Symmetries of integrable open boundaries in the Hubbard model and other spin chains

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Abstract

We proceed to study the symmetries of integrable open boundaries in the one dimensional Hubbard model, the Heisenberg XXX spin chain and Inozemtsev's hyperbolic spin chain.

For the Hubbard model, we show that when placed on the left half-line, the known integrable open boundaries (a magnetic field and chemical potential) break the bulk Yangian symmetry to a twisted Yangian corresponding to the $(\mathfrak{sl}_2, \mathfrak{u}_1)$ symmetric pair. Furthermore, we consider two additional boundaries, corresponding to the symmetric pairs $(\mathfrak{so}_4, \mathfrak{sl}_2)$ and $(\mathfrak{sl}_2, \mathfrak{sl}_2)$ and construct their twisted Yangian symmetries. This provides a step forward in the classification of integrable boundaries of the open Hubbard model. We conclude our study of this model by examining the symmetries of its bulk and open $SU(n)$ generalisation.

For the Heisenberg XXX spin chain and Inozemtsev's hyperbolic spin chain we construct a procedure to, given the integrable bulk models, systematically obtain their integrable boundaries and corresponding Yangian symmetries for the symmetric pairs $(\mathfrak{sl}_2, \mathfrak{u}_1), (\mathfrak{so}_4, \mathfrak{sl}_2)$ and $(\mathfrak{sl}_2, \mathfrak{sl}_2)$. We call this method 'folding', and it is motivated by the wish to study integrable boundaries for long-range spin chains. We test this procedure by first applying it on the Heisenberg XXX spin chain and confirming it reproduces well known results. We then apply the folding to Inozemtsev's hyperbolic spin chain and classify its integrable open boundaries and their twisted Yangian symmetries.

Contents

Introduction

This thesis studies the symmetries governing the integrable structure of several models in 1+1 dimensions. More specifically, we will be looking at the different ways one can impose a open boundary condition which preserves integrability in the Hubbard model, the Heisenberg XXX spin chain and Inozemtsev's hyperbolic spin chain.

The study of symmetries has always had a immense role in our attempt at understanding reality. This role is explicitly stated in what is arguably the most powerful theorem in mathematical physics: Noether's theorem. Noether proved that for every continuous symmetry of a physical model there is a corresponding quantity that is conserved in time. The consequences of this theorem range across all of physics, from the solution of the spinning top in classical mechanics to restriction and prediction of particle interactions in the Standard Model of Particle Physics.

The ability to solve – i.e find the quantities of interest of, like energy or momentum and the spectra of their values – any of these systems depends on how many symmetries it possesses. The more symmetries, the more conditions in the equations of motion, and if one has as many conditions as degrees of freedom, the system can be solved, and it is known as a solvable, or integrable, system.

Integrable systems appear everywhere in physics and mathematics. The most famous one in statistical physics is the Heisenberg XXX spin chain, which is used to study the properties of materials at the quantum level. Bethe found a procedure to solve this model in 1931, known today as the Bethe ansatz method. As expected, this method has implications for the symmetry type of the model which, as we will see soon, allows one to construct an infinite set of conserved quantities.

The study of integrable systems via algebraic methods flourished in the 1970s and 80s, when it was found that the existence of infinitely many conserved quantities was deeply connected to the scattering (or R-) matrix of the model in question. This enabled the unification of previous work started by Bethe and algebraic properties studied in solvable systems by Yang, Baxter and many others into the Quantum Inverse Scattering Method (QISM).

All these models possess a common algebraic structure derived from their R-matrix. However, the slightest perturbation of their Lagrangian may have non-trivial consequences so as to destroy integrability. Yet integrability is surprisingly frequent in fundamental physics. The most natural of these perturbations is caused by the imposition of a boundary. Which boundaries preserve integrability and their symmetries is the focus of this thesis. It is organised as follows:

Chapter 1: The QISM and Yangian symmetries. In the first Chapter we will introduce the QISM and several integrable models, all of which are relevant for the results in the next two Chapters. Starting from the Yang-Baxter equation, we will show how an integrable model is obtained, and present the XXZ spin chain as a first example. We will then use the Heisenberg spin chain to show how the Algebraic Bethe Ansatz works, and unravel the full set of symmmetries behind its solvability: the Yangian. To conclude this Chapter, we will introduce the boundary QISM and show how, for some special open boundary conditions, there is still a remnant of such symmetry that ensures integrability: a twisted Yangian symmetry. We will look at a specific integrable boundary for the Heisenberg spin chain, but leave the additional possible cases for future Chapters.

Chapter 2: Twisted Yangian symmetry of the open Hubbard model. This Chapter will deal with one of the most intriguing and rich integrable models in condensed matter physics: the Hubbard model. Its exoticity comes with a price: it exhibits a bizarre integrable structure, which is why its R-matrix took much longer to construct than that of any other known solvable model. We will give an overview of how the model was conceived, its symmetries and Shastry's construction of its R-matrix. We will then introduce the $SU(n)$ generalisation of the Hubbard model, which will be used to point out how special the $n = 2$ case is. Then, we will make connections between the Hubbard model and AdS/CFT integrability – in particular, the role of the partial particle-hole transformation. Until this point, the vast majority of the material is not the author's work, excluding the Yangian symmetry of the $SU(n)$ Hubbard model in Section 2.3 and the role of the partial particle-hole transformation in connecting its R-matrix to that of the $\mathrm{Ad}S_5 \times S^5$ superstring in Section 2.4. Some introductory material aside, Sections 2.5, 2.6 and 2.7 are based on papers [121, 89] and some unpublished work by the author and collaborators, where we constructed twisted Yangian symmetries corresponding to three different open boundary conditions in the half-infinite Hubbard chain.

Chapter 3: Folding a spin chain. This Chapter closely follows the paper by the author and collaborators [137]. We explore a procedure to obtain integrable boundary Hamiltonians and their corresponding twisted Yangian symmetries for integrable models in the bulk. This method is shown to work for known boundaries of the Heisenberg spin chain and proves to be remarkably useful when applied to long-range spin chains, where no other clear treatment of integrable boundaries is known.

Chapter 4: Conclusion and Outlook. In this Chapter we sum up our results and list several directions this research could take next. This is based on the concluding statements of the papers by the author and collabotators [89, 121, 137].

Acknowledgements

I am thankful beyond words to my PhD supervisor Niall MacKay, who has always looked after my interests as a researcher and introduced me to the beautiful subject of integrable systems. Additionally, I am grateful to have had Ed Corrigan and Gustav Delius in my supervisory panel. They have always provided useful guidance and suggestions.

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The greatest thanks of all go to the actual heroes of the story, who have been in for the ride since I left my home eleven years ago: my parents **Mercedes** and **Juan**. I would have not made it this far without your unlimited love and support.

Author's Declaration

The work presented in this thesis is based on research carried out in the Department of Mathematics at the University of York. This work has not previously been presented for an award at this or any other University.

The construction of the twisted Yangian for the boundary magnetic field, chemical potential and the achiral boundary of the Hubbard model in Sections 2.5 and 2.6 is based on the following papers by the author and collaborators:

- A. De La Rosa Gomez and N. J. MacKay, *Twisted Yangian symmetry of the open Hubbard model*, J. Phys. **A47** 305203 (2014), `arXiv:hep-th/1404.2095`.
- A. De La Rosa Gomez, *The particle-hole transformation, supersymmetry and achiral boundaries of the open Hubbard model*, JHEP **1504** 063 (2015), `arXiv:1412.3292`.

In addition, the role of the partial particle-hole transformation in connecting the Hubbard model to the AdS/CFT correspondence is based on the first part of the latter paper. Chapter 3 is based on

- A. De La Rosa Gomez, N. J. MacKay and V. Regelskis, *How to fold a spin chain: integrable boundaries of the Heisenberg XXX and Inozemtsev hyperbolic models*, Phys. Lett. **A381** 1340-1348 (2017), `arXiv:math-ph/1610.01523`.

Chapter 4 is based on the conclusions of the above papers. Furthermore, the bulk and twisted Yangian symmetry of the $SU(n)$ Hubbard model from Sections 2.8 and 2.3 respectively, and the free boundary of the Hubbard model from Section 2.7 is unpublished original work from the author.

Chapter 1

The QISM and Yangian symmetries

1.1 Integrability in classical physics

Before venturing ourselves into the world of quantum integrable models, it is useful to briefly state how these arise in classical physics for motivational purposes. At the classical level [1, 2], integrable models live in a symplectic manifold $(\mathcal{M}, \omega)$. This is a manifold $\mathcal{M}$ of dimension $2n$ equipped with a closed, non-degenerate 2-form: a map

$$\omega : T\mathcal{M} \times T\mathcal{M} \to \mathbb{R}, \tag{1.1.1}$$

where $T\mathcal{M}$ is the tangent bundle of $\mathcal{M}$. In other words, at any point $p \in \mathcal{M}$, ω maps a pair of vectors in the tangent space $T_p\mathcal{M} \times T_p\mathcal{M}$ to a number. Locally, one can assign coordinates $(q_1, \ldots q_n, p_1, \ldots p_n)$ to each point in $\mathcal{M}$ such that

$$\omega = \sum_{i=1}^{n} \mathrm{d}q_i \wedge \mathrm{d}p_i \qquad i = 1, ...n. \tag{1.1.2}$$

Non-degeneracy of ω implies there is a one-to-one correspondence of one-forms and vector fields through the following statement: for every differentiable function $H : \mathcal{M} \to \mathbb{R}$, there exists an unique vector field X_H such that

$$\mathrm{d}H(Y) = \omega(X_H, Y) \quad \forall\, Y \in T\mathcal{M}. \tag{1.1.3}$$

We call X_H the Hamiltonian vector field of the Hamiltonian H. Using (1.1.3) and writing $X_H = \sum_{i=1}^{n} \left(a_i \frac{\partial}{\partial q_i} + b_i \frac{\partial}{\partial p_i} \right)$, where a_i and b_i are functions of the local coordinates, we have that

$$\mathrm{d}H(Y) = \frac{\partial H}{\partial q_i} \mathrm{d}q_i(Y) + \frac{\partial H}{\partial p_i} \mathrm{d}p_i(Y) \tag{1.1.4}$$

and

$$\omega(X_H, Y) = \left(\sum_{i=1}^{n} \mathrm{d}q_i \wedge \mathrm{d}p_i\right)(X_H, Y) = \sum_{i=1}^{n} a_i \mathrm{d}p_i(Y) - b_i \mathrm{d}q_i(Y), \tag{1.1.5}$$

so that, in terms of the coordinates p_i and q_i, the Hamiltonian vector field is given by

$$X_H = \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}. \tag{1.1.6}$$

If $x(t) = (p_i(t), q_i(t))$ is an integral curve of X_H for any i then $dH(X_H(x(t))) = 0$, so that Hamilton's equations hold:

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \dot{q}_i = \frac{\partial H}{\partial p_i} \quad \forall\ i \tag{1.1.7}$$

and thus the Hamiltonian H is constant along the integral curves. The physical interpretation is conservation in time of energy of a system with n degrees of freedom characterised by the positions $q_i(t)$ and corresponding momenta $p_i(t)$ on the phase space $\mathcal{M}$. Additionally, H determines how other functions on $\mathcal{M}$ evolve in time: using (1.1.6) and (1.1.7), we obtain that for any function F on $\mathcal{M}$,

$$\frac{dF}{dt} = \sum_{i=1}^{n} \frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial p_i}\dot{p}_i = \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\frac{\partial F}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial F}{\partial p_i} = X_H(F) = \{H, F\} \tag{1.1.8}$$

where $\{H, F\}$ is the Poisson bracket of H and F. This is a binary operation on any two differentable functions F and G on $\mathcal{M}$, given by

$$\{F, G\} = \sum_{i=1}^{n} \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i}. \tag{1.1.9}$$

One can check this operation is skew-symmetric and satisfies the Jacobi identity, thus the set of differentiable functions on $\mathcal{M}$ with the Poisson bracket is a Lie algebra. The coordinates p_i and q_i are canonical, meaning they satisfy

$$\{p_i, q_j\} = \delta_{ij}, \quad \{q_i, q_j\} = \{p_i, p_j\} = 0. \tag{1.1.10}$$

The system described by the Hamiltonian H is Liouville integrable, or integrable in the Liouville sense, if there are at least $n-1$ quantities $F_1, F_2 \ldots F_{n-1}$ which Poisson-commute with each other and with the Hamiltonian

$$\{F_i, F_j\} = 0, \quad \{F_i, H\} = 0 \quad \forall\ i \tag{1.1.11}$$

and thus these quantities are not only in involution, but also conserved in time. If one then constructs a canonical – (1.1.10)-preserving – transformation $(p_i, q_i) \mapsto (F_i, \psi_i)$ (details on how to do this can be found in [1]) then Hamilton's equations can be easily solved

$$\begin{aligned} \dot{F}_i = 0 \qquad\qquad &\Longrightarrow \quad F_i(t) = F_i(0), \\ \dot{\psi}_i = \frac{\partial H}{\partial F_i} = \Theta_i \quad &\Longrightarrow \quad \psi_i(t) = \psi_i(0) + \Theta_i t, \end{aligned} \tag{1.1.12}$$

the second equation being a consequence of the possibility of writing H as a linear combination of all charges in involution.

From Noether's theorem [3], the conserved quantities $\{F_i\}$ should be connected to continuous symmetries of the system, and hence integrability is achieved due to a large number of symmetries which highly constrain the possible solutions. An illustrative example of a system with this property is the Kepler 2-body system, which is actually 'superintegrable': due to the conservation of the Laplace-Runge-Lenz vector, it possesses a number of conserved quantities higher than number of degrees of freedom. In fact, it is 'maximally superintegrable': it possesses $2n-1$ conserved charges for n degrees of freedom (where $n = 3$). This is a consequence of its $\mathfrak{so}_3$ symmetry being enhanced by hidden symmetries of inverse square forces to $\mathfrak{so}_4$ [6].

A considerable amount of work has been devoted, and still is, to the study of classical integrable systems. Like the aforementioned example, these appear in everyday classical models like the harmonic oscillator and the Euler top. In 1+1D classical field theory, other relevant examples are the sine-Gordon model, Toda field theory and the Korteweg–de Vries (KdV) equation [1, 4]. A variety of tools, including Lax pairs and the Inverse Scattering Transform [1, 5], are employed to find their conserved quantities and solutions.

At the quantum level, however, there are new things to consider other than simply the usual procedure to transition from the classical to the quantum world, which is given by the change of functions to operators and the Poisson bracket of functions to the commutator of operators:

$$\begin{aligned} F \quad &\mapsto \quad \widehat{F} \\ \{F, G\} \quad &\mapsto \quad \frac{1}{i\hbar}[\widehat{F}, \widehat{G}]. \end{aligned} \tag{1.1.13}$$

The problem we face in this transition is due to the infinite degrees of freedom a quantum theory may possess, making it difficult to define what it means to be integrable i.e. even in the case where the model possesses an infinite set of commuting operators, how can we use it to find solutions? The goal is to discover a method of 1) finding and/or constructing a large – possibly infinite – set of operators $\{\widehat{F}_i\}$ which commute with the Hamiltonian operator $\widehat{H}$ and each other, and 2) such that solutions of the system can be found. This is known as the Quantum Inverse Scattering Method (QISM) [19, 21].

Starting in the next Section, we introduce the fundamental objects which play a role in the QISM, beginning with its protagonist: the R-matrix. Many reviews have been done in the subject matter, with their applications in high energy physics and condensed matter physics [7, 8, 9, 10] and the introductory Section of [11]. We will follow the common structure of these reviews and introduce additional concepts when relevant.

1.2 The Yang-Baxter equation

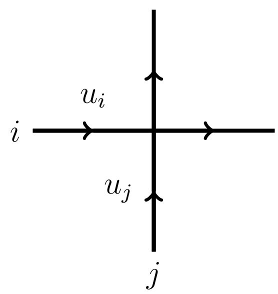
The main subject of study in this Section is the R-matrix $R(u, u')$, which is a map

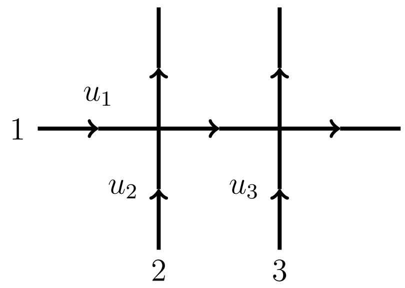
$$R(u, u') : V \otimes V' \to V \otimes V'. \tag{1.2.1}$$

Here V and V' are vector spaces and $u, u' \in \mathbb{C}$ are the so-called spectral parameters [7]. When acting on the space $V_1 \otimes V_2 \otimes \ldots \otimes V_L$, where V_i are Hilbert spaces for $i = 1 \ldots L$, we will denote by $R_{ij}(u_i, u_j)$ the object acting as $R(u_i, u_j)$ on the space $V_i \otimes V_j$ and the identity everywhere else. For example:

$$R_{12}(u_1, u_2) = R(u_1, u_2) \otimes \mathbf{1}_{V_3} \otimes \ldots \otimes \mathbf{1}_{V_L} \tag{1.2.2}$$

where $\mathbf{1}_{V_i}$ is the identity map in $\mathrm{End}(V_i)$. For computational and visualisation purposes, it is convenient to represent $R_{ij}(u_i, u_j)$ pictorially [11], with different spaces corresponding to different lines

$R_{ij}(u_i, u_j) =$  (1.2.3) and the arrows specify the order in which the product of R-matrices act e.g.

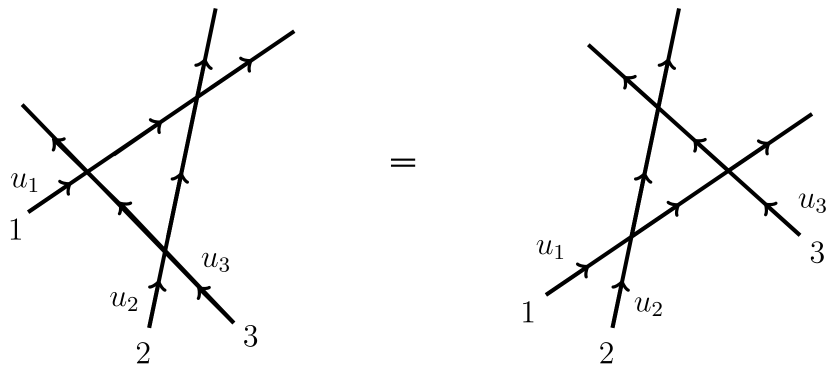
$R_{13}(u_1, u_3)R_{12}(u_1, u_2) =$ 

(1.2.4)

Although it is a priori not clear how to approach integrability for a 1+1D QFT with an number of degrees of freedom which could be infinite, one can obtain quantum integrable models from R-matrices that satisfy a special property. It was discovered [13, 14, 15, 16] that, for one and two dimensional systems in statistical mechanics, the ability to calculate every eigenvalue of the transfer matrix required an associated R-matrix to satisfy the star-triangle relation, which is now called the Yang-Baxter equation (YBE)

$$R_{12}(u_1, u_2)R_{13}(u_1, u_3)R_{23}(u_2, u_3) = R_{23}(u_2, u_3)R_{13}(u_1, u_3)R_{12}(u_1, u_2). \tag{1.2.5}$$

This is an equality between two maps acting on $V_1 \otimes V_2 \otimes V_3$, and can be expressed pictorially as



(1.2.6)

The simplest R-matrix solution of the YBE for $\mathrm{End}(V^{\otimes 2})$ is the permutation operator P

$$P : x \otimes y \mapsto y \otimes x \quad x, y \in V \tag{1.2.7}$$

As before, we denote P_{ij} as the object acting as P on the i-th and j-th spaces in the tensor product and the identity everywhere else. One can easily check that P_{ij} satisfies the YBE:

$$P_{12}P_{13}P_{23}(v_1 \otimes v_2 \otimes v_3) = (v_3 \otimes v_2 \otimes v_1) = P_{23}P_{13}P_{12}(v_1 \otimes v_2 \otimes v_3). \tag{1.2.8}$$

In fact, if $R(u,u')$ satifies the YBE, then $\check{R}(u,u') = PR(u,u')$ provides an equivalent way of writing the YBE acting on $V^{\otimes 3}$ [8]

$$(\check{R}(u,u') \otimes \mathbf{1}_V)(\mathbf{1}_V \otimes \check{R}(u,u''))(\check{R}(u',u'') \otimes \mathbf{1}_V) = \\ = (\mathbf{1}_V \otimes \check{R}(u',u''))(\check{R}(u,u'') \otimes \mathbf{1}_V)(\mathbf{1}_V \otimes \check{R}(u,u')). \tag{1.2.9}$$

Indeed, the two most trivial solutions of the YBE are the identity, where $\check{R} = P$, and the permutation operator, so we denote by (u_0, u_0') the pair of points in the spectral parameters such that $\check{R}(u_0, u_0') = \mathbf{1}_{V\otimes V}$ or

$$R(u_0, u_0') = P. \tag{1.2.10}$$

Using all the concepts we have introduced, we will proceed to show that an R-matrix which satisfies the YBE leads to an integrable model and a method of solving it: the QISM [7, 19, 21]. We will assume that the R-matrix acts on a Hilbert space $\mathcal{V}$ given by the L-fold tensor product of finite-dimensional vector spaces V:

$$\mathcal{V} = \bigotimes_{i=1}^{L} V_{\rho_i} \tag{1.2.11}$$

where the subscript denotes that the space V_{ρ_i} is in a representation ρ_i of a Lie algebra $\mathfrak{g}$. Thus $R_{ij}(u,u') \in \mathrm{End}(V_{\rho_i} \otimes V_{\rho_j})$. In particular, the fundamental representation will be denoted by the label ρ_a. In the case where we are dealing with more than one space in the fundamental representation, these will be labeled V_{ρ_a}, V_{ρ_b}, etc. The Lax matrix, or Lax operator, is defined as

$$L_j(u) := R_{aj}(u, u_0') \in \mathrm{End}(V_{\rho_a} \otimes V_{\rho_j}). \tag{1.2.12}$$

Since

$$R_{ab}(u_1,u_2)R_{a1}(u_1,u_0')R_{b1}(u_2,u_0') = R_{b1}(u_2,u_0')R_{a1}(u_1,u_0')R_{ab}(u_1,u_2), \tag{1.2.13}$$

we have that the Lax matrix satisfies the RLL relation

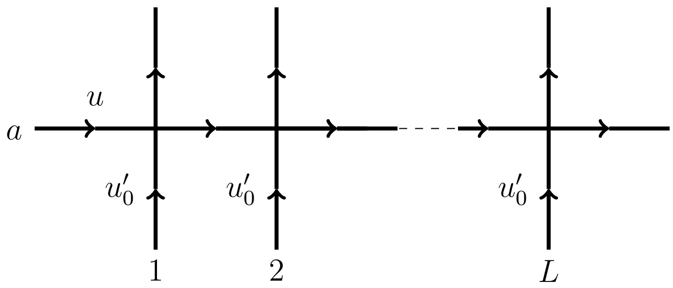
$$R(u,u')(L_j(u) \otimes \mathbf{1}_{V_{\rho_b}})(\mathbf{1}_{V_{\rho_a}} \otimes L_j(u')) = (\mathbf{1}_{V_{\rho_a}} \otimes L_j(u'))(L_j(u) \otimes \mathbf{1}_{V_{\rho_b}})R(u,u'). \tag{1.2.14}$$

We can construct the monodromy matrix $T_a(u)$ from Lax matrices

$$T_a(u) = L_L(u)L_{L-1}(u)...L_1(u) \in \mathrm{End}(V_{\rho_a} \otimes \bigotimes_{j=1}^{L} V_{\rho_j}) \tag{1.2.15}$$

Here the space in the fundamental representation is known as the auxiliary space, and the L-fold tensor product to the right of it as the quantum space [7]. We note that the auxiliary space could be in any other representation as long as it is a common space in all Lax matrices. The pictorial expression of the monodromy matrix follows from (1.2.4)

$T_a(u) =$



(1.2.16)

Using this picture, one can easily show the defining relation of the Yang-Baxter algebra (or the so-called RTT-relation)

$$R_{ab}(u, u')T_a(u)T_b(u') = T_b(u')T_a(u)R_{ab}(u, u') \tag{1.2.17}$$

by using the RLL relation [19] or, equivalently, proving the equivalence between the diagrams corresponding to the right and left expressions using the picture of the YBE [11].

At the special value $u = u_0$, the trace of the monodromy matrix over the auxiliary space becomes the shift operator $\mathcal{U}$

$$\mathcal{U} = \mathrm{tr}_a P_{a,L}P_{a,L-1} \dots P_{a1} = P_{12}P_{23} \dots P_{L-1,L} \tag{1.2.18}$$

which shifts elements of the quantum space from positions i to $i-1$ in the tensor product, with periodicity $0 \equiv L$. Being a translation operator, we naturally define the momentum operator $\hat{p}$ of the quantum model via the formula $e^{\mathrm{i}\hat{p}} = \mathcal{U}$, where $\mathrm{i} = \sqrt{-1}$.

We shall assume the R-matrix to be invertible. Acting on both sides of the RTT relation by $R^{-1}_{ab}(u, u')$, we have that

$$R^{-1}_{ab}(u, u')T_b(u)T_a(u') = T_a(u')T_b(u)R^{-1}_{ab}(u, u') \tag{1.2.19}$$

and thus, for some scalar function $\mu(u,u')$, we see that

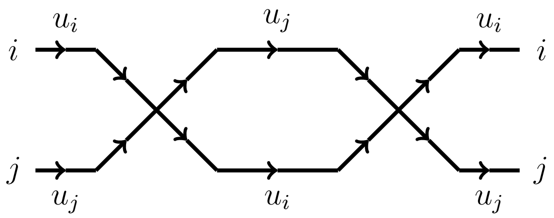
$$R_{ij}^{-1}(u_i,u_j) = \mu(u_i,u_j)R_{ji}(u_j,u_i). \tag{1.2.20}$$

Since multiplying the R-matrix by a scalar factor does not affect the YBE, $R(u,u')$ can always be rescaled by a function $\nu(u,u')$ such that

$$\nu(u,u')\nu(u',u) = \frac{1}{\mu(u,u')} \tag{1.2.21}$$

and $R(u,u')$ satisfies the unitarity condition:

$$R_{ji}(u_j,u_i)R_{ij}(u_i,u_j) =$$



$$= \mathbf{1}_{V_{\rho_i}\otimes V_{\rho_j}} \tag{1.2.22}$$

There is still freedom to multiply the R-matrix by an additional scalar function $\kappa(u,u')$ such that $\kappa(u,u')\kappa(u',u) = 1$, so if one wishes to restrict its form even more, we may require a condition which arises in $2d$ QFTs known as crossing unitarity [12]

$$R_{ij}^{tr_i}(-u_i,u_j)R_{ji}^{tr_i}(u_j,-u_i) = \mathbf{1}_{V_{\rho_i}\otimes V_{\rho_j}}. \tag{1.2.23}$$

where tr_i denotes the transpose of the i-th space.

If we act on the RTT-relation from the left (or right) by $R_{ab}^{-1}(u,u')$ and take the trace over auxiliary spaces a and b, using the cyclicity of the trace, we obtain

$$[\tau(u),\tau(u')] = 0 \tag{1.2.24}$$

where the operator $\tau(u) = \mathrm{tr}_a T_a(u)$ is the transfer matrix [16]. Consider the power series of $\tau(u)$ around the point u_0

$$\tau(u) = \sum_{n=0}^{\infty} t_n (u-u_0)^n. \tag{1.2.25}$$

From the definition of the monodromy matrix, $\tau(u)$ is a polynomial of at least degree L. Then, from (1.2.24), we have that there is a family of at least L operators t_n that commute with one another

$$[t_n,t_m] = 0. \tag{1.2.26}$$

Picking one of these operators to be the Hamiltonian with periodic boundary conditions – induced by the shift operator $\mathcal{U}$ – the quantum model it describes has at least as many conserved quantities t_n as degrees of freedom, and thus it must be integrable. We pick the Hamiltonian of the model to be its logarithmic derivative at the point $u = u_0$, which is the linear term in the expansion of $\tau(u_0)^{-1}\tau(u)$:

$$\mathcal{H} = \frac{d}{dt} \ln \tau(u)|_{u=u_0}. \tag{1.2.27}$$

Picking this Hamiltonian instead of the linear term in $\tau(u)$ does not affect the number of conserved charges as $[\tau(u_0)^{-1}\tau(u), \tau(u_0)^{-1}\tau(u')] = 0$ follows from (1.2.24).

From now on, if an R-matrix satisfies the YBE, we will say it is the R-matrix of the model whose Hamiltonian appears in its the logarithmic derivative of its transfer matrix.

1.3 The XXZ spin chain

Solutions of the YBE are, in general, difficult to find. To make life easier, we can impose certain conditions on the R-matrix which come from the model itself. For example, let the Hilbert space of a 1+1D quantum model be $\mathcal{V} = V^{\otimes L}$ with $V = \mathbb{C}^2$. The basis for each of these vector spaces is given by 'spin up' $|\uparrow\rangle$ and 'spin down' $|\downarrow\rangle$, which are eigenvectors of the spin operator $S^z = \frac{\hbar}{2}\sigma^z$ with eigenvalues $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$ respectively. These can be intepreted as the basis of spin $\frac{1}{2}$ states. The R-matrix is then a member of $\mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$. Furthermore, let us assume the R-matrix is of 'difference form' i.e. $R(u, u') = R(u-u')$[1]. Upon setting $u = u_1 - u_3$ and $u' = u_2 - u_3$, the YBE becomes

$$R_{12}(u - u')R_{13}(u)R_{23}(u') = R_{23}(u')R_{13}(u)R_{12}(u - u') \tag{1.3.1}$$

and the RTT relation becomes

$$R_{ab}(u - u')T_a(u)T_b(u') = T_b(u')T_a(u)R_{ab}(u - u'). \tag{1.3.2}$$

In the system described, an orthonormal basis in $\mathbb{C}^2 \otimes \mathbb{C}^2$ is given by $|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$, where $|jk\rangle = |j\rangle \otimes |k\rangle$. Due to S^z symmetry (conservation of spin), we

[1]We shall make a note, however, that although very common, this is not true in general. For example: the R-matrix of the Hubbard model does not satisfy this property.

have that the action of the R-matrix on this basis is

$$R(u)\left|\uparrow\uparrow\right\rangle = a(u)\left|\uparrow\uparrow\right\rangle, \quad R(u)\left|\uparrow\downarrow\right\rangle = b(u)\left|\uparrow\downarrow\right\rangle + c(u)\left|\downarrow\uparrow\right\rangle$$
$$R(u)\left|\downarrow\downarrow\right\rangle = f(u)\left|\downarrow\downarrow\right\rangle, \quad R(u)\left|\downarrow\uparrow\right\rangle = d(u)\left|\uparrow\downarrow\right\rangle + e(u)\left|\downarrow\uparrow\right\rangle$$

where $a(u)$ through $f(u)$ is a set of complex functions, and we have omitted the subscripts of $R(u)$ because we are considering its action on 2 spaces of our choice. Since we picked one of two ways we could label the basis of $\mathbb{C}^2$, the system must remain unchanged under the relabeling $\uparrow \leftrightarrow \downarrow$, so we must have that $f(u) = a(u)$, $d(u) = c(u)$ and $e(u) = b(u)$. Thus, the form of the R-matrix acting on the column vector $(\left|\uparrow\uparrow\right\rangle, \left|\uparrow\downarrow\right\rangle, \left|\downarrow\uparrow\right\rangle, \left|\downarrow\downarrow\right\rangle)^T$ is

$$R(u) = \begin{pmatrix} a(u) & 0 & 0 & 0 \\ 0 & b(u) & c(u) & 0 \\ 0 & c(u) & b(u) & 0 \\ 0 & 0 & 0 & a(u) \end{pmatrix}. \tag{1.3.3}$$

Having significantly reduced the number of independent entries, it is relatively easy to check that $R(u)$ obeys the YBE if

$$\Delta(u) = \Delta(u') = \Delta(u - u') \tag{1.3.4}$$

where

$$\Delta(u) = \frac{a(u)^2 + b(u)^2 - c(u)^2}{2a(u)b(u)}. \tag{1.3.5}$$

This implies that $\Delta(u)$ does not depend on u, and thus shall be from now on denoted as Δ_q, where $q \in \mathbb{C}$. An elegant set of functions which satisfy (1.3.4) is given by

$$a(u;q) = 1, \quad b(u;q) = \frac{[u]_q}{[u+1]_q}, \quad c(u;q) = \frac{[1]_q}{[u+1]_q}, \tag{1.3.6}$$

where $[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}$. In this case, $\Delta_q = \frac{q+q^{-1}}{2}$. Hence the R-matrix

$$R(u;q) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{[u]_q}{[u+1]_q} & \frac{[1]_q}{[u+1]_q} & 0 \\ 0 & \frac{[1]_q}{[u+1]_q} & \frac{[u]_q}{[u+1]_q} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.3.7}$$

satisfies the YBE and is therefore the R-matrix of an integrable model [16, 27]. To infer more about the model, let us derive its Hamiltonian. We have that

$$\begin{aligned} R(u;q) &= \frac{1}{2}\left(1+\frac{[u]_q}{[u+1]_q}\right)\sigma^0\otimes\sigma^0+\frac{1}{2}\left(1-\frac{[u]_q}{[u+1]_q}\right)\sigma^z\otimes\sigma^z \\ &+\frac{[1]_q}{[u+1]_q}(\sigma^+\otimes\sigma^-+\sigma^-\otimes\sigma^+), \end{aligned} \tag{1.3.8}$$

where the Pauli matrices, together with the identity form the fundamental representation of $\mathfrak{gl}_2$[2]

$$\sigma^0=\mathbf{1}_2=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \sigma^+=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad \sigma^-=\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad \sigma^z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}, \tag{1.3.9}$$

and satisfy the usual $\mathfrak{gl}_2$ relations

$$[\sigma^z,\sigma^\pm]=\pm2\sigma^\pm,\quad [\sigma^+,\sigma^-]=\sigma^z,\quad \{\sigma^+,\sigma^-\}=\sigma^0. \tag{1.3.10}$$

This R-matrix becomes the permutation operator at $u_0=u_0'=0$. The Lax matrix $L_j(u;q)=R_{aj}(u;q)$ is obtained by mapping elements of the second space of (1.3.8) to the set of spin operators

$$\Sigma_L=\{\sigma_i^+,\sigma_i^-,\sigma_i^z,\sigma_i^0, i=1,\dots L\} \tag{1.3.11}$$

which satisfy

$$[\sigma_i^a,\sigma_j^b]=\delta_{ij}f^{ab}{}_c\sigma_i^c,\quad \{\sigma_i^+,\sigma_j^-\}=\delta_{ij}\sigma_i^0 \tag{1.3.12}$$

where f_{abc} are the $\mathfrak{sl}_2$ structure constants in (1.3.10). More specifically, this map is given by

$$\sigma^a\otimes\sigma^b\mapsto\sigma^a\otimes\sigma_j^b \tag{1.3.13}$$

Thus the Lax matrix is:

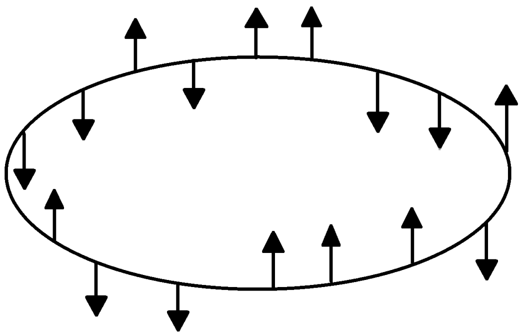
$$L_j(u;q)=\frac{1}{2}\begin{pmatrix}(1+\sigma_j^z)+\frac{[u]_q}{[u+1]_q}(1-\sigma_j^z) & \frac{2[1]_q}{[u+1]_q}\sigma_j^- \\ \frac{2[1]_q}{[u+1]_q}\sigma_j^+ & (1-\sigma_j^z)+\frac{[u]_q}{[u+1]_q}(1+\sigma_j^z)\end{pmatrix}. \tag{1.3.14}$$

[2]From now on, $\mathbf{1}_n$ will denote the $n\times n$ identity matrix.

The monodromy matrix is then $T_a(u;q) = L_L(u;q)L_{L-1}(u;q)\dots L_1(u;q)$. Using (1.2.27), we have that the Hamiltonian for this model is

$$\begin{aligned}\mathcal{H}_q &= \frac{d}{du}\ln\tau(u;q)|_{u=0} \\ &= \frac{4q\log(q)}{q-q^{-1}}\left(\sum_{i=1}^{L}\sigma_i^+\sigma_{i+1}^- + \sigma_i^-\sigma_{i+1}^+ + \frac{\Delta_q}{2}(\sigma_i^z\sigma_{i+1}^z - 1)\right)\end{aligned} \tag{1.3.15}$$

with periodic boundary conditions $\sigma_{L+1}^a = \sigma_1^a$. This is the Hamiltonian of the XXZ spin chain [15, 27]. The integrable structure of this model was one of the main focuses in the introduction of the QISM, and it is used as an example in several introductory reviews on quantum integrability [8, 23] and this thesis. Its physical intepretation is that of a 1 dimensional chain of sites labeled 1 to L populated by spin-$\frac{1}{2}$ particles that are allowed to interact between nearest-neighbouring sites. Each site is occupied by a particle with spin up or spin down:



Note the interaction between spins in the z direction – if $q \neq 1$ – is different than the other two directions; hence the name of the spin chain. The quantity Δ_q is known as the anisotropy parameter, and measures how far the model deviates from the soon to be discussed XXX model. We will drop the scalar multiplying the Hamiltonian and the constant term in (1.3.15) and set the XXZ Hamiltonian to be

$$\mathcal{H}_q = -\left(\sum_{i=1}^{L}\sigma_i^+\sigma_{i+1}^- + \sigma_i^-\sigma_{i+1}^+ + \frac{\Delta_q}{2}\sigma_i^z\sigma_{i+1}^z\right). \tag{1.3.16}$$

It is an easy exercise to check that the two site XXZ Hamiltonian eigenvector-eigenvalue pairs [8] are

$$\{|\uparrow\uparrow\rangle, -\tfrac{\Delta_q}{2}\},\ \ \{|\downarrow\downarrow\rangle, -\tfrac{\Delta_q}{2}\},\ \ \{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle, -1+\tfrac{\Delta_q}{2}\},\ \ \{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle, 1+\tfrac{\Delta_q}{2}\} \tag{1.3.17}$$

and hence the ground state is ferromagnetic when $\Delta_q > 1$ and antiferromagnetic when $\Delta_q < 1$. Furthermore, the system 'prefers' both unaligned ground states when $\Delta_q < -1$.

The main object governing the integrability of this model is the quantum group $U_q(\mathfrak{sl}_2)$ [20, 22, 24, 25]. This is the q-deformed universal enveloping algebra of $\mathfrak{sl}_2$, generated by four elements $K, K^{-1}, X^{\pm}$ satisfying

$$KK^{-1} = K^{-1}K = 1, \quad KX^{\pm} = q^{\pm 2}X^{\pm}K, \quad [X^+, X^-] = \frac{K - K^{-1}}{q - q^{-1}}. \tag{1.3.18}$$

A representation of these generators in terms of the pauli matrices is given by

$$X^{\pm} = \sigma^{\pm}, \quad K = q^{\sigma^z}, \tag{1.3.19}$$

which act on $V = \mathbb{C}^2$. $U_q(\mathfrak{sl}_2)$ acts on $\mathbb{C}^2 \otimes \mathbb{C}^2$ via the coproduct Δ, which is a map $U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$ that preserves the relations (1.3.18) [24]. In particular,

$$\begin{aligned} &\Delta(X^+) = X^+ \otimes 1 + K \otimes X^+, && \Delta(X^-) = X^- \otimes K^{-1} + 1 \otimes X^- \\ &\Delta(K) = K \otimes K, && \Delta(K^{-1}) = K^{-1} \otimes K^{-1}. \end{aligned} \tag{1.3.20}$$

The generators act on the Hilbert space being $\mathcal{V} = (\mathbb{C}^2)^{\otimes L}$ via the L-th coproduct [27]

$$\begin{aligned} &\Delta^L(K) = K^{\otimes L}, && \Delta^L(X^+) = \sum_{i=1}^{L} K^{\otimes(i-1)} \otimes X^+ \otimes 1_2^{\otimes(L-i)}, \\ &\Delta^L(K^{-1}) = (K^{-1})^{\otimes L}, && \Delta^L(X^-) = \sum_{i=1}^{L} 1_2^{\otimes(i-1)} \otimes X^- \otimes (K^{-1})^{\otimes(L-i)}, \end{aligned} \tag{1.3.21}$$

which in terms of the spin operators is given by

$$\begin{aligned} &\Delta^L(K) = \prod_{j=1}^{L} q^{\sigma_j^z}, && \Delta^L(X^+) = \sum_{j=1}^{L} \left(\prod_{k=1}^{j-1} q^{\sigma_k^z} \right), \\ &\Delta^L(K^{-1}) = \prod_{j=1}^{L} q^{-\sigma_j^z}, && \Delta^L(X^-) = \sum_{j=1}^{L} \left(\sigma_j^- \prod_{k=j+1}^{L} q^{-\sigma_k^z} \right). \end{aligned} \tag{1.3.22}$$

If we consider the XXZ spin chain Hamiltonian with the following open boundary conditions

$$\mathcal{H}_q^{open} = -\left(\sum_{i=1}^{L-1} \sigma_i^+ \sigma_{i+1}^- + \sigma_i^- \sigma_{i+1}^+ + \frac{\Delta_q}{2} \sigma_i^z \sigma_{i+1}^z \right) - \frac{q - q^{-1}}{4}(\sigma_1^z - \sigma_L^z) \tag{1.3.23}$$

then we have that $[\mathcal{H}_q^{open}, \Delta^L(g)] = 0$ for $g = X^{\pm}, K$ and K^{-1} [26]. Hence the XXZ spin chain possesses an $U_q(\mathfrak{sl}_2)$ symmetry in this way. In addition, the symmetry of the XXZ Hamiltonian can be enhanced even further to the affine quantum group $U_q(\widehat{\mathfrak{sl}_2})$ when the length of the chain is infinite

$$\mathcal{H}_q^{\infty} = -\left(\sum_{i\in\mathbb{Z}} \sigma_i^+\sigma_{i+1}^- + \sigma_i^-\sigma_{i+1}^+ + \frac{\Delta_q}{2}\sigma_i^z\sigma_{i+1}^z\right). \tag{1.3.24}$$

This is only possible in the antiferromagnetic regime $\Delta_q < -1$ $(-1 < q < 0)$ where one may neglect boundary terms at $\pm\infty$ over the antiferromagnetic vacuum [27, 28, 29, 30].

Quantum groups and their connections to the YBE are very interesting in their own right and a major subject of study by both theoretical physicists and algebraists [24, 31, 32, 33, 34]. I will however abstain from giving a formal review of them as one cannot do so briefly and properly at the same time. Thus details about them will be brought up when necessary. The question remains though: why bring up the symmetry of the XXZ spin chain in the first place? The answer is simple: integrable systems have many conserved charges which, intuitively, should come from extended symmetries. Thus the set of symmetries is a natural starting point in our investigation of the integrable structure of these models. But first, let us look at a method of solving these models using the Yang-Baxter algebra.

1.4 The Algebraic Bethe Ansatz

Integrability provides a set of techniques to find the spectra of quantities of interest of 1+1D quantum models, i.e. solve the equation

$$\mathcal{H}\left|\psi\right\rangle = E\left|\psi\right\rangle, \tag{1.4.1}$$

where $\mathcal{H}$ is the Hamiltonian of the model. A standard method to do this, pioneered by Bethe in 1931, is the Coordinate Bethe Ansatz (CBA) [36], which assumes the form of the n-particle eigenfunctions of the Hamiltonian to be

$$\left|\psi_n\right\rangle = \sum_{1\leq x_1<x_2<\ldots<x_n\leq L}\ \sum_{p\in\mathcal{P}_n} A_p \exp\big(ik_{p(1)}x_1 + \ldots + ik_{p(n)}x_n\big)\left|x_1x_2\ldots x_n\right\rangle \tag{1.4.2}$$

where x_i denotes the position of the particle on the spin chain and k_i its quasimomentum. The second sum runs over all elements of $\mathcal{P}_n$, the permutation group of n elements.

Alternatively, an R-matrix that satisfies the YBE can also be used to solve the underlying model through a technique known as the Algebraic Bethe Ansatz (ABA) [7, 19, 40, 38]. In this Section we will review how to do this for the most famous spin chain model in the physics literature: the Heisenberg XXX spin chain [36, 37], which is the case in (1.3.16) when $q \to 1$:

$$\mathcal{H}^H = J \lim_{q\to 1} \mathcal{H}_q = -J \sum_{i=1}^{L} \left(\sigma_i^+ \sigma_{i+1}^- + \sigma_i^- \sigma_{i+1}^+ + \frac{1}{2} \sigma_i^z \sigma_{i+1}^z \right). \tag{1.4.3}$$

In the future we will refer to this model as simply the Heisenberg spin chain. The coupling constant J does not affect the integrability of the model, but it is relevant physically: depending on its sign, the lowest energy states of this Hamiltonian occur when neighbouring spins are aligned or unaligned. Thus the Heisenberg spin chain describes a ferromagnet when $J > 0$ and antiferromagnet when $J < 0$, giving an explanation for the origins of magnetism at the quantum level. Additionally, one can easily check that the operators

$$\mathcal{E}_0^{\pm} = \sum_{i=1}^{L} \sigma_i^{\pm}, \quad \mathcal{E}_0 = \sum_{i=1}^{L} \sigma_i^z \tag{1.4.4}$$

commute with $\mathcal{H}^H$ and satisfy the $\mathfrak{sl}_2$ defining relations $[\mathcal{E}_0^z, \mathcal{E}_0^{\pm}] = \pm 2\mathcal{E}_0^{\pm}$ and $[\mathcal{E}_0^+, \mathcal{E}_0^-] = \mathcal{E}_0^z$. Hence the Heisenberg spin chain possesses an $\mathfrak{sl}_2$ symmetry [7] [3]. Its R-matrix, $R^H(u)$, is obtained by taking the limit of $R(u;q)$ as $q \to 1$ [34]. We set

$$R^H(u) := (u+1) \lim_{q\to 1} R(u;q) = \begin{pmatrix} u+1 & 0 & 0 & 0 \\ 0 & u & 1 & 0 \\ 0 & 1 & u & 0 \\ 0 & 0 & 0 & u+1 \end{pmatrix} = u\mathbf{1}_4 + \mathbf{P} \tag{1.4.5}$$

where $\mathbf{P}$ is the 4×4 permutation operator.

The existence of an R-matrix allows for a procedure to construct n-particle eigenvectors $|\psi_n\rangle$ of this Hamiltonian. Since $\mathcal{H}^H$ commutes with the transfer matrix, they share the same eigenvectors, and we can thus diagonalise $\mathcal{H}^H$ using the latter. Let us write the Lax matrix of the Heisenberg spin chain with a shift in the spectral parameter so that $L_j^H(\frac{1}{2}) = P_{aj}$:

$$L_j^H(u) = \begin{pmatrix} u + \frac{1}{2}\sigma_j^z & \sigma_j^- \\ \sigma_j^+ & u - \frac{1}{2}\sigma_j^z \end{pmatrix}. \tag{1.4.6}$$

This still leads to a monodromy matrix for the Heisenberg spin chain since, as we

[3]And the trivial symmetry, so that the actual symmetry of the model is $\mathfrak{gl}_2$.

can see in (1.3.2), shifting the spectral parameters u and u' by the same value does not change the argument of $R(u-u')$. Writing the monodromy matrix as

$$T^H(u) = L^H_L(u)L^H_{L-1}(u)\dots L^H_1(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}, \tag{1.4.7}$$

the n-particle eigenvectors $|\psi_n\rangle$ of $\mathcal{H}^H$ must also be eigenvectors of the transfer matrix

$$\tau^H(u) = \mathrm{tr}_a T^H_a(u) = A(u) + D(u). \tag{1.4.8}$$

Recall that in a QFT, $|\psi_n\rangle$ are usually written in terms of creation operators acting on the vacuum. Our choice of pseudo-vacuum for the Hilbert space $\mathcal{V} = (\mathbb{C}^2)^{\otimes L}$ is

$$|\Omega\rangle = \bigotimes_{i=1}^{L} |\uparrow\rangle_i = |\uparrow\uparrow\dots\uparrow\rangle \tag{1.4.9}$$

where a flip of the spin $\uparrow$ to $\downarrow$ in the i-th space would represent a particle excitation in that position. Recall the action of the spin operators on the Hilbert space

$$\begin{aligned} &\sigma^+_i |\downarrow\rangle_i = |\uparrow\rangle_i\,, \quad \sigma^z_i |\uparrow\rangle_i = |\uparrow\rangle_i\,, \qquad \sigma^a_i |\uparrow\rangle_j = |\uparrow\rangle_j\ \sigma^a_i\,, \\ &\sigma^-_i |\uparrow\rangle_i = |\downarrow\rangle_i\,, \quad \sigma^z_i |\downarrow\rangle_i = -|\downarrow\rangle_i\,, \quad \sigma^a_i |\downarrow\rangle_j = |\downarrow\rangle_j\ \sigma^a_i \end{aligned} \tag{1.4.10}$$

and $\sigma^+_i|\uparrow\rangle_i = \sigma^-_i|\downarrow\rangle_i = 0$. It is not difficult to see that the operatorial entries of the monodromy matrix (1.4.7) act on $|\Omega\rangle$ as

$$A(u)\,|\Omega\rangle = (u+\tfrac{1}{2})^L\,|\Omega\rangle\,, \quad D(u)\,|\Omega\rangle = (u-\tfrac{1}{2})^L\,|\Omega\rangle\,, \quad C(u)\,|\Omega\rangle = 0. \tag{1.4.11}$$

The Algebraic Bethe Ansatz (ABA) [7, 19] consists of assuming the $C(u)$ are annihilation operators and the $B(u)$ are creation operators, such that any n-particle eigenvector of the Heisenberg spin chain has the form[4]

$$|\psi_n\rangle = |u_1,\dots,u_n\rangle = B(u_1)\dots B(u_n)\,|\Omega\rangle\,. \tag{1.4.12}$$

To obtain the necessary conditions such that $|\psi_n\rangle$ is an eigenvector of the transfer matrix, we have to study how $A(u)$ and $D(u)$ act on it. Using the RTT relation,

[4]This is equivalent to [36].

we obtain the following commutation relations between $B(u), A(u)$ and $D(u)$:

$$\begin{aligned}
A(u)B(u') &= \frac{1+u'-u}{u'-u}B(u')A(u) - \frac{1}{u'-u}B(u)A(u'),\\
D(u)B(u') &= \frac{1+u-u'}{u-u'}B(u')D(u) - \frac{1}{u-u'}B(u)D(u'),\\
B(u)B(u') &= B(u')B(u).
\end{aligned} \tag{1.4.13}$$

Acting with the transfer matrix on $|\psi_n\rangle$, and using the above commutation relations, we have that

$$\begin{aligned}
\tau^H(u)\,|\psi_n\rangle &= A(u)\,|\psi_n\rangle + D(u)\,|\psi_n\rangle\\
&= \left((u+\tfrac{1}{2})^L\prod_{i=1}^{n}\frac{(u-u_i+1)}{u-u_i} + (u-\tfrac{1}{2})^L\prod_{i=1}^{n}\frac{(u-u_i-1)}{u-u_i}\right)|\psi_n\rangle\\
&\quad + \sum_{i=1}^{n}\left(\frac{(u_i+\frac{1}{2})^L}{(u-u_i)}\prod_{k\neq i}^{n}\frac{(u_i-u_k-1)}{(u_i-u_k)} - \frac{(u_i-\frac{1}{2})^L}{(u-u_i)}\prod_{k\neq i}^{n}\frac{(u_i-u_k+1)}{(u_i-u_k)}\right)\\
&\quad \times |u_1,\ldots u_{i-1},u,u_{i+1},\ldots u_n\rangle\,.
\end{aligned} \tag{1.4.14}$$

So $|\psi_n\rangle$ is an eigenvector of the transfer matrix with eigenvalue

$$\Lambda(u,\{u_i\}_n) = (u+\tfrac{1}{2})^L\prod_{i=1}^{n}\frac{(u-u_i-1)}{u-u_i} + (u-\tfrac{1}{2})^L\prod_{i=1}^{n}\frac{(u-u_i+1)}{u-u_i} \tag{1.4.15}$$

if the second line of (1.4.14) – called the 'unwanted terms' – vanishes i.e. if the following equations hold

$$\left(\frac{u_k+\frac{1}{2}}{u_k-\frac{1}{2}}\right)^L = \prod_{i\neq k}^{n}\frac{u_k-u_i+1}{u_k-u_i-1}, \qquad k=1,\ldots n \tag{1.4.16}$$

which are the famous Bethe equations [36]. Using these it is possible to find an exact solution to the Bethe roots u_i and obtain the exact form of any quantity of the model as function of these. Using (1.4.15) and that the shift operator is related to the momentum operator $\hat{p}$ by the formula $e^{i\hat{p}} = \mathcal{U} = \tau(\frac{1}{2})$, we have that the momentum of an n-particle eigenstate is given by

$$p_n = -i\ln\Lambda(\tfrac{1}{2},\{u_i\}_n) = \mathrm{i}\sum_{i=1}^{n}\ln\left(\frac{u_i-\frac{1}{2}}{u_i+\frac{1}{2}}\right). \tag{1.4.17}$$

Similarly, its energy is given by

$$E_n = -2J\frac{d}{du}\ln\,\Lambda(u,\{u_i\}_n)|_{u=\frac{1}{2}} = -2JL - 2J\sum_{i=1}^{n}\frac{1}{u_i^2-\frac{1}{4}}. \tag{1.4.18}$$

1.5 Yangian symmetries

A natural question to ask about the Heisenberg spin chain is the following: what is the algebraic structure governing its integrability? To answer this question, we have to take a look at the properties of the R-matrix that can be inferred from the RTT relation. Let us direct our attention to the structure of the monodromy matrix: can write $T^H(u)$ as a power series in u

$$T^H(u) = u^L \mathbf{1}_4 + u^L \sum_{a,b} \sigma^a \otimes \left(\sum_{n=0}^{L-1} u^{-n-1} Q_n^b\right) \tag{1.5.1}$$

where the $Q_n^a, a = 0,+,-,z$ are operators built of the spin operators σ_j^a. The (a,b) in the expansion are $(+,-),(-,+),(z,z)$ and $(0,0)$. One can easily see that $Q_0^{\pm} = \mathcal{E}_0^{\pm}$ and $Q_0^z = \mathcal{E}_0^z$, the $\mathfrak{sl}_2$ representation in terms of the spin operators introduced in (1.4.4). The relations between operators Q_n^a with different values of n can be obtained using the RTT relation [42, 55]:

$$[Q_0^a, Q_n^b] = f^{ab}{}_c Q_n^c \tag{1.5.2}$$

and

$$\begin{aligned} [Q_{n+1}^z, Q_m^{\pm}] - [Q_{m+1}^z, Q_n^{\pm}] &= \mp \tfrac{1}{2}([Q_n^z, Q_m^{\pm}] - [Q_m^z, Q_n^{\pm}] + \{Q_m^0, Q_n^{\pm}\} - \{Q_n^0, Q_m^{\pm}\}) \\ [Q_{n+1}^+, Q_m^-] - [Q_n^+, Q_{m+1}^-] &= \tfrac{1}{4}([Q_n^0, Q_m^z] - [Q_m^0, Q_n^z]). \end{aligned} \tag{1.5.3}$$

Furthermore, if one collects terms of the expansion of $T^H(u)\otimes T^H(u')$, taking the tensor product in the quantum space and performing the usual matrix multiplication on the auxiliary space [47, 48], we obtain the following properties:

$$[Q_0^a \otimes 1 + 1 \otimes Q_0^a, R^H(u)] = 0 \tag{1.5.4}$$

and

$$\begin{aligned} &[(Q_1^a + uQ_0^a)\otimes 1 + 1\otimes(Q_1^a + vQ_0^a) + \tfrac{1}{2}f^a{}_{bc}Q_0^b\otimes Q_0^c]R^H(u-v) = \\ &\quad = R^H(u-v)[(Q_1^a + uQ_0^a)\otimes 1 + 1\otimes(Q_1^a + vQ_0^a) - \tfrac{1}{2}f^a{}_{bc}Q_0^b\otimes Q_0^c] \end{aligned} \tag{1.5.5}$$

for $a = \pm, z, 0$. Hence the Heisenberg spin chain enjoys additional relations to a standard $\mathfrak{gl}_2$ symmetry. One may recognise (1.5.2) and (1.5.3) as the defining relations of the Yangian algebra of $\mathfrak{gl}_2$, $\mathcal{Y}(\mathfrak{gl}_2)$, and the objects appearing in (1.5) and (1.5.5) as the coproduct of its generators.

The Yangian $\mathcal{Y}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ first appeared in Drinfeld's work [24, 44] and was also implicit in [45]. If an integrable model has classical symmetry $\mathfrak{g}$, preserved in the quantum theory, then the Yangian $\mathcal{Y}(\mathfrak{g})$ (or $U_q(\widehat{\mathfrak{g}})$ in the case $\mathfrak{g}$ is q-deformed by quantisation [25]) is responsible for its integrable structure. This has been shown for a wide range of models, from the classical version in the Principal Chiral Model (PCM) [46] to that of the $\mathrm{Ad}S_5 \times S^5$ superstring in the planar limit [43]. Several reviews have been done on their significance in the study of 1+1D integrable models [42, 47, 48, 56], which we shall follow closely in the upcoming description. Algebraically, it is defined as follows. Let the Lie algebra $\mathfrak{g}$ be generated by $\{Q_0^a\}$, $a = 1, \ldots, \dim(\mathfrak{g})$ with structure constants f_{abc} and coproduct

$$\begin{aligned} \Delta:\ U\mathfrak{g} &\to U\mathfrak{g} \otimes U\mathfrak{g} \\ Q_0^a &\mapsto Q_0^a \otimes 1 + 1 \otimes Q_0^a \end{aligned} \tag{1.5.6}$$

where $U\mathfrak{g}$ is the universal enveloping algebra, i.e. the set of powers, polynomials and series in Q_0^a subject to the Lie bracket. If $\mathfrak{g}$ is the Lie symmetry of a 1+1D quantum model living on $V^{\otimes L}$ where V is a Hilbert space, the coproduct Δ defines the action of the generators of $\mathfrak{g}$ on $V \otimes V$ and induces a Hopf algebra structure on $\mathfrak{g}$ [44]. For the Lie generators, Δ is trivial; however in general the coproduct is a map $U\mathfrak{g}$ to $U\mathfrak{g} \otimes U\mathfrak{g}$ that is coassociative

$$(\Delta \otimes 1)\Delta(x) = (1 \otimes \Delta)\Delta(x),\ \forall x \in U\mathfrak{g} \tag{1.5.7}$$

and a Lie algebra homomorphism:

$$\Delta([x, y]) = [\Delta(x), \Delta(y)] \quad \forall\ x, y \in U\mathfrak{g}. \tag{1.5.8}$$

The Yangian $\mathcal{Y}(\mathfrak{g})$ [47] is the enveloping algebra generated by $\mathfrak{g}$ and a second set of generators $\{Q_1^a\}$ satisfying

$$[Q_0^a, Q_1^b] = f^{ab}{}_c Q_1^c \tag{1.5.9}$$

with non-trivial coproduct

$$\begin{aligned} \Delta: \ \mathcal{Y}(\mathfrak{g}) &\to \mathcal{Y}(\mathfrak{g}) \otimes \mathcal{Y}(\mathfrak{g}) \\ Q_1^a &\mapsto Q_1^a \otimes 1 + 1 \otimes Q_1^a + \frac{\alpha}{2} f^a{}_{bc} Q_0^c \otimes Q_0^b \end{aligned} \tag{1.5.10}$$

where α is proportional to the coupling constant of the theory. These operators obey the so-called Drinfel'd 'terrible' relations [44]:

$$[Q_1^a, [Q_1^b, Q_0^c]] - [Q_0^a, [Q_1^b, Q_1^c]] = \frac{\alpha^2}{2} a_{abcdeg} \{Q_0^d, Q_0^e, Q_0^g\} \tag{1.5.11}$$

and

$$[[Q_1^a, Q_1^b], [Q_0^l, Q_1^m]] + [[Q_1^l, Q_1^m], [Q_0^a, Q_1^b]] = \alpha^2 (a_{abcdeg} f_{mlc} + a_{lmcdeg} f_{abc}) \{Q_0^d, Q_0^e, Q_1^g\} \tag{1.5.12}$$

where

$$\{x_1, x_2, x_3\} = \sum_{i \neq j \neq k} x_i x_j x_k, \quad a_{abcdeg} = \frac{1}{24} f_{adi} f_{bej} f_{cgk} f_{ijk} \tag{1.5.13}$$

The generators Q_0^a and Q_1^a are referred to as level 0 and level 1 Yangian generators respectively, and level n operators K_n^a – which may be a linear combination of generators up to order n – are obtained by computing the commutator of k and $n-k$ level generators.

Finite dimensional representations of $\mathcal{Y}(\mathfrak{g})$ are realized in one-parameter families via the automorphism

$$\begin{aligned} \phi_\mu : \mathcal{Y}(\mathfrak{g}) &\to \mathcal{Y}(\mathfrak{g}) \\ Q_0^a &\mapsto Q_0^a \\ Q_1^a &\mapsto Q_1^a + \mu Q_0^a \, . \end{aligned} \tag{1.5.14}$$

This automorphism is of great importance. To illustrate this, let us remark that for certain irreducible representations ρ of $\mathfrak{g}$, enumerated in [44], one can construct a representation $\widetilde{\rho}$ of $\mathcal{Y}(\mathfrak{g})$

$$\widetilde{\rho}(Q_0^a) = \rho(Q_0^a), \quad \widetilde{\rho}(Q_1^a) = 0, \tag{1.5.15}$$

which combined with the automorphism (1.5.14), leads to the so called evaluation representation of $\mathcal{Y}(\mathfrak{g})$ with spectral parameter u, denoted $\rho_u := \widetilde{\rho} \circ \phi_{\frac{u\alpha}{2}}$:

$$\rho_u(Q_0^a) = \rho(Q_0^a), \quad \rho_u(Q_1^a) = \frac{u\alpha}{2}\rho(Q_0^a). \tag{1.5.16}$$

This representation will become significant in the next Chapter, because in the context of integrable boundaries, a reflection changes the sign of the spectral parameter.

Finally, let us state that $\mathcal{Y}(\mathfrak{g})$ is equipped with the additional Hopf algebra properties: a co-unit, physically interpreted as a 'vacuum'

$$\begin{aligned} \epsilon : \mathcal{Y}(\mathfrak{g}) &\rightarrow \mathbb{C} \\ Q_n^a &\mapsto 0 \end{aligned} \tag{1.5.17}$$

and an antipode (physically a PT symmetry)

$$\begin{aligned} s : \mathcal{Y}(\mathfrak{g}) &\rightarrow \mathcal{Y}(\mathfrak{g}) \\ Q_0^a &\mapsto -Q_0^a \\ Q_1^a &\mapsto -Q_1^a + \frac{1}{2} f^a{}_{bc} Q_0^b Q_0^c \,. \end{aligned} \tag{1.5.18}$$

For each Yangian algebra there is an associated R-matrix which satisfies the YBE: relations (1.5.2) and (1.5.3) are true iff the RTT relation holds. Given a Yangian $\mathcal{Y}(\mathfrak{g})$, one can obtain a suitable R-matrix $R(u)$ (in this particular case, of difference form) by requiring the intertwining relation [48, 55]

$$(\Delta^{op} x)R(u) = R(u)(\Delta x) \;\forall x \in \mathcal{Y}(\mathfrak{g}). \tag{1.5.19}$$

where $\Delta^{op} = P\Delta$.

For the Heisenberg spin chain, even though the Yangian algebra should be $\mathcal{Y}(\mathfrak{gl}_2)$, we are going to restrict our study to its more interesting subalgebra $\mathcal{Y}(\mathfrak{sl}_2)$. This is because

$$\mathcal{Y}(\mathfrak{gl}_2) \cong Z(\mathcal{Y}(\mathfrak{sl}_2)) \otimes \mathcal{Y}(\mathfrak{sl}_2) \tag{1.5.20}$$

where $Z(\mathcal{Y}(\mathfrak{gl}_2))$ is the center, which will not be relevant in discussing symmetry breakings.

The form of the level 0 and level 1 $\mathcal{Y}(\mathfrak{sl}_2)$ generators can be easily extracted from the expansion of the monodromy matrix of the Heisenberg spin chain. They are in a representation $\rho_L : U(\mathfrak{sl}_2) \to \Sigma_L$ given by $\rho_L(Q_0^{\pm}) = \mathcal{E}_0^{\pm}$, $\rho_L(Q_0^z) = \mathcal{E}_0^z$, and, after rescaling by the coupling constant $-J$,

$$\rho_L(Q_1^{\pm}) = \mathcal{E}_1^{\pm} + J\mathcal{E}_0^{\pm}, \quad \rho_L(Q_1^z) = \mathcal{E}_1^z + \mathcal{E}_0^z \tag{1.5.21}$$

where

$$\mathcal{E}_1^{\pm} = \pm\tfrac{J}{2} \sum_{1\le i<j\le L} (\sigma_i^{\pm}\sigma_j^z - \sigma_j^{\pm}\sigma_i^z), \quad \mathcal{E}_1^z = J \sum_{1\le i<j\le L} (\sigma_i^-\sigma_j^+ - \sigma_j^-\sigma_i^+) \qquad (1.5.22)$$

satisfy all properties of level 1 generators of $\mathcal{Y}(\mathfrak{sl}_2)$ The shift in level 0 generators to get (1.5.22) from (1.5.21) does not affect the integrability of the model since it is an automorphism of the Yangian (1.5.14). Furthermore, one can check that in the evaluation representation (1.5.16) with $\rho_a(Q_0^a) = \sigma^a$,

$$\begin{aligned} \rho_u(\Delta Q_0^a) &= \sigma^a \otimes \mathbf{1}_2 + \mathbf{1}_2 \otimes \sigma^a \\ \rho_u(\Delta Q_1^a) &= \frac{J}{2}(u\sigma^a \otimes \mathbf{1}_2 + \mathbf{1}_2 \otimes u'\sigma^a) + \frac{J}{2} f^a{}_{bc}\sigma^b \otimes \sigma^c \end{aligned} \qquad (1.5.23)$$

satisfy the intertwining relation (1.5.19) with $R^H(u-u')$.

The Yangian $\mathcal{Y}(\mathfrak{g})$ can be obtained from the quantum affine group $U_q(\widehat{\mathfrak{sl}_2})$ in the limit $q \to 1$ [50]. Hence we expect the Yangian generators to be symmetries of the Hamiltonian. However, we have

$$\begin{aligned} [\mathcal{H}^H, \mathcal{E}_1^{\pm}] &= \pm\frac{J^2}{2}\sum_{i=1}^{L-1}(\sigma_{i+1}^{\pm} - \sigma_i^{\pm}) = \pm\frac{J^2}{2}(\sigma_L^{\pm} - \sigma_1^{\pm}), \\ [\mathcal{H}^H, \mathcal{E}_1^z] &= J^2\sum_{i=1}^{L-1}(\sigma_i^z - \sigma_{i+1}^z) = J^2(\sigma_1^z - \sigma_L^z), \end{aligned} \qquad (1.5.24)$$

where $\mathcal{H}^H$ is the Heisenberg spin chain Hamiltonian defined on the segment from $i = 1$ to $i = L$. Thus it turns out that, as in the case of the affine quantum group symmetry of the XXZ spin chain [27, 29], $\mathcal{Y}(\mathfrak{sl}_2)$ is only a symmetry of the Heisenberg spin chain Hamiltonian in the antiferromagnetic regime ($J < 0$) if the length of the chain is infinite and we impose antiferromagnetic boundary conditions at $\pm\infty$, as only then we can neglect the boundary terms (1.5.24). In addition, as seen above, one may neglect these terms when working on asymptotic states as one can intepret them the result of integrating discretised total derivatives over the infinite interval [49]. In this case, commutators of the Yangian generators will produce an infinite tower of symmetries of the Hamiltonian, thus ensuring integrability.

1.6 The reflection equation

So far, we have focused almost completely on models living on the circle and the infinite interival. For the periodic case, a variety of bonundary conditions are at our disposal: if the model has Lie symmetry $\mathfrak{g}$, they are determined by a $\mathfrak{g}$-invariant matrix M acting on the auxiliary space of the monodromy matrix

$$T_a(u)' = M_a T_a(u). \tag{1.6.1}$$

The matrix M will result in the usual periodicity when it is proportional to the identity, and one can obtain the so called integrable twisted periodic boundary conditions for more general forms of M (for example, in the Hubbard model it can be seen in [120]) .

Now we will focus on integrable models with open boundary conditions. We are interested on how these boundaries affect the mathematical structures underlying the integrability of the bulk model. This is, in fact, the main subject of research throughout this thesis. A similar procedure to the QISM, but for models with open boundary conditions was invented in 1987 by Sklyanin: the boundary QISM [61]. It too is based around the construction of a family of mutually commuting transfer matrices, which can be diagonalised using the ABA. This construction led to the analysis of all possible integrable boundary conditions via algebraic study of how the boundary breaks the symmetry of the bulk model - usually a quantum group or a Yangian - and the necessary conditions to preserve enough symmetry so that the model remains integrable. The ability to still generate an infinite tower of conserved charges [52, 53, 54] depends on this boundary symmetry being a coideal subalgebra of a quantum group [56] or twisted Yangian [55]. In this Section we will focus on the latter. Both these symmetries have been studied extensively in many spin chains [57] and D-brane configurations the AdS/CFT correspondence [58, 59, 60, 119, 135].

In addition to the R-matrix, in this Section we will consider the K-matrices

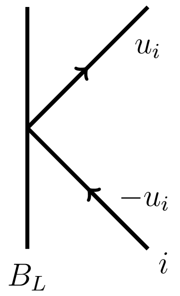
$$K^{\pm}(u) : V \to V \tag{1.6.2}$$

where V is a Hilbert space and u is the spectral parameter. This operator represents the reflection of a particle off a boundary, and it is also referred to as the reflection matrix or the boundary S-matrix. $K^{+}(u)$ governs the reflection off a right boundary and $K^{-}(u)$ that off a left boundary. When acting on $\mathcal{V} = \bigotimes_{i=1}^{L} V_{\rho_i}$, the operator $K_i^{\pm}(u_i)$ acts as $K^{\pm}(u_i)$ on the i-th space and the identity everywhere else. For example,

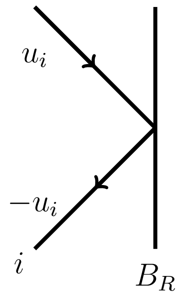
$$K_1^{\pm}(u_1) = K^{\pm}(u_1) \otimes \bigotimes_{i=2}^{L} \mathbf{1}_{\rho_i}. \tag{1.6.3}$$

As in the case of the R-matrix, it is convenient to represent the K-matrices reflecting off a left(right) boundary $B_{L(R)}$ diagrammatically

$$K_i^-(u_i) =$$



$$K_i^+(u_i) =$$



(1.6.4)

The result in obtaining open boundary conditions assumes the R-matrix is of difference form. However, we will not make this assumption and instead adapt the procedure to a general R-matrix. Suppose $R(u, u')$ satisfies the YBE, and $K^{\pm}(u)$ satisfies $K_i^-(u_0) \propto \mathbf{1}_{V_{\rho_i}}$, where $R(u_0, u_0) = P$. Furthemore, let us assume unitarity

$$R_{12}(u_1, u_2) R_{21}(u_2, u_1) = \mathbf{1}_{V_{\rho_1} \otimes V_{\rho_2}} \tag{1.6.5}$$

and crossing unitarity

$$R_{12}^{tr_1}(u_1, -u_2) R_{21}^{tr_1}(-u_2, u_1) = \nu(u_1, u_2) \mathbf{1}_{V_{\rho_1} \otimes V_{\rho_2}} \tag{1.6.6}$$

for some scalar function $\nu(u_1, u_2)$ that is symmetric in u_1 and u_2. Then, if the K-matrix $K^-(u)$ is a solution of the left reflection equation (left RE)

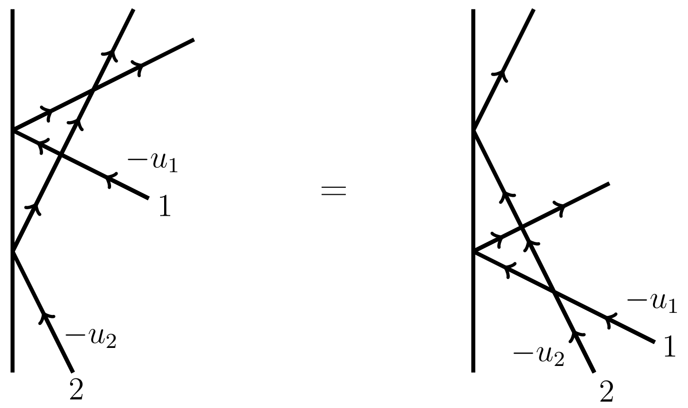
$$\begin{aligned} R_{12}(u_1, u_2) K_1^-(u_1) R_{21}(u_2, -u_1) K_2^-(u_2) = \\ = K_2^-(u_2) R_{12}(u_1, -u_2) K_1^-(u_1) R_{21}(-u_2, -u_1) \end{aligned} \tag{1.6.7}$$

and $K^+(u)$ is a solution of the right reflection equation (right RE)

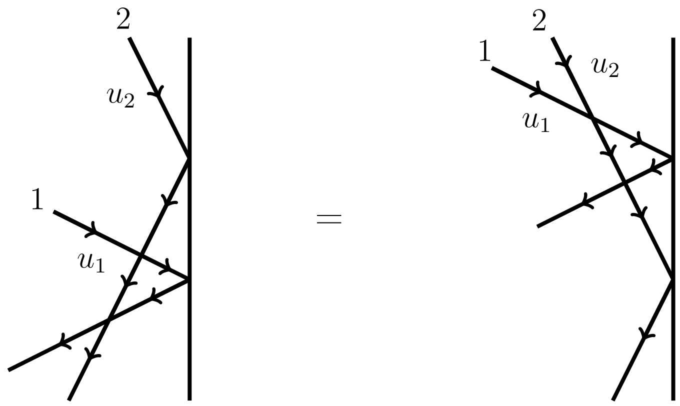
$$\begin{aligned} R_{12}(-u_1, -u_2) K_1^+(u_1) R_{21}(-u_2, u_1) K_2^+(u_2) = \\ = K_2^+(u_2) R_{12}(-u_1, u_2) K_1^+(u_1) R_{21}(u_2, u_1), \end{aligned} \tag{1.6.8}$$

then one can obtain a quantum integrable system with open boundaries from

$R(u,u')$ and $K^{\pm}(u)$ [61]. Both the left and right RE can be expressed pictorially:



(1.6.9)



(1.6.10)

We now proceed to show how to construct a commuting family of transfer matrices using the left and right K-matrices. Let $T_a(u)$ be the bulk monodromy matrix (1.2.15). Then one can show [61] that the right boundary monodromy matrix, defined as

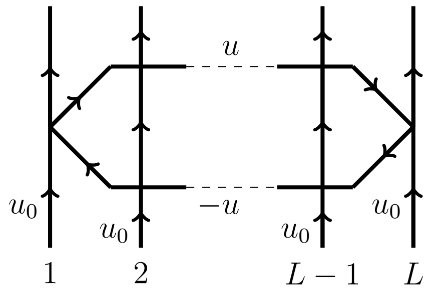
$$T_a^+(u) := T_a^{-1}(-u)K_a^+(u)T_a(u) \tag{1.6.11}$$

satisfies boundary analogue of the RTT relation: the boundary Yang-Baxter algebra

$$R_{ab}(-u,-u')T_a^+(u)R_{ba}(-u',u)T_b^+(u') = T_b^+(u')R_{ab}(-u,u')T_a^+(u)R_{ba}(u',u) \tag{1.6.12}$$

where a and b are auxiliary spaces in the fundamental representation, and the K-matrices act on such spaces. If one defines the boundary or double row monodromy matrix

$$T_a^B(u) := K_a^-(u)T_a^+(u) = \tag{1.6.13}$$



then the boundary transfer matrices $\tau^B(u) = \mathrm{tr}_a T_a^B(u)$ form a commuting family of operators:

$$[\tau^B(u), \tau^B(u')] = 0 \tag{1.6.14}$$

and thus, analogous to the case of periodic boundary conditions, the Hamiltonian

$$\mathcal{H}_B = \frac{d}{du}\ln\big(\tau^B(u)\big)|_{u=u_0} \tag{1.6.15}$$

describes an integrable 1+1D model with open boundary conditions. We have provided the algebraic proof of (1.6.14) in the Appendix A.1. A proof specialised to R-matrices of difference form can be found in the original paper [61] together with the ABA for the open XXZ spin chain. An alternative, more general proof is given in [62].

As an illustrative example of an integrable model with open boundaries, consider the Heisenberg spin chain, with symmetric R-matrix $R(u,u') = R^H(u-u') \in \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$, K-matrices $K^\pm(u) \in \mathrm{End}(\mathbb{C}^2)$ and $u_0 = u_0' = 0$. The REs become

$$\begin{aligned} R^H(u-u')(K^-(u)\otimes \mathbf{1}_2)R^H(u+u')(\mathbf{1}_2 \otimes K^-(u')) = \\ = (\mathbf{1}_2 \otimes K^-(u'))R^H(u+u')(K^-(u)\otimes \mathbf{1}_2)R^H(u-u'), \end{aligned} \tag{1.6.16}$$

and

$$\begin{aligned} R^H(u'-u)(K^+(u)\otimes \mathbf{1}_2)R^H(-u-u'-2\eta)(\mathbf{1}_2 \otimes K^-(u')) = \\ = (\mathbf{1}_2 \otimes K^-(u'))R^H(-u-u'-2\eta)(K^-(u)\otimes \mathbf{1}_2)R^H(u'-u), \end{aligned} \tag{1.6.17}$$

where in the latter we have shifted the arguments of the R-matrix by a constant η. This does not affect the RTT relation, and such constant is used to characterise the R-matrix [61]. For the purpose of simplicity let us set $\eta = 1$. A diagonal solution for the K-matrices of the left and right REs [51] is

$$K^-(u,p_-) = \begin{pmatrix} p_- + u & 0 \\ 0 & p_- - u \end{pmatrix}, \quad K^+(u,p_+) = K^+(-u-1,p_+) \qquad (1.6.18)$$

where $p_\pm$ are constants related to the strength of the boundary fields. These are the isotropic limit of the solutions found in [61]. Using the procedure from the previous Section one obtains the following Hamiltonian, which we have rescaled for convenience

$$\begin{aligned} \mathcal{H}^H_{p,q} &= -\frac{J}{4p_-p_+}\frac{d}{du}\ln\big(\tau^B(u)\big)|_{u=0} \\ &= -J\sum_{i=1}^{L-1}(\sigma_i^+\sigma_{i+1}^- + \sigma_i^-\sigma_{i+1}^+ + \tfrac{1}{2}\sigma_i^z\sigma_{i+1}^z) + \mu_L\sigma_1^z + \mu_R\sigma_L^z \end{aligned} \qquad (1.6.19)$$

where $\mu_L = \frac{-J}{2p_-}$ and $\mu_R = \frac{-J}{2p_+}$. This is the well-known Hamiltonian of the Heisenberg spin chain with a boundary magnetic fields. The strength of the fields is given by μ_L at $i = 1$ and μ_R at $i = L$. This type of integrable boundary is solvable by ABA [64]. The boundary terms break the $\mathfrak{sl}_2$ symmetry of the model to $\mathfrak{u}_1$, generated by the cartan generator $\mathcal{E}_0^z$ (1.4.4). Since this model is integrable, we expect it to preserve a remnant of the original Yangian symmetry, which we will proceed to construct in the next Section.

1.7 Twisted Yangian symmetry

Consider the antiferromagnetic Heisenberg spin chain living on the left half-line with a boundary, so that the Hilbert space is now half-infinite, and assume that the boundary vector space in one-dimensional

$$\mathcal{V} = (\bigotimes_{i\leq 0} V_{\rho_i}) \otimes V_B, \quad V_i = \mathbb{C}^2 \quad \forall i, \quad V_B = \mathbb{C}. \qquad (1.7.1)$$

The resulting integrable Hamiltonian, obtained from the right boundary monodromy matrix (1.6.11), possesses only the right boundary magnetic field [145]

$$\mathcal{H}^H_\mu = -J\sum_{i<0}(\sigma_i^+\sigma_{i+1}^- + \sigma_i^-\sigma_{i+1}^+ + \tfrac{1}{2}\sigma_i^z\sigma_{i+1}^z) + \mu\sigma_0^z, \quad J<0. \qquad (1.7.2)$$

One can interpret this model as the left side of (1.6.19) when the right and left boundaries are really far apart. This model also breaks the $\mathfrak{sl}_2$ symmetry of the bulk model into an $\mathfrak{u}_1$ generated by $(\mathcal{E}_0^z)^{\leftarrow}$, where the left arrow upperscript denotes that the sum (1.4.4) runs only over the sites $i \leq 0$.

What happens to the Yangian symmetry of the infinite chain when one constraints it to the half-line is more intricate. The natural reaction is to think that $\mathcal{Y}(\mathfrak{sl}_2)$ breaks to $\mathcal{Y}(\mathfrak{u}_1)$ – however, this does not not accurately represent its full action on the Hilbert space, because if $x \in \mathcal{Y}(\mathfrak{u}_1)$

$$\Delta x \in \mathcal{Y}(\mathfrak{u}_1) \otimes \mathcal{Y}(\mathfrak{u}_1) \tag{1.7.3}$$

and the left hand side of the coproduct should belong to $\mathcal{Y}(\mathfrak{sl}_2)$. Thus, $\mathcal{Y}(\mathfrak{sl}_2)$ must break to an object which we will denote as $\mathcal{Y}(\mathfrak{sl}_2, \mathfrak{u}_1)$ such that, if $x \in \mathcal{Y}(\mathfrak{sl}_2, \mathfrak{u}_1)$, then x is a symmetry of the antiferromagnetic $\mathcal{H}^H_\mu$ (neglecting boundary terms at $-\infty$) and

$$\Delta x \in \mathcal{Y}(\mathfrak{sl}_2) \otimes \mathcal{Y}(\mathfrak{sl}_2, \mathfrak{u}_1). \tag{1.7.4}$$

Such a property defines a right coideal subalgebra of $\mathcal{Y}(\mathfrak{sl}_2)$. This is called the twisted Yangian $\mathcal{Y}(\mathfrak{sl}_2, \mathfrak{u}_1)$ [47], also denoted by $\mathcal{Y}^+(\mathfrak{sl}_2)$ [55]. For this particular model, it is generated by $(\mathcal{E}^z_0)^{\leftarrow}$ and [146, 147]

$$\mathcal{X}^\pm = (\mathcal{E}^\pm_1)^{\leftarrow} \pm \tfrac{J}{2}(\mathcal{E}^\pm_0)^{\leftarrow}(\mathcal{E}^z_0)^{\leftarrow} + \tfrac{J}{2}\big(1 \mp \tfrac{J}{\mu}\big)(\mathcal{E}^\pm_0)^{\leftarrow}. \tag{1.7.5}$$

In other words, although $\mathcal{E}^\pm_0$ and $\mathcal{E}^\pm_1$ are broken symmetries, there is a combination of them and $\mathcal{E}^z_0$ which - up to terms at $-\infty$ which may be neglected in the antiferromagnetic regime - still commutes with the boundary Hamiltonian (1.7.2) and generates an infinite set of conserved quantities. The latter property is only possible if the relations of the twisted Yangian $\mathcal{Y}(\mathfrak{sl}_2, \mathfrak{u}_1)$, generated by Q^z_0 and $\widehat{Q}^\pm_1$, are satisfied:

$$\begin{aligned} &[Q^z_0, \widehat{Q}^\pm_1] = \pm 2\widehat{Q}^\pm_1, \\ &[\widehat{Q}^\pm_1, [\widehat{Q}^\pm_1, [\widehat{Q}^\mp_1, \widehat{Q}^\pm_1]]] = 12J^2\widehat{Q}^\pm_1(Q^z_0 + c)\widehat{Q}^\pm_1. \end{aligned} \tag{1.7.6}$$

The twisted Yangian generators Q^z_0 and $\widehat{Q}^\pm_1$ can be written in terms of operators from the original Yangian as follows. Let $\alpha^\pm \in \mathbb{C}$ be such that $\alpha^+ - \alpha^- = 2c$. Then there is a natural embedding $\varphi^+ : \mathcal{Y}(\mathfrak{sl}_2, \mathfrak{u}_1) \hookrightarrow \mathcal{Y}(\mathfrak{sl}_2)$ of algebras:

$$Q^z_0 \mapsto Q^z_0, \quad \widehat{Q}^\pm_1 \mapsto Q^\pm_1 \pm \tfrac{J}{2}Q^\pm_0 Q^z_0 + J\alpha^\pm Q^\pm_0. \tag{1.7.7}$$

Set $c = -J/\mu$. Let Σ^-_∞ denote the set of spin operators acting on the left half-line

$$\Sigma^-_\infty = \{\sigma^a_i; i \leq 0, a = +, -, z\}. \tag{1.7.8}$$

The representation of the twisted Yangian $\mathcal{Y}(\mathfrak{sl}_2, \mathfrak{u}_1)$ on the left half-infinite Heisenberg spin chain is given by the map $\rho^+_\infty : \mathcal{Y}(\mathfrak{sl}_2, \mathfrak{u}_1) \to \Sigma^-_\infty$ defined by

$$Q^z_0 \mapsto (\mathcal{E}^z_0)^{\leftarrow}, \quad \widehat{Q}^\pm_1 \mapsto \mathcal{X}^\pm. \tag{1.7.9}$$

One can check that, up to terms at $-\infty$ which we may neglect, these operators commute with the Hamiltonian (1.7.2) and generate an infinite tower of symmetries via the commutator.

Twisted Yangians $\mathcal{Y}(\mathfrak{g}, \mathfrak{h})$ can be used, given a theory on the line, to classify all possible left and right integrable boundary conditions and construct an infinite set of non-local symmetries, under the suitable conditions in which boundaries at $-\infty$ (for a right boundary) and ∞ (for a left boundary) may be neglected e.g. the XXX and XXZ spin chains must be in the antiferromagnetic regime. We have to consider how the boundary at the site $i = 0$ breaks the symmetry of the model. Suppose a 1+1D quantum model on the line has Yangian symmetry $\mathcal{Y}(\mathfrak{g})$ in this way. If the model is on the half-line, the boundary will typically break $\mathfrak{g}$ to a subalgebra $\mathfrak{h}$. But if the boundary condition preserves integrability, $\mathfrak{h}$ must be invariant under a graded involution i [53]. One can split $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ under i such that $i(\mathfrak{h}) = +1$ and $i(\mathfrak{m}) = -1$. Then

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}\,, \tag{1.7.10}$$

and $(\mathfrak{g}, \mathfrak{h})$ are said to form a symmetric pair. This property, together with orthogonality with respect to the Killing form, $\kappa(\mathfrak{h}, \mathfrak{m}) = 0$, guarantees the coideal property: that the coproduct of any twisted Yangian charge $\widehat{Q}$ preserved at the boundary must be in the tensor product of the bulk and twisted Yangian,

$$\Delta\widehat{Q} \in \mathcal{Y}(\mathfrak{g}) \otimes \mathcal{Y}(\mathfrak{g}, \mathfrak{h}) \tag{1.7.11}$$

where $\mathcal{Y}(\mathfrak{g}, \mathfrak{h})$ is generated by $\mathfrak{h}$ and a deformation of the grade-1 $\mathfrak{m}$ generators (indexed here by p) [52, 54], given by

$$\widehat{Q}^p_1 = Q^p_1 + \frac{\alpha}{4}[C_\mathfrak{h}, Q^p_0] + \beta Q^p_0 \tag{1.7.12}$$

where $C_\mathfrak{h}$ is the Casimir operator of $\mathfrak{g}$ restricted to $\mathfrak{h}$, and α and β are model dependent constants. These deformed generators obey commutation relations analogous to (1.5.11) and additional Drinfel'd terrible relations [65, 66].

Hence for an open boundary condition to preserve integrability, one must have the Lie symmetry $\mathfrak{g}$ broken to a subalgebra $\mathfrak{h}$ thereof, and $(\mathfrak{g}, \mathfrak{h})$ must form a symmetric pair. Thus it is possible to list all possible integrable open boundary conditions even before attempting to construct them and their twisted Yangian symmetries. The formula above for the symmetric pair $(\mathfrak{sl}_2, \mathfrak{u}_1)$ in (1.7.7). For example, if $\mathfrak{g} = \mathfrak{sl}_n$, all possible symmetric pairs are given by $\mathfrak{h} = \mathfrak{so}_n$, $\mathfrak{sp}_n$ (if n is even) and $\mathfrak{sl}_{n-m} \times \mathfrak{sl}_m$. For all of these breakings, the boundary conditions are integrable and twisted Yangians can be constructed [52, 53, 54]. They will be discussed in three different models in future Chapters.

Analogous to the case of the Yangian, the existence of the twisted Yangian is equivalent to the boundary Yang-Baxter relation. It can often be more useful to use these relations to compute K-matrices rather than to explicitly solve the REs, and they have recently been used for several types of boundaries in the AdS/CFT correspondence [119, 135, 58]. Suppose the twisted Yangian charges are in the evaluation representation. Then these completely fix he K-matrix $K^-(u)$ via the boundary intertwining relation [55, 56]

$$\rho_u(x) K^-(u) = K^-(u) \rho_{-u}(x) \quad \forall\, x \in \mathcal{Y}(\mathfrak{g}, \mathfrak{h}). \tag{1.7.13}$$

Examples of this relation for the twisted Yangians $\mathcal{Y}(\mathfrak{sl}_2, \mathfrak{u}_1)$ and $\mathcal{Y}(\mathfrak{sl}_2, \mathfrak{sl}_2)$ used to yield solutions for the RE of the Heisenberg spin chain are given in [68]. For a classification of K-matrices for all symmetric pairs, see [69].

Chapter 2

Twisted Yangian symmetries of the open Hubbard model

2.1 Introducing the 1D Hubbard model

Now that we have provided the necessary tools from integrability, it is time to take a look at one of the most intriguing integrable models in statistical physics: the Hubbard model. The Hubbard model is used to describe strongly correlated electrons, and the physical changes materials undergo when they transition from being insulators to conductors and vice versa. It was first introduced by John Hubbard in 1963 through a series of papers on electron correlations in narrow energy bands [70, 71, 72, 73, 74, 75]. In the definitive guide [76] a vast amount of information on its physical properties and algebraic structure in one spatial dimension is available. The latter was shown to be integrable by Lieb and Wu via CBA [77] shortly after its first appearance. However, the ABA approach took a lot longer to be applied [78] because of the unusual structure of its R-matrix, which we will cover in the next Section.

Hubbard's original construction of the Hubbard model, restricted to a single band, goes as follows [70]. Consider a static, one dimensional lattice of L ions numbered from 1 to L, where N electrons can jump between the ions. Their Hamiltonian is given by

$$H = \sum_{i=1}^{N} \left(\frac{p_i^2}{2m_e} + V_I(x_i) \right) + \sum_{1 \leq i < j \leq N} V_C(x_i - x_j), \tag{2.1.1}$$

where p_i is the momentum of the i-th electron, $V_I(x)$ is the periodic potential of the ions, m_e is the elecron mass and $V_C(x)$ is the usual Coulomb potential between electrons

$$V_C(x_i - x_j) = \frac{e^2}{|x_i - x_j|}, \tag{2.1.2}$$

where e is the electron charge. Because of the long-range nature of interactions between electrons, this N-body problem is much too complicated to study. To make things easier, one can add an auxiliary potential $V_A(x)$ to the one particle part of H and substract it from the Coulomb potential such that the range of two-particle interactions becomes much smaller [76]. This method is part of the so called mean field approximation [79]. Explicitly, we have that

$$H = \sum_{i=1}^{N} h_1(p_i, x_i) + \sum_{1\leq i<j\leq N} U(x_i, x_j) \tag{2.1.3}$$

where

$$h_1(p_i, x_i) = \frac{p_i^2}{2m_e} + V_A(x_i) + V_I(x_i), \quad U(x_i, x_j) = V_C(x_i - x_j) - \tfrac{1}{N-1}(V_A(x_i) + V_A(x_j)). \tag{2.1.4}$$

Since h_1 is periodic, its eigenfunctions are Bloch states [82]:

$$\psi_k(x) = e^{\mathrm{i}kx} u_k(x) \tag{2.1.5}$$

where $u_k(x)$ is periodic in x and k is the quasimomentum. A complementary set of mutually orthogonal eigenfunctions of h_1 is given by the Wannier functions $\phi(x - R_i)$ [80, 81], where R_i is a lattice vector and

$$\phi(x - R_i) = \frac{1}{\sqrt{L}} \sum_k e^{-\mathrm{i}kR_i} \psi_k(x). \tag{2.1.6}$$

Now we can express the first set of eigenfunctions as sums over the ions on the band

$$\psi_k(x) = \frac{1}{\sqrt{L}} \sum_{i=1}^{L} e^{\mathrm{i}kR_i} \phi(x - R_i). \tag{2.1.7}$$

We wish to second-quantise the Hamiltonian and express it in terms of the usual fermionic creation and annihilation operators $\mathrm{c}^\dagger$ and c. In addition, we wish to associate one of two spins $\sigma = \uparrow, \downarrow$ to each electron. Take the Fourier transform

$$\mathrm{c}^\dagger_{i,\sigma} = \frac{1}{\sqrt{L}} \sum_k e^{-\mathrm{i}kR_i} \mathrm{c}^\dagger_{k,\sigma}, \tag{2.1.8}$$

and the operators satisfy the usual fermionic anti-commutation relations

$$\{\mathrm{c}^\dagger_{i\sigma}, \mathrm{c}_{j\tau}\} = \delta_{\sigma\tau}\delta_{ij}, \quad \{\mathrm{c}^\dagger_{i\sigma}, \mathrm{c}^\dagger_{j\tau}\} = \{\mathrm{c}_{i\sigma}, \mathrm{c}_{j\tau}\} = 0. \tag{2.1.9}$$

Thus the fermionic fields we want to use can be expressed as sums over both the quasimomenta and the ions on the band

$$\Psi_\sigma^\dagger(x) = \sum_k \psi_k^*(x) \mathrm{c}_{k,\sigma}^\dagger = \sum_{i=1}^L \phi_k^*(x) \mathrm{c}_{i,\sigma}^\dagger. \tag{2.1.10}$$

Finally, performing standard second-quantization [83], we obtain

$$\begin{aligned} \mathcal{H}^h &= \sum_{\sigma=\uparrow,\downarrow} \int \mathrm{d}x\ \Psi_\sigma^\dagger(x) h_1 \Psi_\sigma(x) + \tfrac{1}{2} \sum_{\sigma,\tau=\uparrow,\downarrow} \int \mathrm{d}x\mathrm{d}y\ \Psi_\sigma^\dagger(x)\Psi_\tau^\dagger(y) U(x,y) \Psi_\tau(y)\Psi_\sigma(x) \\ &= \sum_{i,j=1}^L \sum_{\sigma=\uparrow,\downarrow} t_{ij} \mathrm{c}_{i\sigma}^\dagger \mathrm{c}_{j\sigma} + \sum_{i,j,k,l=1}^L U_{ijkl} \mathrm{c}_{i\uparrow}^\dagger \mathrm{c}_{j\downarrow}^\dagger \mathrm{c}_{k\downarrow} c_{l\uparrow} \end{aligned} \tag{2.1.11}$$

where the matrix coefficients are given by

$$\begin{aligned} t_{ij} &= \int \mathrm{d}x\ \phi^*(x-R_i)(h_1\phi)(x-R_j), \\ U_{ijkl} &= \int \mathrm{d}x\mathrm{d}y\ \phi^*(x-R_i)\phi^*(x-R_j)U(x,y)\phi(x-R_k)\phi(x-R_l). \end{aligned} \tag{2.1.12}$$

Under the asumption that electrons are only allowed to move between neighbouring ions, and the Coulomb force only affects electrons if they are located on the same ion, we set

$$t_{ij} = t\delta_{|i-j|,1}, \quad U_{ijkl} = U\delta_{ij}\delta_{jk}\delta_{kl} \tag{2.1.13}$$

and define the number operator $\mathrm{n}_{i\sigma} = \mathrm{c}_{i\sigma}^\dagger \mathrm{c}_{i\sigma}$. The Hamiltonian of the Hubbard model then reads [70]

$$\mathcal{H}^h = -t \sum_{i=1}^L \sum_{\sigma=\uparrow,\downarrow} \mathrm{c}_{i\sigma}^\dagger \mathrm{c}_{i+1\sigma} + \mathrm{c}_{i+1\sigma}^\dagger \mathrm{c}_{i\sigma} + U \sum_{i=1}^L \left(\mathrm{n}_{i\uparrow} - \frac{1}{2}\right)\left(\mathrm{n}_{i\downarrow} - \frac{1}{2}\right) \tag{2.1.14}$$

with periodic boundary conditions $c_{L+1,\sigma} = c_{1,\sigma}$. We have performed a shift $\mathrm{n}_{i\sigma} \to \mathrm{n}_{i\sigma} - \frac{1}{2}$, which produces an external chemical potential that controls the particle occupancy on the chain sites. In this case the model is at 'half-filling', meaning there are half as many electrons living in it as the maximum number, and interestingly, this model turns into the Heisenberg spin chain in the large U limit (see Appendix 2 in [76] and [85]). Here t measures the amplitude of the motion of electrons between neighbouring ions, and U is the strength of the Coulomb repulsion between electrons on the same ion. This is a periodic spin chain with L sites, which are the ions. The first sum, known as the hopping term, is the discrete analogue of a kinetic term in a continuous Hamiltonian, where particles are annihilated in a site and created in a neighbouring one i.e. they 'hop' from site to site. The ratio t/U measures the conductivity of the material it describes. The Hubbard model has been used in condensed matter physics in one and several dimensions to describe, among other phenomena, the electronic properties of solids with narrow bands, band magnetism in several metals, the Mott metal-insulator transition and electronic properties of high critical temperature cuprates in the normal state [76]. In one dimension, a classification of the solutions to the Lieb-Wu equations [77] was proposed by Takahashi in 1972 [85] under the so called 'string hypothesis', which led to a better picture of the thermodynamics of the Hubbard model via the Thermodynamic Bethe Ansatz [86, 87, 88].

However, despite the apparent simplicity of its Hamiltonian, there is no general procedure for studying the Hubbard model in more than one dimension, and even in one dimension it is not clear how to approach much of its structure. Nevertheless, this Chapter hopes to shed some light in the integrable structure of the open Hubbard model. In the first Section, we will describe the set of symmetries of its Hamiltonian. Section 2.2 will be dedicated to reviewing Shastry's construction of its R-matrix from that of two free fermion models. Then we will introduce a generalisation of the Hubbard model to higher symmetries, for which the Yangian was constructed by the author. In section 2.4, we will point out some connections between the Hubbard model and AdS/CFT correspondence. The construction of the twisted Yangian symmetries in Sections 2.5 - 2.8 are results by the author and collaborators, where three different types of boundaries are studied, and Sections 2.5 and 2.6 are based on papers [121] and [89].

2.2 Symmetries of the bulk Hubbard model

Let us analyse the symmetries of the Hubbard Hamiltonian. We will set $t = 1$ from now on. This is equivalent to rescaling the Hamiltonian by $1/t$ and defining U to be the ratio of the original U and t. The Hilbert space of the Hubbard model is given by $\mathcal{V} = \bigotimes_{i=1}^{L} V_i$ where the vector spaces V_i are all four dimensional:

$$V_i = \{|0\rangle_i \,, \mathsf{c}^\dagger_{i\downarrow} |0\rangle_i := |{\downarrow}\rangle_i \,, \mathsf{c}^\dagger_{i\uparrow} |0\rangle_i := |{\uparrow}\rangle_i \,, \mathsf{c}^\dagger_{i\downarrow}\mathsf{c}^\dagger_{i\uparrow} |0\rangle_i := |{\downarrow\uparrow}\rangle_i\} \ \forall i. \tag{2.2.1}$$

The model possesses a $\mathfrak{sl}_2^s \times \mathfrak{sl}_2^c$ symmetry (which, via its inclusion in the $\mathfrak{sl}(2|2) \ltimes \mathbb{R}^2$ symmetry of worldsheet scattering in AdS/CFT [108], is the source of the renewed recent interest from the string theory community [128, 129] and thereby in new generalizations such as [98, 130, 131]). The labels s and c correspond to the the given names of these symmetries: the 'spin' and the 'charge' $\mathfrak{sl}_2$ respectively. One must note that the latter symmetry is only present if the number of spin chain sites is even, half-infinite or infinite [93]. If one defines the following operators

$$\mathcal{E}^s_{i,n} = \mathsf{c}^\dagger_{i\uparrow}\mathsf{c}_{i+n\downarrow} \quad \mathcal{F}^s_{i,n} = \mathsf{c}^\dagger_{i\downarrow}\mathsf{c}_{i+n\uparrow} \quad \mathcal{H}^s_{i,n} = \mathsf{c}^\dagger_{i\uparrow}\mathsf{c}_{i+n\uparrow} - \mathsf{c}^\dagger_{i\downarrow}\mathsf{c}_{i+n\downarrow}, \quad n \in \mathbb{Z} \tag{2.2.2}$$

then $\mathfrak{sl}_2^s$ is generated by

$$E^s_0 = \sum_{i=1}^{L} \mathcal{E}^s_{i,0} \quad F^s_0 = \sum_{i=1}^{L} \mathcal{F}^s_{i,0} \quad H^s_0 = \sum_{i=1}^{L} \mathcal{H}^s_{i,0} \tag{2.2.3}$$

satisfying the relations $[H^s_0, E^s_0] = 2E^s_0$, $[H^s_0, F^s_0] = -2F^s_0$ and $[E^s_0, F^s_0] = H^s_0$ i.e. they provide a representation of $\mathfrak{sl}_2$ in terms of two sets of fermionic oscillators. Similarly, if one defines the following operators

$$\mathcal{E}^c_{i,n} = (-1)^{i+n}\mathsf{c}^\dagger_{i\uparrow}\mathsf{c}^\dagger_{i+n\downarrow} \quad \mathcal{F}^c_{i,n} = (-1)^{i}\mathsf{c}_{i\downarrow}\mathsf{c}_{i+n\uparrow} \quad \mathcal{H}^c_{i,n} = \mathsf{c}^\dagger_{i\uparrow}\mathsf{c}_{i+n\uparrow} + (-1)^n\mathsf{c}^\dagger_{i\downarrow}\mathsf{c}_{i+n\downarrow} - \delta_{n,0}, \tag{2.2.4}$$

then $\mathfrak{sl}_2^c$ is generated by

$$E^c_0 = \sum_{i=1}^{L} \mathcal{E}^c_{i,0} \quad F^c_0 = \sum_{i=1}^{L} \mathcal{F}^c_{i,0} \quad H^c_0 = \sum_{i=1}^{L} \mathcal{H}^c_{i,0}. \tag{2.2.5}$$

Interestingly, there is a connection between the charge and the spin symmetry. The charge symmetry can be obtained from the spin symmetry through either of the partial particle-hole transformations (PHTs) $\mathcal{P}_\downarrow$ and $\mathcal{P}_\uparrow$ [76, 89]. These are the following maps for $\sigma, \tau = \uparrow, \downarrow$:

$$\mathcal{P}_\sigma : \quad (\mathrm{c}_{i\sigma}, \mathrm{c}^\dagger_{i\sigma}, \mathrm{c}_{i\tau}, \mathrm{c}^\dagger_{i\tau}, U) \;\mapsto\; ((-1)^i\mathrm{c}^\dagger_{i\sigma}, (-1)^i\mathrm{c}_{i\sigma}, \mathrm{c}_{i\tau}, \mathrm{c}^\dagger_{i\tau}, -U) \quad \sigma \neq \tau. \tag{2.2.6}$$

These operators act on states living in a single site according to the following diagram (where $\mathcal{P}_{i\sigma}$ is the partial PHT restricted to one site):

$$
\begin{tikzcd}
{|0\rangle_i} \arrow[loop above, "\mathcal{H}^c_{i,0}"] \arrow[rr, bend left, "\mathcal{E}^c_{i,0}"] \arrow[dd, leftrightarrow, bend right, "\mathcal{P}_{i\downarrow}"'] \arrow[ddrr, leftrightarrow, "\mathcal{P}_{i\uparrow}"] & & {|\uparrow\downarrow\rangle_i} \arrow[loop above, "\mathcal{H}^c_{i,0}"] \arrow[ll, bend left, "\mathcal{F}^c_{i,0}"] \arrow[dd, leftrightarrow, bend left, "\mathcal{P}_{i\downarrow}"] \arrow[ddll, leftrightarrow] \\
& & \\
{|\downarrow\rangle_i} \arrow[loop below, "\mathcal{H}^s_{i,0}"'] \arrow[rr, bend left, "\mathcal{E}^s_{i,0}"] & & {|\uparrow\rangle_i} \arrow[loop below, "\mathcal{H}^s_{i,0}"'] \arrow[ll, bend left, "\mathcal{F}^s_{i,0}"]
\end{tikzcd}
\tag{2.2.7}
$$

One can see that $\mathcal{P}_\downarrow$ maps (2.2.3) to (2.2.5), effectively changing the label from s to c. $\mathcal{P}_\uparrow$ is equivalent to the same change of label but composed with the automorphism of the algebra $\{E^c_0, F^c_0, H^c_0\} \mapsto \{F^c_0, E^c_0, -H^c_0\}$. One may think of P_σ as 'almost' supersymmetries of the model, aside from changing the sign of U. The full PHT, $\mathcal{P}_\uparrow\mathcal{P}_\downarrow$, is indeed a symmetry of the Hubbard Hamiltonian. These transformations are interesting on their own and will become important in making connections to the AdS/CFT correspondence and constructing an integrable boundary in Sections 2.4 and 2.6 respectively.

As expected, the Hubbard model possesses a symmetry of Yangian type, which has been additionally explored in the context of AdS/CFT [43]. This Yangian was constructed for the infinite chain [90] and is composed of two copies of $\mathcal{Y}(\mathfrak{sl}_2)$. One of the copies, $\mathcal{Y}(\mathfrak{sl}_2)^s$, is generated by $\mathfrak{sl}^s_2$ and the following level 1 generators

$$
\begin{aligned}
E^s_1 &= \sum_{-L<i<L} (\mathcal{E}^s_{i,1} - \mathcal{E}^s_{i+1,-1}) + \tfrac{U}{2} \sum_{-L\le i<j\le L} (\mathcal{E}^s_{i,0}\mathcal{H}^s_{j,0} - \mathcal{E}^c_{j,0}\mathcal{H}^s_{i,0}), && (2.2.8)\\
F^s_1 &= \sum_{-L<i<L} (\mathcal{F}^s_{i,1} - \mathcal{F}^s_{i+1,-1}) - \tfrac{U}{2} \sum_{-L\le i<j\le L} (\mathcal{F}^s_{i,0}\mathcal{H}^s_{j,0} - \mathcal{F}^s_{j,0}\mathcal{H}^s_{i,0}), && (2.2.9)\\
H^s_1 &= \sum_{-L<i<L} (\mathcal{H}^s_{i,1} - \mathcal{H}^s_{i+1,-1}) - U \sum_{-L\le i<j\le L} (\mathcal{E}^s_{i,0}\mathcal{F}^s_{j,0} - \mathcal{E}^s_{j,0}\mathcal{F}^s_{i,0}). && (2.2.10)
\end{aligned}
$$

The other copy, $\mathcal{Y}(\mathfrak{sl}_2)^c$, can be obtained through $\mathcal{P}_\downarrow$ or $\mathcal{P}_\uparrow$. These are symmetries of $\mathcal{H}^h$ defined on the infinite chain ($L \to \infty$) limit over the antiferromagnetic, empty band vacuum [91]

$$|\mathrm{vac}\rangle = \bigotimes_{k\in\mathbb{Z}} |0\rangle_k \,, \tag{2.2.11}$$

where the monodromy matrix - which we shall construct shortly - is infrared renormalisable and boundary terms may be neglected [92].

2.3 Shastry's R-matrix

After the discovery of the QISM it quickly became apparent that most known solvable models had an associated R-matrix that satisfied the YBE and from which their Hamiltonian could be extracted. The Hubbard model, however, proved to be much harder to crack, and it was conjectured [39] that finding its R-matrix was so troublesome because it is not of difference form, i.e. it cannot be written as a function of the difference between the spectral parameters appearing in the YBE. It was not until nearly a decade later that Shastry found a suitable R-matrix for the Hubbard model. His first results relied on heavy computations [94, 95], while later he constructed such R-matrix through a remarkably beautiful procedure [96]. The latter was mainly clever guesswork, and it is the one that we will follow in this Section.

Let us introduce the free fermion or XX model [35]. This the case when $q = i$ (or $\Delta_q = 0$) in (1.3.16):

$$\mathcal{H}^{XX} = \sum_{i=1}^{L} \left(\sigma_i^+ \sigma_{i+1}^- + \sigma_i^- \sigma_{i+1}^+\right). \tag{2.3.1}$$

We would like to write $\mathcal{H}^{XX}$ in terms of fermionic creation and annihilation operators, which are members of the fermionic oscillator algebra $\mathrm{Osc}_L = \{\mathrm{c}_i^\dagger, \mathrm{c}_i; 1 \leq i \leq L\}$ and satisfy the usual relations

$$\{\mathrm{c}_i^\dagger, \mathrm{c}_j\} = \delta_{ij}, \quad \{\mathrm{c}_i^\dagger, \mathrm{c}_j^\dagger\} = \{\mathrm{c}_i, \mathrm{c}_j\} = 0. \tag{2.3.2}$$

This is achieved through a Jordan-Wigner transformation JW [41], which maps Osc_L to the set of spin operators Σ_L in the following way:

$$\begin{aligned} \mathrm{JW} : \mathrm{Osc}_L \quad &\to \quad \Sigma_L \\ \mathrm{c}_i \quad &\mapsto \quad \left(\prod_{k=1}^{i-1} \sigma_k^z\right) \sigma_i^- \\ \mathrm{c}_i^\dagger \quad &\mapsto \quad \left(\prod_{k=1}^{i-1} \sigma_k^z\right) \sigma_i^+ \end{aligned} \tag{2.3.3}$$

The Hamiltonian then becomes

$$\mathcal{H}^{XX} = \sum_{i=1}^{L} \mathrm{c}_i^\dagger \mathrm{c}_{i+1} + \mathrm{c}_{i+1}^\dagger \mathrm{c}_i, \tag{2.3.4}$$

where, to switch to periodic boundary conditions in (2.3.4) we require

$$\left(\prod_{k=1}^{L} \sigma_k^z\right) \sigma_{L+1}^- = \sigma_1^-, \quad \left(\prod_{k=1}^{L} \sigma_k^z\right) \sigma_{L+1}^+ = \sigma_1^+. \tag{2.3.5}$$

This is the one of the simplest quantum many-body models one can construct: a purely discrete kinetic (or 'hopping') term on a chain of sites going from 1 to L, with periodic boundary conditions $c_{L+1} = c_1$. It exhibits an $U_q(\mathfrak{sl}_2)$ symmetry with $q = i$ [100]. Interestingly, an extended version of (2.3.4) given by

$$\mathcal{H}^{XX} = \sum_{i=1}^{L} \mathrm{c}_i^\dagger \mathrm{c}_{i+1} + \mathrm{c}_{i+1}^\dagger \mathrm{c}_i + \mathrm{c}_i^\dagger \mathrm{c}_i + \mathrm{c}_{i+1}^\dagger \mathrm{c}_{i+1} \tag{2.3.6}$$

commutes with the following operators

$$\mathfrak{Q} = \sum_{i=1}^{L} \mathrm{c}_i, \quad \mathfrak{G} = \sum_{i=1}^{L} \mathrm{c}_i^\dagger, \quad \mathfrak{K} = \sum_{i=1}^{L} \mathrm{c}_i^\dagger \mathrm{c}_i, \quad \mathfrak{C} = 1. \tag{2.3.7}$$

These form a representation of the $\mathfrak{gl}_{1|1}$ superalgebra in terms of fermionic oscillators, which has the defining relations [97]

$$[\mathfrak{K}, \mathfrak{Q}] = -\mathfrak{Q}, \quad [\mathfrak{K}, \mathfrak{G}] = \mathfrak{G}, \quad \{\mathfrak{Q}, \mathfrak{G}\} = \mathfrak{C}. \tag{2.3.8}$$

This algebra is extensively studied in the AdS/CFT correspondence – it appears as a symmetry of the AdS_3 and AdS_2 superstrings [59].

One can see that the Hubbard model when $U = 0$ consists of two uncoupled XX models. Hence, the natural point to start in constructing the R-matrix of the Hubbard model is to look at that of the XX model

$$\check{R}^{xx}(u) = \cos(u) P R(u; q \to i) = \begin{pmatrix} \cos(u) & 0 & 0 & 0 \\ 0 & 1 & \sin(u) & 0 \\ 0 & \sin(u) & 1 & 0 \\ 0 & 0 & 0 & \cos(u) \end{pmatrix} \tag{2.3.9}$$

which satisfies the YBE in the form (1.3.1). Using the Pauli matrices, one can write this R-matrix in the form

$$\check{R}^{xx}(u) = \sum_{a,b} \check{R}^{xx}(u)^{ab} \sigma^a \otimes \sigma^b, \quad a, b = 0, +, -, z \tag{2.3.10}$$

with $\check{R}^{xx}(u)^{00} = \frac{1}{2}(\cos(u) + 1)$, $\check{R}^{xx}(u)^{zz} = \frac{1}{2}(\cos(u) - 1)$, $\check{R}^{xx}(u)^{+-} = \check{R}^{xx}(u)^{-+} = \sin(u)$ and every other coeffient is zero. One can build a 16×16 R-matrix $\check{r}(u)$ that satisfies the YBE by 'fusing' two XX $R-$matrices

$$\check{r}(u) = \check{R}^{xx}_{\uparrow}(u) \check{R}^{xx}_{\downarrow}(u) \in \mathrm{End}((\mathbb{C}^2 \otimes \mathbb{C}^2) \otimes (\mathbb{C}^2 \otimes \mathbb{C}^2)) \tag{2.3.11}$$

where

$$\check{R}^{xx}_{\uparrow}(u) = \sum_{a,b} \check{R}^{xx}(u)^{ab} (\sigma^a \otimes \mathbf{1}_2) \otimes (\sigma^b \otimes \mathbf{1}_2), \tag{2.3.12}$$

$$\check{R}^{xx}_{\downarrow}(u) = \sum_{a,b} \check{R}^{xx}(u)^{ab} (\mathbf{1}_2 \otimes \sigma^a) \otimes (\mathbf{1}_2 \otimes \sigma^b). \tag{2.3.13}$$

To construct the Lax matrix, we must take the second space of $\check{r}(u)$ to spin operators in the following way

$$\begin{aligned} (\sigma^a \otimes \mathbf{1}_2) &\mapsto \sigma^a_j \\ (\mathbf{1}_2 \otimes \sigma^a) &\mapsto \tau^a_j \end{aligned} \tag{2.3.14}$$

where σ^a_j and τ^a_j are two independent sets of $\mathfrak{sl}_2$ spin operators, i.e.

$$[\sigma^a_i, \sigma^b_j] = \delta_{ij} f^{ab}{}_c \sigma^c_j, \quad [\tau^a_i, \tau^b_j] = \delta_{ij} f^{ab}{}_c \tau^c_j, \quad [\sigma^a_i, \tau^b_j] = 0. \tag{2.3.15}$$

The explicit form of the Lax matrix is

$$l_j(u) = \sum_{a,b,a',b'} \check{R}^{xx}(u)^{ab} \check{R}^{xx}(u)^{a'b'} (\sigma^a \otimes \sigma^{a'}) \sigma^b_j \tau^{b'}_j \tag{2.3.16}$$

and, consequently, $\check{r}(u)$ satisfies the Yang-Baxter algebra:

$$\check{r}(u - u')(l_j(u) \otimes l_j(u')) = (l_j(u') \otimes l_j(u)) \check{r}(u - u'). \tag{2.3.17}$$

The expansion of the corresponding transfer matrix is $\tau_{\sigma,\tau}(u) = 1 + h_{\sigma,\tau} u + O(u^2)$ where $h_{\sigma,\tau}$ is the periodic Hamiltonian of uncoupled XX models in terms of spin operators

$$h_{\sigma,\tau} = \sum_{i=1}^{L} \left(\sigma_i^+\sigma_{i+1}^- + \sigma_i^-\sigma_{i+1}^+ + (\sigma \to \tau)\right). \tag{2.3.18}$$

Following Shastry's work [96], we will use the XX R-matrix to construct that of the Hubbard model. It is natural to guess that the Hubbard model Lax matrix, $L_i^h(u)$, is given by

$$L_i^h(u) = G(h(u))\ l_i(u)\ G(h(u)) \tag{2.3.19}$$

where $G(h(u))$ depends on the spectral parameter, acts on the auxiliary space and it will be responsible for the appearance of the Coulomb interaction in the transfer matrix. For (2.3.19) to be true, we need

$$\tau^h(u) = \ln\bigl(\mathrm{tr} L_L^h(u)...L_1^h(u)\bigr) = 1 + u\mathcal{H}^h + ... \tag{2.3.20}$$

and $\mathcal{H}^h$ must commute to all higher charges in the expansion. It should appear in terms of the spin operators, an expression that must coincide with (2.1.14) after the following Jordan-Wigner transformation [95]:

$$c_{i,\uparrow} \mapsto \left(\prod_{k=1}^{i-1}\sigma_k^z\right)\sigma_i^-, \quad c_{i,\downarrow} \mapsto \left(\prod_{i=1}^{L}\sigma_i^z\right)\left(\prod_{k=1}^{i-1}\tau_k^z\right)\tau_i^-. \tag{2.3.21}$$

which yields

$$\mathcal{H}^h = \sum_{i=1}^{L} \sigma_i^+\sigma_{i+1}^- + \sigma_i^-\sigma_{i+1}^+ + (\sigma \to \tau) + \frac{U}{4}\sum_{i=1}^{L}\sigma_i^z\tau_i^z \tag{2.3.22}$$

Shastry's procedure was to construct a second charge $\mathcal{J}^h$ that commutes with the Hamiltonian and impose that it appears in the quadratic term of the transfer matrix, hence limiting the form that $G(h(u))$ can take. This charge is

$$\mathcal{J}^h = \sum_{i=1}^{L}(\sigma_{i+1}^-\sigma_i^z\sigma_{i-1}^+ - \mathrm{h.c.}) + \frac{U}{2}\sum_{i=1}^{L}(\sigma_i^+\sigma_{i+1}^- - \sigma_i^-\sigma_{i+1}^+) \times (\tau_i^z + \tau_{i+1}^z) + (\sigma \leftrightarrow \tau). \tag{2.3.23}$$

Simultaneously, by taking a linear combination of two equations in $\check{r}(u)$, he constructed $\check{R}^h(u,u')$ such that

$$\check{R}^h(u,u')(L_i^h(u)\otimes L_i^h(u')) = (L_i^h(u')\otimes L_i^h(u))\check{R}^h(u-u'). \tag{2.3.24}$$

Let us explain how this is achieved. Introduce the conjugation matrix $C_i^\alpha, \alpha = \uparrow, \downarrow$.

This is such that

$$\check{R}^{xx}_{\alpha,12}(u)C^{\alpha}_{i} = C^{\alpha}_{i}\check{R}^{xx}_{\alpha,12}(-u), \quad i = 1,2,\ \alpha =\uparrow,\downarrow \tag{2.3.25}$$

and in the fundamental representation, $C^{\uparrow} = \sigma^{z} \otimes \mathbf{1}_2$ and $C^{\downarrow} = \mathbf{1}_2 \otimes \sigma^{z}$, with

$$C^{\uparrow}_{i} = (\mathbf{1}_4)^{\otimes(i-1)} \otimes (\sigma^{z} \otimes \mathbf{1}_2) \otimes (\mathbf{1}_4)^{\otimes L-i}, \quad C^{\downarrow}_{i} = (\mathbf{1}_4)^{\otimes(i-1)}(\mathbf{1}_2 \otimes \sigma^{z}) \otimes (\mathbf{1}_4)^{\otimes L-i}. \tag{2.3.26}$$

If one takes the YBE for $R^{xx}_{\alpha}(u)$ and acts with $C^{\alpha}_{1}C^{\alpha}_{2}$ from the left and C^{α}_{3} from the right one obtains the decorated YBE [96], originally called by Shastry the decorated star-triangle relation

$$R^{xx}_{\alpha,12}(u+u')C^{\alpha}_{1}R^{xx}_{\alpha,13}(u)R^{xx}_{\alpha,23}(u') = \tag{2.3.27}$$
$$= R^{xx}_{\alpha,23}(u')R^{xx}_{\alpha,13}(u)C^{\alpha}_{1}R^{xx}_{\alpha,12}(u+u'), \quad \alpha =\uparrow,\downarrow. \tag{2.3.28}$$

This results in two equations $\check{r}(u)$ must satisfy: the defining relation of the Yang-Baxter algebra

$$\check{r}(u-u')(l_i(u) \otimes l_i(u')) = (l_i(u') \otimes l_i(u))\check{r}(u-u') \tag{2.3.29}$$

and the decorated Yang-Baxter algebra

$$\check{r}(u+u')(C^{\uparrow}C^{\downarrow} \otimes \mathbf{1}_4)(l_i(u) \otimes l_i(u')) = (l_i(u') \otimes l_i(u))(\mathbf{1}_4 \otimes C^{\uparrow}C^{\downarrow})\check{r}(u+u'). \tag{2.3.30}$$

Although in this case the latter is a consequence of the YBE, it is in general an independent equation, which includes a particle-antiparticle map. These two equations will be used to obtain a new R-matrix. We will furthermore assume $G(h(u))G(-h(u)) = \mathbf{1}_4$. Then the Lax matrix $l_i(u) = G(-h(u))\ L^{h}_{i}(u)\ G(-h(u))$. Set $h(u) = h$ and $h(u') = h'$, and define $G(a,b) := G(a) \otimes G(b)$. Now take a linear combination of the YBE and decorated YBE. After some algebraic manipulation we have that

$$G(h',h)[\alpha\check{r}(u-u') + \beta\check{r}(u+u')(C^{\uparrow}C^{\downarrow} \otimes \mathbf{1}_4)]G(-h,-h')(L^{h}_{i}(u) \otimes L^{h}_{i}(u')) =$$
$$= (L^{h}_{i}(u') \otimes L^{h}_{i}(u))G(-h',-h)[\alpha\check{r}(u-h') + \beta(C^{\uparrow}C^{\downarrow} \otimes \mathbf{1}_4)\check{r}(u+u')]G(h,h')$$

where α and β are scalar functions of the spectral parameters. Using this equation, the R-matrix

$$R^{h}(u,u') = G(2h',2h)[\alpha(u,u')\check{r}(u-u') + \beta(u,u')\check{r}(u+u')(C^{\uparrow}C^{\downarrow} \otimes \mathbf{1}_4)]G(-2h,-2h') \tag{2.3.31}$$

satisfies the Yang-Baxter algebra with the Lax matrix $L_i^h(u)$ if

$$G(2h', 2h)[\alpha\check{r}(u-u') + \beta\check{r}(u+u')(C^{\uparrow}C^{\downarrow} \otimes \mathbf{1}_4)]G(-2h, -2h') = \\ = [\alpha\check{r}(u-u') + \beta(C^{\uparrow}C^{\downarrow} \otimes \mathbf{1}_4)\check{r}(u+u')]. \quad (2.3.32)$$

This condition, together with the requirement that $\mathcal{H}^h$ and $\mathcal{J}^h$ must appear as the first two non-trivial terms in the expansion of the transfer matrix (2.3.20), leads to the following results for $G(h(u))$:

$$G(h(u)) = \exp\bigl(h(u)C^{\uparrow}C^{\downarrow} \otimes \mathbf{1}_4\bigr), \quad \frac{\sinh(2h(u))}{\sin(2u)} = \frac{\sinh(2h(u'))}{\sin(2u')} = \frac{U}{4} \quad (2.3.33)$$

and the functions $\alpha(u, u')$ and $\beta(u, u')$

$$\alpha(u, u') = 1, \quad \beta(u, u') = \frac{\cos(u+u')}{\cos(u-u')}\tanh(h(u) - h(u')) \quad (2.3.34)$$

and thus the R-matrix of the Hubbard model, or Shastry's R-matrix, is

$$R^h(u, u') = G(2h', 2h)[\check{r}(u-u') + \beta(u, u')\check{r}(u+u')(C^{\uparrow}C^{\downarrow} \otimes \mathbf{1}_4)]G(-2h, -2h'). \quad (2.3.35)$$

It is difficult not to admire the ingenuity of Shastry's procedure. This, however, does not explicitly prove that $R^h(u, u')$ satisfies the YBE. This was shown later using the tetrahedron Zamolodchikov algebra [99, 100] for both the Hubbard model and its $SU(n)$ generalisation [101], which we will also discuss in a future Section. This actually led to the study of ways in which one could 'couple' (in the sense that we have seen) XX R-matrices such that the resultant R-matrix satisfied the YBE [99]. In fact, the Hubbard model is only one of many models that can be obtained this way. All these models have the property that their R-matrix is not of difference form. It is also worth noting this was done for the Hubbard model fermionic Lax operator [102], which we will introduce in Section 2.5.

If one attempts to find the symmetry of $R^h(u, u')$ from scratch i.e. find matrices $\mathcal{M}$ such that,

$$(\mathcal{M} \otimes \mathbf{1}_4 + \mathbf{1}_4 \otimes \mathcal{M})R^h(u, u') = R^h(u, u')(\mathcal{M} \otimes \mathbf{1}_4 + \mathbf{1}_4 \otimes \mathcal{M}), \quad (2.3.36)$$

one obtains

$$\mathcal{M} = \begin{pmatrix} \mathcal{C}_1 & 0 & 0 & \mathcal{C}_2 \\ 0 & \mathcal{S}_1 & \mathcal{S}_2 & 0 \\ 0 & \mathcal{S}_3 & \mathcal{S}_4 & 0 \\ \mathcal{C}_3 & 0 & 0 & \mathcal{C}_4 \end{pmatrix} \tag{2.3.37}$$

where the entries satisfy $\mathcal{S}_1\mathcal{S}_4 - \mathcal{S}_2\mathcal{S}_3 = \mathcal{C}_1\mathcal{C}_4 - \mathcal{C}_2\mathcal{C}_3 = \gamma$. We can set $\gamma = 1$ and we have:

$$\mathcal{M}^s = \begin{pmatrix} \mathcal{S}_1 & \mathcal{S}_2 \\ \mathcal{S}_3 & \mathcal{S}_4 \end{pmatrix} \in SL(2)^s \quad \mathcal{M}^c = \begin{pmatrix} \mathcal{C}_1 & \mathcal{C}_2 \\ \mathcal{C}_3 & \mathcal{C}_4 \end{pmatrix} \in SL(2)^c. \tag{2.3.38}$$

Then one obtains a 4×4 representation of the $\mathfrak{sl}_2^s \times \mathfrak{sl}_2^c$ symmetry of the Hubbard model, which we will denote as ρ^h, given by

$$\rho^h(E_0^s) = \sigma^+ \otimes \sigma^-, \quad \rho^h(F_0^s) = \sigma^- \otimes \sigma^+, \quad \rho^h(H_0^s) = \sigma^z \otimes 1 - 1 \otimes \sigma^z \tag{2.3.39}$$

$$\rho^h(E_0^c) = \sigma^+ \otimes \sigma^+, \quad \rho^h(F_0^c) = \sigma^- \otimes \sigma^-, \quad \rho^h(H_0^c) = \sigma^z \otimes 1 + 1 \otimes \sigma^z \tag{2.3.40}$$

which is equivalent to performing the Jordan-Wigner transformation (2.3.21) on the operators (2.2.3) for a single site, and using the inverse of (2.3.14).

2.4 The $SU(n)$ Hubbard model

In this Section we will proceed to present an integrable generalisation of the $\mathfrak{sl}_2^s \times \mathfrak{sl}_2^c$ Hubbard model. Its Hamiltonian and R-matrix were constructed by Maassarani and collaborators [103, 105] by coupling two generalisations of XX models (also constructed by the same author in [104]) using Shastry's procedure and showing their ansatz for the Hamiltonian appears in the expansion of the transfer matrix. This generalisation is refered to in the literature as 'the $SU(n)$ Hubbard model', however we would like to point out this is not due to its symmetry algebra but the dimension of the objects that are used to build it. Hence, when we refer to it as such, it is only to avoid confusion when the material presented is compared to its sources.

The construction of this model goes as follows. Let $\mathtt{E}^{ab}$ be the $n \times n$ matrix with a 1 in the (a, b) entry and zeros everywhere else. The set of these matrices satisfy the usual $\mathfrak{su}_n$ relations

$$[\mathtt{E}^{ab}, \mathtt{E}^{cd}] = \delta_{bc}\mathtt{E}^{ad} - \delta_{ad}\mathtt{E}^{cb}. \tag{2.4.1}$$

In terms of these matrices, the free fermion R-matrix and Hamiltonian can be written as

$$R^{xx}(u) = \cos(u) \sum_{a=1}^{2} \mathtt{E}^{aa} \otimes \mathtt{E}^{aa} + \sin(u) \sum_{\substack{a,b=1 \\ a\neq b}}^{2} \mathtt{E}^{aa} \otimes \mathtt{E}^{bb} + \sum_{\substack{a,b=1 \\ a\neq b}}^{2} \mathtt{E}^{ab} \otimes \mathtt{E}^{ba} \tag{2.4.2}$$

and

$$\mathcal{H}^{xx} = \sum_{i=1}^{L} \sum_{\substack{a,b=1 \\ a\neq b}}^{2} (\mathtt{E}_i^{ab} \mathtt{E}_{i+1}^{ba}), \quad \mathtt{E}_i^{ab} = \mathbf{1}_n^{\otimes i-1} \otimes \mathtt{E}^{ab} \otimes \mathbf{1}_n^{\otimes L-i}, \tag{2.4.3}$$

respectively. It was shown in [104] that a generalisation of this model to $\mathfrak{su}_n$ matrices is possible so that integrability is not broken and it contains the usual XX model as the $n=2$ case as defined by the R-matrix

$$\begin{aligned} \mathtt{R}^{xx}(u) \quad &= \quad \cos(u)\big(\mathtt{E}^{nn} \otimes \mathtt{E}^{nn} + \sum_{a,b<n} \mathtt{E}^{aa} \otimes \mathtt{E}^{bb}\big) + \sin(u) \sum_{a<n} (\mathtt{E}^{nn} \otimes \mathtt{E}^{aa} + \mathtt{E}^{aa} \otimes \mathtt{E}^{nn}) \\ &+ \sum_{a<n} (\mathtt{E}^{na} \otimes \mathtt{E}^{an} + \mathtt{E}^{an} \otimes \mathtt{E}^{na}). \end{aligned} \tag{2.4.4}$$

One can easily check that this satisfies the YBE and its corresponding Hamiltonian is

$$\mathtt{H}^{xx} = \sum_{i=1}^{L} \sum_{a<n} (\mathtt{E}_i^{na} \mathtt{E}_{i+1}^{an} + \mathtt{E}_i^{an} \mathtt{E}_{i+1}^{na}). \tag{2.4.5}$$

We would like to stress that although it is built using $\mathfrak{su}_n$ matrices, the symmetry of this model is not $\mathfrak{su}_n$ (as $\mathfrak{su}_2$ is not the symmetry of the $d=2$ case but its quantum deformed version for a special value of $q=i$ (1.3.18)). Its symmetry is actually $\mathfrak{su}_{n-1} \times \mathfrak{u}_1$, given by

$$\mathtt{I}^{ab} = \sum_{i=1}^{L} \sum_{a<n} \mathtt{E}_i^{ab} \in \mathfrak{su}_{n-1}, \quad \mathtt{C} = \sum_{i=1}^{L} \left(\sum_{a<n} \mathtt{E}_i^{aa} - (n-1) \mathtt{E}_i^{nn} \right) \in \mathfrak{u}_1. \tag{2.4.6}$$

To construct a generalisation of the Hubbard model analogous to that of the XX model, we need to add an extra label $\alpha = \sigma, \tau$ to the matrices $\mathtt{E}_{i\alpha}^{ab}$, which now satisfy the relations

$$[\mathtt{E}_{i\alpha}^{ab}, \mathtt{E}_{j\beta}^{cd}] = \delta_{\alpha\beta} \delta_{ij} (\delta_{bc} \mathtt{E}_{i\alpha}^{ad} - \delta_{ad} \mathtt{E}_{i\alpha}^{cb}). \tag{2.4.7}$$

Maassarani's generalisation of the Hubbard model [103] is built by coupling two $SU(n)$ XX models in the most intuitive way

$$\mathtt{H}^{h} = \sum_{i=1}^{L} \sum_{a<n} \sum_{\alpha=\sigma\tau} \mathtt{E}_{i\alpha}^{an} \mathtt{E}_{i\alpha}^{na} + \mathtt{E}_{i\alpha}^{na} \mathtt{E}_{i\alpha}^{an} + \sum_{i=1}^{L} \left(\mathtt{C}_{i\sigma} + \frac{n-2}{2} \right) \left(\mathtt{C}_{i\tau} + \frac{n-2}{2} \right) \tag{2.4.8}$$

where $\mathtt{C}_{i\alpha} = \sum_{a<n} \mathtt{E}^{aa}_{i\alpha} - (n-1)\mathtt{E}^{nn}_{i\alpha}$ is the largest Cartan generator of $\mathfrak{su}_n$. For the case $n = 2$, taking $\sigma^+ = \mathtt{E}^{12}_\sigma, \sigma^- = \mathtt{E}^{21}_\sigma, \sigma^z = \mathtt{E}^{11}_\sigma - \mathtt{E}^{22}_\sigma$ and $(\sigma \to \tau)$, we recover the usual Hubbard model Hamiltonian after the Jordan-Wigner transformation (2.3.21). This Hamiltonian (2.4.8) exhibits a $(\mathfrak{su}_{n-1} \times \mathfrak{u}_1)_\sigma \times (\mathfrak{su}_{n-1} \times \mathfrak{u}_1)_\tau$ symmetry given by the following operators

$$\mathtt{I}^{ab}_\alpha = \sum_{i=1}^{L} \sum_{a,b<n} \mathtt{E}^{ab}_{i\alpha}, \quad \mathtt{C}_\alpha = \sum_{i=1}^{L} \left(\sum_{a<n} \mathtt{E}^{aa}_{i\alpha} - (n-1)\mathtt{E}^{nn}_{\alpha i} \right), \quad \alpha = \sigma, \tau. \tag{2.4.9}$$

Its R-matrix is constructed following of Shastry's method for the $n = 2$ case, using both the YBE and decorated YBE, and it has been shown to satisfy the YBE using the tetrahedron Zamolodchikov algebra [101]. The solutions of this model via the Coordinate Bethe Ansatz were computed in [106].

Note how in the $n = 2$ case the symmetry is $\mathfrak{u}_1^4$, but we know from previous Sections that the full symmetry of the Hubbard Hamiltonian is $\mathfrak{sl}_2^s \times \mathfrak{sl}_2^c$. The reason for this is that $n = 2$ is a special case, where matrices with labels σ and τ can be combined to form operators that generate a larger algebra. This is due to proposition (3.3) in [107] which, unfortunately, is not true for the general n case. Nevertheless, the model being integrable, it implies the $(\mathfrak{su}_{n-1} \times \mathfrak{u}_1)^2$ symmetry can be extended to two copies of the Yangian $\mathcal{Y}(\mathfrak{su}_{n-1} \times \mathfrak{u}_1)$. The non-trivial level 1 generators are given by

$$\mathtt{J}^{ab}_\alpha = U \sum_{i<j} \sum_{c<n} \left(\mathtt{E}^{ac}_{i\alpha} \mathtt{E}^{cb}_{\alpha j} - \mathtt{E}^{ac}_{\alpha j} \mathtt{E}^{cb}_{i\alpha} \right). \tag{2.4.10}$$

These satisfy the usual Yangian relations

$$[\mathtt{I}^{ab}_\alpha, \mathtt{J}^{cd}_\beta] = \delta_{\alpha\beta}(\delta_{bc}\mathtt{J}^{ad}_\alpha - \delta_{ad}\mathtt{J}^{cb}_\alpha) \tag{2.4.11}$$

and the coproduct

$$\Delta(\mathtt{J}^{ab}_\alpha) = \mathtt{J}^{ab}_\alpha \otimes 1 + 1 \otimes \mathtt{J}^{ab}_\alpha + U \sum_{c<n} (\mathtt{I}^{ac}_\alpha \otimes \mathtt{I}^{cb}_\alpha - \mathtt{I}^{cb}_\alpha \otimes \mathtt{I}^{ac}_\alpha). \tag{2.4.12}$$

These commute with the $SU(n)$ Hubbard Hamiltonian on the infinite chain if the infinite interval limit is taken over the empty band vacuum, where one may neglect boundary terms at $\pm\infty$. If one interprets the $\mathtt{E}^{na}_i$ and $\mathtt{E}^{an}_i$ for $a < n$ as creation and annihilation operators of a particle a at site i respectively [106], the empty band vacuum is given by $|\mathtt{vac}\rangle = |0\rangle^{\otimes L}$ as $L \to \infty$, where $|0\rangle$ is the following vector of size n:

$$|0\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}. \tag{2.4.13}$$

Again, note the difference in structure between these Yangian generators and the $n = 2$ ones (2.2.10). They are effectively the two XX Yangian copies, so the its structure is unaltered by the coupling. This is unexpected because the coupled and uncoupled R-matrices are different.

2.5 The Hubbard model and AdS/CFT

Aside from its applications in condensed matter physics and unusual integrable structure, the Hubbard model has gained interest lately from the high energy physics community due to its connections with the AdS/CFT correspondence [98, 108, 110]. In this Section we will review the role integrability plays in this correspondence and, more specifically, how the Hubbard model fits into the story. As this thesis does not focus on this subject, we will keep this Section rather brief and concise, to merely convey ideas of integrability that have added more value to the field of high energy physics. The author would recommend a series of reviews [9, 113] for a more detailed, thorough explanation on how the results mentioned here were obtained.

The $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence is a gravity/gauge duality between Type IIB superstring theory living in a $\mathrm{Ad}S_5 \times S^5$ background and $\mathcal{N} = 4$ Super Yang Mills theory (SYM) in 4D Minkowski spacetime [112]. The latter is a conformal field theory (CFT) with $SU(N)$ gauge group, and as such, the form of its observables is very restricted. It is found to exist at the 4D boundary of type IIB in $\mathrm{Ad}S^5$. The relevant parameters of the string theory are the effective tension $T = R^2/2\pi\alpha'$ and the string coupling g_{Str}, and those of the gauge theory are the dimension of the gauge group N and the 't Hooft coupling $\lambda = g_{YM}^2 N$. Here α' is the string tension, R is the radius of the S^5 and g_{YM} is the CFT coupling. The correspondence relates these as follows:

$$\lambda = 4\pi^2 T^2, \quad \frac{1}{N} = \frac{g_{str}}{4\pi^2 T^2}. \tag{2.5.1}$$

The discovery of this duality by Maldacena in 1997 encouraged the search for others in smaller dimensions, which we now know exist for $\mathrm{Ad}S_d/\mathrm{CFT}_{d-1}$ for $d = 4, 3, 2$ [114, 115, 116], even if in some cases it is still unclear what the corresponding CFT is. Nevertheless, this correspondence allows for a better understanding of features of a gravity theory and make progress in heavy computations by studying its dual gauge theory and viceversa.

The set of isometries of $\mathrm{Ad}S_5 \times S^5$ is $SO(2,4) \times SO(6)$, which corresponds to the bosonic symmetries of the theory. The existence of 32 supersymmetries enhances this symmetry to $PSU(2,2|4)$. The particle content of the theory is composed of an $SU(N)$ gauge field, 6 scalars transforming under the fundamental representation of $SO(6)$, 4 chiral and 4 anti-chiral fermions transforming under the fundamental and anti-fundamental representations of $SU(4) \cong SO(6)$.

The techniques in integrability can only be applied to 1+1D and in some cases, 2+1D quantum models. Hence one cannot seek to solve the aforementioned theories in general. However, in the $\lambda \to \infty$ limit, the gauge theory becomes planar i.e. only 1+1D diagrams – or single trace operators –are allowed. Additionally, the dilatation operator, which is measures the scaling dimension of operators in the CFT, becomes the Hamiltonian of the $SO(6)$ Heisenberg spin chain [117]. Hence one can compute the spectrum of planar scaling dimensions, and in string theory this is dual to the energy spectrum of free strings. Thus integrability allows for an agreement in the perturbative regime of both theories.

To fit AdS/CFT integrability in the context of the QISM, we need to find the R-matrix of $\mathrm{AdS}_5/\mathrm{CFT}_4$ in the planar limit. The full symmetry algebra being two copies of $\mathfrak{psl}_{2,2|4}$, picking light-cone gauge restricts the symmetry to two copies of $\mathfrak{sl}_{2|2} \ltimes \mathbb{R}^2$, each of which is generated by six bosonic operators $\{\mathcal{L}^{\alpha}{}_{\beta}, \mathcal{R}^{a}{}_{b}\}$ and eight fermiomic ones $\{\mathcal{Q}^{\alpha}{}_{a}, \mathcal{G}^{b}{}_{\beta}\}$, satisfying the following relations

$$
\begin{aligned}
&[\mathcal{L}^{\alpha}{}_{\beta}, \mathcal{L}^{\gamma}{}_{\xi}] = \delta^{\gamma}_{\beta}\mathcal{L}^{\alpha}{}_{\xi} - \delta^{\alpha}_{\xi}\mathcal{L}^{\gamma}{}_{\beta}, && [\mathcal{R}^{a}{}_{b}, \mathcal{R}^{c}{}_{d}] = \delta^{c}_{b}\mathcal{R}^{a}{}_{d} - \delta^{a}_{d}\mathcal{R}^{c}{}_{b} \\
&[\mathcal{L}^{\alpha}{}_{\beta}, \mathcal{Q}^{\gamma}{}_{b}] = \delta^{\gamma}_{\beta}\mathcal{Q}^{\alpha}{}_{b} - \frac{1}{2}\delta^{\alpha}_{\beta}\mathcal{Q}^{\gamma}{}_{b}, && [\mathcal{L}^{\alpha}{}_{\beta}, \mathcal{G}^{a}{}_{\gamma}] = -\delta^{\alpha}_{\gamma}\mathcal{G}^{\alpha}{}_{\beta} + \frac{1}{2}\delta^{\alpha}_{\beta}\mathcal{G}^{a}{}_{\gamma} \\
&\{\mathcal{Q}^{\alpha}{}_{a}, \mathcal{Q}^{\beta}{}_{b}\} = \epsilon^{\alpha\beta}\epsilon_{ab}\mathcal{P}, && \{\mathcal{G}^{a}{}_{\alpha}, \mathcal{G}^{b}{}_{\beta}\} = \epsilon^{ab}\epsilon_{\alpha\beta}\mathcal{K} \\
&\{\mathcal{Q}^{\alpha}{}_{a}, \mathcal{G}^{b}{}_{\beta}\} = \delta^{b}_{a}\mathcal{L}^{\alpha}{}_{\beta} + \delta^{\alpha}_{\beta}\mathcal{R}^{b}{}_{a} + \delta^{b}_{a}\delta^{\beta}_{\alpha}\mathcal{C}
\end{aligned} \tag{2.5.2}
$$

where $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{K}$ are central elements. The superalgebra acts on two bosonic $|\phi^a\rangle$ and two fermionic $|\psi^\alpha\rangle$ states, $a, \alpha = 1, 2$ in the following way:

$$
\mathcal{R}^{a}{}_{b}\,|\phi^a\rangle = \delta^{c}_{b}\,|\phi^a\rangle - \frac{1}{2}\delta^{a}_{b}\,|\phi^c\rangle\,, \qquad \mathcal{L}^{\alpha}{}_{\beta}\,|\psi^\gamma\rangle = \delta^{\gamma}_{\beta}\,|\psi^\alpha\rangle - \frac{1}{2}\delta^{\alpha}_{\beta}\,|\psi^\gamma\rangle \tag{2.5.3}
$$

$$
\mathcal{Q}^{\alpha}{}_{a}\,|\phi^b\rangle = \mathbf{a}\delta^{b}_{a}\,|\psi^\alpha\rangle \qquad \mathcal{Q}^{\alpha}{}_{a}\,|\psi^\beta\rangle = \mathbf{b}\epsilon^{\alpha\beta}\epsilon_{ab}\,|\phi^b\rangle \tag{2.5.4}
$$

$$
\mathcal{G}^{a}{}_{\alpha}\,|\psi^\beta\rangle = \mathbf{c}\epsilon^{ab}\epsilon_{\alpha\beta}\,|\psi^\beta\rangle \qquad \mathcal{G}^{a}{}_{\alpha}\,|\phi^b\rangle = \mathbf{d}\delta^{\beta}_{\alpha}\,|\psi^\alpha\rangle \tag{2.5.5}
$$

where $\mathbf{a}$,$\mathbf{b}$,$\mathbf{c}$ and $\mathbf{d}$ are complex numbers and the closure of the algebra requires that

$\mathbf{ad} - \mathbf{bc} = 1$, which implies

$$\mathcal{C} = \frac{\mathbf{ad} + \mathbf{bc}}{2}, \quad \mathcal{P} = \mathbf{ab}, \quad \mathcal{K} = \mathbf{cd}. \tag{2.5.6}$$

These generators admit a representation in terms of the fermionic oscillators [100]. The bosonic generators can be written as

$$\begin{aligned} &\mathcal{R}^1{}_1 = -\mathcal{R}^2{}_2 = \frac{1}{2}(\mathrm{n}_\uparrow + \mathrm{n}_\downarrow - 1), \qquad &&\mathcal{R}^1{}_2 = (\mathcal{R}^2{}_1)^\dagger = \mathrm{c}^\dagger_\downarrow \mathrm{c}^\dagger_\uparrow \\ &\mathcal{L}^1{}_1 = -\mathcal{L}^2{}_2 = \frac{1}{2}(\mathrm{n}_\uparrow - \mathrm{n}_\downarrow), &&\mathcal{L}^1{}_2 = (\mathcal{L}^2{}_1)^\dagger = \mathrm{c}^\dagger_\uparrow \mathrm{c}_\downarrow \end{aligned} \tag{2.5.7}$$

and, defining $\mathbb{P}$ to be the spin permutation operator, we can write the fermionic generators as

$$\begin{aligned} \mathcal{Q}^1{}_1 = \ &(\mathbf{a} + (\mathbf{b} - \mathbf{a})\mathrm{n}_\downarrow)\mathrm{c}^\dagger_\uparrow = \ \mathbb{P}\mathcal{Q}^2{}_1, \quad \mathcal{Q}^1{}_2 = \ -(\mathbf{b} + (\mathbf{a} - \mathbf{b})\mathrm{n}_\uparrow)\mathrm{c}_\downarrow = \ -\mathbb{P}\mathcal{Q}^2{}_2 \\ \mathcal{S}^1{}_1 = \ &(\mathbf{d} + (\mathbf{c} - \mathbf{d})\mathrm{n}_\downarrow)\mathrm{c}_\uparrow = \ \mathbb{P}\mathcal{S}^1{}_2, \quad \mathcal{S}^2{}_1 = \ -(\mathbf{c} + (\mathbf{d} - \mathbf{c})\mathrm{n}_\uparrow)\mathrm{c}^\dagger_\downarrow = \ -\mathbb{P}\mathcal{S}^2{}_2. \end{aligned} \tag{2.5.8}$$

The form of these generators should already remind one of the symmetries of the Hubbard model (2.2.3) and (2.2.5). The parameters $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are usually written in terms of the 'AdS/CFT variables' x^+, x^- and g [43],

$$\mathbf{a} = \sqrt{\frac{g}{2}}\eta, \quad \mathbf{b} = \sqrt{\frac{g}{2}}\frac{i}{\eta}\left(\frac{x^+}{x^-} - 1\right), \quad \mathbf{c} = \sqrt{\frac{g}{2}}\frac{\eta}{x^+}, \quad \mathbf{d} = \sqrt{\frac{g}{2}}i(x^- - x^+) \tag{2.5.9}$$

where $\eta = \sqrt{i(x^- - x^+)}$ and x^+, x^- are connected to the momentum of the particle as

$$\frac{x^+}{x^-} = \mathrm{e}^{\mathrm{i}p}. \tag{2.5.10}$$

The connection between AdS/CFT and the Hubbard model was initially found in [109]. Indeed, if one considers only the even part of this algebra, this is just the $\mathfrak{sl}_2^s \times \mathfrak{sl}_2^c$ symmetry of the Hubbard model, and if one attempts to construct the R-matrix for AdS/CFT by requiring invariance under the full algebra, the discovery is that it is not of difference form and its corresponding Hamiltonian is that of an extended Hubbard model [110, 100]. Furthermore, one finds that Shastry's and the AdS/CFT R-matrix are related by a similarity transformation and the following identification of variables [111]:

$$g = \frac{1}{U}, \quad x^+ = \frac{a(u)}{b(u)}\mathrm{e}^{2h(u)}, \quad x^- = \frac{b(u)}{a(u)}\mathrm{e}^{2h(u)}, \tag{2.5.11}$$

with $a(u)^2 - b(u)^2 = 1$ and $a(u)b(u) = 2i\sinh(2h(u))U^{-1}$. The standard choice of $a(u)$ and $b(u)$ is

$$a(u) = \cos(u), \quad b(u) = -i\sin(u). \tag{2.5.12}$$

But the Hubbard model is not supersymmetric, and even in the AdS/CFT side, the bosonic symmetries $\mathfrak{sl}_2^s \times \mathfrak{sl}_2^c$ [110] can only be enhanced to $\mathfrak{sl}_{2|2}$ if

$$\left(\frac{x^+}{x^-}\right)^{L/2} = 1. \tag{2.5.13}$$

where L is the length of the chain. Note that this requires that the length of the chain is even, so that the second $\mathfrak{sl}_2$ symmetry is present, when it is also a requirement in the Hubbard model (2.2.5). This condition is derived using the Bethe ansatz method for a spin chain with $\mathfrak{sl}_{2|2} \ltimes \mathbb{R}^2$ symmetry before identifying variables according to (2.5.11) [110]. We know that, for the case of $U = 0$, the Hubbard model becomes two uncoupled XX models and hence has a $U_i(\mathfrak{sl}_2)\times U_i(\mathfrak{sl}_2)$ symmetry, which is too a case considered in [110]. However, we would like to shed some light on the role of supersymmetry by carefully studying the full symmetry algebra of the Hubbard model and an operator we have encountered before, which relates the two copies of $\mathfrak{sl}_2$. These are the partial PHTs (2.2.6)

$$\mathcal{P}_\downarrow : \quad (c_{i\downarrow}, c_{i\downarrow}^\dagger, c_{i\uparrow}, c_{i\uparrow}^\dagger) \;\mapsto\; ((-1)^i c_{i\downarrow}^\dagger, (-1)^i c_{i\downarrow}, c_{i\uparrow}, c_{i\uparrow}^\dagger). \tag{2.5.14}$$

and

$$\mathcal{P}_\uparrow : \quad (c_{i\downarrow}, c_{i\downarrow}^\dagger, c_{i\uparrow}, c_{i\uparrow}^\dagger) \;\mapsto\; (c_{i\downarrow}, c_{i\downarrow}^\dagger, (-1)^i c_{i\uparrow}^\dagger, (-1)^i c_{i\uparrow}). \tag{2.5.15}$$

$\mathcal{P}_\sigma$ is not necessarily a symmetry of the Hubbard model. However, both the Hamiltonian and the R-matrix satisfy

$$\mathcal{P}_\sigma Z(u,u',U) = Z(u,u',-U)\mathcal{P}_\sigma, \qquad Z = \mathcal{H}^h, R^h(u,u'). \tag{2.5.16}$$

and hence the map $\mathcal{P}_\sigma$ combined with a change of sign in U is an additional symmetry of the model – more specifically, a supersymmetry.

To see how this operator enhances the symmetry to $\mathfrak{sl}_{2|2}$, let us write down the supersymmetry generators in a complatible representation to (2.3.40) and relate these to the partial PHT, for which there are a number of choices:

$$\begin{aligned} \mathcal{P}_\downarrow^\pm(\mathbf{a}_1) &= \mathbf{a}_1(\mathbf{1}_2 \otimes (\sigma^+ \pm \sigma^-)), &(2.5.17)\\ \mathcal{P}_\uparrow^\pm(\mathbf{a}_2) &= \mathbf{a}_2((\sigma^+ \pm \sigma^-) \otimes \mathbf{1}_2), &(2.5.18)\end{aligned}$$

where $\mathbf{a}_1$ and $\mathbf{a}_2$ are nonzero complex numbers. The supercharges are

$$\begin{aligned}
&\mathcal{Q}^1{}_1(\mathbf{a},\mathbf{b}) = ((\mathbf{a}+\mathbf{b})\mathbf{1}_2 - (\mathbf{a}-\mathbf{b})\sigma^z)\otimes\sigma^-, && \mathcal{Q}^2{}_2(\mathbf{a},\mathbf{b}) = ((\mathbf{a}+\mathbf{b})\mathbf{1}_2 + (\mathbf{a}-\mathbf{b})\sigma^z)\otimes\sigma^+,\\
&\mathcal{Q}^1{}_2(\mathbf{a},\mathbf{b}) = \sigma^+\otimes((\mathbf{a}-\mathbf{b})\mathbf{1}_2 + (\mathbf{a}+\mathbf{b})\sigma^z), && \mathcal{Q}^2{}_1(\mathbf{a},\mathbf{b}) = -\sigma^-\otimes((\mathbf{b}-\mathbf{a})\mathbf{1}_2 + (\mathbf{a}+\mathbf{b})\sigma^z),\\
&\mathcal{G}^1{}_1(\mathbf{c},\mathbf{d}) = \mathcal{Q}^2{}_2(\mathbf{c},\mathbf{d}), && \mathcal{G}^2{}_2(\mathbf{c},\mathbf{d}) = \mathcal{Q}^1{}_1(\mathbf{c},\mathbf{d}),\\
&\mathcal{G}^1{}_2(\mathbf{c},\mathbf{d}) = -\mathcal{Q}^1{}_2(\mathbf{c},\mathbf{d}), && \mathcal{G}^2{}_1(\mathbf{c},\mathbf{d}) = -\mathcal{Q}^2{}_1(\mathbf{c},\mathbf{d}),
\end{aligned} \tag{2.5.19}$$

where the bold variables are nonzero complex numbers satisfying $\mathbf{ad} - \mathbf{bc} = 1$. It is now easy to see that the operators $\mathcal{P}^{\pm}_{\sigma}$ are sums of these supercharges with a specific choice of variables:

$$\mathcal{P}^{\pm}_{\downarrow}(\mathbf{a}) \;=\; \mathcal{Q}^1{}_1(\mathbf{a},\pm\mathbf{a}) + \mathcal{Q}^2{}_2(\mathbf{a},\pm\mathbf{a}) = \mathcal{G}^2{}_2(\mathbf{a},\pm\mathbf{a}) + \mathcal{G}^2{}_2(\mathbf{a},\pm\mathbf{a}), \tag{2.5.20}$$

$$\mathcal{P}^{\pm}_{\uparrow}(\mathbf{c}) \;=\; \mathcal{Q}^1{}_2(\mathbf{c},\mp\mathbf{c}) + \mathcal{Q}^2{}_1(\mathbf{c},\mp\mathbf{c}) = \mathcal{G}^1{}_2(\mathbf{c},\mp\mathbf{c}) + \mathcal{G}^2{}_1(\mathbf{c},\mp\mathbf{c}), \tag{2.5.21}$$

and hence the supercharges can be obtained by computing commutators of the partial PHT with the generators of the bosonic subalgebra. This is equivalent to condition (2.5.13). In this case however, imposing the condition $\mathbf{ad} - \mathbf{bc} = 1$ is equivalent to the following relation among the free parameters:

$$\mathbf{c} = \pm\frac{1}{2\mathbf{a}}. \tag{2.5.22}$$

Consequently, the superalgebra generated by $\mathfrak{sl}^s_2\times\mathfrak{sl}^c_2$ and $\mathcal{P}^{\pm}_{\sigma}$ is $\mathfrak{sl}_{2|2}$. This symmetry lacks the central extension that governs the scattering of the $AdS_5\times S^5$ superstring. Instead, the possible central charges $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{K}$ generated by the supersymmetries (see A.1) are

$$\langle\mathcal{C},\mathcal{P},\mathcal{K}\rangle = \langle\ \frac{\mathbf{ad}+\mathbf{bc}}{2},\ \mathbf{ab},\ \mathbf{cd}\ \rangle = \langle\ 0, \mp\mathbf{a}^2, \pm\frac{1}{4\mathbf{a}^2}\ \rangle. \tag{2.5.23}$$

We can see that the relations $\mathbf{b} = \mp\,\mathbf{a}$ and $\mathbf{d} = \pm\,\mathbf{c}$ are equivalent to condition (2.5.13), which is ultimately due to the existence of the partial particle-hole transformation. Thus we conclude that the existence of $\mathcal{P}_\sigma$ and a careful choice of parameters is what allows us to connect the Hubbard model, which lacks supersymmetry, with the integrable structure of the $AdS_5\times S^5$ superstring.

2.6 Boundary magnetic field and chemical potential

As one can gather by studying the literature on the 1-D Hubbard model [76], a lot less is known about the symmetries governing its integrable structure with boundary conditions compared to the case of periodic boundary conditions and infinite interval. This is due to the size and unusual structure of its R-matrix and symmetries thereof. In the context of the QISM, only a boundary magnetic field and chemical potential have been studied in detail [120] and solved via the ABA [132, 124]. These boundaries break one of the $\mathfrak{sl}_2$ symmetries to a $\mathfrak{u}_1$. This suggests the question: does this boundary possess a twisted Yangian symmetry? Additionally, are there other integrable boundaries corresponding to different symmetric pairs? And if so, can they be constructed?

From this Section until the end of this Chapter we will proceed to answer these questions. We will start with a review of the known construction of the fermionic Lax operator of the Hubbard model and solutions to the graded reflection equation [120, 122, 123]. We then present the results based on papers by the author and collaborators [89, 121] at the end of this Section and 2.6 and additional unpublished work on a new open integrable boundary in Section 2.7. This provides a classification and construction of twisted Yangian symmetries and integrable boundaries for the open Hubbard model for three types of symmetric pairs and twisted Yangians.

In Section 2.2, we showed how Shastry constructed a Lax operator and R-matrix from which the Hamiltonian of the periodic Hubbard model could be extracted. If we perform the Jordan-Wigner transformation (2.3.21) on such a Lax matrix we obtain the so called fermionic Lax operator [125, 126]

$$L_j^f(u) = \begin{pmatrix} -\mathrm{e}^{h(u)} f_{j\uparrow}(u) f_{j\downarrow}(u) & -f_{j\uparrow}(u)\mathsf{c}_{j\downarrow} & \mathrm{i}\mathsf{c}_{i\uparrow} f_{j\downarrow}(u) & \mathrm{i}\mathrm{e}^{h(u)}\mathsf{c}_{i\uparrow}\mathsf{c}_{j\downarrow} \\ -\mathrm{i} f_{j\uparrow}(u)\mathsf{c}_{j\downarrow}^\dagger & \mathrm{e}^{-h(u)} f_{j\uparrow}(u) g_{j\downarrow}(u) & \mathrm{e}^{-h(u)}\mathsf{c}_{j\uparrow}\mathsf{c}_{j\uparrow}^\dagger & \mathrm{i}\mathsf{c}_{j\uparrow} g_{j\downarrow}(u) \\ \mathsf{c}_{j\uparrow}^\dagger f_{j\downarrow}(u) & \mathrm{e}^{-h(u)}\mathsf{c}_{j\uparrow}^\dagger\mathsf{c}_{j\downarrow} & \mathrm{e}^{-h(u)} g_{j\uparrow}(u) f_{j\downarrow}(u) & g_{j\uparrow}(u)\mathsf{c}_{j\downarrow} \\ -\mathrm{i}\mathrm{e}^{h(u)}\mathsf{c}_{j\uparrow}^\dagger\mathsf{c}_{j\downarrow}^\dagger & \mathsf{c}_{j\uparrow}^\dagger g_{j\downarrow}(u) & \mathrm{i} g_{j\uparrow}(u)\mathsf{c}_{j\downarrow}^\dagger & -\mathrm{e}^{h(u)} g_{j\uparrow}(u) g_{j\downarrow}(u) \end{pmatrix} \tag{2.6.1}$$

where the functions $f_{j\sigma}(u)$ and $g_{j\sigma}(u)$ are

$$\begin{aligned} f_{j\sigma}(u) &= \sin(u) + \mathrm{e}^{iu}\mathsf{n}_{j\sigma}, \\ g_{j\sigma}(u) &= \cos(u) - \mathrm{e}^{iu}\mathsf{n}_{j\sigma}. \end{aligned} \tag{2.6.2}$$

Because this is an object with fermionic entries acting on 2 fermionic states and 2 bosonic ones, one cannot perform usual matrix operations i.e. this is not an actual matrix, but a $\mathbb{Z}_2$ graded object. The above is only useful for visualising the components of such object, but formally, one should write [127]

$$L_j^f(u) = \sum_{1\le\alpha,\beta\le 4} \Theta^{\alpha\beta}(u)\mathbb{E}_{\alpha\beta} \tag{2.6.3}$$

where $\Theta^{\alpha\beta}(u)$ is the (α,β) entry of (2.6.1) and the $\mathbb{E}_{\alpha\beta}$ are the set of supermatrices, satisfying the commutation relations

$$[\mathbb{E}_{\alpha\beta}, \mathbb{E}_{\gamma\delta}] = \delta_{\beta\gamma}\mathbb{E}_{\alpha\delta} - (-1)^{(\deg(\alpha)+\deg(\beta))(\deg(\gamma)+\deg(\delta))}\delta_{\delta\alpha}\mathbb{E}_{\gamma\beta}. \tag{2.6.4}$$

with grading

$$\deg(1) = \deg(4) = 0, \quad \deg(2) = \deg(3) = 1. \tag{2.6.5}$$

However, a trick to get around this problem is to define the graded tensor product:

$$[A \underset{s}{\otimes} B]_{ac,bd} = (-1)^{[\deg(a)+\deg(b)]\deg(c)} A_{ab}B_{cd}. \tag{2.6.6}$$

Instead of dealing with Shastry's R-matrix, we will focus on the so called 'fermionic R-matrix' $R^f(u,u')$, whose relation to $R^h(u,u')$ is given in appendix B.1. The fermionic R-matrix satisfies the Yang-Baxter algebra for graded models with the fermionic Lax operator [122, 123]:

$$\check{R}^f(u,u')(L_j^f(u) \underset{s}{\otimes} L_j^f(u')) = (L_j^f(u') \underset{s}{\otimes} L_j^f(u))\check{R}^f(u,u'). \tag{2.6.7}$$

The monodromy matrix is constructed as usual

$$T_a^f(u) := L_L^f(u)L_{L-1}^f(u)\dots L_1^f(u) \tag{2.6.8}$$

and it satisfies the graded Yang-Baxter algebra, or graded RTT relation

$$\check{R}^f(u,u')(T^f(u) \underset{s}{\otimes} T^f(u')) = (T^f(u') \underset{s}{\otimes} T^f(u))\check{R}^f(u,u'). \tag{2.6.9}$$

When obtaining the transfer matrix, one must take the trace over the supermatrices: the supertrace $\tau^f(u) = \mathrm{str}(T^f(u)) = \mathrm{tr}(\sigma^z \otimes \sigma^z\ T^f(u))$. Then

$$[\tau^f(u), \tau^f(u')] = 0 \tag{2.6.10}$$

and the periodic Hubbard Hamiltonian (2.1.14) can be extracted from its expansion [125, 126].

Now we will proceed to comment on the real diagonal K-matrix solutions of the Hubbard model. These were computed in [120] and used for the ABA [124]. The graded reflection equation for the Hubbard model, as shown in the literature is obtained from (1.6.8) and (1.6.7) by using the following property of fermionic R-matrix

$$R_{ij}^f(u_i,u_j) = R_{ji}^f(-u_j,-u_i)^* \tag{2.6.11}$$

Now $K_1^{\pm}(u_1) = K^{\pm}(u_1) \otimes_s \mathbf{1}_4$ and $K_2^{\pm}(u_2) = \mathbf{1}_4 \otimes_s K^{\pm}(u_2)$. The left graded reflection equation is identical to (1.6.7)

$$\begin{aligned} &R^f_{12}(u_1,u_2)K_1^-(u_1)R^f_{21}(u_2,-u_1)K_2^-(u_2) = \\ &\quad = K_2^-(u_2)R^f_{12}(u_1,-u_2)K_1^-(u_1)R^f_{21}(-u_2,-u_1) \end{aligned} \tag{2.6.12}$$

and the right graded reflection equation is obtained by taking the complex conjugate of (1.6.8) and using (2.6.11):

$$\begin{aligned} &R^f_{21}(u_2,u_1)K_1^+(u_1)R^f_{12}(-u_1,u_2)K_2^+(u_2) = \\ &\quad = K_2^+(u_2)R^f_{21}(-u_2,u_1)K_1^+(u_1)R^f_{12}(-u_1,-u_2). \end{aligned} \tag{2.6.13}$$

We shall assume the K-matrix solutions to be diagonal and not operatorial:

$$K^{\pm}(u,p_{\pm}) = \begin{pmatrix} x_1^{\pm}(u) & 0 & 0 & 0 \\ 0 & x_2^{\pm}(u) & 0 & 0 \\ 0 & 0 & x_3^{\pm}(u) & 0 \\ 0 & 0 & 0 & x_4^{\pm}(u) \end{pmatrix}. \tag{2.6.14}$$

For both left and right boundary, we have two solutions for the K-matrix [120]. One of them breaks the spin symmetry $\mathfrak{sl}_2^s$ into $\mathfrak{u}_1^s$: for the left boundary it is

$$\begin{aligned} x_1^-(u) &= x_4^-(u) = \mathrm{e}^{2h}\left(1+p_-\mathrm{e}^{-2h}\tan u\right)\left(1-p_-\mathrm{e}^{-2h}\tan u\right), \\ x_2^-(u) &= \left(1-p_-\mathrm{e}^{-2h}\tan u\right)\left(1-p_-\mathrm{e}^{2h}\tan u\right), \\ x_3^-(u) &= \left(1+p_-\mathrm{e}^{-2h}\tan u\right)\left(1+p_-\mathrm{e}^{2h}\tan u\right). \end{aligned} \tag{2.6.15}$$

and for the right boundary it is

$$\begin{aligned} x_1^+(u) &= x_4^+(u) = \mathrm{e}^{-2h}\left(p_+ + \mathrm{e}^{2h}\tan u\right)\left(p_+ - \mathrm{e}^{2h}\tan u\right), \\ x_2^+(u) &= \left(p_+ + \mathrm{e}^{2h}\tan u\right)\left(p_+ + \mathrm{e}^{-2h}\tan u\right), \\ x_3^+(u) &= \left(p_+ - \mathrm{e}^{2h}\tan u\right)\left(p_+ - \mathrm{e}^{-2h}\tan u\right). \end{aligned} \tag{2.6.16}$$

The other breaks the charge symmetry $\mathfrak{sl}_2^c$ into $\mathfrak{u}_1^c$: for the left boundary we have

$$\begin{aligned} x_1^-(u) &= \left(1-p_-\mathrm{e}^{-2h}\tan u\right)\left(1-p_-\mathrm{e}^{2h}\tan u\right), \\ x_2^-(u) &= x_3^-(u) = \mathrm{e}^{-2h}\left(1+p_-\mathrm{e}^{2h}\tan u\right)\left(1-p_-\mathrm{e}^{2h}\tan u\right), \\ x_4^-(u) &= \left(1+p_-\mathrm{e}^{2h}\tan u\right)\left(1+p_-\mathrm{e}^{-2h}\tan u\right). \end{aligned} \tag{2.6.17}$$

and for the right boundary it is

$$\begin{aligned} x_1^+(u) &= \left(p_+ + \mathrm{e}^{2h}\tan u\right)\left(p_+ + \mathrm{e}^{-2h}\tan u\right), \\ x_2^+(u) &= x_3^+(u) = \mathrm{e}^{2h}\left(p_+ + \mathrm{e}^{-2h}\tan u\right)\left(p_+ - \mathrm{e}^{-2h}\tan u\right) \\ x_4^+(u) &= \left(p_+ - \mathrm{e}^{2h}\tan u\right)\left(p_+ - \mathrm{e}^{-2h}\tan u\right). \end{aligned} \tag{2.6.18}$$

Using these K-matrices one can obtain a commuting family of conserved quantities by expanding the transfer matrix

$$\tau_B^f(u) = \mathrm{str}\, K^-(u)(T^f(-u)))^{-1}K^+(u)T^f(u) = c_1 + c_2 u + \mathcal{H}^h_{open}u^2 + O(u^3) \tag{2.6.19}$$

where c_1 and c_2 are constants. The reason the Hamiltonian appears in the quadratic term in the expansion is due to additional properties of the K-matrix [120]. There are four different integrable choices for the Hamiltonian $\mathcal{H}^h_{open}$

$$\begin{aligned} \mathcal{H}^h_{open} &= -\sum_{i=1}^{L}\sum_{\sigma=\uparrow,\downarrow} \mathrm{c}^\dagger_{i\sigma}\mathrm{c}_{i+1\sigma} + \mathrm{c}^\dagger_{i+1\sigma}\mathrm{c}_{i\sigma} + U\sum_{i=1}^{L}(\mathrm{n}_{i\uparrow}-\tfrac{1}{2})(\mathrm{n}_{i\downarrow}-\tfrac{1}{2}) \\ &\quad -p^c_-(\mathrm{n}_{1\uparrow}+\mathrm{n}_{1\downarrow}-1) - p^s_-(\mathrm{n}_{1\uparrow}-\mathrm{n}_{1\downarrow}) \\ &\quad -p^c_+(\mathrm{n}_{L\uparrow}+\mathrm{n}_{L\downarrow}-1) - p^s_+(\mathrm{n}_{L\uparrow}-\mathrm{n}_{L\downarrow}) \end{aligned} \tag{2.6.20}$$

given by

$$(p^c_-, p^s_-, p^c_+, p^s_+) = \begin{cases} (p^c_-, 0, p^c_+, 0) & \mathfrak{sl}^s_2 \times \mathfrak{u}^c_1 \\ (p^c_-, 0, 0, p^s_+) & \mathfrak{u}^s_1 \times \mathfrak{u}^c_1 \\ (0, p^s_-, 0, p^s_+) & \mathfrak{u}^s_1 \times \mathfrak{sl}^c_2 \\ (0, p^s_-, p^c_+, 0) & \mathfrak{u}^s_1 \times \mathfrak{u}^c_1 \end{cases}$$

One can also obtain off-diagonal boundaries which possess symmetries isomorphic to these by performing an $\mathfrak{sl}^c_2 \times \mathfrak{sl}^c_2$ rotation on the K-matrices [120].

We would like to construct the twisted Yangian symmetry for these boundaries [121]. Since there are both right and left boundary reflection equations, we can focus on left and right boundaries independently. Furthermore, since the left and right K-matrices are related by a simple transformation (in Appendix B.4), we can direct our attention to the right boundary alone. Let us set the Hubbard model on a half-infinite chain with a right integrable boundary:

$$\mathcal{H}^h_p = -\sum_{i<0}\sum_{\sigma=\uparrow,\downarrow} \mathrm{c}^\dagger_{i\sigma}\mathrm{c}_{i+1\sigma} + \mathrm{c}^\dagger_{i+1\sigma}\mathrm{c}_{i\sigma} + U\sum_{i\le 0}(\mathrm{n}_{i\uparrow}-\tfrac{1}{2})(\mathrm{n}_{i\downarrow}-\tfrac{1}{2}) - p\mathcal{B}_0 \tag{2.6.21}$$

where

$$\mathcal{B}_0 = \begin{cases} \mathcal{H}^s_{0,0} = \mathrm{n}_{0\uparrow} - \mathrm{n}_{0\downarrow} & \text{for a right boundary magnetic field} \\ \mathcal{H}^c_{0,0} = \mathrm{n}_{0\uparrow} + \mathrm{n}_{0\downarrow} - 1 & \text{for a right boundary chemical potential} \end{cases}$$

and p is the boundary field strength. Clearly, the Lie symmetry of the model has now been broken to either $\mathfrak{u}^s_1 \times \mathfrak{sl}^c_2$ or $\mathfrak{sl}^s_2 \times \mathfrak{u}^c_1$, but integrability is unaffected [132]. This and the fact that $(\mathfrak{sl}_2, \mathfrak{u}_1)$ form a symmetric pair hint at the existence of a boundary twisted Yangian symmetry, which we shall construct shortly. We will focus on the case of a boundary magnetic field; the chemical potential case can be obtained via the partial PHT. Specifically, the Lie symmetry breaking is

$$\begin{array}{ccc} \mathfrak{sl}^s_2 & \to & \mathfrak{u}^s_1 \\ \{E^s_0, F^s_0, H^s_0\} & \to & \{H^s_0\} \end{array} \tag{2.6.22}$$

so we expect $\mathcal{Y}(\mathfrak{sl}_2)^s$ to break to the twisted Yangian $\mathcal{Y}(\mathfrak{sl}_2, \mathfrak{u}_1)^s$, generated by H^s_0 and two other operators

$$\widehat{E}^s_1 = E^s_1 + \frac{U}{2} E^s_0 H^s_0 \,, \qquad \widehat{F}^s_1 = F^s_1 - \frac{U}{2} F^s_0 H^s_0 \tag{2.6.23}$$

constructed using (1.7.12). These agree with the twisted Yangian found in [65], and hence satisfy the terrible relations (1.7.6).

However, there exists a subtlety in the symmetry generators which cannot be derived by any other method other than guesswork and computation of commutators. For commutation with the Hamiltonian, one has to make use both of the evaluation automorphism (1.5.14) and of the freedom to make it site-dependent, by adding a boundary term. Doing so, we find (details in Appendix B.2) that

$$\begin{aligned} \widetilde{E}^s_1 &= \widehat{E}^s_1 + \mu_+ E^s_0 - p\mathcal{E}^s_{0,0} \,, \\ \widetilde{F}^s_1 &= \widehat{F}^s_1 + \mu_- F^s_0 + p\mathcal{F}^s_{0,0} \,, \end{aligned} \tag{2.6.24}$$

where

$$\mu_\pm = \frac{U}{2} \pm \left(p - \frac{1}{p}\right) , \tag{2.6.25}$$

together with H^s_0, are the charges which commute with the half-infinite Hubbard Hamiltonian over the antiferromagnetic vacuum (2.2.11) with right boundary $-p\mathcal{H}^s_{0,0}$ after neglecting boundary terms at $-\infty$. These modifications do not change the twisted Yangian defining relations, which in this particular case are (1.7.6) but with functions of the coupling constant of the Hubbard model instead of that of the

Heisenberg spin chain. Namely, the operators (2.6.24) satisfy

$$[H_0^s, \widetilde{E}_1^s] = 2\widetilde{E}_1^s, \quad [H_0^s, \widetilde{F}_1^s] = -2\widetilde{F}_1^s \tag{2.6.26}$$

which are preserved by the coproduct (Appendix B.3). Additionally, they satisfy the property (1.7.6):

$$\begin{aligned}
[\widetilde{E}_1^s, [\widetilde{E}_1^s, [\widetilde{E}_1^s, \widetilde{F}_1^s]]] &= -12U^2 \widetilde{E}_1^s \left(\frac{1}{U} \left(p - \frac{1}{p} \right) + H_0^s \right) \widetilde{E}_1^s \\
[\widetilde{F}_1^s, [\widetilde{F}_1^s, [\widetilde{F}_1^s, \widetilde{E}_1^s]]] &= 12U^2 \widetilde{F}_1^s \left(\frac{1}{U} \left(p - \frac{1}{p} \right) + H_0^s \right) \widetilde{F}_1^s .
\end{aligned} \tag{2.6.27}$$

These charges satisfy the coideal property,

$$\begin{aligned}
\Delta H_0^s &= H_0^s \otimes 1 + 1 \otimes H_0^s \\
\Delta \widetilde{E}^s &= (\widehat{E}^s + \mu_+ E_0^s) \otimes 1 + 1 \otimes (\widehat{E}^s + \mu_+ E_0^s - p\mathcal{E}_{0,0}^s) + U E_0^s \otimes H_0^s \\
\Delta \widetilde{F}^s &= (\widehat{F}^s + \mu_- F_0^s) \otimes 1 + 1 \otimes (\widehat{F}^s + \mu_- F_0^s + p\mathcal{F}_{0,0}^s) - U F_0^s \otimes H_0^s
\end{aligned} \tag{2.6.28}$$

i.e. all these coproducts belong to $\mathcal{Y}(\mathfrak{sl}_2)^s \otimes \mathcal{Y}(\mathfrak{sl}_2, \mathfrak{u}_1)^s$ (details in Appendix B.3). Thus the modified $\mathcal{Y}(\mathfrak{sl}_2, \mathfrak{u}_1)$ generated by $H_0^s, \widetilde{E}^s$ and $\widetilde{F}^s$ is the boundary twisted Yangian of the half-infinite Hubbard chain in the presence of a right boundary magnetic field. Further, taking the 4×4 $\mathfrak{sl}_2$ representation given in (2.3.40), the reflection matrix for a boundary magnetic field at site $i = 0$ satisfies the boundary intertwining relation (1.7.13) (see Appendix B.4).

2.7 Achiral boundary

Now we will proceed to construct an integrable boundary which is characteristic of models with a symmetry that can be represented by two copies of the same symmetry algebra [89]. Suppose a 1+1D physical theory has symmetry algebra $\mathfrak{g}_L \times \mathfrak{g}_R$, where $\mathfrak{g}_L$ and $\mathfrak{g}_R$ are generated by $\{Q_0^{a,L}\}$ and $\{Q_0^{a,R}\}$ respectively (the superscript's only use is to label the copies). One can also decompose this symmetry into $\mathfrak{g}_+ \oplus \mathfrak{g}_-$, where $Q_0^{\pm,a} = Q_0^{a,L} \pm Q_0^{a,R}$. If we were to impose an achiral boundary condition on the real line [135], which satisfies $\alpha(Q_0^{a,L}) = Q_0^{a,R}$ and $\alpha^2 = id$, $\mathfrak{g}_L \times \mathfrak{g}_L$ would break to the subalgebra $\mathfrak{g}_+$.

One can check that $\mathfrak{g}_L \times \mathfrak{g}_R$ and $\mathfrak{g}_+$ form a symmetric pair, and hence an integrable system with this type of boundary condition is expected to possess a remnant of the original $\mathcal{Y}(\mathfrak{g} \times \mathfrak{g})$ symmetry. This is not $\mathcal{Y}(\mathfrak{g})$, but rather, the twisted Yangian $\mathcal{Y}(\mathfrak{g}_L \times \mathfrak{g}_R, \mathfrak{g}_+)$ [135]. Now the task is to construct its generators. It is generated by

$\mathfrak{g}_+$ and a deformation of the grade 1 generators $Q_1^{a,-} = Q_1^{a,L} - Q_1^{a,R}$ [135] given by:

$$\widehat{Q}_1^{a,-} = Q_1^{a,-} + \alpha[\mathbf{C}_+, Q_0^{a,-}] \tag{2.7.1}$$

where α is a deformation parameter fixed by the theory (a coupling constant) and $\mathbf{C}_+$ is the quadratic Casimir operator of $\mathfrak{g}_L \times \mathfrak{g}_R$ restricted to $\mathfrak{g}_+$. These operators need to satisfy additional relations [65, 66], which for the $(\mathfrak{so}_4, \mathfrak{sl}_2^{\Delta})$ symmetric pair are given by

$$[Q_0^{a,+}, \widehat{Q}_1^{b,-}] = f^{ab}{}_c \widehat{Q}^{c,-}$$

and

$$[\widehat{Q}_1^{z,-}, [\widehat{Q}_1^{+,-}, \widehat{Q}_1^{-,-}]] = \lambda^2(\widehat{Q}_1^{-,-} Q_0^{+,+} - Q_0^{-,+} \widehat{Q}_1^{+,-}) Q_0^{z,+} \tag{2.7.2}$$

This is the case of the Hubbard model on a chain of even length. The $\mathfrak{so}_4$ algebra may be generated by operators $\mathcal{A}^a$ and $\mathcal{B}^a$, $a = +, -, z$, satisfying the following relations

$$[\mathcal{A}^a, \mathcal{A}^b] = f^{ab}{}_c \mathcal{A}^c, \quad [\mathcal{B}^a, \mathcal{B}^b] = f^{ab}{}_c \mathcal{A}^c, \quad [\mathcal{A}^a, \mathcal{B}^b] = f^{ab}{}_c \mathcal{B}^c, \tag{2.7.3}$$

where $f^{ab}{}_c$ are the $\mathfrak{sl}_2$ structure constants. Note that $\{\mathcal{A}^a\}$ generate a full diagonal $\mathfrak{sl}_2$ algebra, which we will denote as $\mathfrak{sl}_2^{\Delta}$. Since $\mathfrak{so}_4 \cong \mathfrak{sl}_2^2$, $\mathcal{A}^a$ and $\mathcal{B}^b$ can be constructed via the $\mathfrak{sl}_2^s \times \mathfrak{sl}_2^c$ generators in the following way

$$\mathcal{A}_0^+ = E_0^s + E_0^c, \quad \mathcal{A}_0^- = F_0^s + F_0^c, \quad \mathcal{A}_0^z = H_0^s + H_0^c, \tag{2.7.4}$$

$$\mathcal{B}_0^+ = E_0^s - E_0^c, \quad \mathcal{B}_0^- = F_0^s - F_0^c, \quad \mathcal{B}_0^z = H_0^s - H_0^c. \tag{2.7.5}$$

The level 1 generators of the Yangian symmetry are constructed similarly, changing the level label from 0 to 1. $Q_n^{a,+}$ and $Q_n^{a,-}$ correspond to $\mathcal{A}_n^a$ and $\mathcal{B}_n^a$ respectively.

Let us now direct our attention to the left half-infinite Hubbard Hamiltonian, with the limit as the left side of the chain goes to $-\infty$ taken over the antiferromagnetic vaccum. We would like to come up with a boundary term that only preserves $\mathfrak{sl}_2^{\Delta}$, which will correspond to the symmetric pair $(\mathfrak{so}_4, \mathfrak{sl}_2^{\Delta})$ and hence allow for the construction of a twisted Yangian. Interestingly, we have already encountered this term: it is nothing other than the partial PHT (2.2.6). As an operator in terms of fermionic oscillators, this can be represented as

$$\mathcal{P}_{\downarrow} = \prod_j \mathcal{P}_{j\downarrow}, \quad \mathcal{P}_{j\downarrow} = (\mathsf{c}_{j\downarrow} - (-1)^j \mathsf{c}_{j\downarrow}^{\dagger}) \tag{2.7.6}$$

which satisfies

$$\mathcal{P}_{\downarrow} X_0^s \mathcal{P}_{\downarrow}^{\dagger} = X_0^c \text{ for } X = E, F, H, \quad \mathcal{P}_{\downarrow}^2 = 1 \tag{2.7.7}$$

and thus

$$[\mathcal{P}_{\downarrow}, \mathcal{A}_0^a] = 0. \tag{2.7.8}$$

Hence the following half-infinite Hubbard Hamiltonian

$$\mathcal{H}_{\Delta}^h = (\mathcal{H}^h)^{\leftarrow} + p\mathcal{P}_{0\downarrow}, \tag{2.7.9}$$

no longer possesses a full $\mathfrak{so}_4$ symmetry but it is broken to $\mathfrak{sl}_2^{\Delta}$. Here $(\mathcal{H}^h)^{\leftarrow}$ denotes the Hubbard Hamiltonian with sites on the left half-line. The boundary term $p\mathcal{P}_{0\downarrow}$ acts on the basis of states by reflecting a particle as a hole and vice versa at site 0. In doing this, the states gain a factor of p, which is interpreted as a change in phase, requiring $|p|= 1$. This is then an achiral boundary condition, and since $\mathfrak{so}_4$ and $\mathfrak{sl}_2^{\Delta}$ form a symmetric pair, the model is expected to possess a twisted Yangian symmetry $\mathcal{Y}(\mathfrak{so}_4, \mathfrak{sl}_2^{\Delta})$. Naively, one would attempt to construct the deformed level 1 generators using (2.7.1) and obtain, for example,

$$\widehat{\mathcal{B}}^{+} = \mathcal{B}_1^{+} + \frac{U}{4}(\mathcal{B}_0^{+}\mathcal{A}_0^{z} - \mathcal{B}_0^{z}\mathcal{A}_0^{+}). \tag{2.7.10}$$

However, just as in the case of other integrable open boundaries [121], there exist two subtleties. First, one must make use of the automorphism (1.5.14). Secondly, we note that the right Yangian copy is not only obtained through the operator (2.7.6) but also by changing U to $-U$, and the boundary term in the Hamiltonian does not 'know' about this. Hence, to satisfy the coideal property, the level 1 generators which must be deformed are $\mathcal{A}_1^a, a = +, -, z$. One then finds that the operator

$$\widetilde{\mathcal{B}}_1^{+} = \mathcal{A}_1^{+} + \frac{U}{2}\mathcal{B}_0^{+} + \frac{U}{4}(\mathcal{B}_0^{+}\mathcal{A}_0^{z} - \mathcal{B}_0^{z}\mathcal{A}_0^{+}) \tag{2.7.11}$$

commutes with $\mathcal{H}_{\Delta}^h$ over the antiferromagnetic vacuum upon neglecting boundary terms at $-\infty$. Simlarly, the other twisted level 1 charges are:

$$\begin{aligned} \widetilde{\mathcal{B}}_1^{-} &= \mathcal{A}_1^{-} + \frac{U}{2}\mathcal{B}_0^{-} - \frac{U}{4}(\mathcal{B}_0^{-}\mathcal{A}_0^{z} - B_0^{z}A_0^{-}), \\ \widetilde{\mathcal{B}}_1^{z} &= \mathcal{A}_1^{z} + \frac{U}{2}\mathcal{B}_0^{z} - \frac{U}{2}(\mathcal{B}_0^{+}\mathcal{A}_0^{-} - \mathcal{B}_0^{-}\mathcal{A}_0^{+}). \end{aligned} \tag{2.7.12}$$

Their coproducts are

$$\begin{aligned}
\Delta\widetilde{\mathcal{B}}_1^+ &= \widetilde{\mathcal{B}}^+\otimes 1 + 1\otimes\widetilde{\mathcal{B}}^+ + \frac{U}{2}(\mathcal{B}_0^+\otimes\mathcal{A}_0^z - \mathcal{B}_0^z\otimes\mathcal{A}_0^+),\\
\Delta\widetilde{\mathcal{B}}_1^- &= \widetilde{\mathcal{B}}^-\otimes 1 + 1\otimes\widetilde{\mathcal{B}}^- - \frac{U}{2}(\mathcal{B}_0^-\otimes\mathcal{A}_0^z - \mathcal{B}_0^z\otimes\mathcal{A}_0^-),\\
\Delta\widetilde{\mathcal{B}}_1^z &= \widetilde{\mathcal{B}}^z\otimes 1 + 1\otimes\widetilde{\mathcal{B}}^z - U(\mathcal{B}_0^+\otimes\mathcal{A}_0^- - \mathcal{B}_0^-\otimes\mathcal{A}_0^+).
\end{aligned} \tag{2.7.13}$$

Thus the coideal property is satisfied and hence $\mathcal{Y}(\mathfrak{so}_4,\mathfrak{sl}_2^\Delta) = \langle\mathcal{A}_0^a,\widetilde{\mathcal{B}}_1^b\rangle$ forms a coideal subalgebra of $\mathcal{Y}(\mathfrak{so}_4)$. Thus a partial PHT is an achiral boundary in the half-infinite Hubbard chain, and possesses a twisted Yangian symmetry.

2.8 Free boundary

Although it may seem like we have exhausted all possible symmetric pairs, we still have the case where the preserved subalgebra is $\mathfrak{sl}_2$ itself: the trivial symmetric pair $(\mathfrak{sl}_2,\mathfrak{sl}_2)$. At the level of the open Hamiltonian, this is achieved by imposing an object belonging to the center of $\mathfrak{sl}_2$ at the boundary - which we may as well pick the quadratic casimir $C = Q_0^+Q_0^- + Q_0^-Q_0^+ + \frac{1}{2}(Q_0^z)^2$

$$\mathcal{H}_o^h = (\mathcal{H}^h)^{\leftarrow} + C, \tag{2.8.1}$$

We call this type of boundary the free boundary[1]. One can check that all Lie symmetry generators still commute with $\mathcal{H}_o^h$. As in Section 2.1, we will focus on $\mathfrak{sl}_2^s$ since all results for the other $\mathfrak{sl}_2$ copy can be obtained through the partial PHT. Although the full Lie symmetry is preserved, the free boundary breaks all of the Yangian level 1 generators. Due of the preservation of the Lie symmetry, one cannot add a level 0 quadratic term to the level 1 generators – as in the case of the $\mathfrak{u}_1$ boundary – to restore the Yangian symmetry. This means all odd level Yangian elements are broken by the free boundary. However, a modified version of the even level generators is preserved [58, 137], which lets us build an infinite tower of symmetries and thus ensure integrability of the model. This is the twisted Yangian $\mathcal{Y}(\mathfrak{sl}_2,\mathfrak{sp}_2)\cong\mathcal{Y}(\mathfrak{sl}_2,\mathfrak{sl}_2)$, written in [55] as $\mathcal{Y}^-(\mathfrak{sl}_2)$. It is generated by the

[1]Called the open boundary in [137], but this could be confused with 'open boundary conditions' and hence we have decided to change the name here and in the next Chapter.

elements $Q_0^{\pm}$, Q_0^z and $\widehat{Q}_2^{\pm}$, $\widehat{Q}_2^z$ satisfying

$$\begin{aligned}
&[Q_0^z, Q_0^{\pm}] = \pm 2Q_0^{\pm}, \quad [Q_0^+, Q_0^-] = Q_0^z, \\
&[\widehat{Q}_2^z, Q_0^{\pm}] = [Q_0^z, \widehat{Q}_2^{\pm}] = \pm 2Q_2^{\pm}, \quad [\widehat{Q}_2^{\pm}, Q_0^{\mp}] = \pm\widehat{Q}_2^z, \\
&[\widehat{Q}_2^z, [\widehat{Q}_2^+, \widehat{Q}_2^-]] = 4\alpha^2\big(\{Q_0^+, \widehat{Q}_2^-, \widehat{Q}_2^z\} \\
&\qquad\qquad - \{Q_0^-, \widehat{Q}_2^+, \widehat{Q}_2^z\}\big),
\end{aligned} \tag{2.8.2}$$

where α is a model dependent constant. If one takes the level 2 generators of $\mathcal{Y}(\mathfrak{sl}_2)$ to be

$$Q_2^{\pm} = \pm\tfrac{1}{2}[Q_1^z, Q_1^{\pm}], \quad Q_2^z = [Q_1^+, Q_1^-]. \tag{2.8.3}$$

then the twisted level 2 generators of $\mathcal{Y}(\mathfrak{sl}_2, \mathfrak{sl}_2)$ can be obtained by the embedding $\varphi_- : \mathcal{Y}(\mathfrak{sl}_2, \mathfrak{sl}_2) \hookrightarrow \mathcal{Y}(\mathfrak{sl}_2)$ of algebras given by

$$Q_0^a \mapsto Q_0^a, \quad \widehat{Q}_2^a \mapsto Q_2^a + \alpha[Q_1^a, C] + \beta Q_0^a, \tag{2.8.4}$$

where β is another model dependent constant. In the case of the Hubbard model, the relevant operators are represented in terms of the fermionic operators of spin σ Osc_L^{σ} by the map $\rho_{\mathrm{Osc}} : \mathfrak{sl}_2^s \times \mathfrak{sl}_2^c \to \mathrm{Osc}_L^{\uparrow} \times \mathrm{Osc}_L^{\downarrow}$ as follows

$$\rho_{\mathrm{Osc}}(Q_n^+) = E_n^s, \quad \rho_{\mathrm{Osc}}(Q_n)^- = F_n^s, \qquad \rho_{\mathrm{Osc}}(Q_n^z) = H_n^s. \tag{2.8.5}$$

Using the formula (2.8.4), the modified level 2 generators which come the closest to commuting with $\mathcal{H}_o^h$ are

$$\begin{aligned}
\widehat{H}_2^s &= H_2^s - U(E_1^s F_0^s - E_0^s F_1^s), \\
\widehat{E}_2^s &= E_2^s - \tfrac{U}{2}(H_1^s E_0^s - H_0^s E_1^s), \\
\widehat{F}_2^s &= F_2^s + \tfrac{U}{2}(H_1^s F_0^s - H_0^s F_1^s).
\end{aligned} \tag{2.8.6}$$

One must add an extra term at the right boundary to obtain operators which commute with $\mathcal{H}_o^h$ over the antiferromagnetic vacuum

$$\widetilde{H}_2^s = \widehat{H}_2^s - 2\mathcal{H}_{0,0}^s, \quad \widetilde{E}_2^s = \widehat{E}_2^s - 2\mathcal{E}_{0,0}^s, \quad \widetilde{F}_2^s = \widehat{F}_2^s - 2\mathcal{F}_{0,0}^s. \tag{2.8.7}$$

Details of this computation are given in Appendix B.7. Note the use of the site-dependent evaluation automorphism, which is also present in the case of a boundary magnetic field and chemical potential from Section 2.5. It seems that this automorphism is of significant importance in the open Hubbard model. The coproduct of these generators is given by

$$\begin{aligned}
\Delta\widetilde{H}_2^s &= \widehat{H}_2^s \otimes 1 + 1 \otimes \widetilde{H}_2^s - 2U(E_1^s \otimes F_0^s - F_1^s \otimes E_0^s) + U^2(...), \\
\Delta\widetilde{E}_2^s &= \widehat{E}_2^s \otimes 1 + 1 \otimes \widetilde{E}_2^s - U(H_1^s \otimes E_0^s - E_1^s \otimes H_0^s) + U^2(...), \\
\Delta\widetilde{F}_2^s &= \widehat{F}_2^s \otimes 1 + 1 \otimes \widetilde{F}_2^s + U(H_1^s \otimes F_0^s - F_1^s \otimes E_0^s) + U^2(...),
\end{aligned} \tag{2.8.8}$$

where the U^2 terms are all level 0 and thus commute with all of $\mathcal{H}_o^h$. One can see that generators to the right of the coproduct are symmetries of the boundary and those to the left of the coproduct are symmetries of the bulk, and thus this coproduct acts on the Hilbert space properly. This result can be derived from the results in Appendix B.3.

2.9 The open $SU(n)$ Hubbard model

We would like to explore the twisted Yangian symmetry after imposing different boundary conditions on the $SU(n)$ generalisation of the Hubbard model. Since a remnant of the original Yangian is only present if the original Lie symmetry and the subalgebra to which it is broken by the boundary form a symmetric pair, this narrows down the types of integrable boundary conditions we can impose. More specifically, we can add a boundary term $p\mathtt{B}$ in the half-infinite Hamiltonian as

$$\mathtt{H}_B^h = (\mathtt{H}^h)^{\leftharpoonup} + p\mathtt{B}. \tag{2.9.1}$$

The structure of all these twisted Yangians follow straightforwardly from [55] and therefore we will only illustrate their construction with the so called ‘grassmanian’ case [52, 54]: the breaking of $\mathfrak{su}_{n-1}$ to $\mathfrak{su}_m \times \mathfrak{su}_{n-1-m}$. Additionally we will focus on a single spin label (and therefore remove it from our calculations), as the extra twisted Yangian copy can be obtained by simply changing the spin label. We set the boundary term to be

$$\mathtt{B} = k_1 \sum_{a=1}^{m} \mathtt{E}_0^{aa} + k_2 \sum_{a=m+1}^{n-1} \mathtt{E}_0^{aa} + k_3 \mathtt{E}^{nn}, \tag{2.9.2}$$

where $k_1 \neq k_2$ and k_3 are nonzero complex numbers. Adding this boundary term to an $\mathfrak{su}_{n-1}$ invariant Hamiltonian breaks the symmetry to $\mathfrak{su}_m \times \mathfrak{su}_{n-1-m} \times \mathfrak{u}_1$. Note that if either k_1 or k_2 were zero the symmetry would break to $\mathfrak{su}_{n-1-m}$ or $\mathfrak{su}_n$ respectively, and if $k_1 = k_2$ the symmetry would remain unbroken as this would just give the quadratic casimir operator of $\mathfrak{su}_{n-1}$.

This breaks the grade 0 generators of the form $\mathtt{I}^{m+a,m-b}$ and $\mathtt{I}^{m-a,m+b}$ where a and b are positive. It will also break the level 1 generators with such labels, but this can be fixed by adding a quadratic term in the level 0 generators. Indeed, the operator

$$\widehat{\mathtt{J}}^{\,m+a,m-b} = \mathtt{J}^{m+a,m-b} + U(m-n+1)\mathtt{I}^{m+a,m-b} + U\left(\sum_{c=1}^{m} - \sum_{c=m+1}^{n-1}\right)\mathtt{I}^{m+a,c}\mathtt{I}^{c,m-b} \tag{2.9.3}$$

commutes with $\mathtt{H}_B^h$ if the limit as the left side of the model goes to $-\infty$ is taken over the empty band vacuum $|\mathtt{vac}\rangle$ (2.4.13) - only then we may neglect boundary terms at $-\infty$. The coproduct satisfies the coideal property

$$\begin{aligned}\Delta(\widehat{\mathtt{J}}^{\,m+a,m-b}) \;&=\; \widehat{\mathtt{J}}^{\,m+a,m-b}\otimes 1 + 1\otimes\widehat{\mathtt{J}}^{m+a,m-b} \\ &\quad -2U\left(\sum_{c=1}^{m}\mathtt{I}^{m+a,c}\otimes\mathtt{I}^{c,m-b} - \sum_{c=m+1}^{n-1}\mathtt{I}^{c,m-b}\otimes\mathtt{I}^{m+a,c}\right)\end{aligned} \tag{2.9.4}$$

These operators, together with the unbroken subalgebra, form a coideal subalgebra of the original Yangian: the twisted Yangian $\mathcal{Y}(\mathfrak{su}_{n-1}, \mathfrak{su}_m\times\mathfrak{su}_{n-1-m}\times\mathfrak{u}_1)$. Using the evaluation representation of the level 1 generators, the $K-$matrix can be obtained from the boundary intertwining relation (1.7.13)

$$\rho_u(\widehat{J}^{m+a,m-b})K(u) = K(u)\rho_{-u}(\widehat{J}^{m+a,m-b}) \tag{2.9.5}$$

Solving this relation, one finds:

$$K(u) = \nu(u)\mathrm{diag}(1_m, -1_{n-m-1}, n(u)) \tag{2.9.6}$$

where $\nu(u)$ is some prefactor such that $\nu(\theta)\nu(-\theta) = 1$, 1_m is a sequence of m 1's and $n(\theta)$ is an arbitrary element of the $K-$matrix at the (n,n) entry due to the $\mathfrak{u}_1$ symmetry. This element is not fixed by the $\mathcal{Y}(\mathfrak{su}_{n-1})$, and hence we need to check the RE to obtain it. In the form

$$R(u-v)(K(u)\otimes\mathbf{1}_n)R(u+v)(\mathbf{1}_n\otimes K(v)) = (\mathbf{1}_n\otimes K(v))R(u+v)(K(u)\otimes\mathbf{1}_n)R(u-v), \tag{2.9.7}$$

this imposes $n(u) = -1$. Hence $K(u) = \nu(u)\mathrm{diag}(1_m, -1_{n-m})$ which agrees with [53].

Chapter 3

Folding a spin chain

The boundary QISM [61] contains an implicit idea of “folding”. As we have seen, it begins with YBE and REs and their associated R- and K-matrices, and uses the latter to construct a boundary transfer matrix from its bulk parent. This process contains an implied folding of the infinite line (or chain) back on itself to create a half-line, and thereby a boundary-integrable model on this half-line. This folding is only rarely made explicit in the literature [138].

However, this process can be difficult to implement in explicit cases. It does not begin with the Hamiltonian but rather extracts it, together with other conserved quantities and symmetries, from the transfer matrix. If we instead begin with a bulk Hamiltonian and wish to discover integrable boundaries and their symmetries, a different, “bottom-up” procedure is needed. This procedure, which we refer to as “folding”, is a map denoted by f (and $\overline{f}$ in the double-row case – see below) which sends the spin operators and conserved charges of a model defined on an infinite chain to those defined on a semi-infinite chain. The purpose of this Chapter is to detail this procedure and apply it first to a classic and then to an overarching new case.

We begin with the elementary examples of the classic Heisenberg spin chain and a “double-row” model of two Heisenberg spin chains uncoupled except by the boundary. The latter is motivated by a similar structure which emerges in AdS/CFT [135] and also serves as a toy model for the open Hubbard chain with an achiral boundary [89]. We then go on to construct integrable boundaries for the Inozemtsev long-range infinite spin chain [139] and its doubling. In each case we emphasize the Yangian symmetry of the bulk model, and from it derive a twisted Yangian symmetry of the model with an integrable boundary.

The Heisenberg and Inozemtsev spin chains are the natural choices to work with. The former is the most famous, prototypical spin chain in the physics literature. It allows us to check that the results obtained in this paper are in agreement with well-known ones, and also introduces the reader to our procedure through a relatively simple example.

The Inozemtsev chain, by contrast, is less well-known, but may be the more fundamental. All famous $\mathfrak{sl}_2$ spin chains are limiting cases of it (see Section 3.3). It also possesses striking thermodynamic properties of its own [140]. But most importantly for modern fundamental physics, it appears in the context of AdS/CFT – in particular, the expression for the dilatation operator of $\mathcal{N} = 4$ SYM in the planar limit coincides with its conserved charges up to three loops [141, 142].

The main motivation for our folding procedure is that, to construct integrable boundaries for long-range spin chains like Inozemtsev's, one cannot use the REs in the usual way and must instead rely on Dunkl operators [143, 144]. This is where our bottom-up approach becomes useful: starting with a long-range Hamiltonian defined on the infinite line, our folding procedure allows us to systematically construct integrable boundaries without the explicit use of a monodromy matrix.

This Chapter is organized as follows. In Section 3.1 we set up the chain and explain its folding. In Section 3.2 we study folding of the infinite Heisenberg spin chain. In Section 3.3 we review the basics of Inoztemsev's spin chain. The methods obtained are then used in Section 3.4 to fold an Inozemtsev hyperbolic spin chain. Section 3.5 contains concluding remarks and a discussion of relevant open questions.

Most of the results presented were computed using the `Wolfram Mathematica` computer algebra system. For readers' convenience we have detailed explicitly some of the computations that explain the folding of the Hamiltonian and Yangian operators.

3.1 The folding procedure

Fix $L \in \mathbb{N}$ and consider a one-dimensional lattice with $2L$ sites that can be occupied by spin-1/2 particles. Each lattice site is identified with a two-dimensional vector space $V_i \cong \mathbb{C}^2$ spanned by vectors

$$V_i = \mathrm{span}_{\mathbb{C}}\{\, |\uparrow\rangle_i\,,\ |\downarrow\rangle_i\}, \tag{3.1.1}$$

where $-L < i \leq L$ is the index of the site in the lattice. The entire lattice is the $2L$-fold tensor product $V := \bigotimes_{-L<i\leq L} V_i$.

To describe dynamics of such a lattice, we employ spin operators σ_i^+, σ_i^-, σ_i^z and the identity operator σ_i^0 that satisfy the usual (anti-)commutation relations (1.3.12) We require that, for $i \neq j$, the operators act on the states as (1.4.10).

The σ_i^a provide a unitary representation of the universal envelope $U(\mathfrak{sl}_2)$ of the $\mathfrak{sl}_2$

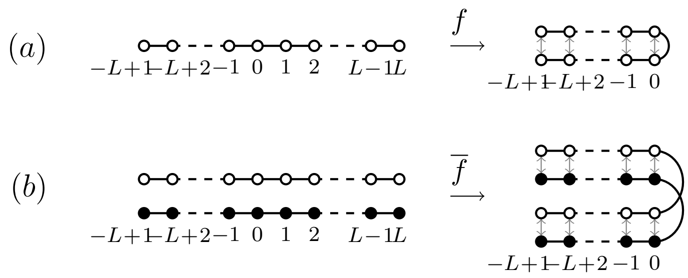


Figure 3.1: Folding: (a) Single-row lattice, (b) Double-row lattice.

Lie algebra

$$\rho_L \;:\; Q_0^{\pm} \mapsto \sum_{-L<i\leq L} \sigma_i^{\pm}, \quad Q_0^z \mapsto \sum_{-L<i\leq L} \sigma_i^z, \tag{3.1.2}$$

where $Q_0^{\pm}, Q_0^z$ are the standard generators of the $\mathfrak{sl}_2$ Lie algebra satisfying $[Q_0^+, Q_0^-] = Q_0^z$ and $[Q_0^z, Q_0^{\pm}] = \pm 2Q_0^{\pm}$. The map (3.1.2) together with (1.4.10) turns the vector space V into a left $U(\mathfrak{sl}_2)$-module.

Folding. We fold the lattice by identifying sites labelled by indices $1 \leq i \leq L$ with those labelled by $1-i$ as shown in Figure 3.1 (a). We say that the lattice is folded over a link.

Let us explain how the folding acts on the matrices σ_i^a. Recall from the fundamental representation of σ_i^a that

$$\sigma_i^{\pm}\sigma_i^z = \mp\sigma_i^{\pm}, \quad \sigma_i^z\sigma_i^{\pm} = \pm\sigma_i^{\pm}, \quad \sigma_i^z\sigma_i^z = \sigma_i^0,$$
$$\sigma_i^{\pm}\sigma_i^{\pm} = 0, \quad \sigma_i^{\pm}\sigma_i^{\mp} = \tfrac{1}{2}(\sigma_i^0 \pm \sigma_i^z),$$

which imply that any polynomial in σ_i^a can be written as a linear combinations of monomials

$$\prod_{-L<i\leq L} \sigma_i^{a_i} \quad \text{with} \quad a_i \in \{\pm, z, 0\}, \tag{3.1.3}$$

or in other words the monomials (3.1.3) provide a vector space basis of $\Sigma_L = \{\sigma_i^0, \sigma_i^a : a \in \{\pm, z\},\, -L < i \leq L\,\rangle$ over the field of complex numbers $\mathbb{C}$. Note that elements of Σ are also elements of End$V$; the element $\prod_{-L<i\leq L} \sigma_i^0$ is the identity map. Set $\Sigma_L^- = \{\sigma_i^0, \sigma_i^a : a \in \{\pm, z\},\, -L < i \leq 0\,\rangle$. We define the multiplicative folding $f : \Sigma_L \to \Sigma_L^-$ acting on monomials (3.1.3) by

$$f : \prod_{-L<i\leq L} \sigma_i^{a_i} \mapsto \prod_{-L<i\leq 0} k^{a_i a_{1-i}}\, \sigma_i^{a_i}\sigma_i^{a_{1-i}}, \tag{3.1.4}$$

where $k^{a_i a_{1-i}} \in \mathbb{C}$ are model-dependent folding constants that will be specified in the examples studied below.

3.2 Integrable boundaries of the Heisenberg spin chain

It is well known that the Hamiltonian of the Heisenberg spin chain

$$\mathcal{H}^H = -J \sum_{-L<i\leq L} \left(\sigma_i^+\sigma_{i+1}^- + \sigma_i^-\sigma_{i+1}^+ + \tfrac{1}{2}\sigma_i^z\sigma_{i+1}^z\right) \tag{3.2.1}$$

commutes with the Lie operators $\mathcal{E}_0^\pm = \rho_L(Q_0^\pm)$ and $\mathcal{E}_0^z = \rho_L(Q_0^z)$. We say that the Hamiltonian $\mathcal{H}^H$ exhibits a $U(\mathfrak{sl}_2)$ Lie algebra symmetry. In previous Chapters, we resctricted to only talking about $\mathfrak{sl}_2$ as the symmetry of this model, but for the folding procedure, we need to consider its universal envelope.

Let us recall from Section 1.5 that when the chain is infinitely long, i.e. $L \to \infty$, the Hamiltonian $\mathcal{H}^H$ for $J < 0$ additionally exhibits a Yangian symmetry. More precisely, it commutes, up to terms at $\pm\infty$ that may be neglected, with the operators

$$\begin{aligned} \mathcal{E}_1'^{\pm} &= \pm\tfrac{J}{2}\sum_{i<j}\sigma_i^\pm\sigma_j^z, \quad \mathcal{E}_1'^{z} = J\sum_{i<j}\sigma_i^+\sigma_j^-, \\ \mathcal{E}_1''^{\pm} &= \mp\tfrac{J}{2}\sum_{i<j}\sigma_i^z\sigma_j^\pm, \quad \mathcal{E}_1''^{z} = -J\sum_{i<j}\sigma_i^-\sigma_j^+, \end{aligned} \tag{3.2.2}$$

which, combined to

$$\mathcal{E}_1^\pm = \mathcal{E}_1'^\pm + \mathcal{E}_1''^\pm, \qquad \mathcal{E}_1^z = \mathcal{E}_1'^z + \mathcal{E}_1''^z, \tag{3.2.3}$$

satisfy the defining relations of the Yangian $\mathcal{Y}(\mathfrak{sl}_2)$ (1.7.6).

Magnetic boundary. Let us now focus on the antiferromagnetic, semi-infinite Heisenberg spin chain with a boundary magnetic field from Section 1.7 [145]

$$\mathcal{H}_\mu^H = (\mathcal{H}^H)^{\leftarrow} + \mu\sigma_0^z, \quad J < 0, \tag{3.2.4}$$

and recall that the presence of the boundary term in (3.2.4) breaks the $\mathcal{Y}(\mathfrak{sl}_2)$ Yangian symmetry down to the $\mathcal{Y}(\mathfrak{sl}_2, \mathfrak{u}_1)$ twisted Yangian. In particular, the Hamiltonian $\mathcal{H}_\mu^H$ commutes with $(\mathcal{E}_0^z)^{\leftarrow}$ and, up to terms at $-\infty$ that can be neglected in this regime, with twisted Yangian operators $\mathcal{X}^\pm$ defined by [146, 147]

$$\mathcal{X}^\pm = (\mathcal{E}_1^\pm)^{\leftarrow} \pm \tfrac{J}{2}(\mathcal{E}_0^\pm)^{\leftarrow}(\mathcal{E}_0^z)^{\leftarrow} + \tfrac{J}{2}\left(1 \mp \tfrac{J}{\mu}\right)(\mathcal{E}_0^\pm)^{\leftarrow}, \tag{3.2.5}$$

that are elements in $\Sigma_\infty^{\leftarrow}$ and satisfy the defining relations of $\mathcal{Y}(\mathfrak{sl}_2, \mathfrak{u}_1)$ (1.7.6). It is worth noting that the operators

$$\begin{aligned} \mathcal{X}'^{\pm} &= (\mathcal{E}_1'^{\pm})^{\leftarrow} \mp \tfrac{J^2}{4\mu}(\mathcal{E}_0^{\pm})^{\leftarrow}, \\ \mathcal{X}''^{\pm} &= (\mathcal{E}_1''^{\pm})^{\leftarrow} \pm \tfrac{J}{2}(\mathcal{E}_0^{\pm})^{\leftarrow}(\mathcal{E}_0^{z})^{\leftarrow} + \tfrac{J}{2}\big(1 \mp \tfrac{J}{2\mu}\big)(\mathcal{E}_0^{\pm})^{\leftarrow}, \end{aligned} \tag{3.2.6}$$

satisfying $\mathcal{X}^{\pm} = \mathcal{X}'^{\pm} + \mathcal{X}''^{\pm}$, are also symmetries of the antiferromagnetic $\mathcal{H}^H_\mu$ upon neglecting terms at $-\infty$. They can be viewed as analogues of the symmetries (3.2.2) of $\mathcal{H}^H$.

Our goal is to demonstrate the method of obtaining the Hamiltonian $\mathcal{H}^H_\mu$ from $\mathcal{H}^H$ and twisted Yangian operators (3.2.5) from those in (3.2.3) by employing the folding (3.1.4). The first step is to impose the following constraints on the folding constants:

$$k^{\pm 0} = -k^{0\pm} = k^{z0} = k^{0z} = 1, \quad k^{\pm z} = k^{z\pm}, \tag{3.2.7}$$

which ensure that Lie symmetries of $\mathcal{H}^H$ are projected to those of $\mathcal{H}^H_\mu$. Recall that $U(\mathfrak{sl}_2)$, as a vector space, is linearly spanned by the monomials $f^l h^m e^n$ with $l, m, n \in \mathbb{Z}_{\geq 0}$. Thus we must make sure that any monomial $(\mathcal{E}_0^-)^l(\mathcal{E}_0^z)^m(\mathcal{E}_0^+)^n$ for any $l, m, n \in \mathbb{Z}_{\geq 0}$, each being a symmetry of $\mathcal{H}^H$, is folded into a symmetry of $\mathcal{H}^H_\mu$, which exhibits a $U(\mathfrak{u}_1) \subset U(\mathfrak{sl}_2)$ symmetry only. The first constraint in (3.2.7) yields

$$\begin{aligned} f(\mathcal{E}_0^{\pm}) &= f\big(\sum_i \sigma_i^{\pm}\big) = (k^{\pm 0} + k^{0\pm}) \sum_{i\leq 0} \sigma_i^{\pm} = 0, \\ f(\mathcal{E}_0^{z}) &= f\big(\sum_i \sigma_i^{\pm}\big) = (k^{z0} + k^{0z}) \sum_{i\leq 0} \sigma_i^{z} = 2\sum_{i\leq 0} \sigma_i^{z} = 2(\mathcal{E}_0^z)^{\leftarrow}, \end{aligned}$$

while the second constraint in (3.2.7) additionally ensures that any monomial $(\mathcal{E}_0^-)^l(\mathcal{E}_0^z)^m(\mathcal{E}_0^+)^n$ is folded into a symmetry of $\mathcal{H}^H_\mu$. In particular, for any $l, m, n \in \mathbb{Z}_{\geq 0}$, we have that

$$f((\mathcal{E}_0^-)^l(\mathcal{E}_0^z)^m(\mathcal{E}_0^+)^n) = \delta_{ln} \sum_{0\leq r\leq l+m} c_r\, ((\mathcal{E}_0^z)^{\leftarrow})^r$$

for some $c_r \in \mathbb{C}$. Note that $k^{\pm\pm}$ do not play a role in the folding, since $\sigma_i^{\pm}\sigma_i^{\pm} = 0$. We also set $k^{zz} = 1$, so that $f(\rho_L((Q_0^z)^l)) = f(\rho_L((Q_0^z)^m)) f(\rho_L((Q_0^z)^n))$ for any $l, m, n \in \mathbb{Z}_+$ satisfying $l = m + n$. (We will comment on this property in Section 3.5).

Next, using (3.2.7) and splitting the sum $\sum_i$ into three terms as $\sum_i = \sum_{i<0} + \delta_{i0} +$

$\sum_{i>0}$, we fold the Hamiltonian $\mathcal{H}^H$ of the infinite chain:

$$\begin{aligned}
&f(\mathcal{H}^H) = \\
&= -J\Bigg(\sum_{i<0} \left(k^{+0}k^{-0}(\sigma_i^+\sigma_{i+1}^- + \sigma_i^-\sigma_{i+1}^+) + \tfrac{1}{2}(k^{z0})^2\sigma_i^z\sigma_{i+1}^z\right) \\
&\qquad + k^{+-}\sigma_0^+\sigma_0^- + k^{-+}\sigma_0^-\sigma_0^+ + \tfrac{1}{2}k^{zz}\sigma_0^z\sigma_0^z \\
&\qquad + \sum_{i>0} \left(k^{0+}k^{0-}(\sigma_{1-i}^+\sigma_{-i}^- + \sigma_{1-i}^-\sigma_{-i}^+) + \tfrac{1}{2}(k^{0z})^2\sigma_{1-i}^z\sigma_{-i}^z\right)\Bigg) \\
&= 2(\mathcal{H}^H)^{\leftarrow} - \tfrac{J}{2}\left(\left(k^{+-} - k^{-+}\right)\sigma_0^z + \left(1 + k^{+-} + k^{-+}\right)\right). \qquad (3.2.8)
\end{aligned}$$

Choosing $k^{-+} - k^{+-} = \frac{4\mu}{J}$ we have that $f(\mathcal{H}^H) = 2\mathcal{H}_\mu^H$ up to a constant term.

In order to fold the Yangian operators (3.2.3) we first split the sum $\sum_{i<j}$ into four terms

$$\textstyle\sum_{i<j\leq 0} + \delta_{i+j\neq 1}\sum_{i\leq 0<j} + \delta_{i+j=1}\sum_{i\leq 0<j} + \sum_{0<i<j}\,. \qquad (3.2.9)$$

By doing so for (3.2.3) and folding each sum individually we find

$$\begin{aligned}
f(\mathcal{E}_1^z) = J\Bigg(&\sum_{i<j\leq 0} \left(k^{+0}k^{-0}\sigma_i^+\sigma_j^- - k^{-0}k^{+0}\sigma_i^-\sigma_j^+\right) \\
&+ k^{0-}\Big((\mathcal{E}_0^+)^{\leftarrow}(\mathcal{E}_0^-)^{\leftarrow} - \sum_{i\leq 0}\sigma_i^+\sigma_i^-\Big) \\
&- k^{0+}\Big((\mathcal{E}_0^-)^{\leftarrow}(\mathcal{E}_0^+)^{\leftarrow} - \sum_{i\leq 0}\sigma_i^-\sigma_i^+\Big) \\
&+ \sum_{i\leq 0}\left(k^{+-}\sigma_i^+\sigma_i^- - k^{-+}\sigma_i^-\sigma_i^+\right) \\
&+ \sum_{0<i<j}\left(k^{0+}k^{0-}\sigma_{1-i}^+\sigma_{1-j}^- - k^{0-}k^{0+}\sigma_{1-i}^-\sigma_{1-j}^+\right)\Bigg) \\
= \tfrac{J}{2}&L(k^{-+} - k^{+-}) - \tfrac{J}{2}(k^{+-} + k^{-+})(\mathcal{E}_0^z)^{\leftarrow}, \qquad (3.2.10)
\end{aligned}$$

which commutes with $f(\mathcal{H}^H)$; and

$$\begin{aligned}
f(\mathcal{E}_1^{\pm}) = \pm\tfrac{J}{2}\Bigg(&\sum_{i<j\leq 0} \big(k^{\pm 0}k^{z0}\sigma_i^{\pm}\sigma_j^{z} - k^{z0}k^{\pm 0}\sigma_i^{z}\sigma_j^{\pm}\big) \\
&+ k^{0z}\big((\mathcal{E}_0^{\pm})^{\leftarrow}(\mathcal{E}_0^{z})^{\leftarrow} - \sum_{i\leq 0}\sigma_i^{\pm}\sigma_i^{z}\big) \\
&- k^{0\pm}\big((\mathcal{E}_0^{z})^{\leftarrow}(\mathcal{E}_0^{\pm})^{\leftarrow} - \sum_{i\leq 0}\sigma_i^{z}\sigma_i^{\pm}\big) \\
&+ \sum_{i\leq 0}\big(k^{\pm z}\sigma_i^{\pm}\sigma_i^{z} - k^{z\pm}\sigma_i^{z}\sigma_i^{\pm}\big) \\
&+ \sum_{0<i<j}\big(k^{0\pm}k^{0z}\sigma_{1-i}^{\pm}\sigma_{1-j}^{z} - k^{0z}k^{0\pm}\sigma_{1-i}^{z}\sigma_{1-j}^{\pm}\big)\Bigg) \\
= 2\big((\mathcal{E}_1^{\pm})^{\leftarrow} &\pm \tfrac{J}{2}(\mathcal{E}_0^{\pm})^{\leftarrow}(\mathcal{E}_0^{z})^{\leftarrow} + \tfrac{J}{2}(1-k^{z\pm})(\mathcal{E}_0^{\pm})^{\leftarrow}\big),
\end{aligned} \tag{3.2.11}$$

which commute with $f(\mathcal{H}^H)$, up to the terms at infinity, only if

$$k^{z\pm} = \mp\frac{4}{k^{+-}-k^{-+}} = \pm\frac{J}{\mu}, \tag{3.2.12}$$

in which case we obtain $f(\mathcal{E}_1^{\pm}) = 2\mathcal{X}^{\pm}$, as expected. We also have that $f(\mathcal{E}_1'^{\pm}) = 2\mathcal{X}'^{\pm}$ and $f(\mathcal{E}_1''^{\pm}) = 2\mathcal{X}''^{\pm}$, so that the summetries (3.2.2) of $\mathcal{H}^H$ are folded into the symmetries (3.2.6) of $(\mathcal{H}^H)^{\leftarrow}$. Thus we have demonstrated that with a suitable choice of the folding constants, which were deduced from the symmetry arguments, the Hamiltonian $\mathcal{H}^H$ of the infinite spin chain and its symmetries can be folded into the Hamiltonian $\mathcal{H}_\mu^H$ of a semi-infinite spin chain with a magnetic boundary and its symmetries.

In the remaining parts of this Section we will demonstrate how to obtain the semi-infinite spin chain with a free boundary and a semi-infinite double-row spin chain with a achiral boundary. The obtained results will then be used in Section 3.4 to obtain the corresponding boundary models for the Inozemtsev hyperbolic spin chain.

Free boundary. Setting the boundary magnetic field strength to $\mu = 0$ in (3.2.4) we obtain a semi-infinite spin chain with a free boundary, namely

$$\mathcal{H}_0^H = (\mathcal{H}^H)^{\leftarrow}. \tag{3.2.13}$$

This Hamiltonian exhibits a $U(\mathfrak{sl}_2)$ symmetry by commuting with operators $(\mathcal{E}_0^{\pm})^{\leftarrow}$ and $(\mathcal{E}_0^{z})^{\leftarrow}$, but does not commute with those in (3.2.3) viewed as elements in $\Sigma_\infty^{\leftarrow}$.

This is the same situation as in the free boundary of the Hubbard model in Section 2.7. In a similar fashion, upon defining higher-order Yangian operators

$$\mathcal{E}_2^{\pm} = \pm\tfrac{1}{2}[\mathcal{E}_1^z, \mathcal{E}_1^{\pm}], \quad \mathcal{E}_2^z = [\mathcal{E}_1^+, \mathcal{E}_1^-] \tag{3.2.14}$$

the Hamiltonian $\mathcal{H}_0^H$ commutes, up to the terms at $-\infty$ which may be neglected in the antiferromagnetic regime, with the operators[1]

$$\begin{aligned}
\mathcal{G}^z &= (\mathcal{E}_2^z)^{\leftarrow} - J\left((\mathcal{E}_1^+)^{\leftarrow}(\mathcal{E}_0^-)^{\leftarrow} - (\mathcal{E}_0^+)^{\leftarrow}(\mathcal{E}_1^-)^{\leftarrow}\right) - \tfrac{J^2}{4}(\mathcal{E}_0^z)^{\leftarrow},\\
\mathcal{G}^{\pm} &= (\mathcal{E}_2^{\pm})^{\leftarrow} \mp \tfrac{J}{2}\left((\mathcal{E}_1^z)^{\leftarrow}(\mathcal{E}_0^{\pm})^{\leftarrow} - (\mathcal{E}_0^z)^{\leftarrow}(\mathcal{E}_1^{\pm})^{\leftarrow}\right) - \tfrac{J^2}{4}(\mathcal{E}_0^{\pm})^{\leftarrow},
\end{aligned} \tag{3.2.15}$$

instead, that, together with $(\mathcal{E}_0^{\pm})^{\leftarrow}$ and $(\mathcal{E}_0^z)^{\leftarrow}$, satisfy the defining relations of the $\mathcal{Y}(\mathfrak{sl}_2, \mathfrak{sl}_2)$ twisted Yangian (2.8.2). To see how these twisted Yangian yields a solution to the RE, see [68].

We now use the folding to obtain the Hamiltonian $\mathcal{H}_0^H$ and its symmetries $(\mathcal{E}_0^a)^{\leftarrow}$ and $\mathcal{G}^a$ with $a \in \{\pm, z\}$. Since the model exhibits a $U(\mathfrak{sl}_2)$ symmetry it is natural to choose $k^{ab} = 1$ for all $a, b \in \{\pm, z, 0\}$. This gives

$$\begin{aligned}
f(\mathcal{E}_0^{\pm}) &= f(\sum_i \sigma_i^{\pm}) = (k^{\pm 0} + k^{0\pm})\sum_{i\le 0}\sigma_i^{\pm} = 2\sum_{i\le 0}\sigma_i^{\pm} = 2(\mathcal{E}_0^{\pm})^{\leftarrow},\\
f(\mathcal{E}_0^{z}) &= f(\sum_i \sigma_i^{\pm}) = (k^{z0} + k^{0z})\sum_{i\le 0}\sigma_i^{z} = 2\sum_{i\le 0}\sigma_i^{z} = 2(\mathcal{E}_0^{z})^{\leftarrow}.
\end{aligned}$$

In a similar way one can check that with this choice of folding constants any monomial $(\mathcal{E}_0^-)^l(\mathcal{E}_0^z)^m(\mathcal{E}_0^+)^n$ for any $l, m, n \in \mathbb{Z}_{\ge 0}$ is folded into a symmetry of $\mathcal{H}_0^H$.

By folding the Hamiltonian $\mathcal{H}^H$ we get

$$f(\mathcal{H}^H) = 2(\mathcal{H}^H)^{\leftarrow} - J\big(\sigma_0^+\sigma_0^- + \sigma_0^-\sigma_0^+ + \tfrac{1}{2}\sigma_0^z\sigma_0^z\big), \tag{3.2.16}$$

which equals $2(\mathcal{H}^H)^{\leftarrow} - \frac{3}{2}J$ and thus agrees with (3.2.13) up to the constant term.

Folding Yangian operators (3.2.3) we find $f(\mathcal{E}_1^a) = -J(\mathcal{E}_0^a)^{\leftarrow}$, which can be easily deduced from (3.2.10) and (3.2.11), and is in agreement with the fact that the $\mathcal{Y}(\mathfrak{sl}_2, \mathfrak{sl}_2)$ twisted Yangian has elements of even grading only. Finally, we want to obtain the operators (3.2.15). By folding the higher-order Yangian operators (3.2.14) we obtain symmetries of $\mathcal{H}_0^H$: folded operators $f(\mathcal{E}_2^{\pm})$ and $f(\mathcal{E}_2^z)$ commute with $\mathcal{H}_0^H$, upon neglecting terms at $-\infty$ in the antiferromagnetic regime. However, the obtained operators do not coincide with those in (3.2.15). It turns out that we

[1] It seems likely that these symmetries were observed before; however, we have been unable to locate them in the literature available to us.

need to fold the following operators

$$\begin{aligned}
\widetilde{\mathcal{E}}_2^+ &= \mathcal{E}_2^+ + \tfrac{1}{3}\big([\mathcal{E}_1'^z, \mathcal{E}_1''^+] + [\mathcal{E}_1''^z, \mathcal{E}_1'^+]\big) + \tfrac{J^2}{3}\big(\mathcal{E}_0^+\mathcal{E}_0^-\mathcal{E}_0^+ - \tfrac{9}{4}\mathcal{E}_0^+\big), \\
\widetilde{\mathcal{E}}_2^- &= \mathcal{E}_2^- - \tfrac{1}{3}\big([\mathcal{E}_1'^z, \mathcal{E}_1'^-] + [\mathcal{E}_1''^z, \mathcal{E}_1''^-]\big) + \tfrac{J^2}{3}\big(\mathcal{E}_0^-\mathcal{E}_0^+\mathcal{E}_0^- - \tfrac{9}{4}\mathcal{E}_0^-\big), \\
\widetilde{\mathcal{E}}_2^z &= \mathcal{E}_2^z + \tfrac{2}{3}\big([\mathcal{E}_1'^+, \mathcal{E}_1'^-] + [\mathcal{E}_1''^+, \mathcal{E}_1''^-]\big) + \tfrac{J^2}{6}\big((\mathcal{E}_0^z)^3 - \tfrac{7}{2}\mathcal{E}_0^z\big),
\end{aligned} \tag{3.2.17}$$

instead. The additional terms in the expressions above are symmetries of $\mathcal{H}_o^H$ and are tailored in such a way that the operators $\widetilde{\mathcal{E}}_2^\pm$ and $\widetilde{\mathcal{E}}_2^z$ fold precisely to those in (3.2.15), up to an overall scalar factor,

$$f(\widetilde{\mathcal{E}}_2^\pm) = \tfrac{8}{3}\mathcal{G}^\pm, \quad f(\widetilde{\mathcal{E}}_2^z) = \tfrac{8}{3}\mathcal{G}^z. \tag{3.2.18}$$

The explicit form of computations in (3.2.18) is very similar to those presented in (3.2.10) and (3.2.11), only the expressions are much more lengthy; thus we have not written them out explicitly. It will be shown in Section 3.4 that long-range analogues of $\widetilde{\mathcal{E}}_2^\pm$ and $\widetilde{\mathcal{E}}_2^z$ fold into twisted Yangian symmetries of the long-range free boundary model.

As the final remark, we note that the free boundary model also exhibits a number of additional symmetries in the antiferromagnetic regime that are obtained by folding quadratic combinations of the operators in (3.2.2).

Double-row chain with an achiral boundary. Our third example of an integrable boundary model arises in the context of the double-row model consisting of two uncoupled Heisenberg spin chains. The energy transport in the anisotropic version of this model was studied in [149]. The Hamiltonian is given by

$$(\mathcal{H}^H)_{\circ\bullet} = -J\sum_{\alpha=\circ,\bullet}\sum_{-L<i\leq L}\big(\sigma_{i\alpha}^+\sigma_{i+1,\alpha}^- + \sigma_{i\alpha}^-\sigma_{i+1,\alpha}^+ + \tfrac{1}{2}\sigma_{i\alpha}^z\sigma_{i+1,\alpha}^z\big), \quad J<0. \tag{3.2.19}$$

In the $L\to\infty$ limit taken over the antiferromagnetic vacuum this model exhibits a $\mathcal{Y}_\circ(\mathfrak{sl}_2)\otimes\mathcal{Y}_\bullet(\mathfrak{sl}_2)\cong\mathcal{Y}(\mathfrak{so}_4)$ symmetry expressed in terms of the Lie operators $\mathcal{E}_{0\alpha}^a$ and $\mathcal{E}_{1\alpha}^a$ with $a\in\{\pm,z\}$ and $\alpha\in\{\circ,\bullet\}$ that are the natural analogues of $\mathcal{E}_0^a$ and $\mathcal{E}_1^a$ for the double-row model. This resembles the case of the Hubbard model with an achiral boundary from Section 2.6. As in the latter, we introduce linear combinations of Lie operators

$$\mathcal{A}_n^a = \mathcal{E}_{n,\circ}^a + \mathcal{E}_{n,\bullet}^a, \quad \mathcal{B}_n^a = \mathcal{E}_{n,\circ}^a - \mathcal{E}_{n,\bullet}^a \tag{3.2.20}$$

for all $a \in \{\pm, z\}$ and $n \in \{0, 1\}$. Then the semi-infinite double-row Hamiltonian with a achiral boundary

$$\mathcal{H}^H_\Delta = (\mathcal{H}^H_{\circ\bullet})^{\leftarrow} - J\big(\sigma^+_{0,\circ}\sigma^-_{0,\bullet} + \sigma^-_{0,\circ}\sigma^+_{0,\bullet} + \tfrac{1}{2}\sigma^z_{0,\circ}\sigma^z_{0,\bullet}\big), \quad J < 0 \tag{3.2.21}$$

exhibits a diagonal $U(\mathfrak{sl}^\Delta_2) \subset U(\mathfrak{sl}^\circ_2) \otimes U(\mathfrak{sl}^\bullet_2)$ symmetry; it commutes with operators $(\mathcal{A}^a_0)^{\leftarrow}$ only. The boundary term couples the two, otherwise uncoupled, spin-chains and can be viewed as a permutation operator; a similar boundary in the context of the Hubbard model was studied in [89]. Moreover, the double-row model with an achiral boundary can also be viewed as an infinite spin-chain with a defect located at the middle of the chain.

The Hamiltonian $\mathcal{H}^H_\Delta$ additionally commutes, up to the terms at $-\infty$ which may be neglected in the antiferromagnetic regime, with the twisted Yangian operators

$$\begin{aligned} \mathcal{Y}^\pm &= (\mathcal{B}^\pm_1)^{\leftarrow} \pm \tfrac{J}{4}\big((\mathcal{B}^\pm_0)^{\leftarrow}(\mathcal{A}^z_0)^{\leftarrow} - (\mathcal{A}^\pm_0)^{\leftarrow}(\mathcal{B}^z_0)^{\leftarrow}\big), \\ \mathcal{Y}^z &= (\mathcal{B}^z_1)^{\leftarrow} - \tfrac{J}{2}\big((\mathcal{B}^+_0)^{\leftarrow}(\mathcal{A}^-_0)^{\leftarrow} - (\mathcal{A}^+_0)^{\leftarrow}(\mathcal{B}^-_0)^{\leftarrow}\big) \end{aligned} \tag{3.2.22}$$

that, together with $(\mathcal{A}^a_0)^{\leftarrow}$, satisfy the defining relations of the $mcY(\mathfrak{so}_4, \mathfrak{sl}^\Delta_2)$ twisted Yangian (2.7.2).

As for the free boundary case, we set $k^{ab} = 1$ for all $a, b \in \{\pm, z, 0\}$. Then a straightforward computation shows that the folding $\overline{f}$ acts on the operators defined in (3.2.20) by

$$\overline{f}(\mathcal{A}^a_0) = 2(\mathcal{A}^a_0)^{\leftarrow}, \quad \overline{f}(\mathcal{B}^a_0) = 0,$$

and on the Hamiltonian (3.2.19) by

$$\begin{aligned} &\overline{f}(\mathcal{H}^H_{\circ\bullet}) = \\ &\quad = -J \sum_{\alpha=\circ,\bullet}\sum_{i<0}\big(\sigma^+_{i,\alpha}\sigma^-_{i+1,\alpha} + \sigma^-_{i,\alpha}\sigma^+_{i+1,\alpha} + \tfrac{1}{2}\sigma^z_{i,\alpha}\sigma^z_{i+1,\alpha}\big) \\ &\qquad - 2J\big(\sigma^+_{0,\circ}\sigma^-_{0,\bullet} + \sigma^-_{0,\circ}\sigma^+_{0,\bullet} + \tfrac{1}{2}\sigma^z_{0,\circ}\sigma^z_{0,\bullet}\big) \\ &\qquad - J \sum_{\alpha=\circ,\bullet}\sum_{i<0}\big(\sigma^+_{1-i,\alpha}\sigma^-_{-i,\alpha} + \sigma^-_{1-i,\alpha}\sigma^+_{-i,\alpha} + \tfrac{1}{2}\sigma^z_{1-i,\alpha}\sigma^z_{-i,\alpha}\big) \\ &\quad = 2(\mathcal{H}^H_{\circ\bullet})^{\leftarrow} - 2J\big(\sigma^+_{0,\circ}\sigma^-_{0,\bullet} + \sigma^-_{0,\circ}\sigma^+_{0,\bullet} + \tfrac{1}{2}\sigma^z_{0,\circ}\sigma^z_{0,\bullet}\big), \end{aligned}$$

thus exactly reproducing (3.2.21).

Applying the folding to the operators $\mathcal{B}^a_1$ we recover the ones defined in (3.2.22).

In particular

$$\begin{aligned}
\overline{f}(\mathcal{B}_1^{\pm}) = \pm\tfrac{J}{2}\Big(&\sum_{i<j\leq 0} (\sigma_{i,\circ}^{\pm}\sigma_{j,\circ}^{z} - \sigma_{i,\circ}^{z}\sigma_{j,\circ}^{\pm} - \sigma_{i,\bullet}^{\pm}\sigma_{j,\bullet}^{z} + \sigma_{i,\bullet}^{z}\sigma_{j,\bullet}^{\pm}) \\
&+ 2\big((\mathcal{E}_{0,\circ}^{\pm})^{\leftarrow}(\mathcal{E}_{0,\bullet}^{z})^{\leftarrow} - (\mathcal{E}_{0,\bullet}^{\pm})^{\leftarrow}(\mathcal{E}_{0,\circ}^{z})^{\leftarrow} \\
&- \sum_{i\leq 0}(\sigma_{i,\bullet}^{z}\sigma_{i,\circ}^{\pm} - \sigma_{i,\circ}^{z}\sigma_{i,\bullet}^{\pm})\big) \\
&+ 2\sum_{i\leq 0}(\sigma_{i,\circ}^{\pm}\sigma_{i,\bullet}^{z} - \sigma_{i,\bullet}^{\pm}\sigma_{i,\circ}^{z}) \\
&+ \sum_{\alpha=\circ,\bullet}\sum_{0<i<j} (\sigma_{1-i,\alpha}^{\pm}\sigma_{1-j,\alpha}^{z} - \sigma_{1-i,\alpha}^{z}\sigma_{1-j,\alpha}^{\pm})\Big) \\
= 2\mathcal{B}_1^{\pm} \pm J\big(&(\mathcal{E}_{0,\circ}^{\pm})^{\leftarrow}(\mathcal{E}_{0,\bullet}^{z})^{\leftarrow} - (\mathcal{E}_{0,\bullet}^{\pm})^{\leftarrow}(\mathcal{E}_{0,\circ}^{z})^{\leftarrow}\big) = 2\mathcal{Y}^{\pm}
\end{aligned}$$

and

$$\begin{aligned}
\overline{f}(\mathcal{B}_1^{z}) = J\Big(&\sum_{i<j\leq 0} (\sigma_{i,\circ}^{+}\sigma_{j,\circ}^{-} - \sigma_{i,\circ}^{-}\sigma_{j,\circ}^{+} - \sigma_{i,\bullet}^{+}\sigma_{j,\bullet}^{-} + \sigma_{i,\bullet}^{-}\sigma_{j,\bullet}^{+}) \\
&+ 2\big((\mathcal{E}_{0,\circ}^{+})^{\leftarrow}(\mathcal{E}_{0,\bullet}^{-})^{\leftarrow} - (\mathcal{E}_{0,\bullet}^{+})^{\leftarrow}(\mathcal{E}_{0,\circ}^{-})^{\leftarrow} \\
&- \sum_{i\leq 0}(\sigma_{i,\bullet}^{-}\sigma_{i,\circ}^{+} - \sigma_{i,\circ}^{-}\sigma_{i,\bullet}^{+})\big) \\
&+ 2\sum_{i\leq 0}(\sigma_{i,\circ}^{+}\sigma_{i,\bullet}^{-} - \sigma_{i,\bullet}^{+}\sigma_{i,\circ}^{-}) \\
&+ \sum_{\alpha=\circ,\bullet}\sum_{0<i<j} (\sigma_{1-i,\alpha}^{+}\sigma_{1-j,\alpha}^{-} - \sigma_{1-i,\alpha}^{-}\sigma_{1-j,\alpha}^{+})\Big) \\
= 2\mathcal{B}_1^{z} - 2J\big(&(\mathcal{E}_{0,\circ}^{+})^{\leftarrow}(\mathcal{E}_{0,\bullet}^{-})^{\leftarrow} - (\mathcal{E}_{0,\bullet}^{+})^{\leftarrow}(\mathcal{E}_{0,\circ}^{-})^{\leftarrow}\big) = 2\mathcal{Y}^{z}.
\end{aligned}$$

Repeating the same steps for $\mathcal{A}_1^a$ we find

$$\overline{f}(\mathcal{A}_1^{\pm}) = \overline{f}(\mathcal{A}_1^{z}) = 0, \tag{3.2.23}$$

as expected. We conclude this Section with a remark that the antiferromagnetic double-row model on the infinite interval also exhibits additional symmetries that are natural analogues of those in (3.2.2).

3.3 Inozemtsev's spin chain

Infinite chain. The Inozemtsev elliptic spin chain is the long-range analogue of the Heisenberg spin chain with Hamiltonian

$$\mathcal{H}^\kappa = -\tfrac{J}{2} \sum_{\substack{-L<i,j\leq L \\ i\neq j}} \wp_L(i-j)\big(\sigma_i^+\sigma_j^- + \sigma_i^-\sigma_j^+ + \tfrac{1}{2}\sigma_i^z\sigma_j^z\big), \tag{3.3.1}$$

where $\wp_L$ is the Weierstraß elliptic function with periods L and $\mathrm{i}\pi/\kappa$ for $\kappa \in \mathbb{R}_{\geq 0}$. This model exhibits a $U(\mathfrak{sl}_2)$ symmetry identical to its nearest-neighbour counterpart. By taking an appropriate limit of the parameter κ and the length L this model specializes to the Haldane-Shastry, Heisenberg and Inozemtsev hyperbolic (also called "infinite") spin chain. To see this, we need to rescale the hopping matrix of (3.3.1)

$$\widehat{\wp}_L(z) := \frac{\sinh^2(\kappa)}{\kappa^2}\Big(\wp_L(z) + \frac{2\kappa}{\mathrm{i}\pi}\zeta_L\Big(\frac{\mathrm{i}\pi}{2\kappa}\Big)\Big), \tag{3.3.2}$$

where ζ_L is the Weierstraß ζ-function with quasiperiods L and $\mathrm{i}\pi/\kappa$. In the $\kappa \to \infty$ limit one has

$$\lim_{\kappa\to\infty} \widehat{\wp}_L(z) = \delta_{z \bmod L,1}, \tag{3.3.3}$$

which recovers the Heisenberg spin chain [148]. One can also take the $\kappa \to 0$ limit to obtain the Haldane-Shastry hopping matrix [150]:

$$\lim_{\kappa\to 0} \widehat{\wp}_L(z) = \frac{\pi^2}{L^2\sin^2(\pi z/L)}. \tag{3.3.4}$$

The limit we are interested in is when the length of the chain becomes infinite [140]. In this case,

$$p_z := \lim_{L\to\infty} \widehat{\wp}_L(z) = \frac{\sinh^2(\kappa)}{\sinh^2(\kappa z)} \tag{3.3.5}$$

and, in the antiferromagnetic regime ($J<0$), the $U(\mathfrak{sl}_2)$ symmetry can be enhanced to the $\mathcal{Y}(\mathfrak{sl}_2)$ Yangian by introducing the operators

$$\mathcal{E}^{\pm}_{\kappa,1} = \pm\frac{J}{2}\sum_{i,j} w_{i-j}\sigma_i^\pm\sigma_j^z, \qquad \mathcal{E}^{z}_{\kappa,1} = J\sum_{i,j} w_{i-j}\sigma_i^-\sigma_j^+, \tag{3.3.6}$$

where $w_z = -\coth(\kappa z)$ when $z \neq 0$ and $w_0 = 0$. These operators commute with the Hamiltonian, up to the terms at $\pm\infty$ which may be neglected in this regime, and satisfy the defining relations of the $\mathcal{Y}(\mathfrak{sl}_2)$ Yangian.

The Hamiltonian also commutes, in this way, with operators $\mathcal{E}'^{a}_{\kappa,1}$ and $\mathcal{E}''^{a}_{\kappa,1}$ defined analogously to $\mathcal{E}^{a}_{\kappa,1}$,

$$w'_z = \frac{\mathrm{e}^{-\kappa z}}{\mathrm{e}^{-\kappa z} - \mathrm{e}^{\kappa z}} \quad \text{and} \quad w''_z = \frac{\mathrm{e}^{\kappa z}}{\mathrm{e}^{-\kappa z} - \mathrm{e}^{\kappa z}}$$

respectively. We also set $w'_0 = w''_0 = 0$, so that $w_z = w'_z + w''_z$. These operators are the long-range analogues of those in (3.2.2). In particular,

$$\lim_{\kappa\to\infty} w'_z = \delta_{z<0}, \qquad \lim_{\kappa\to\infty} w''_z = -\delta_{z>0}.$$

In the remaining part of this Section we will use foldings f and $\overline{f}$ studied in Section 3.3 to obtain integrable long-range boundary Hamiltonians and operators that commute with them. From now on, $\mathcal{H}^\kappa$ will denote

$$\mathcal{H}^\kappa = -\tfrac{J}{2}\sum_{i\neq j} p_{i-j}\big(\sigma_i^+\sigma_j^- + \sigma_i^-\sigma_j^+ + \tfrac{1}{2}\sigma_i^z\sigma_j^z\big), \tag{3.3.7}$$

so that $\lim_{\kappa\to\infty}\mathcal{H}^\kappa = \mathcal{H}^H$ when $L\to\infty$. It is worth noting that the Haldane-Shastry model on the circle also exhibits a $\mathcal{Y}(\mathfrak{sl}_2)$ Yangian symmetry [151] and thus the folding could be applied to its Hamiltonian to obtain integrable boundary long-range Hamiltonians on a segment. Symmetries of the latter model using the transfer matrix techniques were studied in [143].

3.4 Integrable boundaries of Inozemtsev's hyperbolic spin chain

Magnetic boundary. Our goal is to construct a long-range analogue of the Hamiltonian (3.2.4), which exhibits a $\mathcal{Y}(\mathfrak{sl}_2,\mathfrak{u}_1)$ twisted Yangian symmetry. We will achieve this by applying the folding f to $\mathcal{H}_\kappa$ and setting folding constants to the same values as for the semi-infinite Heisenberg spin chain with magnetic boundary, i.e. those given by (3.2.7) and

$$k^{zz} = 1, \quad k^{-+} - k^{+-} = \tfrac{4\mu}{J}, \quad k^{\pm z} = k^{z\pm} = \pm\tfrac{J}{\mu}. \tag{3.4.1}$$

To avoid repetition, let us assume that the model is in the antiferromagnetic regime ($J<0$) for the rest of this chapter due to the reasons stated at the end of Sections 1.5 and 1.7 - it allows us to neglect boundary terms at $\pm\infty$ that may appear as a result of commutators. Introduce the operators

$$\mathcal{H}_\kappa^{\leftrightarrows} = \tfrac{J}{2} \sum_{\substack{i\neq j\\ i,j\leq 0}} p_{i+j-1}\big(\sigma_i^+\sigma_j^- + \sigma_i^-\sigma_j^+ + \tfrac{1}{2}\sigma_i^z\sigma_j^z\big), \tag{3.4.2}$$

$$\mathcal{M}_\kappa^\mu = -\tfrac{J}{2} \sum_{\substack{i\neq j\\ i,j\leq 0}} p_{i+j-1}\sigma_i^z\sigma_j^z + \mu \sum_{i\leq 0} p_{2i-1}\sigma_i^z, \tag{3.4.3}$$

satisfying $\lim\limits_{\kappa\to\infty}(\mathcal{H}^\kappa)^{\leftrightarrows} = 0$ and $\lim\limits_{\kappa\to\infty}\mathcal{M}_\mu^\kappa = \mu\sigma_0^z$. Then similar computations to those in (3.2.8) yield

$$f(\mathcal{H}^\kappa) = 2((\mathcal{H}^\kappa)^{\leftarrow} + (\mathcal{H}^\kappa)^{\leftrightarrows} + \mathcal{M}_\mu^\kappa) + \tfrac{J}{2}(1 + k^{+-} + k^{-+}) \sum_{i\leq 0} p_{2i-1}. \tag{3.4.4}$$

Let us explain the meaning of operators listed above: $(\mathcal{H}^\kappa)^{\leftarrow}$ is the Hamiltonian (3.3.7) restricted to a half-line, $(\mathcal{H}^\kappa)^{\leftrightarrows}$ is the free boundary operator describing the long-range interaction between the sites labelled i and j via the boundary, i.e. at the distance $i + j - 1$, and $\mathcal{M}_\mu^\kappa$ is the the long-range magnetic boundary operator; for both $(\mathcal{H}^\kappa)^{\leftrightarrows}$ and $\mathcal{M}_\kappa^\mu$ their numerical values decay exponentially moving away from the boundary. Hence, by neglecting the constant term in (3.4.4), we conclude that

$$\mathcal{H}_\mu^\kappa := (\mathcal{H}^\kappa)^{\leftarrow} + (\mathcal{H}^\kappa)^{\leftrightarrows} + \mathcal{M}_\mu^\kappa \tag{3.4.5}$$

is the Hamiltonian of the open Inozemtsev hyperbolic spin chain with a magnetic boundary. It will be shown below that it is integrable, i.e. .exhibits a twisted Yangian symmetry, only if $\mu = \pm J$.

We already know that $f(\mathcal{E}_0^a) = 2\delta_{az}(\mathcal{E}_0^a)^{\leftarrow}$, which remains the only Lie symmetry of $\mathcal{H}_\mu^\kappa$. Under f, the operators (3.3.6) are mapped to

$$\begin{aligned} f(\mathcal{E}_{\kappa,1}^\pm) &= \\ &= \pm\tfrac{J}{2}\Bigg(\sum_{i,j\leq 0} k^{\pm 0}k^{z0} w_{i-j}\sigma_i^\pm\sigma_j^z + \sum_{i,j>0} k^{0z}k^{0\pm} w_{i-j}\sigma_{1-i}^\pm\sigma_{1-j}^z \\ &\qquad + \sum_{\substack{i\leq 0, j>0\\ i\neq 1-j}} k^{0z} w_{i-j}\sigma_i^\pm\sigma_{1-j}^z + \sum_{\substack{j\leq 0, i>0\\ j\neq 1-i}} k^{0\pm} w_{i-j}\sigma_{1-i}^\pm\sigma_j^z \\ &\qquad + \sum_{i\leq 0} k^{\pm z} w_{2i-1}\sigma_i^\pm\sigma_i^z + \sum_{i\leq 0} k^{z\pm} w_{1-2i}\sigma_i^z\sigma_i^\pm \Bigg) \\ &= 2\mathcal{E}_1^\pm \pm J \sum_{\substack{i\neq j\\ i,j\leq 0}} w_{i+j-1}\sigma_i^\pm\sigma_j^z - \tfrac{J}{2}(k^{z\pm} + k^{\pm z}) \sum_{i\leq 0} w_{2i-1}\sigma_i^\pm \end{aligned}$$

and

$$
\begin{aligned}
&f(\mathcal{E}^{z}_{\kappa,1}) = \\
&= J\Bigg(\sum_{i,j\leq 0} k^{+0}k^{-0}w_{i-j}\sigma_i^+\sigma_j^- + \sum_{i,j>0} k^{0-}k^{0+}w_{i-j}\sigma_{1-i}^+\sigma_{1-j}^- \\
&\quad + \sum_{\substack{i\leq 0, j>0\\ i\neq 1-j}} k^{0-}w_{i-j}\sigma_i^+\sigma_{1-j}^- + \sum_{\substack{j\leq 0, i>0\\ j\neq 1-i}} k^{0+}w_{i-j}\sigma_{1-i}^+\sigma_j^- \\
&\quad + \sum_{i\leq 0} k^{+-}w_{2i-1}\sigma_i^+\sigma_i^- + \sum_{i\leq 0} k^{-+}w_{1-2i}\sigma_i^-\sigma_i^+\Bigg) \\
&= -\tfrac{J}{2}(k^{+-}+k^{-+})\sum_{i\leq 0} w_{2i-1}\sigma_i^z - \tfrac{J}{2}(k^{+-}-k^{-+})\sum_{i\leq 0} w_{2i-1}.
\end{aligned}
$$

The folded operators $f(\mathcal{E}^{\pm}_{\kappa,1})$ satisfy the defining relations of the $\mathcal{Y}(\mathfrak{sl}_2,\mathfrak{u}_1)$ twisted Yangian provided (3.4.1) holds. It remains to verify if they are symmetries of $\mathcal{H}^{\kappa}_{\mu}$. It is straightforward to see that $[\mathcal{H}^{\kappa}_{\mu}, f(\mathcal{E}^{z}_{\kappa,1})] = 0$. By computing the commutator $[\mathcal{H}^{\kappa}_{\mu}, f(\mathcal{E}^{\pm}_{\kappa,1})]$ we find that it equals to zero over the antiferromagnetic vacuum in the limit as the left side of the chain goes to $-\infty$ only and provided (3.2.7) and the following constraints hold:

$$k^{+-} = -k^{-+} = \pm 2, \quad k^{z-} = -k^{z+} = \mp\tfrac{1}{2}. \tag{3.4.6}$$

In other words, $f(\mathcal{E}^{a}_{\kappa,1})$ are symmetries of $\mathcal{H}^{\kappa}_{\mu}$ only if $\mu = \pm J$ and $J < 0$, thus implying the aforementioned integrability condition for the long-range Hamiltonian $\mathcal{H}^{\kappa}_{\mu}$. In particular, its twisted Yangian symmetries are

$$\mathcal{X}^{\pm}_{\kappa} = (\mathcal{E}^{\pm}_{\kappa,1})^{\leftarrow} \pm \tfrac{J}{2}\sum_{\substack{i\neq j\\ i,j\leq 0}} w_{i+j-1}\sigma_i^{\pm}\sigma_j^{z} \pm \tfrac{J^2}{2\mu}\sum_{i\leq 0} w_{2i-1}\sigma_i^{\pm}, \tag{3.4.7}$$

with $\mu = J$ or $\mu = -J$ the two cases being related to each other via the Lie algebra automorphism $\theta : \sigma^{\pm} \mapsto \sigma^{\mp}$, $\sigma^z \mapsto -\sigma^z$. This automorphism leaves the Hamiltonian $\mathcal{H}^{\kappa}$ (and $(\mathcal{H}^{\kappa})^{\leftarrow}$, $(\mathcal{H}^{\kappa})^{\circlearrowleft}$) invariant, but maps $\mathcal{H}^{J}_{\kappa}$ to $\mathcal{H}^{-J}_{\kappa}$ and $\mathcal{X}^{\pm}_{\kappa}$ to $\mathcal{X}^{\mp}_{\kappa}$.

We conclude this Section with two remarks. First, by applying the same folding procedure to the symmetries $\mathcal{E}'^{\pm}_{\kappa,1}$ and $\mathcal{E}''^{\pm}_{\kappa,1}$ of $\mathcal{H}_{\kappa}$ we obtain operators $\mathcal{X}'^{\pm}_{\kappa} = f(\mathcal{E}'^{\pm}_{\kappa,1})$ and $\mathcal{X}''^{\pm}_{\kappa} = f(\mathcal{E}''^{\pm}_{\kappa,1})$ that are symmetries of $\mathcal{H}^{\mu}_{\kappa}$ in the antiferromagnetic regime provided (3.4.6) holds. They are long-range analogues of the operators (3.2.6). Second, assuming that $\mu \in \mathbb{C}$ is arbitrary and taking the $\kappa \to \infty$ limit, operators $\mathcal{X}^{\pm}_{\kappa}$ and $\mathcal{X}'^{\pm}_{\kappa}$, $\mathcal{X}''^{\pm}_{\kappa}$ specialize to their nearest-neighbour counterparts given in (3.2.5) and (3.2.6).

Free boundary. We want to construct a long-range analogue of the Hamiltonian (3.2.13), which exhibits a $\mathcal{Y}(\mathfrak{sl}_2, \mathfrak{sl}_2)$ twisted Yangian symmetry. We will achieve this by applying the folding f with $k^{ab} = 1$ to $\mathcal{H}^\kappa$. In particular, we find that

$$f(\mathcal{H}^\kappa) = 2((\mathcal{H}^\kappa)^{\leftarrow} + \mathcal{H}^{\leftrightarrows}_\kappa) - \tfrac{3}{2}J\sum_{i\le 0} p_{2i-1}, \tag{3.4.8}$$

which, after dropping the constant term, is the free boundary Hamiltonian as expected from (3.4.4).

To obtain twisted Yangian symmetries of the long-range free boundary model we need to fold the long-range analogues of the operators (3.2.17):

$$\begin{aligned}
\widetilde{\mathcal{E}}^+_{\kappa,2} &= \mathcal{E}^+_{\kappa,2} + \tfrac{1}{3}\big([\mathcal{E}'^z_{\kappa,2}, \mathcal{E}''^+_{\kappa,2}] + [\mathcal{E}''^z_{\kappa,2}, \mathcal{E}'^+_{\kappa,2}]\big) + \tfrac{J^2}{3}\big(\mathcal{E}^+_0\mathcal{E}^-_0\mathcal{E}^+_0 - \tfrac{9}{4}\mathcal{E}^+_0\big),\\
\widetilde{\mathcal{E}}^-_{\kappa,2} &= \mathcal{E}^-_{\kappa,2} - \tfrac{1}{3}\big([\mathcal{E}'^z_{\kappa,2}, \mathcal{E}'^-_{\kappa,2}] + [\mathcal{E}''^z_{\kappa,2}, \mathcal{E}''^-_{\kappa,2}]\big) + \tfrac{J^2}{3}\big(\mathcal{E}^-_0\mathcal{E}^+_0\mathcal{E}^-_0 - \tfrac{9}{4}\mathcal{E}^-_0\big),\\
\widetilde{\mathcal{E}}^z_{\kappa,2} &= \mathcal{E}^z_{\kappa,2} + \tfrac{2}{3}\big([\mathcal{E}'^+_{\kappa,2}, \mathcal{E}'^-_{\kappa,2}] + [\mathcal{E}''^+_{\kappa,2}, \mathcal{E}''^-_{\kappa,2}]\big) + \tfrac{J^2}{6}\big((\mathcal{E}^z_0)^3 - \tfrac{7}{2}\mathcal{E}^z_0\big).
\end{aligned}$$

By doing so we find

$$\begin{aligned}
f(\widetilde{\mathcal{E}}^+_{\kappa,2}) &= \tfrac{16}{3}(\mathcal{E}^\pm_{\kappa,2})^{\leftarrow}\\
&\quad + \tfrac{J^2}{3}\sum_{i,j,k} a_{ijk}\big(\sigma^z_i\sigma^z_j\sigma^\pm_k + 4\sigma^+_i\sigma^-_j\sigma^\pm_k\big) + \tfrac{2J^2}{3}\sum_{i,j} b_{ij}\sigma^\pm_i,\\
f(\widetilde{\mathcal{E}}^z_{\kappa,2}) &= \tfrac{16}{3}(\mathcal{E}^z_{\kappa,2})^{\leftarrow}\\
&\quad + \tfrac{J^2}{3}\sum_{i,j,k} a_{ijk}\big(\sigma^z_i\sigma^z_j\sigma^z_k + 4\sigma^+_i\sigma^-_j\sigma^z_k\big) + \tfrac{2J^2}{3}\sum_{i,j} b_{ij}\sigma^z_i,
\end{aligned}$$

where

$$\begin{aligned}
a_{ijk} &= 2 - w_{i-j}(w_{j-k} + w_{i+k-1} - w_{i-k} - w_{j+k-1})\\
&\quad - w_{i+j-1}(w_{i-k} + w_{j-k} + w_{i+k-1} + w_{j+k-1}),\\
b_{ij} &= 5 + w^2_{i-j} - \frac{1}{4}w^2_{1-2i} - w_{i+j-1}(w_{i+j-1} - 4w_{1-2j})\\
&\quad - 2w_{i-j}(w_{i+j-1} + 2w_{1-2j}).
\end{aligned}$$

The operators $\mathcal{G}^a_\kappa = \frac{3}{8}f(\widetilde{\mathcal{E}}^a_{\kappa,2})$ together with $(\mathcal{E}^a_0)^{\leftarrow}$ satisfy the defining relations of the $\mathcal{Y}(\mathfrak{sl}_2, \mathfrak{sl}_2)$ twisted Yangian and commute with the Hamiltonian $\mathcal{H}^\kappa_0 = +(\mathcal{H}^\kappa)^{\leftrightarrows}$, up to the terms at $-\infty$ which may be neglected since we are working in the antiferromagnetic regime.

We also have that $\lim_{\kappa\to\infty} \mathcal{G}^a_\kappa = \mathcal{G}^a$ and there are a number of additional symmetries of $\mathcal{H}^\kappa_0$ that are obtained by folding quadratic combinations of the symmetries $\mathcal{E}'^a_{\kappa,1}$

and $\mathcal{E}''^{a}_{\kappa,1}$ of $\mathcal{H}_\kappa$.

Double-row chain with a achiral boundary. Let us now focus on the model consisting of two uncoupled Inozemtsev hyperbolic spin chains described by the Hamiltonian

$$\mathcal{H}^{\kappa}_{\circ\bullet} = -\tfrac{J}{2} \sum_{\alpha=\circ,\bullet} \sum_{i\neq j} p_{i-j}\big(\sigma^{+}_{i\alpha}\sigma^{-}_{j,\alpha} + \sigma^{-}_{i\alpha}\sigma^{+}_{j\alpha} + \tfrac{1}{2}\sigma^{z}_{i\alpha}\sigma^{z}_{j\alpha}\big) \tag{3.4.9}$$

as the double-row Heisenberg spin chain this model exhibits a $\mathcal{Y}(\mathfrak{so}_4)$ Yangian symmetry generated by the Lie operators $\mathcal{E}^{a}_{0\alpha}$ and the double-row analogues $\mathcal{E}^{a}_{\kappa 1\alpha}$ of the ones defined in (3.3.6).

We use the folding $\overline{f}$ with $k^{ab}=1$ to obtain an integrable long-range analogue of the Hamiltonian (3.2.19) exhibiting a $\mathcal{Y}(\mathfrak{so}_4, \mathfrak{sl}_2^{\Delta})$ twisted Yangian symmetry.

Introduce the operator

$$\mathcal{D}^{\kappa} = -\tfrac{J}{2} \sum_{\substack{\alpha\neq\beta \\ \alpha,\beta=\circ,\bullet}} \sum_{i,j\leq 0} p_{i+j-1}\big(\sigma^{+}_{i\alpha}\sigma^{-}_{j\beta} + \sigma^{-}_{i\alpha}\sigma^{+}_{j\beta} + \tfrac{1}{2}\sigma^{z}_{i\alpha}\sigma^{z}_{j\beta}\big). \tag{3.4.10}$$

Proceeding in a similar way as for the double-row Heisenberg spin chain we have that

$$\overline{f}(\mathcal{H}^{\kappa}_{\circ\bullet}) = 2\big((\mathcal{H}^{\kappa}_{\circ\bullet})^{\leftarrow} + \mathcal{D}^{\kappa}\big).$$

The operator $\mathcal{D}^{\kappa}$ is the long-range achiral boundary operator for the semi-infinite long-range double-row model; it can also be viewed as a double-row analogue of the free boundary operator $(\mathcal{H}^{\kappa})^{\leftrightarrows}$. In the $\kappa\to\infty$ limit $\mathcal{D}^{\kappa}$ specializes to the boundary term in (3.2.21).

Next we fold the the long-range analogues of the operators (3.2.20). Similarly as before we have that $\overline{f}(\mathcal{A}^{a}_{\kappa,1}) = 0$ and

$$\begin{aligned}
\overline{f}(\mathcal{B}^{\pm}_{\kappa,1}) &= 2(\mathcal{B}^{\pm}_{\kappa,1})^{\leftarrow} \pm J \sum_{i,j\leq 0} w_{i+j-1}(\sigma^{\pm}_{i,\circ}\sigma^{z}_{j,\bullet} - \sigma^{\pm}_{j,\bullet}\sigma^{z}_{i,\circ}),\\
\overline{f}(\mathcal{B}^{z}_{\kappa,1}) &= 2(\mathcal{B}^{z}_{\kappa,1})^{\leftarrow} - 2J \sum_{i,j\leq 0} w_{i+j-1}(\sigma^{+}_{i,\circ}\sigma^{-}_{j,\bullet} - \sigma^{+}_{j,\bullet}\sigma^{-}_{i,\circ}).
\end{aligned}$$

The operators $\mathcal{Y}^{a}_{\kappa,1} = \frac{1}{2}\overline{f}(\mathcal{B}^{a}_{\kappa,1})$ together with $\mathcal{A}^{a}_{0}$ satisfy the defining relations of the $\mathcal{Y}^{\Delta}(\mathfrak{sl}_2)$ twisted Yangian and commute with the Hamiltonian $\mathcal{H}^{\kappa}_{\Delta} = (\mathcal{H}^{\kappa}_{\circ\bullet})^{\leftarrow} + \mathcal{D}^{\kappa}$ up to the terms at $-\infty$, which may be neglected in the antiferromagnetic regime. In the $\kappa\to\infty$ limit $\mathcal{Y}^{a}_{\kappa,1}$ specialize to those given in (3.2.22). We also remark that there exists a number of symmetries of $\mathcal{H}_{\Delta}^{\kappa}$ that are obtained by folding the double-row analogues of the operators $\mathcal{E}'^{a}_{\kappa,1}$ and $\mathcal{E}''^{a}_{\kappa,1}$. These also specialize to those of the double-row Heisenberg spin chain.

Chapter 4

Conclusions and outlook

In this thesis we have explored the symmetries of integrable boundaries in the Hubbard model, the Heisenberg spin chain and Inozemtsev's hyperbolic long-range spin chain. As stressed in [76] (p284), the boundary Hubbard model continues to require a deeper understanding of its algebraic structure. In principle, this should be deducible from the supersymmetric structures of AdS/CFT string worldsheet scattering, but in the absence of this we hope to have provided a useful step by constructing explicitly the twisted Yangian symmetry for the open Hubbard model with integrable boundary conditions in the form of a magnetic field, chemical potential, achiral boundary and free boundary. We found that the twisted grade-1 generators include a boundary field term not observed in similar constructions for other models.

From the point of view of the intrinsic study of the Hubbard model, the construction of these 'modified' twisted Yangians lays the foundation for extended study of the boundary scattering and associated bound states. The latter have been analyzed using the Bethe ansatz (see [76], Sect. 8.3), but the K-matrix has hitherto been computable only via the RE [120], and any boundary bound state scattering would have to be computed by fusion. With the boundary's hidden charges now known, the linear conservation equations may be used instead.

In the context of the richness of the Hubbard model's connections with other topics in theoretical physics, and especially in AdS/CFT, our construction provides a spur to further work in various directions. First, the connections with the twisted Yangian of the $Y = 0$ and $Z = 0$ maximal giant gravitons [119, 58], that of the $D5$ brane [135] and the deformed Hubbard model [98, 60] should be understood. Secondly, the mathematics of the connection with the boundary analogue of the tetrahedron equation should be established [133]. Finally, it would be interesting to search for similar constructions in related models [130, 131, 134].

We have also shown that the partial PHT plays a crucial role in relating the integrable structure of the Hubbard model to that of the $AdS_5 \times S_5$ superstring. Furthermore, we have shown that a particle-hole reflection is an achiral boundary in the half-infinite Hubbard chain, and constructed its corresponding twisted Yangian symmetry. It would be interesting to see if extended Hubbard chains – possessing an arbitrary symmetry group [130] – or those with variable range hopping [134] can give rise to interesting integrable boundary theories, and whether these have any relation to other integrable structures in AdS/CFT.

We should also attempt to construct the K-matrices for the achiral and free boundary using invariance under the twisted Yangians, as it was done in [119, 135]. As in the latter case, we expect the free boundary K-matrix to be a solution of the 'vector boundary reflection equation' [68], where the boundary space does not transform as a singlet of the algebra and the K-matrix has the form $K_{ij}(u) \in \mathrm{End}(V_{\rho_i} \otimes V_{\rho_j})$.

Finally, we presented a method for constructing integrable boundaries for $\mathfrak{sl}_2$-symmetric spin chains and their doublings without relying on the reflection. This method, which we refer to as "folding", consists in a map denoted by f (and $\overline{f}$ in the doubled case) which sends the operators of a model defined on the infinite line to those on the half-line.

More precisely, given a Hamiltonian $\mathcal{H}$ of an infinite spin chain and a family of operators $\{\mathcal{Q}_\alpha\}_{\alpha\in I}$ indexed by a set I and commuting with the Hamiltonian, $[\mathcal{H}, \mathcal{Q}_\alpha] = 0$ for all $\alpha \in I$ in a particular regime where boundary terms at $\pm\infty$ may be neglected, the folding identifies the positive half-line with the negative half-line in such a way that, for a suitable choice of the "folding constants", the folded Hamiltonian $f(\mathcal{H})$, which now describes a semi-infinite spin chain, commutes with the folded operators, namely $[f(\mathcal{H}), f(\mathcal{Q}_\alpha)] = 0$ for all $\alpha \in I$.

The choice of the folding constants is dictated by the symmetry properties of the Hamiltonian $\mathcal{H}$ and the would-be symmetries of the folded Hamiltonian $f(\mathcal{H})$. Integrability is then ensured by the existence of an infinite number of conserved quantities, i.e. operators commuting with the Hamiltonian in this regime and satisfying the defining relations of an infinite-dimensional algebra [47]. In the case when the Hamiltonian exhibits a $\mathcal{Y}(\mathfrak{sl}_2)$ Yangian symmetry there are three non-equivalent boundary integrable models that can be obtained: a spin chain with a magnetic boundary, a spin chain with an free boundary, and a double-row model with a achiral boundary. These models exhibit $\mathcal{Y}(\mathfrak{sl}_2, \mathfrak{u}_1)$, $\mathcal{Y}(\mathfrak{sl}_2, \mathfrak{sl}_2)$ and $\mathcal{Y}(\mathfrak{so}_4, \mathfrak{sl}_2^{\triangle})$ twisted Yangian symmetries, respectively. For the Heisenberg spin chain the corresponding models are well-studied. However, this is not the case for the Inozemtsev hyperbolic spin chain. Integrable boundary Hamiltonians for the latter were constructed in [143, 144] using the Dunkl operators and, although similar in form, the results obtained in *loc. cit.* differ from ours. It remains to be shown whether folding can yield those boundary Hamiltonians and if they exhibit any twisted Yangian symmetries. This is natural to expect, since Hamiltonians of such type were shown to obey infinite dimensional symmetries [152, 153].

The method presented in Chapter 3 can be easily applied to any integrable spin chains. Let $\mathfrak{g}$ be any simple Lie algebra of $\mathrm{rank}(\mathfrak{g}) \geq 2$ and let $\mathcal{H}_{\mathfrak{g}}$ be a spin chain Hamiltonian exhibiting $\mathcal{Y}(\mathfrak{g})$ Yangian symmetry. Let $\theta : \mathfrak{g} \to \mathfrak{g}$ be an involutive automorphism of $\mathfrak{g}$. Denote by $\mathfrak{h} = \mathfrak{g}^{\theta}$ the θ-fixed subalgebra, so that $(\mathfrak{g}, \mathfrak{h})$ is a symmetric pair. For such a pair there exists an infinite dimensional algebra, the $\mathcal{Y}(\mathfrak{g}, \mathfrak{h})$ twisted Yangian, which is a coideal subalgebra of $\mathcal{Y}(\mathfrak{g})$ [154, 66], and there must exist a boundary-integrable spin chain exhibiting such a symmetry in a particular regime where terms at $-\infty$ may be neglected, which can be constructed using the folding method. While this might be rather straightforward for spin chains with nearest neighbor interactions only, since the boundary term for such models in many cases is a symmetry breaking term exhibiting $\mathfrak{h}$-symmetry only, this is no longer true for the long-range spin chains, as we have shown in Chapter 3. Moreover, obtaining long-range Hamiltonians using the techniques of the inverse scattering method is a rather challenging task, as was shown in [143, 144]; thus the "bottom-up" approach provides a short-cut for constructing such models.

It is important to note that the folding f is generally not an algebra homomorphism. It can only be so if $\mathfrak{h}$ is a commutative subalgebra of $\mathfrak{g}$. The only symmetric pair satisfying this requirement is $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{sl}_2, \mathfrak{u}_1)$, which we have studied in this letter. In all other cases the map f is effectively a projector.

Another important thing to note is that folding is only a good method of constructing boundary-integrable models if it is defined over a link. If we instead fold at a site, symmetry arguments force the folding constants associated with that site to be zero, thus effectively turning folding over a site into folding over a link.

We finish by noting that a very interesting subject for our folding method would be the Hubbard model [76] and its long-range analogue [134]. It would be interesting to see if one can gain further insight into the unusual structure of the Hubbard model's known integrable boundaries [89, 120, 121] and perhaps obtain new ones.

Appendix A

Appendix to Chapter 1

A.1 Commutativity of boundary transfer matrices

We would like to show that the boundary transfer matrices for R-matrices that are not of difference form, as given in Section 2.6, commute. Suppose $R(u, u')$ satisfies the YBE, unitarity

$$R_{12}(u_1, u_2)R_{21}(u_2, u_1) = \mathbf{1}_{V\otimes V} \tag{A.1.1}$$

and crossing unitarity

$$R_{12}^{tr_1}(u_1, -u_2)R_{21}^{tr_1}(-u_2, u_1) = \nu_{12}\mathbf{1}_{V\otimes V} \tag{A.1.2}$$

for some scalar symmetric function ν_{12}. Label the auxiliary spaces 1 and 2. Then

$$\begin{aligned}
\tau_B(u)\tau_B(v) &= \mathrm{tr}_1(K_1^-(u_1)T_1^+(u_1))\mathrm{tr}_2(K_2^-(u_2)T_2^+(u_2)) \\
&= \mathrm{tr}_1(K_1^-(u_1)^{tr}T_1^+(u_1)^{tr})\mathrm{tr}_2(K_2^-(u_2)T_2^+(u_2)) \\
&= \mathrm{tr}_{12}(K_1^-(u_1)^{tr}T_1^+(u_1)^{tr}K_2^-(u_2)T_2^+(u_2)) \\
&= \mathrm{tr}_{12}(K_2^-(u_2)K_1^-(u_1)^{tr}T_1^+(u_1)^{tr}T_2^+(u_2)) \\
&= \mathrm{tr}_{12}(K_2^-(u_2)K_1^-(u_1)^{tr}(R_{21}^{tr_1}(-u_2, u_1))^{-1}R_{21}^{tr_1}(-u_2, u_1)T_1^+(u_1)^{tr}T_2^+(u_2)) \\
&= \mathrm{tr}_{12}((K_2^-(u_2)K_1^-(u_1)^{tr}(R_{21}^{tr_1}(-u_2, u_1))^{-1})^{tr_1}(R_{21}^{tr_1}(-u_2, u_1)T_1^+(u_1)^{tr}T_2^+(u_2))^{tr_1}) \\
&= \mathrm{tr}_{12}(K_2^-(u_2)(R_{21}^{tr_1}(-u_2, u_1))^{-1})^{tr_1}K_1^-(u_1)T_1^+(u_1)R_{21}(-u_2, -u_1)T_2^+(u_2)) \\
&= \nu_{12}\mathrm{tr}_{12}(K_2^-(u_2)R_{12}(u_1, -u_2)K_1^-(u_1)T_1^+(u_1)R_{21}(-u_2, u_1)T_2^+(u_2))\dots
\end{aligned} \tag{A.1.3}$$

Now if one inserts $\mathbf{1}_{V\otimes V} = R_{21}(-u_2, -u_1)R_{12}(-u_1, -u_2)$ in between $K_1(u_1)$ and $T_1^+(u_1)$, uses both (1.6.7) and (1.6.12) and the cyclicity of the trace, one obtains

$$\dots = \nu_{12}\mathrm{tr}_{12}(K_1^-(u_1)R_{21}(u_2, -u_1)K_2^-(u_2)T_2^+(u_2)R_{12}(-u_1, u_2)T_1^+(u_1)) \tag{A.1.4}$$

which is exactly (A.1.3) with 1 and 2 interchanged. This is just $\tau_B(v)\tau_B(u)$, and thus the boundary transfer matrices commute.

Appendix B

Appendix to Chapter 2

B.1 Shastry's R-matrix

Shastry's R-matrix $R^h(u,u')$ is related to the fermionic R-matrix $R^f(u,u')$ as follows (with exact same parametrisation as in [120])

$$\check{R}^h(u,u') = W^{-1}\check{R}^f(u,u')W, \tag{B.1.1}$$

where W is a diagonal 16×16 matrix

$$W = \mathrm{diag}(1,1,-\mathrm{i},-\mathrm{i},-\mathrm{i},-\mathrm{i},1,1,1,-1,-1,\mathrm{i},\mathrm{i},\mathrm{i},\mathrm{i},-1,-1).$$

and

$$R^f(u,u') =$$
$$\begin{pmatrix}
r_1^+ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\mathrm{i}r_2^+ & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\mathrm{i}r_2^+ & 0 & 0 & 0 & 0 & 0 & r_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -r_3^+ & 0 & 0 & \mathrm{i}r_6 & 0 & 0 & -\mathrm{i}r_6 & 0 & 0 & r_4^+ & 0 & 0 & 0 \\
0 & e & 0 & 0 & \mathrm{i}r_2^- & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -r_1^- & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mathrm{i}r_6 & 0 & 0 & c^- & 0 & 0 & -r_4^- & 0 & 0 & -\mathrm{i}r_6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{i}r_2^- & 0 & 0 & 0 & 0 & 0 & e & 0 & 0 \\
0 & 0 & e & 0 & 0 & 0 & 0 & 0 & \mathrm{i}r_2^- & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\mathrm{i}r_6 & 0 & 0 & -r_4^- & 0 & 0 & r_3^- & 0 & 0 & \mathrm{i}r_6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -r_1^- & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{i}r_2^- & 0 & 0 & e & 0 \\
0 & 0 & 0 & d^+ & 0 & 0 & -\mathrm{i}r_6 & 0 & 0 & \mathrm{i}r_6 & 0 & 0 & -c^+ & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & r_5 & 0 & 0 & 0 & 0 & 0 & -\mathrm{i}r_2^+ & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e & 0 & 0 & -\mathrm{i}r_2^+ & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r_1^+
\end{pmatrix}, \tag{B.1.2}$$

where

$$\begin{aligned}
r_1^{\pm} &= \cos^2(u-u')\left\{1 \pm \tanh(h_1-h_2)\frac{\cos(u+u')}{\cos(u-u')}\right\}, \\
r_2^{\pm} &= \sin(u-u')\cos(u-u')\left\{1 \pm \tanh(h_1-h_2)\frac{\sin(u+u')}{\sin(u-u')}\right\} \\
&= \sin(u-u')\cos(u-u')\left\{1 \pm \tanh(h_1+h_2)\frac{\cos(u+u')}{\cos(u-u')}\right\}, \\
r_3^{\pm} &= \sin^2(u-u')\left\{1 \pm \tanh(h_1+h_2)\frac{\sin(u+u')}{\sin(u-u')}\right\}, \\
r_4^{\pm} &= 1 \pm \tanh(h_1-h_2)\frac{\cos(u-u')}{\cos(u+u')}, \\
&= 1 \pm \tanh(h_1+h_2)\frac{\sin(u-u')}{\sin(u+u')}, \\
r_5 &= \frac{\cos(u-u')}{\cosh(h_1-h_2)}, \qquad r_6 = \frac{\sin(u-u')}{\cosh(h_1+h_2)}.
\end{aligned} \tag{B.1.3}$$

B.2 Obtaining $\mu_{\pm}$ for $\mathcal{Y}(\mathfrak{sl}_2, \mathfrak{u}_1)^s$

Let $\widetilde{E}_1^s = E_1^s + \mu_+ E_0^s + \frac{\alpha}{2} E_0^s H_0^s - p\,\mathcal{E}_{0,0}^s$. We will obtain the value of μ_+ and α for which $\widetilde{E}_1^s$ commutes with the Hamiltonian $\mathcal{H}_p^h \equiv -K + UV - p\mathcal{H}_{0,0}^s$. (We already know that $\alpha = U$, since it is the constant which determines the strength of the interaction, but we will see it arise naturally from this computation). The limit as the left side of the chain tends to $-\infty$ is taken over the antiferromagnetic vacuum $|\mathrm{vac}\rangle$ (2.2.11), so if the commutator results in boundary terms at $-\infty$, we may neglect them and set the result to zero. We divide the computation of the full commutator into smaller components:

$$\begin{aligned}
&[-K+UV, E_1^s] = -2\mathcal{E}_{0,0}^s \\
&[-K, -p\mathcal{E}_{0,0}^s] = p(\mathcal{E}_{-1,1}^s - \mathcal{E}_{0,-1}^s) \\
&[-K, \mu_+ E_0^s + \frac{\alpha}{2} E_0^s H_0^s] = 0 \\
&[UV, \mu_+ E_0^s + \frac{\alpha}{2} E_0^s H_0^s - p\mathcal{E}_{0,0}^s] = 0 \\
&[-p\mathcal{H}_{0,0}^s, E_1^s] = -p((\mathcal{E}_{-1,1}^s - \mathcal{E}_{0,-1}^s) - U(\mathcal{E}_{0,0}^s + \mathcal{E}_{0,0}^s H_0^s)) \\
&[-p\mathcal{H}_{0,0}^s, \mu_+ E_0^s + \frac{\alpha}{2} E_0^s H_0^s - p\mathcal{E}_{0,0}^s] = -2p((-p+\mu_+)\mathcal{E}_{0,0}^s + \frac{\alpha}{2}\mathcal{E}_{0,0}^s H_0^s)
\end{aligned}$$

Hence, for $[\mathcal{H}^h_p, \widetilde{E}^s_1] = 0$,

$$\alpha = U, \quad \mu_+ = \frac{U}{2} + \left(p - \frac{1}{p}\right), \tag{B.2.1}$$

so that $\widetilde{E}^s_1 = E^s_1 + \mu_+ E^s_0 - p\mathcal{E}^s_{0,0} + \frac{U}{2} E^s_0 H^s_0$ is a conserved charge. Similarly, for $\widetilde{F}^s_1 = F^s_1 + \mu_- F^s_0 - \frac{\alpha}{2} F^s_0 H^s_0 + p\mathcal{F}^s_{0,0}$ to commute with $\mathcal{H}^h_p$, we require the same value of α, and $\mu_- = \frac{U}{2} - \left(p - \frac{1}{p}\right)$.

B.3 Computation of $\mathcal{Y}(\mathfrak{sl}_2, \mathfrak{u}_1)^s$ coproducts

Rewrite E^s_1 as

$$E^s_1 = \sum_{i\le 0} (\mathcal{E}^s_{i-1,1} - \mathcal{E}^s_{i,-1}) + \frac{U}{2} \sum_{i,j\le 0} \mathrm{sgn}(j-i) \mathcal{E}^s_{i,0} \mathcal{H}^s_{j,0} \tag{B.3.1}$$

The coproduct of the first sum is trivial but that of the second is not. However, we can use a standard trick and, for any $x \in \mathbb{Z} + \frac{1}{2}$, split such a sum into left ($i < x$ and $j < x$) and right ($i > x$ and $j > x$) factors, interpreting these as left and right factors of the coproduct accordingly. The boundary terms $\mathcal{E}^s_{0,0}$ and $\mathcal{F}^s_{0,0}$ therefore appear in the right factor only. For the quadratic term,

$$\sum_{i,j\le 0} \mathrm{sgn}(j-i)\mathcal{E}_{i,0}\mathcal{H}_{j,0} = \left(\sum_{\substack{\imath<x\le 0\\ j<x\le 0}} + \sum_{\substack{x<i\le 0\\ x<j\le 0}}\right) \mathrm{sgn}(j-i)\mathcal{E}_{i,0}\mathcal{H}_{j,0} + \sum_{\substack{i<x\le 0\\ x<j\le 0}} \mathcal{E}_{i,0}\mathcal{H}_{j,0} - \sum_{\substack{x<i\le 0\\ j<x\le 0}} \mathcal{E}_{i,0}\mathcal{H}_{j,0}$$

Hence

$$\begin{aligned} \Delta\left(\sum_{i,j} \mathrm{sgn}(j-i)\mathcal{E}^s_{i,0}\mathcal{H}^s_{j,0}\right) &= \sum_{i,j\le 0} \mathrm{sgn}(j-i)\mathcal{E}^s_{i,0}\mathcal{H}^s_{j,0} \otimes 1 + 1 \otimes \sum_{i,j\le 0} \mathrm{sgn}(j-i)\mathcal{E}^s_{i,0}\mathcal{H}^s_{j,0} \\ &\quad + E^s_0 \otimes H^s_0 - H^s_0 \otimes E^s_0 \end{aligned} \tag{B.3.2}$$

and finally,

$$\begin{aligned} \Delta\widetilde{E}^s &= \Delta E^s_1 + \frac{U}{2} \Delta E^s_0 \Delta H^s_0 + \mu_+ \Delta E^s_0 - p(1 \otimes \mathcal{E}^s_{0,0}) \\ &= (E^s_1 + \mu_+ E^s_0) \otimes 1 + 1 \otimes (E^s_1 + \mu_+ E^s_0 + p\mathcal{E}_{0,0}) + \frac{U}{2}(E^s_0 \otimes H^s_0 - H^s_0 \otimes E^s_0) \\ &\quad + \frac{U}{2}(E^s_0 H^s_0 \otimes 1 + 1 \otimes E^s_0 H^s_0 + E^s_0 \otimes H^s_0 + H^s_0 \otimes E^s_0) \\ &= \widetilde{E}^s \otimes 1 + 1 \otimes \widehat{E}^s + U E^s_0 \otimes H^s_0\,. \end{aligned} \tag{B.3.3}$$

One obtains $\Delta\widetilde{F}_1^s$ similarly, by replacing (p, U, μ_+) with $(-p, -U, \mu_-)$ (without changing the sign of U in μ_-). We will check that Δ is a preserves the Lie bracket using the latter result:

$$\begin{aligned}
[\Delta H_0^s, \Delta\widetilde{E}_1^s] &= [H_0^s \otimes 1 + 1 \otimes H_0^s, (\widehat{E}_1^s + \mu_+ E_0^s) \otimes 1 + 1 \otimes (\widehat{E}_1^s + \mu_+ E_0^s - p\mathcal{E}_{0,0}^s) - UE_0 \otimes H_0] \\
&= ([H_0^s, \widehat{E}_1^s] + \mu_+[H_0^s, E_0^s]) \otimes 1 + 1 \otimes ([H_0^s, \widehat{E}_1^s] + \mu_+[H_0^s, E_0^s] - p[H_0^s, \mathcal{E}_{0,0}^s]) \\
&\quad - U[H_0^s, E_0^s] \otimes H_0^s \\
&= 2((\widehat{E}_1^s + \mu_+ E_0^s) \otimes 1 + 1 \otimes (\widehat{E}_1^s + \mu_+ E_0^s - p\mathcal{E}_{0,0}^s)) - 2E_0^s \otimes H_0^s \\
&= 2\Delta\widetilde{E}_1^s.
\end{aligned} \tag{B.3.4}$$

B.4 Generators of $\mathcal{Y}(\mathfrak{sl}_2, \mathfrak{u}_1)^s$ satisfying the boundary intertwining relation with the K-matrix

Take the K-matrix $K^+(u, p)$ (2.6.18). We have that combining the evaluation representation with the 4×4 representation ρ^h into a representation ρ_u

$$\rho_u(E_1^s) = \left(\frac{\sqrt{1 + \frac{U^2}{16}\sin^2 2u}}{\tan u} \right) \rho^h(E_0^s), \quad \rho_u(\mathcal{E}_{0,0}^s) = \left(1 + \frac{1}{\tan^2 u}\right) \rho^h(E_0^s). \tag{B.4.1}$$

This is equivalent to the evaluation representation of the Yangian generators of AdS/CFT R-matrix [43] upon setting

$$u = \frac{p}{4}, \quad U = 16g. \tag{B.4.2}$$

These satisfy the boundary intertwining relation (1.7.13) with the K-matrix,

$$K^+(u, p)\rho_u(\widetilde{E}_1^s) = \rho_{-u}(\widetilde{E}_1^s)K^+(u, p)\,. \tag{B.4.3}$$

To interpret the $u = 0$ pole of the p-dependent term, consider our original reason for adding $\mathcal{E}_{0,0}^s$ to the twisted Yangian generator: that the quadratic deformation in terms of level 0 generators alone cannot not fix the commutativity of E_1^s with the open Hubbard Hamiltonian,

$$[\mathcal{H}_p^H, E_1^s + \frac{U}{2}E_0^s H_0^s] = p[E_1^s, \mathcal{H}_p^H] \neq 0\,. \tag{B.4.4}$$

But by adding $-p\mathcal{E}^s_{0,0}$ to the generator one obtains

$$[\mathcal{H}^H_p, \widetilde{E}^s_1] = p[E^s_1, \mathcal{H}^H_p] + \left[\sum_{\sigma=\uparrow,\downarrow} c^\dagger_{-1\sigma} c_{0\sigma} + c^\dagger_{0\sigma} c_{-1\sigma}, p\mathcal{E}^s_{0,0} \right] = 0\,. \tag{B.4.5}$$

Thus the existence of the conserved twisted Yangian generator is due to the interplay of its p-dependent term with the hopping term in the Hamiltonian acting on the site $i = 0$ and its neighbor $i = -1$. This hopping action corresponds to the annihilation of a particle at one site and its creation at a neighboring site, interpreted as motion of the particle. If such a particle has rapidity $u = 0$ at $i = 0$ then it is static at the boundary, and the hopping term acting on site 0 must vanish. At that point the K-matrix becomes trivial, and, expectedly, the $\widetilde{E}^s_1$ symmetry degenerates to E^s_0,

To conclude, note that for the left boundary, with K-matrices given in [120], one can obtain the evaluation representation of the symmetry generators for these by making use of the relation between left and right K-matrices

$$K^-(u,p) = p^2 K^+(-u, \tfrac{1}{p}). \tag{B.4.6}$$

This corresponds to a weak $\leftrightarrow$ strong exchange $p \leftrightarrow \frac{1}{p}$ and a reversal of the direction of the rapidity. Hence, if $Q(u,p)$ is a symmetry of $K^-(u,p)$, then $Q(-u, \frac{1}{p})$ is a symmetry of $K^+(u,p)$.

B.5 $\mathcal{Y}(\mathfrak{so}_4, \mathfrak{sl}_2^{\Delta})$ as a symmetry of $\mathcal{H}_o^h$

We will proceed to show that $\widetilde{\mathcal{B}}^+_1$ commutes with the achiral Hamiltonian $\mathcal{H}^h_\Delta$, on the half-infinite interval limit taken over the antiferromagnetic vacuum. Here we will construct the half-infinite Hubbard chain by folding an infinite one at a spin site, say N, and identifying sites $N+n$ and $N-n$. Such identification commutes with the partial PHT. Hence, since all components of $\widetilde{B}^+$ are already conserved charges of an infinite Hubbard chain, we only need to show that

$$[\mathcal{P}_{0\downarrow}, \widetilde{\mathcal{B}}^+_1] = 0. \tag{B.5.1}$$

It is helpful to divide the commutator into components. First, let us compute $[\mathcal{P}_{0\downarrow}, \mathcal{A}^+_1]$ by dividing $\mathcal{A}^+_1$ into U-independent and dependent components A^{+0}_1 and A^{+U}_1. For the commutator with A^{+0}_1, it is convenient to write $\mathcal{P}_{0\downarrow}$ in the fermionic representation:

$$\mathcal{P}_{0\downarrow} = c^\dagger_{0\downarrow} - c_{0\downarrow}. \tag{B.5.2}$$

Then we find that

$$\begin{aligned}
[\mathcal{P}_{0\downarrow}, \mathcal{A}_1^{+0}] &= [c_{0\downarrow}^\dagger - c_{0\downarrow}, c_{-1\uparrow}^\dagger c_{0\downarrow} - c_{0\uparrow}^\dagger c_{-1\downarrow} + (c_{-1\uparrow}^\dagger c_{0\downarrow}^\dagger + c_{0\uparrow}^\dagger c_{-1\downarrow}^\dagger) \\
&= -(c_{-1\uparrow}^\dagger - c_{-1\uparrow}^\dagger) \\
&= 0. \qquad (B.5.3)
\end{aligned}$$

For $\mathcal{A}_2^{+U}$, as we will see, it is not necessary to compute the commutator of $\mathcal{P}_{0\downarrow}$ with the different operators, but rather it is sufficient to know that $[\mathcal{P}_{0\downarrow}, \mathcal{B}_0^+] = 2[\mathcal{P}_{0\downarrow}, E_0^s]$, which can be inferred from the relation $(\mathcal{P}_{0\downarrow})^{-1}\mathcal{E}_{0,0}^c\mathcal{P}_{0\downarrow} = \mathcal{E}_{0,0}^s$. We find that

$$[\mathcal{P}_{0\downarrow}, \mathcal{A}_2^{+U}] = \frac{U}{2}\sum_{i<0}([\mathcal{P}_{0\downarrow}, \mathcal{E}_{0,0}^s](\mathcal{H}_{i,0}^s + \mathcal{H}_{0,0}^s) - [\mathcal{P}_{0\downarrow}, \mathcal{H}_{0,0}^s](\mathcal{E}_{i,0}^s + \mathcal{E}_{i,0}^0)). \qquad (B.5.4)$$

If one makes the ansatz that the quadratic modification must be of the form $X_B^+ = \mu\mathcal{B}_0^+ + k(\mathcal{B}_0^+\mathcal{A}_0^z - \mathcal{B}_0^z\mathcal{A}_0^+)$, then

$$[\mathcal{P}_{0\downarrow}, X_B^+] = -(4k-2\mu)[\mathcal{P}_{0\downarrow}, \mathcal{E}_{0,0}^s] + 2k\sum_{i<N}([\mathcal{P}_{0\downarrow}, \mathcal{E}_{0,0}^s](\mathcal{H}_{i,0}^s + \mathcal{H}_{0,0}^s) - [\mathcal{P}_{0\downarrow}, \mathcal{H}_{0,0}^s](\mathcal{E}_{i,0}^s + \mathcal{E}_{i,0}^s)) \qquad (B.5.5)$$

Hence we arrive at the conclusion that $[\mathcal{P}_{0\downarrow}, \mathcal{A}_1^+ + X_B^+] = 0$ if

$$k = \frac{U}{4}, \quad \mu = \frac{U}{2}. \qquad (B.5.6)$$

B.6 Computation of $\mathcal{Y}(\mathfrak{so}_4, \mathfrak{sl}_2^\Delta)$ coproducts

We will proceed to compute $\Delta\widetilde{\mathcal{B}}_1^+$. Define

$$\mathcal{A}_{i,n}^+ = \mathcal{E}_{i,n}^s + \mathcal{E}_{i,n}^c, \quad \mathcal{A}_{i,n}^- = \mathcal{F}_{i,n}^s + \mathcal{F}_{i,n}^c, \quad \mathcal{A}_{i,n}^z = \mathcal{H}_{i,n}^s + \mathcal{H}_{i,n}^c \qquad (B.6.1)$$

and

$$\mathcal{B}_{i,n}^+ = \mathcal{E}_{i,n}^s - \mathcal{E}_{i,n}^c, \quad \mathcal{B}_{i,n}^- = \mathcal{F}_{i,n}^s - \mathcal{F}_{i,n}^c, \quad \mathcal{B}_{i,n}^z = \mathcal{H}_{i,n}^s - \mathcal{H}_{i,n}^c \qquad (B.6.2)$$

so that we can rewrite

$$\begin{aligned}
\mathcal{A}_1^+ &= \sum_i(\mathcal{E}_{i-1,1}^s - \mathcal{E}_{i,-1}^s + \mathcal{E}_{i-1,1}^c - \mathcal{E}_{i,-1}^c) + \frac{U}{2}\sum_{i,j}\mathrm{sgn}(j-i)(\mathcal{E}_{i,0}^s\mathcal{H}_{j,0}^s - \mathcal{E}_{i,0}^c\mathcal{H}_{j,0}^c) \\
&= \sum_i(\mathcal{A}_{i-1,1}^+ - \mathcal{A}_{i,-1}^+) + \frac{U}{4}\sum_{i,j}\mathrm{sgn}(j-i)(\mathcal{B}_{i,0}^+\mathcal{A}_{j,0}^z - \mathcal{B}_{i,0}^z\mathcal{A}_{j,0}^+) \qquad (B.6.3)
\end{aligned}$$

Using the trick from Appendix B.3, one can show that

$$\Delta\mathcal{A}_1^+ = \mathcal{A}_1^+\otimes 1 + 1\otimes\mathcal{A}_1^+ + \frac{U}{4}(\mathcal{B}_0^+\otimes\mathcal{A}_0^z + \mathcal{A}_0^+\otimes\mathcal{B}_0^z - \mathcal{A}_0^z\otimes\mathcal{B}_0^+ - \mathcal{B}_0^z\otimes\mathcal{A}_0^+) \quad \text{(B.6.4)}$$

Since Δ is a homomorphism,

$$\begin{aligned}
\Delta\widetilde{\mathcal{B}}_1^+ &= \Delta\mathcal{A}_1^+ - \frac{U}{2}\Delta\mathcal{B}_0^+ + \frac{U}{4}(\Delta\mathcal{B}_0^+\Delta\mathcal{A}_0^z - \Delta\mathcal{B}_0^z\Delta\mathcal{A}_0^+)\\
&= \mathcal{A}_1^+\otimes 1 + 1\otimes\mathcal{A}_1^+ + \frac{U}{4}(2\mathcal{B}_0^+\otimes\mathcal{A}_0^z - 2\mathcal{B}_0^z\otimes\mathcal{A}_0^+ + (\mathcal{B}_0^+\mathcal{A}_0^z - \mathcal{B}_0^+\mathcal{A}_0^z)\otimes 1\\
&\quad +1\otimes(\mathcal{B}_0^+\mathcal{A}_0^z - \mathcal{B}_0^+\mathcal{A}_0^z)) + \frac{U}{2}(\mathcal{B}_0^+\otimes 1 + 1\otimes\mathcal{B}_0^+)\\
&= \widetilde{\mathcal{B}}^+\otimes 1 + 1\otimes\widetilde{\mathcal{B}}^+ + \frac{U}{2}(\mathcal{B}_0^+\otimes\mathcal{A}_0^z - \mathcal{B}_0^z\otimes\mathcal{A}_0^+) \qquad \text{(B.6.5)}
\end{aligned}$$

B.7 Construction of the $\mathcal{Y}(\mathfrak{sl}_2,\mathfrak{sl}_2)^s$ generators

Here we will show the construction of the free boundary twisted Yangian generators. As in previous boundary Hamiltonians, we will be working in the antiferromagnetic regime and assuming the limit as the left side of the chain goes to $-\infty$ to be taken over the Hubbard model antiferromagnetic vacuum, so we can neglect boundary terms at $-\infty$ which may appear as a result of commutators.

Let us first compute $[\mathcal{H}_o^h, H_2^s]$. Using the Jacobi identity,

$$\begin{aligned}
[\mathcal{H}_o^h, H_2^s] &= [\mathcal{H}_o^h,[E_1^s,F_1^s]]\\
&= -[E_1^s,[F_1^s,\mathcal{H}_o^h]] - [F_1^s,[\mathcal{H}_o^h,E_1^s]]\\
&= -2\left([E_1^s,\mathcal{F}_{0,0}^s] - [F_1^s,\mathcal{E}_{0,0}^s]\right) \qquad \text{(B.7.1)}
\end{aligned}$$

Now $[E_1^s,\mathcal{F}_{0,0}^s]$:

$$\begin{aligned}
[E_1^s,\mathcal{F}_{0,0}^s] &= \left[\sum_{i\le 0}(\mathcal{E}_{i-1,1}^s - \mathcal{E}_{i,-1}^s),\mathcal{F}_{0,0}^s\right] + \frac{U}{2}\left[\sum_{i<j\le 0}(\mathcal{E}_{i,0}^s\mathcal{H}_{j,0}^s - \mathcal{E}_{j,0}^s\mathcal{H}_{i,0}^s),\mathcal{F}_{0,0}^s\right]\\
&= \mathsf{c}_{0\downarrow}^\dagger\mathsf{c}_{-1\downarrow} + \mathsf{c}_{-1\uparrow}^\dagger\mathsf{c}_{0\uparrow} - \frac{U}{2}\left(\mathcal{H}_{0,0}^s\sum_{i<0}\mathcal{H}_{i,0}^s + 2\mathcal{F}_{0,0}^s\sum_{i<0}\mathcal{E}_{i,0}^s\right) \qquad \text{(B.7.2)}
\end{aligned}$$

and $[F_1^s, \mathcal{E}_{0,0}^s]$:

$$
\begin{aligned}
[F_1^s, \mathcal{E}_{0,0}^s] &= \left[\sum_{\leq 0}(\mathcal{F}_{i-1,1}^s - \mathcal{F}_{i,-1}^s), \mathcal{E}_{0,0}^s\right] + \frac{U}{2}\left[\sum_{i<j}(\mathcal{F}_{i,0}^s\mathcal{H}_{j,0}^s - \mathcal{F}_{j,0}^s\mathcal{H}_{i,0}^s), \mathcal{E}_{0,0}^s\right] \\
&= \mathsf{c}_{0\uparrow}^\dagger \mathsf{c}_{-1\uparrow} + \mathsf{c}_{-1\downarrow}^\dagger \mathsf{c}_{0\downarrow} - \frac{U}{2}\left(\mathcal{H}_{0,0}^s\sum_{i<0}\mathcal{H}_{i,0}^s + 2\mathcal{E}_{0,0}^s\sum_{i<0}\mathcal{F}_{i,0}^s\right), \qquad \text{(B.7.3)}
\end{aligned}
$$

so that the full commutator is

$$
\begin{aligned}
[\mathcal{H}_o^h, H_2^s] &= -2\left(\mathcal{H}_{-1,1}^s - \mathcal{H}_{0,-1}^s - U\sum_{i<0}(\mathcal{E}_{i,0}^s\mathcal{F}_{0,0}^s - \mathcal{F}_{i,0}^s\mathcal{E}_{0,0}^s)\right) \\
&= -2(\mathcal{H}_{-1,1}^s - \mathcal{H}_{0,-1}^s) + 2U\left((E_0^s - \mathcal{E}_{0,0}^s)\mathcal{F}_{0,0}^s - (F_0^s - \mathcal{F}_{0,0}^s)\mathcal{E}_{0,0}^s\right) \\
&= -2(\mathcal{H}_{-1,1}^s - \mathcal{H}_{0,-1}^s) + 2U(E_0^s\mathcal{F}_{0,0}^s - F_0^s\mathcal{E}_{0,0}^s - \mathcal{H}_{0,0}^s) \qquad \text{(B.7.4)}
\end{aligned}
$$

and hence, using the fact that $[\mathcal{H}_o^h, \mathcal{H}_{0,0}^s] = (\mathcal{H}_{-1,1}^s - \mathcal{H}_{0,-1}^s)$, one can see that the twisted level 2 Yangian generator

$$
\widehat{H}_2^s = H_2^s - U(E_1^sF_0^s - E_0^sF_1^s) - 2\mathcal{H}_{0,0}^s \qquad \text{(B.7.5)}
$$

commutes with $\mathcal{H}_o^h$.

Again, for $\widehat{\mathcal{E}}_2^s$, we start by using the Jacobi identity:

$$
\begin{aligned}
[\mathcal{H}_o^h, E_2^s] &= \tfrac{1}{2}[\mathcal{H}_o^h, [H_1^s, E_1^s]] \\
&= -\tfrac{1}{2}\left([H_1^s, [E_1^s, \mathcal{H}_o^h]] + [E_1^s, [\mathcal{H}_o^h, H_1^s]]\right) \\
&= -([H_1^s, \mathcal{E}_{0,0}^s] - [E_1^s, \mathcal{H}_{0,0}^s]) \qquad \text{(B.7.6)}
\end{aligned}
$$

Now $[H_1^s, \mathcal{E}_{0,0}^s]$:

$$
\begin{aligned}
[H_1^s, \mathcal{E}_{0,0}^s] &= \left[\sum_{i\leq 0}(\mathcal{H}_{i-1,1}^s - \mathcal{H}_{i,-1}^s), \mathcal{E}_{0,0}^s\right] + U\left[\sum_{i<j\leq 0}(\mathcal{E}_{i,0}^s\mathcal{F}_{j,0}^s - \mathcal{E}_{j,0}^s\mathcal{F}_{i,0}^s), \mathcal{E}_{0,0}^s\right] \\
&= 2(\mathcal{E}_{-1,1}^s - \mathcal{E}_{0,-1}^s) - U\sum_{i<0}\mathcal{E}_{i,0}^s\mathcal{H}_{0,0}^s \qquad \text{(B.7.7)}
\end{aligned}
$$

and $[E_1^s, \mathcal{H}_{0,0}^s]$:

$$
\begin{aligned}
[E_1^s, \mathcal{H}_{0,0}^s] &= \left[\sum_{i}(\mathcal{E}_{i,1}^s - \mathcal{E}_{i,-1}^s), \mathcal{H}_{0,0}^s\right] + \frac{U}{2}\left[\sum_{i<j\leq 0}(\mathcal{E}_{i,0}^s\mathcal{H}_{j,0}^s - \mathcal{E}_{j,0}^s\mathcal{H}_{i,0}^s), \mathcal{H}_{0,0}^s\right] \\
&= -2(\mathcal{E}_{-1,1}^s - \mathcal{E}_{0,-1}^s) + U\mathcal{E}_{0,0}^s\sum_{i<0}\mathcal{H}_{i,0}^s, \qquad \text{(B.7.8)}
\end{aligned}
$$

so that the full commutator is

$$\begin{aligned}
[\mathcal{H}_0^h, E_2^s] &= -2(\mathcal{E}^s_{-1,1}-\mathcal{E}^s_{0,-1}) + U\sum_{i<0}(\mathcal{E}^s_{i,0}\mathcal{H}^s_{0,0}-\mathcal{H}^s_{i,0}\mathcal{E}^s_{0,0}) \\
&= -2(\mathcal{E}^s_{-1,1}-\mathcal{E}^s_{0,-1}) + U\left((E_0^s-\mathcal{E}^s_{0,0})\mathcal{H}^s_{0,0}-(H_0^s-\mathcal{H}^s_{0,0})\mathcal{E}^s_{0,0}\right) \\
&= -2(\mathcal{E}^s_{-1,1}-\mathcal{E}^s_{0,-1}) + U(E_0^s\mathcal{H}^s_{0,0}-H_0^s\mathcal{E}^s_{0,0}+2\mathcal{E}^s_{0,0}) \qquad \text{(B.7.9)}
\end{aligned}$$

and hence, using the fact that $[\mathcal{H}_o^h,\mathcal{E}^s_{0,0}]=\mathcal{E}^s_{-1,1}-\mathcal{E}^s_{0,-1}$, one can see that the twisted level 2 Yangian generator

$$\widehat{E}_2^s = E_2^s - \frac{U}{2}(H_1^sE_0^s - H_0^sE_1^s) - 2\mathcal{E}^s_{0,0} \qquad \text{(B.7.10)}$$

commutes with $\mathcal{H}_o^h$. Similarly, for $\widehat{\mathcal{F}}_2^s$, let's first compute $[\mathcal{H}_o^h, F_2^s]$:

$$\begin{aligned}
[\mathcal{H}_o^h, F_2^s] &= -\tfrac{1}{2}[\mathcal{H}_o^h,[H_1^s,F_1^s]] \\
&= \tfrac{1}{2}\left([H_1^s,[F_1^s,\mathcal{H}_o^h]]+[F_1^s,[\mathcal{H}_o^h,H_1^s]]\right) \\
&= ([H_1^s,\mathcal{F}^s_{0,0}]-[F_1^s,\mathcal{H}^s_{0,0}]) \qquad \text{(B.7.11)}
\end{aligned}$$

Now $[H_1^s,\mathcal{F}^s_{0,0}]$:

$$\begin{aligned}
[H_1^s,\mathcal{F}^s_{0,0}] &= \left[\sum_{i\le 0}(\mathcal{H}^s_{i-1,1}-\mathcal{H}^s_{i,-1}),\mathcal{F}^s_{0,0}\right] + U\left[\sum_{i<j\le 0}(\mathcal{E}^s_{i,0}\mathcal{F}^s_{j,0}-\mathcal{E}^s_{j,0}\mathcal{F}^s_{i,0}),\mathcal{F}^s_{0,0}\right] \\
&= -2(\mathcal{F}^s_{-1,1}-\mathcal{F}^s_{0,-1}) - U\sum_{i<0}\mathcal{H}^s_{0,0}\mathcal{F}^s_{i,0} \qquad \text{(B.7.12)}
\end{aligned}$$

and $[F_1^s,\mathcal{H}^s_{0,0}]$:

$$\begin{aligned}
[F_1^s,\mathcal{H}^s_{0,0}] &= \left[\sum_{i\le 0}(\mathcal{F}^s_{i-1,1}-\mathcal{F}^s_{i,-1}),\mathcal{H}^0_{NL}\right] - \frac{U}{2}\left[\sum_{i<j\le 0}(\mathcal{F}^s_{i,0}\mathcal{H}^s_{j,0}-\mathcal{F}^s_{j,0}\mathcal{H}^s_{i,0}),\mathcal{H}^s_{0,0}\right] \\
&= 2(\mathcal{F}^s_{-1,1}-\mathcal{F}^s_{0,-1}) + U\mathcal{F}^s_{0,0}\sum_{i<0}\mathcal{H}^s_{i,0}, \qquad \text{(B.7.13)}
\end{aligned}$$

so that the full commutator is

$$\begin{aligned}
[\mathcal{H}_o^h, H_2^s] &= -2(\mathcal{F}^s_{-1,1}-\mathcal{F}^s_{0,-1}) - U\sum_{i<0}(\mathcal{F}^s_{i,0}\mathcal{H}^s_{0,0}-\mathcal{H}^s_{i,0}\mathcal{F}^s_{0,0}) \\
&= -2(\mathcal{F}^s_{-1,1}-\mathcal{F}^s_{0,-1}) - U\left((F_0^s-\mathcal{F}^s_{0,0})\mathcal{H}^s_{0,0}-(H_0^s-\mathcal{H}^s_{0,0})\mathcal{F}^s_{0,0}\right) \\
&= -2(\mathcal{F}^s_{-1,1}-\mathcal{F}^s_{0,-1}) - U(F_0^s\mathcal{H}^s_{0,0}-H_0^s\mathcal{F}^s_{0,0}+2\mathcal{F}^s_{0,0}) \qquad \text{(B.7.14)}
\end{aligned}$$

and hence, using the fact that $[\mathcal{H}_o^s,\mathcal{F}^s_{0,0}]=\mathcal{F}^s_{-1,1}-\mathcal{F}^s_{0,-1}$, one can see that the twisted level 2 Yangian generator

$$\widehat{F}_2^s = F_2^s + \frac{U}{2}(H_1^s F_0^s - H_0^s F_1^s) - 2\mathcal{F}_{0,0}^s \tag{B.7.15}$$

commutes with $\mathcal{H}_o^s$.

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