Essays on Commodity Futures

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To my parents: Fei Shu and Xiujuan Yu

谨以此篇献给

我的父亲舒飞

及我的母亲俞秀娟

Abstract

This thesis intends to study the mechanism behind the commodity futures term structure, and the interaction between commodity markets, particularly the crude oil market, and the macroeconomic indicators in the real economy. The first part of thesis comprises a comprehensive review of the relevant literature, revealing that, although there has been extensive investigation into commodity prices and their term structure modelling, based on either "pure macro" or "pure finance" perspectives, the discussion of their joint application, remains very limited. The subsequent preliminary data analysis highlights some other concerns in respect to this subject area, such as the effect of the unit root, commonly observed in the commodity price related models, and its possible solution. On the basis of these observations, I propose two models to add to the existing literature.

The second part of this thesis proposes a joint affine term structure model for multiple commodity futures contracts. In this model, the instantaneous short rate factor is a pure latent variable, and is jointly determined by several commodity markets. The empirical evidence, presented in this part, suggests that the path of this "commodity market implied short rate factor" is consistent with the policy rate. It reveals that the expectation in respect to the interest rate in the commodity market reflects and anticipates developments in monetary policy.

The third part of this thesis presents a macro-finance model for the economy and the oil market, allowing us to study interactions between the convenience yield, the spot and futures markets, monetary policy and macroeconomic variables. I use the Kalman filter to represent latent variables that handle the effects of exogenous shocks to inflation and the oil price, and to deal with missing observations. Traditional models use latent variables, with little economic meaning, to explain commodity futures, while this model makes the effect of macroeconomic variables explicit. I find a significant interaction between the economy and the oil markets, including an important link in the monetary transmission mechanism, running from the policy interest rate to the convenience yield, oil price and hence inflation and policy transmission.

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Declaration

I declare that this thesis is a presentation of original work and I am the sole author. This work has not previously been presented for an award at this, or any other, University. All sources are acknowledged as References.

Haicheng Shu

Chapter 1

Introduction

A futures contract is a legal agreement to buy or sell commodities, currencies, or other financial instruments at a predetermined price and on a pre-specified time in the future. The term structure of the commodity futures is defined as the relationship between the spot price and the futures price of the commodity for any delivery date. It provides useful information for various market participants, because it synthesises the information available in the market and, therefore, reflects the market's expectations of the financial and economic condition in the future. Over the past decades, the literatures on the commodity market, commodity futures term structure, and the joint behaviour of the financial market and the macroeconomy has experienced dramatic developments. The considerable progress in these subject areas further provides many new opportunities to researchers in various fields, such as macroeconomists, financial economists, policy makers and others.

Previous literature has tended to study the commodity market from two main perspectives. The traditional "pure macro" perspective uses popular macroeconometric approaches to investigate the role of the spot commodity price in the macroeconomy. This strand of literature usually emphasises the strong connection between commodity prices, particularly the spot crude oil price, and other macroeconomic variables. For example, it is widely suggested that, crude oil price shocks have, in various ways, triggered a range of economic recessions post-

World War II (see Hamilton 1983, Kilian 2006, 2008 and others). Besides, Sims (1992), Eichumbaum (1992) and others also found that the inclusion of the spot commodity price could be sufficient to improve traditional macro-econometric modelling, in terms of solving the confusing stylised fact called the "price puzzle", which often appears in vector autoregressive models (VARs). This is because the spot commodity price usually contains useful information about future inflationary pressure that monetary authorities can observe and react to much faster. Although the extensive "pure macro" literature on the spot commodity price has yielded remarkable development over the years, I have yet to witness thorough discussions on the mechanism of the commodity futures term structure, from this perspective.

On the basis of traditional views, such as the "Theory of Normal Backwardation" by Keynes (1930) and Hicks (1939) and the "Theory of Storage" by Kaldor (1939), Working (1949), and Brennan (1958), the "pure finance" perspective, on the other hand, uses a reduced-form Gaussian factor model for commodity futures, focusing on the time series behaviour of the commodity futures term structure. It suggests that, under the no arbitrage condition, commodity futures prices, with any remaining term to maturity at any point in time, are determined by a time-invariant linear function of several unobservable common state variables. Specifically, previous studies tend to recognise three latent common factors that determine the commodity futures term structure, namely, the spot commodity price, the convenience yield, and the instantaneous short rate factor. This model is popular in the literature since it is tractable with a closed-form solution, and flexible with few extra restrictions. There has been long-standing concern about the lack of macroeconomic consideration in these latent state variables, however. Therefore, a challenging task to interpret these latent variables with a macroeconomic rationale.

The growing interest in the structural relationship between the interest rate term structure and the macroeconomy has spawned a relatively modern theory, known as the "macro-finance" model, which combines the "pure macro" and the "pure finance" perspectives (see Ang and Piazzesi 2003, Diebold, Rudebusch and Aruoba 2006 and others). The later development of the semi-structural central bank model (CBM) further extends the "error correction" specification to the standard macro-finance model framework (see Kozicki and Tinsley 2005, Dewachter and Lyrio 2006, Dewachter, Lyrio and Maes 2006, Spencer 2008, Liu and Spencer 2010 and others), enabling the evaluation of the long term underlying relationships between variables in the model. Although the macro-finance model has received extensive attention from academia and policy makers since its establishment, its main focus is, rather narrowly, on the structural relationship between interest rate related term structures and their relevant variables. Hence, the development of the macro-finance model for the commodity futures term structure, particularly the crude oil futures term structure, remains very limited, despite the evidence of the effect of oil shocks on the macroeconomy.

In this thesis, I extend the above discussion to a comprehensive literature review in Chapter 2. I also highlight some other concerns on this subject area in the subsequent preliminary data analysis using standard macro-econometric approaches; for example, the effect of the unit root, commonly observed in commodity price-related models, and its possible solution by applying the vector error correction model. Motivated by these prior discussions, I propose two models to add to the existing literature.

Chapter 3 of this thesis proposes a joint affine term structure model for multiple commodity futures contracts, which falls into the category of the "pure finance" perspective. Generally speaking, this model yields two main novelties. First, as suggested by the observation in Casassus and Collin-Dufresne (2005), the short rate factor estimates in each of the four separately estimated commodity classes are, as quoted, "misspecified"; since for the same short rate factor, in each case; for they receive four different sets of parameters estimates from the empirical model. My joint affine term structure model in Chapter 3 identifies the instantaneous short rate as a factor that is common to all selected commodity markets. This avoids estimating different sets of parameters for the same interest rate factor in different commodity markets.

Second, previous empirical literature (Schwartz 1997, Casassus and Collin-Dufresne 2005) has frequently employed the practice of separately estimating the latent instantaneous short rate with a single factor Gaussian model. This approach helps to identify the instantaneous short rate factor, which is otherwise, a difficult task. By taking this action however, the short rate factor in the commodity market is in essence pinned down to a time series that is defined externally, meaning that it is impossible to access the actual measurement of the short rate factor that is suggested by the commodity market. The model in Chapter 3 of this thesis, however, does not attempt to apply this practice. Instead, the instantaneous short rate factor is a pure latent variable, jointly identified and determined by several benchmark commodity markets, such that it can be regarded as a commodity market implied short rate factor.

All state variables are unobservable in the empirical model, and maximum likelihood estimation with the Kalman filter algorithm is used to estimate the latent variables in this chapter. The dataset contains term structures of crude oil, copper, gold, and corn futures contracts. The time span is from the end of 1989 to the end of 2015. The the model is found to perform well empirically, with a reasonably small root mean squared error (RMSE) and stationary residuals. The behaviour of the commodity term structures and their state variable estimates is in line with conventional wisdom and theoretical rationale. The dynamic of the commodity market implied short rate factor generally tracks the US Federal Funds rate very well, though it is more volatile because of the additional variation that is introduced from the commodity term structure. Meanwhile, the Granger causality test result shows that these two time series significantly Granger cause each other, which implies that the expectation of the interest rate in the commodity market also reflects and anticipates developments in monetary policy.

Chapter 4 of this thesis proposes a macro-finance model for the crude oil futures term structure. The semi-structural CBM approach is followed, which incorporates the behaviour of observable macroeconomic indicators in terms of the output gap, inflation and the short term interest rate. The novelty of this model is that it is able to study the crude oil futures term structure, and also introduces the spot crude oil price to the macro system. This enables a study of the interaction between the oil market and the macroeconomy. The error correction component in the model naturally accommodates the potential unit root problem that appears in the preliminary data analysis. It also reveals that two vitally important unobservable variables relate to the long term inflation asymptote and the underlying spot oil price, due to the cointegration relationship of the observable variables.

One important consideration of this model is that, it is constructed in order to evaluate the situation in the real economy. Unlike bond prices, however, which are the relative prices of money in different periods, futures prices are absolute prices which tend to increase over time due to the effects of inflation. It is difficult to allow for this without using the spot price as an explicit factor as in this model. Moreover, there is an important arbitrage identity linking the spot oil price, the convenience yield and the interest rate that would be difficult to incorporate into a model that did not identify these explicitly.

I use maximum likelihood estimation with the Kalman filter algorithm to estimate the empirical model. The application of the Kalman filter handles the missing observation problem, which is serious in the case because a long sample is used in order to include the 1970s oil shocks that the crude oil futures price does not cover. It also represents latent variables that handle the effects of exogenous shocks to inflation and the oil price.

The empirical result is consistent with the theoretical priors. The model reflects the work of Kilian (2005) and others. The strength of the economy in the run up to the oil price shocks of the 1970s explains why these were persistent, and the weakness of the economy in the run up to the first Gulf war, in 1991, explains why the effect of that shock was only temporary. The results are also consistent with the existing macro-finance literature in underlining the importance of both macroeconomic variables and latent factors.

This model also yields some other findings. First, the convenience yield naturally plays a key role in determining the futures term structure, but it also plays an important role in the macro system. The convenience yield can be viewed as a proxy for oil inventory, which serves as a buffer, damping the effect of oil and economic shocks on the real economy. I find that the convenience yield provides an important channel, through which policy interest rates affect oil prices, activity and inflation. Second, as originally documented in Sims (1992), policy makers anticipate inflationary developments which cause the inflation level to rise. The use of a commodity prices index helps to resolve the "price puzzle", because it contains useful information about future inflationary pressure. The use of Kalman filters in this case, reflects this argument since they are able to pick up the effect of unobservable expectational influences related to the oil price and the macro system, influences that serve an a important role in channelling effects between monetary policy and the macroeconomy.

Chapter 2

Literature review and preliminary data analysis

2.1 Introduction

In this chapter, I review different strands of literature that try to explain variations in the commodity futures term structure. I start with some traditional theories such as the "Theory of Normal Backwardation" and the "Theory of Storage" that date back to the 1930s. The former suggests that the hedging/speculative position drives the variation in the futures curve, while the latter explains the role of spot inventory, and explains the convenience yield factor. On the basis of these traditional theories, there has been remarkable development in the reduced-form Gaussian factor model for the commodity futures term structure. This model intends to capture the time series behaviour of the commodity futures term structure with latent variables, which is very similar to the strand of literature that models the term structure dynamic of the bond yield. These latent variables models are sometimes referred to as "pure finance" models, and there has been long-standing criticism of their lack of economic interpretation. In order to address this, the recent development of macro-finance modelling links this "pure finance" model with the macro-econometric literature. This relatively new strand of literature combines dynamic term structure models with macroeconomic interpretation. Although we have witnessed tremendous development in the macro-finance modelling for interest rate term structure, the current progress of its application in the commodity futures term structure remains very limited. This is despite the long-standing concern in respect to the significant interrelationships between the commodity price, particularly the crude oil price, and the macroeconomy, discussed in numerous studies over the years.

In order briefly to investigate these interrelationships from a prefatory empirical perspective, I undertake a preliminary data analysis using standard macroeconometric methods. Several variables are examined that are also used in the later chapters of this thesis, including the crude oil convenience yield, the log real spot oil price and three other new Keynesian macro indicators, namely: the US output gap, the US inflation rate, and the Fed Funds rate. This analysis yields two main findings. First, the result of the bivariate Granger causality test verifies strong interactions between these variables. Second, using an unrestricted vector autoregressive (VAR) model with one lag, the impulse response functions of the testing variables appear to be significantly affected by the data non-stationarity. It seems clear that the application of a vector error correction model (VECM), that properly accommodates long-run cointegration relationships in the underlying variables, is more appropriate than an unrestricted VAR model, in this context.

2.2 Literature review

2.2.1 The traditional views on commodity futures

2.2.1.1 The Theory of Normal Backwardation

The traditional view of commodity futures prices was proposed by Keynes (1930) and Hicks (1939). This view is called the Theory of Normal Backwardation and suggests that price variations in commodity futures contracts, are primarily driven by the position of hedgers and speculators. Suppliers of underlying commodity products are hedgers in the commodity market, and normally have long positions in physical commodity inventories. In order to hedge against the risk induced by the price uncertainty of the value of their physical inventories, they take short positions in the futures markets. To offset speculators, thus take a long position on these futures contracts and require a positive return for bearing the associated underlying price risk. This is usually reflected in the price discount of futures contracts. The associated risk premium, commensurate with the price risk, declines with maturity. In general, therefore, the futures price should rise through the life of the contract, and reach the expected spot price at expiration. As the name of the theory suggests, this situation is referred to as *Normal Backwardation*.

The opposite scenario to Normal Backwardation is called *Contango* and refers to a situation in which the commodity futures contract price is higher than the spot commodity price expected in the future. The Keynes-Hicks view implies that the commodity futures markets usually exhibit normal backwardation. Contango contradicts this theory, and is thus abnormal in the commodity futures markets.

> *"...in normal conditions the spot price exceeds the forward price i.e. there is backwardation. In other words, the normal supply price on the spot includes remuneration for the risk of price fluctuation during the period of production, whilst the forward price excludes this."* (Keynes 1930)

The empirical literature, however, has found this Keynes-Hicks view to have questionable validity. In the early research, structural models often obtained conflicting results. For example, Telser (1958) and Cootner (1960) used the same data to test the validity of normal backwardation. The former found support for the Theory of Normal Backwardation, while the later found no evidence for this theory.

The long-standing conclusion in later research suggests that the state of normal backwardation is uncommon in commodity futures contracts. Even if it is observed, it only exists in some particular markets. Rockwell (1967) studied 25 different commodity markets and found that, normal backwardation only existed in one market, with a further 11 out of the 25 commodity markets giving inconsistent evidence for normal backwardation. He concluded that normal backwardation is not a characteristic for the majority of commodity markets, and that the theory clearly does not have general applicability for all futures markets. Similarly, in Kolb (1992), who studied 29 different futures contracts, positive average returns only existed in nine commodities, and four commodities exhibited statistically significant negative average returns.

Empirical evidence in Dusak (1973) investigated wheat, corn and soybean futures, between 1952 to 1967, and rejected the Theory of Normal Backwardation. Dusak suggested that the average return from these futures contracts for speculators is very close to zero. Chang (1985) used non-parametric statistical procedures to investigate the same markets as in Dusak (1973), but drew different conclusions. He suggested that normal backwardation can be observed if the appropriate empirical methodology is applied and more realistic assumptions are made. He also pointed out that the degree of validity of the theory differs in different commodity markets and different time periods. Bessembinder (1992) studied 22 different financial, foreign currency, agricultural and metal futures markets from 1967 to 1989. His results did not support normal backwardation in markets such as financial and metal futures, but did show that it had some merit in explaining agricultural and currency futures. Miffre (2000) was in favour of the theory of normal backwardation. By testing expected futures returns as time-variant variables, he suggested that the assumption that expected futures returns are constant potentially leads to incorrect inferences regarding the existence of the futures risk premium over the years.

2.2.1.2 The Theory of Storage

The Theory of Storage was developed from the critiques of the Theory of Normal Backwardation by Kaldor (1939), Working (1949) and Brennan (1958).

It provides an alternative perspective on the interpretation of the futures curves. The theory suggests that the equilibrium/no-arbitrage relationship between the commodity spot and futures prices is determined by the storage decisions of the physical inventory holders. Specifically, the price difference between commodity at present and future delivery (also known as the "basis" of the commodity), equals the cost minus benefit of carrying the physical inventory of such a commodity product. Denote $F_{i,\tau,t}$ as the futures price of commodity i at time t for τ term to maturity, and where $S_{i,t}$ is the nominal immediate spot price of commodity i at time t, I have the following no-arbitrage relationship implied by The Theory of Storage:

$$F_{i,\tau,t} = S_{i,t}e^{(r_t+w-\delta_{i,t})\tau} \tag{2.1}$$

where r_t is the instantaneous short rate factor and w is the fixed cost of storage. In practice, w is usually regarded as a combination of various fixed costs, including costs in warehousing, insurance, maintenance, spoilage, obsolescence and so on. On this basis, $r_t + w$ stands for the cost of holding the commodity inventory, comprising the financial cost component, r_t, and the marginal storage cost component, w. $\delta_{i,t}$ is the convenience yield of commodity i at time t. It refers to the unobservable marginal benefit of the physical storage holders of commodity inventories.

> *"The convenience yield is attributed to the advantage (in terms of less delay and lower costs) of being able to keep regular customers satisfied or of being able to take advantage of a rise in demand and price without resorting to a revision of the production schedule."* (Brennan 1958)

The convenience yield is negatively related to the level of commodity inventory. In the case of inventory shortage, the commodity storage holders are in a more "convenient" position to utilise the physical inventories, which means they are enjoying a higher convenience yield, with the opposite happening if inventories are large.

The Theory of Storage is congruent with the Theory of Normal Backwardation by Keynes and Hicks. Normal backwardation appears when the basis of the commodity is negative (i.e. the commodity spot price is higher than the commodity futures price). This happens when the inventory of the commodity tightens, which leads to the benefit of holding the physical inventory, or the convenience yield, exceeding the interest rate and other costs. In this case, people are willing to pay more for the spot commodity, or any futures contracts near it, rather than to purchase futures contracts with longer terms to maturity. On the other hand, contango appears when the basis of the commodity is positive. This happens when there is a sufficient supply of that commodity in the economy, or relatively slack demand. In such a situation, the convenience yield is unable to subsidise the costs. Working's theory leads us to expect a negative relationship between the convenience yield and physical commodity inventories.

A number of empirical papers have investigated the theory of storage and its implications. Brennan (1991) found a negative relationship between the convenience yield and the inventory level in different commodity futures such as metals, crude oil, lumber, etc. Pindyck (1994) got a similar finding in other commodities. However, Gorton, Hayashi and Rouwenhorst (2012), however, studied over 30 commodity markets and found that, the slope of the futures curve (i.e. the commodity basis), is positively related to the level of inventories. Fama and French (1987) argued that the Theory of Storage is less controversial in explaining the commodity futures price behaviour than the Theory of Normal Backwardation.

> *"...there are two popular views of commodity futures prices. The Theory of Storage explains the difference between contemporaneous spot and futures prices in terms of foregone interest in storing a commodity, warehousing costs, and a convenience yield in inventory. The alternative view splits a futures price into an expected risk premium and a forecast of future spot price. The Theory of Storage is not controversial."* (Fama and French 1987)

They also found that the convenience yield for most agricultural commodities varies seasonally. This seasonal effect, however, is not found in metal commodities.

2.2.1.3 Commodity prices and macro-econometric modelling

Over the past decades, we have witnessed the profound influence of commodity prices on the real economy and monetary policy, especially prices for major energy products like crude oil. There has therefore been a growing literature attempting to understand the role of commodity price in macro-econometric modelling, as well as the implications of interactions between the commodity prices and the macroeconomy.

The price puzzle

The most common approach, structural vector autoregressive (SVAR) analysis, often produces confusing "stylised facts" in monetary policy studies. As first documented by Sims (1992), it often shows that the level of inflation responds positively to monetary contractions, at least in the short run. This contradicts the standard theoretical expectation. The presence of this anomaly casts serious doubt on the ability of SVARs to model monetary policy shocks appropriately, as well as on their effectiveness in capturing the monetary transmission mechanism.

Sims (1992), however, also demonstrates that this anomaly can be largely alleviated by the inclusion of the commodity price index in the SVAR. The rationale for this is that, monetary authorities, in practice, use information from forecasts of inflationary pressure when deciding on their policies, which reflects the effects of commodity prices on inflation. Monetary authorities observe and react much faster to the current value of commodities than the price level does, therefore, tight monetary policies precede and thus appear to cause inflation. Eichumbaum (1992) first labelled this anomaly as the *"price puzzle"*, he also provided empirical evidence to support Sims's commodity price argument about the price puzzle. This view is also supported by Christiano, Eichenbaum, and Evans (1994), Balke and Emery (1994), Leeper, Sims, and Zha (1996), Zha (1997), Castelnuovo and

Surico (2010) and many others.

> *"...monetary authority often has information regarding inflationary pressures, not captured in the history of the variables included in the VAR. Acting on the basis of such knowledge, policy makers may raise interest rates in an effort to forestall inflation. Under these circumstances the econometricians would find that innovations to interest rates are followed by increases in the price level and interest rates as well as declines in aggregate output. "* (Eichumbaum 1992)

The presence of the price puzzle has motivated a considerable literature. Giordani (2004) provided an extended overview of this topic. He suggested that the central banks use of inflation forecasts, rather than inflation may not be the only reason for the price puzzle. It could also be caused by the omission of an accurate measure of output. He suggested that using the output gap, instead of log real GDP, mitigates the price puzzle. He further pointed out that the inclusion of the commodity price index alleviates the price puzzle not only because it indicates the future expectation of inflationary pressure, but also because it contains useful information about the output gap. Bernanke, Boivin and Eliasz (2005) suggest that the approach of a Factor Augmented Vector Autoregressive (FAVAR) model could be useful in solving the price puzzle, because it includes extra informational variables such as the commodity price indicator. They argued that, policy makers take into account hundreds of variables when making monetary policies, whereas standard VAR analysis uses only three to six variables. This may lead to omitted variable bias, and such misspecification could contribute to the presence of the price puzzle. They further argued that, although the inclusion of a commodity price index is useful to solve the price puzzle, it is also impractical. Unless such an ad hoc choice of commodity price index can be justified by a theoretical model, there will be uncertainty about the underlying mechanism.

Hansen (2004) examined different variables that are capable of forecasting inflationary pressure. He found it difficult to justify the relationship between forecasting power and mitigation of the price puzzle. He also found that the significance of the price puzzle variese in different periods, and that the effectiveness of using “informational variables” to solve the price puzzle also differs in different historical periods. Rusnak, Havranek, and Horvath (2013) tested over 200 different VAR models in the literature, and cast doubt on these price puzzle solutions. The empirical evidence indicates that the possibility of solving the price puzzle, using existing approaches in the literature, remains questionable. The puzzle is only properly solved by imposing sign restrictions on parameters in the VAR model.

Oil price in macro-econometric studies

Besides the popular debate on the solution of the price puzzle using commodity price indicators, a large number of macro-econometric studies also focus on the interrelationship between the commodity price and the macroeconomy. Because crude oil accounts for over 70% of the consumption of all traded commodities, and is also the most important commodity product in terms of its impact on the real economy, the vast majority of the literature on commodity markets focuses only on the crude oil market.

The standard macro-econometric literature studies the relationship between the spot oil price and macroeconomic indicators. It has found that oil price shocks induce economic recession, and influence the output and the inflationary pressure in the macroeconomy. Hamilton (1983) studied the post World War II oil price effects on the US macroeconomy using a standard VAR model. He found that the overwhelming majority of post-war oil price shocks led to serious recessions in the US economy. Specifically, they severely decreased the aggregate measure of output, and created substantial inflationary pressure. Consequently, these post-war oil price shocks had a profound influence on the dynamic pattern of monetary policy and the economic regime. Gisser and Goodwin (1986) suggested that the crude oil price has both real and inflationary effects. It determines a broad range of macroeconomic indicators, which are often excluded in monetary policy, and always excluded in fiscal policy. Raymond and Rich (1997) found that the oil price has played an important role in the post-war recessions but that with the exception of recession periods, the oil price's effect on output measures such as GDP growth was modest and overstated. Kilian (2006) studied the exogenous oil supply shock in different periods. He found that oil price shocks in the 1970s were persistent due to the contemporaneous strength of the economy. When the economy was weaker, however,as it was, for example, during the Gulf War in 1991, the oil shock had only a temporary effect.

Another strand of macro-econometric literature investigates the links between oil prices and monetary policy. Bernanke, Gurtler and Watson (1997) supported the traditional view, that oil price shocks are at least mildly recessionary without the intervention of monetary policy makers. They confirmed the ability of monetary policy to dampen the recessionary effect introduced by an oil price shock. They also suggest that the cause of aggregate fluctuations in the US economy is the response of systematic monetary policy to the actual or potential inflationary pressures triggered by oil price shocks. Hamilton and Herrera (2004) pointed out that there is a lag between the beginning of an oil shock and the point at which it is most influential. This, however, is not taken into account by Bernanke, Gurtler and Watson (1997).

Kilian (2006, 2008) implied two main channels of transmission of oil price effects to the macroeconomy. The first is the adverse aggregate supply shock, which is caused by the higher costs of imports. The second is the adverse aggregate demand shock, which is caused by lower domestic purchasing power. Notably, the demand side channel of transmission seems to be much more effective than the supply side channel of transmission. He further stressed that policy makers do not respond directly to the oil price innovations, but indirectly respond to the underlying demand and supply shocks that drive the real price of oil along with other macroeconomic variables. A more recent study by Ano Sujithan, Avouyidovi, and Koliai (2013) found new evidence to support the response of monetary policy instruments to shocks from commodity price movements. They concluded that the linkage between commodity markets and the monetary policy indicators has become much stronger, especially since the Lehman crisis.

Other macro-econometric literature about interactions between the crude oil price and the macroeconomy includes: Burbidge and Harrison (1984), Santini (1994), Hooker (1996), Daniel (1997), Carruth, Hooker and Oswald (1998), Muellbauer and Nunziata (2001), Hooker (2002), Leduc and Swell (2004), Carlstrom and Fuerst (2006), Nakov and Pescatori (2010), Kilian (2011) and others. These studies have exclusively investigated the role of the spot price of crude oil in the macroeconomy. However, we have yet to see an extensive discussion of the linkage between crude oil futures prices and the macroeconomy, especially since the much more active trading of crude oil futures contracts in recent decades, compared to the spot crude oil product in recent decades. The crude oil futures market also reflects the risk premium and market expectation in respect to the spot oil price, which tends to provide more fruitful implications than the conventional strand of the literature.

2.2.2 The dynamic term structure model of commodity futures

Another more modern view in the literature on commodity futures is the reduced-form Gaussian factor model. It focuses on explaining the time series behaviour of the commodity futures term structure. It suggests that at any point in time, under a no-arbitrage condition, there exists a linear relationship between several unobservable stochastic Gaussian factors and commodity futures prices. These models are popular in the literature as they are both tractable with closed-form solutions, and flexible with few restrictions on the determining variables. Since the early 1980s they have formed the fundamental framework for term structure research on both bond interest rates, and commodity futures.

Based on the benchmark option pricing framework by Black and Scholes (1973), Merton (1973), and Cox and Ross (1976), the first attempt at a reduced-form Gaussian model for commodity markets can be attributed to Schwartz

(1982). He developed a model for pricing a commodity-linked bond with a closed-form solution, in which the commodity price is a determining latent factor to the commodity-linked bond, and is specified as a simple geometric Brownian motion under the risk-neutral measure.

Brennan and Schwartz (1985) first set up a single factor model with a tractable analytical solution for commodity futures contracts, where the spot price of the commodity is assumed to be the only source of price uncertainty under the risk-neutral measure. Although Brennan and Schwartz (1985) recognised the presence of the convenience yield and the interest rate in the valuation of commodity futures following the analysis of Ross (1978), they assumed that these two variables are only either constant terms on their own, or constant terms proportional to the spot price of the commodity.

The one factor approach has been questioned by studies such as Hilliard and Reis (1998), which suggested that it drastically mis-prices oil futures because the constant convenience yield is not specified appropriately. They also pointed out that the one factor model is unable to explain effectively more complex cross-sectional conditions in commodity futures during the term of the contract.

In light of the Theory of Storage advanced by Kaldor (1939), Working (1949), and Brennan (1958), studies such as Ross (1978), Brennan (1986), Fama and French (1987) and others discovered that the inclusion of a stochastic inventory measure substantially improves the empirical performance of the commodity futures pricing model. Paddock, Siegel, and Smith (1988), in particular, addressed the necessity to include an equilibrium model for the underlying inventory. Empirically, this reduces possible measurement errors in commodity futures modelling, as in Brennan and Schwartz (1985). They accommodated this equilibrium model by specifying the commodity inventory as a mean reverting stochastic process, and the same specification for the commodity inventory was found in Morck, Schwartz and Stangeland (1989).

Gibson and Schwartz (1992) argued that the assumption of constant convenience yield can only hold in a very restricted environment. Relaxing this assumption satisfies the previous proposition in Paddock, Siegel, and Smith (1988) and Morck, Schwartz and Stangeland (1989), since the negative relationship between the level of inventory and convenience yield is widely known as posited by the Theory of Storage. Standing on this ground, Gibson and Schwartz (1992) formally proposed a two factor commodity futures pricing model. In this model, the commodity futures price is determined by two stochastic factors under the risk-neutral measure: the spot price of the commodity and the convenience yield. They build on the earlier discussions in Gibson and Schwartz (1989), which explained that the mean reverting nature of the convenience yield is a result of the cyclic nature of the inventory level, therefore being specified as an Ornstein-Uhlenbeck process. As typical in the literature, the spot price of the commodity is defined as a geometric Brownian motion. Specifically:

$$dS_t = \mu_s^Q S_t dt + \sigma_S S dW_{S,t}^Q \tag{2.2}$$

$$d\delta_t = \kappa_\delta^Q(\theta_\delta^Q - \delta_t)dt + \sigma_\delta dW_{\delta,t}^Q \tag{2.3}$$

where S_t and δ_t are the spot price of the commodity and the stochastic convenience yield, μ_s^Q is the drift in the geometric Brownian motion, and σ_S and σ_δ are the volatilities in S_t and δ_t. S_t and δ_t maybe correlated: $dW_S dW_\delta = \rho_{S,\delta} dt$, where $\rho_{S,\delta}$ is their correlation coefficient.

Applying Ito's Lemma yields the following closed-form solution for the price of the commodity futures contract:

$$lnF_{\tau,t} = \alpha_\tau + lnS_t - \delta_t \frac{1 - e^{-\kappa_\delta^Q \tau}}{\kappa_\delta^Q} \tag{2.4}$$

where

$$\begin{aligned} \alpha_\tau = & \left(r - (\theta_\delta^Q - \frac{\lambda_\delta}{\kappa_\delta^Q}) + \frac{\sigma_\delta^2}{2\kappa_\delta^{Q^2}} - \frac{\sigma_S \sigma_\delta \rho_{S,\delta}}{\kappa_\delta^Q} \right) \tau + \frac{\sigma_\sigma^2(1 - e^{-2\kappa_\delta^Q \tau})}{4\kappa_\delta^{Q^3}} \\ & + \left((\theta_\delta^Q - \frac{\lambda_\delta}{\kappa_\delta^Q})\kappa_\delta^Q + \sigma_S \sigma_\delta \rho_{S,\delta} - \frac{\sigma_\delta^2}{\kappa_\delta^Q} \right) \frac{1 - e^{-2\kappa_\delta^Q \tau}}{\kappa_\delta^{Q^2}} \end{aligned} \tag{2.5}$$

and λ_δ is the parameter for the price of risk attributed to the convenience yield.

Empirically, the introduction of the convenience yield to the model genuinely improves the goodness-of-fit, but Gibson and Schwartz (1989) also reported that the model only fits well for futures contracts with short term maturities, while mispricing is still documented in contracts with long term maturities. It seems that two questionable assumptions in this model might be to blame for this imperfection. First, the interest rate is assumed to be a constant factor. Second, the risk premium is also assumed to be constant.

Schwartz (1997) relaxes the assumption of a constant interest rate factor, and proposes a three factor model for commodity futures pricing. The convenience yield is specified in the same form as in the Gibson and Schwartz (1992) two factor model. The interest rate factor in this context is viewed as the instantaneous short rate. Similar to the specification of the convenience yield, it is also assumed to have a mean reverting feature and, therefore, follows an Ornstein-Uhlenbeck process as in Vasicek (1977). In summary, we have the following stochastic processes which determine the commodity futures price under the risk-neutral measure:

$$dS_t = (r_t - \delta_t)S_t dt + \sigma_S S_t dW^Q_{S,t} \tag{2.6}$$

$$d\delta_t = \kappa^Q_\delta(\theta^Q_\delta - \delta_t)dt + \sigma_\delta dW^Q_{\delta,t} \tag{2.7}$$

$$dr_t = \kappa^Q_r(\theta^Q_r - r_t)dt + \sigma_r dW^Q_{r,t} \tag{2.8}$$

where r_t is the instantaneous short rate. Considering the convenience yield as the conceptual analogue of the "dividend yield" to the commodity asset, as suggested by Ross (1978), the drift term in equation (2.6) should be defined as $r_t - \delta_t$. S_t, δ_t and r_t maybe correlated as: $dW_{S,t}dW_{\delta,t} = \rho_{S,\delta}dt$, $dW_{S,t}dW_{r,t} = \rho_{S,r}dt$, $dW_{\delta,t}dW_{r,t} = \rho_{\delta,r}dt$.

Applying Ito's Lemma yields a closed-form solution as follows:

$$lnF_{\tau,t} = A_\tau + lnS_t - \delta_t \frac{1 - e^{-\kappa^Q_\delta \tau}}{\kappa^Q_\delta} + r_t \frac{1 - e^{-\kappa^Q_r \tau}}{\kappa^Q_r} \tag{2.9}$$

where:

$$
\begin{aligned}
A_\tau = & \frac{(\kappa_\delta^Q(\theta_\delta^Q - \frac{\lambda_\delta}{\kappa_\delta^Q}) + \sigma_S\sigma_\delta\rho_{S\delta})((1-e^{-\kappa_\delta^Q\tau}) - \kappa_\delta^Q\tau)}{\kappa_\delta^{Q^2}} \\
& - \frac{\sigma_\delta^2(4(1-e^{-\kappa_\delta^Q\tau}) - (1-e^{-\kappa_\delta^Q\tau}) - \kappa_\delta^Q\tau)}{\kappa_\delta^{Q^3}} \\
& - \frac{(\kappa_r^Q(\theta_r^Q - \frac{\lambda_r}{\kappa_r^Q}) + \sigma_S\sigma_r\rho_{Sr})((1-e^{-\kappa_r^Q\tau}) - \kappa_r^Q\tau)}{\kappa_r^{Q^2}} \\
& - \frac{\sigma_2^r(4(1-e^{-\kappa_r^Q\tau}) - (1-e^{-\kappa_r^Q\tau}) - 2\kappa_r^Q\tau)}{\kappa_r^{Q^3}} \\
& + \sigma_\delta\sigma_r\rho_{\delta r}\left(\begin{array}{c} \frac{(1-e^{-\kappa_\delta^Q\tau}) + (1-e^{-\kappa_r^Q\tau}) - (1-e^{-(\kappa_\delta^Q+\kappa_r^Q)\tau})}{\kappa_\delta^Q\kappa_r^Q(\kappa_\delta^Q+\kappa_r^Q)} \\ + \frac{\kappa_\delta^{Q^2}(1-e^{-\kappa_r^Q\tau}) + \kappa_r^{Q^2}(1-e^{-\kappa_\delta^Q\tau}) - \kappa_\delta^Q\kappa_r^{Q^2}\tau - \kappa_r^Q\kappa_\delta^{Q^2}\tau}{\kappa_\delta^{Q^2}\kappa_r^{Q^2}(\kappa_\delta^Q+\kappa_r^Q)} \end{array}\right)
\end{aligned}
\tag{2.10}
$$

where λ_δ and λ_r are the parameters for the price of risk attributed to the convenience yield δ_t and short rate r_t.

Schwartz (1997) compared the three factor model with the previous one factor and two factor model. The empirical evidence indicated that a one factor model performs poorly and is unable to properly capture cross-sectional variations in the commodity futures term structure. Although the two factor model and three factor model are almost the same in the estimation of commodity futures with short term maturities, the three factor model clearly outperforms the two factor model for the estimation of those with long term maturities in terms of the goodness-of-fit.

Comparing equations (2.4) and (2.9), it is not hard to see that the three factor commodity futures pricing model also nests the two factor commodity futures pricing model. At any point in time, the commodity futures price is exponentially affine to the three state variables, namely:

$$F_{\tau,t} = exp(A_\tau + B'_\tau X_t) \tag{2.11}$$

where A_τ is defined previously in equation (2.10), $X_t = (lnS_t, \delta_t, r_t)'$, that is the as vector of state variables which determines the variations in the commodity futures term structure, and B'_τ is the loading of state variables that affects the commodity futures prices as:

$$B_\tau = \begin{pmatrix} 1 \\ -\dfrac{1 - e^{-\kappa_\delta^Q \tau}}{\kappa_\delta^Q} \\ \dfrac{1 - e^{-\kappa_r^Q \tau}}{\kappa_r^Q} \end{pmatrix}, \tag{2.12}$$

Casassus and Collin-Dufresne (2005) extended the Schwartz (1997) three factor model. They formally specified the affine framework and its general closed-form solution. In addition to this, they contributed to the specification of the time invariant price of risk in their model, based on Duffee (2002), which greatly improved the empirical performance of their model. They also found that the convenience yield factor is in fact composed of an idiosyncratic component, and the weighted average of the other two latent factors:

$$\delta_t = \beta_r r_t + \beta_S S_t + \hat{\delta}_t \tag{2.13}$$

where β_r and β_S are the weights on the effects of the interest rate factor, r_t, and the spot commodity price factor, S_t, on the convenience yield[1]. $\hat{\delta}_t$ is the idiosyncratic convenience yield component, which is specified as a standard mean reverting process, the same as in equations (2.3) and (2.7) in the Gibson and Schwartz (1992) two factor model and Schwartz (1997) three factor model:

$$d\hat{\delta}_t = \kappa_{\hat{\delta}}^Q(\theta_{\hat{\delta}}^Q - \hat{\delta}_t)dt + \sigma_{\hat{\delta}} dW_{\hat{\delta},t}^Q \tag{2.14}$$

This leads to a novel specification of the stochastic process of convenience

[1]Please refer to Appendix D in Casassus and Collin-Dufresne (2005) for detailed specification of β_r and β_S.

yield δ_t under a no-arbitrage condition as:

$$d\delta_t = (\kappa^Q_{\delta,0} + \kappa^Q_{\delta,r} r_t + \kappa^Q_{\delta}\delta_t + \kappa^Q_{\delta,S} S_t)dt + \sigma_\delta dW^Q_t \quad (2.15)$$

where $\kappa^Q_{\delta,S}$ and $\kappa^Q_{\delta,r}$ are, respectively, the effects from the spot commodity price and interest rate factor on the convenience yield under the risk-neutral measure. This reveals the fact that the dynamic of the convenience yield is not a standalone process as the conventional models suggest, but is actually influenced by the other two latent factors. The conventional models, therefore, are restricted versions, since the convenience yield specification in Casassus and Collin-Dufresne (2005) degenerates to the one in the Schwartz (1997) three factor model if $\alpha_r = \kappa^Q_{\delta,r} = 0$ and $\alpha_s = \kappa^Q_{\delta,s} = 0$.

2.2.3 Developments in macro-finance modelling

There has been extensive development in mainstream finance models of bond yield curves and commodity futures curves, which only use latent variables to determine the variations in the term structure. These models share the advantage of being tractable and flexible, and are usually referred to as "pure finance" models. Although they are favoured by researchers due to their useful statistical descriptions of term structure dynamics, there has been long-standing criticism of the lack of economic interpretation given to their latent factors. The integration of the structural relationship between the financial term structure and the macroeconomy is of interest to researchers in various fields, such as macroeconomists, financial economists, policy makers and market participants. The growing desire to introduce macroeconomic insight into the canonical affine term structure framework has motivated the rapid development of macro-finance modelling.

2.2.3.1 The mainstream macro-finance modelling

In this spirit, macro-finance models that target the bond interest rate were initially developed. The pioneering research of Ang and Piazzesi (2003) incorporated macroeconomic variables in the affine term structure model (ATSM) of zero coupon bond yields. They chose three unobservable variables, also known as the “level”, “slope”, and “curvature” factors of the interest rate term structure, and two observable macroeconomic indicators, representing inflation and the real activity of the economy; combining these to explain the interaction between the term structure of zero coupon yields and the macroeconomy. They found that macroeconomic indicators explain as much as 85% of the variation in the bond yield. Macro factors are particularly good at providing descriptions of the behaviour of the yields in the short term. Nevertheless, it is also necessary to retain latent variables in the model, because the macro factors are ineffective at explaining long term yields, while the latent variables still account for most of the movements of the yield curves at the long end. The parsimonious version of the general ATSM framework is considered to be the Nelson-Siegel model. Diebold, Rudebusch and Aruoba (2006) extended the dynamic Nelson-Siegel representation introduced by Diebold and Li (2006), and established another example of a macro-finance model for US Treasury yields. They found that the first latent factor, also known as the “level” factor, is highly correlated with the level of inflation. The second factor, which is normally referred to as the “slope” factor, is closely related to the real activity. Yet there seems to be little evidence to suggest a connection between the third latent factor, also labelled as the “curvature” factor, and key macroeconomic indicators.

Other papers provided more perspectives on general macro-finance modelling of interest rate term structures, such as Hordahl, Tristani and Vestin (2006) and Rudebusch and Wu (2008), Joslin, Priebsch and Singleton (2014) and others. These studies give sensible descriptions of the behaviour of yield curves, as well as interpretations of the structural relationships between macro variables and yield curves. However, they do not provide an adequate description of the long term linkage between variables in the macro-finance framework.

Initiated by Svensson (1999), Smets (2002), Rudebusch (2002), Kozicki and Tinsley (2005) and others, subsequent macro-finance research has used the semi-structural "central bank model" (CBM). This research suggests that the latent variables represent exogenous shocks to the inflation target or underlying inflation rate of the central bank. Further studies, such as Dewachter and Lyrio (2006), Dewachter, Lyrio and Maes (2006), Spencer (2008), Liu and Spencer (2010), adopt a vector error correction model (VECM) specification for the macro-finance framework in order to capture the underlying features that drive the yield curves and the macroeconomy. They show that a common non-stationary trend, which seems to be the underlying rate of inflation, characterises the variations in the interest rate term structure at the long end.

2.2.3.2 Macro-finance modelling of the commodity futures term structure

The mainstream macro-finance model focuses on term structures of the interest rate, yet developments in the macro-finance model that targets the commodity futures term structure have progressed only very slowly in the literature. In addition, while the subsequent CBM represents the behaviour of the macroeconomy in terms of the output, inflation, and the short term interest rate, there is no role for the commodity price, particularly the oil price, in the basic model, despite the evidence of the effect of oil shocks on the macroeconomy.

To the best of my knowledge, the only attempt to address this in the literature is Heath's (2016) paper studying crude oil futures prices, which followed the general set-up in Joslin, Priebsch and Singleton (2014). In Heath's work, the cost of carry is modelled directly, instead of being separated into the convenience yield and the interest rate factor. The latent system that consists of the nominal spot oil price and the cost of carry is assumed to follow a simple autoregressive scheme under the risk-neutral measure, independently of macro variables. The cross-section of oil futures is then affine to these two latent factors only. The macro system contains two observable macroeconomic variables that measure real activity and inventory levels. The real world time series dynamic is modelled using a VAR with a state vector that includes the latent variables alongside the macroeconomic variables. The macroeconomic variables are therefore "unspanned" in the sense that they only have a lagged dynamic effect on the futures curve, not a contemporaneous one.

Conceptually, Heath (2016) follows the existing strand of macro-finance modelling in a rigorous manner. This approach is unable, however, fully to specify the features that are related to the commodity futures term structure. One major drawback is that, it does not properly specify the dynamic of the spot oil price under the risk-neutral measure, which is defined by the no-arbitrage relationship, as suggested by Working's theory. Instead, it only defines the spot oil price as a standard autoregressive process. Another drawback is that nominal spot oil prices are used in the model. Unlike bond prices however, which are the relative prices of money in different periods, futures prices tend to increase over time due to the effect of inflation, which imparts a strong secular uptrend. The real spot oil price, therefore, which removes the effect of inflation on nominal prices, tends to be a better candidate than the nominal spot oil price, in terms of providing more meaningful implications for the structural relationship between variables in the real economy. Another concern with respect to Heath (2016) is that the cost of carry is treated as a single variable, even though the model does not include any proxy for the interest rate in the model. This model setup, therefore, disables the model's ability to evaluate the effects of monetary policy and other macro shocks on the spot oil price and the convenience yield.

2.3 Preliminary data analysis

This section will provide the preliminary analysis for the log real spot price of crude oil and some macro indicators. These are the same observable data that are used in the chapter 4. By evaluating the results, it will be possible to get an initial idea of how these variables interact with each other under the preliminary standard macro-econometric framework. This will allow an understanding of the implications for the macro-finance model for commodity futures that will be developed later in the thesis.

2.3.1 Data

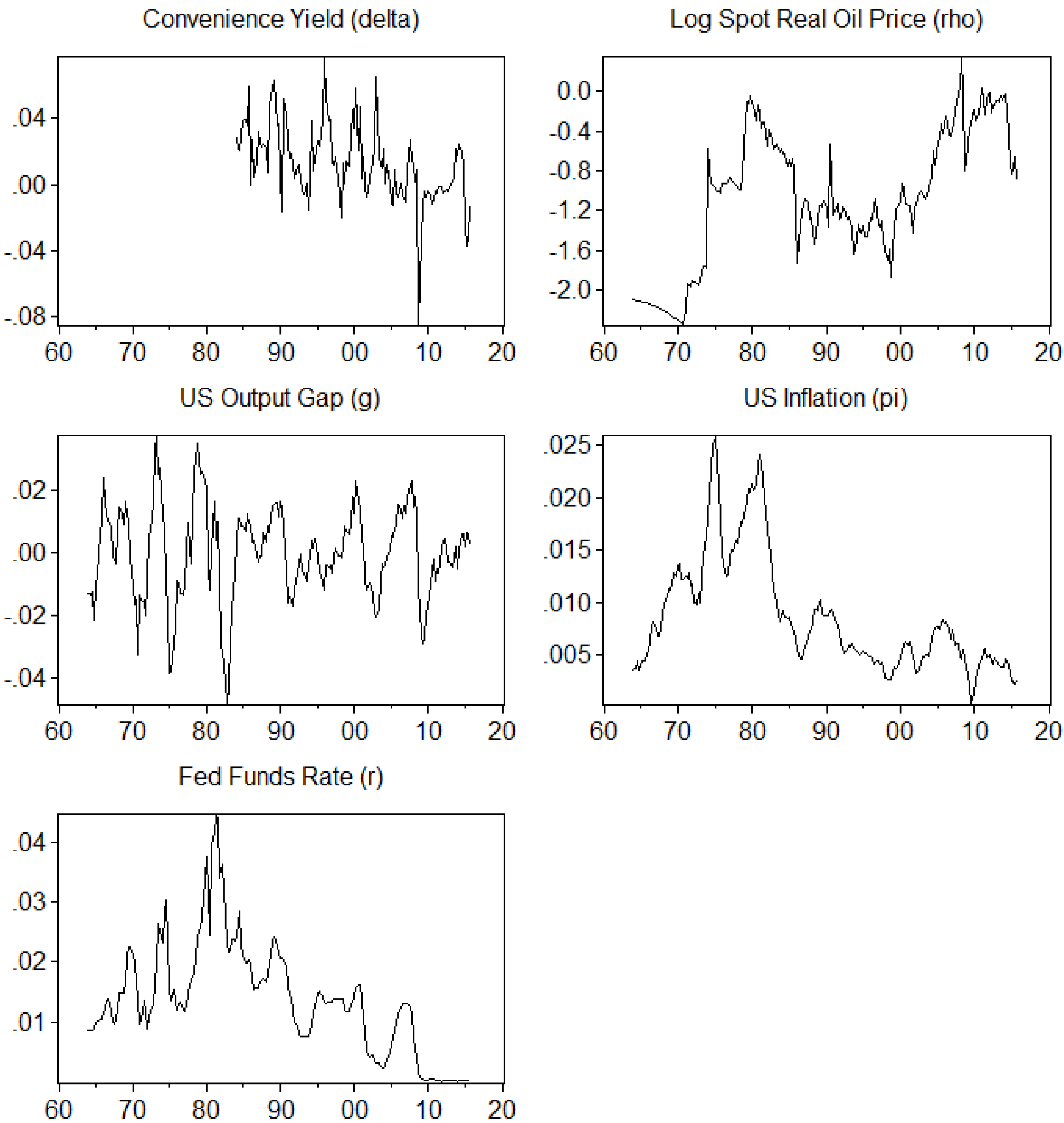
Quarterly data is used in this analysis for the convenience yield of crude oil (δ), the spot price of crude oil in the real term (ρ), the US output gap (g), US inflation rate (π), and US interest rate (r). All data are downloaded from the Thompson Reuters DataStream.

The US output gap (g) is generated by applying the Hodrick-Prescott (HP)[2] filter to the log US GDP that is seasonally adjusted at a constant price. The US inflation rate (π) is the quarterly log difference of the seasonally adjusted US implicit price deflater (IPD) of GDP, that is $\pi_t = lnIPD_t - lnIPD_{t-4}$. The US Federal Funds rate is used as a proxy for the US interest rate, specified as a quarterly decimal fraction (i.e. the annual rate as a % divided by 400). The time span of all these data series starts from the first quarter of the year 1964 to the fourth quarter of 2015, which allows the analysis to cover oil shocks during the 1970s.

The spot oil price (S) is a composite series. Although the full series of the West Texas Intermediate (WTI) spot oil price only starts from the first quarter of 1984, the spot price series for London Brent crude oil dates back to the first quarter of 1970. Since the WTI and the Brent oil market are well known for their high level of co-movement over the years, and they have only deviated in recent years due to the development of shale oil, I make a plausible assumption that the WTI spot oil price was essentially the same as the London Brent spot oil price between the years 1970 to 1984. In this spirit, the two series are spliced together to enable the spot crude oil price series to be constructed from 1970 to 2015. Another assumption is that the spot crude oil price is constant over time before the year 1970, which means that the spot oil price is set at that of the first quarter of 1970 for the period, from 1964 to 1970. This is because before OPEC overtook the power of global oil pricing at the beginning of the 1970s, international crude oil was deliberately set at a fixed price by the major crude oil monopolies in the US after World War II.

[2] I use a standard "lambda" of 1600 in the HP filtering procedure

Figure 2.1: The data



This figure plots the five variables that I will investigate in this section. Convenience yield is derived using the no-arbitrage equation of commodity futures pricing. Spot oil price is a composite series that combines the spot price of Brent crude oil and WTI crude oil. The output gap is generated using the Hodrick-Prescott (HP) filter to the log US GDP. Inflation is the log difference of the US implicit price deflater of GDP. The Federal Funds rate proxies the US interest rate.

This composite spot oil price series is specified in normal terms. Since this work is interested in the relationship between the log spot price of the crude oil and other macro variables in the real economy, however, the log spot oil price is generated in real terms by subtracting the log nominal spot oil price from the log GDP price level:

$$\rho_t = s_t - p_t \tag{2.16}$$

where ρ_t is the real spot oil price at time t taking the natural logarithm, s_t is the log spot oil price in the nominal term, and p_t is the price level that is proxied by the US implicit price deflator of GDP. This log real spot oil price (ρ_t) is used in the later analysis.

Table 2.1: Summary Statistics

Variables	Mean	S.D.	Skew	Kurt	ADF	Obs.
Convenience yield (δ)	0.0123	0.023	-0.123	1.948	0.002	128
Log real spot oil price (ρ)	-1.067	0.678	-0.204	-0.878	0.296	208
US output gap (g)	3.3×10^{-11}	0.015	-0.360	0.523	0.000	208
US inflation (π)	8.6×10^{-4}	5.5×10^{-4}	1.240	0.831	0.541	208
Fed Funds rate (r)	0.014	9.2×10^{-4}	0.708	0.830	0.432	208

This table presents the summary statistics of the dataset. These data were supplied by Datasteam and are discussed in the text. Mean denotes the arithmetic mean of the sample, S.D. standard deviation; Skew. and Kurt. report skewness and excess kurtosis, standard measures of the third and fourth moments. Obs. reports the number of observations. ADF reports the p-value of the Augmented Dickey Fuller test statistic under the null hypothesis of non-stationarity. The lag lengths of the ADF test are determined by the Akaike information criterion.

The crude oil convenience yield is an implied series generated by reversing the well-known no-arbitrage equation of commodity futures pricing:

$$F_{\tau,t} = S_t e^{(r_t - \delta_t)\tau} \tag{2.17}$$

where $F_{\tau,t}$ is the WTI crude oil futures price at time t for τ term to maturity. In this case, the three-months crude oil futures price is used; therefore, τ is explicitly equals to 3/12. S_t is the nominal spot price of oil. r_t is the short rate that is proxied by the Federal Funds rate. δ_t is the implied convenience yield that I intend to derive from this no-arbitrage equation. Since the three-months

WTI crude oil futures price data are only available from 1984 to 2015, therefore, differing from the other four data series, which start from the first quarter of 1964, the implied convenience yield has the time span from the first quarter of 1984 to the fourth quarter of 2015.

All data are plotted in Figure 2.1. Summary statistics for these data are also presented in Table 2.1. The ADF test result in Table 2.1 reports the p-value of the test statistic. The null hypothesis ($H0$), which states "the variable has a unit root", is tested against the alternative hypothesis ($H1$), which states "the variable does not have a unit root". The p-value of the ADF test statistics for the implied convenience yield and the US output gap are reported as 0.002 and 0.000 respectively. This indicates that there is sufficient evidence to reject the $H0$ and accept the $H1$ in both cases under the 5% confidence level. This means that statistically, both of these variables have a unit root. In other words, the implied convenience yield and the US output gap are stationary time series. This is, however, different from the cases of the log real spot oil price, the US inflation rate, and the Federal Funds rate. The p-values of the ADF test statistics for these data series are reported as 0.296, 0.541, and 0.432, respectively, which indicates that there is insufficient statistical evidence to reject the $H0$ and accept the $H1$ under the 5% confidence level. In other words, each of these three variables has a unit root, and therefore is a non-stationary time series.

2.3.2 The Granger causality test

This section discuss the Granger causality test result. The Granger causality test shows how the lag of one variable is able to forecast another, thus revealing the presence of interconnections between variables. The test result reports a p-value of the test statistic. The null hypothesis ($H0$), which states that "one variable does not Granger cause the other one", is tested against the alternative hypothesis ($H1$) , which states that "one variable Granger causes the other one". Specifically, a Granger causality test is undertaken for the log real spot oil price and other macro variables, namely: the US output gap, the US inflation rate and the Federal Funds rate, with the intention of finding out the strengths of the linkages between these variables.

Table 2.2 presents the results for the overall period from 1964Q1 to 2015Q4. The p-values indicate there is sufficient statistical evidence to reject the null hypothesis of the statements: "Log real spot oil price $\nRightarrow$ US inflation" and "US inflation $\nRightarrow$ Log real spot oil price" under the 95% confidence level. I can also reject the null hypothesis of the statements: "Log real spot oil price $\nRightarrow$ Fed Funds rate" under the 90% confidence level. Nevertheless, there is not sufficient statistical evidence to reject all the other null hypotheses. This indicates that the log real spot oil price is actively related to the other US macro variables. Specifically, the real spot oil price significantly Granger causes the US inflation level. This is in line with conventional wisdom, because crude oil is the fundamental material for various industrial products. The real oil price, therefore, determines the fundamental cost of the industrial production at every level. The real spot oil price also significantly Granger causes the US Fed Funds rate. The Fed Funds rate is determined by the Federal Open Market Committee (FOMC) in the Federal Reserve system. As a policy rate, it has strong implication for monetary policy in the US. These results show that the real spot oil price is able to infer the variation in the US policy rate. It therefore seems possible that the monetary authority takes more information from the oil price when carrying out their policies.

Table 2.2: Bivariate Granger Causality Test for the Log Real Spot Oil Price and Other Variables

Null hypothesis	p-value
Time period	*1964Q1-2015Q4*
Log real spot oil price $\nRightarrow$ US output gap	0.433
US output gap $\nRightarrow$ Log real spot oil price	0.132
Log real spot oil price $\nRightarrow$ US inflation	0.000
US inflation $\nRightarrow$ Log real spot oil price	0.001
Log real spot oil price $\nRightarrow$ Fed Funds rate	0.075
Fed Funds rate $\nRightarrow$ Log real spot oil price	0.633

Table 2.2 shows the bivariate Granger causality test result of the log real spot oil price and the US output gap, the US inflation and the Fed Funds rate, from 1964Q1 to 2015Q4. Arrow notation "$\nRightarrow$" in the null hypothesis statement, stands for: "does not Granger causes". The number of lags/degree of freedom in each case is decided using the Akaike information criterion.

Although it is counter intuitive to find no obvious interaction between the log real spot oil price and the US output gap in this period, these results are in line with empirical evidence in previous studies. In Hooker (1996) and Hamilton (1996), bivariate interactions between different measurements of output and oil price could only be statistically verified in the post-World War II period before the early 1970s, which is largely excluded in my sample data. For the periods after the early 1970s, however, these interactions become very weak. Hooker (1996) suggests that the reason for this was mainly because the oil price became endogenous after the early 1970s. To be more specific, after the early 1970s, most information about the oil price in the future was included in the measurements of output, hence, there is a much smaller unforecastable component of the oil price to be captured by the Granger causality test. Notwithstanding this, as pointed out by Kilian (2006, 2008) and others, the possible underlying real oil effect on output could be transmitted by other channels more indirectly in recent decades. These channels include aggregate supply and demand and, subsequently, inflation and monetary policy, which are not specified in the bivariate Granger causality test.

Table 2.3: Bivariate Granger Causality Test for the Convenience Yield and Other Variables

Null hypothesis *Time period*	p-value *1984Q1-2015Q4*
Implied convenience yield $\nRightarrow$ Log real spot oil price	0.001
Log real spot oil price $\nRightarrow$ Implied convenience yield	0.000
Implied convenience yield $\nRightarrow$ US output gap	0.629
US output gap $\nRightarrow$ Implied convenience yield	0.965
Implied convenience yield $\nRightarrow$ US inflation	0.425
US inflation $\nRightarrow$ Implied convenience yield	0.159
Implied convenience yield $\nRightarrow$ Fed Funds rate	0.816
Fed Funds rate $\nRightarrow$ Implied convenience yield	0.011

Table 2.3 shows the bivariate Granger causality test result of the implied convenience yield and the other four variables for the period: 1984Q1 to 2015Q4. Arrow notation: $\nRightarrow$ in the null hypothesis statement, stands for: "does not Granger causes". The number of lags/degrees of freedom in each case is decided using the Akaike information criterion.

Table 2.3 shows the Granger causality test results for the convenience yield and other variables. Because the convenience yield only starts from the first quarter of 1984, it is only possible to test period from 1984Q1 to 2015Q4. The p-values indicate that there is sufficient statistical evidence to reject the following $H0$s: "Implied convenience yield $\nRightarrow$ Log real spot oil price", "Log real spot oil price $\nRightarrow$ Implied convenience yield", and "Fed Funds rate $\nRightarrow$ Implied convenience yield". However, there is no sufficient statistical evidence to reject the other $H0$s. Casassus and Collin-Dufresne (2005) numerically proved that the convenience yield is affected by the spot commodity price and the interest rate variable, although it is under the risk-neutral measure. These results also empirically verify these interactions under the real world measure. Furthermore, there is also strong evidence that convenience yield significantly determines the log real spot oil price. These findings will be useful for specifications in future models.

Table 2.4: Bivariate Granger Causality Test for the US Output Gap, the US Inflation, and the Fed Funds Rate

Null hypothesis	p-value
Time period	*1964Q1-2015Q4*
US Inflation $\nRightarrow$ US output gap	0.039
US output gap $\nRightarrow$ US inflation	0.000
Fed Funds rate $\nRightarrow$ US output gap	0.000
US output gap $\nRightarrow$ Fed Funds rate	0.000
Fed Funds rate $\nRightarrow$ US inflation	0.001
US inflation $\nRightarrow$ Fed Funds rate	0.008

Table 2.4 shows the bivariate Granger causality test result for the US output gap, US inflation and the Fed Funds rate, for the period: 1964Q1 to 2015Q4. Arrow notation: $\nRightarrow$ in the null hypothesis statement, stands for: "does not Granger causes". The number of lags/degrees of freedom in each case is decided using the Akaike information criterion.

Finally, Table 2.4 tests the Granger causality of the bivariate US output gap, US inflation, and the Federal Funds rate for the overall period between 1964Q1 and 2015Q4. The p-values indicate that there is sufficient statistical evidence to reject all null hypotheses in the table. This indicates that the interactions of these variables are all strong and significant, as implied by conventional wisdom.

2.3.3 Impulse response functions

These interactions are now further evaluated using the impulse response functions, which describe the lag response of one variable to a one time impulse from another. These impulse response functions are here specified orthogonal innovations generated from an unrestricted vector autoregressive model with one lag (VAR(1)) only. Four variables are tested in this exercise: the log real spot oil price (ρ_t), the US output gap (g_t), US inflation (π_t) and the Fed Funds rate (r_t). The VAR(1) model for the state vector, $X_t = (\rho_t, g_t, \pi_t, r_t)'$, is therefore defined as:

$$X_t = K + \Phi X_{t-1} + W_t \qquad\qquad W_t \sim (0, R) \tag{2.18}$$

where K is a 4×1 vector of constant terms, Φ is a 4×4 matrix that specifies the interrelationships between state variables, R is a 4×4 matrix that can be Cholesky decomposed as $R = \Sigma\Sigma$, and Σ is a 4×4 lower triangular matrix. The ordering in X_t is based on the descending order of exogenity of the variables.

Figure 2.2 presents the results for the overall period between 1964Q1 to 1984Q1. The impulse responses of the output gap to the real oil price shock shows that there is an immediate positive response of the output to the oil price shock, but this later becomes negative after about five quarters. This is in line with the description in Hamilton and Herrera (1994), who suggested that although oil shocks drag output, and sometimes introduce post-war recessions, their effect is normally delayed for a few quarters. Impulse responses of inflation and the interest rate to shocks from the output gap are positive. This indicates that growing industrial production induces higher inflation and triggers policy makers to react positively. Meanwhile, inflation negatively responds to policy shocks, which means that monetary policy is effective in reducing pressure from inflation. One would expect the real oil price to respond positively to the shock from output due to higher demand, yet the positive response from the real spot oil price is very weak and quickly turns to negative in the long-run. The policy shock has an immediate negative effect on the output, however this effect later becomes positive due to a lower level of inflation that is maintained by the higher policy rate. The impulse responses to inflation shocks are mixed. Policy makers raise the interest rate as a reaction to the inflation shock in order to reduce the inflationary pressure if the effect of the inflation shock persists. Meanwhile, this drags down output because higher inflation creates higher costs for industrial production. Nevertheless, the response of the real spot oil price to inflation is not obvious. The immediate and persistent negative response of inflation to the real oil price shock in these findings is therefore difficult to explain. This also happens to the Fed Funds rate's response to the real oil price. It seems to be the case that impulse response functions are also puzzled by unit roots in the non-stationary data series for the time period.

Figure 2.2: Impulse Response Functions of Unrestricted VAR with One Lag for the Period Between Year 1964Q1 to 2015Q4

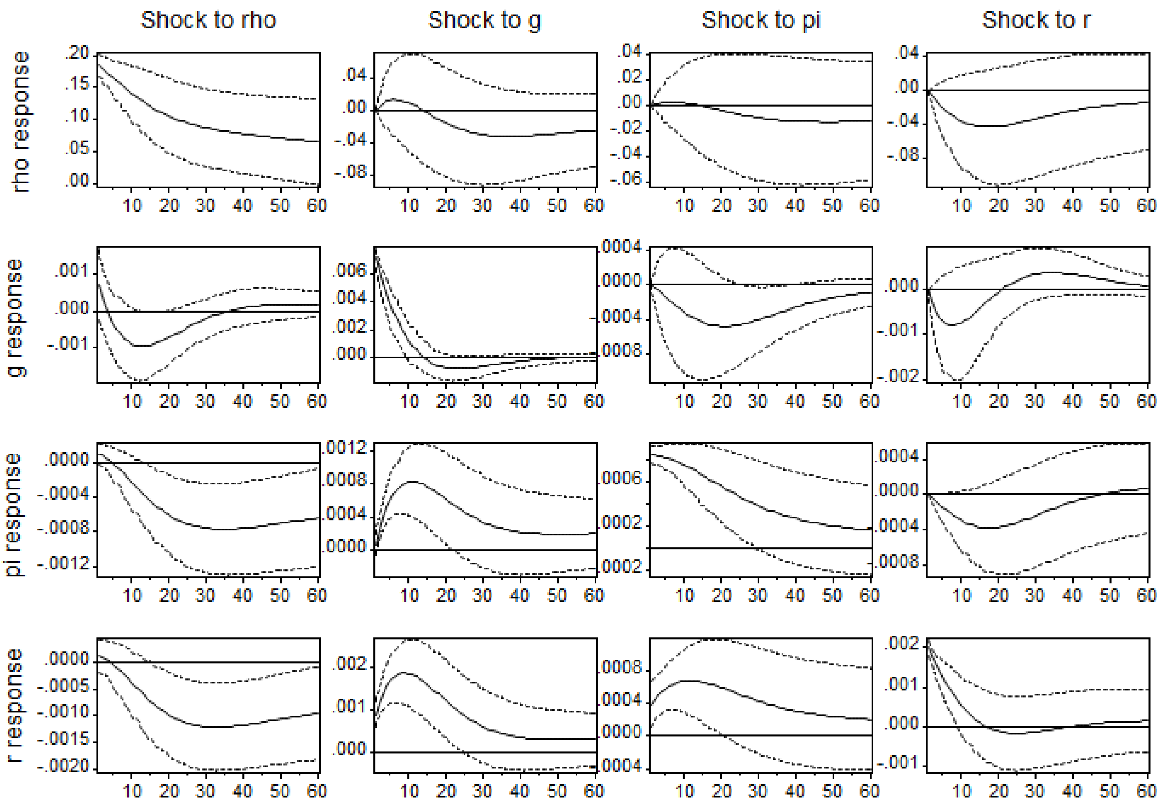


Figure 2.2 shows the impulse response functions for the overall period: 1964Q1 to 2015Q4. Each panel shows the effect of Choleskey one standard error (S.D.) innovations on the state variables (solid lines). This figure also displays the confidence bound at ± two standard error (S.E.), i.e. the 95% confidence interval (dashed lines) in each case. Elapsed time is measured in calendar quarters.

The influence of unit roots in the VAR system drives the impulse response functions to display confusing stylised facts, which cast serious doubt on the ability of the VAR model to specify the interrelationships between the tested time series appropriately. Phillips (1998) thoroughly investigated this issue and found that the impulse response matrices estimated by the VAR system with non-stationary data were inconsistent at a long horizon because they tended to provide random impulse responses rather than the true impulse responses of the variables. He further concluded that one must inevitably expect large uncertainty in the relevant analyse using impulse response functions estimated by an unrestricted VAR system that has a non-stationary characteristic.

One popular solution is to use the vector error correction model (VECM) instead of an unrestricted VAR when encountering a non-stationary data series. If the non stationarity is caused by a long-run stochastic trend in the underlying variables, that has linear combination which is integrated of the order zero, these variables are said to be cointegrated, and the VECM is particularly useful in estimating both short-term, and long-term cointegrated relationships between time series. Empirically, Phillips (1998) also pointed out that the VECM presents highly accurate impulse response functions, and much better forecasts than the unrestricted VAR model. In this spirit, chapter 4 of this thesis will further demonstrates the application of the VECM to the macro-finance model of crude oil market.

2.4 Concluding remarks

This chapter first reviewed the various strands in the previous literature that are relevant to the commodity futures term structure modelling. It has shown that the mainstream macro-finance modelling pays the vast majority of its attention to the bond prices and interest rate term structure. Relatively, very little attention has been devoted to the commodity futures term structure, regardless of the long-standing discussion on the significant interrelationship between the commodity market and the macroeconomy that has been revealed by numerous studies over the years.

In order to provide prefatory empirical implications for the later chapters, a preliminary data analysis model was carried out, which tested several variables in this analysis, such as the crude oil convenience yield, the log real oil price, the US output gap, the US inflation rate, and the Fed Funds rate. The bivariate Granger causality test suggests that these tested variables have strong interactions. The impulse response functions generated by the basic unconstrained VAR(1) model are severely puzzled by the data's non-stationarity. It seems clear that the VECM model would be a better candidate than an unconstrained VAR model in terms of properly specifying the long-run conintegrated relationships in these variables.

Chapter 3

The Interest Rate Factor in the Commodity Markets

3.1 Introduction

Commodity futures markets have seen a considerable expansion in activity in the past a few decades, with estimated investment inflows to various commodity futures indices, from early 2000 to 2008, totalling $200 billion (Cheng and Xiong 2014). In parallel with this growth, a rapid development of reduced-form Gaussian models has been observed to investigate the time series behaviour of the commodity futures term structure (Brennan and Schwartz 1985, Gibson and Schwartz 1990, Schwartz 1997, Casassus and Collin-Dufresne 2005, Liu and Tang 2011). These Gaussian models suggest that commodity futures prices are affine to three unobservable stochastic factors; the spot price, the convenience yield, and the interest rate factor. Other branches of research provide insights into the economic interpretation of these stochastic factors, including the "Theory of Normal Backwardation" (see Keynes 1923, Hicks 1939) and the "Theory of Storage" (see Working 1933, Working 1934, Kaldor 1939, Working 1949, Brennan 1958 and others). The former suggests that the optimal hedging/speculative position drives the variation in the futures curve, while the latter explains the role of spot inventory, and introduces the convenience yield factor.

The spot price and convenience yield factors account for the vast majority of the early research into the economic interpretation of the unobserved stochastic factors. By comparison, relatively little attention has been paid to the interest rate factor. The conventional reduced-form Gaussian models, as discussed previously, generally accept the proposition in Schwartz (1997). This proposition regards the interest rate factor as equivalent to the short rate. In the empirical literature, previous studies have also frequently employed the practice of pinning down the latent interest rate factor to certain “observable” short rate series[1]. While this practice has the advantage of helping to identify the variable, which is, as documented in Schwartz (1997), otherwise a difficult task, it comes at the expense of eliminating the possibility of modelling the full set of dynamics that may exist for this variable. This leaves open the possibility that there might be more appropriate ways to measure the short rate that is applied to the commodity markets. By taking the approach of proxying the unobservable short rate by the observed short rate, we are, for example, unable to use these models to evaluate the version of the interest rate that is implied by the commodity markets. This might suggest forms of the commodity market implied latent short rate that are different from existing observable series.

A further consideration is that there are different commodity markets for different types of commodity products. In the existing empirical research, however, these markets are normally modelled seperately. Brennan and Schwartz (1985), Gibson and Schwartz (1990), and Liu and Tang (2011), for example, focus on single commodity markets in their empirical analysis; and although in Schwartz (1997) and Casassus and Collin-Dufresne (2005) several commodity classes, such as oil, copper, gold, and silver, are considered, these markets are still modelled individually. Casassus and Collin-Dufresne (2005) document that the estimates of the interest rate parameters are different in each separate model. Given this observation, a model that nests multiple commodity classes is needed in order

[1] The “observable” short rate series is normally a product that is separately measured by the interest rate term structure estimation using a single factor Gaussian model (as in Schwartz 1997, Casassus and Collin-Dufresne 2005, Liu and Tang 2011 and others).

to inspect the wider information content in the commodity markets for the interest rate. In this spirit, Paschke and Prokopczuk (2007) present a joint affine study of three highly related commodity classes: crude oil, heating oil, and gasoline; finding a non-stationary common component that drives all of their selected commodities. However further discussion of the joint affine model of commodity futures is, in general, limited.

This thesis, therefore, presents a joint affine term structure model, which is in the reduced-form Gaussian modelling category. This allows us to model multiple commodity markets to be modelled together. The theoretical model in this chapter is based on the work in Casassus and Collin-Dufresne (2005). The novelty of this joint model is that, it identifies the instantaneous short rate as a factor that is common to all commodity markets. This avoids estimating different sets of parameters for the same interest rate factor in different commodity markets, as noted by Casassus and Collin-Dufresne (2005). I make no attempt to pin down this variable to any observable data. In other words, the short rate factor will be a pure latent factor. This common interest rate will be jointly identified by multiple commodity markets. In turn, it will be shown that it also plays an important role in the estimation of commodity futures prices. Since the selected commodity markets are also the benchmark markets in each commodity class, the short rate factor implied by these markets can be interpreted as the commodity market's implied instantaneous short rate. It thus reflects the expectation that benchmark commodity markets have of the interest rate.

For the empirical model, monthly data on crude oil, copper, gold and corn futures contracts from January,1989 to November,2015 are used. As discussed in Casassus and Collin-Dufresne (2005), the model is estimated using maximum likelihood in order to take full advantage of the Gaussian affine structure. All state variables in this chapter will be unobservable, and thus the Kalman filter algorithm is used to estimate the latent factors.

In terms of the implicit commodity implied short rate factor, the analysis yields a number of findings. Specifically, its dynamics appear to be consistent with the dynamics of observable interest rates, and the correlation coefficient between this time series and the US policy rate is remarkably high. Test statistics however, suggest that they are Granger causing each other, which is supported by the fact that the commodity implied short rate factor is different from the US policy rate. It seems that the expectation of the interest rate in the commodity market both reflects and anticipates developments in monetary policy.

The remainder of this chapter will be organised as follows. Section 3.2 introduces the theoretical model, and the derivation and the closed form solution for the discrete time affine term structure. Section 3.3 introduces the joint affine term structure model for multiple commodity markets. Section 3.4 describes the data and explains the empirical methodology, while also discussing the preferred model selection. Section 3.5 discusses the empirical results of this model estimation. Section 3.6 concludes.

3.2 A discrete time affine term structure model for individual commodity futures

3.2.1 The state dynamic

Conventional models suggest that, for any commodity i, their futures term structure is driven by three latent factors. These can be further defined as their (nominal) spot commodity price ($S_{i,t}$) and convenience yield ($\delta_{i,t}$), and the instantaneous short rate (r_t).

Assuming that $S_{i,t}$ is lognormal under the risk-neutral measure Q, denote $s_{i,t} = ln(S_{i,t})$, I define $s_{i,t}$ follows the stochastic process as:

$$s_{i,t+1} = \kappa^Q_{s_i} + s_{i,t} + r_t - \delta_{i,t} + \epsilon^Q_{s_i,t+1} \qquad \epsilon^Q_{s_i,t+1} \sim N(0, \sigma^2_{s_i}) \tag{3.1}$$

where, $\kappa^Q_{s_i} = -\frac{1}{2}\sigma^2_{s_i}$. The complete derivation for equation (3.1) can be found in the Appendix A.1.

As implied by Casassus and Collin-Dufresne (2005), the definition of δ_t follows a Gaussian process as:

$$\delta_{i,t+1} = \kappa^Q_{\delta_i} + \phi^Q_{\delta_i,s_i} s_{i,t} + \phi^Q_{\delta_i}\delta_{i,t} + \phi^Q_{\delta_i,r} r_t + \epsilon^Q_{\delta_i,t+1} \qquad \epsilon^Q_{\delta_i,t+1} \sim N(0, \sigma^2_{\delta_i}) \tag{3.2}$$

which allows the convenience yield factor to be affected by the spot commodity price and short rate factor under the measure Q.

The short rate factor r_t follows a independent Gaussian process:

$$r_{t+1} = \kappa^Q_r + \phi^Q_r r_t + \epsilon^Q_{r,t+1} \qquad \epsilon^Q_{r,t+1} \sim N(0, \sigma^2_r) \tag{3.3}$$

Stacking equations (3.1), (3.2) and (3.3), I can write the state dynamic of any commodity class under the measure Q as:

$$X_{i,t+1} = K^Q_i + \Phi^Q_i X_{i,t} + W^Q_{i,t} \qquad W^Q_{i,t} \sim N(0, R_i) \tag{3.4}$$

where,

$$X_{i,t+1} = \begin{pmatrix} s_{i,t+1} \\ \delta_{i,t+1} \\ r_{i,t+1} \end{pmatrix} \qquad K^Q_i = \begin{pmatrix} \kappa^Q_{s_i} \\ \kappa^Q_{\delta_i} \\ \kappa^Q_{r_i} \end{pmatrix} \qquad \Phi^Q_i = \begin{pmatrix} 1 & -1 & 1 \\ \phi^Q_{\delta_i,s_i} & \phi^Q_{\delta_i} & \phi^Q_{\delta_i,r} \\ 0 & 0 & \phi^Q_r \end{pmatrix} \tag{3.5}$$

$R_i = \Sigma_i \Sigma'_i$, and Σ_i is a lower triangular matrix.

3.2.2 Generating the cross-sectional factor loadings under the risk-neutral measure

I denote $F_{i,\tau,t}$ as the futures price for commodity i at time t with τ units of time to its maturity. It is well-known that $F_{i,\tau,t}$ follows a martingale under the risk-neutral measure Q.

$$F_{i,\tau,t} = E^Q_t(F_{i,\tau-1,t+1}) \qquad \tau \geq 1. \tag{3.6}$$

This is because commodity futures contracts do not yield dividends or convenience yields, and do not have a cost of carry. The maturity value of the futures price is equal to the future spot price. So for the special case of $\tau = 0$:

$$F_{i,0,t+1} = S_{i,t+1}. \tag{3.7}$$

Assuming futures prices of commodity i at any point in time with any remaining term to maturity are exponentially affine to the three state variables, denote $f_{i,\tau,t} = ln(F_{i,\tau,t})$, I can adopt the trial solution for the log futures price of commodity i with τ term to maturity at time t:

$$f_{i,\tau,t} = \eta_{i,\tau} + \psi_{i,s,\tau} s_{i,t} + \psi_{i,\delta,\tau}\delta_{i,t} + \psi_{i,r,\tau} r_t \tag{3.8}$$

$$= \eta_{i,\tau} + \Psi_{i,\tau} X_{i,t} \tag{3.9}$$

where $\Psi_{i,\tau} = (\psi_{i,s,\tau}, \psi_{i,\delta,\tau}, \psi_{i,\pi,\tau})'$ contains the cross-sectional parameters in the measurement equation solved by the affine term structure setting.

In the special case of $\tau = 1$, using equation (3.7): $f_{i,0,t} = s_{i,t}$, which gives the starting values:

$$\psi_{i,s,0} = 1 \qquad \psi_{i,\delta,0} = \psi_{i,r,0} = 0. \tag{3.10}$$

To verify the trial solution (3.8) and find its parameters, I take logs of equation (3.6) to get:

$$\begin{aligned} f_{i,\tau,t} &= lnE_t^Q(F_{i,\tau-1,t+1}) \\ &= E_t^Q(f_{i,\tau-1,t+1}) + \frac{1}{2}Var(f_{i,\tau-1,t+1}) \end{aligned} \tag{3.11}$$

substituting equation (3.1), (3.2), (3.3), and (3.8) into equation (3.11) gives:

$$\begin{aligned} E_t^Q(f_{i,\tau-1,t+1}) =& \eta_{i,\tau-1} + \psi_{i,s,\tau-1}(\kappa_{s_i}^Q + s_t + r_t - \delta_{i,t}) \\ &+ \psi_{i,\delta,\tau-1}(\kappa_{\delta_i}^Q + \phi_{\delta_i,r}^Q r_t + \phi_{\delta_i}^Q \delta_{i,t} + \phi_{\delta_i,s_i}^Q s_{i,t}) \\ &+ \psi_{i,r,\tau-1}(\kappa_r^Q + \phi_r^Q r_t) \end{aligned} \tag{3.12}$$

$$Var(f_{i,\tau-1,t+1}) = \frac{1}{2}\psi_{i,s,\tau-1}^2 \sigma_{s_i}^2 + \frac{1}{2}\psi_{i,\delta,\tau-1}^2 \sigma_{\delta_i}^2 + \frac{1}{2}\psi_{i,r,\tau-1}^2 \sigma_r^2 \tag{3.13}$$

This verifies that the trial solution in equation (3.8) provides a recursive process for η_τ:

$$\begin{aligned}\eta_{i,\tau} =&\eta_{i,\tau-1} + \psi_{s_i,\tau-1}\kappa^Q_{s_i} + \psi_{\delta_i,\tau-1}\kappa^Q_{\delta_i} + \psi_{r,\tau-1}\kappa^Q_r \\ &+ \frac{1}{2}\psi^2_{s_i,\tau-1}\sigma^2_{s_i} + \frac{1}{2}\psi^2_{\delta_i,\tau-1}\sigma^2_{\delta_i} + \frac{1}{2}\psi^2_{r,\tau-1}\sigma^2_r \\ =&\eta_{i,\tau-1} + \Psi'_{i,\tau-1}K^Q_i + \frac{1}{2}\Psi'_{i,\tau-1}R_i\Psi_{i,\tau-1}\end{aligned} \tag{3.14}$$

and $\psi_{i,s,\tau}$, $\psi_{i,\delta,\tau}$, and $\psi_{i,r,\tau}$ follow recursive processes as below:

$$\psi_{i,s,\tau} =\psi_{i,s,\tau-1} + \psi_{i,\delta,\tau-1}\phi^Q_{\delta_i,s_i} \tag{3.15}$$

$$\psi_{i,\delta,\tau} =-\psi_{i,s_i,\tau-1} + \psi_{i,\delta,\tau-1}\phi^Q_{\delta_i} \tag{3.16}$$

$$\psi_{i,r,\tau} =\psi_{i,s,\tau-1} + \psi_{i,\delta,\tau-1}\phi^Q_{\delta_i,r} + \psi_{i,r,\tau-1}\phi^Q_r \tag{3.17}$$

In matrix form, these equations suggest that Ψ_τ is updated by the following recursive process:

$$\Psi'_{i,\tau} = \Psi'_{i,\tau-1}\Phi^Q_i \tag{3.18}$$

3.3 A joint affine term structure model for multiple commodity term structures

I jointly model four commodity futures: crude oil, copper, gold, and corn. Subscripts o, c, g and n are introduced, in place of subscript i to represent the four selected commodity classes, and the factors that are different in each class of commodity futures are defined as $z_{i,t} = (s_{i,t}, \delta_{i,t})'$ as the factors that are different in each class of commodity futures. In other words, these are *commodity specific factors* which vary in each commodity class, while r_t plays the role of the *common factor* in the model, which is assumed to be the same for all commodity futures. The solution for the cross-sectional recursive process, as demonstrated in equations (3.14) and (3.18), is the same for different classes of commodity futures. I can, therefore, write the *joint* transition equation under the measure

Q as:

$$X_{t+1} = K^Q + \Phi^Q X_t + W_t^Q \qquad W_t^Q \sim N(0, R) \tag{3.19}$$

where:

$$X_t = \begin{pmatrix} z_{o,t} \\ z_{c,t} \\ z_{g,t} \\ z_{n,t} \\ r_t \end{pmatrix} \quad K_t^Q = \begin{pmatrix} K_{z_o}^Q \\ K_{z_c}^Q \\ K_{z_g}^Q \\ K_{z_n}^Q \\ k_r^Q \end{pmatrix} \quad \Phi^Q = \begin{pmatrix} \Phi_{z_o}^Q & 0_{2,2} & 0_{2,2} & 0_{2,2} & \Phi_{z_o,r}^Q \\ 0_{2,2} & \Phi_{z_c}^Q & 0_{2,2} & 0_{2,2} & \Phi_{z_c,r}^Q \\ 0_{2,2} & 0_{2,2} & \Phi_{z_g}^Q & 0_{2,2} & \Phi_{z_g,r}^Q \\ 0_{2,2} & 0_{2,2} & 0_{2,2} & \Phi_{z_n}^Q & \Phi_{z_n,r}^Q \\ 0_{1,2} & 0_{1,2} & 0_{1,2} & 0_{1,2} & \phi_r^Q \end{pmatrix} \tag{3.20}$$

Likewise, W_t^Q and R collect $W_{i,t}^Q$ and R_i in a similar way to these vectors and matrix. The extended definition of X_t, K_t^Q, and Φ^Q can be found in the Appendix B.1.

The VAR is now redefined under measure P. This implies a system that is congruent with the companion form (3.19):

$$X_{t+1} = K + \Phi X_t + W_t \qquad W_t \sim N(0, R) \tag{3.21}$$

where K is the vector for constant terms, Φ specifies the interrelationships between state variables under the measure P, and R is a 9×9 matrix. $R = \Sigma\Sigma'$, where Σ is a 9×9 lower triangular matrix. The extended definition of K and Φ are shown in Appendix B.2. For convenience of notation, I denote K, Φ, W_t, and all parameters contained by K, Φ, W_t without the superscript P representing its probability measure under a real-world dynamic.

This benchmark model has the dynamic matrix Φ, following the specification suggested in Casassus and Collin-Dufresne (2005):

$$\Phi = \begin{pmatrix} \Phi_{z_o} & 0_{2,2} & 0_{2,2} & 0_{2,2} & \Phi_{z_o,r} \\ 0_{2,2} & \Phi_{z_c} & 0_{2,2} & 0_{2,2} & \Phi_{z_c,r} \\ 0_{2,2} & 0_{2,2} & \Phi_{z_g} & 0_{2,2} & \Phi_{z_g,r} \\ 0_{2,2} & 0_{2,2} & 0_{2,2} & \Phi_{z_n} & \Phi_{z_n,r} \\ 0_{1,2} & 0_{1,2} & 0_{1,2} & 0_{1,2} & \phi_r \end{pmatrix} \tag{3.22}$$

where Φ_{z_o}, Φ_{z_c}, Φ_{z_g}, Φ_{z_n}, show how the lagged specific factors of each selected commodity type affect their current specific factors, and ϕ_r represents how the lagged common factor affects the current common factor, and $\Phi_{z_o,r}$, $\Phi_{z_c,r}$, $\Phi_{z_g,r}$ and $\Phi_{z_n,r}$ capture how the lagged common factor affects the current specific factors of each commodity type. The other off diagonal elements are constrained to zeros. This means that, in this benchmark model, the lagged specific factors of each commodity class do not affect the others, or the common factor.

The risk-neutral parameters K^Q and Φ^Q, as defined in (3.19), are already modelled in a parsimonious manner and do not need any further specification. For the real world dynamic, the parameter matrix Φ can be further defined to reflect different model specifications. This restriction can be relaxed, however, in order to achieve an unconstrained model, which allows state factors to be interdependent of each other.

3.4 Empirical implementation

3.4.1 Data

The data set consists of futures contracts on West Texas Intermediate (WTI) crude oil, Chicago Mercantile Exchange (CMX) gold, CMX copper and Chicago Board of Trade (CBT) corn. For all commodities, "end of month" monthly data are used from January 1989 to November 2015. Table 3.1 presents summary statistics for the four commodities. For crude oil and copper contracts with maturities at 1, 2, 3, 6, 9, 12 and 18 months (labelled from f_1 to f_{18}); for gold contracts with maturities at 1, 2, 3, 6, 9 and 12 months, and for corn contracts with maturities at 1, 2, 3, 4, 5 and 6 months. Selections of the starting year and maturities are based on both the alignment and availability of the data. All data are downloaded from the Thompson Reuters DataStream database.

Figure 3.1: Log Futures Prices

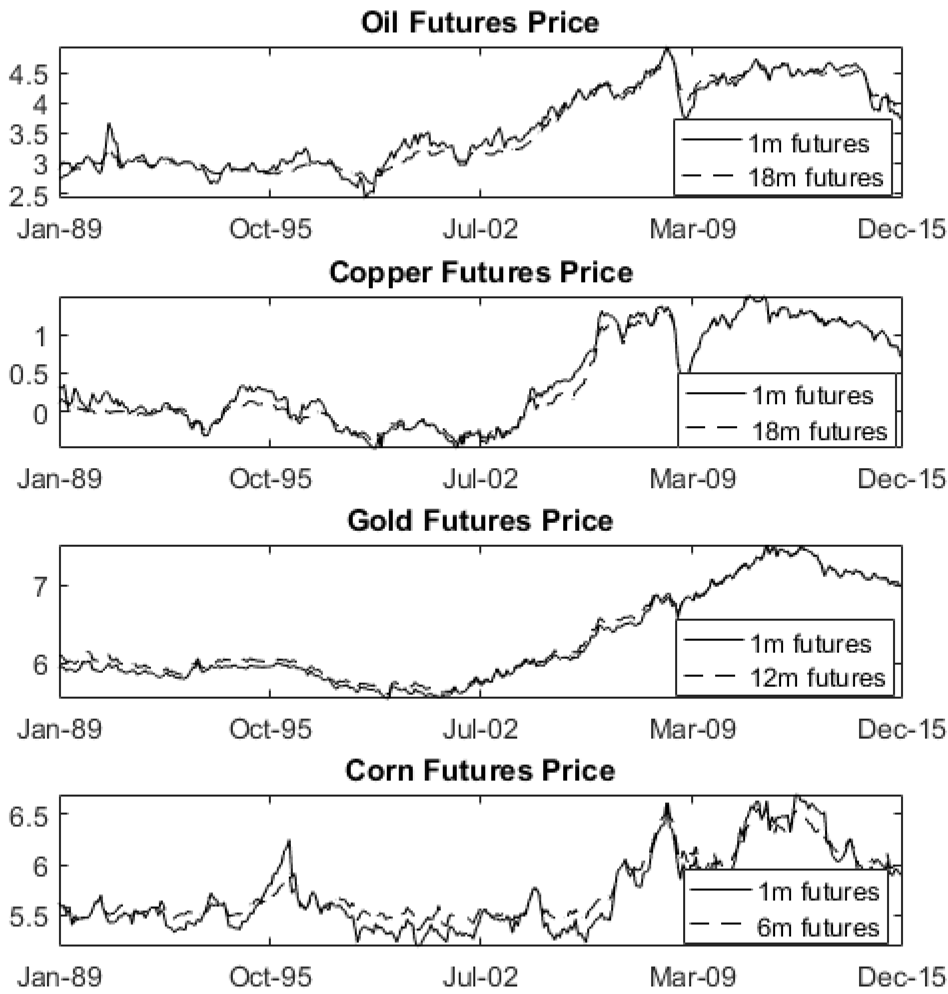


Figure3.1 displays selected time series for each commodity term structure. The futures with the shortest (solid lines) and longest (dashed lines) term to maturity are plotted for each commodity class. All data are taken in natural logarithm.

Figure 3.1 shows the price of commodity contracts for crude oil, copper, corn and gold. The futures contracts with the shortest and longest term to maturity

Table 3.1: Summary Statistics of Futures Prices

	Mean	S.D.	Skewness	Excess Kurtosis	ADF
			WTI Oil		
$f_{o,1}$	3.604	0.669	0.295	-1.399	-1.692
$f_{o,2}$	3.604	0.673	0.299	-1.432	-1.621
$f_{o,3}$	3.602	0.676	0.308	-1.457	-1.558
$f_{o,6}$	3.595	0.683	0.330	-1.511	-1.417
$f_{o,9}$	3.587	0.688	0.347	-1.546	-1.323
$f_{o,12}$	3.581	0.691	0.358	-1.570	-1.256
$f_{o,18}$	3.572	0.693	0.372	-1.601	-1.186
			CMX Copper		
$f_{c,1}$	0.414	0.609	0.345	-1.400	-1.264
$f_{c,2}$	0.412	0.608	0.359	-1.399	-1.256
$f_{c,3}$	0.410	0.607	0.375	-1.400	-1.244
$f_{c,6}$	0.402	0.608	0.422	-1.404	-1.229
$f_{c,9}$	0.393	0.608	0.460	-1.406	-1.207
$f_{c,12}$	0.386	0.607	0.487	-1.406	-1.184
$f_{c,18}$	0.373	0.603	0.528	-1.403	-1.097
			CMX Gold		
$f_{g,1}$	6.273	0.599	0.715	-1.022	-0.044
$f_{g,2}$	6.276	0.598	0.714	-1.023	-0.086
$f_{g,3}$	6.278	0.596	0.713	-1.024	-0.046
$f_{g,6}$	6.293	0.591	0.702	-1.030	-0.036
$f_{g,9}$	6.308	0.585	0.689	-1.036	-0.025
$f_{g,12}$	6.325	0.579	0.676	-1.043	-0.019
			CBT Corn		
$f_{n,1}$	5.717	0.383	0.921	-0.191	-1.473
$f_{n,2}$	5.734	0.371	0.941	-0.174	-2.124
$f_{n,3}$	5.748	0.359	0.956	-0.162	-1.857
$f_{n,4}$	5.757	0.345	0.953	-0.216	-1.798
$f_{n,5}$	5.763	0.334	0.943	-0.310	-1.745
$f_{n,6}$	5.771	0.328	0.921	-0.470	-1.809

Table 3.1 presents the summary statistics of selected futures prices data. All time series are taken as natural logarithm. The Augmented Dickey Fuller (ADF) test results are used for oil futures prices, and lag length is determined by the Akaike information criterion. The critical values for the t-statistic are -3.4496, -2.8699 and -2.5713 for the 1%, 5% and 10% confidence level. Probabilities are based on Mackinnon (1996) one sided p-values.

that is available in the dataset are plotted for each commodity class. It can be seen that these are persistent series over the selected period of time. This persistent feature is also evidenced by the stationarity tests displayed in Table 3.1. If we compare these term structures cross-sectionally, it can be seen that the corn futures price displays much more variation than the other commodities. This is due to the stronger influence of seasonal effects on corn futures, as documented in Fama and French (1987). Finally, the cross-sectional variation of gold contracts appears to be the most stable among the four commodity contracts. This implies that gold contracts have a much lower convenience yield compared with the others, because the price difference between short-term and long-term gold futures contracts is, relatively, much smaller.

3.4.2 Empirical methodology

The state variables here are treated as latent factors, that are estimated along with the parameters of the model. The latent factors are identified by the Φ^Q matrix in equation (3.20). This implies that the spot price and convenience yield are specific to each futures curve, whereas the short rate is common to all. Additionally, for each futures curve, the spot price restriction in equation (3.5) implies that the spot price, convenience yield, and short interest rate factor are identified. In order to identify these latent factors uniquely, I follow Joslin, Singleton and Zhu (2011), and set: $K_i^Q = (\kappa_{s_i}^Q, 0, 0)'$. This means that the mean of the latent factors is normalised under the measure Q.

The model is estimated using maximum likelihood estimation with the Kalman filter algorithm. In terms of the state space form, the joint transition equations are defined under the measures Q and P in equations (3.19) and (3.20).

The affine term structure model in equation (3.9) yields the joint measurement equation:

$$f_t = D + HX_t + e_t \qquad e_t \sim N(0, Q) \tag{3.23}$$

where $f_t = (f_{o,t}, f_{c,t}, f_{g,t}, f_{n,t})'$, which stacks four selected types of observable futures contracts with 1 month to τ months term to maturity. $D = (\eta_o, \eta_c, \eta_g, \eta_n)'$ stacks their recursive process for $\eta_{i,\tau}$ as in equation (3.9). H stacks the recursive solution for $\Psi_{i,\tau}$ in equation (3.9). Since $\Psi_{i,\tau}$ can also be partitioned into $\Psi_{i,z_o,\tau}$ and $\Psi_{i,r,\tau}$, H can be defined as the following expression in order to accommodate the chosen commodity futures data:

$$H = \begin{pmatrix} \Psi_{o,z_o} & 0_{7,2} & 0_{7,2} & 0_{7,2} & \Psi_{o,r} \\ 0_{7,2} & \Psi_{c,z_c} & 0_{7,2} & 0_{7,2} & \Psi_{c,r} \\ 0_{6,2} & 0_{6,2} & \Psi_{g,z_g} & 0_{6,2} & \Psi_{g,r} \\ 0_{6,2} & 0_{6,2} & 0_{6,2} & \Psi_{n,z_n} & \Psi_{n,r} \end{pmatrix} \tag{3.24}$$

All of the commodity data that are used are observed with measurement errors, which are accommodated by Q in equation (3.23). This is specified as a diagonal matrix that stacks the measurement error of oil, copper, gold, and corn contracts sequentially in its diagonal elements. For an extended definition of these vectors and matrices, please refer to Appendix B.1.

I define the parameter set of the state space form as $\theta = (K, \Phi, K^Q, \Phi^Q, R, Q)$, $\theta \in \Theta$, where Θ is the parameter space. This estimation feeds into the following log-likelihood function:

$$\ln \mathcal{L}(\theta|\mathcal{F}_{t-1}) = \sum_{t=1}^{T} \ln f(f_t|\theta, \mathcal{F}_{t-1}) \tag{3.25}$$

where $\mathcal{F}_{t-1}$ stands for the filtration of information at time $t-1$. Appendix B.3 provides more details about the Kalman filter technique and the maximum likelihood estimation.

Possible model specifications are considered by imposing restrictions on the Φ parameter matrix defined earlier. Specifically, I compare and explore the following two physical factor dynamics:

- Model $M0$: The benchmark model where Φ is restricted as in (3.22).
- Model $M1$: The unconstrained model where all off-diagonal elements of Φ

are unrestricted to zero.

I use the Likelihood Ratio (LR) test, to test the null hypothesis of the benchmark model against the alternative hypothesis of the unconstrained model. The benchmark model is preferred, unless the unconstrained model provides a significantly higher log-likelihood value. The parameter set $\theta = (K, \Phi, K^Q, \Phi^Q, R, Q)$ has 122 parameters in total to estimate for the benchmark $M0$ model. The total number of parameters to be estimated for $M1$ is 178. This means that the degree of freedom in the LR test is 56.

The LR test results are presented in Table 3.2. The alternative hypothesis of an unrestricted model is tested against the null hypothesis of a restricted model. The null hypothesis of the benchmark model is accepted at the 90% confidence level. This implies that $M1$ makes little improvement from $M0$ statistically. As a result, $M0$ is chosen as the preferred model in this chapter.

Table 3.2: The Likelihood Ratio Test

Table 3.2a

Model	Description	Log likelihood	Num of parameters
$M0$	Benchmark: Φ restricted	33589.0	122
$M1$	Unconstrained: Φ unrestricted	33611.5	178

Table 3.2b

Hypothesis		Test result		χ^2 critical values		
$H0$	$H1$	LR stat	$d.f.$	0.90	0.95	0.99
$M0$	$M1$	45.00	56	69.91	74.46	83.51

Table 3.2a shows the model specifications with their associated likelihood value and the number of parameters in each specification. Table 3.2b shows the likelihood ratio test, where the null H0 of the benchmark/constrained model (M0) is tested against the alternative H1 of the unconstrained model (M1). The table shows the likelihood ratio test statistics (LR stat), the degree of freedom ($d.f.$), and the upper tail χ^2 distribution critical values for different confidence intervals.

3.5 Empirical results

The parameter estimates are presented in Tables 3.3 and 3.4. It can be seen from Table 3.3 that, the statistical significance of parameters under the measure

Q is generally higher than that in the system under the measure P. This is in line with Cochrane and Piazzesi (2009), who suggested that parameters in the Q dynamic are better defined than those in the P dynamic. The reason for this is because the cross-sectional model fitting depends upon the Q dynamic, which usually has very small measurement errors. On the other hand, the P dynamic is determined by the Kalman-VAR system which has larger forecasting errors. Table 3.4 displays the state covariance estimates. These are parameters contained in the matrix Σ of equations (3.20) and (3.19). I parametrize Σ as a lower triangular matrix. It can be seen from Table 3.4, that many of the off-diagonal parameters are statistically insignificant. It might, therefore, be sensible to specify this matrix in a more parsimonious manner.

The model performs well empirically. The left column of Figure 3.2 plots the actual 1-month contracts price of four different commodity classes, against their fitted series. From this, we can see that the fitted series tracks the actual series very closely. The second column presents the fitting error of the 1-month contracts. These fitting errors appear to be highly stationary with zero mean and time-invariant variances. Table 3.5 presents summary statistics for the fitting errors of all commodity contracts. The means and variances of these fitting errors are very small relative to the level of variance in the data. The ADF test results on these fitting errors also show that they are statistically highly stationary series. A closer look at the right column of Figure 3.2 outlines more observations. Crude oil, copper, and gold appear to have been sensitive to similar shocks in the past. A notable example occurred 2008, where all of their fitting errors spike. This is however, not obvious in the case of corn, where the fitting error is generally more volatile than the other three, regardless of events such as the 2008 Lehman crisis. This implies that the corn market is affected specifically by many other different shocks in the economy, so that financial shocks such as the 2008 Lehman crisis are equally emphasised.

I show estimates of the root mean squared error (RMSE) of each futures contract in Table 3.6. These are economically small with respect to the level of

Table 3.3: Parameters Estimates: the P and Q Dynamics

Parameters	Estimates	t-stats	Parameters	Estimates	t-stats
		Dynamic under the measure P			
κ_{s_o}	0.0627	2.4295	$\phi_{s_c,r}$	2.0570	0.7642
κ_{s_c}	5.5×10^{-15}	7.1×10^{-13}	ϕ_{δ_c,s_c}	-9.3×10^{-6}	-0.0234
κ_{s_g}	6.9×10^{-22}	1.9×10^{-20}	ϕ_{δ_c,δ_c}	0.8659	30.1506
κ_{s_n}	0.1153	1.4001	$\phi_{\delta_c,r}$	0.2889	2.1215
κ_{δ_o}	0.0076	2.3817	ϕ_{s_g,s_g}	1.0004	180.8045
κ_{δ_c}	-3.7×10^{-5}	-0.0923	$\phi_{s_g\delta_g}$	0.4444	0.1738
κ_{δ_g}	-0.0003	-3.4233	$\phi_{s_g,r}$	-0.7092	-0.2544
κ_{δ_n}	-0.0053	-15.9207	ϕ_{δ_g,s_g}	0.0001	51.0077
κ_r	3.2×10^{-5}	0.8215	ϕ_{δ_g,δ_g}	0.8094	23.9173
ϕ_{s_o,s_o}	0.9845	148.1508	$\phi_{\delta_g,r}$	0.1459	3.9546
$\phi_{s_o\delta_o}$	-0.5341	-1.6351	ϕ_{s_n,s_n}	0.9780	79.9842
$\phi_{s_o,r}$	1.5600	0.6687	$\phi_{s_n\delta_n}$	-0.1472	-0.6987
ϕ_{δ_o,s_o}	0.0002	44.8739	$\phi_{s_n,r}$	7.8×10^{-15}	3.1×10^{-14}
ϕ_{δ_o,δ_o}	0.8898	34.5991	ϕ_{δ_n,s_n}	0.9365	45.3643
$\phi_{\delta_o,r}$	0.0129	0.0879	ϕ_{δ_n,δ_n}	0.9259	65.6749
ϕ_{s_c,s_c}	0.9951	130.6671	$\phi_{\delta_n,r}$	7.8×10^{-15}	3.1×10^{-14}
ϕ_{s_c,δ_c}	-0.5808	-1.0524	$\phi_{r,r}$	0.9737	82.2745
		Dynamic under the measure Q			
$\kappa^Q_{s_o}$	0.0068	27.1732	$\phi^Q_{\delta_c,r}$	0.2158	4.6640
$\kappa^Q_{s_c}$	-0.0007	-1.4301	$\phi^Q_{\delta_g,s_g}$	-1.9×10^{-5}	-49.5318
$\kappa^Q_{s_g}$	0.0011	3.6541	$\phi^Q_{\delta_g,\delta_g}$	0.9247	195.6095
$\kappa^Q_{s_n}$	-0.0696	-7.7038	$\phi^Q_{\delta_g,r}$	-0.5719	-21.1259
$\phi^Q_{\delta_o,s_o}$	0.0002	44.8739	$\phi^Q_{\delta_n,s_n}$	4.9×10^{-11}	2.1×10^{-7}
$\phi^Q_{\delta_o,\delta_o}$	0.9049	471.0800	$\phi^Q_{\delta_n,\delta_n}$	0.9259	65.6749
$\phi^Q_{\delta_o,r}$	-0.1066	-3.5478	$\phi^Q_{\delta_n,r}$	9.0×10^{-6}	2.8×10^{-5}
$\phi^Q_{\delta_c,s_c}$	0.0003	7.3513	$\phi^Q_{r,r}$	0.8074	104.4892
$\phi^Q_{\delta_c,\delta_c}$	0.9142	194.8108			

Table 3.3 presents the parameters in the dynamic structure. The upper panel shows the parameters of the transition equation under the measure P, as in equation (3.21), and the lower panel displays the parameters of the transition equation under the measure Q, as in equation (3.19).

Table 3.4: Parameter Estimates: State Covariances

Parameters	Estimates	t-stats	Parameters	Estimates	t-stats
σ_{s_o,s_o}	0.1008	17.9963	σ_{r,s_c}	-0.0001	-1.7699
σ_{δ_o,s_o}	0.0065	13.8139	$\sigma_{\delta_c,\delta_c}$	0.0031	16.7374
σ_{s_c,s_o}	0.0256	5.4122	σ_{s_g,δ_c}	-8.4×10^{-14}	-3.4×10^{-11}
σ_{δ_c,s_o}	-0.0003	-1.1514	$\sigma_{\delta_g,\delta_c}$	-4.5×10^{-13}	-4.5×10^{-9}
σ_{s_g,s_o}	-0.0110	-4.2445	σ_{s_n,δ_c}	1.3×10^{-15}	2.4×10^{-13}
σ_{δ_g,s_o}	1.3×10^{-20}	1.4×10^{-16}	$\sigma_{\delta_n,\delta_c}$	-3.8×10^{-19}	-5.2×10^{-16}
σ_{s_n,s_o}	-0.0071	-1.3556	σ_{r,δ_c}	4.0×10^{-19}	8.4×10^{-15}
σ_{δ_n,s_o}	0.0003	0.4915	σ_{s_g,s_g}	0.0424	12.4616
σ_{r,s_o}	2.6×10^{-5}	0.5811	σ_{δ_g,s_g}	-1.1×10^{-22}	-7.3×10^{-19}
$\sigma_{\delta_o,\delta_o}$	0.0043	49.3894	σ_{s_n,s_g}	-0.0128	-2.2999
σ_{s_c,δ_o}	-3.4×10^{-15}	-6.0×10^{-13}	σ_{δ_n,s_g}	0.0008	1.0834
$\sigma_{\delta_c,\delta_o}$	-0.0003	-1.1829	σ_{r,s_g}	-1.5×10^{-25}	-2.1×10^{-21}
σ_{s_g,δ_o}	-1.5×10^{-15}	-5.6×10^{-13}	$\sigma_{\delta_g,\delta_g}$	0.0009	9.4728
$\sigma_{\delta_g,\delta_o}$	-2.1×10^{-15}	-1.5×10^{-11}	σ_{s_n,δ_g}	7.7×10^{-20}	9.4×10^{-18}
σ_{s_n,δ_o}	4.2×10^{-12}	7.1×10^{-10}	$\sigma_{\delta_n,\delta_g}$	1.7×10^{-10}	1.7×10^{-7}
$\sigma_{\delta_n,\delta_o}$	-0.0007	-0.8740	σ_{r,δ_g}	-0.0003	-6.9586
σ_{r,δ_o}	2.0×10^{-14}	2.9×10^{-10}	σ_{s_n,s_n}	0.0864	20.9964
σ_{δ_c,s_o}	0.0022	8.5526	σ_{δ_n,s_n}	-0.0080	-10.3606
σ_{s_c,s_c}	0.0747	13.8706	σ_{r,s_n}	-4.2×10^{-13}	-1.9×10^{-8}
σ_{s_g,s_c}	-0.0095	-3.5601	$\sigma_{\delta_n,\delta_n}$	0.0058	69.6926
σ_{δ_g,s_c}	-0.0002	-1.5293	σ_{r,δ_n}	8.6×10^{-11}	2.7×10^{-6}
σ_{s_n,s_c}	-0.0075	-1.3316	$\sigma_{r,r}$	0.0003	28.2802
σ_{δ_n,s_c}	4.8×10^{-20}	6.4×10^{-17}			

Table 3.4 presents the state covariance as in the martix R of the transition equation (3.21) and (3.19).

variance in the data. Generally speaking, the model fits better at the long end of the term structures rather than the short end. For instance, the oil contract with the longest term to maturity (18 months) has an RMSE of 0.0580. This is 59.8% lower than the oil contract with the shortest term to maturity (1 month), which has an RMSE of 0.0927. For the case of copper, gold, and corn, this percentage is 18.6%, -0.2% and 39.9%, respectively. This is because short term commodity term contracts carry more variations than those long term ones, since most of the trading activities happen in the short end. Gold is an exception to this. Its RMSE shows little obvious difference between the long end and the short end due to its rather flat cross-sectional futures curve.

I plot the estimated commodity specific factors in Figure 3.3. The first col-

Table 3.5: Summary Statistics for the Fitting Errors

	Mean	S.D.	ADF		Mean	S.D.	ADF
		Oil				Copper	
1m	-0.0013	0.0920	-14.705	1m	0.0046	0.0762	-15.946
2m	0.0010	0.0864	-14.848	2m	0.0048	0.0752	-16.072
3m	0.0018	0.0823	-14.684	3m	0.0048	0.0744	-15.921
6m	0.0020	0.0722	-14.504	6m	0.0045	0.0712	-10.483
9m	0.0014	0.0650	-14.652	9m	0.0041	0.0679	-10.476
12m	0.0014	0.0602	-14.924	12m	0.0046	0.0660	-10.502
18m	0.0025	0.0557	-14.922	18m	0.0039	0.0662	-10.175
		Gold				Corn	
1m	0.0010	0.0448	-20.416	1m	-0.0009	0.0780	-11.329
2m	0.0012	0.0449	-20.380	2m	0.0010	0.0769	-10.463
3m	0.0006	0.0450	-20.410	3m	0.0015	0.0753	-10.059
6m	0.0013	0.0450	-20.564	6m	0.0005	0.0680	-10.481
9m	0.0010	0.0449	-20.639	9m	-0.0006	0.0632	-17.688
12m	0.0012	0.0449	-20.656	12m	0.0026	0.0628	-16.356

Table 3.5 presents the summary statistics for the fitting errors. The lag order selection criteria and t-statistics critical values for ADF tests are outlined in Table 3.1

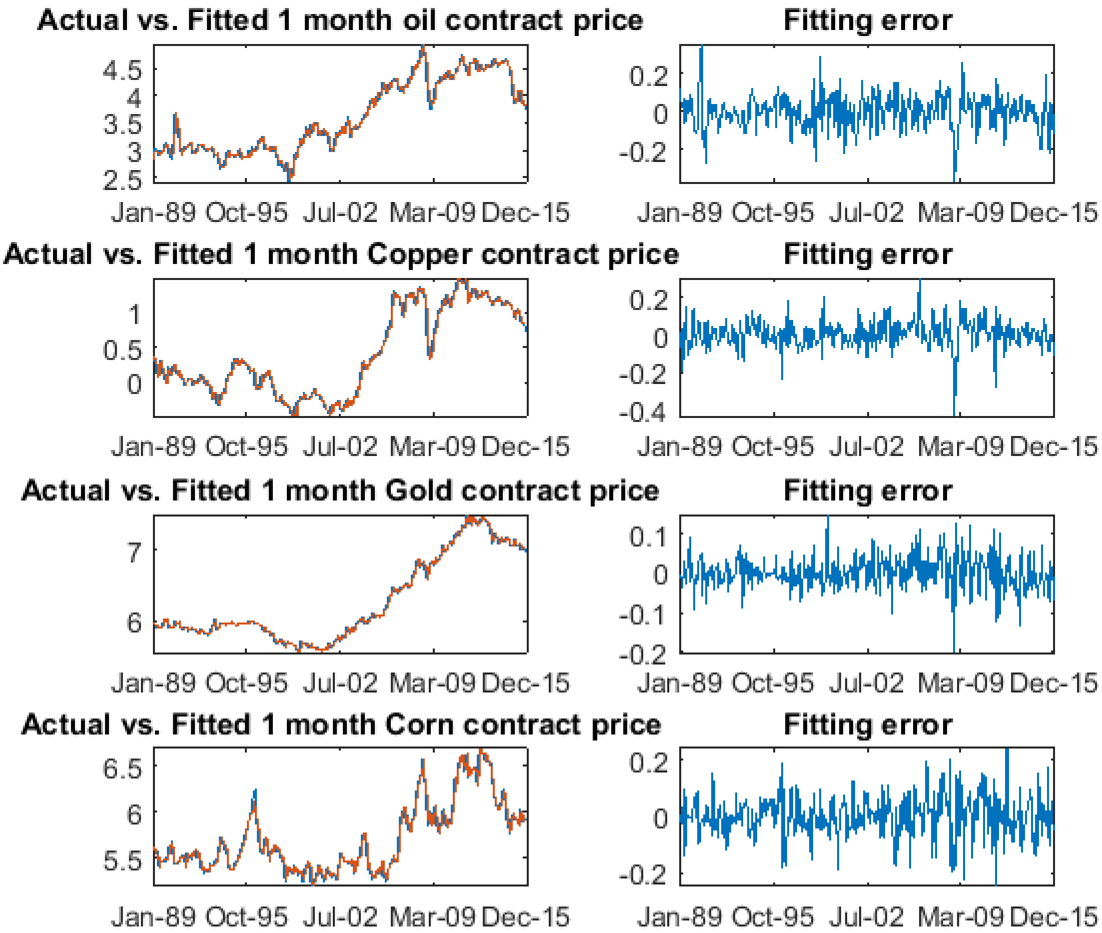
Table 3.6: The Root Mean Squared Error (RMSE) of the Joint Commodity Term Structure Model

τ	1m	2m	3m	6m	9m	12m	18m
Oil	0.0927	0.0874	0.0828	0.0725	0.0660	0.0620	0.0580
τ	1m	2m	3m	6m	9m	12m	18m
Copper	0.0763	0.0747	0.0733	0.0700	0.0677	0.0662	0.0643
τ	1m	2m	3m	6m	9m	12m	
Gold	0.0451	0.0451	0.0452	0.0452	0.0452	0.0452	
τ	1m	2m	3m	4m	5m	6m	
Corn	0.0795	0.0733	0.0680	0.0636	0.0598	0.0568	

Table 3.6 reports the RMSE for oil, copper, gold, and corn futures contracts, using natural logarithm estimates.

umn presents the first specific factors of each commodity class, while the second column shows the second specific factors. It is clear that the first factor, also known as the spot price fact, captures most of the variation in the futures prices, and, therefore, shares high persistence with the futures prices time series in each case. This is evidenced by the ADF test result in the upper panel of Table 3.7. The lower panel of Table 3.7 shows that the convenience yield factors for all commodity classes are statistically stationary. Notice that, among all convenience yields, gold is the most undetermined. It has both a level and a variance

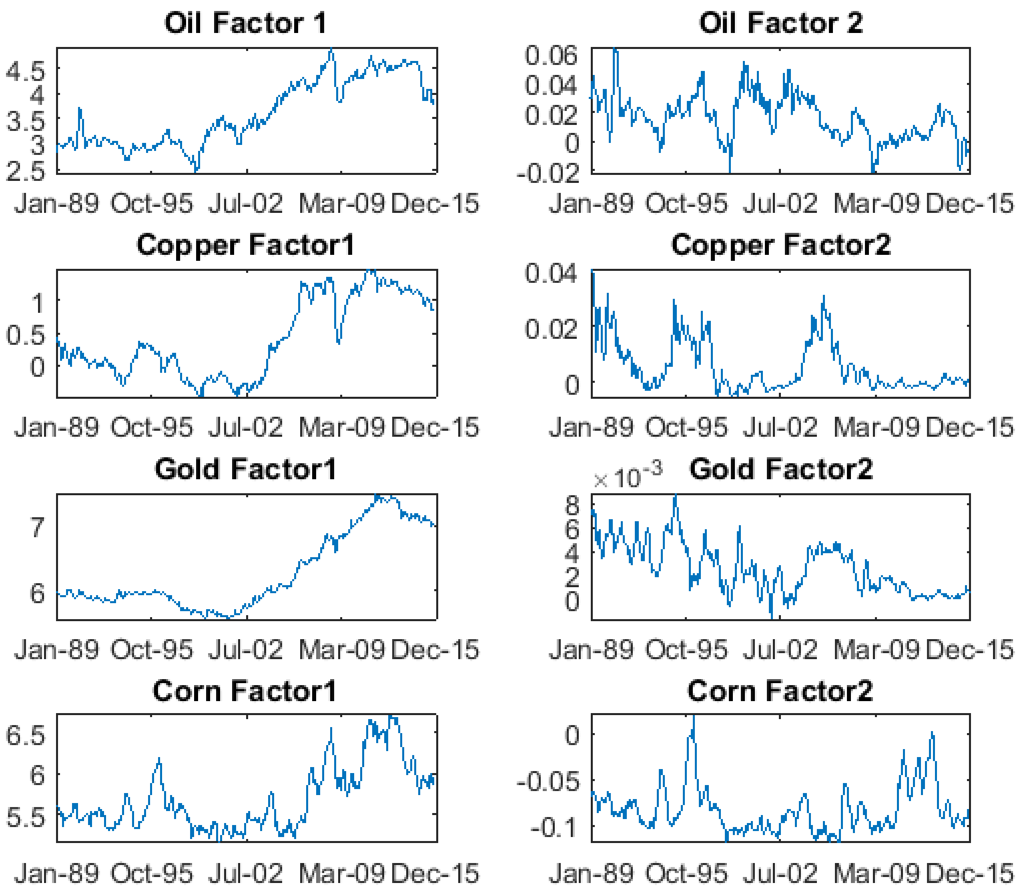
Figure 3.2: The Fitting Errors of the Estimation



The first column plots the actual 1-month contract price of four different commodity classes, against their fitted series, estimated by the joint commodity term structure model. The second column shows the fitting error between them in each case.

that are an order of magnitude smaller than those of other commodities. This is consistent with previous empirical findings that suggest that precious metals do not show significant convenience yields (Schwarts 1997, Casassus and Collin-Dufresne 2005, Liu and Tang 2011, Gospodinov and Ng 2013). The results here also shows that the convenience yield of the corn is the most volatile compared with the convenience yield of other commodity classes. This is again in line with Fama and French (1987), who imply that the variability of the convenience yields

Figure 3.3: The Commodity Specific Factors Estimates



The first column plots the first specific factor, that is the spot price factor, of each commodity class. The second column plots the second specific factor, that is the convenience yield factor, of each commodity class

for agriculture commodities could arise from seasonal effects that serve to adjust supply and demand shocks.

3.5.1 The behaviour of the commodity term structures

The behaviour of the futures term structures of commodity i is depicted by the factor loadings as $\Psi_{i,\tau}$ in equation (3.18). Figure 3.4 shows the loadings of commodity term structures on their specific factors (solid and broken line), and the common factor (dashed line). These factor loadings depend upon the

Table 3.7: Summary Statistics of the Commodity Specific Factors

	$s_{o,t}$	$s_{c,t}$	$s_{g,t}$	$s_{n,t}$
Mean	3.607	0.410	6.271	5.698
S.D.	0.660	0.606	0.599	0.395
ADF	-1.732	-1.049	0.010	-1.901
	$\delta_{o,t}$	$\delta_{c,t}$	$\delta_{g,t}$	$\delta_{n,t}$
Mean	0.016	0.006	0.003	-0.083
S.D.	0.016	0.009	0.002	0.026
ADF	-4.209	-4.409	-3.222	-4.758

Table 3.7 presents the summary statistics of the commodity specific factors for each commodity class. Lag order selection criteria and t-statistics critical values for ADF tests are outlined in Table 3.1.

parameters of the risk-neutral factor dynamics. Dividing these loadings by maturity, gives the factor loadings for the annualised cost of carry $(\Psi_{i,\tau}/\tau)$, as a function of maturity (expressed in months). In each case, the factor loading of the spot price starts from value one, and the loadings on the convenience yield factor and the interest rate factor start from zero. This is decided by the initial condition, as shown in equation (3.10). The factor loading of the spot price reflects its role as a level factor to each commodity term structure. The rest of the cross-sectional variation is accounted for by the combination of the convenience yield factor and the interest rate factor. The loadings of these two factors in the commodity term structure quickly reach the point of inflection. They then slowly decline and converge towards zero. In general, loadings on the convenience yield and the interest rate factor move in different directions from each other. Proportionally, gold contracts take a higher loading on the interest rate in comparison to the other commodities. This is because there is very little use for gold in industrial production, and thus it is predominantly used for investment purposes. As presented by its factor loading, therefore, gold contract prices appear to be more sensitive to variations in the interest rate than other more industrial-based commodities.

Figure 3.4: The Factor Loadings

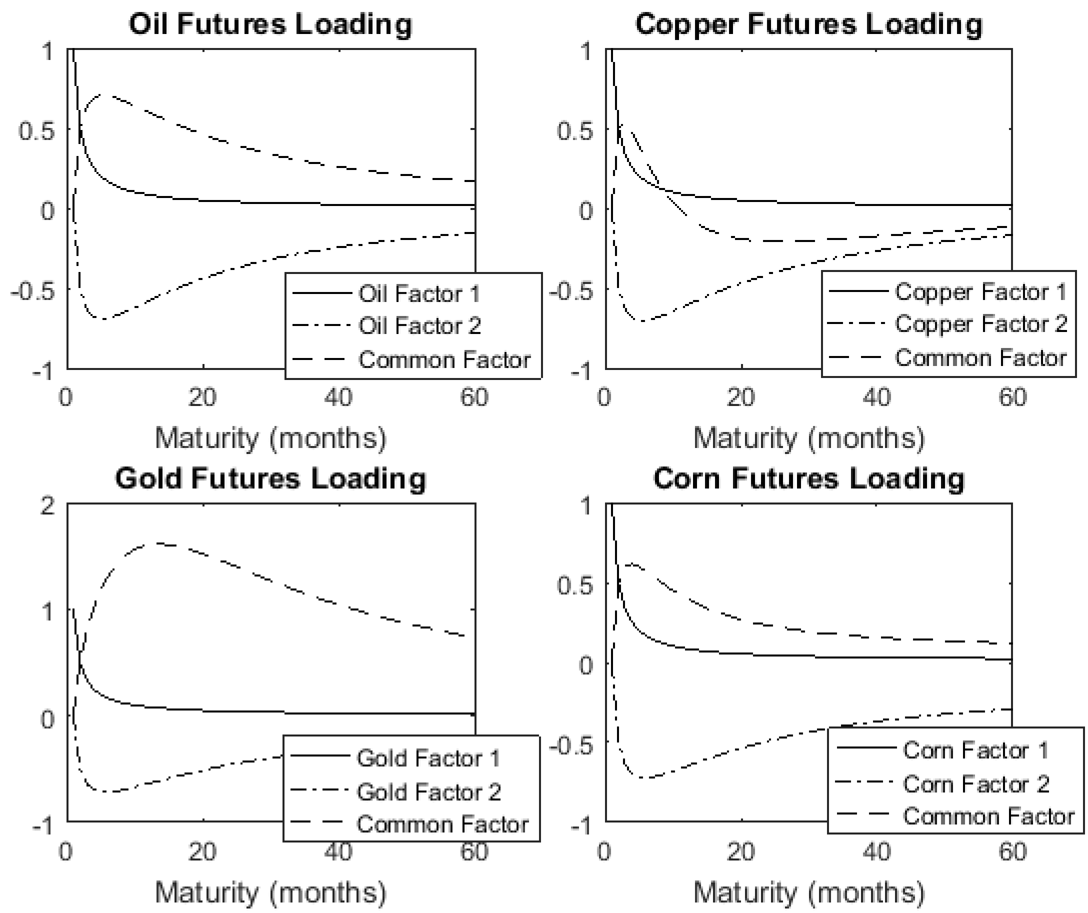


Figure 3.4 shows the loadings of commodity term structures on their specific factors (solid and broken line), and the common factor (dashed line). Each factor loading, when divided by maturity, gives the factor loadings for the annualised cost of carry (Ψ_τ/τ), as a function of monthly maturity.

3.5.2 The commodity market implied short rate factor

The instantaneous short rate factor in this joint commodity futures model is a pure latent factor, and the selected benchmark commodity markets jointly imply its variation. Hence, it reflects the expectation of the global commodity markets as to the US interest rate policy. Conversely, this "commodity market implied short rate factor" also influences the commodity term structures dynamics, together with other specific factors.

Figure 3.5 plots the commodity market implied short rate factor. The upper panel shows the estimate of this factor with confidence bounds at the ± 95% confidence interval. The commodity market implied short rate series moves close to, and later below, the zero lower bound after the year 2008. However, this negativity is not statistically significant because the confidence bound at the upper 95% quantile is always above the zero lower bound. The lower panel of Figure 3.5 shows that the commodity market implied short rate dynamics generally track the US Federal Funds rate very well. The correlation coefficient between the commodity implied short rate factor and the US Fed Funds rate is 0.897. This means that the commodity market implied short rate factor is largely consistent with the US Federal Funds rate. However, it is more volatile than the US Federal Funds rate perhaps due to the additional volatility that characterises commodity markets.

I undertake Granger causality test on these two series to test the null hypothesis that, the commodity market implied short rate does not Granger cause the US Federal Funds rate. Table 3.8 reports the test results. The null hypothesis is rejected at the 99% confidence level. Furthermore, the null hypothesis that the US Federal Funds rate does not Granger cause the commodity market implied short rate is also rejected at the 99% confidence level. The Granger causality, therefore, test shows the significance of the lagged influence of one variable to another. One would expect there to be no influence between the two variables if they are statistically similar. These results therefore provide evidence that the latent short rate factor implied by the commodity market is a different time series to the observable spot interest rate. This also suggests that the monetary policy authority is influenced by the expectations of global commodity markets when deciding on their policies. On the other hand, the expectations of the commodity market about the interest rate are also influenced by interest rate policies.

Figure 3.5: The Commodity Market Implied Interest Rate Factor

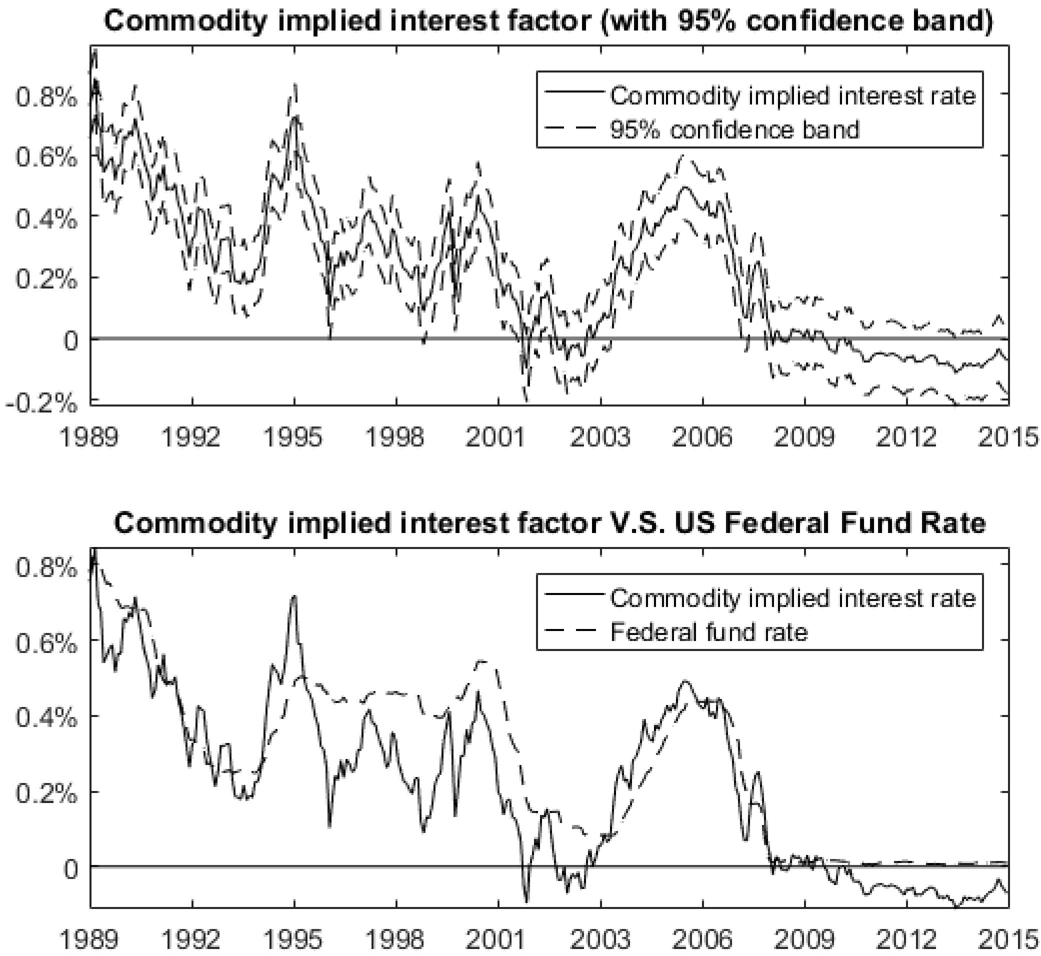


Figure 3.5 plots the commodity market implied interest rate factor (solid line) against the US Fed Fund rate (dashed line) from Jan/1989 to Dec/2015.

Table 3.8: The Granger Causality Test on the Interest Rates

Null Hypothesis	Test stats	p-value	df
Commodity implied interest $\nRightarrow$ Federal Funds rate	23.023	0.003	8
Federal Funds rate $\nRightarrow$ Commodity implied interest	26.981	0.000	8

Table 3.8 shows the Granger causality test results for the commodity market implied short rate factor estimates. Arrow notation: $\nRightarrow$, stands for: "does not Granger cause". The χ^2 critical values for the 90%, 95%, and 99% confidence interval with eight degrees of freedom are 13.362, 15.507 and 20.090, respectively. The number of lags/degrees of freedom is decided using the Akaike information criterion.

3.6 Concluding remarks

This chapter extends the discussion of commodity term structure modelling from individual commodity markets to multiple commodity markets. A joint model of commodity futures is proposed in this chapter. This model performs well in the term structure estimation, with reasonably small and stationary pricing errors. It also contributes substantially to the existing literature in a number of ways.

Previous studies have experienced difficulties identifying the interest rate factor in single market models. The solution that has typically been employed, is effectively to pin down the interest rate to an observable series. This chapter, however, jointly models several commodity futures term structures in order to estimate the pure latent interest rate factor. This is generated from several benchmark commodity markets. A strong connection is found between this factor and observable interest rates. Statistically, these two time series are found to Granger cause each other, suggesting that interest rate expectations in global commodity markets both reflects and anticipates developments in monetary policy.

There are some avenues for future research. For example, the latent short rate factor could be defined as a Cox, Ingersoll and Ross (CIR) process (see Cox, Ingersoll and Ross 1981) instead of as a standard independent Gaussian process so as to avoid the possible negativity in its estimate. It may also be of interest to explore other common factors that are beyond the scope of this chapter, such as the business cycle, inflation rate and others. It would also be possible to extend the selection of commodity categories from the existing choice of four benchmark markets to a broader range. The extra variation introduced by additional commodity term structures might help to further the understanding of the overall commodity market.

Chapter 4

Oil prices in the real economy

4.1 Introduction

This chapter presents a macro-finance model of the US economy and the crude oil market. Its novelty lies in the use of both macroeconomic and oil market data to study interactions between the oil spot and futures markets, monetary policy and the macro-economy.

There is an extensive macro-econometric literature, pioneered by Hamilton (1983), that studies the effect of oil prices on the economy. He argued that most of the post-war recessions in the US have been caused by oil price shocks. Hamilton and his colleagues followed this paper up with many studies documenting the adverse effect of oil price changes on real output and inflation over the last few decades (see Hamilton 1985, Hamilton 2003, Herrera and Hamilton 2004, Hamilton 2008, Hamilton and Wu 2014). Many other authors have also studied this effect (see Raymond and Rich 1997 , Finn 2000, Hooker 1996 and Hooker 2002). These papers focus on the effect of spot oil prices on the economy, neglecting the information that can be gleanced from futures, oil inventory and its convenience yield. Kilian (2006, 2008), however, attributes a much greater role to demand side pressures on the oil price, and argues that the effect of supply side shocks on the economy depends critically upon the tightness of the oil market in the run up to the shock, as reflected in inventories.

As its name suggests, the convenience yield reflects the convenience and security of holding physical stocks of oil rather than an equivalent position in the oil futures market. If inventories are excessive, the convenience yield can turn negative, reflecting insurance and other holding costs. The relationship between the spot and short forward price is governed by an arbitrage relationship that depends upon the spot convenience yield and the spot interest rate. Longer maturity futures prices depend upon the evolution of oil prices and hence the expected interest rate and convenience yield under the risk-neutral measure, which makes an allowance for the risk of holding positions in the futures market.

The literature on commodity futures dates back to the 1930s and the papers by Kaldor (1939), Working (1949), and Brennan (1958) and others. Empirical pricing models date back to the early 1980s (see Schwartz 1982, Brennan and Schwartz 1985, Gibson and Schwartz 1990, Brennan 1991, Cortazar and Schwartz 1994, Schwartz 1997, Casassus and Collin-Dufresne 2003). These empirical models suggest that the term structure of commodity futures prices is similar to the term structure of interest rates in the sense that most of the variation in the cross-section can be explained by three latent factors. However, both the models of futures and of interest rates are silent about the nature of these latent variables and their links with the macro-economy.

Interest in the structural relationship between the yield curve and the economy has led to the development of macro-finance models, following the pioneering research of Ang and Piazzesi (2003), who successfully introduced macroeconomic factors into the term structure model. They found that although these indicators provide a good description of the behaviour of short rates, it was necessary to retain latent variables in order to model long term rates. Subsequent macro-finance research has used the semi-structural “central bank model” (CBM) developed by Svensson (1999), Smets (1999), Kozicki and Tinsley (2005) and others, where this latent variable represents exogenous shocks to the central bank inflation target or underlying rate of inflation (see Kozicki and Tinsley 2005, Dewachter and Lyrio 2006, Dewachter, Lyrio and Maes 2006).

The CBM represents the behaviour of the macroeconomy in terms of the output gap (g_t), inflation (π_t) and the short term interest rate (r_t). There is no role for the oil price in the basic model, despite the evidence of the effect of oil price shocks on the macro-economy. This chapter, therefore, develops a macro-finance model that introduces the oil price and the convenience yield of oil inventory into the CBM. This model is used to study the interaction between the oil market and the macroeconomy and to model the term structure of oil futures prices, following the semi-structural strand of the macro-finance literature in modelling the underlying inflation rate as a latent variable using the Kalman filter. An additional latent variable is used to handle exogenous supply side and other shocks to the oil price, which plays a vital role in this model.

The semi-structural approach is also followed in the use of a long sample of macroeconomic data. This begins in 1964, and thus includes the period of the 1970s oil shocks, thus helping to identify the links between the oil market and the economy. The Kalman filter plays a key role here since it handles missing observations, which other latent variable techniques (such as principal components) find very difficult to handle. This allows a relatively long data set for macro variables and spot oil prices to be combined with a relatively short data set (beginning in 1984) for oil futures prices. Estimates of the convenience yield are provided by the Kalman filter, informed by the futures prices after 1984. Estimates for earlier years are inferred from the macro variables and the way these interact with the oil market in the post-1984 period.

The empirical results are consistent with the theoretical priors. They are also consistent with the existing macro-finance literature in underlining the importance of both macroeconomic variables and latent factors such as the underlying inflation rate, which reflects salient historical episodes such as the Volker disinflation of 1979-81. The latent oil market variables also reflect the impact of exogenous shocks such as the two Gulf wars, the internet bubble at the turn of the millennium and the 2008 financial crisis. The results provide strong evidence of the interaction between the oil market and the US economy. For example, reflecting the work of Kilian (2005) and his colleagues, the strength of the economy in the run up to oil price shocks of the 1970s helps explain why these were persistent, and how the weakness of the economy in the run up to the first Gulf war in 1991 helps explain why that shock had only a temporary effect on the oil price. These effects are illustrated in Figure 4.4.

The convenience yield naturally plays a key role in pricing futures, but it is more of a surprise to find that it also plays an important role in the macro-model. The convenience yield can be viewed as a proxy for oil inventory, which serves as a buffer, damping the effect of oil and economic shocks on the real economy. The results show that the convenience yield plays a key role in the monetary transmission mechanism, providing an important channel through which policy interest rates affect oil prices, activity and inflation. This is of relevance to central bank policy maker since, naturally, when evaluating the effect of their policies on the economy, they focus on the money and bond markets. The oil spot and futures markets are also important, and central bank researchers have indeed been studying these recently (see Chin and Liu 2015, Millard and Shakir 2013, Elekdag et al 2007, Bodenstein, Guerrieri, and Kilian 2012).

These macro and oil market variables are used here as factors in an affine model of the term structure of oil futures prices. This is specified in nominal terms, with the spot price as the baseline, combining the US general price level (the GDP deflator) and the real price from the macro model. Thus, the spot oil price has a unit effect at the short end of the futures curve, but this influence fades with maturity. The GDP deflator is non-stationary and has a permanent unit effect on nominal spot and futures prices. The spot interest rate and convenience yield determine the slope of the short-maturity futures curve. Inflation also has an important effect, but the effect of the output gap is negligible. Futures prices incorporate risk premiums, which this model suggests increase with maturity and are strongly cross-correlated. Oil prices play an important role in the model, increasing the premium. The contribution of the underlying inflation rate is also important, apparently pushing up the premium until inflation began to decline following the Volker experiment and, more recently, depressing it.

As in any macro-finance model, the latent variables are updated in line with surprises in both macro and financial variables. For example, the Kalman variable representing the underlying oil price normally follows the price with a lag, reflecting the delayed effect of exogenous supply-side surprises. It also, however, reflects the effect of surprises in macro variables that influence the demand for oil. For example, the negative macro surprises associated with the recent financial crisis and recession pushed the underlying oil price well below the actual price and partly anticipated the sharp fall seen since mid-2014. A latent factor model of the oil market that only updated these factors in line with surprises to spot and futures prices would miss this effect.

The use of Kalman filters to pick up the effect of unobservable expectational influences helps to solve the price puzzle - the tendency (noted originally by (Sims 1992)) for increases in policy interest rates to anticipate inflationary developments and apparently cause inflation. As Sims showed, the use of a commodity prices model also helps in resolving this. The model results suggest that oil prices and (as noted) the convenience yield are important links in monetary transmission mechanisms, as illustrated by Figure 4.8. Monetary policy also influences inflation through its effect on the output gap, but this figure shows that, once oil market effects are accounted for, this effect is surprisingly weak.

The chapter is organized as follows. The next section specifies the macro-finance dynamic term structure model (DTSM). This specifies the state dynamics under the real-world and risk-neutral probability measures and the risk premium. Section 4.4 sets out the empirical methodology and econometric model, describes the data used,, and discusses the empirical findings and results. Section 4.5 concludes.

4.2 The real-world dynamics

The research strategy is to follow the semi-structural macro-finance literature, which models the dynamics under the risk-neutral and the real world measure in terms of observable variables like the interest rate and oil price, assuming that these are measured without error. Because the spot oil price is used as a factor, it is possible to switch from the nominal prices used in the futures market to the real oil prices relevant for the real economy by dividing by the GDP deflator. As will be shown, this allows future prices to be modelled by specifying both the risk-neutral dynamics of oil prices and the macroeconomy using real variables. An alternative approach, pioneered by Joslin, Singleton and Zhu (2013), and followed by Heath (2016), would be to assume that the cross-section of oil futures prices is affine to a small number of latent factors (like their own principal components). Unlike bond prices, however, which are the relative prices of money in different periods, futures prices tend to increase over time due to the effect of inflation, which it is difficult to allow for without using the spot price as an explicit factor. Moreover, there is an important arbitrage identity linking the spot oil price, the convenience yield and interest rate that would be difficult to incorporate into a model that did not identify these explicitly.

4.2.1 The spot oil market variables

The relationship between the spot (S_t) and one period future ($F_{1,t}$) oil price depends upon the cost of carrying inventory ($r_t - \delta_t$), which can be decomposed into the spot convenience yield of holding physical oil inventories δ_t and the spot interest rate r_t:

$$F_{1,t} = S_t e^{(r_t - \delta_t)}, \tag{4.1}$$

This shows that if the cost of carry is negative the curve is downward sloping. The market is then said to be "in backwardation", as it was in 2012 for example. If the cost of carry is positive, the forward curve is upward sloping and is said to be “in contango”, as it was in 2015 for example.

Many commodity market models, like Heath (2016), treat the cost of carry as a single variable. The model constructed here, however, follows Cassasus and Dufresne (2005), in distinguishing the convenience yield and the spot interest rate. Although it follows these authors in assuming that the convenience yield is a latent variable, it follows the macro-finance approach in assuming that the interest rate is observed without error. As will be shown, the decomposition of the cost of carry plays an important role in this model, consistent with the theory of storage (see Working 1933 , Kaldor 1939 , Working 1949 , Brennan 1958 , Weymar 1968). This theory suggests that the convenience yield is closely related to the level of the commodity stored in inventory. It states that when working inventories (those held by the market for commercial rather than strategic purposes) are tight, the convenience yield will be high; the cost of carry negative and the futures curve in backwardation. On the other hand, when oil inventories are abundant, as they have been recently, the convenience yield will be negative, adding to the interest cost of carry and pushing the futures curve into contango. My model allows the convenience yield and the interest rate to interact, so that an increase in interest rates tends to reduce inventories and push up the convenience yield in compensation.

4.2.2 The other macro variables

The model of the economy is naturally specified in terms of the real rather than the nominal oil price. The log nominal price is denoted by $s_t = \log S_t$ and the log real price by $\rho_t = s_t - p_t$, where p_t is the log implicit GDP deflator. The effect of the business cycle is represented by g_t, the US output gap based on the constant price US GDP series (see Section 4.4). Inflation is represented by π_t, the annual US inflation rate, calculated using the US GDP price deflator, and r_t is represented by the US Federal Funds rate. I collect these in the vector $n_t = (\rho_t \ g_t \ \pi_t \ r_t)'$ and model them as part of the state vector, assuming that they are all measured without error. Table 1 reports the summary statistics for these data. These suggest that g_t is stationary, but that the other variables are non-stationary.

The state vector also includes the convenience yield (δ_t), which is modelled as a latent variable. It also contains two other latent variables, denoted by ρ_t^* and π_t^*, representing the real spot oil price and inflation asymptotes respectively. The identification scheme ensures that ρ_t and π_t converge upon these values. Since the long-run real interest rate is assumed to be constant, it also anchors the long-run value of the nominal interest rate. The non-stationarity of inflation and interest rates is allowed for by assuming that π_t^* is integrated of order one ($I(1)$). Similarly, The non-stationarity of the real oil price is allowed for by assuming that ρ_t^* is ($I(1)$). This ensures that deviations of ρ_t, π_t. and r_t. from their asymptotic values are stationary ($I(0)$).

These latent variables are modelled using the Kalman filter. This updates the estimates of the latent variables optimally in each period using the errors made in predicting the observed variables. This makes an allowance for exogenous political, monetary and other shocks. For example, wars in the Middle East that have the effect of causing the oil price to exceed the model prediction lead to upwards revisions to ρ_t^*, allowing the model to reflect these influences with a one period lag.

4.2.3 The state dynamics

The real spot price and inflation asymptotes are assumed to be mean independent and, as noted, $I(1)$:

$$\rho_t^* = \kappa_{\rho^*} + \rho_{t-1}^* + \epsilon_{\rho^*,t} \qquad \epsilon_{\rho^*,t} \sim N(0, \sigma_{\rho^*}^2), \tag{4.2}$$

$$\pi_t^* = \kappa_{\pi^*} + \pi_{t-1}^* + \epsilon_{\pi^*,t} \qquad \epsilon_{\pi^*,t} \sim N(0, \sigma_{\pi^*}^2). \tag{4.3}$$

These are collected in a vector of asymptotes $l_t^* = (\rho_t^*, \pi_t^*)'$. The convenience yield (δ_t), is modelled as a stationary latent variable that depends upon the contemporaneous values of the other two latent factors and the lagged values of spot oil price and the interest rate:

$$\delta_t = a_\delta + \Upsilon_{\delta,l^*} l_t^* + \phi_{\delta,\delta}\delta_{t-1} + \Upsilon_{\delta,n} n_{t-1} + \epsilon_{\delta,t} \qquad \epsilon_{\delta,t} \sim N(0, \sigma_\delta^2). \tag{4.4}$$

where $\Upsilon_{\delta,l^*} = (\theta_{\delta,\pi^*}, \theta_{\delta\pi^*})$, $\Upsilon_{\delta,n} = (\phi_{\delta,\rho}, 0, 0, \phi_{\delta,r})$. The latent variables are collected in the vector: $z_t = (\rho_t^*, \pi_t^*, \delta_t)'$.

The convenience yield is assumed to affect the real spot oil price (ρ_t) but not the other macro variables:

$$\rho_t = a_\rho + \theta_{\rho,\rho^*}\rho_t^* + \theta_{\rho,\pi^*}\pi_t^* + \phi_{\rho,\delta}\delta_t + \phi_{\rho,\rho}\rho_{t-1} + \phi_{\rho,g}g_{t-1} + \phi_{\rho,\pi}\pi_{t-1} + \phi_{\rho,r}r_{t-1} + \epsilon_{\rho,t} \tag{4.5}$$

Other macro variables depend upon each other (with a lag) as well as the asymptotes:

$$g_t = a_g + \theta_{g,\rho^*}\rho_t^* + \theta_{g,\pi^*}\pi_t^* + \phi_{g,\rho}\rho_{t-1} + \phi_{g,g}g_{t-1} + \phi_{g,\pi}\pi_{t-1} + \phi_{g,r}r_{t-1} + \epsilon_{g,t} \tag{4.6}$$

$$\pi_t = a_\pi + \theta_{\pi,\rho^*}\rho_t^* + \theta_{\pi,\pi^*}\pi_t^* + \phi_{\pi,\rho}\rho_{t-1} + \phi_{\pi,g}g_{t-1} + \phi_{\pi,\pi}\pi_{t-1} + \phi_{\pi,r}r_{t-1} + \epsilon_{\pi,t} \tag{4.7}$$

$$r_t = a_r + \theta_{r,\rho^*}\rho_t^* + \theta_{r,\pi^*}\pi_t^* + \phi_{r,\rho}\rho_{t-1} + \phi_{r,g}g_{t-1} + \phi_{r,\pi}\pi_{t-1} + \phi_{r,r}r_{t-1} + \epsilon_{r,t} \tag{4.8}$$

where, g_t is the US output gap, π_t is the US inflation rate, and r_t is the US Federal Funds rate. Stacking equations (4.5) to (4.8), the dynamics of the observable variables (n_t) are defined as:

$$n_t = A_n + \Theta_{n,l^*} l_t^* + \Phi_{n,\delta}\delta_t + \Phi_{n,n} n_{t-1} + \epsilon_{n,t} \tag{4.9}$$

$$A_n + \Theta_{n,z} z_t + \Phi_{n,n} n_{t-1} + \epsilon_{n,t} \qquad \epsilon_{n,t} \sim N(0, \Sigma_n). \tag{4.10}$$

where: A_n, is a 4×1 vector, Θ_{n,l^*} is a 4×2 matrix, $\Phi_{n,\delta}$ is a 4×1 vector with $\phi_{\delta,\rho}$ as the first element and zeros elsewhere. Hence, $\Theta_{n,z} = (\Theta_{n,l^*}, \Phi_{n,\delta})$ is a 4×3 matrix. $\Phi_{n,n}$ and Σ_n are 4×4 matrices all defined in the Appendix C.1. The matrix Σ_n and $\epsilon_{n,t}$ are factorized using the standard "LDL" factorization :

$$\epsilon_{n,t} = L_n D_n u_{n,t}; \quad u_{n,t} \sim N(0, I_4); \quad \Sigma_n = L_n D_n^2 L_n' \tag{4.11}$$

where D_n, and L_n are defined in the Appendix C.1.

4.2.4 The identification scheme

Following Dewachter, Lyrio and Maes (2006), the first two latent factors are normalised by aligning them with the asymptotic values of the associated observed variables. Stacking equations (4.4) and (4.9):

$$\begin{pmatrix} \delta_t \\ n_t \end{pmatrix} = \begin{pmatrix} a_\delta \\ A_n \end{pmatrix} + \begin{pmatrix} \phi_{\delta,\delta} & \Upsilon_{\delta,n} \\ \Phi_{n,\delta} & \Phi_{n,n} \end{pmatrix} \begin{pmatrix} \delta_{t-1} \\ n_{t-1} \end{pmatrix} + \begin{pmatrix} \Upsilon_{\delta,l^*} \\ \Theta_{n,l^*} \end{pmatrix} l_t^* + \begin{pmatrix} \epsilon_{\delta,t} \\ \epsilon_{n,t} \end{pmatrix} \tag{4.12}$$

Setting the error terms to zero and setting lagged $(t-1)$ values equal to current period (t) values gives the asymptotic values of the variables associated with any values of l_t^*:

$$\begin{pmatrix} \delta_t^* \\ n_t^* \end{pmatrix} = \begin{pmatrix} 1-\phi_{\delta,\delta} & -\Upsilon_{\delta,n} \\ -\Phi_{n,\delta} & I-\Phi_{n,n} \end{pmatrix}^{-1} \begin{pmatrix} a_\delta \\ A_n \end{pmatrix} + \begin{pmatrix} 1-\phi_{\delta,\delta} & -\Upsilon_{\delta,n} \\ -\Phi_{n,\delta} & I-\Phi_{n,n} \end{pmatrix}^{-1} \begin{pmatrix} \Upsilon_{\delta,l^*} \\ \Theta_{n,l^*} \end{pmatrix} l_t^* \tag{4.13}$$

$$= \varphi + R l_t^*.$$

This defines the steady state intercept vector and response matrices φ and R in terms of the dynamic parameters. Given any steady state specification of φ and R, these relationships can be inverted to define the associated short run dynamic parameters. In this chapter R is used to rotate the latent variables ρ_t^* and π_t^* in order to align them (up to the additive adjustments φ) with ρ_t and π_t:

$$\begin{pmatrix} a_\delta \\ A_n \end{pmatrix} = \begin{pmatrix} 1-\phi_{\delta,\delta} & -\Upsilon_{\delta,n} \\ -\Phi_{n,\delta} & I-\Phi_{n,n} \end{pmatrix} \varphi, \quad \begin{pmatrix} \Upsilon_{\delta,l^*} \\ \Theta^*_{n,l^*} \end{pmatrix} = \begin{pmatrix} 1-\phi_{\delta,\delta} & -\Upsilon_{\delta,n} \\ -\Phi_{n,\delta} & I-\Phi_{n,n} \end{pmatrix} R \tag{4.14}$$

where:

$$\varphi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \varphi_r \end{pmatrix} \qquad R = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}. \tag{4.15}$$

This rotation implies: $\delta_t^* = g_t^* = 0$ and that $\rho_t = \rho_t^*$ and $\pi_t^* = \pi_t^*$ in the steady state. The final row of R also ensures that the interest rate asymptote also moves in line with the inflation asymptote $r_t = \varphi_r + \pi_t^*$. Subtracting (4.13) from (4.12) and substituting (4.14) puts (4.12) into an $I(0)$ error correction format:

$$\begin{pmatrix} \delta_t - \delta_t^* \\ n_t - n_t^* \end{pmatrix} = \begin{pmatrix} \phi_{\delta,\delta} & \Upsilon_{\delta,n} \\ \Phi_{n,\delta} & \Phi_{n,n} \end{pmatrix} \begin{pmatrix} \delta_{t-1} - \delta_t^* \\ n_{t-1} - n_t^* \end{pmatrix} + \begin{pmatrix} \epsilon_{\delta,t} \\ \epsilon_{n,t} \end{pmatrix} \tag{4.16}$$

In terms of the structural equations, this system can be written (using (4.15)) as:

$$\begin{aligned}
\delta_t &= \phi_{\delta,\delta}\delta_{t-1} + \phi_{\delta,\rho}(\rho_{t-1} - \rho_t^*) + \phi_{\delta,r}(r_{t-1} - \pi_t^* - \varphi_r) + \epsilon_{\delta,t} \\
\rho_t - \rho_t^* &= \phi_{\rho,\delta}(\rho_{t-1} - \rho_t^*) + \phi_{\rho,\rho}\rho_{t-1} + \phi_{\rho,g}g_{t-1} + \phi_{\rho,\pi}(\pi_{t-1} - \pi_t^*) + \phi_{\rho,r}(r_{t-1} - \pi_t^* - \varphi_r) + \epsilon_{\rho,t} \\
g_t &= \phi_{g,\rho}(\rho_{t-1} - \rho_t^*) + \phi_{g,g}g_{t-1} + \phi_{g,\pi}(\pi_{t-1} - \pi_t^*) + \phi_{g,r}(r_{t-1} - \pi_t^* - \varphi_r) + \epsilon_{g,t} \\
\pi_t - \pi_t^* &= \phi_{\pi,\rho}(\rho_{t-1} - \rho_t^*) + \phi_{\pi,g}g_{t-1} + \phi_{\pi,\pi}(\pi_{t-1} - \pi_t^*) + \phi_{\pi,r}(r_{t-1} - \pi_t^* - \varphi_r) + \epsilon_{\pi,t} \\
r_t - \pi_t^* &= \varphi_r + \phi_{r,\rho}(\rho_{t-1} - \rho_t^*) + \phi_{r,g}g_{t-1} + \phi_{r,\pi}(\pi_{t-1} - \pi_t^*) + \phi_{r,r}(r_{t-1} - \pi_t^* - \varphi_r) + \epsilon_{r,t}.
\end{aligned} \tag{4.17}$$

4.2.5 The transition equation

The Kalman filter uses surprises in the four observed variables n_t to update the estimates of the three latent variables z_t (see equation (C.19) in the Appendix C.4). To allow it to do this, z_t is contemporary with n_t in (4.10). The transition equation uses lagged explanatory variables, however. To specify this, equation (4.2) and (4.3) are first substituted into (4.4) in order to replace $l_t^{*\prime}$ by its lagged value:

$$\delta_t = \kappa_\delta + \Upsilon_{\delta,l^*} l^*_{t-1} + \phi_{\delta,\delta}\delta_{t-1} + \Upsilon_{\delta,n} n_{t-1} + \eta_{\delta,t} \tag{4.18}$$

where $\kappa_\delta = a_\delta + \theta_{\delta,\rho^*}\kappa_{\rho^*} + \theta_{\delta,\pi}\kappa_{\pi^*}$, and $\eta_{\delta,t} = \epsilon_{\delta,t} + \theta_{\delta,\rho^*}\epsilon_{\rho^*,t} + \theta_{\delta,\pi^*}\epsilon_{\pi^*,t}$. where a_δ is defined in equation (4.14). Stacking equations (4.2), (4.3) and (4.18) I can then write the latent variable transition dynamics compactly as:

$$z_t = K_z + \Upsilon_{z,z} z_{t-1} + \Upsilon_{z,n} n_{t-1} + \eta_{z,t} \tag{4.19}$$

where K_z is a 3×1 vector, while $\Upsilon_{z,z}$ and $\Upsilon_{z,n}$ are 3×3 and 3×4 matrices derived from the parameters of equations (4.2) to (4.4). The error vector is factorized as $\eta_{z,t} = L_z D_z u_{z,t}$, where $u_{z,t}$ is another vector of white noise processes. These matrices are all defined in the Appendix C.1.

Next, equation (4.11) and (4.19) are substituted into (4.10):

$$n_t = K_n + \Upsilon_{n,z} z_{t-1} + \Upsilon_{n,n} n_{t-1} + \eta_{n,t}, \tag{4.20}$$

where: $K_n = A_n + \Theta_{n,z} K_z$, $\Upsilon_{n,z} = \Theta_{n,z}\Upsilon_{z,z}$, $\Upsilon_{n,n} = \Phi_{n,n} + \Theta_{n,z}\Upsilon_{z,n}$, and $\eta_{n,t} = L_n D_n u_{n,t} + \Upsilon_{n,z} u_{z,t}$, and A_n and Υ_{n,l^*} are defined in equation (4.14). K_n, $\Upsilon_{n,z}$, and $\Upsilon_{n,n}$ are all defined in the Appendix C.1.

Finally, stacking equations (4.19) and (4.20) gives the transition equation for the state vector $X_t = (z_t', \ n_t')'$:

$$\begin{pmatrix} z_t \\ n_t \end{pmatrix} = \begin{pmatrix} K_z \\ K_n \end{pmatrix} + \begin{pmatrix} \Upsilon_{z,z} & \Upsilon_{z,n} \\ \Upsilon_{n,z} & \Upsilon_{n,n} \end{pmatrix} \begin{pmatrix} z_{t-1} \\ n_{t-1} \end{pmatrix} + \begin{pmatrix} L_z D_z & 0_{3,4} \\ \Upsilon_{n,z} & L_n D_n \end{pmatrix} \begin{pmatrix} u_{z,t} \\ u_{n,t} \end{pmatrix}. \tag{4.21}$$

More compactly:

$$X_t = A + BX_{t-1} + LDU_t \qquad U_t \sim N(0, I) \tag{4.22}$$

$$= A + BX_{t-1} + W_t \qquad W_t \sim (0, \Sigma). \tag{4.23}$$

where A, B, L and D are defined for convenience in the Appendix C.1.

4.3 Futures prices and the risk-neutral dynamics

4.3.1 The transition equation under measure Q

This macro-econometric model can be estimated as a stand-alone vector error correction model (VECM) that models the interaction between ρ_t, δ_t and the other macro variables. Here, however, the model dynamics are respecified under the risk-neutral measure in order to model the behaviour of oil futures prices jointly with the macro variables over the period 1984-2015. The mainstream macro-finance literature is followed in adopting the same macroeconomic structure but allowing the deterministic parameters of this structure to reflect the change of measure. This has the advantage of allowing the information in futures prices to inform the latent variables ρ_t^* and π_t^* employed in the macro-model, since these remain the primary focus of interest.

I use the essentially affine model of Duffee (2002) to change the probability measure. This redefines the deterministic and stochastic parts of the VAR under measure P, in a way that ensures that the expectation of W^Q under the Q measure is zero. This implies a system that is congruent with the companion form (4.22):

$$X_t = A^Q + B^Q X_{t-1} + W^Q \qquad W^Q \sim N(0, \Sigma), \tag{4.24}$$

A^Q is an unrestricted vector. The first two rows of B^Q are restricted to take account of the exclusion restrictions in (4.2) and (4.3). Similarly, the third row takes account of those in (4.4). The fourth row of B^Q imposes the arbitrage identity discussed in the Appendix C.2, which can be written as:

$$\rho_{t+1} = \rho_t - B_\pi^Q X_t + r_t - \delta_t - \frac{1}{2}\sigma_\rho^2 + \epsilon_{\rho,t+1}. \tag{4.25}$$

where $B_\pi^Q X_t$ is the model-implied expectation of π_{t+1}. B^Q is specified formally in the Appendix C.2.

In contrast, Heath (2016) assumes that the nominal spot oil price and cost of carry follow a simple autoregressive scheme under Q, independently of the macro variables. The cross-section of futures prices is then affine in these two factors. The real-world time series dynamics are modelled using a VAR with a state vector that includes these factors alongside macroeconomic variables, which are thus 'unspanned', in the sense that they only have a lagged or dynamic effect on the futures curve, not a contemporaneous one.

This specification raises several concerns, however. First, as the Appendix A.1 shows, the parameters of the spot price equation are determined by an arbitrage identity (see equation (A.7), which is the nominal counterpart of (4.25)) under the risk-neutral measure and should not be freely estimated. Second, Heath's model is specified in terms of the nominal spot price, without allowing for the effect of inflation, which imparts a strong secular uptrend. In contrast, this model is specified in real terms so as to remove the effect of inflation on nominal prices, and macro shocks are also allowed to have a contemporaneous effect on this structure. Since I also split the cost of carry is also into the spot interest rate and convenience yield, monetary policy and other macro shocks can have an additional indirect effect on the convenience yield and spot oil price, working through the interest rate. In practice, it is found that the effect of these macro shocks is typically relatively small, but well-defined, given the 'tiny' measurement errors found in cross-section estimates of financial prices.

The Appendix C.3 shows that A^Q and B^Q in equation (4.24) are related to A^P and B^P in equation (4.22) by:

$$A^Q = A - LDD'\Lambda_1 \tag{4.26}$$

$$B^Q = B - L\Lambda_2 \tag{4.27}$$

where Λ_1 is a 7×1 vector, which is composed of $\Lambda_{1,z}$, $\Lambda_{1,n}$. Λ_2 is a 7×7 matrix of parameter determining the risk premium, it is composed of $\Lambda_{2,z}$, $\Lambda_{2,n}$, $\Lambda_{2,z,n}$ as:

$$\Lambda_1 = \begin{pmatrix} \Lambda_{1,z} \\ \Lambda_{1,n} \end{pmatrix} \qquad \Lambda_2 = \begin{pmatrix} \Lambda_{2,z} & 0_{3,5} \\ \Lambda_{2,z,n} & \Lambda_{2,n} \end{pmatrix}. \tag{4.28}$$

4.3.2 The term structure of futures prices

The state dynamics under the risk-neutral measure Q (equation (4.24)) determine the cross-sectional loadings. First, the affine trial solution for the log futures prices is adopted, as follows:

$$f_{\tau,t} = \alpha_\tau + \Psi_\tau X_t + \psi_{p,\tau} p_t. \tag{4.29}$$

The initial condition, is implied by the special case when $\tau = 0$, in which $f_{0,t} = \rho_t + p_t$, giving the starting values for the first latent factor (ρ_t^*) as:

$$\psi_{\rho,0} = 1 \tag{4.30}$$

$$\psi_{p,0} = 1, \tag{4.31}$$

$$\psi_{\delta,0} = \psi_{\pi^*,0} = \psi_{g,0} = \psi_{\pi,0} = \psi_{r,0} = 0. \tag{4.32}$$

This makes the futures prices exponentially affine in the factors and thus log normal. To verify the trial solution (4.29), and find its parameters, logs are taken of equation (A.1), using the formula for the expectation of a log normal variable, to get:

$$f_{\tau,t} = lnE_t(F_{\tau-1,t+1}) = E_t(f_{\tau-1,t+1}) + \frac{1}{2}Var(f_{\tau-1,t+1}). \tag{4.33}$$

Increasing t and reducing τ in equation (4.29) by one, substituting $p_{t+1} = \pi_{t+1} + p_t$ and then equation (4.24):

$$\begin{aligned} E_t(f_{\tau-1,t+1}) =& \alpha_{\tau-1} + \Psi_{\tau-1} E_t(X_{t+1}) + \psi_{p,\tau-1} E_t(p_{t+1}) \\ =& \alpha_{\tau-1} + \Psi_{\tau-1}(A^Q + B^Q X_t) + \psi_{p,\tau-1}(B^Q_\pi X_t + p_t), \end{aligned} \tag{4.34}$$

$$Var(f_{\tau-1,t+1}) = \Psi_{\tau-1} \Sigma \Psi'_{\tau-1}. \tag{4.35}$$

Substituting these into equation (4.33) using the starting values, equation

(4.31) and (4.32) verifies the trial solution in equation (4.29) provided that:

$$\Psi_\tau = \Psi_{\tau-1}B^Q + B^Q_\pi, \tag{4.36}$$

$$\psi_{p,\tau} = \psi_{p,\tau-1} = 1, \tag{4.37}$$

$$\alpha_\tau = \alpha_{\tau-1} + \Psi_{\tau-1}A^Q + \frac{1}{2}\Psi_{\tau-1}\Sigma\Psi'_{\tau-1}, \tag{4.38}$$

where α is constant. Equation (4.37) shows that all the loadings ψ_p are equal to one, simplifying (4.29) to:

$$h_{\tau,t} = f_{\tau,t} - ip_t = \alpha_\tau + \Psi'_\tau X_t, \tag{4.39}$$

Stacking these equations gives the Affine Term Structure Model (ATSM):

$$h_t = \alpha + \Psi_z z_t + \Psi_n n_t \tag{4.40}$$

4.3.3 The measurement equation

The measurement equation in the state space representation is defined as:

$$y_t = D + GX_t + e_t \qquad e_t \sim N(0, Q) \tag{4.41}$$

assuming that, the commodity futures data $h_t = (h_t, \dots, h_\tau)'$ are observed with measurement errors, and the variables $n_t = (\rho_t, g_t, \pi_t, r_t)'$ are observed without error. To accommodate this, I define the measurement equation (4.41) is defined using equation (4.40) as:

$$\begin{pmatrix} h_t \\ n_t \end{pmatrix} = \begin{pmatrix} \alpha \\ j_0 \end{pmatrix} + \begin{pmatrix} \Psi_z & \Psi_n \\ 0_{4,3} & j_1 \end{pmatrix} \begin{pmatrix} z_t \\ n_t \end{pmatrix} + e_t \qquad e_t \sim N(0, Q),$$

where, $j_0 = (0, 0, 0, 0)'$, and $j_1 = diag(1, 1, 1, 1)$. $Q = diag(q_1^2, \dots, q_\tau^2, 0, 0, 0, 0)$.

4.4 The empirical model

4.4.1 The Kalman filter and the likelihood function

To complete the dynamic term structure model, I now describe the maximum likelihood approach used to estimate the Kalman filter and the model parameters. The filter uses surprises in forecasting the macro variables and the futures prices to update the estimates of the latent variables. The Appendix C.4 sets out the algebra for the revisions as equation (C.19), and derives the log-likelihood function equation (C.20) for this model. The Kalman filter is used instead of the principal components to capture the latent variables since this avoids assuming that any yields, or combination of yields, are measured without error, and it allows the latent variables to be aligned with the asymptotes ρ^* and π^*. Most importantly, however, the Kalman filter nicely resolves the missing variable problem (see Appendix C.4).

The empirical model consists of a heteroscedastic VAR describing the three latent variables and five macroeconomic variables (4.22) and the auxiliary equations describing the representative futures prices (4.40). It is estimated by maximum likelihood and the Kalman filter, which gives optimal estimates of the latent variables in this situation. The likelihood function is derived in Appendix C.4. This section describes the data and the empirical results.

4.4.2 Data sources and description

The model is estimated using quarterly time series of the macro variables and crude oil futures. All data are downloaded from Thompson Reuters Datastream. Summary statistics are presented in Table 4.1. Figure 4.1 shows the West Texas Intermediate (WTI) oil futures prices and Figure 4.2 shows the four observed macro variables. I use data for the US output gap, US inflation and US Federal Funds rate, from Q1 1964 to Q4 2015. This allows the effect of the oil shocks of the 1970s to be analysed. The Fed Funds rate is specified as a quarterly decimal fraction (the annual rate as % divided by 400). We generate the US output gap by applying the HP filter to log US GDP. US inflation is the log difference of the US implicit price deflator.

Table 4.1: Summary Statistics

WTI futures prices							
	Mean	S.D	Skew	Kurt	ADF	KPSS	Obs
h_3	-0.881	0.520	0.409	-1.091	-1.988	0.870	128
h_6	-0.889	0.524	0.524	-1.154	-1.761	0.873	128
h_9	-0.896	0.527	0.527	-1.197	-1.600	0.870	128
h_{12}	-0.902	0.529	0.529	-1.229	-1.483	0.868	128
h_{18}	-0.865	0.555	0.555	-1.504	-1.153	0.888	105
h_{24}	-0.752	0.575	0.575	-1.664	-1.341	0.929	81
Observed state variables							
	Mean	S.D	Skew	Kurt	ADF	KPSS	Obs
ρ^o	-1.067	0.678	-0.204	-0.878	-2.275	0.810	208
g^o	3.37×10^{-11}	0.015	-0.360	0.523	-5.711	0.018	208
π^o	8.67×10^{-4}	5.54×10^{-4}	1.240	0.831	-2.093	0.873	208
r^o	0.014	9.24×10^{-4}	0.708	0.830	-2.154	0.902	208

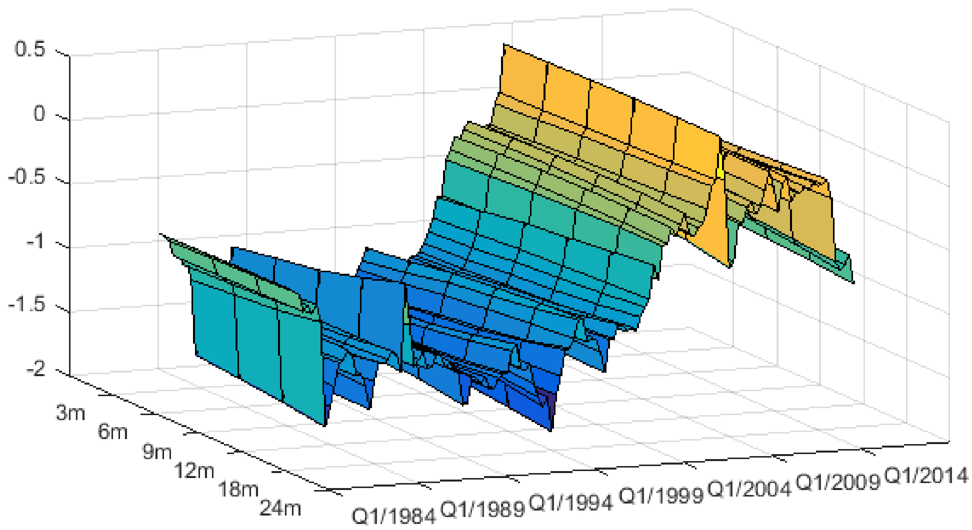
These data were supplied by Datasteam and are discussed in the text. h_τ denotes the τ-month maturity log futures price less the log CPI as in equation (4.39). Mean denotes the sample arithmetic mean, S.D. standard deviation; Skew. and Kurt. report skewness and excess kurtosis, standard measures of the third and fourth moments. Obs. reports the number of observations. ADF shows the Augmented Dickey Fuller test statistic under the null hypothesis of non-stationarity. The lag length is determined by the Akaike information criterion. The critical values are -3.461, -2.875, and -2.574 for the 99%, 95% and 90% confidence level, based on Mackinnon (1996). KPSS shows the Kwiatkowski Phillips Schmidt Shin test statistic under the null hypothesis of stationarity. Providing 0.7390, 0.4630, and 0.3470 for the 99%, 95% and 90% confidence level based on Kwiatkowski Phillips Schmidt Shin (1992).

The spot oil price is a composite series. WTI spot price, which matches the futures data, is available from Q1 1983, while the Brent price, which gives the price of a similar grade, is available from Q1 1970[1]. Since the crude oil price was fixed close to $2.25 per barrel between 1964 and Q1 1970[2], this value is used until then; the Brent price from Q1 1970 to Q4 1982 and WTI thereafter. To represent the term structure of oil futures, the prices of WTI light crude oil futures traded on New York Mercantile Exchange (NYMEX) are used, starting from the year 1984, when these oil futures contracts started trading. Oil futures contracts with

[1]Brent and WTI spot oil price series only diverge significantly in recent years, when the latter went to a discount because of export controls and the development of the US shale hydrocarbon industry.

[2]Before the 1970s, the oil market was monopolized by the major Western oil companies, and the oil price at that time was described by the phrase :“ take the price used by Exxon, add it to that used by Shell and divide the sum by two” (Carollo (2012)).

Figure 4.1: The Term Structure of Log Real WTI Oil Futures Contracts



This figure shows the term structure of (log) WTI light crude oil futures, which started trading on NYMEX in 1984. The data are from Datastream. The 1, 2, 3, 6, 9 and 12 months maturities are available from Q1 1984, the 18 month contract from Q3 1989; and the 24 month contract from Q1 1995.

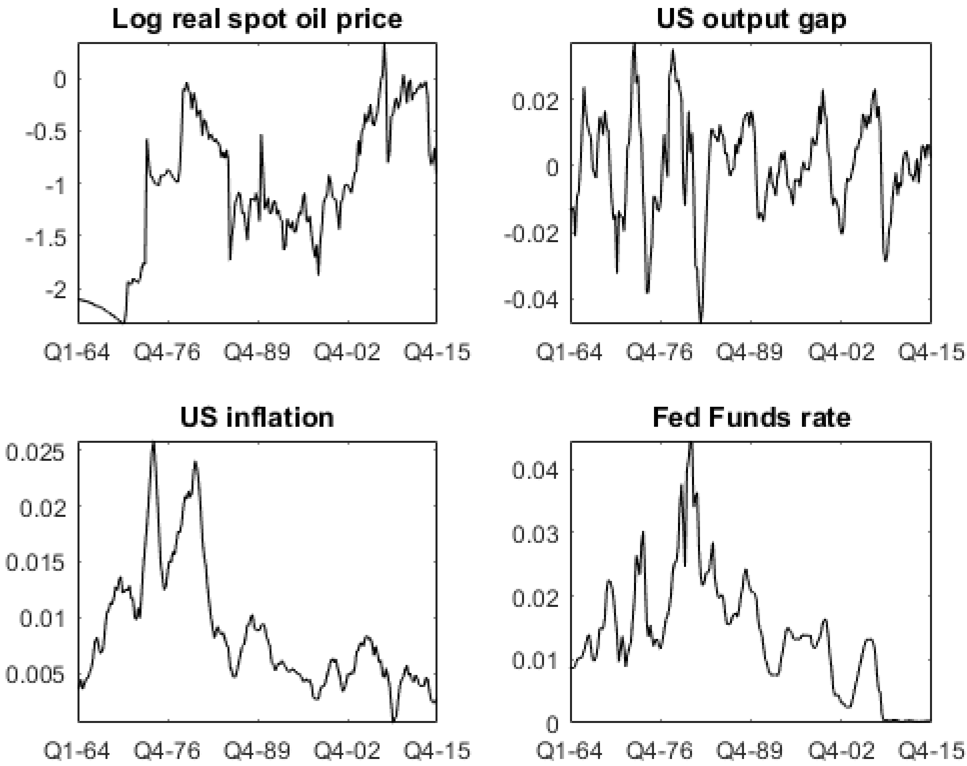
1, 2, 3, 6, 9, 12, 18 and 24 month maturities are studied. The series for the prices of oil futures with 1, 2, 3, 6, 9 and 12 month maturities are available from Q1 1984, the 18 month contract from Q3 1989; and the 24 month contract from Q1 1995.

4.4.3 The behaviour of the macro and spot oil market variables

Figure 4.3 shows the estimates of the latent and observed state variables included in l_t and n_t respectively. In the latter case, these estimates are shown alongside their observed values. Table 4.2 shows their root mean squared errors (RMSEs). The long term inflation asymptote (π^*) in figure (4.3) picks up the secular trends in inflation and interest rates. This variable resembles the inflation target identified by Ireland (2007) (his figure 4), which largely accommodates the oil price hikes in the 1970s but then falls back sharply after the Volker deflation in the early 1980s.

The top panel of Figure 4.4 plots the real spot price against the latent

Figure 4.2: Observed Variables



This figure shows the convenience yield and the four macro variables. The convenience yield is extracted from the Kalman filter pre-1984Q1 and the oil futures data post-1984Q1 using equation (4.1). The US output gap is obtained by applying the HP filter to log real US GDP. US inflation is the log annual difference in the US implicit GDP price deflator. The US interest rate is represented by the Fed Funds rate. All data are taken from Thompson Reuters DataStream.

Table 4.2: The Root Mean Squared Error (RMSE) of the Joint Commodity Term Structure Model

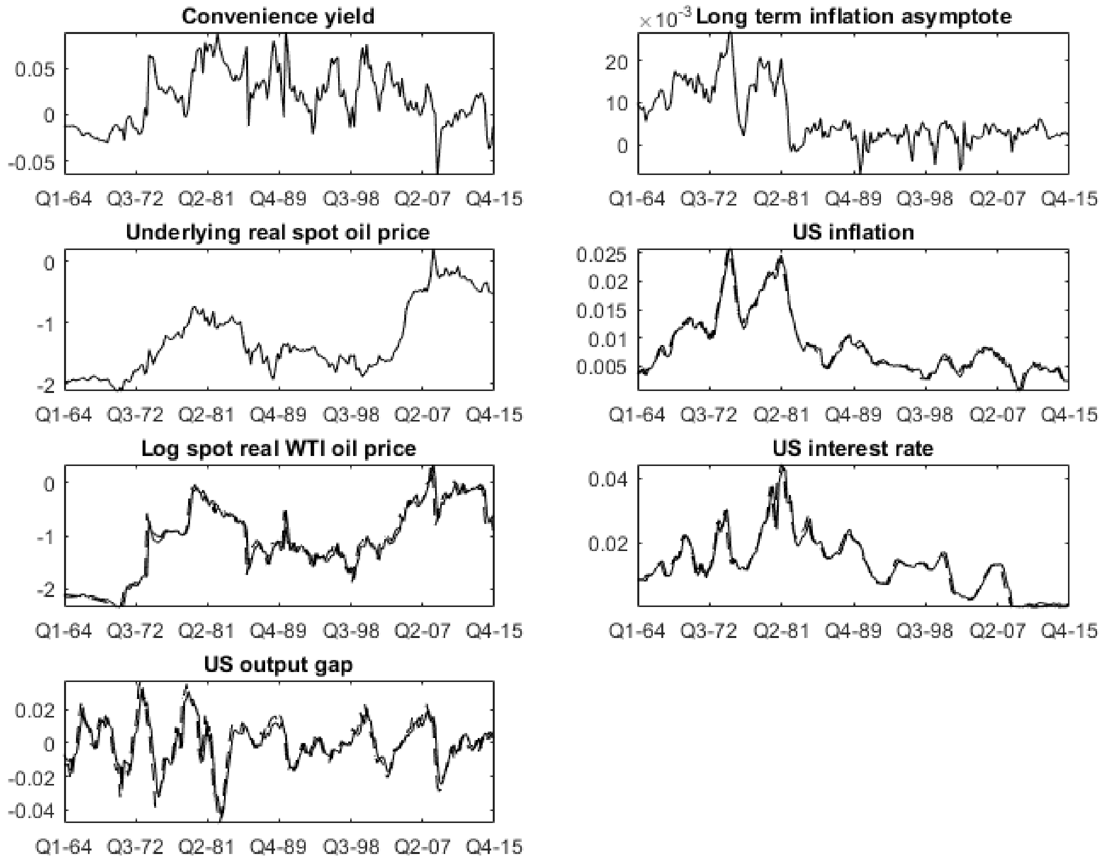
n_t^o	ρ_t	g_t	π_t	r_t
	0.185	7.2×10^{-3}	8.3×10^{-4}	2.2×10^{-3}

$h_{\tau,t}$	3m	6m	9m	12m	18m	24m
	0.184	0.165	0.151	0.141	0.125	0.119

The RMSE of the estimates for the log real spot oil price (ρ), output gap (g) interest (r) and inflation rates (π) for the period 1964-2015 are reported in the top panel. The bottom panel shows the RMSEs for estimates of log futures price less the log CPI, as in equation (4.39). These are for the periods since 1983, when they became available.

variable (ρ^*) representing the underlying real spot oil price. The underlying price normally follows the price with a lag, reflecting the lagged effect of oil price

Figure 4.3: State Variables Representing the macroeconomy and Oil Markets

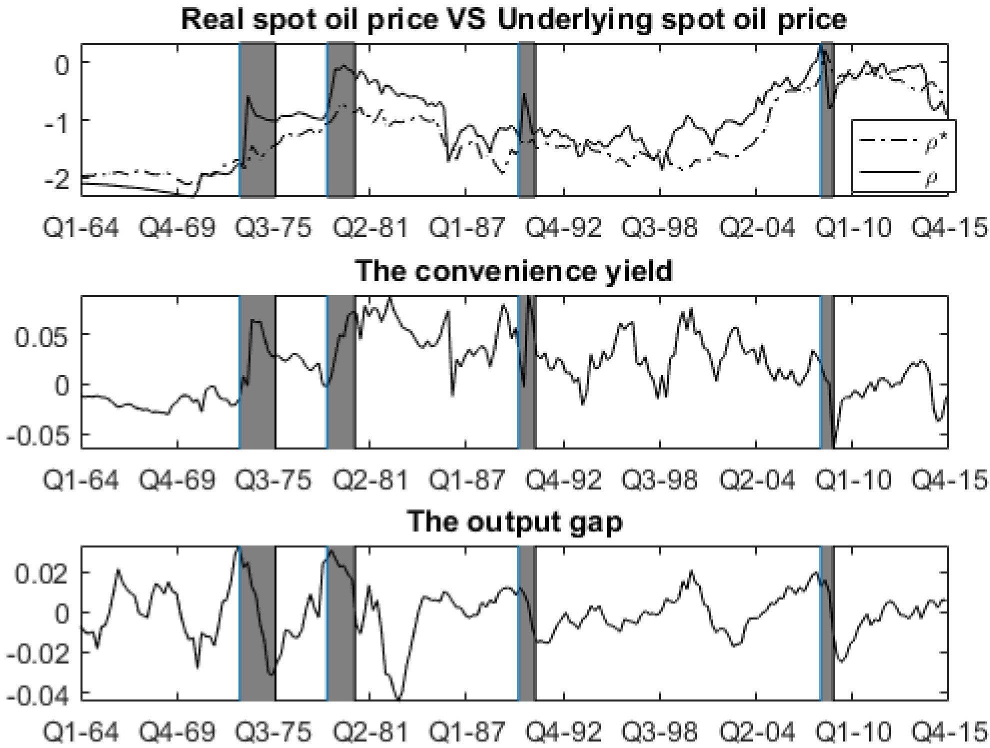


The solid lines in this figure show model estimates of the state variables, while the dashed lines show their observed values (if the variable is observable in the model). The right-hand panel depicts the inflation and interest rates, showing how the latent variable representing the long term inflation asymptote (π^*) picks up their secular trends. The left-hand panel shows the oil market variables and the output gap. The relationships between these variables are analysed in the next figure.

surprises. ρ^* also reflects the effect of surprises in macro variables, however, and in particular those associated with the recent period of financial crisis and recession, which pushed it well below the price by the end of 2013, anticipating about half of the sharp fall seen since mid-2014.

Figure 4.4 also shows how oil prices interact with the output gap. The output gap was very high before both of the oil shocks of the 1970s, indicated by the left sides of the first two vertical bars, reflecting the strength of the US economy. This

Figure 4.4: The Relationship Between the Strength of the Economy and the Oil Price



The top panel of this figure plots the real spot price (ρ, continuous line) against the latent variable (ρ^*, dashed line) representing the underlying real spot oil price. The underlying price normally follows the price with a lag, but anticipates the sharp fall seen since mid-2014. The third panel shows how these prices interact with the output gap, which reflects the strength of economic activity. This had the effect of tightening the oil market before the two 1970s oil shocks, resulting in persistent price increases, which then pushed the economy into recession. In contrast, the economy was not as strong prior to the price hike seen at the time of the first Gulf war in 1992. Consequently, this was much less persistent and was not followed by a recession. The US economy was also strong as the oil price peaked in 2008, but the ensuing recession was arguably due to the financial crisis rather than the high oil price.

helped tighten the oil market, causing the underlying oil price to trend upwards. These price increases were quite persistent, provoking a sharp fall in the output gap as the economy moved into recession. In contrast, the economy was not as strong prior to the oil price spike seen at the time of the first Gulf war in 1992, which left ρ^* well below ρ, as indicated by the third vertical bar. This was much less persistent, represented by a spike rather than a step increase, and was not followed by a serious slowdown. These episodes reflect the observations of Killian

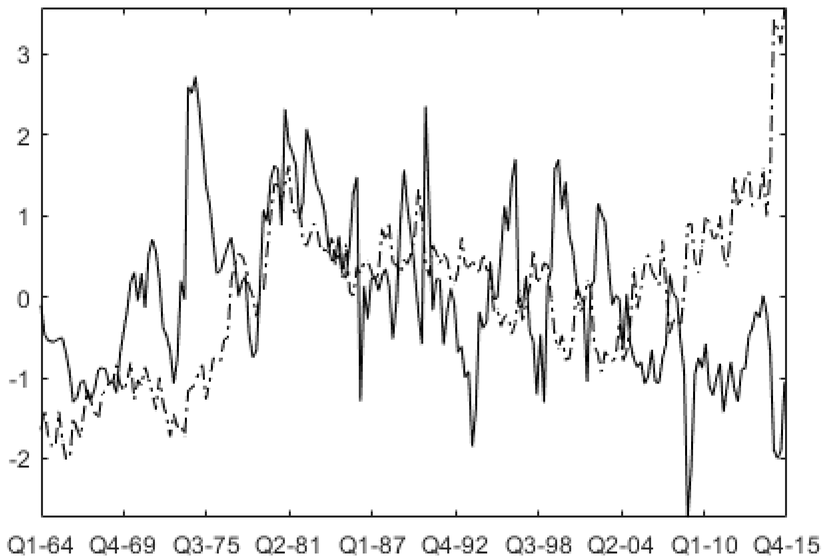
(2005), suggesting that oil shocks caused by political or other exogenous events only have a big impact on the real economy if they are strong and the underlying oil price trend is upward. The US economy was also strong when the oil price peaked in 2008, the fourth vertical line, but the ensuing recession was arguably due to the financial crisis rather than the high oil price, which fell back sharply as the recession took hold. These shocks are also reflected in the convenience yield (δ), estimated using the Kalman filter. This estimate picks up the tightness of the market in 1974 and 1980 quite nicely, as well as market developments since 1984.

Figure 4.5 shows the inverse correlation between the estimated convenience yield and the US oil inventory (excluding the US Strategic Petroleum Reserve (SPR)), in other words the oil inventory that is held by the market. Although inventory is not part of the model, this panel shows that short-run swings in the convenience yield and the oil inventory tend to be inversely related. However, there are some notable spikes in the estimated convenience yield (δ) that are not reflected in the inventory. For example, there is a sharp spike in Q2 1974, which arguably reflects rationing and other effects designed to conserve oil stocks and help shield the economy from the Arab oil embargo[3].

Table 4.3 reports the estimates of the parameters obtained from the Kalman-VAR under the measure P. The estimates of the key parameters conform to economic priors and are generally statistically significant. The estimates of $\phi_{\rho,g}$ and $\phi_{\rho,\pi}$, are significant, capturing the effect of the economy on the real oil price, while the significance of $\phi_{\delta,\rho}$, $\phi_{\delta,r}$ captures the effect on the convenience yield, in line with the results of Casassus and Collin-Dufresne (2003). As one would expect, the estimate of $\phi_{\pi,\rho}$ indicates that the real oil price has a significant short run impact on inflation, although the impact on activity is not significant.

[3]For example, the US Congress passed the Emergency Highway Energy Conservation Act to impose a national maximum speed limit of 55 mph in 1974, with similar restrictions imposed in European countries. In the UK, petrol coupons were issued in preparation for petrol rationing, although this was not actually implemented.

Figure 4.5: The Convenience Yield and US Oil Inventories



This figure presents the model estimates of the convenience yield ($\hat{\delta}$, blue line) alongside the data for US oil inventories (green line). These exclude the Strategic Petroleum Reserve (SPR). It shows that short-run swings in the convenience yield and the oil inventory tend to be inversely related. The breakdown of this relationship in 1970 is discussed in the text.

Table 4.3: Estimates of the Parameters of the Real World Dynamics

Parameters	Estimates	t-stats	Parameters	Estimates	t-stats
k_{ρ^*}	6.4×10^{-3}	6.217	$\phi_{g,\rho}$	1.6×10^{-3}	0.804
k_{π^*}	-3.8×10^{-5}	-0.932	$\phi_{g,g}$	0.858	25.433
φ_r	4.8×10^{-3}	170.179	$\phi_{g,\pi}$	-0.154	-0.822
$\phi_{\delta,\delta}$	0.318	4.594	$\phi_{g,r}$	-0.099	-1.111
$\phi_{\delta,\rho}$	0.048	5.878	$\phi_{\pi,\rho}$	9.4×10^{-4}	4.197
$\phi_{\delta,r}$	0.739	28.761	$\phi_{\pi,g}$	0.014	5.141
$\phi_{\rho,\delta}$	0.176	0.412	$\phi_{\pi,\pi}$	0.897	53.516
$\phi_{\rho,\rho}$	0.818	25.678	$\phi_{\pi,r}$	1.4×10^{-33}	1.5×10^{-31}
$\phi_{\rho,g}$	2.335	3.043	$\phi_{r,\rho}$	1.0×10^{-3}	1.780
$\phi_{\rho,\pi}$	10.834	15.710	$\phi_{r,g}$	0.037	11.410
$\phi_{\rho,r}$	-2.256	-15.008	$\phi_{r,\pi}$	-0.011	-0.220
			$\phi_{r,r}$	0.922	31.861

This table presents the parameters of the dynamic structure equation (4.23) defined under the real world measure P, with their asymptotic t-statistics.

4.4.3.1 Impulse response functions

The dynamics of these interaction effects can be seen from Figures 4.6 and 4.7, which depict the impulse response functions. These show the dynamic ef-

Table 4.4: Estimates of the Parameters of the Risk-neutral Dynamics

Parameters	Estimates	t-stats	Parameters	Estimates	t-stats
$k^Q_{\rho^*}$	-0.063	-6.383	$\phi^Q_{\rho,r}$	0.522	8.650
$k^Q_{\pi^*}$	3.0×10^{-3}	352.241	ϕ^Q_{g,ρ^*}	-0.027	-1.299
k^Q_{δ}	0.065	129.712	ϕ^Q_{g,π^*}	13.480	1.879
k^Q_{ρ}	-7.2×10^{-3}	-6.324	$\phi^Q_{g,\rho}$	1.345	38.091
k^Q_{g}	0.175	5.403	$\phi^Q_{g,g}$	-0.286	-12.261
k^Q_{π}	1.5×10^{-3}	13.942	$\phi^Q_{g,\pi}$	5.666	3.339
k^Q_{r}	0.055	125.304	$\phi^Q_{g,r}$	4.813	17.617
$\phi^Q_{\rho^*,\rho^*}$	0.805	27.445	ϕ^Q_{π,ρ^*}	-3.2×10^{-3}	-0.775
$\phi^Q_{\pi^*,\pi^*}$	0.151	2.883	ϕ^Q_{π,π^*}	1.178	2.051
ϕ^Q_{δ,ρ^*}	-0.021	-1.753	$\phi^Q_{\pi,\rho}$	0.061	26.738
ϕ^Q_{δ,π^*}	-4.514	-5.390	$\phi^Q_{\pi,g}$	-0.098	-402.235
$\phi^Q_{\delta,\delta}$	0.437	13.556	$\phi^Q_{\pi,\pi}$	1.094	13.001
$\phi^Q_{\delta,\rho}$	0.238	23.989	$\phi^Q_{\pi,r}$	0.478	70.881
$\phi^Q_{\delta,r}$	0.334	39.191	ϕ^Q_{r,ρ^*}	0.037	6.920
ϕ^Q_{ρ,s^*}	3.2×10^{-3}	0.249	ϕ^Q_{r,π^*}	-0.240	-0.133
ϕ^Q_{ρ,π^*}	-1.178	-0.610	$\phi^Q_{r,\rho}$	0.174	9.427
$\phi^Q_{\rho,\delta}$	-1.000	-26.693	$\phi^Q_{r,g}$	2.3×10^{-3}	2.945
$\phi^Q_{\rho,\rho}$	0.939	62.364	$\phi^Q_{r,\pi}$	0.544	8.017
$\phi^Q_{\rho,g}$	0.098	23.245	$\phi^Q_{r,r}$	0.666	21.227
$\phi^Q_{\rho,\pi}$	-1.094	-9.997			

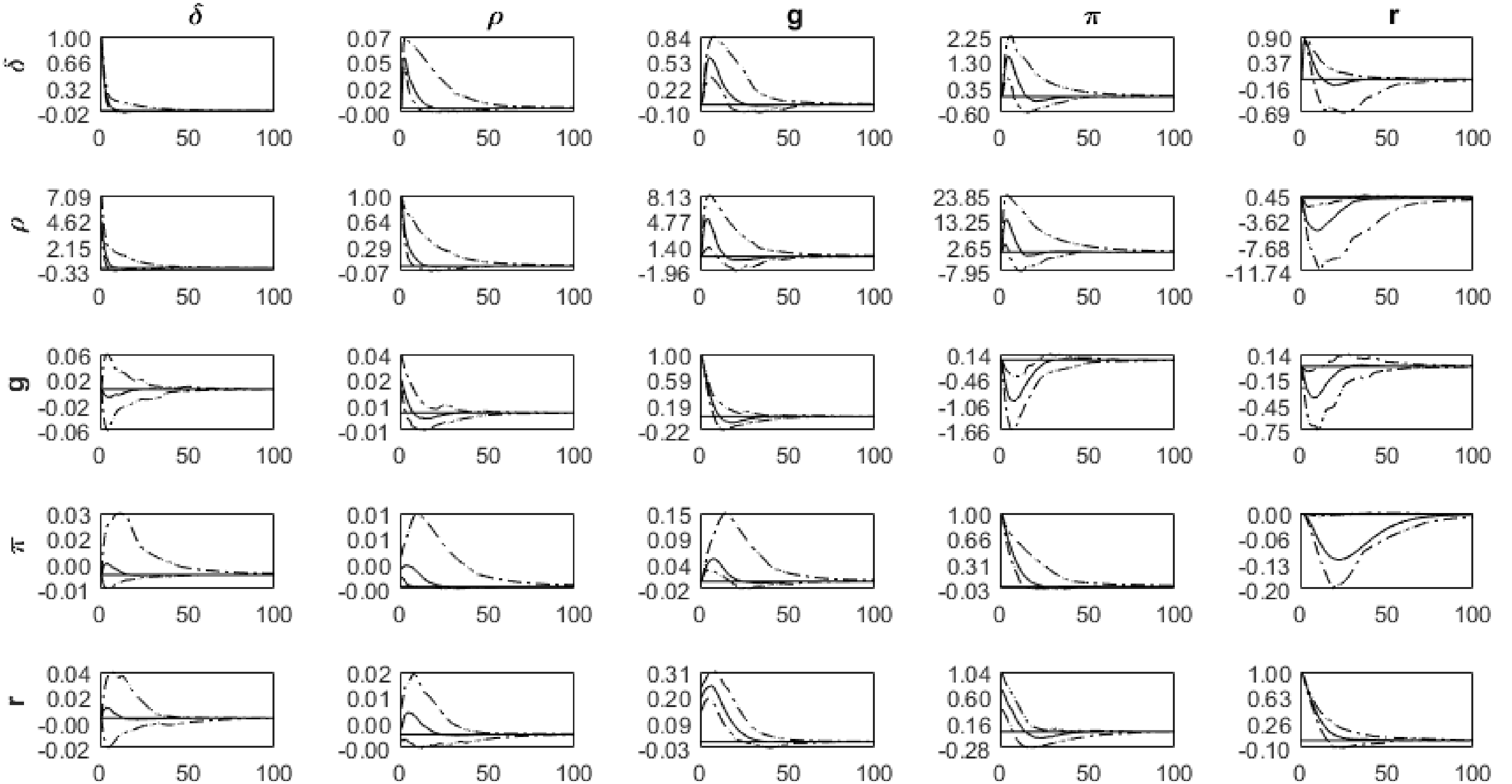
This table presents the parameters of the dynamic structure equation (4.24) defined under the risk-neutral measure Q, with their asymptotic t-statistics.

fects of innovations in the macroeconomic variables. Because these innovations are correlated empirically, orthogonalized innovations using the triangular factorization defined in section (4.2.3) are applied here. The orthogonalized impulse responses show the effect on the macroeconomic system of increasing each of these innovations by one percentage point for just one period using the Wald representation of the system. Each column shows the effect of a unit shock to a macro variable, while the rows show their effects

The relationships between the output gap, inflation and interest rates are in line with economic priors and similar to those seen in previous macro-finance models. Figure 4.6 presents impulse response functions of these variables with 95% confidence bounds[4]. The final row of Figure 4.6 shows that the US Fed

[4]The confidence bound of impulse response functions in this chapter takes 95% quantile of 1000 times simulated sample of parametric wild bootstrapping. The detailed methodology follows section 3 in Lütkepohl (2000) and Appendix B in Coroneo (2016)

Figure 4.6: Impulse response functions for the oil and macro variables



This figure and Figure 4.7 depict the impulse response functions, which show the dynamic effects of innovations in the state variables. The confidence bound at 95% confidence interval is displayed as dashed lines in each case. Elapsed time is measured in calendar quarters. Each column shows the effect of a unit shock to a state variable, while the rows show their effects. The relationships between the output gap, inflation and interest rates are in line with economic priors and similar to those seen in previous macro-finance models. The novelty here is the introduction of oil prices and the convenience yield. As I would expect, this figure shows that oil price innovations act as supply-side shocks that increase inflation and interest rates. In turn, the oil price responds positively to the output gap and inflation and negatively to the interest rate. The final column shows that an increase in interest rates pushes up the convenience yield. A plausible explanation is that interest rates increase the cost of carry and reduce inventories. The fall in inventories then increases the convenience yield and depresses the spot price, and hence inflation. This effect is analysed in Figure 4.8. Monetary policy also affects inflation through its effect on the output gap.

Table 4.5: Risk premium parameter estimates

Parameters	Estimates	t-stat	Parameters	Estimates	t-stat
$\lambda_{1,\delta}$	-0.069	-488.346	$\lambda_{2,g,\rho}$	-1.344	-71.313
λ_{1,ρ^*}	0.069	53.823	$\lambda_{2,g,g}$	1.144	70.250
λ_{1,π^*}	-3.1×10^{-3}	-423.968	$\lambda_{2,g,\pi}$	-5.820	-14.591
$\lambda_{1,\rho}$	0.019	32.327	$\lambda_{2,g,r}$	-4.912	-14.891
$\lambda_{1,g}$	-0.175	-19.006	λ_{2,π,ρ^*}	0.002	2.200
$\lambda_{1,\pi}$	-1.5×10^{-3}	-4.004	λ_{2,π,π^*}	-1.075	-1.549
$\lambda_{1,r}$	-0.055	-54.708	$\lambda_{2,\pi,\rho}$	-0.060	-58.137
$\lambda_{2,\rho^*,\rho^*}$	0.195	12.177	$\lambda_{2,\pi,g}$	0.112	52.791
λ_{2,π^*,π^*}	0.849	12.057	$\lambda_{2,\pi,\pi}$	-0.197	-2.345
$\lambda_{2,\delta,\rho^*}$	-0.027	-7.483	$\lambda_{2,\pi,r}$	-0.478	-30.864
λ_{2,δ,π^*}	3.775	1.867	λ_{2,r,ρ^*}	-0.038	-3.922
$\lambda_{2,\delta,\delta}$	-0.119	-1.647	λ_{2,r,π^*}	0.329	0.158
$\lambda_{2,\delta,\rho}$	-0.190	-19.393	$\lambda_{2,r,\rho}$	-0.173	-10.500
$\lambda_{2,\delta,r}$	0.404	32.782	$\lambda_{2,r,g}$	0.034	32.840
λ_{2,g,ρ^*}	0.025	1.204	$\lambda_{2,r,\pi}$	-0.555	-7.837
λ_{2,g,π^*}	-13.227	-1.731	$\lambda_{2,r,r}$	0.256	7.147

These parameters allow for risk using the essentially affine specification of Duffee (2002). They connect, the parameters shown in Tables 4.3 and 4.4, estimated respectively under the measures P and Q. (See equations (4.26) and (4.27)).

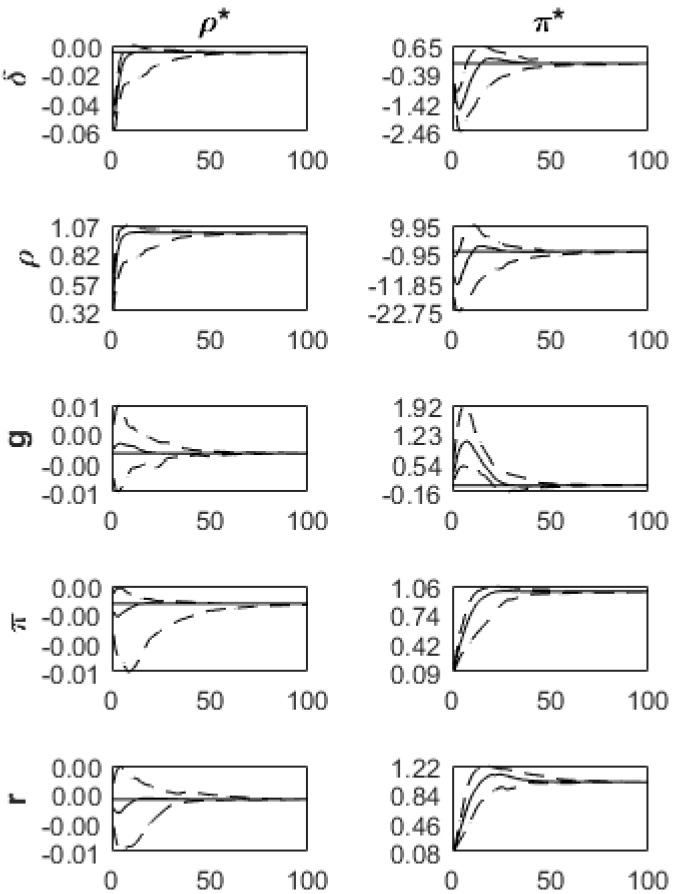
Table 4.6: Volatility parameter estimates

Parameters	Estimates	t-stat	Parameters	Estimates	t-stat
$c_{\delta,\rho}$	4.211	8.856	d_{ρ^*}	0.106	12.866
$c_{g,\rho}$	0.009	1.857	d_{π^*}	0.003	7.946
$c_{\pi,\rho}$	0.001	2.871	d_{δ}	0.027	14.825
$c_{\pi,g}$	0.000	0.000	d_{ρ}	0.132	19.600
$c_{r,\rho}$	0.001	1.031	d_{g}	0.007	18.688
$c_{r,g}$	0.118	5.745	d_{π}	-0.001	-11.733
$c_{r,\pi}$	0.414	1.874	d_{r}	0.002	17.581

These parameters are used in the "LDL" decomposition of the covariance matrix (section (4.2.3)).

changes interest rates in response to inflation and economic activity, consistent with the Taylor rule, which suggests that the central bank adjusts the policy interest rate in order to maintain a stable rate of inflation. The final column shows that output and inflation in turn fall in response to the higher interest rate. The use of Kalman filters to pick up the effect of unobservable expectational influences helps to solve the notorious price puzzle - the tendency (noted originally by Sims (1992)) for increases in policy interest rates to anticipate inflationary developments and thus apparently cause inflation.

Figure 4.7: Impulse Response Functions for the Latent State Variables



This figure shows the dynamic effects of innovations in the latent variables. The confidence bound at 95% confidence interval is displayed as dashed lines in each case. Elapsed time is measured in calendar quarters. Each column shows the effect of a unit shock to a state variable, while the rows show their effects (see notes to Figure 4.6). By construction π^* has a unit long-run effect on inflation and interest rates, with no effect on other variables. Similarly, ρ^* only affects the real oil price in the long run.

The introduction of the oil market variables in this model also plays a key role in resolving the price puzzle. As one would expect, Figure 4.6 shows that oil price innovations act as supply-side shocks that increase inflation and interest rates. In turn, the oil price responds positively to the output gap and inflation, and negatively to the interest rate as part of the monetary transmission mechanism. My results suggest that oil inventories and the convenience yield are also important links in this chain. The final column shows that an increase in interest rates pushes up the convenience yield. A plausible explanation is that interest rates increase the cost of carry and reduce inventories. The fall in inventories

Figure 4.8: The Role of the Oil Market in the Monetary Transmission Mechanism

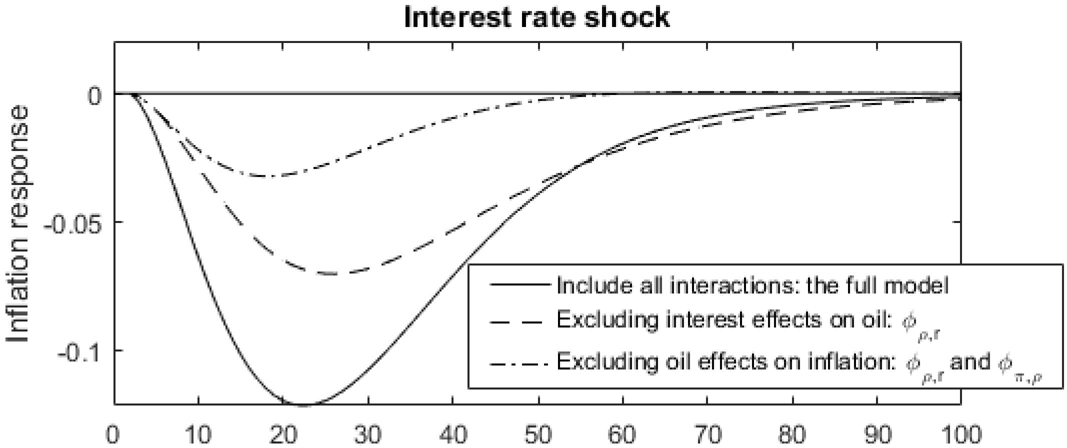


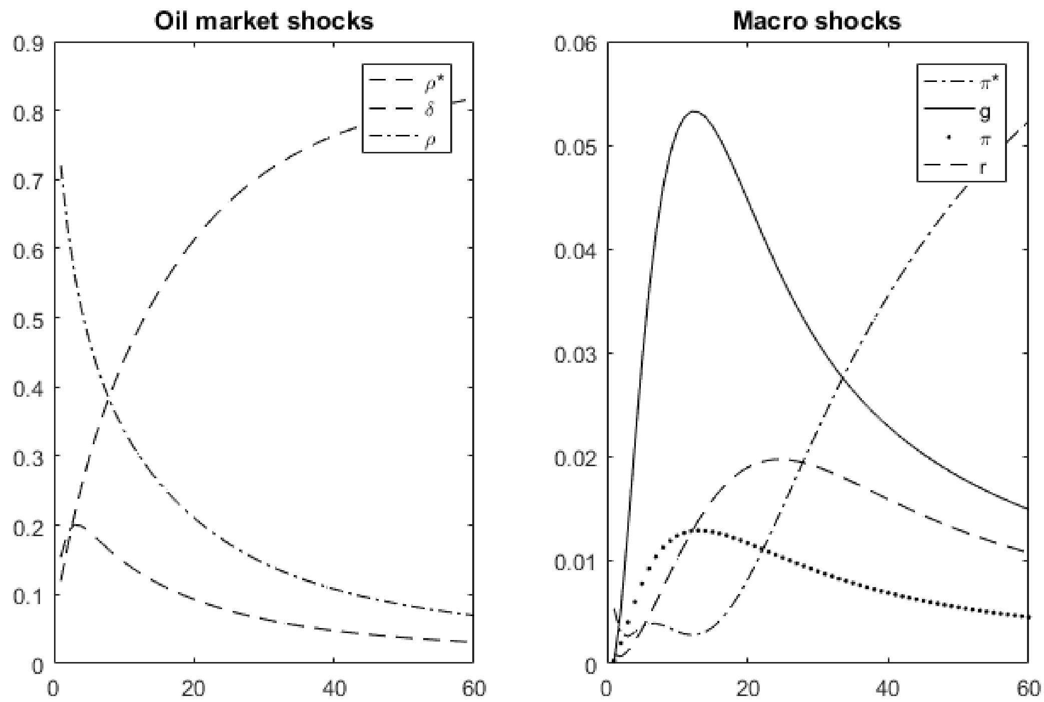
Figure 4.7 showed that the policy interest rate affects inflation through its effect on the oil market as well as the output gap. To decompose this effect, this figure shows the response of inflation to interest rate changes when different parts of the monetary transmission mechanism are shut off. The negative response of inflation is maximized if I keep all of these links, as indicated by the continuous line, as in Figure 4.7. Removing the direct effect of the interest rate on the oil price by setting the parameter $\phi_{\delta,r}$ to zero halves the response, as indicated by the dashed line. Removing the indirect effects (working through the effect of interest rates and output and hence the real oil price and inflation) by setting the parameter $\phi_{\pi,s}$, to zero, gives the dashed-dotted line, further reducing the response of inflation to the interest rate shock

increases the convenience yield and depresses the spot price, and hence inflation, as indicated in Figure 4.6. To illustrate these effects, Figure 4.8 shows how this impulse response changes as I set different parameters to zero in order to shut off different parts of this monetary transmission mechanism. Monetary policy also affects inflation through its effect on the output gap, but this figure shows that once oil market effects are accounted for, this effect is surprisingly weak.

4.4.3.2 Analysis of Variance

The real-world dynamics are reflected in Figures 4.9 and 4.10. These report the results of the Analysis of Variance (ANOVA) exercise and show the share of the total variance attributable to the innovations at different lag lengths. These are also obtained using the Wald representation of the system, as described in Cochrane and Piazzesi (2009). They indicate the contribution each innovation would make to the volatility of each model variable if the error process was suddenly started (having been dormant previously). As such, they reflect the variances of the shocks and well as the impulse responses.

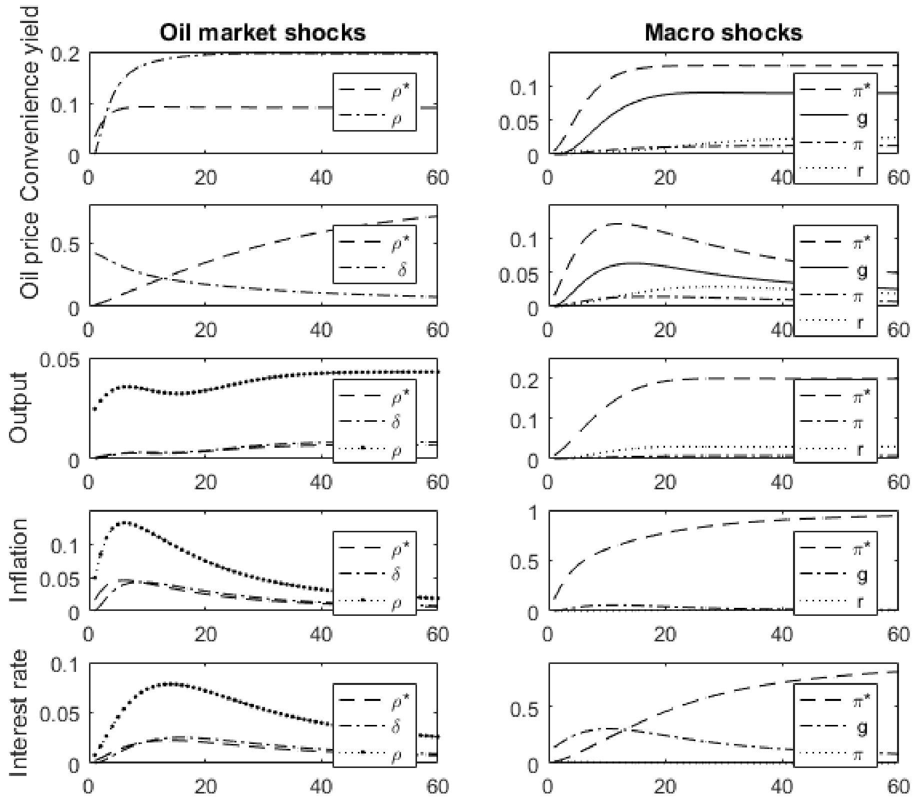
Figure 4.9: Variance Decomposition for the 12 Month Crude Oil Futures



This figure reports the results of the Analysis of Variance (ANOVA) exercise and shows the share of the total variance in the 12 month futures price attributable to the innovations at different lag lengths (see notes to Figure 4.9). These effects are dominated by the effect of shocks to the spot rate (shown in Figure 4.9).

The first column of Figure 4.9 shows the effects of oil market shocks while the second shows the effects of macro shocks. The first two rows show that the variance in the convenience yield and oil price is dominated by oil market shocks. Although the impulse responses show that macro shocks have a significant effect on oil prices, their relatively low variance means that these effects are dwarfed by the effect of the high volatility oil shocks. The variance of the oil price is naturally dominated by its asymptote after 60 months. Nevertheless, macro shocks account for 10% of the variance in the convenience yield after 24 months and about 10% of the variance of the oil price after 12 months. The remaining rows show that oil market shocks have a significant effect on the short run variance of inflation,

Figure 4.10: Variance Decompositions for the State Variables



This figure (and Figure 4.10) report the results of the Analysis of Variance (ANOVA) exercise and show the share of the total variance in state variables (and futures prices) attributable to the innovations at different lag lengths. The inflation asymptote (π^*) is important in explaining variations of all these variables. The underlying oil price (ρ^*), together with the inflation target (π^*) and the convenience yield (δ) explain most of the variation in the real oil price. The final row of this figure shows that variations in the interest rate are strongly determined by changes in the inflation target (π^*), as well as shocks to output and the oil market. The variance of output and inflation are also significantly affected by real oil price shocks. The inflation response is rapid but then tends to fade as monetary policy counteracts the influence of oil shocks. Exogenous shocks to economic activity (g) also contribute to the variation of the real oil price, although most of the variation is explained by the latent variables ρ^*, π^* and δ.

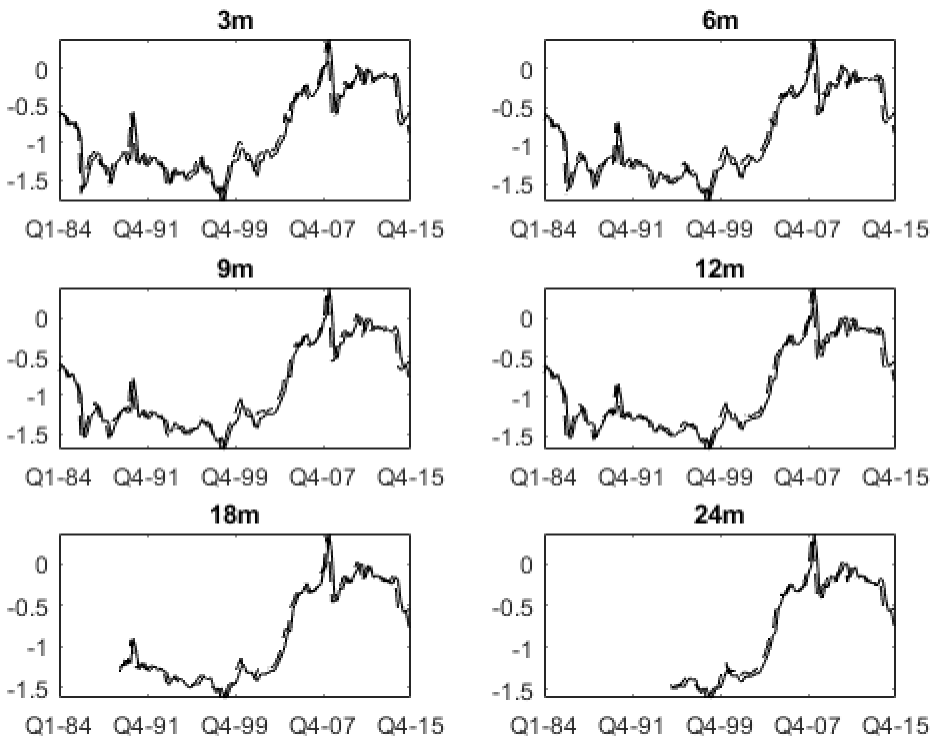
accounting for nearly 10% of the variance after 12 months. The effects of oil shocks on interest rates are much smaller. The longer run variances of inflation and interest rates are naturally dominated by the inflation asymptote.

4.4.4 The behaviour of the futures market

Figure 4.11 shows the estimates of the futures prices alongside their observed values. Table 4.2 shows the root mean squared error of the futures prices. Table

4.4 reports the parameters estimated under the measure Q, and embedded in the affine model of the term structure of futures prices (equation 4.40). Again, these are nicely in line with their priors. They are related to those of the Kalman-VAR by the prices of risk and variance estimates (equation 4.28) shown in Table 4.4. The significance of the parameter estimates under measure Q is generally higher than those of the Kalman-VAR under measure P. Cochrane and Piazzesi (2009) suggest that Q parameters are precise because they come from the cross-section that has "tiny" measurement errors, while the P parameters come from the Kalman-VAR, which has large forecasting errors.

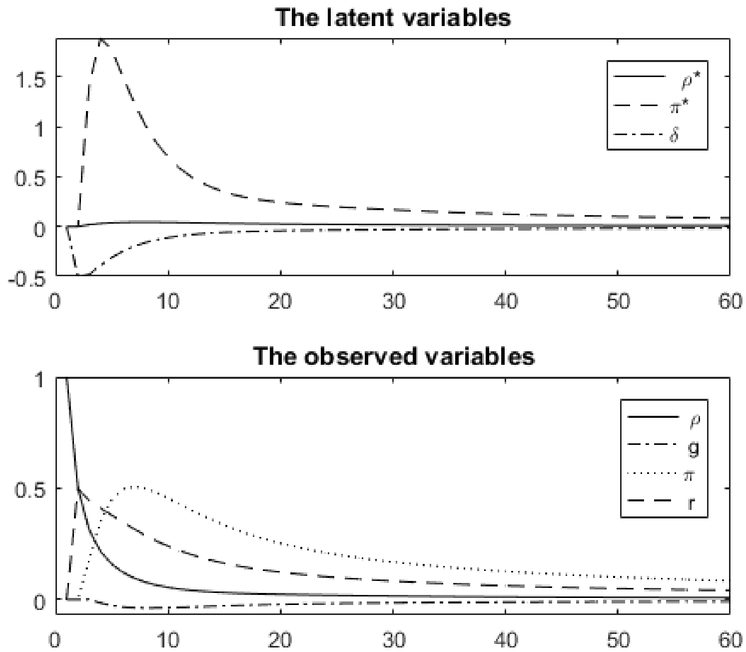
Figure 4.11: Log WTI Crude Oil Contract Estimates



This figure shows the estimates of the futures prices (solid line) alongside their observed values, where these data are available (dashed). The estimates for the missing 18 and 24 month maturity data are extracted from the Kalman filter, with the horizontal lines indicating the first available data point.

The behaviour of the futures curve is dictated by the factor loadings, which

Figure 4.12: Factor Loadings Showing the Effect of the State Variables on the Term Structure of Cost of Carry

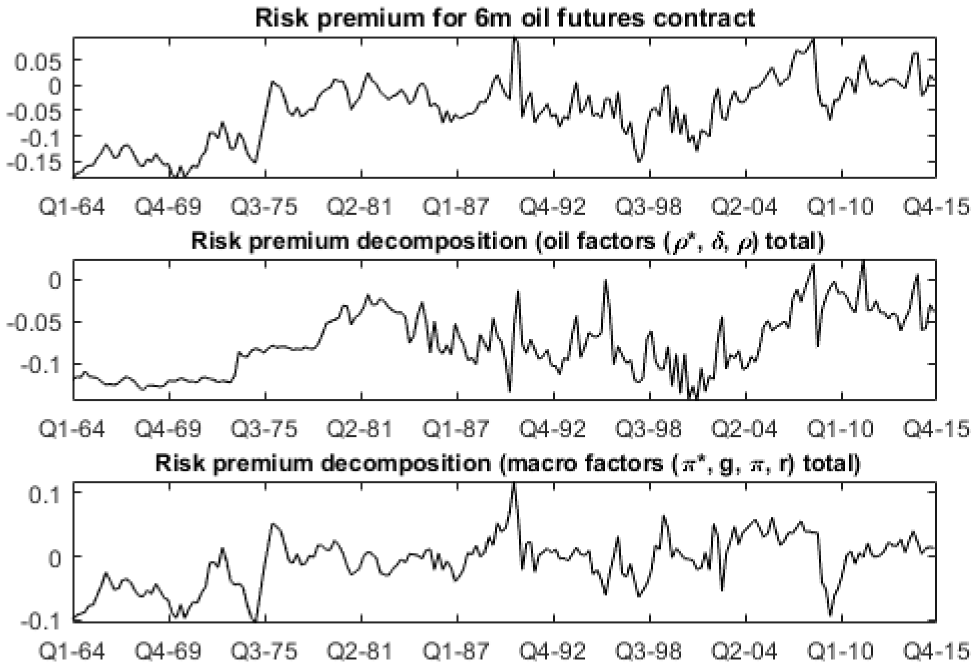


The behaviour of the futures curve is dictated by the factor loadings. This figure shows the loadings of the cost of carry Ψ_τ/τ, (expressed in quarters). These loadings depend upon the parameters of the risk-neutral factor dynamics (Section 4.3.2). The first panel shows the loadings on the three latent variables. The inflation asymptote π^* and δ^* are important influences, but ρ^* does not appear to be relevant. The second panel shows the loadings on the macro variables. The convenience yield δ, oil price ρ and the interest rate r, have a direct impact on the oil market under both measures and have relatively large loadings. The spot oil price has a unit effect at the short end of the futures curve, but its influence fades with maturity. The spot interest rate and convenience yield determine the slope of the short-term yield curve. Since this is a model of the nominal prices, inflation also has some effect, but the effect of the output gap is negligible.

depend in turn upon the parameters of the risk-neutral factor dynamics (Section 4.3.2). Empirically, this system has a single unit root under Q that is closely associated with the underlying inflation rate π^* (which has a unit root under P by assumption) and means that the loadings of the futures on the factors (Ψ_τ) increase with maturity (τ). Dividing these loadings by maturity gives the factor loadings for the annualized cost of carry (Ψ_τ/τ). These loadings are depicted in Figure 4.12, as a function of maturity (expressed in quarters). The first panel shows the loadings on the three latent variables and the second those on the observed variables.

Figure 4.13 presents the model estimate of the risk premium in the 6-month

Figure 4.13: The Risk Premium in Futures Contracts



This figure describes the risk premium in 6-months futures contract. Commodity futures incorporate risk premiums, which are the difference between the real world and risk-neutral expectations of the future spot price (refer to equation (C.14)). This shows the 6-month premium. The model suggests that risk premiums increase with maturity and are strongly cross-correlated. Oil market variables play an important role here, with the underlying price having a strong positive effect on the premium. The correlation between the contribution of the oil factors (ρ^*, ρ, δ) and the 6-month risk premium is 0.8038. However, macro variables are also influential. The rate of inflation has a strong positive effect, which boosted the premium until the Volker disinflation, and also helped explain the fall in the premium in 2009-10. The correlation between the total macro contribution (π^*, g, π, r) and the 6-month risk premium is 0.8257.

futures market. Recall that I only have futures data for the period since 1984, and that the estimates before then depend up inferences about the convenience yield obtained from the Kalman filter. Risk premiums are the difference between the real world and risk-neutral expectations of the future spot price (equation (C.14)). Recall that the futures price is the risk neutral expectation. Producers want to sell futures to hedge against future price falls. When they are dominant, as they have been historically, this depresses the futures price relative to the expected spot rate until arbitrageurs are prepared take the other side of the market. Hamilton and Wu (2014), however, argue that buying pressure from commodity index funds has recently had the opposite effect. The model presented in this chapter suggests that risk premiums increase with maturity and are strongly cross-correlated. The decomposition of the 6-month future into oil and macro factors shows that oil market variables play an important role here, with the underlying price having a strong positive effect on the premium. The correlation between the contribution of the oil factors (ρ^*, ρ, δ) and the 6-month risk premium is 0.8038. Macro variables are also influential, however. The rate of inflation has a strong positive effect, which boosted the premium until the Volker disinflation, and helps explain the fall in the premium in 2009-10. The correlation between the total macro contribution (π^*, π, g, r) and the 6-month risk premium is 0.8257.

4.5 Concluding remarks

This chapter presents a macro-finance model that includes oil prices, the convenience yield and makes the crude oil futures exponential-affine in the state variables. The expectation had been to find significant links between oil prices and the macroeconomy, therefore it was a surprise to find that the convenience yield also plays an important role, acting as a buffer between oil prices and the economy. The model also throws light on the notorious 'price puzzle', indicating the importance of the link between US monetary policy, commodity prices and inflation. The convenience yield also plays an important role here, transmitting monetary signals to the real economy, influencing output and inflation. The macro-finance framework would seem to offer practitioners and academic researchers an important tool for understanding the effects of monetary policy on the commodity markets and the economy.

Chapter 5

Conclusions

This thesis studies commodity futures term structures, and the interaction between commodity market, particularly the crude oil markets, and the real economy, focusing on the dynamic factor model of the term structure and its application to the macro-finance model.

A comprehensive literature review was presented in Chapter 2, and this introduced some studies of the relationship between commodity prices and the macroeconomy, as well as the commodity futures term structures. Nevertheless, the analysis of these two areas seems to be somewhat separated. The traditional view tends to investigate the macroeconomic interaction of the spot commodity price using conventional macro-econometric methods. It is generally silent, however, about the role of the commodity futures market. Meanwhile, the mainstream dynamic term structure model for commodity futures only use unobservable variables. This approach, however, has been the subject of long-standing concerns as to its ability to provide sufficient economic interpretation of the latent factors.

In contrast, the macro-finance model combines the standard macro-econometric methodology and the dynamic term structure model, and has yielded numerous findings, and become a popular topic in the literature. The vast majority of the the macro-finance studies only focus on the interest rate term structure. The discussion of the commodity futures term structure, however, remains extremely limited, disregarding the increasingly influential interaction between the commodity futures market and the real economy. In addition, the preliminary data analysis further highlighted the effect of the unit root that is commonly observed in models of commodity prices, and proposed the vector error correction model (VECM), as the sensible solution to this problem.

These concerns motivate the remainder of this thesis. The goal of Chapter 3 was further to explore the mainstream dynamic term structure model for commodity futures. The chapter presented a joint affine term structure model for four benchmark commodity futures contracts, which extends the discussion of the commodity term structure from each separate commodity market, to multiple commodity markets jointly. In this model, the instantaneous short rate is a pure latent factor that is commonly applied to all selected commodity contracts. In turn, its dynamic is jointly determined by these commodity markets. The empirical evidence that suggests this model performs well in terms of term structure fitting and estimation. The result further indicates that, this unobservable common interest rate factor, which can be interpreted as the "commodity market implied short rate factor", generally tracks the policy rate very well.

I present a macro-finance model for the economy and the oil market in Chapter 4 of this thesis, that allows interactions between the convenience yield, the spot and futures markets, monetary policy and macroeconomic variables to be studied. I use the Kalman filter to represent latent variables to handle the effects of exogenous shocks to inflation and the oil price, and to deal with missing observations. Traditional models use latent variables, with little economic meaning, to explain commodity futures, but this model makes the effect of macroeconomic variables explicit. An important interaction was found between the economy and the oil markets, including an important link in the monetary transmission mechanism, running from the policy interest rate to the convenience yield, oil price and hence inflation and policy transmission.

There are some avenues for future research. Most of the focus of the macro-finance literature is on the interactions between macroeconomic indicators and a stand-alone financial market, such as bond or other financial instrument. Very few study the joint behaviour of several financial markets together with the macroeconomy. Chin and Liu (2015) attempt to build a joint affine model, which combines bond prices and crude oil futures prices with latent, not macro, factors. In the future, I could develop a macro finance model of multiple financial markets.

Another aspect that one could further investigate is that, in this thesis, I focus on the US economy and its WTI crude oil price. I do this for two reasons. First, it would be difficult to develop a global GDP aggregate reflecting global demand for oil. Second, because in many respects, the US energy market has traded independently of world markets, as evidenced by the divergence between US and UK energy prices, for example, WTI and Brent crude oil price. The growth in international oil trade, the relatively recent discovery and exploitation of US shale hydrocarbons, and the removal of the US oil export ban, are arguments for developing a global demand model explaining a price like the Brent crude oil price, which is more representative of the price of oil traded in the world market.

Appendix A

General appendix

A.1 The arbitrage relationship

This section specifies the arbitrage relationships relating to the futures prices to the real and nominal spot price, convenience yield and interest rate. Following Campbell, Lo and Mackinlay (1997), I start with the well-known property of nominal futures prices, $F_{\tau,t}$ is denoted as the futures price at time t with τ units of time to its maturity. They follow a martingale under the risk-neutral measure Q:

$$F_{\tau,t} = E_t^Q(F_{\tau-1,t+1}) \qquad\qquad \tau \geq 1. \tag{A.1}$$

This is essentially because these contracts do not yield dividends or convenience benefits (Cox, Ingersoll and Ross (1981)). The maturity value of the futures price will also equal the future spot price. S_t is denoted as the spot commodity price at time t. So, for the special case of $\tau = 1$: $F_{0,t+1} = S_{t+1}$. Substituting this into (A.1):

$$F_{1,t} = E_t^Q(S_{t+1}). \tag{A.2}$$

The risk-neutral spot oil price dynamics follow by combining this with the standard arbitrage condition for a *forward* price. Importantly, for the special case of $\tau = 1$, the interest rate is known. Denote δ_t and r_t is the convenience yield and interest rate at time t. So this relationship also holds for the futures price: $F_{\tau,t} = S_t e^{(r_t - \delta_s)}$. Taking logs:

$$\ln E_t^Q(S_{t+1}) = s_t + r_t - \delta_t. \tag{A.3}$$

where: $s_t = \ln S_t$. Finally, suppose that S_{t+1} is log normal under Q so that taking logs again:

$$s_{t+1} = s_t + \mu_t + \epsilon^Q_{s,t+1} \qquad \epsilon^Q_{s,t+1} \sim N(0, \sigma_s^2). \tag{A.4}$$

where $\epsilon^Q_{s,t+1}$ is a risk-neutral oil price shock and:

$$E_t^Q(S_{t+1}) = S_t e^{(\mu_t + \frac{1}{2}\sigma_s^2)}. \tag{A.5}$$

Taking logs and substituting into (A.3) gives:

$$\mu_t = r_t - \delta_t - \frac{1}{2}\sigma_s^2. \tag{A.6}$$

Finally, substituting this back into equation (A.4) gives the dynamic equation for the nominal spot price under probability measure Q:

$$s_{t+1} = s_t + r_t - \delta_t - \frac{1}{2}\sigma_s^2 + \epsilon^Q_{s,t+1} \qquad \epsilon^Q_{s,t+1} \sim N(0, \sigma_s^2) \tag{A.7}$$

The macro model works with the real oil price ρ_{t+1}, which is the nominal price s_{t+1} less the log price level p_{t+1}. This follows a real arbitrage relationship derived by adjusting (A.7) for inflation by subtracting $p_{t+1} = p_t - \pi_{t+1}$ from both sides:

$$\rho_{t+1} = \rho_t - \pi_{t+1} + r_t - \delta_t - \frac{1}{2}\sigma_s^2 + \epsilon^Q_{s,t+1}. \tag{A.8}$$

where $\pi_{t+1} = p_{t+1} - p_t$ is inflation: the first difference of the log price level. Strictly speaking, this is the monthly change, which is difficult to model, so I follow mainstream macro-finance studies in approximating this by a twelfth of the annual rate. This introduces an additional error term into (A.8), but this is small relative to the volatility of the real oil price.

A.2 The ADACX cluster for computational intensive tasks

The ADACX cluster in the Alcuin Research Resource Centre (ARRC) at the University of York (United Kingdom) is used to compute model iteration in multiple experiments in parallel for empirical exercises in this thesis.

The ADACX cluster is a facility for computationally intensive tasks. It is provided within Research Centre for Social Sciences (RCSS) by a cluster of 25 application servers, each having 2 oct-core Intel E5-2690 processors (2.9GHz, 16 physical processor cores per server, 32 including hyperthreading). Each of the servers has 192GB 1600MHz RAM and a 10Gbit network connection to the local file server. All are running 64 bit Windows Server 2008 R2 Enterprise and Citrix XenApp 6.

Appendix B

Appendices to Chapter 3

B.1 The joint transition equation under the measure Q

The stacked *joint* transition equation under the measure Q is:

$$X_{t+1} = K^Q + \Phi^Q X_t + W_t^Q \qquad W_t^Q \sim N(0, R) \tag{B.1}$$

where X_t is a 9×T the state vector, which first enumerates specific factors for each different commodity class, and then includes interest rate as the common factor in the end. Specifically:

$$X_t = \begin{pmatrix} s_{o,t} & \delta_{o,t} & s_{c,t} & \delta_{c,t} & s_{g,t} & \delta_{g,t} & s_{n,t} & \delta_{n,t} & r_t \end{pmatrix}^\top,$$

Setting $\kappa_{s_i}^Q = -\frac{1}{2}\sigma_{s_i}^2$. In the meantime, applying Joslin, Singleton and Zhu (2011) as discussed in equation (3.4). K^Q can be parsimoniously defined as the following:

$$K^Q = \begin{pmatrix} k_{o,s}^Q & 0 & k_{c,s}^Q & 0 & k_{g,s}^Q & 0 & k_{n,s}^Q & 0 & 0 \end{pmatrix}^\top,$$

To accommodates specifications of state dynamics under the measure Q

suggested by equation (3.1), (3.2), and (3.3) across commodity classes. We have the following stack form for Φ^Q:

$$\Phi^Q = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \phi^Q_{\delta_o s_o} & \phi^Q_{\delta_o} & 0 & 0 & 0 & 0 & 0 & 0 & \phi^Q_{\delta_o r} \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \phi^Q_{\delta_c s_c} & \phi^Q_{\delta_c} & 0 & 0 & 0 & 0 & \phi^Q_{\delta_c r} \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \phi^Q_{\delta_g s_g} & \phi^Q_{\delta_g} & 0 & 0 & \phi^Q_{\delta_g r} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \phi^Q_{\delta_n s_n} & \phi^Q_{\delta_n} & \phi^Q_{\delta_n r} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \phi^Q_r \end{pmatrix}$$

where Φ^Q allows spot price and convenience yield to be specific factors to each individual commodity class, and interest rate to be a common factor which affects all in the same manner across commodity classes.

B.2 The joint state space representation under the measure P

The state space representation was specified based on equation (3.9) and equation (3.4) under the measure P is:

$$f_t = D + HX_t + e_t \qquad e_t \sim N(0, Q) \tag{B.2}$$

$$X_t = K + \Phi X_{t-1} + W_t \qquad W_t \sim N(0, R) \tag{B.3}$$

f_t in the measurement equation (B.2) is a N×1 vector, it aligns sample observations of the four different commodity futures term structure, as:

$$f_t = \begin{pmatrix} f_{o,1,t} & \dots & f_{o,\tau,t}, & f_{c,1,t} & \dots & f_{c,\tau,t}, & f_{g,1,t} & \dots & f_{g,\tau,t}, & f_{n,1,t} & \dots & f_{n,\tau,t} \end{pmatrix}^\top,$$

H is a N×9 matrix, accommodating recursive solution for Ψ_τ as in equation (3.18) for different commodity classes. It has non-zero values only for the corresponding elements, mapping state variables specific to each commodity classes.

$$H = \begin{pmatrix} \psi_{o,s,1} & \psi_{o,\delta,1} & 0 & 0 & 0 & 0 & 0 & 0 & \psi_{o,r,1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \psi_{o,s,\tau} & \psi_{o,\delta,\tau} & 0 & 0 & 0 & 0 & 0 & 0 & \psi_{o,r,\tau} \\ 0 & 0 & \psi_{c,s,1} & \psi_{c,\delta,1} & 0 & 0 & 0 & 0 & \psi_{c,r,1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \psi_{c,s,\tau} & \psi_{c,\delta,\tau} & 0 & 0 & 0 & 0 & \psi_{c,r,\tau} \\ 0 & 0 & 0 & 0 & \psi_{g,s,1} & \psi_{g,\delta,1} & 0 & 0 & \psi_{g,r,1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \psi_{g,s,\tau} & \psi_{g,\delta,\tau} & 0 & 0 & \psi_{g,r,\tau} \\ 0 & 0 & 0 & 0 & 0 & 0 & \psi_{n,s,1} & \psi_{n,\delta,1} & \psi_{n,r,1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \psi_{n,s,\tau} & \psi_{n,\delta,\tau} & \psi_{n,r,\tau} \end{pmatrix}$$

Since specific factors, such as the spot price and convenience yield of each commodity class, have no impact to others under the risk-neutral measure. Therefore, this joint affine setting shows that specific commodity factors for one commodity class are unspanned factors for other commodity classes.

Q is a $N \times N$ diagonal matrix that accommodates measurement errors of the observed sample time series on the left hand side of the measurement equation.

$$Q = diag\left(\begin{array}{cccccccccccc} q^2_{o,1} & \dots & q^2_{o,\tau}, & q^2_{c,1} & \dots & q^2_{c,\tau}, & q^2_{g,1} & \dots & q^2_{g,\tau}, & q^2_{n,1} & \dots & q^2_{n,\tau}, \end{array} \right),$$

The state vector X_t the same as in the state dynamic (B.1). K is defined as the following vector:

$$K = \left(\begin{array}{ccccccccc} k_{o,s} & k_{o,\delta} & k_{c,s} & k_{c,\delta} & k_{g,s} & k_{g,\delta} & k_{n,s} & k_{n,\delta} & k_r \end{array} \right)^\top,$$

Corresponding to specification of Φ^Q in equation (3.19), Φ of the benchmark model, as displayed in equation (3.22) is specified as:

$$\Phi = \begin{pmatrix} \phi_{s_o} & \phi_{s_o\delta_o} & 0 & 0 & 0 & 0 & 0 & 0 & \phi_{s_o,r} \\ \phi_{\delta_o s_o} & \phi_{\delta_o} & 0 & 0 & 0 & 0 & 0 & 0 & \phi_{\delta_o,r} \\ 0 & 0 & \phi_{s_c} & \phi_{s_c\delta_c} & 0 & 0 & 0 & 0 & \phi_{s_c,r} \\ 0 & 0 & \phi^Q_{\delta_c s_c} & \phi_{\delta_c} & 0 & 0 & 0 & 0 & \phi_{\delta_c,r} \\ 0 & 0 & 0 & 0 & \phi_{s_g} & \phi_{s_g\delta_g} & 0 & 0 & \phi_{s_g,r} \\ 0 & 0 & 0 & 0 & \phi^Q_{\delta_g s_g} & \phi_{\delta_g} & 0 & 0 & \phi_{\delta_g,r} \\ 0 & 0 & 0 & 0 & 0 & 0 & \phi_{s_n} & \phi_{s_n\delta_n} & \phi_{s_n,r} \\ 0 & 0 & 0 & 0 & 0 & 0 & \phi^Q_{\delta_n s_n} & \phi_{\delta_n} & \phi_{\delta_n,r} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \phi_r \end{pmatrix},$$

B.3 The Kalman filter and the log-likelihood function with latent variables

We start with the state space representation based on equation (3.9) and equation (3.4) under the measure P as:

$$f_t = D + HX_t + e_t \qquad e_t \sim N(0, Q) \tag{B.4}$$

$$X_t = K + \Phi X_{t-1} + W_t \qquad W_t \sim N(0, R) \tag{B.5}$$

where all vectors and matrices in equation (B.4) and (B.5) are extensively specified in Appendix B.2

As implied by Harvey (1989), Hamilton (1994) , Tsay (2012) and others. The short hand notation is used to denote the expectation of X at time $t+1$ conditional on existing information $\mathcal{F}$ at time t as:

$$\hat{X}_{t+1|t} = E(X_{t+1}|\mathcal{F}_t) \tag{B.6}$$

where $\mathcal{F}_t = (f_t, f_{t-1}, f_{t-2}, \ldots, f_T, X_t, X_{t-1}, X_{t-2}, \ldots, X_T)$ stands for the filtration at time t, representing relevant existing information at the time

Following state space representation as equation (B.4) and (B.5). The following conditional expectations of the state variables and observed variables are specified:

$$\hat{X}_{t+1|t} = K + \Phi X_{t|t} \tag{B.7}$$

$$\hat{f}_{t+1|t} = D + H\hat{X}_{t+1|t} \tag{B.8}$$

Let $\hat{P}_{t+1|t}$ and $\hat{F}_{t+1|t}$ be the one step ahead covariance matrix of forecast error of $\hat{X}_{t+1|t}$ and $\hat{f}_{t+1|t}$, conditional on past information. Their expressions can be specified as following:

$$\hat{P}_{t+1|t} = E(X_{t+1} - \hat{X}_{t+1|t})(X_{t+1} - \hat{X}_{t+1|t})' = \Phi \hat{P}_{t|t} \Phi + R \tag{B.9}$$

$$\hat{F}_{t+1|t} = E(f_{t+1} - \hat{f}_{t+1|t})(f_{t+1} - \hat{f}_{t+1|t})' = H \hat{P}_{t+1|t} H + Q \tag{B.10}$$

Denoting $X_{t|t}$ in equation (B.7) as the expectation of X at time t conditional on information $\mathcal{F}$ at time t, which can be written as:

$$\begin{aligned}
\hat{X}_{t|t} &= E_t(X_t|\mathcal{F}_t) = E_t(X_t|\mathcal{F}_{t-1}, f_t) \\
&= E_t(X_t|\mathcal{F}_{t-1}) + Cov(X_t, f_t)(Var(f_t - \hat{f}_{t|t-1}))^{-1}(f_t - \hat{f}_{t|t-1}) \\
&= \hat{X}_{t|t-1} + \hat{P}_{t|t-1} H^{'} (H \hat{P}_{t|t-1} H + Q)^{-1} (f_t - \hat{f}_{t|t-1})
\end{aligned} \tag{B.11}$$

where the covariance of the forecast error between X_t and f_t can be derived as:

$$\begin{aligned}
Cov(X_t, f_t) &= E((X_t - \hat{X}_{t|t-1})(f_t - \hat{f}_{t|t-1})) \\
&= E((X_t - \hat{X}_{t|t-1})(X_t - \hat{X}_{t|t-1})^{'} H) = \hat{P}_{t|t-1} H^{'}
\end{aligned} \tag{B.12}$$

Finally, let $\hat{P}_{t|t}$ be the the covariance matrix of forecast error of $\hat{X}_{t|t}$ in equation (B.9), to give the following expression:

$$\begin{aligned}
\hat{P}_{t|t} &= Var(X_t|\mathcal{F}_t) = Var(X_t|\mathcal{F}_{t-1}, f_t) \\
&= Var(X_t|\mathcal{F}_{t-1}) - Cov(X_t, f_t)(Var(f_t - \hat{f}_{t|t-1})^{-1}) Cov(X_t, f_t)' \\
&= \hat{P}_{t|t-1} - \hat{P}_{t|t-1} H^{'} (H \hat{P}_{t|t-1} H + Q)^{-1} H \hat{P}_{t|t-1}
\end{aligned} \tag{B.13}$$

One step ahead estimation, such as $\hat{X}_{t+1|t}$, $\hat{f}_{t+1|t}$, $\hat{P}_{t+1|t}$, $\hat{F}_{t+1|t}$, is specified, as in the " predicting equations", and $\hat{X}_{t|t}$, $\hat{P}_{t|t}$ as in the " updating equations" processes in order to satisfy the recursive process:

Prediction update	Correction update
$\hat{X}_{t+1\|t} = K + \Phi \hat{X}_{t\|t}$	$\hat{P}_{t\|t} = \hat{P}_{t\|t-1} - \hat{P}_{t\|t-1} H' (H\hat{P}_{t\|t-1}H + Q)^{-1} H\hat{P}_{t\|t-1}$
$\hat{f}_{t+1\|t} = D + H\hat{X}_{t+1\|t}$	$\hat{X}_{t\|t} = \hat{X}_{t\|t-1} + \hat{P}_{t\|t-1} H' (H\hat{P}_{t\|t-1}H + Q)^{-1} (f_t - \hat{f}_{t\|t-1})$
$\hat{P}_{t+1\|t} = \Phi \hat{P}_{t\|t} \Phi + R$	
$\hat{F}_{t+1\|t} = H\hat{P}_{t+1\|t} H + Q$	

The predicting equations projects the current state estimate ahead in time and the updating equations adjusts the projected estimates by the actual observation at that time.

Equation $(B.8)$ and $(B.10)$ imply that, observed data f_t conditional on the parameter set $\theta = (K, \Phi, K^Q, \Phi^Q, R, Q)$ follows asymptotic distribution:

$$f_t | \theta, \mathcal{F}_{t-1} \sim N(D + H\hat{X}_{t|t-1}, H\hat{P}_{t|t-1}H' + Q) \tag{B.14}$$

such that the likelihood function is defined as:

$$\begin{aligned} f(f_t|\theta, \mathcal{F}_{t-1}) =& (2\pi)^{-n/2} |H\hat{P}_{t|t-1}H' + Q|^{-1/2} \\ & \exp\left(\frac{1}{2}(f_t - D - H\hat{X}_{t|t-1})'(H\hat{P}_{t|t-1}H' + Q)^{-1}(f_t - D - H\hat{X}_{t|t-1})\right) \end{aligned} \tag{B.15}$$

and the joint log-likelihood function with respect to the parameter set θ is:

$$\begin{aligned} \ln \mathcal{L}(\theta|\mathcal{F}_{t-1}) =& \sum_{t=1}^{T} \ln f(f_t|\theta, \mathcal{F}_{t-1}) \\ =& -\frac{nT}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{T} \ln(|H\hat{P}_{t|t-1}H' + Q|) \\ & - \frac{1}{2}\sum_{t=1}^{T}\left((f_t - D - H\hat{X}_{t|t-1})'(H\hat{P}_{t|t-1}H' + Q)^{-1}(f_t - D - H\hat{X}_{t|t-1})\right) \end{aligned} \tag{B.16}$$

B.4 Parameter estimation

The expected log-likelihood conditional on the data and the estimates is computed from the previous iteration (equation (B.16)) using the Kalman filter, as Appendix B.3 describes, with initial parameters of the model. The parameters are then updated, maximizing the expected log-likelihood with respect to θ:

$$\hat{\theta} = arg \max_{\theta} \ln \mathcal{L}(\theta, \mathcal{F}_{t-1}) \tag{B.17}$$

where $\hat{\theta}$ is the estimator for the parameter value that makes the observed data most probable.

The starting values of parameters are initialized in K^Q, Φ^Q, K, Φ, and the diagonal elements in Σ with parameters estimates from separate models for each individual commodity classes. ϕ_r^Q and ϕ_r in matrix Φ^Q and Φ are excluded and randomized, however, and it is assumed that they are uniformly distributed between 0 and 1. For all the rest of the parameters in Q and Σ, their starting values are assumed to be uniformly distributed between -0.001 and 0.001.

I randomize 300 sets of initial values of parameters, and each separately run each of them until the model is converged. I choose the directory with the highest log-likelihood was chosen at convergence to present in this thesis. The log-likelihood of the model reported in this thesis starts from 652313 and converges at $(-)$33589. This is converged using both the "fminsearch" and "fminunc" optimizer in Matlab. The "fminsearch" optimizer uses the Nelder-Mead simplex algorithm, as described in Lagarias et al (1998). The "fminnuc" optimizer uses the quasi-Newton algorithm, as described in Kelley (1999).

B.5 Confidence bounds of variable estimation

Confidence bands are calculated for the commodity market implied short rate factor as in Figure 3.5. Following the derivation and corresponding notations of the full state-space representation and Kalman Filter in Appendix B.3, the expectation of r at time t conditional on existing information set: $\mathcal{F}_t = (f_t, f_{t-1}, f_{t-2}, \ldots, f_1)$ at time t is:

$$\hat{r}_{t|t} = E(r_t|\mathcal{F}_t)$$

denote $\hat{P}_{rr,t|t}$ as the associate mean square error of $\hat{r}_{t|t}$, it is defined as:

$$\hat{P}_{rr,t|t} = E(r_t - \hat{r}_{t|t})(r_t - \hat{r}_{t|t})'$$

The $(1-\alpha)$ confidence interval is therefore constructed for the variable $\hat{r}_{t|t}$ as:

$$CI(\hat{r}_{t|t})_{1-\alpha} = (\hat{r}_{t|t} - \Phi^{-1}(1-\frac{\alpha}{2})\sqrt{\hat{P}_{rr,t|t}}, \hat{r}_{t|t} + \Phi^{-1}(1-\frac{\alpha}{2})\sqrt{\hat{P}_{rr,t|t}})$$

where $CI(\hat{r}_{t|t})_{1-\alpha}$ denotes the confidence interval of $\hat{r}_{t|t}$ at $(1-\alpha)$ significance level. $\Phi^{-1}(1-\frac{\alpha}{2})$ denotes the $(1-\frac{\alpha}{2})$ quantile of the standard normal distribution.

Appendix C

Appendices to Chapter 4

C.1 The transition equations

The dynamics of the observations $n_t = (\rho_t, g_t, \pi_t, r_t)'$ under P are described by equation (4.10) where:

$$A_n = \begin{pmatrix} a_\rho \\ a_g \\ a_\pi \\ a_r \end{pmatrix} \qquad \Theta_{n,z} = \begin{pmatrix} \theta_{\rho,\rho^*} & \theta_{\rho,\pi^*} & \phi_{\rho,\delta} \\ \theta_{g,\rho^*} & \theta_{g,\pi^*} & 0 \\ \theta_{\pi,\rho^*} & \theta_{\pi,\pi^*} & 0 \\ \theta_{r,\rho^*} & \theta_{r,\pi^*} & 0 \end{pmatrix} \qquad \Phi_{n,n} = \begin{pmatrix} \phi_{\rho,\rho} & \phi_{\rho,g} & \phi_{\rho,\pi} & \phi_{\rho,r} \\ \phi_{g,\rho} & \phi_{g,g} & \phi_{g,\pi} & \phi_{g,r} \\ \phi_{\pi,\rho} & \phi_{\pi,g} & \phi_{\pi,\pi} & \phi_{\pi,r} \\ \phi_{r,\rho} & \phi_{r,g} & \phi_{r,\pi} & \phi_{r,r} \end{pmatrix}$$

and where the error term $\epsilon_{n,t}$ can be decomposed using (4.11) where:

$$L_n = \begin{pmatrix} 1 & 0 & 0 & 0 \\ c_{g,\rho} & 1 & 0 & 0 \\ c_{\pi,\rho} & c_{\pi,g} & 1 & 0 \\ c_{r,\rho} & c_{r,g} & c_{r,\pi} & 1 \end{pmatrix} \qquad D_n = \begin{pmatrix} d_\rho & 0 & 0 & 0 \\ 0 & d_g & 0 & 0 \\ 0 & 0 & d_\pi & 0 \\ 0 & 0 & 0 & d_r \end{pmatrix} \qquad u_{n,t} = \begin{pmatrix} u_{\rho,t} \\ u_{g,t} \\ u_{\pi,t} \\ u_{r,t} \end{pmatrix}$$

The dynamics of the latent vector, $z_t = (\rho_t^*, \pi_t^*, \delta_t)'$ are described by equation

(4.19), where:

$$K_z = \begin{pmatrix} \kappa_{\rho^*} \\ \kappa_{\pi^*} \\ a_\delta + \theta_{\delta,\rho^*}\kappa_{\rho^*} + \theta_{\delta,\pi}\kappa_{\pi^*} \end{pmatrix} \quad \Upsilon_{z,z} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \theta_{\delta,\rho^*} & \theta_{\delta,\pi^*} & \phi_{\delta,\delta} \end{pmatrix} \quad \Upsilon_{z,n} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \phi_{\delta,\rho} & 0 & 0 & \phi_{\delta,r} \end{pmatrix}$$

and where the error term $\eta_{z,t}$ can be decomposed using a similar LDL factorization, where:

$$L_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \theta_{\delta,\rho^*} & \theta_{\delta,\pi^*} & 1 \end{pmatrix} \quad D_z = \begin{pmatrix} d_{\rho^*} & 0 & 0 \\ 0 & d_{\pi^*} & 0 \\ 0 & 0 & d_\delta \end{pmatrix} \quad u_{z,t} = \begin{pmatrix} u_{\rho^*,t} \\ u_{\pi^*,t} \\ u_{\delta,t} \end{pmatrix}$$

Recall that equation (4.11) and (4.19) are substituted into (4.10) to introduce a lag to the latent vector z_t. This yields equation (4.20), where:

$$K_n = \begin{pmatrix} a_\rho + \theta_{\rho,\rho^*}\kappa_{\rho^*} + \theta_{\rho,\pi}\kappa_{\pi^*} + (a_\delta + \theta_{\delta,\rho^*}\kappa_{\rho^*} + \theta_{\delta,\pi}\kappa_{\pi^*})\phi_{\rho,\delta} \\ a_g + \theta_{g,\rho^*}\kappa_{\rho^*} + \theta_{g,\pi}\kappa_{\pi^*} \\ a_\pi + \theta_{\pi,\rho^*}\kappa_{\rho^*} + \theta_{\pi,\pi}\kappa_{\pi^*} \\ a_r + \theta_{r,\rho^*}\kappa_{\rho^*} + \theta_{r,\pi}\kappa_{\pi^*} \end{pmatrix}$$

$$\Upsilon_{n,z} = \begin{pmatrix} \theta_{\rho,\rho^*} + \theta_{\delta,\rho^*}\phi_{\rho,\delta} & \theta_{\rho,\pi^*} + \theta_{\delta,\pi^*}\phi_{\rho,\delta} & \phi_{\rho,\delta}\phi_{\delta,\delta} \\ \theta_{g,\rho^*} & \theta_{g,\pi^*} & 0 \\ \theta_{\pi,\rho^*} & \theta_{\pi,\pi^*} & 0 \\ \theta_{r,\rho^*} & \theta_{r,\pi^*} & 0 \end{pmatrix}$$

$$\Upsilon_{n,n} = \begin{pmatrix} \phi_{\rho,\rho} + \phi_{\delta,\rho}\phi_{\rho,\delta} & \phi_{\rho,g} & \phi_{\rho,\pi} & \phi_{\rho,r} + \phi_{\delta,r}\phi_{\rho,\delta} \\ \phi_{g,\rho} & \phi_{g,g} & \phi_{g,\pi} & \phi_{g,r} \\ \phi_{\pi,\rho} & \phi_{\pi,g} & \phi_{\pi,\pi} & \phi_{\pi,r} \\ \phi_{r,\rho} & \phi_{r,g} & \phi_{r,\pi} & \phi_{r,r} \end{pmatrix}$$

Finally, stacking equation (4.19) and (4.20) gives the transition equation (4.22), defined under the measure P, where:

$$A = \begin{pmatrix} K_z \\ K_n \end{pmatrix} = \begin{pmatrix} \kappa_{\rho^*} \\ \kappa_{\pi^*} \\ \kappa_\delta \\ \kappa_\rho \\ \kappa_g \\ \kappa_\pi \\ \kappa_r \end{pmatrix} = \begin{pmatrix} \kappa_{\rho^*} \\ \kappa_{\pi^*} \\ a_\delta + \theta_{\delta,\rho^*}\kappa_{\rho^*} + \theta_{\delta,\pi}\kappa_{\pi^*} \\ a_\rho + \theta_{\rho,\rho^*}\kappa_{\rho^*} + \theta_{\rho,\pi}\kappa_{\pi^*} + (a_\delta + \theta_{\delta,\rho^*}\kappa_{\rho^*} + \theta_{\delta,\pi}\kappa_{\pi^*})\phi_{\rho,\delta} \\ a_g + \theta_{g,\rho^*}\kappa_{\rho^*} + \theta_{g,\pi}\kappa_{\pi^*} \\ a_\pi + \theta_{\pi,\rho^*}\kappa_{\rho^*} + \theta_{\pi,\pi}\kappa_{\pi^*} \\ a_r + \theta_{r,\rho^*}\kappa_{\rho^*} + \theta_{r,\pi}\kappa_{\pi^*} \end{pmatrix}$$

$$B = \begin{pmatrix} \Upsilon_{z,z} & \Upsilon_{z,n} \\ \Upsilon_{n,z} & \Upsilon_{n,n} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \theta_{\delta,\rho^*} & \theta_{\delta,\pi^*} & \phi_{\delta,\delta} & \phi_{\delta,\rho} & 0 & 0 & \phi_{\delta,r} \\ \theta_{\rho,\rho^*} + \theta_{\delta,\rho^*}\phi_{\rho,\delta} & \theta_{\rho,\pi^*} + \theta_{\delta,\pi^*}\phi_{\rho,\delta} & \phi_{\rho,\delta}\phi_{\delta,\delta} & \phi_{\rho,\rho} + \phi_{\delta,\rho}\phi_{\rho,\delta} & \phi_{\rho,g} & \phi_{\rho,\pi} & \phi_{\rho,r} + \phi_{\delta,r}\phi_{\rho,\delta} \\ \theta_{g,\rho^*} & \theta_{g,\pi^*} & 0 & \phi_{g,\rho} & \phi_{g,g} & \phi_{g,\pi} & \phi_{g,r} \\ \theta_{\pi,\rho^*} & \theta_{\pi,\pi^*} & 0 & \phi_{\pi,\rho} & \phi_{\pi,g} & \phi_{\pi,\pi} & \phi_{\pi,r} \\ \theta_{r,\rho^*} & \theta_{r,\pi^*} & 0 & \phi_{r,\rho} & \phi_{r,g} & \phi_{r,\pi} & \phi_{r,r} \end{pmatrix}$$

$$L = \begin{pmatrix} L_z & 0_{3,4} \\ \Upsilon_{n,z} & L_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \theta_{\delta,\rho^*} & \theta_{\delta,\pi^*} & 1 & 0 & 0 & 0 & 0 \\ \theta_{\rho,\rho^*} & \theta_{\rho,\pi^*} & c_{\rho,\delta} & 1 & 0 & 0 & 0 \\ \theta_{g,\rho^*} & \theta_{g,\pi^*} & 0 & c_{g,\rho} & 1 & 0 & 0 \\ \theta_{\pi,\rho^*} & \theta_{\pi,\pi^*} & 0 & c_{\pi,\rho} & c_{\pi,g} & 1 & 0 \\ \theta_{r,\rho^*} & \theta_{r,\pi^*} & 0 & c_{r,\rho} & c_{r,g} & c_{r,\pi} & 1 \end{pmatrix}.$$

$$D = \begin{pmatrix} D_z & 0_{3,4} \\ 0_{4,3} & D_n \end{pmatrix} = \begin{pmatrix} d_{\rho^*} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & d_{\pi^*} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d_\delta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d_g & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_\pi & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & d_r \end{pmatrix} \qquad u_t = \begin{pmatrix} u_{z,t} \\ u_{n,t} \end{pmatrix} = \begin{pmatrix} u_{\rho^*,t} \\ u_{\pi^*,t} \\ u_{\delta,t} \\ u_{s,t} \\ u_{g,t} \\ u_{\pi,t} \\ u_{r,t} \end{pmatrix}$$

C.2 The risk-neutral dynamics

The dynamics under measure Q are given by equation (4.24), where:

$$A^Q = \begin{pmatrix} k^Q_{\rho^*} \\ k^Q_{\pi^*} \\ k^Q_{\delta} \\ k^Q_{\rho} \\ k^Q_{g} \\ k^Q_{\pi} \\ k^Q_{r} \end{pmatrix} \qquad B^Q = \begin{pmatrix} \phi^Q_{\rho^*,\rho^*} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \phi^Q_{\pi^*,\pi^*} & 0 & 0 & 0 & 0 & 0 \\ \phi^Q_{\delta,\rho^*} & \phi^Q_{\delta,\pi^*} & \phi^Q_{\delta,\delta} & \phi^Q_{\delta,\rho} & 0 & 0 & \phi^Q_{\delta,r} \\ -\phi^Q_{\pi,\rho^*} & -\phi^Q_{\pi,\pi^*} & -1 & 1-\phi^Q_{\pi,\rho} & -\phi^Q_{\pi,g} & -\phi^Q_{\pi,\pi} & 1-\phi^Q_{\pi,r} \\ \phi^Q_{g,\rho^*} & \phi^Q_{g,\pi^*} & 0 & \phi^Q_{g,\rho} & \phi^Q_{g,g} & \phi^Q_{g,\pi} & \phi^Q_{g,r} \\ \phi^Q_{\pi,\rho^*} & \phi^Q_{\pi,\pi^*} & 0 & \phi^Q_{\pi,\rho} & \phi^Q_{\pi,g} & \phi^Q_{\pi,\pi} & \phi^Q_{\pi,r} \\ \phi^Q_{r,\rho^*} & \phi^Q_{r,\pi^*} & 0 & \phi^Q_{r,\rho} & \phi^Q_{r,g} & \phi^Q_{r,\pi} & \phi^Q_{r,r} \end{pmatrix}$$

C.3 The change of probability measure

The analogue of equation (4.19) under measure Q is:

$$z_t = K^Q_z + \Upsilon^Q_{z,z} z_{t-1} + \Upsilon^Q_{z,n} n_{t-1} + u^Q_{z,t} \tag{C.1}$$

Following Duffee (2002), the essential affine specification, implies:

$$u^Q_{z,t} = u_{z,t} + L_z D_z \Lambda_{z,t-1}, \tag{C.2}$$

where $\Lambda_{z,t}$ is affine in the state variables:

$$\Lambda_{z,t} = D_z \Lambda_{1,z} + D_z^{-1} \Lambda_{2,z} z_t + D_z^{-1} \Lambda_{2,z,n} n_t, \tag{C.3}$$

and where the price of risk parameters are defined in the main text. It is assumed that $\Lambda_{2,z,n} = 0_{3,4}$, in order to preserve the mean-independent dynamics of z_t and keep the system recursive. Substituting (C.2) and (C.3) into (C.1) and comparing this with (4.19) gives:

$$K_z^Q = K_z - L_z D_z D_z' \Lambda_{1,z} \tag{C.4}$$

$$\Upsilon_{z,z}^Q = \Upsilon_{z,z} - L_{z,z}\Lambda_{2,z}, \tag{C.5}$$

Similarly, the analogue of (4.20) under measure Q is:

$$n_t = K_n^Q + \Upsilon_{n,z}^Q z_{t-1} + \Upsilon_{n,n}^Q n_{t-1} + \eta_{n,t}^Q, \tag{C.6}$$

Defining:

$$\eta_{n,t}^Q = \eta_{n,t} + L_n D_n \Lambda_{n,t-1}$$

$$\Lambda_{n,t} = D_n \Lambda_{1,n} + D_n^{-1}\Lambda_{2,n} n_t + D_n^{-1}\Lambda_{2,n,z} z_t$$

and substituting these into (C.6) gives:

$$K_n^Q = K_n - L_n D_n D_n' \Lambda_{1,n} \tag{C.7}$$

$$\Upsilon_{n,n}^Q = \Upsilon_{n,n} - L_n \Lambda_{2,n} \tag{C.8}$$

$$\Upsilon_{n,z}^Q = \Upsilon_{n,z} - \Psi_{n,z}\Lambda_{2,n,z} \tag{C.9}$$

Stacking these relationships gives equation (4.26) and (4.27) of the main text:

$$A^Q = A - LDD'\Lambda_1 \tag{C.10}$$

$$B^Q = B - L\Lambda_2 \tag{C.11}$$

Another way to summarize the implication of these results is to calculate the risk premium of a $\tau-$period contract, denoted as $\varrho_{\tau,t}$. This is the difference between the log expected price of contract in the next period under the measure P, and its log price at time t. In this case, since I model $h_{\tau,t}$, let $H_{\tau,t} = exp(h_{\tau,t})$, we have:

$$\varrho_{\tau,t} = lnE^P(H_{\tau-1,t+1}) - ln(H_{\tau,t}) \tag{C.12}$$

The martingale property of nominal futures prices, as represented by equation (A.1), also applies to $H_{\tau,t}$, namely:

$$H_{\tau,t} = E_t^Q(H_{\tau-1,t+1}) \qquad \tau \geq 1. \tag{C.13}$$

Substituting equation (C.13) into (C.12) yields:

$$\varrho_{\tau,t} = lnE^P(H_{\tau-1,t+1}) - lnE_t^Q(H_{\tau-1,t+1})$$

This means that the $\tau-$period risk premium can be identified from differences between the expectation of the log futures price in the real world and risk-neutral measures. Assuming $H_{\tau,t}$ is log normally distributed, I have:

$$\begin{aligned}\varrho_{\tau,t} =&(E^P(h_{\tau-1,t+1}) + \frac{1}{2}Var(h_{\tau-1,t+1})) - (E^Q(h_{\tau-1,t+1}) + \frac{1}{2}Var(h_{\tau-1,t+1}))\\ =&E^P(h_{\tau-1,t+1}) - E^Q(h_{\tau-1,t+1}) \end{aligned} \tag{C.14}$$

Since $h_{\tau,t} = \alpha_\tau + \Psi'_\tau X_t$ as shown by equation (A.1):

$$\begin{aligned}\varrho_{\tau,t} =&(\alpha_{\tau-1} + \Psi_{\tau-1}E^P(X_{t+1})) - (\alpha_{\tau-1} + \Psi_{\tau-1}E^Q(X_{t+1}))\\ =&(\alpha_{\tau-1} + \Psi_{\tau-1}(A^P + B^P X_t)) - (\alpha_{\tau-1} + \Psi_{\tau-1}(A^Q + B^Q X_t))\\ =&\Psi_{\tau-1}(LDD'\Lambda_1 + L\Lambda_2 X_t). \end{aligned} \tag{C.15}$$

C.4 The Kalman filter and maximum likelihood estimation in the macro-finance application

Expectations conditional upon the available information are represented at time t with a 'hat' (so that $\hat{z}_t = E_t(z_t)$; $\hat{z}_{\rho|t} = E_t(z_\rho)$; $\rho \geq t$) and define the following conditional covariances:

$$\begin{aligned}
P_{zz} &= E_t(z_t - \hat{z}_t)(z_t - \hat{z}_t)' = \hat{V}_t; \\
P_{nn} &= E_t(n_{t+1} - n_{t+1|t})(n_{t+1} - n_{t+1|t})' = \Theta_{n,l^*}\hat{V}_{t+1|t}\Theta'_{n,l^*} + \Sigma_n \\
P_{hh} &= E_t(h_{t+1|t} - \hat{h}_{t+1|t})(h_{t+1|t} - \hat{h}_{t+1|t})' \\
&= (\Psi_z + \Psi_n\Theta_{n,l^*})\hat{V}_{t+1|t} + \Psi_n\Sigma_n\Psi'_n + Q. \\
P_{zh} &= E_t(z_{t+1} - \hat{z}_{t+1|t})(h_{t+1} - \hat{h}_{t+1|t}) = (\Psi_z + \Psi_n\Theta_{n,l^*})\hat{V}_{t+1|t}, \\
P_{nh} &= E_t(n_{t+1} - n_{t+1|t})(h_{t+1} - \hat{h}_{t+1|t}) = \Sigma_n\Psi'_n + \Theta_{n,l^*}\hat{V}_{t+1|t}(\Psi'_z + \Psi'_n\Theta'_{n,l^*}), \\
P_{nz} &= E_t(n_{t+1} - n_{t+1|t})(z_{t+1} - \hat{z}_{t+1|t}) = \Theta_{n,l^*}\hat{V}_{t+1|t},
\end{aligned} \tag{C.16}$$

where $\hat{z}_{t+1|t}$, $n_{t+1|t}$, and $h_{t+1|t}$ follow equation (4.19), (4.20), and (4.40).

In the pre-1984 period there are no futures data and the estimates of the latent variables (which include the convenience yield δ) are updated in terms of the surprises in n_t to as:

$$\begin{aligned}
\hat{z}_{t+1} &= \hat{z}_{t+1|t} + P_{zn}P_{nn}^{-1}(n_{t+1} - n_{t+1|t}), \\
\hat{V}_{t+1} &= \hat{V}_{t+1|t} - P_{zn}P_{nn}^{-1}P_{zn}.
\end{aligned} \tag{C.17}$$

The log-likelihood function for period $t+1$ is:

$$\ln \mathcal{L}(\theta, \mathcal{F}_{t-1}) = -\frac{4T}{2}ln(2\pi) - \frac{1}{2}\ln|P_{nn}| - \frac{1}{2}(n_{t+1} - n_{t+1|t})'P_{nn}^{-1}(n_{t+1} - n_{t+1|t}) \tag{C.18}$$

where $\theta = (A, B, \Lambda_1, \Lambda_2, L, D, Q)$ is the parameter set of the model, $\theta \in \Theta$, where Θ is the parameter space. $\mathcal{F}_{t-1}$ stands for filtration of all available information at time $t-1$.

In the post-1983 period, the futures allow estimates of the latent variables to be updated using:

$$\hat{z}_{t+1} = \hat{z}_{t+1|t} + \Gamma \begin{pmatrix} h_{t+1} - \hat{h}_{t+1|t} \\ n_{t+1} - n_{t+1|t} \end{pmatrix}, \tag{C.19}$$

$$\hat{V}_{t+1} = \hat{V}_{t+1|t} - \Gamma \begin{pmatrix} P_{zh} \\ P_{zn} \end{pmatrix}.$$

where:

$$\Gamma = \begin{pmatrix} P_{zh} & P_{zn} \end{pmatrix} \begin{pmatrix} P_{hh} & P_{hn} \\ P_{nh} & P_{nn} \end{pmatrix}^{-1}$$

and Γ is the Kalman gain matrix.

The number of futures prices that are available to construct h_{t+1} increase from four in 1984 to six by 2015. This means that the number N of observed variables increases from nine to eleven. Taking account of this, the log-likelihood function for period $t+1$ can be written as:

$$\ln \mathcal{L}(\theta, \mathcal{F}_{t-1}) = -\frac{NT}{2} ln(2\pi) - \frac{1}{2} \ln \left(Det \begin{pmatrix} P_{hh} & P_{hn} \\ P_{nh} & P_{nn} \end{pmatrix} \right) - \frac{1}{2} \begin{pmatrix} h_{t+1} - h_{t+1|t} & n_{t+1} - n_{t+1|t} \end{pmatrix}' \begin{pmatrix} P_{hh} & P_{hn} \\ P_{nh} & P_{nn} \end{pmatrix}^{-1} \begin{pmatrix} h_{t+1} - \hat{h}_{t+1|t} \\ n_{t+1} - n_{t+1|t} \end{pmatrix} \tag{C.20}$$

The likelihood for the full sample follows by substituting (C.17) into (C.18) or (C.19) into (C.20) appropriately, followed by (C.16), (4.19), (4.20) and (4.40) and then summing over $t = 0, ...T-1$. It is optimized with respect to the parameters of (4.19), (4.20) and (4.40).

C.5 Parameter estimation

The expected log-likelihood conditional on the data and the estimates is computed from the previous iteration (equation (C.20)) with initial parameters of the model using the Kalman filter as in Appendix C.4. The parameters are then updated maximizing the expected log-likelihood with respect to θ:

$$\hat{\theta} = arg\ \max_{\theta}\ \ln \mathcal{L}(\theta, \mathcal{F}_{t-1}) \tag{C.21}$$

To get the initial values of parameters of the model, diagonal parameters are assumed in matrix B of the transition equation under P as equation (4.23) and are uniformly distributed between 0 and 1. All the rest of the parameters in the model are assumed to be uniformly distributed between -0.1 and 0.1.

I randomize 300 sets of initial values of parameters, and each run separately until the model is converged. The result presented in this thesis is developed from the directory with the highest log-likelihood at convergence. The log-likelihood starts from 248646 and converges at $(-)$6804.76. This is converged using both “fminsearch” and “fminunc” optimizer in Matlab. The “fminsearch” optimizer uses the Nelder-Mead simplex algorithm, as described in Lagarias et al (1998). The “fminnuc” optimizer uses the quasi-Newton algorithm, as described in Kelley (1999).

Bibliography

Ang, Andrew and Monika Piazzesi, "A no-arbitrage vector autoregression of term structure dynamics with macroeconomic and latent variables," *Journal of Monetary Economics*, 2003, *50*, 745–787.

Balke, Nathan S and Kenneth M Emery, "Understanding the price puzzle," *Economic Review Federal Reserve Bank of Dallas*, 1994, p. 15.

Bernanke, Ben S, Jean Boivin, and Piotr Eliasz, "Measuring the effects of monetary policy: a factor-augmented vector autoregressive (FAVAR) approach," *The Quarterly Journal of Economics*, 2005, *120* (1), 387–422.

___ **, Mark Gertler, Mark Watson, Christopher A Sims, and Benjamin M Friedman**, "Systematic monetary policy and the effects of oil price shocks," *Brookings Papers on Economic Activity*, 1997, *1997* (1), 91–157.

Black, Fischer and Myron S Scholes, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 1973, *81* (3), 637–54.

Bodenstein, Martin, Luca Guerrieri, and Lutz Kilian, "Monetary policy responses to oil price fluctuations," *IMF Economic Review*, 2012, *60* (4), 470–504.

Brennan, Michael J, "The Supply of Storage," *The American Economic Review*, 1958, *48* (06), 50–72.

——, "A theory of price limits in futures markets," *Journal of Financial Economics*, 1986, *16* (2), 213–233.

——, "The price of convenience and the valuation of commodity contingent claims," *Stochastic Models and Option Values*, 1991, *200*, 33–71.

—— **and Eduardo S Schwartz**, "Evaluating Natural Resource Investments," *The Journal of Business*, 1985, *58* (2), 135–57.

Burbidge, John and Alan Harrison, "Testing for the effects of oil-price rises using vector autoregressions," *International Economic Review*, 1984, pp. 459–484.

Campbell, John Y, Andrew Wen-Chuan Lo, and Archie Craig MacKinlay, *The econometrics of financial markets*, Vol. 2, princeton University press Princeton, NJ, 1997.

Carlstrom, Charles T and Timothy Stephen Fuerst, "Oil prices, monetary policy, and counterfactual experiments," *Journal of Money, Credit, and Banking*, 2006, *38* (7), 1945–1958.

Carollo, Salvatore, *Evolution of the Price of Crude Oil from the 1960s up to 1999*, John Wiley & Sons, Ltd., 2011.

Carruth, Alan A, Mark A Hooker, and Andrew J Oswald, "Unemployment equilibria and input prices: Theory and evidence from the United States," *Review of Economics and Statistics*, 1998, *80* (4), 621–628.

Casassus, Jaime and Pierre Collin-Dufresne, "Stochastic convenience yield implied from commodity futures and interest rates," *The Journal of Finance*, 2005, *60* (5), 2283–2331.

—— **and** ——, "Stochastic Convenience Yield Implied from Commodity Futures and Interest Rates," *Journal of Finance*, 2005, *60* (5), 2283–2331.

Castelnuovo, Efrem and Paolo Surico, "Monetary policy, inflation expectations and the price puzzle," *The Economic Journal*, 2010, *120* (549), 1262–1283.

Cheng, Ing-Haw and Wei Xiong, "Financialization of commodity markets," *Annu. Rev. Financ. Econ.*, 2014, *6* (1), 419–441.

Chin, Michael and Zhuoshi Liu, "A joint affine model of commodity futures and US Treasury yields," *Bank of England Working Paper*, 2015.

Christiano, Lawrence J, Martin Eichenbaum, and Charles Evans, "The effects of monetary policy shocks: some evidence from the flow of funds," Technical Report, National Bureau of Economic Research 1994.

Cochrane, John H and Monika Piazzesi, "Decomposing the Yield Curve," *SSRN Working Paper Series*, 2009.

Coroneo, Laura, "TIPS Liquidity Premium and Quantitative Easing," *SSRN Working Paper Series*, 2016.

Cortazar, Gonzalo and Eduardo S Schwartz, "The valuation of commodity contingent claims," *The Journal of Derivatives*, 1994, *1* (4), 27–39.

Cox, John C, Jonathan E Ingersoll, and Stephen A Ross, "The relation between forward prices and futures prices," *Journal of Financial Economics*, 1981, *9* (4), 321–346.

Dewachter, Hans and Marco Lyrio, "A Structural Macro Model of the Yield Curve," Computing in Economics and Finance 2006, Society for Computational Economics 2006.

___ **and** ___ , "Macro factors and the term structure of interest rates," *Journal of Money, Credit and Banking*, 2006, pp. 119–140.

—, —, **and Konstantijn Maes**, "A multi-factor model for the valuation and risk managment of demand deposits," Working Paper Research, National Bank of Belgium 2006.

—, —, **and** —, "A joint model for the term structure of interest rates and the macroeconomy," *Journal of Applied Econometrics*, 2006, *21* (4), 439–462.

Diebold, Francis X and Canlin Li, "Forecasting the term structure of government bond yields," *Journal of Econometrics*, 2006, *130* (2), 337–364.

—, **Glenn D Rudebusch, and S Boragan Aruoba**, "The macroeconomy and the yield curve: a dynamic latent factor approach," *Journal of Econometrics*, 2006, *131* (1), 309–338.

Duffee, Gregory R, "Term premia and interest rate forecasts in affine models," *The Journal of Finance*, 2002, *57* (1), 405–443.

Duffie, Darrell and Rui Kan, "A Yield-Factor Model Of Interest Rates," *Mathematical Finance*, 1996, *6* (4), 379–406.

—, **Jun Pan, and Kenneth Singleton**, "Transform Analysis and Asset Pricing for Affine Jump-Diffusions," *Econometrica*, 2000, *68* (6), 1343–1376.

Durbin, James and Siem Jan Koopman, *Time series analysis by state space methods* number 38, Oxford University Press, 2012.

Dusak, Katherine, "Futures trading and investor returns: An investigation of commodity market risk premiums," *Journal of Political economy*, 1973, *81* (6), 1387–1406.

Eichenbaum, M, "Comments on'Interpreting the Macroeconomic Time Series Facts: the Effects of Monetary Policy'by Christopher Sims," *European Economic Review*, 1992, *36*, 1001–1011.

Elekdag, Selim, Rene Lalonde, Douglas Laxton, Dirk Muir, and Paolo Pesenti, "Oil price movements and the global economy: A model-based assessment," Technical Report, National Bureau of Economic Research 2008.

Fama, Eugene F and Kenneth R French, "Commodity futures prices: Some evidence on forecast power, premiums, and the theory of storage," *Journal of Business*, 1987, pp. 55–73.

Finn, Mary G, "Perfect competition and the effects of energy price increases on economic activity," *Journal of Money, Credit and banking*, 2000, pp. 400–416.

Gibson, Rajna and Eduardo S Schwartz, " Stochastic Convenience Yield and the Pricing of Oil Contingent Claims," *Journal of Finance*, 1990, *45* (3), 959–76.

___ **and** ___ , "Stochastic convenience yield and the pricing of oil contingent claims," *The Journal of Finance*, 1990, *45* (3), 959–976.

Giordani, Paolo, "An alternative explanation of the price puzzle," *Journal of Monetary Economics*, 2004, *51* (6), 1271–1296.

Gisser, Micha and Thomas H Goodwin, "Crude oil and the macroeconomy: Tests of some popular notions: Note," *Journal of Money, Credit and Banking*, 1986, *18* (1), 95–103.

Gospodinov, Nikolay and Serena Ng, "Commodity prices, convenience yields, and inflation," *Review of Economics and Statistics*, 2013, *95* (1), 206–219.

Hamilton, James D, "Oil and the macroeconomy since World War II," *The Journal of Political Economy*, 1983, pp. 228–248.

——, "Historical causes of postwar oil shocks and recessions," *The Energy Journal*, 1985, *6* (1), 97–116.

——, *Time Series Analysis*, Princeton University Press, 1994.

——, "This is what happened to the oil price-macroeconomy relationship," *Journal of Monetary Economics*, 1996, *38* (2), 215–220.

——, "What is an oil shock?," *Journal of Econometrics*, 2003, *113* (2), 363–398.

——, "Understanding crude oil prices," Technical Report, National Bureau of Economic Research 2008.

—— **and Ana Maria Herrera**, "Oil shocks and aggregate macroeconomic behavior: the role of monetary policy: a comment," *Journal of Money, Credit, and Banking*, 2004, *36* (2), 265–286.

—— **and Jing Cynthia Wu**, "Risk premia in crude oil futures prices," *Journal of International Money and Finance*, 2014, *42*, 9–37.

Hanson, Michael S, "The "price puzzle" reconsidered," *Journal of Monetary Economics*, 2004, *51* (7), 1385–1413.

Harvey, Andrew C., *Forecasting, Structural Time Series Models and the Kalman Filter* number 9780521321969. In 'Cambridge Books.', Cambridge University Press, October 1990.

Heath, Davidson, "Macroeconomic factors in oil futures markets," 2016.

Herrera, Ana Maria and James D Hamilton, "Oil shocks and aggregate macroeconomic behavior: the role of monetary policy," *Department of Economics, UCSD*, 2001.

Hicks, John Richard, *Value and capital*, Oxford University Press, 1939.

Hilliard, Jimmy E and Jorge Reis, "Valuation of commodity futures and options under stochastic convenience yields, interest rates, and jump diffusions in the spot," *Journal of Financial and Quantitative Analysis*, 1998, *33* (1), 61–86.

Hooker, Mark A, "What happened to the oil price-macroeconomy relationship?," *Journal of Monetary Economics*, 1996, *38* (2), 195–213.

——, "Are oil shocks inflationary?: Asymmetric and nonlinear specifications versus changes in regime," *Journal of Money, Credit, and Banking*, 2002, *34* (2), 540–561.

Hördahl, Peter, Oreste Tristani, and David Vestin, "A joint econometric model of macroeconomic and term-structure dynamics," *Journal of Econometrics*, 2006, *131* (1), 405–444.

Ireland, Peter N, "Changes in the Federal Reserve's inflation target: Causes and consequences," *Journal of Money, credit and Banking*, 2007, *39* (8), 1851–1882.

Joslin, Scott, Kenneth J Singleton, and Haoxiang Zhu, "A new perspective on Gaussian dynamic term structure models," *Review of Financial Studies*, 2011, *24* (3), 926–970.

——**, Marcel Priebsch, and Kenneth J Singleton**, "Risk premiums in dynamic term structure models with unspanned macro risks," *The Journal of Finance*, 2014, *69* (3), 1197–1233.

Kaldor, Nicholas, "Speculation and economic stability," *The Review of Economic Studies*, 1939, *7* (1), 1–27.

Kelley, Carl T, *Iterative methods for optimization*, SIAM, 1999.

Keynes, John Maynard, "Some aspects of commodity markets," *Manchester Guardian Commercial: European Reconstruction Series*, 1923, *13*, 784–786.

___ , *Treatise on money*, Macmillan and Company Limited, 1930.

Kilian, Lutz, "Not all oil price shocks are alike: Disentangling demand and supply shocks in the crude oil market," 2006.

___ , "Exogenous oil supply shocks: how big are they and how much do they matter for the US economy?," *The Review of Economics and Statistics*, 2008, *90* (2), 216–240.

___ **and Logan T Lewis**, "Does the Fed respond to oil price shocks?," *The Economic Journal*, 2011, *121* (555), 1047–1072.

Kozicki, Sharon and Peter A Tinsley, "Shifting endpoints in the term structure of interest rates," *Journal of Monetary Economics*, 2001, *47* (3), 613–652.

___ **and** ___ , "Permanent and transitory policy shocks in an empirical macro model with asymmetric information," *Journal of Economic Dynamics and Control*, 2005, *29* (11), 1985–2015.

Lagarias, Jeffrey C, James A Reeds, Margaret H Wright, and Paul E Wright, "Convergence properties of the Nelder–Mead simplex method in low dimensions," *SIAM Journal on optimization*, 1998, *9* (1), 112–147.

Leduc, Sylvain and Keith Sill, "A quantitative analysis of oil-price shocks, systematic monetary policy, and economic downturns," *Journal of Monetary Economics*, 2004, *51* (4), 781–808.

Leeper, Eric M, Christopher A Sims, Tao Zha, Robert E Hall, and Ben S Bernanke, "What does monetary policy do?," *Brookings papers on economic activity*, 1996, *1996* (2), 1–78.

Liu, Peng and Ke Tang, "The stochastic behavior of commodity prices with heteroskedasticity in the convenience yield," *Journal of Empirical Finance*, 2011, *18* (2), 211–224.

Lütkepohl, Helmut, "Bootstrapping impulse responses in VAR analyses," in "COMPSTAT" Springer 2000, pp. 109–119.

Miffre, Joëlle, "Normal backwardation is normal," *Journal of Futures Markets*, 2000, *20* (9), 803–821.

Millard, Stephen and Tamarah Shakir, "Oil shocks and the UK economy: the changing nature of shocks and impact over time," *Bank of England Working Paper*, 2013.

Morck, Randall, Eduardo Schwartz, and David Stangeland, "The valuation of forestry resources under stochastic prices and inventories," *Journal of Financial and Quantitative Analysis*, 1989, *24* (4), 473–487.

Muellbauer, John and Luca Nunziata, "Credit, the stock market and oil: Forecasting US GDP," 2001.

Nakov, Anton and Andrea Pescatori, "Oil and the great moderation," *The Economic Journal*, 2010, *120* (543), 131–156.

Paddock, James L, Daniel R Siegel, and James L Smith, "Option valuation of claims on real assets: The case of offshore petroleum leases," *The Quarterly Journal of Economics*, 1988, *103* (3), 479–508.

Paschke, Raphael and Marcel Prokopczuk, "Integrating multiple commodities in a model of stochastic price dynamics," 2007.

Phillips, Peter CB, "Impulse response and forecast error variance asymptotics in nonstationary VARs," *Journal of Econometrics*, 1998, *83* (1), 21–56.

Raymond, Jennie E and Robert W Rich, "Oil and the macroeconomy: A Markov state-switching approach," *Journal of Money, Credit, and Banking*, 1997, pp. 193–213.

Ross, Stephen A, "A simple approach to the valuation of risky streams," *Journal of Business*, 1978, pp. 453–475.

Rudebusch, Glenn D, "Assessing nominal income rules for monetary policy with model and data uncertainty," *The Economic Journal*, 2002, *112* (479), 402–432.

——— **and Tao Wu**, "A Macro-Finance Model of the Term Structure, Monetary Policy and the Economy," *The Economic Journal*, 2008, *118* (530), 906–926.

Rusnák, Marek, Tomas Havranek, and Roman Horváth, "How to solve the price puzzle? A meta-analysis," *Journal of Money, Credit and Banking*, 2013, *45* (1), 37–70.

Santini, Danilo J, "Verification of energy's role as a determinant of US economic activity," Technical Report, Argonne National Lab., IL (USA) 1987.

Schwartz, Eduardo S, "Option pricing theory and its application," *The Journal of Finance*, 1982, *37* (2).

———, "The Pricing of Commodity-Linked Bonds," *The Journal of Finance*, 1982, *37* (2), 525–539.

———, " The Stochastic Behavior of Commodity Prices: Implications for Valuation and Hedging," *Journal of Finance*, 1997, *52* (3), 923–73.

Sims, Christopher A, "Interpreting the macroeconomic time series facts: The effects of monetary policy," *European Economic Review*, 1992, *36* (5), 975–1000.

Smets, Frank, "Output gap uncertainty: does it matter for the Taylor rule?," *Empirical Economics*, 2002, *27* (1), 113–129.

Spencer, Peter and Zhuoshi Liu, "An open-economy macro-finance model of international interdependence: The OECD, US and the UK," *Journal of Banking & Finance*, 2010, *34* (3), 667–680.

Spencer, Peter D, "Stochastic Volatility in a Macro-Finance Model of the US Term Structure of Interest Rates 1961–2004," *Journal of Money, Credit and Banking*, 2008, *40* (6), 1177–1215.

Sujithan, Kuhanathan Ano, Sanvi Avouyi-Dovi, and Lyes Koliai, "Does Monetary Policy Respond to Commodity Price Shocks?," in "62nd annual meeting of the AFSE" 2013, p. 52.

Svensson, Lars EO, "Inflation targeting as a monetary policy rule," *Journal of Monetary Economics*, 1999, *43* (3), 607–654.

Tsay, Ruey S., *Analysis of financial time series* Wiley series in probability and statistics, Hoboken (N.J.): Wiley-Interscience, 2005.

Vasicek, Oldrich, "An equilibrium characterization of the term structure," *Journal of Financial Economics*, 1977, *5* (2), 177–188.

Weymar, F Helmut, "The dynamics of the world cocoa market.," *The dynamics of the world cocoa market.*, 1968.

Working, Holbrook, "Price Relations Between July And September Wheat Futures At Chicago Since 1885," *Wheat Studies*, 1933, *9* (06).

——, "Price Relations Between May and New-Crop Wheat Futures at Chicago Since 1885," *Wheat Studies*, 1934, *10* (05).

——— , "The Theory of Price of Storage working," *The American Economic Review*, 1949, *39* (06).

Zha, Tao, "Identifying monetary policy: a primer," *Economic Review-Federal Reserve Bank of Atlanta*, 1997, *82* (2), 26.