentations

To construct a group presentation for a group G, first we define the *normal closure* of any subset K of a group G to be the intersection of all normal subgroups of G containing K. Clearly the normal closure of K is a smallest normal subgroup of G containing K. Hence, the normal closure of K is often called the *normal subgroup generated by K*.

Let $\psi : G \to H$ be a group homomorphism. The kernel of ψ is the normal subgroup of G defined by

$$\ker\psi = \{x \in G : x\psi = e_H\}.$$

The above definition of $\ker\psi$ is related to the definition of $\operatorname{Ker}\psi$ as a relation, since

$$
\begin{aligned}
u \ker\psi = v\ker\psi &\Leftrightarrow uv^{-1} \in \ker\psi \\
&\Leftrightarrow (uv^{-1})\psi = e_H \\
&\Leftrightarrow (u)\psi(v^{-1})\psi = e_H \qquad \text{(as } \psi \text{ is a group homomorphism)} \\
&\Leftrightarrow (u\psi)(v\psi)^{-1} = e_H \\
&\Leftrightarrow u\psi = v\psi \\
&\Leftrightarrow u \operatorname{Ker}\psi\, v \\
&\Leftrightarrow u \operatorname{Ker}\psi = v \operatorname{Ker}\psi \qquad \text{(as Ker } \psi \text{ is a congruence).}
\end{aligned}
$$

A group presentation for a group G is usually defined to be a pair $\langle X : R\rangle$, where X is a free set of generators of a free group $FG(X)$ and $R = \{u_i v_i^{-1} : i \in I\} \subseteq FG(X)$. To say that a group G has presentation $\langle X : R\rangle$ is to say that $G \cong FG(X)/R^\sharp$, where $R^\sharp$ is the smallest normal subgroup of $FG(X)$ generated by R or, equivalently there is a group epimorphism $\psi : FG(X) \to G$ where $R^\sharp$ generates $\operatorname{Ker}\psi$, i.e., $\operatorname{Ker}\psi = R^\sharp$. If such an epimorphism exists, then G has presentation $\langle X : R\rangle$ via ψ.

If a group G has a group presentation $\langle X : R\rangle$ then G has a monoid presentation $\langle X \cup X^{-1} : R'\rangle$, where X^{-1} is defined in page 16,

$$R' = R \cup \{xx^{-1} = \epsilon = x^{-1}x : x \in X\}$$

and ϵ is the empty word in $FG(X)$. Now, as a group G has a monoid presentation $\langle X \cup X^{-1} : R'\rangle$ we know from Subsection 1.5.2 that we can construct a semigroup presentation for G. On the other hand each semigroup presentation (monoid presentation) for G yields a group presentation for G.

1.6 Semidirect products of semigroups, monoids and groups

Definition 1.6.1. Let S and T be semigroups, then

$$S \times T = \{(s,t) : s \in S, t \in T\},$$

with the binary operation

$$(s_1, t_1)(s_2, t_2) = (s_1 s_2, t_1 t_2)$$

is a semigroup, called a *direct product* of semigroups S and T.

If S and T are monoids, then $S \times T$ is a monoid with identity $(1_S, 1_T)$. And $S \times T$ is a *direct product of monoids S and T*.

If G and H are groups, then $G \times H$ with the above binary operation is a group called the *external direct product* of groups G and H. The inverse of an element $(g, h) \in G \times H$ is (g^{-1}, h^{-1}).

Definition 1.6.2. Let $(S, \cdot)$ be a semigroup. We define $(S^{opp}, *)$ to be the *opposite* semigroup of $(S, \cdot)$ such that

$$a * b = b \cdot a.$$

Hence, if S is a semigroup of mappings, as we see later the semigroup S^{opp} consists of the same mappings, composed in the opposite order.

Definition 1.6.3. Let T be semigroup and X be a set. Then T acts on X on the left if there exists a map $T \times X \to X$, $(t, x) \mapsto t \cdot x$, such that for all $x \in X$, and $t_1, t_2 \in T$,

(S1) $t_1 \cdot (t_2 \cdot x) = (t_1 t_2) \cdot x$.

This is equivalent to saying that there is a *semigroup* morphism $\lambda : T \to \mathcal{T}_X^{opp}$ such that for any $t \in T$ and operation $\lambda_t : X \to X$ given by $\lambda_t(x) = t \cdot x$, and

$$t\lambda = \lambda_t,$$

where $\mathcal{T}_X$ is the full transformation monoid on X under composition from left to right. We give an explicit description for $\mathcal{T}_X$ later on in Chapter 3.

We only need to prove λ is a semigroup morphism, which means $(st)\lambda = (s\lambda)(t\lambda)$, for all $s, t \in T$. Notice that, $(st)\lambda = \lambda_{st}$ and $(s\lambda)(t\lambda) = \lambda_s\lambda_t$, for all $s, t \in T$. By using (S1) we obtain for any $x \in X$

$$((s\lambda)(t\lambda))(x) = (\lambda_s\lambda_t)(x) = \lambda_s(\lambda_t(x)) = \lambda_s(t\cdot x) = s\cdot(t\cdot x) = (st)\cdot x = \lambda_{st}(x) = (st)\lambda(x),$$

so that $(st)\lambda = (s\lambda)(t\lambda)$.

If T is a monoid (or, indeed, a group) and acts on a set X, then for a monoid action we insist that for all $x \in X$

(S2) $1_T \cdot x = x$.

This is equivalent to saying that $\lambda : T \to \mathcal{T}_X^{opp}$ as above is a *monoid* morphism, i.e.,

$$1_T\lambda = I_X.$$

Conversely, if T is a semigroup such that $\theta : T \to \mathcal{T}_X^{opp}$ is a semigroup morphism, define

$$t \cdot x = (t\theta)(x),$$

then T acts X on left.

To show that, first we check (S1) holds. Then for all $s, t \in T$ and $x \in X$

$$\begin{aligned}(st) \cdot x &= \big((st)\theta\big)(x) \\ &= ((s\theta)(t\theta))(x) \qquad \text{(as } \theta \text{ is a semigroup morphism)} \\ &= (s\theta)((t\theta)(x)) \\ &= (s\theta)(t \cdot x) \\ &= s \cdot (t \cdot x).\end{aligned}$$

Further, if T is a monoid and θ is a monoid morphism, we need to prove that (S2) holds. Hence, for all $x \in X$

$$1_T \cdot x = (1_T\theta)(x) = x.$$

Therefore, T acts X on left as required.

Just to mention, where convenient we will omit "$\cdot$" in expression such as $s \cdot x$ and write this simply as sx.

Definition 1.6.4. Let T be semigroup (monoid) and S be a semigroup, such that T acts on S on the left. Then T acts S on the left by endomorphisms, if for all $t, \in T$ and $s_1, s_2 \in S$ we have

(S3) $t \cdot (s_1 s_2) = (t \cdot s_1)(t \cdot s_2)$.

This is equivalent to saying that there is a semigroup (monoid) morphism $\theta : T \to (\operatorname{End} S)^{opp}$, where $\operatorname{End} S$ is the endomorphism monoid of semigroup morphisms. Recall that in $\operatorname{End} S \subseteq \mathcal{T}_S$ we compose from left to right, so that in $(\operatorname{End} S)^{opp}$ we compose from right to left, so that for $\theta, \psi \in (\operatorname{End} S)^{opp}$, to compute $\theta\psi$ we first do ψ then θ.

If T is a monoid (semigroup) and if S is a monoid then we insist

(S4) $t \cdot 1_S = 1_S$,

so that the action of T is by monoid morphisms of S.

If T and S are monoids, then T acts on S by endomorphisms if $\theta : T \to (\operatorname{End} S)^{opp}$ is a monoid morphism, in this case $(\operatorname{End} S)^{opp}$ is the endomorphism monoid of monoid morphisms, such that

$$(\theta(t))(1_S) = 1_S \quad \text{for all} \quad t \in T.$$

This translates to $t \cdot 1_S = 1_S$ for all $t \in T$.

Remark, the restriction of θ to the group of units, H_1 of T, is such that

$$\theta|_{H_1} : H_1 \to \operatorname{Aut} S.$$

The semidirect product of two semigroups was used for the first time by Neumann [35] to construct wreath products of semigroups.

Definition 1.6.5. Suppose that T and S are semigroups such that T acts on the left of S by endomorphisms. Define a binary operation on $S \times T$ by

$$(s, t)(s', t') = (s(t \cdot s'), tt').$$

Then $S \times T$ with the above binary operation is a semigroup, called a *semidirect product* of S by T, and denoted by $S \rtimes T$.

If T and S are monoids and T acts on the left of S by endomorphisms satisfying (S1), (S2), (S3) and (S4), then $S \rtimes T$ is a monoid with identity $(1_S, 1_T)$.

This is because

$$\begin{aligned}(1_S, 1_T)(s,t) &= (1_S(1_T \cdot s), 1_T t)\\ &= (1_S s, 1_T t) \qquad \text{(by using (S2))}\\ &= (s,t),\end{aligned}$$

and

$$\begin{aligned}(s,t)(1_S, 1_T) &= (s(t \cdot 1_S), t1_T)\\ &= (s1_S, t1_T) \qquad \text{(by using (S4))}\\ &= (s,t).\end{aligned}$$

Let S and T be semigroups and $S \rtimes T$ be the semidirect product of S by T. If U is a subsemigroup of T, then U acts S on left by endomorphisms, and clearly $S \rtimes U$ is a subsemigroup of $S \rtimes T$.

Now, let M and K be monoids and $M \rtimes K$ be the semidirect product of M by K. If H is a submonoid of K, then H acts M on left by endomorphisms, and it is clear that $M \rtimes H$ is a submonoid of $M \rtimes K$ with identity $(1_M, 1_K)$.

Lemma 1.6.6. *Let G and H be groups, such that G acts on the left of H by endomorphisms. Then G acts on the left of H by automorphisms.*

Proof. Let $\theta_g : H \to H$ given by $\theta_g(h) = g \cdot h$ for all $g \in G$ and $h \in H$. As G acts on the left of H by endomorphisms, this means $g \cdot (h_1 h_2) = (g \cdot h_1)(g \cdot h_2)$ and $g \cdot 1_H = 1_H$ by (S3) and (S4), respectively. To prove G acts on the left of H by automorphism we only need to prove that θ_g is bijection.

To prove θ_g is one to one. Let $h, k \in H$ and suppose $\theta_g(h) = \theta_g(k)$, that implies $g \cdot h = g \cdot k$. As G is a group then $g^{-1} \in G$ and hence $g^{-1} \cdot (g \cdot h) = g^{-1} \cdot (g \cdot k)$, and then $(g^{-1}g) \cdot h = (g^{-1}g) \cdot k$ by using (S1), this implies $1_G \cdot h = 1_G \cdot k$, and by using (S2) we obtain $h = k$. Hence, θ_g is one to one.

To prove θ_g is onto. Let $k \in H$, then there exist $g^{-1} \cdot k \in H$ such that

$$\begin{aligned}
\theta_g(g^{-1} \cdot k) &= g \cdot (g^{-1} \cdot k) \\
&= (gg^{-1}) \cdot k \qquad \text{by (S1)} \\
&= 1_G \cdot k \\
&= k \qquad \text{by (S2).}
\end{aligned}$$

So, θ_g is onto. □

Notice that, if G and H are groups such that G acts on the left of H by automorphisms, then $H \times G$ with the binary operation that defined in Definition 1.6.5 is a group, called an *external semidirect product* of H by G, and denoted by $H \rtimes G$. The inverse of an element $(h, g) \in H \rtimes G$ is $(g^{-1} \cdot h^{-1}, g^{-1})$.

This is because

$$\begin{aligned}
(h, g)(g^{-1} \cdot h^{-1}, g^{-1}) &= (h(g \cdot (g^{-1} \cdot h^{-1})), gg^{-1}) \\
&= (h((gg^{-1}) \cdot h^{-1}), 1_G) \qquad \text{(by using (S1))} \\
&= (h(1_G \cdot h^{-1}), 1_G) \\
&= (hh^{-1}, 1_G) \qquad \text{(by using (S2))} \\
&= (1_H, 1_G)
\end{aligned}$$

and

$$\begin{aligned}
(g^{-1} \cdot h^{-1}, g^{-1})(h, g) &= ((g^{-1} \cdot h^{-1})(g^{-1} \cdot h), g^{-1}g) \\
&= (g^{-1} \cdot (h^{-1}h), 1_G) \qquad \text{(by using (S3))} \\
&= (g^{-1} \cdot 1_H, 1_G) \\
&= (1_H, 1_G) \qquad \text{(by using (S4)).}
\end{aligned}$$

Chapter 2

Preliminaries II: Universal algebras and independence algebras

This chapter is devoted to the study of *universal algebras* and *independence algebras*. The formal definition of a (universal) algebra will be given in Section 2.1. In Section 2.2 we define *independence algebras*, the special class of algebras in which we are interested.

We recommend [2], [3], [5], [7], [17], [18], [19], [32] and [34] as references for this chapter.

2.1 Universal algebras

We need some basic ideas from the field of universal algebra. Specially, the notion of algebra, subalgebra and homomorphism.

Definition 2.1.1. [7, 32] An *algebra* $\mathbf{A}$ is an ordered pair $\mathbf{A}=\langle A, F\rangle$ such that A is a non-empty set and $F = \langle F_i : i \in I\rangle$ where F_i is a finitary operation on A for each $i \in I$. The set A is called the *universe* of $\mathbf{A}$, F_i is referred to as a *fundamental* or *basic operation* of $\mathbf{A}$ for each $i \in I$, and I is called the *index set* or *the set of* *operation symbols of* **A**.

Algebras are usually denoted by bold face letters, while the underlying universes are denoted by the corresponding standard face letters.

We now explain the notion of the rank of an operation on A. Let A be a non-empty set and $n \in \mathbb{N}^0$, where $\mathbb{N}^0 = \{0, 1, 2, \cdots\}$. An *$n$-ary operation* or (an *operation of rank n*) on A is a function

$$A^n \to A.$$

We have some special terminology for n-ary operation where $n \in \{0, 1, 2, 3\}$. The operation of rank 0 on A is a function

$$A^0 \to A,$$

as A^0 is a singleton this takes one value and we call this operation *a nullary operation*. We often associate a nullary operation with the single element in its image. An operation of rank 1 on A is a called a *unary operation* and it is a function

$$A \to A.$$

An operation of rank 2 on A is called a *binary operation* and it is a function

$$A^2 \to A.$$

Similarly, a *ternary operation* is an operation of rank 3 and it is a function

$$A^3 \to A.$$

By a *finitary operation* on A we mean n-ary operation on A of some $n \in \mathbb{N}^0$.

The rank function or similarity type of **A** where $\mathbf{A} = \langle A, F \rangle$ and $F = \langle F_i : i \in I \rangle$

is the function

$$\rho : I \to \mathbb{N}^0$$

given by $\rho(i) =$ rank of F_i. A *partial operation* of rank n on A is a function from a subset of A^n into A.

We will use $Q^{\mathbf{A}}$ to stand for the fundamental operation of $\mathbf{A}$ indexed by Q, where Q is an operation symbol of $\mathbf{A}$, and we will say that $Q^{\mathbf{A}}$ is the *interpretation* of Q in $\mathbf{A}$.

Moreover, where Q is an operation symbol of the algebra $\mathbf{A}$ with rank r, and $a_1, \cdots, a_r \in A$, we often use the expression

$$Q^{\mathbf{A}}(a_1, \cdots, a_r),$$

and sometimes we replace it by $Q^{\mathbf{A}}(\overline{a})$, where $\overline{a}$ stands for the a tuple of elements of A of the correct length.

Definition 2.1.2. [7, 32] Let $\mathbf{A}=\langle A, F_i\rangle$ be an algebra and let F_i be an operation of rank r on the non-empty set A, and let X be a subset of A. We say that X is closed with respect to F_i (also that F_i preserves X and that X is invariant under F_i) if and only if

$$F_i\,(a_0, a_1, \cdots, a_{r-1}) \in X \quad \text{for all} \quad a_0, \cdots, a_{r-1} \in X.$$

Remark 2.1.3. [7, 32] If F_i is constant (F_i an operation of rank 0), this means that X is closed with respect to F_i if and only if $F_i \in X$. Hence the empty set $\emptyset$ is closed with respect to every operation on A of positive rank, however it is not closed with respect to any operation of rank 0.

Algebras $\mathbf{A}$ and $\mathbf{B}$ are said to be *similar* if and only if they have the same rank function. The similarity relation between algebras is an equivalence relation whose equivalence classes will be called *similarity classes*.

Definition 2.1.4. [7, 32] Let **A** be an algebra. A subset of the universe A of **A**, which is closed with respect to each fundamental operation of **A**, is called a *subuniverse* of **A**. The algebra **B** is said to be a *subalgebra* of **A** if and only if **A** and **B** are similar, the universe B of **B** is a subuniverse of **A**, and $Q^{\mathbf{B}}$ is the restriction to B of $Q^{\mathbf{A}}$, for each operation symbol Q of **A**.

The set **SubA** denotes the set of all subuniverses of **A**.

Remark 2.1.5. [7, 32] The empty set $\emptyset$ is a subuniverse of **A** if and only if **A** does not contain a constant (i.e., there are no operations of rank 0 on **A**).

If **A** does not contain a constant, we will consider the empty set $\emptyset$ to be a subalgebra of **A**, although the empty set $\emptyset$ is not the universe of any algebra and operations are not defined.

Definition 2.1.6. [7, 32] Let **A** and **B** be similar algebras and Q be an operation symbol of rank r. A function ψ from A to B is said to respect Q if and only if

$$\left(Q^{\mathbf{A}}(a_0, a_1, \cdots, a_{r-1})\right)\psi = Q^{\mathbf{B}}\left((a_0)\psi, \cdots, (a_{r-1})\psi\right) \quad \text{for all} \quad a_0, \cdots, a_{r-1} \in A.$$

Definition 2.1.7. [7, 32] Let **A** and **B** be similar algebras. A function ψ from A to B is called a *homomorphism* from A to B if and only if ψ respects every basic operation symbol of **A**.

Hom(A, B) denotes the set of all homomorphisms from **A** to **B**.

Let **A** and **B** be similar algebras. Each of

$$h : \mathbf{A} \to \mathbf{B}$$

$$\mathbf{A} \xrightarrow{\mathbf{h}} \mathbf{B}$$

$$h \in \mathrm{Hom}(\mathbf{A}, \mathbf{B})$$

denotes that h is a homomorphism from **A** to **B**.

Both

$$h : \mathbf{A} \rightarrowtail \mathbf{B}$$

and

$$\mathbf{A} \overset{\mathbf{h}}{\rightarrowtail} \mathbf{B}$$

denote that h is a one-to-one homomorphism from $\mathbf{A}$ into $\mathbf{B}$. We call such homomorphisms *embeddings*. Likewise, both

$$h : \mathbf{A} \twoheadrightarrow \mathbf{B}$$

and

$$\mathbf{A} \overset{\mathbf{h}}{\twoheadrightarrow} \mathbf{B}$$

denote that h is a homomorphism from $\mathbf{A}$ onto $\mathbf{B}$, and in this case we say that $\mathbf{B}$ is the *homomorphic image of* $\mathbf{A}$ *under* h. Further, $\mathbf{A} \overset{\mathbf{h}}{\cong} \mathbf{B}$ denotes that h is a one-to-one homomorphism from $\mathbf{A}$ onto $\mathbf{B}$. We call such homomorphisms *isomorphisms*. The algebras $\mathbf{A}$ and $\mathbf{B}$ are said to be *isomorphic*, which we denote by $\mathbf{A} \cong \mathbf{B}$, if and only if there is an isomorphism from $\mathbf{A}$ onto $\mathbf{B}$.

Isomorphism is an equivalence relation between algebras, and is a finer equivalence relation than similarity, in the sense that if two algebras are isomorphic, then they are also similar.

Definition 2.1.8. [7, 32] Let n, $k \in \mathbb{N}^0$, the *composition of operations* is the construction of an n-ary operation h from k given n-ary operations $f_1, f_2, \cdots, f_k$ and a k-ary operation g, by the rule

$$h(\overline{x}) = g(f_1(\overline{x}), \cdots, f_k(\overline{x})).$$

All of these must be operations on the same set A.

Definition 2.1.9. [7, 32] For any $k, n \in \mathbb{N}$ such that $k \leq n$, the *k-th projection operations* on a set A is an n-ary operations $p_k^n(x_1, \cdots, x_n) = x_k$.

Definition 2.1.10. [7, 32] A *clone* on a non-empty set A is set of operations on A that contains the projection operations and is closed under all compositions. The clone of all operations on A will be denoted by $Clo\,A$, while the set of all n-ary operations on A, where $n \in \mathbb{N}^0$ will be written as $Clo_n\,A$.

Definition 2.1.11. [7, 32] The *clone of term operations* of an algebra $\mathbf{A}$, denoted by $Clo\,\mathbf{A}$, is the smallest clone on the base set A that contains the basic operations of $\mathbf{A}$. The set of n-ary operations in $Clo\,\mathbf{A}$ is denoted by $Clo_n\,\mathbf{A}$. The member of $Clo\,\mathbf{A}$ are called *term operations* of the algebra $\mathbf{A}$.

In the following lemma we describe how the map in any universal algebra preserves all the term operations:

Lemma 2.1.12. [7, 32] *Let* $\mathbf{A}$=$\langle A, F\rangle$ *be an algebra with* $F = \langle F_i : i \in I\rangle$. *Then* $\alpha : A \to A$ *preserves each* F_i *if and only if* $\alpha : A \to A$ *preserves all term operations.*

Proof. By mathematical induction on n, suppose

$$\begin{aligned} F_0 &= Proj(A) = \{p_k^m : k \leq m,\, m \in \mathbb{N}\}; \\ F_{n+1} &= F_n \cup \{f(g_1, \cdots, g_k) : \text{either}\, f = Proj(A)\,\text{or},\ f \in F_n,\ \text{rank}\, f = k, \\ &\quad \text{and}\ g_1, \cdots, g_k \in F_n\ \text{where rank}\, g_i = s\}. \end{aligned}$$

Let $f \in F_n$ be an m-ary operation. If $n = 0$, then f is a projection, so $f = p_k^m$ where $k \leq m$ and $f(x_1, \cdots, x_m)\alpha = x_k\alpha = f(x_1\alpha, \cdots, x_m\alpha)$.

Suppose $n \in \mathbb{N}^0$ and α preserves all operations in F_n. Let $h \in F_{n+1} \setminus F_n$. Then either h is a projection or, $h = f(g_1, \cdots, g_k)$ where $f \in F_n$ and $g_1, \cdots, g_k \in F_n$ with rank $g_i = s$. The first case has already been considered. In the second as α preserves each operation F_i, we have

$$
\begin{aligned}
h(x_1,\cdots,x_s)\alpha &= f(g_1(x_1,\cdots,x_s),\cdots,g_k(x_1,\cdots,x_s))\alpha \\
&= f(g_1(x_1,\cdots,x_s)\alpha,\cdots,g_k(x_1,\cdots,x_s)\alpha) \\
&= f(g_1(x_1\alpha,\cdots,x_s\alpha),\cdots,g_k(x_1\alpha,\cdots,x_s\alpha)) \\
&= h(x_1\alpha,\cdots,x_s\alpha),
\end{aligned}
$$

as required.

□

Theorem 2.1.13. [7, 32] *Let* $\mathbf{A}$ *be an algebra and let* S *be any non-empty collection of subuniverses of* $\mathbf{A}$. *Then* $\cap S$ *is a subuniverse of* $\mathbf{A}$.

Proof. Evidently $\cap S$ is a subset of A. Let F be any basic operation of $\mathbf{A}$ and suppose that r is the rank of F. To see that $\cap S$ is closed under F, pick any $a_0,\cdots,a_{r-1}\in\cap S$. For all $B\in S$ we know that $a_0,\cdots,a_{r-1}\in B$ but then $F(a_0,\cdots,a_{r-1})\in B$, since B is a subuniverse. Therefore $F(a_0,\cdots,a_{r-1})\in\cap S$ and $\cap S$ is closed under F. □

Definition 2.1.14. [7, 32] Let $\mathbf{A}$ be an algebra and let $X\subseteq A$. The *subuniverse of* $\mathbf{A}$ *generated by* X is the set $\bigcap\{B : X\subseteq B \text{ and } B \text{ is a subuniverse of } \mathbf{A}\}$.

The set $\mathbf{Sg}^{\mathbf{A}}(X)$ denotes the subuniverse of $\mathbf{A}$ generated by X.

Since $X\subseteq A$ and A is a subuniverse of $\mathbf{A}$, Theorem 2.1.13 justifies calling $\mathbf{Sg}^{\mathbf{A}}(X)$ a subuniverse of $\mathbf{A}$.

Lemma 2.1.15. *The subuniverses of* $\mathbf{A}$ *are precisely those subsets* X *of* A *such that* $X=\mathbf{Sg}^{\mathbf{A}}(X)$.

Theorem 2.1.16. *Let* $\gamma:\mathbf{A}\to\mathbf{B}$ *be a homomorphism. The image and inverse image under* γ *of subuniverses are subuniverses.*

Proof. Let C be a subuniverse of $\mathbf{A}$, and D be a subuniverse of $\mathbf{B}$. Suppose that for all $i = 1, \cdots, n$ we have $a_i \in C$ so that $a_1\gamma, a_2\gamma, \cdots, a_n\gamma$ elements of $C\gamma$. Let Q be an operation symbol of rank n, then $Q^{\mathbf{B}}(a_1\gamma, \cdots, a_n\gamma) = (Q^{\mathbf{A}}(a_1, \cdots, a_n))\gamma \in C\gamma$, since all $a_i \in C$ and C is subuniverse of $\mathbf{A}$. So, $C\gamma$ is a subuniverse of $\mathbf{B}$.

Conversely, where D be a subuniverse of $\mathbf{B}$, then the inverse image of D under γ is $D\gamma^{-1} = \{a \in C : a\gamma \in D\}$. If $a_1, \cdots, a_n \in D\gamma^{-1}$, then $a_i\gamma \in D$ for all $i = 1, 2, \cdots, n$. As γ is a homomorphism and taking Q as before, we get $(Q^{\mathbf{A}}(a_1, \cdots, a_n))\gamma = Q^{\mathbf{B}}(a_1\gamma, \cdots, a_n\gamma) \in D$, since D is a subuniverse. So $Q^{\mathbf{A}}(a_1, \cdots, a_n) \in D\gamma^{-1}$. Hence $D\gamma^{-1}$ is a subuniverse of $\mathbf{A}$. □

2.2 Independence algebras

2.2.1 Basic definitions

Let V be a vector space over a field, and let $\mathrm{End}\,(V)$ be the monoid of all linear maps $\alpha : V \to V$ with multiplication being composition of mappings. Then we say that $\mathrm{End}\,(V)$ is the full linear monoid over V. Let $X_n = \{1, 2, \cdots, n\}$. The full transformation monoid under the semigroup operation of composition is denoted by $\mathcal{T}_n$. We give an explicit description for $\mathcal{T}_n$ later on in Chapter 3.

If V is a vector space of $\dim n$, then $\mathrm{End}\,(V)$ and $\mathcal{T}_n$ are each a union of finitely many ideals I_k ($0 \le k \le n$ for $\mathrm{End}\,(V)$ and $1 \le k \le n$ for $\mathcal{T}_n$) such that the Rees quotients I_k/I_{k-1} are completely 0-simple, and I_k consists of all elements of rank $\le k$. For $\alpha \in \mathrm{End}\,(V)$, $\mathrm{rank}\,\alpha$ is $\dim(\mathrm{Im}\,\alpha)$ and for $\alpha \in \mathcal{T}_n$, $\mathrm{rank}\,\alpha$ is $|\mathrm{Im}\,\alpha|$.

This led Gould to ask [17]

"What do vector spaces and sets have in common which forces $\mathrm{End}\,(V)$ and $\mathcal{T}_n$ to support a similar pleasing structure?"

To answer the above, Gould [17] investigated the endomorphism monoid of a class of universal algebras, called an *independence algebras* previously known as

v^*-algebras [34].

Let A be a set and $C : P(A) \to P(A)$ be a function, where $P(A)$ is the set of all subsets of A. Then C is a *closure operator* on A if for all $X, Y \in P(A)$:

(i) $X \subseteq C(X)$;

(ii) if $X \subseteq Y$ then $C(X) \subseteq C(Y)$;

(iii) $C(X) = C(C(X))$.

For any universal algebra $\mathbf{A}$ the function

$$\mathrm{Sg}^{\mathbf{A}} : P(A) \to P(A),$$

where

$$X \mapsto \mathrm{Sg}^{\mathbf{A}}(X) = \langle X \rangle$$

is a closure operator.

Note that, $\mathrm{Sg}^{\mathbf{A}}(\emptyset) = \emptyset$ if and only if $\mathbf{A}$ has no constants.

A closure operator C on a set B is *algebraic* if for all $X \subseteq B$

$$C(X) = \bigcup \{C(Y) : Y \subseteq X, |Y| < \infty\}.$$

Also, for any universal algebra $\mathbf{A}$ is , $\mathrm{Sg}^{\mathbf{A}}$ is an algebraic closure operator. We can write this as

$$\mathrm{Sg}^{\mathbf{A}} = \langle X \rangle = \bigcup_{\substack{Y \subseteq X \\ |Y| < \infty}} \langle Y \rangle .$$

We need only to show $\langle X \rangle \subseteq \bigcup_{\substack{Y \subseteq X \\ |Y| < \infty}} \langle Y \rangle$. Let $a \in \langle X \rangle$ this implies $a = t(x_1, \cdots, x_n)$ where $\{x_1, \cdots, x_n\} \subseteq X$. So, $a \in \langle \{x_1, \cdots, x_n\} \rangle$, which means $a \in \bigcup_{\substack{Y \subseteq X \\ |Y| < \infty}} \langle Y \rangle$.

Definition 2.2.1. [17] The *Exchange Property* (EP) for a closure operator C on a set A is defined as follows:

(EP) For all $X \subseteq A$ and $x, y \in A$, if $x \notin C(X)$ but $x \in C(X \cup \{y\})$, then $y \in C(X \cup \{x\})$.

Definition 2.2.2. [17] Let C be a closure operator on a set A, and $X \subseteq A$. Then X is C-independent if for all $x \in X$, $x \notin C(X \setminus \{x\})$.

If $\mathbf{A}$ is an algebra then we refer to $\mathrm{Sg}^{\mathbf{A}}$-independent sets more simply as *independent* sets.

It is obvious if $\mathbf{A}$ is any algebra such that every subset of A is a subuniverse, then every subset of A is independent. This is because, where $\mathbf{A}$ is any algebra then $\mathrm{Sg}^{\mathbf{A}}$ is a closure operator such that $C(X) = \mathrm{Sg}^{\mathbf{A}} = \langle X \rangle = X$. So, if $X \subseteq A$ is a subuniverse of $\mathbf{A}$, then for all $x \in X = \langle X \rangle$ we have $x \notin X \setminus \{x\} = \langle X \setminus \{x\} \rangle$. Therefore, X is independent.

Definition 2.2.3. [17] Let $\mathbf{A}$ be an algebra, and $X \subseteq A$, then X is a basis of $\mathbf{A}$ if X generates $\mathbf{A}$ and is independent.

We remark that any algebra satisfying (EP) has a basis, and in such an algebra a subset X is a basis if and only if X is a minimal generating set if and only if X is the maximal independent set. All basis of $\mathbf{A}$ have the same cardinality, called the rank of $\mathbf{A}$. Furthermore, any independent subset X can be extended to be a basis of $\mathbf{A}$.

Definition 2.2.4. [17] An algebra $\mathbf{A}$ has the *Free basis* property (F) if:

(F) For any basis X of A and function $\alpha : X \to A$, α can be extended to an element of $\mathrm{End}\,\mathbf{A}$.

Definition 2.2.5. [17] An algebra $\mathbf{A}$ satisfying (EP) and (F) is called an *independence algebra.*

2.2.2 Endomorphism monoids of independence algebras

Let $\mathbf{A}$ be an independence algebra. From Chapter 1, $\mathrm{End}\,\mathbf{A}$ is the endomorphism monoid of $\mathbf{A}$ and $\mathrm{Aut}\,\mathbf{A}$ is the automorphism group of $\mathbf{A}$. Clearly $\mathrm{Aut}\,\mathbf{A}$ is the group of units of $\mathrm{End}\,\mathbf{A}$.

If $\alpha \in \operatorname{End}\mathbf{A}$ then $\operatorname{Ker}\alpha = \{(x,y) \in \mathbf{A}\times\mathbf{A} : x\alpha = y\alpha\}$ is a subalgebra of $\mathbf{A}\times\mathbf{A}$ and a congruence on $\mathbf{A}$, and $\operatorname{Im}\alpha = \{a\alpha : a \in \mathbf{A}\}$ is a subalgebra of $\mathbf{A}$.

The rank of α[1], written as $\rho(\alpha)$, is defined to be $\rho(\operatorname{Im}\alpha)$. If $\alpha, \beta \in \operatorname{End}\mathbf{A}$ with $\operatorname{Im}\alpha = \operatorname{Im}\beta$, then $\rho(\alpha) = \rho(\beta)$. On the other hand, if $\operatorname{Ker}\alpha = \operatorname{Ker}\beta$ then $\operatorname{Im}\alpha \cong A/\operatorname{Ker}\alpha = A/\operatorname{Ker}\beta \cong \operatorname{Im}\beta$. So, $\rho(\alpha) = \rho(\beta)$.

In [17] Gould, obtained explicit descriptions of Green's relations on $\operatorname{End}\mathbf{A}$, where $\mathbf{A}$ is an independence algebra.

Lemma 2.2.6. [17] *Let* $\mathbf{A}$ *be an independence algebra. For all* $\alpha, \beta \in End\,\mathbf{A}$*, we have the following:*

(*i*) $\alpha \leq_{\mathcal{L}} \beta$ *if and only if* $Im\,\alpha \subseteq Im\,\beta$*;*

(*ii*) $\alpha \leq_{\mathcal{R}} \beta$ *if and only if* $Ker\,\beta \subseteq Ker\,\alpha$*;*

(*iii*) $\alpha\ \mathcal{D}\ \beta$ *if and only if* $\rho(\alpha) = \rho(\beta)$*;*

(*iv*) $\alpha \leq_{\mathcal{J}} \beta$ *if and only if* $\rho(\alpha) \leq \rho(\beta)$*;*

(*v*) $\mathcal{D} = \mathcal{J}$*.*

Corollary 2.2.7. [17] *Let* $\mathbf{A}$ *be an independence algebra. For all* $\alpha, \beta \in End\,\mathbf{A}$*, then:*

(*i*) $\alpha\ \mathcal{L}\ \beta$ *if and only if* $Im\,\alpha = Im\,\beta$*;*

(*ii*) $\alpha\ \mathcal{R}\ \beta$ *if and only if* $Ker\,\alpha = Ker\,\beta$*;*

(*iii*) $\alpha\ \mathcal{J}\ \beta$ *if and only if* $\rho(\alpha) = \rho(\beta)$ *if and only if* $\alpha\ \mathcal{D}\ \beta$*.*

An example of an independence algebra is the free G-act on n free generators over a group G, which will be the main goal of Chapter 4.

[1] Please note, this is a different notion of rank to that of the rank of an operation as given earlier in this chapter.

Chapter 3

Transformation semigroups

The *symmetric groups* $\mathcal{S}_X$, the full *transformation semigroups* $\mathcal{T}_X$ and the partial *transformation semigroups* $\mathcal{PT}_X$ are amongst the most interesting semigroups in semigroup theory. We consider these semigroups (which are, in fact, all monoids), in this chapter. If X is a set with n elements, then it will be convenient to denote the set X as the set $X_n = \{1, 2, \cdots, n\}$. If $X = X_n$ we will write $\mathcal{S}_n$ for $\mathcal{S}_X$, $\mathcal{T}_n$ for $\mathcal{T}_X$ and $\mathcal{PT}_n$ for $\mathcal{PT}_X$. The set $\text{Sing}_n = \mathcal{T}_n \setminus \mathcal{S}_n$ is a subsemigroup (indeed, an ideal) of $\mathcal{T}_n$, which consists of all non-invertible (i.e., *singular*) transformations on X_n.

Transformation semigroups are found everywhere in semigroup theory: Cayley's Theorem states that every semigroup S embeds in some transformation semigroup $\mathcal{T}_X$. If S is a group, the Cayley representation maps S into the symmetric group $\mathcal{S}_X \subseteq \mathcal{T}_X$, which is the group of units of $\mathcal{T}_X$, and consists of all permutations on X. In the case S does not have an identity element, the Cayley representation maps S into $\text{Sing}_X = \mathcal{T}_X \setminus \mathcal{S}_X$.

The aim of this chapter is to give basic definitions for $\mathcal{S}_n$, $\mathcal{T}_n$, $\mathcal{PT}_n$ and Sing_n. In Section 3.1 we recall the basic concepts for $\mathcal{S}_n$. In Section 3.2 we define the basic concepts for $\mathcal{T}_n$, and some of its general properties such as Green's relations and idempotents. Section 3.3 is devoted to the study of the $\mathcal{PT}_n$. Also, we verify that $\mathcal{PT}_n$ is isomorphic to $\overline{\mathcal{T}_{n,0}}$ which is the submonoid of the full transformation semigroup $\mathcal{T}_{n,0}$ on $X_{n,0} = \{0, 1, 2, \cdots, n\}$. Finally, the definition of the semigroup of all singular selfmaps Sing_n is given in Section 3.4. Howie's famous result [24], tells us that Sing_n is generated by its idempotents of rank $n-1$.

Further material can be found in [15], [24] and [39].

3.1 Symmetric groups $\mathcal{S}_X$ and $\mathcal{S}_n$

The *symmetric groups* $\mathcal{S}_X$ on a non-empty set X is the group of all bijections from X into itself under composition of functions.

If $\alpha \in \mathcal{S}_n$ we can describe α in a general way

$$\alpha = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1\alpha & 2\alpha & \cdots & n\alpha \end{pmatrix}.$$

The elements of $\mathcal{S}_n$ can be specified in various ways, such as, if $\beta \in \mathcal{S}_4$, we either give an explicit description for β by stating $1\beta = 2$, $2\beta = 1$, $3\beta = 4$ and $4\beta = 3$ or, write β by using "two row" notation,

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}.$$

As β is one-one, that is, the entries of the second row is a re-arrangement or *permutation* of the first row.

Lemma 3.1.1. *The group $\mathcal{S}_n$ contains $n!$ elements.*

Denoting the order of a semigroup S by $|S|$, by Lemma 3.1.1, we have $|\mathcal{S}_n| = n!$.

For simplicity, the operation of composition $\circ$ is replaced by the operation of multiplication i.e., for any $\alpha, \beta \in \mathcal{S}_X$

$$x(\alpha \circ \beta) = x(\alpha\beta) = (x\alpha)\beta \quad \text{for all} \quad x \in X.$$

If $\sigma \in \mathcal{S}_4$ such that

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix},$$

then

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix},$$

is the inverse of σ, such that

$$\sigma\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = I_4,$$

where I_4 is the identity function of $\mathcal{S}_4$.

The following definition explains the *cycle* notation concept that helps us study the behaviour of elements of $\mathcal{S}_n$.

Definition 3.1.2. Let $a_1, a_2, \cdots, a_m \in X_n = \{1, 2, \cdots, n\}$ be distinct. Then

$$(a_1 a_2 \cdots a_m)$$

is *cycle* and represents the function

$$a_1 \mapsto a_2 \mapsto \cdots \mapsto a_{m-1} \mapsto a_m \mapsto a_1$$
$$x \mapsto x \quad \text{for all} \quad x \notin \{a_1, a_2, \cdots, a_m\}.$$

Further, the length of any cycle is coinciding with its order, such that the order of a cycle of length m is m.

For example, if

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix} = (1\,2\,4) \quad \text{and} \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 2 & 1 & 5 \end{pmatrix} = (1\,3\,2\,4),$$

then σ and τ are cycles, and the order of σ and τ is 3 and 4, respectively.

Note that, there are various ways to write σ and τ, such as

$$\sigma = (1\,2\,4) = (2\,4\,1) = (4\,1\,2) \quad \text{and} \quad \tau = (1\,3\,2\,4) = (3\,2\,4\,1) = (2\,4\,1\,3) = (4\,1\,3\,2).$$

Two cycles are *disjoint*, if they have no elements in common.

Not every elements of $\mathcal{S}_n$ is a cycle, for example β above.

Lemma 3.1.3. *Every element of $\mathcal{S}_n$ is a product of disjoint cycles.*

So by this lemma and β as above we have $\beta = (1\,2)(3\,4)$.

Moreover, the identity function of $\mathcal{S}_n$ is $I_n = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 1 & 2 & 3 & 4 & \cdots & n \end{pmatrix}$. We think of I_n as being *empty product.*

The decomposition of an element as a product of disjoint cycles is called *cycle decomposition* and is unique except for the order in which the cycles are written.

Proposition 3.1.4. *Let α, β be disjoint cycles in $\mathcal{S}_n$. Then $\alpha\beta = \beta\alpha$.*

The order of an element in $\mathcal{S}_n$ is the least common multiple (*l.c.m.*) of the lengths of the cycles in its cycle decomposition.

For example,

$$\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 4 & 5 & 3 & 6 & 7 \end{pmatrix} = (1\,2)\,(3\,4\,5),$$

the order of γ is *l.c.m.* of 2 and 3, i.e., 6.

Definition 3.1.5. A *transposition* is a cycle of length 2.

Remark here that, for any transposition $\sigma = (u\,v)$, σ has order 2, $\sigma^2 = I_n$, and $\sigma = \sigma^{-1}$.

Any cycle is a product of transposition for

$$(a_1 a_2 \cdots a_m) = (a_1 a_2)(a_1 a_3) \cdots (a_1 a_m).$$

As $(\mathcal{S}_n, \circ)$ is a semigroup, so our composition of functions starts from (a_1a_2) and compose from left to right, therefore, the last function we do is (a_1a_m).

In general functions do not commute. Notice that, these transposition are not disjoint.

We define the rank of an element α of $\mathcal{S}_n$ to be the cardinality of $\operatorname{Im}\alpha$, (i.e., $|\operatorname{Im}\alpha| = \operatorname{rank}\alpha$), and as $\mathcal{S}_n$ is the group of all bijection maps then $|\operatorname{Im}\alpha| = \operatorname{rank}\alpha = n$.

3.2 Full transformation semigroups $\mathcal{T}_X$ and $\mathcal{T}_n$

An important example of a semigroup is the *full transformation semigroup* $\mathcal{T}_X$ on a non-empty set X, which is the set of all transformations on X, (i.e., all functions $X \to X$), under the semigroup operation of composition. As before, we often write the operation of composition as a multiplication, i.e., for any $\alpha, \beta \in \mathcal{T}_X$,

$$x(\alpha \circ \beta) = x(\alpha\beta) = (x\alpha)\beta \quad \text{for all} \quad x \in X.$$

The transformation $I_X : X \to X$ is the identity transformation on X, so that for any $\alpha \in \mathcal{T}_X$ we have $I_X\alpha = \alpha I_X$. Hence $\mathcal{T}_X$ with the operation of composition is a monoid.

If $\alpha \in \mathcal{T}_X$, α can be written as

$$\alpha = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1\alpha & 2\alpha & \cdots & n\alpha \end{pmatrix}.$$

Observe that not every element in the full transformation monoid $\mathcal{T}_n$ is one-one, hence the second row in α above is not a permutation of the first row.

Lemma 3.2.1. [25] *The full transformation monoid $\mathcal{T}_n$ contains n^n elements.*

We define the rank of an element α of $\mathcal{T}_X$ to be the cardinality of $\operatorname{Im}\alpha$, (i.e., $|\operatorname{Im}\alpha| = \operatorname{rank}\alpha$).

Observe

$$\mathcal{S}_n = \{\alpha \in \mathcal{T}_n : |\operatorname{Im}\alpha| = n\}.$$

The next lemma describes Green's relations for the full transformation monoid $\mathcal{T}_X$.

Lemma 3.2.2. [25] *Let $\mathcal{T}_X$ be the full transformation monoid on a non-empty set X. For all $\alpha, \beta \in \mathcal{T}_X$, we have the following:*

(i) $\alpha\,\mathcal{L}\,\beta$ *if and only if* $Im\,\alpha = Im\,\beta$;

(ii) $\alpha\,\mathcal{R}\,\beta$ *if and only if* $Ker\,\alpha = Ker\,\beta$;

(iii) $\alpha\,\mathcal{D}\,\beta$ *if and only if* $rank\,\alpha = rank\,\beta$;

(iv) $\mathcal{D} = \mathcal{J}$.

3.2.1 Idempotents in $\mathcal{T}_X$ and $\mathcal{T}_n$

The following lemma gives a rather useful characterization of the idempotents of a transformation monoid.

Lemma 3.2.3 (The $E(\mathcal{T}_X)$ Lemma)**.** *An element $\varepsilon \in \mathcal{T}_X$ is idempotent if and only if $\varepsilon|_{Im\,\varepsilon} = I_{Im\,\varepsilon}$.*

Proof. Since $\varepsilon|_{\operatorname{Im}\varepsilon} = I_{\operatorname{Im}\varepsilon}$, that is, $y\varepsilon = y$ for all $y \in \operatorname{Im}\varepsilon$. Of course, for each $x \in X$, $x\varepsilon \in \operatorname{Im}\varepsilon$. Then

$$\begin{aligned}
\varepsilon \in E(\mathcal{T}_X) &\Leftrightarrow \varepsilon^2 = \varepsilon, \\
&\Leftrightarrow x\varepsilon^2 = x\varepsilon && \text{for all } x \in X, \\
&\Leftrightarrow (x\varepsilon)\varepsilon = x\varepsilon && \text{for all } x \in X, \\
&\Leftrightarrow y\varepsilon = y && \text{for all } y \in \operatorname{Im}\varepsilon, \\
&\Leftrightarrow \varepsilon|_{\operatorname{Im}\varepsilon} = I_{\operatorname{Im}\varepsilon}.
\end{aligned}$$

□

The following corollary is folklore, [15]. The result may also be obtained by using *generating functions* see [24]. We also show how generating functions may be used when enumerating the number of idempotents in certain wreath products (see underneath Lemma 4.3.2).

Corollary 3.2.4. [15] *The number $E(\mathcal{T}_n)$ of idempotents in the semigroup $\mathcal{T}_n$ equals*

$$E(\mathcal{T}_n) = \sum_{k=1}^{n} \binom{n}{k} k^{n-k}.$$

Proof. There exist $\binom{n}{k}$ possible images for an idempotent μ with $|\mathrm{Im}\,\mu| = k$. Suppose those images are $Y_1, Y_2, \cdots, Y_{\binom{n}{k}}$, for each of those images there exist k^{n-k} idempotents with this image. This is true since for any $Y_j = \{i_1, \cdots, i_k\}$ where $1 \leq i_1 < \cdots < i_k \leq n$, $\mu = \mu^2$ has image Y_j if and only if $i_l\mu = i_l$ for all $l \in \{1, \cdots, k\}$. Any values for $i\mu \in Y_j$, $i \notin Y_j$ will give a suitable idempotent. There are therefore k choices for each $i \notin Y_j$ and so k^{n-k} possible choices for $\mu = \mu^2$ with $\mathrm{Im}\,\mu = Y_j$. Hence we have $\binom{n}{k}k^{n-k}$ idempotents of $\mathcal{T}_n$ with k elements in the image. By using the sum rule the proof will be complete. □

3.3 Partial transformation semigroups $\mathcal{PT}_X$ and $\mathcal{PT}_n$

Let $\mathcal{PT}_X = \{\alpha :\, \alpha : Y \to Z, \text{ where}\, Y, Z \subseteq X\}$. Then $\mathcal{PT}_X$ under composition of partial functions is a monoid, called *the partial transformation monoid* on X. Notice that $\mathcal{T}_X \subset \mathcal{PT}_X$. If $\alpha : X \to Y$ is any map, then the *preimage* or *inverse image* of a set $B \subseteq Y$ under α is the subset of X defined by

$$(B)\alpha^{-1} = \{x \in X :\, (x)\alpha \in B\}.$$

We know from Chapter 1 it is not necessary α^{-1} to be a partial map. Therefore, α^{-1} does not necessarily exist.

Lemma 3.3.1. [25] *Let X be a non-empty set. If $\alpha \in \mathcal{PT}_X$, then $\alpha^{-1} \in \mathcal{PT}_X$ if and only if α is one-one.*

If α, $\beta \in \mathcal{PT}_X$, then

$$\operatorname{Dom} \alpha\beta = [\operatorname{Im} \alpha \cap \operatorname{Dom} \beta]\alpha^{-1},$$

$$\operatorname{Im} \alpha\beta = [\operatorname{Im} \alpha \cap \operatorname{Dom} \beta]\beta,$$

and for all $x \in \operatorname{Dom} \alpha\beta$, then $x(\alpha\beta) = (x\alpha)\beta$.

If $X_n = \{1, \cdots, n\}$, we usually write $\mathcal{PT}_n$ for $\mathcal{PT}_X$. The element $\alpha \in \mathcal{PT}_n$ can be illustrate in the following form:

$$\alpha = \begin{pmatrix} 1 & 2 & \cdots & n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix},$$

where

$$y_i = \begin{cases} - & i \notin \operatorname{Dom} \alpha; \\ i\alpha & i \in \operatorname{Dom} \alpha. \end{cases}$$

Notice that, $\mathcal{PT}_X$ has a zero, " *the empty map*", and in case $X_n = \{1, \cdots, n\}$, this element is

$$0 = \begin{pmatrix} 1 & 2 & \cdots & n \\ - & - & \cdots & - \end{pmatrix}.$$

For example, if $X_2 = \{1, 2\}$, then we can list the elements of $\mathcal{PT}_2$ as follows:

$$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 2 \\ 1 & - \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & - \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ - & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ - & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ - & - \end{pmatrix}.$$

The first row of the above list consists of total transformations and lists all elements in $\mathcal{T}_2$. The above example demonstrates $\mathcal{T}_X \subset \mathcal{PT}_X$.

Lemma 3.3.2. [15] *The partial transformation monoid $\mathcal{PT}_n$ contains $(n+1)^n$ elements.*

3.3.1 Green's relations on $\mathcal{PT}_X$

It is well-known that when **A** and **B** are algebras (of the same type), and $\alpha : \mathbf{A} \to \mathbf{B}$ is a homomorphism, then $\operatorname{Im}\alpha = \{a\alpha : a \in \mathbf{A}\}$ is a subalgebra of **B**.
Also,

$$\operatorname{Ker}\alpha = \{(x, y) \in \mathbf{A} \times \mathbf{A} : x\alpha = y\alpha\},$$

is a subalgebra of $\mathbf{A} \times \mathbf{A}$ and a congruence on **A**.

Definition 3.3.3. Let $\alpha \in \mathcal{PT}_X$. Define binary relations $\operatorname{Ker}\alpha$ on $\operatorname{Dom}\alpha$ and π_α on X by the rules:

$$x \operatorname{Ker}\alpha\, y \text{ if and only if } x, y \in \operatorname{Dom}\alpha \text{ and } x\alpha = y\alpha,$$

and

$$x\, \pi_\alpha\, y \text{ if and only if } x, y \in \operatorname{Dom}\alpha \text{ and } x\alpha = y\alpha \text{ or}\,, x, y \in X \setminus \operatorname{Dom}\alpha.$$

Notice that $\operatorname{Ker}\alpha$ is the usual kernel relation on the domain of α, and π_α is $\operatorname{Ker}\alpha \cup \omega_{X\setminus \operatorname{Dom}\alpha}$, where ω_T is the universal relation on a set T. Clearly, $\operatorname{Ker}\alpha$ and π_α are equivalence relations on $\operatorname{Dom}\alpha$ and X, respectively.

The proof of the following is given in the case of finite X in [15]; ours is essentially the same.

Lemma 3.3.4. [15] *For all α, $\beta \in \mathcal{PT}_X$, we have the following:*

(i) $\alpha \leq_{\mathcal{L}} \beta$ *if and only if* $\operatorname{Im}\alpha \subseteq \operatorname{Im}\beta$*;*

(ii) $\alpha \leq_{\mathcal{R}} \beta$ *if and only if* $\operatorname{Dom}\alpha \subseteq \operatorname{Dom}\beta$ *and* $\pi_\beta \subseteq \pi_\alpha$*.*

Proof. (i) If $\alpha \leq_{\mathcal{L}} \beta$ then $\alpha = \gamma\beta$ for some $\gamma \in \mathcal{PT}_X$; hence $\operatorname{Im}\alpha = \operatorname{Im}\gamma\beta \subseteq \operatorname{Im}\beta$.

Conversely, suppose that $\operatorname{Im}\alpha \subseteq \operatorname{Im}\beta$. For each $a \in \operatorname{Im}\alpha$ we have $a \in \operatorname{Im}\beta$ so $a = y_a\beta$ for some $y_a \in \operatorname{Dom}\beta$. Define $\gamma \in \mathcal{PT}_X$ by $\operatorname{Dom}\gamma = \operatorname{Dom}\alpha$ and $x\gamma = y_{x\alpha}$, for each $x \in \operatorname{Dom}\alpha$. Then $\operatorname{Im}\gamma \subseteq \operatorname{Dom}\beta$, and so $\operatorname{Dom}\gamma\beta = \operatorname{Dom}\gamma = \operatorname{Dom}\alpha$, and for any $x \in \operatorname{Dom}\alpha$, $x\gamma\beta = y_{x\alpha}\beta = x\alpha$ so, $\alpha = \gamma\beta$.

(ii) If $\alpha \leq_{\mathcal{R}} \beta$ in $\mathcal{PT}_X$ then $\alpha = \beta\gamma$ for some $\gamma \in \mathcal{PT}_X$. For arbitrary $x \in \operatorname{Dom}\alpha$, we have $x\alpha = x\beta\gamma$, thus $x \in \operatorname{Dom}\beta$. Hence $\operatorname{Dom}\alpha \subseteq \operatorname{Dom}\beta$.

Let $(x, y) \in \pi_\beta$.

Case (1) If $x \in \operatorname{Dom}\alpha$ then $x \in \operatorname{Dom}\beta$ so as $(x, y) \in \pi_\beta$, we must have $y \in \operatorname{Dom}\beta$. Then as $x\beta = y\beta$, $x\alpha = x\beta\gamma = y\beta\gamma = y\alpha$ so $y \in \operatorname{Dom}\alpha$ also; and $x\alpha = y\alpha$ so $(x, y) \in \pi_\alpha$.

Case (2) If $x, y \notin \operatorname{Dom}\alpha$, so $(x, y) \in \pi_\alpha$.

Hence, $\pi_\beta \subseteq \pi_\alpha$.

Conversely, suppose that $\operatorname{Dom}\alpha \subseteq \operatorname{Dom}\beta$ and $\pi_\beta \subseteq \pi_\alpha$. Let $U = (\operatorname{Dom}\alpha)\beta$ and define $\gamma : U \to \operatorname{Im}\alpha$ by $(a\beta)\gamma = a\alpha$ for any $a\beta \in U$ with $a \in \operatorname{Dom}\alpha$. If $a, a' \in \operatorname{Dom}\alpha$ and $a\beta = a'\beta$ then as $\pi_\beta \subseteq \pi_\alpha$ we have $a\alpha = a'\alpha$, so γ is well-defined. As $\operatorname{Dom}\gamma = U = (\operatorname{Dom}\alpha)\beta$, $\gamma \in \mathcal{PT}_X$. Certainly $\operatorname{Dom}\gamma \subseteq \operatorname{Im}\beta$. Let $c \in \operatorname{Dom}\beta\gamma$, so that $c \in \operatorname{Dom}\beta$ and $c\beta \in \operatorname{Dom}\gamma$. It follows that $c\beta = a\beta$ for some $a \in \operatorname{Dom}\alpha$. Thus $(a, c) \in \pi_\beta \subseteq \pi_\alpha$ so that as $a \in \operatorname{Dom}\alpha$ we have $c \in \operatorname{Dom}\alpha$. Hence $\operatorname{Dom}\beta\gamma \subseteq \operatorname{Dom}\alpha$ and clearly the converse is true by definition of γ. Thus $\operatorname{Dom}\alpha = \operatorname{Dom}\beta\gamma$ and then it is immediate that $\beta\gamma = \alpha$. Thus $\alpha \leq_{\mathcal{R}} \beta$ in $\mathcal{PT}_X$.

□

Corollary 3.3.5. *For all α, $\beta \in \mathcal{PT}_X$, we have:*

(i) $\alpha\,\mathcal{L}\,\beta$ *if and only if* $Im\,\alpha = Im\,\beta$;

(*ii*) $\alpha \mathcal{R} \beta$ *if and only if* $Dom\,\alpha = Dom\,\beta$ *and* $Ker\,\alpha = Ker\,\beta$*;*

(*iii*) $\alpha \mathcal{H} \beta$ *if and only if* $Im\,\alpha = Im\,\beta$*,* $Dom\,\alpha = Dom\,\beta$ *and* $Ker\,\alpha = Ker\,\beta$*.*

Proof. (i) This follows immediately from Lemma 3.3.4.

(ii) From Lemma 3.3.4, $\alpha\ \mathcal{R}\ \beta$ if and only if $\mathrm{Dom}\,\alpha = \mathrm{Dom}\,\beta$ and $\pi_\alpha = \pi_\beta$. This gives

$$\mathrm{Ker}\,\alpha = \pi_\alpha \cap (\mathrm{Dom}\,\alpha \times \mathrm{Dom}\,\alpha) = \pi_\beta \cap (\mathrm{Dom}\,\beta \times \mathrm{Dom}\,\beta) = \mathrm{Ker}\,\beta.$$

Conversely, if $\mathrm{Dom}\,\alpha = \mathrm{Dom}\,\beta$ and $\mathrm{Ker}\,\alpha = \mathrm{Ker}\,\beta$, then

$$\begin{aligned}\pi_\alpha &= \mathrm{Ker}\,\alpha \cup ((X \setminus \mathrm{Dom}\,\alpha) \times (X \setminus \mathrm{Dom}\,\alpha))\\ &= \mathrm{Ker}\,\beta \cup ((X \setminus \mathrm{Dom}\,\beta) \times (X \setminus \mathrm{Dom}\,\beta))\\ &= \pi_\beta.\end{aligned}$$

(iii) This follows immediately from (i) and (ii).

□

3.3.2 Idempotents in $\mathcal{PT}_X$ and $\mathcal{PT}_n$

The following theorem describes the idempotents elements in $\mathcal{PT}_X$.

Theorem 3.3.6. [15] *An* $\alpha \in \mathcal{PT}_X$ *is an idempotent if and only if* $Im\,\alpha \subseteq Dom\,\alpha$ *and the restriction* $\alpha|_{Im\,\alpha} = I_{Im\,\alpha}$*.*

Proof. Let $\alpha \in \mathcal{PT}_X$ such that $\alpha^2 = \alpha$. Let $x \in \mathrm{Dom}\,\alpha$ then as $\alpha^2 = \alpha$ so $x \in \mathrm{Dom}\,\alpha^2$ and $x\alpha = x\alpha^2 = (x\alpha)\alpha$ so $x\alpha \in \mathrm{Dom}\,\alpha$ therefore, $\mathrm{Im}\,\alpha \subseteq \mathrm{Dom}\,\alpha$, and for each $y \in \mathrm{Im}\,\alpha$ we have $y\alpha = y$. Conversely, if α acts as the identity on $\mathrm{Im}\,\alpha$ and $\mathrm{Im}\,\alpha \subseteq \mathrm{Dom}\,\alpha$ then for $x \in \mathrm{Dom}\,\alpha$, $x\alpha \in \mathrm{Im}\,\alpha \subseteq \mathrm{Dom}\,\alpha$. Then we have $x \in \mathrm{Dom}\,\alpha^2$ and so as $\mathrm{Dom}\,\alpha^2 \subseteq \mathrm{Dom}\,\alpha$ always, we have $\mathrm{Dom}\,\alpha^2 = \mathrm{Dom}\,\alpha$. Further, $(x\alpha)\alpha = x\alpha$ which means $x\alpha^2 = x\alpha$ for any $x \in \mathrm{Dom}\,\alpha$ so that $\alpha^2 = \alpha$. □

Corollary 3.3.7. [15] *The number $E(\mathcal{PT}_n)$ of idempotents in the semigroup $\mathcal{PT}_n$ equals*

$$E(\mathcal{PT}_n) = \sum_{k=0}^{n} \binom{n}{k} (k+1)^{n-k}.$$

Proof. There exist $\binom{n}{k}$ possible images for an idempotent α with $|\mathrm{Im}\,\alpha| = k$. Suppose those images are $Z_1, Z_2, \cdots, Z_{\binom{n}{k}}$; for each of those images there exist $(k+1)^{n-k}$ idempotents with this image. This is true since for any $Z_j = \{i_1, \cdots, i_k\}$ where $1 \leq i_1 < \cdots < i_k \leq n$, $\alpha = \alpha^2$ has image Z_j if and only if $i_l\alpha = i_l$ for all $l \in \{1, \cdots, k\}$, and for all $i \in \mathrm{Dom}\,\alpha \setminus Z_j$, $i\alpha \in Z_j$. Let $i \in \{1, \cdots, n\} \setminus Z_j$. Either $i \notin \mathrm{Dom}\,\alpha$ or, $i \in \mathrm{Dom}\,\alpha$ and there are k possibilities $i_1, \cdots, i_k$ for $i\alpha$. There are therefore, $k+1$ choices for each $i \notin Z_j$ and so $(k+1)^{n-k}$ possible choices for $\alpha = \alpha^2$ with $\mathrm{Im}\,\alpha = Z_j$. Hence we have $\binom{n}{k}(k+1)^{n-k}$ idempotents of $\mathcal{PT}_n$ with k elements in the image. By using the sum rule the proof will be complete. □

Consider the finite set $X_{n,0} = \{0, 1, 2, \cdots, n\}$, and

$$\mathcal{T}_{n,0} = \{\alpha : \alpha \text{ is a transformation on } X_{n,0}\}$$

be the full transformation semigroup on $X_{n,0}$. Notice that $\mathcal{T}_{n,0} \cong \mathcal{T}_{n+1}$.

Let

$$\overline{\mathcal{T}_{n,0}} = \{\alpha \in \mathcal{T}_{n,0} : 0\alpha = 0\}.$$

Lemma 3.3.8. *The subset $\overline{\mathcal{T}_{n,0}}$ is a submonoid of $\mathcal{T}_{n,0}$.*

Proof. Let α, $\beta \in \overline{\mathcal{T}_{n,0}}$ so α, $\beta \in \mathcal{T}_{n,0}$, $0\alpha = 0$, and $0\beta = 0$. We want $\alpha\beta \in \overline{\mathcal{T}_{n,0}}$, that is, $\alpha\beta \in \mathcal{T}_{n,0}$ and $0(\alpha\beta) = 0$. Under usual composition $\alpha\beta \in \mathcal{T}_{n,0}$ and since $0(\alpha\beta) = (0\alpha)\beta = 0\beta = 0$, we obtain $\alpha\beta \in \overline{\mathcal{T}_{n,0}}$. It is clear that $I_{n,0} \in \mathcal{T}_{n,0}$ is such that $0I_{n,0} = 0$, therefore, $I_{n,0} \in \overline{\mathcal{T}_{n,0}}$. Hence $\overline{\mathcal{T}_{n,0}}$ is a submonoid of $\mathcal{T}_{n,0}$ as required. □

For completeness we give the proof of the following result. The key idea is to add an extra element 0 to the domain, such that for any $\alpha \in \mathcal{PT}_n$ we extend the domain of α in such a way that α sends any elements for which α was not previously defined to 0, and 0 maps to 0.

Lemma 3.3.9. *For $n \in \mathbb{N}$, $\mathcal{PT}_n$ is isomorphic to $\overline{\mathcal{T}_{n,0}}$.*

Proof. Define $\psi : \mathcal{PT}_n \to \overline{\mathcal{T}_{n,0}}$ by

$$\alpha\psi = \overline{\alpha}$$

such that

$$i\overline{\alpha} = \begin{cases} 0 & \text{if} \quad i = 0; \\ i\alpha & \text{if} \quad i \in \operatorname{Dom}\alpha; \\ 0 & \text{if} \quad i \in \{1, 2, \cdots, n\} \setminus \operatorname{Dom}\alpha. \end{cases}$$

To prove ψ is bijection. Let $\alpha\psi = \beta\psi$, this implies that $i(\alpha\psi) = i(\beta\psi)$ for all i. For all i with $i \neq 0$, we have $i \in \{1, 2, \cdots, n\} \setminus \operatorname{Dom}\alpha$ if and only if $i(\alpha\psi) = 0$ if and only if $i(\beta\psi) = 0$ if and only if $i \in \{1, 2, \cdots, n\} \setminus \operatorname{Dom}\beta$. Hence, $\operatorname{Dom}\alpha = \operatorname{Dom}\beta$. Moreover, if $i \in \operatorname{Dom}\alpha = \operatorname{Dom}\beta$ that implies $i\alpha = i(\alpha\psi) = i(\beta\psi) = i\beta$, so that $\alpha = \beta$ and, therefore, ψ is one to one.

To prove ψ is onto let $\mu \in \overline{\mathcal{T}_{n,0}}$. Define $\mu' \in \mathcal{PT}_n$ by

$$\operatorname{Dom}\mu' = \{i \in \{1, \cdots, n\} : i\mu \neq 0\},$$

and for all $i \in \operatorname{Dom}\mu'$, $i\mu' = i\mu$.

To show $\mu'\psi = \mu$. We have

$$i(\mu'\psi) = i\overline{\mu'} = \begin{cases} 0 & \text{if} \quad i = 0; \\ i\mu' & \text{if} \quad i \in \operatorname{Dom}\mu'; \\ 0 & \text{if} \quad i \in \{1, 2, \cdots, n\} \setminus \operatorname{Dom}\mu'. \end{cases}$$

$$= \begin{cases} 0 & \text{if} \quad i = 0; \\ i\mu & \text{if} \quad i \in \operatorname{Dom}\mu'; \\ 0 & \text{if} \quad i \in \{1, 2, \cdots, n\} \setminus \operatorname{Dom}\mu'. \end{cases}$$

For $i \in \{1, 2, \cdots, n\} \setminus \operatorname{Dom} \mu'$, we have by definition of $\operatorname{Dom} \mu'$ that $i\mu = 0$.

Therefore,

$$i(\mu'\psi) = i\overline{\mu'} = \begin{cases} 0 & \text{if} \quad i = 0; \\ i\mu & \text{if} \quad i \in \operatorname{Dom} \mu'; \\ i\mu & \text{if} \quad i \in \{1, 2, \cdots, n\} \setminus \operatorname{Dom} \mu'. \end{cases}$$

So for all value of i, $i(\mu'\psi) = i\overline{\mu'} = i\mu$, hence $\mu'\psi = \mu$, so that ψ is onto.

To show that ψ is homomorphism it is enough to prove that $(\alpha\beta)\psi = \alpha\psi\beta\psi$. Let α, $\beta \in \mathcal{PT}_n$. For all i

$$i(\alpha\beta)\psi = i\overline{\alpha\beta} = \begin{cases} 0 & \text{if} \quad i = 0; \\ i(\alpha\beta) & \text{if} \quad i \in \operatorname{Dom} \alpha\beta; \\ 0 & \text{if} \quad i \in \{1, 2, \cdots, n\} \setminus \operatorname{Dom} \alpha\beta. \end{cases}$$

$$= \begin{cases} 0 & \text{if} \quad i = 0; \\ (i\alpha)\beta & \text{if} \quad i \in \operatorname{Dom} \alpha \text{ and } i\alpha \in \operatorname{Dom} \beta; \\ 0 & \text{if} \quad i \in \{1, 2, \cdots, n\} \setminus \operatorname{Dom} \alpha\beta. \end{cases}$$

On the other hand, $i(\alpha\psi\beta\psi) = i(\overline{\alpha}\overline{\beta})$. For all i, if $i = 0$, we get $0(\overline{\alpha}\overline{\beta}) = (0\overline{\alpha})\overline{\beta} = 0\overline{\beta} = 0$. If $i \in \operatorname{Dom} \alpha\beta$, then $i \in \operatorname{Dom} \alpha$ and $i\alpha \in \operatorname{Dom} \beta$, so that we get $i(\overline{\alpha}\overline{\beta}) = (i\overline{\alpha})\overline{\beta} = (i\alpha)\overline{\beta} = (i\alpha)\beta = i(\alpha\beta) = i\overline{\alpha\beta}$. Otherwise, $i \notin \operatorname{Dom} \alpha\beta$. So $i \notin \operatorname{Dom} \alpha$ or $i \in \operatorname{Dom} \alpha$ and $i\alpha \notin \operatorname{Dom} \beta$, and then

$$i(\overline{\alpha}\overline{\beta}) = (i\overline{\alpha})\overline{\beta} = \begin{cases} 0\overline{\beta} & \text{if} \quad i \notin \operatorname{Dom} \alpha; \\ (i\alpha)\overline{\beta} & \text{if} \quad i \in \operatorname{Dom} \alpha \text{ and } i\alpha \notin \operatorname{Dom} \beta. \end{cases}$$

$$= \begin{cases} 0 & \text{if} \quad i \notin \operatorname{Dom}\alpha; \\ 0 & \text{if} \quad i \in \operatorname{Dom}\alpha \text{ and } i\alpha \notin \operatorname{Dom}\beta. \end{cases}$$

Therefore,

$$i(\overline{\alpha}\overline{\beta}) = \begin{cases} 0 & \text{if} \quad i = 0; \\ i(\alpha\beta) & \text{if} \quad i \in \operatorname{Dom}\alpha\beta; \\ 0 & \text{if} \quad i \in \{1, 2, \cdots, n\} \setminus \operatorname{Dom}\alpha\beta. \end{cases}$$

Hence, $(\alpha\beta)\psi = \alpha\psi\beta\psi$, as required. □

3.4 Semigroup of all singular selfmaps Sing_n

As mentioned before, that transformation semigroups are ubiquitous in semigroup theory is due to the Cayley's Theorem, that states that every semigroup S embeds in some transformation semigroup $\mathcal{T}_X$. However, in case that S does not possess an identity element, the Cayley representation maps into $\mathrm{Sing}_X = \mathcal{T}_X \setminus \mathcal{S}_X$, the set of all non-invertible transformation on X.

If $X_n = \{1, 2, \cdots, n\}$, then

$$\mathcal{T}_n \setminus \mathcal{S}_n = \mathrm{Sing}_n = \{\alpha \in \mathcal{T}_n : |\operatorname{Im}\alpha| \leq n - 1\}$$

is a subsemigroup (indeed, an ideal) of $\mathcal{T}_n$, which called the semigroup of all *singular selfmaps* of X_n.

Howie proved in [24] if X is finite then every element of $\mathcal{T}_X$ that is not bijective is expressible as a product of idempotents.

Theorem 3.4.1. [24] *Let $X_n = \{1, 2, \cdots, n\}$, and let $\mathcal{T}_n$ be the full transformation monoid on X_n. Then the subsemigroup of $\mathcal{T}_n$ generated by its non-identity idempotents is* $\mathrm{Sing}_n = \mathcal{T}_n \setminus \mathcal{S}_n$*. In fact, every element of* Sing_n *is a product of idempotent with rank $n - 1$.*

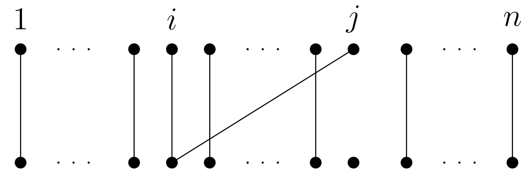
The defect of an element $\alpha \in \mathcal{T}_n$ is defined as $n - |\operatorname{Im}\alpha|$. From the above theorem we deduce that Howie's famous result that states Sing_n is generated by its idempotent of rank $n-1$ (of defect 1).

The later are precisely the maps $\varepsilon_{ij} \in \mathcal{T}_n$ (for $i,\, j \in X_n$ with $i \neq j$) defined by

$$k\varepsilon_{ij} = \begin{cases} k & \text{if} \quad k \neq j; \\ i & \text{if} \quad k = j. \end{cases}$$

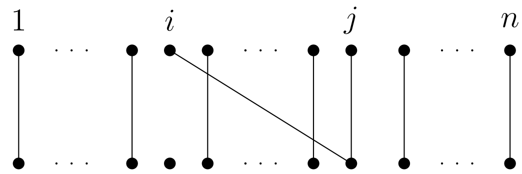
These idempotent may be represented diagrammatically, for $1 \leq i \neq j \leq n$, by

$\varepsilon_{ij} =$



and

$\varepsilon_{ji} =$



.

We write $\mathcal{X} = \{\varepsilon_{ij} : i,\, j \in X_n,\, i \neq j\}$ for the set of all rank $n-1$ idempotents from $\mathcal{T}_n$. By using the property of Green's relations, which follows from the fact that

$$\alpha\,\mathcal{R}\,\beta \Leftrightarrow \operatorname{Ker}\alpha = \operatorname{Ker}\beta \qquad \text{and} \qquad \alpha\,\mathcal{L}\,\beta \Leftrightarrow \operatorname{Im}\alpha = \operatorname{Im}\beta,$$

it is easy to check that for all $i,\, j,\, k,\, l \in X_n$ with $i \neq j$ and $k \neq l$,

$$\varepsilon_{ij}\,\mathcal{R}\,\varepsilon_{kl} \;\Leftrightarrow\; \{i,\, j\} = \{k,\, l\} \qquad \text{and} \qquad \varepsilon_{ij}\,\mathcal{L}\,\varepsilon_{kl} \;\Leftrightarrow\; j = l.$$

Theorem 3.4.1 can be rewrite as follows:

Theorem 3.4.2. [24] *If* $n \geq 2$, *then* $\operatorname{Sing}_n = \langle \mathcal{X} \rangle$.

In Chapter 6, we find a presentations for $M \rtimes S$ and $M \wr_n S$ products, where S is a subsemigroup of the full transformation semigroup $\mathcal{T}_n$. In particular, we interested in the case that $S = \mathrm{Sing}_n$.

Chapter 4

Free (left) S-acts and their endomorphism monoids

Our aim is to study free (left) S-acts for a monoid S, and their endomorphism monoids. In Section 4.1 we address the fundamental concepts and definitions of the endomorphism monoid of a free S-act on n free generators. In Section 4.2 we describe Green's relations on $G \wr_n \mathcal{T}_n$. Thus far the work in this chapter is re-working known results. In Section 4.3 we count the number of idempotents in $\operatorname{End} F_n(S)$ where S is finite: this is new. We first consider the case where S is a finite group and then move on to the general case where S is a finite monoid.

We recommend [26] as a references for Chapter 4.

4.1 $S \wr_n \mathcal{T}_n$ is the endomorphism monoid of a free S-act of rank n

We are already defined the following concept in Chapter 1, towards the end of Definition 1.6.3.

Definition 4.1.1. A non-empty set A is a (left) S-act if there exists a map

$$S \times A \to A,\ (s, a) \mapsto sa$$

such that for all $a \in A$, and $s, t \in S$

$$s(ta) = (st)a \text{ and } 1_S a = a.$$

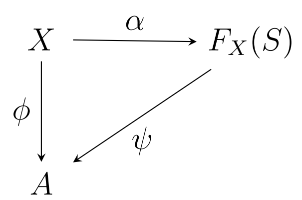
Definition 4.1.2. Let A and B be two left S-acts. A mapping $\alpha : A \to B$ is called a homomorphism of left S-acts or just an S-morphism if $(sa)\alpha = s(a\alpha)$ for all $a \in A$, $s \in S$.

By using the same technique in Chapter 1, in the next definition defines a free S-act.

Definition 4.1.3. [25] Let X be a non-empty set, $F_X(S)$ is a free S-act on X if

(i) there is a map $\alpha : X \to F_X(S)$;

(ii) for every S-act A and every map $\phi : X \to A$ there exists a unique morphism $\psi : F_X(S) \to A$ such that the diagram commutes.



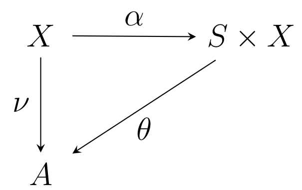
We say that X is the set of (free) generators of $F_X(S)$. From standard universal algebra we know that $F_X(S)$ exists. We now show how to give an explicit construction.

Lemma 4.1.4. [26] *Let $\emptyset \neq X$ be a set. We make $S \times X$ into left S-act by defining an action of S on $S \times X$ by $s(t, x) = (st, x)$. Clearly $S \times X$ is a left S-act. Further $S \times X$ is the free left S-act on X.*

Proof. Let $\alpha : X \longrightarrow S \times X$ be given by $x\alpha = (1_S, x)$. Let A be a left S-act and $\nu : X \longrightarrow A$ be a function. Define $\theta : S \times X \longrightarrow A$ by $(s, x)\theta = s(x\nu)$. Then θ is an S-morphism since for any $s \in S$ and $x \in X$,

$$\begin{aligned}(t(s,x))\theta &= (ts,x)\theta \\ &= (ts)(x\nu) \\ &= t(s(x\nu)) \\ &= t((s,x)\theta).\end{aligned}$$

For any $x \in X$, $x\alpha\theta = (1_S, x)\theta = 1_S(x\nu) = x\nu$ and so $\alpha\theta = \nu$ and the diagram



commutes. It can be check that θ is unique S-morphism from $S \times X$ to A such that the above diagram commutes. Suppose that $\psi : S \times X \longrightarrow A$ is such that $\alpha\psi = \nu$. Then for any $(s, x) \in S \times X$,

$$(s,x)\psi = (s(1_S,x))\psi = s((1_S,x)\psi) = s(x(\alpha\psi)) = s(x\nu) = (s,x)\theta.$$

$\square$

It is convenient to use the following equivalent formulation for the free S-act on X.

Let $T_X(S)$ be the set of all expression of the form sx, $x \in X, s \in S$, where $sx = s'x'$ for $s, s' \in S, x, x' \in X$ if and only if $x = x'$ and $s = s'$. We make $T_X(S)$ a left S-act by putting $t(sx) = (ts)x$, and identify 1_Sx with x. Clearly $S \times X \cong T_X(S)$ where $(s, x) \longmapsto sx$.

From now take $F_X(S) = T_X(S)$. Notice

$$F_X(S) = \dot{\bigcup}_{x \in X} Sx$$

where

$$Sx = \{sx : s \in S\}.$$

If $|X| = n$, we usually write $F_n(S)$ for $F_X(S)$. If $|X| = 1$, say $X = \{x\}$, then $F_1(S) = Sx$ and it is clear that $Sx \cong S$, where S is regarded as a left ideal of S.

Let $\alpha \in \operatorname{End} F_n(S)$. Notice that, each $\alpha \in \operatorname{End} F_n(S)$ depends only on its action on the free generators $\{x_i : i \in \{1, 2, \cdots, n\}\}$ and it is therefore convenient to write

$$x_i\alpha = \omega_\alpha^i x_{i\bar{\alpha}}$$

where $i \in \{1, 2, \cdots, n\}$, $\bar{\alpha} \in \mathcal{T}_n$ and $(\omega_\alpha^1, \cdots, \omega_\alpha^n) \in S^n$. For $s \in S$ and $i \in \{1, 2, \cdots, n\}$,

$$(sx_i)\alpha = s(x_i\alpha) = s\omega_\alpha^i x_{i\bar{\alpha}}.$$

We now define the " wreath product" multiplication $S^n \times \mathcal{T}_n$ by putting

$$(s_1, \cdots, s_n, \eta)(t_1, \cdots, t_n, \mu) = (s_1 t_{1\eta}, \cdots, s_n t_{n\eta}, \eta\mu),$$

under this multiplication, $S^n \times \mathcal{T}_n$ becomes a monoid with identity $(1, 1, \cdots, 1, I_n)$ where I_n is the identity transformation in $\mathcal{T}_n$, and it is denoted $S \wr_n \mathcal{T}_n$. This is a special case of a more general notion of a wreath product (see for example [42]).

According to the next lemma it is easy to show that the wreath product multiplication is associative, which also will be shown in Chapter 6.

We remind the reader that if $\alpha \in \operatorname{End} F_n(S)$, and u, v lie in the same indecomposable component, then $u\alpha$, $v\alpha$ also lie in the same component. Thus α produces a mapping from $\{x_i S : 1 \leq i \leq n\}$ to itself, which corresponds to $\bar{\alpha}$.

Lemma 4.1.5. *For $n \in \mathbb{N}$, $End F_n(S)$ is isomorphic to $S \wr_n \mathcal{T}_n$.*

Proof. Define $\psi : \operatorname{End} F_n(S) \longrightarrow S^n \times \mathcal{T}_n$ by

$$\alpha\psi = (\omega_\alpha^1, \omega_\alpha^2, \cdots, \omega_\alpha^n, \bar{\alpha}).$$

To prove that ψ is bijection. Let $\alpha\psi = \beta\psi$, so,

$$(\omega_\alpha^1, \omega_\alpha^2, \cdots, \omega_\alpha^n, \overline{\alpha}) = (\omega_\beta^1, \cdots, \omega_\beta^n, \overline{\beta})$$

where $x_i\alpha = \omega_\alpha^i x_{i\overline{\alpha}}$, $x_i\beta = \omega_\beta^i x_{i\overline{\beta}}$. Therefore, for any $sx_i \in F_n(S)$ we have,

$$(sx_i)\alpha = s(x_i\alpha) = s(\omega_\alpha^i x_{i\overline{\alpha}}) = s(\omega_\beta^i x_{i\overline{\beta}}) = s(x_i\beta) = (sx_i)\beta$$

so that, $\alpha = \beta$ and ψ is one to one.

To prove ψ is onto let $(s_1, \cdots, s_n, \eta) \in S^n \times \mathcal{T}_n$, and define $\alpha : F_n(S) \longrightarrow F_n(S)$ by $x_i\alpha = s_i x_{i\eta}$. Remark, $\omega_\alpha^i = s_i$, $i\overline{\alpha} = i\eta$, for all i, so $\overline{\alpha} = \eta$, and

$$\alpha\psi = (\omega_\alpha^1, \cdots, \omega_\alpha^n, \overline{\alpha}) = (s_1, \cdots, s_n, \eta),$$

so that ψ is onto.

Now, to prove ψ is homomorphism. Let $\alpha, \beta \in \operatorname{End} F_n(S)$. It is enough to show that $(\alpha\beta)\psi = (\alpha\psi)(\beta\psi)$. For any $i \in \{1, \cdots, n\}$

$$x_i\alpha\beta = (\omega_\alpha^i x_{i\overline{\alpha}})\beta = \omega_\alpha^i (x_{i\overline{\alpha}}\beta) = \omega_\alpha^i \omega_\beta^{i\overline{\alpha}} x_{i\overline{\alpha}\overline{\beta}}.$$

So $(\alpha\beta)\psi = (\omega_\alpha^1 \omega_\beta^{1\overline{\alpha}}, \cdots, \omega_\alpha^n \omega_\beta^{n\overline{\alpha}}, \overline{\alpha}\overline{\beta}) = (\omega_\alpha^1, \cdots, \omega_\alpha^n, \overline{\alpha})(\omega_\beta^1, \cdots, \omega_\beta^n, \overline{\beta}) = (\alpha\psi)(\beta\psi)$, as required.

□

Lemma 4.1.6. *Let $A = (s_1, \cdots, s_n, \tau)$ be an element in $S \wr_n \mathcal{T}_n$. Then A is idempotent if and only if τ is idempotent and $s_i = s_i s_{i\tau}$ for all $i \in \{1, 2, \cdots, n\}$.*

Proof. Let $A = (s_1, \cdots, s_n, \tau)$. Then $A^2 = (s_1 s_{1\tau}, \cdots, s_n s_{n\tau}, \tau^2)$, so that $A = A^2$ if and only if $s_i = s_i s_{i\tau}$, $1 \leq i \leq n$, and $\tau = \tau^2$. □

4.2 Green's relations on $G \wr_n \mathcal{T}_n$

Independence algebras are a class of universal algebras having sets of free generators including free G-acts over any group G. In [17], Gould obtained results characterising Green's relations on End $\mathbf{A}$, where $\mathbf{A}$ is an independence algebra. In this subsection we prove explicitly the description of [17] for $\leq_{\mathcal{L}}$, $\leq_{\mathcal{R}}$, $\leq_{\mathcal{J}}$, $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, $\mathcal{D}$ and $\mathcal{J}$ in the special case of End $F_n(G)$.

To proceed, we describe the subalgebras of $F_n(G)$, as these will be the images of endomorphisms of $F_n(G)$.

Lemma 4.2.1. *A subset A is a subalgebra of $F_n(G)$ if and only if*

$$A = Gx_{i_1} \dot{\bigcup} Gx_{i_2} \dot{\bigcup} \cdots \dot{\bigcup} Gx_{i_m},$$

for some $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ *and* $0 \leq m \leq n$.

Proof. Let A be a subalgebra of $F_n(G)$. We claim $A = Gx_{i_1} \dot{\bigcup} Gx_{i_2} \dot{\bigcup} \cdots \dot{\bigcup} Gx_{i_m}$, where $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ and $0 \leq m \leq n$. Let $gx_k \in A$; as A is a subalgebra we obtain $(hg^{-1})(gx_k) \in A$ for all $h \in G$, hence $hx_k \in A$, which proves one direction of our claim.

The converse is clear. □

For a subalgebra

$$A = Gx_{i_1} \dot{\bigcup} Gx_{i_2} \dot{\bigcup} \cdots \dot{\bigcup} Gx_{i_m},$$

where $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ and $0 \leq m \leq n$, we say that *rank of* A, $\rho(A)$, is m. For $\alpha \in \operatorname{End} F_n(G)$ we define $\rho(\alpha)$ to be $\rho(\operatorname{Im} \alpha)$.

Lemma 4.2.2. *For all α, $\beta \in End\, F_n(G)$, we have the following:*

(*i*) $\alpha \leq_{\mathcal{L}} \beta$ *if and only if* $Im\, \alpha \subseteq Im\, \beta$;

(ii) $\alpha \leq_{\mathcal{R}} \beta$ *if and only if* $Ker\beta \subseteq Ker\alpha$*;*

(iii) $\rho(\alpha\beta) \leq \rho(\alpha),\ \rho(\beta)$.

Proof. (i) It is clear that if $\alpha \leq_{\mathcal{L}} \beta$ in End $F_n(G)$ then $\alpha \leq_{\mathcal{L}} \beta$ in $\mathcal{T}_{F_n(G)}$ and this implies $\alpha = \gamma\beta$ for some $\gamma \in$ End $F_n(G)$ and from this we obtain $\operatorname{Im}\alpha = \operatorname{Im}\gamma\beta \subseteq \operatorname{Im}\beta$.

Conversely, suppose $\operatorname{Im}\alpha \subseteq \operatorname{Im}\beta$, for each $i \in \{1, \cdots, n\}$ we have $x_i\alpha \in \operatorname{Im}\alpha \subseteq \operatorname{Im}\beta$, so we choose $a_i \in F_n(G)$ such that $x_i\alpha = a_i\beta$. Now define $\nu \in$ End $F_n(G)$ by $x_i\nu = a_i$ for $i \in \{1, \cdots, n\}$. Then clearly $x_i\nu\beta = a_i\beta = x_i\alpha$. Hence we obtain $\nu\beta = \alpha$ which means $\alpha \leq_{\mathcal{L}} \beta$.

(ii) If $\alpha \leq_{\mathcal{R}} \beta$ in End $F_n(G)$ then $\alpha \leq_{\mathcal{R}} \beta$ in $\mathcal{T}_{F_n(G)}$ and this implies that $\alpha = \beta\gamma$, for some $\gamma \in$ End $F_n(G)$.
Let $(x, y) \in \operatorname{Ker}\beta$ so $x\beta = y\beta$. Then

$$x\alpha = x(\beta\gamma) = (x\beta)\gamma = (y\beta)\gamma = y(\beta\gamma) = y\alpha.$$

Hence, $(x, y) \in \operatorname{Ker}\alpha$ so that $\operatorname{Ker}\beta \subseteq \operatorname{Ker}\alpha$, as required.

Conversely, suppose $\operatorname{Ker}\beta \subseteq \operatorname{Ker}\alpha$. Define $\gamma : F_n(G) \longrightarrow F_n(G)$ by let $\operatorname{Im}\beta = Gx_{i_1} \dot{\bigcup} \cdots \dot{\bigcup} Gx_{i_m}$ and then define $x_{i_j}\gamma = \omega_j\alpha$, where $\omega_j\beta = x_{i_j}$, and $x_i\gamma = x_i$, for all $i \notin \{i_1, \cdots, i_m\}$.

Now, if $\omega_j\beta = \omega_j^{'}\beta = x_{i_j}$, then $(\omega_j, \omega_j^{'}) \in \operatorname{Ker}\beta \subseteq \operatorname{Ker}\alpha$ so that $\omega_j\alpha = \omega_j^{'}\alpha$, and hence γ is well-defined.

As $F_n(G)$ is free on $X_n = \{x_1, \cdots, x_n\}$, γ must be a G-morphism. Let $w \in F_n(G)$ be such that $w = gx_k$ and let $x_k\beta = hx_{i_j}$, then we have $w\beta = g(x_k\beta) = g(hx_{i_j}) = gh(w_j\beta) = (ghw_j)\beta$. Now, $w\beta\gamma = (ghx_{i_j})\gamma = (gh)(x_{i_j}\gamma) = (gh)(w_j\alpha) = (ghw_j)\alpha$. As $w\beta = (ghw_j)\beta$, and $\operatorname{Ker}\beta \subseteq \operatorname{Ker}\alpha$, we have $w\alpha = (ghw_j)\alpha = w\beta\gamma$. Hence, $\alpha = \beta\gamma$.

(iii) We claim that for any $\tau, \kappa \in \operatorname{End} F_n(G)$ that

$$\rho(\tau\kappa) \leq \rho(\kappa) \text{ and } \rho(\tau\kappa) \leq \rho(\tau).$$

Recall, $\rho(\tau) = \rho(\operatorname{Im}\tau)$ and let $\operatorname{Im}\tau = \dot{\bigcup}_{y\in Y} Gy$, where $Y \subseteq X_n$ so that $\rho(\tau) = |Y|$. Since $\operatorname{Im}(\tau\kappa) = \operatorname{Im}(\tau)\kappa = (\dot{\bigcup}_{y\in Y} Gy)\kappa = \dot{\bigcup}_{y\in Y} G(y\kappa)$ this implies that $\rho(\tau\kappa) \leq |Y| = \rho(\tau)$. Now, $\rho(\kappa\tau) = \rho(\operatorname{Im}\kappa\tau) = \rho((\operatorname{Im}\kappa)\tau)$. As $\operatorname{Im}\kappa\tau \subseteq \operatorname{Im}\tau$ then we have $\rho(\operatorname{Im}\kappa\tau) \leq \rho(\operatorname{Im}\tau)$ this implies that $\rho(\kappa\tau) \leq \rho(\tau)$. So that $\rho(\alpha\beta) \leq \rho(\alpha)$, $\rho(\beta)$.

□

Lemma 4.2.3. *For all α, $\beta \in End\ F_n(G)$, we have the following:*

(i) $\alpha\ \mathcal{L}\ \beta$ *if and only if* $Im\,\alpha = Im\,\beta$;

(ii) $\alpha\ \mathcal{R}\ \beta$ *if and only if* $Ker\,\alpha = Ker\,\beta$;

(iii) $\alpha\ \mathcal{H}\ \beta$ *if and only if* $Im\,\alpha = Im\,\beta$ *and* $Ker\,\alpha = Ker\,\beta$;

(iv) $\alpha\ \mathcal{D}\ \beta$ *if and only if* $\rho(\alpha) = \rho(\beta)$;

(v) $\alpha \leq_{\mathcal{J}} \beta$ *if and only if* $\rho(\alpha) \leq \rho(\beta)$;

(vi) $\alpha\ \mathcal{J}\ \beta$ *if and only if* $\rho(\alpha) = \rho(\beta)$;

(vii) $\mathcal{D} = \mathcal{J}$.

Proof. It is easy to prove (i) and (ii) by using the previous Lemma.

(iii) This is an immediate consequence of (i) and (ii).

$\rho(\alpha) = \rho(\mu\beta\nu) \leq \rho(\mu\beta) \leq \rho(\beta)$ and $\rho(\beta) = \rho(\gamma\alpha\delta) \leq \rho(\gamma\alpha) \leq \rho(\alpha)$, so that $\rho(\alpha) = \rho(\beta)$.

(iv) First to prove if $\rho(\alpha) = \rho(\beta)$ then $\alpha \mathcal{D} \beta$. Let $\rho(\alpha) = \rho(\beta)$, then $\operatorname{Im} \alpha = \dot{\bigcup}_{y \in Y} Gy$, $\operatorname{Im} \beta = \dot{\bigcup}_{z \in Z} Gz$ for some $Y, Z \subseteq X_n$ with $|Y| = |Z| = \rho(\alpha) = \rho(\beta)$. Suppose $\tau : Y \longrightarrow Z$ is a bijection and define $\overline{\tau} : \operatorname{Im} \alpha \longrightarrow \operatorname{Im} \beta$ by $(gy)\overline{\tau} = g(y\tau)$ for all $g \in G$ and $y \in Y$.

It is clear that $\overline{\tau}$ is one to one since if $(gy)\overline{\tau} = (hy^{'})\overline{\tau}$ this implies that $g(y\tau) = h(y^{'}\tau)$ for all $g, h \in G$ and $y, y^{'} \in Y$. So, this forces $y\tau = y^{'}\tau$ and $g = h$. Since τ is a bijection we obtain $y = y^{'}$. Moreover, $\overline{\tau}$ is onto since from the definition of $\overline{\tau}$ we have $(gy)\overline{\tau} = g(y\tau)$ for all $g \in G$, $y \in Y$ and since τ is bijection so for all $gz \in \operatorname{Im} \beta$, pick $y \in Y$ with $y\tau = z$, then $gy \in \operatorname{Im} \alpha$ and that $gz = g(y)\tau = (gy)\overline{\tau}$. Let $\gamma = \alpha\overline{\tau}$ so that $\gamma \in \operatorname{End} F_n(G)$. Note that $\operatorname{Im} \gamma = \operatorname{Im} (\alpha\overline{\tau}) = (\operatorname{Im} \alpha)\overline{\tau} = \operatorname{Im} \beta$, so that $\beta \mathcal{L} \gamma$. Now let $u, v \in F_n(G)$, it is clear $u\alpha = v\alpha$ if and only if $(u\alpha)\overline{\tau} = (v\alpha)\overline{\tau}$, as $\overline{\tau}$ is one-one, so that $\operatorname{Ker} \alpha = \operatorname{Ker} \alpha\overline{\tau} = \operatorname{Ker} \gamma$, therefore, $\alpha \mathcal{R} \gamma$. Hence $\alpha \mathcal{D} \beta$.

Conversely, suppose $\alpha \mathcal{D} \beta$; then $\alpha \mathcal{R} \gamma \mathcal{L} \beta$ for some $\gamma \in \operatorname{End} F_n(G)$. It is clear from (i) and (ii) $\operatorname{Ker} \alpha = \operatorname{Ker} \gamma$ and $\operatorname{Im} \gamma = \operatorname{Im} \beta$. Now $\operatorname{Im} \alpha \cong F_n(G)/\operatorname{Ker} \alpha = F_n(G)/\operatorname{Ker} \gamma \cong \operatorname{Im} \gamma$ and since $\rho(\alpha) = \rho(\operatorname{Im} \alpha) = \rho(\operatorname{Im} \gamma) = \rho(\gamma)$ we obtain $\rho(\alpha) = \rho(\gamma)$, moreover, $\rho(\gamma) = \rho(\operatorname{Im} \gamma) = \rho(\operatorname{Im} \beta) = \rho(\beta)$ so that we obtain $\rho(\alpha) = \rho(\beta)$.

(v) If $\alpha \leq_{\mathcal{J}} \beta$ then $\alpha = \gamma\beta\delta$, so that by Lemma 4.2.2 (iii) we obtain

$$\rho(\alpha) = \rho(\gamma\beta\delta) \leq \rho(\gamma\beta) \leq \rho(\beta).$$

Conversely, suppose $\rho(\alpha) \leq \rho(\beta)$ and let $\operatorname{Im} \alpha = \dot{\bigcup}_{y \in Y} Gy$ and $\operatorname{Im} \beta = \dot{\bigcup}_{z \in Z} Gz$ for some $Y, Z \subseteq X_n$; so that $\rho(\alpha) = |Y|$ and $\rho(\beta) = |Z|$. As $\rho(\alpha) \leq \rho(\beta)$ so that there is a one to one map $\varphi : Y \longrightarrow Z$, now let $W = \operatorname{Im} \varphi$, so $W \subseteq Z$ and $|Y| = |W|$. Fix $w_0 \in W$ and define $\kappa : Z \longrightarrow W$ by

$$z\kappa = z, \quad \text{for all} \quad z \in W,$$

$$z\kappa = w_0, \quad \text{for all} \quad z \in Z \setminus W,$$

so that $\operatorname{Im}\kappa = W$. Now define $\gamma : \operatorname{Im}\beta = \dot{\bigcup}_{z\in Z} Gz \longrightarrow \dot{\bigcup}_{w\in W} Gw$ by $z\gamma = z\kappa$. Clearly γ extends to a G-act morphism so $\beta\gamma \in \operatorname{End} F_n(G)$. Since $\operatorname{Im}\beta\gamma = (\operatorname{Im}\beta)\gamma = (\dot{\bigcup}_{z\in Z} Gz)\gamma = \dot{\bigcup}_{z\in Z} Gz\gamma = \dot{\bigcup}_{z\in Z} Gz\kappa = \dot{\bigcup}_{w\in W} Gw$ we have $\rho(\beta\gamma) = |W| = |Y| = \rho(\alpha)$ and hence $\rho(\beta\gamma) = \rho(\alpha)$, so by (iv) we obtain $\beta\gamma\ \mathcal{D}\ \alpha$ so $\beta\gamma\ \mathcal{J}\ \alpha$ as $\mathcal{D} \subseteq \mathcal{J}$, and hence $\alpha \leq_{\mathcal{J}} \beta$.

(vi) If $\rho(\alpha) = \rho(\beta)$ then by (iv) $\alpha\ \mathcal{D}\ \beta$ so that $\alpha\ \mathcal{J}\ \beta$ as $\mathcal{D} \subseteq \mathcal{J}$.

Conversely, suppose $\alpha\ \mathcal{J}\ \beta$, then $\alpha = \mu\beta\nu$, $\beta = \gamma\alpha\delta$, for some $\mu, \nu, \gamma, \delta \in \operatorname{End} F_n(G)$. By using Lemma 4.2.2 (iii) we obtain $\rho(\alpha) = \rho(\mu\beta\nu) \leq \rho(\mu\beta) \leq \rho(\beta)$ and $\rho(\beta) = \rho(\gamma\alpha\delta) \leq \rho(\gamma\alpha) \leq \rho(\alpha)$, so that $\rho(\alpha) = \rho(\beta)$.

(vii) This is an immediate consequence of (iv) and (vi).

$\square$

4.3 Idempotents in wreath products

In this section, our aim is to count the the number of idempotents for the endomorphism monoid of a free S-act of rank n. We first consider the case where S is a finite group and move on to the general case where S is a finite monoid.

For $A \in S \wr_n \mathcal{T}_n$, we define $\operatorname{rank} A = \operatorname{rank}\alpha$, where $A = (s_1, s_2, \cdots, s_n, \alpha)$. In the case where G is a group, this coincides with the usual notation of rank of an endomorphism of the independence algebra $F_n(G)$.

4.3.1 Idempotents in $G \wr_n \mathcal{T}_n$

Consider the special case where G is a group.

Corollary 4.3.1. *Let G be a group and let $A = (g_1, \cdots, g_n, \mu)$ be an element in $G \wr_n \mathcal{T}_n$. Then A is idempotent if and only if $\mu = \mu^2$ and $g_j = 1$ for all $j \in Im\,\mu$.*

Proof. From Lemma 4.1.6, A is idempotent if and only if $\mu = \mu^2$ and $g_i = g_i g_{i\mu}$, $i \in \{1, 2, \cdots, n\}$. As G is group, the latter is equivalent to $g_{i\mu} = 1$ for all $i \in \{1, 2, \cdots, n\}$, that is, $g_j = 1$ for all $j \in \operatorname{Im}\mu$.

□

Lemma 4.3.2. *Let G be a group. The number $E(n, G)$ of idempotents in $G \wr_n \mathcal{T}_n$ equals*

$$E(n, G) = \sum_{k=1}^{n} \binom{n}{k} k^{n-k} |G|^{n-k}.$$

Proof. Let $A \in G \wr_n \mathcal{T}_n$ such that $A = (g_1, \cdots, g_n, \mu) = A^2$ and $|\operatorname{Im}\mu| = k$ where, $k \in \{1, 2, \cdots, n\}$. There exist $\binom{n}{k}$ possible images for an idempotent μ with $|\operatorname{Im}\mu| = k$. Suppose those images are $Y_1, Y_2, \cdots, Y_{\binom{n}{k}}$, for each of those images there exist k^{n-k} idempotents of $\mathcal{T}_n$ with this image. In virtue of Corollary 3.2.4, we obtain $\binom{n}{k} k^{n-k}$ idempotents of $\mathcal{T}_n$ with k elements in the image. By using Corollary 4.3.1, $A = A^2$ if and only if $\mu = \mu^2$ and $g_i = 1$ for all $i \in \operatorname{Im}\mu$. However, g_j is arbitrary elements in G, for all $j \notin \operatorname{Im}\mu$, so for any fixed $\mu = \mu^2 \in \mathcal{T}_n$, where $|\operatorname{Im}\mu| = k$ there exist $|G|^{n-k}$ choices for $(g_1, \cdots, g_n, \mu)$, and then we obtain $\binom{n}{k} k^{n-k} |G|^{n-k}$ idempotents in $G \wr_n \mathcal{T}_n$, and by using the sum rule the proof will be complete. □

We remark that the technique of [24] may be used to show that the exponential generating function of $E(n, G)$ is

$$\psi(z) \equiv \exp(z e^{z|G|}) = \sum_{n=0}^{\infty} \frac{E(n, G)}{n!} z^n.$$

The next example explain how the previous corollary works:

Example 4.3.3. Let $G = \{1, a, a^2\}$ be a finite group, and $n = 3$. To count the number of idempotent in $G \wr_3 \mathcal{T}_3$ (recall that if $A \in G \wr_3 \mathcal{T}_3$ means $A = (g_1, g_2, g_3, \mu)$, where $g_1, g_2, g_3 \in G$ and $\mu \in \mathcal{T}_3$).

It is clear that

$$\begin{aligned}G^3 =\{&(1,1,1),(a,a,a),(a^2,a^2,a^2),(1,a,a^2),(a,1,a^2),(a,a^2,1),(1,1,a),\\&(1,a,1),(a,1,1),(1,1,a^2),(1,a^2,1),(a^2,1,1),(a,a,1),(a,1,a),\\&(1,a,a),(a,a,a^2),(a,a^2,a),(a^2,a,a),(a^2,a^2,a),(a^2,a,a^2),(a,a^2,a^2),\\&(a^2,a^2,1),(a^2,1,a^2),(1,a^2,a^2),(a^2,a,1),(a^2,1,a),(1,a^2,a)\},\end{aligned}$$

so, $|G^3| = 27 = |G|^3 = 3^3$.

Since $n = 3$ we have 3 cases to find an idempotent in $G \wr_3 \mathcal{T}_3$;

Case 1 If $\mu \in \mathcal{T}_3$ and $\operatorname{rank}\mu = 1$ ($|\operatorname{Im}\mu| = 1$). There are 3 idempotent elements in $\mathcal{T}_3$ having rank 1:

$$c_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \quad c_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix} \quad c_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix}.$$

Now, if (g_1, g_2, g_3, c_1) is idempotent, then by using Corollary 4.3.1, we obtain $g_1 = 1$, and in this case it is obvious that there are 3 choices for g_2 and for g_3, and they are either 1, a or a^2. Therefore, we have 9 idempotent elements in $G \wr_3 \mathcal{T}_3$ of the form (g_1, g_2, g_3, c_1):

$$\begin{aligned}&(1,1,1,c_1),(1,a,a^2,c_1),(1,1,a,c_1),(1,a,1,c_1),\\&(1,1,a^2,c_1),(1,a^2,1,c_1),(1,a^2,a,c_1),(1,a,a,c_1),\\&(1,a^2,a^2,c_1).\end{aligned}$$

Similarly if $\mu = c_2$ or c_3. Hence, if $\operatorname{rank}\mu = 1$ there are 27 idempotent elements in $G \wr_3 \mathcal{T}_3$.

Case 2 If $\mu \in \mathcal{T}_3$ and $\operatorname{rank}\mu = 2$ ($|\operatorname{Im}\mu| = 2$). There are 6 idempotent elements in

$\mathcal{T}_3$ having rank 2:

$$\alpha_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 3 \end{pmatrix} \quad \alpha_3 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$$

$$\alpha_4 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix} \quad \alpha_5 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \end{pmatrix} \quad \alpha_6 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix}.$$

If $\mu = \alpha_1$ this implies $\operatorname{Im}\mu = \{2, 3\}$ so, by Corollary 4.3.1, we obtain $g_2 = g_3 = 1$. Hence, there are 3 choices for g_1. So, there are 3 idempotent elements in $G \wr_3 \mathcal{T}_3$ of the form $(g_1, g_2, g_3, \alpha_1)$:

$$(1, 1, 1, \alpha_1), (a, 1, 1, \alpha_1), (a^2, 1, 1, \alpha_1).$$

Similarly there are 3 idempotent elements (g_1, g_2, g_3, μ) for any $\mu = \mu^2 \in \mathcal{T}_3$ with $\operatorname{rank}\mu = 2$. Hence, if $\operatorname{rank}\mu = 2$ there are 18 idempotent elements in $G \wr_3 \mathcal{T}_3$.

Case 3 If $\mu \in \mathcal{T}_3$ and $\operatorname{rank}\mu = 3$ ($|\operatorname{Im}\mu| = 3$). There is only one idempotent element in $\mathcal{T}_3$ has rank $= 3$, and it is

$$I = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}.$$

Now, if $\mu = I$ this yields $\operatorname{Im}\mu = \{1, 2, 3\}$ and so that, by using Corollary 4.3.1, we will obtain only one idempotent element in $G \wr_3 \mathcal{T}_3$, which is $(1, 1, 1, \mu)$.

In order to count all idempotent elements in $G \wr_3 \mathcal{T}_3$, we must count all idempotent elements in three cases together to obtain $(3 \times 9) + (6 \times 3) + 1 = 27 + 18 + 1 = 46$.

Using the formula in Lemma 4.3.2, we obtain the same result.

In the case $|G| = 1$, that implies $G \wr_n \mathcal{T}_n \cong \mathcal{T}_n$, and Lemma 4.3.2 reduced to the formula in Corollary 3.2.4.

4.3.2 Idempotents in $S \wr_n \mathcal{T}_n$

We now proceed to count the number of idempotent elements in $S \wr_n \mathcal{T}_n$, where S is a finite monoid.

Let $E(S)$ be the set of all idempotent elements in S. For $A \in S \wr_n \mathcal{T}_n$, recall that $\operatorname{rank} A = \operatorname{rank} \tau$, where $A = (s_1, \cdots, s_n, \tau)$. Using Lemma 4.1.6, A is idempotent if and only if $\tau = \tau^2$ and $s_i = s_i s_{i\tau}$ for all $i \in \{1, 2, \cdots, n\}$. Consequently, for $i \in \operatorname{Im} \tau$, it follows $s_i \in E(S)$.

The next result calculates the number of idempotents in $S \wr_n \mathcal{T}_n$.

Theorem 4.3.4. *Let S be a finite monoid. Let $f \in E(S)$ and put*

$$P(f) = |\{s \in S : sf = s\}|.$$

Now define

$$\mathcal{P} = \{P(e) : e \in E(S)\},$$

and for $l \in \mathcal{P}$ let

$$m(l) = |\{e \in E(S) : P(e) = l\}|.$$

The number $I(n, S)$ of idempotents in $S \wr_n \mathcal{T}_n$ equals

$$I(n, S) = \sum_{r=1}^{n} \binom{n}{r} \left(\sum_{(l_1, \cdots, l_r) \in \mathcal{P}^r} m(l_1) \cdot \cdots \cdot m(l_r)(l_1 + \cdots + l_r)^{n-r} \right).$$

Proof. Suppose that $I(n, S)$ be the number of idempotents in $S \wr_n \mathcal{T}_n$, and $I(n, r, S)$ be the number of idempotents of rank r.

Then

$$I\,(n,S) = \sum_{r=1}^{n} I\,(n,r,S).$$

Let

$$I^{'}(n,r,S) = |\{A = (s_1,\cdots,s_n,\tau) : A = A^2, \text{ and } \operatorname{Im}\tau = \{1,2,\cdots,r\}\}|$$

be the number of idempotents $A = (s_1,\cdots,s_n,\tau)$ where $\operatorname{Im}\tau = \{1,\cdots,r\}$.

Note that, $\tau = \tau^2$ so $j\tau \in \{1,\cdots,r\}$ for $r+1 \leq j \leq n$.

Claim,

$$I\,(n,r,S) = \binom{n}{r} I^{'}\,(n,r,S).$$

Clearly, for any subset $T \subseteq \{1,2,\cdots,n\}$, with $|T| = r$, we have

$$I^{'}\,(n,r,S) = |\{A = (s_1,\cdots,s_n,\kappa) : A = A^2, \operatorname{Im}\kappa = T\}|.$$

Since there are $\binom{n}{r}$ possible images for idempotent κ with $\operatorname{Im}\kappa = T$, we have

$$I\,(n,r,S) = \binom{n}{r} I^{'}\,(n,r,S).$$

Our first aim is to show how many $A = A^2$ such that

$$A = \left(e_1,\cdots,e_r,s_{r+1},\cdots,s_n,\tau = \begin{pmatrix} 1 & 2 & 3 & \cdots & r & r+1 & \cdots & n \\ 1 & 2 & 3 & \cdots & r & (r+1)\tau & \cdots & n\tau \end{pmatrix}\right),$$

where $\operatorname{Im}\tau = \{1,2,\cdots,r\}$ and $e_1,\cdots,e_r$ are fixed idempotents.

Let $e_1,e_2,\cdots,e_r \in E(S)$, and

$$I^{'}\,(e_1,e_2,\cdots,e_r,n,r,S) = |\{A = (e_1,e_2,\cdots,e_r,s_{r+1},\cdots,s_n,\tau) :\ A = A^2,$$
$$\operatorname{Im}\tau = \{1,\cdots,r\}\}|.$$

Recall that

$$P(e_i) = |\{s \in S : se_i = s\}|.$$

We are going to prove that

$$I^{'}(e_1, e_2, \cdots, e_r, n, r, S) = (P(e_1) + P(e_2) + \cdots + P(e_r))^{n-r}.$$

Note that any idempotent $\tau \in \mathcal{T}_n$ with Im $\tau = \{1, 2, \cdots, r\}$ determines and is determined by a labelled partition of $\{r+1, \cdots, n\}$ into subsets $N_1, \cdots, N_r$ with $|N_i| = k_i$ and $k_1 + \cdots + k_r = n - r$.

Notice that, we say "*partition*" but we allow some k_i to be 0, i.e., some N_i to be empty. The labelled partition corresponds to $\operatorname{Ker}\tau$ where

$$\operatorname{Ker}\tau = \{\{1\} \cup N_1, \{2\} \cup N_2, \cdots, \{r\} \cup N_r\},$$

so that $N_j\tau = j$ if $N_j \neq \emptyset$.

Given any $k_1, \cdots, k_r \geq 0$ with $k_1 + \cdots + k_r = n - r$, there are $\frac{(n-r)!}{k_1! \cdots k_r!}$ labelled partitions of $\{r+1, \cdots, n\}$, each corresponding to $\tau = \tau^2$ where $N_i\tau = i$ and $|N_i| = k_i$, $1 \leq i \leq r$, [21, 37]. For each τ there are $P(e_1)^{k_1} \cdots P(e_r)^{k_r}$ choices of $s_{r+1}, \cdots, s_n$.

Thus

$$I^{'}(e_1, e_2, \cdots, e_r, n, r, S) = \sum_{k_1 + \cdots + k_r = n-r} \frac{(n-r)!}{k_1! \cdots k_r!} P(e_1)^{k_1} \cdots P(e_r)^{k_r}.$$

From the "*multinomial Formula*", [21, 37],

$$I^{'}(e_1, e_2, \cdots, e_r, n, r, S) = (P(e_1) + \cdots P(e_r))^{n-r}. \qquad (*)$$

For illustration, we now present an alternative way of counting idempotents, to verify directly the formula (*).

We again ask ourselves how to count the number of idempotents

$A = (e_1, \cdots, e_r, s_{r+1}, \cdots, s_n, \tau)$, where $\tau = \begin{pmatrix} 1 & 2 & 3 & \cdots & r & r+1 & \cdots & n \\ 1 & 2 & 3 & \cdots & r & (r+1)\tau & \cdots & n\tau \end{pmatrix}$ and $e_1, \cdots, e_r$ are fixed idempotents.

For $j \in \{r+1, \cdots, n\}$, $j\tau \in \{1, \cdots, r\}$. Given $j\tau$, we know that s_j must be chosen such that $s_j = s_j s_{j\tau}$, and this is the only constraint on the choice of s_j and $j\tau$ for $r+1 \leq j \leq n$.

Thus

$$\begin{aligned} I'(e_1, e_2, \cdots, e_r, n, r, S) &= \text{number of ways of choosing the pairs}\,(s_j, j\tau); \\ &= \prod_{r+1 \leq j \leq n} \qquad (\text{number of ways of choosing}\,(s_j, j\tau)). \end{aligned}$$

Now for any $j \in \{r+1, \cdots, n\}$, if $j\tau = l$ we have $P(e_l)$ choices for s_j.

So there are $P(e_1) + \cdots + P(e_r)$ choices for $(s_j, j\tau)$.

Thus

$$I'(e_1, e_2, \cdots, e_r, n, r, S) = (P(e_1) + \cdots + P(e_r))^{n-r}$$

as required.

Now, returning to the main argument

$$\begin{aligned} I'(n, r, S) &= \sum_{e_1, e_2, \cdots, e_r \in E(S)} I'(e_1, e_2, \cdots, e_r, n, r, S) \\ &= \sum_{(e_1, e_2, \cdots, e_r) \in E(S)^r} (P(e_1) + \cdots + P(e_r))^{n-r}. \end{aligned}$$

As

$$\mathcal{P} = \{P(e) : e \in E(S)\},$$

and for $l \in \mathcal{P}$ let

$$m(l) = |\{e \in E(S) : P(e) = l\}|.$$

Then

$$I^{'}(n, r, S) = \sum_{(l_1, \cdots, l_r) \in \mathcal{P}^r} m(l_1) \cdot \cdots \cdot m(l_r)(l_1 + \cdots + l_r)^{n-r}.$$

From the above, the formula for $I(n, S)$ can be written as

$$I(n, S) = \sum_{r=1}^{n} \binom{n}{r} \left(\sum_{(l_1, \cdots, l_r) \in \mathcal{P}^r} m(l_1) \cdot \cdots \cdot m(l_r)(l_1 + \cdots + l_r)^{n-r} \right).$$

□

Theorem 4.3.4 simplifies substantially in the case that S is a finite group. In this case, $E(S) = \{f\}$ where f is the idempotent of S,

$$P(f) = |\{s \in S : sf = s\}| = |S|,$$

$$\mathcal{P} = \{P(e) : e \in E(S)\} = \{|S|\},$$

$$m(l) = |\{e \in E(S) : P(e) = l\}| = 1.$$

Now in the formula

$$I^{'}(n, r, S) = \sum_{(l_1, \cdots, l_r) \in \mathcal{P}^r} m(l_1) \cdot \cdots \cdot m(l_r)(l_1 + \cdots + l_r)^{n-r}$$

we have first one element $(|S|, \cdots, |S|) \in \mathcal{P}^r$ and, $m(|S|) = 1$, so $I^{'}(n, r, S)$ reduce to

$$\begin{aligned} I^{'}(n, r, S) &= (r|S|)^{n-r} \\ &= r^{n-r}|S|^{n-r}. \end{aligned}$$

Hence,

$$I(n,S) = \sum_{r=1}^{n} \binom{n}{r} r^{n-r} |S|^{n-r},$$

which exactly what we got in Lemma 4.3.2.

4.3.3 Example

To implement the formula, we need only identify the idempotents $E(S)$ of S, and for each $e \in E(S)$, calculate $P(e)$. This immediately gives us $\mathcal{P}$ and $m(l)$ for each $l \in \mathcal{P}$.

We will demonstrate the formula of counting the number of idempotents $S \wr_n \mathcal{T}_n$ by the following worked example:

Example 4.3.5. Let $S = \{0, 1, a\}$, where $a^2 = 1$, and $n = 3$. We want to count the number of idempotents in $S \wr_3 \mathcal{T}_3$.

We know that for $e \in E(S)$,

$$P(e) = |\{s \in S : se = s\}|,$$

so

$$P(e_1) = P(1) = |\{s \in S : s \cdot 1 = s, \}| = 3,$$

and

$$P(e_2) = P(0) = |\{s \in S : s \cdot 0 = s\}| = 1.$$

Also, since

$$\mathcal{P} = \{P(e) : e \in E(S)\},$$

and

$$m(l) = |\{e \in E(S) : P(e) = l\}|,$$

we obtain $\mathcal{P} = \{3, 1\}$, where $l_1 = 3$ and $l_2 = 1$, and $m(l_1) = 1$, $m(l_2) = 1$.

Now we have the parameters $\mathcal{P}, l_1, l_2, m(l_1)$ and $m(l_2)$, we apply the formula to give

$$\begin{aligned} I\,(3,S) &= \binom{3}{1}(3^2+1^2)+\binom{3}{2}((1+1)+(1+3)+(3+1)+(3+3))+\binom{3}{3}(8) \\ &= 30+48+8 \\ &= 86. \end{aligned}$$

For the last summand, notice we are merely counting the choices for $(p,q,r)\in\mathcal{P}^3$. For the illustrations, we now give a detailed verification.

Since $n=3$ we have 3 cases to find an idempotent in $S \wr_3 \mathcal{T}_3$:

Case 1 If $\tau \in \mathcal{T}_3$ and $\operatorname{rank}\tau = 1 = r$. There are 3 idempotent elements in $\mathcal{T}_3$ having rank 1: c_1, c_2, and c_3.

By using the formula to count the number of idempotent where $n=3$, and $r=1$, we have the following:

$$I(3,1,S)=\binom{3}{1}\,I^{'}(3,1,S),$$

where

$$I^{'}(3,1,S)=\sum_{l\in\mathcal{P}^1} m(l)l^2 = 1\cdot 3^2+1\cdot 1^2 = 10.$$

This is because if $A=(s_1,s_2,s_3,c_1)$, then $s_1=s_1^2$, $s_2=s_2s_1$, and $s_3=s_3s_1$. We have two choices for s_1, which are 0 and 1. When $s_1=0$ there is one choice for s_2 and for s_3, which is 0. When $s_1=1$ there are 3 choices for s_2 and for s_3 and they are either $0,1,a$. Therefor, we have 9 idempotent elements in $S \wr_3 \mathcal{T}_3$ of the form $(1,s_2,s_3,c_1)$. Hence we will have 10 idempotent elements in $S \wr_3 \mathcal{T}_3$ of the form (s_1,s_2,s_3,c_1).

Similarly, for $\tau = c_2$ or c_3.

Therefore,

$$I(3,1,S) = \binom{3}{1} I^{'}(3,1,S) = 30,$$

is the number of idempotent in $S \wr_n \mathcal{T}_n$ when rank $\tau = 1$.

Case 2 If $\tau \in \mathcal{T}_3$ and rank $\tau = 2 = r$. There are 6 idempotent elements in $\mathcal{T}_3$ having rank 2. By using the formula we will obtain

$$I(3,2,S) = \binom{3}{2} I^{'}(3,2,S),$$

where

$$\begin{aligned} I^{'}(3,2,S) &= \sum_{(l_1,l_2)\in\mathcal{P}^2} m(l_1)m(l_2)(l_1+l_2)^{3-2} \\ &= m(l_1)m(l_1)(l_1+l_1) + m(l_1)m(l_2)(l_1+l_2) \\ &+ m(l_2)m(l_1)(l_2+l_1) + m(l_2)m(l_2)(l_2+l_2) \\ &= 16. \end{aligned}$$

Because if $A = (s_1, s_2, s_3, \tau)$ and $\tau = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$ for instance, then $s_1 = s_1^2$, $s_2 = s_2^2$ and $s_3 = s_3 s_1$. So, there are two choices for s_1 and for s_2, which are 0 and 1. When $s_1 = 0$, s_2 will be either 0 or 1, and in each case s_3 will be 0. If s_1=1, s_2 will be either 0 or 1 and in each case s_3 will be either 0,1 or a. Therefore, we will obtain 8 idempotent elements in this situation. Now, since there are two idempotents in $\mathcal{T}_3$ have the same image which is $\{1,2\}$, so we have 16 idempotent elements for them together.

The total number of idempotents in $S \wr_3 \mathcal{T}_3$ where rank $\tau = 2 = r$ is

$$I(3,2,S) = \binom{3}{2} I^{'}(3,2,S) = 48.$$

Case 3 If $\tau \in \mathcal{T}_3$ and $\operatorname{rank} \tau = 3 = r$. There is only one idempotent element in $\mathcal{T}_3$ has rank 3, and it is

$$I = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}.$$

By using the formula we will obtain

$$I(3,3,S) = \binom{3}{3} I^{'}(3,3,S),$$

where

$$I^{'}(3,3,S) = \sum_{(l_1,l_2,l_3)\in\mathcal{P}^3} m(l_1)m(l_2)m(l_3)(l_1+l_2+l_3)^{3-3} = 8.$$

As $A = (s_1, s_2, s_3, I)$, such that $s_1 = s_1^2$, $s_2 = s_2^2$ and $s_3 = s_3^2$. Hence, there are two choices for each of s_1, s_2 and s_3 which are either 0 or 1. Hence we will obtain 8 idempotent elements in this situation.

So, the total number of idempotents in $S \wr_3 \mathcal{T}_3$ where $\operatorname{rank} \tau = 3 = r$ is

$$I(3,3,S) = \binom{3}{3} I^{'}(3,2,S) = 8.$$

Now, since

$$I\,(3,S) = \sum_{r=1}^{3} I\,(3,r,S) = 30 + 48 + 8 = 86$$

idempotents in $S \wr_3 \mathcal{T}_3$.

We can find the number of idempotents in $S \wr_3 \mathcal{T}_3$ in another way.

We know that,

$$I\,(n,S) = \sum_{r=1}^{n} I\,(n,r,S)$$

where

$$I\,(n,r,S) = \binom{n}{r} I^{'}\,(n,r,S)$$

and

$$I^{'}(n, r, S) = \sum_{(e_1, e_2, \cdots, e_r) \in E(S)^r} (P(e_1) + \cdots + P(e_r))^{n-r}.$$

So, if $\tau \in \mathcal{T}_3$ and $\operatorname{rank} \tau = 1 = r$ we have

$$I^{'}(3, 1, S) = \sum_{e_r \in E(S)} P(e_r)^{3-1} = 3^2 + 1^2 = 10.$$

So, the number of idempotents in $S \wr_3 \mathcal{T}_3$ where rank $\tau = 1 = r$ is

$$I(3, 1, S) = \binom{3}{1} I^{'}(3, 1, S) = 30.$$

Now, if $\tau \in \mathcal{T}_3$ and $\operatorname{rank} \tau = 2 = r$, this implies that

$$\begin{aligned} I^{'}(3, 2, S) &= \sum_{(e_1, e_2) \in E(S)^2} (P(e_1) + P(e_2))^{3-2} \\ &= P(e_1) + P(e_1) + P(e_1) + P(e_2) \\ &+ P(e_2) + P(e_1) + P(e_2) + P(e_2) \\ &= 16 \end{aligned}$$

and the number of idempotents in $S \wr_3 \mathcal{T}_3$ where $\operatorname{rank} \tau = 2 = r$ is

$$I(3, 2, S) = \binom{3}{2} I^{'}(3, 2, S) = 48.$$

Also, where $\tau \in \mathcal{T}_3$ and $\operatorname{rank} \tau = 3 = r$, we will have

$$I^{'}(3, 3, S) = \sum_{(e_1, e_2, e_3) \in E(S)^3} (P(e_1) + P(e_2) + P(e_3))^{3-3} = 8$$

and the number of idempotents in $S \wr_3 \mathcal{T}_3$ where $\operatorname{rank} \tau = 3 = r$ is

$$I(3, 3, S) = \binom{3}{3} I^{'}(3, 3, S) = 8$$

and so

$$I\,(3,S) = \sum_{r=1}^{3} I\,(3,r,S) = 30 + 48 + 8 = 86,$$

verifying our earlier result for the number of idempotents in $S \wr_3 \mathcal{T}_3$.

Chapter 5

Monoids of partial endomorphisms

The aim of this chapter is to study the monoid $\mathcal{PT}_{\mathbf{A}}$ of partial endomorphisms of an independence algebra $\mathbf{A}$. In the first section we define the monoid $\mathcal{PT}_{\mathbf{A}}$, and verify that the monoid $\mathcal{PT}_{\mathbf{A}}$ is a submonoid of $\mathcal{PT}_{A}$.

In Section 5.2 we focus on the special case where $\mathbf{A}$ is the free left G-act $F_n(G) = \dot{\bigcup}_{i=1}^{n} Gx_i$ of rank n. If G is trivial then clearly $\mathcal{PT}_{\mathbf{F_n(G)}}$ is isomorphic to $\mathcal{PT}_n$. In the case where G is non-trivial, we prove that $\mathcal{PT}_{\mathbf{F_n(G)}}$ is isomorphic to $\operatorname{End} F_n(G)^0$, the endomorphism monoid of the left G-act given by $F_n(G)^0 = F_n(G)\dot{\cup}\{0\}$, where $\{0\}$ is a trivial left G-act. As an alternative description, we show that $\mathcal{PT}_{\mathbf{F_n(G)}}$ is embedded via φ in $G^0 \wr_{n+1} \mathcal{T}_{n,0}$, where G^0 is the group G with 0 adjoined and $\operatorname{Im}\varphi$ is the monoid

$$K_n(G)^0 = \{(0, g_1, \cdots, g_n, \alpha) : i\alpha = 0 \quad \text{if and only if} \quad g_i = 0 \\ \text{where} \quad 1 \leq i \leq n \quad \text{and} \quad \alpha \in \overline{\mathcal{T}_{n,0}}\}.$$

In Subsection, 5.2.1 the formula to count the number of idempotents in $\mathcal{PT}_{\mathbf{F_n(G)}}$ is found. In Subsection 5.2.2 we describe Green's relations on $\mathcal{PT}_{\mathbf{F_n(G)}}$. In Subsection 5.2.3 we describe some of the ideals of $K_n(G)^0$ in terms of the ideals of $\overline{\mathcal{T}_{n,0}}$.

Finally, the formula to count the number of nilpotents in $\mathcal{PT}_{\mathbf{F_n(G)}}$ is given in

Section 5.3.

The recommend references of this chapter are [15], [17], [22], [23], [25] and [40].

5.1 Semigroups $\mathcal{PT}_A$ and $\mathcal{PT}_{\mathbf{A}}$

Let $\mathbf{A}$ be an algebra and A the universe of $\mathbf{A}$. The semigroup of all partial maps from A to A is denoted by $\mathcal{PT}_A$, and the semigroup of all morphisms $\mathbf{B} \to \mathbf{C}$, where $\mathbf{B}$, $\mathbf{C}$ are subalgebras of $\mathbf{A}$ is denoted by $\mathcal{PT}_{\mathbf{A}}$.
The following lemma shows that $\mathcal{PT}_{\mathbf{A}}$ is a submonoid of $\mathcal{PT}_A$.

Lemma 5.1.1. *The semigroup $\mathcal{PT}_{\mathbf{A}}$ is a submonoid of $\mathcal{PT}_A$.*

Proof. Let α, $\beta \in \mathcal{PT}_{\mathbf{A}}$ such that $\alpha : \mathbf{B} \to \mathbf{C}$ and $\beta : \mathbf{D} \to \mathbf{W}$ be morphisms. Since $\operatorname{Im}\alpha$ and $\operatorname{Dom}\beta$ are subuniverses of $\mathbf{C}$ and $\mathbf{D}$, respectively, and by using Theorem 2.1.13 and Theorem 2.1.16, we get $\operatorname{Dom}\alpha\beta = [\operatorname{Im}\alpha \cap \operatorname{Dom}\beta]\alpha^{-1}$, and $\operatorname{Im}\alpha\beta = [\operatorname{Im}\alpha \cap \operatorname{Dom}\beta]\beta$ are subuniverses of $\mathbf{B}$ and $\mathbf{W}$, respectively. We define the composition of α and β as $x(\alpha\beta) = (x\alpha)\beta$ for all $x \in \operatorname{Dom}\alpha\beta$. Thus $\alpha\beta$ is a map between two subuniverses.

Now to show that $\alpha\beta \in \mathcal{PT}_{\mathbf{A}}$, which means for all $b_1, \cdots, b_n \in \operatorname{Dom}\alpha\beta$ and terms $t(b_1, \cdots, b_n)$,

$$(t(b_1, \cdots, b_n))(\alpha\beta) = t(b_1(\alpha\beta), \cdots, b_n(\alpha\beta)).$$

Notice that,

$$\begin{aligned}
(t(b_1, \cdots, b_n))(\alpha\beta) &= (t(b_1, \cdots, b_n)\alpha)\beta \\
&= t(b_1\alpha, \cdots, b_n\alpha)\beta && \text{(as } \alpha \text{ is morphism)} \\
&= t((b_1\alpha)\beta, \cdots, (b_n\alpha)\beta) && \text{(as } \beta \text{ is morphism)} \\
&= t((b_1(\alpha\beta), \cdots, b_n(\alpha\beta)).
\end{aligned}$$

Clearly, the identity I_A of $\mathcal{PT}_A$ is an automorphism, so $I_A \in \mathcal{PT}_{\mathbf{A}}$.

$\square$

It is worth mentioning that $\mathcal{PT}_{\mathbf{A}}$ has the empty map if and only if the empty set $\emptyset$ is a subalgebra if and only if $\mathbf{A}$ has no constants.

5.2 Semigroups $\mathcal{PT}_{\mathbf{F_n(G)}}$

The semigroup of all morphisms $\mathbf{B} \to \mathbf{C}$, where $\mathbf{B}$, $\mathbf{C}$ are subalgebras of $\mathbf{F_n(G)}$, is denoted by $\mathcal{PT}_{\mathbf{F_n(G)}}$.

Let $\alpha \in \mathcal{PT}_{\mathbf{F_n(G)}}$. Then α can be represented by

$$\alpha = \begin{pmatrix} x_{i_1} & \cdots & x_{i_k} \\ g^{\alpha}_{i_1} x_{i_1\overline{\alpha}} & \cdots & g^{\alpha}_{i_k} x_{i_k\overline{\alpha}} \end{pmatrix},$$

where $\overline{\alpha} \in \mathcal{PT}_n$, $g^{\alpha}_{i_1}, \cdots, g^{\alpha}_{i_k} \in G$, and $x_{i_l}\alpha = g^{\alpha}_{i_l} x_{i_l\overline{\alpha}}$. Moreover, every choice of $\overline{\beta} \in \mathcal{PT}_n$ with $\operatorname{Dom}\overline{\beta} = \{j_1, \cdots, j_t\}$, where $1 \leq j_1 < \cdots < j_t \leq n$, $t \geq 0$ and $h^{\beta}_{j_1}, \cdots, h^{\beta}_{j_t} \in G$, gives

$$\beta = \begin{pmatrix} x_{j_1} & \cdots & x_{j_t} \\ h^{\beta}_{j_1} x_{j_{1\overline{\beta}}} & \cdots & h^{\beta}_{j_t} x_{j_{t\overline{\beta}}} \end{pmatrix} \in \mathcal{PT}_{\mathbf{F_n(G)}}.$$

It is obvious that, if G is trivial, then in this case $\operatorname{End} F_n(G)$ will be isomorphic to $\mathcal{T}_n$ and $\mathcal{PT}_{\mathbf{F_n(G)}}$ to $\mathcal{PT}_n$.

Consider the G-act $F_n(G)^0 = F_n(G)\dot{\cup}\{0\}$, where G is non-trivial and $\{0\}$ is a trivial G-act. As each $\alpha \in \operatorname{End} F_n(G)^0$ depends only on its action on the free generators $\{x_i : i \in \{1, 2, \cdots, n\}\}$, therefore,

$$x_i\alpha = g^{\alpha}_i x_{i\overline{\alpha}},$$

for some $g^{\alpha}_i \in G$, where g^{α}_i is uniquely defined if $i\overline{\alpha} \neq 0$ for all i, where $1 \leq i \leq n$.

Note, $0\overline{\alpha} = 0$ so $\overline{\alpha} \in \overline{\mathcal{T}_{n,0}}$. Moreover, for $\alpha, \beta \in \operatorname{End} F_n(G)^0$, $\alpha = \beta$ if and only if $x_i\alpha = x_i\beta$ for all $i \in \{1, \cdots, n\}$.

Remark 5.2.1. Let $G \neq \{e\}$. If $\alpha \in \operatorname{End} F_n(G)^0$, we can not have $0\alpha = gx_i$, for some $g \in G$ and $i \in \{1, 2, \cdots, n\}$, which means any $\alpha \in \operatorname{End} F_n(G)^0$ must fix 0.

Proof. Let $h \in G$, $h \neq e$. If $0\alpha = gx_i$, then $h(gx_i) = h(0\alpha) = (h.0)\alpha = 0\alpha = gx_i$, so we will get $hg = g$, giving $h = e$, and that is a contradiction. □

The proof of the following is similar to the proof of Lemma 3.3.9.

Lemma 5.2.2. *Let $G \neq \{e\}$. For $n \in \mathbb{N}$, $\mathcal{PT}_{\mathbf{F_n(G)}}$ is isomorphic to $\operatorname{End} F_n(G)^0$.*

Proof. Define $\gamma : \mathcal{PT}_{\mathbf{F_n(G)}} \to \operatorname{End} F_n(G)^0$ by

$$\alpha\gamma = \overline{\alpha} \qquad \text{where} \qquad 0\overline{\alpha} = 0$$

and

$$(gx_i)\overline{\alpha} = \begin{cases} (gx_i)\alpha & \text{if} \quad gx_i \in \operatorname{Dom}\alpha; \\ 0 & \text{if} \quad gx_i \in F_n(G) \setminus \operatorname{Dom}\alpha. \end{cases}$$

We show that $\overline{\alpha}$ is a G-act morphism, that means we need $g(a\overline{\alpha}) = (ga)\overline{\alpha}$ for all $g \in G$ and $a \in F_n(G)^0$.

If $a = 0$, then $g(0\overline{\alpha}) = g.0 = 0 = (g.0)\overline{\alpha}$, since $\overline{\alpha}$ fixes 0.

Now consider $a = hx_i$. Observe that $x_i \in \operatorname{Dom}\alpha$ if and only if $hx_i \in \operatorname{Dom}\alpha$ for all $h \in G$. If $hx_i \in \operatorname{Dom}\alpha$, then for any $g \in G$, $ghx_i \in \operatorname{Dom}\alpha$ and so

$$\begin{aligned} (ghx_i)\overline{\alpha} &= (ghx_i)\alpha \\ &= g((hx_i)\alpha) && \text{(as } \alpha \text{ is morphism)} \\ &= g((hx_i)\overline{\alpha}). && \text{(as } hx_i \in \operatorname{Dom}\alpha) \end{aligned}$$

On the other hand, if $hx_i \notin \operatorname{Dom}\alpha$, then $ghx_i \notin \operatorname{Dom}\alpha$ for all $g \in G$, and we have $(ghx_i)\overline{\alpha} = 0 = g0 = g((hx_i)\overline{\alpha})$.

To prove γ is bijection. Let $\alpha\gamma = \beta\gamma$, we know that $\overline{\alpha}$ and $\overline{\beta}$ fix 0. For an element gx_i, $\alpha\gamma = \beta\gamma$ implies that $(gx_i)\alpha\gamma = (gx_i)\beta\gamma$, for all i. We have $gx_i \in F_n(G) \backslash \operatorname{Dom} \alpha$ if and only if $(gx_i)\overline{\alpha} = 0$ if and only if $(gx_i)\overline{\beta} = 0$ if and only if $gx_i \in F_n(G) \setminus \operatorname{Dom} \beta$ for all i. Hence, $\operatorname{Dom} \alpha = \operatorname{Dom} \beta$. Moreover, if $gx_i \in \operatorname{Dom} \alpha = \operatorname{Dom} \beta$ this implies that $(gx_i)\alpha = (gx_i)\overline{\alpha} = (gx_i)\overline{\beta} = (gx_i)\beta$, so that $\alpha = \beta$ and, therefore, γ is one to one.

To prove γ is onto. Let $\eta \in \operatorname{End} F_n(G)^0$. Define $\eta' \in \mathcal{PT}_{\mathbf{F_n(G)}}$ by

$$\operatorname{Dom} \eta' = \{gx_i \in F_n(G) : (gx_i)\eta \neq 0\},$$

and for all $gx_i \in \operatorname{Dom} \eta'$, $(gx_i)\eta' = (gx_i)\eta$. To show $\operatorname{Dom} \eta'$ is a subalgebra of $\mathbf{F_n}(\mathbf{G})$, we claim $\operatorname{Dom} \eta' = Gx_{i_1} \dot{\bigcup} Gx_{i_2} \dot{\bigcup} \cdots \dot{\bigcup} Gx_{i_m}$ where $1 \leq i_1 < i_2 < \cdots < i_m \leq n$. Let $gx_k \in \operatorname{Dom} \eta'$ for some $1 \leq k \leq n$, so that by the definition of $\operatorname{Dom} \eta'$ we have $(gx_k)\eta \neq 0$ and $(gx_k)\eta' = (gx_k)\eta$. Where $h \in G$ we want $hgx_k \in \operatorname{Dom} \eta'$. We have $(hgx_k)\eta = h((gx_k)\eta)$, since $\eta \in \operatorname{End} F_n(G)^0$, therefore, $h((gx_k)\eta) \neq 0$ as $(gx_k)\eta \neq 0$. So $hgx_k \in \operatorname{Dom} \eta'$. Hence $\operatorname{Dom} \eta'$ is a subalgebra of $\mathbf{F_n}(\mathbf{G})$. It is obvious that η' is a G-act morphism, as for any $gx_i \in \operatorname{Dom} \eta'$ and $h \in G$ we have $h((gx_i)\eta') = h((gx_i)\eta) = (hgx_i)\eta = (hgx_i)\eta'$.

Now,

$$(gx_i)\eta'\gamma = (gx_i)\overline{\eta'} = \begin{cases} (gx_i)\eta' & \text{if} \quad gx_i \in \operatorname{Dom} \eta'; \\ 0 & \text{if} \quad gx_i \in F_n(G) \setminus \operatorname{Dom} \eta'. \end{cases}$$

$$= \begin{cases} (gx_i)\eta & \text{if} \quad gx_i \in \operatorname{Dom} \eta'; \\ 0 & \text{if} \quad gx_i \in F_n(G) \setminus \operatorname{Dom} \eta'. \end{cases}$$

For $gx_i \in F_n(G) \setminus \operatorname{Dom} \eta'$, we must have by the definition of $\operatorname{Dom} \eta'$ that $(gx_i)\eta = 0$.

Therefore,

$$(gx_i)\eta'\gamma = (gx_i)\overline{\eta'} = \begin{cases} (gx_i)\eta & \text{if} \quad gx_i \in \operatorname{Dom}\eta'; \\ (gx_i)\eta & \text{if} \quad gx_i \in F_n(G) \setminus \operatorname{Dom}\eta'. \end{cases}$$

Moreover, since η, $\overline{\eta'} \in \operatorname{End} F_n(G)^0$, so η, $\overline{\eta'}$ must fix 0, which means $0 = 0\eta = 0\overline{\eta'}$. So for all elements of $F_n(G)^0$, η and $\overline{\eta'}$ agree, hence $\eta'\gamma = \eta$, so that γ is onto.

To show γ is homomorphism. Let α, $\beta \in \mathcal{PT}_{\mathbf{F_n(G)}}$. We want to prove that $(\alpha\beta)\gamma = \alpha\gamma\beta\gamma$. We have for all i,

$$(gx_i)(\alpha\beta)\gamma = (gx_i)\overline{\alpha\beta} = \begin{cases} (gx_i)\alpha\beta & \text{if} \quad gx_i \in \operatorname{Dom}\alpha\beta; \\ 0 & \text{if} \quad gx_i \in F_n(G) \setminus \operatorname{Dom}\alpha\beta; \end{cases}$$

$$= \begin{cases} (gx_i)\alpha\beta & \text{if} \quad gx_i \in \operatorname{Dom}\alpha \text{ and } (gx_i)\alpha \in \operatorname{Dom}\beta; \\ 0 & \text{if} \quad gx_i \in F_n(G) \setminus \operatorname{Dom}\alpha\beta. \end{cases}$$

Furthermore, since $\overline{\alpha\beta} \in \operatorname{End} F_n(G)^0$, we have $0 = 0\overline{\alpha\beta}$. On the other hand, $(gx_i)(\alpha\gamma\beta\gamma) = (gx_i)\overline{\alpha}\overline{\beta}$. It is clear that $\overline{\alpha}$, $\overline{\beta} \in \operatorname{End} F_n(G)^0$, hence we get $0(\overline{\alpha}\overline{\beta}) = (0\overline{\alpha})\overline{\beta} = 0\overline{\beta} = 0 = 0\overline{\alpha\beta}$. If $gx_i \in \operatorname{Dom}\alpha\beta$, then $gx_i \in \operatorname{Dom}\alpha$ and $(gx_i)\alpha \in \operatorname{Dom}\beta$, so that we get $(gx_i)\overline{\alpha}\overline{\beta} = ((gx_i)\overline{\alpha})\overline{\beta} = ((gx_i)\alpha)\overline{\beta} = ((gx_i)\alpha)\beta = (gx_i)\alpha\beta = (gx_i)\overline{\alpha\beta}$. Otherwise, $gx_i \notin \operatorname{Dom}\alpha\beta$. So $gx_i \notin \operatorname{Dom}\alpha$ or $gx_i \in \operatorname{Dom}\alpha$ and $(gx_i)\alpha \notin \operatorname{Dom}\beta$. Then

$$(gx_i)\overline{\alpha}\overline{\beta} = ((gx_i)\overline{\alpha})\overline{\beta} = \begin{cases} 0\overline{\beta} & \text{if} \quad gx_i \notin \operatorname{Dom}\alpha; \\ ((gx_i)\alpha)\overline{\beta} & \text{if} \quad gx_i \in \operatorname{Dom}\alpha \text{ and } (gx_i)\alpha \notin \operatorname{Dom}\beta. \end{cases}$$

$$= \begin{cases} 0 & \text{if} \quad gx_i \notin \operatorname{Dom}\alpha; \\ 0 & \text{if} \quad gx_i \in \operatorname{Dom}\alpha \text{ and } (gx_i)\alpha \notin \operatorname{Dom}\beta. \end{cases}$$

Therefore,

$$(gx_i)\overline{\alpha}\overline{\beta} = \begin{cases} (gx_i)\alpha\beta & \text{if} \quad gx_i \in \operatorname{Dom}\alpha\beta; \\ 0 & \text{if} \quad gx_i \notin \operatorname{Dom}\alpha\beta, \end{cases}$$

and so $(\alpha\beta)\gamma = \alpha\gamma\beta\gamma$, as required. □

As we explained in Chapter 4, it has long been known that the endomorphism monoid of a free G-act on n free generators is isomorphic to a wreath product $G \wr_n \mathcal{T}_n$. By the wreath product $G^0 \wr_{n+1} \mathcal{T}_{n,0}$ we mean $G^0 \wr_{n+1} \mathcal{T}_{n+1}$, where we are using the set $\{0, 1, \cdots, n\}$ rather than $\{1, \cdots, n+1\}$. So $G^0 \wr_{n+1} \mathcal{T}_{n,0}$ is the monoid consisting of elements of the form $(g_0, g_1, \cdots, g_n, \alpha)$, such that $g_0, g_1, \cdots, g_n \in G^0$ and $\alpha \in \mathcal{T}_{n,0}$. We know from Chapter 3 that the subset $\overline{\mathcal{T}_{n,0}}$ is a submonoid of $\mathcal{T}_{n,0}$, from which we obtain that $G^0 \wr_{n+1} \overline{\mathcal{T}_{n,0}} = \{(g_0, g_1, \cdots, g_n, \alpha) : \alpha \in \overline{\mathcal{T}_{n,0}}\}$ is a submonoid of the wreath product $G^0 \wr_{n+1} \mathcal{T}_{n,0}$.

Lemma 5.2.3. *Let $G \neq \{e\}$. For $n \in \mathbb{N}$, $\mathcal{PT}_{\mathbf{F_n(G)}}$ is embedded in $G^0 \wr_{n+1} \overline{\mathcal{T}_{n,0}}$.*

Proof. Let $\alpha \in \mathcal{PT}_{\mathbf{F_n(G)}}$, where $\overline{\alpha} \in \mathcal{PT}_n$, we described α by

$$\alpha = \begin{pmatrix} x_{i_1} & \cdots & x_{i_k} \\ g^{\alpha}_{i_1} x_{i_{1\overline{\alpha}}} & \cdots & g^{\alpha}_{i_k} x_{i_{k\overline{\alpha}}} \end{pmatrix}.$$

First, let $\alpha' \in \overline{\mathcal{T}_{n,0}}$ be given by, for $0 \leq i \leq n$,

$$i\alpha' = \begin{cases} i\overline{\alpha} & \text{if} \quad i \in \operatorname{Dom}\overline{\alpha} \quad (\text{i.e., } x_i \in \operatorname{Dom}\alpha); \\ 0 & \text{if} \quad i \notin \operatorname{Dom}\overline{\alpha} \quad (\text{i.e., } x_i \notin \operatorname{Dom}\alpha); \\ 0 & \text{if} \quad i = 0. \end{cases}$$

Notice that, g_i^α is defined and lies in G for all i such that $i\alpha' \neq 0$, and we put $g_i^\alpha = 0$ for all i such that $i\alpha' = 0$.

Let $\varphi : \mathcal{PT}_{\mathbf{F_n(G)}} \to G^0 \wr_{n+1} \overline{\mathcal{T}_{n,0}}$ be given by

$$\alpha\varphi = (g_0^\alpha, g_1^\alpha, \cdots, g_n^\alpha, \alpha').$$

To prove φ is an embedding, we have to show that φ is homomorphism and one to one. To prove φ is homomorphism let $\alpha, \beta \in \mathcal{PT}_{\mathbf{F_n(G)}}$, with the aim to show $(\alpha\beta)\varphi = \alpha\varphi\beta\varphi$.

Note that,

$$\alpha\varphi\beta\varphi = (g_0^\alpha g_{0\alpha'}^\beta, g_1^\alpha g_{1\alpha'}^\beta, \cdots, g_n^\alpha g_{n\alpha'}^\beta, \alpha'\beta')$$

and

$$(\alpha\beta)\varphi = (g_0^{\alpha\beta}, g_1^{\alpha\beta}, \cdots, g_n^{\alpha\beta}, (\alpha\beta)').$$

Observe that $\alpha'\beta' = (\alpha\beta)'$ follows from the proof below, together with the fact that $\overline{\alpha} \to \alpha'$ in the standard embedding of $\mathcal{PT}_n$ in $\overline{\mathcal{T}_{n,0}}$.

For $\alpha, \beta \in \mathcal{PT}_{\mathbf{F_n(G)}}$, we want to show that $\overline{\alpha}\overline{\beta} = \overline{\alpha\beta}$. Each side of the equality has domain a subalgebra, so we first need to check

$$i \in \operatorname{Dom}\overline{\alpha}\overline{\beta} \Leftrightarrow i \in \operatorname{Dom}\overline{\alpha\beta},$$

and in this case $i\overline{\alpha}\overline{\beta} = i\overline{\alpha\beta}$.

Now,

$$
\begin{aligned}
i \in \operatorname{Dom} \overline{\alpha}\overline{\beta} &\Leftrightarrow i \in \operatorname{Dom} \overline{\alpha} \text{ and } i\overline{\alpha} \in \operatorname{Dom} \overline{\beta} \\
&\Leftrightarrow x_i \in \operatorname{Dom} \alpha \text{ and } x_{i\overline{\alpha}} \in \operatorname{Dom} \beta \\
&\Leftrightarrow x_i \in \operatorname{Dom} \alpha \text{ and } x_i\alpha \in \operatorname{Dom} \beta \ (\text{as} \operatorname{Dom} \beta \, \text{is a subact}) \\
&\Leftrightarrow x_i \in \operatorname{Dom} \alpha\beta \\
&\Leftrightarrow i \in \operatorname{Dom} \overline{\alpha\beta}.
\end{aligned}
$$

If $i \in \operatorname{Dom} \overline{\alpha}\overline{\beta}$, then $x_i\alpha\beta = (g_i^{\alpha} x_{i\overline{\alpha}})\beta = g_i^{\alpha} g_{i\overline{\alpha}}^{\beta} x_{i\overline{\alpha}\overline{\beta}}$ and $x_i\alpha\beta = g_i^{\alpha\beta} x_{i\overline{\alpha\beta}}$, so $i\overline{\alpha}\overline{\beta} = i\overline{\alpha\beta}$, gives $\overline{\alpha}\overline{\beta} = \overline{\alpha\beta}$.

Clearly, $g_0^{\alpha} g_{0\alpha'}^{\beta} = 0 = g_0^{\alpha\beta}$. Let $1 \leq i \leq n$, then

$$
g_i^{\alpha} g_{i\alpha'}^{\beta} = 0 \Leftrightarrow \begin{cases} g_i^{\alpha} = 0 \quad \text{or;} \\ g_i^{\alpha} \neq 0 \quad \text{and} \quad g_{i\alpha'}^{\beta} = 0 \end{cases}
$$

$$
\Leftrightarrow \begin{cases} x_i \notin \operatorname{Dom} \alpha \quad \text{or;} \\ x_i \in \operatorname{Dom} \alpha \quad \text{and} \quad x_{i\overline{\alpha}} \notin \operatorname{Dom} \beta \end{cases}
$$

$$
\Leftrightarrow \begin{cases} x_i \notin \operatorname{Dom} \alpha \quad \text{or;} \\ x_i \in \operatorname{Dom} \alpha \quad \text{and} \quad x_i\alpha \notin \operatorname{Dom} \beta \end{cases}
$$

$$
\Leftrightarrow x_i \notin \operatorname{Dom} \alpha\beta \Leftrightarrow g_i^{\alpha\beta} = 0.
$$

If $g_i^{\alpha\beta} \neq 0$, then

$$
g_i^{\alpha\beta} x_{i\overline{\alpha\beta}} = x_i\alpha\beta = (x_i\alpha)\beta = (g_i^{\alpha} x_{i\overline{\alpha}})\beta = g_i^{\alpha} g_{i\overline{\alpha}}^{\beta} x_{i\overline{\alpha}\overline{\beta}}.
$$

From this we obtain

$$
g_i^{\alpha\beta} = g_i^{\alpha} g_{i\alpha'}^{\beta}.
$$

Hence φ is homomorphism. It is obvious φ is one to one. Therefore, φ is an embedding.

Note that φ is not a monoid embedding as if I is the identity of $\mathcal{PT}_{\mathbf{F_n(G)}}$ then $I\varphi = (0, e, \cdots, e, I_{n,0})$, where $I_{n,0}$ is the identity of $\overline{\mathcal{T}_{n,0}}$. □

5.2.1 Idempotents in $\mathcal{PT}_{\mathbf{F_n(G)}}$

In this subsection our aim is to count the number of idempotents in $\mathcal{PT}_{\mathbf{A}}$ where $\mathbf{A}$ is $\mathbf{F_n}(\mathbf{G})$ and where G is group.

Let

$$K_n(G)^0 = \{(0, g_1, \cdots, g_n, \alpha) : i\alpha = 0 \quad \text{if and only if} \quad g_i = 0 \\ \text{where} \quad 1 \leq i \leq n \quad \text{and} \quad \alpha \in \overline{\mathcal{T}_{n,0}}\}.$$

Observe that $K_n(G)^0 = \operatorname{Im}\varphi$ in Lemma 5.2.3, so that $K_n(G)^0$ is a monoid with identity $(0, e, \cdots, e, I_{n,0})$ and the multiplication given by

$$(0, k_1, \cdots, k_n, \alpha)(0, u_1, \cdots, u_n, \beta) = (0, k_1u_{1\alpha}, \cdots, k_nu_{n\alpha}, \alpha\beta).$$

Since φ is an embedding, we have $\mathcal{PT}_{\mathbf{F_n(G)}} \cong K_n(G)^0 \cong \operatorname{End} F_n(G)^0$.

Corollary 5.2.4. *Let G^0 be G with a 0 adjoined, and $A = (0, g_1, \cdots, g_n, \alpha)$ be an element in $K_n(G)^0$. The following statements are equivalent:*

(i) A is idempotent;

(ii) α is idempotent and for all i, where $i \in \{1, \cdots, n\}$, with $i\alpha \neq 0$, $g_ig_{i\alpha} = g_i$;

(iii) for all i, where $i \in \{1, \cdots, n\}$, with $i\alpha \neq 0$, $g_{i\alpha} = 1$.

Proof. Let $A = (0, g_1, \cdots, g_n, \alpha) \in K_n(G)^0$, so that $A^2 = (0, g_1g_{1\alpha}, \cdots, g_ng_{n\alpha}, \alpha^2)$.

By using the fact $g_i = 0$ if and only if $i\alpha = 0$, we have

$$\begin{aligned} A = A^2 &\Leftrightarrow \alpha = \alpha^2 \text{ and for all } i \in \{1, \cdots, n\},\ g_i = g_i g_{i\alpha}, \\ &\Leftrightarrow \alpha = \alpha^2 \text{ and for all } i \in \{1, \cdots, n\} \text{ with } i\alpha \neq 0,\ g_i = g_i g_{i\alpha}, \\ &\Leftrightarrow \alpha = \alpha^2 \text{ and for all } i \in \{1, \cdots, n\} \text{ with } i\alpha \neq 0,\ g_{i\alpha} = 1. \end{aligned}$$

□

It is worth mentioning that in $\mathcal{PT}_{\mathbf{F_n(G)}}$ if Im α and Dom β are disjoint then $\alpha\beta$ (under composition of partial functions in $\mathcal{PT}_{\mathbf{F_n(G)}}$) is 0, (where 0 represents the empty map).

Let $(0, g_1, \cdots, g_n, c_0) \in K_n(G)^0$, where $c_0 \in \overline{\mathcal{T}_{n,0}}$ is the zero of $\overline{\mathcal{T}_{n,0}}$, i.e., the constant map with image 0. Then by definition of $K_n(G)^0$ we have $g_i = 0$ for all $i \in \{1, \cdots, n\}$. Thus $(0, \cdots, 0, c_0)$ is the only element of $K_n(G)^0$ with final coordinate c_0. Notice that $(0, \cdots, 0, c_0) \in E\left(K_n(G)^0\right)$ (the set of idempotent elements in $K_n(G)^0$). Remark here that $(0, \cdots, 0, c_0)$ is the zero of $K_n(G)^0$, to show that let $(0, h_1, \cdots, h_n, \alpha)$ be any element in $K_n(G)^0$, then

$$(0, h_1, \cdots, h_n, \alpha)(0, \cdots, 0, c_0) = (0, \cdots, 0, c_0),$$

and

$$(0, \cdots, 0, c_0)(0, h_1, \cdots, h_n, \alpha) = (0, \cdots, 0, c_0).$$

This implies $(0, \cdots, 0, c_0)$ is the zero of $K_n(G)^0$.

Corollary 5.2.5. *Let $G \neq \{e\}$. The number $I(n, G)$ of idempotents in $K_n(G)^0$ equals*

$$I(n, G) = \left[\sum_{k=1}^{n} \sum_{l=k}^{n} \binom{n}{l} \binom{l}{k} k^{l-k} |G|^{l-k}\right] + 1.$$

Proof. We count the possibilities for $A = (0, g_1, \cdots, g_n, \overline{\alpha}) \in K_n(G)^0$, where $\alpha \in \mathcal{PT}_n$ and $\overline{\alpha} \in \overline{\mathcal{T}_{n,0}}$ is defined in Lemma 3.3.9, to be idempotent.

If $|\operatorname{Im}\alpha| = 0$, then $\alpha = \emptyset$, so that $\overline{\alpha} = c_0$ and we already remarked that $(0, \cdots, 0, c_0) \in E(K_n(G)^0)$.

We now count the possibilities for $A = A^2$, where $|\operatorname{Im}\alpha| = k \in \{1, \cdots, n\}$. We know that this entails $\overline{\alpha} = \overline{\alpha}^2$ and hence $\alpha = \alpha^2$. Thus $\operatorname{Im}\alpha \subseteq \operatorname{Dom}\alpha$ and $i\alpha = i$ for all $i \in \operatorname{Im}\alpha$. We therefore count the choices for $A = A^2$ where $|\operatorname{Im}\alpha| = k > 0$ and $|\operatorname{Dom}\alpha| = l$, where $k \leq l \leq n$.

There are $\binom{n}{l}$ choices for $\operatorname{Dom}\alpha$. For each of those there are $\binom{l}{k}$ choices for $\operatorname{Im}\alpha$. For each of these, $i\alpha = i$ for all $i \in \operatorname{Im}\alpha$, and there are k choices for $i\alpha$ for $i \in \operatorname{Dom}\alpha \setminus \operatorname{Im}\alpha$, i.e., $\binom{n}{l}\binom{l}{k}k^{l-k}$ choices for α. For each of these possibilities for α, we know from Corollary 5.2.4 that the only further condition is that $g_j = 1$ for all $j \in \operatorname{Im}\alpha$. By definition of $K_n(G)^0$, w have $g_i = 0$ for all $i \notin \operatorname{Dom}\alpha$. Thus there are $|G|^{l-k}$ choices for the remaining g_is.

We conclude, there are $\binom{n}{l}\binom{l}{k}k^{l-k}|G|^{l-k}$ choices for $A = A^2$, where $|\operatorname{Im}\alpha| = k > 0$ and $|\operatorname{Dom}\alpha| = l$. Thus the total number of idempotents in $K_n(G)^0$ is

$$I(n, G) = \left[\sum_{k=1}^{n}\sum_{l=k}^{n}\binom{n}{l}\binom{l}{k}k^{l-k}|G|^{l-k}\right] + 1.$$

□

Now let us consider the following example which explains how the formula in the Corollary 5.2.5 works:

Example 5.2.6. Let $G = \{1, a\}$. We want to count the numbers of idempotents in $K_4(G)^0$.

Recall that if $A \in K_4(G)^0$ means $A = (0, g_1, g_2, g_3, g_4, \overline{\alpha})$, where $g_1, g_2, g_3, g_4 \in G^0$ and $\overline{\alpha} \in \overline{\mathcal{T}_{4,0}}$, and by the definition of $K_n(G)^0$ we have $g_i = 0$ if and only if $i\overline{\alpha} = 0$ where $1 \leq i \leq n$. Furthermore, we know from Lemma 3.3.9 that $\mathcal{PT}_4$ is isomorphic to $\overline{\mathcal{T}_{4,0}}$, hence where $\alpha \in \mathcal{PT}_4$ we have $\alpha \leftrightarrow \overline{\alpha}$.

For $\alpha \in \mathcal{PT}_4$ suppose that $k = |\operatorname{Im}\alpha|$, and $l = |\operatorname{Dom}\alpha|$.

We have 5 cases to find the numbers of idempotents in $K_4(G)^0$ all of them corresponds to k.

Case (1) If $k = |\operatorname{Im}\alpha| = 1$. In this case we have 4 cases for $l = |\operatorname{Dom}\alpha|$, which are either $l = 1, 2, 3$ or, 4.

Case (i) If $l = |\operatorname{Dom}\alpha| = 1$. There are 4 idempotents in $\mathcal{PT}_4$ having $k = 1$ and $l = 1$, which are as follows:

$$\alpha_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & - & - & - \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ - & 2 & - & - \end{pmatrix}$$

$$\alpha_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ - & - & 3 & - \end{pmatrix} \quad \alpha_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ - & - & - & 4 \end{pmatrix}.$$

Observe that, there are 4 ways to choose $\operatorname{Im}\alpha$, for each of these there is only one way to choose $\operatorname{Dom}\alpha$. Now, if $(0, g_1, g_2, g_3, g_4, \overline{\alpha_1}) \in K_4(G)^0$ then by using Corollary 5.2.4, we obtain $g_1 = 1$, and by the definition of $K_4(G)^0$ we have $g_2 = g_3 = g_4 = 0$ as $2, 3, 4 \notin \operatorname{Dom}\alpha_1$. Therefor, we have only one idempotent in $K_4(G)^0$ of the form $(0, g_1, g_2, g_3, g_4, \overline{\alpha_1})$ which is

$$(0, 1, 0, 0, 0, \overline{\alpha_1}).$$

Similarly, if we have idempotents of the form $(0, g_1, g_2, g_3, g_4, \overline{\alpha_2})$, $(0, g_1, g_2, g_3, g_4, \overline{\alpha_3})$ or, $(0, g_1, g_2, g_3, g_4, \overline{\alpha_4})$.

Case (ii) If $l = |\operatorname{Dom}\alpha| = 2$. There are 12 idempotents in $\mathcal{PT}_4$ having $k = |\operatorname{Im}\alpha| = 1$ and $l = |\operatorname{Dom}\alpha| = 2$ which are as follows:

$$\alpha_5 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & - & - \end{pmatrix} \quad \alpha_6 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & - & 1 & - \end{pmatrix}$$

$$\alpha_7 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & - & - & 1 \end{pmatrix} \quad \alpha_8 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ - & 2 & - & 2 \end{pmatrix}$$

$$\alpha_9 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & - & - \end{pmatrix} \quad \alpha_{10} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ - & 2 & 2 & - \end{pmatrix}$$

$$\alpha_{11} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ - & - & 3 & 3 \end{pmatrix} \quad \alpha_{12} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ - & 3 & 3 & - \end{pmatrix}$$

$$\alpha_{13} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & - & 3 & - \end{pmatrix} \quad \alpha_{14} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ - & - & 4 & 4 \end{pmatrix}$$

$$\alpha_{15} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ - & 4 & - & 4 \end{pmatrix} \quad \alpha_{16} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & - & - & 4 \end{pmatrix}.$$

Observe that, there are 4 ways to choose $\operatorname{Im}\alpha$, for each of these there are 3 ways to choose $\operatorname{Dom}\alpha$. If $\operatorname{Im}\alpha = \{1\}$, suppose $(0, g_1, g_2, g_3, g_4, \overline{\alpha_5}) \in K_4(G)^0$ then by using Corollary 5.2.4, $g_1 = 1$, and as $2 \in \operatorname{Dom}\alpha_5$ there are 2 choices for g_2, which are either 1 or, a. Further by the definition of $K_4(G)^0$ we have $g_3 = g_4 = 0$, as $3, 4 \notin \operatorname{Dom}\alpha_5$. We deduce there are 2 idempotents in $K_4(G)^0$ of the form $(0, g_1, g_2, g_3, g_4, \overline{\alpha_5})$ as follows:

$$(0, 1, 1, 0, 0, \overline{\alpha_5}),\ (0, 1, a, 0, 0, \overline{\alpha_5}).$$

Similarly, if we have idempotent of the form

$$(0, g_1, g_2, g_3, g_4, \overline{\alpha_6}) \quad \text{or,} \quad (0, g_1, g_2, g_3, g_4, \overline{\alpha_7}).$$

So, we have 6 idempotents in $K_4(G)^0$ if $k = 1$, $l = 2$ and $\operatorname{Im}\alpha = \{1\}$. As we have 4 ways to choose $\operatorname{Im}\alpha$ we deduce there are $4 \times 6 = 24$ idempotents in $K_4(G)^0$ in this case.

Case (iii) If $l = |\operatorname{Dom}\alpha| = 3$.

There are 4 ways to choose $\operatorname{Im}\alpha$, for each of these there are 3 ways to choose $\operatorname{Dom}\alpha$. In each way of choosing $\operatorname{Dom}\alpha$ there are 4 idempotents in $K_4(G)^0$ of the form $(0, g_1, g_2, g_3, g_4, \overline{\alpha})$ such that by using Corollary 5.2.4, $g_j = 1$ where $j \in \operatorname{Im}\alpha$, and by the definition of $K_4(G)^0$ we have $g_i = 0$ for all $i \notin \operatorname{Dom}\alpha$. We obtain there are $4 \times 3 = 12$ idempotents in $K_4(G)^0$ in each way of choosing $\operatorname{Im}\alpha$. As there are 4 ways to choose $\operatorname{Im}\alpha$ there are $4 \times 12 = 48$ idempotent in $K_4(G)^0$ in this case.

Case (iv) If $l = |\operatorname{Dom}\alpha| = 4$. There are 4 ways to choose $\operatorname{Im}\alpha$, for each of these there is only one way to choose $\operatorname{Dom}\alpha$. In each way of choosing $\operatorname{Im}\alpha$ there are 8 idempotents in $K_4(G)^0$ of the form $(0, g_1, g_2, g_3, g_4, \overline{\alpha})$ such that by using Corollary 5.2.4, $g_j = 1$ where $j \in \operatorname{Im}\alpha$, and by the definition of $K_4(G)^0$ we have $g_i = 0$ for all $i \notin \operatorname{Dom}\alpha$. As there are 4 ways to choose $\operatorname{Im}\alpha$ there are $4 \times 8 = 32$ idempotents in $K_4(G)^0$ in this case.

We put everything together to obtain $4 + 24 + 48 + 32 = 108$ idempotents in $K_4(G)^0$ corresponding to $k = 1$.

Case 2 If $k = |\operatorname{Im}\alpha| = 2$. In this case we have 3 cases for $l = |\operatorname{Dom}\alpha|$, which are either $l = 2, 3$ or, 4.

Case (i) If $l = |\operatorname{Dom}\alpha| = 2$. There are 6 ways to choose $\operatorname{Im}\alpha$, for each of these there is only one way to choose $\operatorname{Dom}\alpha$. In each way of choosing $\operatorname{Im}\alpha$ there is only one idempotent in $K_4(G)^0$ of the form $(0, g_1, g_2, g_3, g_4, \overline{\alpha})$ such that by using Corollary 5.2.4, $g_j = 1$ where $j \in \operatorname{Im}\alpha$, and by the definition of $K_4(G)^0$ we have $g_i = 0$ for all $i \notin \operatorname{Dom}\alpha$. As there are 6 ways to choose $\operatorname{Im}\alpha$ there are $6 \times 1 = 6$ idempotents in $K_4(G)^0$ in this case.

Case (ii) If $l = |\operatorname{Dom}\alpha| = 3$.

There are 6 ways to choose $\operatorname{Im}\alpha$, for each of these there are 4 ways to choose $\operatorname{Dom}\alpha$. In each way of choosing $\operatorname{Dom}\alpha$ there are 2 idempotents in $K_4(G)^0$ of the form $(0, g_1, g_2, g_3, g_4, \overline{\alpha})$ such that by using Corollary 5.2.4, $g_j = 1$ where $j \in \operatorname{Im}\alpha$, and by the definition of $K_4(G)^0$ we have $g_i = 0$ for all $i \notin \operatorname{Dom}\alpha$. We obtain there are $4 \times 2 = 8$ idempotents in $K_4(G)^0$ in each way of choosing $\operatorname{Im}\alpha$. As there are 6 ways to choose $\operatorname{Im}\alpha$ there are $6 \times 8 = 48$ idempotent in $K_4(G)^0$ in this case.

Case (iii) If $l = |\operatorname{Dom}\alpha| = 4$. There are 6 ways to choose $\operatorname{Im}\alpha$, for each of these there are 4 ways to choose $\operatorname{Dom}\alpha$. In each way of choosing $\operatorname{Dom}\alpha$ there are 4 idempotents in $K_4(G)^0$ of the form $(0, g_1, g_2, g_3, g_4, \overline{\alpha})$ such that by using Corollary 5.2.4, $g_j = 1$ where $j \in \operatorname{Im}\alpha$, and by the definition of $K_4(G)^0$ we have $g_i = 0$ for all $i \notin \operatorname{Dom}\alpha$. We obtain there are $4 \times 4 = 16$ idempotents in $K_4(G)^0$ in each way of choosing $\operatorname{Im}\alpha$. As there are 6 ways to choose $\operatorname{Im}\alpha$ there are $6 \times 16 = 96$ idempotent in $K_4(G)^0$ in this case.

We put everything together to obtain $6+48+96 = 150$ idempotents in $K_4(G)^0$ corresponding to $k = 2$.

Case 3 If $k = |\operatorname{Im}\alpha| = 3$. In this case we have 2 cases for $l = |\operatorname{Dom}\alpha|$, which are either $l = 3$ or, $l = 4$.

Case (i) If $l = |\operatorname{Dom}\alpha| = 3$.

There are 4 ways to choose $\operatorname{Im}\alpha$, for each of these there is only one way to choose $\operatorname{Dom}\alpha$. In each way of choosing $\operatorname{Im}\alpha$ there is only one idempotent in $K_4(G)^0$ of the form $(0, g_1, g_2, g_3, g_4, \overline{\alpha})$ such that by using Corollary 5.2.4, $g_j = 1$ where $j \in \operatorname{Im}\alpha$, and by the definition of $K_4(G)^0$ we have $g_i = 0$ for all $i \notin \operatorname{Dom}\alpha$. As there are 4 ways to choose $\operatorname{Im}\alpha$ there are $4 \times 1 = 4$ idempotents in $K_4(G)^0$ in this case.

Case (ii) If $l = |\operatorname{Dom}\alpha| = 4$. There are 4 ways to choose $\operatorname{Im}\alpha$, for each of these there are 3 ways to choose $\operatorname{Dom}\alpha$. In each way of choosing

Dom α there are 2 idempotents in $K_4(G)^0$ of the form $(0, g_1, g_2, g_3, g_4, \overline{\alpha})$ such that by using Corollary 5.2.4, $g_j = 1$ where $j \in \operatorname{Im}\alpha$, and by the definition of $K_4(G)^0$ we have $g_i = 0$ for all $i \notin \operatorname{Dom}\alpha$. We obtain there are $3 \times 2 = 6$ idempotents in $K_4(G)^0$ in each way of choosing $\operatorname{Im}\alpha$. As there are 4 ways to choose $\operatorname{Im}\alpha$ there are $4 \times 6 = 24$ idempotent in $K_4(G)^0$ in this case.

We put everything together to obtain $4 + 24 = 28$ idempotents in $K_4(G)^0$ corresponding to $k = 3$.

Case 4 If $k = |\operatorname{Im}\alpha| = 4$. In this case there is just one case for $l = |\operatorname{Dom}\alpha|$, which is $l = 4$.

There is only one idempotent element in $\mathcal{PT}_4$ having k, $l = 4$ and which is

$$I = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

In this case $\operatorname{Im} I = \{1, 2, 3, 4\}$ and $\operatorname{Dom}\alpha = \{1, 2, 3, 4\}$. From that we obtain only one idempotent in $K_4(G)^0$ of the form $(0, g_1, g_2, g_3, g_4, I_{n,0})$. By using Corollary 5.2.4, $g_j = 1$ where $j \in \operatorname{Im}\alpha$, we obtain only one idempotent in $K_4(G)^0$, which is $(0, 1, 1, 1, 1, I_{n,0}) \in K_4(G)^0$.

Case 5 If $k = |\operatorname{Im}\alpha| = 0$. In this case there is just one case for $l = |\operatorname{Dom}\alpha|$ which is $l = 0$.

There is one idempotent element in $\mathcal{PT}_4$ having k, $l = 0$ and which is

$$0 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ - & - & - & - \end{pmatrix}.$$

In this case $\operatorname{Im} 0 = \emptyset$ and $\operatorname{Dom} 0 = \emptyset$. By using the definition of $K_4(G)^0$ we have $g_i = 0$ for all $i \notin \operatorname{Dom}\alpha$. Hence we have only one idempotent in $K_4(G)^0$, which is $(0, 0, 0, 0, 0, c_0) \in K_4(G)^0$.

In order to count all idempotents element in $K_4(G)^0$ we must count all idempotent elements in five cases together to obtain $108 + 150 + 28 + 1 + 1 = 288$ idempotents in $K_4(G)^0$.

Notice that by using the formula in Corollary 5.2.5, the same result can be obtained.

5.2.2 Green's relations on $\mathcal{PT}_{\mathbf{F_n(G)}}$

In [17], Gould obtained results characterising Green's relations on End $\mathbf{A}$, where $\mathbf{A}$ is an independence algebra. In this section, we prove the corresponding results for $\mathcal{PT}_{\mathbf{A}}$, where $\mathbf{A}$ is $\mathbf{F_n(G)}$ and G is group. These results we present here are new, although whilst this thesis was under construction, the characterisation of Green's relations in $\mathcal{PT}_{\mathbf{F_n(G)}}$ presented here were extended to the case of $\mathcal{PT}_{\mathbf{A}}$ for an arbitrary independence algebra $\mathbf{A}$ and appeared in [43].

Let $F_n(G) = Gx_1 \dot{\bigcup} Gx_2 \dot{\bigcup} \cdots \dot{\bigcup} Gx_n$. Recall that, for a subalgebra $\mathbf{B}$ of $\mathbf{F_n(G)}$, we say that *rank of* $\boldsymbol{B}$, $\rho(\mathbf{B})$ is m, where $B = Gx_{i_1} \dot{\bigcup} Gx_{i_2} \dot{\bigcup} \cdots \dot{\bigcup} Gx_{i_m}$, and where $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ and $0 \leq m \leq n$. If $\alpha \in \mathcal{PT}_{\mathbf{F_n(G)}}$ we define $\rho(\alpha)$ to be $\rho(\operatorname{Im}\alpha)$.

Lemma 5.2.7. *For all α, $\beta \in \mathcal{PT}_{\mathbf{F_n(G)}}$, we have the following:*

(i) $\alpha \leq_{\mathcal{L}} \beta$ *if and only if* $\operatorname{Im}\alpha \subseteq \operatorname{Im}\beta$*;*

(ii) $\alpha \leq_{\mathcal{R}} \beta$ *if and only if* $\operatorname{Dom}\alpha \subseteq \operatorname{Dom}\beta$ *and* $\pi_\beta \subseteq \pi_\alpha$*;*

(iii) $\rho(\alpha\beta) \leq \rho(\alpha)$ *and* $\rho(\alpha\beta) \leq \rho(\beta)$*.*

Proof. (i) If $\alpha \leq_{\mathcal{L}} \beta$ in $\mathcal{PT}_{\mathbf{F_n(G)}}$, then $\alpha = \gamma\beta$ for some $\gamma \in \mathcal{PT}_{\mathbf{F_n(G)}}$. It follows that $\alpha \leq_{\mathcal{L}} \beta$ in $\mathcal{PT}_{F_n(G)}$ so that $\operatorname{Im}\alpha \subseteq \operatorname{Im}\beta$ by (i) Lemma 3.3.4.

Conversely, suppose that $\operatorname{Dom}\alpha = \dot{\bigcup}_{j\in J} Gx_{x_j}$ and $\operatorname{Im}\alpha \subseteq \operatorname{Im}\beta$. For each $j \in J$ pick $a_j \in \operatorname{Dom}\beta$ with $x_j\alpha = a_j\beta$. Define $\gamma \in \mathcal{PT}_{\mathbf{F_n(G)}}$ by $\operatorname{Dom}\gamma = \operatorname{Dom}\alpha$ and $x_j\gamma = a_j$ for all $j \in J$. Then $\operatorname{Im}\gamma \subseteq \operatorname{Dom}\beta$, so $\operatorname{Dom}\gamma\beta = \operatorname{Dom}\gamma = \operatorname{Dom}\alpha$, and for all $j \in J$, $x_j\gamma\beta = a_j\beta = x_j\alpha$. Hence $\alpha = \gamma\beta$ and $\alpha \leq_{\mathcal{L}} \beta$ in $\mathcal{PT}_{\mathbf{F_n(G)}}$.

(*ii*) If $\alpha \leq_{\mathcal{R}} \beta$ in $\mathcal{PT}_{\mathbf{F_n(G)}}$, then $\alpha = \beta\gamma$ for some $\gamma \in \mathcal{PT}_{\mathbf{F_n(G)}}$. It follows that $\alpha \leq_{\mathcal{R}} \beta$ in $\mathcal{PT}_{F_n(G)}$ and then by (*ii*) Lemma 3.3.4 we obtain $\operatorname{Dom}\alpha \subseteq \operatorname{Dom}\beta$ and $\pi_\beta \subseteq \pi_\alpha$.

Conversely, suppose $\operatorname{Dom}\alpha \subseteq \operatorname{Dom}\beta$ and $\pi_\beta \subseteq \pi_\alpha$. Let $U = (\operatorname{Dom}\alpha)\beta$ and define γ by $\operatorname{Dom}\gamma = U$ and for any $a\beta \in U$ with $a \in \operatorname{Dom}\alpha$, $(a\beta)\gamma = a\alpha$. Notice that if $a, a^{'} \in \operatorname{Dom}\alpha$ and $a\beta = a^{'}\beta$ then as $\pi_\beta \subseteq \pi_\alpha$ we have $a\alpha = a^{'}\alpha$, so that γ is well-defined. If $a\beta \in U$ with $a \in \operatorname{Dom}\alpha$ and $g \in G$, then $ag \in \operatorname{Dom}\alpha$ and $((a\beta)g)\gamma = (ag)\beta\gamma = ((ag)\beta)\gamma = (ag)\alpha = (a\alpha)g = ((a\beta)\gamma)g$, so that $\gamma \in \mathcal{PT}_{\mathbf{F_n(G)}}$. Certainly $\operatorname{Dom}\gamma \subseteq \operatorname{Im}\beta$. Let $c \in \operatorname{Dom}\beta\gamma$, so that $c \in \operatorname{Dom}\beta$ and $c\beta \in \operatorname{Dom}\gamma$. It follows that $c\beta = a\beta$ for some $a \in \operatorname{Dom}\alpha$. Thus $(a, c) \in \pi_\beta \subseteq \pi_\alpha$ so that as $a \in \operatorname{Dom}\alpha$ we have $c \in \operatorname{Dom}\alpha$. Hence $\operatorname{Dom}\beta\gamma \subseteq \operatorname{Dom}\alpha$ and clearly the converse is true by definition of γ. Thus $\operatorname{Dom}\alpha = \operatorname{Dom}\beta\gamma$ and then it is immediate that $\beta\gamma = \alpha$. Thus $\alpha \leq_{\mathcal{R}} \beta$ in $\mathcal{PT}_{\mathbf{F_n(G)}}$.

(*iii*) We claim for any $\tau, \kappa \in \mathcal{PT}_{\mathbf{F_n(G)}}$ that

$$\rho(\tau\kappa) \leq \rho(\kappa) \text{ and } \rho(\tau\kappa) \leq \rho(\tau).$$

Recall, $\rho(\tau) = \rho(\operatorname{Im}\tau)$ and let Im $\tau = \dot{\bigcup}_{y \in Y} Gy$, where $Y \subseteq \{x_1, \cdots, x_n\}$, so that $\rho(\tau) = |Y|$. Since Im $(\tau\kappa) = (\operatorname{Im}\tau)\kappa = (\dot{\bigcup}_{y \in Y} Gy)\kappa = \dot{\bigcup}_{y \in Y} G(y\kappa)$ this implies that $\rho(\tau\kappa) \leq |Y| = \rho(\tau)$. Now, $\rho(\kappa\tau) = \rho(\operatorname{Im}\kappa\tau) = \rho((\operatorname{Im}\kappa)\tau)$. As Im $\kappa\tau \subseteq$ Im τ then we have $\rho(\operatorname{Im}\kappa\tau) \leq \rho(\operatorname{Im}\tau)$ this implies that $\rho(\kappa\tau) \leq \rho(\tau)$.

□

Lemma 5.2.8. *For all α, $\beta \in \mathcal{PT}_{\mathbf{F_n(G)}}$, we have:*

(*i*) $\alpha \,\mathcal{L}\, \beta$ *if and only if* $Im\,\alpha = Im\,\beta$;

(*ii*) $\alpha \,\mathcal{R}\, \beta$ *if and only if* $Dom\,\alpha = Dom\,\beta$ *and* $Ker\,\alpha = Ker\,\beta$;

(*iii*) $\alpha \,\mathcal{H}\, \beta$ *if and only if* $Im\,\alpha = Im\,\beta$, $Dom\,\alpha = Dom\,\beta$ *and* $Ker\,\alpha = Ker\beta$;

(iv) $\alpha \mathcal{D} \beta$ *if and only if* $\rho(\alpha) = \rho(\beta)$*;*

(v) $\alpha \leq_{\mathcal{J}} \beta$ *if and only if* $\rho(\alpha) \leq \rho(\beta)$*;*

(vi) $\alpha \ \mathcal{J} \ \beta$ *if and only if* $\rho(\alpha) = \rho(\beta)$*;*

(vii) $\mathcal{D} = \mathcal{J}$*.*

Proof. (i) This follows immediately from Lemma 5.2.7.

(ii) From Lemma 5.2.7, $\alpha \mathcal{R} \beta$ if and only if $\mathrm{Dom}\,\alpha = \mathrm{Dom}\,\beta$ and $\pi_\alpha = \pi_\beta$. This gives

$$\mathrm{Ker}\,\alpha = \pi_\alpha \cap (\mathrm{Dom}\,\alpha \times \mathrm{Dom}\,\alpha) = \pi_\beta \cap (\mathrm{Dom}\,\beta \times \mathrm{Dom}\,\beta) = \mathrm{Ker}\,\beta.$$

Conversely, if Dom α = Dom β and Ker α = Ker β, then

$$\begin{aligned}\pi_\alpha &= \mathrm{Ker}\,\alpha \cup ((\mathbf{F_n}(\mathbf{G}) \setminus \mathrm{Dom}\,\alpha) \times (\mathbf{F_n}(\mathbf{G}) \setminus \mathrm{Dom}\,\alpha)) \\ &= \mathrm{Ker}\,\beta \cup ((\mathbf{F_n}(\mathbf{G}) \setminus \mathrm{Dom}\,\beta) \times (\mathbf{F_n}(\mathbf{G}) \setminus \mathrm{Dom}\,\beta)) \\ &= \pi_\beta.\end{aligned}$$

(iii) This follows immediately from (i) and (ii).

(iv) Suppose that $\alpha \ \mathcal{D} \ \beta$, so that there exists $\gamma \in \mathcal{PT}_{\mathbf{F_n(G)}}$ with $\alpha \ \mathcal{R} \ \gamma \ \mathcal{L} \ \beta$. From (i) $\mathrm{Im}\,\gamma = \mathrm{Im}\,\beta$ and from (ii) $\mathrm{Dom}\,\alpha = \mathrm{Dom}\,\gamma$ and $\mathrm{Ker}\,\alpha = \mathrm{Ker}\,\gamma$. By the Fundamental Theorem of Monomorphisms we have

$$\mathrm{Im}\,\alpha \cong \mathrm{Dom}\,\alpha/\mathrm{Ker}\,\alpha = \mathrm{Dom}\,\gamma/\mathrm{Ker}\,\gamma \cong \mathrm{Im}\,\gamma = \mathrm{Im}\,\beta,$$

so that $\rho(\alpha) = \rho(\beta)$.

Conversely, suppose that $\rho(\alpha) = \rho(\beta)$. Let $\mathrm{Im}\,\alpha = \dot{\bigcup}_{y \in Y} Gy$ and $\mathrm{Im}\,\beta = \dot{\bigcup}_{z \in Z} Gz$ for some $Y, Z \subseteq X_n$ with $|Y| = |Z| = \rho(\alpha) = \rho(\beta)$ and let $\tau : Y \to Z$ be a bijection.

Then τ lifts to an isomorphism $\overline{\tau} : \operatorname{Im}\alpha \longrightarrow \operatorname{Im}\beta$ given by $(gy)\overline{\tau} = g(y\tau)$.

We have $\operatorname{Dom}\alpha\overline{\tau} = \operatorname{Dom}\alpha$ and as $\overline{\tau}$ is a bijection, $\operatorname{Ker}\alpha\overline{\tau} = \operatorname{Ker}\alpha$, so that $\alpha \,\mathcal{R}\, \alpha\overline{\tau}$. On the other hand, $\operatorname{Im}\alpha\overline{\tau} = \operatorname{Im}\beta$ so that $\alpha\overline{\tau} \,\mathcal{L}\, \beta$. Hence $\alpha \,\mathcal{D}\, \beta$ as required.

(v) If $\alpha \leq_{\mathcal{J}} \beta$ then by Lemma 5.2.7 (iii), $\rho(\alpha) \leq \rho(\beta)$.

Conversely, suppose that $\rho(\alpha) \leq \rho(\beta)$ and let $\operatorname{Im}\alpha = \dot{\bigcup}_{y\in Y} Gy$ and $\operatorname{Im}\beta = \dot{\bigcup}_{z\in Z} Gz$ for some for some $Y, Z \subseteq X_n$ with $|Y| = \rho(\alpha) \leq \rho(\beta) = |Z|$. Then $Z = Y' \dot{\bigcup} Z'$ where $|Y'| = |Y|$. Now fix $y'_0 \in Y'$ and define $\tau : Z \to Y'$ by $y'\tau = y'$ for all $y' \in Y'$, and $z'\tau = y'_0$ for all $z' \in Z'$. Clearly τ lifts to a morphism $\overline{\tau} : \operatorname{Im}\beta \to \dot{\bigcup}_{y'\in Y'} Gy'$. Then $\rho(\alpha) = \rho(\beta\,\overline{\tau})$ so that $\alpha \,\mathcal{D}\, \beta\,\overline{\tau}$ and $\alpha \,\mathcal{J}\, \beta\,\overline{\tau} \leq_{\mathcal{J}} \beta$.

(vi) This follows from (v).

(vii) This is an immediate consequence of (iv) and (v).

□

5.2.3 Ideals of $\mathcal{PT}_{\mathbf{F_n(G)}}$

This subsection devoted to considering the ideals of $K_n(G)^0$ in terms of the ideals of $\overline{\mathcal{T}_{n,0}}$.

The following definition have already been defined in Chapter 1.

Definition 5.2.9. [25] A non-empty subset I of a semigroup S is called *left ideal* (*right ideal*) of S if $SI \subseteq I$ ($IS \subseteq I$), i.e., if for all $s \in S$ and $a \in I$ we have $sa \in I$ ($as \in I$). A subset I is called a (two-sided) ideal of S or simply an ideal provided that it is both a left and right ideal of S, i.e., $IS \cup SI \subseteq I$.

Lemma 5.2.10. *Let I be an ideal of $K_n(G)^0$ and*

$$I' = \{\alpha : \exists(0, g_1, \cdots, g_n, \alpha) \in I\}.$$

Then I' is an ideal of $\overline{\mathcal{T}_{n,0}}$.

Conversely, if J is an ideal of $\overline{\mathcal{T}_{n,0}}$, then putting

$$J' = \{(0, g_1, \cdots, g_n, \alpha) \in K_n(G)^0 : \alpha \in J\},$$

we have J' is an ideal of $K_n(G)^0$.

Proof. Let I be an ideal of $K_n(G)^0$. Let $\alpha \in I'$, so there exist $(0, g_1, \cdots, g_n, \alpha) \in I$.

Let $\beta \in \overline{\mathcal{T}_{n,0}}$, put $h_i = 0$ for all i with $i\beta = 0$ and $h_i = e$ else. Then, $(0, h_1, \cdots, h_n, \beta) \in K_n(G)^0$ and

$$(0, h_1, \cdots, h_n, \beta)(0, g_1, \cdots, g_n, \alpha),\ (0, g_1, \cdots, g_n, \alpha)(0, h_1, \cdots, h_n, \beta) \in I.$$

So $\alpha\beta$, $\beta\alpha \in I'$. Hence I' is an ideal of $\overline{\mathcal{T}_{n,0}}$.

Conversely, suppose J is an ideal of $\overline{\mathcal{T}_{n,0}}$. Let $(0, g_1, \cdots, g_n, \alpha) \in J'$ and $(0, h_1, \cdots, h_n, \beta) \in K_n(G)^0$. Then

$$(0, g_1, \cdots, g_n, \alpha)(0, h_1, \cdots, h_n, \beta) = (0, g_1 h_{1\alpha}, \cdots, g_n h_{n\alpha}, \alpha\beta).$$

As $\alpha \in J$ and J is an ideal of $\overline{\mathcal{T}_{n,0}}$, we obtain $\alpha\beta \in J$. Hence,

$$(0, g_1 h_{1\alpha}, \cdots, g_n h_{n\alpha}, \alpha\beta) \in J'.$$

Also,

$$(0, h_1, \cdots, h_n, \beta)(0, g_1, \cdots, g_n, \alpha) = (0, h_1 g_{1\beta}, \cdots, h_n g_{n\beta}, \beta\alpha)$$

and as $\alpha \in J$ and J is an ideal of $\overline{\mathcal{T}_{n,0}}$, we obtain $\beta\alpha \in J$. Hence,

$$(0, h_1 g_{1\beta}, \cdots, h_n g_{n\beta}, \beta\alpha) \in J'.$$

□

Observe that Lemma 5.2.10 works for 1-sided ideal.

Lemma 5.2.11. *Let I be a left ideal of $K_n(G)^0$, then $(0, g_1, \cdots, g_n, \alpha) \in I$ if and only if $(0, g'_1, \cdots, g'_n, \alpha) \in I$ for any $g'_i \in G^0$ with $g'_i = 0$ if and only if $i\alpha = 0$.*

Proof. Let $(0, g_1, \cdots, g_n, \alpha) \in I$. Let $\varepsilon \in \overline{\mathcal{T}_{n,0}}$ such that

$$i\varepsilon = \begin{cases} i & \text{for all } i \text{ such that } i\alpha \neq 0; \\ 0 & \text{else.} \end{cases}$$

Let $g'_1, \cdots, g'_n \in G^0$ be as given. Then $(0, g'_1 g_1^{-1}, \cdots, g'_n g_n^{-1}, \varepsilon) \in K_n(G)^0$, where $0^{-1} = 0$, and as I is a left ideal of $K_n(G)^0$ we obtain

$$(0, g'_1 g_1^{-1}, \cdots, g'_n g_n^{-1}, \varepsilon)(0, g_1, \cdots, g_n, \alpha) = (0, g'_1 g_1^{-1} g_{1\varepsilon}, \cdots, g'_n g_n^{-1} g_{n\varepsilon}, \varepsilon\alpha).$$

Now, if $i\alpha \neq 0$ we have $g_i \neq 0$ and $i\varepsilon = i$, so $g'_i g_i^{-1} g_{i\varepsilon} = g'_i$, and if $i\alpha = 0$ we will have $g'_i = 0 = g'_i g_i^{-1} g_{i\varepsilon}$. Hence,

$$(0, g'_1 g_1^{-1}, \cdots, g'_n g_n^{-1}, \varepsilon)(0, g_1, \cdots, g_n, \alpha) = (0, g'_1, \cdots, g'_n, \alpha) \in I.$$

□

Notice that, Lemma 5.2.11 does not appear to hold for right or two-sided ideals of $K_n(G)^0$.

5.3 Nilpotents in $\mathcal{PT}_{\mathbf{F_n(G)}}$

In this section we compile the necessary background information for graph theory. We start by providing brief accounts of the basic ideas on graphs, trees, rooted trees and labelled rooted trees. We explain how Cayley's theorem be used to count the number of labelled forest on n nodes which has k labelled rooted trees. The final subsection of this section is devoted to count the number of nilpotents in $\mathcal{PT}_{\mathbf{A}}$, where $\mathbf{A}$ is $\mathbf{F_n(G)}$ and G is a group.

5.3.1 Graphs

Definition 5.3.1. [22, 23] A *Graph* G of order v consists of a finite non-empty set $V = V(G)$ of v vertices together with a prescribed set E of q unordered pairs of distinct vertices of V.

A pair $e = \{x, y\}$ of vertices in E is called an *edge* of G and e is said to *join* x and y. We write $e = xy$ and say that x and y are *adjacent* vertices; the vertex x and edge e are *incident* with each other, as are y and e. If two distinct edges $e, e^{'}$ are incident with a common vertex, then they are *adjacent edges.* A graph with v vertices and q edges is called a (p, q) *graph.* The $(1, 0)$ graph is trivial.

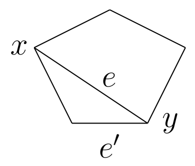


Figure 5.1: Figure G

Remark 5.3.2. [22, 23]

(i) A graph G is *labelled* when the v vertices are distinguished by names such as a, b, c, d, e as an example below:

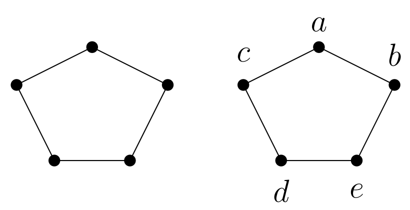


Figure 5.2: Unlabelled graph (left) and labelled graph (right)

(ii) Two labelled graphs G_1 and G_2 are considered the same and called *isomorphic* if and only if there is a one to one map from $V(G_1)$ onto $V(G_2)$ which preserves not only adjacency but also the labelling.

(iii) A *walk* of a graph G is a sequence of vertices $v_0, v_1, \cdots, v_n$ such that v_i is adjacent to v_{i+1} for each i.

(iv) A *closed walk* is a walk such that $v_0 = v_n$, and is open otherwise.

(v) A *cycle* is a closed walk where none of the vertices repeat except for the first and the last (i.e., $i \neq j \Rightarrow v_i \neq v_j$ except when $(i, j) = (0, k)$).

(vi) A graph is *connected* if for all pairs of vertices (v_i, v_j) there exist a walk that begins at v_i and ends at v_j.

(vii) The *degree* of a vertex v_i is the number of edges incident with v_i (i.e., the number of edges attached to it).

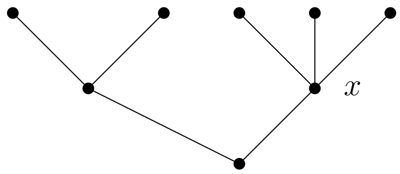


Figure 5.3: The graph of the vertex x with degree 4, denoted by $d(x) = 4$

(viii) The *complete graph* is a graph has every pair of its v vertices adjacent.

(ix) A graph is *acyclic* if it has no cycles.

Definition 5.3.3. [22, 23] A *tree* is a connected acyclic graph. Every non-trivial trees has at least two endpoints of degree 1.

Observe that the last graph is a typical example for the tree.

Definition 5.3.4. [22, 23] Any graph without cycles is a *forest*, which means the components of a forest are trees.

Definition 5.3.5. [22, 23] A *branch* at a vertex v of a tree is a maximal subtree containing v as an endpoint.

Definition 5.3.6. [22, 23] A *rooted tree* is a tree has one vertex which has been distinguished from the other vertices to be a root.

It is clear that all trees whose roots have degree n can be formed from a collection of n rooted trees by joining them by one new point, such that the new point, which will form the new root, is adjacent to each of the roots of the n given rooted trees.

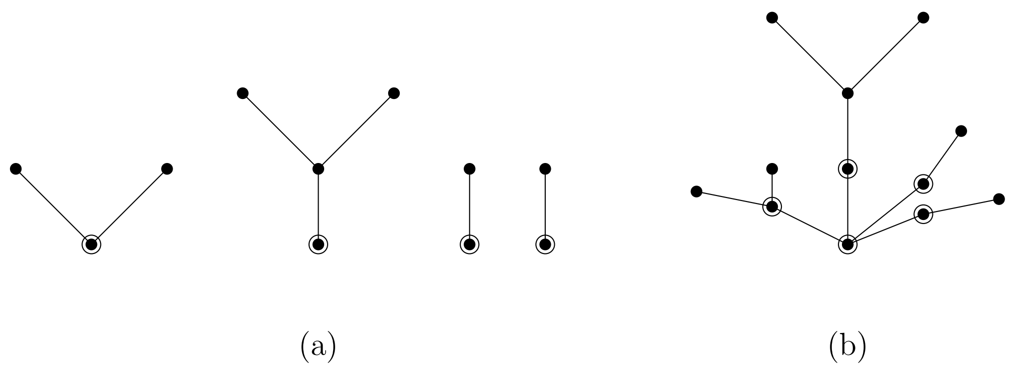


Figure 5.4: Four rooted trees and the corresponding tree whose root has degree 4.

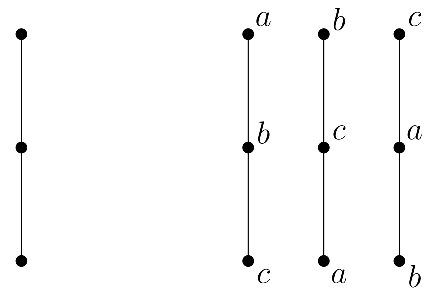
Observe that, in our work we consider labelled rooted trees. Despite, there is only one tree of order 3 if the tree is unlabelled, however in the sense of labelled trees, there are 3 different labelled trees obtaining by marking the inner vertex a, b and c. This can be shown by using *Cayley's* formula.

Theorem 5.3.7. (Cayley)[22, 23] *There are n^{n-2} different labelled trees on n vertices.*

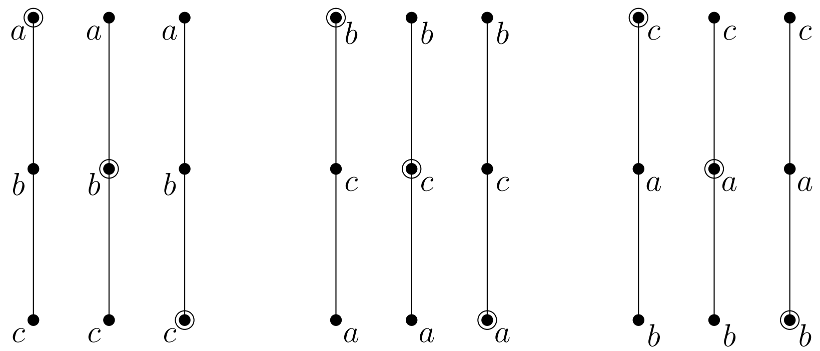
Cayley's formula shows us how many different labelled trees we can construct on n vertices. Moreover, in the concept of labelled trees it is easily to observe that any node in a tree can be considered as a root. Hence we can find the number of labelled rooted trees on n nodes by using the following theorem:

Theorem 5.3.8. [40] *The number of labelled rooted trees on n nodes is n^{n-1}.*

Therefore, there are 9 labelled rooted trees can be obtained from the 3 different labelled trees when we consider the nodes a, b or c as a root respectively, which can be shown in the following figure:



Unlabelled tree   3 different labelled trees



9 different labelled rooted trees

Figure 5.5: Three different types of trees.

In order to determine the number of forests that have n labelled nodes of which k have been designated as roots, we need the following generalization of Cayley's formula:

Theorem 5.3.9. [40] *Given n labelled nodes of which k are designated as roots, the number of forests of k rooted trees that can be formed on these nodes is $k\,n^{n-k-1}$.*

From the above theorem one can be deduce that Cayley's formula is a special case follows from the case $k = 1$.

5.3.2 Nilpotents

In this subsection we explain how to count the number of nilpotents of $\mathcal{PT}_n$ through some examples.

Definition 5.3.10. [15] An element a of a semigroup S with zero 0 is called *nilpotent* or *nilelement* provided that $a^n = 0$ for some $n \in \mathbb{N}$. The minimal n for which $a^n = 0$ is called the *nilpotency degree* or *nilpotency class* of the element a and is denoted by $nd(a)$.

Proposition 5.3.11. [15] *An element $\alpha \in \mathcal{PT}_n$ is nilpotent if and only if the graph of α does not contain cycles.*

An example for the graph of $\alpha \in \mathcal{PT}_n$ can be shown in any of the examples at the end of this subsection.

Proposition 5.3.12. [15] *The nilpotency degree of a nilpotent element $\alpha \in \mathcal{PT}_n$ equals the length of the longest trajectory in the graph of α.*

From the above proposition we deduce that the trajectory of each vertex a in the graph of α must break at some vertex, if α is nilpotent.

Suppose that $N_1(n,s) = s\,n^{n-s-1}$, which represents the number of labelled forests with s designated roots. Notice that, s also represents number of elements not in $\operatorname{Dom}\alpha$.

The formula to count the number of nilpotents in the semigroup $\mathcal{PT}_n$ has been found in [15]. In the following corollary we found our formula to count the number of nilpotents in the semigroup $\mathcal{PT}_n$.

Corollary 5.3.13. *The number $N(n,s)$ of nilpotents in the semigroup $\mathcal{PT}_n$ equals*

$$N(n,s) = \sum_{s=0}^{n} \binom{n}{s} N_1(n,s).$$

Proof. Suppose that $|\operatorname{Dom}\alpha| = l$, where $0 \leq l < n$. Now let $s = n - |\operatorname{Dom}\alpha|$, where $0 \leq s \leq n$, be the number of elements not in $\operatorname{Dom}\alpha$.

We know that the graph of the elements of $\alpha \in \mathcal{PT}_n$ will be a forest and a forest is a disjoint union of trees, and the number of trees will be the number of elements not in $\operatorname{Dom}\alpha$. So there are $\binom{n}{s}$ choices for $s = n - |\operatorname{Dom}\alpha|$, hence we have

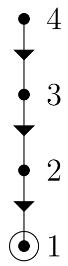
$\binom{n}{s} N_1(n, s)$ nilpotents of $\alpha \in \mathcal{PT}_n$ with s elements not in $\operatorname{Dom}\alpha$. By using the sum rule the proof will be complete. □

Let us have the following examples for $\alpha \in \mathcal{PT}_n$ such that α has one element not in $\operatorname{Dom}\alpha$:

Example 5.3.14. Let $\alpha \in \mathcal{PT}_4$ be

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ - & 1 & 2 & 3 \end{pmatrix}.$$

This has the following graph



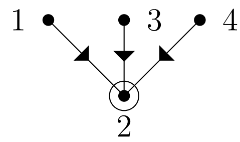
It is obvious that every vertex in the graph of α has unique trajectory and it is break at some vertex. Hence by Propositions 5.3.11 and 5.3.12, α is nilpotent and $\alpha^4 = 0$.

From the above example we deduced that the graph of α is a labelled rooted tree, which rooted at the vertex 1. Furthermore, as $\alpha \in \mathcal{PT}_4$ so we obtain $1 \notin \operatorname{Dom}\alpha$, and then we have one labelled rooted tree, which labelled at the vertex 1. This means, the number of labelled roots equals to the number of elements not in $\operatorname{Dom}\alpha$.

Example 5.3.15. Let $\alpha \in \mathcal{PT}_4$ be

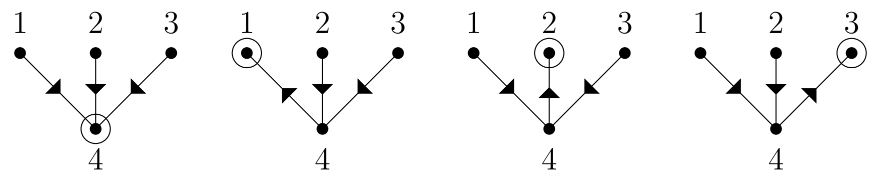
$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & - & 2 & 2 \end{pmatrix}.$$

This has the following labelled rooted tree:



We have just one element not in $\operatorname{Dom}\alpha$ which is 2, this means we have just one labelled rooted tree which is rooted at 2. By Propositions 5.3.11 and 5.3.12, α is nilpotent and $\alpha^2 = 0$.

Since any vertex of the tree can be considered as a root, therefore, for any new root we have a new nilpotent in $\mathcal{PT}_n$ which has one element not in $\operatorname{Dom}\alpha$. Hence, if $\alpha \in \mathcal{PT}_4$, such that α has one element not in $\operatorname{Dom}\alpha$ we will obtain the following labelled rooted trees:



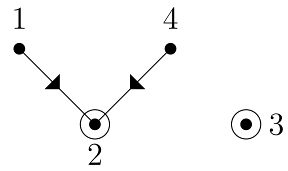
It is clear that by using Theorem 5.3.9 there are $1 \cdot 4^{4-1-1} = 16$ forests of which has one rooted tree. Moreover, by using Corollary 5.3.13 we will obtain $\binom{4}{1} \cdot 1 \cdot 4^{4-1-1} = 64$ nilpotents in $\mathcal{PT}_4$ having one element not in $\operatorname{Dom}\alpha$.

Now consider the following example where $\alpha \in \mathcal{PT}_4$ which has two elements not in $\operatorname{Dom}\alpha$.

Example 5.3.16. Take $\alpha \in \mathcal{PT}_4$ to be

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & - & - & 2 \end{pmatrix}.$$

This has the following labelled rooted trees:



Clearly, there are two distinct labelled trees rooted at 2 and 3, since 2 and 3 are not in $\operatorname{Dom}\alpha$. By Propositions 5.3.11 and 5.3.12, α is nilpotent and $\alpha^2 = 0$.

Since any nodes in any tree can be considered as a root, hence at any new root we could obtain a new nilpotent has two elements not in $\operatorname{Dom}\alpha$.

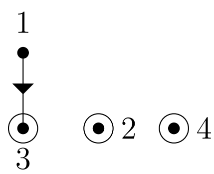
By using Theorem 5.3.9, the number of forests which has two rooted trees is $2 \cdot 4^{4-2-1} = 8$ and then the number of nilpotents in $\mathcal{PT}_4$ having two elements not in $\operatorname{Dom}\alpha$ is $\binom{4}{2} \cdot 2 \cdot 4^{4-2-1}$=$6 \cdot 8 = 48$.

In case there are three elements not in $\operatorname{Dom}\alpha$, we have the following example:

Example 5.3.17. We take an element $\alpha \in \mathcal{PT}_4$ to be

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & - & - & - \end{pmatrix},$$

its graph will be as follows:



It is obvious that α has three elements not in $\operatorname{Dom}\alpha$, which are $2, 3$ and 4. Hence, we have three distinct labelled rooted trees which rooted at $2, 3$ and 4. By Propositions 5.3.11 and 5.3.12, α is nilpotent and $\alpha^2 = 0$.

Since any vertex in any tree can be considered as a root, this means at any new root we could obtain a new distinct nilpotent in $\mathcal{PT}_4$ has three elements not in

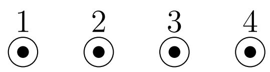
Dom α. By using Theorem 5.3.9, the number of forests which has three rooted trees is $3 \cdot 4^{4-3-1} = 3$ and then the number of nilpotents in $\mathcal{PT}_4$ have three elements not in Dom α is $\binom{4}{3} \cdot 3 \cdot 4^{4-3-1} = 12$.

The following example explain the case where $\alpha \in \mathcal{PT}_4$ such that there are no elements in Dom α which means Dom $\alpha = \emptyset$ (the empty set). In other words, where α is the zero element in $\mathcal{PT}_4$ and we call it $\alpha = 0$.

Example 5.3.18. Let $\alpha \in \mathcal{PT}_4$ such that

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ - & - & - & - \end{pmatrix},$$

this has graph as follows:



It is clear that we have four labelled rooted tree, since there are four elements not in Dom α. By Proposition 5.3.11 and Proposition 5.3.12, α is nilpotent and $\alpha^1 = 0$. By using Theorem 5.3.9, the number of forest which has four rooted trees is $4 \cdot 4^{4-4-1} = 1$, so we have just one forest that has 4 distinct rooted trees. Further the number of nilpotent in $\mathcal{PT}_4$ having no elements in Dom α is $\binom{4}{4} \cdot 4 \cdot 4^{4-4-1} = 1$.

5.3.3 Number of Nilpotent in $\mathcal{PT}_{\mathbf{F_n(G)}}$

This subsection is devoted to finding the number of nilpotents in $\mathcal{PT}_{\mathbf{F_n(G)}}$ where G is a group.

We know from Subsection 5.2.1 that

$$K_n(G)^0 = \{(0, g_1, \cdots, g_n, \alpha) : i\alpha = 0 \text{ if and only if } g_i = 0$$
$$\text{where } 1 \leq i \leq n \text{ and } \alpha \in \overline{\mathcal{T}_{n,0}}\},$$

and $(0, \cdots, 0, c_0)$ is the only element of $K_n(G)^0$ with final co-ordinate c_0 and $(0, \cdots, 0, c_0)$ is the zero of $K_n(G)^0$. Notice that $(0, \cdots, 0, c_0) \in N\left(K_n(G)^0\right)$ (the set of the nilpotent elements in $K_n(G)^0$).

Lemma 5.3.19. *Let G^0 be G with a 0 adjoined, and $A = (0, g_1, \cdots, g_n, \alpha)$ be an element in $K_n(G)^0$. Then A is nilpotent if and only if $\alpha^k = c_0$ (the constant map with image 0 of $\overline{\mathcal{T}_{n,0}}$) for some $k \geq 1$, that is, if and only if α is nilpotent in $\overline{\mathcal{T}_{n,0}}$.*

Proof. Suppose that $A = (0, g_1, \cdots, g_n, \alpha)$ is nilpotent, then $A^k = (0, \cdots, 0, \alpha^k)$. As $(0, \cdots, 0, c_0)$ is the zero of $K_n(G)^0$, then we have $\alpha^k = c_0$, so α is nilpotent in $\overline{\mathcal{T}_{n,0}}$.

Conversely, if $\alpha \in \overline{\mathcal{T}_{n,0}}$ is nilpotent then $\alpha^k = c_0$. Then $A^k = (0, h_1, \cdots, h_n, \alpha^k) = (0, h_1, \cdots, h_n, c_0) \in K_n(G)^0$. As the only element of $K_n(G)^0$ with final co-ordinate c_0 is $(0, \cdots, 0, c_0)$. Hence, $A^k = (0, \cdots, 0, c_0)$ as required. □

Corollary 5.3.20. *Let $G \neq \{e\}$. The number $N(n, G)$ of nilpotents in $K_n(G)^0$ equals*

$$N\left(K_n(G)^0\right) = \sum_{s=0}^{n} \binom{n}{s} N_1(n, s)\, |G|^{n-s},$$

where $N_1(n, s) = s\, n^{n-s-1}$.

Proof. We count the possibilities for $A = (0, g_1, \cdots, g_n, \overline{\alpha}) \in K_n(G)^0$, where $\overline{\alpha} \in \overline{\mathcal{T}_{n,0}}$ and $\alpha \in \mathcal{PT}_n$ is defined in Lemma 3.3.9, to be nilpotent.

To count the possibilities for $A^m = (0, \cdots, 0, \overline{\alpha}^m)$ for some $m \in \mathbb{N}$, suppose that $|\mathrm{Dom}\, \alpha| = l$, where $0 \leq l < n$, be the number of elements in $\mathrm{Dom}\, \alpha$. Now let $s = n - |\mathrm{Dom}\, \alpha|$, where $0 \leq s \leq n$, be the number of elements in not $\mathrm{Dom}\, \alpha$.

In virtue of Corollary 5.3.13, we obtain there are $\binom{n}{s} N_1(n, s)$ nilpotents of $\mathcal{PT}_n$ with s elements not in $\mathrm{Dom}\, \alpha$. By definition of $K_n(G)^0$ we have $g_i = 0$ for all $i \notin \mathrm{Dom}\, \alpha$. Thus there are $|G|^{n-s}$ choices for the remaining g_i's and then we will obtain $\binom{n}{s} N_1(n, s)\, |G|^{n-s}$ nilpotents in $K_n(G)^0$. By using the sum rule the proof will be complete. □

From the above Corollary one can deduce that Corollary 5.3.13 is a special case follows from the case $G = \{e\}$.

Example 5.3.21. Let $G = \{1, a\}$. We want to count the numbers of nilpotent in $K_4(G)^0$, (recall that if $A \in K_4(G)^0$ means $A = (0, g_1, g_2, g_3, g_4, \overline{\alpha})$, where $g_1, \cdots, g_4 \in G^0$ and $\overline{\alpha} \in \overline{\mathcal{T}_{4,0}}$).

In this example we consider $\overline{\alpha} \in \overline{\mathcal{T}_{4,0}}$.

Since $n = 4$, therefore, we have 4 cases to find the number of nilpotent in $K_4(G)^0$.

Case 1 If $\overline{\alpha} \in \overline{\mathcal{T}_{4,0}}$ such that $\overline{\alpha}$ has one element not in $\operatorname{Dom}\overline{\alpha}$. Via Examples 5.3.14 and 5.3.15 the number of nilpotents in $\overline{\mathcal{T}_{4,0}}$ having one element not in $\operatorname{Dom}\overline{\alpha}$ is 64 elements. By definition of $K_n(G)^0$ we have $g_i = 0$ for all $i \notin \operatorname{Dom}\alpha$. Thus there are $|G|^{4-1}$ choices for the remaining g_i's. So we have 512 nilpotents in $K_4(G)^0$ where there is one element not in $\operatorname{Dom}\overline{\alpha}$.

Case 2 If $\overline{\alpha} \in \overline{\mathcal{T}_{4,0}}$ such that $\overline{\alpha}$ has two elements not in $\operatorname{Dom}\overline{\alpha}$. We know from Example 5.3.16 that the number of nilpotents in $\overline{\mathcal{T}_{4,0}}$ having two elements not in $\operatorname{Dom}\overline{\alpha}$ is 48 elements. By definition of $K_4(G)^0$ we have $g_i = 0$ for all $i \notin \operatorname{Dom}\alpha$. Thus there are $|G|^{4-2}$ choices for the remaining g_i's. Therefore, we have 192 nilpotents in $K_4(G)^0$ where there are two elements not in $\operatorname{Dom}\overline{\alpha}$.

Case 3 If $\overline{\alpha} \in \overline{\mathcal{T}_{4,0}}$ such that $\overline{\alpha}$ has three elements not in $\operatorname{Dom}\overline{\alpha}$. By Example 5.3.17, the number of nilpotents in $\overline{\mathcal{T}_{4,0}}$ having three elements not in $\operatorname{Dom}\overline{\alpha}$ is 12 elements. By definition of $K_4(G)^0$ we have $g_i = 0$ for all $i \notin \operatorname{Dom}\alpha$. There are $|G|^{4-3}$ choices for the remaining g_i's. So we obtain 24 nilpotents in $K_4(G)^0$ where there are three elements not in $\operatorname{Dom}\overline{\alpha}$.

Case 4 If $\overline{\alpha} \in \overline{\mathcal{T}_{4,0}}$ such that $\overline{\alpha}$ has four elements not in $\operatorname{Dom}\overline{\alpha}$. We know from Example 5.3.18 that the number of nilpotents in $\overline{\mathcal{T}_{4,0}}$ having four elements not in $\operatorname{Dom}\overline{\alpha}$ is 1 element. By definition of $K_4(G)^0$ we have $g_i = 0$ for all $i \notin \operatorname{Dom}\alpha$. So there are $|G|^{4-4}$ choices for the remaining g_i's. This means we have 1 nilpotent in $K_4(G)^0$ where there are four elements not in $\operatorname{Dom}\overline{\alpha}$.

In order to count out all nilpotent elements in $K_4(G)^0$ we must count all nilpotent elements in four cases together to get 512+192+24+1=729.

Notice that by using the formula in Corollary 5.3.20 we get the same result.

Chapter 6

Presentations of certain subsemigroups of semidirect products

This chapter is devoted to finding presentations for certain subsemigroups of semidirect products.

In the first section we define for a monoid M and for the full transformation semigroup $\mathcal{T}_n$, the wreath product $M \wr_n \mathcal{T}_n$ to be the semidirect product $M^n \rtimes \mathcal{T}_n$ with the coordinatewise action of $\mathcal{T}_n$ on M^n. (Recall we defined the wreath product in Chapter 4).

We find a monoid presentation for $M^n \rtimes \mathcal{T}_n$ from a presentation of M^n and $\mathcal{T}_n$ by using Lavers' technique [27]. We give a general presentation for a semidirect product $M \rtimes S$ which allows us to find a number of presentations for $M \wr_n \mathrm{Sing}_n$. In the case where $M \wr_n \mathrm{Sing}_n$ is idempotent generated, we give a presentation in terms of a particularly natural idempotent generating set: these results are taken from the joint paper [14], to which I contributed in small part. We find a monoid presentation for $(N^1 \wr_n \{1_n\}) \sqcup (M \wr_n \mathrm{Sing}_n)$ where M is a monoid, G is a group of units of M and $N = M \setminus G$ is an ideal of M; this is a minor adjustment of a known result [14]. Finally, we suppose M and T are monoids such that M is a left T-act by endomorphisms and G and H are the groups of units of M and T, respectively. In addition, we suppose $N = M \setminus G$ and $S = T \setminus H$ are ideals of M and T, respectively, with N and G are left S-acts. Then a monoid presentation for $(N^1 \rtimes \{1\}) \sqcup (M \rtimes S)$ is obtained.

We have seen in Chapter 1 that presentations are a means of defining semigroups as homomorphic images of free semigroups. The main motivation for studying semigroup presentations is that they allow us to study certain classes of semigroups in terms of efficient sets of data which are nevertheless sufficient to encode the semigroup operation. This approach includes finite presentations for infinite semigroups, and presentations of semigroups in terms of presentations of constituent parts having an already better understood structure.

6.1 Presentations for semidirect product $M \rtimes S$

The aim of this section is to construct a presentation for $M^n \rtimes \mathcal{T}_n$ from a presentation of M^n and $\mathcal{T}_n$.

Recall from page 65 that, if M is a monoid and $\mathcal{T}_n$ is the full transformation monoid, we define the "wreath product" multiplication on $M^n \times \mathcal{T}_n$ by putting

$$(m_1, \cdots, m_n, \alpha)(h_1, \cdots, h_n, \beta) = (m_1 h_{1\alpha}, \cdots, m_n h_{n\alpha}, \alpha\beta).$$

Under this multiplication, $M^n \times \mathcal{T}_n$ becomes a monoid with identity $(1, \cdots, 1, I_n)$, where I_n is the identity transformation of $\mathcal{T}_n$. The set $M^n \times \mathcal{T}_n$, with the wreath product multiplication, is denoted by $M \wr_n \mathcal{T}_n$.

Any wreath product is a special kind of semidirect product. In this special case we now show this directly.

It is easy to prove that if M is a monoid with identity 1, then a wreath product $M \wr_n \mathcal{T}_n$, is a semidirect product of M^n by $\mathcal{T}_n$, which will be shown in the following lemma:

Lemma 6.1.1. *Let M be a monoid with identity* 1*, then* $M \wr_n \mathcal{T}_n = M^n \rtimes \mathcal{T}_n$.

Proof. Define a left action of $\mathcal{T}_n$ on M^n by $\alpha \cdot (m_1, \cdots, m_n) = (m_{1\alpha}, \cdots, m_{n\alpha})$.

Then we shall prove that

(i) $I_n \cdot (m_1, \cdots, m_n) = (m_1, \cdots, m_n)$;

(ii) $\alpha \cdot (1, \cdots, 1) = (1, \cdots, 1)$;

(iii) $\alpha \cdot (\beta \cdot (m_1, \cdots, m_n)) = \alpha\beta \cdot (m_1, \cdots, m_n)$;

(iv) $\alpha \cdot ((m_1, \cdots, m_n)(h_1, \cdots, h_n)) = (\alpha \cdot (m_1, \cdots, m_n))\,(\alpha \cdot (h_1, \cdots, h_n))$.

for all m_i, $h_i \in M, i \in \{1, \cdots, n\}$ and $\alpha, \beta \in \mathcal{T}_n$.

(i) and (ii) are straightforward.

(iii) As $\alpha \cdot (m_1, \cdots, m_n) = (m_{1\alpha}, \cdots, m_{n\alpha})$, we have

$$
\begin{aligned}
\alpha \cdot (\beta \cdot (m_1, \cdots, m_n)) &= \alpha\,(m_{1\beta}, \cdots, m_{n\beta}) \\
&= \alpha \cdot (h_1, \cdots, h_n) \quad (\text{letting } \ h_i = m_{i\beta} \ \text{ for all } \ i \in \{1, \cdots, n\}) \\
&= (h_{1\alpha}, \cdots, h_{n\alpha}).
\end{aligned}
$$

Let $j = i\alpha$, then $h_{i\alpha} = h_j = m_{j\beta} = m_{i\alpha\beta}$, whence

$$\alpha \cdot (\beta \cdot (m_1, \cdots, m_n)) = (h_{1\alpha}, \cdots, h_{n\alpha}) = (m_{1\alpha\beta}, \cdots, m_{n\alpha\beta}) = \alpha\beta \cdot (m_1, \cdots, m_n).$$

(iv) Here we have

$$
\begin{aligned}
\alpha \cdot ((m_1, \cdots, m_n)(h_1, \cdots, h_n)) &= \alpha \cdot (m_1 h_1, \cdots, m_n h_n) \\
&= \alpha \cdot (v_1, \cdots, v_n) \quad (\text{letting } v_i = m_i h_i \text{ for all } i) \\
&= (v_{1\alpha}, \cdots, v_{n\alpha}) \\
&= (m_{1\alpha} h_{1\alpha}, \cdots, m_{n\alpha} h_{n\alpha}) \\
&= (m_{1\alpha}, \cdots, m_{n\alpha})(h_{1\alpha}, \cdots, h_{n\alpha}) \\
&= (\alpha \cdot (m_1, \cdots, m_n)) (\alpha \cdot (h_1, \cdots, h_n)) \,.
\end{aligned}
$$

We have shown in (iv) that $\mathcal{T}_n$ acts on M^n by endomorphisms, hence we form the semidirect product semigroup $M^n \rtimes \mathcal{T}_n$. Clearly $M \wr_n \mathcal{T}_n = M^n \rtimes \mathcal{T}_n$ as defined earlier. □

6.1.1 Presentation for $\mathcal{S}_n$

The following two propositions state the first two presentations for $\mathcal{S}_n$, which were discovered in 1897 by Moore; see also Coxeter and Moser 1980 [8].

Proposition 6.1.2. [38] *The presentation*

$$\langle a,\, b \,|\, a^2 = b^n = (ba)^{n-1} = (b^{-1}ab)^3 = (ab^{-j}ab^j)^2 = 1 \ (2 \le j \le n-2) \rangle$$

defines $\mathcal{S}_n$ *in terms of generators* (12) *and* $(12 \cdots n)$.

In the next proposition we will consider the larger set $(12), (23), \cdots, (n-1\ n)$ of generators:

Proposition 6.1.3. [38] *The presentation*

$$\langle a_1, \cdots, a_{n-1} \,|\, a_i^2 = (a_j a_{j+1})^3 = (a_k a_l)^2 = 1$$
$$(1 \le i \le n-1,\ 1 \le j \le n-2,\ 1 \le k \le l-2 \le n-3) \rangle$$

defines $\mathcal{S}_n$ *in terms of generators* $(12), (23), \cdots, (n-1\ n)$.

6.1.2 Presentation for $\mathcal{T}_n$

The first presentation for the full transformation semigroup $\mathcal{T}_n$ was given in 1958 by Aizenshtat [1], as follows:

Proposition 6.1.4. [1, 38] *Assume that* $\langle a,\, b \,|\, R\rangle$ *is any semigroup presentation for the symmetric group* $\mathcal{S}_n$ *in terms of generators* $\alpha = (12)$ *and* $\beta = (12\cdots n)$. *Then the presentation*

$$\Big\langle a,\, b,\, t \big|\, at = b^{-2}ab^2tb^{-2}ab^2 = bab^{-1}abtb^{-1}abab^{-1} = (tbab^{-1})^2 = t,$$
$$(b^{-1}abt)^2 = tb^{-1}abt = (tb^{-1}ab)^2,\, (tbab^{-2}ab)^2 = (bab^{-2}abt)^2\Big\rangle$$

defines the full transformation semigroup $\mathcal{T}_n$ *in terms of generators* α, β *and*

$$\tau = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 1 & 3 & \cdots & n \end{pmatrix}.$$

Remark 6.1.5. As $\mathcal{T}_n$ has the above presentation, there exist an epimorphism $\psi : \{a, b, t\}^+ \to \mathcal{T}_n$ defined by $a\psi = \alpha$, $b\psi = \beta$ and $t\psi = \tau$ where $\alpha = (12)$ and $\beta = (12\cdots n)$.

6.1.3 Presentation for M^n and for $M^n \rtimes \{1_n\} = M \wr_n \{1_n\}$

This subsection is devoted to finding the presentation of the monoid M^n, where M is a monoid and has a monoid presentation $\langle K : W\rangle$.

The following lemma is already known and can be found in [30] and [41].

Lemma 6.1.6. *Let* M *be a monoid which has a monoid presentation* $\langle K : W\rangle$.

Then M^n has a monoid presentation $\langle H : R \rangle$ where

$$H = \{\tau_{i,k} : 1 \leq i \leq n, k \in K\}$$

and $R = R_1 \cup R_2$ where

$$R_1 = \{\tau_{i,g}\tau_{j,h} = \tau_{j,h}\tau_{i,g} : 1 \leq i, j \leq n, i \neq j\},$$

and

$$R_2 = \{\omega(\tau_{i,k_1}, \cdots, \tau_{i,k_r}) = \nu(\tau_{i,k_1}, \cdots, \tau_{i,k_r}) : \omega(k_1, \cdots, k_r) = \nu(k_1, \cdots, k_r) \in W\}.$$

We specialise Lemma 6.1.6 to standard presentation for the monoid M as follows:

Corollary 6.1.7. *Let M be a monoid which has a monoid presentation $\left\langle \overline{M} : W \right\rangle$, where*

$$\overline{M} = \{\overline{m} : m \in M\}$$

and

$$W = \{\overline{m}\,\overline{n} = \overline{mn} : \overline{m}, \overline{n} \in \overline{M}\}.$$

Then M^n has a monoid presentation $\left\langle \overline{H} : \overline{R} \right\rangle$ where

$$\overline{H} = \{\tau_{i,\overline{m}} : 1 \leq i \leq n, \overline{m} \in \overline{M}\}$$

and $\overline{R} = \overline{R_1} \cup \overline{R_2}$ where

$$\overline{R_1} = \{\tau_{i,\overline{m}}\tau_{j,\overline{n}} = \tau_{j,\overline{n}}\tau_{i,\overline{m}} : 1 \leq i, j \leq n, i \neq j\},$$

$$\overline{R_2} = \{\tau_{i,\overline{m}}\tau_{i,\overline{n}} = \tau_{i,\overline{mn}} : 1 \leq i \leq n\}.$$

In order to find a presentation for $M^n \rtimes \mathcal{T}_n$, we will utilize T.G. Lavers's [27] technique, which allows us to find a presentation for the semidirect product of two monoids, and is presented in the following lemma:

Lemma 6.1.8. [27, Corollary 2] *Let S and T be monoids. If $S \cong \langle A : Z \rangle$ and $T \cong \langle B : Q \rangle$ via ψ and φ, respectively, and a semidirect product $S \rtimes T$ exists, then*

$$S \rtimes T \cong \langle A \cup B \,:\, Z \cup Q \cup \{(ba,\, (b \cdot a)b) : a \in A,\, b \in B\} \rangle ,$$

where $b \cdot a$ is a representation in A^* of $b\varphi \cdot a\psi$.

Proposition 6.1.9. *The monoid $M^n \rtimes \mathcal{T}_n$ has presentation*

$$\langle H \cup \{a, b, t\} \,:\, R \cup P \cup T \rangle ,$$

where R represents the set of relations in the presentation for M^n, P represents the set of relations in the presentation for $\mathcal{T}_n$ and T is the set of relations:

$(a\,\tau_{1,m}, \tau_{2,m}a),\ (a\,\tau_{2,m}, \tau_{1,m})$	*where*	$m \in M$	(T1a)
$(a\,\tau_{i,m}, \tau_{i,m}a)$	*where*	$i \geq 3,\ m \in M$	(T1b)
$(b\,\tau_{i,m}, \tau_{i-1,m}b)$	*where*	$1 \leq i \leq n-1,\ m \in M$	(T2a)
$(b\,\tau_{n,m}, \tau_{1,m}b)$	*where*	$i = n,\ m \in M$	(T2b)
$(t\,\tau_{1,m}, \tau_{1,m}\,\tau_{2,m}t),\ (t\,\tau_{2,m}, t)$	*where*	$m \in M$	(T3a)
$(t\,\tau_{i,m}, \tau_{i,m}t)$	*where*	$i \geq 3,\ m \in M.$	(T3b)

Proof. We know from Lemma 6.1.6 that M^n has presentation $\langle H : R \rangle$ via

$$\varphi : H^* \to M^n : \tau_{i,m} \mapsto (1, \cdots, 1, \underset{i\text{-th place}}{m}, 1, \cdots, 1)$$

and from Proposition 6.1.4 that $\mathcal{T}_n$ has presentation $\langle \{a, b, t\} : P \rangle$ via

$$\gamma : \{a, b, t\}^+ \to \mathcal{T}_n : a \mapsto \alpha,\ b \mapsto \beta \text{ and } t \mapsto \tau.$$

Hence, by using Lemma 6.1.8 we obtain

$$M^n \rtimes \mathcal{T}_n \cong \langle H \cup \{a, b, t\} \,:\, R \cup P \cup \{(c\tau_{i,m}, (c \cdot \tau_{i,m})c) : \, c \in \{a, b, t\}, 1 \leq i \leq n, \, m \in M\} \rangle \,.$$

It remains to prove that the relation

$$\{(c\tau_{i,m}, (c \cdot \tau_{i,m})c) : \, c \in \{a, b, t\}, 1 \leq i \leq n, \, m \in M\}$$

corresponds to the relation T.

For $i \geq 3, \; a \cdot \tau_{i,m}$ is chosen to be an element of H^* such that

$$a\gamma \cdot (1, \cdots, 1, \underset{i\text{-th place}}{m}, 1, \cdots, 1) = (a \cdot \tau_{i,m})\varphi.$$

Now

$$a\gamma \cdot (1, \cdots, 1, \underset{i\text{-thplace}}{m}, 1, \cdots, 1) = (1, \cdots, 1, \underset{i\text{-th place}}{m}, 1, \cdots, 1),$$

so we can take $a \cdot \tau_{i,m}$ to be $\tau_{i,m}$. This gives the identities (T1b). By using the same technique we can complete the proof.

□

6.1.4 Presentation for Sing_n

Recall that $\mathrm{Sing}_n = \mathcal{T}_n \setminus \mathcal{S}_n$ is the *singular part* of $\mathcal{T}_n$, which consists of all non-invertible (i.e., *singular*) transformation on $X_n = \{1, \cdots, n\}$, where $n \geq 0$.

Where $\mathcal{X} = \{\varepsilon_{ij} : i, j \in X_n, i \neq j\}$, we already have the following result in Chapter 3.

Theorem 6.1.10. [24, Theorem I] *If* $n \geq 2$, *then* $\mathrm{Sing}_n = \langle \mathcal{X} \rangle$.

A presentation for Sing_n was given in [11], in terms of generating set $\mathcal{X}$. Define an alphabet

$$X = \{e_{ij} : i, j \in X_n, i \neq j\},$$

an epimorphism

$$\phi : X^+ \to \mathrm{Sing}_n : e_{ij} \mapsto \varepsilon_{ij},$$

and let R be the set of relations

$$e_{ij}^2 = e_{ij} = e_{ji}e_{ij} \qquad \text{for distinct } i,\, j \tag{R1}$$

$$e_{ij}e_{kl} = e_{kl}e_{ij} \qquad \text{for distinct } i,\, j,\, k,\, l \tag{R2}$$

$$e_{ik}e_{jk} = e_{ik} \qquad \text{for distinct } i,\, j,\, k \tag{R3}$$

$$e_{ij}e_{ik} = e_{ik}e_{ij} = e_{jk}e_{ij} \qquad \text{for distinct } i,\, j,\, k \tag{R4}$$

$$e_{ki}e_{ij}e_{jk} = e_{ik}e_{kj}e_{ji}e_{ik} \qquad \text{for distinct } i,\, j,\, k \tag{R5}$$

$$e_{ki}e_{ij}e_{jk}e_{kl} = e_{ik}e_{kl}e_{li}e_{ij}e_{jl} \qquad \text{for distinct } i,\, j,\, k,\, l. \tag{R6}$$

Theorem 6.1.11. [11, Theorem 6] *For $n \geq 2$, the semigroup Sing_n has presentation $\langle X : R \rangle$ via ϕ.*

6.2 Presentation for semidirect product $M \rtimes S$ and singular wreath product $M \wr_n \mathrm{Sing}_n$

We recall that the main motivation for studying semigroup presentations is that they allow us to study certain classes of semigroups in terms of efficient sets of data which are nevertheless sufficient to encode the semigroup operation.

A general presentation for the endomorphism monoid End **A** of an arbitrary independence algebra **A** is not currently known. But a presentation for a special subclass of algebras, like the endomorphism monoid of a free G-act of finite rank can be described using result of Lavers [27] on general product of monoids, since we know these endomorphism monoids are isomorphic to wreath products of the form $G \wr_n \mathcal{T}_n$. In this section we are interested in the more general problem of finding presentations for wreath products $M \wr_n \mathrm{Sing}_n$. To achieve our aim of finding presentations for wreath products $M \wr_n \mathrm{Sing}_n$, we first prove general results on presentations for arbitrary semidirect products $M \rtimes S$ where M is a monoid and S a semigroup. From these we are able to deduce a number of presentations for $M \wr_n \mathrm{Sing}_n$ that extend the presentations for Sing_n. These results are taken from the joint paper [14]; I do not give proofs, since I am not presenting them as mine, but rather to set the scene for what follows in the next section.

6.2.1 Presentation for semidirect product $M \rtimes S$

Let S be a semigroup and M a monoid with identity 1. (Observe that S might also be a monoid). Suppose S has a left action on M by monoid endomorphisms. Recall that the *semidirect product* $M \rtimes S$ has the underlying set $M \times S = \{(a, s) : a \in M,\ s \in S\}$, and product defined by

$$(a, s)(b,\ t) = (a(s\ \cdot\ b),\ st) \qquad \text{for all } s,\ t \in S \text{ and } a,\ b \in M.$$

Now, suppose that S has presentation $\langle X : R\rangle$ via $\phi : X^+ \to S$. Define an alphabet $X_M = \{x_a : x \in X,\ a \in M\}$. We regard X as a subset of X_M by identifying $x \in X$ with $x_1 \in X_M$. For a word $\omega = x_1 \cdots x_k \in X^+$, and for an element $a \in M$, we define the word $\omega_a = (x_1)_a x_2 \cdots x_k \in X_M^+$. Consider the set $R_M = R_M^1 \cup R_M^2$ of relations over X_M, where R_M^1 and R_M^2 are defined by

$$R_M^1 = \{(u_a,\ v_a) : (u,\ v) \in R,\ a \in M\}$$

and

$$R_M^2 = \{(x_a y_b,\ x_{a(x\phi \cdot b)} y) : x,\ y \in X,\ a,\ b \in M\}.$$

Notice that as $X \subseteq X_M$, we also have $R \subseteq R_M$, via $(u,\ v) = (u_1,\ v_1)$. Define a map

$$\phi_M : X_M^+ \to M \rtimes S \quad \text{by} \quad x_a\phi_M = (a,\ x\phi) \quad \text{for all } x \in X \text{ and } a \in M.$$

It easy to show that $\omega_a\phi_M = (a, \omega\phi)$ for all $a \in M$ and $\omega \in X^+$. Also from the surjectivity of $\phi : X^+ \to S$, one can prove that ϕ_M is surjective. Indeed, let $(b, z) \in M \rtimes S$ where $b \in M$, $z \in S$. As ϕ is surjective, $z = \omega\phi$ for some $\omega \in X^+$. Hence, we have

$$(\omega)_b\phi_M = (b, \omega\phi) = (b, z).$$

Theorem 6.2.1. [14, Theorem 3.1] *With the above notation, $M \rtimes S$ has a semigroup presentation $\langle X_M : R_M\rangle$ via ϕ_M.*

Proof. Let $\approx_M$ be the congruence on X_M^+ generated by the relation R_M. We showed above that ϕ_M is surjective, so we just need to show that $\operatorname{Ker}\phi_M = \approx_M$. First note that for any $(u, v) \in R$ and $a \in M$, $u_a\phi_M = (a, u\phi) = (a, v\phi) = v_a\phi_M$, while for any $x, y \in X$ and $a, b \in M$, $(x_a y_b)\phi_M = (a, x\phi)(b, y\phi) = (a(x\phi \cdot b), (xy)\phi) = (a(x\phi \cdot b), x\phi)(1, y\phi) = (x_{a(x\phi\cdot b)}y)\phi_M$, showing that $\approx_M \subseteq \operatorname{Ker}\phi_M$.

Conversely, suppose $u = (x_1)_{a_1} \cdots (x_k)_{a_k}$, $v = (y_1)_{b_1} \cdots (y_l)_{b_l} \in X_M^+$ are such that $u\phi_M = v\phi_M$. Using relations from R_M^2, we have

$$u \approx_M (x_1)_a x_2 \cdots x_k = (x_1 \cdots x_k)_a$$

and

$$v \approx_M (y_1)_b y_2 \cdots y_l = (y_1 \cdots y_l)_b$$

for some $a, b \in M$. Since $\approx_M \subseteq \operatorname{Ker}\phi_M$, we have

$$(a, (x_1 \cdots x_k)\phi) = (x_1 \cdots x_k)_a\phi_M = u\phi_M = v\phi_M = (y_1 \cdots y_l)_b\phi_M = (b, (y_1 \cdots y_l)\phi).$$

It follows that $a = b$ and $(x_1 \cdots x_k)\phi = (y_1 \cdots y_l)\phi$. Let $\approx$ be the congruence on X^+ generated by the relation R. Since $\operatorname{Ker}\phi = \approx$, it follows that there is a sequence of words $x_1 \cdots x_k = \omega_0, \omega_1, \cdots, \omega_r = y_1 \cdots y_l$ such that, for each $0 \le i \le r - 1$, $\omega_i = \omega_i' u\,\omega_i''$ and $\omega_{i+1} = \omega_i' v\,\omega_i''$ for some $\omega_i', \omega_i'' \in X^*$ and $(u, v) \in R$. But then we see that $(\omega_i)_a \approx_M (\omega_{i+1})_a$, using either $(u, v) \in R \subseteq R_M$ (if ω_i' is non-empty) or

$(u_a, v_a) \in R_M$ (if ω_i' is empty). But then

$$u \approx_M (x_1 \cdots x_k)_a = (\omega_0)_a \approx_M (\omega_1)_a \approx_M \cdots \approx_M (\omega_r)_a = (y_1 \cdots y_l)_a = (y_1 \cdots y_l)_b \approx_M v,$$

completing the proof.

□

The next Corollary follows directly from Theorem 6.2.1.

Corollary 6.2.2. *If S is finitely presented and M is finite, then $M \rtimes S$ is finitely presented.*

We remark here that since $M \wr_n S = M^n \rtimes S$ is a semidirect product, Theorem 6.2.1 leads to a general presentation for $M \wr_n S$, modulo a presentation $\langle X : R \rangle$ for S, but in this subsection we will not state this explicitly, and this will be the subject of the next subsection.

6.2.2 Presentation for $M \wr_n \mathbf{Sing}_n$

A motivating example of a semidirect product is that of a *wreath product*, the subject of the current subsection. The main topic of this subsection is to find the presentation for the singular wreath product $M \wr_n \mathrm{Sing}_n$. The results are taken from [14]; we do not present the proofs or indicate that they are part of this thesis, rather we state them here as a guide, since we will be following a similar pattern in the next subsection.

Let S be a subsemigroup of the full transformation semigroup $\mathcal{T}_n$, and let M be an arbitrary monoid. Recall that S has a natural left action on M^n (the direct product of n copies of M) given by

$$\alpha \cdot (a_1, \cdots, a_n) = (a_{1\alpha}, \cdots, a_{n\alpha}) \qquad \text{for } \alpha \in S \text{ and } a_1, \cdots, a_n \in M.$$

For $i, j \in X_n = \{1, \cdots, n\}$ with $i \neq j$, and for $\mathbf{a} \in M^n$, we define $\varepsilon_{ij;\mathbf{a}} =$

$(\mathbf{a}, \varepsilon_{ij}) \in M \wr_n \mathrm{Sing}_n$. As a special case, for $a, b \in M$, we define

$$\varepsilon_{ij;ab} = \varepsilon_{ij;\mathbf{a}} \qquad \text{where } \mathbf{a} \in M^n \text{ is defined by } a_k = \begin{cases} a & \text{if} \quad k = i \\ b & \text{if} \quad k = j \\ 1 & \text{otherwise.} \end{cases}$$

As a special case of the latter, we define $\varepsilon_{ij;a} = \varepsilon_{ij;1a}$, for $a \in M$. We gather these elements into the sets

$$\begin{aligned} \mathcal{X}_n &= \{\varepsilon_{ij;\mathbf{a}} : i,\, j \in X_n,\, i \neq j,\, \mathbf{a} \in M^n\}, \\ \mathcal{X}_2 &= \{\varepsilon_{ij;ab} : i,\, j \in X_n,\, i \neq j,\, a,\, b \in M\}, \\ \mathcal{X}_1 &= \{\varepsilon_{ij;a} : i,\, j \in X_n,\, i \neq j,\, a \in M\}. \end{aligned}$$

Remark here that, we identify $\varepsilon_{ij} \in \mathrm{Sing}_n$ with $\varepsilon_{ij;1} \in M \wr_n \mathrm{Sing}_n$. Therefore, we have $\mathcal{X} \subseteq \mathcal{X}_1 \subseteq \mathcal{X}_2 \subseteq \mathcal{X}_n$.

As we mentioned before, since $M \wr_n \mathrm{Sing}_n = M^n \rtimes \mathrm{Sing}_n$ is a semidirect product, Theorem 6.2.1 allows us to write down a presentation for $M \wr_n \mathrm{Sing}_n$ in terms of the presentation $\langle X : R \rangle$ for Sing_n from Theorem 6.1.11. To state this presentation, let us define an alphabet

$$X_n = \{e_{ij;\mathbf{a}} : i,\, j \in X_n,\, i \neq j,\, \mathbf{a} \in M^n\},$$

an epimorphism

$$\phi_n : X_n^+ \to M \wr_n \mathrm{Sing}_n : e_{ij;\mathbf{a}} \mapsto \varepsilon_{ij;\mathbf{a}},$$

and let R_n be the set of relations (identifying a letter $e_{ij} \in X$ with $e_{ij;(1,\cdots,1)} \in X_n$)

$$e_{ij;\mathbf{a}}e_{ij} = e_{ij;\mathbf{a}} = e_{ji;\mathbf{a}}e_{ij} \qquad \text{for } \mathbf{a} \in M^n \text{ and for distinct } i,\ j \qquad ((\mathrm{R}n)1)$$

$$e_{ij;\mathbf{a}}e_{kl} = e_{kl;\mathbf{a}}e_{ij} \qquad \text{for } \mathbf{a} \in M^n \text{ and for distinct } i,\ j,\ k,\ l \qquad ((\mathrm{R}n)2)$$

$$e_{ik;\mathbf{a}}e_{jk} = e_{ik;\mathbf{a}} \qquad \text{for } \mathbf{a} \in M^n \text{ and for distinct } i,\ j,\ k \qquad ((\mathrm{R}n)3)$$

$$e_{ij;\mathbf{a}}e_{ik} = e_{ik;\mathbf{a}}e_{ij} = e_{jk;\mathbf{a}}e_{ij} \qquad \text{for } \mathbf{a} \in M^n \text{ and for distinct } i,\ j,\ k \qquad ((\mathrm{R}n)4)$$

$$e_{ki;\mathbf{a}}e_{ij}e_{jk} = e_{ik;\mathbf{a}}e_{kj}e_{ji}e_{ik} \qquad \text{for } \mathbf{a} \in M^n \text{ and for distinct } i,\ j,\ k \qquad ((\mathrm{R}n)5)$$

$$e_{ki;\mathbf{a}}e_{ij}e_{jk}e_{kl} = e_{ik;\mathbf{a}}e_{kl}e_{li}e_{ij}e_{jl} \qquad \text{for } \mathbf{a} \in M^n \text{ and for distinct } i,\ j,\ k,\ l \qquad ((\mathrm{R}n)6)$$

$$e_{ij;\mathbf{a}}e_{kl;\mathbf{b}} = e_{ij;\mathbf{c}}e_{kl} \qquad \text{for } \mathbf{a},\ \mathbf{b} \in M^n \text{ and any } i,\ j,\ k,\ l, \qquad ((\mathrm{R}n)7)$$

where in (Rn)7, $\mathbf{c} = \mathbf{a}(\varepsilon_{ij} \cdot \mathbf{b}) = (c_1, \cdots, c_n)$ satisfies $c_j = a_j b_i$ and $c_k = a_k b_k$ for $k \neq j$. Notice that i, j, k, l are not assumed to be distinct (apart from $i \neq j$ and $k \neq l$) in (Rn)7.

Obviously, the identities (Rn)1 ,$\cdots$, (Rn)6 come from the relation R^1_M in Theorem 6.2.1, because we can find $(u_a, v_a) \in \mathrm{R}n$ such that $(u, v) \in \mathrm{R}$ (where R represents the relations in Theorem 6.1.11). Whilst the relation (Rn)7 can be obtain from relation R^2_M in Theorem 6.2.1 by replacing $x, y, a, b, a(x\phi \cdot b)$ by $e_{ij}, e_{kl}, \mathbf{a}, \mathbf{b}, \mathbf{c}$, respectively. Hence, the next corollary is special case of Theorem 6.2.1.

Corollary 6.2.3. [14, Corollary 5.1] *The semigroup $M \wr_n \mathrm{Sing}_n$ has presentation $\langle X_n : R_n \rangle$ via ϕ_n.*

As the presentation $\langle X_n : R_n \rangle$ utilises the large generating set $\mathcal{X}_n$, in order to simplify this presentation we will use the smaller generating set $\mathcal{X}_2 \subseteq \mathcal{X}_n$. For this define an alphabet

$$X_2 = \{e_{ij;ab} : i,\, j \in X_n,\, i \neq j,\, a,\, b \in M^n\},$$

an epimorphism

$$\phi_2 : X_2^+ \to M \wr_n \mathrm{Sing}_n : e_{ij;ab} \mapsto \varepsilon_{ij;ab},$$

and let R_2 be the set of relations

$$e_{ij;ab}e_{ij;cd} = e_{ij;ac,\,bc} = e_{ji;ba}e_{ij;dc} \qquad \text{for } a,\, b,\, c,\, d \in M \text{ and distinct } i,\, j \qquad \text{((R2)1)}$$

$$e_{ij;ab}e_{kl;cd} = e_{kl;cd}e_{ij;ab} \qquad \text{for } a,\, b,\, c,\, d \in M \text{ and distinct } i,\, j,\, k,\, l \qquad \text{((R2)2)}$$

$$e_{ik;ab}e_{jk;1c} = e_{ik;ab} \qquad \text{for } a,\, b,\, c \in M \text{ and distinct } i,\, j,\, k \qquad \text{((R2)3a)}$$

$$e_{ik;ab}e_{jk;c1} = e_{ki;ba}e_{ji;c1}e_{ik;11} \qquad \text{for } a,\, b,\, c \in M \text{ and distinct } i,\, j,\, k \qquad \text{((R2)3b)}$$

$$e_{ik;aa}e_{jk;b1} = e_{ik;11}e_{jk;b1}e_{ik;a1} \qquad \text{for } a,\, b, \in M \text{ and distinct } i,\, j,\, k \qquad \text{((R2)3c)}$$

$$e_{ij;ab}e_{ik;cd} = e_{ik;ac,\,d}e_{ij;1,\,bc} = e_{jk;bc,\,d}e_{ij;ac,\,1} \qquad \text{for } a,\, b,\, c,\, d \in M \text{ and distinct } i,\, j,\, k \qquad \text{((R2)4a)}$$

$$e_{ij;c,\,ad}e_{ik;1,\,bd} = e_{ik;c,\,bd}e_{ij;1,\,ad} = e_{jk;ab}e_{ij;cd} \qquad \text{for } a,\, b,\, c,\, d \in M \text{ and distinct } i,\, j,\, k \qquad \text{((R2)4b)}$$

$$e_{ki}e_{ij}e_{jk} = e_{ik}e_{kj}e_{ji}e_{ik} \qquad \text{for distinct } i,\, j,\, k \qquad \text{((R2)5)}$$

$$e_{ki}e_{ij}e_{jk}e_{kl} = e_{ik}e_{kl}e_{li}e_{ij}e_{jl} \qquad \text{for distinct } i,\, j,\, k,\, l. \qquad \text{((R2)6)}$$

Theorem 6.2.4. [14, Theorem 5.2] *The semigroup $M \wr_n \mathrm{Sing}_n$ has presentation*

$\langle X_2 : R_2 \rangle$ *via* ϕ_2.

Proof. See [14]. $\square$

Recall that, $\mathcal{X} \subseteq \mathcal{X}_1 \subseteq \mathcal{X}_2 \subseteq \mathcal{X}_n \subseteq M \wr_n \mathrm{Sing}_n$. Also recall that in Chapter 1 we defined the *Green's pre-orders* as follows: for $a, b \in S$

$$a \leq_{\mathcal{R}} b \Leftrightarrow aS^1 \subseteq bS^1, \;\; a \leq_{\mathcal{L}} b \Leftrightarrow S^1 a \subseteq S^1 b, \;\; a \leq_{\mathcal{J}} b \Leftrightarrow S^1\, a\, S^1 \subseteq S^1\, b\, S^1,$$

and the *Green's relations* as follows: for any $a, b \in S$

$$a \;\mathcal{R}\; b \Longleftrightarrow aS^1 = bS^1,\; a \;\mathcal{L}\; b \Longleftrightarrow S^1 a = S^1 b,\; a \;\mathcal{J}\; b \Longleftrightarrow S^1 a S^1 = S^1 b S^1.$$

Remark here that, if M is a monoid and $M/\mathcal{L}$ is the partially ordered set of all $\mathcal{L}$-classes, then

$$\begin{aligned} M/\mathcal{L} \text{ is a chain } &\Leftrightarrow \text{ the principal left ideals of } M \text{ form a chain under inclusion} \\ &\Leftrightarrow \text{ all finitely generated left ideals of } M \text{ are principal.} \end{aligned}$$

In the next theorem we re-prove a classical result of Bulman-Fleming [6] that $M \wr_n \mathrm{Sing}_n$ is idempotent generated if and only if the set $M/\mathcal{L}$ of $\mathcal{L}$-classes of M form a chain under the usual ordering of $\mathcal{L}$-classes. We denote $\delta = \{(i, i) : i \in X_n\}$, the diagonal equivalence relation.

Theorem 6.2.5. [14, Theorem 4.7] *If M is any monoid, then*

(*i*) $M \wr_n \mathrm{Sing}_n = \langle \mathcal{X}_n \rangle = \langle \mathcal{X}_2 \rangle$;

(*ii*) $\langle E(M \wr_n \mathrm{Sing}_n) \rangle = \langle \mathcal{X}_1 \rangle = \{(\mathbf{a}, \alpha) \in M \wr_n \mathrm{Sing}_n : a_i \leq_{\mathcal{L}} a_j$ *for some* $(i, j) \in Ker\,\alpha \setminus \delta\}$;

(*iii*) $M \wr_n \mathrm{Sing}_n$ *is idempotent generated if and only if* $M/\mathcal{L}$ *is a chain.*

Proof. See [14]. $\square$

From Theorem 6.2.5, we deduced that where M is a monoid and $M/\mathcal{L}$ is a chain, then the singular wreath product $M \wr_n \mathrm{Sing}_n$ is generated by its idempotents from the set $\mathcal{X}_1$. Hence our aim is to obtain a presentation for $M \wr_n \mathrm{Sing}_n$ in terms of the idempotent generating set $\mathcal{X}_1$.

With this in mind, define an alphabet

$$X_1 = \{e_{ij;a} : i,\, j \in X_n,\, i \neq j,\, a \in M\},$$

an epimorphism

$$\phi_1 : X_1^+ \to M \wr_n \mathrm{Sing}_n : e_{ij;a} \mapsto \varepsilon_{ij;a},$$

and let R_1 be the set of relations

$$e_{ij;a}e_{ij;b} = e_{ij;a} \quad \text{for } a,\, b \in M \text{ and distinct } i,\, j \tag{(R1)1a}$$

$$e_{ij;1}e_{ji;a}e_{ij;b} = e_{ji;1}e_{ij;ab} \quad \text{for } a,\, b \in M \text{ and distinct } i,\, j \tag{(R1)1b}$$

$$e_{ji;a}e_{ij;c} = e_{ji;b}e_{ij;c} \quad \text{for } a,\, b,\, c \in M \text{ and distinct } i,\, j \text{ with } ac = bc \tag{(R1)1c}$$

$$e_{ij;b}e_{ji;c}e_{ij;1} = e_{ji;a}e_{ij;bc} \quad \text{for } a,\, b,\, c \in M \text{ and distinct } i,\, j \text{ with } abc = c \tag{(R1)1d}$$

$$e_{ji;1}e_{ij;1} = e_{ij;1} \quad \text{for distinct } i,\, j \tag{(R1)1e}$$

$$e_{ij;a}e_{kl;b} = e_{kl;b}e_{ij;a} \quad \text{for } a,\, b \in M \text{ and distinct } i,\, j,\, k,\, l \tag{(R1)2}$$

$$e_{ik;a}e_{jk;b} = e_{ik;a} \quad \text{for } a,\, b \in M \text{ and distinct } i,\, j,\, k \tag{(R1)3a}$$

$$e_{ij;1}e_{jk;a}e_{kj;1} = e_{ji;1}e_{ik;a}e_{ki;1}e_{ij;1} \quad \text{for } a \in M \text{ and distinct } i,\, j,\, k \tag{(R1)3b}$$

$$e_{ij;1}e_{ji;a}e_{ik;b} = e_{ji;1}e_{ik;b}e_{kj;a}e_{jk;1} \quad \text{for } a,\, b \in M \text{ and distinct } i,\, j,\, k \tag{(R1)3c}$$

$$e_{ij;b}e_{ik;ab} = e_{ik;ab}e_{ij;b} = e_{jk;a}e_{ij;b} \quad \text{for } a,\, b \in M \text{ and distinct } i,\, j,\, k \tag{(R1)4}$$

$$e_{ki}e_{ij}e_{jk} = e_{ik}e_{kj}e_{ji}e_{ik} \quad \text{for distinct } i,\, j,\, k \tag{(R1)5}$$

$$e_{ki}e_{ij}e_{jk}e_{kl} = e_{ik}e_{kl}e_{li}e_{ij}e_{jl} \quad \text{for distinct } i,\, j,\, k,\, l. \tag{(R1)6}$$

Note that in relations ((R1)5) and ((R1)6) we have identified X with a subset of X_1. It is important to note that some of the relations above (such as ((R1)1b)) involve a letter in of the form $e_{ij;ab}$ from X_1, where "ab" denotes a single subscript (the product of a and b in M): in particular, $e_{ij;ab}$ does not represent the letter from X_2 where a and b are separate subscripts.

Theorem 6.2.6. [14, Theorem 5.9] *If $M/\mathcal{L}$ is a chain, then the semigroup $M \wr_n \mathrm{Sing}_n$ has presentation $\langle X_1 : R_1 \rangle$ via ϕ_1.*

Proof. See [14]. $\square$

The next Theorem is a special case of Theorem 6.2.6, where M is a group.

Theorem 6.2.7. [14, Theorem 5.12] *If M is a group, then $M \wr_n \mathrm{Sing}_n$ has presentation $\left\langle X_1 : R_1' \right\rangle$ via ϕ_1, where R_1' is obtained from relation R_1 by replacing* ((R1)1a)–((R1)1e) *by*

$$e_{ij;a}e_{ij;b} = e_{ij;a} = e_{ji;a^{-1}}e_{ij;a} \qquad \text{for } a,\, b \in M \text{ and distinct } i,\, j \quad ((\mathrm{R}'1)1a)$$

$$e_{ij;1}e_{ji;a}e_{ij;b} = e_{ji;1}e_{ij;ab} \qquad \text{for } a,\, b \in M \text{ and distinct } i,\, j. \quad ((\mathrm{R}'1)1b)$$

Proof. See [14]. $\square$

6.3 Presentations for $(N^1 \wr_n \{1_n\}) \sqcup (M \wr_n \mathbf{Sing}_n)$ and $(N^1 \rtimes \{1\}) \sqcup (M \rtimes S)$

Let M, T be monoids such that M is a left T-act by endomorphisms, and let G and H be the group of units of M and T, respectively. Suppose $N = M \setminus G$ and $S = T \setminus H$ are ideals of M and T, respectively.

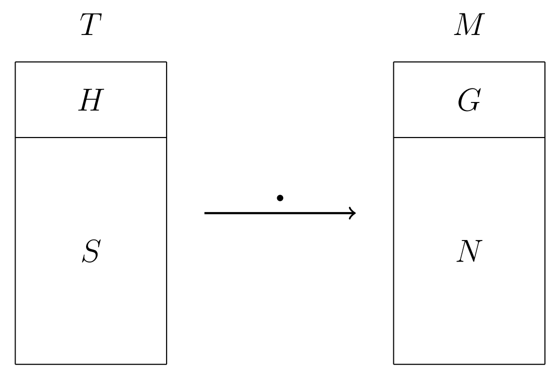


Figure 6.1: The left action of T on M.

In this section, we will write $A = B \sqcup C$ to indicate that A is the disjoint union of B and C, and to avoid confusion, we will write 1 and 1_n for the identity of M and $\mathcal{T}_n$, respectively.

This section is devoted to finding the monoid presentation for $(N^1 \rtimes \{1\}) \sqcup (M \rtimes S)$. We will start with a special case by finding presentation for $(N^1 \wr_n \{1_n\}) \sqcup (M \wr_n \mathrm{Sing}_n)$.

Lemma 6.3.1. *Suppose M, T, G, H, N and S as before. Let $W = M \rtimes T$, and let K be the group of units of W. Then*

(i) $K = \{(m, t) : m \in G, t \in H\}$;

(ii) $V = W \setminus K$ *is an ideal of* W;

(iii) $V_T = \{(m, t) : m \in M, t \in S\} = M \rtimes S$ *is an ideal of* W;

(iv) $V_M = \{(m, t) : m \in N, t \in T\} = N \rtimes T$ *is an ideal of* W;

(v) $V_M \cup V_T = V$;

(vi) $N^1 \rtimes \{1\}$ *is a submonoid of* W;

(vii) $(N^1 \rtimes \{1\}) \sqcup (M \rtimes S)$ *is a subsemigroup of* W;

(viii) $M \rtimes S = (N^1 \rtimes \{1\})(G \rtimes S)$.

Proof. (*i*) Suppose $Z = \{(m,t) : m \in G, t \in H\}$. We defined K to be the group of units of W. We have to show that $Z = K$. Let $(m,t) \in Z$. As G and H are the group of units of M and T respectively, so there exist $m^{-1} \in G$ and $t^{-1} \in H$. In order to prove $(m,t) \in K$, we have to show that $(m, t)(u, t^{-1}) = (1_M, 1_T)$ and $(u, t^{-1})(m,t) = (1_M, 1_T)$ for some $u \in G$. Set $u = t^{-1} \cdot m^{-1}$. Then

$$\begin{aligned}
(m,t)(u,t^{-1}) &= (m(t \cdot u), tt^{-1}) \\
&= (m(t \cdot (t^{-1} \cdot m^{-1})), tt^{-1}) \\
&= (m((tt^{-1}) \cdot m^{-1}), tt^{-1}) \qquad \text{(as } M \text{ is a left } T\text{-act)} \\
&= (m(1_T \cdot m^{-1}), tt^{-1}) \qquad \text{(as } M \text{ is a left } T\text{-act)} \\
&= (1_M, 1_T).
\end{aligned}$$

Also,

$$\begin{aligned}
(u,t^{-1})(m,t) &= (u(t^{-1} \cdot m), t^{-1}t) \\
&= ((t^{-1} \cdot m^{-1})((t^{-1} \cdot m)), t^{-1}t) \\
&= (t^{-1} \cdot (m^{-1}m), t^{-1}t) \qquad \text{(as } T \text{ acts by monoid endomorphisms)} \\
&= (t^{-1} \cdot 1_M, t^{-1}t) \\
&= (1_M, 1_T) \qquad \text{(as } T \text{ acts by monoid endomorphisms).}
\end{aligned}$$

Therefore, $(m,t) \in K$, which implies $Z \subseteq K$.

Now, let $(m,t) \in K$. As K is the group of units of W, there exist $(u,n) \in K$ such that $(m,t)(u,n) = (m(t \cdot u), tn) = (1_M, 1_T)$, which means $m(t \cdot u) = 1_M$, and $tn = 1_T$. If $m \notin G$, then $m \in N \setminus G$, which is an ideal, but then $m(t \cdot u) \in N \setminus G$, and this is a contradiction, as $m(t \cdot u) = 1_M$. Then $m \in G$. By the same manner we obtain $t \in H$. So $K = Z$, as required.

(*ii*) Let $(m,s) \in V$ and let $(n,t) \in W$. If $t \in S$ or, $s \in S$, then $(m,s)(n,t)$,

$(n,t)(m,s) \in V$ as st, $ts \in S$, since S is an ideal. Suppose now that $t, s \in H$. Then $m \in N$ and $(m,s)(n,t) = (m(s \cdot n), st) \in V$ as $m(s \cdot n) \in N$, since N is an ideal. Also $(n,t)(m,s) = (n(t \cdot m), ts) \in V$. Indeed, if $n(t \cdot m) \in G$ then $t \cdot m \in G$, and as G is the group of units of M, there exist $k \in G$ such that $(t \cdot m)k = 1$. As $t \in H$ and H is the group of units of T, we have $t^{-1} \in H$ and

$$
\begin{aligned}
1 = t^{-1} \cdot 1 = t^{-1} \cdot ((t \cdot m)k) \\
&= (t^{-1}t \cdot m)(t^{-1} \cdot k) \qquad \text{(as } T \text{ acts by monoid endomorphisms)} \\
&= m(t^{-1} \cdot k),
\end{aligned}
$$

which forces $m \in G$, and this is a contradiction. Thus $n(t \cdot m) \in N$, which means $(n,t)(m,s) = (n(t \cdot m), ts) \in V$, as required.

(iii) Let $(m,t) \in V_T$ and $(n,s) \in W$, so that $(m,t)(n,s) = (m(t \cdot n), ts)$. As $t \in S$, this gives $ts \in S$ as S is an ideal, so $(m,t)(n,s) \in V_T$. Similarly, $(n,s)(m,t) = (n(s \cdot m), st)$, and as $t \in S$ this gives $st \in S$ as S is an ideal. Hence, $(n,s)(m,t) \in V_T$, which means V_T is an ideal.

(iv) Let $(m,\, t) \in V_M$ and $(n,\, s) \in W$, so that $(m,\, t)(n,\, s) = (m(t \cdot n),\, ts)$. As $m \in N$, this forces $m(t \cdot n) \in N$, hence $(m,\, t)(n,\, s) \in V_M$. Similarly, $(n,\, s)(m,\, t) = (n(s \cdot m), st)$, and as $s \in T, m \in N$, this gives $s \cdot m \in N$, which means $(n,\, s)(m,\, t) \in V_M$, so V_M is an ideal.

(v) Clear.

(vi) To prove $N^1 \rtimes \{1\}$ is a submonoid of W, let $(m,1)$, $(n,1) \in N^1 \rtimes \{1\}$. Then $(m,1)(n,1) = (m(1 \cdot n), 1) = (mn, 1)$. Since $m, n \in N$, and N is an ideal of M, we have $mn \in N$. Hence, $(m,1)(n,1) \in N^1 \rtimes \{1\}$, and as $(1,1) \in N^1 \rtimes \{1\}$, so $N^1 \rtimes \{1\}$ is a submonoid of W.

(vii) To prove $(N^1 \rtimes \{1\}) \sqcup (M \rtimes S)$ is a subsemigroup of W, let $(n,1),(m,s) \in (N^1 \rtimes \{1\}) \sqcup (M \rtimes S)$, such that $n \in N^1, m \in M$, and $s \in S$. Then $(n,1)(m,s) = (n(1 \cdot m), 1s) = (nm, s)$. As $s \in S$, then $(nm, s) \in (N^1 \rtimes \{1\}) \sqcup (M \rtimes S)$.

($viii$) It is clear that $(N^1 \rtimes \{1\})(G \rtimes S) \subseteq M \rtimes S$.

Conversely, let $(m,s) \in M \rtimes S$. Then if $m \in G$, we have $(m,s) = (1,1)(m,s) \in (N^1 \rtimes \{1\})(G \rtimes S)$. If $m \in N$, then $(m,s) = (m,1)(1,s) \in (N^1 \rtimes \{1\})(G \rtimes S)$. Hence, $M \rtimes S = (N^1 \rtimes \{1\})(G \rtimes S)$, as required. □

6.3.1 Presentation for $(N^1 \wr_n \{1_n\}) \sqcup (M \wr_n \mathbf{Sing}_n)$

Our goal in this subsection is to finding a monoid presentation for $(N^1 \wr_n \{1_n\}) \sqcup (M \wr_n \mathrm{Sing}_n)$. We suppose M is a monoid, G is a group of units of M and $N = M \setminus G$ is an ideal of M. Now, if we let $B = (N^1 \wr_n \{1_n\}) \sqcup (M \wr_n \mathrm{Sing}_n)$, it is clear that $N^1 \wr_n \{1_n\}$ is a submonoid of B and $M \wr_n \mathrm{Sing}_n$ is an ideal of B. We follow the argument for [14, Theorem 6.3], making minor adjustments.

The general result [13, Theorem 7.1] provides a way to "stitch together" a monoid presentation for $A = C \sqcup D$ (where C is a submonoid of A and D is an ideal) from presentations for C and D.

Lemma 6.3.2. *Suppose M is a monoid, with group of units G such that $N = M \setminus G$ is an ideal. Then $M \wr_n \mathrm{Sing}_n = (N^1 \wr_n \{1_n\})(G \wr_n \mathrm{Sing}_n)$.*

Proof. Notice that, if $(\mathbf{a}, 1_n) \in N^1 \wr_n \{1_n\}$ and $(\mathbf{b}, \alpha) \in G \wr_n \mathrm{Sing}_n$, then $(\mathbf{a}, 1_n)(\mathbf{b}, \alpha) = (\mathbf{ab}, \alpha) \in M \wr_n \mathrm{Sing}_n$, so $(N^1 \wr_n \{1_n\})(G \wr_n \mathrm{Sing}_n) \subseteq M \wr_n \mathrm{Sing}_n$.

Conversely, assume $(\mathbf{c}, \gamma) \in M \wr_n \mathrm{Sing}_n$. For each $i \in X_n = \{1, \cdots, n\}$, define

$$a_i = \begin{cases} 1 & \text{if} \quad c_i \in G \\ c_i & \text{if} \quad c_i \in N \end{cases} \qquad \text{and} \qquad b_i = \begin{cases} c_i & \text{if} \quad c_i \in G \\ 1 & \text{if} \quad c_i \in N. \end{cases}$$

Since $a_i b_i = c_i$ for each i, it follows that $(\mathbf{c}, \gamma) = (\mathbf{a}, 1_n)(\mathbf{b}, \gamma)$. It is clear that $(\mathbf{b}, \gamma) \in G \wr_n \mathrm{Sing}_n$, while $(\mathbf{a}, 1_n) \in N^1 \wr_n \{1_n\}$ follows from $N^1 = \{1\} \cup N = \{1\} \cup (M \setminus G)$, and this completes the proof. □

Suppose now that N^1 has a monoid presentation $\langle Z : P \rangle$ via $\psi : Z^* \to N^1 : z\psi \mapsto \bar{z}$ for all $z \in Z$. As this is a monoid presentation, it is important to assume $z\psi \neq 1$ for all $z \in Z$ in what follows.

Define new alphabets $Z_{(i)} = \{z_i : z \in Z\}$ for each $i \in X_n$, and put $\mathbf{Z} = Z_{(1)} \cup \cdots \cup Z_{(n)}$. For a word $\omega = z_1 \cdots z_k \in Z^*$, and for $i \in X_n$, define $\omega_{(i)} = (z_1)_{(i)} \cdots (z_k)_{(i)} \in Z^*_{(i)}$. For each $i \in X_n$, write $P_{(i)} = \{(u_{(i)}, v_{(i)}) : (u, v) \in P\}$ and put $\mathbf{P} = P_{(1)} \cup \cdots \cup P_{(n)}$. We also define

$$R^* = \{(x_{(i)} y_{(j)}, y_{(j)} x_{(i)}) : x, y \in Z, i, j \in X_n, i \neq j\}.$$

For $a \in M$ and $i \in X_n$, write $a_{(i)} = (1, \cdots, 1, \underset{i}{a}, 1, \cdots, 1, 1_n)$, where the a is in the ith position.

Define an epimorphism

$$\Psi : \mathbf{Z}^* \to N^1 \wr_n \{1_n\} : z_{(i)} \mapsto (z\psi)_{(i)} = \bar{z}_{(i)}.$$

Remark here that any subsemigroup S of $\mathcal{T}_n$ leads to a wreath product $K \wr_n S$, for any monoid K. In particular, when $S = \{1_n\} \subseteq \mathcal{T}_n$ consists of only the identity transformation, $K \wr_n \{1_n\}$ is isomorphic to the direct product of n copies of K. Hence the next result follows from an obvious result on presentation for direct products of monoids, which is represented in Lemma 6.1.6.

Lemma 6.3.3. *With the above notation, the monoid $N^1 \wr_n \{1_n\}$ has monoid presentation $\langle \mathbf{Z} : \mathbf{P} \cup R^* \rangle$ via Ψ.*

We know from Theorem 6.2.7 that $G \wr_n \mathrm{Sing}_n$ has a semigroup presentation $\langle X_1 : R_1' \rangle$ via $\phi_1 : X_1^+ \to G \wr_n \mathrm{Sing}_n$, where $X_1 = \{e_{ij;a} : i, j \in X_n, i \neq j, a \in G\}$. We will stitch this together with the monoid presentation $\langle \mathbf{Z} : \mathbf{P} \cup R^* \rangle$ for $N^1 \wr_n \{1_n\}$

in order to find a monoid presentation for $B = (N^1 \wr_n \{1\}) \sqcup (M \wr_n \mathrm{Sing}_n)$. From Lemma 6.3.2, and as $B = (N^1 \wr_n \{1\}) \sqcup (M \wr_n \mathrm{Sing}_n)$, we may define an epimorphism

$$\Theta : (\mathbf{Z} \cup X_1)^* \to B : z_{(i)} \mapsto \bar{z}_{(i)},\, e_{ij;a} \mapsto \varepsilon_{ij;a}.$$

Now, we will choose a set of words $\{h_a : a \in N^1\} \subseteq Z^*$ such that $h_a\psi = a$ for all $a \in N^1$. For $a \in N^1$ and $i \in X_n$, define $h_{a;i} = (h_a)_{(i)} \in Z^*_{(i)}$, noting that $h_{a;i}\Theta = h_{a;i}\Psi = a_{(i)}$. Remark here if $a = 1$ we choose $h_a = \varepsilon$, where ε is the empty word in Z^*.

Notice here that as $h_{a_i;i}\Theta = a_{i_{(i)}} = (1, \cdots, 1, \underset{i}{a_i}, 1, \cdots, 1,\, 1_n)$, then

$$\begin{aligned} h_{a_1;1}\Theta &= a_{1_{(1)}} = (\underset{1}{a_1},\, 1,\, \cdots, 1,\, 1_n) \\ h_{a_2;2}\Theta &= a_{2_{(2)}} = (1,\, \underset{2}{a_2}, 1, \cdots, 1,\, 1_n) \\ h_{a_3;3}\Theta &= a_{3_{(3)}} = (1,\, 1,\, \underset{3}{a_3}, 1, \cdots, 1,\, 1_n) \\ &\vdots \\ h_{a_n;n}\Theta &= a_{n_{(n)}} = (1,\, \cdots,\, 1,\, \underset{n}{a_n},\, 1_n). \end{aligned}$$

So

$$\begin{aligned} h_{a_1;1}\Theta \cdots h_{a_n;n}\Theta &= (\underset{1}{a_1},\, 1,\, \cdots, 1,\, 1_n) \cdots (1,\, \cdots,\, 1,\, \underset{n}{a_n},\, 1_n) \\ &= (\underset{1}{a_1},\, \cdots,\, \underset{n}{a_n},\, 1_n) \\ &= (h_{a_1;1} \cdots h_{a_n;n})\Theta \qquad \text{(as } \Theta \text{ is an epimorphism).} \end{aligned}$$

Therefore, if $\omega \in Z^*$ is such that $\omega\Theta = (a_1, \cdots, a_n, 1_n) \in N^1 \wr_n \{1_n\}$, this means ω could be transformed into $h_{a_1;1} \cdots h_{a_n;n}$ using relations $P \cup R^*$ by Lemma 6.3.3.

Now let Q denote the set of relations

$$e_{ij;a} z_{(k)} = \begin{cases} z_{(i)} h_{a\bar{z};j} e_{ij;1} & \text{if} \quad k = i \\ e_{ij;a} & \text{if} \quad k = j \\ z_{(k)} e_{ij;a} & \text{otherwise} \end{cases} \tag{Q1}$$

$$z_{(j)} e_{ij;a} = h_{\bar{z}a;j} e_{ij;1} \tag{Q2}$$

$$z_{(i)} e_{ji;a} e_{ij;b} = h_{\bar{z}ab;i} e_{ij;b}, \tag{Q3}$$

where $z \in Z$ in each relation, and i, j, k, a, b range over all allowable values, subject to the stated constraints.

Remark here that, as $N = M \setminus G$, where G is the group of units of M, and $M \setminus G$ is an ideal, and as we assumed $\bar{z} = z\psi \neq 1$ for all $z \in Z$, we have $a\bar{z}$, $\bar{z}a \in M \setminus G$ for all $z \in Z$ and $a \in G$. Hence the words $h_{a\bar{z}}$, $h_{\bar{z}a}$, $h_{\bar{z}ab}$ appearing in the above relations are well defined.

Our main goal is to provide a proof of the following theorem:

Theorem 6.3.4. *Let M be a monoid with group of units G such that $N = M \setminus G$ is an ideal. With the above notation $B = (N^1 \wr_n \{1_n\}) \sqcup (M \wr_n \mathrm{Sing}_n)$ has a monoid presentation $\langle \mathbf{Z} \cup X_1 \,:\, \mathbf{P} \cup R^* \cup R_1' \cup Q \rangle$ via Θ.*

We will write $\approx_\Theta = (\mathbf{P} \cup R^* \cup R_1' \cup Q)^\sharp$ for the congruence on $(\mathbf{Z} \cup X_1)^*$ generated by the relations $\mathbf{P} \cup R^* \cup R_1' \cup Q$.

To prove Theorem 6.3.4, we need to show that $\approx_\Theta = \mathrm{Ker}\,\Theta$, and some preliminary lemmas are therefore required.

The next result follows by simple diagrammatic check that the relations Q are preserved by Θ.

Lemma 6.3.5. *We have $\approx_\Theta \subseteq Ker\,\Theta$.*

Proof. Notice that we only need to check Θ preserve the relation Q, as Θ already contains Ψ and ϕ_1, which preserve the relations $\mathbf{P} \cup R^*$ and R_1', respectively.

We will exam the left hand side and the right hand side are equal of the relation Q.

For relation $(Q1a)$, we have $e_{ij;a}z_{(k)} = z_{(i)}h_{a\bar{z};j}e_{ij;1}$, if $k = i$. We have to prove that

$$(e_{ij;a}z_{(k)})\Theta = (z_{(i)}h_{a\bar{z};j}e_{ij;1})\Theta.$$

For the left hand side we have

$$\begin{aligned}
(e_{ij;a}z_{(k)})\Theta &= (e_{ij;a})\Theta\,(z_{(k)})\Theta \qquad \text{(as } \Theta \text{ is a morphism)}\\
&= \varepsilon_{ij;a}\bar{z}_{(i)} \qquad \text{(as } k = i)\\
&= (1, \cdots, 1, \underset{j}{a}, 1, \cdots, 1, \varepsilon_{ij})(1, \cdots, 1, \underset{i}{\bar{z}}, 1, \cdots, 1, 1_n)\\
&= (1, \cdots, 1, \underset{j}{a\bar{z}}, 1, \cdots, 1, \underset{i}{\bar{z}}, 1, \cdots, \varepsilon_{ij}).
\end{aligned}$$

For the right hand side we have

$$\begin{aligned}
(z_{(i)}h_{a\bar{z};j}e_{ij;1})\Theta &= (z_{(i)})\Theta\,(h_{a\bar{z};j})\Theta\,(e_{ij;1})\Theta\\
&= \bar{z}_{(i)}(a\bar{z}_{(j)})\varepsilon_{ij;1}\\
&= (1, \cdots, 1, \underset{i}{\bar{z}}, 1, \cdots, 1, 1_n)(1, \cdots, 1, \underset{j}{a\bar{z}}, 1, \cdots, 1, 1_n)(1, \cdots, 1, \varepsilon_{ij})\\
&= (1, \cdots, 1, \underset{i}{\bar{z}}, 1, \cdots, 1, \underset{j}{a\bar{z}}, \cdots, \varepsilon_{ij}).
\end{aligned}$$

For relation (Q1b), we have $e_{ij;a}z_{(k)} = e_{ij;a}$, if $k = j$. We have to prove that

$$(e_{ij;a}z_{(j)})\Theta = (e_{ij;a})\Theta.$$

For the left hand side we have

$$
\begin{aligned}
(e_{ij;a}z_{(j)})\Theta &= (e_{ij;a})\Theta\,(z_{(j)})\Theta \qquad \text{(as } \Theta \text{ is a morphism)}\\
&= \varepsilon_{ij;a}\bar{z}_{(j)}\\
&= (1,\cdots,1,\underset{j}{a},1,\cdots,1,\varepsilon_{ij})(1,\cdots,1,\underset{j}{\bar{z}},1,\cdots,1,1_n)\\
&= (1,\cdots,1,\underset{j}{a},1,\cdots,1,\varepsilon_{ij}).
\end{aligned}
$$

For the right hand side we have

$$(e_{ij;a})\Theta = \varepsilon_{ij;a} = (1,\cdots,1,\underset{j}{a},1,\cdots,1,\varepsilon_{ij}).$$

For relation (Q1c), we have $e_{ij;a}z_{(k)} = z_{(k)}e_{ij;a}$. We want to show that

$$(e_{ij;a}z_k)\Theta = (z_{(k)}e_{ij;a})\Theta.$$

For the left hand side we have

$$
\begin{aligned}
(e_{ij;a}z_{(k)})\Theta &= (e_{ij;a})\Theta\,(z_{(k)})\Theta \qquad \text{(as } \Theta \text{ is a morphism)}\\
&= \varepsilon_{ij;a}\bar{z}_{(k)}\\
&= (1,\cdots,1,\underset{j}{a},1,\cdots,1,\varepsilon_{ij})(1,\cdots,1,\underset{k}{\bar{z}},1,\cdots,1,1_n)\\
&= (1,\cdots,1,\underset{j}{a},1,\cdots,1,\underset{k}{\bar{z}},1,\cdots,1,\varepsilon_{ij}).
\end{aligned}
$$

For the right hand side we have

$$
\begin{aligned}
(z_{(k)}e_{ij;a})\Theta &= (z_{(k)})\Theta\,(e_{ij;a})\Theta \qquad \text{(as } \Theta \text{ is a morphism)}\\
&= \bar{z}_{(k)}\varepsilon_{ij;a}\\
&= (1,\cdots,1,\underset{k}{\bar{z}},1,\cdots,1,1_n)(1,\cdots,1,\underset{j}{a},1,\cdots,1,\varepsilon_{ij})\\
&= (1,\cdots,1,\underset{k}{\bar{z}},1,\cdots,1,\underset{j}{a},1,\cdots,1,\varepsilon_{ij}).
\end{aligned}
$$

For relation (Q2), we have $z_{(j)}e_{ij;a} = h_{\bar{z}a;j}e_{ij;1}$. We want to prove that

$$(z_{(j)}e_{ij;a})\Theta = (h_{\bar{z}a;j}e_{ij;1})\Theta.$$

For the left hand side we have

$$\begin{aligned}
(z_{(j)}e_{ij;a})\Theta &= (z_{(j)})\Theta\,(e_{ij;a})\Theta \qquad \text{(as } \Theta \text{ is a morphism)}\\
&= \bar{z}_{(j)}\varepsilon_{ij;a}\\
&= (1, \cdots, 1, \underset{j}{\bar{z}}, 1, \cdots, 1, 1_n)(1, \cdots, 1, \underset{j}{a}, 1, \cdots, 1, \varepsilon_{ij})\\
&= (1, \cdots, 1, \underset{j}{\bar{z}a}, 1, \cdots, 1, \varepsilon_{ij}).
\end{aligned}$$

For the right hand side we have

$$\begin{aligned}
(h_{\bar{z}a;j}e_{ij;1})\Theta &= (h_{\bar{z}a;j})\Theta\,(e_{ij;1})\Theta \qquad \text{(as } \Theta \text{ is a morphism)}\\
&= (\bar{z}a)_{(j)}\varepsilon_{ij;1}\\
&= (1, \cdots, 1, \underset{j}{\bar{z}a}, 1, \cdots, 1, 1_n)(1, \cdots, 1, \varepsilon_{ij})\\
&= (1, \cdots, 1, \underset{j}{\bar{z}a}, 1, \cdots, 1, \varepsilon_{ij}).
\end{aligned}$$

For relation (Q3), we have $z_{(i)}e_{ji;a}e_{ij;b} = h_{\bar{z}ab;i}e_{ij;b}$. We have to prove

$$(z_{(i)}e_{ji;a}e_{ij;b})\Theta = (h_{\bar{z}ab;i}e_{ij;b})\Theta.$$

For the left hand side we have

$$\begin{aligned}
(z_{(i)}e_{ji;a}e_{ij;b})\Theta &= (z_{(i)})\Theta\,(e_{ji;a})\Theta\,(e_{ij;b})\Theta \qquad \text{(as } \Theta \text{ is a morphism)}\\
&= \bar{z}_{(i)}\varepsilon_{ji;a}\varepsilon_{ij,b}\\
&= (1,\,\cdots,\,1,\,\underset{i}{\bar{z}},\,1,\,\cdots,\,1,\,1_n)(1,\,\cdots,\,1,\,\underset{i}{a},\,1,\,\cdots,\,1,\,\varepsilon_{ji})\\
&\quad (1,\,\cdots,\,1,\,\underset{j}{b},\,1,\,\cdots,\,1,\,\varepsilon_{ij})\\
&= (1,\,\cdots,\,1,\,\underset{i}{\bar{z}a},\,1,\,\cdots,\,1,\,\varepsilon_{ji})(1,\,\cdots,\,1,\,\underset{j}{b},\,1,\,\cdots,\,1,\,\varepsilon_{ij})\\
&= (1,\,\cdots,\,1,\,\underset{i}{\bar{z}ab},\,1,\,\cdots,\,1,\,\cdots,\,1,\,\underset{j}{b},\,1,\,\cdots,\,1,\,\varepsilon_{ij}).
\end{aligned}$$

For the right hand side we have

$$\begin{aligned}
(h_{\bar{z}ab;i}e_{ij;b})\Theta &= (h_{\bar{z}ab;i})\Theta\,(e_{ij;b})\Theta \qquad \text{(as } \Theta \text{ is a morphism)}\\
&= (\bar{z}ab)_{(i)}\varepsilon_{ij;b}\\
&= (1,\,\cdots,\,1,\,\underset{i}{\bar{z}ab},\,1,\,\cdots,\,1,\,1_n)(1,\,\cdots,\,1,\,\underset{j}{b},\,1,\,\cdots,\,1,\,\varepsilon_{ij})\\
&= (1,\,\cdots,\,1,\,\underset{i}{\bar{z}ab},\,1,\,\cdots,\,1,\,\cdots,\,1,\,\underset{j}{b},\,1,\,\cdots,\,1,\,\varepsilon_{ij}).
\end{aligned}$$

Hence, $\approx_\Theta \subseteq \operatorname{Ker}\Theta$ as required. □

To prove the reverse we need to prove the following two lemmas:

Lemma 6.3.6. *If $\omega \in (\mathbf{Z} \cup X_1)^*$, then $\omega \approx_\Theta \omega_1\omega_2$ for some $\omega_1 \in \mathbf{Z}^*$ and $\omega_2 \in X_1^*$. If $\omega \notin \mathbf{Z}^*$, then $\omega_2 \in X_1^+$.*

Proof. For a word $u \in (\mathbf{Z} \cup X_1)^*$, we write $\gamma(u)$ for the number of letters from X_1 appearing in u. We prove the lemma by induction on $\gamma(\omega)$. If $\gamma(\omega) = 0$, then we are already done (with $\omega_1 = \omega$ and $\omega_2 = 1$), so suppose $\gamma(\omega) \geq 1$, and write $\omega = ue_{ij;a}v$, where $u \in (\mathbf{Z} \cup X_1)^*$ and $v \in \mathbf{Z}^*$, so $\gamma(u) = \gamma(\omega) - 1$. By (Q1), we have $e_{ij;a}v \approx_\Theta ze_{ij;b}$ for some $z \in \mathbf{Z}^*$ and some $b \in G$. Since $\gamma(uz) = \gamma(u) = \gamma(\omega) - 1$, the induction hypothesis gives $uz \approx_\Theta u_1u_2$ for some $u_1 \in \mathbf{Z}^*$ and $u_2 \in X_1^*$. So $\omega = ue_{ij;a}v \approx_\Theta uze_{ij;b} \approx_\Theta u_1u_2e_{ij;b}$, and we are done (with $\omega_1 = u_1 \in \mathbf{Z}^*$ and

$\omega_2 = u_2 e_{ij;b} \in X_1^+$). Now, to prove the final assertion in the lemma, suppose $\omega \in (\mathbf{Z} \cup X_1)^* \setminus \mathbf{Z}^*$, such that $\omega \approx_\Theta \omega_1\omega_2$, where $\omega_1 \in \mathbf{Z}^*$ and $\omega_2 \in X_1^*$. If $\omega_2 = \epsilon$, (the empty word in X_1^*), this means $\omega \approx_\Theta \omega_1\epsilon = \omega_1$. As $\omega_1 \in \mathbf{Z}^*$ this gives $\omega \in \mathbf{Z}^*$, a contradiction. Hence, $\omega_2 \in X_1^+$. □

In the case that $\omega \notin \mathbf{Z}^*$, the two words ω_1, ω_2 can be chosen to have a very specific form, and that can be shown only if Lemma 6.3.6 improved as follows:

Lemma 6.3.7. *Let $\omega \in (\mathbf{Z} \cup X_1)^* \setminus \mathbf{Z}^*$, and write $\omega\Theta = (\mathbf{a}, \alpha)$. For $i \in X_n$, define*

$$b_i = \begin{cases} 1 & \text{if} \quad a_i \in G \\ a_i & \text{if} \quad a_i \in N \end{cases} \qquad \text{and} \qquad c_i = \begin{cases} a_i & \text{if} \quad a_i \in G \\ 1 & \text{if} \quad a_i \in N. \end{cases}$$

Then $\omega \approx_\Theta \omega_1\omega_2$ for some $\omega_1 \in \mathbf{Z}^*$ and $\omega_2 \in X_1^+$ with $\omega_1\Theta = (\mathbf{b}, 1_n)$ and $\omega_2\Theta = (\mathbf{c}, \alpha)$.

Proof. By Lemma 6.3.6, the set $\Lambda = \{(\omega_1, \omega_2) \in \mathbf{Z}^* \times X_1^+ : \omega \approx_\Theta \omega_1\omega_2\}$ is non-empty. We define $\zeta : \Lambda \to \mathbb{N}$ as follows. Let $(\omega_1, \omega_2) \in \Lambda$, and write

$$\omega_1\Theta = (\mathbf{p}, 1_n) \qquad \text{and} \qquad \omega_2\Theta = (\mathbf{q}, \alpha),$$

where $\mathbf{p} = (p_1, \cdots, p_n) \in (N^1)^n$ and $\mathbf{q} = (q_1, \cdots, q_n) \in G^n$. Notice that, as $\omega_1 \in \mathbf{Z}^*$, the last coordinate of $\omega_1\Theta$ must be 1_n, it implies the last coordinate of $\omega_2\Theta$ must be α. We then define $\zeta(\omega_1, \omega_2)$ to be the cardinality of the set $\Upsilon(\omega_1, \omega_2) = \{i \in X_n : (p_i, q_i) \neq (b_i, c_i)\}$.

Now choose a pair $(\omega_1, \omega_2) \in \Lambda$ for which $\zeta(\omega_1, \omega_2)$ is minimal. We claim that $\zeta(\omega_1, \omega_2) = 0$. Indeed, suppose to the contrary that $\zeta(\omega_1, \omega_2) \geq 1$. As above, write $\omega_1\Theta = (\mathbf{p}, 1_n)$ and $\omega_2\Theta = (\mathbf{q}, \alpha)$, noting that

$$(\mathbf{a}, \alpha) = \omega\Theta = (\omega_1\Theta)(\omega_2\Theta) = (\mathbf{pq}, \alpha).$$

This gives $p_iq_i = a_i$ for all i. Since $\omega_2 \in X_1^+$, and $\alpha \in \mathrm{Sing}_n$, so we may fix some

$(i,\, j) \in \operatorname{Ker}\alpha$ with $i \neq j$. By relabelling the elements of X_n, if necessary, we may assume that $(i,\, j) = (1,\, 2)$. Define words

$$u_1 = (e_{21;q_1 q_2^{-1}} e_{12;q_2}) \cdot (e_{23;q_3} e_{32;1}) \cdots (e_{2n;q_n} e_{n2;1})$$

and

$$u_2 = (e_{12;q_2 q_1^{-1}} e_{21;q_1}) \cdot (e_{13;q_3} e_{31;1}) \cdots (e_{1n;q_n} e_{n1;1}),$$

and let v be any word over X (regarded as a subset of X_1 as usual) with $v\Theta = (1, \cdots, 1, \alpha)$. It easy to check (diagrammatically) that $u_1\Theta = (\mathbf{q},\, \varepsilon_{12})$ and $u_2\Theta = (\mathbf{q},\, \varepsilon_{21})$. In particular, since $\alpha = \varepsilon_{12}\alpha = \varepsilon_{21}\alpha$, we have

$$(u_1\Theta)(v\Theta) = (u_2\Theta)(v\Theta) = (\mathbf{q},\, \alpha) = \omega_2\Theta.$$

Now as ω_2, $u_1 v$, $u_2 v$ all belong to X_1^+, Theorem 6.2.7 then gives $\omega_2 \approx_\Theta u_1 v \approx_\Theta u_2 v$. Moreover, Lemma 6.3.3 gives $\omega_1 \approx_\Theta h_{p_1;1} \cdots h_{p_n;n}$.

Since $\zeta(\omega_1,\, \omega_2) \geq 1$, we may fix some $r \in \Upsilon(\omega_1,\, \omega_2)$. Notice that if $p_r = 1$, this would imply $a_r = p_r q_r \in G$, which would give $(b_r,\, c_r) = (1,\, a_r) = (p_r,\, q_r)$, contradicting our assumption that $r \in \Upsilon(\omega_1,\, \omega_2)$. In particular, $h_{p_r;r} \neq 1$, so we may write $h_{p_r;r} = (z_1)_{(r)} \cdots (z_k)_{(r)} z_{(r)}$, where $z_1, \cdots, z_k, z \in Z$. Hence we obtain $((z_1)_{(r)} \cdots (z_k)_{(r)} z_{(r)})\Theta = h_{p_r;r}\Theta = (p_r)_{(r)}$, which gives $\bar{z}_1 \cdots \bar{z}_k \bar{z} = p_r$. Note that R^* gives $\omega_1 \approx_\Theta \omega_3 h_{p_r;r}$, where $\omega_3 = h_{p_1;1} \cdots h_{p_{r-1};r-1} h_{p_{r+1};r+1} \cdots h_{p_n;n}$. Note also that, $p_r \neq 1$ implies $p_r \in N = M \setminus G$, and as N is an ideal of M, we have $a_r = p_r q_r \in N$, so $(b_r,\, c_r) = (a_r,\, 1)$.

We now consider separate cases, depending on the value of r.

Case 1. Suppose that $r \geq 3$. Note that

$$\begin{aligned}
h_{p_r;r}u_1 = (z_1)_{(r)} \cdots (z_k)_{(r)} z_{(r)} (e_{21;q_1q_2^{-1}} e_{12;q_2}) \cdot (e_{23;q_3} e_{32;1}) \cdots (e_{2,r-1;q_{r-1}} e_{r-1,2;1}) \\
\times (e_{2r;q_r} e_{r2;1})(e_{2,r+1;q_{r+1}} e_{r+1,2;1}) \cdots (e_{2n;q_n} e_{n2;1}) \\
\approx_\Theta (z_1)_{(r)} \cdots (z_k)_{(r)} (e_{21;q_1q_2^{-1}} e_{12;q_2}) \cdot (e_{23;q_3} e_{32;1}) \cdots (e_{2,r-1;q_{r-1}} e_{r-1,2;1}) \\
\times z_{(r)} (e_{2r;q_r} e_{r2;1})(e_{2,r+1;q_{r+1}} e_{r+1,2;1}) \cdots (e_{2n;q_n} e_{n2;1}) \quad \text{(by } (Q1)) \\
\approx_\Theta (z_1)_{(r)} \cdots (z_k)_{(r)} (e_{21;q_1q_2^{-1}} e_{12;q_2}) \cdot (e_{23;q_3} e_{32;1}) \cdots (e_{2,r-1;q_{r-1}} e_{r-1,2;1}) \\
\times h_{\bar{z}q_r;r} (e_{2r;1} e_{r2;1})(e_{2,r+1;q_{r+1}} e_{r+1,2;1}) \cdots (e_{2n;q_n} e_{n2;1}) \quad \text{(by } (Q2)) \\
\approx_\Theta (z_1)_{(r)} \cdots (z_k)_{(r)} h_{\bar{z}q_r;r} (e_{21;q_1q_2^{-1}} e_{12;q_2}) \cdot (e_{23;q_3} e_{32;1}) \cdots (e_{2,r-1;q_{r-1}} e_{r-1,2;1}) \\
\times (e_{2r;1} e_{r2;1})(e_{2,r+1;q_{r+1}} e_{r+1,2;1}) \cdots (e_{2n;q_n} e_{n2;1}) \qquad \text{(by } (Q1)).
\end{aligned}$$

(In the last step of the previous calculation, recall that $h_{\bar{z}q_r;r}$ involves only letters from $Z_{(r)}$). Note also that

$$((z_1)_{(r)} \cdots (z_k)_{(r)} h_{\bar{z}q_r;r})\Theta = (\bar{z}_1 \cdots \bar{z}_k \bar{z} q_r)_{(r)} = (p_r q_r)_{(r)} = (a_r)_{(r)}.$$

Notice here that as $a_r \in N$, and as we know that $(h_{a_r;r})\Theta = (a_r)_{(r)}$, so Lemma 6.3.3 gives $(z_1)_{(r)} \cdots (z_k)_{(r)} h_{\bar{z}q_r;r} \approx_\Theta h_{a_r;r}$.

Now put

$$\begin{aligned}
u_3 = (e_{21;q_1q_2^{-1}} e_{12;q_2}) \cdot (e_{23;q_3} e_{32;1}) \cdots (e_{2,r-1;q_{r-1}} e_{r-1,2;1})(e_{2r;1} e_{r2;1}) \\
(e_{2,r+1;q_{r+1}} e_{r+1,2;1}) \cdots (e_{2n;q_n} e_{n2;1}).
\end{aligned}$$

The above calculations show that $h_{p_r;r}u_1 \approx_\Theta h_{a_r;r}u_3$, and it follows that

$$\omega \approx_\Theta \omega_1\omega_2 \approx_\Theta (\omega_3 h_{p_r;r})(u_1 v) \approx_\Theta (\omega_3 h_{a_r;r})(u_3 v) = v_1 v_2,$$

where $v_1 = \omega_3 h_{a_r;r} \in \mathbf{Z}^*$ and $v_2 = u_3 v \in X_1^+$. It follows that $(v_1, v_2) \in \Lambda$, and it is easy to check that

$$v_1\Theta = (p_1,\, \cdots,\, p_{r-1},\, a_r,\, p_{r+1}, \cdots,\, p_n,\, 1_n)$$

and

$$v_2\Theta = (q_1,\, \cdots,\, q_{r-1},\, 1,\, q_{r+1}, \cdots,\, q_n,\, \alpha).$$

Since $(b_r,\, c_r) = (a_r,\, 1)$, it follows that $\zeta(v_1,\, v_2) = \zeta(\omega_1,\, \omega_2) - 1$, contradicting the minimality of $\zeta(\omega_1,\, \omega_2)$, and completing the proof of the claim in this case.

Case 2. Suppose $r = 1$.

In this case we have $\omega_1 \approx_\Theta \omega_3 h_{p_1;1}$, where $\omega_3 = h_{p_2;2} \cdots h_{p_n;n}$, and we also have $h_{p_1;1} = (z_1)_{(1)} \cdots (z_k)_{(1)} z_{(1)}$. Note that by using (Q3) we have

$$\begin{aligned} h_{p_1;1}u_1 &= (z_1)_{(1)} \cdots (z_k)_{(1)} z_{(1)} (e_{21;q_1 q_2^{-1}} e_{12;q_2}) \cdot (e_{23;q_3} e_{32;1}) \cdots (e_{2n;q_n} e_{n2;1}) \\ &\approx_\Theta (z_1)_{(1)} \cdots (z_k)_{(1)} h_{\bar{z}q_1;1} e_{12;q_2} \cdot (e_{23;q_3} e_{32;1}) \cdots (e_{2n;q_n} e_{n2;1}). \end{aligned}$$

As in the previous case we have $(z_1)_{(1)} \cdots (z_k)_{(1)} h_{\bar{z}q_1;1} \approx_\Theta h_{a_1;1}$. It quickly follows that $h_{p_1;1}u_1 \approx_\Theta h_{a_1;1}u_4$, where $u_4 = e_{12;q_2} \cdot (e_{23;q_3} e_{32;1}) \cdots (e_{2n;q_n} e_{n2;1})$.

As $\omega_1 \approx_\Theta \omega_3 h_{p_1;1}$ and $\omega_2 \approx_\Theta u_1 v$, we have

$$\omega \approx_\Theta \omega_1 \omega_2 \approx_\Theta (\omega_3 h_{p_1;1})(u_1 v) \approx_\Theta (\omega_3 h_{a_1;1})(u_4 v) = v_1 v_2,$$

where $v_1 = \omega_3 h_{a_1;1} \in \mathbf{Z}^*$ and $v_2 = u_4 v \in X_1^+$. This time we obtain

$$v_1\Theta = (a_1,\, p_2,\, \cdots,\, p_n,\, 1_n) \quad \text{and} \quad v_2\Theta = (1,\, q_2, \cdots, q_n,\, \alpha),$$

and again we have $\zeta(v_1,\, v_2) = \zeta(\omega_1,\, \omega_2) - 1$, a contradiction.

Case 3. Suppose $r = 2$.

This case is almost identical to previous case, but we use the word u_2 (defined above) instead of u_1. Hence now we have $\omega_1 \approx_\Theta \omega_3 h_{p_2;2}$, where $\omega_3 = h_{p_1;1} h_{p_3;3} \cdots h_{p_n;n}$, and we also have $h_{p_2;2} = (z_1)_{(2)} \cdots (z_k)_{(2)} z_{(2)}$. Note that by using (Q3) we have

$$\begin{aligned} h_{p_2;2} u_2 &= (z_1)_{(2)} \cdots (z_k)_{(2)} z_{(2)} (e_{12;q_2 q_1^{-1}} e_{21;q_1}) \cdot (e_{13;q_3} e_{31;1}) \cdots (e_{1,n;q_n} e_{n1;1}) \\ &\approx_\Theta (z_1)_{(2)} \cdots (z_k)_{(2)} h_{\bar{z} q_2;2} e_{21;q_1} \cdot (e_{13;q_3} e_{31;1}) \cdots (e_{1n;q_n} e_{n1;1}). \end{aligned}$$

Moreover,

$$((z_1)_{(2)} \cdots (z_k)_{(2)} h_{\bar{z} q_2;2})\Theta = (\bar{z}_1 \cdots \bar{z}_k \bar{z} q_2)_{(2)} = (p_2 q_2)_{(2)} = (a_2)_{(2)},$$

as $\bar{z}_1 \cdots \bar{z}_k \bar{z} = p_2$. Hence, we have $(z_1)_{(2)} \cdots (z_k)_{(2)} h_{\bar{z} q_2;2} \approx_\Theta h_{a_2;2}$. So $h_{p_2;2} u_2 \approx_\Theta h_{a_2;2} u_5$, where here we have $u_5 = e_{21;q_1} \cdot (e_{13;q_3} e_{31;1}) \cdots (e_{1n;q_n} e_{n1;1}) \in X^+$. As $\omega_1 \approx_\Theta \omega_3 h_{p_2;2}$ and $\omega_2 \approx_\Theta u_2 v$, we have

$$\omega \approx_\Theta \omega_1 \omega_2 \approx_\Theta (\omega_3 h_{p_2;2})(u_2 v) \approx_\Theta (\omega_3 h_{a_2;2})(u_5 v) = v_1 v_2,$$

where $v_1 = \omega_3 h_{a_2;2} \in \mathbf{Z}^*$ and $v_2 = u_5 v \in X_1^+$. It easy to check that

$$v_1 \Theta = (p_1,\, a_2,\, p_3,\, \cdots,\, p_n,\, 1_n) \quad \text{and} \quad v_2 \Theta = (q_1,\, 1,\, q_3,\, \cdots,\, q_n,\, \alpha),$$

and again we have $\zeta(v_1, v_2) = \zeta(\omega_1, \omega_2) - 1$, a contradiction. This completes the proof of the claim that $\zeta(\omega_1, \omega_2) = 0$. And this of course, complete the proof of the lemma. □

Now, we are ready to prove Theorem 6.3.4.

Proof. In order to prove Theorem 6.3.4, we need to prove $\operatorname{Ker}\Theta \subseteq \approx_\Theta$. Let $u, v \in (\mathbf{Z} \cup X_1)^*$ be such that $u\Theta = v\Theta$. Notice that, if $u \in \mathbf{Z}^*$, this means $u\Theta \in N^1 \wr \{1_n\}$, and as $u\Theta = v\Theta$, this will imply $v \in \mathbf{Z}^*$, so in this case $u \approx_\Theta v$ follows from Lemma 6.3.3. Now, if $u \notin \mathbf{Z}^*$, this means $v \notin \mathbf{Z}^*$ as well (as $u\Theta = v\Theta$), so we have $u \in (\mathbf{Z} \cup X_1)^*$ and $u \notin \mathbf{Z}^*$. Lemma 6.3.7 then gives $u \approx_\Theta u_1 u_2$ and $v \approx_\Theta v_1 v_2$ for some u_1, $v_1 \in \mathbf{Z}^*$ and u_2, $v_2 \in X_1^+$ with $u_1\Theta = v_1\Theta$ and $u_2\Theta = v_2\Theta$. Lemma 6.3.3 and Theorem 6.2.7, respectively, then give $u_1 \approx_\Theta v_1$ and $u_2 \approx_\Theta v_2$. Putting this all together we obtain $u \approx_\Theta u_1u_2 \approx_\Theta v_1v_2 \approx_\Theta v$. $\square$

6.3.2 Presentation for $(N^1 \rtimes \{1\}) \sqcup (M \rtimes S)$

In this subsection we will suppose the monoids M and T are as in the beginning of this section, and we will suppose N and G are left S-acts.

This subsection is devoted to finding a presentation for $C = (N^1 \rtimes \{1\}) \sqcup (M \rtimes S)$. By using Lemma 6.3.1 (*viii*) we may write $C = (N^1 \rtimes \{1\}) \sqcup (N^1 \rtimes \{1\})(G \rtimes S)$.

Suppose S has a semigroup presentation $\langle X : R \rangle$ via $\phi : X^+ \to S$. From Subsection 6.2.1, we immediately have semigroup presentation for $M \rtimes S$ and $G \rtimes S$. Then $G \rtimes S$ has a semigroup presentation $\langle X_G : R_G \rangle$ via $\phi_G : X_G^+ \to G \rtimes S : x_g \mapsto (g, x\phi)$, such that $X_G = \{x_g : x \in X, g \in G\}$ and $R_G = R_G^1 \cup R_G^2$ where

$$R_G^1 = \{(u_g, v_g) : (u, v) \in R, g \in G\}$$

and

$$R_G^2 = \{(x_a y_b, x_{a(x\phi,b)} y) : x, y \in X, a, b \in G\}.$$

Clearly $X \subseteq X_G$ and $R \subseteq R_G$.

To find a presentation for $C = (N^1 \rtimes \{1\}) \sqcup (M \rtimes S)$, we will use the same technique that we used in the previous subsection to find the presentation for $B = (N^1 \wr_n \{1_n\}) \sqcup (M \wr_n \mathrm{Sing}_n)$. It is very important to mention that there are some differences between the presentation of B and the presentation of C, as in the presentation of B we have $N^1 \wr_n \{1_n\} = (N^1)^n \rtimes \{1_n\}$, but in a presentation of C we have $N^1 \rtimes \{1\}$. Remark here that $N^1 = (M \setminus G)^1$ but $(N^1)^n \neq (M^n \setminus \overline{G})^1$, where $\overline{G}$ is the group of units of M^n. For example, if $n = 2$, $g \in G$, $g \neq 1$ and $m \in N$, then $(g, m) \in (M^2 \setminus \overline{G})^1$ but $(g, m) \notin (N^1)^2$.

Suppose that N has a semigroup presentation $\langle Y : P \rangle$ via ξ. Then $N = \langle y\xi \rangle$, which means the generating set of N will be $\mathcal{Y} = \{y\xi \,:\, y \in Y\}$. We will write $\sim_\xi = P^\sharp$ for the congruence on Y^+ generated by the relation P. Moreover, since $N^1 = N \cup \{1\}$, we have that N^1 has a monoid presentation $\langle Y \,:\, P\rangle$ via $\xi^* : Y^* \to N^1$, with the same set of generators of N and the same relations.

We will write $\approx_{\xi^*} = P^\sharp$ for the congruence on Y^* generated by the relation P. Now, we only want to prove $\operatorname{Ker}\xi^* = \approx_{\xi^*}$.

Let $\omega, \omega' \in Y^*$ be such that $\omega\xi^* = \omega'\xi^*$. It is clear that as $\omega\xi^* = \omega'\xi^* \in N^1 = N \cup \{1\}$, either $\omega\xi^* = \omega'\xi^* = 1$, which means $\omega = \omega' = \epsilon$ (the empty word in Y^*), and so $\omega \approx_{\xi^*} \omega'$ or, $\omega\xi^* = \omega'\xi^* \in N$, so that $\omega, \omega' \in Y^+$ and $\omega\xi = \omega'\xi$. Hence $\omega \sim_\xi \omega'$, as $\operatorname{Ker}\xi = \sim_\xi$.

Now, as $\omega \sim_\xi \omega'$, and by using Proposition 1.2.4, we have either $\omega = \omega'$, which gives $\omega \approx_{\xi^*} \omega'$ or, for some $n \in \mathbb{N}$, there is a sequence

$$\omega = c_1a_1d_1,\ c_1b_1d_1 = c_2a_2d_2,\ \cdots,\ c_nb_nd_n = \omega'$$

of elementary P-transition connecting ω to ω', where $c_i, d_i \in N^1$ and $(a_i, b_i) \in P$ or $(b_i, a_i) \in P$ for all i, which will give $\omega \approx_{\xi^*} \omega'$. So we obtain $\operatorname{Ker}\xi^* \subseteq \approx_{\xi^*}$.

Conversely, suppose $u, v \in Y^*$ are such that $u \approx_{\xi^*} v$. If $u = v = \epsilon$ then $u\xi^* = v\xi^* = 1$, which gives $\approx_{\xi^*} \subseteq \operatorname{Ker}\xi^*$. If neither u nor v is the empty word ϵ, then $u, v \in Y^+$, so $u\xi = v\xi$ and $u \sim_\xi v$ as N has presentation $\langle Y \,:\, P\rangle$ via ξ. From this we obtain $u\xi^* = v\xi^*$, and so $\approx_{\xi^*} \subseteq \operatorname{Ker}\xi^*$.

Note that, if U is a semigroup (monoid), then there exists a map $U \rtimes \{1\} \to U$, where $(u, 1) \mapsto u$, and $(u, 1)(v, 1) = (uv, 1)$. So it is easy to show that $N^1 \rtimes \{1\}$ has a monoid presentation $\langle Y \,:\, P\rangle$ via $\xi' : Y^* \to N^1 \rtimes \{1\} : y \mapsto y\xi' = (y\xi, 1)$.

As C is generated by $y\xi' \cup x_g\phi_G$, we have that

$$\Phi : (Y \cup X_G)^* \to C\ : y \mapsto (y\xi, 1),\ x_g \mapsto (g, x\phi)$$

is an epimorphism, where Φ extends ξ' and ϕ_G.

For each $y \in Y, x \in X, g \in G$ choose $\omega_{g,\,x,\,y} \in Y^+$ such that

$$(\omega_{g,\,x,\,y})\Phi = (g(x\phi \cdot y\xi)g^{-1}, 1).$$

Note that

$$\begin{aligned}
(x_g y)\Phi &= (x_g)\Phi\,(y)\Phi && \text{(as } \Phi \text{ is a morphism)}\\
&= (g,\, x\phi)(y\xi,\, 1)\\
&= (g(x\phi \cdot y\xi),\, x\phi)\\
&= (g(x\phi \cdot y\xi)g^{-1}g,\, x\phi)\\
&= (g(x\phi \cdot y\xi)g^{-1},\, 1)(g,\, x\phi)\\
&= (\omega_{g,\,x,\,y})\Phi\,(x_g)\Phi\\
&= (\omega_{g,\,x,\,y}\, x_g)\Phi && \text{(as } \Phi \text{ is a morphism).}
\end{aligned}$$

For each $y \in Y, g \in G$ choose $v_{y,g} \in Y^+$ such that

$$v_{y,g}\Phi = ((y\xi)g, 1).$$

Then for any $x \in X$,

$$\begin{aligned}
(v_{y,g}x)\Phi &= v_{y,g}\Phi\, x\Phi && \text{(as } \Phi \text{ is a morphism)}\\
&= ((y\xi)g, 1)(1, x\phi)\\
&= ((y\xi)g, x\phi)\\
&= (y\xi, 1)(g, x\phi)\\
&= y\Phi\, x_g\Phi\\
&= (yx_g)\Phi && \text{(as } \Phi \text{ is a morphism).}
\end{aligned}$$

Let $\mathcal{Q} = \{x_g y = \omega_{g,\,x,\,y} x_g, v_{y,g}x = yx_g : \omega_{g,\,x,\,y}, v_{y,g} \in Y^+, x \in X, y \in Y, g \in G\}$.

Our aim is to prove the following theorem:

Theorem 6.3.8. *Let M and T be monoids such that M is a left T-act by endomorphism, and let G and H be the groups of units of M and T, respectively. Suppose $N = M \setminus G$ and $S = T \setminus H$ are ideals of M and T, respectively. If N and G are left S-acts, then with above notation $C = (N^1 \rtimes \{1\}) \sqcup (M \rtimes S) = (N^1 \rtimes \{1\}) \sqcup (N^1 \rtimes \{1\})(G \rtimes S)$ has a monoid presentation $\langle Y \cup X_G : P \cup R_G \cup \mathcal{Q} \rangle$ via Φ.*

We will write $\approx_\Phi = (P \cup R_G \cup \mathcal{Q})^\sharp$ for the congruence on $(Y \cup X_G)^*$ generated by $P \cup R_G \cup \mathcal{Q}$.

To prove Theorem 6.3.8, we need to show that $\approx_\Phi = \operatorname{Ker}\Phi$, and for this we require preliminary lemmas.

The next result follows immediately from the above discussions.

Lemma 6.3.9. $\approx_\Phi \subseteq$ *$Ker\,\Phi$.*

To prove the reverse containment, we need the following lemma:

Lemma 6.3.10. *If $\omega \in (Y \cup X_G)^*$, then $\omega \approx_\Phi \omega_1\omega_2$ for some $\omega_1 \in Y^*$ and $\omega_2 \in X_G^*$. If $\omega \in (Y \cup X_G)^* \setminus Y^*$, then $\omega \approx_\Phi \omega_1\omega_2$, where $\omega_1 \in Y^*$ and $\omega_2 \in X_G^+$.*

Proof. For a word $u \in (Y \cup X_G)^*$, we write $\Gamma(u)$ for the number of letters from X_G appearing in u. We prove the lemma by induction on $\Gamma(\omega)$. If $\Gamma(\omega) = 0$, then we are already done (with $\omega_1 = \omega$ and $\omega_2 = 1$), so suppose $\Gamma(\omega) \geq 1$, and write $\omega = ux_gv$, where $u \in (Y \cup X_G)^*$ and $v \in Y^*$, so $\Gamma(u) = \Gamma(\omega) - 1$. By repeated applications of the relations from $\mathcal{Q}$, we have $x_gv \approx_\Phi zx_g$ for some $z \in Y^*$. Since $\Gamma(uz) = \Gamma(u) = \Gamma(\omega) - 1$, an induction hypothesis gives $uz \approx_\Phi u_1u_2$ for some $u_1 \in Y^*$ and $u_2 \in X_G^*$. So

$$\omega = ux_gv \approx_\Phi uzx_g \approx_\Phi u_1u_2x_g,$$

and we are done (with $\omega_1 = u_1 \in Y^*$ and $\omega_2 = u_2x_g$).

If $\omega \in (Y \cup X_G)^* \setminus Y^*$, then ω contains at least one symbol x_g. If $\omega \approx_\Phi \omega'$ for any $\omega' \in (Y \cup X_G)^*$, then notice that ω' contains at least one symbol from X_G for at any step $\omega^{''} \to \omega^{'''}$ in the deduction of $\omega^{'}$ from ω, applying a relation from R^1_G always leaves at least one symbol x_g (as R is a set of semigroups identities) and any application of an identities from R^2_G or $\mathcal{Q}$ leaves the number of symbols from X_G fixed.

$\square$

We are now ready to prove Theorem 6.3.8.

Proof. In order to prove Theorem 6.3.8, it suffices to show that $\operatorname{Ker}\Phi \subseteq \approx_\Phi$. Suppose $\omega, \omega^{'} \in (Y \cup X_G)^*$ are such that $\omega\Phi = \omega^{'}\Phi$. If $\omega \in Y^*$, then $\omega\Phi \in N^1 \rtimes \{1\}$, which also gives $\omega^{'} \in Y^*$. Hence in this case $\omega \approx_\Phi \omega^{'}$, as $N^1 \rtimes \{1\}$ has presentation $\langle Y : P \rangle$ via $\xi^{'}$. So suppose $\omega \notin Y^*$, noting that this also forces $\omega^{'} \notin Y^*$. Lemma 6.3.10 then gives $\omega \approx_\Phi uv$ and $\omega^{'} \approx_\Phi u^{'}v^{'}$ for some $u, u^{'} \in Y^*$ and $v, v^{'} \in X_G^+$. By Lemma 6.3.9, we obtain $u\Phi v\Phi = u^{'}\Phi v^{'}\Phi$.

Let $u\Phi = (n, 1)$ and $u^{'}\Phi = (n^{'}, 1)$, where $n, n^{'} \in N^1$. Let $v\Phi = (g, s)$ and $v^{'}\Phi = (g^{'}, s^{'})$, where $g, g^{'} \in G$ and $s, s^{'} \in S$. Hence $(ng, s) = (n^{'}g^{'}, s^{'})$, so $ng = n^{'}g^{'}$ and $s = s^{'}$.

Note also that $u = \epsilon$ if and only if $u^{'} = \epsilon$. Suppose first $u = y_1 \cdots y_t$, $u^{'} = z_1 \cdots z_r$, where $t, r \geq 1$, $y_i, z_j \in Y$. Let $\overline{x} = xx^1 \cdots x^l$, where $l \geq 0$, $x, x^k \in X$. Then $ux_g = y_1 \cdots y_t x_g \approx_\Phi y_1 \cdots y_{t-1} v_{y_t,g} x$ and $u^{'}x_{g^{'}} = z_1 \cdots z_r x_{g^{'}} \approx_\Phi z_1 \cdots z_{r-1} v_{z_r,g^{'}} x$ and as

$$
\begin{aligned}
(y_1 \cdots y_{t-1} v_{y_t,g})\Phi &= (y_1)\Phi \cdots (y_{t-1})\Phi(v_{y_t,g})\Phi \qquad \text{(as } \Phi \text{ is a morphism)} \\
&= (y_1\xi, 1) \cdots (y_{t-1}\xi, 1)((y_t\xi)g, 1) \\
&= ((y_1\xi) \cdots (y_t\xi)g, 1) \\
&= ((u\xi)g, 1) \\
&= (ng, 1) \\
&= (n^{'}g^{'}, 1)
\end{aligned}
$$

$$
\begin{aligned}
&= ((u'\xi)g', 1)\\
&= ((z_1\xi)\cdots(z_r\xi)g', 1)\\
&= (z_1\xi, 1)\cdots(z_{r-1}\xi, 1)((z_r\xi)g', 1)\\
&= (z_1)\Phi\cdots(z_{r-1})\Phi(v_{z_r,g'})\Phi\\
&= (z_1\cdots z_{r-1}v_{z_r,g'})\Phi \qquad\qquad (\text{as}\,\Phi\,\text{is a morphism}),
\end{aligned}
$$

we have $y_1\cdots y_{t-1}v_{y_t,g} \approx_\Phi z_1\cdots z_{r-1}v_{z_r,g'}$.

Also,

$$
\begin{aligned}
v\Phi &= (g, x\phi)(1, x^1\phi)\cdots(1, x^l\phi)\\
&= (g, x\phi)(1, x^1\phi\cdots x^l\phi)\\
&= (g, x\phi)(1, (x^1\cdots x^l)\phi) \qquad (\text{as}\,\phi\,\text{is a morphism})\\
&= x_g\Phi(x^1\cdots x^l)\Phi\\
&= (x_g x^1\cdots x^l)\Phi \qquad\qquad (\text{as}\,\Phi\,\text{is a morphism}),
\end{aligned}
$$

giving $v \approx_\Phi x_g x^1\cdots x^l$ and similarly $v' \approx_\Phi x_{g'}x^1\cdots x^l$.

Finally,

$$
\begin{aligned}
uv &\approx_\Phi y_1\cdots y_t x_g x^1\cdots x^l \approx_\Phi y_1\cdots y_{t-1}v_{y_t,g}xx^1\cdots x^l\\
&\approx_\Phi z_1\cdots z_{r-1}v_{z_r,g'}xx^1\cdots x^l \approx_\Phi u'x_{g'}x^1\cdots x^l \approx_\Phi u'v'.
\end{aligned}
$$

On the other hand, if $u = u' = \epsilon$ then clearly $u \approx_\Phi u'$ and $g = g'$, $s = s'$, so as $v\Phi = v'\Phi$ we have $v \approx_\Phi v'$ and $uv \approx_\Phi u'v'$.

Putting this all together, we obtain that $\omega \approx_\Phi uv \approx_\Phi u'v' \approx_\Phi \omega'$.

□

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