The spectral density for scalar fields in de Sitter space at one-loop

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Abstract

We review the Källén-Lehmann spectral representation in flat spacetime before moving on to de Sitter space. We compute one-loop corrections of de Sitter invariant two-point functions in the Lorentzian signature which are defined by the interacting Euclidean vacuum for scalar fields with cubic interactions. These apply to all massive scalars in the complementary and principal series. Our investigations are motivated by the behaviour of the spectral density at the one-loop level whereby we can find a general expression relating the two-point function and the free propagators. Using well established techniques for treating quantum fields in de Sitter, we compute the spectral density for specific cases, in both complementary and principal series in three dimensions and discuss the nature of particle stability. We also comment on extending this beyond the one-loop level.

Contents

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Lastly, I am indebted to my family for all of their love and support and without whom this would not have been possible.

Dedication

I leave Sisyphus at the foot of the mountain. One always finds one's burden again. But Sisyphus teaches the higher fidelity that negates the gods and raises rocks. He too concludes that all is well. This universe henceforth without a master seems to him neither sterile nor futile. Each atom of that stone, each mineral flake of that night filled mountain, in itself forms a world. The struggle itself toward the heights is enough to fill a man's heart.

One must imagine Sisyphus happy.

- Albert Camus [1]

Declaration

I hereby declare that this thesis has not been and will not be submitted in whole or in part to another University for the award of any other degree.

Callum King

1 Introduction

Quantum field theory is one of the fundamental pillars of modern physics. It successfully combines the principles of Einstein's theory of special relativity with that of quantum mechanics to accurately describe a whole range of theories in Nature. From quantum chromodynamics (QCD) to electroweak interactions and moving beyond the fundamental forces, quantum field theory also has numerous applications to condensed matter physics. There are multiple tools that have been developed to treat quantum fields and calculate observables systematically and consistently; perturbation theory and the renormalisation group to name but two.

Quantum field theory defined for flat, or Minkowski spacetime, are theories for fundamental forces of Nature and have applications for solid state physics . However, we know that the universe is not flat everywhere, there is curvature. General relativity, developed by Albert Einstein over a hundred years ago combines special relativity with Newton's law of gravitation, leading to the idea of gravitation being the geometry of the spacetime [2, 3]. In the absence of gravitational effects we have our standard Minkowski spacetime. Curvature of spacetime is related by the energy and momentum of whatever matter and radiation is present in the system. Quantum field theory in curved spacetime can provide an accurate description of quantum phenomena where the effects of curved spacetime become more pronounced for example, in the vicinity of black holes and of quantum fluctuations in the early universe. In the early universe there is strong evidence for the theory of inflation [4, 5, 6], there was an exponential expansion of space, eventually leading to the formation of our universe. Many models of inflation, where the effects of gravity were much more prevalent, looking at quantum field theory in curved space time.

Taking the model of de Sitter space time, defined by constant positive curvature with its dynamics dominated by a positive cosmological constant, we examine interacting theories [7, 8, 9]. A de Sitter universe is a solution of the field equations of general relativity and a strong candidate for the behaviour of universe at a time close to $t = 10^{-33}$ seconds after the Big Bang theory. It is also a candidate for the universe's ultimate fate in the infinite future, at the end of all things. While free quantum fields are relatively well understood, interacting theories pose problems [9]. Our interest is the Källén-Lehmann spectral representation and the behaviour of the two-point function at the one-loop level [2, 10, 11, 12]. We firstly review the spectral density and the Källén-Lehmann representation for scalar field theory in flat space and illustrate with an example in two dimensions how we can obtain the spectral density at the one-loop level before moving on to a review of de Sitter spacetime as a candidate for the early universe and cosmology.

We investigate scalar field theory in de Sitter space [6, 13], computing one-loop corrections of two-point functions defined by the interacting Euclidean vacuum for scalar fields considering cubic interactions applying to all massive scalars in the complementary and principal series [14]. We investigate spectral functions at the one-loop level whereby we can find a general expression of relating the two-point function and the free propagators by way of the spectral density. Reviewing the work of Marolf and Morrison [10], we adopt the techniques developed to deal with analytic continuation in de Sitter field theory from the Euclidean to the Lorentzian signature, as well as a review of the treatment of ultraviolet (UV) divergences in the de Sitter spacetime and perturbative corrections to the two-point function. These arise in correlation functions due to the denominator having terms both dependent on the angular momentum L and the mass, which in the zero mode and massless limit generate divergences [10, 15, 16, 17, 18, 19]. With masses large enough, infrared (IR) divergences can be avoided at tree level, but still occur in loop diagrams [10].

We explore the perturbative corrections to the propagators [10]. While some previous work has focussed on decays at large distances in de Sitter spaces, we focus on the spectral density and compute it for some cases when the particles masses lie in both the complementary and principal series. Currently, there is debate about the particle instability in de Sitter with some arguing that all particles are unstable [21], while others say that before some critical mass we lose particle stability. We comment on these matters by making some investigations into the nature of the spectral density and its behaviour in de Sitter space [11, 19, 20, 21, 22, 23].

2 Källén-Lehmann representation in flat space

2.1 Introduction to spectral density

Everything we compute in this section relies on principles of quantum mechanics and special relativity. We begin by examining the analytic structure of the time ordered two-point function for scalar field ϕ in flat spacetime before continuing this to curved spacetime. For the free field case the interpretation of a two-point function $\langle 0|T\phi(x)\phi(y)|0\rangle$ is straightforward. It is the amplitude for a particle to propagate from y to x. In the interacting theories however, the interpretation has some differences. For this first chapter we only require the general principles of relativity and quantum mechanics. Calculations will not depend on an expansion in perturbation theory or on the type of interactions [8]. Our examination begins with the two-point function,

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle \tag{1}$$

where Ω is the true vacuum. We begin our examinations with the two-point function, by inserting the identity operator inside our two-point function; introducing a complete set of states. The states are chosen to be eigenstates of a fully interacting Hamiltonian, H, and knowing that H and $\mathbf{P}$ commute, $[H, \mathbf{P}] = 0$, these states can be chosen so they are also the eigenstates of $\mathbf{P}$ through Lorentz invariance [8]. Additionally, we let $|\lambda_0\rangle$ be an eigenstate of H such that $\mathbf{P}|\lambda_0\rangle = 0$, so it is has zero momentum. Therefore, by Lorentz invariance, any boost of the λ_0 state, denoted here as $\lambda_{\mathbf{p}}$ is an eigenstate of H. We can now make use of a useful completeness relation for one particle states, [8]

$$(1)_{1-particle} = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}|p\rangle\langle p| \tag{2}$$

This can be written for the entire Hilbert space. We first assume these states have been relativistically normalised allow $|\lambda_{\mathbf{p}}\rangle$, with momentum p, be a boost of λ_0. The identity for entire Hilbert space has the form,

$$\mathbb{I} = |\Omega\rangle\langle\Omega| + \sum_{\lambda}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}|\lambda_{\mathbf{p}}\rangle\langle\lambda_{\mathbf{p}}| \tag{3}$$

Here $E_{\mathbf{P}} = \sqrt{\mathbf{p}^2 + m_\lambda^2}$ and m_λ is the rest mass of states $|\lambda_{\mathbf{p}}\rangle$. Our two-point function is,

$$T\langle\Omega|\phi(x)\left(|\Omega\rangle\langle\Omega| + \sum_\lambda \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}|\lambda_{\mathbf{p}}\rangle\langle\lambda_{\mathbf{p}}|\right)\phi(y)|\Omega\rangle \tag{4}$$

Our summation here runs over all zero momentum states. The first term $\langle\Omega|\phi(x)|\Omega\rangle\langle\Omega|\phi(y)|\Omega\rangle$, is equal to some constant which can normally be set to zero by symmetry, for scalar fields. Therefore for the purpose of this paper we will neglect it. We also assume $x^0 > y^0$. It then takes the form,

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \sum_\lambda \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\langle\Omega|\phi(x)|\lambda_{\mathbf{p}}\rangle\langle\lambda_{\mathbf{p}}|\phi(y)|\Omega\rangle \tag{5}$$

where we can break up elements accordingly,

$$\langle\Omega|\phi(x)|\lambda_{\mathbf{p}}\rangle = \langle\Omega|e^{iP\cdot x}\phi(0)e^{-iP\cdot x}|\lambda_{\mathbf{p}}\rangle$$

$$\langle\Omega|\phi(x)|\lambda_{\mathbf{p}}\rangle = \langle\Omega|\phi(0)|\lambda_{\mathbf{p}}\rangle e^{-ip\cdot x}|_{p^0=E_p} \tag{6}$$

and using $U^{-1}\phi(0)U = \phi(0)$, where U is the unitary operator which implements a Lorentz boost from $\overrightarrow{p}$ to 0 [8]. This allows us to arrive at,

$$\langle\Omega|\phi(x)|\lambda_{\mathbf{p}}\rangle = \langle\Omega|\phi(x)|\lambda_{\mathbf{0}}\rangle e^{-ip\cdot x}|_{p^0=E_p} \tag{7}$$

As stated previously this is true for scalars but not for cases where we must consider higher spins. Introducing an integration over p^0, again for $x^0 > y^0$, our two-point function takes the form,

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \sum_\lambda \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2 - m_\lambda^2 + i\epsilon}e^{-ip\cdot(x-y)}|\langle\Omega|\phi(0)|\lambda_0\rangle|^2 \tag{8}$$

where we have the Feynman propagator save for a replacement of m with m_λ. However, there is a way of representing the two-point function more succinctly, known as the Källén-

Lehmann representation.

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \int_0^\infty \frac{dM^2}{2\pi}\rho(M^2)\frac{ie^{-ip\cdot(x-y)}}{p^2 - M^2 + i\epsilon} \tag{9}$$

where $\rho(M^2)$ is the spectral density defined by,

$$\rho(M^2) = \sum_\lambda (2\pi)\delta(M^2 - m_\lambda^2)|\langle\Omega|\phi(0)|\lambda_0\rangle|^2 \tag{10}$$

For a general theory, $4m^2 \gtrsim M^2$ we have one-particle states only contributing a delta function. The spectral density, has the form,

$$\rho(M^2) = 2\pi\delta(M^2 - m^2) \cdot Z + (terms\ for\ when\ M^2 \gtrsim 4m^2) \tag{11}$$

where Z is the field strength renormalisation and m is the physical mass. The spectral density is a positive definite quantity [8, 25].

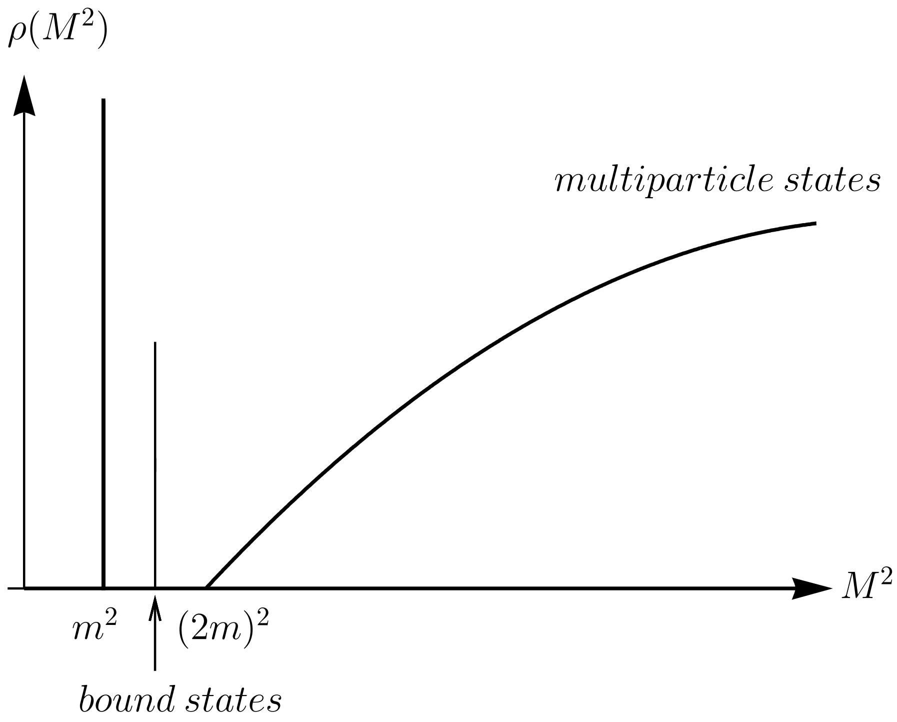


Figure 1: The spectral density $\rho(M^2)$ for some interacting theory. We observe that one-particle states contribute a δ-function at m^2, m being the particle's mass [8]. Bound states, if any, contribute similar poles before we reach a continuous spectrum of multiparticle states.

The Fourier transform of the two-point function is therefore,

$$\int d^4x e^{ip\cdot x}\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \int_0^\infty dM^2\rho(M^2)\frac{i}{p^2-M^2+i\epsilon}$$

$$\begin{aligned}\int d^4x e^{ip\cdot x}\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle &= \frac{iZ}{p^2-m^2+i\epsilon} \\ &+ \int_{4m^2}^\infty dM^2\rho(M^2)\frac{i}{p^2-M^2+i\epsilon} \end{aligned} \tag{12}$$

where, on the complex p^2-plane, we pick up an isolated pole at m^2 coming from the one-particle states. In the event of bound states we acquire more poles. At $(2m)^2$, when we encounter a continuous spectrum of two particle and multiparticle states, we take a branch cut.

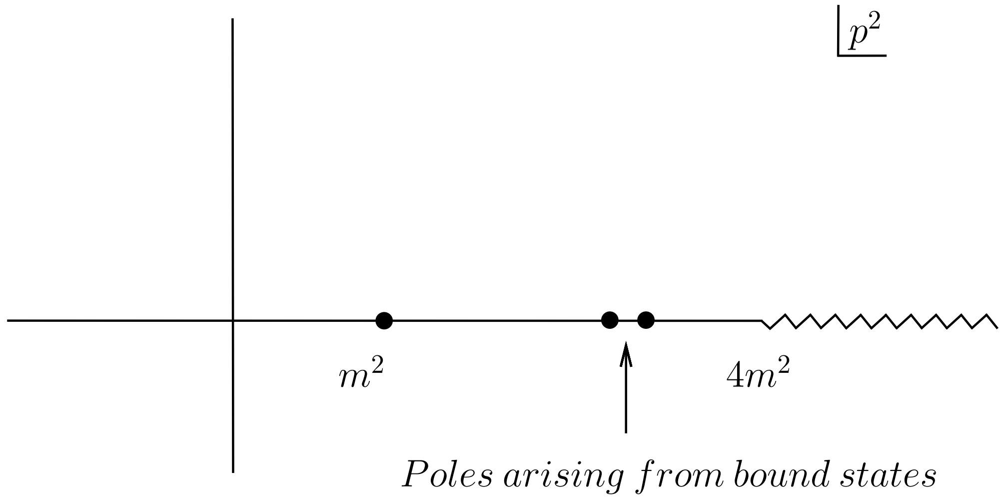


Figure 2: Analytic structure of Fourier transform of the two-point in complex p^2 plane.

In practice, when calculating the spectral density it is often possible to expand the two-point function, compare this expression and simply read off the spectral density. While these calculations can be in principle straightforward to solve, there exists a simpler expression for computing the spectral density. To be precise, we can define the spectral density in a more convenient representation by the propagator and its complex conjugate. It is merely the sum of the two with a factor of i. Defining the propagator,

$$\Delta(p^2) = \frac{i}{p^2 - m^2 - \Sigma(p^2)} = \int \frac{d\sigma^2}{2\pi} \rho(\sigma^2) \frac{i}{p^2 - \sigma^2 + i\epsilon} \tag{13}$$

where $\Sigma(p^2)$ is the sum of all one-particle irreducible (1PI) diagrams. The spectral density can be found by adding this to its complex conjugate, through the formula,

$$\frac{1}{p^2 - \sigma^2 + i\epsilon} - \frac{1}{p^2 - \sigma^2 - i\epsilon} = -2\pi i \delta(p^2 - \sigma^2) \tag{14}$$

therefore,

$$\rho(p^2) = \Delta(p^2) + \Delta(p^2)^* = \frac{i}{p^2 - m^2 - \Sigma(p^2)} - \frac{i}{p^2 - m^2 - \Sigma(p^2)^*}$$

$$\rho(p^2) = \int \frac{d\sigma^2}{2\pi} \rho(\sigma^2) \left[\frac{i}{p^2 - \sigma^2 + i\epsilon} - \frac{i}{p^2 - \sigma^2 - i\epsilon} \right] = \int \frac{d\sigma^2}{2\pi} \rho(\sigma^2) 2\pi i \delta(p^2 - \sigma^2) \tag{15}$$

If we expand our original expression for the propagator out, splitting into both real and imaginary parts of the sum 1PI diagrams, we can show that the spectral density is positive definite,

$$\rho(p^2) = \frac{i}{p^2 - m^2 - Re(\Sigma(p^2)) - iIm(\Sigma(p^2))} - \frac{i}{p^2 - m^2 - Re(\Sigma(p^2)^*) + iIm(\Sigma(p^2)^*)}$$

$$\rho(p^2) = -\frac{2Im\Sigma(p^2)}{(p^2 - m^2 - Re(\Sigma(p^2)))^2 + (Im(\Sigma(p^2)))^2} \tag{16}$$

where $\Sigma(p^2)$ is defined accordingly,

$$\Sigma(p^2) = 2\pi\lambda^2 \int \frac{d\sigma^2}{2\pi} \frac{f(\sigma^2)}{p^2 - \sigma^2 + i\epsilon} \tag{17}$$

with some arbitrary function $f(\sigma^2)$. After some simple rearrangement and algebra the form for the spectral density can be given as,

$$\rho(p^2) = -\frac{2Im\,\Sigma(p^2)}{[p^2 - m^2 - Re\,\Sigma(p^2)]^2 + Im\,\Sigma(p^2)^2} \tag{18}$$

where it can be shown the $Im\,\Sigma(p^2)$ will be negative.

2.2 Computing the spectral density in flat space

Our ultimate aim is to understand the spectral density in curved spacetime but first it is worthwhile comparing it with the flat spacetime case. We can use this as a step on the path to understanding how the spectral density behaves in the curved spacetime case, for massive scalars for Lagrangians with interactions beyond the quadratic order [10, 15, 18]. We wish to examine the Källén-Lehmann spectral representation and see where we have a relationship between the two-point function and the propagators.

Firstly, we consider some Lagrangian with a cubic interaction term of the form,

$$\mathcal{L} = ... - \frac{\lambda}{2}\Phi^2\phi \tag{19}$$

of some $\phi \to \phi$ scattering process, mediated by 2Φ's with momentum k and $p - k$

the reason we are looking at the interactions is because interacting quantum field theory in de Sitter proves more challenging than the free theory.

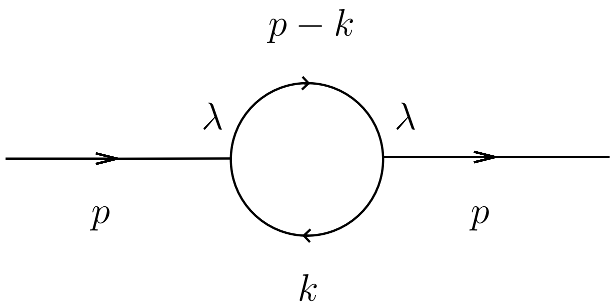


Figure 3: $\phi \to \phi$ scattering mediated by scalar Φ particles

Taking the case where ϕ going to ϕ via some interactions of Φ's with masses m and M respectively, and we assume $m > 2M$ so ϕ can decay into 2Φ particles. Neglecting the external lines, this can be computed accordingly for the simple two dimensional case as,

$$-i\Sigma(p^2) = \lambda^2 \int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2 - M^2 + i\epsilon} \frac{1}{(k-p)^2 - M^2 + i\epsilon} \tag{20}$$

which according to the Feynman parametrisation prescription for evaluating loop integrals, takes the following form,

$$\frac{1}{A_1 A_2 ... A_n} = \int_0^1 dx_1 ... dx_n \; \delta\left(\sum_i x_i - 1\right) \frac{(n-1)!}{[x_1 A_1 + ... x_n A_n]^n} \tag{21}$$

leading to,

$$\frac{1}{A_1 A_2 ... A_n} = \lambda^2 \int \frac{d^2\ell}{(2\pi)^2} \int_0^1 dx \frac{1}{[\ell^2 - p^2 \cdot x(1-x) - M^2 + i\epsilon]^2} = -\frac{i\lambda^2}{4\pi} \int_0^\infty d\ell \int_0^1 dx \frac{2\ell}{[\ell^2 + \Delta]^2}$$

$$\frac{1}{A_1 A_2 ... A_n} = -\frac{i\lambda^2}{4\pi} \int_0^1 dx \frac{1}{[M^2 - x(1-x)p^2]} = -\frac{i\lambda^2}{2\pi} \int_0^{1/2} dy \frac{1}{[M^2 - (\frac{1}{4} - y^2)p^2]} \tag{22}$$

We then make the substitution $\sigma^2 = \frac{M^2}{\frac{1}{4} - y^2}$, where σ^2 is our variable rather than the square of the variable leaving us with,

$$\Sigma(p^2) = -\frac{\lambda^2}{2\pi} \int_{4M^2}^\infty d\sigma^2 \frac{1}{p^2 - \sigma^2} \frac{1}{2M^2 \sqrt{\frac{1}{4} - \frac{M^2}{\sigma^2}}} \tag{23}$$

So our full propagator, to $\mathcal{O}(\lambda^2)$, can be represented in the Källén-Lehmann spectral representation through the two-point function relating the spectral density ρ defined by,

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \int_0^\infty \frac{d\sigma^2}{2\pi}\rho(\sigma^2)D_F(x-y;\sigma^2) \tag{24}$$

with the spectral density ρ,

$$\rho(\sigma^2) = \sum_\lambda (2\pi)\delta(\sigma^2 - m_\lambda^2)|\langle\Omega|T\phi(0)|\lambda_0\rangle|^2 \tag{25}$$

For $p^2 \ll 4M^2$ $\Sigma(p^2)$ is real then we have,

$$\rho(p^2) = i\left[\frac{1}{p^2 - m^2 - \Sigma(p^2) + i\epsilon} - \frac{1}{p^2 - m^2 - \Sigma(p^2) - i\epsilon}\right]$$

$$\rho(p^2) = i\left[-2\pi i\delta\left(p^2 - m^2 - \Sigma(p^2)\right)\right] \tag{26}$$

where we define m_{phys} as the physical mass of the particle, where $m^2 \ll 4M^2$, as,

$$m_{phys} = m^2 + \Sigma(p^2) \tag{27}$$

Giving us,

$$\rho(p^2) = 2\pi Z\delta(p^2 - m_{phys}) \tag{28}$$

For values our $p^2 > M^2$, $\Sigma(p^2)$ is no longer real and we have the spectral density as,

$$\rho(p^2) = i\left[\frac{1}{p^2 - m^2 - \Sigma(p^2)} - \frac{1}{p^2 - m^2 - \Sigma^*(p^2)}\right] \tag{29}$$

and returning to our form for the $\Sigma(p^2)$,

$$\Sigma(p^2) = \frac{\lambda^2}{2\pi}\int_{4M^2}^\infty d\sigma^2 \frac{1}{p^2 - \sigma^2 + i\epsilon}\frac{1}{\sqrt{(\sigma)^2 - 4M^2\sigma^2}} \tag{30}$$

turning first to the imaginary part,

$$i\ Im\ \Sigma(p^2) = \frac{\lambda^2}{2\pi}\int_{4M^2}^\infty d\sigma^2 \frac{1}{\sqrt{(p^2)^2 - 4M^2\sigma^2}} \times [-i\delta(p^2 - \sigma^2)] \tag{31}$$

therefore the imaginary part has the form of,

$$Im\ \Sigma(p^2) = -\frac{\lambda^2}{2\pi}\frac{1}{\sqrt{(p^2)^2 - 4M^2\sigma^2}} \tag{32}$$

The real part of $\Sigma(p^2)$ cannot be computed in closed form. Our spectral density will therefore be,

$$\rho(p^2) = -\frac{-2\left(\frac{\lambda^2}{2\pi}\frac{1}{\sqrt{(p^2)^2-4M^2\sigma^2}}\right)}{\left(p^2 - m^2 - Re\ \Sigma(p^2)\right)^2 + \left(\frac{\lambda^2}{2\pi}\frac{1}{\sqrt{(p^2)^2-4M^2\sigma^2}}\right)^2}$$

$$\rho(p^2) = \frac{\lambda^2}{\pi\left(p^2 - m^2 - Re\ \Sigma(p^2)\right)^2 ((p^2)^2 - 4M^2\sigma^2)^{1/2} + \frac{\lambda^4}{4\pi}\left[(p^2)^2 - 4M^2\sigma^2\right]^{-1/2}} \tag{33}$$

in two dimensions.

The way the spectral density behaves in flat space can be illustrated in the stable case and unstable case. In the stable case for an interacting scalar field, we have a delta function coming from the one-particle states, located at m^2 where m is the mass of the particle. We then have a continuous spectrum of two or more particle states beginning at $(2m)^2$. In the case where we encounter bound states we will encounter additional delta functions in between the physical mass squared and twice the physical mass squared, m_{phys}^2 and $4m_{phys}^2$ after which, we get the multiparticle states.

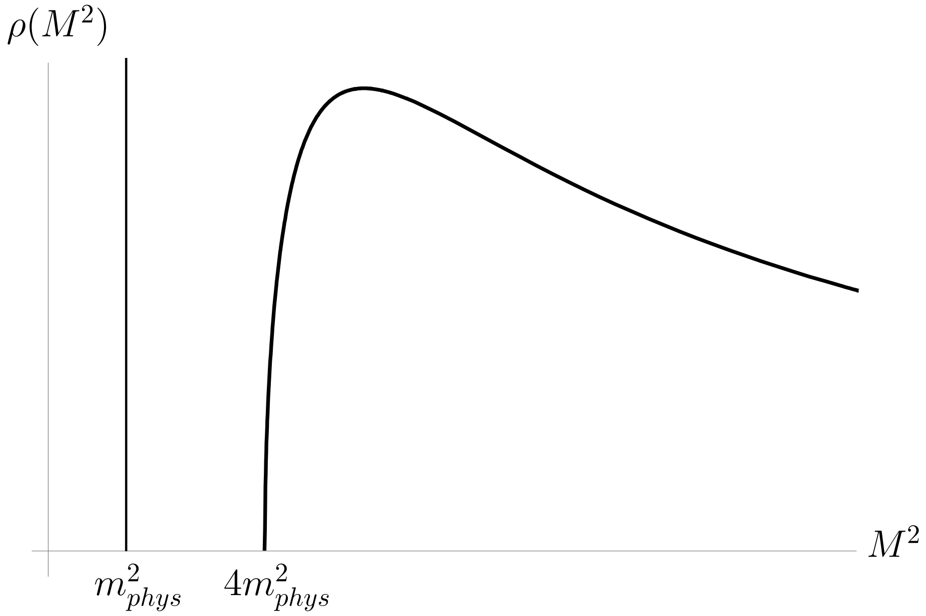


Figure 4: An example of a stable spectral density in flat space for a generic interacting theory in the absence of bound states.

In the unstable case we get a continuous spectrum with the spectral density decaying rapidly as can be seen below.

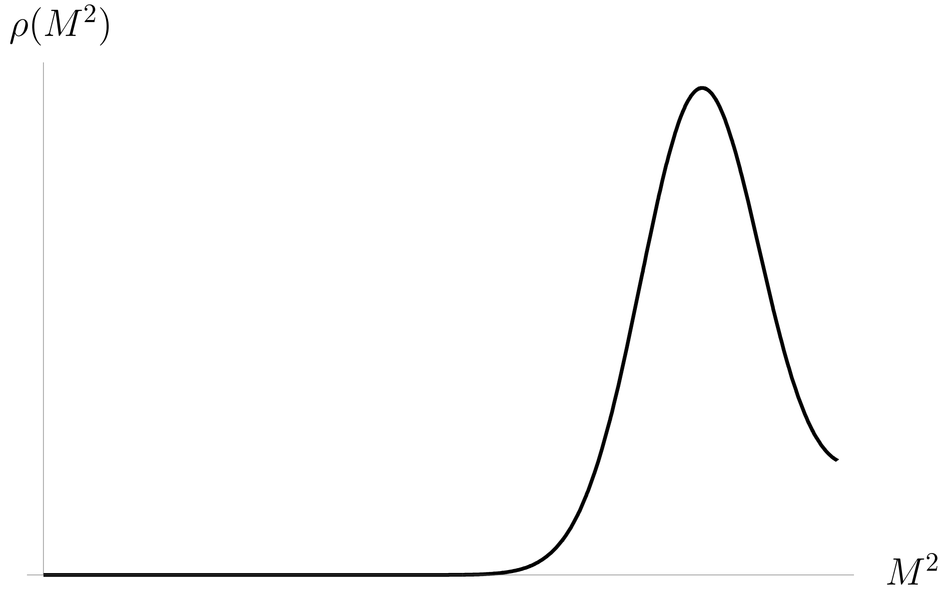


Figure 5: A general spectral density for the unstable particle case in flat space.

3 De Sitter field theory

3.1 Introduction

Here we review some of the foundations of de Sitter spacetime, [2, 4, 5, 9, 20, 24] before reviewing the previous work focusing at the one-loop level on the behaviour at large distances, large Z, while our focus is on computing the spectral density in both the complementary and principal series.

De Sitter space has many resemblances to Minkowski space, as a sphere in Euclidean space. Working in D-dimensions, it is the Lorentzian manifold analogue of a D-sphere which is a specialised case of pseudo-Riemannian manifold. It is maximally symmetric meaning it retains the same number of symmetries as Euclidean. This can also be defined as having,

$$\frac{D}{2}(D+1) \tag{34}$$

linearly independent Killing vectors, D being the dimension; it is also homogeneous [7]. Furthermore, the Riemann tensor obeys the relationship,

$$R_{\mu\nu\rho\sigma} = \frac{R}{D(D-1)}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right) \tag{35}$$

R here being the Ricci scalar curvature and $g_{\mu\nu}$ being the metric. In de Sitter space it is maximally symmetric and its scalar curvature is both positive and constant.
Construction of a de Sitter space can be done as follows. Consider first a $D = 5$ flat space with a metric,

$$g_{\mu\nu} = diag(1, -1, -1, -1, -1) \tag{36}$$

5-vectors X^μ can define a four dimensional space satisfying,

$$-X^\mu X^\nu g_{\mu\nu} = \frac{1}{H^2} \tag{37}$$

where H is the Hubble parameter which determines the rate of spatial expansion [9].
We then induce a metric by considering this as a subspace of $(\mathbb{R}, g_{\mu\nu})$ and the space we

have can be said to be a de Sitter space. De Sitter space is also simply connected for $D \geq 3$ meaning all paths between two-points can be transformed continuously but remain in that topological space. It is represented by the hyperboloid given below.

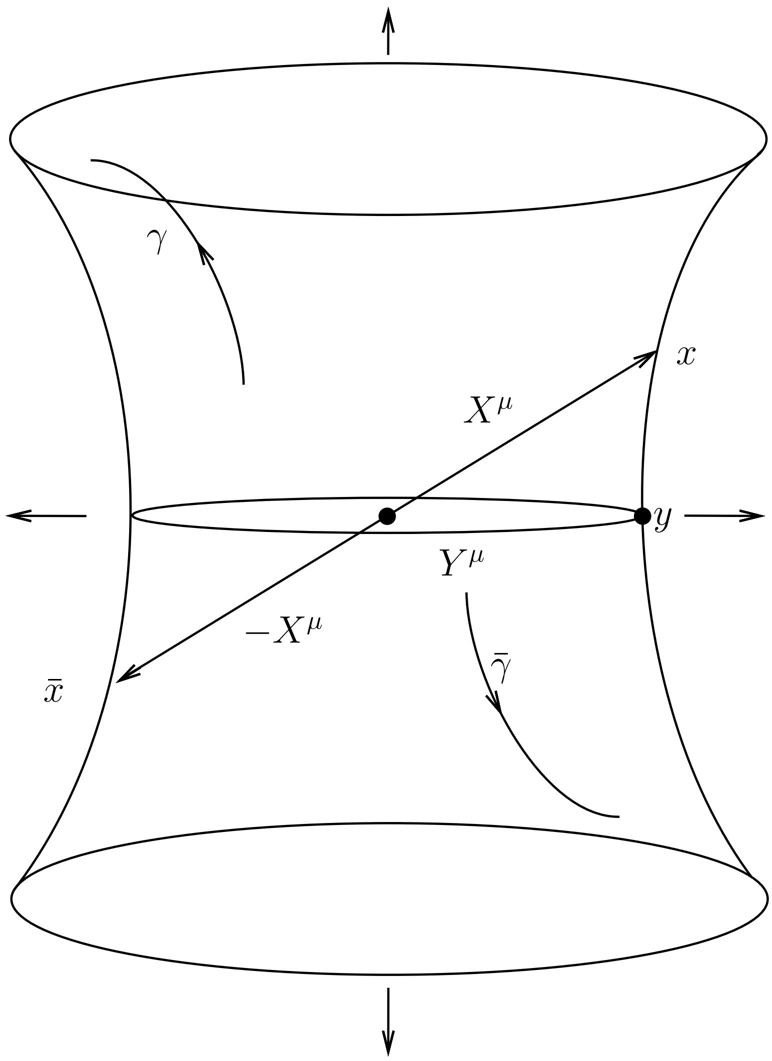


Figure 6: De Sitter hyperboloid where with antipodal point $\bar{x}$. Future directed paths denoted by γ are sent to a past directed curve $\bar{\gamma}$

Comparing it with the sphere we find many similarities with every point having an antipodal point denoted by a bar with the following relationship,

$$X^\mu(\bar{x}) = -X^\mu(x) \tag{38}$$

where X^μ is defined as,

$$X^\mu = (t, -\overrightarrow{X}) \tag{39}$$

$d(x, y)$ is defined by,

$$d(x, y) = H^{-1} \arccos(Z(x, y)) \tag{40}$$

where $Z(x, y)$ is given as,

$$Z(x, y) \equiv H^2 g_{\mu\nu} X^\mu X^\nu \tag{41}$$

which displays similarities to the sphere replacing $\arccos(Z)$ with $\arccos(\theta)$, therefore $\arccos(Z)$ is the hyperbolic angle between points x and y. Here $Z(x, y)$ has the property that when we replace x with $\bar{x}$ we get an overall minus sign.

$$Z(\bar{x}, y) = -Z(x, y) \tag{42}$$

$$Z(\bar{x}, \bar{y}) = Z(x, y) \tag{43}$$

3.2 Interacting scalar fields in de Sitter

Free quantum fields in de Sitter behave very well while interacting field theories pose a number of challenges. While particles in Minkowski space cannot decay into a heavier product of daughter particles due to energy conservation, in de Sitter this can be the case.

The reason this, initially alarming, phenomenon is possible is due to the lack of a globally timelike Killing vector field. This means that the notion of, positive definite, energy conservation no longer applies as it does in flat space. To that end let us first consider the D-dimensional de Sitter space dS_D metric,

$$ds^2 = \ell^2(-dt^2 + \cosh^2 t \, d\Omega_d^2) \tag{44}$$

where l is the de Sitter length scale [10], which is the inverse of the Hubble parameter H,

$$l = \frac{1}{H} \tag{45}$$

so we can see de Sitter is a contracting and expanding spacetime more clearly,

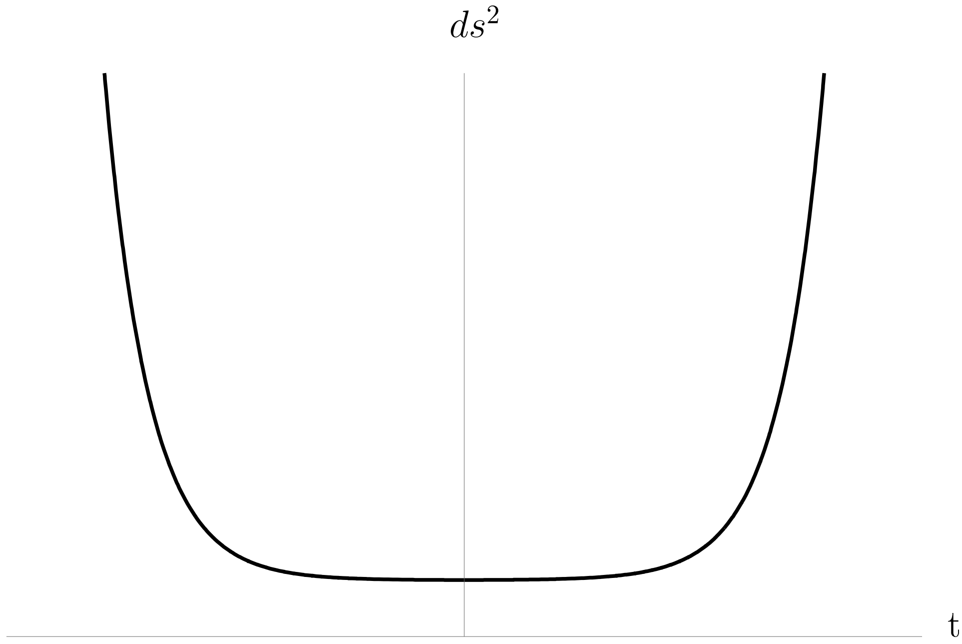


Figure 7: A generic de Sitter spacetime where we observe the contraction and expansion for early past and late future.

as well as $d\Omega_d^2$ is the metric on the unit d where $D = d + 1$ sphere. For a free theory, our scalar fields define a representation in the connected de Sitter group $SO_0(D, 1)$, obeying the Klein-Gordon equation,

$$\Box\phi - M^2\phi = 0 \tag{46}$$

with some mass M. By representation here we mean a group representation where our group $SO_0(D, 1)$ is defined in terms linear transformations of vector spaces. Rescaling the mass with a dimensionless parameter, σ, so that,

$$-\sigma(\sigma + d) := M^2\ell^2 \tag{47}$$

choosing the positive root

$$\sigma := -\frac{d}{2} + \sqrt{\left(\frac{d}{2}\right)^2 - M^2\ell^l} \tag{48}$$

with three de Sitter representations,

- Complementary series where $-\frac{d}{2} < \sigma < 0$
- Principal series where $\sigma = -\frac{d}{2} + i\rho\ ,\ \rho \in \mathbb{R},\ \rho \geq 0$
- Discrete series for $\sigma \in \mathbb{N}_0$

We ignore the discrete series as they are not physical, our work focusing instead on the complementary and principal series [10]. Here our σ will correspond to a unitary irreducible representation in our two cases; the complementary and principal series [10]. An irreducible representation is one that cannot be expressed by any subrepresentations. Heavier fields belong in the principal series while the lighter fields lie in the complementary case. Green's functions can be defined in the three dimensional sphere and analytically continue to de Sitter space denoted Δ^{σ}_{xy} denoting arguments x and y. These Green's functions are invariant under de Sitter transformations. In both principal and complementary series and fixing one argument, while taking the other as $|t| \to \infty$, terms in the propagators fall off as $e^{\sigma|t|}, e^{-(\sigma+d)|t|}$.

In spite of this exponential decay of propagators in de Sitter, we get exponential growth from the contracting and expanding Nature of the de Sitter volume from the $(\cosh(t))^d$. Multiple products of propagators decay slowly enough so that tree level diagrams diverge, even with σ near to zero.

In the principal series, IR divergences emerge from loop diagrams [10]. This is a diagram with two external lines where we fix the end points at x_1 and x_2. We need to fix the relative positions of vertices and then integrate over dS_D. We will pick up exponential factors with argument $d \cdot t$, which for $t \to \infty$ from the measure, with the integrand being suppressed by the propagators which decay at most exponential factors $-d \cdot t$. Every $\int dt$ will diverge proportional to powers of t. The form chosen for treating these IR divergences is by analytic continuation from Euclidean signature.

3.3 Analytic continuation in de Sitter space

With the metric for D-dimensional de Sitter space,

$$ds^2 = \ell^2(-dt^2 + \cosh^2 t \, d\Omega_d^2) \tag{49}$$

we can relate it for the D-sphere in Euclidean space via the following Wick rotation, by the transformation,

$$t = i\left(\tau - \frac{\pi}{2}\right) \tag{50}$$

with the metric for our Euclidean sphere transforms as,

$$d\Omega_D^2 = \ell^2 \left[-(i \, d\tau)^2 + \left(\frac{e^{i(\tau - \frac{\pi}{2})} + e^{-i(\tau - \frac{\pi}{2})}}{2} \right)^2 d\Omega_d^2 \right] \tag{51}$$

through Euler's formula,

$$d\Omega_D^2 = \ell^2 \left[d\tau^2 + \left(\cos(\tau - \frac{\pi}{2}) \right)^2 d\Omega_d^2 \right]$$

$$= \ell^2(d\tau^2 + \sin^2 \tau \, d\Omega_d^2) \tag{52}$$

No IR divergences occur from integrating over the Euclidean sphere S^D due to it being compact [9, 10, 11, 16, 19, 23]. This is valid provided we are only considering massive scalars, which we are in accordance with [10]. In this case, the Feynman diagrams in Euclidean signature converge to define an interacting state on the sphere that is $SO(D+1)$ invariant and therefore satisfy Schwinger-Dyson equations in the Euclidean signature where, for some action S in the presence of some source J takes the form,

$$\frac{\delta S}{\delta \phi(x)} \left[-i \frac{\delta}{\delta J(x)} \right] Z[J] + J(x) Z[J] = 0 \tag{53}$$

with Z here being the generating functional. That this is satisfied means when we analytically continue this from the Euclidean to the Lorentzian signature, these equations are also satisfied and also invariant under $SO(D, 1)$ to all orders of . We can therefore evolve the correlators over time, t, starting at $t = 0$ where the Lorentzian correlators are identical to the Euclidean save for factors of i coming from the derivatives. Our Euclidean-signature Feynman diagrams converge and define interacting states on the sphere which are invariant under $SO(D, 1)$ satisfying the Euclidean Schwinger-Dyson equations. To perform the analytic continuations, we employ two tools, the embedding distance and Watson-Sommerfield transformations [10].

The Euclidean two-point correlation functions denoted $\langle\phi(x_i)\phi(x_j)\rangle$ can be written as functions of geodesic distance between x_i and x_j,[10] by parametrising this using embedding distance [10]. This is the length of chord in some ambient space $\mathbb{R}^{D+1}$ between two points x and y [10]. Ambient space being the space that surrounds the object we are examining; while chord length here is defined as a line segment on the sphere which end points both lie on the edge of the sphere. It is not a length as such. Embedding distance is given in terms of coordinates on the sphere as,

$$Z_{ij} := Z(x_i, x_j) = \cos\tau_i \cos\tau_j + \sin\tau_i \sin\tau_j(\overrightarrow{x_i} \cdot \overrightarrow{x_j}) \tag{54}$$

where we restrict to $Z \in [-1, 1]$ when analytically continued, using equation (3.17), from the sphere,[10] becomes the de Sitter embedding distance

$$Z_{ij} = -\sinh\tau_i \sinh\tau_j + \cosh\tau_i \cosh\tau_j(\overrightarrow{x_i} \cdot \overrightarrow{x_j}) \tag{55}$$

embedded into the manifold $\mathbb{M}^{(D,1)}$ which on dS_D has a range of values over $\mathbb{R}$ [10]. We then perform analytic continuation to the Lorentzian by continuing Z_{ij} from $[-1, 1]$ to $\mathbb{R}$. This is because it satisfies the following conditions,

- for spacelike separations $Z_{ij} \in [-1, 1)$
- null for $Z_{ij} = 1$
- for timelike case $|Z_{ij}| > 1$

The time ordered two-point correlation functions in the Lorentzian will be,

$$\langle T\phi(x_i)\phi(x_j)\rangle_L := \langle\phi_m\phi_n(\bar{Z}_{ij})\rangle$$

$$= \langle \phi_m \phi_n (Z_{ij} + i\epsilon) \rangle \tag{56}$$

Generalising this we can define Wightman two-point function in Lorentzian to be,

$$\langle T\phi(x_i)\phi(x_j)\rangle_L := \langle \phi_m \phi_n (\tilde{Z}_{ij}) \rangle \tag{57}$$

where $\tilde{Z}_{ij}$ may be defined as,

$$\tilde{Z}_{ij} = Z_{ij} \pm i\epsilon \tag{58}$$

depending on whether $x_1^0 > x_2^0$ or $x_2^0 > x_1^0$ respectively.

3.4 Spherical harmonics, Watson-Sommerfield transformations and Gegenbauer polynomials

Calculating Feynman diagrams, working in the Euclidean signature in the basis $L^2(S^D)$ given by spherical harmonics $Y_{\overrightarrow{L}}$. Here $\overrightarrow{L}$ is a D-dimensional vector satisfying $L_D \geq L_{D-1} \geq \cdots \geq L_2 \geq |L_1|$ and spherical harmonics satisfying the following relations,

$$-\ell^2 \nabla_x^2 Y_{\overrightarrow{L}}(x) = -L(L+d) Y_{\overrightarrow{L}} \tag{59}$$

$$\ell^D \tilde{\delta}(x_i, x_j) = \sum_{\overrightarrow{L}} Y_{\overrightarrow{L}}(x_i) Y^*_{\overrightarrow{L}}(x_j) \tag{60}$$

$$\ell^D \delta_{\overrightarrow{L},\overrightarrow{M}} = \int Y_{\overrightarrow{L}}(x_i) Y^*_{\overrightarrow{M}}(x_i) \tag{61}$$

Here our ∇_x^2 is just the Laplacian on the Euclidean sphere defined by our metric $d\Omega_D^2$ [10, 15, 18]. The last two equations are orthonormality and completeness relations. $\overrightarrow{L} = (L_D, L_{D-1}, ..., L_1)$ represents the set of angular momentum on the D-sphere and have the property of,

$$L_D \geq L_{D-1} \geq ... \geq |L_1| \tag{62}$$

There is also the relation for when $\overrightarrow{L} = (L, \overrightarrow{j})$,

$$\sum_{\overrightarrow{j}} Y_{L\overrightarrow{j}}(x_i) Y^*_{L\overrightarrow{j}}(x_j) = \frac{\Gamma(\frac{d}{2})(2L+d)}{4\pi^{d/2+1}} C_L^{d/2}(Z_{x_i x_j}) \tag{63}$$

where $\Gamma(d/2)$ is a gamma function which is defined for complex t with positive real part (so that it is absolutely convergent),

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx \tag{64}$$

specifying to the case for positive integer values, $n \in \mathbb{N}$, however the Γ-function has the form,

$$\Gamma(n) = (n-1)! \tag{65}$$

$C_L^{d/2}(Z_{x_i x_j})$ is a Gegenbauer polynomial and the Gegenbauer function $C_\lambda^\alpha(z)$ is a polynomial provided that λ is a non-negative integer. Feynman diagrams on S^D can be expressed as sums over spherical harmonics which we can express in terms of Gegenbauer polynomials. Looking at its expression in terms of a hypergeometric function, we find the hypergeometric series will terminate. $C_L^{d/2}(Z_{x_i x_j})$ defined in terms of the hypergeometric function ${}_2F_1$ as,

$$C_\lambda^\alpha(z) :=_2 F_1\left(-\lambda, \lambda + 2\alpha, \alpha + \frac{1}{2}; \frac{1-z}{2}\right) \Gamma\begin{bmatrix} 2\alpha + \lambda \\ 1 + \lambda, 2\alpha \end{bmatrix} \tag{66}$$

where this notation for gamma functions follows a shorthand prescription defined as,

$$\Gamma\begin{bmatrix} 2\alpha + \lambda \\ 1 + \lambda, 2\alpha \end{bmatrix} = \frac{\Gamma(2\alpha + \lambda)}{\Gamma(1+\lambda)\Gamma(2\alpha)} \tag{67}$$

${}_2F_1$ is the Gaussian hypergeometric function given by,

$${}_2F_1(a,b;c;z) = 1 + \frac{ab}{c} z + \frac{a(a+1)b(b+1)}{c(c+1)2!} z^2 ... + \frac{a(a+1)...(a+n-1)b(+1)...(b+n-1)}{c(c+1)...(c+n-1)n!} z^n + ...$$

$${}_2F_1(a,b;c;z) = 1 + \frac{ab}{c} z + \cdots \frac{(a)_n (b)_n}{(c)_n} z^n \tag{68}$$

and where $(a)_n$ represents the Pochhammer symbol defined by,

$$(a)_n = \Gamma\begin{bmatrix} a+n \\ a \end{bmatrix} = a(a+1)\cdots(a+n-1) \tag{69}$$

for any complex a and $n \in \mathbb{N}_0$. The Gegenbauer polynomials obey useful relations,

$$(\lambda+\alpha)C_\lambda^\alpha(Z) = \alpha\left[C_\lambda^{\alpha+1}(Z) - C_{\lambda-2}^{\alpha+1}(Z)\right]$$

$$\frac{d^n}{dZ^n}C_\lambda^\alpha(Z) = 2^n(\alpha)_n C_{\lambda-n}^{\alpha+n}(Z) \tag{70}$$

$$C_L^\alpha(Z) = (-1)^L C_L^\alpha(Z)$$

the first two being recursion relations and the final one being the Gegenbauer reflection formula. Summations over Gegenbauer polynomials can, by analytic continuation, transform into contour integrals in the complex plane. Returning to our sum, while it is possible to analytically continue such sums over the polynomials using (3.25), this diverges for large Z_{xy}. We therefore use Watson-Sommerfield transformations which can be thought of as an analytic continuation in momentum space.

3.5 Watson-Sommerfield transformations

Consider some function, $f(z)$, so that it is analytic $\forall z = n \in \mathbb{Z}$ and otherwise arbitrary, which decays at a minimum like,

$$\frac{1}{|z|^2} \quad as \quad |z| \to \infty \tag{71}$$

Then for an infinite sum,

$$N = \sum_{n=-\infty}^{\infty} f(n) \tag{72}$$

and consider a function F related to f by,

$$F(z) = \pi f(z)\cot(\pi z) \tag{73}$$

which has simple poles for $\forall z = n \in \mathbb{Z}$ with all residues as,

$$Res\,(F(z), z = n) = f(n) \tag{74}$$

An integral over a circle with radius $R \to \infty$ centred at the origin will be zero due to its behaviour as $|z| \to \infty$

$$\oint F(z)dz = 0 \tag{75}$$

By the residue theorem however we know,

$$\oint_C F(z)dz = 2\pi i \left\{ \sum_{n=-\infty}^{\infty} Res\left(F(z), z = n\right) + \sum_i Res\left(F(z), z = z_i\right) \right\} = 0 \tag{76}$$

with z_i being the poles coming from the function $f(n)$. Using these equations we can rearrange to get,

$$\sum_{n=-\infty}^{\infty} f(n) = -\sum_i Res\left[F(z), z = z_i\right] \tag{77}$$

While this a simple example, the general case a sum $S = \sum_L s(L)$, we define a function $\tilde{s}(L)$ with the following two key features. Firstly, that they must agree with $s(L)$ $\forall$ L appearing in the sum. Secondly, $\tilde{s}(L)$ must be analytic in some open neighbourhood of our complex L-plane at each of the aforementioned points where they agree. We then multiply $\tilde{s}(L)$ by some kernel function $k(L)$, which is meromorphic meaning holomorphic except at a series of isolated poles. A suitable contour of integration is chosen, C_0, resulting in our original series now being represented as,

$$S = \sum_L s(L) = \oint_{C_0} \frac{dL}{2\pi i} k(L)\tilde{s}(L) \tag{78}$$

We can then deform our contour to another over which we may perform calculations more easily, $C_0 \to C$, and have greater control over. As can be seen, our sum contains a Gegenbauer polynomial which we analytically continue to a function of our complex L, denoted Γ_P associated with the principal series for our dimensionless parameter σ defined previously.

3.6 Free Klein-Gordon equation

For illustration of these techniques we review the free Klein-Gordon field. Beginning with the free propagator, Δ^{σ}_{xy}, there exists a unique solution to the Klein-Gordon equation,

$$-(\nabla^2_x - M^2)\Delta^{\sigma}_{xy} = -(\nabla^2_y - M^2)\Delta^{\sigma}_{xy} = \delta_{xy} \tag{79}$$

In terms of spherical harmonics, our propagator Δ^{σ}_{xy} has the form,

$$\Delta^{\sigma}_{xy} = \ell^{2-D} \sum_{\vec{L}} \frac{Y_{\vec{L}}(x) Y_{\vec{L}}(y)}{M^2\ell^2 + L(L+d)} \tag{80}$$

in terms of our dimensionless mass parameter σ and rearranging for $M^2\ell^2$,

$$M^2\ell^2 = -\sigma^2 - \sigma d \tag{81}$$

So our denominator transforms accordingly denoted $\lambda_{L\sigma}$,

$$\lambda_{L\sigma} = L(L+d) + M^2\ell^2 = L(L+d) - \sigma^2 - \sigma d = (L-\sigma)(L+\sigma+d) \tag{82}$$

so our propagator has the form,

$$\Delta^{\sigma}_{xy} = \ell^{2-D} \sum_{\vec{L}} \frac{Y_{\vec{L}}(x) Y_{\vec{L}}(y)}{\lambda_{L\sigma}} \tag{83}$$

which gives us a spectral representation of our propagator on $(x, y) \in S^D \times S^D$. We now employ the relations developed above between spherical harmonics as well as Gegenbauer functions summing over which yields us an expression for the propagator of the form,

$$\Delta^{\sigma} = \ell^{2-D} \frac{\Gamma(\frac{d}{2})}{4\pi^{d/2+1}} \sum_{L=0}^{\infty} \frac{2L+d}{\lambda_{L\sigma}} C_L^{d/2}(Z) \tag{84}$$

which gives us the spectral representation of the propagator over $Z \in [-1, 1]$.
We are now free to compute the final sum letting $\tilde{s}(L)$ be,

$$\tilde{s}(L) = \frac{(2L+d)}{\lambda_{L\sigma}} e^{-i\pi L} C_L^{d/2}(Z) \tag{85}$$

which we obtain from the reflection formula for Gegenbauer polynomials from our propagator,

$$C_L^{\alpha}(Z) = (-1)^L C_L^{\alpha}(-Z) \tag{86}$$

and the kernel function,

$$k(L) = \frac{\pi e^{i\pi L}}{\sin(\pi L)} = -e^{i\pi L}\Gamma[-L, L+1] \tag{87}$$

which gives us poles of unit residue,

$$\Delta^{\sigma}(Z) = \ell^{2-D}\frac{\Gamma(\frac{d}{2})(-1)}{4\pi^{d/2+1}\Gamma(d)}\frac{1}{2\pi i}\oint_{C_1} dL_2F_1\left(-L, L+d; \frac{d+1}{2}; \frac{1+Z}{2}\right)\Gamma[-L, L+d]\frac{2L+d}{\lambda_{L\sigma}} \tag{88}$$

The hypergeometric function is singular at $Z = 1$ and our contour integral has poles $L \in \mathbb{N}_0$, $L = -s, -(d+1), \cdots$ as well as $L = \sigma, -(\sigma+d)$. We deform the contour C_1, integrating around poles encountered, to a straight line passing through $L = -d/2$ deforming through either the $L = \sigma$ or $L = -(\sigma+d)$. We therefore acquire a residue, which is equal to the other, while the remaining integral vanishes because our integrand is antisymmetric under the transformation $L \to -(L+d)$ and we obtain a form for the propagator, which gives us poles of unit residue,

$$\Delta^{\sigma}(Z) = \ell^{2-D}\frac{\Gamma(\frac{d}{2})}{4\pi^{d/2+1}\Gamma(d)}\times Res\left[\frac{2L+d}{\lambda_{L\sigma}}\Gamma[-L, L+d]_2F_1\left(-L, L+d; \frac{d+1}{2}; \frac{1+Z}{2}\right)\right]_{L=L_0} \tag{89}$$

where L_0 is the pole σ or $-(\sigma+d)$ taking $L = \sigma$.

For illustration we choose the pole $L_0 = \sigma$,

$$\Delta^{\sigma}(Z) = \frac{\ell^{2-D}}{4\pi^{d/2+1}}\times_2F_1\left(-\sigma, \sigma+d; \frac{d+1}{2}; \frac{1+Z}{2}\right)\Gamma\left[\begin{matrix}\frac{d}{2}, -\sigma, \sigma+d \\ d\end{matrix}\right] \tag{90}$$

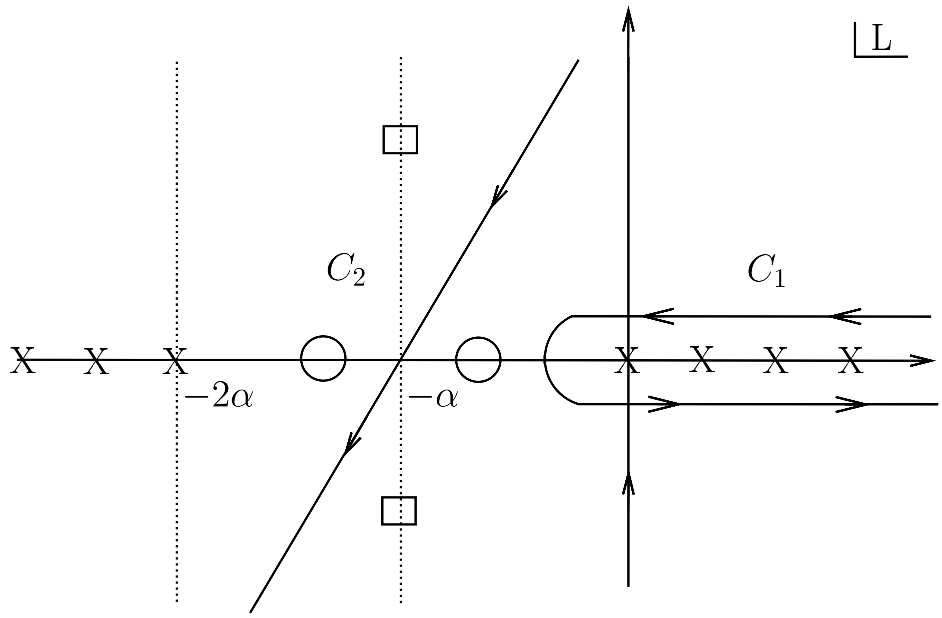


Figure 8: Example of the location of the poles in the complex L-plane when computing the propagator. $C1$ is the original curve which is deformed, while C_2 is some arbitrary straight line going through reflection point $L = -d/2$. Retaining the notation of previous literature, poles in the principal series are denoted by boxes while those in the complementary are represented by circles.

Reviewing Watson-Sommerfield transformations, we see how propagators that can be expressed in terms of spherical harmonics and Gegenbauer polynomials, can be recast as contour integrals in the complex L-plane. The reason we develop this machinery for computing propagators is to apply this to correlation functions and then the spectral density function in de Sitter space.

4 Perturbative corrections in de Sitter

4.1 Corrections to $\mathcal{O}(\mu^2)$

4.2 Introduction

Following investigations into the previous literature in de Sitter space [10], corrections to propagators through interactions of the form, $V_{int} = \mu\phi_1(x)\phi_2(x)$ at tree level, with some coupling constant μ with mass dimension $[\mu] = \frac{6-D}{2}$, gives us interactions of form,

$$\langle\phi_1(x_1)\phi_2(x_2)\rangle = -\mu \int_{y\in S^D} \Delta^{\sigma_1}_{1y}\Delta^{\sigma_2}_{y2} + \mathcal{O}(\mu^3) \tag{91}$$

which, in the Euclidean signature takes the form,

$$\langle\phi_1(x_1)\phi_2(x_2)\rangle = \frac{\mu}{M_1^2 - M_2^2}\left[\Delta^{\sigma_1}(Z_{12}) - \Delta^{\sigma_2}(Z_{12})\right] + \mathcal{O}(\mu^3) \tag{92}$$

due to the fact that there are no surface terms upon integration by parts. This is not true however, in de Sitter space [10]. At the one-loop level for our case, we present the two-point function which will be made up of three and four particle interactions only. We are considering three particle interactions as they provide more interesting features while computations for four particle interactions can be found in the literature [10]. Our interaction will now take the form,

$$V_{int} = \mu\phi_1(x)\phi_2(x)\phi_3(x) + \sum_{i=1}^{3}\left[-\frac{1}{2}\phi_i(x)[(\delta\phi_i)\nabla_x^2 - (\delta M_i^2)]\phi_i(x)\right] \tag{93}$$

with $\delta\phi_i$ and δM_i^2 are the counterterms to the field and mass renormalisation respectively. Respectively, they have mass dimension 0 and +2. For the corrections to the two-point function we sum diagrams of terms and counter terms at one-loop level, we define them as,

$$\langle\phi_1(x_1)\phi_1(x_2)\rangle^{(2)} = (I) + (II) + (III) \tag{94}$$

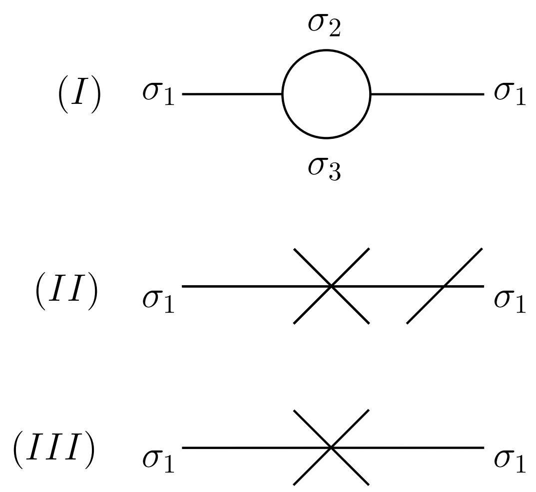


Figure 9: Perturbative corrections to the two-point function to $\mathcal{O}(\mu^2)$. Here, $(I), (II)$ and (III) are the one-loop contribution, counterterm arising from renormalisation of the field and mass renormalisation respectively. The slash in diagram (II) represents the action of ∇^2.

will give us one term which is the one-loop contribution and two counter terms: the field and mass renormalisations, denoted (I), (II) and (III) respectively. We also use a shorthand for the spacetime dimension, $\alpha := d/2 = (D-1)/2$.

$$(I) = \mu^2 \int_{x \in S^D} \int_{y \in S^D} \Delta^{\sigma_1}_{1x} \Delta^{\sigma_2}_{xy} \Delta^{\sigma_3}_{xy} \Delta^{\sigma_1}_{y2} \tag{95}$$

where the product of two propagators is defined through the spherical harmonics,

$$\Delta^{\sigma_1}\Delta^{\sigma_2}(Z_{12}) = \sum_{\vec{L}} \rho_{\sigma_1\sigma_2}(L) Y_{\vec{L}}(x_1) Y^*_{\vec{L}}(x_2)$$

so,

$$\Delta^{\sigma_1}\Delta^{\sigma_2}(Z_{12}) = \ell^{4-2D} \frac{\Gamma(\alpha)}{2\pi^{\alpha+1}} \sum_{L=0}^{\infty} (L+\alpha)\ \sigma_1\sigma_2(L) C^\alpha_L(L) \tag{96}$$

and where $\rho_{\sigma_1\sigma_2}(L)$ is a spectral function not to be confused with the spectral density. It is defined through the integral of three Gegenbauer polynomials,

$$\rho_{\sigma_1\sigma_2} := \ell^{2D-4} \frac{2\pi^{\alpha+1}}{\Gamma(\alpha)(L+\alpha)} \frac{1}{A^\alpha_L} \int_{-1}^{1} dZ (1-Z^2)^{\alpha-1/2} C^\alpha_L(Z) \Delta^{\sigma_1}(Z) \Delta^{\sigma_2}(Z) \tag{97}$$

which converges for $0 < \alpha < \frac{3}{2}$ with normalisation A^{α}_{L} defined as,

$$A^{\alpha}_{L}\delta^{LM} := \int_{-1}^{1} dZ(1-Z^2)^{\alpha-1/2}C^{\alpha}_{L}(Z)C^{\alpha}_{M}(Z)\delta^{LM} \tag{98}$$

There are several features, which we will see later prove useful, that our spectral function remains invariant under the following transformations,

$$\sigma_1 \to -(\sigma_1 + 2\alpha), \quad \sigma_2 \to -(\sigma_2 + 2\alpha), \quad \sigma_1 \leftrightarrow \sigma_2 \tag{99}$$

$$\overline{\rho_{\sigma_1\sigma_2}(L)} = \rho_{\overline{\sigma_1\sigma_2}}(\overline{L}) \tag{100}$$

$\rho_{\sigma_1\sigma_2}$ has the feature of being absolutely convergent and provided we are working with on shell masses σ_1 and σ_2,

$$\overline{\rho_{\sigma_1\sigma_2}(L)} = \rho_{\sigma_1\sigma_2}(\overline{L}) \tag{101}$$

are invariant under these complex conjugations [10]. This means $\rho_{\sigma_1\sigma_2}$ will only be complex if L is complex. Turning now to our counterterms, these are given as,

$$(II) = (\delta\phi_1)\int_{x\in S^D} \Delta^{\sigma_1}_{1x}\Box_x\Delta^{\sigma_1}_{x2} = -\ell^{2-D}\frac{\Gamma(\alpha)}{2\pi^{\alpha+1}}\sum_{L=0}^{\infty}\frac{(L+\alpha)(\delta\phi_i)L(L+2\alpha)}{(\lambda_{L\sigma_1})^2}C^{\alpha}_{L}(Z_{12}) \tag{102}$$

$$(III) = -(\delta M_1^2)\int_{x\in S^D} \Delta^{\sigma_1}_{1x}\Delta^{\sigma_1}_{x2} = -\ell^{4-D}\frac{\Gamma(\alpha)}{2\pi^{\alpha+1}}\sum_{L=0}^{\infty}\frac{(L+\alpha)(\delta M_1^2)L(L+2\alpha)}{\lambda^2_{L\sigma_1}}C^{\alpha}_{L}(Z_{12}) \tag{103}$$

Summing (I), (II), (III) together yields,

$$\langle\phi_1(x_1)\phi_1(x_1)\rangle^{(2)} = \ell^{2-D}\frac{\Gamma(\alpha)}{2\pi^{\alpha+1}}\sum_{L=0}^{\infty}\frac{\mu^2\ell^{6-D\rho_{\sigma_2\sigma_3}}(L) - \ell^2(\delta M_1^2) - L(L+2\alpha)(\delta\phi_1)}{(\lambda_{L\sigma_1})^2} \tag{104}$$

which we simplify,

$$\langle\phi_1(x_1)\phi_1(x_1)\rangle^{(2)} = \ell^{2-D}\frac{\Gamma(\alpha)}{2\pi^{\alpha+1}}\sum_{L=0}^{\infty}\frac{\Pi(L)}{(\lambda_{L\sigma_1})^2}(L+\alpha)C^{\alpha}_{L} \tag{105}$$

$\Pi(L)$ being the dimensionless self-energy defined as,

$$\Pi(L) = \mu^2\ell^{6-2D}\rho_{\sigma_2\sigma_3}(L) - \ell^2(\delta M_1^2) - L(L+2\alpha)(\delta\phi_1) \tag{106}$$

and finally putting this into its simplest form,

$$\langle \phi_1(x_1)\phi_1(x_1)\rangle^{(2)} = \ell^{2-D}\frac{\Gamma(\alpha)}{2\pi^{\alpha+1}}\sum_{L=0}^{\infty} f(L)(L+\alpha)C_L^{\alpha}(L) \tag{107}$$

with $f(L)$ being defined accordingly,

$$f(L) := \frac{\mu^2\ell^{6-D\rho_{\sigma_2\sigma_3}}(L) - \ell^2(\delta M_1^2) - L(L+2\alpha)(\delta\phi_1)}{(\lambda_{L\sigma_1})^2} = \frac{\Pi(L)}{(\lambda_{L\sigma_1})^2} \tag{108}$$

4.3 Watson-Sommerfield transformations applied to perturbative corrections

With all of our shorthand notation now defined, we can now implement the Watson-Sommerfield transformation, transforming our corrections using some kernel function and integrating around some contour, C, in order to analytically continue this to the Lorentzian signature using Watson Sommerfield transformation [10]. This contour has poles $\forall L \in \mathcal{N}_0$,

$$\langle \phi_1(x_1)\phi_1(x_1)\rangle^{(2)} = -2\oint_C \frac{dL}{2\pi i} f(L)(L+\alpha)\Delta^L(Z_{12}) \tag{109}$$

Our contour of integration may be shifted away from our curve C to a new contour Γ to the line Γ_P whereby the real part of our momentum lies on the line $-\alpha$, choosing the contour to pass on the left side of the poles we will encounter. There will be multiple poles acquiring their residues. The poles lie at,

$$L = \sigma_1, \quad -(\sigma_1+2\alpha), \quad L = n, \quad L = -(n+2\alpha) \quad for \quad n \in \mathbb{N}_0 \tag{110}$$

The first two are simple poles coming from our distribution of $\Delta^L(Z)$ the last two being double poles in $(\lambda_{L\sigma_1})^2$ as well as simple poles arising from the spectral function in complex plane L,

$$L = \sigma_1+\sigma_2-2n, \quad -\sigma_1+\sigma_2-2n, \quad +\sigma_1-\sigma_1-2n, \quad -\sigma_1-\sigma_2-4\alpha-2n \tag{111}$$

However, relatively few poles are acquired when we deform our contour accordingly $C \to \Gamma$. This will be affected whether our ϕ_i's are in the complementary or principal series [10, 11, 22, 23]. For the complementary series case, if just ϕ_1 is in the complementary series our pole is at $L = \sigma_1$ with our mass lying between, $-\alpha < \sigma_1 < 0$. If σ_2 and σ_3 lie in the complementary series, our poles lie at $L = \sigma_2 + \sigma_3$, and $L = \sigma_2 + \sigma_3 = 2$, with the range of masses lying in the $-\alpha < \sigma_2 + \sigma_3 < 0$ and possibly $-\alpha < \sigma_2 + \sigma_3 - 2 < 0$. In these cases all poles lie to the right of the contour Γ. When ϕ_1 is in the principal series, both poles, $L = \sigma_1$ and $L = -(\sigma_1 + 2\alpha)$, are on Γ_P. Our corrections to order $\mathcal{O}(\mu^2)$ are then,

$$\langle \phi_1(x_1)\phi_1(x_2)\rangle^{(2)} = 2Res\left[f(L)(L+\alpha)\Delta^L(Z_{12})\right]_{L=\sigma_1,-(\sigma_1+2\alpha),(\sigma_2+\sigma_3),(\sigma_2+\sigma_3-2)}$$

$$+2\int_\Gamma \frac{dL}{2\pi i} f(L)(L+\alpha)\Delta^L(Z_{12}) \tag{112}$$

The last three poles in the first line need not be considered unless $Re(L) \geq -\alpha$. Returning to our example of ϕ_1 being in the complementary case, the first term in our previous equation will be just the residue computed at $L = \sigma_1$,

$$R_1 = 2Res\left[f(L)(L+\alpha)\Delta^L(Z_{12})\right]_{L=\sigma_1} \tag{113}$$

Expanding this out we can rewrite it as,

$$R_1 = -\frac{\partial}{\partial M^2}\left[\left(\mu^2 Re[\ell^{4-D}\rho_{\sigma_2\sigma_3}(\sigma(M^2))] - (\delta M_1^2) + M^2(\delta\phi_1)\right)\right]_{M^2=M_1^2} \tag{114}$$

The integral over our contour Γ is a little more difficult however. We first let $L = -\alpha + i\nu$ so our integral,

$$I := 2\int_{\Gamma_P} \frac{dL}{2\pi i}(L+\alpha)f(L)\Delta^L(Z_{12})$$

$$I = \frac{i}{\pi}\int_{-\infty}^{\infty} d\nu \quad \nu\left[f(-\alpha+i\nu)\Delta^{-\alpha+i\nu}(Z_{12})\right] \tag{115}$$

Our contour is symmetric under complex conjugation so we can rewrite it so our integral is real for real Z_{12}.

$$I = \frac{i}{\pi}\int_0^\infty d\nu \quad \nu\left[f(-\alpha+i\nu)\Delta^{-\alpha+i\nu}(Z_{12}) - f(-\alpha-i\nu)\Delta^{-\alpha-i\nu}(Z_{12})\right] \tag{116}$$

Expanding out, our counterterms cancel and we are left with just an expression in terms of the propagator and the spectral function $\rho_{\sigma_2\sigma_3}$, which is absolutely convergent. There are also some simplifications owing to the fact that $\lambda_{-\alpha+i\nu,\sigma_1} = \lambda_{-\alpha-i\nu,\sigma_1}$ as well as $\Delta^{-\alpha+i\nu}(Z_{12}) = \Delta^{-\alpha-i\nu}(Z_{12})$

$$I = \frac{i\mu^2\ell^{6-D}}{\pi}\int_0^\infty d\nu \frac{\nu\left[\rho_{\sigma_2\sigma_3}(-\alpha+i\nu) - \rho_{\sigma_2\sigma_3}(-\alpha-i\nu)\right]}{(\lambda_{-\alpha+i\nu,\sigma_1})^2}$$

$$I = -2\frac{\mu^2\ell^{6-D}}{\pi}\int_0^\infty d\nu \frac{\nu Im\left[\rho_{\sigma_2\sigma_3}(-\alpha+i\nu)\right]}{(M_1^2\ell^2 - M_{-\alpha+i\nu}^2\ell^2)^2}\Delta^{-\alpha+i\nu}(Z_{12})$$

$$I = -2\frac{\mu^2\ell^{2-D}}{\pi}\int_0^\infty d\nu \frac{\nu Im\left[\rho_{\sigma_2\sigma_3}(-\alpha+i\nu)\right]}{(M_1^2 - M_{-\alpha+i\nu}^2)^2}\Delta^{-\alpha+i\nu}(Z_{12}) \tag{117}$$

In the case of $\phi_1(x)$ being in the principal series with $\sigma_1 = -\alpha + i\tau$ for $\tau \in \mathbb{R}$, both poles lie along Γ_P. The residues sum to twice our result in the complementary case, $2R_1$. The last case considered is when ϕ_2 and ϕ_3 have light enough masses, therefore lying in the complementary series. We then specify to,

$$-\alpha < \sigma_2 + \sigma_3 < 0 \tag{118}$$

and perhaps also,

$$-\alpha < \sigma_2 + \sigma_3 - 2 < 0 \tag{119}$$

Deforming the contour, we encounter the poles $L = \sigma_2 + \sigma_3, \sigma_2 + \sigma_3 - 2$, whose residues when evaluated give,

$$R_2 = 2Res\left[f(L)(L+\alpha)\Delta^L(Z_{12})\right]_{L=\sigma_2+\sigma_3} = \frac{\mu^2\ell^{6-D}}{4\pi^{\alpha+1}(\lambda_{\sigma_2+\sigma_3})^2}\Gamma\begin{bmatrix} -\sigma_2, \sigma_2+\alpha, -\sigma_3, \sigma_3+\alpha \\ -\sigma_2-\sigma_3, \sigma_2+\sigma_3+\alpha \end{bmatrix}$$

as well as,

$$\times\Delta^{\sigma_2+\sigma_3}(Z_{12}) \tag{120}$$

$$R_3 = 2Res\left[f(L)(L+\alpha)\Delta^L(Z_{12})\right]_{L=\sigma_2+\sigma_3-2} = \frac{\mu^2\ell^{6-D}}{\pi^{\alpha+1}(\lambda_{\sigma_2+\sigma_3-2})^2}\frac{\alpha(\sigma_2+\sigma_3+2\alpha-2)}{\sigma_2+\sigma_3+\alpha-1}$$

$$\times\Gamma\begin{bmatrix}1-\sigma_2, \sigma_2+\alpha-1, 1-\sigma_3, \sigma_3+\alpha-1\\ 2-\sigma_2-\sigma_3, \sigma_2+\sigma_3+\alpha-2\end{bmatrix}\Delta^{\sigma_2+\sigma_3-2}(Z_{12}) \tag{121}$$

Our corrections to order $\mathcal{O}(\mu^2)$ are the sum of these residues,

$$\langle\phi_1(x_1)\phi_1(x_2)\rangle^{(2)} = R_1 + P(I) + R_2 + R_3 \tag{122}$$

The last two terms R_2 and R_3 exist only for $-\alpha < \sigma_2+\sigma_3 < 0$ and $-\alpha < \sigma_2+\sigma_3-2 < 0$, respectively; when their masses are sufficiently light. The $P(I)$ represents that we are taking the principal part of the branch whereby we are integrating poles that lie on the axis. Our correlator in the Lorentz signature, $\langle T\phi_1(x_1)\phi_1(x_2)\rangle$, is obtained by going from $Z_{12} \to \tilde{Z}_{12}$ while our corrections $\langle\phi_1(x_1)\phi_1(x_2)\rangle_L^{(2)}$ is identical to our result for the Euclidean save for $Z_{12} \to \bar{Z}_{12}$. This is related to the spectral density we encounter with the Källén-Lehmann representation,

$$\langle\phi_1(x_1)\phi_1(x_2)\rangle_L^{(2)} = \int_0^\infty dM^2\rho(M^2)\Delta^{M^2}(Z_{12}) \tag{123}$$

We need to bring all terms in $\langle\phi_1(x_1)\phi_1(x_2)\rangle^{(2)}$ into a form whereby they are integrals over our masses lying in the principal series. Our expression for I is by definition already in this form, so the remaining terms, R_1, R_2 and R_3 also need to be brought into this form. These terms however, can brought into an integral over M^2 by multiplying them by delta functions.

4.4 Gegenbauer polynomials and normalisation

Gegenbauer polynomials $C_L^\alpha(Z)$ obey some useful relations so we define A_L^α as the normalisation for our Gegenbauer polynomials integral,

$$A_L^\alpha\delta^{LM} := \int_{-1}^1 dZ(1-Z^2)^{\alpha-1/2}C_L^\alpha(Z)C_M^\alpha(Z)\delta^{LM} \tag{124}$$

as well as the integral of three Gegenbauer polynomials with a common degree, in our case denoted α. This integral defined as $D(\alpha; L, M, N)$,

$$D(\alpha; L, M, N) := \int_{-1}^{1} dZ(1-Z^2)^{\alpha-1/2} C_L^{\alpha}(Z) C_M^{\alpha}(Z) C_N^{\alpha}(Z) \tag{125}$$

which is non-zero only when,

$$J := \frac{L+M+N}{2} \in \mathbb{N}_0 \tag{126}$$

where J is a natural number, giving us the form,

$$D(\alpha; L, M, N) = \frac{2^{1-2\alpha}\pi}{\Gamma^4(\alpha)} \Gamma\begin{bmatrix} J+2\alpha, J-L+\alpha, J-M+\alpha, J-N+\alpha \\ J+\alpha+1, J-L+1, J-M+1, J-N+1 \end{bmatrix} \tag{127}$$

4.5 Spectral function calculations

For computing the spectral function, which differs from the spectral density [10], we incorporate the above restrictions of requiring $J \in \mathbb{N}$ and the triangle inequalities.

$$\Lambda_{L\sigma} := \frac{2(L+\alpha)}{\lambda_{L\sigma}} = \frac{2(L+\alpha)}{(L-\sigma)(L+\sigma+2\alpha)} = \frac{1}{L-\sigma} + \frac{1}{L+\sigma+2\alpha} \tag{128}$$

$$\rho_{\sigma_1\sigma_2} = \frac{2\pi^{\alpha+1}}{\Gamma(\alpha)(L+\alpha)A_L^{\alpha}} \frac{\Gamma^2(\alpha)}{(4\pi^{\alpha+1})^2}$$

$$\times \sum_{M=0}^{\infty}\sum_{N=0}^{\infty} \Lambda_{M\sigma_1}\Lambda_{N\sigma_2} \int_{-1}^{1} dZ(1-Z^2)^{\alpha-1/2} C_L^{\alpha}(Z) C_M^{\alpha}(Z) C_N^{\alpha}(Z)$$

$$\begin{aligned} \rho_{\sigma_1\sigma_2} &= \frac{\Gamma(\alpha)}{8\pi^{\alpha+1}(L+\alpha)A_L^{\alpha}} \sum_{M=0}^{\infty}\sum_{N=0}^{\infty} \Lambda_{M\sigma_1}\Lambda_{N\sigma_2} D(\alpha; L, M, N) \\ &=: \frac{1}{8\pi^{\alpha+1}} \Gamma\begin{bmatrix} L+1 \\ \alpha, L+2\alpha \end{bmatrix} S_{\sigma_1\sigma_2} \end{aligned} \tag{129}$$

where $S_{\sigma_1\sigma_2}$ is defined as,

$$S_{\sigma_1\sigma_2} := \sum_{M,N} \Lambda_{M,\sigma_1}\Lambda_{N,\sigma_2} \Gamma\begin{bmatrix} J+2\alpha, J-L+\alpha, J-M+\alpha, J-N+\alpha \\ J+\alpha+1, J-L+1, J-M+1, J-N+1 \end{bmatrix} \tag{130}$$

L, M and N satisfy the triangle inequalities

$$|L - M| \leq N \leq L + M \tag{131}$$

$$|L - N| \leq M \leq L + N \tag{132}$$

Following the previous prescription [10] we make a change of variables which allow us to encapsulate the conditions.

$$G := \frac{-L + M + N}{2} = J - L, \qquad K := \frac{L + M - N}{2} = J - N \tag{133}$$

Re-expressing $S_{\sigma_1\sigma_2}$ in terms of these new variables G and K yields,

$$S_{\sigma_1\sigma_2} = \sum_{G=0}^{\infty}\sum_{K=0}^{L} \Lambda_{G+K,\sigma_1}\Lambda_{G+L-K,\sigma_2}\Gamma\begin{bmatrix} K+\alpha, L-K+\alpha, G+\alpha, G+L+2\alpha \\ K+1, L-K+1, G+1, G+L+\alpha+1 \end{bmatrix} \tag{134}$$

Performing the sum over K first gives us,

$$H(L;G) := \sum_{K=0}^{L} \Lambda_{G+K,\sigma_1}\Lambda_{G+L-K,\sigma_2}\Gamma\begin{bmatrix} K+\alpha, L-K+\alpha, G+\alpha, G+L+2\alpha \\ K+, L-K+\alpha \end{bmatrix} \tag{135}$$

To treat this sum, previous literature has attempted to solve this problem by means of contour integration in the complex K-plane. To do this our integral is multiplied by $\pi\cot(\pi K)$, which has poles for $K \in \mathbb{N}_0$. This obeys the relationship

$$\pi\cot(\pi K) = -\cos(\pi K)\Gamma[-K, K+1] \tag{136}$$

Our contour integral I now becomes,

$$I := -\frac{1}{2\pi i}\oint_{\infty} dK \cos(\pi K)\Lambda_{G+K,\sigma_1}\Lambda_{G+L-K,\sigma_2}\frac{\Gamma[K+\alpha, L-K+\alpha]}{(K)_{L+1}} \tag{137}$$

with the contour of integration $\oint_{\infty}$ chosen to be an arc where the modulus of momentum is near infinity. It is assumed $\sigma_i \neq -\alpha + \mathbb{Z}$ so as to ensure poles do not overlap. This integral has multiple simple poles at,

- $K = 0, 1, ..., L$ from the $(K)_{L+1}$ Pochhammer symbol for complex K

- $K=-\alpha-n, n\in\mathbb{N}_0$ coming a $\Gamma(K+\alpha)$

- $K=L+\alpha+n$ from the other $\Gamma(L-K+\alpha)$ in the numerator

- $K=-G+\sigma_1$ and $K=-G-\sigma_1-2\alpha$ from Λ_{G+L,σ_1}

- Poles from the other Λ_{G+L-K,σ_2} located at $K=G+L-\sigma_2$ and $K=G+\sigma_2+2\alpha$

These will all sum to zero but nevertheless allow us to get a concrete answer for $H(L;G)$. The solutions to these poles are given, respectively as,

- The Pochhammer poles are chosen sum to $-H(L;G)$ by construction

- The infinite series $\cos(\pi\alpha)\times\sum_{n=0}^{\infty}\Lambda_{G-n-\alpha,\sigma_1}\Lambda_{G+n+L+\alpha,\sigma_2}\Gamma\left[\begin{smallmatrix}n+\alpha,n+L+2\alpha\\ n+1,n+L+\alpha+1\end{smallmatrix}\right]$

- Another infinte series $\cos(\pi\alpha)\times\sum_{n=0}^{\infty}\Lambda_{G+n+L+\alpha,\sigma_1}\Lambda_{G-n-L-\alpha,\sigma_2}\Gamma\left[\begin{smallmatrix}n+\alpha,n+L+2\alpha\\ n+1,n+L+\alpha+1\end{smallmatrix}\right]$

- $\frac{\pi\cos(\pi\sigma_1)}{\sin\pi(\sigma_1+\alpha)}\times\Lambda_{2G+L-\sigma_1,\sigma_2}\times\Gamma\left[\begin{smallmatrix}G-\sigma_1,G+L-\sigma_1+\alpha\\ G+L+1-\sigma_1,G+1-\sigma_1-\alpha\end{smallmatrix}\right]+(\sigma_1\to-(\sigma_1+2\alpha))$

- $\frac{\pi\cos(\pi\sigma_2)}{\sin\pi(\sigma_2+\alpha)}\times\Lambda_{2G+L-\sigma_2,\sigma_1}\times\Gamma\left[\begin{smallmatrix}G-\sigma_2,G+L-\sigma_2+\alpha\\ G+L+1-\sigma_2,G+1-\sigma_2-\alpha\end{smallmatrix}\right]+(\sigma_2\to-(\sigma_2+2\alpha))$

combining these results yields,

$$H(L;G)=\left[\frac{\pi\cos\pi\sigma_1}{sin(\sigma_1+\alpha)}\Lambda_{2G+L-\sigma_1,\sigma_2}\Gamma\left[\begin{matrix}G-\sigma_1,G+L-\sigma_1+\alpha\\ G+L+1-\sigma_1,G+1-\sigma_1-\alpha\end{matrix}\right]+3\;sym\right]$$

$$+\cos(\pi\alpha)\sum_{n=0}^{\infty}[(\Lambda_{G-n-\alpha,\sigma_1}\Lambda_{G+n+L+\alpha,\sigma_2}+\Lambda_{G+n+L+\alpha,\sigma_1}\Lambda_{G-n-\alpha,\sigma_2})] \tag{138}$$

where our 3-sym contains three terms referring to our original term save for the following alterations: $\sigma_1\to-(\sigma_1+2\alpha)$, $\sigma_1\leftrightarrow\sigma_2$ and $\sigma_1\to-(\sigma_2+2\alpha)$ with $\sigma_2\to\sigma_1$ simultaneously. The rest of the function S_{σ_1,σ_2} will involve computing the G-sum,

$$S_{\sigma_1\sigma_2}=\sum_{G=0}^{\infty}\Gamma\left[\begin{matrix}G+\alpha,G+L+2\alpha\\ G+1,G+L+\alpha+1\end{matrix}\right]H(L;G) \tag{139}$$

The infinite series will give a term proportional to,

$$\sum_{G=0}^{\infty}\sum_{n=0}^{\infty}\Bigg[(\Lambda_{G-n-\alpha,\sigma_1}\Lambda_{G+n+L+\alpha,\sigma_2}+\Lambda_{G+n+L+\alpha,\sigma_1}\Lambda_{G-n-\alpha,\sigma_2})$$

$$\times\Gamma\left[\begin{matrix}G+\alpha,G+L+2\alpha,n+\alpha,n+L+2\alpha\\G+\alpha,G+L+2\alpha,n+\alpha,n+L+2\alpha\end{matrix}\right]\Bigg] \tag{140}$$

This is invariant under the transformation $G \leftrightarrow n$ as is our $\Lambda_{G+n+L+\alpha,\sigma_i}$ although for $G-n$ in our Λ's we pick up an additional minus sign, i.e.,

$$\Lambda_{G-n-\alpha,\sigma_i} = -\Lambda_{n-G-\alpha,\sigma_i} \tag{141}$$

giving each an overall (-1) under these transformations, which results in the double sum vanishing and valid $\forall\alpha \in \mathbb{R}$ as well as $\forall\sigma_i \in \mathbb{C}$. This leaves us with our $S_{\sigma_1\sigma_2}(L)$ being,

$$S_{\sigma_1\sigma_2}(L) = \frac{\pi\cos(\pi\sigma_1)}{\sin\left(\pi(\sigma_1+\alpha)\right)}\sum_{G=0}^{\infty}\Bigg[\Lambda_{2G+L-\sigma_1,\sigma_2}$$

$$\times\Gamma\left[\begin{matrix}G+\alpha,G+L+2\alpha,G-\sigma_1,G+L-\sigma_1+\alpha\\G+\alpha,G+L+2\alpha,n+\alpha,n+L+2\alpha\end{matrix}\right]\Bigg]+3\,sym \tag{142}$$

We will be considering this in the case of $\alpha = 1$, so it is useful to consider that in this case it simplifies down with the Γ-functions completely cancelling out and we are left with,

$$S_{\sigma_1\sigma_2}(L) = \frac{\pi\cos(\pi\sigma_1)}{sin\pi(\sigma_1+1)}\sum_{G=0}^{\infty}[\Lambda_{2G+L-\sigma_1,\sigma_2}+3\,syms]$$

$$S_{\sigma_1\sigma_2}(L) = -\pi\cot(\pi\sigma_1)\sum_{G=0}^{\infty}[\Lambda_{2G+L-\sigma_1,\sigma_2}+3\,syms]$$

$$S_{\sigma_1\sigma_2}(L) = -\pi\cot(\pi\sigma_1)\sum_{G=0}^{\infty}\left[\frac{1}{2G+L-\sigma_1-\sigma_2}+\frac{1}{2G+L-\sigma_1+\sigma_2+2}+3\,syms\right]$$

$$= -\frac{\pi}{2}\cot(\pi\sigma_1)\sum_{G=0}^{\infty}\left[\frac{1}{G+\frac{L-\sigma_1-\sigma_2}{2}}+\frac{1}{G+\frac{L-\sigma_1+\sigma_2+2}{2}}+3\,syms\right] \tag{143}$$

which can be rendered into the form of a digamma or ψ-function, which is defined by the derivative of the logarithmic gamma function,

$$\psi(x) = \frac{d}{dx}\, ln\left[\Gamma(x)\right] = \left[\frac{\Gamma'(x)}{\Gamma(x)}\right]$$

and has several useful relations that listed in the appendix along with the unabridged calculation, which we employ. Recalling that our spectral function, $\rho_{\sigma_1\sigma_2}$, just contains a prefactor the spectral function $\rho_{\sigma_1\sigma_2}$ is,

$$\rho_{\sigma_1,\sigma_2} = \frac{1}{8\pi^{\alpha+1}}\Gamma\left[\begin{matrix} L+1 \\ \alpha, L+2\alpha\end{matrix}\right] S_{\sigma_1\sigma_2} \tag{144}$$

which for $\alpha = 1$ gives us,

$$\rho_{\sigma_1\sigma_2} = \frac{1}{8\pi^2}\frac{1}{L+1}S_{\sigma_1\sigma_2} \tag{145}$$

which put in its simplest form becomes,

$$\rho_{\sigma_1\sigma_2} = \frac{1}{16\pi(L+1)}\left[\left(\frac{\sin(\pi(\sigma_1+\sigma_2))}{\sin(\pi\sigma_1)\sin(\pi\sigma_2)}\psi\left(\frac{L-\sigma_1-\sigma_2}{2}\right) + \sigma\ syms\right) + 2\pi\right] \tag{146}$$

where our $\sigma\ syms$ is $\sigma_1 \to -(\sigma_1 + 2\alpha)$, $\sigma_1 \leftrightarrow \sigma_2$ and finally $\sigma_1 \to -(\sigma_2 + 2\alpha)$ with $\sigma_2 \to \sigma_1$ in conjuncture. We observe that it is in agreement with prior work [10]. This means that our dimensionless self free energy in $\alpha = 1$ will be,

$$\Pi(L) = \mu^2 \rho_{\sigma_2\sigma_3}(L) \tag{147}$$

where because $D = d + 1 = 3$ our counterterms are not needed as IR divergences do not arise in the $\alpha = 1$ case. We wish to compute ρ_{σ_2,σ_3} for an interaction of the form found in figure 4.1 and when we compare this with the literature for our $\alpha = 1$ case, specialising to $\sigma_2 = \sigma_3 = -\frac{1}{2}$,

$$\rho_{-\frac{1}{2},-\frac{1}{2}} = \frac{1}{8(L+1)} \tag{148}$$

This will be the case that we use when extending to calculations of the spectral density in both the complementary and principal series. This is because the term will contain our dimensionless self energy defined above. For our case for $\alpha = 1$ and $\sigma_2 = \sigma_3 = -\frac{1}{2}$ will be,

$$\Pi(L) = \frac{\mu^2}{8(L+1)} \tag{149}$$

4.6 Generalising to n-loop chain diagrams

While the main focus of this body of work concerns the spectral density at the one-loop level, it is possible to extend this to any number of loop diagrams. To do this we make several assumptions, firstly that the loop is mediated by σ_2 and σ_3 noting that the identical particle case will have a symmetry factor of two to our coupling μ. The second is we are simply working in $\alpha = 1$ and therefore can neglect our counter terms.

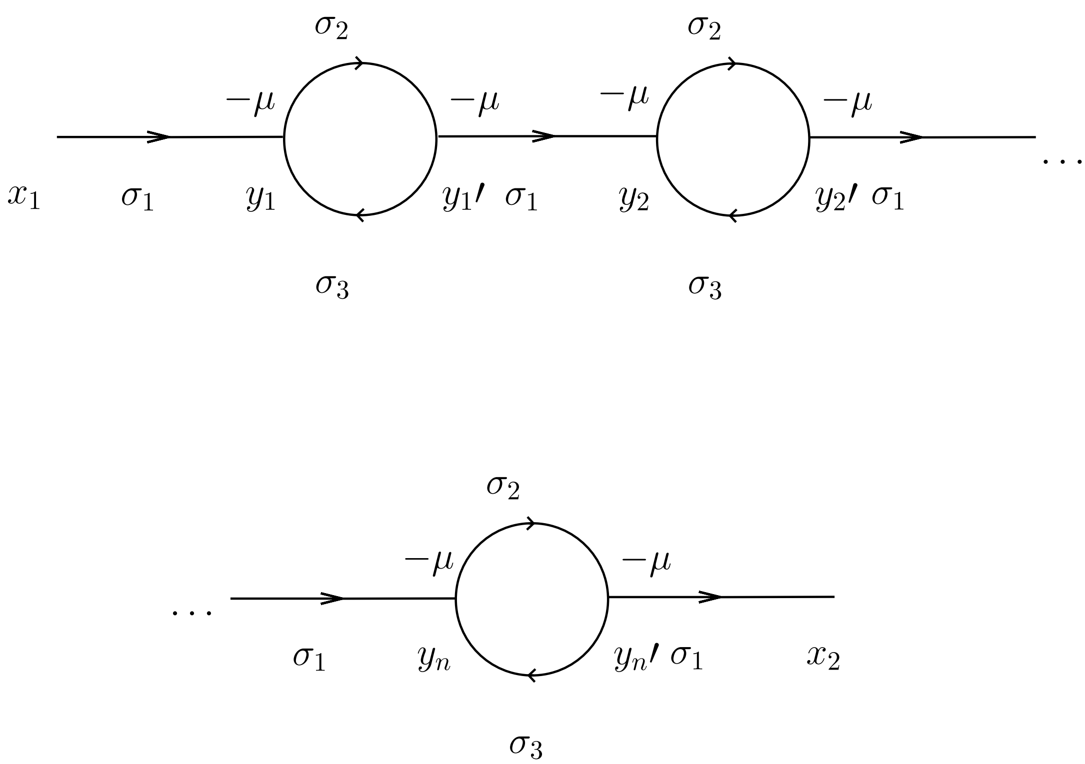


Figure 10: Summing over n-loop diagrams of σ_1 being mediated by σ_2 and σ_3. For $\sigma_2 = \sigma_3$ we get a symmetry factor of two with our coupling.

Using the relationships defined previously for the spherical harmonics, there will be $2n$ δ-functions which will simplify our calculation enormously. With n-loops we will have

$(n+1)$-propagators and again we specialise to $\alpha = 1$.

$$\langle\phi(x_1)\phi(x_2)...\phi(x_n)\rangle = \ell^{-1}\sum_{\vec{L}}\frac{Y_{\vec{L}}(x_1)Y^*_{\vec{L}}(x_2)}{\lambda_{L\sigma_1}}\sum_{n=0}^{\infty}\left[\frac{\ell^2\mu^2\rho_{\sigma_2\sigma_3}(L)}{\lambda_{L\sigma_1}}\right]^n \tag{150}$$

where again the full derivation of this is found in the appendix. This can be simplified down as a final summation over n, and making use of the relations for spherical harmonics, shown above, gives us

$$\langle\phi(x_1)\phi(x_2)...\phi(x_n)\rangle = \frac{1}{2\pi^2\ell}\sum_{L=0}^{\infty}\frac{L+1}{\lambda_{L\sigma_1} - \mu^2\rho_{\sigma_2\sigma_3}}C^1_L(Z_{12}) \tag{151}$$

our definition for $f(L)$ being,

$$f(L) = \frac{1}{\lambda_{L\sigma_1} - \mu^2\rho_{\sigma_2\sigma_3}} \tag{152}$$

Putting this all together in the Källén-Lehmann spectral representation, we find that,

$$\langle\phi_1(x_1)\phi_1(x_2)\rangle^{(2)} = \ell^{-1}\sum_{\vec{L}}\frac{Y_{\vec{L}}(x_1)Y^*_{\vec{L}}(x_2)}{\lambda_{L\sigma_1} - \mu^2\rho_{\sigma_2\sigma_3}}$$

$$= \int_0^{\infty}\frac{dM^2}{2\pi}\rho(M^2)\sum_{\vec{L}}\frac{1}{L(L+2\alpha) + M^2 - \mu^2\rho_{\sigma_2\sigma_3}}Y_{\vec{L}}(x_1)Y^*_{\vec{L}}(x_2) \tag{153}$$

4.7 Summary

In summary, we have reviewed the work found in [10] for computing perturbative corrections to propagators in de Sitter space. Our motivation is to apply this machinery to the spectral density at the one-loop level, computing it in both the complementary and principal series. Through reviewing Gegenbauer polynomials and the digamma function in conjunction with the techniques of Watson-Sommerfield transformations can we see how propagators can be recast as contour integrals in the complex plane. We then computed the spectral density function, $\rho_{\sigma_2\sigma_3}$, not to be confused with the spectral density, for the case of $\alpha = 1$ finding it in agreement with previous results [10]. As was shown, this spectral function is related to the dimensionless-self free energy which we require in order to compute the spectral density for the case of $\alpha = 1$ in the next chapter.

Lastly, we made attempts to extend this to the n-loop level which as was shown presented several challenges in the Lorentzian regime. We leave this as an open question as to why efforts to extend this prove challenging.

5 Computing the spectral density in the complementary and principal series

Now we arrive at the focus of our work, computing the spectral densities in both the lighter, complementary series, and, heavier fields, in the principal series. We will present our calculations in the d-dimensional form before restricting ourselves to the case of $\alpha = (D-1)/2 = d/2 = 1$ and $\sigma_2 = \sigma_3 = -\frac{1}{2}$.

We present these results for the spectral density and comment on its nature comparing it with the flat space case. We also find, for the complementary series, an interesting result which appears to be in conflict with previous results [10, 14, 19, 21] and support arguments found in [11, 23].

5.1 Complementary

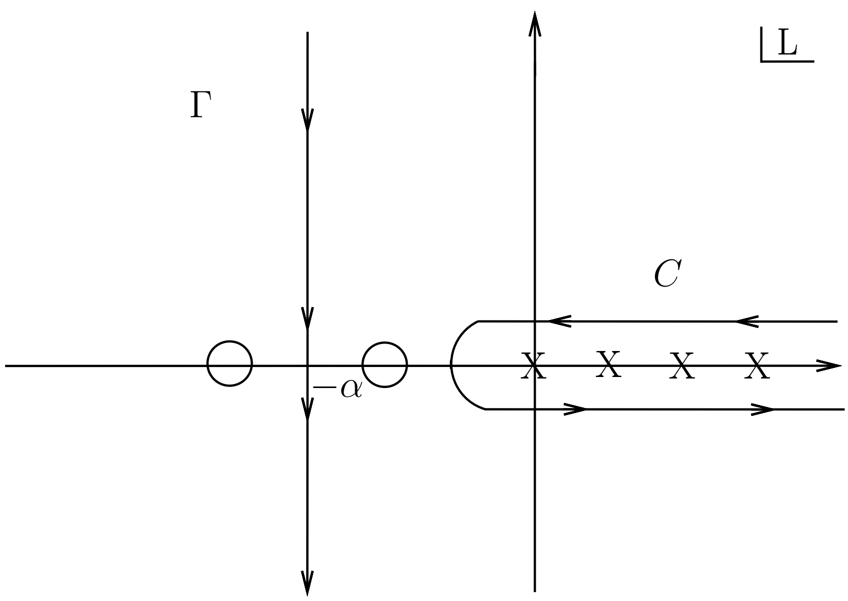


Figure 11: Contour in the complementary series picking up only one pole as it is deformed.

In the complementary series $M^2\ell^2 < 1$, we deform our contour away from the poles for $L = 0, 1, ...$ and we change the direction of the curve Γ resulting in us multiplying by -1.

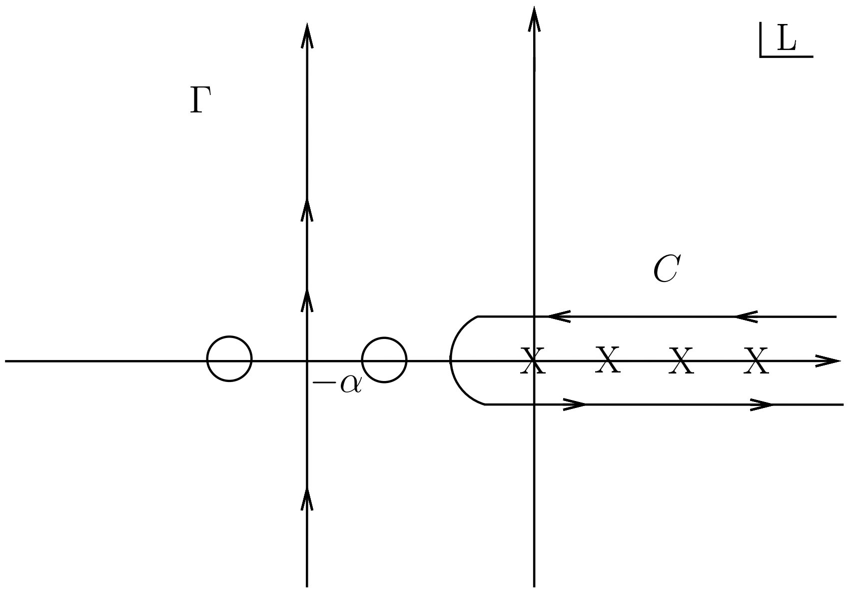


Figure 12: Contour in the complementary series with the direction reversed.

$$\langle\phi_1(x_1)\phi_1(x_2)\rangle = -2\oint_C \frac{dL}{2\pi i}\frac{L+\alpha}{\lambda_{L\sigma_1}-\Pi(L)}\Delta^L(Z_{12}) \tag{154}$$

for the simple case for illustration where $\Pi(L)=0$ we just get,

$$\langle\phi_1(x_1)\phi_1(x_2)\rangle = -2\oint_C \frac{dL}{2\pi i}\frac{L+\alpha}{(L-\sigma_1)(L+\sigma_1+2\alpha)}\Delta^L(Z_{12})$$

$$\langle\phi_1(x_1)\phi_1(x_2)\rangle = +2\times\frac{\sigma_1+\alpha}{2\sigma_1+2\alpha}\Delta^{\sigma_1}(Z_{12}) = \Delta^{\sigma_1}(Z_{12}) \tag{155}$$

which is just the propagator. Returning to the case at hand, where $\Pi(L)=\mu^2\ell^{6-2D}\rho_{\sigma_2\sigma_3}\neq 0$ in the complementary series our one-loop propagator has the form,

$$\langle 0|\phi_1(x_1)\phi_1(x_2)|0\rangle = -2\oint_C \frac{dL}{2\pi i}\frac{L+\alpha}{\lambda_{L\sigma_1}-\Pi(L)}\Delta^L(Z_{12})$$

$$\begin{aligned}\langle 0|\phi_1(x_1)\phi_1(x_2)|0\rangle &= 2\lim_{L\to L_0}\frac{(L-L_0)(L+\alpha)}{\lambda_{L\sigma_1}-\Pi(L)}\Delta^{L_0}(Z_{12})\\ &+2\int_{-\frac{d}{2}-i\infty}^{-\frac{d}{2}+i\infty}\frac{dL}{2\pi i}\frac{L+\alpha}{\lambda_{L\sigma_1}-\Pi(L)}\Delta^L(Z_{12})\end{aligned} \tag{156}$$

The first term corresponding to the first term in equation (2.11) [8, 10]. L_0 is our pole encountered at,

$$L_0 = -\alpha + \sqrt{M_0^2\ell^2 - \alpha^2} \tag{157}$$

The location of poles can be computed exactly, but this is quite lengthy and perturbative corrections give us simpler results. The exact locations of the poles have been calculated, using Vieta's substitution method, and are found in the appendix. We have that, to $\mathcal{O}(\mu^2)$,

$$\lambda_{L\sigma_1} - \Pi(L) = L(L+2) - \sigma_1(\sigma_1+2) - \frac{\mu^2}{8(L+1)}$$

$$\lambda_{L\sigma_1} - \Pi(L) = L(L+2) - -\sigma_1(\sigma_1+2) - \frac{\mu^2}{8(\sigma_1+1)} - \frac{\mu^2}{8}\left[\frac{1}{L+1} - \frac{1}{\sigma_1+1}\right]$$

simplifying we get,

$$\lambda_{L\sigma_1} - \Pi(L) = (L-L_0)(L+2+L_0) + \frac{\mu^2(L-\sigma_1)}{8(L+1)(\sigma_1+1)} \tag{158}$$

where L_0 is,

$$L_0 = \sigma_1 + \frac{\mu^2}{16(\sigma_1+1)^2} \tag{159}$$

Therefore, we have that,

$$\lambda_{L\sigma_1} - \Pi(L) \approx (L-L_0)(L+2+L_0)\frac{\mu^2}{8(\sigma_1+1)^2}(L-L_0)$$

$$\lambda_{L\sigma_1} - \Pi(L) = (L-L_0)\left[L+2+L_0 + \frac{\mu^2}{8(\sigma_1+1)^2}\right] \tag{160}$$

Applying this to 5.3 we get,

$$= 2\lim_{L\to L_0} \frac{(L-L_0)(L+1)}{(L-L_0)\left[L+2+L_0+\frac{\mu^2}{8(\sigma_1+1)^2}\right]}$$

$$= \frac{2\left[\sigma_1 + \frac{\mu^2}{16(\sigma_1+1)^2}\right]}{2\left[\sigma_1 + \frac{\mu^2}{16(\sigma_1+1)^2}\right] + \frac{\mu^2}{8(\sigma_1+1)^2}} \approx \frac{1}{1+\frac{\mu^2}{8\sigma_1(\sigma_1+1)^2}}$$

$$\approx 1 - \frac{\mu^2}{8\sigma_1(\sigma_1+1)^2} \tag{161}$$

For now we return to the second term in equation (5.3) where we will be able to generate an expression for the spectral density. In order to find the spectral density in the complementary series, we first decompose the integral into two parts over the space.

$$2\int_{-\frac{d}{2}-i\infty}^{-\frac{d}{2}+i\infty}\frac{dL}{2\pi i}\frac{L+\alpha}{\lambda_{L\sigma_1}-\Pi(L)}\Delta^L(Z_{12}) = 2\int_{-\frac{d}{2}}^{-\frac{d}{2}+i\infty}\frac{dL}{2\pi i}\frac{L+\alpha}{\lambda_{L\sigma_1}-\Pi(L)}\Delta^L(Z_{12})$$

$$+2\int_{-\frac{d}{2}-i\infty}^{-\frac{d}{2}}\frac{dL}{2\pi i}\frac{L+\alpha}{\lambda_{L\sigma_1}-\Pi(L)}\Delta^L(Z_{12}) \qquad (162)$$

Making a change of variables $L = iL_E - \alpha$ will allow us to form this into the Källén-Lehmann spectral representation leaving our integrand as a function of L for simplicity,

$$2\int_{-\frac{d}{2}-i\infty}^{-\frac{d}{2}+i\infty}\frac{dL}{2\pi i}\frac{L+\alpha}{\lambda_{L\sigma_1}-\Pi(L)}\Delta^L(Z_{12}) = 2\int_0^{\infty}\frac{dL}{2\pi}\frac{iL_E}{\lambda_{L,\sigma_1}-\Pi(L)}\Delta^L(Z_{12}) \qquad (163)$$

$$+2\int_{-\infty}^{0}\frac{dL_E}{2\pi}\frac{iL_E}{\lambda_{L\sigma_1}-\Pi(L)}\Delta^L(Z_{12}) \qquad (164)$$

We then make use of the fact that the second term is invariant under the transformation $iL_E \to -iL_E$ and rearranging our limits to yield,

$$2\int_{-\frac{d}{2}-i\infty}^{-\frac{d}{2}+i\infty}\frac{dL}{2\pi i}\frac{L+\alpha}{\lambda_{L\sigma_1}-\Pi(L)}\Delta^L(Z_{12}) = 2\int_0^{\infty}\frac{dL}{2\pi}\frac{iL_E}{\lambda_{L,\sigma_1}-\Pi(L)}-2\int_0^{\infty}\frac{dL_E}{2\pi}\frac{-iL_E}{\lambda_{L\sigma_1}-\Pi(L)}\Delta^L(Z_{12})$$

$$2\int_{-\frac{d}{2}-i\infty}^{-\frac{d}{2}+i\infty}\frac{dL}{2\pi i}\frac{L+\alpha}{\lambda_{L\sigma_1}-\Pi(L)}\Delta^L(Z_{12}) = 2\int_0^{\infty}\frac{dL_E}{2\pi}iL_E\left[\frac{1}{\lambda_{L,\sigma_1}-\Pi(L)}-\frac{1}{\lambda_{L,\sigma_1}+\Pi(L)}\right]\Delta^L(Z_{12})$$

$$2\int_{-\frac{d}{2}-i\infty}^{-\frac{d}{2}+i\infty}\frac{dL}{2\pi i}\frac{L+\alpha}{\lambda_{L\sigma_1}-\Pi(L)}\Delta^L(Z_{12}) = 4\int_0^{\infty}\frac{dL_E}{2\pi}iL_E\left[\frac{\Pi(L)}{\lambda_{L,\sigma_1}^2-\Pi(L)^2}\right]\Delta^L(Z_{12}) \qquad (165)$$

Recalling that our values of L must range over all values of,

$$L = -\left(\frac{d}{2}\right)+\left[\left(\frac{d}{2}\right)^2 - M^2\ell^2\right]^{1/2} = -\alpha + i\left[M^2\ell^2-\alpha^2\right]^{1/2} \qquad (166)$$

therefore our variable L_E now will range over,

$$L_E = \left(M^2\ell^2-\alpha^2\right)^{1/2}$$

$$dL_E = \frac{\ell^2 dM^2}{2\left(M^2\ell^2 - \alpha^2\right)^{1/2}} \tag{167}$$

where our variable will be changed once more, this time to M^2 to put it into the Källén-Lehmann spectral representation. Therefore our function will now be of the form,

$$2\int_{-\frac{d}{2}-i\infty}^{-\frac{d}{2}+i\infty} \frac{dL}{2\pi i} \frac{L+\alpha}{\lambda_{L\sigma_1} - \Pi(L)} \Delta^L(Z_{12}) = i\ell^2 \int_{\ell^{-2}}^{\infty} \frac{dM^2}{4\pi} \left[\frac{\frac{\mu^2}{8i(M^2\ell^2-1)^2}}{\ell^4(M^2 - M_1^2)^2 + \frac{\mu^4}{(64(M^2\ell^2-1))}} \right] \Delta^{M^2}(Z_{12})$$

so we get,

$$2\int_{-\frac{d}{2}-i\infty}^{-\frac{d}{2}+i\infty} \frac{dL}{2\pi i} \frac{L+\alpha}{\lambda_{L\sigma_1} - \Pi(L)} \Delta^L(Z_{12}) = \int_{\ell^{-2}}^{\infty} \frac{dM^2}{2\pi} \left[\frac{8\ell^2\mu^2(M^2\ell^2-1)^{1/2}}{64\ell^4(M^2 - M_1^2)^2(M^2\ell^2-1) + \mu^4} \right] \Delta^{M^2}(Z_{12}) \tag{168}$$

again specifying $\alpha = 1$, $\sigma_2 = \sigma_3 = -\frac{1}{2}$ and recalling our calculation for $\Pi(L)$ use $\Pi(L) = \mu^2/8(L+1) = \mu^2/8iL_E$, for the specific $\alpha = 1$ case. Putting this all together, our two-point function when we recombine with the first term has the form,

$$\begin{aligned} \langle \phi_1(x_1)\phi_1(x_2) \rangle = &\oint_C \frac{dL}{2\pi i} \frac{L+1}{\lambda_{L\sigma_1} - \Pi(L)} \Delta^L(Z_{12}) \\ &+ \int_{\ell^{-2}}^{\infty} \frac{dM^2}{2\pi} \left[\frac{8\ell^2\mu^2(M^2\ell^2-1)^{1/2}}{64\ell^4(M^2 - M_1^2)^2(M^2\ell^2-1) + \mu^4} \right] \Delta^{M^2}(Z_{12}) \end{aligned} \tag{169}$$

Therefore our spectral density, $\rho(M^2)$, for the case where $\alpha = 1$, $\sigma_2 = \sigma_3 = -\frac{1}{2}$ will have the form,

$$\rho(M^2) = \frac{8\ell^2\mu^2(M^2\ell^2-1)^{1/2}}{64\ell^4(M^2 - M_1^2)^2(M^2\ell^2-1) + \mu^4} \tag{170}$$

for $M^2\ell^2 < 1$.

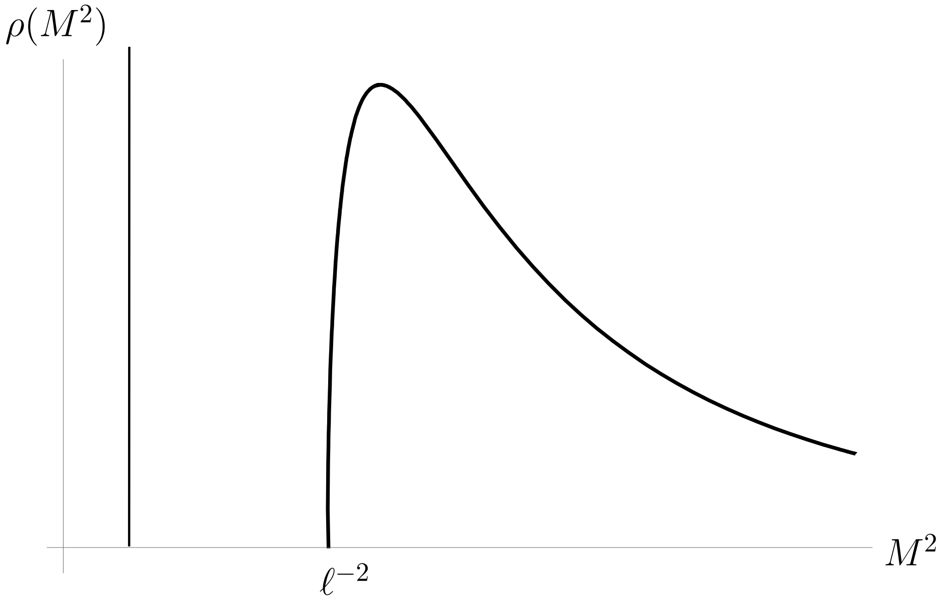


Figure 13: Graph of the spectral density in the complementary case in de Sitter space. It exhibits the delta function from one-particle states and continuous spectrum after ℓ^{-2}.

From (5.8) we determine therefore that, for the complementary case we have,

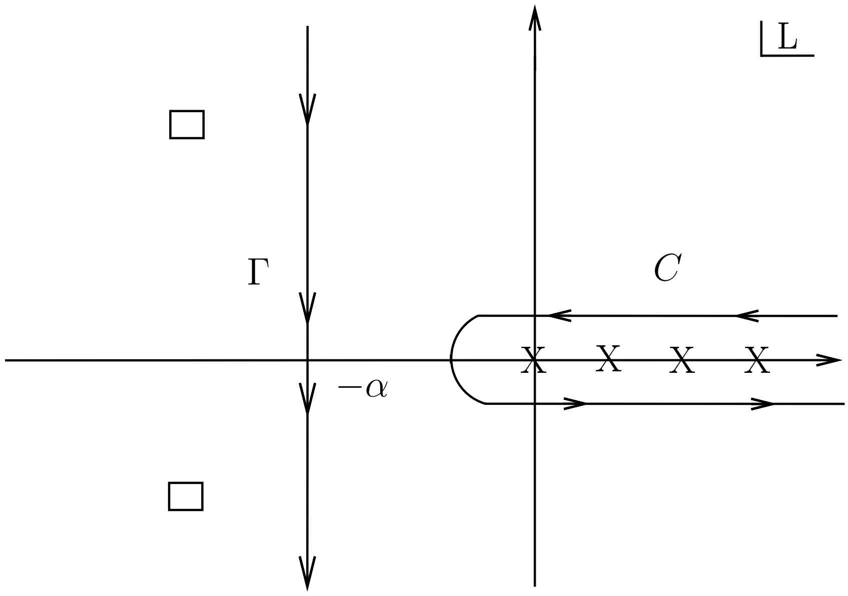
$$\frac{\mu^2}{8\sigma_1(\sigma_1+1)^2} = \ell^2 \int_{l^{-2}}^{\infty} \frac{dM^2}{4\pi} \frac{8\mu^2(M^2\ell^2-1)^{1/2}}{64(M^2\ell^2-1)\ell^4(M^2-M_1^2)^2+\mu^4} \tag{171}$$

The right hand side will be just equal to one if we are in the principal series where, $M_1^2 > \ell^{-2}$.

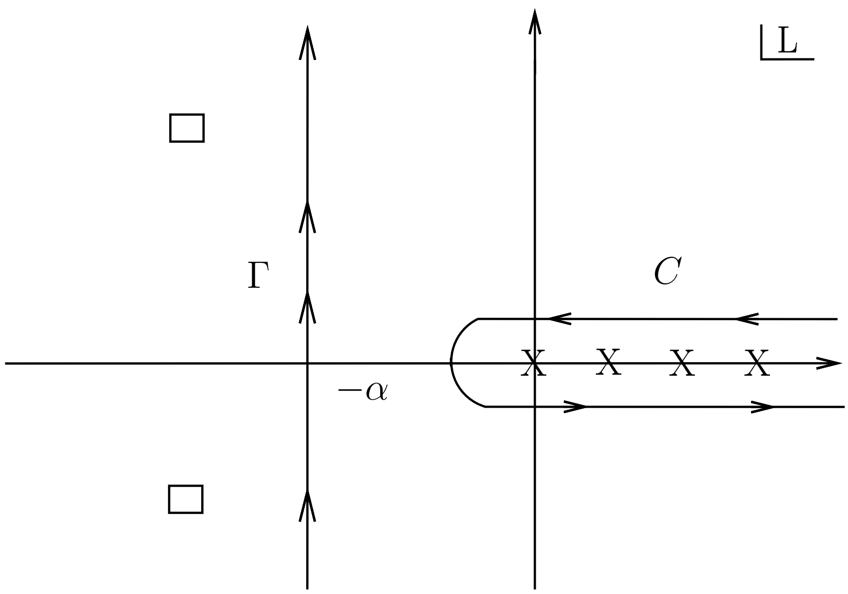
This behaves remarkably similar to the stable case in flat spacetime. There has been an ongoing debate about particle stability and particle decay in de Sitter space. Some argue that the concept of particle stability is not present in de Sitter space at all [10, 19, 21], while others argue that in the case of the complementary series, we find that particle stability is possible, before a certain critical mass (m_c) [11, 23]. This work appears to support the latter argument although we must stress that, working in $\alpha = 1$ specifying to the case, $\sigma_2 = \sigma_3 = -\frac{1}{2}$, we are very tightly constrained and beyond this case our results may change.

5.2 Principal series

Now we compute the spectral density for heavier masses lying in the principal series. Due to the perturbative corrections we will not cross any poles. Our work here will only focus on the shift in $\lambda_{L\sigma_1}$.



again changing the direction of our contour therefore introducing an overall minus sign,



Here our masses lie in the range,

$$\sigma = -\alpha + i\nu \tag{172}$$

for $\nu \in \mathbb{R}$, $\nu \geq 0$. From our definition of our dimensionless mass parameter, we can obtain a value of ν,

$$\sigma = -\alpha + \left(\alpha^2 - M^2\ell^2\right)^{1/2}$$

$$\sigma = -\alpha + i\left(M^2\ell^2 - \alpha^2\right)^{1/2} = -\alpha + i\nu \tag{173}$$

The location of the poles which lie in the principal series is shifted in such a way that we never encounter the poles as we deform our contour away from C_1 to Γ. Our spectral density for $\alpha = 1$, $\sigma_2 = \sigma_3 = -\frac{1}{2}$ is therefore given as,

$$\rho(M^2) = \frac{8\ell^2\mu^2(M^2\ell^2 - 1)^{1/2}}{64\ell^4(M^2 - M_1^2)^2(M^2\ell^2 - 1) + \mu^4} \tag{174}$$

however, this time $M^2\ell^2 > 1$, so this changes the nature of the spectral density. We pass close to the pole while never actually encountering it. Our spectral density therefore grows very large as it nears the pole but remains finite. We note that this looks similar to our unstable flat space case.

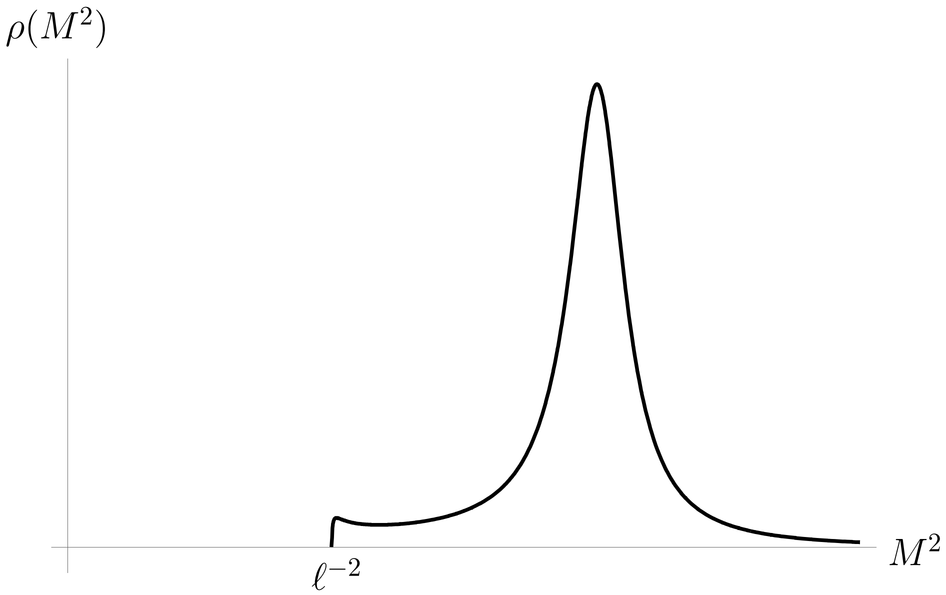


Figure 14: Graph of the spectral density in the principal series in de Sitter space. We pass close to the pole while never actually encountering it. Our spectral density becomes very large at the pole, but never reaches infinity.

5.3 Summary

To conclude, we have, by restricting ourselves to the case of $\alpha = 1$, computed the spectral density to $\mathcal{O}(\mu^2)$ noting the qualitative and quantitative differences for our two cases where the mass lies in both the complementary and principal series. We note that the calculations made in these cases appear to give very different behaviour for the spectral density. In the principal series, we note that it behaves very similar to the unstable case we examined in flat space.

However, in the complementary series, we find that it behaves very similar to the stable flat space scenario. There have been two competing arguments about the nature of particle stability in de Sitter space developing recently. One argument put forward postulates that all particles behave like the unstable flat space case in both complementary and principal series cases [10, 14, 19, 21]. The other, conflicting argument put forth is that in the case of the complementary series, for certain masses the concept of particle stability is possible and the spectral density behaves like the stable flat space case [11, 23, 27]. Our work appears to support the latter argument and in moving forward it would be interesting to see how our result would be affected by expanding our work beyond the $\alpha = 1$ case. Our calculation here points out a mathematical fact, which in our simple set up can be naively interpreted as stability of scalar particles with lower masses and some coupling in $D = 3$. It remains unclear as to the physical interpretation of this because we do not know how to extract physics from the spectral density function in de Sitter space. Given more time we would have liked to consider the wider implications of this result.

Conclusions

In conclusion, we have studied aspects of scalar field theory in flat space as well as de Sitter to gain a better understanding of the spectral density functions in de Sitter space. In de Sitter due to the lack of a globally timelike Killing vector field, there is no notion of positive definite energy conservation [7, 9, 10, 11, 19, 23]. Physically this means that particles in de Sitter space can decay into heavier daughter particles. We examined scalar fields with cubic interactions in de Sitter space and the spectral density in de Sitter for a variety of masses, also examining the UV divergences which arise in de Sitter. We presented the relationship between the spectral density and the two-point function in flat space and how it behaves in both stable and unstable particle cases in chapter 2.

In chapter 3, we presented an introduction to de Sitter space and reviewed work examining large distance behaviour adapting it for our own investigations of the spectral density and its behaviour in curved spacetime [10]. We showed how we analytically continue correlation functions from Euclidean signature to Lorentzian signature quantum field theory. We presented how we get equations of motion and how they can be solved in terms of spherical harmonics on the D-sphere. We then explained the process of Watson-Sommerfield transformations and how we used them to obtain forms for the propagators in the complex L-plane.

We then moved to computing perturbative corrections in de Sitter space ranging over both the lighter, complementary series, and heavier fields lying in the principal series, while choosing to ignore discrete or tachyonic masses and how we compute the spectral function, $\rho_{\sigma_2\sigma_3}$, and its relation with the spectral density, $\rho(M^2)$. This was done for the case of $\alpha = 1$ where we neglected our counterterms, at the one-loop level. As an extension we also expand this to n-loop chain diagrams. Efforts to extend this into the Lorentzian signature proved difficult and we leave this an open problem in de Sitter as well as the reason why this proves challenging.

Finally, we presented some numerical calculations of the spectral density making conservative approximations and specifying some values for isolated cases. We computed the spectral density function in both complementary and principal series representations highlighting the qualitative and quantitative differences for the case of $\alpha = 1$, $\sigma_2 = \sigma_3 = -1/2$. We observed that the spectral density appears to behave similar to the stable case in flat space when working in the complementary series and like that of the unstable (flat space) case when working in the principal series. Investigations into this are highly specific, and moving beyond our case of $\alpha = 1$ we must acknowledge and stress that not only the consideration of UV divergences, but also other dynamics, may complicate matters further in the physical picture developed here.

There has been some debate about particle stability in de Sitter space for the complementary and principal series cases which we have reviewed and commented upon. The difference related to particle stability in the complementary series. Previous work has claimed that particles in de Sitter space will decay in both the complementary and principal series [10, 14, 19, 21, 26], but our work appears to support arguments made to the contrary [11, 23, 27] referring to the behaviour in the complementary series.

It is the second argument which postulates that, for specific cases in the complementary series, it is indeed possible to recover the notion of particle stability in de Sitter space. This is a hotly debated topic in the field and while they both agree for the principal series, it is in the lighter masses case where the discrepancy lies. This can be observed from the behaviour of our spectral density in both our cases and compared with that of the flat space developed in chapter 2. As we showed in chapter 5, for the complementary series case, we found the spectral density looks very similar to the stable flat space case contrary to arguments made in [10, 21]. This points to the idea that we might be able to recover the notion of particle stability in de Sitter space in the complementary series in agreement with [11, 23, 27].

This result is rather surprising given that the machinery we have reviewed and developed is put forth by those who argue against the result we obtain [10]. As stated previously, the lack of a globally timelike Killing vector field means that the notion of positive definite energy conservation does not exist in de Sitter. Again we need to stress that we are simply pointing out a mathematical fact, that can be interpreted, naively, as stability of scalar particles in lower-mass, but non-zero, limit in $D = 3$ for a particular coupling. The physical interpretation remains unclear as it is not clear how to extract physics from the spectral density function in de Sitter space. Given more time we would have liked to consider the wider implications of this result. As previously stated, we note that this may be a feature of the approximations made in our calculations with the restrictions we have imposed. It would be interesting to consider whether this is an artefact of the assumptions we have made, or is a genuine physical feature of de Sitter space for lower-masses scalar particles in the complementary series.

Appendix

Computing the spectral function

with the ψ-function is defined as

$$\psi(x) = \frac{d}{dx}\, ln\left[\Gamma(x)\right] = \left[\frac{\Gamma'(x)}{\Gamma(x)}\right]$$

with some useful properties we employ,

$$\psi(x+1) = \psi(x) + \frac{1}{x}$$

$$= -\gamma + \int_0^{\infty}\left[\frac{e^{-t}}{t} - \frac{e^{-xt}}{1-e^{-t}}\right] dt \tag{175}$$

where γ is the Euler?Mascheroni constant.

$$\psi\left(\frac{3}{4} - n\right) = \psi\left(\frac{1}{4} + n\right) + \pi \tag{176}$$

$$\sum_{n=0}^{\infty}\left[\frac{1}{n+a} - \frac{1}{n+b}\right] = \psi(a) - \psi(b) \tag{177}$$

After employing these we are rendered a solution of the form,

$$S_{\sigma_1\sigma_2}(L) = -\frac{\pi}{2}\cot(\pi\sigma_1)\sum_{G=0}^{\infty}\left[\frac{1}{G + \frac{L-\sigma_1-\sigma_2}{2}} + \frac{1}{G + \frac{L-\sigma_1+\sigma_2+2}{2}} + 3\, syms\right] \tag{178}$$

expanding out our 3-syms getting,

$$\begin{aligned}
S_{\sigma_1\sigma_2}(L) = -\frac{\pi}{2}\cot(\pi\sigma_1)\sum_{G=0}^{\infty}\left[\frac{1}{G+\frac{L-\sigma_1-\sigma_2}{2}}+\frac{1}{G+\frac{L-\sigma_1+\sigma_2+2}{2}}+3syms\right] \\
= -\frac{\pi}{2}\cot(\pi\sigma_1)\left[\psi\left(\frac{L-\sigma_1-\sigma_2}{2}\right)-\psi\left(\frac{L-\sigma_1+\sigma_2+2}{2}\right)\right. \\
\left.+\psi\left(\frac{L-\sigma_1-\sigma_2+2}{2}\right)-\psi\left(\frac{L+\sigma_1-\sigma_2}{2}\right)\right] \\
-\frac{\pi}{2}\cot(\pi\sigma_2)\left[\psi\left(\frac{L-\sigma_1-\sigma_2}{2}\right)-\psi\left(\frac{L-\sigma_1+\sigma_2+2}{2}\right)\right. \\
\left.+\psi\left(\frac{L-\sigma_1+\sigma_2+2}{2}\right)-\psi\left(\frac{L-\sigma_1+\sigma_2}{2}\right)\right] \\
= -\frac{\pi}{2}\frac{\sin(\pi(\sigma_1+\sigma_2))}{\sin(\pi\sigma_1)\sin(\pi\sigma_2)}\left[\psi\left(\frac{L-\sigma_1-\sigma_2}{2}\right)-\psi\left(\frac{L-\sigma_1+\sigma_2+2}{2}\right)\right. \\
\left.+\psi\left(\frac{L-\sigma_1+\sigma_2+2}{2}\right)-\psi\left(\frac{L-\sigma_1+\sigma_2}{2}\right)\right] \\
= -\frac{\pi}{2}\frac{\sin(\pi(\sigma_1+\sigma_2))}{\sin(\pi\sigma_1)\sin(\pi\sigma_2)}\left[\psi\left(\frac{L-\sigma_1-\sigma_2}{2}\right)-\psi\left(\frac{L-\sigma_1+\sigma_2+2}{2}\right)\right. \\
+\psi\left(\frac{L-\sigma_1+\sigma_2+2}{2}\right)-\psi\left(\frac{L-\sigma_1+\sigma_2}{2}\right) \\
\left.+\pi\cot\left[\pi\left(\frac{L+\sigma_1+\sigma_2+2}{2}\right)\right]-\pi\cot\left[\pi\left(\frac{L-\sigma_1+\sigma_2+2}{2}\right)\right]+2\pi\right]
\end{aligned} \tag{179}$$

using the relationship between $S_{\sigma_2\sigma_3}$ and $\rho_{\sigma_1\sigma_2}$ which put in its simplest form becomes,

$$\rho_{\sigma_1\sigma_2} = \frac{1}{16\pi(L+1)}\left[\left(\frac{\sin(\pi(\sigma_1+\sigma_2))}{\sin(\pi\sigma_1)\sin(\pi\sigma_2)}\psi\left(\frac{L-\sigma_1-\sigma_2}{2}\right)+\sigma\ syms\right)+2\pi\right] \tag{180}$$

$$= \ell^{-1} \sum_{\vec{L}_1} \frac{Y_{\vec{L}_1}(x_1) Y^*_{\vec{L}_1}(y_1)}{\lambda_{L_1\sigma_1}} \cdot \ell \sum_{\vec{L}'_1} \mu^2 \rho_{\sigma_2\sigma_3}(L'_1) Y_{\vec{L}'_1}(y_1) Y^*_{\vec{L}_1}(y'_1)$$

$$\times \ell^{-1} \sum_{\vec{L}_2} \frac{Y_{\vec{L}_2}(y'_1) Y^*_{\vec{L}_2}(y_2)}{\lambda_{L_2\sigma_1}} \cdot \ell \sum_{\vec{L}'_2} \mu^2 \rho_{\sigma_2\sigma_3}(L'_2) Y_{\vec{L}'_2}(y_2) Y^*_{\vec{L}'_2}(y'_2) \cdots$$

$$\cdots \times \ell^{-1} \sum_{\vec{L}_n} \frac{Y_{\vec{L}_n}(y'_{n-1}) Y^*_{\vec{L}_n}(y_n)}{\lambda_{L_n\sigma_1}} \cdot \ell \sum_{\vec{L}'_n} \mu^2 \rho_{\sigma_2\sigma_3}(L'_n) Y_{\vec{L}'_n}(y_n) Y^*_{\vec{L}'_n}(y'_n)$$

$$\times \ell^{-1} \sum_{\vec{L}_{n+1}} \frac{Y_{\vec{L}_{n+1}}(y'_n) Y^*_{\vec{L}_{n+1}}(y_n)}{\lambda_{L_{n+1}\sigma_1}}$$

using the relations for spherical harmonics developed in chapter 3 and 4,

$$= \ell^{-1} \sum_{\vec{L}} \frac{Y_{\vec{L}}(x_1) Y^*_{\vec{L}}(x_2)}{\lambda_{L\sigma_1}} \times \frac{1}{1 - \frac{\mu^2 \rho_{\sigma_2\sigma_3}}{\lambda_{L\sigma_1}}}$$

$$= \ell^{-1} \sum_{\vec{L}} \frac{Y_{\vec{L}}(x_1) Y^*_{\vec{L}}(x_2)}{\lambda_{L\sigma_1} - \mu^2 \rho_{\sigma_2\sigma_3}}$$

$$= \ell^{-1} \sum_{L=0}^{\infty} \frac{1}{\lambda_{L\sigma_1} - \mu^2 \rho_{\sigma_2\sigma_3}} \times \frac{2(L+1)}{4\pi^2} C^1_L(Z_{12})$$

Exact location of the poles

Vieta's substitution

The exact location of the pole must satisfy the cubic equation,

$$L^3 + 3L^2 + (2 - 2\sigma - 8\sigma_1^2)L - 2\sigma_1 - 2\sigma_1^2 - \frac{\mu^2}{8} = 0 \tag{181}$$

where for some cubic satisfying,

$$z^3 + a_2 z^2 + a_1 z + a_0 = 0 \tag{182}$$

by the substitution,

$$z = x - \frac{a_2}{3} \tag{183}$$

our cubic takes the following form,

$$x^3 + px - q = 0 \tag{184}$$

with p and q being,

$$\begin{aligned} p &= a_1 - \frac{a_2^2}{3} \\ q &= \frac{a_1 a_2}{3} - a_0 - \frac{2a_2^3}{27} \end{aligned} \tag{185}$$

We then make the substitution known as Vieta's substitution which allows our cubic to be rendered into the form of a quadratic which we then can solve for our pole L_0,

$$x = \omega - \frac{p}{3\omega} \tag{186}$$

giving us,

$$\begin{aligned} \omega^3 - q - \frac{p^3}{27\omega^3} = 0 \to (\omega^3)^2 - q\omega^3 - \frac{p^3}{27} = 0 \\ \omega^3 = \frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3} \end{aligned} \tag{187}$$

therefore ω is just,

$$\omega = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} \tag{188}$$

in order to return to the original pole z_0 in terms of the variables p and q, the second term for our formula for x can be expressed as,

$$-\frac{p^3}{27} = \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} \tag{189}$$

Exact location of poles

The location of our poles will be therefore be,

$$z_0 = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} - \frac{a_2}{3} \tag{190}$$

The pole in the first term L_0 of equation (5.3) is found as,

$$L_0 = -1 + \frac{\sqrt[3]{\left(\frac{\mu^2}{8} + \frac{16}{9}\right)\sqrt{\frac{1}{4}\left(\frac{\mu^2}{8} + \frac{16}{9}\right)^2 + \frac{1}{27}\left(-\sigma_1^2 - 2\sigma_1 - 1\right)^3}}}{\sqrt[3]{2}}$$

$$+\sqrt[3]{\sqrt{\frac{1}{4}\left(\frac{\mu^2}{8} + \frac{16}{9}\right)^2 + \frac{1}{27}\left(-\sigma_1^2 - 2\sigma_1 - 1\right)^3} + \frac{1}{2}\left(\frac{\mu^2}{8} + \frac{16}{9}\right)} \tag{191}$$

In the second term, we get our pole, denoted $L_0^{(2)}$, from the second term of the two-point function, which has the same form as the solution for the first term save for our p and q as,

$$p = 2H^2M_1^2 - \frac{1}{3}\left(H^2 + 2M_1^2\right)^2 + (M_1^2)^2$$

$$q = -\frac{1}{64}H^6\mu^4 + H^2(M_1^2)^2 + \frac{1}{3}\left(H^2 + 2M_1^2\right)\left(2H^2M_1^2 + (M_1^2)^2\right) - \frac{2}{27}\left(H^2 + 2M_1^2\right) \tag{192}$$

where H is the Hubble parameter and the pole lies at,

$$
\begin{aligned}
L_0^{(2)} = -1 + 2^{-1/3}\Bigg[\Big(&H^2(M_1^2)^2 - 2/27(H^2+2M_1^2) + 1/3(H^2+2M_1^2)(2H^2M_1^2+(M_1^2)^2) \\
&-\Big(\frac{H^6\mu^4}{64}\Big)\Big) \\
&+\Big[\frac{1}{27}\left(2H^2M_1^2 - \frac{1}{3}\left(H^2+2M_1^2\right)^2 + (M_1^2)^2\right)^3 + \frac{1}{4}\Big(-\frac{1}{64}H^6\mu^4 \\
&+H^2(M_1^2)^2 \\
&+\frac{1}{3}\left(H^2+2M_1^2\right)\left(2H^2M_1^2+(M_1^2)^2\right) - \frac{2}{27}\left(H^2+2M_1^2\right)\Big)^2\Big]^{1/2}\Bigg]^{1/3} \\
+\Bigg[\frac{1}{2}\Big(&-\frac{1}{64}H^6\mu^4 + H^2(M_1^2)^2 + \frac{1}{3}\left(H^2+2M_1^2\right)\left(2H^2M_1^2+(M_1^2)^2\right) - \frac{2}{27}\left(H^2+2M_1^2\right)\Big) \\
&+\Big(1/27(2H^2M_1^2+(M_1^2)^2 - 1/3(H^2+2M_1^2)^2)^3 \\
&+1/4(H^2(M_1^2)^2 - 2/27(H^2+2M_1^2) + 1/3(H^2+2M_1^2)(2H^2M_1^2+(M_1^2)^2) \\
&-(\frac{H^6\mu^4}{64}))^2\Big)^{1/2}\Bigg]^{1/3}
\end{aligned}
\tag{193}
$$

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“ It’s gone. It’s done” - Frodo Baggins

“Yes Mr Frodo it’s over now” - Samwise Gamgee [28]