Homogeneity and $\aleph_0$-categoricity of semigroups

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Abstract

In this thesis we study problems in the theory of semigroups which arise from model theoretic notions. Our focus will be on $\aleph_0$-categoricity and homogeneity of semigroups, a common feature of both of these properties being *symmetricity*. A structure is *homogeneous* if every local symmetry can be extended to a global symmetry, and as such it will have a rich automorphism group. On the other hand, the Ryll-Nardzewski Theorem dictates that $\aleph_0$-categorical structures have oligomorphic automorphism groups. Numerous authors have investigated the homogeneity and $\aleph_0$-categoricity of algebras including groups, rings, and of relational structures such as graphs and posets. The central aim of this thesis is to forge a new path through the model theory of semigroups.

The main body of this thesis is split into two parts. The first is an exploration into $\aleph_0$-categoricity of semigroups. We follow the usual semigroup theoretic method of analysing Green's relations on an $\aleph_0$-categorical semigroup, and prove a finiteness condition on their classes. This work motivates a generalization of characteristic subsemigroups, and subsemigroups of this form are shown to inherit $\aleph_0$-categoricity. We also explore methods for building $\aleph_0$-categorical semigroups from given $\aleph_0$-categorical structures.

In the second part we study the homogeneity of certain classes of semigroups, with an emphasis on completely regular semigroups. A complete description of all homogeneous bands is achieved, which shows them to be regular bands with homogeneous structure semilattices. We also obtain a partial classification of homogeneous inverse semigroups. A complete description can be given in a number of cases, including inverse semigroups with finite maximal subgroups, and periodic commutative inverse semigroups. These results extend the classification of homogeneous semilattices by Droste, Truss, and Kuske [27]. We pose a number of open problems, that we believe will open up a rich subsequent stream of research.

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Preface

This thesis is a study into a pair of model theoretic properties applied to semigroups, namely $\aleph_0$-categoricity and homogeneity. A summary of the work in this thesis is given below.

In Chapter 1 we present the background model theory required for this thesis. The chapter ends on a result which gives a link between the properties of $\aleph_0$-categoricity and homogeneity, and provides further motivation for much of our work. In Chapter 2 we give some preliminaries from semigroup theory. Emphasis is made on exploring the structure of completely regular semigroups, and how isomorphisms between them can be constructed.

Our main work begins in Chapter 3 with the study into the $\aleph_0$-categoricity of structures, with emphasis on semigroups. In Section 3.1 we give an account of the historical background to $\aleph_0$-categoricity, showing it to be a popular area of research in the last 50 years. In Section 3.2 we introduce the fundamental result on $\aleph_0$-categoricity: the Ryll-Nardzewski Theorem (RNT). We give a number of known consequences of the RNT, such as that every $\aleph_0$-categorical structure is uniformly locally finite. Our various methods for proving $\aleph_0$-categoricity are outlined, each centring around the RNT. For illustration we use countable rectangular bands as our main example, showing that any such semigroup is $\aleph_0$-categorical.

In Section 3.3 we consider a generalization to a characteristic substructure which arises from the model theoretic concept of a definable set. This is used to show that $\aleph_0$-categoricity is inherited by every Green's class which forms a subsemigroup and, in particular, by maximal subgroups. If $\mathcal{K}$ is one of the Green's relations, then the set of cardinals of $\mathcal{K}$-classes of an $\aleph_0$-categorical semigroup is shown to be finite. We finish the section by examining when the $\aleph_0$-categoricity of a semigroup passes to its quotients. In particular, we show that $\aleph_0$-categoricity passes to any quotient of a semigroup by a Green's relation which forms a congruence. These results are then applied to the principal factors of an arbitrary $\aleph_0$-categorical semigroup in Section 3.4. The principal factors of an $\aleph_0$-categorical semigroup are shown to be $\aleph_0$-categorical completely (0)-simple or null semigroups, and the set of principal factors is finite, up to isomorphism. This naturally leads us to consider the $\aleph_0$-categoricity of Rees matrix semigroups in Section 3.5. We follow a common method devised by Graham of constructing a bipartite graph from the sandwich matrix of a Rees matrix semigroup S, and show that it inherits the $\aleph_0$-categoricity of S. Examples of $\aleph_0$-categorical Rees matrix semigroups are then constructed from known $\aleph_0$-categorical groups and bipartite graphs. Our central result is that the $\aleph_0$-categoricity of an orthodox Rees matrix semigroup depends only on the $\aleph_0$-categoricity of its maximal subgroups and its induced bipartite graph.

In Sections 3.6 and 3.7 we examine the $\aleph_0$-categoricity of a pair of well known semigroup constructs: 0-direct unions and strong semilattices of semigroups. The former construct allows a generalization of our results on $\aleph_0$-categorical Rees matrix semigroups to primitive regular semigroups. Finally, in Section 3.8 we discuss open problems and future directions in the work on $\aleph_0$-categorical semigroups.

The rest of this thesis is concerned with homogeneity of semigroups. Chapter 4 introduces the property of homogeneity from a general setting. A literature review is given in Section 4.1, with an emphasis on the homogeneity of groups. The seminal work of Fraïssé is introduced in Section 4.2, which allows us to build homogeneous structures from certain classes of finitely generated structures.

In Section 4.3 we discuss the importance of choice of signature for considering homogeneity. Many of the semigroups we study can alternatively be considered as I-semigroups, that is, semigroups with an additional unary operation satisfying certain laws. Our choice of signature for a class of semigroups can often be naturally dictated by the variety of semigroups or I-semigroups in which the class belongs. Our motivating example is the class of completely regular semigroups. We also describe a stronger form of homogeneity of a completely regular semigroup, which further takes into consideration the automorphism group of the induced structure semilattice.

The substructure of an arbitrary homogeneous structure is assessed in Section 4.4, and later applied to the case of semigroups. The main result of this section is that the maximal subgroups of a homogeneous semigroup are homogeneous groups, and are pairwise isomorphic. We then examine the homogeneity of non-periodic semigroups in Section 4.5, and show that a completely regular non-periodic homogeneous semigroup is completely simple. The importance of the homogeneity of completely simple semigroups is therefore pivotal, and is the subject of Section 4.6.

The results of Chapter 4 are used throughout Chapters 5, 6, and 7, where the homogeneity of bands, inverse semigroups, and orthodox completely regular semigroups are respectively studied. Our results are obtained by using a mix of semigroup theory brute force and Fraïssé's method. A complete description of homogeneous bands is achieved in Chapter 5. An immediate consequence is that each homogeneous band is a regular band and has a homogeneous structure semilattice. On the other hand, a classification of homogeneous inverse semigroups is shown to be a greater challenge, although a number of partial classifications are given. In particular, all homogeneous inverse semigroups with finite maximal subgroups are described, along with periodic homogeneous commutative inverse semigroups.

Every homogeneous inverse semigroup is shown to be either bisimple or Clifford, and in the latter case a decomposition into a spined product is obtained.

In Chapter 7 we combine our results on homogeneous bands and Clifford semigroups to produce examples of homogeneous orthodox completely regular semigroups. This thesis ends with a discussion into the key open problems which have arisen during the work on homogeneity, and other directions that future research into homogeneity may take.

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Declaration

The work presented in this thesis is based on research carried out in the Department of Mathematics, University of York. This work has not previously been presented for an award at this, or any other, University.

Chapters 1 and 2 presents a review of the background material underlying the research in this thesis. The main results contained in Chapters 3, 4, 5, 6 and 7 are my own, unless referenced otherwise. Chapter 3 is devoted to work that will form a joint paper with Victoria Gould. Most of the content of Chapters 4, 5, 6 and 7 are from [75] and [76].

Chapter 1

Preliminaries I: An introduction to model theory

In this chapter we outline the basic model theory required for this thesis. The results and definitions are taken from a number of introductory text books, including [12], [51] and [69]. In these books one can find the original citations.

Throughout this thesis, we write maps on the right of their arguments, so that the composition of mappings are from the left to the right.

1.1 First order structures

Model theory can be seen as the study of structures from a logical viewpoint. Defining model theory in such a succinct way is possibly controversial, and certainly likely to annoy a handful of model theorists. Most famously is Chang and Keisler's attempt at a pithy definition, given in their 1990 text [12], which states:

> "universal algebra + logic = model theory."

This definition was soon seen as dated, as the field of model theory quickly grew and evolved. The introduction to Hodges 1993 text [51] best sums up the trepidation of defining such a changing theory:

> "Should I begin by defining 'model theory'? This might be unsafe..."

Here I shelter myself behind his attempt at a definition:

> "Model theory is the study of the construction and classification of structures within specified classes of structures."

For our work, 'specified class of structures' will mostly be the class of semigroups, although we built our framework from complete generality. By 'construction', Hodges means the building of structures which satisfy some desired property, such as having a large automorphism group or being a commutative semigroup. Finally, by 'classifying' he means subdividing a class of structures into subclasses in a meaningful way. A famous example from semigroup theory is the classification of inverse completely 0-simple semigroups: they are classified by showing that every inverse completely 0-simple semigroup is determined, up to isomorphism, by its maximal subgroups and cardinality of its subset of idempotents.

There is often a dividing line between relational structures, such as graphs and partial orderings, and algebraic structures, such as semigroups and groups. A key difference between these two classes of structures is examined in the subsequent section. However, we begin by introducing the theory of structures from a general setting, and one which can be seen to encompass both algebraic and relational structures. It relies on the following description of relations and functions on a set.

Let X be a set and n a non-negative integer. A subset R of X^n is called a *finitary relation of arity* n or an n*-ary relation*. A map from X^n to X is called a *finitary function of arity* n or an n*-ary function*.

Definition 1.1.1. A (first order) *structure* $\mathcal{M}$ is a non-empty set M, called the *universe*, together with:

(i) a set of finitary relations on M;

(ii) a set of finitary functions on M;

(iii) a set of elements of M called *constant elements*.

Each n-ary relation is named by an n-ary relational symbol, and if R is a relation symbol then we denote $R^{\mathcal{M}}$ as the relation named by R. Similarly, we denote $f^{\mathcal{M}}$ as the n-ary function named by a n-ary function symbol f, and $c^{\mathcal{M}}$ as the constant element named by a constant symbol c. We call $R^{\mathcal{M}}$, $f^{\mathcal{M}}$ and $c^{\mathcal{M}}$ the *interpretations* of the symbols R, f and c, respectively. The *cardinality* of $\mathcal{M}$ is defined as the cardinality of its universe M.

The structure $\mathcal{M}$ will often be written as

$$\mathcal{M} = (M, R^{\mathcal{M}}, f^{\mathcal{M}}, c^{\mathcal{M}} : R \in \mathfrak{R}, f \in \mathfrak{F}, c \in \mathfrak{C}),$$

where $\mathfrak{R}$, $\mathfrak{F}$ and $\mathfrak{C}$ denote the sets of relational symbols, functional symbols and constant symbols of $\mathcal{M}$, respectively. Where no confusion may arise, we will not distinguish between the relation $R^{\mathcal{M}}$ and its named relational symbol R. The set $L = \mathfrak{R} \cup \mathfrak{F} \cup \mathfrak{C}$ is called the *signature* of $\mathcal{M}$, and $\mathcal{M}$ is called an L*-structure*. The *cardinality* of the signature L is defined as the cardinality $|L|$ of L.

If L is a signature without functions or constant symbols, then an L-structure is called a *relational structure*. If L is a language without relational symbols, then an L-structure is called an *algebraic structure*.

Each signature gives rise to a *language*. The language consists of the symbols from the signature together with logical symbols and punctuation, and forms the framework of first order logic. This will be the main topic of the subsequent section.

We use standard notation by letting M, N, X etc. denote the universes of the structures $\mathcal{M}, \mathcal{N}$ and $\mathcal{X}$. Where no confusion can arise, we occasionally abuse notation by simply referring to a structure $\mathcal{M}$ as M. Signatures will usually be denoted by L, or a subscript may be used to distinguish signatures that will be key to this thesis.

Example 1.1.2. Let $L_S = \{\cdot\}$ be the signature consisting of a single binary function symbol, which we call the *signature of semigroups*. Then a semigroup $(S, \cdot^S)$ can be regarded as an L_S-structure, where $\cdot^S$ is an interpretation of $\cdot$. Note that $(\mathbb{Z}, -)$ is also an L_S-structure, despite it not being a semigroup.

Example 1.1.3. We may extend L_S by adding a constant symbol 1 to obtain the *signature of monoids* $L_{Mo} = \{\cdot, 1\}$.

Example 1.1.4. Now extend L_{Mo} by adding a unary function symbol $^{-1}$, we obtain the *signature of groups* $L_G = \{\cdot, ^{-1}, 1\}$. A group $(G, \cdot^G, {}^{(-1)^G}, 1^G)$ can be considered as an L_G-structure, where the group operation $\cdot^G$ interprets $\cdot$, the inverse function ${}^{(-1)^G}$ interprets $^{-1}$ and 1^G is the group identity interpreting 1.

When fixing a signature for groups, we could have defined a function symbol for, say, the commutator. Alternatively, we could consider a group as an L_S-structure, where we think of this as 'forgetting' the symbols $^{-1}$ and 1. This is an example of *reduction*, which we now formalise. Let L and L' be a pair of signatures such that $L \subseteq L'$. Then every L'-structure $\mathcal{M}$ forms an L-structure simply by removing the symbols in $L' \setminus L$. We observe that no elements of M are removed, despite the fact that constants in $\mathcal{M}$ may no longer be constants in the new structure. The resulting L-structure is called the *L-reduct* of $\mathcal{M}$, denoted $\mathcal{M}|L$, and $\mathcal{M}$ is called an *expansion* of $\mathcal{M}|L$. For example, the L_S-reduct of an L_G-structure $(G, \cdot, ^{-1}, 1)$ is $(G, \cdot)$.

The choice of signature for a structure is central to its study, and in most cases should be done in a way such that fundamental concepts, such as morphisms and substructures, agree with the corresponding concept from the relevant branch of mathematics. Before giving an example of this phenomenon, we first define the concepts raised here: morphisms and substructures. We fix the following standard notation for maps.

Notation 1.1.5. Let $\phi : A \to B$ be a map between sets A and B. If A' is a subset of A then we denote the restriction of ϕ to A' as $\phi|_{A'}$.

Notation 1.1.6. Let A and B be sets and $\{A_i : i \in I\}$ be a partition of A. If $\phi_i : A_i \to B$ is a map for each $i \in I$, then we let $\bigcup_{i\in I} \phi_i$ denote the map $\phi : A \to B$ given by

$$a\phi = a\phi_i \quad \text{if } a \in A_i.$$

Definition 1.1.7. Let $\mathcal{M}$ and $\mathcal{N}$ be L-structures with universes M and N, respectively. An *L-morphism* $\phi : \mathcal{M} \to \mathcal{N}$ is a map from M to N that preserves the relations, functions and constants, that is, such that

(i) if $R \in \mathfrak{R}$ is of arity n and $x_1, \dots, x_n \in M$ then

$$(x_1, \dots, x_n) \in R^{\mathcal{M}} \Rightarrow (x_1\phi, \dots, x_n\phi) \in R^{\mathcal{N}};$$

(ii) if $f \in \mathfrak{F}$ is of arity n and $x_1, \dots, x_n \in M$ then

$$((x_1, \dots, x_n)f^{\mathcal{M}})\phi = (x_1\phi, \dots, x_n\phi)f^{\mathcal{N}};$$

(iii) $c^{\mathcal{M}}\phi = c^{\mathcal{N}}$ for all $c \in \mathfrak{C}$.

An L-morphism from a substructure $\mathcal{M}$ to itself is called an L-endomorphism.

An *L-embedding* $\phi : \mathcal{M} \to \mathcal{N}$ is an injective L-morphism such that

$$(x_1, \dots, x_n) \in R^{\mathcal{M}} \Leftrightarrow (x_1\phi, \dots, x_n\phi) \in R^{\mathcal{N}}$$

for all $R \in \mathfrak{R}$ of arity n and $x_1, \dots, x_n \in M$. A bijective L-embedding is called an *L-isomorphism*. An L-isomorphism from $\mathcal{M}$ to $\mathcal{M}$ is called an L-automorphism, and the set of all L-automorphisms of $\mathcal{M}$ forms a group under composition, denoted by $\mathrm{Aut}(\mathcal{M})$. Each L-structure $\mathcal{M}$ possesses a *trivial automorphism*, denoted $1_{\mathcal{M}}$, given by $m1_{\mathcal{M}} = m$ for all $m \in M$, which is the identity of $\mathrm{Aut}(\mathcal{M})$.

For example, let G and H be a pair of groups in the signature L_G, and $\phi : G \to H$ be an L_G-morphism. Then ϕ preserves both the group operations and the inverses, and also maps the identity of G to the identity of H, that is,

$$(gg')\phi = (g\phi)(g'\phi), \quad (g\phi)^{-1} = (g^{-1})\phi, \quad e_G\phi = e_H,$$

for all $g, g' \in G$, where e_G and e_H are the identities of G and H, respectively. We therefore have the usual concept of a morphism of groups. Note that every map between groups which preserves multiplication gives rise to a group morphism. It follows that even in the signature L_S, every L_S-morphism is a group morphism. We will see an example of morphisms between relational structures in the subsequent section.

Definition 1.1.8. Let L be a signature and let $\mathcal{M}$ and $\mathcal{N}$ be a pair of L-structures.

Then we call $\mathcal{N}$ a *morphic image* of $\mathcal{M}$ if there exists a surjective L-morphism $\phi : \mathcal{M} \to \mathcal{N}$.

Definition 1.1.9. Let $\mathcal{M}$ be an L-structure with subset N. We call N *characteristic* if it is invariant under automorphisms of $\mathcal{M}$, that is if $\phi \in \mathrm{Aut}(\mathcal{M})$ then $N\phi = N$.

The automorphism group of a structure $\mathcal{M}$ has a natural action on M^n, where elements of $\mathrm{Aut}(\mathcal{M})$ act component-wise on the set M^n. That is, if $\phi \in \mathrm{Aut}(\mathcal{M})$ and $(a_1, \ldots, a_n) \in M^n$ then define

$$(a_1, \ldots, a_n)\phi := (a_1\phi, \ldots, a_n\phi).$$

Definition 1.1.10. Let L be a signature and $\mathcal{M}$ an L-structure. A *substructure* of $\mathcal{M}$ is an L-structure $\mathcal{N}$ such that $N \subseteq M$ and the inclusion map $\iota : N \to M$ is an L-embedding.

It follows that if $\mathcal{N}$ is a substructure of $\mathcal{M}$, then $R^{\mathcal{N}} = R^{\mathcal{M}} \cap N^n$ for each n-ary relation symbol R, $f^{\mathcal{N}} = f^{\mathcal{M}}|_{N^n}$ for each n-ary function symbol f and $c^{\mathcal{N}} = c^{\mathcal{M}}$ for each constant symbol c. In particular, a substructure of an algebraic structure $\mathcal{M}$ is a subset of M which is closed under the operations on $\mathcal{M}$.

Returning again to groups in the signature L_G, then every substructure is a subgroup, further highlighting the naturalness of this choice of signature. On the other hand, by regarding a group in the signature L_S, the substructures need only be subsets closed under the binary operation, that is, be subsemigroups. As such the signature L_S is seen as a less natural choice. Note however that a finite subsemigroup of a group is a group, since each element of the subsemigroup will have its inverse and the identity element as a power.

We now describe a method for constructing substructures from arbitrary subsets of the universe. It relies on the fact that the intersection of a collection of substructures is, if non-empty, a substructure.

Let $\mathcal{M}$ be an L-structure and A a subset of M. The *substructure of $\mathcal{M}$ generated by A* is the uniquely determined substructure with universe

$$\bigcap\{N : A \subseteq N, \mathcal{N} \textit{ is a substructure of } \mathcal{M}\}$$

which we denote as $\langle A\rangle_{\mathcal{M}}$. We say that $\mathcal{M}$ is *generated by* A if $\langle A\rangle_{\mathcal{M}} = \mathcal{M}$, and if further A is finite then we call $\mathcal{M}$ *finitely generated* (f.g.) or $|A|$-*generated*. The set A is called a *generating set* of $\mathcal{M}$.

In later chapters there will be a number of exceptions to this notation, most prominently $\langle A\rangle$ will simply denote the substructure generated by A in the signature of semigroups L_S.

Example 1.1.11. In the group $\mathcal{G} = (\mathbb{Z}, +, ^{-1}, 0)$, with signature L_G, we have

$$\langle 2 \rangle_{\mathcal{G}} = (2\mathbb{Z}, +, ^{-1}, 0).$$

The following is a simple consequence of [96, Lemma 1.1.7], and shows that L-isomorphisms between structures induces isomorphisms between their substructures.

Corollary 1.1.12. *Let $\phi : \mathcal{M} \to \mathcal{N}$ be an L-morphism and, for $A \subseteq M$, let $\mathcal{A} = \langle A \rangle_{\mathcal{M}}$. Then ϕ induces an L-morphism*

$$\phi|_{\mathcal{A}} : \mathcal{A} \to \langle \{a\phi : a \in A\} \rangle_{\mathcal{M}}$$

determined by $a \mapsto a\phi$ for each $a \in A$. Moreover, if ϕ is an L-isomorphism then so is $\phi|_{\mathcal{A}}$.

We may now define the first main property which is studied in this thesis.

Definition 1.1.13. An L-structure $\mathcal{M}$ is *homogeneous* if every L-isomorphism between f.g. substructures extends to an L-automorphism of $\mathcal{M}$.

We may study a weaker form of homogeneity by defining a structure $\mathcal{M}$ to be *n-homogeneous* for some $n \in \mathbb{N}$, if every isomorphism between substructures of cardinality n extends to an automorphism of $\mathcal{M}$. A homogeneous structure is clearly n-homogeneous for each $n \in \mathbb{N}$.

Definition 1.1.14. A structure $\mathcal{M}$ is *locally finite* if each of its f.g. substructures is finite.

We study a stronger property than locally finiteness as follows.

Definition 1.1.15. We call $\mathcal{M}$ *uniformly locally finite* (ULF) if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for every substructure $\mathcal{N}$ of $\mathcal{M}$, if $\mathcal{N}$ has a generating set of cardinality at most n, then $\mathcal{N}$ has cardinality at most $f(n)$.

There are numerous ways of building new structures 'from old', and we now study one of the more well known examples: direct products. The direct product of algebraic structures such as groups and semigroups is well known, although for relational structures such as graphs it is arguably less so. It will be fruitful to define a notion of the direct product for arbitrary structures, and in such a way that generalizes the algebraic direct product.

Let L be a signature and I a non-empty index set. For each $i \in I$, let $\mathcal{M}_i$ be an L-structure with universe M_i. Let X be the Cartesian product of the sets M_i ($i \in I$), that is, the set of all maps $\theta : I \to \bigcup_{i \in I} M_i$ such that $i\theta \in M_i$ for each $i \in I$. We build an L-structure $\mathcal{N}$ with universe X in the following way:

(1) for each n-ary relation symbol R of L, and n-tuple $\underline{a} = (\theta_1, \dots, \theta_n)$ of X, we define $\underline{a} \in R^{\mathcal{N}}$ if and only if $(i\theta_1, \dots, i\theta_n) \in R^{\mathcal{M}_i}$ for each $i \in I$;

(2) for each n-ary function symbol f of L, and n-tuple $\underline{a} = (\theta_1, \dots, \theta_n)$ of X, we define $\underline{a}f^{\mathcal{N}}$ to be the element ψ of X such that $i\psi = (i\theta_1, \dots, i\theta_n)f^{\mathcal{M}_i}$ for each $i \in I$;

(3) for each constant symbol c of L, we define $c^{\mathcal{N}}$ to be the element φ of X such that $i\varphi = c^{\mathcal{M}_i}$ for each $i \in I$.

Then $\mathcal{N}$ forms an L-structure, which we denote as $\prod_{i \in I} \mathcal{M}_i$, or simply $\prod_{i \in I} M_i$ where no confusion may arise.

If $|I| = r$ is finite, then we may adapt the construction above by letting X be the simplified form of the Cartesian product

$$\prod_{1 \leq i \leq r} M_i = \{(a_1, \dots, a_r) : a_i \in M_i\}.$$

The relations, functions and constants on the direct product $\mathcal{N}$ are then given by:

(1) for each n-ary relation symbol R of L, and n-tuple $\underline{a} = (a_1, \dots, a_n)$ of X, where $a_k = (a_{k1}, \dots, a_{kr})$, we define $\underline{a} \in R^{\mathcal{N}}$ if and only if $(a_{1i}, \dots, a_{ni}) \in R^{\mathcal{M}_i}$ for each $1 \leq i \leq r$;

(2) for each n-ary function symbol f of L, and n-tuple $\underline{a} = (a_1, \dots, a_n)$ of X, where $a_k = (a_{k1}, \dots, a_{kr})$, we define $\underline{a}f^{\mathcal{N}}$ to be the element $(b_1, \dots, b_r)$ of X such that $b_i = (a_{1i}, \dots, a_{ni})f^{\mathcal{M}_i}$ for each $i \in I$;

(3) for each constant symbol c of L, we define $c^{\mathcal{N}}$ to be the element $(c_1^{\mathcal{M}_1}, \dots, c_r^{\mathcal{M}_i})$ of X.

In this case the direct product $\mathcal{N}$ will be simply denoted by $\mathcal{M}_1 \times \mathcal{M}_2 \times \cdots \times \mathcal{M}_r$.

We will be working with tuples of sets throughout this thesis, and it is worth fixing the following notation.

Notation 1.1.16. Given a pair of tuples $\underline{a} = (a_1, \dots, a_n)$ and $\underline{b} = (b_1, \dots, b_m)$, we denote $(\underline{a}, \underline{b})$ as the $(n+m)$-tuple given by

$$(a_1, \dots, a_n, b_1, \dots, b_m),$$

and extend this notation for $(\underline{a}, \underline{b}, \underline{c}, \dots)$ etc. If $\underline{x}$ is an r-tuple, then we write $|\underline{x}| = r$.

1.1.1 Relational structures: graphs and posets

In this subsection we define a number of key relational structures for our work: graphs, posets and linear orderings. In addition, this discussion provides further examples of morphisms and substructures, defined in the previous subsection.

A (simple) graph $G = (V, E)$ is a set V of *vertices* together with a set E of *edges*, where each edge is a set of two distinct vertices. Here, our graphs are undirected, and have no loops or multiple edges. A graph (V, E) can be naturally considered in the signature $L_{Gr} = \{R\}$, comprising a single binary relation symbol R, where R is interpreted as the edge relation. That is, (V, E) becomes the L_{Gr}-structure $\mathcal{G} = (V, R^{\mathcal{G}})$, where $(u, v) \in R^{\mathcal{G}}$ if and only if $\{u, v\} \in E$. Where no confusion may arise, will usually write the symbol R as E, and call L_{Gr} the *signature of graphs*.

Example 1.1.17. Let $\mathcal{G} = (\{1, 2, 3\}, E)$ and $\mathcal{G}' = (\{4, 5\}, E')$ be the graphs given by

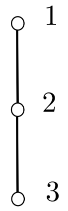

Figure 1.1: The graph $\mathcal{G}$.

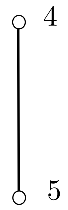

Figure 1.2: The graph $\mathcal{G}'$.

We observe that an L_{Gr}-morphism is a map between the vertices of a pair of graphs which preserves edges, while an L_{Gr}-embedding is an injective map which preserves edges *and* non-edges. For example, the map $\phi : \mathcal{G} \to \mathcal{G}'$ given by

$$1\phi = 3\phi = 4 \text{ and } 2\phi = 5$$

is an L_{Gr}-morphism, since it preserves edges. However, ϕ is not an L_{Gr}-embedding since it is not injective.

The graph $(\{1, 2\}, \{\{1, 2\}\})$ is a subgraph of the graph $\mathcal{G}$, and is the subgraph generated by the subset $\{1, 2\}$ of $\{1, 2, 3\}$. However, the empty graph on two vertices $(\{1, 2\}, \emptyset)$ is not a subgraph of $\mathcal{G}$ as $\{1, 2\} \in E$.

It is important to notice that any subset A of the graph (V, E) gives rise to a subgraph (A, E'), where $E' \subseteq E$.

In general, if $\mathcal{M} = (M, \mathfrak{R})$ is a relational structure and A is a subset of M, then

$$\langle A \rangle_{\mathcal{M}} = (A, \mathfrak{R})$$

where, for each $R \in \mathfrak{R}$, we have $R^{\langle A \rangle_{\mathcal{M}}} = R^{\mathcal{M}} \cap A^n$. It clearly then follows that every relational structure with finite signature is ULF (a result which is given as an exercise in [52, Exercise 1.2.6]).

This is one of the fundamental differences between relational structures and arbitrary structures. Indeed, the existence of functions can add elements to our generated set and, as seen in Example 1.1.11, even a single element generating set can generate an infinite substructure.

Posets are our second example of relation structures, and are fundamental to the study of semigroups. As such, a more leisurely exposition is required.

Given a binary relation σ on a set X, we often denote $(x, y) \in \sigma$ as $x \,\sigma\, y$. We define a binary relation σ on a set X to be an *equivalence relation* if:

(i) $(x, x) \in \sigma$ for all $x \in X$ (σ is *reflexive*);

(ii) if $(x, y) \in \sigma$ then $(y, x) \in \sigma$ for any $x, y \in X$ (σ is *symmetric*);

(iii) if $(x, y) \in \sigma$ and $(y, z) \in \sigma$ then $(x, z) \in \sigma$ for any $x, y, z \in X$ (σ is *transitive*).

If σ is an equivalence relation on a set X and $x \in X$ then we denote $x\tau$ as the τ-class containing x:

$$x\tau = \{y \in X : (x, y) \in \sigma\}.$$

If Y is a subset of X, then τ restricts to an equivalence relation on Y, and we abuse notation somewhat by denoting Y/τ as the set of equivalence classes of Y under the restriction of τ. That is,

$$Y/\tau = \{y\tau \cap Y : y \in Y\}.$$

A binary relation σ on a set X is called *anti-symmetric* if

$$(\forall x, y \in X) \quad (x, y) \in \sigma \text{ and } (y, x) \in \sigma \Rightarrow x = y.$$

We call the binary relation σ a *partial order* if it is reflexive, transitive and anti-symmetric. Given a partial order σ, we use the standard convention of writing $(x, y) \in \sigma$ as either $x \leq y$ or $y \geq x$. We write correspondingly $x < y$ or $y > x$ if $(x, y) \in \sigma$ and $x \neq y$. We call $(X, \leq)$ a *partially ordered set*, or simply a *poset*. As with graphs, we natural consider posets in the signature $L_P = \{\leq\}$ consisting of a single binary relation symbol $\leq$.

Example 1.1.18. Every set X forms a poset by letting $\leq$ be the equality relation, that is,

$$x \leq y \Leftrightarrow x = y.$$

A poset with partial order being equality is called an *antichain*.

Notation 1.1.19. Let $(X, \leq)$ be a poset with subsets Y and Z. We denote the property that $y \geq z$ for all $y \in Y$, $z \in Z$ as $Y \geq Z$, and say that Y *is an upperbound of* Z. If $y > z$ for all $y \in Y, z \in Z$ then we write $Y > Z$. If $Y = \{y_1, \ldots, y_n\}$ is finite, then we simplify this as $y_1, \ldots, y_n > Z$, and similarly for $Y > z_1, \ldots, z_n$. Given $x, x' \in X$, if $x \not\geq x'$ and $x' \not\geq x$ then we say that x *and* x' *are incomparable*, denoted $x \perp x'$.

A partial order with the property that

$$(\forall x, y \in X) \quad x \leq y \text{ or } y \leq x$$

is called a *linear order* or a *chain*.

An element x of a poset X is called *minimal* (*maximal*) if there are no elements of X strictly less (greater) than x under the partial order. An element x of X is *minimum* (*maximum*) if $x \leq X$ ($x \geq X$), and if such an element exists it is unique. The set of all minimal and maximal elements of X are called the *endpoints* of X, and may not be unique. Given $x, y \in X$, we call x an *upper (lower) cover* of y if $x > y$ ($x < y$) and whenever $x \geq z \geq y$ ($x \leq z \leq y$), then $x = z$ or $z = y$. If every element of X has an upper and lower cover then X is called *discrete*. On the other hand, X is called *dense* if, whenever $x > y$ in X, then there exists $z \in X$ such that $x > z > y$. In particular, X is dense if and only if no element has an upper and lower cover.

Example 1.1.20. The integers $\mathbb{Z}$ form a discrete linear order under the natural order. On the other hand the rationals $\mathbb{Q}$ form a dense linear order under the natural order, and we have the following well known result.

Theorem 1.1.21. *Every countable dense linear order without endpoints is isomorphic to the rationals* $\mathbb{Q}$ *under the natural order.*

1.2 Formulas and models

In this section we explore the machinery used by model theorists for studying and classifying structures. When defining a poset in the previous section, we gave a list of axioms which needed to be satisfied. This is a common occurrence in mathematics. For example, most introductory group theory courses will begin by listing the three axioms of groups. Each of the axioms use only the symbols from L_G, as well as quantifiers, logical connectives and so on.

We begin by formalizing this concept by using the symbols of a signature L, together with the usual logical symbols, to build formulas which are interpreted in any L-structure.

Definition 1.2.1. Given a signature L, an *L-formula* is a finite string of symbols built from

(i) the symbols from L;

(ii) the logical connectives $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$;

(iii) quantifiers $\exists, \forall$;

(iv) variables $x_0, x_1, \ldots$;

(v) the equality symbol $=$;

(vi) parentheses) and (.

Of course, not every finite string of these symbols is a formula, and the set of formulas is defined inductively by certain syntactic rules. However, we may think of a string of symbols to be a formula if it 'makes sense', as the following examples highlight.

Example 1.2.2. The string $(\vee x(\rightarrow \exists$ is clearly not a formula.

Example 1.2.3. In the signature of semigroups $L_S = \{\cdot\}$, the string

$$(\exists x)\,[(x \cdot x = x) \wedge (x \cdot y = y \cdot x)]$$

is a formula.

Given an L-formula ϕ, we say that variable x *is a free variable* in ϕ if it is not bound by a quantifier, and we call ϕ *bound* otherwise. A formula without free variables is called a *sentence*, while a formula in which all variables are free is called *quantifier-free*. As standard notation, we will often write $\phi(x_1, \ldots, x_n)$ to make explicit that $x_1, \ldots, x_n$ are precisely the free variables in ϕ, and will denote $P^n(L)$ as the set of all L-formulae with exactly n free variables.

Example 1.2.4. The formula in Example 1.2.3 has x as a bound variable and y as a free variable. By bounding y by a quantifier we obtain a sentence, for example

$$(\exists x)(\forall y)\,[(x \cdot x = x) \wedge (x \cdot y = y \cdot x)].$$

Any L-formula $\phi(x_1, \ldots, x_n)$ has a natural interpretation in an L-structure $\mathcal{M}$, and can be seen to express a property of elements of M^n. We can therefore introduce the notion of the *truth* of an L-formula in an L-structure. Much like the definition of a formula, the truth of $\phi(a_1, \ldots, a_n)$ in $\mathcal{M}$ for some $(a_1, \ldots, a_n) \in M^n$ is defined inductively in a natural way, and denoted $\mathcal{M} \models \phi(a_1, \ldots, a_n)$. We will say that $\underline{a} \in M^n$ *has first order property* $\phi(x_1, \ldots, x_n)$ if $\mathcal{M} \models \phi(\underline{a})$.

An L-sentence can be interpreted as either true or false in an L-structure. We call a property of $\mathcal{M}$ *first order* if it can be written as an L-sentence ϕ such that $\mathcal{M} \models \phi$.

Example 1.2.5. In the signature L_S, the property of commutativity is first order, and is defined by the sentence

$$\phi_{\text{comm}} := (\forall x)(\forall y)\,[x \cdot y = y \cdot x].$$

For example, $(\mathbb{Z}, \times) \models \phi_{\text{comm}}$.

Example 1.2.6. The formula in L_S,

$$\phi(y) := (\forall x)\,[x \cdot y = y \cdot x]$$

can be interpreted as the property of central elements, that is, elements which commute with all other elements. For example, in a group G with identity 1 we have $(G, \cdot) \models \phi(1)$, and if G is abelian then $(G, \cdot) \models \phi(g)$ for all $g \in G$.

Example 1.2.7. Both the property of being finite of cardinality n and the property of being infinite, can be expressed using formulae. To see this, let ϕ_n be the sentence given by

$$\phi_n := (\exists x_1) \cdots (\exists x_n) \bigwedge_{i \neq j} \neg[x_i = x_j],$$

for each $n \in \mathbb{N}$. Then we interpret ϕ_n as the property of the universe of a structure having at least n elements. Hence, for any signature L and L-structure $\mathcal{M}$, we have $|M| = n$ if and only if

$$\mathcal{M} \models \phi_n \wedge \neg\phi_{n+1}.$$

On the other hand, M is infinite if and only if $\mathcal{M} \models \phi_n$ for all $n \in \mathbb{N}$.

Not every property of an L-structure can be expressed as an L-formula. Following the example given by Rosenstein [84, p437] in the context of groups, to express that an infinite semigroup is generated by a single element we cannot write

$$(\exists x)(\forall y)(\exists n)[y = x^n]$$

or

$$(\exists x)(\forall y)[y = x^0 \vee y = x^1 \vee y = x^2 \vee \cdots]$$

since these expressions are not L-formulae. Indeed, in the first case a quantifier ranges over $\mathbb{N}$ and not variables, and in the second infinite disjunction occurs. A note of caution should be made, since this does not actually prove that the property of being generated by a single element is not first order. The property is not first order, but a proof would require machinery outside the scope of this thesis.

Definition 1.2.8. Given a signature L, then we define an L-*theory* T to be a set of L-sentences, where T is permitted to be finite or infinite. We say that an L-structure $\mathcal{M}$ *models* T, denoted $\mathcal{M} \models T$, if all sentences in T are true in $\mathcal{M}$, that is $M \models \phi$ for all $\phi \in T$.

A class $\mathcal{K}$ of L-structures is called *axiomatizable* if there exists a theory T such that $\mathcal{K}$ is precisely the class of L-structures which model T. Great progress has been made into axiomatizability of S-acts (see [36] and [37], for example).

Example 1.2.9. The *theory of semigroups*, denoted T_S, in L_S consists of the single sentence

$$(\forall x)(\forall y)(\forall z)\,[x(yz) = (xy)z],$$

which is interpreted as the property of associativity. While an L_S-structure need not be a semigroup, we have that $\mathcal{M} \models T_S$ if and only if $\mathcal{M}$ is a semigroup. That is, the class of all semigroups in the signature L_S is axiomatizable by T_S.

Example 1.2.10. The *theory of linear orders without endpoints* T_{LO}, in the signature $L_P = \{\leq\}$, consists of the following L-sentences:

$$\begin{aligned}
&(\forall x)[(x \leq x)] && \text{(reflexive)},\\
&(\forall x)(\forall y)[x \leq y \wedge y \leq x \to x = y] && \text{(anti-symmetric)},\\
&(\forall x)(\forall y)(\forall z)[(x < y \wedge y < z) \to x < z] && \text{(transitive)},\\
&(\forall x)(\forall y)[x \leq y \vee y \leq x] && \text{(linear order)},\\
&(\forall x)(\exists y)(\exists z)[y < x \wedge x < z] && \text{(no endpoints)}.
\end{aligned}$$

The class of all linear orders without endpoints is axiomatized by T_{LO}. If we add the sentence

$$(\forall x)(\forall y)\big[(x < y) \to [(\exists z)(x < z \wedge z < y)]\big] \tag{1.1}$$

then we have the *theory of dense linear orders without endpoints* T_{DLO}.

Similarly we form the *theory of groups* T_G in the signature L_G, the *theory of graphs* T_{Gr} in the signature L_{Gr}, and the *theory of posets* T_P in the signature L_P, etc.

Definition 1.2.11. Let T be an L-theory and ϕ an L-sentence. We call ϕ a *logical consequence* of T, denoted $T \models \phi$, if $\mathcal{M} \models \phi$ whenever $\mathcal{M} \models T$.

Example 1.2.12. The sentence ϕ_{comm} from Example 1.2.5 is not a logical consequence of T_S, since there exist non-commutative semigroups. However the sentence

$$(\forall x)[x^2 = x \to x^3 = x]$$

is a logical consequence of T_S, since any idempotent of a semigroup is equal to any of its powers.

Given an L-structure $\mathcal{M}$, then the *full theory* of $\mathcal{M}$, denoted $\mathrm{Th}(\mathcal{M})$, is the set of all L-sentences ϕ such that $\mathcal{M} \models \phi$. Notice that for any semigroup $\mathcal{M}$ in the signature L_S,

$$T_S \subseteq \mathrm{Th}(\mathcal{M}),$$

and similarly for T_G, T_{Gr} etc.

Definition 1.2.13. An L-theory T is called *satisfiable* if there exists an L-structure $\mathcal{M}$ such that $\mathcal{M} \models T$. An L-theory T is called *complete* if, for any L-sentence ϕ, either $T \models \phi$ or $T \models \neg\phi$.

Example 1.2.14. The full theory of a structure is complete.

Example 1.2.15. The theory T_S is not complete. For example, since there exist both commutative and non-commutative semigroups, we have $T_S \not\models \phi_{\mathrm{comm}}$ and $T_S \not\models \neg\phi_{\mathrm{comm}}$.

The full theory of a structure is, in general, too difficult to determine. We can often overcome this problem by finding a 'simpler' complete L-theory T such that $\mathcal{M} \models T$. Indeed, let T be a complete theory such that $\mathcal{M} \models T$. Then for any L-sentence ϕ, if $T \not\models \phi$ then $T \models \neg\phi$ by completeness, and so $\mathcal{M} \models \neg\phi$. It follows that $\mathcal{M} \models \phi$ if and only if $T \models \phi$.

Definition 1.2.16. A pair of L-structures $\mathcal{M}$ and $\mathcal{N}$ are *elementary equivalent*, denoted $\mathcal{M} \equiv \mathcal{N}$, if they satisfy the same L-sentences, that is $\mathrm{Th}(\mathcal{M})=\mathrm{Th}(\mathcal{N})$.

We observe that a pair of isomorphic structures are clearly elementary equivalent, but the converse need not be true.

Example 1.2.17. In the signature of posets $L_P = \{\leq\}$ we have that $(\mathbb{Q}, \leq)$ and $(\mathbb{Z}, \leq)$ are not elementary equivalent, since the property of a poset being dense is first order by (1.1).

The following result is immediate from the definitions given above, and provides a useful test for the completeness of a theory.

Theorem 1.2.18. *Let T be a satisfiable theory. Then T is complete if and only if for each $\mathcal{M}, \mathcal{N} \models T$ we have $\mathcal{M} \equiv \mathcal{N}$.*

Example 1.2.19. The theory of dense linear orders without endpoints is complete [69, Theorem 2.4.1], and so $(\mathbb{Q}, <) \equiv (\mathbb{R}, <)$ by the theorem above.

One of the most fundamental questions in model theory is the following:

> Given a satisfiable theory T, how many countable models of T exist, up to isomorphism?

In [98], Vaught studied this question for complete theories, and showed a somewhat surprising result that, assuming that the Continuum Hypothesis is true, no complete theory has precisely 2 non-isomorphic countable models. On the other hand, for any $n \in \mathbb{N} \setminus \{2\}$, there exist complete theories with precisely n non-isomorphic countable models. The case when $n = 1$ is of particular interest, and studied in greater detail in Chapter 3:

Definition 1.2.20. An L-theory T is *$\aleph_0$-categorical* if all countable models of T are isomorphic. A countable L-structure $\mathcal{M}$ is *$\aleph_0$-categorical* if $\mathrm{Th}(\mathcal{M})$ is an $\aleph_0$-categorical theory.

Hence if $\mathcal{M}$ is an $\aleph_0$-categorical L-structure then for every countable L-structure $\mathcal{N}$ we have

$$\mathcal{N} \equiv \mathcal{M} \Leftrightarrow \mathcal{N} \cong \mathcal{M}.$$

1.2.1 Definable sets

Let $\mathcal{M}$ be an L-structure and let A be a fixed subset of M^n for some $n \in \mathbb{N}$. Then we call A *definable* if there exists an L-formula $\phi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ and $\underline{b} \in M^m$ such that

$$A = \{\underline{a} \in M^n : \mathcal{M} \models \phi(\underline{a}, \underline{b}).$$

We say that $\phi(x_1, \ldots, x_n, \underline{b})$ *defines* A. Given a fixed subset X of M, then A is called *X-definable* or *definable with parameters from X* if there exists an L-formula $\psi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ and $\underline{b} \in X^m$ such that $\psi(x_1, \ldots, x_n, \underline{b})$ defines A.

Example 1.2.21. Let $(S, \cdot)$ be a semigroup in L_S. Then the set of all central elements of S is defined by the formula $\phi(y) = (\forall x)\,[xy = yx]$ as

$$\{a \in S : as = sa \text{ for all } s \in S\} = \{a \in S : (S, \cdot) \models \phi(a)\}.$$

In a later chapter we will show that the property of a structure being $\aleph_0$-categorical translates to a property of its automorphism group. This is a common occurrence in model theory, and is further highlighted in the following method for proving that a subset is not definable.

Given an automorphism ϕ of a structure $\mathcal{M}$ with subset A, then we say that ϕ *fixes A pointwise* if $a\phi = a$ for all A. We say that ϕ *fixes A setwise* if $A\phi = A$.

Proposition 1.2.22. [69, Proposition 1.3.5] *Let $\mathcal{M}$ be an L-structure and let A be an X-definable subset of M^n for some $n \in \mathbb{N}$. Then any automorphism ϕ of M which fixes X pointwise fixes A setwise.*

On the class of $\aleph_0$-categorical structures we have a partial converse to the proposition above:

Proposition 1.2.23. [73] *Let $\mathcal{M}$ be an $\aleph_0$-categorical L-structure, X a finite subset of M and $A \subseteq M^n$ for some $n \in \mathbb{N}$. Then A is X-definable if and only if any automorphism of $\mathcal{M}$ which fixes X pointwise fixes A setwise.*

1.2.2 Quantifier elimination

Quantifier-free formulae are, in most cases, far simpler to work with than formulae with quantifiers. For example, for any L-formula ϕ and variable x not given in ϕ, we have $\mathcal{M} \models \phi$ if and only if $\mathcal{M} \models (\forall x)\phi$ for all L-structures $\mathcal{M}$. It therefore makes more sense to work with ϕ rather than $(\forall x)\phi$.

Similarly, sets defined by quantifier-free formulae tend to be simpler to describe. It is therefore often useful to find, if possible, a quantifier-free formula which is 'equivalent' to our given formula. A classical example in the language of fields $L = \{+, -, \cdot, 0, 1\}$, where $+, -$ and $\cdot$ are binary function symbols and 0 and 1 are constants, is the quadratic solution formula $\phi(a, b, c)$ given by

$$(\exists x)[ax^2 + bx + c = 0].$$

Then in the field of complex numbers we have

$$(\mathbb{C}, +, -, \cdot, 0, 1) \models \phi(a, b, c) \leftrightarrow [(\neg a = 0) \vee (\neg b = 0) \vee (c = 0)].$$

Definition 1.2.24. A theory T has *quantifier elimination* if, for every formula ϕ, there exists a quantifier-free formula ψ such that

$$T \models \phi \leftrightarrow \psi.$$

Example 1.2.25. The theory of dense linear orders without endpoints T_{DLO} has quantifier elimination. Every formula is equivalent to a formula built in following way. Let $\sigma : \{(i, j) : 1 \leq i, j \leq n\} \to 3$, and $\phi_\sigma(x_1, \ldots, x_n)$ be the formula given by

$$\bigwedge_{\sigma(i,j)=0} x_i = x_j \wedge \bigwedge_{\sigma(i,j)=1} [x_i \leq x_j \wedge \neg x_i = x_j] \wedge \bigwedge_{\sigma(i,j)=2} [x_j \leq x_i \wedge \neg x_i = x_j].$$

Although quantifier elimination is in no way central to this thesis, it serves both as a model theoretic motivation for studying $\aleph_0$-categoricity and homogeneity, and gives a vital link between the two concepts:

Theorem 1.2.26. [51] *Let L be a finite signature and $\mathcal{M}$ a countable L-structure. Then the following are equivalent:*

(i) *$\mathcal{M}$ is homogeneous and ULF;*

(ii) $\mathcal{M}$ *is* $\aleph_0$*-categorical and has quantifier elimination.*

An immediate consequence is that if $\mathcal{M}$ is a homogeneous relational structure with finite signature L, then it is $\aleph_0$-categorical.

We end this chapter by fixing the following notation.

Notation 1.2.27. In a given section of this thesis, we will predominantly be working in a fixed signature. Where no confusion may arise, the prefix L will then be dropped from the concepts introduced in this section. For example, we write *structure* instead of L-structure, *formula* instead of L-formula and *theory* instead of L-theory.

Chapter 2

Preliminaries II: An introduction to semigroup theory

In this section we outline the basic semigroup theory required in this thesis. The definitions and results are taken from the standard books on introductory semigroup theory: [20], [21], [55] and [72]. Here we mostly study semigroups as L_S-structures. Concepts such as morphisms and subsemigroups then follow by the general definitions given in the previous chapter, but are given here due to their importance in this thesis.

2.1 Monoids and zeros

A *semigroup* $(S, \cdot)$ is a non-empty set S together with an associative binary operation $\cdot$ defined on S, so that if $x, y, z \in S$ then

$$(x \cdot y) \cdot z = x \cdot (y \cdot z).$$

We follow the usual convention of denoting the product $x \cdot y$ by juxtaposition xy. We say that S is *commutative* if $xy = yx$ for all $x, y \in S$. An element u of S is called a *left identity* if $ua = a$ for each $a \in S$, and called a *right identity* if $au = a$ for each $a \in S$. A left and right identity is called an *identity*, and it is unique (if it exists). A semigroup with an identity is called a *monoid*. Dually, an element 0 of S is called a *left (right) zero* of S if $0a = 0$ ($a0 = 0$) for each $a \in S$. A left and right zero is called a *zero*, and it is unique (if it exists). If S contains a zero then we call S a *semigroup with zero*.

If S is not a monoid, then we can adjoin an identity 1 to S to form a monoid. That is, we take some $1 \notin S$ and extend the binary operation on S to $S \cup \{1\}$ by defining $1x = x1 = x$ for all $x \in S^1$. We then define

$$S^1 = \begin{cases} S & \text{if } S \text{ has an identity element,} \\ S \cup \{1\} & \text{else.} \end{cases}$$

We call S^1 the *monoid obtained from S by adjoining an identity if necessary.*

Similarly, if S does not contain a zero, then we adjoin a zero 0 to S to form a semigroup with zero, and take

$$S^0 = \begin{cases} S & \text{if } S \text{ has a zero,} \\ S \cup \{0\} & \text{else.} \end{cases}$$

We call S^0 the *semigroup obtained from S by adjoining a zero if necessary.*

Example 2.1.1. A trivial semigroup $\{e\}$ is a semigroup with cardinality one, so that $e^2 = e$. In this case e is both a zero and an identity element.

Example 2.1.2. A *null* semigroup is a semigroup in which the product of any pair of elements is zero. That is, a semigroup N with zero is null if $xy = 0$ for each $x, y \in N$.

If A and B are subsets of a semigroup S, then we define the product of A and B in the natural way as $AB = \{ab : a \in A, b \in B\}$. For singleton sets we simplify our notation by writing, for example, aB instead of $\{a\}B$. For $a \in S$ we then have three key subsets of S given by

(i) $S^1a = Sa \cup \{a\}$;

(ii) $aS^1 = aS \cup \{a\}$;

(iii) $S^1aS^1 = SaS \cup Sa \cup aS \cup \{a\}$,

where the multiplication of subsets is taking place inside S^1.

A non-empty subset T of S is called a *subsemigroup* if it is closed under the operation of S, that is, if $xy \in T$ for each $x, y \in T$. If T also forms a group under the restriction of the operation of S to T, then T is called a *subgroup* of S. A subsemigroup T of S is called a *left ideal* if $ST \subseteq T$, a *right ideal* if $TS \subseteq T$ and a (two-sided) *ideal* if it is both a left and right ideal. For example, S forms an ideal of itself, and if S contains a zero then $\{0\}$ is an ideal. An ideal T such that T is non-zero and $T \subset S$ is called a *proper ideal.*

Example 2.1.3. For any $a \in S$, the sets S^1a, aS^1 and S^1aS^1 are left, right and two-sided ideals of S, respectively, which we call the *principal left, right and two-sided ideals of S generated by a.* They are, respectively, the smallest left, right and two-sided ideals containing a.

An element e of S is called an *idempotent* if $e^2 = e$, and we denote the set of idempotents of S as $E(S)$. We observe that $\{e\}$ is a trivial subsemigroup (indeed, subgroup) of S for any $e \in E(S)$. The set $E(S)$ comes equipped with a partial order $\leq$ defined by

$$e \leq f \text{ if and only if } ef = fe = e.$$

We call $\leq$ the *natural order* on $E(S)$.

If $E(S) = S$ then we call S a *band*. A commutative band is called an (algebraic) *semilattice*, and the natural order simplifies to

$$e \leq f \text{ if and only if } ef = e.$$

For any non-empty subset A of S, the intersection of all subsemigroups of S containing A forms a subsemigroup of S, which we simply denote as $\langle A \rangle$, called the *subsemigroup of S generated by A*. For example, if $A = \{a\}$ is a singleton set, then

$$\langle a \rangle = \{a, a^2, a^3, \dots\},$$

which we call the *monogenic subsemigroup of S generated by a*. If S is a monoid, then by working instead in the signature of monoids L_{Mo}, the *submonoid* of S generated by a non-empty set A is defined as the intersection of all the submonoids of S containing A, and denoted by $\langle A \rangle_{Mo}$. For example, if S is a monoid and $a \in S$ then

$$\langle a \rangle_{Mo} = \{1, a, a^2, a^3, \dots\}.$$

We define the *order* of an element a of a semigroup S as the cardinality of $\langle a \rangle$. Note that, even if S is a monoid, our definition of order relies on $\langle a \rangle$ not $\langle a \rangle_{Mo}$. If a has finite order then the sequence $(a^n)_{n \in \mathbb{N}}$ contains repetitions, and we can define the *index* of a, say m, to be the least element of

$$\{x \in \mathbb{N} : (\exists y \in \mathbb{N})\, a^x = a^y, x \neq y\}.$$

It then follows that the set

$$\{x \in \mathbb{N} : a^{m+x} = a^m\}$$

is non-empty, and has a least element r, known as the *period* of a.

Lemma 2.1.4. [55, Theorem 1.2.2] *Let a be an element of a semigroup S of index m and period r, so that $a^{m+r} = a^m$. Then:*

(i) $a^{m+qr} = a^m$ *for all* $q \in \mathbb{N}$*;*

(ii) $\langle a \rangle = \{a, a^2, \dots, a^{m+r-1}\}$ *and the order of a is* $m + r - 1$*;*

(iii) *the subset* $K_a = \{a^m, \ldots, a^{m+r-1}\}$ *of* $\langle a \rangle$ *is a cyclic group of order* r. *In particular,* K_a *contains an idempotent of* S.

On the other hand, if $a \in S$ has infinite order then there are no repetitions in $a, a^2, a^3, \ldots$, and it follows that $\langle a \rangle$ is isomorphic to the semigroup $(\mathbb{N}, +)$ of natural numbers under addition.

A semigroup in which all elements have finite order is called *periodic*, otherwise it is called *non-periodic*.

2.2 Morphisms, congruences and direct products

By applying Definition 1.1.7 to the signature L_S, we obtain the concept of morphisms between semigroups as follows. Given a pair of semigroups S and T, a map $\phi : S \to T$ is a morphism if it preserves the operation on S, that is, if

$$(\forall x, y \in S) \quad (x\phi)(y\phi) = (xy)\phi.$$

Note that if S is a semigroup and ϕ is an endomorphism of S then we can extend ϕ to an endomorphism of S^1, simply by fixing 1. That is, take the map $\phi' : S^1 \to S^1$ given by $\phi'|_S = \phi$ and $1\phi = 1$. Dually we may extend endomorphisms of S to endomorphisms of S^0.

A relation σ on a semigroup S is *left compatible* (with the operation on S) if

$$(\forall x, y, z \in S) \quad (x, y) \in \sigma \Rightarrow (zx, zy) \in \sigma.$$

Dually, σ is *right compatible* if

$$(\forall x, y, z \in S) \quad (x, y) \in \sigma \Rightarrow (xz, yz) \in \sigma,$$

and is called *compatible* if

$$(\forall x, y, x', y' \in S) \quad [(x, y) \in \sigma \text{ and } (x', y') \in \sigma] \Rightarrow (xx', yy') \in \sigma.$$

An equivalence relation which is (left/right) compatible is called a *(right/left) congruence*. Equivalently, an equivalence relation ρ is a congruence if and only if it is a left and a right congruence.

Let σ be a relation on a set X. Then the intersection of all congruences on X containing σ is a congruence, denoted $\sigma^\sharp$, and is the unique minimum congruence on X containing σ. We call $\sigma^\sharp$ the congruence *generated by* σ. We say that a congruence ρ on a semigroup is *finitely generated* (f.g.) if there exits a finite relation σ such that $\rho = \sigma^\sharp$.

There exists a particularly useful description of the congruence $\sigma^\sharp$, which we now examine. Let σ be a relation on a semigroup S. If $x, y \in S$ are such that

$$x = uav, \quad y = ubv,$$

for some $u, v \in S^1$, where either $(a, b) \in \sigma$ or $(b, a) \in \sigma$, then we say that x is connected to y by an *elementary σ-transition*, denoted $x \to y$.

Proposition 2.2.1. [55, Proposition 1.5.9] *Let σ be a relation on a semigroup S, and let $x, y \in S$. Then $(x, y) \in \sigma^\sharp$ if and only if either $x = y$ or, for some $n \in \mathbb{N}$, there is a sequence*

$$x = z_1 \to z_2 \to \cdots \to z_n = y$$

of elementary σ-transitions connecting x to y.

If ρ is a congruence on a semigroup S then the binary operation on the set of equivalence classes S/ρ given by

$$(a\rho)(b\rho) = (ab)\rho.$$

is well-defined and associative [55, Section 1.5]. Hence S/ρ forms a semigroup, called a *quotient semigroup*. Clearly if S is a monoid, then so is S/ρ.

Given a proper ideal I of a semigroup S, we define a relation ρ_I on S by

$$a \,\rho_I\, b \Leftrightarrow \text{ either } a, b \in I \text{ or } a = b.$$

Then ρ_I forms a congruence, called the *Rees congruence on S modulo I*, and S/ρ_I is called a *Rees factor semigroup*. Moreover, S/ρ_I is a semigroup with zero element I, and can be regarded as consisting of I together with the elements of $S \setminus I$ with product $*$ given by

$$s * t = \begin{cases} st & \text{if } s, t, st \in S \setminus I, \\ I & \text{else.} \end{cases}$$

We write S/I instead of S/ρ_I where no confusion can arise.

We apply our general definition of direct product of structures to L_S as follows. Given a pair of semigroups S_1 and S_2, then the direct product of S_1 and S_2 is the set $S_1 \times S_2$ together with the (associative) operation

$$(s, t)(s', t') = (ss', tt'),$$

so that $S_1 \times S_2$ forms a semigroup.

If $M = S \times T$ and $M' = S' \times T'$ are a pair of direct products of semigroups, and

$\phi : S \to S'$ and $\psi : T \to T'$ are morphisms, then the map $\Phi : M \to M'$ given by

$$(s,t)\Phi = (s\phi, t\psi)$$

is a morphism, which we denote as $\phi \times \psi$. Clearly if ϕ and ψ are injective/surjective then so is $\phi \times \psi$.

We end the section by describing a second type of product of semigroups: spined product. Let P and Q be a pair of semigroups and $\theta : P \to M$ and $\phi : Q \to M$ a pair of surjective morphisms. Following [7], we define the *spined product of P and Q w.r.t. M* to be the subset

$$P \bowtie_{M,\theta,\phi} Q = \{(a,b) : a \in P, b \in Q, a\theta = b\phi\}$$

of $P \times Q$. The spined product $P \bowtie_{M,\theta,\phi} Q$ forms a subsemigroup of $P \times Q$.

2.3 Posets and semilattices

Let $(X, \leq)$ be a poset. Then, given a subset Y of X, we call an element x of X a *lower bound* of Y if $x \leq Y$. If the set of lower bounds of Y is non-empty and has a maximum element y then we call y the *meet* of Y. If y exists, it is unique, and we denote it as

$$y = \bigwedge Y.$$

If $Y = \{a, b\}$ then we simply write $y = a \wedge b$.

A *lower semilattice* is a poset in which the meet of any pair of elements exists. If $(Y, \leq)$ is a lower semilattice, then it is easily verifiable that $(Y, \wedge)$ forms a semigroup. Furthermore, since $x \wedge x = x$ and $x \wedge y = y \wedge x$ for all $x, y \in Y$, the semigroup $(Y, \wedge)$ is an (algebraic) semilattice. We have proven the first half of the following proposition.

Proposition 2.3.1. [55, Proposition 1.3.2] *Let $(Y, \leq)$ be a lower semilattice. Then $(Y, \wedge)$ is an algebraic semilattice and*

$$(\forall x, y \in Y) \quad x \leq y \text{ if and only if } x \wedge y = x.$$

Conversely, if Y is an algebraic semilattice with natural order $\leq$, then $(Y, \leq)$ is a lower semilattice, where $x \wedge y = xy$.

A lower semilattice $(Y, \leq)$ can alternatively be considered as the structure

$$(Y, \leq, \wedge)$$

in the signature $L_{LS} = \{\leq, \wedge\}$, where $\leq$ is a binary relation symbol, interpreted as the natural order, and $\wedge$ is a binary function symbol, interpreted as the meet. A consequence of Proposition 2.3.1 is that, for many of the model theoretic notions introduced in this thesis, it makes no difference to consider lower semilattices on L_S or L_{LS}.

Every linearly ordered set is a semilattice, where the meet function $\wedge$ is the minimum of two elements. That is, every linear order $(P, \leq)$ in the signature of posets L_P may be regarded instead as a semilattice $(P, \leq, \wedge)$ in L_{LS}.

2.4 Green's relations and regular elements

We now introduce a collection of relations on an arbitrary semigroup S, known as *Green's relations*. Introduced by Green in [41], they describe the ideal structure of S, and as such are fundamental to the study of semigroups.

Define the binary relations $\leq_l, \leq_r$ and $\leq_j$ on S by

$$\begin{aligned} a \leq_l b &\quad \text{if and only if} \quad S^1 a \subseteq S^1 b, \\ a \leq_r b &\quad \text{if and only if} \quad aS^1 \subseteq bS^1, \\ a \leq_j b &\quad \text{if and only if} \quad S^1 a S^1 \subseteq S^1 b S^1. \end{aligned}$$

The relations $\leq_l, \leq_r$ and $\leq_j$ are called the *Green's left, left and two-sided orders*, respectively. Each relation is a reflective and transitive binary relation, which is known as a *quasi-order*. Moreover, $\leq_l / \leq_r$ is right/left compatible with the operation on S. This follows immediately from the fact that

$$L_{xa} \leq L_a \quad \text{and} \quad R_{ax} \leq R_s \tag{2.1}$$

for any $a \in S$ and $x, y \in S^1$.

The five Green's relations are then given by:

(i) $a \,\mathcal{L}\, b$ if and only if $S^1 a = S^1 b$;

(ii) $a\mathcal{R}b$ if and only if $aS^1 = bS^1$;

(iii) $a \,\mathcal{H}\, b$ if and only if $a \,\mathcal{R}\, b$ and $a \,\mathcal{L}\, b$;

(iv) $a \,\mathcal{D}\, b$ if and only if $\exists c$ such that $a \,\mathcal{R}\, c \,\mathcal{L}\, b$;

(v) $a \,\mathcal{J}\, b$ if and only if $S^1 a S^1 = S^1 b S^1$.

Each relation is an equivalence relation, with $\mathcal{L}, \mathcal{R}$ and $\mathcal{J}$ being the corresponding equivalence relations associated with $\leq_l$,$\leq_r$ and $\leq_j$, respectively. Note that $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ and $\mathcal{D}$ is the least equivalence relation containing $\mathcal{L}$ and $\mathcal{R}$. Moreover, $\mathcal{L}$ and $\mathcal{R}$ commute, with

$$\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}.$$

It is clear that $\mathcal{L} \subseteq \mathcal{J}$ and $\mathcal{R} \subseteq \mathcal{J}$. Hence as $\mathcal{D}$ is the smallest equivalence relation containing $\mathcal{L}$ and $\mathcal{R}$ we have $\mathcal{D} \subseteq \mathcal{J}$. Figure 2.4 shows the corresponding *Hasse diagram.*

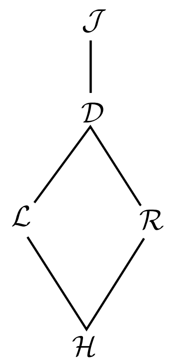

Figure 2.1: Hasse diagram of Green's relations.

We call the equivalence classes of the Green's relations the $\mathcal{L}-, \mathcal{R}-$ etc classes of S. For each $a \in S$ we let L_a denote the $\mathcal{L}$-class containing a, and similarly for R_a, H_a, D_a, J_a. The three Green's orders induce partial orders on the set of equivalence classes of their corresponding Green's relation as follows.

$$L_a \leq L_b \quad \text{if and only if} \quad a \leq_l b,$$
$$R_a \leq R_b \quad \text{if and only if} \quad a \leq_r b,$$
$$J_a \leq J_b \quad \text{if and only if} \quad a \leq_j b.$$

The $\mathcal{D}$-classes of S are a union of the $\mathcal{L}$-classes, and also a union of $\mathcal{R}$-classes. By the definition of the relation $\mathcal{D}$ we have

$$a \, \mathcal{D} \, b \Leftrightarrow L_a \cap R_b \neq \emptyset \Leftrightarrow R_a \cap L_b \neq \emptyset.$$

It therefore pays to visualize a $\mathcal{D}$-class as an 'eggbox'- a term coined by Clifford and Preston. An eggbox is a grid in which each column represents an $\mathcal{L}$-class, each row represents a $\mathcal{R}$-class, and each cell represents a $\mathcal{H}$-class.

The following pair of results on the $\mathcal{H}$-classes of S contained in the same eggbox are vital to the study of arbitrary $\mathcal{D}$-classes of semigroups.

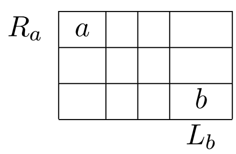

Figure 2.2: $\mathcal{D}$-class.

Lemma 2.4.1. [55, Lemma 2.2.3] *Let a, b be $\mathcal{D}$-equivalent elements in a semigroup S. Then $|H_a| = |H_b|$.*

Theorem 2.4.2 (The Maximal Subgroup Theorem)**.** [55, Corollary 2.2.6] *If e is an idempotent of a semigroup S then H_e is a group with identity element e. No $\mathcal{H}$-class contains more than one idempotent.*

An element a of a semigroup S is called *regular* if there exists b in S such that $a = aba$. A semigroup is called *regular* if each of its elements is regular. Note that if $a = aba$ then ab and ba are idempotent as

$$(ab)(ab) = (aba)b = ab, \quad (ba)(ba) = b(aba) = ba.$$

Example 2.4.3. Every group is regular as $gg^{-1}g = g$. Every band is regular since $eee = e$ for any idempotent e.

If a is a regular element, then it can be shown that every element in D_a is regular, and so we may speak of *regular $\mathcal{D}$-classes* without ambiguity. In particular, it follows from Example 2.4.3 that every $\mathcal{D}$-class containing an idempotent is regular.

Proposition 2.4.4. [55, Propositions 2.3.2, 2.3.3] *In a regular $\mathcal{D}$-class, each $\mathcal{L}$-class and each $\mathcal{R}$-class contains an idempotent. Moreover, every idempotent e is a right identity for L_e and a left identity for R_e.*

If a is an element of a semigroup S, then we say that a' is an *inverse* of a if

$$a = aa'a, \quad a' = a'aa'.$$

Of course every element with an inverse is regular. Conversely, regular elements possess inverses, since if $a = aba$ is regular then $a' = bab$ can be shown to be an inverse of a. For each $a \in S$ we let $V(a)$ denote the set of inverses of a.

Example 2.4.5. If S is a group then $V(a) = \{a^{-1}\}$ for any $a \in S$. If N is a null semigroup, then 0 is regular since it is an idempotent, and if $a \in N \setminus \{0\}$ then $aba = 0$ for any $b \in S$. It follows that $V(a) = \emptyset$ for any $a \in N \setminus \{0\}$ and $V(0) = \{0\}$.

The following theorem gives a useful method for locating inverses of elements in a semigroup.

Theorem 2.4.6. [55, Theorem 2.3.4] *Let a be an element of a semigroup S contained in a regular $\mathcal{D}$ class D.*

(i) *If $a' \in V(a)$, then $a' \in D$ and the $\mathcal{H}$-classes $R_a \cap L_{a'}$ and $L_a \cap R_{a'}$ contain the idempotents aa' and $a'a$, respectively.*

(ii) *If $b \in D$ is such that $R_a \cap L_b$ and $L_a \cap R_b$ contain idempotents e, f, respectively, then H_b contains an inverse a'' of a such that $aa'' = e$ and $a''a = f$.*

(iii) *No $\mathcal{H}$-class contains more than one inverse of a.*

We end this section by giving a pair of results which are of considerable use in our later work. They dictate how the idempotents and the maximal subgroups of a semigroup behave within a $\mathcal{D}$-class.

Proposition 2.4.7. [55, Proposition 2.3.5] *Let e, f be a pair of idempotents in a semigroup S. Then $e \mathcal{D} f$ if and only if there exists $a \in S$ and $a' \in V(a)$ such that*

$$aa' = e, \quad a'a = f.$$

Proposition 2.4.8. [55, Proposition 2.3.6] *If H and H' are a pair of group $\mathcal{H}$-classes in the same $\mathcal{D}$-class then $H \cong H'$.*

2.5 0-simple semigroups and principal factors

A semigroup is called *simple* if it has no proper ideals. This is equivalent to a semigroup having a single $\mathcal{J}$-class. A simple semigroup is *completely simple* if it contains an idempotent which is minimal within the set of idempotents $E(S)$ of S under the natural order. That is, if it contains an idempotent e such that

$$(\forall f \in E(S)) \quad ef = fe = f \Rightarrow f = e.$$

A semigroup with a single $\mathcal{D}$-class is called *bisimple*. Clearly every bisimple semigroup is simple.

A semigroup with zero S is called *0-simple* if $\{0\}$ and S are its only ideals and $S^2 \neq \{0\}$. This is equivalent to S not being a null semigroup and $\{0\}$ and $S \setminus \{0\}$ being its only $\mathcal{J}$-classes. A 0-simple semigroup is called *completely 0-simple* if it contains an idempotent which is minimal within the set of non-zero idempotents. That is, if it contains an idempotent e such that

$$(\forall f \in E(S)) \quad ef = fe = f \neq 0 \Rightarrow e = f.$$

We call such an idempotent *primitive*. It is known that a finite 0-simple semigroup is completely 0-simple, and every completely (0-)simple semigroup is regular.

We now describe a well known decomposition theorem of an arbitrary semigroup, which highlights the importance of 0-simple and simple semigroups to the theory of semigroups. For each element a of a semigroup S, let $J(a) = S^1 a S^1$ and consider the set

$$I(a) = J(a) \setminus J_a.$$

If $I(a)$ is empty, then $J(a) = J_a$ is the unique minimal ideal of S, which we call the *kernel of* S, and is denoted $K(S)$. Note that such an ideal may not exist. If $I(a)$

is non-empty then it forms an ideal of S, and thus also of $J(a)$. The Rees factor semigroups $J(a)/I(a)$ ($a \in S$) and $K(S)$ are called the *principal factors* of S.

Theorem 2.5.1. [20, Lemma 2.39] *The principal factors of a semigroup are either 0-simple, simple or null. The only simple principal factor is the kernel, if it exists.*

Of course, for Theorem 2.5.1 to be of use we require a deeper understanding of (0-)simple semigroups. The most famous breakthrough in this area came in 1940 by Rees [78], where a recipe for constructing all completely 0-simple semigroups was given. This result is commonly known as the *Rees Theorem*, which we now state.

Theorem 2.5.2 (The Rees Theorem)**.** *Let G be a group, let I and Λ be non-empty index sets and let $P = (p_{\lambda,i})$ be an $\Lambda \times I$ matrix with entries in $G \cup \{0\}$. Suppose no row or column of P consists entirely of zeros (that is, P is regular). Let $S = (I \times G \times \Lambda) \cup \{0\}$, and define multiplication $*$ on S by*

$$(i, g, \lambda) * (j, h, \mu) = \begin{cases} (i, gp_{\lambda,j}h, \mu) & \text{if } p_{\lambda,j} \neq 0 \\ 0 & \text{else} \end{cases}$$
$$0 * (i, g, \lambda) = (i, g, \lambda) * 0 = 0 * 0 = 0.$$

Then S is a completely 0-simple semigroup, denoted $\mathcal{M}^0[G; I, \Lambda; P]$, and is called a (regular) Rees matrix semigroup (over G). Conversely, every completely 0-simple semigroup is isomorphic to a Rees matrix semigroup.

The matrix P is called the *sandwich matrix* of S. The regularity of the matrix P ensures that S forms a regular semigroup.

The strength in the Rees Theorem is that it permits a relatively simple isomorphism theorem, which follows from [55, Section 3.4].

Theorem 2.5.3. *Let $S_1 = \mathcal{M}^0[G_1; I_1, \Lambda_1; P]$ and $S_2 = \mathcal{M}^0[G_2; I_2, \Lambda_2; Q]$ be a pair of Rees matrix semigroups where $P = (p_{\lambda,i})$ and $Q = (q_{\mu,j})$. Let $\psi_I : I_1 \to I_2$ and $\psi_\Lambda : \Lambda_1 \to \Lambda_2$ be bijections, let $\theta : G_1 \to G_2$ be an isomorphism and let u_i and v_λ be elements of G_2 for each $i \in I_1, \lambda \in \Lambda_1$. Then the mapping $\phi : S_1 \to S_2$ given by*

$$(i, g, \lambda)\phi = (i\psi_I, u_i \cdot (g\theta) \cdot v_\lambda, \lambda\psi_\Lambda)$$

is an isomorphism if and only if

$$p_{\lambda,i}\,\theta = v_\lambda \cdot q_{\lambda\psi,i\psi} \cdot u_i.$$

Moreover, every isomorphism from S_1 to S_2 can be described in this way.

Consequently, if $S = \mathcal{M}^0[G; I, \Lambda; P]$ is a Rees matrix semigroup then the rows and columns of P can be permuted to obtain an isomorphic Rees matrix semigroup. Formally, let ψ_I and ψ_Λ be bijections of I and Λ, respectively, and let $Q = (q_{\lambda,i})$ be the $\Lambda \times I$ matrix with $q_{\lambda,i} = p_{\lambda\psi_\Lambda^{-1}, i\psi_I^{-1}}$ for each $i \in I, \lambda \in \Lambda$. Let $u_i = 1 = v_\lambda$ for each $i \in I, \lambda \in \Lambda$ and let θ be the identity automorphism of G. Then the mapping $\phi : S \to \mathcal{M}^0[G; I, \Lambda; Q]$ given by

$$(i, g, \lambda)\phi = (i\psi_I, u_i(g\theta)v_\lambda, \lambda\psi_\Lambda) = (i\psi_I, g, \lambda\psi_\Lambda)$$

is an isomorphism by Theorem 2.5.3 since $p_{\lambda,i}\theta = p_{\lambda,i} = v_\lambda q_{\lambda\psi_\Lambda, i\psi_I} u_i$.

A non-trivial completely simple semigroup may be regarded as a completely 0-simple semigroup simply by adjoining a zero. However, it is worth noting that not every completely 0-simple semigroups arises in this way, as the Rees Theorem indicates. A simplified Rees Theorem for completely simple semigroups then follows:

Theorem 2.5.4. *Let G be a group, let I and Λ be non-empty index sets and let $P = (p_{\lambda,i})$ be an $\Lambda \times I$ matrix with entries in G. Let $S = I \times G \times \Lambda$, and define multiplication $*$ on S by*

$$(i, g, \lambda) * (j, h, \mu) = (i, gp_{\lambda,j}h, \mu)$$

Then S is a completely simple semigroup, denoted $\mathcal{M}[G; I, \Lambda; P]$. Conversely, every completely simple semigroup is isomorphic to a semigroup constructed in this way.

Notice that by adjoining a zero to a completely simple semigroup $\mathcal{M}[G; I, \Lambda; P]$ we may apply a suitably simplified version of Theorem 2.5.3.

2.6 Semigroup and monoid presentations

Let A be a non-empty set. Let A^+ be the set of all finite, non-empty words $a_1a_2\cdots a_n$ formed from the *alphabet* A. Then A^+ forms a semigroup with respect to the binary operation of juxtaposition of words

$$(a_1a_2\cdots a_n)(b_1b_2\cdots b_m) = a_1a_2\cdots a_nb_1b_2\cdots b_m,$$

called the *free semigroup* on A. The set A is the unique minimal generating set of A^+. By adjoining an identity 1 to the free semigroup A^+, we attain the *free monoid*, denoted A^*. The element 1 corresponds to the empty product of elements of A.

If S is a semigroup generated by a set A, then there exists a congruence ρ on A^+ such that S is isomorphic to A^+/ρ. If A is finite and there exists a finite set

$$R = \{(u_1, v_1), \ldots, (u_r, v_r)\} \in A^+ \times A^+$$

such that $S \cong A^+/R^\sharp$ then S is called *finitely presented*, and that it has *finite* *presentation*

$$\langle A : u_1 = v_1, \ldots, u_r = v_r \rangle.$$

We define a *finitely presented monoid* by replacing A^+ with A^* in the definition above.

Example 2.6.1. An example paramount to our study is the *bicyclic monoid* B, a finitely presented monoid presented by

$$B = \langle p, q : pq = 1 \rangle_{Mo},$$

so that $B = \{p, q\}^*/R^\sharp$ where $R = \{(pq, 1)\}$. Note that B is simple, regular (indeed inverse - see later) but not completely simple as its idempotents form an infinite descending chain, and thus no principal idempotents exist.

2.7 Semilattices of semigroups

In this section we introduce the most important semigroup construction in this thesis, and certainly the most used. Let Y be a semilattice. A semigroup S is a *semilattice Y of semigroups S_α* $(\alpha \in Y)$ if S is the disjoint union of the semigroups S_α and, for each $\alpha, \beta \in Y$, we have

$$S_\alpha S_\beta \subseteq S_{\alpha\beta}. \tag{2.2}$$

The semilattice Y is called the *structure semilattice of S*. We follow the usual convention of denoting an element a of S_α as a_α.

If $S = \bigcup_{\alpha \in Y} S_\alpha$ is a semilattice of semigroups, then the map $\sigma_S : S \to Y$ given by $s_\alpha \sigma_S = \alpha$ is a morphism since if $\alpha, \beta \in Y$ and $x_\alpha \in S_\alpha, y_\beta \in S_\beta$, then $x_\alpha y_\beta \in S_{\alpha\beta}$, so that

$$(x_\alpha y_\beta)\sigma_S = \alpha\beta = (x_\alpha \sigma_S)(y_\beta \sigma_S).$$

Let $T = \bigcup_{\alpha \in Y} T_\alpha$ be a semilattice of semigroups, with the same structure semilattice as S. Then the spined product of S and T w.r.t. Y is given by

$$\{(s,t) : s \in S, t \in T, s\sigma_S = t\sigma_T\} = \{(s_\alpha, t_\alpha) : s_\alpha \in S_\alpha, t_\alpha \in T_\alpha, \alpha \in Y\},$$

which we denote by $S \bowtie T$.

Notice that, by (2.2), we understand the 'global' structure of a semilattice Y of semigroups S_α, but not its local structure. That is, given $x \in S_\alpha$ and $y \in S_\beta$, we know that xy lies in $S_{\alpha\beta}$, but not its exact location. One method for describing a 'local' structure is as follows.

Let Y be a semilattice and let $\leq$ be the natural order on Y. To each $\alpha \in Y$ associate a semigroup S_α, and assume that $S_\alpha \cap S_\beta = \emptyset$ if $\alpha \neq \beta$. For each pair $\alpha, \beta \in Y$ with $\alpha \geq \beta$, let $\psi_{\alpha,\beta} : S_\alpha \to S_\beta$ be a morphism, and assume that the following conditions hold:

$$\text{for all } \alpha \in Y,\ \psi_{\alpha,\alpha} = 1_{S_\alpha}, \tag{2.3}$$

$$\text{for all } \alpha, \beta, \gamma \in Y \text{ such that } \alpha \geq \beta \geq \gamma, \tag{2.4}$$

$$\psi_{\alpha,\beta}\psi_{\beta,\gamma} = \psi_{\alpha,\gamma}.$$

On the set $S = \bigcup_{\alpha \in Y} S_\alpha$ define a multiplication by

$$a * b = (a\psi_{\alpha,\alpha\beta})(b\psi_{\beta,\alpha\beta})$$

for $a \in S_\alpha, b \in S_\beta$, and denote the resulting structure by $S = [Y; S_\alpha; \psi_{\alpha,\beta}]$. Then S is a semigroup, and is called a *strong semilattice Y of the semigroups S_α $(\alpha \in Y)$*. The semigroups S_α are often referred to as the *components* of S. Note that S is certainly a semilattice of the semigroups S_α $(\alpha \in Y)$.

The idempotents of $S = [Y; S_\alpha; \psi_{\alpha,\beta}]$ are given by $E(S) = \bigcup_{\alpha \in Y} E(S_\alpha)$, and if $E(S)$ forms a subsemigroup of S then

$$E(S) = [Y; E(S_\alpha); \psi_{\alpha,\beta}|_{E(S_\alpha)}].$$

We build morphisms between strong semilattices of semigroups in a natural way as follows. The result is well known, but is proven here for completeness.

Theorem 2.7.1. *Let $S = [Y; S_\alpha; \psi_{\alpha,\beta}]$ and $T = [Z; T_\alpha; \varphi_{\alpha,\beta}]$ be a pair of strong semilattices of semigroups. Let $\pi : Y \to Z$ be a morphism and, for each $\alpha \in Y$, let $\theta_\alpha : S_\alpha \to T_{\alpha\pi}$ be a morphism. Assume further that for any $\alpha \geq \beta$, the diagram*

$$\begin{tikzcd} S_\alpha \arrow[r, "\theta_\alpha"] \arrow[d, "\psi_{\alpha,\beta}"] & T_{\alpha\pi} \arrow[d, "\varphi_{\alpha\pi,\beta\pi}"] \\ S_\beta \arrow[r, "\theta_\beta"] & T_{\beta\pi} \end{tikzcd} \tag{2.5}$$

commutes. Then the map $\theta = \bigcup_{\alpha \in Y} \theta_\alpha$ is a morphism from S into T, denoted $\theta = [\theta_\alpha, \pi]_{\alpha \in Y}$. Moreover, θ is injective/surjective if and only if π and each θ_α are injective/surjective.

Proof. Let θ be constructed as above. For $a \in S_\alpha$ and $b \in S_\beta$ we have

$$\begin{aligned}(a\theta)(b\theta) &= (a\theta_\alpha)(b\theta_\beta) = (a\theta_\alpha\varphi_{\alpha\pi,(\alpha\beta)\pi})(b\theta_\beta\varphi_{\beta\pi,(\alpha\beta)\pi}) \\ &= (a\psi_{\alpha,\alpha\beta}\theta_{\alpha\beta})(b\psi_{\beta,\alpha\beta}\theta_{\alpha\beta}) = \big((a\psi_{\alpha,\alpha\beta})(b\psi_{\beta,\alpha\beta})\big)\theta_{\alpha\beta} \\ &= (ab)\theta_{\alpha\beta} = (ab)\theta,\end{aligned}$$

and so θ is a morphism. The final result is easily shown. □

We denote the diagram (2.5) by $[\alpha, \beta; \alpha\pi, \beta\pi]$, or $[\alpha, \beta; \alpha\pi, \beta\pi]^S$ if the semigroup S needs highlighting. The morphism π is called the *induced (semilattice) morphism* from Y to Z. Note that for any morphism $\pi : Y \to Z$ and morphisms $\theta_\alpha : S_\alpha \to T_{\alpha\pi}$ $(\alpha \in Y)$ then the diagram $[\alpha, \alpha; \alpha\pi, \alpha\pi]$ commutes for any $\alpha \in Y$ since

$$\psi_{\alpha,\alpha}\theta_\alpha = 1_{S_\alpha}\theta_\alpha = \theta_\alpha = \theta_\alpha 1_{T_{\alpha\pi}} = \theta_\alpha\varphi_{\alpha\pi,\alpha\pi}.$$

Consequently, we need only check that the diagram $[\alpha, \beta; \alpha\pi, \beta\pi]$ commutes for each $\alpha > \beta$. We use this fact throughout this thesis.

Unfortunately, not all morphisms between strong semilattices of semigroups can be constructed as in Theorem 2.7.1. We call a class $\mathcal{K}$ of strong semilattices of semigroups *morphism-pure* if every morphism (and thus every isomorphism) between members of $\mathcal{K}$ can be constructed as in Theorem 2.7.1. We call a strong semilattice of semigroups S *automorphism-pure* if every automorphism of S can be constructed as in Theorem 2.7.1. Hence if $\mathcal{K} = \{S\}$ is morphism-pure then S is *automorphism-pure.*

2.8 Inverse semigroups

A semigroup S is *inverse* if every element has a unique inverse. Every group is inverse, but the class of inverse semigroups is far broader than the class of groups. The property of being inverse has a number of useful equivalent statements:

Theorem 2.8.1. [55, Theorem 5.1.1] *Let S be a semigroup. Then the following are equivalent:*

(i) *S is inverse;*

(ii) *S is regular, and its idempotents commute;*

(iii) *every $\mathcal{L}$-class and every $\mathcal{R}$-class contains a unique idempotent.*

The following collection of basic facts of inverse semigroups is from [55, Chapter 5].

Proposition 2.8.2. *Let S be a semigroup with set of idempotents $E(S)$. Then*

(i) *$E(S)$ forms a semilattice;*

(ii) *$(a^{-1})^{-1} = a$ and $(a_1a_2 \dots a_n)^{-1} = a_n^{-1} \cdots a_2^{-1}a_1^{-1}$ for every $a, a_1, \dots, a_n$ in S*

(iii) *$a \,\mathcal{R}\, b$ if and only if $aa^{-1} = bb^{-1}$; $a \,\mathcal{L}\, b$ if and only if $a^{-1}a = b^{-1}b$;*

(iv) *if* $e, f \in E(S)$ *then* $e\,\mathcal{D}\,f$ *if and only if there exists* $a \in S$ *such that* $aa^{-1} = e$ *and* $a^{-1}a = f$.

Every inverse semigroup S comes equipped with a partial order $\leq$ defined by

$$a \leq b \text{ if and only if there exists } e \in E(S) \text{ such that } a = eb.$$

We call $\leq$ the *natural order on* S. Equivalently, $a \leq b$ if and only if $a = aa^{-1}b$, and we refer the reader to [55, Proposition 5.2.1] for a number of other characterizations of $\leq$. Note that $\leq$ reduces to the natural order on the semilattice $E(S)$. Moreover, the natural order restricts to equality on maximal subgroups, since if $a, b \in H_e$ for some $e \in E(S)$ then

$$a \leq b \Leftrightarrow a = aa^{-1}b \Leftrightarrow a = eb \Leftrightarrow a = b.$$

The natural order is compatible with the multiplication of S, so that,

$$a \leq b \text{ and } c \in S \Rightarrow ac \leq bc \text{ and } ca \leq cb.$$

Example 2.8.3. The bicyclic monoid $B = \langle p, q : pq = 1 \rangle_{Mo}$ is an inverse semigroup with a single $\mathcal{D}$-class and $p^{-1} = q$. The idempotents of B form a chain given by

$$1 = q > qp > q^2p^2 > q^3p^3 > \cdots .$$

The second class of inverse semigroups we study are Clifford semigroups. An inverse semigroup S is called *Clifford* if $E(S)$ is central, that is, idempotents commute with every element of S. The property of being Clifford has a number of alternative statements, and the following list is in no way complete.

Theorem 2.8.4. [55, Theorem 4.2.1] *Let* S *be a semigroup. Then the following statements are equivalent:*

(i) S *is a Clifford semigroup;*

(ii) S *is a semilattice of groups;*

(iii) S *is a strong semilattice of groups;*

(iv) S *is regular and each* $\mathcal{D}$*-class is a group;*

(v) S *is inverse and* $xx^{-1} = x^{-1}x$ *for all* $x \in S$.

Let $S = [Y; G_\alpha; \psi_{\alpha,\beta}]$ be a Clifford semigroup. Then $\mathcal{H}$ forms a congruence on S with $S/\mathcal{H} \cong E(S)$. Furthermore, the natural order $\leq$ on S is equivalent to

$$a_\alpha \geq b_\beta \text{ if and only if } \alpha \geq \beta \text{ and } a_\alpha \psi_{\alpha,\beta} = b_\beta$$

for each $a_\alpha, b_\beta \in S$.

2.9 Unary semigroups, I-semigroups and varieties

A semigroup equipped with a unary function $a \mapsto a'$ is called a *unary semigroup*. The *signature of unary semigroups* is defined to be the signature $L_{US} = \{\cdot,'\}$, where $\cdot$ is a binary function symbol and $'$ is a unary function symbol.

A unary semigroup S is called an *I-semigroup* if

$$(a')' = a, \quad aa'a = a,$$

for all $a \in S$, so that a' is an inverse of a. Since we are studying I-semigroups in the signature L_{US}, concepts such as substructure, morphisms and direct products of I-semigroups can be deduced from Preliminaries I.

Example 2.9.1. By Proposition 2.8.2, an inverse semigroup forms an I-semigroup, with unary function $a \mapsto a'$. A band can trivially be regarded as an I-semigroup, with identity unary function $e \mapsto e$.

A non-empty class $\mathcal{V}$ of I-semigroups is a *variety* if it is closed under morphic images, I-subsemigroups and direct product, that is:

(1) if $S \in \mathcal{V}$ and $\phi : S \to T$ is a morphism, then $T \in \mathcal{V}$;

(2) if $S \in \mathcal{V}$ and T is an I-subsemigroup of S, then $T \in \mathcal{V}$;

(3) if $S_i \in \mathcal{V}$ $(i \in I)$, then $\prod_{i \in I} S_i \in \mathcal{V}$.

A *subvariety* of a variety $\mathcal{V}$ of I-semigroups is a subclass of $\mathcal{V}$ which is itself a variety of I-semigroups.

Let A be a non-empty set, and let $F_{2,1}(A)$ be the set of all finite, non-empty words in the alphabet $A \cup \{(,),'\}$, defined by the rules:

(1) $A \subseteq F_{2,1}(A)$;

(2) if $a \in F_{2,1}(A)$ then $(a)' \in F_{2,1}(A)$;

(3) if $a, b \in F_{2,1}(A)$ then $(a)(b) \in F_{2,1}(A)$.

Let $u, v \in F_{2,1}(A)$, and let S be an I-semigroup. Then every map $\phi : A \to S$ can be shown to extend to a morphism $\bar{\phi} : F_{2,1}(A) \to S$. We say that S *satisfies the identity* $u = v$ if $u\bar{\phi} = v\bar{\phi}$ for every map $\phi : A \to S$. That is, S satisfies $u = v$ if we obtain equality in S for every substitution in u and v by elements of S.

Let $\mathcal{E}$ be a class of I-semigroups. Suppose there exists a countable set A and $R \subseteq F_{2,1}(A) \times F_{2,1}(A)$ such that, for any I-semigroup S, we have $S \in \mathcal{E}$ if and only if S satisfies the identity $u = v$ for each $(u, v) \in R$. Then $\mathcal{E}$ is called an *equational class*, defined by the identities $u = v$ for each $(u, v) \in R$. In the context of I-semigroups, the properties of a class of I-semigroups being a variety and being an equational class are equivalent. We denote the variety defined by the identities $u_1 = v_1, u_2 = v_2, \dots$ as $[u_1 = v_1, u_2 = v_2, \dots]$. When listing the identities of a variety of I-semigroups, the identities

$$x(yz) = (xy)z, \quad (x')' = x, \quad xx'x = x,$$

are taken as read. Examples include:

completely simple semigroups:	$\mathcal{CS} = [xx' = x'x, xyx(xyx)' = xx']$;
inverse semigroups:	$\mathcal{I} = [xx'yy' = yy'xx']$;
Clifford semigroups:	$\mathcal{CL} = [xx' = x'x, xx'yy' = yy'xx']$;
groups:	$\mathcal{G} = [xx' = yy']$.

Our work on varieties of I-semigroups can be generalized, and in particular can be simplified to consider varieties of semigroups. For varieties of semigroups the identity $x(yz) = (xy)z$ is taken as read:

commutative semigroups:	$\mathcal{C} = [xy = yx]$;
semilattices:	$\mathcal{SL} = [x^2 = x, xy = yx]$;
null semigroup:	$\mathcal{Z} = [xy = zt]$.

However, the class of inverse semigroups does not form a *variety of semigroups* since a subsemigroup of an inverse semigroup need not be inverse. This is crucial in the context of homogeneity, which is explored in Chapter 6.

2.10 Bands

Much of the early work on bands was to determine their lattice of varieties; a feat that was independently completed by Biryukov [9], Fennemore [31] and Gerhard [33]. In addition, Fennemore determined all identities on bands, showing that every variety of bands can be defined by a single identity. The lower part of the lattice of varieties of bands, as shown in Figure 2.3, contains the following varieties which are required for our work:

left zero bands:	$\mathcal{LZ} = [xy = x]$;
right zero bands:	$\mathcal{RZ} = [xy = y]$;
rectangular bands:	$\mathcal{RB} = \mathcal{LZ} \vee \mathcal{RZ} = [xyx = x]$;
semilattices:	$\mathcal{SL} = [xy = yx]$;
left normal bands:	$\mathcal{LN} = [zxy = zyx]$;
right normal bands:	$\mathcal{RN} = [xyz = yxz]$;
normal bands:	$\mathcal{N} = \mathcal{LN} \vee \mathcal{RN} = [zxyz = zyxz]$;
left regular bands:	$\mathcal{LG} = [xy = xyx]$;
right regular bands:	$\mathcal{RG} = [xy = yxy]$;
regular bands:	$\mathcal{GG} = \mathcal{LG} \vee \mathcal{RG} = [zxyz = zxzyz]$,

where the given relation characterizes the variety in the variety of bands, so the identity $x^2 = x$ is given as read. The varieties $[zxy = zxzy]$ and $[yxz = yzxz]$ are known as the varieties of *left quasi-normal bands* and *right quasi-normal bands*, respectively, and are not required for this thesis.

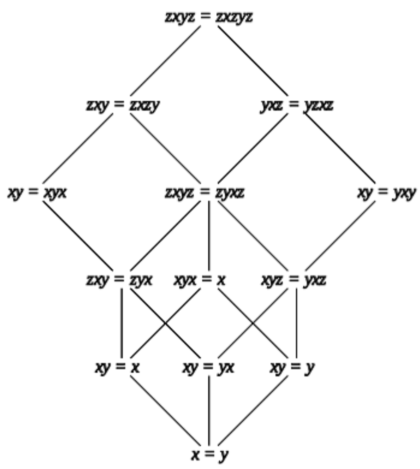

Figure 2.3: Lower part of the lattice of varieties of bands.

We proceed to give alternative descriptions of a number of these varieties. Along with semilattices, a variety of bands required for the construction of an arbitrary band are *rectangular bands*, that is, bands satisfying the identity $xyx = x$. A band is rectangular if and only if it contains a single $\mathcal{D}$-class, and is thus simple. Similarly, *left zero bands* are precisely the bands with a single $\mathcal{L}$-class, dually for *right zero bands*.

The first fact on rectangular bands given in the proposition below is taken from

[55, Theorem 1.1.3], the others are easily shown.

Proposition 2.10.1. *Let L be a left zero semigroup and R a right zero semigroup. Then $B_{L,R} = L \times R$ forms a rectangular band, and the operation is given by*

$$(i, j) \cdot (k, \ell) = (i, \ell).$$

Conversely, every rectangular band is isomorphic to some $B_{L,R}$. The left and right Green's relations on $B_{L,R}$ simplify to

$$(i, j)\, \mathcal{R}\, (k, \ell) \Leftrightarrow i = k \text{ and } (i, j)\, \mathcal{L}\, (k, \ell) \Leftrightarrow j = \ell,$$

and $B_{L,R}$ forms an antichain under the natural order, so that

$$e \leq f \Leftrightarrow e = f.$$

Consequently, rectangular bands are completely simple, and we can alternatively consider the semigroup $B_{L,R}$ as $\mathcal{M}[\{1\}; |L|, |R|; P]$, where $\{1\}$ is the trivial group, so that $p_{\lambda,i} = 1$ for all $i \in I, \lambda \in \Lambda$ (although this form will not be used here). An isomorphism theorem for rectangular bands follows immediately from [55, Corollary 4.4.3], and is stated below.

Proposition 2.10.2. *A pair of rectangular bands $B_{L,R}$ and $B_{L',R'}$ are isomorphic if and only if $|L| = |L'|$ and $|R| = |R'|$. Moreover, if $\phi_L : L \to L'$ and $\phi_R : R \to R'$ are a pair of bijections, then the map $\phi : B_{L,R} \to B_{L',R'}$ defined by*

$$(i, j)\phi = (i\phi_L, j\phi_R)$$

is an isomorphism. Every isomorphism from $B_{L,R}$ to $B_{L',R'}$ can be constructed in this way, and is denoted $\phi = \phi_L \times \phi_R$.

For each $n, m \in \mathbb{N}^* = \mathbb{N} \cup \{\aleph_0\}$, we let $B_{n,m}$ denote the unique, up to isomorphism, rectangular band with n $\mathcal{R}$-classes and m $\mathcal{L}$-classes.

A structure theorem for bands was achieved by McLean in [63]:

Proposition 2.10.3. *Let B be an arbitrary band. Then $\mathcal{D}$ is a congruence on B and $Y = S/\mathcal{D}$ is a semilattice. Moreover, $B = \bigcup_{\alpha \in Y} B_\alpha$ is a semilattice of rectangular bands B_α, which are the $\mathcal{D}$-classes of B.*

By studying strong semilattices of rectangular bands we obtain alternative descriptions of the varieties $\mathcal{LN}, \mathcal{RN}$ and $\mathcal{N}$.

Lemma 2.10.4. [55, Section 4.6] *A band is normal if and only if it is isomorphic to a strong semilattice of rectangular bands. A band is left (right) normal if and only if it is a normal band with $\mathcal{D}$-classes being left (right) zero, that is, if and only if it is isomorphic to a strong semilattice of left (right) zero bands.*

2.11 Completely regular semigroups

In this section we describe a class of semigroups which generalize both Clifford semigroups and bands.

A semigroup S is called *completely regular* if every $\mathcal{H}$-class is a group. Every element a of a completely regular semigroup S has an inverse with which it commutes, namely its inverse in the group H_a. We denote such an inverse as a^{-1}. Consequently, every completely regular semigroup S has a unary operation $a \mapsto a^{-1}$ so that S may be regarded as an I-semigroup. Furthermore, as we have remarked, the class of completely regular semigroups forms an I-variety, which we call the *variety of completely regular semigroups*, defined by the identity $xx' = x'x$.

Theorem 2.11.1. [72, Theorem II.1.4] *A semigroup S is completely regular if and only if S is a semilattice of completely simple semigroups.*

If $S = \bigcup_{\alpha \in Y} S_\alpha$ is a completely regular semigroup then $S/\mathcal{D} \cong Y$ and the completely simple semigroups S_α are the $\mathcal{D}$-classes of S. Each Green's relation will be shown to be preserved under morphisms in Chapter 3, and we thus have the following result on morphisms between completely regular semigroups.

Proposition 2.11.2. *Let $S = \bigcup_{\alpha \in Y} S_\alpha$ and $T = \bigcup_{\alpha' \in Y'} T_{\alpha'}$ be a pair of completely regular semigroups, and $\phi : S \to T$ a morphism between them. Then there exists a morphism $\pi : Y \to Y'$ and morphisms $\phi_\alpha : S_\alpha \to T_{\alpha\pi}$ for each $\alpha \in Y$ such that $\phi = \bigcup_{\alpha \in Y} \phi_\alpha$. Moreover, if ϕ is surjective/injective, then so are π and ϕ_α.*

Following Theorem 2.7.1, we call the morphism π the *induced semilattice morphism* of ϕ. We may build isomorphisms between spined products of completely regular semigroups via isomorphisms of their induced semilattices as follows.

Proposition 2.11.3. *Let $S_i = \bigcup_{\alpha \in Y} S_\alpha^{(i)}$ and $T_i = \bigcup_{\alpha' \in Y'} T_\alpha^{(i))}$ be completely regular semigroups for $i = 1, 2$. Let $\phi_i : S_i \to T_i$ $(i = 1, 2)$ be a pair of isomorphisms, both with induced semilattice isomorphism $\phi : Y \to Y'$. Then the map $\chi : S_1 \bowtie S_2 \to T_1 \bowtie T_2$ given by*

$$(g_\alpha, h_\alpha)\chi = (g_\alpha\phi_1, h_\alpha\phi_2) \quad (g_\alpha \in S_\alpha^{(1)}, h_\alpha \in S_\alpha^{(2)}, \alpha \in Y)$$

is an isomorphism.

A completely regular semigroup in which $\mathcal{H}$ is a congruence is called a *cryptogroup*. A *normal cryptogroup* is a cryptogroup S in which $S/\mathcal{H}$ forms a normal band.

Example 2.11.4. Let S be a Clifford semigroup. Then the quotient $S/\mathcal{H}$ is a semilattice, and is therefore a normal band. Hence every Clifford semigroup is a normal cryptogroup.

Example 2.11.5. The $\mathcal{H}$-relation on a band is trivial, and so every band is a cryptogroup. It is then trivial that a band is a normal cryptogroup if and only if it is normal.

The fact that Clifford semigroups and normal bands are normal cryptogroups are not random occurrences, and point towards the following alternative description of normal cryptogroups.

Theorem 2.11.6. [72, Theorem IV.1.6] *A semigroup is a normal cryptogroup if and only if it is a strong semilattice of completely simple semigroups.*

The justification for studying normal cryptogroups, and not strong semilattices of arbitrary semigroups, is immediate from the following lemma, which follows from [72, Lemma IV.1.8].

Lemma 2.11.7. *The class of all strong semilattices of completely simple semigroups is morphism-pure. Consequently, the same is true for the class of strong semilattices of normal bands and the class of strong semilattices of groups.*

Chapter 3

$\aleph_0$-categorical semigroups

This chapter investigates the form of an $\aleph_0$-categorical semigroup, as well as methods for constructing $\aleph_0$-categorical semigroups from $\aleph_0$-categorical components.

Throughout the rest of this thesis, all structures are assumed to be countable.

3.1 Historical background

In this section we give a brief historical survey of $\aleph_0$-categorical structures. Originally, the concept of $\aleph_0$-categoricity was purely of interest to logicians from a model theoretic standpoint. A shift in direction came in 1959 with the much celebrated Ryll-Nardzewski Theorem. This result gives a number of characterisations of $\aleph_0$-categoricity, and in particular translates the concept to an algebraic viewpoint. Since the publication of the Ryll-Nardzewski Theorem, it has motivated both model theorists and algebraists to examine $\aleph_0$-categoricity for a variety of structures.

The $\aleph_0$-categoricity of linear orders were studied by Rosenstein in [83], and a complete characterisation was achieved. Every $\aleph_0$-categorical linear order was shown to be finitely axiomatizable. The $\aleph_0$-categoricity of arbitrary posets were considered by Grzegorczyk [43], although no full description has been found. Examples of $\aleph_0$-categorical relational structures also arose from studies into homogeneous relational structures by Theorem 1.2.26.

For algebraic structures, the difficulty in achieving full classifications was apparent from the start, although great progress has been made for groups and rings by a number of authors. The early work of $\aleph_0$-categorical rings was started by Baldwin and Rose in [5], where the Jacobson radical of an $\aleph_0$-categorical ring was analysed, and by Macintyre and Rosenstein in [67], where a complete classification of $\aleph_0$-categorical rings with 1 and without non-zero nilpotent elements was achieved. Further studies were made in [14, 15, 60].

It is worthwhile going over the history of $\aleph_0$-categorical groups in greater detail, since the $\aleph_0$-categoricity of a semigroup will be seen to pass to its maximal subgroups. Rosenstein was one of the first to examine $\aleph_0$-categoricity for groups in 1973 in his seminal paper [84]. Here, Rosenstein considered the $\aleph_0$-categoricity of direct products of groups, direct limits of groups, and Burnside groups. Additionally, a complete classification of $\aleph_0$-categorical abelian groups were determined as follows.

Proposition 3.1.1. *An abelian group G is $\aleph_0$-categorical if and only if it is of finite exponent, that is, there exists a positive integer n such that $g^n = 1$ for all $g \in G$; the least such n is the exponent of G.*

This gives us a useful pool of infinite $\aleph_0$-categorical groups, such as the group $\bigoplus_{\mathbb{N}} \mathbb{Z}_2$, and indeed any countably infinite direct sum of a finite abelian group.

Non-abelian examples of $\aleph_0$-categorical groups arose from the work of Sabbagh [89]. Here, Sabbagh showed that the group $GL_n(R)$, of invertible $n \times n$ matrices with coefficients in R, is $\aleph_0$-categorical when R is an $\aleph_0$-categorical ring.

In [70], Olin constructed an example of an $\aleph_0$-categorical group with a non $\aleph_0$-categorical subgroup. In [1], Apps reduced the problem of classifying $\aleph_0$-categorical characteristically simple groups (that is, groups which contains no proper characteristic subgroups) to studying non-abelian p-groups, and it is conjectured that no non-abelian $\aleph_0$-categorical p-groups exist. Further studies were made in [2, 4, 13, 30, 85]. Examples of $\aleph_0$-categorical non-abelian groups also arise from studies into homogeneous groups, which will be discussed in the next chapter. Of course, unlike with relational structures, we are required to restrict our attention to ULF homogeneous groups.

Our final example is of a preservation theorem, attained by Grzegorczyk in [42], which states that $\aleph_0$-categoricity is preserved under finite direct product.

Proposition 3.1.2. *Let M and N be a pair of $\aleph_0$-categorical L-structures. Then the L-structure $M \times N$ is $\aleph_0$-categorical.*

However little is known in the case of semigroups, and this chapter is an attempt to bridge this gap in knowledge. Unless stated otherwise, we will assume throughout this chapter that semigroups are L_S-structures.

3.2 Methods for proving $\aleph_0$-categoricity

Recall that a (countable) structure M is *$\aleph_0$-categorical* if every countable model of $\mathrm{Th}(M)$ is isomorphic to M. In particular, a semigroup is $\aleph_0$-categorical if it can be characterized, within the class of countable semigroups, by its first order properties up to isomorphism. To show that a semigroup S is $\aleph_0$-categorical it suffices to find a list T of first order properties of S which no non-isomorphic, countable semigroup shares. The set T can be thought of as a set of axioms of S, or as a first order definition of S.

Example 3.2.1. A countably infinite null semigroup N is $\aleph_0$-categorical. To see this, we note that two null semigroups are isomorphic if and only if they have the same cardinality. Hence, if we define an infinite theory T by

(1) $(\forall a)(\forall b)(\forall c)[a(bc) = (ab)c]$;

(2) $(\exists a)(\forall x)(\forall y)[xy = a]$;

(3) $\neg(\exists x_1)(\exists x_2)\cdots(\exists x_n)(\forall y)[y = x_1 \vee y = x_2 \vee \cdots \vee y = x_n]$ for each $n \geq 1$;

then N models T and any countable model M of T is a null semigroup by (1) and (2), and is infinite by (3), so that $N \cong M$.

However, at this stage proving that a semigroup is not $\aleph_0$-categorical would require computing its full theory. We instead turn to the aforementioned Ryll-Nardzewski Theorem (RNT), proven independently by Engeler [28], Ryll-Nardzewski [88], and Svenonius [95]. For our study we require only one of the given characterisations of $\aleph_0$-categoricity, which relies on the following terminology.

Given a structure M, and n-tuples $\underline{a} = (a_1, \dots, a_n)$ and $\underline{b} = (b_1, \dots, b_n)$ of M, then we say that $\underline{a}$ is *automorphically equivalent to/has the same n-automorphism type as* $\underline{b}$ (in M) if there exists an automorphism ϕ of M such that $\underline{a}\phi = \underline{b}$ (so that $a_i\phi = b_i$ for each i). That is, a pair of n-tuples are automorphically equivalent if they lie in the same orbit of the natural group action of $\mathrm{Aut}(M)$ on M^n. We denote this equivalence relation by $\underline{a} \sim_{M,n} \underline{b}$. We call $\mathrm{Aut}(M)$ *oligomorphic* if there are only finitely many orbits in its action on M^n for each $n \geq 1$.

Theorem 3.2.2. (Ryll-Nardzewski Theorem) *A structure M is $\aleph_0$-categorical if and only if $|M^n/\sim_{M,n}|$ is finite for each $n \geq 1$, that is, if Aut(M) is oligomorphic.*

To prove that a structure M is $\aleph_0$-categorical, it thus suffices to show that, for each $n \geq 1$, there exists a finite list of n-tuples $\underline{a}_1, \dots, \underline{a}_{\pi(n)}$ of M such that every n-tuple of M is automorphically-equivalent to an element of the list.

On the other hand, to show that M is not $\aleph_0$-categorical it suffices to show that there exists, for some n, an infinite set $\{\underline{a}_i \mid i \in \mathbb{N}\}$ of n-tuples of M such that $\underline{a}_i \sim_{M,n} \underline{a}_j$ if and only if $i = j$.

An immediate consequence of the RNT is that all finite structures are $\aleph_0$-categorical, and as such our interest is in determining the $\aleph_0$-categoricity of infinite structures. A second consequence is that $\aleph_0$-categoricity is preserved under reducts. The result is well known, but it will be insightful outline a proof.

Corollary 3.2.3. *Let L and L' be signatures with $L \subseteq L'$. If M is an $\aleph_0$-categorical L'-structure, then its L-reduct $M|L$ is an $\aleph_0$-categorical L-structure.*

Proof. If $\phi : N \to N'$ is an L'-morphism of L'-structures N and N', then clearly ϕ is also an L-morphism $\phi : N|L \to N'|L$ between the L-reducts of N and N', respectively. The result then follows immediately from the RNT. □

An immediate consequence of Theorem 1.2.26 is that an $\aleph_0$-categorical structure with quantifier elimination is ULF. We now show, via the RNT, that the statement still holds without the condition of quantifier elimination. This result is given in [51, Corollary 7.3.2], but is proven here only using our simplified RNT for completeness.

Corollary 3.2.4. *Let M be an $\aleph_0$-categorical structure. Then M is ULF.*

Proof. We first show that M is locally finite. Suppose, seeking a contradiction, that $X = \langle x_1, \dots, x_n \rangle_M$ is infinite, and take an infinite list $w_1, w_2, \dots$ of distinct elements of X. For each $i \in \mathbb{N}$, let $\underline{w_i}$ be the $(n+1)$-tuple of M given by

$$\underline{w_i} = (x_1, \dots, x_n, w_i).$$

Then as $|M^{n+1}/\sim_{M,n+1}|$ is finite by the RNT, there exist $i \neq j$ and an automorphism θ of M such that $\underline{w_i}\theta = \underline{w_j}$. Since each generator of X is fixed by θ, the substructure X is pointwise fixed by Corollary 1.1.12. However, $w_i\theta = w_j$ and so $w_i = w_j$, and we arrive at our desired contradiction.

Let $A = \langle a_1, \dots, a_n \rangle_M$ and $B = \langle b_1, \dots, b_n \rangle_M$ be a pair of n-generated substructure of M. If $(a_1, \dots, a_n) \sim_{M,n} (b_1, \dots, b_n)$ via $\phi \in \text{Aut}(M)$, say, then it follows again by Corollary 1.1.12 that $A\phi = B$. Hence the number of distinct cardinalities of n-generated substructures of M is bound by $|M^n/\sim_{M,n}|$, which is finite by the RNT, and so M is ULF. □

The following lemma is immediate from a simple counting argument, and as such the proof is omitted.

Lemma 3.2.5. *Let X be a set and $\gamma_1, \dots, \gamma_r$ be a finite list of equivalence relations on X with $\gamma_1 \cap \gamma_2 \cap \dots \cap \gamma_r$ contained in an equivalence relation σ on X. Then*

$$|X/\sigma| \leq \prod_{1 \leq i \leq r} |X/\gamma_i|.$$

We use the RNT in conjunction with Lemma 3.2.5 to prove that a structure M is $\aleph_0$-categorical in the following way. For each $n \in \mathbb{N}$, let $\gamma_1, \dots, \gamma_r$ be a finite list of equivalence relations on M^n such that M^n/γ_i is finite for each $1 \leq i \leq r$ and

$$\gamma_1 \cap \gamma_2 \cap \dots \cap \gamma_r \subseteq \sim_{M,n}.$$

A consequence of the two aforementioned results that M is $\aleph_0$-categorical.

This method is used throughout this chapter, and is particular suited for the building of an $\aleph_0$-categorical structure from a given list of $\aleph_0$-categorical structures.

For simplicity we will reference this method as Lemma 3.2.5. Moreover, where no confusion may arise, it will often be used in a less formal way as follows. A *condition* imposed on n-tuples of M will naturally translate to an equivalence relation, and we will say that a condition has *finitely many choices* if its corresponding equivalence relation has finitely many equivalence classes.

Example 3.2.6. Given a structure M, we may impose a condition on a pair of n-tuples of M which states that if a pair of entries in one of the tuples are equal then the same is true for the other tuple, and conversely. Formally, we define an equivalence $\natural_n$ on M^n by

$$(a_1, \dots, a_n) \natural_n (b_1, \dots, b_n) \text{ if and only if } [a_i = a_j \Leftrightarrow b_i = b_j, \text{ for each } i, j]. \tag{3.1}$$

it is easy to see that a pair of n-tuples $\underline{a}$ and $\underline{b}$ are $\natural_n$-equivalent if and only if there exists a bijection ϕ of M such that $\underline{a}\phi = \underline{b}$. Notice that

$$\underline{a} \sim_{M,n} \underline{b} \Rightarrow \underline{a} \natural_n \underline{b},$$

and the number of $\natural_n$-classes of M^n is equal to the number of ways of partitioning a set of size n, which is finite[1].

Example 3.2.7. If M is a set, regarded as an $\emptyset$-structure, then automorphisms of M are simply bijections. Since any bijection between subsets of a set can be extended to a bijection of the whole set, it follows that $\sim_{M,n} = \natural_n$. Hence, as $|M^n/\natural_n|$ is finite for each n, the set M is $\aleph_0$-categorical.

Let M be an L-structure with subsets M_i $(i \in I)$, where I is a countable set. For each $i \in I$, let Q_i be a unary relation symbol, where Q_i is interpreted as M_i, and set $L' = L \cup \{Q_1, Q_2, \dots\}$. We denote $\bar{M} := (M; M_1, M_2, \dots)$ as the L'-structure such that its L-reduct is M. Then the universes of M and $\bar{M}$ are equal and

$$\text{Aut}(\bar{M}) = \{\phi \in \text{Aut}(M) : M_i\phi = M_i \text{ for each } i \in I\}.$$

Moreover, by Corollary 3.2.3, if $\bar{M}$ is $\aleph_0$-categorical, then so too is its reduct M. We call $\bar{M}$ the $\{M_1, M_2, \dots\}$-*extension of* M or simply a *set-extension of* M.

Lemma 3.2.8. *Let M be a structure and $\{M_i : i \in I\}$ a set of pairwise disjoint subsets of M. If $\bar{M} = (M; M_1, M_2, \dots)$ is $\aleph_0$-categorical, then I is finite.*

Proof. Fix $x_i \in M_i$ for each $i \in I$. If $x_i \sim_{\bar{M},1} x_j$ for some $i, j \in I$, via $\phi \in \text{Aut}(\bar{M})$, say, then $M_i\phi = M_i$ and $x_i\phi = x_j \in M_j$. Since the subsets M_i are pairwise disjoint, this forces $i = j$, and so $|I|$ is bound by the number of 1-automorphism types of $\bar{M}$, which is finite by the RNT. □

[1] The number of ways of partitioning a finite set of size n is denoted by B_n and is called the *nth Bell number*, named after E. T. Bell (for a formulation, see [87]).

Corollary 3.2.9. *If B is a rectangular band and $B_1, \dots, B_r$ is a finite list of subbands of B, then $\bar{B} = (B; B_1, \dots, B_r)$ is $\aleph_0$-categorical. In particular, a rectangular band is $\aleph_0$-categorical.*

Proof. Let $B = L \times R$ be a rectangular band, where L is a left zero semigroup and R is a right zero semigroup. For each $1 \leq k \leq r$, let

$$B_k^L = \{i \in L : (i,j) \in B_k \text{ for some } j \in R\},$$
$$B_k^R = \{j \in R : (i,j) \in B_k \text{ for some } i \in L\}.$$

Define a pair of equivalence relations σ_L and σ_R on L and R, respectively, by

$$i \, \sigma_L \, j \Leftrightarrow [i \in B_k^L \Leftrightarrow j \in B_k^L, \text{ for each } k],$$
$$i \, \sigma_R \, j \Leftrightarrow [i \in B_k^R \Leftrightarrow j \in B_k^R, \text{ for each } k].$$

The equivalence classes of σ_L are simply the set $L \backslash \bigcup_{1 \leq k \leq r} B_k^L$ together with certain intersections of the sets B_k^L. Since r is finite, it follows that L/σ_L is finite, and similarly R/σ_R is finite. Let $\underline{a} = ((i_1, j_1), \dots, (i_n, j_n))$ and $\underline{b} = ((k_1, \ell_1), \dots, (k_n, \ell_n))$ be a pair of n-tuples of B under the four conditions that

(1) $i_s \, \sigma_L \, k_s$ for each $1 \leq s \leq n$,

(2) $j_s \, \sigma_R \, \ell_s$ for each $1 \leq s \leq n$,

(3) $(i_1, \dots, i_n) \natural_n (k_1, \dots, k_n)$,

(4) $(j_1, \dots, j_n) \natural_n (\ell_1, \dots, \ell_n)$,

where $\natural_n$ is the equivalence relation given by (3.1) (in fact, $\natural_n$ is used twice, on different ground sets). By conditions (3) and (4), there exist bijections

$$\phi_L : \{i_1, \dots, i_n\} \to \{k_1, \dots, k_n\} \text{ and } \phi_R : \{j_1, \dots, j_n\} \to \{\ell_1, \dots, \ell_n\}$$

given by $i_s\phi_L = k_s$ and $j_s\phi_R = \ell_s$ for each $1 \leq s \leq n$. By condition (1), we can pick a bijection Φ_L of L which extends ϕ_L and fixes each σ_L-classes setwise, and similarly construct Φ_R. Then $\Phi = \Phi_L \times \Phi_R$ is an automorphism of B by Proposition 2.10.2. Moreover, if $(i,j) \in B_k$ then $i \in B_k^L$ and as $i \, \sigma_L \, (i\Phi_L)$ we have $i\Phi_L \in B_k^L$. Dually, $j \in B_k^R$ and as $j \, \sigma_R \, (j\Phi_R)$ we have $j\Phi_R \in B_k^R$. Hence there exist $\ell \in L$ and $r \in R$ such that $(i\Phi_L, r)$ and $(\ell, j\Phi_R)$ are in B_k, so that

$$(i\Phi_L, r)(\ell, j\Phi_R) = (i\Phi_L, j\Phi_R) \in B_k$$

as B_k is a subband. We have thus shown that $(i,j)\Phi = (i\Phi_L, j\Phi_R) \in B_k$, and so $B_k\Phi \subseteq B_k$. We observe that $\Phi^{-1} = \Phi_L^{-1} \times \Phi_R^{-1}$ is also an automorphism of B with Φ_L^{-1} and Φ_R^{-1} setwise fixing the σ_L-classes and σ_R-classes, respectively. Following our previous argument we have $B_k\Phi^{-1} \subseteq B_k$, and so $B_k\Phi = B_k$ for each k. Thus Φ is an automorphism of $\bar{B}$, and is such that

$$(i_s, j_s)\Phi = (i_s\Phi_L, j_s\Phi_R) = (i_s\phi_L, j_s\phi_R) = (k_s, \ell_s)$$

for each $1 \leq s \leq n$, so that $\underline{a} \sim_{\bar{B},n} \underline{b}$. Hence, as each of the four conditions on $\underline{a}$ and $\underline{b}$ have finitely many choices, it follows that $\bar{B}$ is $\aleph_0$-categorical by Lemma 3.2.5. □

Let M be a set, and $M_1, \dots, M_r$ be subsets of M. Equip M with a binary operation $\cdot$ such that $x \cdot y = x$ for all $x, y \in M$, so that $(M, \cdot)$ is a left zero semigroup. Since left zero semigroups are rectangular bands, and as subsets of left zero semigroups are easily shown to form subbands, we have that $\bar{B} = ((M, \cdot); M_1, \dots, M_r)$ is $\aleph_0$-categorical by the Corollary 3.2.9. Hence by taking the $\{Q_1, \dots, Q_r\}$-reduct of the $\{\cdot, Q_1, \dots, Q_r\}$-structure $\bar{B}$, the same is true for $(M; M_1, \dots, M_r)$ by Lemma 3.2.3. We have proven the following result.

Corollary 3.2.10. *Let M be a set, and $M_1, \dots, M_r$ be a finite list of subsets of M. Then $(M; M_1, \dots, M_r)$ is $\aleph_0$-categorical.*

Let M be a structure and Ψ a subgroup of $\mathrm{Aut}(M)$. Then we say that M is $\aleph_0$-*categorical over* Ψ if Ψ has only finitely many orbits in its action on M^n for each $n \geq 1$. We denote the resulting equivalence relation on M^n as $\sim_{M,\Psi,n}$.

We observe that if $M_1, M_2, \dots$ are subsets of M and $\bar{M} = \{M; M_1, M_2, \dots\}$ then the equivalence relations $\sim_{M,\mathrm{Aut}(\bar{M}),n}$ and $\sim_{\bar{M},n}$ coincide (on the universe of M). That is, $\aleph_0$-categoricity over subgroups of the automorphism group of M can be seen as generalizing $\aleph_0$-categoricity of set-extensions of M.

The following simple consequence of the RNT is a generalization of Exercise 7.3.1 in [51]:

Lemma 3.2.11. *Let M be a structure and $T = \{t_1, \dots, t_r\}$ a finite subset of M. Let Ψ be the subgroup of Aut(M) consisting of automorphisms of M which fix T pointwise. Then for any subset X of M, we have that $|X^n/\sim_{M,n}|$ is finite for all $n \geq 1$ if and only if $|X^n/\sim_{M,\Psi,n}|$ is finite for all $n \geq 1$. In particular, M is $\aleph_0$-categorical if and only if M is $\aleph_0$-categorical over Ψ.*

Proof. Suppose $|X^n/\sim_{M,n}|$ is finite for all $n \geq 1$. Let $\underline{a} = (a_1, \dots, a_n)$ and $\underline{b} = (b_1, \dots, b_n)$ be n-tuples of X such that

$$(\underline{a}, t_1, \dots, t_r) \sim_{M,n+r} (\underline{b}, t_1, \dots, t_r)$$

via $\phi \in \mathrm{Aut}(M)$, say. Then ϕ fixes T pointwise, so that $\phi \in \Psi$. Moreover, $a\phi = b$, so that

$$|X^n/\sim_{M,\Psi,n}| \leq |X^{n+r}/\sim_{M,n+r}| < \aleph_0.$$

The converse and the case when $M = X$ are immediate. $\square$

If $T = \{t_1, \dots, t_n\}$ then the lemma above may be restated, albeit in a rather clumsy way, as M being $\aleph_0$-categorical if and only if $(M; \{t_1\}, \dots, \{t_n\})$ is $\aleph_0$-categorical.

Corollary 3.2.12. *Let M be a structure and T a finite subset of M. Then M is $\aleph_0$-categorical if and only if, for each $n \in \mathbb{N}$, $|(M \setminus T)^n / \sim_{M,n}|$ is finite.*

Proof. If M is $\aleph_0$-categorical then $|M^n / \sim_{M,n}|$ is finite by the RNT, and thus so is $|(M \setminus T)^n / \sim_{M,n}|$.

For the converse, we first fix some notation. Let X be a subset of M and $\underline{x} = (x_1, \dots, x_n)$ an n-tuple of M. Then we let

$$\underline{x}[X] := \{k \in \{1, \dots, n\} : x_k \in X\}$$

be the set of entries of $\underline{x}$ which lie in X. If $\underline{x}[X] = \{k_1, \dots, k_r\}$ is such that $k_1 < k_2 < \dots < k_r$ then we obtain an r-tuple of X given by

$$\underline{x}^X := (x_{k_1}, \dots, x_{k_n}).$$

We also let Ψ be the subgroup of $\mathrm{Aut}(M)$ consisting of automorphisms of M which fix T pointwise.

Let $\underline{a}$ and $\underline{b}$ be n-tuples of M under the conditions that

(1) $\underline{a}[T] = \underline{b}[T]$,

(2) $\underline{a}^T = \underline{b}^T$,

(3) $\underline{a}^{M\setminus T} \sim_{M,\Psi,|\underline{a}^{M\setminus T}|} \underline{b}^{M\setminus T}$.

Conditions (1) and (2) have a total of $(|T|+1)^n$ choices, which is finite since T is. By Lemma 3.2.11, condition (3) also has finitely many choices since $|(M \setminus T)^m / \sim_{M,m}|$ is finite for each m by our hypothesis. The total number of choices is therefore finite. By condition (3) there exists an automorphism ϕ of M fixing T pointwise and with $\underline{a}^{M\setminus T}\phi = \underline{b}^{M\setminus T}$. Since T is fixed pointwise we have $\underline{a}^T\phi = \underline{a}^T = \underline{b}^T$ and it follows that $\underline{a}\phi = \underline{b}$. The result is then immediate from Lemma 3.2.5. $\square$

Our final method for proving $\aleph_0$-categoricity will be applied to cases where we can build automorphisms of our structure via isomorphisms between certain substructures. For example, for a strong semilattice of semigroups $S = [Y; S_\alpha; \psi_{\alpha,\beta}]$, we can construct automorphisms of S from certain isomorphisms between the semigroups S_α by Theorem 2.7.1. In this example we also require an automorphism between the index set of the semigroups, that is, an automorphism of the semilattice Y. Occurrences of automorphisms with this additional property will need to be considered in our method, so that it may be flexibly used for a variety of semigroups.

Notation 3.2.13. Given a pair of structures M and M' of the same signature, we let $\mathrm{Iso}(M; M')$ denote the set of all isomorphisms from M onto M'.

Definition 3.2.14. Let M be an L-structure with fixed substructure M'. Let $\mathcal{A} = \{M_i : i \in N\}$ be a set of substructures of M' indexed by some K-structure N such that $M' = \bigcup_{i\in N} M_i$. Let $N_1, \ldots, N_r$ be a finite partition of N. Set $\bar{N} = (N; N_1, \ldots, N_r)$. For each $i, j \in N$, let $\Psi_{i,j}$ be a subset of $\mathrm{Iso}(M_i; M_j)$ under the conditions that

(A) if $i, j \in N_k$ for some $1 \leq k \leq r$ then $\Psi_{i,j} \neq \emptyset$,

(B) if $\phi \in \Psi_{i,j}$ and $\phi' \in \Psi_{j,\ell}$ then $\phi\phi' \in \Psi_{i,\ell}$,

(C) if $\phi \in \Psi_{i,j}$ then $\phi^{-1} \in \Psi_{j,i}$,

(D) if $\pi \in \mathrm{Aut}(\bar{N})$ and $\phi_i \in \Psi_{i,i\pi}$ for each $i \in N$, then there exists an automorphism of M extending the ϕ_i.

Then $\mathcal{A}$ is called an $(M, M'; \bar{N}; \Psi)$-system (in M), where $\Psi = \bigcup_{i,j\in N} \Psi_{i,j}$. If $M' = M$ then we may simply refer to this as an $(M; \bar{N}; \Psi)$-system.

By Condition (A) if $i, j \in N_k$ for some k, then $M_i \cong M_j$. Hence the number of isomorphism types in $\mathcal{A}$ is bounded by r. Moreover, by Conditions (A), (B) and (C) that $\Psi_{i,i}$ is a subgroup of $\mathrm{Aut}(M_i)$ for each $i \in N$. If the sets M_i are not pairwise disjoint, then Condition (D) should be met with caution. Indeed, if $x \in M_i \cap M_j$ then taking π to be the identity map of $\bar{N}$, we have that $x\phi_i \in M_i \cap M_j$ for all automorphisms ϕ_i of M_i (dually for j). However, for our work the sets M_i will mostly be pairwise disjoint, or will all intersect at a fixed point of M, which is also fixed by every isomorphism between the M_i. For example, M could be a semigroup containing a zero, and 0 is the intersection of each of the sets M_i.

Note also that no link needs to exist between the signatures L and K, and for most of our examples they will be the signature of semigroups and the signature of sets (the empty signature), respectively.

For the remainder of the chapter, we will reference condition (A) as *Condition 3.2.14(A)*, and similarly for conditions (B),(C) and (D).

Lemma 3.2.15. *Let M be a structure, and $\mathcal{A} = \{M_i : i \in N\}$ be an $(M, M'; \bar{N}; \Psi)$-system for some substructure M' of M. If $\bar{N}$ is $\aleph_0$-categorical and each M_i is $\aleph_0$-categorical over $\Psi_{i,i}$ then*

$$|(M')^n/\sim_{M,n}| < \aleph_0$$

for each $n \geq 1$.

Proof. Let $\Psi = \bigcup_{i,j\in N} \Psi_{i,j}$. Let $\bar{N} = (N; N_1, \dots, N_r)$ and, for each $1 \leq k \leq r$, fix some $m_k \in N_k$. For each $i \in N_k$, let $\theta_i \in \Psi_{i,m_k}$, noting that such an element exists by Condition 3.2.14(A) on Ψ. Let $\underline{a} = (a_1, \dots, a_n)$ and $\underline{b} = (b_1, \dots, b_n)$ be a pair of n-tuples of M', with $a_t \in M_{i_t}$ and $b_t \in M_{j_t}$, and such that

$$(i_1, \dots, i_n) \sim_{\bar{N},n} (j_1, \dots, j_n)$$

via $\pi \in \mathrm{Aut}(\bar{N})$, say. For each $1 \leq k \leq r$, let $i_{k1}, i_{k2}, \dots, i_{kn_k}$ be the entries of $(i_1, \dots, i_n)$ belonging to N_k, where $k1 < k2 < \cdots < kn_k$, and set

$$\underline{a}_k = (a_{k1}, \dots, a_{kn_k}) \in (M')^{n_k}.$$

We similarly form each $\underline{b}_k$, observing that as $i_t\pi = j_t$ for each $1 \leq t \leq n$ we have that $j_{k1}, j_{k2}, \dots, j_{kn_k}$ are precisely the entries of $(j_1, \dots, j_n)$ belonging to N_k, so that $\underline{b}_k = (b_{k1}, \dots, b_{kn_k})$ for some $b_{kt} \in M'$. Notice that as $N_1, \dots, N_r$ partition N we have $n = n_1 + n_2 + \cdots + n_r$. Since $i_{kt}, j_{kt} \in N_k$ for each $1 \leq t \leq n_k$, we have that $a_{kt}\theta_{i_{kt}}$ and $b_{kt}\theta_{j_{kt}}$ are elements of M_{m_k}. We may thus suppose further that

$$(a_{k1}\theta_{i_{k1}}, \dots, a_{kn_k}\theta_{i_{kn_k}}) \sim_{M_{m_k},\Psi_{m_k,m_k},n_k} (b_{k1}\theta_{j_{k1}}, \dots, b_{kn_k}\theta_{j_{kn_k}})$$

via $\sigma_k \in \Psi_{m_k,m_k}$, say (where if $\underline{a}_k$ is a 0-tuple, then we take any $\sigma_k \in \Psi_{m_k,m_k}$). For each $1 \leq k \leq r$ and each $i \in N_k$, let

$$\phi_i = \theta_i\sigma_k\theta_{i\pi}^{-1} : M_i \to M_{i\pi},$$

noting that $\phi_i \in \Psi_{i,i\pi}$ by Conditions 3.2.14(B) and 3.2.14(C) on Ψ, since θ_i, σ_k and $\theta_{i\pi}$ are elements of Ψ. Hence, by Condition 3.2.14(D) on Ψ, there exists an automorphism ϕ of M extending each ϕ_i. For any $1 \leq k \leq r$ and any $1 \leq t \leq n_k$ we have

$$a_{kt}\phi = a_{kt}\phi_{i_{kt}} = a_{kt}\theta_{i_{kt}}\sigma_k\theta_{i_{kt}\pi}^{-1} = b_{kt}\theta_{j_{kt}}\theta_{j_{kt}}^{-1} = b_{kt},$$

and so $\underline{a} \sim_{M,n} \underline{b}$ via ϕ. Since $\bar{N}$ is $\aleph_0$-categorical and each M_i are $\aleph_0$-categorical over $\Psi_{i,i}$, the conditions imposed on the tuples $\underline{a}$ and $\underline{b}$ have finitely many choices, and so by from Lemma 3.2.5 we have $|(M')^n / \sim_{M,n}|$ is finite. □

In particular, by Corollary 3.2.10, the structure N in the lemma above can simply be an indexing set. In most cases we take $M' = M$, and the result simplifies accordingly by the RNT.

Corollary 3.2.16. *Let M be a structure, and $\mathcal{A} = \{M_i : i \in N\}$ be an $(M; \bar{N}; \Psi)$-system in M, where $\Psi = \bigcup_{i,j\in N} \Psi_{i,j}$. If $\bar{N}$ is $\aleph_0$-categorical and each M_i is $\aleph_0$-categorical over $\Psi_{i,i}$ then M is $\aleph_0$-categorical.*

3.3 Relatively characteristic subsets and substructures

We now study the substructure of an $\aleph_0$-categorical structure. Our first interest is in determining classes of substructures which inherit $\aleph_0$-categoricity, and applying these general results to the case of semigroups. As remarked in Section 3.1, $\aleph_0$-categoricity is not inherited by every substructure, and an example is given in [70].

We begin by considering characteristic substructures, that is, substructures which are invariant under automorphisms of the structure.

Example 3.3.1. For any semigroup S, the subsemigroup generated by the idempotents, $\langle E(S)\rangle$, is a characteristic subsemigroup. Indeed, for any automorphism ϕ of S and $e \in E(S)$,

$$(e\phi)^2 = e^2\phi = e\phi$$

so that $E(S)\phi \subseteq E(S)$. Similarly $E(S)\phi^{-1} \subseteq E(S)$, and so $E(S)\phi = E(S)$. Hence $\phi|_{\langle E(S)\rangle}$ is an automorphism of $\langle E(S)\rangle$ by Corollary 1.1.12 as required.

This easily generalizes as follows. If A is a characteristic subset of a structure M then $\langle A\rangle_M$ is a characteristic substructure of M.

It is clear that the $\aleph_0$-categoricity of a structure passes to characteristic substructures. However the condition on a subset/substructures to be characteristic is too restrictive, since many key subsemigroups of a semigroup, such as maximal subgroups and principal ideals, are excluded. We instead study a weaker condition, but one in which $\aleph_0$-categoricity is still preserved. Our definition is motivated by the Green's classes of a semigroup.

Definition 3.3.2. Let M be a structure and, for some fixed $t \in \mathbb{N}$, let $\{\underline{X}_i : i \in I\}$ be a collection of t-tuples of M. Let $\{A_i : i \in I\}$ be a collection of subsets of M with the property that for any automorphism ϕ of M such that there exist $i, j \in I$ with $\underline{X}_i\phi = \underline{X}_j$, then $\phi|_{A_i}$ is a bijection from A_i onto A_j. Then we call $\mathcal{A} = \{(A_i, \underline{X}_i) : i \in I\}$ a *system of t-pivoted pairwise relatively characteristic (t-p.p.r.c.) subsets* of M. The t-tuple $\underline{X}_i$ is called the *pivot* of A_i ($i \in I$). If further each A_i forms a substructure of M, then we call $\mathcal{A}$ a *system of t-p.p.r.c. substructures* of M. If $|I| = 1$ then, letting $A_1 = A$ and $\underline{X}_1 = \underline{X}$, we write $\{(A, \underline{X})\}$ simply as $(A, \underline{X})$, and call A an *$\underline{X}$-pivoted relatively characteristic ($\underline{X}$-p.r.c.) subset/substructure* of M.

Clearly if $\{(A_i, \underline{X}_i) : i \in I\}$ forms a system of t-p.p.r.c. subsets of M and J is a subset of I then $\{(A_j, \underline{X}_j) : j \in J\}$ is also a system of t-p.p.r.c. subsets of M. In particular, each A_i is an $\underline{X}_i$-p.r.c. subset of M.

Moreover, if A is an $\underline{X}$-p.r.c. subset in M then A is a union of orbits of the set of automorphisms of M which fixes $\underline{X}$. Indeed, if $a \in A$ and $\phi \in \mathrm{Aut}(M)$ fixes $\underline{X}$ then $A\phi = A$, so that $a\phi \in A$.

Definition 3.3.2 has strong links with the model theoretic concept of definably, which we briefly outlined in Subsection 1.2.1. Let M be an $\aleph_0$-categorical structure, A a subset of M, and $\underline{X} = (x_1, \ldots, x_n)$ an n-tuple of M. Then A is an $\underline{X}$-p.r.c. subset if and only if A is an $\{x_1, \ldots, x_n\}$-definable subset of M by Proposition 1.2.23. In fact much of the work in this section could be given in terms of definable sets, but in keeping with our algebraic viewpoint at this stage it is more natural to use Definition 3.3.2.

The following lemma will be useful in many applications throughout this chapter.

Lemma 3.3.3. *Let M be a structure and, for some fixed $t \in \mathbb{N}$, let $\{\underline{X}_i : i \in I\}$ be a collection of t-tuples of M. Then for any collection $\{A_i : i \in I\}$ of subsets of M, the following are equivalent:*

(i) $\{(A_i, \underline{X}_i) : i \in I\}$ is a system of t-p.p.r.c. subsets/substructures of M;

(ii) if $\phi \in Aut(M)$ is such that there exist $i, j \in I$ with $\underline{X}_i\phi = \underline{X}_j$, then $x\phi \in A_j$ for all $x \in A_i$.

Proof. We prove the result for the case where each A_i is a substructure of M, the case where the A_i are subsets is then immediate.

(i) $\Rightarrow$ (ii). Immediate.

(ii) $\Rightarrow$ (i). Let $\phi \in \mathrm{Aut}(M)$ be such that there exist $i, j \in I$ with $\underline{X}_i\phi = \underline{X}_j$. Then by our hypothesis $\phi|_{A_i}$ is a map from A_i to A_j and, being a restriction of an automorphism, is an injective morphism. Moreover, as $\underline{X}_j\phi^{-1} = \underline{X}_i$ and $\phi^{-1} \in \mathrm{Aut}(M)$, we have that $x\phi^{-1} \in A_i$ for all $x \in A_j$. Hence $\phi|_{A_i}$ is surjective, and thus an automorphism. $\square$

We observe that if $\{(A_i, \underline{X}_i) : i \in I\}$ is a system of t-p.p.r.c. subsets of a structure M then $\{(\langle A_i \rangle_M, \underline{X}_i) : i \in I\}$ forms a system of t-p.p.r.c. substructures of M. For if $\phi \in \mathrm{Aut}(S)$ is such that $\underline{X}_i\phi = \underline{X}_j$ for some $i, j \in I$ then $A_i\phi = A_j$, and so $\langle A_i \rangle_M \phi \subseteq \langle A_j \rangle_M$. The result follows by Lemma 3.3.3.

Example 3.3.4. Let S be a semigroup. Then $\{(S^1aS^1, a) : a \in S\}$ forms a system of 1-p.p.r.c. subsets of S. To see this, let $\phi \in$ Aut(S) be such that $a\phi = b$, and let $x \in S^1aS^1$. Then there exist $u, v \in S^1$ with $x = uav$, and so by interpreting 1ϕ as 1 we have

$$x\phi = (u\phi)(a\phi)(v\phi) = (u\phi)b(v\phi) \in S^1bS^1,$$

and the result follows by Lemma 3.3.3. A similar result also holds for principal left/right ideals of a semigroup.

Proposition 3.3.5. *Let M be an $\aleph_0$-categorical structure and $\{(A_i, \underline{X}_i) : i \in I\}$ be a system of t-p.p.r.c. substructures. Then each A_i is $\aleph_0$-categorical.*

Proof. Let $\underline{X}_i = (x_{i1}, \ldots, x_{it})$ $(i \in I)$. Let $\underline{a} = (a_1, \ldots, a_n)$ and $\underline{b} = (b_1, \ldots, b_n)$ be pair of n-tuples of A_i such that

$$(\underline{a}, \underline{X}_i) \sim_{M,n+t} (\underline{b}, \underline{X}_i)$$

via $\phi \in \mathrm{Aut}(M)$, say. Then $\underline{X}_i\phi = \underline{X}_i$ and so $\phi|_{A_i}$ is an automorphism of A_i as $\{(A_i, \underline{X}_i) : i \in I\}$ is a system of t-p.p.r.c. substructures. Moreover, $\underline{a}\phi|_{A_i} = \underline{a}\phi = \underline{b}$ and so $\underline{a} \sim_{A_i,n} \underline{b}$. We have thus shown that

$$|A_i^n/\sim_{A_i,n}| \leq |M^{n+t}/\sim_{M,n+t}| < \aleph_0$$

for each $n \geq 1$, since M is $\aleph_0$-categorical. Hence A_i is $\aleph_0$-categorical by the RNT. □

A partial converse to Corollary 3.2.10 can be achieved by restricting our subsets to r.c. subsets as follows.

Lemma 3.3.6. *Let M be an $\aleph_0$-categorical structure and $A_1, \ldots, A_r$ a finite list of subsets of M. Suppose there exist finite tuples $\underline{X}_1, \ldots, \underline{X}_r$ of M such that A_i forms an $\underline{X}_i$-p.r.c. subset of M $(1 \leq i \leq r)$. Then $\bar{M} = (M; A_1, \ldots, A_r)$ is $\aleph_0$-categorical.*

Proof. Suppose $\underline{X}_i \in M^{t_i}$ for each $1 \leq i \leq r$, and let $k(n) = n + \sum_{i=1}^{r} t_i$. Let $\underline{a}$ and $\underline{b}$ be a pair of n-tuples of $\bar{M}$ such that

$$(\underline{a}, \underline{X}_1, \ldots, \underline{X}_r) \sim_{M,k(n)} (\underline{b}, \underline{X}_1, \ldots, \underline{X}_r)$$

via $\phi \in \mathrm{Aut}(M)$, say. Then for each $1 \leq i \leq r$ we have $\underline{X}_i\phi = \underline{X}_i$, so that $A_i\phi = A_i$ as A_i is an $\underline{X}_i$-p.r.c subset. Hence $\phi \in \mathrm{Aut}(\bar{M})$, and is such that $\underline{a}\phi = \underline{b}$. We have thus shown that

$$|\bar{M}^n/\sim_{\bar{M},n}| \leq |M^{k(n)}/\sim_{M,k(n)}| < \aleph_0$$

for each n since M is $\aleph_0$-categorical. Hence $\bar{M}$ is $\aleph_0$-categorical. □

We now give a method for constructing systems of t-p.p.r.c. subsets of a structure via certain equivalence relations. Let $\phi : M \to N$ be an isomorphism between structures M and N, and τ_M and τ_N be equivalence relations on M and N, respectively. We call τ_M and τ_N *preserved under* ϕ if $a\,\tau_M\,b$ if and only if $a\phi\,\tau_N\,b\phi$ for each $a, b \in M$. This is clearly equivalent to

$$(x\tau_M)\phi = (x\phi)\tau_N \quad (\forall x \in M),$$

where $(x\tau_M)\phi = \{y\phi : y \in x\tau_M\}$. If $M = N$ then we say that τ_M is *preserved under* ϕ.

Note that if τ is an equivalence relation on a structure M then

$$\mathrm{Aut}(M)[\tau] := \{\phi \in \mathrm{Aut}(M) : \tau \text{ is preserved under } \phi\}$$

is a subgroup of $\mathrm{Aut}(M)$. Indeed, if $\phi, \psi \in \mathrm{Aut}(M)[\tau]$ and $a, b \in M$ then

$$\begin{aligned}(a\phi\psi^{-1})\,\tau\,(b\phi\psi^{-1}) &\Leftrightarrow (a\phi\psi^{-1})\psi\,\tau\,(b\phi\psi^{-1})\psi && (\text{as } \psi \in \mathrm{Aut}(M)[\tau])\\ &\Leftrightarrow (a\phi)\,\tau\,(b\phi)\\ &\Leftrightarrow a\,\tau\,b && (\text{as } \phi \in \mathrm{Aut}(M)[\tau]).\end{aligned}$$

Hence $\phi\psi^{-1} \in \mathrm{Aut}(M)[\tau]$ as required.

If $\mathrm{Aut}(M) = \mathrm{Aut}(M)[\tau]$ then we call τ *preserved under automorphisms* (of M). The following lemma is then immediate.

Lemma 3.3.7. *Let M be a structure and τ be an equivalence relation on M, preserved under automorphisms of M. Then $\{(x\tau, x) : x \in M\}$ forms a system of 1-p.p.r.c. subsets of M.*

For example, each Green's relation is preserved by all isomorphisms between semigroups, and thus by automorphisms. We prove the result for $\mathcal{R}$, the other cases being proved similarly. Let $\phi \in \mathrm{Iso}(S;T)$ for some semigroups S and T. By interpreting 1ϕ as 1, we have for $a, b \in S$,

$$\begin{aligned}a\,\mathcal{R}\,b &\Leftrightarrow a = bu, b = av \text{ for some } u, v \in S^1\\ &\Leftrightarrow a\phi = b\phi u\phi, b\phi = a\phi v\phi \text{ for some } u\phi, v\phi \in S^1\\ &\Leftrightarrow a\phi\,\mathcal{R}\,b\phi.\end{aligned}$$

Consequently, for any semigroup S, each of $\{(R_a, a) : a \in S\}$, $\{(L_a, a) : a \in S\}$, $\{(H_a, a) : a \in S\}$, $\{(D_a, a) : a \in S\}$ and $\{(J_a, a) : a \in S\}$ form systems of 1-p.p.r.c. subsets of S. Hence, by the Maximal Subgroup Theorem, $\{(H_e, e) : e \in E(S)\}$ forms a system of 1-p.p.r.c. subsemigroups of S. The following result is then immediate from Proposition 3.3.5.

Corollary 3.3.8. *The maximal subgroups of an $\aleph_0$-categorical semigroup are $\aleph_0$-categorical.*

This raises the following question: given an equivalence relation τ of an $\aleph_0$-categorical structure, is there a bound on the set of cardinals of the τ-classes?

This will be of importance in later sections when examining the $\aleph_0$-categoricity of semigroups built from possibly infinitely many subsemigroups, such as strong semilattices of semigroups over an infinite semilattice. We consider the question here in a more general setting.

Proposition 3.3.9. *Let M be an $\aleph_0$-categorical structure and $\{(A_i, \underline{X}_i) : i \in I\}$ be a system of t-p.p.r.c. subsets. Then $\{|A_i| : i \in I\}$ is finite. Moreover, if each A_i forms a substructure of M, then $\{A_i : i \in I\}$ is finite, up to isomorphism.*

Proof. Suppose for some $i \neq j$ we have $\underline{X}_i \sim_{M,t} \underline{X}_j$ via $\phi \in \mathrm{Aut}(M)$, say. Then $A_i\phi = A_j$ and it follows that $\{|A_i| : i \in I\}$ is bound by the number of t-automorphism types of M, which is finite by the $\aleph_0$-categoricity of M. The case where each A_i is a substructure of M also follows. $\square$

Corollary 3.3.10. *An $\aleph_0$-categorical semigroup has only finitely many maximal subgroups, up to isomorphism.*

We now mirror our generalization of characteristic subsets to automorphism preserving equivalence relations.

Definition 3.3.11. Let τ be an equivalence relation on a structure M and $\underline{X}$ a finite tuple of M. Then we call τ an *$\underline{X}$-relatively automorphism preserved ($\underline{X}$-r.a.p.) equivalence relation with pivot $\underline{X}$*, if whenever $\phi \in \mathrm{Aut}(M)$ is such that $\underline{X}\phi = \underline{X}$, then τ preserves ϕ.

Example 3.3.12. Clearly if τ is an automorphism preserving equivalence relation, then τ is an $\underline{X}$-r.a.p. equivalence relation for any finite tuple $\underline{X}$ of M.

More inspiring examples will be given in due course. We note that, as with $\underline{X}$-p.r.c. subsets, there are connections between definable sets and $\underline{X}$-r.a.p. equivalence relations. Indeed, if τ is an $\underline{X}$-r.a.p. equivalence relation on an $\aleph_0$-categorical structure M with pivot $\underline{X} = (x_1, \dots, x_t)$ then τ, considered as a set of ordered pairs, is an $\{x_1, \dots, x_t\}$-definable subsets of M^2 by Proposition 1.2.23.

Lemma 3.3.13. *Let M be a structure and τ an $\underline{X}$-r.a.p. equivalence relation on M, where $\underline{X} \in M^t$. For each $a \in M$, let $\underline{X}_a$ be the $(t+1)$-tuple given by $(\underline{X}, a)$. Then $\{(a\tau, \underline{X}_a) : a \in M\}$ forms a system of $(t+1)$-p.p.r.c. subsets of M.*

Proof. Let ϕ be an automorphism of M such that $\underline{X}_a\phi = \underline{X}_b$ for some $a, b \in M$. Then $\underline{X}\phi = X$ so that τ is preserved under ϕ, and $a\phi = b$. Hence

$$(a\tau)\phi = (a\phi)\tau = b\tau.$$

$\square$

We now assess when the $\aleph_0$-categoricity of a semigroup passes to its quotients, and in particular to its Rees factor semigroups. Our work relies on the following method for constructing isomorphisms between certain quotient semigroups.

Proposition 3.3.14. *Let $\phi : S \to T$ be an isomorphism between semigroups S and T. Let ρ_S and ρ_T be a congruences on S and T, respectively, which are preserved under ϕ. Then the map ψ from S/ρ_S to T/ρ_T given by*

$$(a\rho_S)\psi = (a\phi)\rho_T \quad (a\rho_S \in S/\rho_S), \tag{3.2}$$

is an isomorphism.

Proof. The map ψ is well-defined and injective as

$$\begin{aligned}(a\rho_S)\psi = (b\rho_S)\psi &\Leftrightarrow (a\phi)\rho_T = (b\phi)\rho_T \\ &\Leftrightarrow a\rho_S = b\rho_S.\end{aligned}$$

Let $t\rho_T \in T/\rho_T$. Then as ϕ is surjective, there exists $s \in S$ such that $t = s\phi$, so that $t\rho_T = (s\phi)\rho_T = (s\rho_S)\psi$. Hence ψ is surjective, and is a morphism as

$$\begin{aligned}(a\rho_S)\psi(b\rho_S)\psi &= (a\phi)\rho_T(b\phi)\rho_T = (a\phi b\phi)\rho_T \\ &= ((ab)\phi)\rho_T = ((ab)\rho_S)\psi \\ &= (a\rho_S b\rho_S)\psi.\end{aligned}$$

Thus ψ is an isomorphism as desired. □

Proposition 3.3.15. *Let S be an $\aleph_0$-categorical semigroup and ρ an $\underline{X}$-r.a.p. congruence on S. Then S/ρ is $\aleph_0$-categorical.*

Proof. Suppose $\underline{X} \in S^t$ and let $\underline{a} = (a_1\rho, \ldots, a_n\rho)$ and $\underline{b} = (b_1\rho, \ldots, b_n\rho)$ be a pair of n-tuples of S/ρ such that

$$(a_1, \ldots, a_n, \underline{X}) \sim_{S,n+t} (b_1, \ldots, b_n, \underline{X})$$

via $\phi \in \mathrm{Aut}(S)$, say. Then $\underline{X}\phi = \underline{X}$, so that ρ is preserved under the automorphism ϕ. By Proposition 3.3.14, we can construct an automorphism ψ of S/ρ given by

$$(a\rho)\psi = (a\phi)\rho \quad (a\rho \in S/\rho).$$

Since $(a_k\rho)\psi = (a_k\phi)\rho = b_k\rho$ for each $1 \leq k \leq n$ we have $\underline{a} \sim_{S/\rho,n} \underline{b}$, and so

$$|(S/\rho)^n/ \sim_{S/\rho,n}| \leq |S^{n+t}/ \sim_{S,n+t}| < \aleph_0.$$

as S is $\aleph_0$-categorical. Hence S/ρ is $\aleph_0$-categorical. □

This naturally generalizes to any universal algebra, but requires a level of background material that we cannot justify explaining here, and is not needed for this work.

If we drop the condition on Proposition 3.3.15 that the congruence is relatively automorphism preserving then the statement is no longer true. An example of an $\aleph_0$-categorical group with a non $\aleph_0$-categorical quotient group is given by Rosenstein [85].

Corollary 3.3.16. *Let ρ be a finitely generated congruence on an $\aleph_0$-categorical semigroup S. Then S/ρ is $\aleph_0$-categorical.*

Proof. Let $\sigma = \{(u_1, v_1), \ldots, (u_r, v_r)\}$ be a finite relation on S and let $\rho = \sigma^\sharp$. We may assume w.l.o.g. that σ is symmetric (adding (v_i, u_i) for each $1 \leq i \leq r$ if necessary). We claim that ρ is an $\underline{X}$-r.a.p. congruence with pivot

$$\underline{X} = (u_1, v_1, u_2, v_2, \ldots, u_r, v_r).$$

Let ϕ be an automorphism of S which fixes $\underline{X}$ and let $a, b \in S$. Then by Proposition 2.2.1 and the symmetricity of σ, we have that $a \,\rho\, b$ if and only if for some $n \geq 0$, there exist $c_1, \ldots, c_n, d_1, \ldots, d_n$ in S^1, and $(u_{i_1}, v_{i_1}), \ldots, (u_{i_n}, v_{i_n}) \in \sigma$ such that

$$\begin{aligned} a &= c_1 \cdot u_{i_1} \cdot d_1, \\ c_1 \cdot v_{i_1} \cdot d_1 &= c_2 \cdot u_{i_2} \cdot d_2, \\ c_2 \cdot v_{i_2} \cdot d_2 &= c_3 \cdot u_{i_3} \cdot d_3, \\ &\vdots \\ c_n \cdot v_{i_n} \cdot d_n &= b. \end{aligned}$$

Applying ϕ and ϕ^{-1}, this occurs if and only if there exist $c_1\phi, \ldots, c_n\phi, d_1\phi, \ldots, d_n\phi$ in S^1 (where we interpret 1ϕ as 1) and $(u_{i_1}, v_{i_1}), \ldots, (u_{i_n}, v_{i_n}) \in \sigma$ such that

$$\begin{aligned} a\phi &= c_1\phi \cdot u_{i_1} \cdot d_1\phi, \\ c_1\phi \cdot v_{i_1} \cdot d_1\phi &= c_2\phi \cdot u_{i_2} \cdot d_2\phi, \\ c_2\phi \cdot v_{i_2} \cdot d_2\phi &= c_3\phi \cdot u_{i_3} \cdot d_3\phi, \\ &\vdots \\ c_n\phi \cdot v_{i_n} \cdot d_n\phi &= b\phi. \end{aligned}$$

for some $n \geq 0$, since $\underline{X}\phi = \underline{X}$, so that each u_{i_k} and v_{i_k} is fixed by ϕ. Hence $a \,\rho\, b$ if and only if $a\phi \,\rho\, b\phi$, thus completing the proof of the claim. The result follows by Proposition 3.3.15. □

We now apply our recent results to the case of Rees factor semigroups. If I is a characteristic ideal of S then it is easily shown that the Rees congruence ρ_I is preserved under automorphisms of S, and so the $\aleph_0$-categoricity of S passes to the Rees factor semigroup S/I by Proposition 3.3.15. On the other hand, if I is relatively characteristic then, although ρ_I may no longer be preserved under all automorphisms of S, we can find a pivot $\underline{X}$ for ρ_I such that ρ_I is an $\underline{X}$-r.a.p. congruence as follows.

Lemma 3.3.17. *Let S be an $\aleph_0$-categorical semigroup and I an $\underline{X}$-p.r.c. ideal of S. Then S/I is $\aleph_0$-categorical.*

Proof. We claim that ρ_I is an $\underline{X}$-r.a.p. congruence. Let ϕ be an automorphism of S which fixes $\underline{X}$, so that $I\phi = I$ since I is an $\underline{X}$-p.r.c ideal. Then, for any $a, b \in S$, we have

$$a \,\rho_I\, b \Leftrightarrow [a = b \text{ or } a, b \in I] \Leftrightarrow [a\phi = b\phi \text{ or } a\phi, b\phi \in I] \Leftrightarrow a\phi \,\rho_I\, b\phi,$$

and so ϕ preserves ρ_I as required. The result follows by Proposition 3.3.15. $\square$

We end this section by studying a final class of equivalence relations on a semigroup: those with finite equivalence classes. Let M be a structure and τ an equivalence relation on M. Define an equivalence relation $\sim_{M,\tau,n}$ on the set $(M/\tau)^n$ by $(m_1\tau, \dots, m_n\tau) \sim_{M,\tau,n} (m'_1\tau, \dots, m'_n\tau)$ if and only if there exists an automorphism ϕ of M such that $(m_k\tau)\phi = m'_k\tau$ for each $1 \leq k \leq n$. Note that the automorphism ϕ does not have to be τ preserving. Moreover, by taking τ to be the identity relation ι we recover the usual definition of automorphic equivalence of the tuples $(m_1, \dots, m_n)$ and $(m'_1, \dots, m'_n)$.

Proposition 3.3.18. *Let M be a structure and τ an equivalence on M with each τ-class being finite. Then $|(M/\tau)^n/ \sim_{M,\tau,n} |$ is finite for each $n \geq 1$ if and only if M is $\aleph_0$-categorical and $A = \{|m\tau| : m \in M\}$ is finite.*

Proof. Given an n-tuple $\underline{a} = (a_1, \dots, a_n)$ of M, we let $\underline{a}\tau$ denote the n-tuple of M/τ given by

$$\underline{a}\tau := (a_1\tau, \dots, a_n\tau).$$

Suppose that $|(M/\tau)^n/ \sim_{M,\tau,n} |$ is finite for each $n \geq 1$. Let $Z = \{\underline{a}_i : i \in \mathbb{N}\}$ be an infinite set of n-tuples of M, where $\underline{a}_i = (a_{i1}, \dots, a_{in})$. Since $|(M/\tau)^n/ \sim_{M,\tau,n} |$ is finite, there exists an infinite subset $\{\underline{a}_i : i \in I\}$ of Z such that $\underline{a}_i\tau \sim_{M,\tau,n} \underline{a}_j\tau$ for each $i, j \in I$. In particular, for each $i \in I$ there exists an automorphism ϕ_i of M with $(\underline{a}_i\tau)\phi_i = \underline{a}_1\tau$. Hence $a_{ik}\phi_i \in a_{ik}\tau\phi = a_{1k}\tau$ for each $1 \leq k \leq n$, and so

$$\underline{a}_i\phi_i \in \{(z_1, \dots, z_n) : z_k \in a_{1k}\tau\}.$$

Notice that set $\{(z_1, \dots, z_n) : z_k \in a_{1k}\tau\}$ is finite since each τ-class is finite. Consequently, there exist distinct $i, j \in I$ such that $\underline{a}_i\phi_i = \underline{a}_j\phi_j$, so that $\underline{a}_i\phi_i\phi_j^{-1} = \underline{a}_j$. Hence $\underline{a}_i$ and $\underline{a}_j$ are automorphically equivalent. It follows that M contains no infinite set of distinct n-automorphism types, and is thus $\aleph_0$-categorical by the RNT. Furthermore, by our usual argument we have that A is bound by $|(M/\tau)/ \sim_{M,\tau,1} |$.

Conversely, suppose M is $\aleph_0$-categorical and A is finite. Let $\underline{m} = (m_1\tau, \dots, m_n\tau)$ and $\underline{m}' = (m'_1\tau, \dots, m'_n\tau)$ be a pair of n-tuples of $(M/\tau)^n$, under the condition that $|m_k\tau| = |m'_k\tau|$ for each k. Since each entry of an n-tuple of $(M/\tau)^n$ has $|A|$ potential cardinalities, it follows that this condition has $|A|^n$ choices. For each $1 \leq k \leq n$, let $m_k\tau = \{a_{k1}, \dots, a_{ks_k}\}$ and $m'_k\tau = \{b_{k1}, \dots, b_{ks_k}\}$, and let $T(n) = s_1 + s_2 + \cdots + s_n$. Suppose further that

$$(a_{11}, \dots, a_{1s_1}, a_{21}, \dots, a_{2s_2}, \dots, a_{ns_n}) \sim_{M,T(n)} (b_{11}, \dots, b_{1s_1}, b_{21}, \dots, b_{2s_2}, \dots, b_{ns_n}),$$

via $\phi \in \mathrm{Aut}(S)$, say. Then $(m_k\tau)\phi = m'_k\tau$ for each k, since $a_{kr}\phi = b_{kr}$ for each $1 \leq r \leq s_k$. Hence $\underline{m} \sim_{M,\tau,n} \underline{m}'$, and so $|(M/\tau)^n/\sim_{M,\tau,n}|$ is finite by Lemma 3.2.5 since $|A|^n$ and each $|M^{T(n)}/\sim_{M,T(n)}|$ are finite for each $n \geq 1$, thus completing the proof. □

Corollary 3.3.19. *Let S be a regular semigroup with each maximal subgroup being finite. Then S is $\aleph_0$-categorical if and only if $|E(S)^n/\sim_{S,n}|$ is finite for each $n \geq 1$.*

Proof. If S is $\aleph_0$-categorical, then

$$|E(S)^n/\sim_{S,n}| \leq |S^n/\sim_{S,n}| < \aleph_0$$

for each $n \geq 1$ by the RNT.

Conversely, suppose $|E(S)^n/\sim_{S,n}|$ is finite for each $n \geq 1$ and consider a pair of n-tuples of $S/\mathcal{H}$ given by $\underline{a} = (H_{a_1}, \dots, H_{a_n})$ and $\underline{b} = (H_{b_1}, \dots, H_{b_n})$. Since S is regular, there exist idempotents $e_i, f_i, \bar{e}_i, \bar{f}_i$ of S with $e_i \,\mathcal{R}\, a_i \,\mathcal{L}\, f_i$ and $\bar{e}_i \,\mathcal{R}\, b_i \,\mathcal{L}\, \bar{f}_i$ for each $1 \leq i \leq n$, by Proposition 2.4.4. Suppose further that

$$(e_1, f_1, e_2, f_2, \dots, e_n, f_n) \sim_{S,2n} (\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \dots, \bar{e}_n, \bar{f}_n),$$

via $\phi \in \mathrm{Aut}(S)$, say. Then as $\mathcal{R}$ and $\mathcal{L}$ are automorphism preserving we have that $R_{e_i}\phi = R_{\bar{e}_i}$ and $L_{f_i}\phi = L_{\bar{f}_i}$ for each i, so that

$$H_{a_i}\phi = (R_{a_i} \cap L_{a_i})\phi = (R_{e_i} \cap L_{f_i})\phi = R_{e_i}\phi \cap L_{f_i}\phi = R_{\bar{e}_i} \cap L_{\bar{f}_i} = H_{b_i}.$$

Hence $\underline{a} \sim_{S,\mathcal{H},n} \underline{b}$, and we have thus shown that

$$|(S/\mathcal{H})^n/\sim_{S,\mathcal{H},n}| \leq |E(S)^{2n}/\sim_{S,2n}| < \aleph_0.$$

Since each maximal subgroup of the regular semigroup S is finite, it follows from Lemma 2.4.1 and Proposition 2.4.4 that every $\mathcal{H}$-class of S is finite. Hence S is $\aleph_0$-categorical by Proposition 3.3.18. □

3.4 Principal factors of an $\aleph_0$-categorical semigroups

Our interest in this section is in determining how $\aleph_0$-categoricity effects the principal factors of a semigroup. We observe first that the principal factors of an $\aleph_0$-categorical semigroup behave in much the same way as the maximal subgroups:

Theorem 3.4.1. *The principal factors of an $\aleph_0$-categorical semigroup S are $\aleph_0$-categorical, and either completely 0-simple, completely simple or null. Moreover, S has only finitely many principal factors, up to isomorphism.*

Proof. Since S is $\aleph_0$-categorical, the ideals $J(a) = S^1aS^1$ are $\aleph_0$-categorical by Example 3.3.4 and Proposition 3.3.5. Let ϕ be an automorphism of S such that $a\phi = b$. Then $J(a)\phi = J(b)$ as $\{(J(a), a) : a \in S\}$ is a system of 1-p.p.r.c. subsemigroups of S. Moreover, as $\mathcal{J}$ is preserved under automorphisms we have $J_a\phi = J_b$, and so

$$I(a)\phi = (J(a) \setminus J_a)\phi = J(b) \setminus J_b = I(b).$$

Consequently, $\{(I(a), a) : a \in S\}$ is a system of 1-p.p.r.c. subsemigroups of S and, in particular, $I(a)$ is an a-p.r.c. ideal of $J(a)$ for each $a \in S$. Hence $J(a)/I(a)$ is $\aleph_0$-categorical by Lemma 3.3.17. If the kernel $K(S)$ of S exists, it is a $\mathcal{J}$-class of S, and is thus $\aleph_0$-categorical. Hence each principal factor of S is $\aleph_0$-categorical.

Moreover, as $\phi|_{J(a)}$ is an isomorphism from $J(a)$ to $J(b)$ with $I(a)\phi|_{J(a)} = I(b)$, it follows that the isomorphism $\phi|_{J(a)}$ preserves $\rho_{I(a)}$ and $\rho_{I(b)}$, and so $J(a)/I(a)$ is isomorphic to $J(b)/I(b)$ by Proposition 3.3.14. Hence the set

$$\{J(a)/I(a) : a \in S\}$$

of non minimal ideal principal factors of S has at most $|S/\sim_{S,1}|$ elements, up to isomorphism. Since $K(S)$ is the unique minimal ideal of S, if it exists, S has only finitely many principal factors, up to isomorphism.

By Theorem 2.5.1, the principal factors of a semigroup S are either 0-simple, simple or null. A periodic 0-simple semigroup is completely 0-semigroup by [20, Corollary 2.56]. If M is a periodic simple semigroup then M^0 is completely 0-simple, so that $M = M^0 \setminus \{0\}$ contains a minimal idempotent under the natural order. Hence M is completely simple. Since every ULF semigroup is periodic, each principal factor is either completely 0-simple, completely simple or null by Corollary 3.2.4. □

By Example 3.2.1 we have that every null semigroup is $\aleph_0$-categorical. To understand the $\aleph_0$-categoricity of an arbitrary semigroup it is therefore essential to examine the completely simple and completely 0-simple cases.

3.5 $\aleph_0$-categorical Rees matrix semigroups

When studying the model theoretic properties of a semigroup S with zero it can be important to distinguish which signature we are working over: the signature of semigroups L_S or the signature of semigroups with zero, $L_0 = L_S \cup \{0\}$, where 0 is a constant symbol. When applying the RNT to L_0, we are studying the action of automorphisms of a semigroup S which fix 0, on n-tuples of S. However all L_S-automorphisms of S necessarily fix 0, and so in the context of $\aleph_0$-categoricity it makes no difference which language we use. We have thus proven the following proposition.

Proposition 3.5.1. *Let S be a semigroup with zero. Then S is $\aleph_0$-categorical as a semigroup if and only if it is $\aleph_0$-categorical as a semigroup with zero.*

In keeping with previous sections, we continue to work over the signature L_S. We also remark that much of the early work in this section can easily be transferred to the signature of monoids L_{Mo} for working with the $\aleph_0$-categoricity of monoids.

Given a semigroup S with zero, we denote $S^* = S \setminus \{0\}$. The following result is then immediate from Lemma 3.2.12.

Corollary 3.5.2. *A semigroup with zero S is $\aleph_0$-categorical if and only if*

$$|(S^*)^n / \sim_{S,n}| < \aleph_0,$$

for each $n \geq 1$.

On the other hand, $\aleph_0$-categorical semigroups can be built from a known $\aleph_0$-categorical semigroup simply by adjoining a zero:

Lemma 3.5.3. *A semigroup without zero S is $\aleph_0$-categorical if and only if S^0 is $\aleph_0$-categorical.*

Proof. Recall that every automorphism of S extends to an automorphism of S^0, simply by fixing 0. Consequently, if S is $\aleph_0$-categorical then so is S^0. Conversely, if S^0 is $\aleph_0$-categorical, then so is its characteristic subsemigroup S. $\square$

We remark that the lemma above still holds when we *force* a zero. That is, if $S = S^0$ has a zero, then we may adjoin a new zero, say $\sharp$, to S by defining $\sharp s = s\sharp = \sharp$ for all $s \in S$. This way, we can build new $\aleph_0$-categorical semigroups by repeatedly forcing a zero.

Motivated by the previous section, we now examine the $\aleph_0$-categoricity of a completely (0-)simple semigroup. Note that if S is a completely simple semigroup, then S^0 is isomorphic to a Rees matrix semigroup with sandwich matrix without zero entries [55, Section 3.3]. Consequently, by the Rees Theorem and Lemma 3.5.3, to examine the $\aleph_0$-categoricity of both completely simple and completely 0-simple semigroups, it suffices to study Rees matrix semigroups.

Given a Rees matrix semigroup $S = \mathcal{M}^0[G; I, \Lambda; P]$ with $P = (p_{\lambda,i})$, we let $G(P)$ denote the subset of G of all non-zero entries of P, that is,

$$G(P) := \{p_{\lambda,i} : p_{\lambda,i} \neq 0\}.$$

The idempotents of S are easily described [55, Page 71]:

$$E(S) = \{(i, p_{\lambda,i}^{-1}, \lambda) : p_{\lambda,i} \neq 0\}.$$

Since there exists a relatively simple isomorphism theorem for Rees matrix semigroups (Theorem 2.5.3), we should be hopeful of achieving a thorough understanding of $\aleph_0$-categorical Rees matrix semigroups via the RNT. However, from the isomorphism theorem it is not clear how the $\aleph_0$-categoricity of the semigroup $\mathcal{M}^0[G; I, \Lambda; P]$ effects the sets I and Λ. We instead follow a technique of Graham [38] and Houghton [53] of constructing a graph from the sets I and Λ. We first give a brief outline of the material required for this construction.

A *bipartite graph* is a (simple) graph whose vertices can be split into two disjoint non-empty sets L and R such that every edge connects a vertex in L to a vertex in R. The sets L and R are called the *left set* and the *right set*, respectively. Formally, a bipartite graph is a triple $\Gamma = \langle L, R, E\rangle$ such that L and R are non-empty trivially intersecting sets and

$$E \subseteq \{\{x, y\} : x \in L,\ y \in R\}.$$

We call $L \cup R$ the set of *vertices* of Γ and E the set of *edges*. An isomorphism between a pair of bipartite graphs $\Gamma = \langle L, R, E\rangle$ and $\Gamma' = \langle L', R', E'\rangle$ is a bijection $\psi : L \cup R \to L' \cup R'$ such that $L\psi = L'$, $R\psi = R'$ (so, ψ is the union of bijections from L to L' and from R to R') and $\{l, r\} \in E$ if and only if $\{l\psi, r\psi\} \in E'$. We are therefore regarding bipartite graphs in the signature $L_{BG} = \{Q_L, Q_R, E\}$, where Q_L and Q_R are unary relations, which we interpret as the sets L and R, respectively, and E is a binary relation interpreted as the edge relation (recalling our convention of letting E denote the edge relation *and* the set of edges).

Let $\Gamma = \langle L, R, E\rangle$ be a bipartite graph. Then Γ is called *complete* if, for all $x \in L$, $y \in R$, we have $\{x, y\} \in E$. If $E = \emptyset$ then Γ is called *empty*. If each vertex of Γ is incident to exactly one edge, then Γ is called a *perfect matching*. The *complement* of Γ is the bipartite graph $\langle L, R, E'\rangle$ with

$$E' = \{\{x, y\} : x \in L, y \in R, \{x, y\} \notin E\}.$$

Hence an empty bipartite graph is the complement of a complete bipartite graph, and vice-versa. We call Γ *random* if, for each $k, \ell \in \mathbb{N}$, and for every distinct

$x_1, \ldots, x_k, y_1, \ldots, y_\ell$ in L (in R) there exist infinitely many $u \in R$ ($u \in L$) such that $\{u, x_i\} \in E$ but $\{u, y_j\} \notin E$ for each $1 \leq i \leq k$ and $1 \leq j \leq \ell$.

It can be easily shown that, for each pair $n, m \in \mathbb{N}^*$, there exists a unique (up to isomorphism) complete bipartite graph with left set of size n and right set of size m, which we denote as $K_{n,m}$. There also exists a unique, up to isomorphism, perfect matching with left and right sets of size n, denoted P_n. Similar uniqueness holds for the empty bipartite graph $E_{n,m}$ with left set of size n and right set of size m, and complements of the perfect matching P_n, which we denote as CP_n. Less obviously, any pair of random bipartite graphs are isomorphic [29].

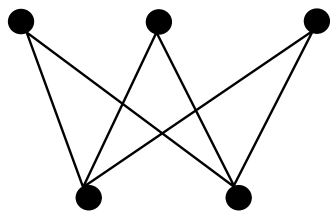

Figure 3.1: $K_{3,2}$.

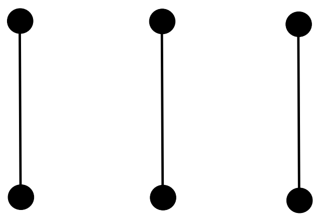

Figure 3.2: P_3.

Homogeneous bipartite graphs have been classified by Goldstern in [35].

Theorem 3.5.4. *A bipartite graph is homogeneous if and only if it is isomorphic to one of:*

(i) *the complete bipartite graph* $K_{n,m}$*,*

(ii) *the empty bipartite graph* $E_{n,m}$*,*

(iii) *a perfect matching* P_n*,*

(iv) *the complement of a perfect matching* CP_n*,*

(v) *a random bipartite graph,*

for some $n, m \in \mathbb{N}^*$*.*

Since bipartite graphs are relational structures and thus ULF, homogeneous bipartite graphs are $\aleph_0$-categorical by Theorem 1.2.26. Unfortunately, no complete classification of $\aleph_0$-categorical bipartite graphs exists.

Let $\Gamma = \langle L, R, E \rangle$ be a bipartite graph. A *path* $\mathfrak{p}$ in Γ is a finite sequence of vertices

$$\mathfrak{p} = (v_1, v_2, \ldots, v_n)$$

such that v_i and v_{i+1} are adjacent for each $1 \leq i \leq n-1$. For example, if $\{x, y\}$ is an edge in E then both (x, y) and (y, x) are paths in Γ. A pair of vertices x and y

are *connected*, denoted $x \bowtie y$, if and only if there exists a path $(v_1, v_2, \ldots, v_n)$ in Γ such that $v_1 = x$ and $v_n = y$. It is clear that $\bowtie$ is an equivalence relation on the set of vertices of Γ, and we call the equivalence classes the *connected components* of Γ. We observe that each connected component is a sub-bipartite graph of Γ under the induced structure. We let $\mathcal{C}(\Gamma)$ denote the set of connected components of Γ.

Let Γ be a bipartite graph with $\mathcal{C}(\Gamma) = \{\Gamma_i : i \in A\}$. For any automorphism ϕ of Γ and $x, y \in \Gamma$ we have that $(x, v_2, \ldots, v_{n-1}, y)$ is a path in Γ if and only if $(x\phi, v_2\phi, \ldots, v_{n-1}\phi, y\phi)$ is a path in Γ, since ϕ preserves edges and non-edges. Hence $x \bowtie y$ if and only if $x\phi \bowtie y\phi$, and so there exists a bijection π of A such that $\Gamma_i\phi = \Gamma_{i\pi}$ for each $i \in I$. We have thus proven the backward direction to the following result, the converse being immediate.

Proposition 3.5.5. *Let $\Gamma = \langle L, R, E\rangle$ be a bipartite graph with connected components $\mathcal{C}(\Gamma) = \{\Gamma_i : i \in A\}$. Let π be a bijection of A and $\phi_i : \Gamma_i \to \Gamma_{i\pi}$ an isomorphism for each $i \in A$. Then $\bigcup_{i\in I} \phi_i$ is an automorphism of Γ. Conversely, every automorphism of Γ can be constructed in this way.*

Proposition 3.5.6. *Let $\Gamma = \langle L, R, E\rangle$ be a bipartite graph with connected components $\mathcal{C}(\Gamma) = \{\Gamma_i : i \in A\}$. Then Γ is $\aleph_0$-categorical if and only if each connected component is $\aleph_0$-categorical and $\mathcal{C}(\Gamma)$ is finite, up to isomorphism.*

Proof. ($\Rightarrow$) By Proposition 3.5.5 we have that, for any choice of $x_i \in \Gamma_i$ $(i \in A)$, the set $\{(\Gamma_i, x_i) : i \in A\}$ forms a system of 1-p.p.r.c. sub-bipartite graphs of Γ. The result then follows from Propositions 3.3.5 and 3.3.9.

($\Leftarrow$) First we show that $\mathcal{C}(\Gamma)$ forms a $(\Gamma; \bar{A}; \Psi)$-system in Γ for some $\bar{A}$ and Ψ. Let $A_1, \ldots, A_r$ be the finite partition of A corresponding to the isomorphism types of the connected components of Γ, that is, $\Gamma_i \cong \Gamma_j$ if and only if $i, j \in A_k$ for some k. Fix $\bar{A} = (A; A_1, \ldots, A_r)$. For each $i, j \in A$, let $\Psi_{i,j} = \mathrm{Iso}(\Gamma_i; \Gamma_j)$ and fix $\Psi = \bigcup_{i,j\in A} \Psi_{i,j}$. Then Ψ clearly satisfy Conditions 3.2.14(A), 3.2.14(B) and 3.2.14(C). Let $\pi \in \mathrm{Aut}(\bar{A})$ and, for each $i \in A$, let $\phi_i \in \Psi_{i,i\pi}$. Then by Proposition 3.5.5, $\phi = \bigcup_{i\in A} \phi_i$ is an automorphism of Γ, and so Ψ satisfies Condition 3.2.14(D). Hence $\mathcal{C}(\Gamma)$ forms an $(\Gamma; \bar{A}; \Psi)$-system. Each Γ_i is $\aleph_0$-categorical (over $\Psi_{i,i} = \mathrm{Aut}(\Gamma_i)$) and $\bar{A}$ is $\aleph_0$-categorical by Corollary 3.2.10, and so Γ is $\aleph_0$-categorical by Corollary 3.2.16. $\square$

Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be a Rees matrix semigroup with $P = (p_{\lambda,i})$. Then we form a bipartite graph $\Gamma(P) = \langle I, \Lambda, E\rangle$ with edge set

$$E = \{\{i, \lambda\} : p_{\lambda,i} \neq 0\},$$

which we call the *induced bipartite graph of S*.

This construct has long been fundamental to the study of Rees matrix semigroups, and has its roots in a paper by Graham in [38]. Here, it is used to describe the maximal nilpotent subsemigroups of a Rees matrix semigroup, where a semigroup is *nilpotent* if some power is equal to $\{0\}$. All maximal subsemigroups of a finite Rees matrix semigroup were described in the same paper, a result which was later extended in [39] to arbitrary finite semigroups. In [54], Howie used the induced bipartite graph to describe the subsemigroup of a Rees matrix semigroup generated by its idempotents. Finally, in [53], Houghton described the homology of the induced bipartite graph, and a detailed overview of his work is given in [80].

Example 3.5.7. Let $S = \mathcal{M}^0[G; \{1, 2, 3\}, \{\lambda, \mu\}; P]$ where

$$P = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} \lambda \\ \mu \end{array} & \begin{bmatrix} a & b & 0 \\ 0 & c & d \end{bmatrix} \end{array}.$$

Then the induced bipartite graph of S is given in Figure 3.3.

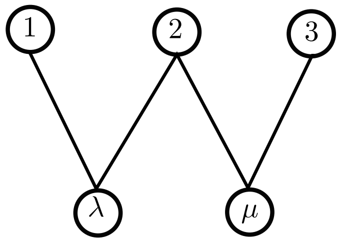

Figure 3.3: Induced bipartite graph.

Example 3.5.8. Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be such that P has no zero entries, so that S is isomorphic to a completely simple semigroup with zero adjoined. Then $\Gamma(P)$ is a complete bipartite graph.

Notation 3.5.9. Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be a Rees matrix semigroup. Given an n-tuple $\underline{a} = ((i_1, g_1, \lambda_1), \dots, (i_n, g_n, \lambda_n))$ of S^*, we denote $\Gamma(\underline{a})$ as the $2n$-tuple $(i_1, \lambda_1, \dots, i_n, \lambda_n)$ of $\Gamma(P)$.

Following [3], we adapt the isomorphism theorem for Rees matrix semigroups (Theorem 2.5.3) to explicitly highlight the roll of the induced bipartite graph:

Theorem 3.5.10. *Let $S_1 = \mathcal{M}^0[G_1; I_1, \Lambda_1; P_1]$ and $S_2 = \mathcal{M}^0[G_2; I_2, \Lambda_2; P_2]$ be a pair of Rees matrix semigroups with sandwich matrices $P_1 = (p_{\lambda,i})$ and $P_2 = (q_{\mu,j})$, respectively. Let $\psi : \Gamma(P_1) \to \Gamma(P_2)$ and $\theta : G_1 \to G_2$ be isomorphisms, and $u_i, v_\lambda \in G_2$ for each $i \in I_1, \lambda \in \Lambda_1$. Then the mapping $\phi : S_1 \to S_2$ given by*

$$(i, g, \lambda)\phi = (i\psi, u_i(g\theta)v_\lambda, \lambda\psi)$$

is an isomorphism if and only if

$$p_{\lambda,i}\,\theta = v_\lambda \cdot q_{\lambda\psi,i\psi} \cdot u_i, \text{ whenever } p_{\lambda,i} \neq 0.$$

Moreover, every isomorphism from S_1 to S_2 can be described in this way. □

The isomorphism ϕ will be denoted as $(\theta, \psi, (u_i)_{i\in I}, (v_\lambda)_{\lambda\in\Lambda})$. We also denote the induced group isomorphism θ as ϕ_G, and the induced bipartite graph isomorphism ψ as $\psi_{\Gamma(P)}$, so that $\phi = (\phi_G, \psi_{\Gamma(P)}, (u_i)_{i\in I}, (v_\lambda)_{\lambda\in\Lambda})$.

The composition and inverses of isomorphisms between Rees matrix semigroups behave in a natural way as follows.

Corollary 3.5.11. *Let $S_k = \mathcal{M}^0[G_k; I_k, \Lambda_k; P_k]$ $(k = 1, 2, 3)$ be Rees matrix semigroups. Then for any pair of isomorphisms $\phi = (\theta, \psi, (u_i)_{i\in I_1}, (v_\lambda)_{\lambda\in\Lambda_1}) \in Iso(S_1; S_2)$ and $\phi' = (\theta', \psi', (u'_j)_{j\in I_2}, (v'_\mu)_{\mu\in\Lambda_2}) \in Iso(S_2; S_3)$ we have*

(i) $\phi\phi' = \big(\theta\theta', \psi\psi', (u'_{i\psi}(u_i\theta'))_{i\in I_1}, ((v_\lambda\theta')v'_{\lambda\psi})_{\lambda\in\Lambda_1}\big)$,

(ii) $\phi^{-1} = (\theta^{-1}, \psi^{-1}, ((u_{i\psi^{-1}})^{-1}\theta^{-1})_{i\in I_2}, ((v_{\lambda\psi^{-1}})^{-1}\theta^{-1})_{\lambda\in\Lambda_2})$.

Proof. If $(i, g, \lambda) \in S_1$ then

$$\begin{aligned}
(i, g, \lambda)\phi\phi' &= (i\psi, u_i(g\theta)v_\lambda, \lambda\psi)\phi' \\
&= \Big(i\psi\psi', u'_{i\psi}\big[\big(u_i(g\theta)v_\lambda\big)\theta'\big]v'_{\lambda\psi}, \lambda\psi\psi'\Big) \\
&= \Big(i\psi\psi', \big(u'_{i\psi}(u_i\theta')\big)\big(g\theta\theta'\big)\big((v_\lambda\theta')v'_{\lambda\psi}\big), \lambda\psi\psi'\Big) \\
&= (i, g, \lambda)\big(\theta\theta', \psi\psi', \big(u'_{i\psi}(u_i\theta')\big)_{i\in I_1}, \big((v_\lambda\theta')v'_{\lambda\psi}\big)_{\lambda\in\Lambda_1}\big),
\end{aligned}$$

and so the first result holds.

Now let $\varphi = (\theta^{-1}, \psi^{-1}, ((u_{i\psi^{-1}})^{-1}\theta^{-1})_{i\in I_2}, ((v_{\lambda\psi^{-1}})^{-1}\theta^{-1})_{\lambda\in\Lambda_2})$. Then by the previous part we have

$$\begin{aligned}
\phi\varphi &= \Big(\theta\theta^{-1}, \psi\psi^{-1}, \Big(\big((u_{i\psi\psi^{-1}})^{-1}\theta^{-1}\big)\big(u_i\theta^{-1}\big)\Big)_{i\in I_1}, \Big(\big(v_\lambda\theta^{-1}\big)\big(v_{\lambda\psi\psi^{-1}})^{-1}\theta^{-1}\big)\Big)_{\lambda\in\Lambda_1}\Big) \\
&= \big(1_{G_1}, 1_{\Gamma(P_1)}, ((u_i^{-1}u_i)\theta^{-1})_{i\in I_1}, ((v_\lambda v_\lambda^{-1})\theta^{-1})_{\lambda\in\Lambda_1}\big) \\
&= (1_{G_1}, 1_{\Gamma(P_1)}, (1)_{i\in I_1}, (1)_{\lambda\in\Lambda_1}) \\
&= 1_{S_1}
\end{aligned}$$

and similarly

$$\begin{aligned}
\varphi\phi &= \big(\theta^{-1}\theta, \psi^{-1}\psi, \big(u_{i\psi^{-1}}\big((u_{i\psi^{-1}})^{-1}\theta^{-1}\theta\big)\big)_{i\in I_2}, \big(\big((v_{\lambda\psi^{-1}})^{-1}\theta^{-1}\theta\big)v_{\lambda\psi^{-1}}\big)_{\lambda\in\Lambda_2} \\
&= (1_{G_2}, 1_{\Gamma(P_2)}, (1)_{i\in I_2}, (1)_{\lambda\in\Lambda_2}) \\
&= 1_{S_2}
\end{aligned}$$

and so $\varphi = \phi^{-1}$ as required. □

Let $\Gamma = \langle L, R, E\rangle$ be a bipartite graph. For each $n \in \mathbb{N}$, we let $\sigma_{\Gamma,n}$ be the equivalence relation on Γ^n given by

$$(x_1, \ldots, x_n)\, \sigma_{\Gamma,n}\, (y_1, \ldots, y_n) \Leftrightarrow [x_i \in L \Leftrightarrow y_i \in L, \text{ for each } 1 \leq i \leq n].$$

Since each entry of an n-tuple lies in either L or R we have that

$$|\Gamma^n/\sigma_{\Gamma,n}| = 2^n,$$

for each n. Due to the automorphisms of Γ fixing the sets L and R, it easily follows that

$$\sim_{\Gamma,n} \subseteq \sigma_{\Gamma,n}.$$

Proposition 3.5.12. *If $S = \mathcal{M}^0[G; I, \Lambda; P]$ is $\aleph_0$-categorical, then G and $\Gamma(P)$ are $\aleph_0$-categorical.*

Proof. Given $p_{\lambda,i} \neq 0$, we have that $\{(i, g, \lambda) : g \in G\}$ is a maximal subgroup of S isomorphic to G. Hence G is $\aleph_0$-categorical by Corollary 3.3.8. Now let $\underline{a} = (a_1, \ldots, a_n)$ and $\underline{b} = (b_1, \ldots, b_n)$ be a pair of $\sigma_{\Gamma,n}$-related n-tuples of $\Gamma(P)$. Let $i_1 < i_2 < \cdots < i_s$ and $j_1 < j_2 < \cdots < j_t$ be the indexes of entries of $\underline{a}$ lying in L and R, respectively. Since $\underline{a}\, \sigma_{\Gamma,n}\, \underline{b}$ we also have that $i_1 < i_2 < \cdots < i_s$ and $j_1 < j_2 < \cdots < j_t$ are the indexes of entries of $\underline{b}$ lying in L and R, respectively. Suppose further that there exist $i \in I, \lambda \in \Lambda$ such that the n-tuples

$$\begin{aligned}&((a_{i_1}, 1, \lambda), \ldots, (a_{i_s}, 1, \lambda), (i, 1, a_{j_1}), \ldots, (i, 1, a_{j_t})) \quad \text{and}\\ &((b_{i_1}, 1, \lambda), \ldots, (b_{i_s}, 1, \lambda), (i, 1, b_{j_1}), \ldots, (i, 1, b_{j_t})),\end{aligned}$$

are automorphically equivalent via $\phi \in \mathrm{Aut}(S)$, say. Then $a_{i_r}\phi_{\Gamma(P)} = b_{i_r}$ and $a_{j_{r'}}\phi_{\Gamma(P)} = b_{j_{r'}}$, for each $1 \leq r \leq s$ and $1 \leq r' \leq t$ by Theorem 3.5.10. Hence $\underline{a} \sim_{\Gamma(P),n} \underline{b}$ via $\phi_{\Gamma(P)}$, and we have thus shown that

$$|\Gamma(P)^n/ \sim_{\Gamma(P),n}| \leq 2^n \cdot |S^n/ \sim_{S,n}|.$$

Hence $\Gamma(P)$ is $\aleph_0$-categorical by the $\aleph_0$-categoricity of S. $\square$

In the next subsection we construct a counterexample to the converse of Proposition 3.5.12. Our method will be to transfer the concept of the connected components of bipartite graphs to corresponding subsemigroups of Rees matrix semigroups.

3.5.1 Connected Rees components

Let $S_k = \mathcal{M}^0[G; I_k, \Lambda_k; P_k]$ $(k \in A)$ be a collection of Rees matrix semigroups with $P_k = (p^{(k)}_{\lambda,i})$ and $S_k \cap S_\ell = \{0\}$ for each $k, \ell \in A$. Then we may form a single Rees matrix semigroup $S = \mathcal{M}^0[G; I, \Lambda; P]$, where $I = \bigcup_{k\in A} I_k$, $\Lambda = \bigcup_{k\in A} \Lambda_k$ and

$P = (p_{\lambda,i})$ is the Λ by I matrix defined by

$$p_{\lambda,i} = \begin{cases} p^{(k)}_{\lambda,i} & \text{if } \lambda, i \in \Gamma(P_k), \text{ for some } k \\ 0 & \text{else.} \end{cases}$$

That is, P is the block matrix

$$P = \begin{bmatrix} P_1 & 0 & 0 & \cdots \\ 0 & P_2 & 0 & \cdots \\ 0 & 0 & P_3 & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix}. \tag{3.3}$$

We denote S by $\circledast^G_{k\in A} S_k$. The subsemigroups S_k of S are called the *Rees components of S*. Notice that each $\Gamma(P_k)$ is a union of connected components of $\Gamma(P)$. The subsemigroup S_k will be called a *connected Rees component of S* if $\Gamma(P_k)$ is connected (and is therefore a connected component of $\Gamma(P)$).

Conversely, for any Rees matrix semigroup $S = \mathcal{M}^0[G; I, \Lambda; P]$ there exist partitions $\{I_k : k \in A\}$ and $\{\Lambda_k : k \in A\}$ of I and Λ, respectively, such that $\mathcal{C}(\Gamma(P)) = \{\Lambda_k \cup I_k : k \in A\}$. Consequently, for each $k \in A$, the subsemigroup $S_k = \mathcal{M}^0[G; I_k, \Lambda_k; P_k]$ of S is a connected Rees component, where P_k is the $\Lambda_k \times I_k$ submatrix of P. It then follows by the observation following Theorem 2.5.3 that, by permuting the rows and columns of P if necessary, we may assume w.l.o.g. that P is a block matrix of the form (3.3).

Note that if S is a Rees matrix semigroup with connected Rees components $\{S_k : k \in A\}$ then

$$E(S) = \bigcup_{k\in A} E(S_k). \tag{3.4}$$

To see this, note that $0 \in E(S_k)$ for each $k \in A$, and if $(i, p^{-1}_{\lambda,i}, \lambda)$ is a non-zero idempotent of S then $p_{\lambda,i} \neq 0$. Hence i and λ are adjacent in the induced bipartite graph of S, and thus lie in the same connected component. It is then immediate that $(i, p^{-1}_{\lambda,i}, \lambda)$ is contained in some connected Rees component of S.

Since automorphisms of $\Gamma(P)$ arise as collections of isomorphisms between its connected components, we reach the following alternative description of automorphisms of a Rees matrix semigroups.

Corollary 3.5.13. *Let $S = \circledast^G_{k\in A} S_k = \mathcal{M}^0[G; I, \Lambda; P]$ be a Rees matrix semigroup such that each $S_k = \mathcal{M}^0[G; I_k, \Lambda_k; P_k]$ is a connected Rees component of S. Let π be a bijection of A and, for each $k \in A$, let $\phi_k = (\theta, \psi_k, (u^{(k)}_i)_{i\in I_k}, (v^{(k)}_\lambda)_{\lambda\in\Lambda_k})$ be an isomorphism from S_k to $S_{k\pi}$. Then $\phi = (\theta, \psi, (u_i)_{i\in I}, (v_\lambda)_{\lambda\in\Lambda})$ is an automorphism of S, where $\psi = \bigcup_{k\in A} \psi_k$, and if $i, \lambda \in \Gamma(P_k)$ then $u_i = u^{(k)}_i$ and $v_\lambda = v^{(k)}_\lambda$. Moreover, every automorphism of S can be described in this way.*

Proof. Let S and ϕ be constructed as in the hypothesis of the corollary. Then the map ψ is an automorphism of $\Gamma(P)$ by Proposition 3.5.5. Let $i \in I$ and $\lambda \in \Lambda$ be such that $p_{\lambda,i} \neq 0$. Then $i, \lambda \in \Gamma(P_k)$ for some $k \in A$, and so by Theorem 3.5.10 we have

$$p_{\lambda,i}\theta = v_\lambda^{(k)} p_{\lambda\psi_k, i\psi_k} u_i^{(k)} = v_\lambda p_{\lambda\psi, i\psi} u_i,$$

since ϕ_k is an isomorphism and ψ extends ψ_k. Hence, again by Theorem 3.5.10, ϕ is an automorphism of S as required.

Conversely, if $(\theta', \psi', (u'_i)_{i\in I}, (v'_\lambda)_{\lambda\in\Lambda})$ is an automorphism of S then by Proposition 3.5.5 there exists a bijection π' of A and isomorphisms $\psi'_k : \Gamma(P_k) \to \Gamma(P_{k\pi'})$, for each $k \in A$, such that $\psi' = \bigcup_{k\in A} \psi'_k$. Hence $(\theta', \psi'_k, (u'_i)_{i\in I_k}, (v'_\lambda)_{\lambda\in\Lambda_k})$ is an isomorphism from S_k to $S_{k\pi'}$ for each $k \in A$. □

We observe that the induced group automorphisms of the isomorphisms ϕ_k are all equal (to θ).

Proposition 3.5.14. *Let $S = \circledast_{k\in A}^G S_k$ be an $\aleph_0$-categorical Rees matrix semigroup such that each S_k is a connected Rees component of S. Then each S_k is $\aleph_0$-categorical and S has finitely many connected Rees components, up to isomorphism.*

Proof. We claim that $\{(S_k, a_k) : k \in A\}$ is a system of 1-p.p.r.c. subsemigroups of S for any $a_k \in S_k^*$, to which the result follows by Propositions 3.3.5 and 3.3.9. Indeed, let ϕ be an automorphism of S such that $a_k\phi = a_l$ for some k, l. Then, by Corollary 3.5.13, there exists a bijection π of A with $S_k\phi = S_{k\pi} = S_l$ as required. □

Our interest is now in attaining a converse to the proposition above, since it would provide us with a new method for building $\aleph_0$-categorical Rees matrix semigroups from 'old'. With the aid of Lemma 3.2.15, we prove that a converse exists in the class of Rees matrix semigroups over finite groups. The case where the maximal subgroups are infinite is an open problem.

Given a pair $S = \mathcal{M}^0[G; I, \Lambda; P]$ and $S' = \mathcal{M}^0[G; I', \Lambda'; Q]$ of Rees matrix semigroups over a group G, we denote $\mathrm{Iso}(S; S')(1_G)$ as the set of isomorphisms between S and S' with trivial induced group isomorphism. That is,

$$\mathrm{Iso}(S; S')(1_G) := \{(\theta, \psi, (u_i)_{i\in I}, (v_\lambda)_{\lambda\in\Lambda}) \in \mathrm{Iso}(S; S') : \theta = 1_G\}.$$

If $S = S'$ we denote this simply as $\mathrm{Aut}(S)(1_G)$, and notice that $\mathrm{Aut}(S)(1_G)$ is a subgroup of $\mathrm{Aut}(S)$ by Corollary 3.5.11.

Lemma 3.5.15. *Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be a Rees matrix semigroup over a finite group G. Then S is $\aleph_0$-categorical if and only if if S is $\aleph_0$-categorical over $Aut(S)(1_G)$.*

Proof. Let S be $\aleph_0$-categorical with $G = \{g_1, \dots, g_r\}$. Let $\underline{a}$ and $\underline{b}$ be a pair of n-tuples of S. For some fixed $p_{\mu,j} \neq 0$, let $\underline{g}$ be the r-tuple of S given by

$$\underline{g} = ((j, g_1, \mu), \dots, (j, g_r, \mu)),$$

and suppose $(\underline{a}, \underline{g}) \sim_{S,n+r} (\underline{b}, \underline{g})$ via $\phi = (\theta, \psi, (u_i)_{i\in I}, (v_\lambda)_{\lambda\in\Lambda})$, say. Then, for each $1 \leq k \leq r$, we have

$$(j, g_k, \mu)\phi = (j\psi, u_j(g_k\theta)v_\mu, \mu\psi) = (j, g_k, \mu),$$

so that $g_k\theta = u_j^{-1} g_k v_\mu^{-1}$. For each $i \in I, \lambda \in \Lambda$, let $\bar{u}_i = u_i u_j^{-1}$ and $\bar{v}_\lambda = v_\mu^{-1} v_\lambda$. Then, since $\underline{g}\phi = \underline{g}$, we have

$$\begin{aligned}(i\psi, \bar{u}_i g_k \bar{v}_\lambda, \lambda\psi) &= (i\psi, (u_i u_j^{-1}) g_k (v_\mu^{-1} v_\lambda), \lambda\psi) \\ &= (i\psi, u_i(g_k\theta)v_\lambda, \lambda\psi) \\ &= (i, g_k, \lambda)\phi,\end{aligned}$$

for any $(i, g_k, \lambda) \in S$. It follows that $\phi = (1_G, \psi, (\bar{u}_i)_{i\in I}, (\bar{v}_\lambda)_{\lambda\in\Lambda}) \in \mathrm{Aut}(S)(1_G)$, so that

$$(\underline{a}, \underline{g}) \sim_{S,\mathrm{Aut}(S)(1_G),n+r} (\underline{b}, \underline{g})$$

and in particular $\underline{a} \sim_{S,\mathrm{Aut}(S)(1_G),n} \underline{b}$. We have thus shown that

$$|S^n / \sim_{S,\mathrm{Aut}(S)(1_G),n}| \leq |S^{n+r} / \sim_{S,n+r}| < \aleph_0,$$

as S is $\aleph_0$-categorical. Hence S is $\aleph_0$-categorical over $\mathrm{Aut}(S)(1_G)$.

The converse is immediate. □

We now prove the converse to Proposition 3.5.14 in the case where the maximal subgroups are finite.

Theorem 3.5.16. *Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be a Rees matrix semigroup such that G is finite. Then S is $\aleph_0$-categorical if and only if each connected Rees component of S is $\aleph_0$-categorical and S has only finitely many connected Rees components, up to isomorphism.*

Proof. ($\Leftarrow$) Let $\{S_k : k \in A\}$ be the connected Rees components of S, which is finite up to isomorphism and with each S_k being $\aleph_0$-categorical. Define a relation η on A by $i \,\eta\, j$ if and only if $\mathrm{Iso}(S_i; S_j)(1_G) \neq \emptyset$. Hence by Corollary 3.5.11 we have that η is an equivalence relation.

We prove that A/η is finite. Suppose, seeking a contradiction, that there exists an infinite set X of pairwise η-inequivalent elements of A. Since S has finitely many connected components up to isomorphism, there exists an infinite subset $\{i_r : r \in \mathbb{N}\}$ of X such that $S_{i_n} \cong S_{i_m}$ for each n, m. Fix an isomorphism $\phi_{i_n} : S_{i_n} \to S_{i_1}$ for each

$n \in \mathbb{N}$. Since Aut(G) is finite, there exist distinct n, m such that $\phi_{i_n}^G = \phi_{i_m}^G$, and so $\phi_{i_n}\phi_{i_m}^{-1} \in \text{Iso}(S_{i_n}; S_{j_m})(1_G)$ by Corollary 3.5.11. Hence $i_n \, \eta \, i_m$, a contradiction, and so A/η is finite.

Let $S' = \bigcup_{k \in A} S_k$, noting that S' is a subsemigroup of S as $S_k S_l = 0$ for each $k \neq l$ in A. We prove that $\{S_k : k \in A\}$ forms an $(S; S'; \bar{A}; \Psi)$-system in S for some $\bar{A}$ and Ψ.

For each $i, j \in A$, let $\Psi_{i,j} = \text{Iso}(S_i; S_j)(1_G)$ and fix $\Psi = \bigcup_{i,j \in A} \Psi_{i,j}$. Let $A/\eta = \{A_1, \ldots, A_r\}$ and set $\bar{A} = (A; A_1, \ldots, A_r)$. Then, by our construction, if $i, j \in A_m$ for some m then $\Psi_{i,j} \neq \emptyset$, and so Ψ satisfies Condition 3.2.14(A). Furthermore, it follows immediately from Corollary 3.5.11 that Ψ satisfies Conditions 3.2.14(B) and 3.2.14(C). Finally, take any $\pi \in \text{Aut}(\bar{A})$ and, for each $k \in A$, let $\phi_k \in \Psi_{k,k\pi}$. Then as $\phi_k^G = 1_G$ for each $k \in A$, we may construct an automorphism ϕ of S from the set of isomorphisms $\{\phi_k : k \in A\}$ by Corollary 3.5.13. Since ϕ extends each ϕ_k by construction, we have that $\{S_k : k \in A\}$ forms an $(S; S'; \bar{A}; \Psi)$-system as required. Since S_k is $\aleph_0$-categorical, it is $\aleph_0$-categorical over $\Psi_{k,k} = \text{Aut}(S_k)(1_G)$ by Lemma 3.5.15. By Corollary 3.2.10, $\bar{A}$ is $\aleph_0$-categorical, and so

$$|(S')^n / \sim_{S,n}| < \aleph_0$$

by Lemma 3.2.15. Given that $E(S) \subseteq S'$ by (3.4), we therefore have that

$$|E(S)^n / \sim_{S,n}| \leq |(S')^n / \sim_{S,n}| < \aleph_0.$$

Hence S, being regular with finite maximal subgroups, is $\aleph_0$-categorical by Corollary 3.3.19.

($\Rightarrow$) Immediate from Proposition 3.5.14. $\square$

Those familiar with semigroup theory will observe that the subsemigroup S' of S in the proof above is an example of a 0-direct union of semigroups, which will be the topic of the subsequent section.

We now construct a counterexample to the converse of Proposition 3.5.12. By Proposition 3.5.14, it suffices to find a Rees matrix semigroup over an $\aleph_0$-categorical group with $\aleph_0$-categorical induced bipartite graph, but with infinitely many non-isomorphic connected Rees components.

Example 3.5.17. Let G be an $\aleph_0$-categorical infinite abelian group with identity element 1, and $\{g_i : i \in \mathbb{N}\}$ be an enumeration of its non-identity elements (such a group exists by Proposition 3.1.1). Let $I_k = \{i_s^k : s \in \mathbb{N}\}$ and $\Lambda_k = \{\lambda_t^k : t \in \mathbb{N}\}$ be infinite sets for each $k \in \mathbb{N}$. Let $P_k = (p^{(k)}_{\lambda_s^k, i_t^k})$ be the $\Lambda_k \times I_k$ matrix such that

$p^{(k)}_{\lambda^k_m, i^k_m} = g_m$ for each $1 \leq m \leq k$, and all other entries being 1, that is,

$$P_k = \begin{bmatrix} g_1 & 1 & 1 & \cdots & 1 & 1 & \cdots \\ 1 & g_2 & 1 & \cdots & 1 & 1 & \cdots \\ 1 & 1 & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 & 1 & \cdots \\ 1 & 1 & \cdots & 1 & g_k & 1 & \cdots \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then each $\Gamma(P_k)$ is a complete bipartite graph, isomorphic to $K_{\aleph_0, \aleph_0}$, and is thus $\aleph_0$-categorical by Theorem 3.5.4. For each $k \in \mathbb{N}$, let S_k be the connected Rees matrix semigroup $[G; I_k, \Lambda_k; P_k]$, and set

$$\circledast^G_{k \in \mathbb{N}} S_k = \mathcal{M}^0[G; I, \Lambda; P].$$

Then $\Gamma(P)$, being the disjoint union of the pairwise isomorphic $\aleph_0$-categorical bipartite graphs $\Gamma(P_k)$, is $\aleph_0$-categorical by Theorem 3.5.6.

We claim that $S_k \cong S_\ell$ if and only if $k = l$. Let $(\theta, \psi, (u_i)_{i \in I_k}, (v_\lambda)_{\lambda \in \Lambda_k})$ be an isomorphism between S_k and S_ℓ, and assume w.l.o.g. that $k \geq \ell$. Since there exist only finitely many rows of P_k and P_ℓ which have non-identity entries, there exists $\lambda^k_s \in \Lambda_k$ such that both row λ^k_s of P_k and row $\lambda^k_s \psi$ of P_ℓ consist entirely of identity entries. Then, for each $i^k_t \in I_k$,

$$p^{(k)}_{\lambda^k_s, i^k_t} \theta = 1\theta = 1 = v_{\lambda^k_s} \cdot p^{(\ell)}_{\lambda^k_s \psi, i^k_t \psi} \cdot u_{i^k_t} = v_{\lambda^k_s} u_{i^k_t}$$

by Theorem 3.5.10, so that

$$v^{-1}_{\lambda^k_s} = u_{i^k_1} = u_{i^k_2} = \cdots = u,$$

say. Dually, by considering the columns of P_k and P_ℓ, we have

$$v_{\lambda^k_1} = v_{\lambda^k_2} = \cdots = u^{-1},$$

since $v^{-1}_{\lambda^k_s} = u$. Hence, for each $1 \leq m \leq k$,

$$g_m \theta = p^{(k)}_{\lambda^k_m, i^k_m} \theta = u^{-1} \cdot p^{(\ell)}_{\lambda^k_m \psi, i^k_m \psi} \cdot u = p^{(\ell)}_{\lambda^k_m \psi, i^k_m \psi} \in \{g_1, \ldots, g_\ell\}$$

as G is abelian. It follows that the automorphism θ maps $\{g_1, \ldots, g_k\}$ to $\{g_1, \ldots, g_\ell\}$. Since $k \geq \ell$, this means that $k = l$, thus proving the claim. We have shown that $\mathcal{M}^0[G; I, \Lambda; P]$ has infinitely many non-isomorphic connected Rees components, and is therefore not $\aleph_0$-categorical by Proposition 3.5.14.

A natural question is to ask whether the converse of Proposition 3.5.12 holds for the class of Rees matrix semigroups with finitely many connected Rees components. A negative answer can be obtained by our usual method, by taking $G = \{1, a\} \cong \mathbb{Z}_2$, and letting P be the $\mathbb{N} \times \mathbb{N}$ matrix given by

$$p_{i,j} = \begin{cases} 1 & \text{if } i \geq j, \\ a & \text{if } i < j. \end{cases}$$

That is,

$$P = \begin{bmatrix} 1 & a & a & \cdots & a & a & \cdots \\ 1 & 1 & a & \cdots & a & a & \cdots \\ 1 & 1 & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & a & a & \cdots \\ 1 & 1 & \cdots & 1 & 1 & a & \cdots \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then $\Gamma(P)$ is isomorphic to the $\aleph_0$-categorical complete bipartite graph $K_{\aleph_0,\aleph_0}$. However $\mathcal{M}^0[G; \mathbb{N}, \mathbb{N}; P]$ is not $\aleph_0$-categorical, since $\{((1,1,1), (i,1,1)) : i \in \mathbb{N}\}$ can be shown to be an infinite set of distinct 2-automorphism types.

3.5.2 Labelled bipartite graphs

The problem arising in Example 3.5.17 is that by shifting from the sandwich matrix $P = (p_{\lambda,i})$ to the induced bipartite graph $\Gamma(P)$ we have "forgotten" the value of the entries $p_{\lambda,i}$. In this subsection we extend the construction of the induced bipartite graph of a Rees matrix semigroup to build classes of $\aleph_0$-categorical Rees matrix semigroups. This, together with the method devised in the previous subsection for constructing $\aleph_0$-categorical Rees matrix semigroups from sets of $\aleph_0$-categorical (connected) Rees matrix semigroups with finite maximal subgroups, will allow further examples of $\aleph_0$-categorical Rees matrix semigroups to be built.

Definition 3.5.18. Let $\Gamma = \langle L, R, E \rangle$ be a bipartite graph, Σ a set, and $f : E \to \Sigma$ a surjective map. Then the triple (Γ, Σ, f) is called a Σ-*labelled (by f) bipartite graph*, which we denote as Γ^f.

There is a natural signature in which to regard Σ-labelled bipartite graphs. For each $\sigma \in \Sigma$, take a binary relation symbol E_σ and let

$$L_{BG\Sigma} = L_{BG} \cup \{E_\sigma : \sigma \in \Sigma\}.$$

Then we call $L_{BG\Sigma}$ the *signature of Σ-labelled bipartite graphs*, where we interpret $(x, y) \in E_\sigma$ if and only if $\{x, y\} \in E$ and $\{x, y\}f = \sigma$.

Let $\Gamma^f = (\Gamma, \Sigma, f)$ and $\Gamma^{f'} = (\Gamma', \Sigma, f')$ be a pair of Σ-labelled bipartite graphs. Then, applying Definition 1.1.7 to the signature $L_{BG\Sigma}$, we have that Γ^f and $\Gamma^{f'}$ are isomorphic if there exists an isomorphism $\psi : \Gamma \to \Gamma'$ which preserves labels, that is, such that

$$\{x, y\}f = \sigma \Leftrightarrow \{x\psi, y\psi\}f' = \sigma.$$

Let Γ^f be a Σ-labelled bipartite graph. Then for any set Σ' and bijection $g : \Sigma \to \Sigma'$, we can form a Σ'-labelling of Γ simply by taking Γ^{fg}, which we call a *relabelling of* Γ^f. Notice that if ψ is an automorphism of Γ, then $\psi \in \mathrm{Aut}(\Gamma^f)$ if and only if $\psi \in \mathrm{Aut}(\Gamma^{fg})$. Indeed, if $\psi \in \mathrm{Aut}(\Gamma^f)$ then for any edge $\{x, y\}$ of Γ we have

$$\{x, y\}fg = \sigma' \Leftrightarrow \{x, y\}f = \sigma' g^{-1} \Leftrightarrow \{x\psi, y\psi\}f = \sigma' g^{-1} \Leftrightarrow \{x\psi, y\psi\}fg = \sigma',$$

since g is a bijection. The converse is proven similarly, and the following result is then immediate.

Lemma 3.5.19. *Let Γ^f be a Σ-labelling of a bipartite graph Γ. Then Γ^f is $\aleph_0$-categorical if and only if any relabelling of Γ^f is $\aleph_0$-categorical.*

Lemma 3.5.20. *If $\Gamma^f = (\Gamma, \Sigma, f)$ is an $\aleph_0$-categorical labelled bipartite graph then Σ is finite and Γ is $\aleph_0$-categorical.*

Proof. For each $\sigma \in \Sigma$, let $\{x_\sigma, y_\sigma\}$ be an edge in Γ such that $\{x_\sigma, y_\sigma\}f = \sigma$. Then $\{(x_\sigma, y_\sigma) : \sigma \in \Sigma\}$ is a set of distinct 2-automorphism types of Γ^f, and so Σ is finite by the RNT. Since Γ is the L_{BG}-reduct of Γ^f, the final result is immediate from Corollary 3.2.3. $\square$

A consequence of the previous pair of lemmas is that, in the context of $\aleph_0$-categoricity, it suffices to consider finitely labelled bipartite graphs, with labelling set $\mathbf{m} = \{1, 2, \ldots, m\}$ for some $m \in \mathbb{N}$. We now construct examples of $\aleph_0$-categorical labelled bipartite graphs.

Lemma 3.5.21. *Let $\Gamma^f = (\langle L, R, E\rangle, \mathbf{m}, f)$ be an $\mathbf{m}$-labelled bipartite graph such that either L or R are finite. Then Γ^f is $\aleph_0$-categorical.*

Proof. Without loss of generality assume that $L = \{l_1, l_2, \ldots, l_r\}$ is finite. Define a relation τ on R by $y \,\tau\, y'$ if and only if y and y' are adjacent to the same elements in L and $\{l_i, y\}f = \{l_i, y'\}f$ for each such $l_i \in L$. Note that since both L and $\mathbf{m}$ are finite, R has finitely many τ-classes, say $R_1, \ldots, R_t$. Fix $\mathcal{A} = (R; R_1, \ldots, R_t)$.

Since L is finite, to prove that Γ^f is $\aleph_0$-categorical it suffices to show that $(\Gamma^f \setminus L)^n = R^n$ has finitely many $\sim_{\Gamma^f,n}$-classes for each $n \in \mathbb{N}$ by Lemma 3.2.12. Let $\underline{a} = (r_1, \ldots, r_n)$ and $\underline{b} = (r_1', \ldots, r_n')$ be n-tuples of R such that $\underline{a} \sim_{\mathcal{A},n} \underline{b}$ via $\psi \in \mathrm{Aut}(\mathcal{A})$, say. We claim that the map $\bar{\psi} : \Gamma^f \to \Gamma^f$ which fixes L and is such that $\bar{\psi}|_R = \psi$ is an automorphism of Γ^f. Indeed, as ψ setwise fixes the τ-classes, we have $(\lambda, \lambda\psi) \in \tau$ for each $\lambda \in R$. Hence λ and $\lambda\psi$ are adjacent to the same elements in L, and so

$$\{l_i, \lambda\} \in E \Leftrightarrow \{l_i, \lambda\psi\} \in E \Leftrightarrow \{l_i\bar{\psi}, \lambda\bar{\psi}\} \in E,$$

so that $\bar{\psi}$ is an automorphism of Γ. Similarly $\{l_i, \lambda\}f = \{l_i, \lambda\psi\}f = \{l_i\bar{\psi}, \lambda\bar{\psi}\}f$, so that $\bar{\psi}$ preserves labels. This proves the claim.

For each $1 \leq k \leq n$ we have $r_k\bar{\psi} = r_k\psi = r'_k$, so that $\underline{a} \sim_{\Gamma^f,n} \underline{b}$. We have thus shown that

$$|(\Gamma^f \setminus L)^n / \sim_{\Gamma^f,n}| \leq |\mathcal{A}^n / \sim_{\mathcal{A},n}|.$$

However $\mathcal{A}$ is $\aleph_0$-categorical by Corollary 3.2.10, and so $|\mathcal{A}^n / \sim_{\mathcal{A},n}|$ is finite for each $n \geq 1$. Hence Γ^f is $\aleph_0$-categorical by Lemma 3.2.12. □

Lemma 3.5.22. *Let $\Gamma^f = (\langle L, R, E\rangle, \mathbf{m}, f)$ be such that there exists $p \in \mathbf{m}$ with $\{x, y\}f = p$ for all but finitely many edges in Γ. Then Γ^f is $\aleph_0$-categorical if and only if Γ is $\aleph_0$-categorical.*

Proof. Let $p \in \mathbf{m}$ be such that $\{x, y\}f = p$ for all but finitely many edges in Γ. Suppose Γ is $\aleph_0$-categorical, and that $\{l_1, r_1\}, \dots, \{l_t, r_t\}$ are precisely the edges of Γ such that $\{l_k, r_k\}f \neq p$, where $l_i \in L$ and $r_i \in R$. Let $\underline{a}$ and $\underline{b}$ be n-tuples of Γ^f such that

$$(\underline{a}, l_1, r_1, \dots, l_t, r_t) \sim_{\Gamma,n+2t} (\underline{b}, l_1, r_1, \dots, l_t, r_t)$$

via $\psi \in \mathrm{Aut}(\Gamma)$, say. We claim that ψ is an automorphism of Γ^f. For each $1 \leq k \leq t$ we have $l_k\psi = l_k$ and $r_k\psi = r_k$ so that

$$\{l_k, r_k\}f = \{l_k\psi, r_k\psi\}f.$$

It follows that $\{l, r\}f = p$ if and only if $\{l\psi, r\psi\}f = p$, and so ψ preserves all labels, thus proving the claim. We have thus shown that $\underline{a} \sim_{\Gamma^f,n} \underline{b}$ via ψ, so that

$$|(\Gamma^f)^n / \sim_{\Gamma^f,n}| \leq |\Gamma^{n+2t} / \sim_{\Gamma,n+2t}| < \aleph_0$$

by the $\aleph_0$-categoricity of Γ. Hence Γ^f is $\aleph_0$-categorical.

The converse is immediate from Lemma 3.5.20. □

Given a Rees matrix semigroup $S = \mathcal{M}^0[G; I, \Lambda; P]$, we form a $G(P)$-labelling of the induced bipartite graph $\Gamma(P) = \langle I, \Lambda, E\rangle$ of S in the natural way by taking the labelling $f : E \to G(P)$ given by

$$\{i, \lambda\}f = p_{\lambda,i}.$$

We denote $\Gamma(P)^f$ by $\Gamma(P)^l$, which we call the *induced labelled bipartite graph* of S.

Note that, unlike the induced bipartite graph $\Gamma(P)$, the induced labelled bipartite graph $\Gamma(P)^l$ obtained from S is not uniquely defined up to isomorphism. That is, there exist isomorphic Rees matrix semigroups with non-isomorphic induced labelled bipartite graphs. For example, let G be a non-trivial group and P and Q be $\mathbf{1} \times \mathbf{2}$ matrices over G^0 given by

$$P = \begin{pmatrix} 1 & a \end{pmatrix} \quad Q = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

where $a \notin \{0, 1\}$. Let $S = \mathcal{M}^0[G; \mathbf{2}, \mathbf{1}; P]$ and $T = \mathcal{M}^0[G; \mathbf{2}, \mathbf{1}; Q]$, noting that $\Gamma(P) = \Gamma(Q)$ (and are isomorphic to $K_{2,1}$). Then $(1_G, 1_{\Gamma(P)}, (u_i)_{i \in \mathbf{2}}, (v_\lambda)_{\lambda \in \mathbf{1}})$ is an isomorphism from S to T, where $u_1 = 1 = v_1$, and $u_2 = a$. However, since $\Gamma(P)^l$ and $\Gamma(Q)^l$ have different labelling sets, they are not isomorphic.

Proposition 3.5.23. *Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be such that G and $\Gamma(P)^l$ are $\aleph_0$-categorical. Then S is $\aleph_0$-categorical.*

Proof. Since $\Gamma(P)^l$ is $\aleph_0$-categorical, the set of group entries of P, $G(P)$, is finite by Lemma 3.5.20, say $G(P) = \{x_1, \dots, x_r\}$. Let $\underline{a} = ((i_1, g_1, \lambda_1), \dots, (i_n, g_n, \lambda_n))$ and $\underline{b} = ((j_1, h_1, \mu_1), \dots, (j_n, h_n, \mu_n))$ be a pair of n-tuples of S under the pair of conditions that

(1) $(g_1, \dots, g_n, x_1, \dots, x_r) \sim_{G,n+r} (h_1, \dots, h_n, x_1, \dots, x_r)$,

(2) $\Gamma(\underline{a}) \sim_{\Gamma(P)^l,2n} \Gamma(\underline{b})$,

via $\theta \in \mathrm{Aut}(G)$ and $\psi \in \mathrm{Aut}(\Gamma(P)^l)$, respectively (noting the use of Notation 3.5.9 here). We claim that $\phi = (\theta, \psi, (1)_{i \in I}, (1)_{\lambda \in \Lambda})$ is an automorphism of S. Indeed, if $p_{\lambda,i} \neq 0$ for some $i \in I, \lambda \in \Lambda$, then $p_{\lambda,i} = x_k$ for some k, so that $\{i, \lambda\}f = \{i\psi, \lambda\psi\}f = x_k$. Consequently,

$$p_{\lambda,i}\theta = x_k\theta = x_k = p_{\lambda\psi, i\psi},$$

and claim follows by Proposition 3.5.10. Hence

$$(i_t, g_t, \lambda_t)\phi = (i_t\psi, g_t\theta, \lambda_t\psi) = (j_t, h_t, \mu_t)$$

for each $1 \le t \le n$, and we have thus shown that

$$|S^n / \sim_{S,n}| \le |G^{n+r} / \sim_{G,n+r}| \cdot |(\Gamma(P)^l)^{2n} / \sim_{\Gamma(P)^l,2n}| < \aleph_0,$$

as G and $\Gamma(P)^l$ are $\aleph_0$-categorical. Hence S is $\aleph_0$-categorical. □

The converse however fails to hold in general, and we will construct a counterexample at the end of the section. Despite this, the proposition above enables us to produce examples of $\aleph_0$-categorical Rees matrix semigroups. For example, the result below is immediate from Lemma 3.5.21.

Corollary 3.5.24. *Let S be a Rees matrix semigroup over an $\aleph_0$-categorical group having sandwich matrix P with finitely many rows or columns, and $G(P)$ being finite. Then S is $\aleph_0$-categorical.*

We are now concerned with how Lemma 3.5.22 may be used in conjunction with Proposition 3.5.23.

Following [58], we call a completely 0-simple semigroup S *pure* if it is isomorphic to a Rees matrix semigroup with sandwich matrix over $\{0, 1\}$. In [53], Houghton considered *trivial cohomology classes* of Rees matrix semigroups, a property which is proven in Section 2 of the article to be equivalent to being pure. Hence, by [53, Theorem 5.1], a completely 0-simple semigroup is pure if and only if, for each $a, b \in S$,

$$[a, b \in \langle E(S) \rangle \text{ and } a\,\mathcal{H}\,b] \Rightarrow a = b.$$

It follows that all orthodox completely 0-simple semigroups are necessarily pure, but the converse is not true. Indeed, a completely 0-simple semigroup is orthodox if and only if it is isomorphic to a Rees matrix semigroup with sandwich matrix over $\{0, 1\}$ and with induced bipartite graph a disjoint union of complete bipartite graphs [45, Theorem 6].

We observe that if the sandwich matrix of a Rees matrix semigroup is over $\{0, 1\}$ then $\Gamma(P)^l$ is simply labelled by $\{1\}$. Therefore all automorphisms of $\Gamma(P)$ automatically preserve the labelling, and so $\Gamma(P)^l$ is $\aleph_0$-categorical if and only if $\Gamma(P)$ is $\aleph_0$-categorical. The following result is then immediate from Proposition 3.5.12 and Lemma 3.5.22.

Corollary 3.5.25. *A pure Rees matrix semigroup $\mathcal{M}^0[G; I, \Lambda; P]$ is $\aleph_0$-categorical if and only if G and $\Gamma(P)$ are $\aleph_0$-categorical.*

Since complete bipartite graphs are $\aleph_0$-categorical by Theorem 3.5.4, a disjoint union of complete bipartite graphs is $\aleph_0$-categorical if and only if it has finitely many connected components, up to isomorphism, by Proposition 3.5.6. The corollary above thus reduces in the orthodox case as follows.

Corollary 3.5.26. *Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be an orthodox Rees matrix semigroup. Then S is $\aleph_0$-categorical if and only if G is $\aleph_0$-categorical and $\Gamma(P)$ has finitely many connected components, up to isomorphism.*

We can further restrict our conditions on our Rees matrix semigroups by studying inverse completely 0-simple semigroups. These are necessarily orthodox, and are isomorphic to a Rees matrix semigroup of the form $\mathcal{M}^0[G; I, I; P]$ where P is the identity matrix, that is, $p_{ii} = 1$ and $p_{ij} = 0$ for each $i \neq j$ in I (see [55, Page 151], for example). Rees matrix semigroups formed in this way are called *Brandt semigroups*, denoted $\mathcal{B}^0[G; I]$. Since the induced bipartite graph of a Brandt semigroup is a perfect matching, it is $\aleph_0$-categorical by Theorem 3.5.4. Corollary 3.5.26 then simplifies.

Corollary 3.5.27. *A Brandt semigroup $\mathcal{B}^0[G;I]$ is $\aleph_0$-categorical if and only if G is $\aleph_0$-categorical.*

We are now able to construct a simple counterexample to the converse of Proposition 3.5.23. Let $G = \{g_i : i \in \mathbb{N}\}$ be an infinite $\aleph_0$-categorical group. Let

$$S = \mathcal{M}^0[G;\mathbb{N},\mathbb{N};P] = \mathcal{B}^0[G;\mathbb{N}] \text{ and } T = \mathcal{M}^0[G;\mathbb{N},\mathbb{N};Q],$$

where $Q = (q_{i,j})$ is such that $q_{i,i} = g_i$ and $q_{i,j} = 0$ for each $i \neq j$. Then $\Gamma(P) = \Gamma(Q)$ (and isomorphic to $P_{\mathbb{N}}$) and $(1_G, 1_{\Gamma(P)}, (g_i^{-1})_{i\in\mathbb{N}}, (1)_{\lambda\in\mathbb{N}})$ is an isomorphism from S to T by Proposition 3.5.10 since

$$p_{i,i}1_G = 1 = g_i g_i^{-1} = 1 \cdot q_{i,i} \cdot g_i^{-1},$$

for each $i \in \mathbb{N}$. Since S is $\aleph_0$-categorical by Corollary 3.5.27, the same is true of T. However, $\Gamma(Q)^l$ is a G-labelling, and is thus not $\aleph_0$-categorical by Lemma 3.5.20, and so T is our desired counterexample. This leads to the following open problem.

Open Problem 1. Does there exist an $\aleph_0$-categorical connected Rees matrix semigroup over a finite group which is not isomorphic to a Rees matrix semigroup with $\aleph_0$-categorical induced labelled bipartite graph?

We could have introduced Houghton's [53] stronger notion of an *induced group labelled bipartite graph*, although this does not appear to be a first order structure. A group labelled bipartite graph is a G-labelled bipartite graph $\Gamma^f = (\langle L,R,E\rangle, G, f)$, for some group G, where an automorphism of Γ^f is a pair $(\psi,\theta) \in \mathrm{Aut}(\Gamma)\times\mathrm{Aut}(G)$ such that, for each $\ell \in L, r \in R$,

$$(\ell,r)f = g \Leftrightarrow (\ell\psi, r\psi)f = g\theta.$$

The induced group labelled bipartite graph of a Rees matrix semigroup $S = \mathcal{M}^0[G;I,\Lambda;P]$ is simply the G-labelled bipartite graph $\Gamma(P)^f$, with automorphisms being pairs $(\psi,\theta) \in \mathrm{Aut}(\Gamma)\times\mathrm{Aut}(G)$ such that $p_{\lambda\psi,i\psi} = p_{\lambda,i}\theta$ for each $i \in I, \lambda \in \Lambda$. Clearly every automorphism of the induced group labelled bipartite graph produces an automorphism of S, although we do not in general obtain all of $\mathrm{Aut}(S)$ in this way. Similar problems therefore arise of when $\aleph_0$-categoricity of S 'passes' to its induced group labelled bipartite graph (by which we mean the induced group labelled bipartite graph has an oligomorphic automorphism group).

3.6 0-direct unions and primitive semigroups

In this section we study a well known decomposition of an arbitrary semigroup with zero which was remarked upon in the previous section, and assess how $\aleph_0$-

categoricity effects such a decomposition. The basic definitions and results are taken from [8].

A semigroup with zero S is a *0-direct union* or *orthogonal sum* of the semigroups S_i $(i \in A)$, if the following hold:

(1) $S_i \neq \{0\}$ for each $i \in A$;

(2) $S = \bigcup_{i \in A} S_i$;

(3) $S_i \cap S_j = S_i S_j = \{0\}$ for each $i \neq j$.

We denote S as $\bigsqcup^0_{i \in A} S_i$. The family $\mathcal{S} = \{S_i : i \in A\}$ is called a *0-direct decomposition* of S, and the S_i are called the *summands* of S. Note that each summand of S forms an ideal of S. If $\mathcal{S}$ and $\mathcal{S}'$ are a pair of 0-direct decompositions of S, then we say that $\mathcal{S}$ is *greater than* $\mathcal{S}'$ if each member of $\mathcal{S}$ is a subsemigroup of some member of $\mathcal{S}'$. We say that S is *0-directly indecomposable* if $\{S\}$ is the unique 0-direct decomposition of S.

Example 3.6.1. Let S be a Rees matrix semigroup with connected Rees components S_i $(i \in A)$ and consider the subsemigroup $S' = \bigcup_{i \in A} S_i$ of S. Then $S_i \cap S_j = S_i S_j = 0$ for each $i \neq j$ and so $S' = \bigsqcup^0_{i \in A} S_i$.

A subset A of a semigroup S is *consistent* if, for $x, y \in S$, $xy \in A$ implies that $x, y \in A$. A subset A of a semigroup with zero is *0-consistent* if $A^* = A \setminus \{0\}$ is consistent. The integral connection between 0-consistency and 0-direct decompositions is that a semigroup with zero S is 0-direct indecomposable if and only if S has no proper 0-consistent ideals [8, Lemma 4]. Consequently, every Rees matrix semigroup, being 0-simple, is 0-direct indecomposable.

The main result of [8] is in proving that every semigroup with zero has a greatest 0-direct decomposition, and the summands of such a decomposition are precisely the 0-direct indecomposable ideals. The importance of the existence of a greatest 0-direct decomposition for $\aleph_0$-categoricity is highlighted in the following proposition.

Proposition 3.6.2. *Let S be a semigroup with zero and let $\mathcal{S} = \{S_i : i \in A\}$ be the greatest 0-direct decomposition of S. Let $\pi : A \to A$ be a bijection and $\phi_i : S_i \to S_{i\pi}$ an isomorphism for each $i \in A$. Then the map $\phi : S \to S$ given by*

$$s_i \phi = s_i \phi_i; \quad (s_i \in S_i)$$

is an automorphism of S, denoted $\phi = \bigsqcup^0_{i \in A} \phi_i$. Moreover, every automorphism of S can be constructed in this way.

Proof. Let ϕ be constructed as in the hypothesis of the proposition. Since $0\phi_i = 0$ for each $i \in A$ the map is well-defined, and it is clearly bijective. Let $a \in S_i$ and $b \in S_j$. If $i = j$ then

$$(ab)\phi = (ab)\phi_i = (a\phi_i)(b\phi_i) = (a\phi)(b\phi),$$

and if $i \neq j$ then

$$(ab)\phi = 0\phi = 0 = (a\phi_i)(b\phi_j) = (a\phi)(b\phi).$$

Hence ϕ is an isomorphism.

Conversely, if ϕ' is an automorphism of S, then it easily follows that

$$\mathcal{S}\phi' = \{S_i\phi' : i \in A\}$$

is a 0-direct decomposition of S. For each summand S_i there exists $k \in A$ such that $S_i \subseteq S_k\phi'$ since $\mathcal{S}$ is the greatest 0-direct decomposition. If $S_i \subseteq S_k\phi' \cap S_{k'}\phi'$ then $S_i = \{0\}$ as $\mathcal{S}\phi'$ is a 0-direct decomposition of S, a contradiction. Hence the element k is unique. Suppose $S_i, S_j \subseteq S_k\phi'$. Then $S_i\phi'^{-1}, S_j\phi'^{-1} \subseteq S_k$, and so as $\{S_i\phi'^{-1} : i \in A\}$ is also a 0-direct decomposition of S, we have that $i = j$ since S_k is 0-direct indecomposable. Hence there exists a bijection π' of A such that $S_i\phi' = S_{i\pi'}$ for each $i \in A$ as required. □

Proposition 3.6.3. *Let S be a semigroup with zero and let $\mathcal{S} = \{S_i : i \in A\}$ be the greatest 0-direct decomposition of S. Then S is $\aleph_0$-categorical if and only if each S_i is $\aleph_0$-categorical and $\mathcal{S}$ is finite, up to isomorphism.*

Proof. It follows immediately from Proposition 3.6.2 that $\{(S_i, x_i) : i \in A\}$ forms a system of 1-p.p.r.c subsemigroups of S for any $x_i \in S_i^*$. Hence if S is $\aleph_0$-categorical then each S_i is $\aleph_0$-categorical and $\mathcal{S}$ is finite, up to isomorphism, by Propositions 3.3.5 and 3.3.9.

Conversely, we shall prove that $\mathcal{S}$ forms an $(S; \bar{A}; \Psi)$-system in S for some $\bar{A}$ and Ψ. Let $A_1, \dots, A_r$ be a partition of A corresponding to the isomorphism types of summands of S, so that $S_i \cong S_j$ if and only if $i, j \in A_k$ for some k. Let $\bar{A} = (A; A_1, \dots, A_r)$. For each $i, j \in A$, let $\Psi_{i,j} = \text{Iso}(S_i; S_j)$ and fix $\Psi = \bigcup_{i,j \in A} \Psi_{i,j}$. Then Conditions 3.2.14(A), 3.2.14(B) and 3.2.14(C) are trivially satisfied by Ψ. Take any $\pi \in \text{Aut}(\bar{A})$ and, for each $i \in A$, let $\phi_i \in \Psi_{i,i\pi}$. Then, as $S_i \cong S_{i\pi}$ by our partition of $\mathcal{S}$, we have that $\phi = \bigsqcup^0_{i\in A} \phi_i$ is an automorphism of S extending each ϕ_i by Proposition 3.6.2, and so Condition 3.2.14(D) is satisfied. Hence $\mathcal{S}$ forms an $(S; \bar{A}; \Psi)$-system. Moreover, $\bar{A}$ is $\aleph_0$-categorical by Corollary 3.2.10, and each S_i is $\aleph_0$-categorical (over $\text{Aut}(S_i) = \Psi_{i,i}$). Hence S is $\aleph_0$-categorical by Lemma 3.2.16. □

When studying $\aleph_0$-categorical semigroups with zero, it therefore suffices to examine 0-direct indecomposable semigroups.

We observe that without the condition of $\mathcal{S}$ being the greatest 0-direct decomposition of S, the converse of Proposition 3.6.3 need not be true. For example, for each $n \in \mathbb{N}$, let N_n be a null semigroup on n non-zero elements. Then $N = \bigsqcup^0_{i\in \mathbb{N}} N_i$

is a countably infinite null semigroup, and is thus $\aleph_0$-categorical by Example 3.2.1. However the set of summands of N is not finite, up to isomorphism.

Since each Rees matrix semigroup is 0-direct indecomposable, we attain the following immediate consequence to Proposition 3.6.3.

Corollary 3.6.4. *Let $S_i = \mathcal{M}^0[G_i; I_i, \Lambda_i, P_i]$ $(i \in A)$ be a collection of Rees matrix semigroups. Then $\bigsqcup^0_{i\in A} S_i$ is $\aleph_0$-categorical if and only if each S_i is $\aleph_0$-categorical and $\{S_i : i \in A\}$ is finite, up to isomorphism.*

Note that a Rees matrix semigroup is a 0-direct union of its connected Rees matrix components if and only if it is a connected Rees matix semigroup. Consequently, the corollary above does not imply Theorem 3.5.16, nor give us a method for proving its generalization.

A semigroup S with zero is called *primitive* if each of its non-zero idempotents is primitive. It follows from the work of Hall in [46] that a regular semigroup S is primitive if and only if S is isomorphic to a 0-direct union of Rees matrix semigroups. A classification of primitive regular $\aleph_0$-categorical semigroups via its Rees matrix ideals then follows.

Corollary 3.6.5. *A primitive regular semigroup S is $\aleph_0$-categorical if and only $S \cong \bigsqcup^0_{i\in A} \mathcal{M}^0[G_i; I_i, \Lambda_i; P_i]$ with each $\mathcal{M}^0[G_i; I_i, \Lambda_i; P_i]$ being $\aleph_0$-categorical, and $\{\mathcal{M}^0[G_i; I_i, \Lambda_i; P_i] : i \in A\}$ being finite, up to isomorphism.*

In particular, since a primitive inverse semigroup is isomorphic to a 0-direct union of Brandt semigroups, the corollary above simplifies accordingly:

Corollary 3.6.6. *A primitive inverse semigroup S is $\aleph_0$-categorical if and only if $S \cong \bigsqcup^0_{i\in A} \mathcal{B}^0[G_i; I_i]$ with each G_i being $\aleph_0$-categorical and the sets $\{G_i : i \in A\}$ and $\{I_i : i \in A\}$ being finite up to isomorphism and bijection, respectively.*

Proof. By Corollary 3.5.27 the Brandt semigroups $B_i = \mathcal{B}^0[G_i; I_i]$ are $\aleph_0$-categorical if and only if the groups G_i are $\aleph_0$-categorical. Since a pair of perfect matchings are isomorphic if and only if they are of the same cardinality, we have by Proposition 3.5.10 that $B_i \cong B_j$ if and only if $G_i \cong G_j$ and $|I_i| = |I_j|$ (a result which is also stated in [65, Section 3.3]). The result then follows by Corollary 3.6.5. $\square$

3.7 $\aleph_0$-categorical strong semilattices of semigroups

We end the new results in this chapter by studying the $\aleph_0$-categoricity of strong semilattices of semigroups.

Let $S = [Y; S_\alpha; \psi_{\alpha,\beta}]$ be a strong semilattice of semigroups. We denote the equivalence relation on Y corresponding to isomorphism types of the semigroups

S_α by η_S, so that

$$\alpha \, \eta_S \, \beta \Leftrightarrow S_\alpha \cong S_\beta.$$

Let $Y/\eta_S = \{Y_1, Y_2, \dots\}$. Denote Y^S as the Y/η_S-extended structure $(Y; Y_1, Y_2, \dots)$ of Y (so that Y^S is a semilattice with distinguished subsets Y_i).

Lemma 3.7.1. *Let $S = [Y; S_\alpha; \psi_{\alpha,\beta}]$ be a strong semilattice of semigroups such that Y^S is $\aleph_0$-categorical. Then Y/η_S is finite.*

Proof. Since η_S is an equivalence relation, the equivalence classes are pairwise disjoint, and so the result is immediate from Lemma 3.2.8. □

Recall that a strong semilattice of semigroups S is automorphism-pure if every automorphism of S can be constructed as in Theorem 2.7.1.

Proposition 3.7.2. *Let $S = [Y; S_\alpha; \psi_{\alpha,\beta}]$ be automorphism-pure and $\aleph_0$-categorical. Then each S_α is $\aleph_0$-categorical and Y^S is $\aleph_0$-categorical.*

Proof. For each $\alpha \in Y$, fix $x_\alpha \in S_\alpha$. We claim that $\{(S_\alpha, x_\alpha) : \alpha \in Y\}$ forms a system of 1-p.p.r.c subsemigroups of S. Indeed, let θ be an automorphism of S such that $x_\alpha\theta = x_\beta$ for some $\alpha, \beta \in Y$. Since S is automorphism-pure, there exists $\pi \in \mathrm{Aut}(Y)$ and $\theta_\alpha \in \mathrm{Iso}(S_\alpha; S_{\alpha\pi})$ such that $\theta = [\theta_\alpha, \pi]_{\alpha \in Y}$. Hence $S_\alpha\theta = S_\beta$, and the claim follows. Consequently, by the $\aleph_0$-categoricity of S, each S_α is $\aleph_0$-categorical by Proposition 3.3.5.

Let $\underline{a} = (\alpha_1, \dots, \alpha_n)$ and $\underline{b} = (\beta_1, \dots, \beta_n)$ be a pair of n-tuples of Y^S such that there exist $a_{\alpha_k} \in S_{\alpha_k}$ and $b_{\beta_k} \in S_{\beta_k}$ with $(a_{\alpha_1}, \dots, a_{\alpha_n}) \sim_{S,n} (b_{\beta_1}, \dots, b_{\beta_n})$ via $[\theta'_\alpha, \pi']_{\alpha \in Y} \in \mathrm{Aut}(S)$, say. Since $\pi \in \mathrm{Aut}(Y)$ and $S_\alpha \cong S_{\alpha\pi'}$ for each $\alpha \in Y$, it follows that $\pi' \in \mathrm{Aut}(Y^S)$. Moreover, $\alpha_k\pi' = \beta_k$ for each k, so that $\underline{a} \sim_{Y^S,n} \underline{b}$ via π'. We have thus shown that

$$|(Y^S)^n / \sim_{Y^S,n}| \leq |S^n / \sim_{S,n}| < \aleph_0,$$

as S is $\aleph_0$-categorical. Hence Y^S is $\aleph_0$-categorical. □

In this chapter we will only be concerned with the $\aleph_0$-categoricity of strong semilattices of semigroups in which all connecting morphisms are either constant maps or all are injective maps. For arbitrary connecting morphisms, the problem of assessing $\aleph_0$-categoricity is extremely difficult, and this is discussed further at the end of the chapter. We first consider the constant maps case.

Suppose that Y is a semilattice and, for each $\alpha \in Y$, S_α is a semigroup containing an idempotent e_α. For each $\alpha > \beta$ in Y, let $\psi_{\alpha,\beta}$ be the constant map with image $\{e_\beta\}$. It is easy to check that (with $\psi_{\alpha,\alpha} = 1_{S_\alpha}$ for all $\alpha \in Y$) we have $\psi_{\alpha,\beta}\psi_{\beta,\gamma} = \psi_{\alpha,\gamma}$ for all $\alpha \geq \beta \geq \gamma$ in Y. We follow the notation of [99] and let $\psi_{\alpha,\beta} = C_{\alpha,e_\beta}$ for all $\alpha > \beta$ in Y. We have shown that $S = [Y; S_\alpha; C_{\alpha,e_\beta}]$ is a strong semilattice of semigroups, which we call a *constant strong semilattice of semigroups*, denoted $S = [Y; S_\alpha; e_\alpha; C_{\alpha,e_\beta}]$.

Example 3.7.3. Let $Y = \{0, i : i \in A\}$ be a primitive semilattice with zero, that is, such that $ii = i$ and $ij = 0 = ji$ for all $i \neq j$ in A. Let $\{S_i : i \in A\}$ be a family of pairwise disjoint semigroups with $E(S_i) \neq \emptyset$ $(i \in A)$, and set $S_0 = \{0\}$. Then we form a strong semilattice of semigroups by taking $S = [Y; S_\alpha; e_\alpha; C_{\alpha,e_\beta}]$. We claim that S is a 0-direct union of the subsemigroups $\bar{S}_\alpha = S_\alpha \cup \{0\}$. Indeed, if $s_i \in \bar{S}_i$ and $s_j \in \bar{S}_j$ where $i \neq j$, then

$$s_i s_j = (s_i C_{i,e_0})(s_j C_{j,e_0}) = 00 = 0$$

so that $\bar{S}_i \cap \bar{S}_j = \bar{S}_i \bar{S}_j = \{0\}$, and the claim follows.

Clearly not every 0-direct union of semigroups can be written as a (constant) strong semilattice of semigroups over a non-trivial semilattice, thus justifying the previous section. A simple example is any 0-direct union of a pair of semigroups without non-zero idempotents.

Notation 3.7.4. If $S = [Y; S_\alpha; e_\alpha; C_{\alpha,e_\beta}]$ is a constant strong semilattice of semigroups, then we denote the subset of $\text{Iso}(S_\alpha; S_\beta)$ consisting of those isomorphisms which map e_α to e_β as $\text{Iso}(S_\alpha; S_\beta)^{[e_\alpha;e_\beta]}$. Notice that the set $\text{Iso}(S_\alpha; S_\alpha)^{[e_\alpha;e_\alpha]}$ is simply the subgroup $\text{Aut}(S_\alpha; \{e_\alpha\})$ of $\text{Aut}(S_\alpha)$.

Definition 3.7.5. Let $S = [Y; S_\alpha; e_\alpha; C_{\alpha,e_\beta}]$ be a constant strong semilattice of semigroups. Define a relation υ_S on Y by

$$\alpha \, \upsilon_S \, \beta \Leftrightarrow \text{Iso}(S_\alpha; S_\beta)^{[e_\alpha;e_\beta]} \neq \emptyset,$$

so that $\upsilon_S \subseteq \eta_S$.

Then υ_S is reflexive since $1_{S_\alpha} \in \text{Aut}(S_\alpha; \{e_\alpha\})$ for each $\alpha \in Y$, and it easily follows that υ_S forms an equivalence relation on Y.

Proposition 3.7.6. *Let $S = [Y; S_\alpha; e_\alpha; C_{\alpha,e_\beta}]$ be such that $Y/\upsilon_S = \{Y_1, \dots, Y_r\}$ is finite, $\mathcal{Y} = (Y; Y_1, \dots, Y_r)$ is $\aleph_0$-categorical and each S_α is $\aleph_0$-categorical. Then S is $\aleph_0$-categorical.*

Proof. We prove that $\{S_\alpha : \alpha \in Y\}$ forms an $(S; \mathcal{Y}; \Psi)$-system for some Ψ. For each $\alpha, \beta \in Y$, let $\Psi_{\alpha,\beta} = \text{Iso}(S_\alpha; S_\beta)^{[e_\alpha;e_\beta]}$ and fix $\Psi = \bigcup_{\alpha,\beta \in Y} \Psi_{\alpha,\beta}$. Then Conditions 3.2.14(A), 3.2.14(B) and 3.2.14(C) are satisfied since υ_S forms an equivalence relation on Y. Let $\pi \in \text{Aut}(\mathcal{Y})$ and, for each $\alpha \in Y$, let $\theta_\alpha \in \Psi_{\alpha,\alpha\pi}$. We claim that $\theta = [\theta_\alpha, \pi]_{\alpha \in Y}$ is an automorphism of S. Indeed, for any $s_\alpha \in S_\alpha$ and any $\beta < \alpha$ we have

$$s_\alpha C_{\alpha,e_\beta} \theta_\beta = e_\beta \theta_\beta = e_{\beta\pi} = s_\alpha \theta_\alpha C_{\alpha\pi,e_{\beta\pi}}$$

so that the diagram $[\alpha, \beta; \alpha\pi, \beta\pi]$ commutes. The claim then follows by Theorem 2.7.1. Since θ extends each θ_α, we have that $\{S_\alpha : \alpha \in Y\}$ is an $(S; \mathcal{Y}; \Psi)$-system. Moreover, as S_α is $\aleph_0$-categorical, it is $\aleph_0$-categorical over $\Psi_{\alpha,\alpha} = \text{Aut}(S_\alpha; \{e_\alpha\})$ by Lemma 3.2.11. Hence S is $\aleph_0$-categorical by Corollary 3.2.16. $\square$

Examining our main two classes of automorphism-pure strong semilattices of semigroups; strong semilattices of groups and of rectangular bands, the results of this section reduce accordingly. If $S = [Y; G_\alpha; e_\alpha; C_{\alpha,e_\beta}]$ is a constant strong semilattice of groups, then e_α is the identity of G_α, and so $\text{Iso}(G_\alpha; G_\beta) = \text{Iso}(G_\alpha; G_\beta)^{[e_\alpha;e_\beta]}$ for each $\alpha, \beta \in Y$. On the other hand, if $S = [Y; B_\alpha; e_\alpha; C_{\alpha,e_\beta}]$ is a constant strong semilattice of rectangular bands, then

$$\text{Iso}(B_\alpha; B_\beta) \neq \emptyset \Leftrightarrow \text{Iso}(B_\alpha; B_\beta)^{[e_\alpha;e_\beta]} \neq \emptyset \text{ for any } e_\alpha \in B_\alpha, e_\beta \in B_\beta,$$

by Proposition 2.10.2. In both cases, we therefore have $\upsilon_S = \eta_S$ and so by Lemma 3.7.1 and Proposition 3.7.6 we attain a converse to Proposition 3.7.2 in the case of constant strong semilattices. Moreover, by Lemma 3.2.9, each rectangular band B_α is $\aleph_0$-categorical, and we have thus proven the following result.

Corollary 3.7.7. *Let $S = [Y; S_\alpha; e_\alpha; C_{\alpha,e_\beta}]$ be a constant strong semilattice of rectangular bands (groups). Then S is $\aleph_0$-categorical if and only if Y^S is $\aleph_0$-categorical (and each group S_α is $\aleph_0$-categorical).*

Consider now a strong semilattice of semigroups $[Y; S_\alpha; \psi_{\alpha,\beta}]$ with each connecting morphism being injective. For each $\alpha > \beta$ in Y, we abuse notation somewhat by denoting the isomorphism $\psi^{-1}_{\alpha,\beta}|_{Im\, \psi_{\alpha,\beta}}$ simply by $\psi^{-1}_{\alpha,\beta}$. We observe that if $\alpha > \beta > \gamma$ and $x_\gamma \in \text{Im } \psi_{\alpha,\gamma}$, say $x_\gamma = x_\alpha\psi_{\alpha,\gamma}$, then

$$x_\gamma\psi^{-1}_{\alpha,\gamma}\psi_{\alpha,\beta} = x_\alpha\psi_{\alpha,\gamma}\psi^{-1}_{\alpha,\gamma}\psi_{\alpha,\beta} = x_\alpha\psi_{\alpha,\beta} = x_\gamma\psi^{-1}_{\beta,\gamma}. \tag{3.5}$$

Hence, on the restricted domain Im $\psi_{\alpha,\gamma}$, we have $\psi^{-1}_{\alpha,\gamma}\psi_{\alpha,\beta} = \psi^{-1}_{\beta,\gamma}$.

Notice that an element of a semilattice Y is minimum under the natural order if and only if it is a zero. If Y has a zero 0 we may define an equivalence relation ξ_S on Y by $\alpha \,\xi_S\, \beta$ if and only if $S_\alpha\psi_{\alpha,0} = S_\beta\psi_{\beta,0}$. If $\alpha \,\xi_S\, \beta$ then $\psi_{\alpha,0}\psi^{-1}_{\beta,0}$ is an isomorphism from S_α onto S_β, and so $\xi_S \subseteq \eta_S$.

Proposition 3.7.8. *Let $S = [Y; S_\alpha; \psi_{\alpha,\beta}]$ be such that each $\psi_{\alpha,\beta}$ is injective. Let Y be a semilattice with zero and $Y/\xi_S = \{Y_1, \ldots, Y_r\}$ be finite, with*

$$\{S_\alpha\psi_{\alpha,0} : \alpha \in Y\} = \{T_1, \ldots, T_r\}.$$

Let $\mathcal{Y} = (Y; Y_1, \ldots, Y_r)$ be a set-extension of Y and $\mathcal{S}_0 = (S_0; T_1, \ldots, T_r)$ a set-extension of S_0. Then S is $\aleph_0$-categorical if $\mathcal{Y}$ and $\mathcal{S}_0$ are $\aleph_0$-categorical. Moreover, *if S is automorphism-pure and $\aleph_0$-categorical, then conversely $\mathcal{Y}$ and $\mathcal{S}_0$ are $\aleph_0$-categorical.*

Proof. Suppose first that $\mathcal{Y} = (Y; Y_1, \dots, Y_r)$ and $\mathcal{S}_0 = (S_0; T_1, \dots, T_r)$ are $\aleph_0$-categorical. Let $\underline{a} = (a_{\alpha_1}, \dots, a_{\alpha_n})$ and $\underline{b} = (b_{\beta_1}, \dots, b_{\beta_n})$ be n-tuples of S with $(\alpha_1, \dots, \alpha_n) \sim_{\mathcal{Y},n} (\beta_1, \dots, \beta_n)$ via $\pi \in \mathrm{Aut}(\mathcal{Y})$, say. Suppose further that

$$(a_{\alpha_1}\psi_{\alpha_1,0}, \dots, a_{\alpha_n}\psi_{\alpha_n,0}) \sim_{\mathcal{S}_0,n} (b_{\beta_1}\psi_{\beta_1,0}, \dots, b_{\beta_n}\psi_{\beta_n,0})$$

via $\theta_0 \in \mathrm{Aut}(\mathcal{S}_0)$, say. Then for each $\alpha \in Y$ we have $S_\alpha\psi_{\alpha,0} = S_{\alpha\pi}\psi_{\alpha\pi,0}$, and so we can take $\theta_\alpha \in \mathrm{Iso}(S_\alpha; S_{\alpha\pi})$ given by

$$\theta_\alpha = \psi_{\alpha,0}\,\theta_0\,\psi^{-1}_{\alpha\pi,0}.$$

For each $\alpha \geq \beta$ in Y, the diagram $[\alpha, \beta; \alpha\pi, \beta\pi]$ commutes as

$$\begin{aligned}
\psi_{\alpha,\beta}\,\theta_\beta &= \psi_{\alpha,\beta}\,(\psi_{\beta,0}\,\theta_0\,\psi^{-1}_{\beta\pi,0}) \\
&= \psi_{\alpha,0}\,\theta_0\,\psi^{-1}_{\beta\pi,0} \\
&= \psi_{\alpha,0}\,\theta_0\,(\psi^{-1}_{\alpha\pi,0}\,\psi_{\alpha\pi,\beta\pi}) \\
&= \theta_\alpha\,\psi_{\alpha\pi,\beta\pi},
\end{aligned}$$

where the penultimate equality is due to (3.5) as $\mathrm{Im}\ \psi_{\alpha\pi,0} = \mathrm{Im}\ \psi_{\alpha,0} = (\mathrm{Im}\ \psi_{\alpha,0})\theta_0$. Hence $\theta = [\theta_\alpha, \pi]_{\alpha\in Y}$ is an automorphism of S by Theorem 2.7.1. Furthermore,

$$a_{\alpha_k}\theta = a_{\alpha_k}\theta_{\alpha_k} = a_{\alpha_k}\psi_{\alpha_k,0}\,\theta_0\,\psi^{-1}_{\alpha_k\pi,0} = b_{\beta_k}\psi_{\beta_k,0}\,\psi^{-1}_{\beta_k,0} = b_{\beta_k}$$

for each $1 \leq k \leq n$, so that $\underline{a} \sim_{S,n} \underline{b}$ via θ. We thus have that

$$|S^n/\sim_{S,n}| \leq |\mathcal{Y}^n/\sim_{\mathcal{Y},n}| \cdot |\mathcal{S}_0^n/\sim_{\mathcal{S}_0,n}| < \aleph_0$$

and so S is $\aleph_0$-categorical.

Conversely, suppose S is automorphism-pure and $\aleph_0$-categorical. For each $1 \leq k \leq r$, fix some $\gamma_k \in Y_k$, where we assume w.l.o.g. that $S_{\gamma_k}\psi_{\gamma_k,0} = T_k$. For each $\alpha \in Y$, fix some $x_\alpha \in S_\alpha$. Let $\underline{a} = (\alpha_1, \dots, \alpha_n)$ and $\underline{b} = (\beta_1, \dots, \beta_n)$ be n-tuples of $\mathcal{Y}$ such that

$$(x_{\alpha_1}, \dots, x_{\alpha_n}, x_{\gamma_1}, \dots, x_{\gamma_r}) \sim_{S,n+r} (x_{\beta_1}, \dots, x_{\beta_n}, x_{\gamma_1}, \dots, x_{\gamma_r}),$$

via $\theta \in \mathrm{Aut}(S)$, say. Since S is automorphism-pure there exist $\pi \in \mathrm{Aut}(Y)$ and $\theta_\alpha \in \mathrm{Iso}(S_\alpha; S_{\alpha\pi})$ such that $\theta = [\theta_\alpha, \pi]_{\alpha\in Y}$. The automorphism π fixes each γ_k, so that $S_{\gamma_k}\theta = S_{\gamma_k}$. Hence, as the diagram $[\gamma_k, 0; \gamma_k, 0]$ commutes for each k, we have

$$T_k = S_{\gamma_k}\psi_{\gamma_k,0} = (S_{\gamma_k}\theta_{\gamma_k})\psi_{\gamma_k,0} = S_{\gamma_k}\psi_{\gamma_k,0}\theta_0 = T_k\theta_0 = T_k\theta.$$

If $\alpha \in Y_k$ then by the commutativity of the diagram $[\alpha; 0; \alpha\pi, 0]$ we therefore have

$$S_\alpha \psi_{\alpha,0} = T_k = T_k \theta_0 = S_\alpha \psi_{\alpha,0} \theta_0 = S_\alpha \theta_\alpha \psi_{\alpha\pi,0} = S_{\alpha\pi} \psi_{\alpha\pi,0},$$

and so $\pi \in \mathrm{Aut}(\mathcal{Y})$. We have thus shown that

$$|\mathcal{Y}^n / \sim_{\mathcal{Y},n}| \leq |S^{n+r} / \sim_{S,n+r}| < \aleph_0$$

and so $\mathcal{Y}$ is $\aleph_0$-categorical. Now suppose $\underline{c}$ and $\underline{d}$ are n-tuples of $\mathcal{S}_0$ such that

$$(\underline{c}, x_{\gamma_1}, \dots, x_{\gamma_r}) \sim_{S,n+r} (\underline{d}, x_{\gamma_1}, \dots, x_{\gamma_r}),$$

via $\theta' = [\theta'_\alpha, \pi']_{\alpha \in Y} \in \mathrm{Aut}(S)$, say. Then arguing as before we have that $T_k \theta' = T_k$ for each k, and it follows that $\theta'_0 \in \mathrm{Aut}(\mathcal{S}_0)$ and is such that $\underline{c}\theta'_0 = \underline{d}$. Hence

$$|\mathcal{S}_0^n / \sim_{\mathcal{S}_0,n}| \leq |S^{n+r} / \sim_{S,n+r}| < \aleph_0$$

and so $\mathcal{S}_0$ is $\aleph_0$-categorical. □

Note that if Y is finite, then it has a zero (as the meet of all the elements of Y). Then any set extension of Y is finite, and thus $\aleph_0$-categorical, and so the result above simplifies accordingly in this case:

Corollary 3.7.9. *Let $S = [Y; S_\alpha; \psi_{\alpha,\beta}]$ be such that each $\psi_{\alpha,\beta}$ is injective and Y is a semilattice with zero. Let $Y/\xi_S = \{Y_1, \dots, Y_r\}$ with*

$$\{S_\alpha \psi_{\alpha,0} : \alpha \in Y\} = \{T_1, \dots, T_r\}.$$

Let $\mathcal{Y} = (Y; Y_1, \dots, Y_r)$ be a set-extension of Y and $\mathcal{S}_0 = (S_0; T_1, \dots, T_r)$ a set-extension of S_0. If $\mathcal{S}_0$ is $\aleph_0$-categorical then S is $\aleph_0$-categorical. Moreover, if S is automorphism-pure and $\aleph_0$-categorical then conversely $\mathcal{S}_0$ is $\aleph_0$-categorical.

Example 3.7.10. An inverse semigroup with semilattice of idempotents E is called *E-unitary* if, for all $e \in E$ and all $s \in S$,

$$es \in E \Rightarrow s \in E.$$

A Clifford semigroup $S = [Y; G_\alpha; \psi_{\alpha,\beta}]$ is E-unitary if and only each $\psi_{\alpha,\beta}$ is injective by [55, Exercise 5.20]. Since Clifford semigroups are automorphism-pure by Lemma 2.11.7, we have the following simplification of Proposition 3.7.8.

Corollary 3.7.11. *Let $S = [Y; G_\alpha; \psi_{\alpha,\beta}]$ be an E-unitary Clifford semigroup. Let Y be a semilattice with zero and $Y/\xi_S = \{Y_1, \dots, Y_r\}$ be finite, with*

$$\{S_\alpha \psi_{\alpha,0} : \alpha \in Y\} = \{T_1, \dots, T_r\}.$$

Let $\mathcal{Y} = (Y; Y_1, \dots, Y_r)$ be a set-extension of Y and $\mathcal{S}_0 = (S_0; T_1, \dots, T_r)$ a set-extension of S_0. Then S is $\aleph_0$-categorical if and only if $\mathcal{Y}$ and $\mathcal{S}_0$ are $\aleph_0$-categorical.

We can also consider a stronger condition on a strong semilattice of semigroups $S = [Y; S_\alpha; \psi_{\alpha,\beta}]$ by taking each connecting morphism to be an isomorphism. In this case $Y/\xi_S = \{Y\}$, and so the result above simplifies accordingly. However we can prove a more general result directly (without the condition that Y has a zero). We first extend our connecting morphism notation by defining, for each $\alpha, \beta \in Y$, the morphism $\psi_{\alpha,\beta}$ by

$$\psi_{\alpha,\beta} = \psi_{\alpha,\alpha\beta}(\psi_{\beta,\alpha\beta})^{-1}.$$

We observe that if $\alpha \geq \beta$ then $\psi_{\alpha,\beta}$ is the same as our original connecting morphism. Furthermore, if $\alpha, \gamma \in Y$ then

$$\psi_{\alpha,\gamma} = \psi_{\alpha,\alpha\gamma}(\psi_{\gamma,\alpha\gamma})^{-1} = (\psi_{\gamma,\alpha\gamma}(\psi_{\alpha,\alpha\gamma})^{-1})^{-1} = (\psi_{\gamma,\alpha})^{-1}. \tag{3.6}$$

A key property of our extended set of connecting morphisms is that transitivity still holds:

Lemma 3.7.12. *For each $\alpha, \beta, \gamma \in Y$ we have $\psi_{\alpha,\gamma} = \psi_{\alpha,\beta}\psi_{\beta,\gamma}$.*

Proof. Let $\alpha, \beta \in Y$ and suppose $\delta \leq \alpha, \beta$. We claim that $\psi_{\alpha,\beta} = \psi_{\alpha,\delta}\psi_{\delta,\beta}$. Since $\alpha \geq \alpha\beta \geq \delta$ we have

$$\psi_{\alpha,\delta} = \psi_{\alpha,\alpha\beta}\psi_{\alpha\beta,\delta}$$

and so

$$\psi_{\alpha,\alpha\beta} = \psi_{\alpha,\delta}\psi^{-1}_{\alpha\beta,\delta} = \psi_{\alpha,\delta}\psi_{\delta,\alpha\beta}$$

by (3.6). Hence

$$\begin{aligned}
\psi_{\alpha,\beta} &= \psi_{\alpha,\alpha\beta}\psi_{\alpha\beta,\beta} \\
&= \psi_{\alpha,\delta}\psi_{\delta,\alpha\beta}\psi_{\alpha\beta,\beta} \\
&= \psi_{\alpha,\delta}\psi_{\delta,\beta},
\end{aligned}$$

thus completing the proof of the claim. Let $\gamma \in Y$ and fix $\tau \in Y$ such that $\tau \leq \alpha, \beta, \gamma$. Then by the claim above,

$$\begin{aligned}
\psi_{\alpha,\beta}\psi_{\beta,\gamma} &= (\psi_{\alpha,\tau}\psi_{\tau,\beta})(\psi_{\beta,\tau}\psi_{\tau,\gamma}) \\
&= \psi_{\alpha,\tau}(\psi_{\tau,\beta}\psi^{-1}_{\tau,\beta})\psi_{\tau,\gamma} \\
&= \psi_{\alpha,\tau}\psi_{\tau,\gamma} \\
&= \psi_{\alpha,\gamma}
\end{aligned}$$

as required. $\square$

The $\aleph_0$-categoricity of strong semilattices of semigroups with connecting morphisms being isomorphisms follows quickly from the next result. We remark that only the first half of the result is required here, however the necessary and sufficient statement will be used in later chapters.

Proposition 3.7.13. *Let $S = [Y; S_\alpha; \psi_{\alpha,\beta}]$ be such that each $\psi_{\alpha,\beta}$ is an isomorphism. Then there exists a semigroup $\bar{S}$ such that $\bar{S} \cong S_\alpha$ for each $\alpha \in Y$ and $S \cong \bar{S} \times Y$. Conversely, if T is a semigroup and Z is a semilattice then $T \times Z$ is isomorphic to a strong semilattice of semigroups such that each connecting morphism is an isomorphism.*

Proof. For any $\alpha, \beta \in Y$, the map $\psi_{\alpha,\beta}$ is an isomorphism, and so the semigroups S_α are pairwise isomorphic. Fix $\delta \in Y$. Then the map $\theta : S \to S_\delta \times Y$ given by

$$x_\alpha \theta = (x_\alpha \psi_{\alpha,\delta}, \alpha) \quad (x_\alpha \in S)$$

is a bijection. If $a_\alpha, b_\gamma \in S$ then, using Lemma 3.7.12, we have

$$\begin{aligned}(a_\alpha\, b_\gamma)\psi_{\alpha\gamma,\delta} &= (a_\alpha \psi_{\alpha,\alpha\gamma}\, b_\gamma \psi_{\gamma,\alpha\gamma})\psi_{\alpha\gamma,\delta} \\ &= (a_\alpha \psi_{\alpha,\alpha\gamma}\psi_{\alpha\gamma,\delta})(b_\gamma \psi_{\gamma,\alpha\gamma}\psi_{\alpha\gamma,\delta}) \\ &= (a_\alpha \psi_{\alpha,\delta})(b_\gamma \psi_{\gamma,\delta}),\end{aligned}$$

since $\psi_{\alpha\gamma,\delta}$ is a morphism. It follows that

$$\begin{aligned}(a_\alpha b_\gamma)\theta &= ((a_\alpha b_\gamma)\psi_{\alpha\gamma,\delta}, \alpha\gamma) \\ &= ((a_\alpha \psi_{\alpha,\delta})(b_\gamma \psi_{\gamma,\delta}), \alpha\gamma) \\ &= (a_\alpha \psi_{\alpha,\delta}, \alpha)(b_\gamma \psi_{\gamma,\delta}, \gamma) \\ &= a_\alpha \theta\, b_\gamma \theta.\end{aligned}$$

Hence θ is an isomorphism as required.

Conversely, let $T_\alpha = \{(a, \alpha) : a \in T\}$ for each $\alpha \in Z$. Clearly each T_α is a semigroup isomorphic to T. For each $\alpha \geq \beta$ in Z, let $\varphi_{\alpha,\beta} : T_\alpha \to T_\beta$ be the isomorphism given by

$$(a, \alpha)\varphi_{\alpha,\beta} = (a, \beta).$$

Then it is easily shown that $[Z; T_\alpha; \varphi_{\alpha,\beta}]$ forms a strong semilattice of semigroups, and is isomorphic to $T \times Z$ by the forward direction to the proof. □

Corollary 3.7.14. *Let $S = [Y; S_\alpha; \psi_{\alpha,\beta}]$ be such that each $\psi_{\alpha,\beta}$ is an isomorphism. If S_α and Y are $\aleph_0$-categorical, then S is $\aleph_0$-categorical. Moreover, if S is automorphism-pure then the converse holds.*

Proof. By Proposition 3.7.13, S is isomorphic to $S_\alpha \times Y$ for any $\alpha \in Y$. The first half of the result then follows by Proposition 3.1.2.

Suppose S is automorphism-pure. Since the components S_α are pairwise isomorphic, we have $Y^S = (Y;Y)$, so clearly $\mathrm{Aut}(Y^S)$=$\mathrm{Aut}(Y)$. Hence Y^S is $\aleph_0$-categorical if and only if Y is $\aleph_0$-categorical, and so the converse holds by Proposition 3.7.2. $\square$

3.8 Further work

The study into the $\aleph_0$-categoricity of semigroups described in this chapter is in no way complete. In particular, we would like to be able to answer Open problem 1, and further describe the $\aleph_0$-categoricity of Rees matrix semigroups with 'more complicated' sandwich matrices.

We have seen in this chapter that the property of $\aleph_0$-categoricity passes to a wide range of subsemigroups. Conversely however, building an $\aleph_0$-categorical semigroup from its $\aleph_0$-categorical 'parts' is difficult, even for relatively easily described semigroups, such as Rees matrix semigroups. One possible direction which we now take is to apply Theorem 1.2.26 by switching our interest to (ULF) homogeneous structures. This will allow more interesting examples of $\aleph_0$-categorical semigroups to be constructed in subsequent chapters.

Chapter 4

Homogeneous structures

Recall that a structure is *homogeneous* if every local symmetry is a part of a global symmetry. Homogeneous structures are therefore highly symmetrical, and tend to have rich automorphism groups. There are two main reasons why we are interested in the property of homogeneity. The first comes from an algebraic viewpoint, where the definition of homogeneity not only arises naturally, but is seemingly strong enough to allow for full classifications. The second is the aforementioned link between homogeneity and $\aleph_0$-categoricity, given in Theorem 1.2.26.

The rest of this thesis is concerned with homogeneity of structures, the focus being on semigroups. We proceed as follows. A literature review is given in Section 4.1, and in Section 4.2 a well known construction of Fraïssé is described. In Section 4.3 we discuss how our choice of signature impacts on the homogeneity of a semigroup, in particular for monoids and completely regular semigroups. In Section 4.4 we describe substructures of a homogeneous structure and, by applying these results to the signature L_S, show how these translate to the semigroup context. In Section 4.5, the homogeneity of non-periodic semigroups is examined, our main result being that a completely regular non-periodic homogeneous semigroup is completely simple. This chapter ends with a brief discussion on the homogeneity of completely simple semigroups, and finite regular homogeneous semigroups are shown to be completely simple. The results of this chapter are then used throughout Chapters 5, 6 and 7, where the homogeneity of bands, inverse semigroups and orthodox completely regular semigroups are studied, respectively.

It should be noted that the order of the chapters does not reflect the order of research. I began my study into homogeneity with bands, followed by inverse semigroups. Much of the material of this chapter and Chapter 7 was produced when attempting to place our results on homogeneous bands and inverse semigroups into a general setting (completely regular semigroups). As such, completely simple semigroups and arbitrary completely regular semigroups have been the least investigated from the point of view of homogeneity, although we will highlight a number of interesting open problems that naturally arise.

4.1 Literature review

The concept of homogeneity was introduced by Fraïssé in 1954 in his seminal paper [32]. Here he described a method for building homogeneous structures from certain classes of finite structures. While he restricted his work to relational structures over a finite signature, his construction was easily generalized to arbitrary structures. His work is regarded as some of the most fundamental in model theory.

Since the work of Fraïssé, there has been a continuous interest in homogeneous structures. The following literature review is in no way complete, and centres on classifications which in some way relate to, or are used in, our research. Much of the early work was on the homogeneity of relational structures. There are a number of reasons for this, the first being a natural continuation to Fraïssé's relational structure viewpoint. Secondly, the homogeneity of relational structures can be considered the most 'natural', for it is easier to picture a highly symmetric graph than, say, a symmetrical semigroup. In addition, the f.g. substructures of a relational structure are normally far easier to understand than for algebraic structures, since they arise simply as the finite subsets under the induced structure. Due to this, a complete classification of homogeneous relational structures such as graphs and posets seems more likely to be obtainable than for algebraic structures. History certainly backs up this point.

Finite homogeneous graphs were determined by Gardiner in [11], a result which was later extended to all homogeneous graphs by Lachlan and Woodrow in [61]. Lachlan [62] classified homogeneous tournaments, and homogeneous posets were determined by Schmerl [92]. The classification of homogeneous bipartite graphs by Goldstern was given in Theorem 3.5.4. The weaker property of n-homogeneity has been studied for graphs in [26] and for posets in [24].

There has also been much progress in the classification of homogeneous non-relational structures. For groups and rings, the interest in homogeneity was kick started by Macintyre [66] in 1971, where quantifier eliminable fields are described. For finite structures, quantifier elimination is equivalent to homogeneity by Theorem 1.2.26, but in general is far more restrictive. Macintyre's work led to a burst of research on quantifier elimination for classes of groups and rings, for example see [16] and [91]. These results were later transferred to the homogeneous setting. One occurrence of this transferal was in 1979, where interest in the quantifier elimination of solvable groups was started by Cherlin and Felgner, whose work in this area continued throughout the 1980s. By the late 80s, their viewpoint was switched to the homogeneity of solvable groups in [17], and the classification of homogeneous solvable groups was reduced to the case of nilpotent groups of class 2 and exponent 4. We refer the reader to [17] for a fantastic historical account of the problems and successes during the 1970s and 80s in the researching of homogeneous groups and rings. It is worth noting that Cherlin and Felgner were not alone in investigating quantifier eliminable and homogeneous solvable groups. Indeed, the theory was developed by Saracino [90] in 1982, where Fraïssé's method was used to prove the existence of uncountably many homogeneous nilpotent groups of class 2 and exponent 4. Furthermore, in 1984 Neumann independently classified all finite homogeneous solvable groups. This feat was soon eclipsed, and a description of all finite homogeneous groups can be found in [19] and [64], although as Cherlin states in [17],

> "The history of the results in the finite case is fairly complicated."

However, very little is known about the homogeneity of semigroups, with the exception of the classification of homogeneous semilattices by Droste, Truss and Kuske [27], and a brief discussion on normals bands in [10]. The work in this thesis aims to bridge this gap in knowledge.

4.2 Fraïssé's Theorem

Our methods for proving homogeneity come in two forms: either we prove it directly with the help of certain isomorphism theorems, or we use the general method of Fraïssé. In this section we describe the latter method. All background material is taken from [51, Chapter 7].

Let L be a signature and M an L-structure. The *age* of M is the class of all f.g. L-structures which can be embedded in M.

Let $\mathcal{K}$ be a class of f.g. L-structures. Then we say

(1) $\mathcal{K}$ is *countable* if it contains only countably many isomorphism types.

(2) $\mathcal{K}$ is *closed under isomorphism* if whenever $A \in \mathcal{K}$ and $B \cong A$ then $B \in \mathcal{K}$.

(3) $\mathcal{K}$ has the *hereditary property* (HP) if given $A \in \mathcal{K}$ and B a f.g. substructure of A then $B \in \mathcal{K}$.

(4) $\mathcal{K}$ has the *joint embedding property* (JEP) if given $B_1, B_2 \in \mathcal{K}$, then there exist $C \in \mathcal{K}$ and embeddings $f_i : B_i \to C$ $(i = 1, 2)$.

(5) $\mathcal{K}$ has the *amalgamation property*[1] (AP) if given $A, B_1, B_2 \in \mathcal{K}$, where A is non-empty, and embeddings $f_i : A \to B_i$ $(i = 1, 2)$, then there exist $D \in \mathcal{K}$

[1] This is also known as the *weak amalgamation property*.

and embeddings $g_i : B_i \to D$ such that

$$f_1 \circ g_1 = f_2 \circ g_2.$$

For example, the age of any structure can be seen to be closed under isomorphism and have HP and JEP. We may now state Fraïssé's Theorem, which can be found in [51, Theorem 7.1.2].

Theorem 4.2.1 (Fraïssé's Theorem). *Let L be a countable signature and $\mathcal{K}$ a non-empty countable class of f.g. L-structures which is closed under isomorphism and satisfies HP, JEP and AP. Then there exists a unique, up to isomorphism, countable homogeneous L-structure M such that $\mathcal{K}$ is the age of M. Conversely, the age of a countable homogeneous L-structure is countable, closed under isomorphism, and satisfies HP, JEP and AP.*

We call M the *Fraïssé limit* of $\mathcal{K}$.

Example 4.2.2. The class of all finite graphs is a Fraïssé class [51, Lemma 7.4.3], and its Fraïssé limit is a countably infinite graph called the *random graph*. This is arguable the most famous example of a Fraïssé limit, due to its numerous beautiful properties and descriptions. An in-depth study of the random graph is given by Cameron [11, Chapter VII].

Example 4.2.3. The class of all finite bipartite graphs is a Fraïssé class [35]. The Fraïssé limit is the random bipartite graph, discussed in Section 3.5.

Example 4.2.4. In 1959, Hall proved in [44] the existence of a unique, up to isomorphism, locally finite homogeneous group which embeds every finite group. This group is the Fraïssé limit of the class of finite groups, and is known as *Hall's universal group.*

Example 4.2.5. The class of all finite (inverse) semigroups does not satisfy the AP [48]. As such, there does not exist an analogy of Hall's universal group for semigroups or inverse semigroups. In [23], a weaker form of homogeneity is examined, and an analogy can be constructed in this case.

Example 4.2.6. A famous solved problem in group theory [49] was the existence of uncountably many 2-generated groups, up to isomorphism. Hence the class of all f.g. groups, and thus the class of all f.g. semigroups, does not form a Fraïssé class.

4.3 Choosing our signature: L_S versus L_{US}

When studying the homogeneity of a semigroup, it is important to distinguish which signature we are working over. For example, we could consider the homogeneity of a monoid S either in the signature of semigroups L_S or the signature of monoids L_{Mo}. In the context of homogeneity, the key difference between these two signatures is substructure: in L_S we consider f.g. subsemigroups, while in L_{Mo} we consider f.g. submonoids. This distinction is particularly important for the idempotents of S. Indeed, for any $e \in E(S)$ we have that $\langle e \rangle = \{e\}$ is isomorphic to $\langle 1 \rangle = \{1\}$, although if $e \neq 1$ then no automorphism of S can extend the unique isomorphism between them (since automorphisms of S must fix 1). It follows that if S is homogeneous in L_S then 1 is its unique idempotent. On the other hand, in the signature of monoids L_{Mo} we have that $\langle e \rangle_{Mo} = \{e, 1\}$ and $\langle 1 \rangle_{Mo} = \{1\}$ are no longer isomorphic, and so no such problem arises if S is homogeneous in L_{Mo}. This occurrence is similar for semigroups with zero, considered either in L_S or L_0. It is worth highlighting the following result that we have proven here:

Lemma 4.3.1. *Let S be either a monoid or a semigroup with zero, which is homogeneous in L_S. Then S contains a unique idempotent.*

Our third example comes from studying I-semigroups, where we restrict our choice to either L_S or the signature of unary semigroups L_{US}. If S is a semigroup with a unary operation such that S is a member of a variety of I-semigroups, then it is more natural to consider it in the signature L_{US} rather than L_S, since here the f.g. substructures are f.g. I-subsemigroups and thus belong to the variety, and isomorphisms are of the 'correct type'. For example, the homogeneity of inverse semigroups in L_S would amount to considering f.g. subsemigroups, which need not be inverse. On the other hand, a substructure of an inverse semigroup in L_{US} is clearly an inverse subsemigroup (since closure under the unary operation $x \mapsto x^{-1}$ gives rise to inverses).

Given that we are considering the homogeneity of both semigroups and I-semigroups (the latter certainly being semigroups) we need to set up some clear labelling conventions. First, if S is an I-semigroup, we will always make it clear whether we are dealing with S in L_S or in L_{US}. If P is an adjective describing a property of I-semigroups, and S has property P, then we say that S is a *homogeneous P semigroup* if S is homogeneous in L_{US}, and S is a *P homogeneous semigroup* if S is a P semigroup that is homogeneous in L_S. The fundamental example of completely regular semigroups is considered in the next subsection.

It is also worth fixing some notation for generating sets of I-semigroups. Let S be an I-semigroup and A a subset of S. Then we denote $\langle A \rangle_I$ as the I-subsemigroup of S generated by A. Much like the convention for our notation for $\langle \cdot \rangle$, this goes against the general convention of generating substructures, but no confusion should arise.

4.3.1 The homogeneity of completely regular semigroups

While studying the homogeneity of bands and inverse semigroups, we will mostly be working with completely regular semigroups. The task of choosing a suitable signature for the homogeneity of completely regular semigroups is therefore pivotal.

Recall that completely regular semigroups form a variety of I-semigroups, with unary operation $a \mapsto a^{-1}$, where a^{-1} is the inverse of $a \in S$ contained in H_a. We therefore have the concept of *homogeneous completely regular semigroups* (in the signature L_{US}). Since the class of all completely simple semigroups forms a subvariety of the variety of completely regular semigroups, as given in Section 2.9, we can also write *homogeneous completely simple semigroups* (again in L_{US}).

The difference between considering completely regular homogeneous semigroups and homogeneous completely regular semigroup lies solely in the f.g. substructures (either f.g. subsemigroups or f.g. completely regular subsemigroups, respectively) and not the isomorphisms. Indeed, if S and T are completely regular semigroups and $\phi : S \to T$ a semigroup morphism, then by [72, Lemma II.2.4] ϕ also preserves the unary operation, so that $a^{-1}\phi = (a\phi)^{-1}$ for all $a \in S$. Hence all semigroup morphisms are also morphisms in L_{US}.

Not every f.g. completely regular semigroup is a f.g. semigroup, and an example is the *free completely regular semigroup of rank 2*, discussed further in [94]. In the non-periodic case we will later show that our two concepts of homogeneity for a completely regular semigroup differ. On the other hand, for periodic completely regular semigroups we have the following.

Lemma 4.3.2. *Let S be a periodic completely regular semigroup. Then S is a homogeneous semigroup if and only if S is a homogeneous completely regular semigroup.*

Proof. Suppose $S = \bigcup_{\alpha \in Y} S_\alpha$ is a periodic completely regular semigroup. Let $T = \langle a_1, \dots, a_n \rangle$ be a f.g. subsemigroup of S. Then by [72, Lemma II.2.6], T is a completely regular subsemigroup if and only if $a^{-1} \in T$ for each $a \in T$. However, as H_a is a periodic group for each $a \in T$, some power of a is equal to a^{-1}, and thus $a^{-1} \in T$. Hence T is a completely regular subsemigroup, and we thus have that

$$\langle a_1, \dots, a_n \rangle = \langle a_1, \dots, a_n \rangle_I.$$

Consequently, every semigroup isomorphism between f.g. subsemigroups of S is a unary semigroup isomorphism between f.g. completely regular subsemigroups of S, and conversely. The result is immediate. □

We now define a stronger notion of homogeneity on a completely regular semigroup: structure-homogeneity. This will later be used for constructing homogeneous completely regular semigroups from a spined product of structure-homogeneous completely regular semigroups. The definition emerges from the following result, which is immediate from [72, Lemma II.3.8].

Lemma 4.3.3. *Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a completely regular semigroup and T a completely regular subsemigroup of S. Then there exists a subsemilattice Y' of Y such that $T = \bigcup_{\alpha \in Y'} T_\alpha$, where T_α is a completely simple subsemigroup of S_α.*

It follows that an isomorphism between a pair of completely regular subsemigroups of a completely regular semigroup induce an isomorphism between their structure semilattices, and the following property can therefore be defined.

Definition 4.3.4. Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a completely regular semigroup. Then S is called a *structure-homogeneous completely regular semigroup* if, given any pair of f.g. completely regular subsemigroups $T = \bigcup_{\alpha \in Z} T_\alpha$ and $T' = \bigcup_{\alpha' \in Z'} T'_{\alpha'}$ and any isomorphism $\theta = [\theta_\alpha, \pi]_{\alpha \in Z}$ from T to T', then for any automorphism $\hat{\pi}$ extending π, there exists an automorphism $\hat{\theta} = [\hat{\theta}_\alpha, \hat{\pi}]_{\alpha \in Y}$ of S extending θ.

It is clear from the definition that if S is a structure-homogeneous completely regular semigroup then it is a homogeneous completely regular semigroup. Moreover, if S is completely simple then, as S is a trivial semilattice of completely simple semigroups, every homogeneous completely simple semigroup is structure-homogeneous. On the other hand, a semilattice Y forms a semilattice of trivial semigroups (which are completely simple) since $Y = \bigcup_{\alpha \in Y} \{\alpha\}$, and so structure-homogeneity and homogeneity in L_{US} are also equivalent in this case.

Lemma 4.3.5. *Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a structure-homogeneous completely regular semigroup. Then for every automorphism π of Y, there exists an automorphism of S with induced semilattice automorphism π.*

Proof. Let π be an automorphism of Y and fix $\alpha \in Y$. Then for any $e_\alpha \in E(S_\alpha)$ and $e_{\alpha\pi} \in E(S_{\alpha\pi})$, the isomorphism ϕ between the trivial subsemigroups $\{e_\alpha\}$ and $\{e_{\alpha\pi}\}$ has induced semilattice isomorphism $\pi|_{\{\alpha\}} : \{\alpha\} \to \{\alpha\pi\}$. Since π extends $\pi|_{\{\alpha\}}$ and S is structure-homogeneous, there exists an automorphism of S with induced semilattice automorphism π as required. □

We end this section by constructing a class of structure-homogeneous completely regular semigroups, which will be vital to both the classification of homogeneous bands and homogeneous inverse semigroups. Let Y be a semilattice and T be a completely simple semigroup. Then $S = Y \times T$ is completely regular, and by Proposition 3.7.13, is isomorphic to a strong semilattice of completely simple semigroups $[Y; S_\alpha; \psi_{\alpha,\beta}]$ with $S_\alpha \cong T$ and with each connecting morphism being an isomorphism. We use the extended notation introduced in Section 3.7 by defining a connecting morphism $\psi_{\alpha,\beta} := \psi_{\alpha,\alpha\beta}\psi^{-1}_{\beta,\alpha\beta}$ for every $\alpha, \beta \in Y$. We aim to prove that S is structure-homogeneous if Y and T are homogeneous. We rely upon the following description of the automorphisms of S.

Lemma 4.3.6. *Let $S = [Y; S_\alpha; \psi_{\alpha,\beta}]$ be a completely regular semigroup such that each connecting morphism $\psi_{\alpha,\beta}$ is an isomorphism. Let $\pi \in Aut\,(Y)$ and, for a fixed $\alpha^* \in Y$, let $\theta_{\alpha^*} \in Iso(S_{\alpha^*}; S_{\alpha^*\pi})$. For each $\delta \in Y$, let $\theta_\delta : S_\delta \to S_{\delta\pi}$ be given by*

$$\theta_\delta = \psi_{\delta,\alpha^*}\theta_{\alpha^*}\psi_{\alpha^*\pi,\delta\pi}. \tag{4.1}$$

Then $\theta = [\theta_\alpha, \pi]_{\alpha\in Y}$ is an automorphism of S. Conversely, every automorphism of S can be so constructed.

Proof. Let π and θ_δ ($\delta \in Y$) be defined as in the hypothesis of the lemma. If $\delta \geq \gamma$ in Y, then by Lemma 3.7.12 we have

$$\begin{aligned}\psi_{\delta,\gamma}\,\theta_\gamma &= \psi_{\delta,\gamma}\,\psi_{\gamma,\alpha^*}\,\theta_{\alpha^*}\,\psi_{\alpha^*\pi,\gamma\pi}\\ &= \psi_{\delta,\alpha^*}\,\theta_{\alpha^*}\,\psi_{\alpha^*\pi,\delta\pi}\,\psi_{\delta\pi,\gamma\pi}\\ &= \theta_\delta\,\psi_{\delta\pi,\gamma\pi}.\end{aligned}$$

Hence the diagram $[\delta, \gamma; \delta\pi, \gamma\pi]$ commutes, and so θ is an automorphism of S by Theorem 2.7.1.

Conversely, suppose $\theta = [\theta_\alpha, \pi]_{\alpha\in Y}$ is an automorphism of S, and fix any $\alpha^* \in Y$. Then for each $\delta \in Y$, since the connecting morphisms are isomorphisms and both the diagrams $[\alpha^*, \alpha^*\delta; \alpha^*\pi, (\alpha^*\delta)\pi]$ and $[\delta, \alpha^*\delta; \delta\pi, (\alpha^*\delta)\pi]$ commute, we have

$$\psi_{\alpha^*,\alpha^*\delta}\,\theta_{\alpha^*\delta} = \theta_{\alpha^*}\,\psi_{\alpha^*\pi,(\alpha^*\delta)\pi}$$

and

$$\psi_{\delta,\alpha^*\delta}\,\theta_{\alpha^*\delta} = \theta_\delta\,\psi_{\delta\pi,(\alpha^*\delta)\pi}.$$

This gives

$$\psi_{\alpha^*\delta,\alpha^*}\,\theta_{\alpha^*}\,\psi_{\alpha^*\pi,(\alpha^*\delta)\pi} = \theta_{\alpha^*\delta} = \psi_{\alpha^*\delta,\delta}\,\theta_\delta\,\psi_{\delta\pi,(\alpha^*\delta)\pi},$$

by (3.6). Hence, again by Lemma 3.7.12 we have

$$\begin{aligned}\theta_\delta &= \psi^{-1}_{\alpha^*\delta,\delta}\,\psi_{\alpha^*\delta,\alpha^*}\,\theta_{\alpha^*}\,\psi_{\alpha^*\pi,(\alpha^*\delta)\pi}\,\psi^{-1}_{\delta\pi,(\alpha^*\delta)\pi}\\ &= \psi_{\delta,\alpha^*\delta}\,\psi_{\alpha^*\delta,\alpha^*}\,\theta_{\alpha^*}\,\psi_{\alpha^*\pi,(\alpha^*\delta)\pi}\,\psi_{(\alpha^*\delta)\pi,\delta\pi}\\ &= \psi_{\delta,\alpha^*}\,\theta_{\alpha^*}\,\psi_{\alpha^*\pi,\delta\pi}\end{aligned}$$

as required. ∎

The following useful lemma is merely a simple extension of the homogeneity of a structure, but we prove it for completeness.

Lemma 4.3.7. *Let M and M' be isomorphic homogeneous L-structures for some signature L. Then any isomorphism between f.g. substructures A and A' of M and M', respectively, can be extended to an isomorphism between M and M'.*

Proof. Let A and A' be f.g. substructures of M and M', respectively. Let $\theta : A \to A'$ be an isomorphism and fix some isomorphism $\phi : M \to M'$. Then

$$\theta(\phi^{-1}|_{A'}) : A \to A'\phi^{-1}$$

is an isomorphism between f.g. subgroups of M, which can thus be extended to an automorphism χ of M. The isomorphism $\chi\phi : M \to M'$ extends θ, since if $g \in A$ then

$$g\chi\phi = g(\theta(\phi^{-1}|_{A'}))\,\phi = g\theta.$$

□

Proposition 4.3.8. *Let $S = [Y; S_\alpha; \psi_{\alpha,\beta}]$ be a strong semilattice of completely simple semigroups S_α with each connecting morphism being an isomorphism. Let Y be a homogeneous semilattice and each S_α be a homogeneous completely simple semigroup. Then S is a structure-homogeneous completely regular semigroup.*

Proof. Let A and A' be a pair of f.g. completely regular subsemigroups of S given by

$$A = [Z; A_\alpha; \psi^A_{\alpha,\beta}]$$
$$A' = [Z'; A'_{\alpha'}; \psi^{A'}_{\alpha',\beta'}]$$

where $\psi^A_{\alpha,\beta}$ and $\psi^{A'}_{\alpha',\beta'}$, being restrictions of isomorphisms, are embeddings. Let $\theta = [\theta_\alpha, \pi]_{\alpha \in Z}$ be an isomorphism from A to A', noting that all isomorphisms are of this form by Lemma 2.11.7, and let $\hat{\pi}$ be an automorphism of Y extending π. Denote the minimum elements of Z and Z' as α^* and β^*, respectively. Then $\alpha^*\pi = \beta^*$, and for each $\delta \in Z$, the diagram

$$\begin{tikzcd} A_\delta \arrow[r, "\theta_\delta"] \arrow[d, "\psi^A_{\delta,\alpha^*}"] & A'_{\delta\pi} \arrow[d, "\psi^{A'}_{\delta\pi,\beta^*}"] \\ A_{\alpha^*} \arrow[r, "\theta_{\alpha^*}"] & A'_{\beta^*} \end{tikzcd} \tag{4.2}$$

commutes by Theorem 2.7.1. By the homogeneity of each S_α, we may extend θ_{α^*} to an isomorphism $\hat{\theta}_{\alpha^*} : S_{\alpha^*} \to S_{\beta^*}$ by Lemma 4.3.7. For each $\delta \in Y$, let $\hat{\theta}_\delta : S_\delta \to S_{\delta\hat{\pi}}$ be the isomorphism given by

$$\hat{\theta}_\delta = \psi_{\delta,\alpha^*}\,\hat{\theta}_{\alpha^*}\,\psi_{\beta^*,\delta\hat{\pi}}.$$

Then $\hat{\theta} = [\hat{\theta}_\delta, \hat{\pi}]_{\delta \in Y}$ is an automorphism of S by Lemma 4.3.6. Moreover, $\hat{\theta}_\delta$ extends

θ_δ for each $\delta \in Z$, since by (4.2)

$$\theta_\delta = \psi^{A}_{\delta,\alpha^*}\,\theta_{\alpha^*}\,(\psi^{A'}_{\delta\pi,\beta^*})^{-1}|_{\mathrm{Im}\,\psi^{A'}_{\delta\pi,\beta^*}},$$

and ψ_{δ,α^*} extends $\psi^{A}_{\delta,\alpha^*}$, $\hat{\theta}_{\alpha^*}$ extends θ_{α^*}, and $\psi_{\beta^*,\delta\hat{\pi}}$ extends $(\psi^{A'}_{\delta\pi,\beta^*})^{-1}|_{\mathrm{Im}\,\psi^{A'}_{\delta\pi,\beta^*}}$. Hence $\hat{\theta}$ extends θ, and S is structure-homogeneous. □

Hence if Y is a homogeneous semilattice and T is a homogeneous completely simple semigroup, then by Proposition 3.7.13 we have that $Y \times T$ is isomorphic to a structure-homogeneous completely regular semigroup. In this case, we will often write that $Y \times T$ is *structure-homogeneous*, where no confusion can arise.

4.4 Substructure of a homogeneous structure

Mirroring our early study into $\aleph_0$-categorical structures, we will now briefly examine examples of substructures which inherit the property of homogeneity. For example, it can be easily shown that the homogeneity of a structure will pass to characteristic substructures, and the result for groups is given in [17, Lemma 1]. We instead view a larger class of substructures: quasi-characteristic.

Definition 4.4.1. Let M be a structure with substructure A. Suppose for any automorphism ϕ of M such that there exist $a, b \in A$ with $a\phi = b$, the map $\phi|_A$ is an automorphism of A. Then we call A a *quasi-characteristic substructure of* M.

Consequently, a substructure A of a structure M is a quasi-characteristic substructure of M if and only if $\{(A, a) : a \in A\}$ forms a system of 1-p.p.r.c substructures. The following result is then immediate from Lemma 3.3.3.

Lemma 4.4.2. *Let M be a structure with substructure A. Then the following are equivalent:*

(i) *A is a quasi-characteristic substructure of M;*

(ii) *if $\phi \in Aut(M)$ is such that there exist $a, b \in A$ with $a\phi = b$, then $x\phi \in A$ for all $x \in A$.*

Remark 4.4.3. Every characteristic substructure is clearly quasi-characteristic.

Example 4.4.4. Let τ be an equivalence relation on a structure M which is preserved under automorphisms. Suppose A is an equivalence class of τ which is a substructure of M, and $\phi \in \mathrm{Aut}(M)$ is such that $A\phi \cap A \neq \emptyset$. Then $A\phi \subseteq A$ and so A is quasi-characteristic.

Lemma 4.4.5. *Let M be a homogeneous structure with a quasi-characteristic substructure A. Then A is homogeneous.*

Proof. Let ϕ be an isomorphism between f.g. substructures N and N' of A. Then N and N' are f.g. substructures of M, and so we may extend ϕ to $\bar{\phi} \in \mathrm{Aut}(M)$. Since $N\bar{\phi} = N'$ and A is quasi-characteristic, we have $\bar{\phi}|_A \in \mathrm{Aut}(A)$, and so A is homogeneous. $\square$

We proved in Proposition 3.3.9 that if τ is an automorphism preserving equivalence relation on an $\aleph_0$-categorical structure M, then $\{|x\tau| : x \in M\}$ is finite. The analogous result for homogeneous structure is as follows.

Corollary 4.4.6. *Let τ be an equivalence relation on a homogeneous structure M which is preserved under automorphisms. Let $x, y \in M$ be such that there exists an isomorphism $\phi : \langle x \rangle_M \to \langle y \rangle_M$ with $x\phi = y$. Then $|x\tau| = |y\tau|$, and if $x\tau$ forms a substructure of M then $x\tau \cong y\tau$.*

Proof. By the homogeneity of M we can extend the isomorphism ϕ to an automorphism θ of M. Since θ preserves τ we have $(x\tau)\theta = (x\theta)\tau = y\tau$, and the result follows (noting that by θ, we have that $x\tau$ forms a substructure if and only $y\tau$ does). $\square$

We now apply our results on quasi-characteristic substructures to the case of semigroups in the signature of semigroups L_S and L_{US}. First, since Green's relations are preserved under automorphisms of a (I-)semigroup, we have the following result by Example 4.4.4 and Lemma 4.4.5.

Corollary 4.4.7. *Let S be a homogeneous (I-)semigroup. Then any $\mathcal{H}/\mathcal{R}/\mathcal{L}/\mathcal{D}/\mathcal{J}$-class of S which forms a (I-)subsemigroup of S is a homogeneous (I-)semigroup. Consequently, the maximal subgroups of S are homogeneous (I-)semigroups.*

Note that a homogeneous group might not be a homogeneous semigroup, a problem which we study in further detail in Chapter 6.

The set of idempotents $E(S)$ of a (I-)semigroup S form a characteristic subset of S by Example 3.3.1. Hence $E(S)$ generates a characteristic (I-)subsemigroup, and we arrive at the corollary below.

Corollary 4.4.8. *Let S be a homogeneous (I-)semigroup. Then $\langle E(S) \rangle$ ($\langle E(S) \rangle_I$) is a homogeneous (I-)semigroup.*

Given a subset N of a structure M, we say that $\mathrm{Aut}(M)$ *acts transitively* on N if for any $a, b \in N$, there exists an automorphism ϕ of M such that $a\phi = b$.

Lemma 4.4.9. *If S be a homogeneous semigroup. Then $Aut(S)$ acts transitively on $E(S)$.*

Proof. Given $e, f \in E(S)$, we have $\langle e \rangle = \{e\} \cong \{f\} = \langle f \rangle$. By extending the unique isomorphism from $\{e\}$ to $\{f\}$ to an automorphism of S gives the result. $\square$

Let S be a homogeneous semigroup and τ an equivalence relation preserved by automorphisms of S. Then as any pair of idempotents e and f of S generate isomorphic subsemigroups of S, it follows by Corollary 4.4.6 that $|e\tau| = |f\tau|$, and if $e\tau$ is a subsemigroup of S then $e\tau \cong f\tau$. By applying this to Green's relations we obtain the following.

Corollary 4.4.10. *Let S be a homogeneous semigroup and $\mathcal{K}$ be a Green's relation on S. Then $|K_e| = |K_f|$ for all $e, f \in E(S)$. Moreover, if for some $e \in E(S)$ the set K_e forms a subsemigroup of S, then $K_e \cong K_f$ for all $f \in E(S)$. Consequently, the maximal subgroups of S are pairwise isomorphic.*

While the corollary above may not hold for all homogeneous I-semigroups, it clearly will hold in the case where $e' = e$ for all idempotents e, since in this case $\langle e \rangle_I = \{e\}$. Our main example of this occurrence is completely regular semigroups, where the unary operation sends idempotents to the inverse in their maximal subgroup, and are thus fixed. Since the $\mathcal{D}$-classes of a completely regular semigroup $\bigcup_{\alpha \in Y} S_\alpha$ are the completely simple semigroups S_α, each of which contain an idempotent, we therefore have the following consequence of Corollaries 4.4.7 and 4.4.10.

Proposition 4.4.11. *Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a homogeneous completely regular semigroup. Then each S_α is a homogeneous completely simple semigroup, and $S_\alpha \cong S_\beta$ for each $\alpha, \beta \in Y$.*

4.5 Non-periodic homogeneous semigroups

We now begin our study into the homogeneity of certain classes of semigroups, starting in this section with non-periodic semigroups.

Given a semigroup S, we denote the set of elements of infinite order as

$$\mathrm{Inf}(S) := \{a \in S : |\langle a \rangle| = \aleph_0\}.$$

We observe that if S is a homogeneous semigroup then $\mathrm{Aut}(S)$ acts transitively on $\mathrm{Inf}(S)$. Indeed, for each $a, b \in \mathrm{Inf}(S)$, we have

$$\langle a \rangle \cong (\mathbb{N}, +) \cong \langle b \rangle,$$

and the result then follows by the homogeneity of S. We claim that either all elements of $\mathrm{Inf}(S)$ lie in subgroups of S, or none of them do. Indeed, if $a, b \in \mathrm{Inf}(S)$ are such that $a \in H_e$ for some $e \in E(S)$, then by taking an automorphism of S sending a to b we have that $b \in H_{e\theta}$ since $\mathcal{H}$ is preserved under automorphisms.

Lemma 4.5.1. *Let S be a homogeneous semigroup with a non-periodic element contained in a maximal subgroup of S. Then $(E(S), \leq)$ is an anti-chain, where $\leq$ is the natural order on $E(S)$.*

Proof. The maximal subgroups of S are isomorphic by Corollary 4.4.10, and so each maximal subgroup of S is non-periodic. Let $e, f \in E(S)$ be such that $e \geq f$, and let $x \in H_f \cap \mathrm{Inf}(S)$. Then

$$ex = e(fx) = (ef)x = fx = x = xf = x(fe) = (xf)e = xe$$

and so the map

$$\phi : \langle e, x \rangle \to \langle f, x \rangle$$

determined by $e\phi = f$ and $x\phi = x$ is an isomorphism. By the homogeneity of S, extend ϕ to an automorphism $\bar{\phi}$ of S. Since $\mathcal{H}$ is preserved under $\bar{\phi}$, we have $H_e\bar{\phi} = H_f$ and $H_f\bar{\phi} = H_x\bar{\phi} = H_x = H_f$. Hence $H_e = H_f$ and so $e = f$ as required. □

A regular semigroup S in which $(E(S), \leq)$ forms an anti-chain is necessarily completely simple since all idempotents are minimal, and so the following corollary to Lemma 4.5.1 is immediate.

Corollary 4.5.2. *Let S be a regular homogeneous semigroup. If S contains a non-periodic element in a subgroup of S then S is completely simple. In particular, non-periodic completely regular homogeneous semigroups are completely simple.*

Open Problem 2. Do there exist a regular homogeneous semigroup with an element of infinite order not contained in a subgroup of S?

This open problem can be extended by dropping the non-periodic condition. That is, does there exist a regular homogeneous semigroup which is not completely regular? Similarly, is a regular homogeneous I-semigroup completely regular? We conjecture that a regular homogeneous (I-)semigroup is completely regular, and results of the subsequent chapters back this stance.

4.6 The homogeneity of completely simple semigroups

In this chapter we have shown that understanding the homogeneity of completely simple semigroups, both in L_S and L_{US}, is vital for the homogeneity of completely regular semigroups. Indeed, completely simple semigroups appear as $\mathcal{D}$-classes of completely regular semigroups, and are also key in comparing our two concepts of homogeneity on a completely regular semigroup. Indeed, by Lemma 4.3.2 the two properties of homogeneity can only disagree on non-periodic completely regular semigroups, and by Corollary 4.5.2 non-periodic completely regular homogeneous semigroups are necessarily completely simple. Hence a completely regular homogeneous semigroup which is not a homogeneous completely regular semigroup is completely simple.

Open Problem 3. How does the homogeneity of a completely simple semigroup in L_S or L_{US} differ?

We end by giving a third motivation for a further study into the homogeneity of completely simple semigroups:

Proposition 4.6.1. *Let S be a regular homogeneous semigroup with finite set of idempotents $E(S)$. Then S is completely simple.*

Proof. Suppose, seeking a contradiction, that there exists an element $e \in E(S)$ which is not minimal in E under the natural order $\leq$ on $E(S)$. Since $E(S)$ is finite there exists a minimal element $f \in E(S)$ with $f < e$. By Lemma 4.4.9 there exists an automorphism of S sending e to f, which clearly contradicts the minimality of f. Hence every idempotent of S is minimal, and so S is completely simple. □

An attempt was made to classify homogeneous completely simple semigroups, and great progress was made in both the idempotent-generated case, and the finite case. In Chapter 7, the classification of orthodox homogeneous completely simple semigroups will be given. However, this is still ongoing work, and time did not permit a further discussion.

Chapter 5

Homogeneous bands

This chapter investigates the homogeneity of bands. Since bands form a variety of semigroups and of completely regular semigroups (where the unary operation is trivial), we could consider homogeneity in L_S or in L_{US}. However, by Lemma 4.3.2 the two concepts of homogeneity intersect, and we may simply write *homogeneous band* without ambiguity. This allows us to use the results and concepts introduced in Subsection 4.3.1, and in particular write *structure-homogeneous band* to mean a band which is a structure-homogeneous completely regular semigroup, again without ambiguity. It follows from the work of Lean [63], that bands are ULF, and so we need only look at isomorphisms between finite subbands.

Our main result is a complete description of all homogeneous bands, showing them to be regular bands. We also examine how our results fit in with known classifications, in particular showing that the structure semilattice of a homogeneous band is itself homogeneous. The classification of homogeneous bands is therefore an extension of the classification of homogeneous semilattices.

Interest in the homogeneity of bands began in [10], where Byleen states the existence of a universal normal band which is homogeneous, although no formal proof is given. The open problem of finding a representation of this band is also stated. We aim to formalise Byleen's brief work on homogeneous normal bands, and obtain a number of properties of the universal normal band.

5.1 Homogeneous semilattices

The homogeneity of semilattices was first studied by Droste in [24] and, together with Truss and Kuske in [27]. Note that both articles consider the homogeneity of semilattices in the signature of lower semilattices $L_{LS} = \{\leq, \wedge\}$. We first show that their work effectively considers homogeneity of (algebraic) semilattices, a result that will be immediate from the following simple consequence of Proposition 2.3.1.

Corollary 5.1.1. *Let $(Y, \leq, \wedge)$ and $(Y', \leq, \wedge)$ be a pair of lower semilattices and $\phi : Y \to Y'$ a map. Then ϕ is a lower semilattice morphism if and only if it is a semigroup morphism from $(Y, \wedge)$ to $(Y', \wedge)$.*

Proof. Let ϕ be a semigroup morphism from $(Y, \wedge)$ to $(Y', \wedge)$. For any $e, f \in Y$, we have

$$e \leq f \Rightarrow ef = e \Rightarrow e\phi f\phi = e\phi \Rightarrow e\phi \leq f\phi,$$

and so ϕ is a morphism preserving $\leq$, and is thus a morphism between lower semilattices $(Y, \leq, \wedge)$ and $(Y', \leq, \wedge)$ as required.

The converse is trivial. □

Lemma 5.1.2. *A semilattice $(Y, \wedge)$ is a homogeneous band if and only if $(Y, \wedge, \leq)$ is a homogeneous lower semilattice.*

Proof. Let $(Y, \wedge)$ be a semilattice which is a homogeneous band. Let $(A, \wedge, \leq)$ and $(A', \wedge, \leq)$ be a pair of f.g. lower subsemilattices of $(Y, \wedge, \leq)$, and $\phi : A \to A'$ a lower semilattice isomorphism. Then by the corollary above ϕ is an isomorphism between the subsemilattices $(A, \wedge)$ and $(A', \wedge)$ of $(Y, \wedge)$, which can thus extend to an automorphism of $(Y, \wedge)$. Applying Corollary 5.1.1 again gives $(Y, \wedge, \leq)$ to be homogeneous. The converse is proven similarly. □

We therefore simply refer to a *homogeneous semilattice* to mean homogeneous in L_S or L_{LS}, without ambiguity. Note however that a homogeneous semilattice need not be homogeneous as a poset. Indeed, $(\mathbb{Q}, \leq)$ is the unique homogeneous semilattice that is also homogeneous as a poset by Schmerl's classification of homogeneous posets [92].

A semilattice Y with natural partial order $\leq$ is called a *semilinear order* if $(Y, \leq)$ is non-linear and, for all $\alpha \in Y$, the set $\{\beta \in Y : \beta \leq \alpha\}$ is linearly ordered. This is equivalent to Y not containing a *diamond*, where a diamond is a collection of distinct $\delta, \alpha, \gamma, \beta \in Y$ such that $\delta > \{\alpha, \gamma\} > \beta$ and $\alpha \perp \gamma$ with $\alpha\gamma = \beta$.

The class of all finite semilattices forms a Fraïssé class, and its Fraïssé limit is called the *universal semilattice*. It was shown [27] that every distinct pair of elements in the universal semilattice has an upper bound, that is, an element strictly greater than both elements, and that the upper bound is never unique.

Lemma 5.1.3. *Let Y be the universal semilattice and Z a finite subsemilattice of Y. Then for any finite semilattice Z' in which Z embeds, there exists $X \subset Y \setminus Z$ such that $Z \cup X \cong Z'$.*

Proof. Let Z' be a finite semilattice and $\theta : Z \to Z'$ an embedding. Since Y is universal, there exists an embedding $\phi : Z' \to Y$. Hence $\theta\phi$ is an isomorphism

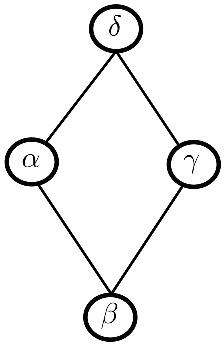

Figure 5.1: A diamond.

between Z and $Z\theta\phi$, which we can extend to an automorphism χ of Y, since the universal semilattice is homogeneous. Then Z is a subsemilattice of $(Z'\phi)\chi^{-1}$ since

$$Z\chi = Z\theta\phi \subseteq Z'\phi.$$

Moreover, $(Z'\phi)\chi^{-1}$ is isomorphic to Z', and the result follows by taking $X = (Z'\phi)\chi^{-1} \setminus Z$. □

In [24], every homogeneous semilattice forming a semilinear order was constructed, which led to the following classification.

Proposition 5.1.4 (Droste, Kuske, Truss [24, 27])**.** *A non-trivial homogeneous semilattice is isomorphic to either* $(\mathbb{Q}, \leq)$*, a semilinear order, or the universal semilattice.*

Note that not every semilinear order is a homogeneous semilattice. Moreover, a non-trivial homogeneous semilattice is dense, a property of homogeneous semilattices which we use throughout these final chapters without reference.

5.2 *The homogeneity of an arbitrary band*

In this section we recall some basic properties of bands, which are used to further understand the homogeneity of an arbitrary band. By Proposition 2.10.3 a band B is a semilattice of rectangular bands, and these rectangular bands form the $\mathcal{D}$-classes of B. We let $\leq$ denote the natural order on B, given by $e \leq f$ if and only if $ef = fe = e$, where $e, f \in B$. We are interested in understanding the $\mathcal{D}$-classes, the natural order, and the structure semilattice of a homogeneous band.

We first give the ideal structure on a band B, which is taken from [72]. Green's left and right quasi-orders simplify as

$$e \leq_l f \Leftrightarrow ef = e, \quad e \leq_r f \Leftrightarrow fe = e,$$

for each $e, f \in B$. The Green's relations on B are then given by:

$$\begin{aligned}
&e\,\mathcal{L}\,f \Leftrightarrow ef = e, fe = f;\\
&e\,\mathcal{R}\,f \Leftrightarrow ef = f, fe = e;\\
&e\,\mathcal{H}\,f \Leftrightarrow e = f;\\
&e\,\mathcal{D}\,f \Leftrightarrow e\,\mathcal{J}\,f \Leftrightarrow efe = e, fef = f,
\end{aligned}$$

for each $e, f \in B$.

After semilattices, the second variety of bands required for the construction of an arbitrary band are rectangular bands. Determining the homogeneity of rectangular bands will therefore be vital for a general study. Recall that $B_{n,m}$ denotes the unique, up to isomorphism, rectangular band with n $\mathcal{R}$-classes and m $\mathcal{L}$-classes.

Proposition 5.2.1. *The rectangular band $B_{n,m}$ is a homogeneous band for any $n, m \in \mathbb{N}^* = \mathbb{N} \cup \{\aleph_0\}$.*

Proof. Let $B_{n,m} = L \times R$ be a rectangular band, and A_1 and A_2 a pair of subbands of B. Since the class of rectangular bands forms a variety, each A_i is a rectangular band and $A_i = L_i \times R_i$ for some $L_i \subseteq L$, $R_i \subseteq R$. Let $\theta : A_1 \to A_2$ be an isomorphism. Then, by Proposition 2.10.2, there exist bijections $\theta_{L_1} : L_1 \to L_2$ and $\theta_{R_1} : R_1 \to R_2$ such that $\theta = \theta_{L_1} \times \theta_{R_1}$. Extend θ_{L_1} to a bijection θ_L of L, and similarly construct the bijection θ_R of R. Then $\hat{\theta} = \theta_L \times \theta_R$ extends θ as required. □

The $\mathcal{D}$-classes of an arbitrary band are therefore homogeneous. However, note that not every band is homogeneous, for homogeneous bands are restricted to having isomorphic $\mathcal{D}$-classes by Corollary 4.4.10. Since bands are completely regular, the isomorphisms between a pair of bands can be obtained from Proposition 2.11.2 as follows.

Proposition 5.2.2. *Let $B = \bigcup_{\alpha \in Y} B_\alpha$ and $B' = \bigcup_{\alpha' \in Y'} B'_{\alpha'}$ be a pair of bands. Then, for any isomorphism $\theta : B \to B'$, there exists an isomorphism $\pi : Y \to Y'$ and an isomorphism $\theta_\alpha : B_\alpha \to B'_{\alpha\pi}$ for every $\alpha \in Y$, such that $\theta = \bigcup_{\alpha \in Y} \theta_\alpha$.*

We abuse notation somewhat by denoting θ as $[\theta_\alpha, \pi]_{\alpha \in Y}$. This notation is normally reserved for strong semilattices of semigroups, but is used for arbitrary semilattices of rectangular bands where no confusion may arise.

We fix a number of useful subsets of an arbitrary band $B = \bigcup_{\alpha \in Y} B_\alpha$. If $\alpha > \beta$ in Y and $e_\alpha \in B_\alpha$ then we let

(i) $B_\beta(e_\alpha) := \{e_\beta \in B_\beta : e_\beta < e_\alpha\}$;

(ii) $B_{\alpha,\beta} := \bigcup_{f_\alpha \in B_\alpha} B_\beta(f_\alpha) = \{e_\beta \in B_\beta : e_\beta < f_\alpha \text{ for some } f_\alpha \in B_\alpha\}$;

(iii) $\mathcal{R}(B_\beta(e_\alpha)) := \{f_\beta \in B_\beta : f_\beta <_r e_\alpha\}$;

and dually for $\mathcal{L}(B_\beta(e_\alpha))$. Note first that

$$\mathcal{R}(B_\beta(e_\alpha)) = \{f_\beta \in B_\beta : \exists e_\beta \in B_\beta(e_\alpha), f_\beta \, \mathcal{R} \, e_\beta\}.$$

Indeed, we claim that if $f_\beta \in B_\beta$ and $e_\alpha \in B_\alpha$, then $f_\beta <_r e_\alpha$ if and only if $f_\beta \, \mathcal{R} \, e_\alpha f_\beta e_\alpha$, to which the result follows as $e_\alpha f_\beta e_\alpha < e_\alpha$. If $f_\beta <_r e_\alpha$ then $f_\beta = e_\alpha f_\beta$, and so

$$f_\beta(e_\alpha f_\beta e_\alpha) = (f_\beta e_\alpha)(f_\beta e_\alpha) = f_\beta e_\alpha = e_\alpha f_\beta e_\alpha$$

so that $f_\beta >_r e_\alpha f_\beta e_\alpha$. Hence f_β and $e_\alpha f_\beta e_\alpha$, being elements of B_β, are $\mathcal{R}$-related. Conversely, if $f_\beta \, \mathcal{R} \, e_\alpha f_\beta e_\alpha$ then

$$f_\beta = (e_\alpha f_\beta e_\alpha) f_\beta = e_\alpha f_\beta$$

and the claim holds.

We observe that the set $B_\beta(e_\alpha)$ is non-empty for each $e_\alpha \in B_\alpha$, since for any $e_\beta \in B_\beta$ we have $e_\alpha > e_\alpha e_\beta e_\alpha \in B_\beta$. Moreover, each of the sets defined above are subbands of B_β. Indeed, if $e_\alpha > e_\beta$ and $f_\alpha > f_\beta$ then

$$e_\beta f_\beta e_\alpha f_\alpha = e_\beta (f_\beta f_\alpha) e_\alpha f_\alpha = e_\beta f_\beta (f_\alpha e_\alpha f_\alpha) = e_\beta f_\beta f_\alpha = e_\beta f_\beta,$$

and similarly $e_\alpha f_\alpha e_\beta f_\beta = e_\beta f_\beta$. Hence $e_\alpha f_\alpha > e_\beta f_\beta \in B_{\alpha,\beta}$, and so $B_{\alpha,\beta}$ is a subband. By taking $e_\alpha = f_\alpha$ gives $B_\beta(e_\alpha)$ to be a subband. Finally $\mathcal{R}(B_\beta(e_\alpha))$, being a collection of $\mathcal{R}$-classes of B_β, is a subband.

Corollary 5.2.3. *Let $B = \bigcup_{\alpha \in Y} B_\alpha$ be a homogeneous band where $B_\alpha = L_\alpha \times R_\alpha$. Then, for all $\alpha > \beta$ and $\alpha' > \beta'$ in Y, $e_\alpha \in B_\alpha$ and $e_{\alpha'} \in B_{\alpha'}$, we have*

(i) *Aut(B) acts transitively on B;*

(ii) *Y is dense and without maximal or minimal elements;*

(iii) *$B_\alpha \cong B_{\alpha'}, L_\alpha \cong L_{\alpha'}$ and $R_\alpha \cong R_{\alpha'}$;*

(iv) *$B_\beta(e_\alpha) \cong B_{\beta'}(e_{\alpha'})$.*

Proof. (i) Immediate from Lemma 4.4.9.

(ii) Suppose, seeking a contradiction, that δ is maximal, and let $\tau < \delta$ in Y. Then for any $e_\delta \in B_\delta$ and $e_\tau \in B_\tau$, there exists by (i) an automorphism θ of B mapping e_δ to e_τ. Hence by Proposition 5.2.2, the induced semilattice automorphism of θ maps δ to τ, contradicting δ being maximal. The result is proven similarly for minimal elements.

Now suppose $\alpha > \beta$ in Y. Since β is not minimal, there exists $\gamma \in Y$ such that $\gamma < \beta$. Let $e_\alpha \in B_\alpha, e_\beta \in B_\beta(e_\alpha)$ and $e_\gamma \in B_\gamma(e_\beta)$, so $e_\gamma \in B_\gamma(e_\alpha)$. Then by extending the isomorphism from $\{e_\alpha, e_\gamma\}$ to $\{e_\alpha, e_\beta\}$ to an automorphism of B, it follows by taking the image of β under the induced automorphism of Y that there exists $\gamma' \in Y$ such that $\alpha > \gamma' > \beta$, and so Y is dense.

(iii) Each B_α is a $\mathcal{D}$-class of B, and so by Proposition 4.4.11 we have $B_\alpha \cong B_{\alpha'}$ for all $\alpha, \alpha' \in Y$. The results for L_α and R_α are then immediate from Proposition 2.10.2.

(iv) Let $e_\beta \in B_\beta(e_\alpha)$ and $e_{\beta'} \in B_{\beta'}(e_{\alpha'})$. Since $\{e_\alpha, e_\beta\}$ and $\{e_{\alpha'}, e_{\beta'}\}$ are isomorphic subbands, the result follows by extending the unique isomorphism between them to an automorphism of B. $\square$

If B is a band with non-trivial structure semilattice, then by the proof of (ii), it is clear that B, regarded as a poset under its natural order, cannot have maximal or minimal elements, since the natural order is preserved under automorphisms of B. On the other hand, if B has trivial structure semilattice then it is a rectangular band, and the ordering is an anti-chain.

One of our fundamental questions in this chapter is whether or not the homogeneity of a band is inherited by its structure semilattice. The answer is yes, but surprisingly we have not been able to find a direct proof. For now we are only able to partially answer this question:

Proposition 5.2.4. *If $B = \bigcup_{\alpha\in Y} B_\alpha$ is a homogeneous band, then its structure semilattice Y is 2-homogeneous. Consequently, if Y is linearly or semi-linearly ordered then Y is homogeneous.*

Proof. Since the unique (up to isomorphism) 2 element semilattice is a chain, it suffices to consider a pair $\alpha_i > \beta_i$ $(i = 1, 2)$ in Y. Fix $e_{\alpha_i} \in B_{\alpha_i}$ for each $i = 1, 2$ and let $e_{\beta_i} \in B_{\beta_i}(e_{\alpha_i})$. By extending the isomorphism between $\{e_{\alpha_1}, e_{\beta_1}\}$ and $\{e_{\alpha_2}, e_{\beta_2}\}$ to an automorphism of B, it follows by Proposition 5.2.2 that Y is 2-homogeneous. The final result is then immediate from [27, Proposition 2.1]. $\square$

To avoid falling into already complete classifications, unless stated otherwise *we assume throughout this chapter that Y is non-trivial* (so B is not a rectangular band) and *each $\mathcal{D}$-class is non-trivial* (so B is not a semilattice).

5.3 Regular bands

In this section we consider three of the varieties of bands given in Figure 2.3 which we have so far neglected: left/right regular bands. We later prove that a homogeneous band is necessarily regular. As such, it will be useful to obtain, in much the same way as with normal bands, an alternative description of regular bands in which a relatively simple isomorphism theorem arises.

Kimura showed in [59] that a band B is regular if and only if it is a spined product of a left regular and right regular band (known as the *left and right component* of B, respectively). He additionally showed that a band is left (right) regular if and only if it is a semilattice of left zero (right zero) semigroups.

Let $B = L \bowtie R$ be a regular band. Then as the classes of left regular, right regular and regular bands form varieties, every subband A of B is regular. Hence by Lemma 4.3.3 we have that there exist subbands L' of L and R' of R such that $A = L' \bowtie R'$.

The following isomorphism theorem, which gives a converse to Proposition 2.11.3 in the case of regular bands, was proven by Kimura, but given in the form below (in the general context of morphisms) in [72, Lemma V.1.10]:

Proposition 5.3.1. *Let $B = L \bowtie R$ and $B' = L' \bowtie R'$ be regular bands with structure semilattices Y and Y', respectively. Let $\theta^l : L \to L'$ and $\theta^r : R \to R'$ be isomorphisms which induce the same semilattice isomorphism $\pi : Y \to Y'$. Define a mapping θ by*

$$(l, r)\theta = (l\theta^l, r\theta^r) \quad ((l, r) \in B).$$

Then θ is an isomorphism from B onto B', denoted $\theta = \theta^l \bowtie \theta^r$, and every isomorphism from B to B' can be so constructed for unique θ^l and θ^r.

In general, a pair of regular bands with isomorphic left and right components need not be isomorphic. Indeed, by the proposition above, there are required to be isomorphisms between the left components and between the right components with equal induced semilattice isomorphism. The ensuing lemma gives a condition on the components of the regular bands which forces them to be isomorphic:

Corollary 5.3.2. *Let $B = L \bowtie R$ and $B' = L' \bowtie R'$ be a pair of regular bands with structure semilattices Y and Y', respectively, and with L and L' structure-homogeneous. Then $B \cong B'$ if and only if $L \cong L'$ and $R \cong R'$ (dually for R and R').*

Proof. Let $\theta^r : R \to R'$ and $\theta^l : L \to L'$ be isomorphisms with induced isomorphisms π_r and π_l of Y into Y', respectively. Then as L' is structure-homogeneous there exists an automorphism ϕ^l of L' with induced automorphism $\pi_l^{-1}\pi_r$ of Y'. Hence $\theta^l\phi^l$ is an isomorphism from L to L' with induced isomorphism $\pi_l(\pi_l^{-1}\pi_r) = \pi_r$, and so $\theta^l\phi^l \bowtie \theta^r$ is an isomorphism from B to B' by Proposition 5.3.1.

The converse is immediate from the proposition above. $\square$

We are now able to give our first justification for studying the stronger property of structure-homogeneity:

Corollary 5.3.3. *Let B be the spined product of a homogeneous left regular band L and homogeneous right regular band R. If either L or R are structure-homogeneous, then B is homogeneous. Moreover, if both L and R are structure-homogeneous, then so is B.*

Proof. Suppose w.l.o.g. that L is structure-homogeneous, with structure semilattice Y. Let $\theta = \theta^l \bowtie \theta^r$ be an isomorphism between finite subbands $A_1 = L_1 \bowtie R_1$ and $A_2 = L_2 \bowtie R_2$ of B. Then by Proposition 5.3.1, the isomorphisms θ^l and θ^r both induce an isomorphism π between the structure semilattices Y_1 and Y_2 of A_1 and A_2, respectively. Since R is homogeneous, we can extend $\theta^r : R_1 \to R_2$ to an automorphism $\bar{\theta}^r$ of R, with induced automorphism $\bar{\pi}$ of Y extending π. Since L is structure-homogeneous, there exists an automorphism $\bar{\theta}^l$ of L extending θ^l and with induced automorphism $\bar{\pi}$ of Y. Hence $\bar{\theta}^l \bowtie \bar{\theta}^r$ is an automorphism of B, which extends θ as required. The final result is proven in a similar fashion. □

5.4 Homogeneous normal bands

In this section we classify homogeneous normal bands. Our aim is helped by not only a classification theorem for normal bands which gives the local structure, but also a relatively simple isomorphism theorem, since strong semilattices of rectangular bands are morphism-pure by Lemma 2.11.7. Theorem 2.7.1 then simplifies:

Theorem 5.4.1. *Let $B = [Y; B_\alpha; \psi_{\alpha,\beta}]$ and $B' = [Y'; B'_{\alpha'}; \psi'_{\alpha',\beta'}]$ be a pair of normal bands. Let $\pi : Y \to Y'$ be an isomorphism, and for every $\alpha \in Y$, let $\theta_\alpha : B_\alpha \to B_{\alpha\pi}$ be an isomorphism such that for any $\alpha \geq \beta$ in Y, the diagram $[\alpha, \beta; \alpha\pi, \beta\pi]$ commutes, that is,*

$$\begin{array}{ccc} B_\alpha & \xrightarrow{\theta_\alpha} & B'_{\alpha\pi} \\ \downarrow{\scriptstyle \psi_{\alpha,\beta}} & & \downarrow{\scriptstyle \psi'_{\alpha\pi,\beta\pi}} \\ B_\beta & \xrightarrow{\theta_\beta} & B'_{\beta\pi} \end{array} \tag{5.1}$$

commutes. Then $\bigcup_{\alpha \in Y} \theta_\alpha = [\theta_\alpha, \pi]_{\alpha \in Y}$ is an isomorphism from B into B'. Conversely, every isomorphism of B into B' can be so constructed for unique π and θ_α.

To understand the homogeneity of normal bands, we require a better understanding of the finite subbands. Since the class of all normal bands forms a variety, the following lemma is immediate from Lemma 4.3.3.

Lemma 5.4.2. *Let A be a subband of a normal band $B = [Y; B_\alpha; \psi_{\alpha,\beta}]$. Then $A = [Z; A_\alpha; \psi^A_{\alpha,\beta}]$, for some subsemilattice Z of Y, subbands A_α of B_α ($\alpha \in Z$) and $\psi^A_{\alpha,\beta} = \psi_{\alpha,\beta}|_A$ for each $\alpha, \beta \in Z$ with $\alpha \geq \beta$.*

Given a normal band $B = [Y; B_\alpha; \psi_{\alpha,\beta}]$, we denote Im $\psi_{\alpha,\beta}$ as $I_{\alpha,\beta}$, or $I^B_{\alpha,\beta}$ if we need to distinguish the band B.

Lemma 5.4.3. *Let $B = [Y; B_\alpha; \psi_{\alpha,\beta}]$ be a homogeneous normal band. Then $B_\beta = \bigcup_{\alpha>\beta} I_{\alpha,\beta}$ for each $\beta \in Y$.*

Proof. As a consequence of Corollary 5.2.3, B contains no maximal elements under its natural order. The result then follows as $e_\alpha \geq e_\beta$ if and only if $\alpha \geq \beta$ and $e_\alpha \psi_{\alpha,\beta} = e_\beta$. □

Lemma 5.4.4. *Let $B = [Y; B_\alpha; \psi_{\alpha,\beta}]$ be a homogeneous normal band. If $\alpha_i > \beta_i$ $(i = 1, 2)$ in Y then there exist isomorphisms $\theta_{\alpha_1} : B_{\alpha_1} \to B_{\alpha_2}$ and $\theta_{\beta_1} : B_{\beta_1} \to B_{\beta_2}$ such that*

$$\theta_{\alpha_1}\psi_{\alpha_2,\beta_2} = \psi_{\alpha_1,\beta_1}\theta_{\beta_1}$$

In particular $I_{\alpha_1,\beta_1} \cong I_{\alpha_2,\beta_2}$ and ψ_{α_1,β_1} is surjective/injective if and only if ψ_{α_2,β_2} is also.

Proof. Let $e_{\alpha_i} \in B_{\alpha_i}$ $(i = 1, 2)$ be fixed. By extending the unique isomorphism between the 2 element subbands $\{e_{\alpha_1}, e_{\alpha_1}\psi_{\alpha_1,\beta_1}\}$ and $\{e_{\alpha_2}, e_{\alpha_2}\psi_{\alpha_2,\beta_2}\}$ to an automorphism of B, the diagram $[\alpha_1, \beta_1; \alpha_2, \beta_2]$ commutes by Proposition 5.4.1, and the result easily follows. □

Since a normal band is regular, it can be regarded as the spined product of a left regular and right regular band. The following results and subsequent Proposition 5.4.5 are taken from [55, Proposition 4.6.17]. Let $B = [Y; B_\alpha; \psi_{\alpha,\beta}]$ be a normal band, where $B_\alpha = L_\alpha \times R_\alpha$ for some left zero semigroup L_α and right zero semigroup R_α. Then the connecting morphisms determine morphisms $\psi^l_{\alpha,\beta} : L_\alpha \to L_\beta$ and $\psi^r_{\alpha,\beta} : R_\alpha \to R_\beta$ such that

$$(l_\alpha, r_\alpha)\psi_{\alpha,\beta} = (l_\alpha\psi^l_{\alpha,\beta}, r_\alpha\psi^r_{\alpha,\beta}) \tag{5.2}$$

for every $(l_\alpha, r_\alpha) \in B_\alpha$. Moreover, $L = \bigcup\{L_\alpha : \alpha \in Y\}$ becomes a strong semilattice of left zero semigroups $[Y; L_\alpha; \psi^l_{\alpha,\beta}]$ under $\circ$, where for $l_\alpha \in L_\alpha$ and $l_\beta \in L_\beta$,

$$l_\alpha \circ l_\beta = (l_\alpha\psi^l_{\alpha,\alpha\beta})(l_\beta\psi^l_{\beta,\alpha\beta}) = l_\alpha\psi^l_{\alpha,\alpha\beta}$$

since $L_{\alpha\beta}$ is left zero (dually for R). Hence by (5.2) we have $B = L \bowtie R$, and we arrive at the subsequent proposition.

Proposition 5.4.5. *Every normal band B is isomorphic to a spined product of a left normal and a right normal band.*

Consequently, by Proposition 5.3.1, a pair of normal bands $L \bowtie R$ and $L' \bowtie R'$ are isomorphic if and only if there exists an isomorphism from L to L' and R to R' with the same induced isomorphism between the structure semilattices.

A normal band is called an *image-trivial normal band* if the images of the non-identity connecting morphisms all have cardinality 1. A normal band is called a *surjective normal band* if each connecting morphism is surjective. Note that a normal band is both image-trivial and surjective if and only if it is a semilattice. Moreover, a normal band $L \bowtie R$ is an image-trivial/surjective normal band if and only if both L and R are likewise image-trivial/surjective.

Lemma 5.4.6. *Let $B = [Y; L_\alpha; \psi^l_{\alpha,\beta}] \bowtie [Y; R_\alpha; \psi^r_{\alpha,\beta}] = L \bowtie R$ be a homogeneous normal band. Then R is either an image-trivial or surjective right normal band (dually for L).*

Proof. If R is a semilattice then B is isomorphic to L, and so the result is immediate. Assume instead that $|R_\alpha| > 1$ for some $\alpha \in Y$. Then $|R_\alpha| > 1$ for all $\alpha \in Y$ by Corollary 5.2.3 (iii). Suppose there exists $\alpha > \beta$ in Y such that $I^R_{\alpha,\beta} \neq R_\beta$. Let $r_\alpha \psi^r_{\alpha,\beta} = r_\beta$, $s_\alpha \psi^r_{\alpha,\beta} = s_\beta$ (with $r_\alpha \neq s_\alpha$) and $t_\beta \notin I^R_{\alpha,\beta}$. Fix $l_\alpha \in L_\alpha$ and let $l_\alpha \psi^l_{\alpha,\beta} = l_\beta$. Note that for any $x_\beta \in R_\beta$ we have

$$(l_\alpha, r_\alpha)(l_\beta, x_\beta) = (l_\beta, r_\beta x_\beta) = (l_\beta, x_\beta) \quad \text{and} \quad (l_\beta, x_\beta)(l_\alpha, r_\alpha) = (l_\beta, r_\beta).$$

Hence if $x_\beta \neq r_\beta$ then $\langle (l_\alpha, r_\alpha), (l_\beta, x_\beta)\rangle = \{(l_\alpha, r_\alpha), (l_\beta, x_\beta), (l_\beta, r_\beta)\}$ is a 3 element subband. In particular, if $s_\beta \neq r_\beta$ then the map

$$\phi : \langle (l_\alpha, r_\alpha), (l_\beta, s_\beta)\rangle \to \langle (l_\alpha, r_\alpha), (l_\beta, t_\beta)\rangle$$

fixing (l_α, r_α) and such that $(l_\beta, s_\beta)\phi = (l_\beta, t_\beta)$ is an isomorphism. Extend ϕ to an automorphism $\theta^l \bowtie \theta^r$ of B. Then as $\theta^r = [\theta^r_\alpha, \pi]_{\alpha \in Y}$ is an automorphism of R we have, by the commutativity of $[\alpha, \beta; \alpha, \beta]$ in R,

$$(s_\alpha \theta^r_\alpha)\psi^r_{\alpha,\beta} = (s_\alpha \psi^r_{\alpha,\beta})\theta^r_\beta = s_\beta \theta^r_\beta = t_\beta,$$

contradicting $t_\beta \notin I^R_{\alpha,\beta}$. Thus $s_\beta = r_\beta$, so that $I_{\alpha,\beta}$ has cardinality 1, and so R is an image-trivial normal band by Lemma 5.4.4. The dual gives the result for left normal bands. $\square$

Hence if $B = L \bowtie R$ is a homogeneous normal band then B is either an image-trivial normal band (if L and R are image-trivial), or the images of the connecting morphisms are a single $\mathcal{L}/\mathcal{R}$-class (if L/R is a surjective normal band and R/L is an image-trivial normal band) or $\mathcal{D}$-class (if L and R are surjective normal bands).

We split our classification of homogeneous normal bands into three parts. In Section 5.4.1 we classify homogeneous image-trivial normal bands, and in Section 5.4.2 homogeneous surjective normal bands. Using the results attained in these sections, the final case (and its dual) is easily achieved at the end of Section 5.4.2.

5.4.1 Image-trivial normal bands

In this section we are concerned with the classification of image-trivial homogeneous normal bands. Following the notation of Section 3.7, we shall denote an image-trivial normal band $[Y; B_\alpha; \psi_{\alpha,\beta}]$ such that $I_{\alpha,\beta} = \{\epsilon_{\alpha,\beta}\}$ for each $\alpha > \beta$, as $[Y; B_\alpha; \epsilon_{\alpha,\beta}; \psi_{\alpha,\beta}]$. Note that if $\alpha, \gamma > \beta$ in Y are such that $\alpha\gamma > \beta$ then

$$B_\alpha\psi_{\alpha,\beta} = (B_\alpha\psi_{\alpha,\alpha\gamma})\psi_{\alpha\gamma,\beta} = \{\epsilon_{\alpha\gamma,\beta}\} = (B_\gamma\psi_{\gamma,\alpha\gamma})\psi_{\alpha\gamma,\beta} = B_\gamma\psi_{\gamma,\beta}$$

and so

$$\epsilon_{\alpha,\beta} = \epsilon_{\alpha\gamma,\beta} = \epsilon_{\gamma,\beta}. \tag{5.3}$$

Notice that (5.3) automatically holds if $\alpha > \gamma > \beta$.

Note that if $Y = \mathbb{Q}$ (under the natural ordering) then for any $\beta \in \mathbb{Q}$ and $\alpha, \gamma > \beta$ we have $\epsilon_{\alpha,\beta} = \epsilon_{\gamma,\beta}$ by (5.3). Hence any $e_\beta \in B_\beta \setminus \{\epsilon_{\alpha,\beta}\}$ is a maximal element in the poset $(B, \leq)$. Consequently, an image-trivial homogeneous normal band with a linear structure semilattice is isomorphic to $\mathbb{Q}$ by Corollary 5.2.3.

While the following lemma is stronger than what is required in this section, it will be vital for later results, and the generalization adds little extra work.

Lemma 5.4.7. *Let $B = [Y; B_\alpha; \psi_{\alpha,\beta}] = L \bowtie R$ be a homogeneous normal band such that either L or R is a non-semilattice image-trivial normal band. Then Y is a homogeneous semilinear order.*

Proof. Assume w.l.o.g. that $L = [Y; L_\alpha; \epsilon^l_{\alpha,\beta}; \psi^l_{\alpha,\beta}]$ is a non-semilattice image-trivial normal band, so that $|L_\alpha| > 1$ for all $\alpha \in Y$ by Corollary 5.2.3 (iii). Note that $R = [Y; R_\alpha; \psi^r_{\alpha,\beta}]$ is image-trivial or surjective by Lemma 5.4.6. Seeking a contradiction, suppose that Y contains a diamond $D = \{\delta, \alpha, \gamma, \beta\}$, where $\delta > \{\alpha, \gamma\} > \beta$. Fix $e_\delta = (l_\delta, r_\delta) \in B_\delta$ and let

$$\begin{aligned} e_\alpha &= e_\delta\psi_{\delta,\alpha} = (\epsilon^l_{\delta,\alpha}, r_\delta\psi^r_{\delta,\alpha}), \\ e_\gamma &= e_\delta\psi_{\delta,\gamma} = (\epsilon^l_{\delta,\gamma}, r_\delta\psi^r_{\delta,\gamma}), \\ e_\beta &= e_\delta\psi_{\delta,\beta} = (\epsilon^l_{\delta,\beta}, r_\delta\psi^r_{\delta,\beta}), \end{aligned}$$

noting that $\epsilon^l_{\alpha,\beta} = \epsilon^l_{\delta,\beta} = \epsilon^l_{\gamma,\beta}$ by (5.3). By construction $\{e_\delta, e_\alpha, e_\gamma, e_\beta\}$ is isomorphic to D. If there exists $l_\beta \in L_\beta \setminus \{\epsilon^l_{\delta,\beta}\}$, then by Lemma 5.4.3 there exists $\tau > \beta$ such that $l_\beta = \epsilon^l_{\tau,\beta}$. Note that $\alpha\tau = \beta$, since if $\alpha\tau > \beta$ then $l_\beta = \epsilon^l_{\tau,\beta} = \epsilon^l_{\alpha,\beta}$ by (5.3). Let $\kappa < \beta$ and $e_\kappa = e_\beta\psi_{\beta,\kappa} = (\epsilon^l_{\beta,\kappa}, r_\delta\psi^r_{\delta,\kappa})$. Extend the unique isomorphism between the 3-chains $e_\delta > e_\alpha > e_\beta$ and $e_\alpha > e_\beta > e_\kappa$ to an automorphism $\theta = [\theta_\alpha, \pi]_{\alpha\in Y}$ of B. Let $\rho \in Y$ be such that $e_\rho = e_\gamma\theta > e_\beta\theta = e_\kappa$. Then $\alpha > \rho > \kappa$ (since $\delta > \gamma > \beta$),

$\rho\beta = \kappa$ (since $\gamma\alpha = \beta$) and

$$\rho\tau = (\rho\alpha)\tau = \rho(\alpha\tau) = \rho\beta = \kappa.$$

We claim that there exists $e_\tau \in B_\tau$ such that $e_\tau > e_\kappa$. For if R is also image-trivial and $B = [Y; B_\alpha; \epsilon_{\alpha,\beta}; \psi_{\alpha,\beta}]$, then the claim holds for any e_τ by (5.3), as $\tau > \beta > \kappa$, so that $\epsilon_{\tau,\kappa} = \epsilon_{\beta,\kappa} = e_\kappa$. On the other hand, if R is surjective, then there exists $r_\tau \in R_\tau$ such that $r_\tau\psi^r_{\tau,\kappa} = r_\delta\psi^r_{\delta,\kappa}$. Thus, for any l_τ, we have

$$(l_\tau, r_\tau)\psi_{\tau,\kappa} = (\epsilon^l_{\tau,\kappa}, r_\tau\psi^r_{\tau,\kappa}) = (\epsilon^l_{\beta,\kappa}, r_\delta\psi^r_{\delta,\kappa}) = e_\kappa,$$

and so the claim is proven. Fix some $e_\tau > e_\kappa$. By extending any isomorphism between the 3 element non-chain semilattices $\langle e_\rho, e_\tau, e_\kappa\rangle$ and $\langle e_\alpha, e_\gamma, e_\beta\rangle$, it follows that there exists $\sigma > \rho, \tau$ (as $\delta > \alpha, \gamma$).

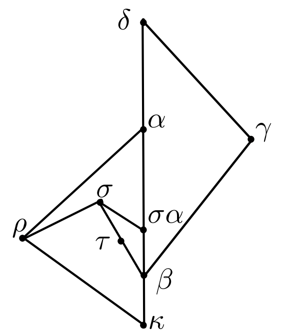

Figure 5.2: A subsemilattice of Y.

Since $\sigma > \tau > \beta$ and $\alpha > \beta$ we have $\sigma\alpha \geq \beta$. If $\sigma\alpha = \beta$ then $\beta \geq \rho$ (as $\sigma, \alpha > \rho$), and so as $\rho\beta = \kappa$ we have $\rho = \kappa$, a contradiction. Hence $\sigma\alpha > \beta$ and we thus arrive at the subsemilattice of Y in Figure 5.2. Moreover,

$$\epsilon^l_{\alpha,\beta} = \epsilon^l_{\sigma\alpha,\beta} = \epsilon^l_{\sigma,\beta} = \epsilon^l_{\tau,\beta} = l_\beta$$

by (5.3), contradicting $l_\beta \neq \epsilon^l_{\delta,\beta} = \epsilon^l_{\alpha,\beta}$. Hence no such l_β exists, and so L_β is trivial, a contradiction of L being a non-semilattice. Hence, by Proposition 5.2.4, Y is a homogeneous linear or semilinear order. Since B is a non-semilattice, the result follows by the note above the lemma. □

In particular, an image-trivial homogeneous normal band has a semilinear structure semilattice. It is therefore crucial to better understand the structure of homogeneous semilinear orders.

Let Y be a dense semilinear order. We call a set $Z \subseteq Y$ *connected* if for any $x, y \in Z$ there exist $z_1, \ldots, z_n \in Z$ ($n \in \mathbb{N}$) with $z_1 = x$, $z_n = y$, and $z_i \leq z_{i+1}$ or $z_{i+1} \leq z_i$ for all $1 \leq i \leq n-1$. Given $\alpha \in Y$, we call the maximal connected subsets of $\{\gamma \in Y : \gamma > \alpha\}$ the *cones of* α, and let $C(\alpha)$ denote the set of all cones of α.

Remark 5.4.8 ([24, Remark 5.11])**.** Let $\alpha \in Y$, $A \in C(\alpha)$ and $\gamma \in A$. Then for any $\delta \in Y$ we have $\delta \in A$ if and only if $\alpha < \delta\gamma$.

Consequently, the cones of $\alpha \in Y$ partition the set $\{\gamma \in Y : \gamma > \alpha\}$. If there exists $r \in \mathbb{N}^*$ such that $|C(\alpha)| = r$ for all $\alpha \in Y$, then r is known as the *ramification order* of Y. Each homogeneous semilinear order has a ramification order [24].

Let $B = [Y; B_\alpha; \epsilon_{\alpha,\beta}; \psi_{\alpha,\beta}]$ be an image-trivial normal band, where Y is a semilinear order. Since B is image-trivial, we can define a cone of $e_\alpha \in B_\alpha$ as a maximal connected subset of $\{\gamma \in Y : \gamma > \alpha, B_\gamma\psi_{\gamma,\alpha} = e_\alpha\}$. Let $C(e_\alpha)$ denote the set of all cones of e_α. Let $\gamma, \gamma' > \alpha$ and suppose γ is connected to γ'. From Remark 5.4.8 we have that $\gamma\gamma' > \alpha$ and so by (5.3) $B_\gamma\psi_{\gamma,\alpha} = B_{\gamma'}\psi_{\gamma',\alpha}$. Consequently, the set $\{\gamma \in Y : \gamma > \alpha, B_\gamma\psi_{\gamma,\alpha} = e_\alpha\}$ is a union of cones of α, and $C(\alpha) = \bigcup_{e_\alpha \in B_\alpha} C(e_\alpha)$.

If there exists $k \in \mathbb{N}^*$ such that $|C(e_\alpha)| = k$ for all $e_\alpha \in B$, then k is called the *ramification order* of B. If B is homogeneous then (as Y is homogeneous and B is 1-homogeneous) the ramification orders exist for Y and B, say, r and k respectively, and they are related according to $r = k \cdot |B_\alpha|$. Moreover, by Lemma 5.4.3, $B_\beta = \bigcup_{\alpha>\beta} \epsilon_{\alpha,\beta}$ for each $\beta \in Y$.

As shown in [24, Theorem 6.21], there exists for each $r \in \mathbb{N}^*$ a unique (up to isomorphism) countable homogeneous semilinear order of ramification order r, denoted T_r. Moreover, a semilinear order is isomorphic to T_r if and only if it is dense and has ramification order r.

We can reconstruct T_r from any $\alpha \in T_r$ inductively by following the proof of Theorem 6.16 in [24], as we now explain, but omitting the proof. Consider an enumeration of T_r given by $T_r = \{a_i : i \in \mathbb{N}\}$, where $a_1 = \alpha$. Let $Y_0 = \emptyset$ and $Y_1 = Z_0$ be a maximal chain in T_r which contains a_1. Suppose for some $i \in \mathbb{N}$, the semilattices Y_j and posets Z_{j-1} $(j \leq i)$ have already been defined such that the following conditions hold for each $1 \leq j \leq i$:

(i) $Y_j = Y_{j-1} \sqcup Z_{j-1}$ and $a_j \in Y_j$ (where $\sqcup$ denotes the disjoint union);

(ii) if $z \in Z_{j-1}$, then there exists a unique maximal chain C in T_r with $z \in C \subseteq Y_j$ and $\{c \in C : z \leq c\} \subseteq Z_{j-1}$;

(iii) if $z \in Z_{j-2}\,(j \geq 2)$ and D is any cone of z disjoint to Y_{j-1}, then $D \cap Z_{j-1} \neq \emptyset$.

It follows from (ii) that whenever $1 \leq j \leq i$, $z \in Y_j$, $y \in T_r$ and $y < z$, then $y \in Y_j$. Moreover, the conditions above trivially hold for the case $i = 1$.

If $a_{i+1} \notin Y_i$ then it is shown that there exists $z \in Z_{i-1}$ such that a_{i+1} belongs to some (unique) cone A_z of z which is disjoint to Y_i. For each $\beta \in Z_{i-1}$ take a maximal subchain of each cone $A \in C(\beta)$ such that $Y_i \cap A = \emptyset$, where if $\beta = z$ then we take a maximal subchain of A_z which contains a_{i+1}. By condition (ii) the set

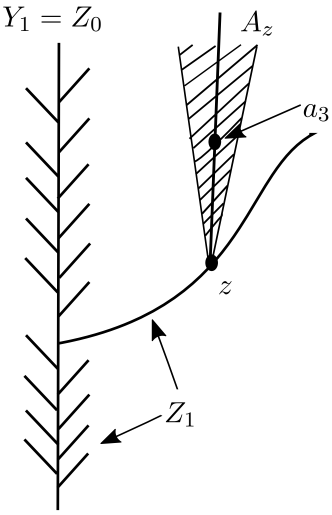

Figure 5.3: The case $i = 2$ [24, Page 68].

$\{y \in Y_i : \beta < y\}$ is a chain, and thus contained in a single cone of β, and so only one cone will intersect Y_i non-trivially.

Let C_β be the disjoint union of the $r-1$ (or r if r is infinite) maximal subchains constructed. We construct Y_{i+1} by adjoining at each $\beta \in Z_{i-1}$ the set C_β, that is, let

$$Z_i = \bigsqcup_{\beta \in Z_{i-1}} C_\beta \quad \text{and} \quad Y_{i+1} = Y_i \sqcup Z_i.$$

Then conditions (i), (ii) and (iii) are shown to hold, and $\bigcup_{i \in \mathbb{N}} Y_i = T_r$ as desired.

We can use this construction to describe automorphisms of T_r. Suppose we also reconstruct T_r from $\alpha' \in T_r$ via sets Y_i', Z_i', C_β' (so that $\bigcup_{i \in \mathbb{N}} Y_i' = T_r$). Let $\pi_1 : Y_1 \to Y_1'$ be an isomorphism such that $\alpha\pi_1 = \alpha'$ (such an isomorphism exists as maximal chains are isomorphic to $\mathbb{Q}$, and $\mathbb{Q}$ is homogeneous). Suppose the isomorphism $\pi_i : Y_i \to Y_i'$ has already been defined for some $i \in \mathbb{N}$. Extend π_i to $\pi_{i+1} : Y_{i+1} \to Y_{i+1}'$ as follows. For each $\beta \in Z_{i-1}$ the posets C_β and $C_{\beta\pi_i}$ are both disjoint unions of the same number of copies of $\mathbb{Q}$, and are thus isomorphic (as posets). Let $\phi_\beta : C_\beta \to C_{\beta\pi_i}$ be an isomorphism, and let

$$\pi_{i+1} = \pi_i \sqcup \bigsqcup_{\beta \in Z_{i-1}} \phi_\beta : Y_{i+1} \to Y_{i+1}'.$$

Then π_{i+1} is an isomorphism, and so $\pi = \bigcup_{i \in \mathbb{N}} \pi_i$ is an automorphism of T_r.

Before classifying image-trivial homogeneous normal bands, it is worth giving a simplified isomorphism theorem, which follows easily from Proposition 5.4.1.

Corollary 5.4.9. *Let $B = [Y; B_\alpha; \epsilon_{\alpha,\beta}; \psi_{\alpha,\beta}]$ and $B' = [Y'; B'_{\alpha'}; \epsilon'_{\alpha',\beta'}; \psi'_{\alpha',\beta'}]$ be a pair of image-trivial normal bands. Let $\pi : Y \to Y'$ be an isomorphism, and for each*

$\alpha \in Y$ let $\theta_\alpha : B_\alpha \to B'_{\alpha\pi}$ be an isomorphism. Then $\bigcup_{\alpha\in Y} \theta_\alpha$ is an isomorphism from B into B' if and only if $\epsilon_{\alpha,\beta}\theta_\beta = \epsilon'_{\alpha\pi,\beta\pi}$ for each $\alpha > \beta$ in Y.

Given a subset A of a band B, we define the *support of* A as

$$\mathrm{supp}(A) := \{\gamma \in Y : A \cap B_\gamma \neq \emptyset\}.$$

If A is a subband of B then clearly $\mathrm{supp}(A)$ is simply the structure semilattice of A.

A subsemigroup A of an image-trivial normal band $[Y; B_\alpha; \epsilon_{\alpha,\beta}; \psi_{\alpha,\beta}]$ is called a *maximal chain* if A is a semilattice and $\mathrm{supp}(A)$ is a maximal chain in Y. Note that if Y is a homogeneous semilinear order and A is a maximal chain in B then

$$A = \bigcup_{\alpha>\beta \text{ in } \mathrm{supp}(A)} \epsilon_{\alpha,\beta} \cong \mathbb{Q},$$

as A is a semilattice, so that $|B_\beta \cap A| \leq 1$ for all $\beta \in Y$. We use the construction of T_r above to prove the following:

Proposition 5.4.10. *Let $B = [T_r; B_\alpha; \epsilon_{\alpha,\beta}; \psi_{\alpha,\beta}]$ and $B' = [T_r; B'_{\alpha'}; \epsilon'_{\alpha',\beta'}; \psi'_{\alpha',\beta'}]$ be a pair of image-trivial normal bands with ramification order k such that there exist $n, m \in \mathbb{N}^*$ with $B_\alpha \cong B'_{\alpha'} \cong B_{n,m}$ for all $\alpha, \alpha' \in T_r$. Let $e \in B$ and $f \in B'$, and consider a pair of sub-rectangular bands $M \subseteq B$ and $N \subseteq B'$ with $M > e$ and $N > f$. Then for any isomorphism $\Phi : M \cup \{e\} \to N \cup \{f\}$, there exists an isomorphism $\theta : B \to B'$ extending Φ. Consequently, B is 1-homogeneous.*

Proof. Assume $r > 1$, else B and B' are isomorphic to $\mathbb{Q}$. Let $e_\sigma, e_{\sigma'}$ be elements of B, M a rectangular subband of B_α, and N a rectangular subband of B_δ, with $M > e_\sigma$ and $N > e_{\sigma'}$ for some $\sigma, \sigma', \alpha, \delta \in T_r$. Let Φ be an isomorphism given by

$$\Phi : M \cup \{e_\sigma\} \to N \cup \{e_{\sigma'}\},$$

so that $M\Phi = N$ and $e_\sigma\Phi = e_{\sigma'}$. Since rectangular bands are homogeneous, we may extend $\Phi|_M$ to an isomorphism $\Phi' : B_\alpha \to B'_\delta$ by Lemma 4.3.7. Fix some $e_\alpha \in M$ and let $e_\alpha\Phi = e_\delta$. Let $\{a_i : i \in \mathbb{N}\}$ and $\{b_i : i \in \mathbb{N}\}$ be a pair of enumerations of T_r such that $\alpha = a_1$ and $\delta = b_1$. Let A be a maximal chain in B such that $e_\sigma, e_\alpha \in A$, and let $Y_1 = Z_0 = \mathrm{supp}(A)$ (so $Y_1 \cong A$). Similarly obtain $e_{\sigma'}, e_\delta \in \hat{A}$ and $\hat{Y}_1 = \hat{Z}_0 = \mathrm{supp}(\hat{A})$ (so $\hat{Y}_1 \cong \hat{A}$). Take an isomorphism $\pi_1 : Y_1 \to \hat{Y}_1$ such that $\sigma\pi_1 = \sigma'$ and $\alpha\pi_1 = \delta$ (again this is possible as Y_1 and $\hat{Y}_1$ are isomorphic to $\mathbb{Q}$). For each $\beta \in Y_1 \setminus \{\alpha\}$, take any isomorphism $\theta_\beta : B_\beta \to B'_{\beta\pi_1}$ such that $(B_\beta \cap A)\theta_\beta = B'_{\beta\pi_1} \cap \hat{A}$ (such an isomorphism exists by Proposition 2.10.2, or simply by the homogeneity of rectangular bands), and let $\theta_\alpha = \Phi'$. Letting $D_1 = [Y_1; B_\alpha; \epsilon_{\alpha,\beta}; \psi_{\alpha,\beta}]$ and $\hat{D}_1 = [\hat{Y}_1; B'_{\alpha'}; \epsilon'_{\alpha',\beta'}; \psi'_{\alpha',\beta'}]$, the map

$$\theta_1 = [\theta_\beta, \pi_1]_{\beta\in Y_1} : D_1 \to \hat{D}_1$$

is an isomorphism by Corollary 5.4.9, since $B_\beta \cap A = \{\epsilon_{\gamma,\beta}\}$ for all $\gamma > \beta$ in Y_1, and $B_{\beta\pi_1} \cap \hat{A} = \{\epsilon_{\gamma\pi_1,\beta\pi_1}\}$ for all $\gamma\pi_1 > \beta\pi_1$ in $\hat{Y_1}$.

Suppose for some $i \in \mathbb{N}$ the semilattices $Y_j, \hat{Y}_j$, posets $Z_{j-1}, \hat{Z}_{j-1}$, bands

$$D_j = [Y_j; B_\alpha; \epsilon_{\alpha,\beta}; \psi_{\alpha,\beta}],\ \hat{D}_j = [\hat{Y}_j; B'_{\alpha'}; \epsilon'_{\alpha',\beta'}; \psi'_{\alpha',\beta'}]$$

and isomorphisms $\pi_j : Y_j \to \hat{Y}_j$, $\theta_j = [\theta_\alpha, \pi_j]_{\alpha \in Y_j} : D_j \to \hat{D}_j$ have already been defined for each $j \leq i$, and are such that Y_j, Z_{j-1} and $\hat{Y}_j, \hat{Z}_{j-1}$ satisfy conditions (i), (ii) and (iii). As in the semilattice construction, if $a_{i+1} \notin Y_i$ then we can fix $z \in Z_{i-1}$ such that a_{i+1} belongs to some cone of z which is disjoint to Y_i, and let $B_{a_{i+1}}\psi_{a_{i+1},z} = e_z$.

Consider the subset $X_{i-1} = \bigcup_{\gamma \in Z_{i-1}} B_\gamma$ of B. For each $e_\beta \in X_i$ (so $\beta \in Z_{i-1}$), take a maximal subchain of each cone $C \in C(e_\beta)$ such that $Y_i \cap C = \emptyset$. If $e_\beta = \epsilon_{y,\beta}$ for (any) y in the chain $\{y \in Y_i : \beta < y\}$, then by condition (ii) precisely one cone will intersect Y_i non-trivially. Otherwise, all cones of e_β intersects Y_i trivially. Moreover, if $a_{i+1} \notin Y_i$ and $\beta = z$, we further require the maximal subchain of a cone of $C(e_z)$ to contain a_{i+1}.

Let C_{e_β} be the disjoint union of the k (or $k-1$ if $e_\beta = \epsilon_{y,\beta}$ for some $y \in Y_i$, and k is finite) maximal subchains and let

$$Z_i = \bigsqcup_{e_\beta \in X_{i-1}} C_{e_\beta}.$$

Let $Y_{i+1} = Y_i \sqcup Z_i$, and note that $\gamma \leq \gamma'$ for $\gamma, \gamma' \in Y_{i+1}$ if and only if

either $\gamma, \gamma' \in Y_i$ and $\gamma \leq \gamma'$ in Y_i;
or $\gamma, \gamma' \in C_{e_\beta}$ for some $e_\beta \in X_{i-1}$ and $\gamma \leq \gamma'$ in C_{e_β};
or $\gamma \in Y_i, \gamma' \in C_{e_\beta}$ for some $e_\beta \in X_{i-1}$ and $\beta \geq \gamma$ in Y_i.

Similarly obtain $\hat{C}_{e_{\beta'}}$, $\hat{Z}_i$ and $\hat{Y}_{i+1}$, noting that as B has ramification order k the set $\hat{C}_{e_{\beta'}}$ will likewise be formed from k (or $k-1$ if $e_{\beta'} = \epsilon'_{y',\beta'}$ for some $y' \in \hat{Y}_i$, and k is finite) maximum subchains. Let $D_{i+1} = [Y_{i+1}; B_\alpha; \epsilon_{\alpha,\beta}; \psi_{\alpha,\beta}]$ and $\hat{D}_{i+1} = [\hat{Y}_{i+1}; B'_{\alpha'}; \epsilon'_{\alpha',\beta'}; \psi'_{\alpha',\beta'}]$.

Recall that $C(\beta) = \bigcup_{e_\beta \in B_\beta} C(e_\beta)$ for all $\beta \in T_r$. Hence as $\bigcup_{e_\beta \in B_\beta} C_{e_\beta}$ is a set of maximal subchains of the $r-1$ (or r if r is infinite) cones of $C(\beta)$ which intersect Y_i trivially, it follows that conditions (i), (ii) and (iii) are satisfied and $\bigcup_{i \in \mathbb{N}} Y_i = T_r$ (similarly for $\hat{Y}_i$). Consequently, $B = \bigcup_{i \in \mathbb{N}} D_i$ and $B' = \bigcup_{i \in \mathbb{N}} \hat{D}_i$.

For each $e_\beta \in X_{i-1}$, let $\sigma_{e_\beta} : C_{e_\beta} \to \hat{C}_{e_\beta\theta_i}$ be an isomorphism (as posets), and let

$$\pi_{i+1} = \pi_i \sqcup \bigsqcup_{e_\beta \in X_{i-1}} \sigma_{e_\beta} : Y_{i+1} \to \hat{Y}_{i+1}.$$

By the order on Y_{i+1} defined above, the map π_{i+1} is an isomorphism, and so the map $\pi = \bigcup_{i \in \mathbb{N}} \pi_i$ is an automorphism of T_r. For each $\gamma \in C_{e_\beta}$ ($e_\beta \in X_{i-1}$), let $\theta_\gamma : B_\gamma \to B_{\gamma\pi_{i+1}}$ be an isomorphism such that $\epsilon_{\gamma',\gamma}\theta_\gamma = \epsilon_{\gamma'\pi_{i+1},\gamma\pi_{i+1}}$ for (any) $\gamma' \in C_{e_\beta}$ with $\gamma' > \gamma$. We claim that the map

$$\theta_{i+1} = \theta_i \sqcup \bigsqcup_{\substack{\gamma \in C_{e_\beta} \\ e_\beta \in X_{i-1}}} \theta_\gamma = [\theta_\alpha, \pi_{i+1}]_{\alpha \in Y_{i+1}} : D_{i+1} \to \hat{D}_{i+1}$$

is an isomorphism. Suppose $\gamma, \gamma' \in Y_{i+1}$ are such that $\gamma \leq \gamma'$. If $\gamma, \gamma' \in Y_i$, then as θ_i preserves the images of the connecting morphisms from Y_i, we have

$$\epsilon_{\gamma',\gamma}\theta_{i+1} = \epsilon_{\gamma',\gamma}\theta_i = \epsilon_{\gamma'\pi_i,\gamma\pi_i} = \epsilon_{\gamma'\pi_{i+1},\gamma\pi_{i+1}}.$$

Similarly, if $\gamma, \gamma' \in C_{e_\beta}$ for some $e_\beta \in X_{i-1}$ then by construction we have that $\epsilon_{\gamma',\gamma}\theta_\gamma = \epsilon_{\gamma'\pi_{i+1},\gamma\pi_{i+1}}$. Finally, if $\gamma \in Y_i$, $\gamma' \in C_{e_\beta}$ for some $e_\beta \in X_{i-1}$ and $\beta \geq \gamma$, then

$$\epsilon_{\gamma',\gamma}\theta_{i+1} = \epsilon_{\beta,\gamma}\theta_{i+1} = \epsilon_{\beta,\gamma}\theta_i = \epsilon_{\beta\pi_i,\gamma\pi_i} = \epsilon_{\beta\pi_{i+1},\gamma\pi_{i+1}} = \epsilon_{\gamma'\pi_{i+1},\gamma\pi_{i+1}}$$

by (5.3) since $\gamma' > \beta \geq \gamma$ and $\gamma'\pi_{i+1} > \beta\pi_{i+1} \geq \gamma\pi_{i+1}$. The claim then follows from Corollary 5.4.9. Hence $\theta = \bigcup_{i \in \mathbb{N}} \theta_i$ is an isomorphism from B to B' which extends Φ. Taking $B = B'$ shows that B is 1-homogeneous. □

As a result, for each collection $r, k, n, m \in \mathbb{N}^*$ such that $r = knm$, there exists a unique, up to isomorphism, image-trivial normal band $[T_r; B_\alpha; \psi_{\alpha,\beta}; \epsilon_{\alpha,\beta}]$ with ramification order k and $B_\alpha \cong B_{n,m}$ for all $\alpha \in T_r$. We denote such a band $T_{n,m,k}$, where $r = nmk$.

Proposition 5.4.11. *An image-trivial homogeneous normal band is isomorphic to $T_{n,m,k}$ for some $n, m, k \in \mathbb{N}^*$. Conversely, every band $T_{n,m,k}$ is homogeneous.*

Proof. By Lemma 5.4.7 an image-trivial homogeneous normal band has a semilinear structure semilattice, and has a ramification order by 1-homogeneity. By the preceding results, it therefore suffices to prove that the band $T_{n,m,k} = [T_r; B_\alpha; \epsilon_{\alpha,\beta}; \psi_{\alpha,\beta}]$ is homogeneous. Since $T_{n,m,k}$ is 1-homogeneous by the proposition above, we proceed by induction, by supposing all isomorphisms between finite subbands of size $j-1$ extend to an automorphism of B. Let $M = [Z_1; M_\alpha; \psi^M_{\alpha,\beta}], N = [Z_2; N_\alpha; \psi^N_{\alpha,\beta}]$ be a pair of finite subbands of B of size j, and $\theta = [\theta_\alpha, \pi]_{\alpha \in Z_1}$ an isomorphism from M to N. By Proposition 5.4.10 we assume that Z_1 and Z_2 are non-trivial, so that N, M are not rectangular bands. Let δ be maximal in Z_1, and $\delta\pi = \delta'$. Then by the inductive hypothesis the isomorphism $\theta|_{M \setminus M_\delta} : M \setminus M_\delta \to N \setminus N_{\delta'}$ extends to an automorphism $\theta^* = [\theta^*_\alpha, \pi^*]_{\alpha \in T_r}$ of B.

Since Z_1 is a finite semilinear order, there exists a unique $\beta \in Z_1$ which is the lower cover of δ. As θ is an isomorphism, it follows from Corollary 5.4.9 that

$$\epsilon_{\delta,\beta}\theta^*_\beta = \epsilon_{\delta,\beta}\theta = \epsilon_{\delta',\beta\pi} = \epsilon_{\delta',\beta\pi^*}.$$

For each $e_\tau \in B$, let $[e_\tau]$ be the subsemilattice of T_r given by

$$[e_\tau] = \{\gamma \in T_r : B_\gamma\psi_{\gamma,\tau} = e_\tau\} = \text{supp}\{e \in B : e > e_\tau\}.$$

Note that $[e_\tau]$ is the union of the cones of e_τ, and so $T_r \setminus [e_\tau]$ forms a subsemilattice of T_r. Then $\hat{\pi}^* = \pi^*|_{T_r\setminus[\epsilon_{\delta,\beta}]} : T_r \setminus [\epsilon_{\delta,\beta}] \to T_r \setminus [\epsilon_{\delta',\beta\pi^*}]$ is an isomorphism, and we now aim to extend the isomorphism

$$[\theta^*_\gamma, \hat{\pi}^*]_{\gamma\in T_r\setminus[\epsilon_{\delta,\beta}]} : \bigcup_{\gamma\in T_r\setminus[\epsilon_{\delta,\beta}]} B_\gamma \to \bigcup_{\gamma\in T_r\setminus[\epsilon_{\delta',\beta\pi^*}]} B_\gamma$$

to an automorphism of B which extends θ.

Since θ maps $M_\delta \cup \{\epsilon_{\delta,\beta}\}$ to $N_{\delta\pi} \cup \{\epsilon_{\delta',\beta\pi}\}$, we can extend the isomorphism $\theta|_{M_\delta\cup\{\epsilon_{\delta,\beta}\}}$ to an automorphism $\bar{\theta}^* = [\bar{\theta}^*_\gamma, \bar{\pi}^*]_{\gamma\in T_r}$ of B by Proposition 5.4.10. Then as $\beta\hat{\pi}^* = \beta\bar{\pi}^*$, the bijection $\tilde{\pi} = \hat{\pi}^*|_{T_r\setminus[\epsilon_{\delta,\beta}]} \sqcup \bar{\pi}^*|_{[\epsilon_{\delta,\beta}]}$ is an automorphism of T_r.

We claim that $\tilde{\theta} = [\tilde{\theta}_\gamma, \tilde{\pi}]_{\gamma\in T_r}$, where

$$\tilde{\theta}_\gamma = \begin{cases} \theta^*_\gamma & \text{if } \gamma \in T_r \setminus [\epsilon_{\delta,\beta}], \\ \bar{\theta}^*_\gamma & \text{if } \gamma \in [\epsilon_{\delta,\beta}], \end{cases}$$

is an automorphism of $T_{n,m,k}$. By Corollary 5.4.9 it is sufficient to prove that $\epsilon_{\gamma',\gamma}\tilde{\theta} = \epsilon_{\gamma'\tilde{\pi},\gamma\tilde{\pi}}$ for any $\gamma' \geq \gamma$ in T_r. Note if $\gamma \in [\epsilon_{\delta,\beta}]$ then $\gamma' \geq \gamma > \beta$ and so $\gamma' \in [\epsilon_{\delta,\beta}]$ by (5.3). Hence, as θ^* and $\bar{\theta}^*$ are automorphisms of B (and by the construction of $\tilde{\theta}$), we only need consider the case where $\gamma' \in [\epsilon_{\delta,\beta}]$ and $\gamma \in T_r\setminus[\epsilon_{\delta,\beta}]$. If $\gamma \neq \beta$ then $\gamma' > \beta > \gamma$, so $\epsilon_{\gamma',\gamma} = \epsilon_{\beta,\gamma}$ by (5.3), and so as $\beta, \gamma \in T_r \setminus [\epsilon_{\delta,\beta}]$,

$$\epsilon_{\gamma',\gamma}\tilde{\theta} = \epsilon_{\beta,\gamma}\tilde{\theta}_\gamma = \epsilon_{\beta,\gamma}\theta^*_\gamma = \epsilon_{\beta\hat{\pi}^*,\gamma\hat{\pi}^*} = \epsilon_{\beta\tilde{\pi},\gamma\tilde{\pi}} = \epsilon_{\gamma'\tilde{\pi},\gamma\tilde{\pi}}$$

with the final equality holding since $\gamma'\tilde{\pi} > \beta\tilde{\pi} > \gamma\tilde{\pi}$. Finally, if $\gamma = \beta$ then

$$\epsilon_{\gamma',\gamma}\tilde{\theta} = \epsilon_{\gamma',\beta}\tilde{\theta}_\beta = \epsilon_{\gamma',\beta}\theta^*_\beta = \epsilon_{\delta,\beta}\theta^*_\beta = \epsilon_{\delta',\beta\hat{\pi}^*} = \epsilon_{\gamma'\bar{\pi}^*,\beta\hat{\pi}^*} = \epsilon_{\gamma'\tilde{\pi},\beta\tilde{\pi}}$$

since $\gamma'\bar{\pi}^* \in [\epsilon_{\delta',\beta\pi^*}] = [\epsilon_{\delta',\beta\hat{\pi}^*}]$. Thus $\tilde{\theta}$ is indeed an automorphism of B, and extends θ by construction. □

It is worth noting that the spined product of a left and right image-trivial homogeneous normal band need not be homogeneous. For example, suppose, seeking a contradiction, that $B = T_{2,1,1} \bowtie T_{1,2,1}$ is homogeneous, and thus isomorphic to some $T_{n,m,k}$. Then $n = 2, m = 2$ and as B has structure semilattice T_2, so must $T_{2,2,k}$, and so $2.2.k = 4k = 2$, contradicting $k \in \mathbb{N}^*$. Our aim is now to prove that the converse holds, that is, if an image-trivial normal band $L \bowtie R$ is homogeneous, then so are L and R.

Corollary 5.4.12. *Let $B = L \bowtie R$ be a homogeneous normal band such that L is image-trivial. Then L is homogeneous (dually for R).*

Proof. Let $B = [Y; L_\alpha; \epsilon^l_{\alpha,\beta}; \psi^l_{\alpha,\beta}] \bowtie [Y; R_\alpha; \psi^r_{\alpha,\beta}]$ be homogeneous. Then by Corollary 5.2.3 (iii) there exists $n \in \mathbb{N}^*$ such that $L_\alpha \cong B_{n,1}$ for all $\alpha \in Y$, and by Lemma 5.4.7 we assume $Y = T_r$, where $r = nk$ for some $k \in \mathbb{N}^*$. Moreover, L has a ramification order, since if $l_\alpha, k_\beta \in L$, then by fixing any $r_\alpha \in R_\alpha$, $s_\beta \in R_\beta$, there exists an automorphism $\theta^l \bowtie \theta^r$ of B sending (l_α, r_α) to (k_β, s_β) as B is 1-homogeneous by Corollary 5.2.3 (i). In particular, $l_\alpha \theta^l = k_\beta$ and so $|C(l_\alpha)| = |C(k_\beta)| = k$. Hence $L \cong T_{n,1,k}$ by Proposition 5.4.10, and is thus homogeneous. $\square$

5.4.2 Surjective normal bands

We now study the homogeneity of surjective normal bands.

Lemma 5.4.13. *Let $B = [Y; B_\alpha; \psi_{\alpha,\beta}]$ be a homogeneous surjective normal band. Then for any finite subsemilattice Z of Y, there exists a subband $A = [Z; \{e_\alpha\}; \psi^A_{\alpha,\beta}]$ of B isomorphic to Z.*

Proof. Suppose first that Y is a linear or semilinear order. The result trivially holds for the case where $|Z| = 1$ by taking A to be a trivial subband. Proceed by induction by assuming that the result holds for all subsemilattices of size $n-1$, and let Z be a subsemilattice of Y of size $n \in \mathbb{N}$. Let δ be maximal in Z, so $Z' = Z \setminus \{\delta\}$ is a subsemilattice of Y of size $n-1$. By the inductive hypothesis, there exists a subband $A' = [Z'; \{e_\alpha\}; \psi^{A'}_{\alpha,\beta}]$ and an isomorphism $\phi : A' \to Z'$. Since Y is linearly or semilinearly ordered and Z is finite, there is a unique $\beta \in Z'$ such that β is a lower cover of δ. Let $\{e_\beta\} = A' \cap B_\beta$. Since $\psi_{\delta,\beta}$ is surjective, there exists $e_\delta \in B_\delta$ such that $e_\delta \psi_{\delta,\beta} = e_\beta$. Let ϕ' be the map from $A' \cup \{e_\delta\}$ to Z given by $A'\phi' = A'\phi$ and $e_\delta \phi' = \delta$. Then ϕ' is clearly an isomorphism, and the inductive step is complete.

Suppose instead that Y contains a diamond $\beta < \{\tau, \gamma\} < \sigma$. We claim that any pair $\alpha, \delta \in Y$ with $\alpha \perp \delta$ has an upper cover. Let $e_{\alpha\delta} \in B_{\alpha\delta}$ be fixed. Since the connecting morphisms are surjective there exist $e_\alpha \in B_\alpha$ and $e_\delta \in B_\delta$ such that

$$e_{\alpha\delta} = e_\alpha \psi_{\alpha,\alpha\delta} = e_\delta \psi_{\delta,\alpha\delta}$$

Similarly for τ and γ, we have $e_\beta = e_\tau \psi_{\tau,\beta} = e_\gamma \psi_{\gamma,\beta}$. The claim then follows by extending the isomorphism from $\{e_\beta, e_\tau, e_\gamma\}$ to $\{e_{\alpha\delta}, e_\alpha, e_\delta\}$ to an automorphism of B. By a simple inductive argument we have that every finite subsemilattice of Y

has an upperbound. Let Z be a finite subsemilattice of Y and $\alpha \in Y$ be such that $\alpha > Z$. Then for any $e_\alpha \in B_\alpha$,

$$\{e_\alpha \psi_{\alpha,\beta} : \beta \in Z\} \cong Z,$$

as required. □

Corollary 5.4.14. *Let $B = [Y; B_\alpha; \psi_{\alpha,\beta}]$ be a homogeneous normal band. Then Y is homogeneous.*

Proof. Suppose first that B is a surjective normal band. Let $\pi : Z \to Z'$ be an isomorphism between finite subsemilattices of Y. By Lemma 5.4.13, there exist subbands $A = [Z; \{e_\alpha\}; \psi^A_{\alpha,\beta}]$ and $A' = [Z'; \{e_{\alpha'}\}; \psi^{A'}_{\alpha',\beta'}]$ isomorphic to Z and Z', respectively. Hence $[\theta_\alpha, \pi]_{\alpha \in Z}$ is an isomorphism from A to A', where θ_α maps e_α to $e_{\alpha\pi}$, and the result follows by the homogeneity of B.

Now let $B = L \bowtie R$ be an arbitrary homogeneous normal band. By Lemma 5.4.6, L and R are either image-trivial or surjective normal bands. If both L and R are surjective, then clearly so too is B, and so Y is homogeneous by the first part. Otherwise, Y is homogeneous by Lemma 5.4.7. □

Corollary 5.4.15. *Let $B = L \bowtie R$ be a homogeneous surjective normal band. Then L and R are homogeneous.*

Proof. Since B is a surjective normal band, the normal bands $L = [Y; L_\alpha; \psi^l_{\alpha,\beta}]$ and $R = [Y; R_\alpha; \psi^r_{\alpha,\beta}]$ are also surjective. Let $L_i = [Z_i; L^i_\alpha; \psi^{L_i}_{\alpha,\beta}]$ $(i = 1, 2)$ be a pair of finite subbands of L and $\theta^l = [\theta^l_\alpha, \pi]_{\alpha \in Z_1}$ an isomorphism from L_1 to L_2. By Lemma 5.4.13, there exist subbands $A_i = \{(l^i_\alpha, r^i_\alpha) : \alpha \in Z_i\}$ of B isomorphic to Z_i. Hence $R_i = \{r^i_\alpha : \alpha \in Z_i\}$ is a subband of R isomorphic to Z_i for each i, and the map $\theta^r : R_1 \to R_2$ given by $r^1_\alpha \theta^r = r^2_{\alpha\pi}$ is an isomorphism. By Proposition 5.3.1, $\theta^l \bowtie \theta^r$ is an isomorphism from $L_1 \bowtie R_1$ to $L_2 \bowtie R_2$, which we may extend to an automorphism $\hat{\theta} = \hat{\theta}^l \bowtie \hat{\theta}^r$ of B. Then $\hat{\theta}^l$ extends θ^l and so L is homogeneous. Dually for R. □

Consider now the case where $B = [Y; B_\alpha; \psi_{\alpha,\beta}]$ is such that there exists $\alpha > \beta$ in Y with $\psi_{\alpha,\beta}$ an isomorphism. Then by Lemma 5.4.4 every connecting morphism is an isomorphism, and so B is isomorphic to $Y \times B_{n,m}$ for some $n, m \in \mathbb{N}^*$ by Proposition 3.7.13. Since each $B_{n,m}$ is homogeneous by Proposition 5.2.1, the following result is then immediate from Proposition 4.3.8 and Corollary 5.4.14.

Proposition 5.4.16. *Let Y be a semilattice and $n, m \in \mathbb{N}^*$. Then $Y \times B_{n,m}$ is structure-homogeneous if and only if Y is homogeneous.*

Let R be a right normal band with homogeneous structure semilattice Y. Then as $Y \times B_{n,1}$ is structure-homogeneous for any $n \in \mathbb{N}^*$, it follows from Corollary 5.3.2

that we can let $(Y \times B_{n,1}) \bowtie R$ denote the unique (up to isomorphism) normal band with left component isomorphic to $(Y \times B_{n,1})$ and right component isomorphic to R. We observe that

$$Y \times B_{n,m} \cong (Y \times B_{n,1}) \bowtie (Y \times B_{1,m}).$$

Furthermore, for any $n \in \mathbb{N}^*$ and homogeneous bands R and L, where R is right normal and L is left normal, the bands $(Y \times B_{n,1}) \bowtie R$ and $L \bowtie (Y \times B_{1,n})$ are homogeneous by Corollary 5.3.3.

Finally, we examine the case where the connecting morphisms are surjective but not injective (so that the $\mathcal{D}$-classes are infinite). Let $B = [Y; B_\alpha; \psi_{\alpha,\beta}]$ be a surjective normal band. For each $\alpha > \beta$, let $K_{\alpha,\beta}$ denote the congruence Ker $\psi_{\alpha,\beta}$ on B_α, or $K^B_{\alpha,\beta}$ if we need to distinguish the band B. Note that if $\alpha > \beta > \gamma$ then $K_{\alpha,\beta} \subseteq K_{\alpha,\gamma}$, for if $e_\alpha \psi_{\alpha,\beta} = f_\alpha \psi_{\alpha,\beta}$ then

$$e_\alpha \psi_{\alpha,\gamma} = e_\alpha \psi_{\alpha,\beta} \psi_{\beta,\gamma} = f_\alpha \psi_{\alpha,\beta} \psi_{\beta,\gamma} = f_\alpha \psi_{\alpha,\gamma}.$$

The dual of Lemma 5.4.7 is then obtained:

Lemma 5.4.17. *Let $B = L \bowtie R$ be a homogeneous normal band such that the connecting morphisms of either L or R are surjective but not injective. Then Y is the universal semilattice.*

Proof. Suppose w.l.o.g. that $R = [Y; R_\alpha; \psi^r_{\alpha,\beta}]$ has surjective but not injective connecting morphisms, so $|R_\alpha| = \aleph_0$ for all $\alpha \in Y$. Suppose, seeking a contradiction, that Y is a linear or semilinear order. Let $e_\alpha, f_\alpha, g_\alpha \in R_\alpha$ be such that $(e_\alpha, f_\alpha) \in K^R_{\alpha,\beta}$ but $(e_\alpha, g_\alpha) \notin K^R_{\alpha,\beta}$, noting that such elements exist as $\psi^r_{\alpha,\beta}$ is surjective but not injective. For any $l_\alpha \in L_\alpha$, extend the automorphism of the right zero subband $\{(l_\alpha, e_\alpha), (l_\alpha, f_\alpha), (l_\alpha, g_\alpha)\}$ which fixes (l_α, e_α) and swaps (l_α, f_α) and (l_α, g_α) to an automorphism $\theta = [\theta_\alpha, \pi]_{\alpha \in Y}$ of B. Then by Proposition 5.3.1 we have $\theta = \theta^l \bowtie \theta^r$ for some automorphisms $\theta^l = [\theta^l_\alpha, \pi]_{\alpha \in Y}$ and $\theta^r = [\theta^r_\alpha, \pi]_{\alpha \in Y}$ of L and R, respectively. It follows by the commutativity of the diagram $[\alpha, \beta; \alpha, \beta\pi]^R$ (in R) that

$$(e_\alpha, g_\alpha) \in K_{\alpha,\beta\pi} \text{ and } (e_\alpha, f_\alpha) \notin K_{\alpha,\beta\pi}.$$

However as $\{\gamma : \gamma < \alpha\}$ is a chain, either $\beta < \beta\pi$ or $\beta > \beta\pi$, which both contradict the note above the lemma. Hence Y contains a diamond and, being homogeneous by Corollary 5.4.14, is thus the universal semilattice. $\square$

To complete the classification of homogeneous surjective normal bands, we switch our methods to Fraïssé's Theorem. For the signature of semigroups, Fraïssé's Theorem becomes:

Theorem 5.4.18 (Fraïssé's Theorem for semigroups)**.** *Let $\mathcal{K}$ be a non-empty countable class of f.g. semigroups which is closed under isomorphism and satisfies HP,* *JEP and AP. Then there exists a unique, up to isomorphism, countable homogeneous semigroup S such that $\mathcal{K}$ is the age of S. Conversely, the age of a countable homogeneous semigroup is closed under isomorphism, is countable and satisfies HP, JEP and AP.*

Let $\mathcal{K}$ be a Fraïssé class contained in a variety of bands $\mathcal{V}$ defined by the identity $a_1a_2\cdots a_n = b_1b_2\cdots b_m$. Then the Fraïssé limit S of $\mathcal{K}$ is a member of $\mathcal{V}$. Indeed, if $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m \in S$ then

$$\langle x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m\rangle \in \mathcal{K}$$

and so $x_1x_2\cdots x_n = y_1y_2\cdots y_m$ as required.

Example 5.4.19. The rectangular band $B_{\aleph_0,\aleph_0}$ is homogeneous by Proposition 5.2.1, and clearly its age is the class of all finite rectangular bands. It follows that the class of all finite rectangular bands forms a Fraïssé class (with Fraïssé limit $B_{\aleph_0,\aleph_0}$).

Example 5.4.20. Let $\mathcal{K}$ be the class of all finite bands. Since the class of all bands forms a variety, $\mathcal{K}$ is closed under both substructure and (finite) direct product, and thus has JEP. However, it was shown by Imaoka [57, Page 12] that AP does not hold.

Consequently, there does not exist a universal homogeneous band, that is, one which embeds every finite band. However, if we refine our class to certain normal bands, AP is shown to hold. To this end, let $\mathcal{K}_N, \mathcal{K}_{RN}$ and $\mathcal{K}_{LN}$ be the classes of finite normal, finite right normal and finite left normal bands, respectively.

Lemma 5.4.21. *The classes $\mathcal{K}_N, \mathcal{K}_{RN}$ and $\mathcal{K}_{LN}$ form Fraïssé classes.*

Proof. Since the class of (left/right) normal bands forms a variety, it is clear that the classes are closed under subbands and have JEP. The weak amalgamation property follows from [57, Section 2] by taking all bands to be finite. Finally, since bands are ULF there exist only finitely many bands, up to isomorphism, of each finite cardinality, and so each class is countable. □

Let $\mathcal{B}_N, \mathcal{B}_{RN}$ and $\mathcal{B}_{LN}$ be the Fraïssé limits of $\mathcal{K}_N, \mathcal{K}_{RN}$ and $\mathcal{K}_{LN}$, respectively. We prove that $\mathcal{B}_{RN}$ is the unique homogeneous right normal band with surjective but not injective connecting morphisms. This will follow quickly from the subsequent result.

Lemma 5.4.22. *Let $R = [Y; R_\alpha; \psi_{\alpha,\beta}]$ be a homogeneous right normal band, where each connecting morphism is surjective but not injective. Let $\beta_1, \dots, \beta_r$ be elements of Y be such that $\beta_i \perp \beta_j$ for all $i \neq j$, where $r \in \mathbb{N}$. Then for any $\alpha > \beta_1, \dots, \beta_r$, and any $e_{\beta_i} \in B_{\beta_i}$ such that $\langle e_{\beta_i} : 1 \leq i \leq r\rangle$ forms a semilattice, we have*

$$|\{e_\alpha \in R_\alpha : e_\alpha\psi_{\alpha,\beta_i} = e_{\beta_i} \textit{ for all } 1 \leq i \leq r\}| = \aleph_0.$$

Proof. Let $\alpha > \beta_1, \ldots, \beta_r$. We observe that Y is the universal semilattice by Lemma 5.4.17, and so every pair of elements has an upper cover. We first prove the result for $r = 1$ (relabelling β_1 simply as β). Since the connecting morphisms are surjective, there exists $e_\alpha \in B_\alpha$ such that $e_\alpha \psi_{\alpha,\beta} = e_\beta$. Suppose, seeking a contradiction, that

$$e_\alpha K_{\alpha,\beta} = \{f_\alpha : (e_\alpha, f_\alpha) \in K_{\alpha,\beta}\}$$

has finite cardinality n. Note that $n \neq 1$ since the connecting morphisms are not injective and $|e_\alpha K_{\alpha,\beta}| = |e_{\alpha'} K_{\alpha',\beta'}|$ for all $\alpha' > \beta'$ and $e_{\alpha'} \in R_{\alpha'}$, by a simple application of homogeneity. Hence for any $\gamma < \beta$ we have that $|e_\alpha K_{\alpha,\beta}| = |e_\alpha K_{\alpha,\gamma}|$ and $K_{\alpha,\beta} \subseteq K_{\alpha,\gamma}$, and so $e_\alpha K_{\alpha,\beta} = e_\alpha K_{\alpha,\gamma}$ as n is finite. Let $e_\beta \psi_{\beta,\gamma} = e_\gamma$. Then choosing any $f_\beta \in e_\beta K_{\beta,\gamma}$ with $f_\beta \neq e_\beta$ there exists $f_\alpha \in R_\alpha$ such that $f_\alpha \psi_{\alpha,\beta} = f_\beta$, and thus

$$f_\alpha \psi_{\alpha,\gamma} = f_\beta \psi_{\beta,\gamma} = e_\beta \psi_{\beta,\gamma} = e_\gamma.$$

Hence $f_\alpha \in e_\alpha K_{\alpha,\gamma}$, but $f_\alpha \notin e_\alpha K_{\alpha,\beta}$, a contradiction and thus n is infinite.

We now consider the result for arbitrary $r \in \mathbb{N}$. Let $f_\alpha \in R_\alpha$, and let $f_\alpha \psi_{\alpha,\beta_i} = f_{\beta_i}$ for some f_{β_i}. Note that $\langle f_{\beta_i} : 1 \leq i \leq r \rangle$ is a semilattice, and is isomorphic to $\langle e_{\beta_i} : 1 \leq i \leq r \rangle$. We can therefore extend the isomorphism between $\langle f_{\beta_i} : 1 \leq i \leq r \rangle$ and $\langle e_{\beta_i} : 1 \leq i \leq r \rangle$ which sends f_{β_i} to e_{β_i} for each i, to an automorphism of B, to obtain some $\delta > \beta_i$ and $e_\delta \in B_\delta$ such that $e_\delta \psi_{\delta,\beta_i} = e_{\beta_i}$ for each i. Since every pair of elements in the universal semilattice has an upper bound, there exists $\tau \in Y$ with $\tau > \alpha, \delta$. Let e_τ be such that $e_\tau \psi_{\tau,\delta} = e_\delta$, and suppose $e_\tau \psi_{\tau,\alpha} = e_\alpha$. Then

$$e_\alpha \psi_{\alpha,\beta_i} = e_\tau \psi_{\tau,\alpha} \psi_{\alpha,\beta_i} = e_\tau \psi_{\tau,\beta_i} = e_\tau \psi_{\tau,\delta} \psi_{\delta,\beta_i} = e_\delta \psi_{\delta,\beta_i} = e_{\beta_i}.$$

By the case where $r = 1$ the set $e_\tau K_{\tau,\delta}$ is infinite, and thus so is the set

$$\{e_\tau \in R_\tau : e_\tau \psi_{\tau,\beta_i} = e_{\beta_i} \text{ for all } 1 \leq i \leq r\}.$$

The result then follows by extending the isomorphism between the semilattices $\langle e_\tau, e_{\beta_i} : 1 \leq i \leq r \rangle$ and $\langle e_\alpha, e_{\beta_i} : 1 \leq i \leq r \rangle$, which sends e_τ to e_α and fixes all other elements, to an automorphism of R. $\square$

Lemma 5.4.23. *Let $R = [Y; R_\alpha; \psi_{\alpha,\beta}]$ be a homogeneous right normal band, where each connecting morphism is surjective but not injective. Then R is isomorphic to $\mathcal{B}_{RN}$ (dually for $\mathcal{B}_{LN}$).*

Proof. We prove that all finite right normal bands embed in R. We proceed by induction, the base case being trivially true, by supposing that all right normal bands of size $n - 1$ embed in R, and let $A = [Z; A_\alpha; \phi_{\alpha,\beta}]$ be of size n. Let α be maximal in Z and fix $e_\alpha \in A_\alpha$. Suppose α is the upper cover of $\beta_1, \ldots, \beta_r$ in Z, and suppose $e_\alpha \phi_{\alpha,\beta_i} = e_{\beta_i}$. Then $A' = A \setminus \{e_\alpha\} = [\bar{Z}; A'_\alpha; \phi'_{\alpha,\beta}]$ is a right normal band of size $n-1$, and so there exists an embedding $\theta : A' \to R$ (which induces an embedding $\pi : \bar{Z} \to Y$). Since Y is the universal semilattice by Lemma 5.4.17, there exists $\delta \in Y$ such that $\bar{Z}\pi \cup \{\delta\} \cong Z$ by Lemma 5.1.3, where we choose $\delta = \alpha\pi$ if $|A_\alpha| > 1$, that is, if $\bar{Z} = Z$. Then by the previous lemma, we may pick an element e_δ of R_δ such that $e_\delta \notin A'\theta$ and $e_\delta\psi_{\delta,\beta_i\pi} = e_{\beta_i}\theta$. Then it is easily verifiable that $A'\theta \cup \{e_\delta\}$ is isomorphic to A, and so the result follows by induction. By Fraïssé's Theorem R is isomorphic to the Fraïssé limit of $\mathcal{K}_{RN}$. □

Corollary 5.4.24. *The band $\mathcal{B}_N$ is isomorphic to $\mathcal{B}_{LN} \bowtie \mathcal{B}_{RN}$.*

Proof. Let $L \bowtie R$ be a finite normal band with structure semilattice Z. Then there exist embeddings $\theta^l : L \to \mathcal{B}_{LN}$ and $\theta^r : R \to \mathcal{B}_{RN}$ with induced embeddings π_l and π_r from Z to Y, respectively. Hence $\pi = (\pi_l)^{-1}|_{Z\pi_l}\pi_r$ is an isomorphism between $Z\pi_l$ to $Z\pi_r$. By Lemma 5.4.13 there exist subbands $A = \{e_\alpha : \alpha \in Z\pi_l\}$ and $A' = \{f_\alpha : \alpha \in Z\pi_r\}$ of $\mathcal{B}_{LN}$ isomorphic to $Z\pi_l$ and $Z\pi_r$, respectively. Consequently, the map $\phi : A \to A'$ given by $e_\alpha\phi = f_{\alpha\pi}$ is an isomorphism, which we extend to an automorphism $\hat{\theta}^l = [\hat{\theta}^l_\alpha, \hat{\pi}]$ of $\mathcal{B}_{LN}$. In particular, $\hat{\pi}$ extends π and $\theta^l\hat{\theta}^l$ is an embedding of L into $\mathcal{B}_{LN}$, with induced embedding $\pi_l\hat{\pi} = \pi_l(\pi_l)^{-1}|_{Z\pi_l}\pi_r = \pi_r$ of Z into Y. Hence $\theta^l\hat{\theta}^l \bowtie \theta^r : L \bowtie R \to \mathcal{B}_{LN} \times \mathcal{B}_{RN}$ is an embedding, and so $\mathcal{B}_{LN} \bowtie \mathcal{B}_{RN}$ embeds all finite normal bands as required. □

We summarise our findings in this subsection.

Proposition 5.4.25. *A surjective normal band is homogeneous if and only if it is isomorphic to either $Y \times B_{n,m}$, $(U \times B_{n,1}) \bowtie \mathcal{B}_{RN}, \mathcal{B}_{LN} \bowtie (U \times B_{1,n})$ or $\mathcal{B}_N$, for some homogeneous semilattice Y and some $n, m \in \mathbb{N}^*$, where U is the universal semilattice.*

Proof. Suppose first that $B = L \bowtie R$ is a homogeneous surjective normal band. Then by Corollary 5.4.14 and Lemma 5.4.15, each of Y, L and R are homogeneous. If a non-trivial connecting morphism of L is an isomorphism, then L is isomorphic to $Y \times B_{n,1}$ by Lemma 5.4.4 and Proposition 3.7.13. Otherwise, the connecting morphisms of L are non-injective and L is isomorphic to $\mathcal{B}_{LN}$ by Lemma 5.4.23. Dually for R. Since the band $Y \times B_{n,m}$ is structure homogeneous for any homogeneous semilattice Y by Proposition 5.4.16, the result follows by Corollary 5.3.2, Lemma 5.4.17 and Corollary 5.4.24.

Conversely, $\mathcal{B}_N, \mathcal{B}_{RN}, \mathcal{B}_{LN}$ are homogeneous by Fraïssé's Theorem. Since each $Y \times B_{n,m}$ is structure-homogeneous, the final cases are homogeneous by Corollary 5.3.3. □

For a complete classification of homogeneous normal bands, it thus suffices to consider the spined product of an image-trivial normal band with a surjective normal band.

To this end, let $B = L \bowtie R$ be a homogeneous normal band, where L is image-trivial and R is surjective. We assume that L and R are not semilattices, since otherwise B would be image-trivial or surjective. Then L is homogeneous by Corollary 5.4.12, and so $L \cong T_{n,1,k}$ for some $n, k \in \mathbb{N}^*$. Since the structure semilattice of B is a semilinear order by Lemma 5.4.7, it follows from Lemma 5.4.17 that the connecting morphisms of R must be isomorphisms. Hence we may assume that $R = T_{nk} \times B_{1,m}$ for some $m \in \mathbb{N}^*$ by Proposition 5.4.16 and that $L = T_{n,1,k}$ by Corollary 5.3.2. Conversely, $T_{n,1,k} \bowtie (T_{nk} \times B_{1,m})$ is homogeneous for any $n, m, k \in \mathbb{N}^*$ by Proposition 5.4.16 and Corollary 5.3.3.

This, together with Propositions 5.4.11 and 5.4.25, gives a complete list of homogeneous normal bands. In the classification theorem below, the three cases (up to duality) are given by: image-trivial normal bands in (i), surjective normal bands in (ii), (iii), (iv),(v), and finally the spined product of an image-trivial normal band with a surjective normal band in (vi) and (vii).

Theorem 5.4.26 (Classification theorem of homogeneous normal bands)**.** *A normal band is homogeneous if and only if it is isomorphic to one of:*

(i) $T_{n,m,k}$;

(ii) $Y \times B_{n,m}$;

(iii) $\mathcal{B}_{LN} \bowtie (U \times B_{1,m})$;

(iv) $(U \times B_{n,1}) \bowtie \mathcal{B}_{RN}$;

(v) $\mathcal{B}_N$;

(vi) $(T_{mk} \times B_{n,1}) \bowtie T_{1,m,k}$;

(vii) $T_{n,1,k} \bowtie (T_{nk} \times B_{1,m})$;

for some homogeneous semilattice Y and some $n, m, k \in \mathbb{N}^*$, where U is the universal semilattice.

We finish this section by giving a complete classification of structure-homogeneous normal bands.

Proposition 5.4.27. *A normal band is structure-homogeneous if and only if isomorphic to $Y \times B_{n,m}$ for some homogeneous semilattice Y and $n, m \in \mathbb{N}^*$.*

Proof. Let $B = [Y; B_\alpha; \psi_{\alpha,\beta}]$ be a structure-homogeneous, so that Y is homogeneous by Corollary 5.4.14. We show that each connecting morphism is an isomorphism, so that the result will follow by Proposition 3.7.13. Suppose first that there exists $\alpha > \beta$ in Y such that $\psi_{\alpha,\beta}$ is not surjective, say, $e_\beta \notin I_{\alpha,\beta}$. Let $f_\alpha \in B_\alpha$ with $f_\alpha \psi_{\alpha,\beta} = f_\beta$. Then by extending the isomorphism between $\{e_\beta\}$ and $\{f_\beta\}$ (with induced isomorphism the trivial map fixing β) to an automorphism of B with induced automorphism 1_Y, a contradiction is achieved. Hence each connecting morphism is surjective.

Suppose $e_\alpha\psi_{\alpha,\beta} = e_\beta = f_\alpha\psi_{\alpha,\beta}$, and fix some $\delta > \alpha$. Then as $\psi_{\delta,\alpha}$ is surjective there exist $e_\delta, f_\delta \in B_\delta$ with $e_\delta\psi_{\delta,\alpha} = e_\alpha$ and $f_\delta\psi_{\delta,\alpha} = f_\alpha$. Let π be an automorphisms of Y such that $\alpha\pi = \beta$ and $\delta\pi = \delta$ (such a map exists by the homogeneity of Y). Extend the isomorphism swapping $\{e_\delta\}$ and $\{f_\delta\}$ to an automorphism $\theta = [\theta_\alpha, \pi]_{\alpha\in Y}$ of B. Then as the diagram $[\delta, \alpha; \delta, \beta]$ commutes,

$$e_\alpha\theta_\alpha = e_\delta\psi_{\delta,\alpha}\theta_\alpha = e_\delta\theta_\delta\psi_{\delta,\beta} = f_\delta\psi_{\delta,\beta} = e_\beta$$

and similarly $f_\alpha\theta_\alpha = e_\beta$. Hence $e_\alpha = f_\alpha$, and so $\psi_{\alpha,\beta}$ is injective.

The converse follows from Proposition 5.4.16. $\square$

5.5 Homogeneous linearly ordered bands

We call a band $B = \bigcup_{\alpha\in Y} B_\alpha$ *linearly ordered* if Y is a linear order. A homogeneous linearly ordered band B has structure semilattice $\mathbb{Q}$ by Proposition 5.2.4. We observe that if B is not normal, then there exist $\alpha > \beta$ and $e_\alpha \in B_\alpha$ such that the subband $B_\beta(e_\alpha)$ of B_β contains more than one $\mathcal{L}$-class or $\mathcal{R}$-class. Hence if B is homogeneous, then by Corollary 5.2.3 (iv) the same is true for $B_\gamma(e_\delta)$ for any $\delta > \gamma$ and $e_\delta \in B_\delta$.

Lemma 5.5.1. *Let B be a linearly ordered homogeneous non-normal band. If $B_\beta(e_\alpha)$ intersects more than one $\mathcal{L}$-class, then $B_\beta(e_\alpha) = \mathcal{R}(B_\beta(e_\alpha))$. Dually, If $B_\beta(e_\alpha)$ intersects more than one $\mathcal{R}$-class, then $B_\beta(e_\alpha) = \mathcal{L}(B_\beta(e_\alpha))$.*

Proof. Suppose, seeking a contradiction, that there exists an element $g_\beta \in B_\beta$ such that $g_\beta <_r e_\alpha$ but $g_\beta \not< e_\alpha$. Then $g_\beta = e_\alpha g_\beta$, so that

$$g_\beta e_\alpha = e_\alpha g_\beta e_\alpha \,\mathcal{R}\, g_\beta \quad \text{and} \quad g_\beta e_\alpha \in B_\beta(e_\alpha).$$

Since the subband $B_\beta(e_\alpha)$ contains more than one $\mathcal{L}$-class, there exists $f_\beta \,\mathcal{R}\, g_\beta e_\alpha$ such that $f_\beta \in B_\beta(e_\alpha) \setminus \{g_\beta e_\alpha\}$. Extend the automorphism of the right zero subsemigroup $\{f_\beta, g_\beta, g_\beta e_\alpha\}$ which fixes $g_\beta e_\alpha$ and swaps f_β and g_β to $\theta \in \mathrm{Aut}(B)$. Then $e_{\alpha'} = e_\alpha\theta > g_\beta e_\alpha, g_\beta$ and

$$g_\beta e_\alpha = (g_\beta e_\alpha)\theta = g_\beta\theta e_\alpha\theta = f_\beta e_{\alpha'}.$$

Hence

$$f_\beta(e_\alpha e_{\alpha'} e_\alpha) = f_\beta e_{\alpha'} e_\alpha = g_\beta e_\alpha e_\alpha = g_\beta e_\alpha$$

and so $f_\beta \not\leq e_\alpha e_{\alpha'} e_\alpha$. If $\alpha' \geq \alpha$ then $e_\alpha e_{\alpha'} e_\alpha = e_\alpha$, contradicting $e_\alpha > f_\beta$. Hence $\alpha' < \alpha$, so that $e_{\alpha'} e_\alpha e_{\alpha'} = e_{\alpha'}$ and

$$g_\beta = g_\beta e_{\alpha'} = g_\beta e_{\alpha'} e_\alpha e_{\alpha'} = g_\beta e_\alpha e_{\alpha'} = g_\beta e_\alpha,$$

a contradiction. $\square$

Let B be a linearly ordered homogeneous non-normal band. By the lemma above, if $B_\beta(e_\alpha)$ contains a square (that is, it intersects more than one $\mathcal{L}$ and $\mathcal{R}$-class) then $B_\beta(e_\alpha) = B_\beta$. Hence $B_\beta(e_\alpha)$ is a single $\mathcal{K}$-class, where $\mathcal{K} = \mathcal{L}, \mathcal{R}$ or $\mathcal{D}$. It follows from Corollary 5.2.3 (iv) that $B_{\beta'}(e_{\alpha'})$ is also a single $\mathcal{K}$-class for all $\alpha' > \beta'$ and $e_{\alpha'} \in B_{\alpha'}$. If $\mathcal{K} = \mathcal{D}$, then B has the following property, which we follow Petrich [71] in calling *$\mathcal{D}$-covering*:

$$\text{If } e, f \in B, \text{ then either } e \,\mathcal{D}\, f \text{ or } e > f \text{ or } e < f.$$

Proposition 5.5.2. *A homogeneous linearly ordered band is regular.*

Proof. Let $B = \bigcup_{\alpha \in \mathbb{Q}} B_\alpha$ be a homogeneous non-normal linearly ordered band (noting that if B was normal, then it would automatically be regular). By Lemma 5.5.1 we may assume first that for all $\alpha > \beta$ and $e_\alpha \in B_\alpha$ we have that $B_\beta(e_\alpha)$ is a union of $\mathcal{R}$-classes. Hence $\mathcal{L}(B_\beta(e_\alpha)) = B_\beta$, and so $f_\beta e_\alpha = f_\beta$ for all $f_\beta \in B_\beta$. Given any $\gamma, \tau, \sigma \in Y$ and any elements $e_\gamma \in B_\gamma$, $f_\tau \in B_\tau$ and $g_\sigma \in B_\sigma$, to show B is a regular band it suffices to show that

$$e_\gamma f_\tau e_\gamma g_\sigma e_\gamma = e_\gamma f_\tau g_\sigma e_\gamma. \tag{5.4}$$

If $\tau < \gamma$ then $f_\tau e_\gamma = f_\tau$, while if $\gamma < \tau$ then $e_\gamma f_\tau = e_\gamma$, and (5.4) is seen to hold in both cases. Assume instead that $\tau = \gamma$ and $\gamma > \sigma$ (since if $\gamma \leq \sigma$ then both sides of (5.4) cancel to e_γ). Then $e_\gamma g_\sigma \,\mathcal{L}\, g_\sigma \,\mathcal{L}\, f_\gamma g_\sigma \,\mathcal{L}\, e_\gamma f_\gamma g_\sigma$ and

$$(e_\gamma g_\sigma)(e_\gamma f_\gamma g_\sigma) = e_\gamma g_\sigma f_\gamma g_\sigma = e_\gamma g_\sigma$$

so that $e_\gamma g_\sigma \,\mathcal{R}\, e_\gamma f_\gamma g_\sigma$, and thus $e_\gamma g_\sigma = e_\gamma f_\gamma g_\sigma$. By post-multiplying by e_γ and noting that $e_\gamma = e_\gamma f_\tau e_\gamma$ we attain (5.4), and thus B is regular. The case where each $B_\beta(e_\alpha)$ is a union of $\mathcal{L}$-classes is proven dually. $\square$

Let $B = L \bowtie R$ be a homogeneous non-normal linearly ordered band, where $L = \bigcup_{\alpha \in \mathbb{Q}} L_\alpha$ and $R = \bigcup_{\alpha \in \mathbb{Q}} R_\alpha$. Then for any finite chain $\alpha_1 > \alpha_2 > \cdots > \alpha_n$ in $\mathbb{Q}$, we pick $l_{\alpha_1} \in L_{\alpha_1}$ to construct a chain $l_{\alpha_1} > l_{\alpha_2} > \cdots > l_{\alpha_n}$ in L. By an identical argument to the proof of Corollary 5.4.15, we have that R is homogeneous, and dually so is L. Hence by Lemma 5.5.1, each $L_\beta(l_\alpha)$ is a single $\mathcal{R}$ or $\mathcal{L}$-class of L. Since L_β is left zero, the first case is equivalent to L being normal, and so by the classification theorem for homogeneous normal bands we have $L \cong \mathbb{Q} \times B_{n,1}$ for some $n \in \mathbb{N}^*$. Otherwise, each $L_\beta(l_\alpha)$ is a single $\mathcal{L}$-class, so that $L_\beta(l_\alpha) = L_\beta$ and L satisfies $\mathcal{D}$-covering.

Consequently, it suffices to consider the homogeneity of linearly ordered bands satisfying $\mathcal{D}$-covering.

Proposition 5.5.3. *Let $B = \bigcup_{\alpha \in \mathbb{Q}} B_\alpha$ and $B' = \bigcup_{\alpha \in \mathbb{Q}} B'_\alpha$ be bands satisfying $\mathcal{D}$-covering such that $B_\alpha \cong B_\beta$ and $B'_\alpha \cong B'_\beta$ for all $\alpha, \beta \in \mathbb{Q}$. If $\pi \in Aut\ (\mathbb{Q})$ and $\theta_\alpha : B_\alpha \to B'_{\alpha\pi}$ an isomorphism for each α, then $\theta = \bigcup_{\alpha \in \mathbb{Q}} \theta_\alpha$ is an isomorphism from B to B'. Moreover, every isomorphism can be constructed in this way.*

Proof. Clearly θ is an bijection. If $\alpha > \beta$ then, for any $e_\alpha \in B_\alpha$ and $e_\beta \in B_\beta$,

$$(e_\alpha e_\beta)\theta = e_\beta \theta_\beta = (e_\alpha \theta_\alpha)(e_\beta \theta_\beta) = (e_\alpha \theta)(e_\beta \theta)$$

and similarly $(e_\beta e_\alpha)\theta = (e_\beta \theta)(e_\alpha \theta)$. Since each of the maps θ_α are morphisms, it then follows that θ is a morphism as required.

The converse is immediate from Proposition 5.2.2. □

We denote $D_{n,m}$ as the unique, up to isomorphism, linearly ordered band with structure semilattice $\mathbb{Q}$, satisfying $\mathcal{D}$-covering, and such that $B_\alpha \cong B_{n,m}$ for all $\alpha \in \mathbb{Q}$, where $n, m \in \mathbb{N}^*$. We observe that, by this uniqueness property, we have $D_{n,m} \cong D_{n,1} \bowtie D_{1,m}$.

Corollary 5.5.4. *The band $D_{n,m}$ is structure-homogeneous for any $n, m \in \mathbb{N}$.*

Proof. Let $A = \bigcup_{1 \leq i \leq k} A_{\alpha_i}$ and $A' = \bigcup_{1 \leq i \leq k} A'_{\beta_i}$ be a finite subband of $D_{n,m}$, where $\alpha_1 > \alpha_2 > \cdots > \alpha_k$ and $\beta_1 > \beta_2 > \cdots > \beta_k$. Then $A_{\alpha_i} > A_{\alpha_j}$ if and only if $\alpha_i > \alpha_j$, and similarly for A'. Let $\theta : A \to A'$ be an isomorphism, so that there exist isomorphisms $\theta_i : A_{\alpha_i} \to A'_{\beta_i}$ such that $\theta = \bigcup_{1 \leq i \leq k} \theta_i$. Let $\pi \in$ Aut $(\mathbb{Q})$ extend the unique isomorphism between $\{\alpha_1, \ldots, \alpha_k\}$ and $\{\beta_1, \ldots, \beta_k\}$. By Proposition 4.3.7, we can extend each θ_i to an isomorphism $\hat{\theta}_{\alpha_i} : B_{\alpha_i} \to B_{\beta_i}$. For each $\alpha \notin \{\alpha_1, \ldots, \alpha_k\}$, fix an isomorphism $\hat{\theta}_\alpha : B_\alpha \to B_{\alpha\pi}$. Then $\hat{\theta} = \bigcup_{\alpha \in \mathbb{Q}} \hat{\theta}_\alpha$ is an automorphism of $D_{n,m}$ by the previous proposition, and extends θ as required. □

Now let $B = L \bowtie R$ be a homogeneous non-normal linearly ordered band not satisfying $\mathcal{D}$-covering. If $L \cong \mathbb{Q} \times B_{n,1}$ then, as shown after Proposition 5.5.2, R satisfies $\mathcal{D}$-covering since B is not normal. Hence $R \cong D_{1,m}$ for some $m \in \mathbb{N}^*$, and so $B \cong (\mathbb{Q} \times B_{n,1}) \bowtie D_{1,m}$ by Corollary 5.3.2 as $D_{1,m}$ is structure-homogeneous. Dually for the case $R \cong \mathbb{Q} \times B_{1,n}$.

Conversely, the bands $(\mathbb{Q} \times B_{n,1}) \bowtie D_{1,m}$ and $D_{n,1} \bowtie (\mathbb{Q} \times B_{1,m})$ are structure-homogeneous, and thus homogeneous, by Corollary 5.3.3. We have therefore achieved a complete classification of homogeneous linearly ordered bands, which is summarised below.

Theorem 5.5.5. *The following are equivalent for a linearly ordered band B:*

(i) *B is homogeneous;*

(ii) *B is structure-homogeneous;*

(iii) *B is isomorphic to either $D_{n,m}, (\mathbb{Q} \times B_{n,1}) \bowtie D_{1,m}, D_{n,1} \bowtie (\mathbb{Q} \times B_{1,m})$ or $\mathbb{Q} \times B_{n,m}$, for some $n, m \in \mathbb{N}^*$.*

5.6 The final case

The final case is to consider homogeneous bands which are non-normal and which are also not linearly ordered. Our aim to show that these do not exist.

Throughout this section we let $B = \bigcup_{\alpha \in Y} B_\alpha$ be a non-normal band, where Y is non-linear, and fix a three element non-chain α, γ, β, where $\alpha\gamma = \beta$. For $e_\alpha \in B_\alpha$ and $e_\gamma \in B_\gamma$, let A be the subband of B given by

$$A = \langle e_\alpha, e_\gamma \rangle = \{e_\alpha, e_\gamma, e_\alpha e_\gamma, e_\gamma e_\alpha, e_\alpha e_\gamma e_\alpha, e_\gamma e_\alpha e_\gamma\},$$

as shown in Figure 5.6. Then A is isomorphic to one of 4 bands, depending on if $A \cap B_\beta$ is trivial, a left zero or right zero band of size 2, or a 2 by 2 square. We will show that none of these possibilities can occur if B is homogeneous.

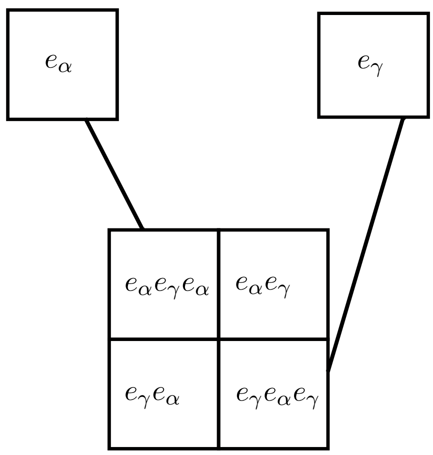

Figure 5.4: The subband A.

Lemma 5.6.1. *For $e_\alpha \in B_\alpha$ and $e_\gamma \in B_\gamma$ we have $|B_\beta(e_\alpha) \cap B_\beta(e_\gamma)| = 1$ if and only if $B_\beta(e_\alpha) \cap B_\beta(e_\gamma) \neq \emptyset$ if and only if $|A \cap B_\beta| = 1$.*

Proof. Suppose that $e_\beta < e_\alpha, e_\gamma$. Then $e_\alpha e_\gamma \in B_\beta$ and, as $\leq$ is compatible with multiplication, $e_\beta \leq e_\alpha e_\gamma$, so that $e_\beta = e_\alpha e_\gamma$. Hence

$$B_\beta(e_\alpha) \cap B_\beta(e_\gamma) = \{e_\alpha e_\gamma\} = \{e_\gamma e_\alpha\}$$

and the result follows. $\square$

Lemma 5.6.2. *For $e_\alpha \in B_\alpha$ and $e_\gamma \in B_\gamma$ we have*

(i) $|\langle e_\alpha, e_\gamma\rangle| = 6$ *if and only if*

$$\mathcal{R}(B_\beta(e_\alpha)) \cap \mathcal{R}(B_\beta(e_\gamma)) = \emptyset = \mathcal{L}(B_\beta(e_\alpha)) \cap \mathcal{L}(B_\beta(e_\gamma));$$

(ii) $|\langle e_\alpha, e_\gamma\rangle| = 4$ *with* $e_\alpha e_\gamma e_\alpha = e_\gamma e_\alpha$ *if and only if* $\mathcal{R}(B_\beta(e_\alpha)) \cap \mathcal{R}(B_\beta(e_\gamma))$ *is non-empty and*

$$\mathcal{L}(B_\beta(e_\alpha)) \cap \mathcal{L}(B_\beta(e_\gamma)) = \emptyset.$$

Moreover, in this case

$$\mathcal{R}(B_\beta(e_\alpha)) \cap \mathcal{R}(B_\beta(e_\gamma)) \subseteq R_{e_\alpha e_\gamma} = R_{e_\gamma e_\alpha}.$$

Dually for $|\langle e_\alpha, e_\gamma\rangle| = 4$ *with* $e_\alpha e_\gamma e_\alpha = e_\alpha e_\gamma$.

Proof. We first show that $e_\gamma e_\alpha = e_\alpha e_\gamma e_\alpha$ if and only if

$$e_\gamma e_\alpha \in \mathcal{R}(B_\beta(e_\alpha)) \cap \mathcal{R}(B_\beta(e_\gamma)).$$

Since $e_\gamma e_\gamma e_\alpha = e_\gamma e_\alpha$ we automatically have $e_\gamma e_\alpha <_r e_\gamma$. Hence if $e_\gamma e_\alpha = e_\alpha e_\gamma e_\alpha$ then $e_\gamma e_\alpha \in B_\beta(e_\alpha)$. The converse holds trivially.

Now suppose $e_\beta <_r e_\alpha, e_\gamma$, so that $e_\beta \leq_r e_\gamma e_\alpha$, as $\leq_r$ is left compatible with multiplication. Hence $e_\beta \,\mathcal{R}\, e_\gamma e_\alpha$, and so $\mathcal{R}(B_\beta(e_\alpha)) \cap \mathcal{R}(B_\beta(e_\gamma))$ is contained in $R_{e_\alpha e_\gamma}$. In particular, we have shown that $\mathcal{R}(B_\beta(e_\alpha)) \cap \mathcal{R}(B_\beta(e_\gamma))$ is non-empty if and only if it contains $e_\gamma e_\alpha$. This, together with the first part of the proof gives the results. □

Lemma 5.6.3. *Suppose that there exist $\sigma > \delta > \tau$ in Y and $e_\sigma > e_\delta$ in B such that $B_\tau(e_\sigma) = B_\tau(e_\delta)$. Then B is not homogeneous.*

Proof. Suppose, seeking a contradiction, that B is homogeneous and $B_\tau(e_\sigma) = B_\tau(e_\delta)$ for some $\sigma > \delta > \tau$ and $e_\sigma > e_\delta$. Let $\sigma' > \delta' > \tau'$ in Y and $e_{\sigma'} > e_{\delta'}$. Then by extending the isomorphism from $e_\sigma > e_\delta > e_\tau$ to $e_{\sigma'} > e_{\delta'} > e_{\tau'}$, for some $e_\tau, e_{\tau'}$, it follows by the homogeneity of B that $B_{\tau'}(e_{\sigma'}) = B_{\tau'}(e_{\delta'})$. The semilattice Y is a semilinear order, since if $\eta > \{\mu, \epsilon\} > \zeta$ is a diamond in Y then for any $e_\eta \in B_\eta$ with $e_\eta > e_\mu, e_\epsilon$ we have

$$B_\zeta(e_\mu) = B_\zeta(e_\eta) = B_\zeta(e_\epsilon)$$

contradicting Lemma 5.6.1, as B is not normal. Hence Y is homogeneous by Proposition 5.2.4. Suppose w.l.o.g. that $B_\tau(e_\sigma)$ has more than 1 $\mathcal{R}$-class. We claim that there exists $g_\tau \in \mathcal{L}(B_\tau(e_\sigma)) \setminus B_\tau(e_\sigma)$. Seeking a contradiction, suppose that no such g_τ exists. Then $\mathcal{R}(B_\tau(e_\sigma)) = B_\tau$, so that for any $\nu \in Y$ with $\nu\sigma = \tau$ and $e_\nu \in B_\nu$

we would have
$$\mathcal{R}(B_\tau(e_\sigma)) \cap \mathcal{R}(B_\tau(e_\nu)) = \mathcal{R}(B_\tau(e_\nu)) = B_\tau$$
by the homogeneity of B. Hence B_τ has 1 $\mathcal{R}$-class by the previous pair of lemmas, a contradiction, and thus the claim holds.

Let $g_\tau \in \mathcal{L}(B_\tau(e_\sigma)) \setminus B_\tau(e_\sigma)$. Then as $g_\tau <_l e_\sigma$ we have $g_\tau e_\sigma = g_\tau$, $e_\sigma g_\tau < e_\sigma$ and $e_\sigma g_\tau \,\mathcal{L}\, g_\tau$. Letting $e_\sigma g_\tau = e_\tau$, then as $B_\tau(e_\sigma)$ has more than one $\mathcal{R}$-class, we may pick $f_\tau \in B_\tau(e_\sigma)$ with $f_\tau \,\mathcal{L}\, e_\tau$ and $f_\tau \neq e_\tau$. Let $A = \{e_\tau, f_\tau, g_\tau\}$, a left zero subsemigroup of B. By extending the automorphism θ of A which fixes e_τ and swaps f_τ and g_τ, to an automorphism $\bar{\theta}$ of B, we have $e_\sigma\bar{\theta} = e_{\sigma'} > e_\tau, g_\tau$ and $e_{\sigma'} f_\tau = e_\tau$.

If $|B_\tau(e_\sigma)| > 2$ then there exists $x_\tau \notin \{e_\tau, f_\tau\}$ with $x_\tau \in B_\tau(e_\sigma)$ and x_τ being $\mathcal{L}$- or $\mathcal{R}$-related to e_τ. We may assume that $\bar{\theta}$ also extends the automorphism of $A \cup \{x_\tau\}$ which extends θ and fixes x_τ. By the homogeneity of B we have $e_{\sigma'} > e_\tau, x_\tau$, so that $\sigma\sigma' > \tau$ to avoid contradicting Lemma 5.6.1. Then
$$e_\sigma e_{\sigma'} e_\sigma \cdot f_\tau = e_\sigma e_{\sigma'} f_\tau = e_\sigma e_\tau = e_\tau$$
so that $f_\tau \not< e_\sigma e_{\sigma'} e_\sigma$ and $\sigma \neq \sigma\sigma'$ (else $f_\tau \not< e_\sigma e_{\sigma'} e_\sigma = e_\sigma$). Hence $e_\sigma e_{\sigma'} e_\sigma < e_\sigma$ and $B_\tau(e_\sigma) = B_\tau(e_\sigma e_{\sigma'} e_\sigma)$, contradicting $f_\tau \notin B_\tau(e_\sigma e_{\sigma'} e_\sigma)$.

It follows that $B_\tau(e_\sigma) = \{e_\tau, f_\tau\}$, $B_\tau(e_{\sigma'}) = \{e_\tau, g_\tau\}$, $\sigma\sigma' = \tau$ and
$$e_\sigma e_{\sigma'} e_\sigma = e_{\sigma'} e_\sigma e_{\sigma'} = e_\tau$$
by Lemma 5.6.1. Now extend the automorphism of A which fixes g_τ and swaps e_τ and f_τ to an automorphism ϕ of B. Then $e_\sigma\phi = e_{\bar{\sigma}} > e_\tau, f_\tau$, so that $\bar{\sigma}\sigma > \tau$ and $e_{\bar{\sigma}} g_\tau = f_\tau$ since $e_\sigma g_\tau = e_\tau$. Since $\bar{\sigma}, \sigma' > \tau$ we have $\bar{\sigma}\sigma' \geq \tau$. Suppose, seeking a contradiction, that $\bar{\sigma}\sigma' > \tau$. Then we claim that $\bar{\sigma} > \{\bar{\sigma}\sigma', \bar{\sigma}\sigma\} > \tau$ forms a diamond. Notice that $\bar{\sigma}\sigma' \neq \bar{\sigma}\sigma$, since otherwise $\bar{\sigma}\sigma = \bar{\sigma}\sigma\bar{\sigma}\sigma' = \tau$, a contradiction. If $\bar{\sigma} = \bar{\sigma}\sigma'$ then $\sigma\bar{\sigma} = \sigma\bar{\sigma}\sigma' = \tau$ since $\sigma\sigma' = \tau$, and so $\bar{\sigma} \neq \bar{\sigma}\sigma'$, similarly $\bar{\sigma} \neq \bar{\sigma}\sigma$. Thus, as the elements are distinct, the set forms a diamond as claimed, which contradicts Y being a semilinear order. Hence $\bar{\sigma}\sigma' = \tau$. Now $e_{\bar{\sigma}}, e_{\sigma'} > e_\tau$, so that $e_{\bar{\sigma}} e_{\sigma'} = e_{\sigma'} e_{\bar{\sigma}} = e_\tau$ and so
$$e_{\bar{\sigma}} g_\tau = e_{\bar{\sigma}}(e_{\sigma'} g_\tau) = e_\tau g_\tau = e_\tau$$
as $e_\tau \,\mathcal{L}\, g_\tau$. However this contradicts $\bar{\sigma} g_\tau = f_\tau$, and B is therefore not homogeneous.

□

Lemma 5.6.4. *Let B be homogeneous and α, γ, β be distinct elements of Y with $\alpha\gamma = \beta$. Then for any $e_\alpha, f_\alpha \in B_\alpha$ and $e_\gamma \in B_\gamma$ such that $e_\alpha, f_\alpha > e_\alpha e_\gamma e_\alpha, f_\alpha e_\gamma f_\alpha$, we have $e_\alpha e_\gamma e_\alpha = f_\alpha e_\gamma f_\alpha$.*

Proof. Let $\sigma < \beta$ and choose $e_\sigma, f_\sigma \in B_\sigma(e_\alpha e_\gamma e_\alpha)$ such that $\langle e_\sigma, f_\sigma\rangle$ is isomorphic to $\langle e_\alpha e_\gamma e_\alpha, f_\alpha e_\gamma f_\alpha\rangle$, noting that elements of this form exist by Corollary 5.2.3 (iv). Extend the isomorphism from

$$\langle e_\alpha, f_\alpha, e_\sigma, f_\sigma\rangle \text{ to } \langle e_\alpha, f_\alpha, e_\alpha e_\gamma e_\alpha, f_\alpha e_\gamma f_\alpha\rangle$$

which maps the generators in order, to an automorphism of B. Then there exist $\tau \in Y$ and $e_\tau \in B_\tau$ (as the image of $e_\alpha e_\gamma e_\alpha$) such that $\alpha > \tau > \beta$ and

$$\{e_\alpha, f_\alpha\} > e_\tau > \{e_\alpha e_\gamma e_\alpha, f_\alpha e_\gamma f_\alpha\}.$$

Then

$$e_\alpha e_\gamma e_\alpha = e_\tau(e_\alpha e_\gamma e_\alpha)e_\tau = (e_\tau e_\alpha)e_\gamma(e_\alpha e_\tau) = e_\tau e_\gamma e_\tau,$$

and similarly $f_\alpha e_\gamma f_\alpha = e_\tau e_\gamma e_\tau$, and the result follows. □

Lemma 5.6.5. *If B is homogeneous, $e_\alpha \in B_\alpha$ and $e_\gamma \in B_\gamma$ then $|\langle e_\alpha, e_\gamma\rangle| \neq 6$.*

Proof. Suppose, seeking a contradiction, that $|\langle e_\alpha, e_\gamma\rangle| = 6$ and let $e_\beta \in B_\beta(e_\alpha)$. Note that any rectangular band D satisfies the identity $xyz = xz$ since if $x, y, z \in D$ then

$$xyz = (xy)(zxz) = x(yz)xz = xz.$$

We therefore have that

$$\begin{aligned} e_\beta e_\gamma e_\beta &= e_\beta(e_\alpha e_\gamma e_\alpha)e_\beta = e_\beta \\ e_\gamma e_\beta e_\gamma &= (e_\gamma e_\alpha)e_\beta(e_\alpha e_\gamma) = (e_\gamma e_\alpha)(e_\alpha e_\gamma) = e_\gamma e_\alpha e_\gamma. \end{aligned}$$

By Lemma 5.6.2 (i), the element e_β is not $\mathcal{L}$- or $\mathcal{R}$-related to $e_\gamma e_\alpha e_\gamma$, so the subband

$$\langle e_\gamma, e_\beta\rangle = \{e_\gamma, e_\beta, e_\gamma e_\beta, e_\beta e_\gamma, e_\gamma e_\alpha e_\gamma\}$$

contains no repetitions. Hence for any $e_\beta, f_\beta < e_\alpha$ we have $\langle e_\gamma, e_\beta\rangle \cong \langle e_\gamma, f_\beta\rangle$. In particular, the map fixing e_γ and swapping some $e_\beta \in B_\beta(e_\alpha)\setminus\{e_\alpha e_\gamma e_\alpha\}$ with $e_\alpha e_\gamma e_\alpha$ is an isomorphism, which can extended to $\theta \in \mathrm{Aut}(B)$. Then $e_\alpha > e_\beta, e_\alpha e_\gamma e_\alpha$ gives $e_\alpha\theta = e_{\alpha'} > e_\alpha e_\gamma e_\alpha, e_\beta$, so that $\alpha\alpha' > \beta$ by Lemma 5.6.1. Moreover, $(e_\alpha e_\gamma e_\alpha)\theta = e_{\alpha'} e_\gamma e_{\alpha'} = e_\beta$, so

$$(e_\alpha e_{\alpha'} e_\alpha)e_\gamma(e_\alpha e_{\alpha'} e_\alpha) = (e_\alpha e_{\alpha'})(e_\alpha e_\gamma e_\alpha)(e_{\alpha'} e_\alpha) = e_\alpha e_\gamma e_\alpha$$

since $\leq$ is compatible with multiplication, and similarly

$$(e_{\alpha'} e_\alpha e_{\alpha'})e_\gamma(e_{\alpha'} e_\alpha e_{\alpha'}) = e_{\alpha'} e_\gamma e_{\alpha'}.$$

Hence $\{e_\alpha e_{\alpha'} e_\alpha, e_{\alpha'} e_\alpha e_{\alpha'}\} > \{e_\alpha e_\gamma e_\alpha, e_\beta\}$ and

$$\{e_\alpha e_\gamma e_\alpha, e_\beta\} = \{(e_\alpha e_{\alpha'} e_\alpha)e_\gamma(e_\alpha e_{\alpha'} e_\alpha), (e_{\alpha'} e_\alpha e_{\alpha'})e_\gamma(e_{\alpha'} e_\alpha e_{\alpha'})\},$$

Since $(\alpha\alpha')\gamma = \beta$ with $\alpha\alpha' \neq \gamma$ we have $e_\alpha e_\gamma e_\alpha = e_\beta$ by Lemma 5.6.4, a contradiction. $\square$

Lemma 5.6.6. *If B is homogeneous, $e_\alpha \in B_\alpha$ and $e_\gamma \in B_\gamma$ then $|\langle e_\alpha, e_\gamma\rangle| \neq 4$.*

Proof. Suppose, seeking a contradiction, that $|\langle e_\alpha, e_\gamma\rangle| = 4$, and assume w.l.o.g. that $e_\alpha e_\gamma e_\alpha = e_\gamma e_\alpha$, so $e_\gamma e_\alpha e_\gamma = e_\alpha e_\gamma$. By Lemma 5.6.2 (ii) we have

$$\mathcal{L}(B_\beta(e_\alpha)) \cap \mathcal{L}(B_\beta(e_\gamma)) = \emptyset \text{ and } \mathcal{R}(B_\beta(e_\alpha)) \cap \mathcal{R}(B_\beta(e_\gamma)) \subseteq R_{e_\gamma e_\alpha} = R_{e_\alpha e_\gamma}.$$

Suppose $B_\beta(e_\alpha)$ has more than 1 $\mathcal{L}$-class, so there exists $e_\beta \in B_\beta(e_\alpha)$ such that $e_\beta \,\mathcal{R}\, e_\gamma e_\alpha$ but $e_\beta \neq e_\gamma e_\alpha$, noting that $e_\beta \neq e_\alpha e_\gamma$ as $|\langle e_\alpha, e_\gamma\rangle| \neq 3$ (see Figure 5.6).

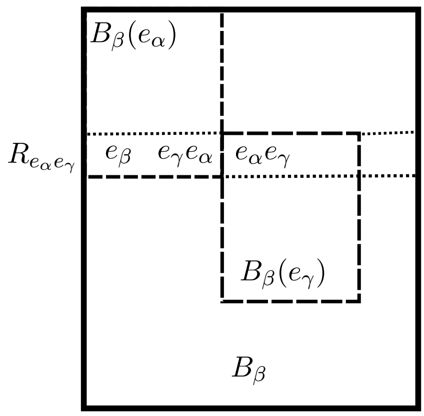

Figure 5.5: The rectangular band B_β.

Since $e_\beta, e_\gamma e_\alpha <_r e_\gamma$ and $e_\beta e_\gamma = e_\beta e_\alpha e_\gamma = e_\alpha e_\gamma$ we have that the subband

$$C = \langle e_\gamma, e_\beta, e_\gamma e_\alpha\rangle = \{e_\gamma, e_\beta, e_\gamma e_\alpha, e_\alpha e_\gamma\}$$

contains no repetitions. Extend the automorphism of C which fixes e_γ and $e_\alpha e_\gamma$ and swaps e_β and $e_\gamma e_\alpha$, to an automorphism θ of B. Then $e_{\alpha'} = e_\alpha\theta > e_\beta, e_\gamma e_\alpha$, so that $\alpha\alpha' > \beta$, and $(e_\gamma e_\alpha)\theta = e_\gamma e_{\alpha'} = e_\beta$. Following the proof of the previous lemma we have

$$\{e_\alpha e_{\alpha'} e_\alpha, e_{\alpha'} e_\alpha e_{\alpha'}\} > \{(e_\alpha e_{\alpha'} e_\alpha)e_\gamma(e_\alpha e_{\alpha'} e_\alpha), (e_{\alpha'} e_\alpha e_{\alpha'})e_\gamma(e_{\alpha'} e_\alpha e_{\alpha'})\}$$

and so $e_\alpha e_\gamma e_\alpha = e_\gamma e_\alpha = e_\beta$, a contradiction. Hence $B_\beta(e_\alpha)$ is a left zero band.

Let $\tau \in Y$ and $e_\tau \in B_\tau$ be such that $\beta < \tau < \alpha$, so $\tau\gamma = \beta$, and $e_\tau < e_\alpha$, so that $B_\beta(e_\tau) \subseteq B_\beta(e_\alpha)$. If $e_\tau \not> e_\gamma e_\alpha$, then $\mathcal{R}(B_\beta(e_\tau)) \cap \mathcal{R}(B_\beta(e_\gamma)) = \emptyset$ as $B_\beta(e_\tau)$ is also left zero by Corollary 5.2.3 (iv). Hence $|\langle e_\tau, e_\gamma\rangle| = 6$ by Lemma 5.6.2, a contradiction, and thus $e_\tau > e_\gamma e_\alpha$. Suppose, seeking a contradiction, that $e_\tau \not> f_\beta$, for some $f_\beta \in B_\beta(e_\alpha)$. Then by extending the automorphism of $\{e_\alpha, f_\beta, e_\gamma e_\alpha\}$ which fixes e_α and swaps f_β with $e_\gamma e_\alpha$ to an automorphism of B, we obtain some e_σ, as the image of e_τ, such that $\alpha > \sigma > \beta$ and $e_\sigma \not> e_\gamma e_\alpha$, a contradiction. Hence $e_\tau > f_\beta$, and so $B_\beta(e_\alpha) = B_\beta(e_\tau)$. By Lemma 5.6.3, B is not homogeneous. □

Lemma 5.6.7. *If B is a non-normal homogeneous band then it is linearly ordered.*

Proof. Let $B = \bigcup_\alpha B_\alpha$ be a non-normal homogeneous band. Suppose, seeking a contradiction, that Y contains a three element non-chain α, γ, β, where $\alpha\gamma = \beta$. Then by the preceding lemmas we have $\langle e_\alpha, e_\gamma\rangle = \{e_\alpha, e_\gamma, e_\alpha e_\gamma\}$ for any $e_\alpha \in B_\alpha$ and $e_\gamma \in B_\gamma$. Hence $B_\beta(e_\alpha) \cap B_\beta(e_\gamma) = \{e_\alpha e_\gamma\}$ and

$$e_\alpha e_\gamma = e_\gamma e_\alpha = e_\alpha e_\gamma e_\alpha = e_\gamma e_\alpha e_\gamma.$$

For any $\alpha > \delta > \beta$ and $e_\alpha > e_\delta$ we have $e_\delta > e_\alpha e_\gamma$. Indeed, if $e_\delta \not> e_\alpha e_\gamma$ then as

$$B_\beta(e_\delta) \cap B_\beta(e_\gamma) \subseteq B_\beta(e_\alpha) \cap B_\beta(e_\gamma) = \{e_\alpha e_\gamma\}$$

we have $B_\beta(e_\delta) \cap B_\beta(e_\gamma) = \emptyset$ and so $|\langle e_\delta, e_\gamma\rangle| > 3$, a contradiction. For any $e_\beta \in B_\beta(e_\alpha) \setminus B_\beta(e_\delta)$, extend the automorphism of $\{e_\alpha, e_\alpha e_\gamma, e_\beta\}$ which fixes e_α and swaps $e_\alpha e_\gamma$ and e_β to an automorphism θ of B. Letting $e_\delta\theta = e_\tau$, then $e_\tau < e_\alpha = e_\alpha\theta$ and $e_\tau \not> e_\alpha e_\gamma$, a contradiction. Thus $B_\beta(e_\alpha) = B_\beta(e_\delta)$, contradicting Lemma 5.6.3. □

This, together with the classification theorem for homogeneous normal bands and Theorem 5.5.5 gives:

Theorem 5.6.8 (Classification theorem for homogeneous bands)**.** *A band is homogeneous if and only if isomorphic to either a homogeneous normal band or a homogeneous linearly ordered band.*

An immediate consequence is that the structure semilattice of a homogeneous band is homogeneous. We would be interested in obtaining a direct proof:

Open Problem 4. Prove directly that the homogeneity of a band is inherited by its structure semilattice.

By Proposition 5.4.27 and Theorem 5.5.5 we achieve a complete list of structure-homogeneous bands:

Theorem 5.6.9 (Classification theorem for structure-homogeneous bands)**.** *A band is structure-homogeneous if and only if isomorphic to either $D_{n,m}, (\mathbb{Q} \times B_{n,1}) \bowtie D_{1,m}, D_{n,1} \bowtie (\mathbb{Q} \times B_{1,m})$ or $Y \times B_{n,m}$, for some homogeneous semilattice Y and $n, m \in \mathbb{N}^*$.*

Chapter 6

Homogeneity of inverse semigroups

After classifying homogeneous bands, and working in the setting of completely regular semigroups, it may seem natural to examine the homogeneity of Clifford semigroups. However, in this chapter we work over a larger variety of I-semigroups; inverse semigroups. Since inverse semigroups form a variety of I-semigroups, we have the concept of a *homogeneous inverse semigroup* (HIS) in L_{US}. We will study both homogeneous inverse semigroups and inverse homogeneous semigroups, and show when they are equivalent. Additionally, we describe the homogeneity of certain classes of inverse semigroups, such as inverse semigroups with finite maximal subgroups and periodic commutative inverse semigroups. Our results may be viewed as extending both the classification of homogeneous semilattices and the classification of certain classes of homogeneous groups, in particular homogeneous finite groups and homogeneous abelian groups.

6.1 Properties of Homogeneity

Let S be an inverse semigroup. Given a subset $A = \{a_1, \ldots, a_n\}$ of S, it follows from Lemma 2.8.2 that

$$\langle A \rangle_I = \langle a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1} \rangle.$$

Hence all f.g. inverse semigroups are f.g. semigroups, and so we obtain:

Lemma 6.1.1. *Every inverse homogeneous semigroup is a HIS.*

We later show that the converse to the lemma above does not hold, that is, the class of HIS is more extensive than the class of inverse homogeneous semigroups. Throughout this chapter, we define the *order* of an element a of S, denoted $o(a)$, to be the cardinality of the monogenic inverse subsemigroup $\langle a\rangle_I$. While this differs from our previous notion of order, the two definitions coincide for Clifford semigroups. Moreover, it follows by the work of Preston [74] that, for any $a \in S$, we have $\langle a\rangle_I$ is finite if and only if $\langle a\rangle$ is finite. Hence we may call an inverse semigroup *periodic* without ambiguity.

We now show that the results on homogeneous semigroups at the end of Section 4.4 also hold for HISs. Let S be an inverse semigroup. Then from Corollaries 4.4.7 and 4.4.8 we have the following lemma.

Lemma 6.1.2. *Let S be a HIS. Then the maximal subgroups of S are pairwise isomorphic HISs and the semilattice of idempotents, $E(S)$, is a HIS.*

We now consider the property of homogeneity as an inverse semigroup for the two key classes of inverse semigroups in Lemma 6.1.2: groups and semilattices. We observe first that, for a group, the inverse of an element coincides with the group inverse. Furthermore, since a finitely generated inverse subsemigroup of a group will contain a unique idempotent, it will be a subgroup [55]. Hence a group is a HIS if and only if it is a homogeneous group.

Given a semilattice Y, then $\langle e_1, \dots, e_n\rangle_I = \langle e_1, \dots, e_n\rangle$ for any $e_1, \dots, e_n$ in Y since $e_i^{-1} = e_i$. Hence a semilattice is a homogeneous semilattice if and only if it is a HIS. Consequently, Lemma 6.1.2 may be restated in a more pleasing manor:

Corollary 6.1.3. *Let S be a HIS with semilattice of idempotents Y. Then Y is a homogeneous semilattice and the maximal subgroups of S are pairwise isomorphic homogeneous groups.*

Since a homogeneous finite semilattice is trivial by Proposition 5.1.4, and an inverse semigroup with a unique idempotent is a group, we have the following.

Corollary 6.1.4. *Let S be a finite inverse semigroup. Then S is a HIS if and only if it is a homogeneous group.*

Lemma 6.1.5. *If S is a HIS then $Aut(S)$ acts transitively on $E(S)$.*

Proof. Given $e, f \in E(S)$, we have $\langle e\rangle_I = \{e\} \cong \{f\} = \langle f\rangle_I$, and so the result follows by the homogeneity of S. $\square$

The inverse semigroup S is *completely semisimple* if no distinct $\mathcal{D}$-related idempotents are related under the natural order on $E(S)$. This is equivalent to S not containing a copy of the bicyclic monoid, for if $e, f \in E(S)$ are such that $e > f$ and $e\,\mathcal{D}\,f$ then there exists $x \in S$ with $xx^{-1} = e$ and $x^{-1}x = f$, and so $\langle x\rangle_I$ is isomorphic to the bicyclic monoid (for further details, see [34]). The converse is immediate.

Theorem 6.1.6. *Let S be a HIS. If S is completely semisimple then it is Clifford, otherwise S is bisimple.*

Proof. Let S be completely semisimple HIS. Suppose, seeking a contradiction, that there exist distinct $\mathcal{D}$-related idempotents e, f, so that $e \perp f$. Since $\mathcal{D}$ is preserved by automorphisms of S, it follows by Lemma 6.1.5 that each $\mathcal{D}$-class contains the same number of idempotents. Indeed, if D_u and D_v are $\mathcal{D}$-classes of S, where $u, v \in E(S)$, then there exists an automorphism θ of S with $u\theta = v$. Hence $D_u\theta = D_v$, so that $E(D_u)\theta = E(D_v)$ and in particular $|E(D_u)| = |E(D_v)|$.

In particular, there exists $g \in E(S)$ with $g \,\mathcal{D}\, ef$ and $g \neq ef$. Then $e > ef$ as $e \perp f$, and so $D_e \neq D_g$ by the semisimplicity of S. We claim that $e > g$. If $g > e$ then $g > ef$, contradicting S being completely semisimple. If $e \perp g$ then there exists an isomorphism between $\langle e, f\rangle_I = \{e, f, ef\}$ and $\langle e, g\rangle_I = \{e, g, eg\}$, which fixes e and sends f to g. Extending to an automorphism ϕ of S, we have

$$D_e\phi = D_e \neq D_g = D_f\phi,$$

contradicting $D_e = D_f$, and the claim holds. Similarly, $f > g$, so that

$$e, f > ef \geq g,$$

and so $ef = g$, a contradiction. Hence $e = f$ and S is Clifford.

Suppose instead that S is not completely semisimple, so that there exist $\mathcal{D}$-related idempotents e', f' with $e' > f'$. Let $h, k \in E(S)$. If $h > k$ or $k > h$ then $\{h, k\} \cong \{e', f'\}$ and so $h \,\mathcal{D}\, k$ by homogeneity. On the other hand, if $h \perp k$ then $\{h, hk\} \cong \{e', f'\} \cong \{k, hk\}$ yields $h \,\mathcal{D}\, hk \,\mathcal{D}\, k$. Thus S is bisimple. □

Proposition 6.1.7. *Let S be a non group bisimple HIS. Then each maximal subgroup of S is infinite and $\mathcal{H}$ is not a congruence.*

Proof. Since S is bisimple, there exists an element x of S with $\langle x\rangle_I$ isomorphic to the bicyclic monoid, with chain of idempotents

$$xx^{-1} > x^{-1}x > x^{-2}x^2 > x^{-3}x^3 > \cdots .$$

For each $n > 2$, by the homogeneity of S, there exists an automorphism θ_n of S extending the unique isomorphism between the chain of idempotents

$$\{xx^{-1}, x^{-1}x, x^{-2}x^2\} \quad \text{and} \quad \{xx^{-1}, x^{-1}x, x^{-n}x^n\}.$$

For each $n > 2$, let $x\theta_n = y_n$. Then $(xx^{-1})\theta_n = y_n y_n^{-1} = xx^{-1}$ and similarly $y_n^{-1}y_n = x^{-1}x$, so that $x \,\mathcal{H}\, y_n$ for each n. Furthermore,

$$(x^{-2}x^2)\theta_n = y_n^{-2}y_n^2 = x^{-n}x^n, \tag{6.1}$$

so that if $y_n = y_m$ then $x^{-n}x^n = x^{-m}x^m$, and so $n = m$. Hence $\{y_n : n > 2\}$ is an infinite subset of H_x, and thus each $\mathcal{H}$-class (and in particular each maximal subgroup) is infinite by Lemma 2.4.1.

Suppose, seeking a contradiction, that $\mathcal{H}$ is a congruence on S. Then as $x \,\mathcal{H}\, y_3$ we have $x^2 \,\mathcal{H}\, y_3^2$, and so by (6.1)

$$x^{-3}x^3 = y_3^{-2}y_3^2 = x^{-2}x^2,$$

a contradiction. $\square$

Open Problem 5. Is a bisimple HIS a group?

We end this section by describing Fraïssé's Theorem for the class of inverse semigroups. This will be of particular use when we examining the homogeneity of commutative inverse semigroups in Section 6.3.

Theorem 6.1.8 (Fraïssé's Theorem for inverse semigroups)**.** *Let $\mathcal{K}$ be a non-empty countable class of f.g. inverse semigroups which is closed under isomorphism and satisfies HP, JEP and AP. Then there exists a unique, up to isomorphism, countable HIS S such that $\mathcal{K}$ is the age of S. Conversely, the age of a countable HIS is closed under isomorphism, is countable and satisfies HP, JEP and AP.*

Example 6.1.9. Let $\mathcal{K}$ be a Fraïssé class of commutative inverse semigroups. Then the Fraïssé limit S of $\mathcal{K}$ is commutative inverse, for if $a, b \in S$ then $\langle a, b\rangle_I \in \mathcal{K}$, and so $ab = ba$. This easily generalises to arbitrary varieties of inverse semigroups.

Example 6.1.10. Let $\mathcal{K}$ be the class of all f.g. Clifford semigroups. Then $\mathcal{K}$ is closed under both substructure and (finite) direct product, and thus has JEP. However it was shown in [48] that AP does not hold.

6.2 The Clifford case

In this section we consider the homogeneity of Clifford semigroups. Since the class of Clifford semigroups forms a variety of completely regular semigroups, we could follow our usual convention of writing *homogeneous completely regular semigroup*, or simply *homogeneous Clifford semigroup*, instead of *Clifford HIS*. We can therefore draw upon the results and definitions in Section 4.3.1. However, in keeping with the previous section we continue writing 'Clifford HIS', and we call a Clifford semigroup a *structure-HIS* if it is a structure-homogeneous completely regular semigroup.

To understand homogeneity of Clifford semigroups, we require a better understanding of their f.g. inverse subsemigroups. The following result is a consequence of Lemma 4.3.3, but is proven here for completeness.

Lemma 6.2.1. *Let $S = [Y; G_\alpha; \psi_{\alpha,\beta}]$ be a Clifford semigroup with inverse subsemigroup T. Then there exist a subsemilattice Y' of Y, and subgroups H_α of G_α for each $\alpha \in Y'$ such that*

$$T = [Y'; H_\alpha; \psi_{\alpha,\beta}|_{H_\alpha}].$$

Proof. Since T is an inverse subsemigroup, there exists a subsemilattice Y' of Y such that $E(T) = \{e_\alpha : \alpha \in Y'\}$. If $g_\alpha, h_\alpha \in T$ then $h_\alpha^{-1} \in T$ since T is inverse, so $g_\alpha h_\alpha^{-1} \in T$. Hence the maximal subgroup of T containing e_α is a subgroup H_α of G_α. Moreover, if $\alpha > \beta$ in Y' then $e_\beta \in T$ and so if $g_\alpha \in T$ then

$$g_\alpha e_\beta = (g_\alpha \psi_{\alpha,\beta})(e_\beta \psi_{\beta,\beta}) = (g_\alpha \psi_{\alpha,\beta})(e_\beta) = g_\alpha \psi_{\alpha,\beta} \in T,$$

and so $H_\alpha \psi_{\alpha,\beta} \subseteq H_\beta$. Hence the homomorphism $\psi_{\alpha,\beta}|_{H_\alpha} : H_\alpha \to H_\beta$ is well-defined, and the result follows. $\square$

Since Clifford semigroups are morphism-pure by Lemma 2.11.7 we have the following isomorphism theorem.

Theorem 6.2.2. *Let $S = [Y; G_\alpha; \psi_{\alpha,\beta}]$ and $T = [Z; H_\gamma; \varphi_{\gamma,\delta}]$ be a pair of Clifford semigroups. Let $\pi : Y \to Z$ be an isomorphism and let $\theta_\alpha : G_\alpha \to H_{\alpha\pi}$ be an isomorphism for each $\alpha \in Y$. Assume further that for any $\alpha \geq \beta$, the diagram $[\alpha, \beta; \alpha\pi, \beta\pi]$ commutes, that is,*

$$\begin{tikzcd} G_\alpha \arrow[r, "\theta_\alpha"] \arrow[d, "\psi_{\alpha,\beta}"] & H_{\alpha\pi} \arrow[d, "\varphi_{\alpha\pi,\beta\pi}"] \\ G_\beta \arrow[r, "\theta_\beta"] & H_{\beta\pi} \end{tikzcd} \tag{6.2}$$

commutes. Then $\theta = \bigcup_{\alpha \in Y} \theta_\alpha = [\theta_\alpha, \pi]_{\alpha \in Y}$ is an isomorphism from S to T. Conversely, every isomorphism from S to T can be so constructed for unique π and θ_α.

Remark 6.2.3. If $S = [Y; G_\alpha; \psi_{\alpha,\beta}]$ and $H_\alpha \cong G_\alpha$ for each $\alpha \in Y$ then the isomorphism theorem above can be used to construct a Clifford semigroup isomorphic to S with maximal subgroups H_α. Formally, if $\theta_\alpha : G_\alpha \to H_\alpha$ is an isomorphism for each $\alpha \in Y$ then $\theta = [\theta_\alpha, 1_Y]_{\alpha \in Y}$ is an isomorphism from S to $T = [Y; H_\alpha; \varphi_{\alpha,\beta}]$, where

$$\varphi_{\alpha,\beta} = \theta_\alpha^{-1} \psi_{\alpha,\beta} \theta_\beta.$$

In particular, maximal subgroups which can be written as a direct sum (d.s.) or a direct product (d.p.) of groups, can be regarded as an internal or external d.s./d.p. without problems arising.

We adopt a non standard notation by denoting the internal d.s. and internal d.p. of a pair of groups H and H' as $H \oplus H'$ and $H \otimes H'$, respectively. We denote the internal direct sum of n copies of a group H by H^n, where $n \in \mathbb{N}^* = \mathbb{N} \cup \{\aleph_0\}$. Unless stated otherwise, we assume that all d.s.'s of groups are internal.

If $S = [Y; G_\alpha; \psi_{\alpha,\beta}]$ is a HIS, then as the groups G_α are the maximal subgroups of S and $Y \cong E(S)$ we then obtain by Corollary 6.1.3:

Corollary 6.2.4. *If $S = [Y; G_\alpha; \psi_{\alpha,\beta}]$ is a HIS then Y is homogeneous and the groups G_α are pairwise isomorphic homogeneous groups.*

Hence, if $S = [Y; G_\alpha; \psi_{\alpha,\beta}]$ is a HIS and $G_\alpha \cong G$ then, by Remark 6.2.3, each group G_α can be taken as a labelling of G, and the morphisms $\psi_{\alpha,\beta}$ to be a labelling of an endomorphism of G.

A subset T of a Clifford semigroup S will be called *order-characteristic* if whenever T contains an element of order n, then every element of order n in S belongs to T.

Given a group G with subset A, then we set

$$o(A) = \{n : \text{there exists } a \in A \text{ such that } o(a) = n\}.$$

Note that if $a_\alpha \in S$ then, as a_α is contained in the group G_α, the inverse subsemigroup $\langle a_\alpha \rangle_I$ is a cyclic group. In particular, our definition of the order of an element intersects with the group theory definition, that is $o(a_\alpha)$ is the minimal $n > 1$ such that $a_\alpha^n = e_\alpha$. Hence, as cyclic groups of the same cardinality are isomorphic, the following generalization of [19, Lemma 1] and its corollary are easily verifiable:

Lemma 6.2.5. *Let S a Clifford HIS with characteristic subset T. Then T is order-characteristic.*

Corollary 6.2.6. *Let S and S' be a pair of isomorphic Clifford HISs with characteristic inverse subsemigroups T and T', respectively such that $o(T) = o(T')$. Then $T \cong T'$, and if $S = S'$ then $T = T'$.*

Lemma 6.2.7. *Let $S = [Y; G_\alpha; \psi_{\alpha,\beta}]$ be a Clifford semigroup. For each $\alpha \in Y$, let H_α be an order-characteristic subgroup of G_α such that $H_\alpha \cong H_\beta$ for all $\alpha, \beta \in Y$. Then*

$$T = [Y; H_\alpha; \psi_{\alpha,\beta}|_{H_\alpha}]$$

is an order-characteristic inverse subsemigroup of S. In particular, if S is a HIS then so is T.

Proof. Notice that as each H_α are isomorphic order-characteristic subgroups, and as $o(H_\alpha\psi_{\alpha,\beta}) \subseteq o(H_\beta)$, it follows that $H_\alpha\psi_{\alpha,\beta} \subseteq H_\beta$ for all $\alpha \geq \beta$ in Y, and so T is well defined. The result is then immediate. □

In particular, if S in Lemma 6.2.7 is a HIS, then the result holds if H_α is characteristic by Lemma 6.2.5.

A pair of subsets A and B of a group G are of *coprime order* if $o(A) \cap o(B) \subseteq \{1\}$. If G is periodic then this is equivalent to being of *relatively prime exponent*, defined in [17], but does not require the theory of supernatural numbers. Note that if $G = A \otimes B$ where A and B are periodic of coprime order, then clearly A and B are order-characteristic subgroups of G, and so the lemma above may be used in this case.

Corollary 6.2.8. *Let G be a homogeneous group with characteristic subsets H and K such that $H \cap K \subseteq \{1\}$. Then H and K are of coprime order.*

Proof. If $h \in H$ and $k \in K$ both have order $n \in \mathbb{N}^*$, then by Lemma 6.2.5 H and K both contain all elements of order n. Since H and K intersect trivially, it follows that $n = 1$, and so the subsets are coprime. □

The subsequent pair of lemmas arise from basic group theory and proofs will be omitted:

Lemma 6.2.9. *Let $G = H \otimes K$ be a group with H and K periodic of coprime order. Then, for each subgroup G' of G, there exist subgroups H' and K' of H and K, respectively, such that $G' = H' \otimes K'$.*

Lemma 6.2.10. *Let $G_1 = H_1 \otimes K_1$ and $G_2 = H_2 \otimes K_2$ be a pair of groups with the H_i and K_i periodic of coprime orders for each $i = 1, 2$, and $H_1 \cong H_2$, $K_1 \cong K_2$. Let $G_1' = H_1' \otimes K_1'$ and $G_2' = H_2' \otimes K_2'$ be subgroups of G_1 and G_2, respectively, and $\theta^H : H_1' \to H_2'$ and $\theta^K : K_1' \to K_2'$ be a pair of morphisms. Then the map θ given by*

$$(hk)\theta = (h\theta^H)(k\theta^K) \quad (h \in H_1', k \in K_2')$$

is a morphism from G_1' to G_2', and every morphism can be so constructed.

The homomorphism θ in the lemma above will often be denoted as $\theta^H \otimes \theta^K$. We observe that Lemmas 6.2.9 and 6.2.10 fail in general if we drop the periodic condition.

If $G = H \otimes K$ is a group with H and K periodic of coprime order then clearly H and K are characteristic subgroups. The following simplification of [17, Lemma 1] then follows from the pair of lemmas above.

Corollary 6.2.11. *Let $G = H \otimes K$ be a group with the H and K periodic of coprime order. Then G is homogeneous if and only if H and K are homogeneous.*

Given a group $G = H \otimes K$ where H and K are periodic of coprime order, let $S = [Y; G_\alpha; \psi_{\alpha,\beta}]$ be the Clifford semigroup with $G_\alpha \cong G$ for each $\alpha \in Y$. Then $G_\alpha = H_\alpha \otimes K_\alpha$ where $H_\alpha \cong H$ and $K_\alpha \cong K$, and by Lemma 6.2.10 we may let $\psi_{\alpha,\beta} = \psi^H_{\alpha,\beta} \otimes \psi^K_{\alpha,\beta}$ where $\psi^H_{\alpha,\beta} : H_\alpha \to H_\beta$ and $\psi^K_{\alpha,\beta} : K_\alpha \to K_\beta$. It follows that the sets

$$S^H := [Y; H_\alpha; \psi^H_{\alpha,\beta}] \text{ and } S^K := [Y; K_\alpha; \psi^K_{\alpha,\beta}]$$

are characteristic inverse subsemigroups of S by Lemma 6.2.7.

Corollary 6.2.12. *Let $S = [Y; H_\alpha \otimes K_\alpha; \psi_{\alpha,\beta}]$ be a periodic Clifford semigroup, where each H_α and K_α are of coprime order. Let π be an automorphism of Y, and $\theta^H = [\theta^H_\alpha, \pi]_{\alpha \in Y}$ and $\theta^K = [\theta^K_\alpha, \pi]_{\alpha \in Y}$ be automorphisms of S^H and S^K, respectively. Letting $\theta_\alpha = \theta^H_\alpha \otimes \theta^K_\alpha$, then $\theta = [\theta_\alpha, \pi]_{\alpha \in Y}$ is an automorphism of S, and all automorphisms of S can be constructed in this way.*

Proof. We show first that θ is an automorphism of S. By Lemma 6.2.10 each θ_α is an isomorphism, so it remains to prove that the diagram $[\alpha, \beta; \alpha\pi, \beta\pi]$ commutes for any $\alpha > \beta$. Let $g_\alpha \in G_\alpha$, say, $g_\alpha = h_\alpha k_\alpha$ $(h_\alpha \in H_\alpha, k_\alpha \in K_\alpha)$. Then

$$\begin{aligned} g_\alpha \theta_\alpha \psi_{\alpha\pi,\beta\pi} &= (h_\alpha \theta^H_\alpha \psi^H_{\alpha\pi,\beta\pi})(k_\alpha \theta^K_\alpha \psi^K_{\alpha\pi,\beta\pi}) \\ &= (h_\alpha \psi^H_{\alpha,\beta} \theta^H_\beta)(k_\alpha \psi^K_{\alpha,\beta} \theta^K_\beta) \\ &= g_\alpha \psi_{\alpha,\beta} \theta_\beta \end{aligned}$$

since $[\alpha, \beta; \alpha\pi, \beta\pi]^{S^H}$ and $[\alpha, \beta; \alpha\pi, \beta\pi]^{S^K}$ commutes. Hence $[\alpha, \beta; \alpha\pi, \beta\pi]^S$ commutes and θ is an automorphism of S. The converse follows from Theorem 6.2.2 and the fact that S^H and S^K are characteristic inverse subsemigroups of S. ☐

Proposition 6.2.13. *Let $S = [Y; H_\alpha \otimes K_\alpha; \psi_{\alpha,\beta}]$ be a periodic Clifford semigroup, where each H_α and K_α are of coprime order. Then S is a structure-HIS if and only if S^H and S^K are structure-HISs.*

Proof. If S is a structure-HIS then S^H and S^K, being characteristic inverse subsemigroups with structure semilattice Y, are also structure-HISs.

Conversely, suppose S^H and S^K are structure-HISs. Let A and B be a pair of f.g. inverse subsemigroups of S. From Lemmas 6.2.4, 6.2.9 and 6.2.10 we have that

$$\begin{aligned} A &= [Z'; H'_\gamma \otimes K'_\gamma; \psi^{H'}_{\gamma,\delta} \otimes \psi^{K'}_{\gamma,\delta}] \\ B &= [Z''; H''_\tau \otimes K''_\tau; \psi^{H''}_{\tau,\sigma} \otimes \psi^{K''}_{\tau,\sigma}] \end{aligned}$$

where H'_γ and K'_γ are subgroups of H_γ and K_γ, respectively, $\psi^{H'}_{\gamma,\delta} = \psi_{\gamma,\delta}|_{H'_\gamma}$ and $\psi^{K'}_{\gamma,\delta} = \psi_{\gamma,\delta}|_{K'_\gamma}$. Similarly for B.

Let $\theta = [\theta_\gamma, \pi]_{\gamma \in Z'} : A \to B$ be an isomorphism, and $\hat{\pi}$ an automorphism of Y which extends π. Then for each $\gamma \in Z'$, we have $\theta_\gamma = \theta^{H'}_\gamma \otimes \theta^{K'}_\gamma$ for some isomorphisms $\theta^{H'}_\gamma : H'_\gamma \to H''_{\gamma\pi}$ and $\theta^{K'}_\gamma : K'_\gamma \to K''_{\gamma\pi}$. Hence $\theta^{H'} = [\theta^{H'}_\gamma, \pi]_{\gamma \in Z'}$ is an isomorphism from $[Z'; H'_\gamma; \psi^{H'}_{\gamma,\delta}]$ to $[Z''; H''_\tau; \psi^{H''}_{\tau,\sigma}]$, and similarly for the isomorphism $\theta^{K'} = [\theta^{K'}_\gamma, \pi]_{\gamma \in Z'}$. Since $\theta^{H'}$ is an isomorphism between f.g. inverse subsemigroups of the structure-HIS S^H, we can extend $\theta^{H'}$ to an automorphism $[\theta^H_\alpha, \hat{\pi}]_{\alpha \in Y}$ of S^H, and similarly extend $\theta^{K'}$ to an automorphism $[\theta^K_\alpha, \hat{\pi}]_{\alpha \in Y}$ of S^K. By Corollary 6.2.12 the map $[\theta^H_\alpha \otimes \theta^K_\alpha, \hat{\pi}]_{\alpha \in Y}$ is an automorphism of S, and extends θ as required. ☐

A simple adaptation of the proof above gives the following result.

Proposition 6.2.14. *Let $S = [Y; H_\alpha \otimes K_\alpha; \psi_{\alpha,\beta}]$ be a periodic Clifford semigroup, where H_α and K_α are of coprime order, and S^H is a structure-HIS. Then S is a HIS if and only if S^K is a HIS.*

Given a Clifford semigroup $[Y; G_\alpha; \psi_{\alpha,\beta}]$ then, for each $\alpha > \beta$, we follow the notation given for normal bands by setting

$$\begin{aligned} I_{\alpha,\beta} &:= \text{Im } \psi_{\alpha,\beta} = \{a_\beta \in G_\beta : \exists a_\alpha \in G_\alpha, a_\alpha\psi_{\alpha,\beta} = a_\beta\}, \\ K_{\alpha,\beta} &:= \text{Ker } \psi_{\alpha,\beta} = \{a_\alpha \in G_\alpha : a_\alpha\psi_{\alpha,\beta} = e_\beta\}, \end{aligned}$$

as the image and kernel of the connecting morphism $\psi_{\alpha,\beta}$, respectively. Given $\alpha > \gamma > \beta$ in Y and $k_\alpha \in K_{\alpha,\gamma}$, then

$$k_\alpha\psi_{\alpha,\beta} = (k_\alpha\psi_{\alpha,\gamma})\psi_{\gamma,\beta} = e_\gamma\psi_{\gamma,\beta} = e_\beta,$$

and so $k_\alpha \in K_{\alpha,\beta}$. Thus $K_{\alpha,\gamma} \subseteq K_{\alpha,\beta}$, and similarly $I_{\alpha,\beta} \subseteq I_{\gamma,\beta}$.

We define the *absolute image* I^*_α and the *absolute kernel* K^*_α of $\alpha \in Y$ as the subsets of G_α given by

$$\begin{aligned} I^*_\alpha &:= \{g_\alpha \in I_\alpha : o(g_\alpha) = o(g_\alpha\psi_{\alpha,\beta}) \text{ for all } \beta < \alpha\}, \\ K^*_\alpha &:= \{a_\alpha \in G_\alpha : a_\alpha\psi_{\alpha,\beta} = e_\beta \text{ for all } \beta < \alpha\} = \bigcap_{\beta<\alpha} K_{\alpha,\beta}. \end{aligned}$$

The set K^*_α, being an intersection of subgroups of G_α, forms a subgroup, while I^*_α may not.

Notation 6.2.15. *Throughout the remainder of this subsection, $S = [Y; G_\alpha; \psi_{\alpha,\beta}]$ denotes a Clifford HIS, so that Y is a homogeneous semilattice and the G_α are pairwise isomorphic homogeneous groups.*

The following lemma will be vital in our understanding of the images and kernels of the connecting morphisms.

Lemma 6.2.16. *Let $\alpha, \alpha', \beta \in Y$ be such that $\alpha, \alpha' > \beta$, and let $g_\beta, h_\beta \in G_\beta$ be of the same order. Then the map*

$$\phi : \langle e_\alpha, g_\beta\rangle_I \to \langle e_{\alpha'}, h_\beta\rangle_I$$

given by $e_\alpha\phi = e_{\alpha'}$ and $g^z_\beta\phi = h^z_\beta$ for $z \in \mathbb{Z}$, is an isomorphism.

Proof. Note that $\langle g_\beta\rangle_I$ and $\langle h_\beta\rangle_I$ are isomorphic cyclic groups. Moreover, e_α is the identity in $\langle e_\alpha, g_\beta\rangle_I$ since $e_\alpha g_\beta = (e_\alpha\psi_{\alpha,\beta})(g_\beta\psi_{\beta,\beta}) = e_\beta g_\beta = g_\beta = g_\beta e_\alpha$ and so

$$\langle e_\alpha, g_\beta\rangle_I = \{e_\alpha\} \cup \langle g_\beta\rangle_I.$$

Similarly for $\langle e_{\alpha'}, h_\beta\rangle_I$, and it is routine to check that ϕ is an isomorphism. $\square$

Lemma 6.2.17. *Let $\alpha, \alpha', \beta \in Y$ be such that $\alpha, \alpha' > \beta$. Then $I_{\alpha,\beta} = I_{\alpha',\beta}$.*

Proof. Let $g_\beta \in I_{\alpha,\beta}$, say $g_\beta = g_\alpha \psi_{\alpha,\beta}$. By the lemma above, there is an isomorphism $\phi : \langle e_\alpha, g_\beta \rangle_I \to \langle e_{\alpha'}, g_\beta \rangle_I$ determined by $e_\alpha \phi = e_{\alpha'}$ and $g_\beta \phi = g_\beta$. Extend ϕ to an automorphism $\theta = [\theta_\alpha, \pi]_{\alpha \in Y}$ of S, so that $\alpha\pi = \alpha'$ and $\beta\pi = \beta$. Then

$$g_\alpha \theta_\alpha \psi_{\alpha',\beta} = g_\alpha \psi_{\alpha,\beta} \theta_\beta = g_\beta \theta_\beta = g_\beta$$

since the diagram $[\alpha, \beta; \alpha', \beta]$ commutes. Hence $g_\beta \in I_{\alpha',\beta}$ and $I_{\alpha,\beta} \subseteq I_{\alpha',\beta}$. The dual gives equality. □

For each $\alpha \in Y$, we let I_α denote the subgroup $I_{\delta,\alpha}$ for (any) $\delta > \alpha$. Since Y has no maximal elements, I_α is non-empty for all $\alpha \in Y$.

Lemma 6.2.18. *For each $\alpha \in Y$, the subgroups I_α and K_α^* are characteristic subgroups of G_α, and are thus homogeneous. Moreover, for each $\alpha > \beta$, the subgroup $K_{\alpha,\beta}$ is homogeneous.*

Proof. Let $\varphi \in \text{Aut}(G_\alpha)$ and $g_\alpha = g_\delta \psi_{\delta,\alpha} \in I_\alpha$ for some $\delta > \alpha$. Then, by Lemma 6.2.16, there exists an isomorphism $\phi : \langle e_\delta, g_\alpha \rangle_I \to \langle e_\delta, g_\alpha \varphi \rangle_I$ fixing e_δ and with $g_\alpha \phi = g_\alpha \varphi$. Extending ϕ to $\theta = [\theta_\alpha, \pi]_{\alpha \in Y} \in \text{Aut}(S)$, then as $[\delta, \alpha; \delta, \alpha]$ commutes, we have

$$g_\alpha \varphi = g_\alpha \phi = g_\alpha \theta_\alpha = g_\delta \psi_{\delta,\alpha} \theta_\alpha = g_\delta \theta_\delta \psi_{\delta,\alpha} \in I_\alpha,$$

and so I_α is characteristic. Now let $k_\alpha \in K_\alpha^*$, and extend the isomorphism between $\langle k_\alpha \varphi \rangle_I$ and $\langle k_\alpha \rangle_I$ which sends $k_\alpha \varphi$ to k_α, to $\bar{\theta} = [\bar{\theta}_\alpha, \bar{\pi}]_{\alpha \in Y} \in \text{Aut}(S)$. Then as $[\alpha, \beta; \alpha, \beta\bar{\pi}]$ commutes for any $\beta < \alpha$, and as $k_\alpha \in K_{\alpha,\beta\bar{\pi}}$, we have

$$(k_\alpha \varphi) \psi_{\alpha,\beta} \theta_\beta = (k_\alpha \varphi) \theta_\alpha \psi_{\alpha,\beta\bar{\pi}} = k_\alpha \psi_{\alpha,\beta\bar{\pi}} = e_{\beta\bar{\pi}}.$$

Hence $k_\alpha \varphi \in K_{\alpha,\beta}$ for all $\beta < \alpha$, that is, $k_\alpha \varphi \in K_\alpha^*$, and so K_α^* is characteristic.

Finally, let ϕ be an isomorphism between f.g. subgroups A_α and A'_α of $K_{\alpha,\beta}$. Then the map $\phi' : A_\alpha \cup \{e_\beta\} \to A'_\alpha \cup \{e_\beta\}$ such that $A_\alpha \phi' = A_\alpha \phi$ and $e_\beta \phi' = e_\beta$ is an isomorphism between f.g. inverse subsemigroups of S. By extending ϕ' to an automorphism of S, the result follows from Theorem 6.2.2. □

We now determine the Clifford semigroup form of Lemma 5.4.4 as follows.

Lemma 6.2.19. *If $\alpha > \beta$ and $\alpha' > \beta'$ in Y then there exists a pair of isomorphisms $\theta_\alpha : G_\alpha \to G_{\alpha'}$ and $\theta_\beta : G_\beta \to G_{\beta'}$ such that $\psi_{\alpha,\beta} = \theta_\alpha \psi_{\alpha',\beta'} \theta_\beta^{-1}$. In particular, if $\psi_{\alpha,\beta}$ is injective/surjective then so is $\psi_{\alpha',\beta'}$, and $I_\beta \cong I_{\beta'}$, $K_{\alpha,\beta} \cong K_{\alpha',\beta'}$ and $K_\alpha^* \cong K_{\alpha'}^*$.*

Proof. Clearly the map $\phi : \{e_\alpha, e_\beta\} \to \{e_{\alpha'}, e_{\beta'}\}$ given by $e_\alpha \phi = e_{\alpha'}$ and $e_\beta \phi = e_{\beta'}$ is an isomorphism. By extending ϕ to an automorphism $\theta = [\theta_\alpha, \pi]_{\alpha \in Y}$ of S, then the first result follows immediately from the commutativity of $[\alpha,\beta;\alpha\pi,\beta\pi] = [\alpha,\beta;\alpha',\beta']$. The injective/surjective properties of the connecting morphisms follow.

We observe that

$$I_\beta\theta_\beta = (G_\alpha\psi_{\alpha,\beta})\theta_\beta = G_\alpha\theta_\alpha\psi_{\alpha',\beta'} = G_{\alpha'}\psi_{\alpha',\beta'} = I_{\beta'}$$

and so $I_\beta \cong I_{\beta'}$. If $k_\alpha \in K_{\alpha,\beta}$ then

$$k_\alpha\theta_\alpha\psi_{\alpha',\beta'} = k_\alpha\psi_{\alpha,\beta}\theta_\beta = e_\beta\theta_\beta = e_{\beta'},$$

so that $K_{\alpha,\beta}\theta \subseteq K_{\alpha',\beta'}$. If $x_{\alpha'} \in K_{\alpha',\beta'}$, then there exists $y_\alpha \in G_\alpha$ with $y_\alpha\theta_\alpha = x_{\alpha'}$, so that

$$y_\alpha\psi_{\alpha,\beta}\theta_\beta = x_{\alpha'}\psi_{\alpha',\beta'} = e_{\beta'}.$$

Hence $y_\alpha\psi_{\alpha,\beta} = e_\beta$, and so $y_\alpha \in K_{\alpha,\beta}$. We have thus shown that $K_{\alpha,\beta}\theta = K_{\alpha',\beta'}$, and so $K_{\alpha,\beta} \cong K_{\alpha',\beta'}$. Finally,

$$K^*_\alpha\theta_\alpha = (\bigcap_{\gamma<\alpha} K_{\alpha,\gamma})\theta_\alpha = \bigcap_{\gamma<\alpha}(K_{\alpha,\gamma}\theta_\alpha) = \bigcap_{\gamma\pi<\alpha} K_{\alpha',\gamma\pi} = K_{\alpha^*}$$

since π is an automorphism of Y. Thus $K^*_\alpha \cong K^*_{\alpha'}$ as required. $\square$

We say that a subset A of a group G is *closed under prime powers* if, whenever $p \in o(A)$ for some prime p, then every power of p in $o(G)$ also lies $o(A)$.

Lemma 6.2.20. *The subgroups I_α and K^*_α are closed under prime powers and $I_\alpha \cap K^*_\alpha = \{e_\alpha\}$. Moreover, every element in G_α of prime order is contained in either I_α or K^*_α.*

Proof. Let $p \in o(K^*_\alpha)$. Proceeding by induction, assume that

$$p, p^2, \ldots, p^{r-1} \in o(K^*_\alpha)$$

for some $r \in \mathbb{N}$. Then by Lemma 6.2.5, every element of G_α of order p^k is in K^*_α for $1 \leq k \leq r-1$. Let $g_\alpha \in G_\alpha$ be of order p^r. Then g_α^p is of order p^{r-1}, so that $g_\alpha^p \in K^*_\alpha$. In particular, for any $\beta < \alpha$ we have $(g_\alpha\psi_{\alpha,\beta})^p = e_\beta$. If $o(g_\alpha\psi_{\alpha,\beta}) = p$ then for any $\alpha' \in Y$ with $\alpha > \alpha' > \beta$ we have $o(g_\alpha\psi_{\alpha,\alpha'}) = p$ and thus $g_\alpha\psi_{\alpha,\alpha'} \in K^*_{\alpha'}$, since $K^*_{\alpha'} \cong K^*_\alpha$ by Lemma 6.2.19. Hence $(g_\alpha\psi_{\alpha,\alpha'})\psi_{\alpha',\beta} = g_\alpha\psi_{\alpha,\beta} = e_\beta$, a contradiction, and so $g_\alpha \in K^*_\alpha$. This completes the inductive step, and so K^*_α is closed under prime powers.

Suppose, seeking a contradiction, that $p \in o(I_\alpha) \cap o(K^*_\alpha)$ for some prime p. Then $I_\alpha \cup K^*_\alpha$ contains every element of G_α of order p by Lemma 6.2.5. Let $g_\alpha \in G_\alpha$ be of order p, so that if $\delta > \alpha$ then there exists $g_\delta \in G_\delta$ with $g_\delta\psi_{\delta,\alpha} = g_\alpha$. Suppose first that $o(g_\delta) = p^n m$ is finite, where $\mathrm{hcf}(p^n, m) = 1$. Then $g_\delta^m\psi_{\delta,\alpha} = g_\alpha^m$ has order p and g_δ^m has order p^n. Since K^*_δ is closed under prime order we have $g_\delta^m \in K^*_\delta$, a contradiction. It follows that the pre-images of elements of order p under the connecting morphisms are all of infinite order. Let $\delta > \tau > \alpha$ and let $h_\tau \in G_\tau$ be of order p. The map from $\langle e_\delta, g_\alpha\rangle_I$ to $\langle e_\delta, h_\tau\rangle_I$ which fixes e_δ and sends g_α to h_τ is an isomorphism by Lemma 6.2.16. By extending the isomorphism to an automorphism θ of S, we have that $g_\delta\theta = h_\delta \in G_\delta$ is such that $h_\delta\psi_{\delta,\tau} = h_\tau$, so that $o(h_\delta) = \aleph_0$ and $h_\delta \in K_{\delta,\alpha}$. Since $g_\delta^p \in K_{\delta,\alpha}$, both $\langle h_\delta, e_\alpha\rangle_I$ and $\langle g_\delta^p, e_\alpha\rangle_I$ are isomorphic to an infinite cyclic group with zero adjoined. Hence the isomorphism from $\langle h_\delta, e_\alpha\rangle_I$ and $\langle g_\delta^p, e_\alpha\rangle_I$ fixing e_α and sending h_δ to g_δ^p is an isomorphism, which we extend to an automorphism $[\theta_\alpha, \pi]_{\alpha\in Y}$ of S. Then $\delta = \delta\pi > \tau\pi > \alpha\pi = \alpha$ and, by the commutativity of $[\delta, \tau; \delta, \tau\pi]$,

$$h_\tau\theta_\tau = h_\delta\psi_{\delta,\tau}\theta_\tau = h_\delta\theta_\delta\psi_{\delta,\tau\pi} = g_\delta^p\psi_{\delta,\tau\pi}.$$

Hence $g_\delta^p\psi_{\delta,\tau\pi}$ is of order p, however $(g_\delta\psi_{\delta,\tau\pi})\psi_{\tau\pi,\alpha} = g_\alpha$, so that $g_\delta\psi_{\delta,\tau\pi}$ is of infinite order, and our desired contradiction is achieved.

Now suppose $x_\alpha \in I_\alpha \cap K_\alpha^*$ has order $n \in \mathbb{N}^*$. If n is finite, then there exists a prime p with $p|n$, and so $x_\alpha^{n/p} \in I_\alpha \cap K_\alpha^*$ has order p, a contradiction. If n is infinite then there exist $\delta > \alpha$ and $x_\delta \in G_\delta$ with $x_\delta\psi_{\delta,\alpha} = x_\alpha$, so that x_δ is of infinite order. Since the absolute kernels are pairwise isomorphic we have $\aleph_0 \in o(K_\sigma^*)$ for each $\sigma \in Y$. Hence K_δ^* contains every element of infinite order in G_δ by Lemma 6.2.5, and so $x_\delta\psi_{\delta,\alpha} = e_\alpha$, a contradiction. We have thus shown that I_α and K_α^* have trivial intersection.

We now prove that I_α is closed under prime powers. Let $p \in o(I_\alpha)$ for some prime p, and let $z_\alpha \in G_\alpha$ be of order p^r. If $o(z_\alpha\psi_{\alpha,\beta}) < p^r$ for all $\beta < \alpha$ then $z_\alpha^{p^{r-1}} \in I_\alpha \cap K_\alpha^*$, a contradiction. Hence there exists β with $z_\alpha\psi_{\alpha,\beta}$ of order p^r, so that $p^r \in o(I_\beta)$. By Lemma 6.2.19 $o(I_\beta) = o(I_\alpha)$ and so I_α is closed under prime powers.

Finally, let $a_\alpha \in G_\alpha$ be of prime order p. If $a_\alpha \notin K_\alpha^*$ then $a_\alpha\psi_{\alpha,\beta}$ has order p for some $\beta < \alpha$, and so by the usual argument $p \in o(I_\alpha)$, and the final result holds. □

Consequently, by corollary 6.2.8, the subgroups I_α and K_α^* are of coprime order. Furthermore, since I_α and K_α^* are characteristic subgroups of G_α, and in particular are invariant under inner automorphisms of G_α, they are normal subgroups. Hence $\langle I_\alpha, K_\alpha^*\rangle_I = I_\alpha \otimes K_\alpha^*$.

Lemma 6.2.21. *If G_α is periodic then $G_\alpha = I_\alpha \otimes K_\alpha^*$. If G_α is non-periodic then either $G_\alpha = I_\alpha$ or $G_\alpha = K_\alpha^*$ or $I_\alpha \otimes K_\alpha^*$ is the set of elements of finite order in G_α.*

Proof. If $g_\alpha \in G_\alpha$ has finite order $n = p_1^{n_1} \dots p_r^{n_r}$ for some primes p_i then, by the Fundamental Theorem of Finite Abelian Groups, $g_\alpha = g_{\alpha,1}g_{\alpha,2}\dots g_{\alpha,r}$ for some $g_{\alpha,i} \in G_\alpha$ of order $p_i^{n_i}$. By the previous corollary we have $g_{\alpha,i} \in I_\alpha \cup K_\alpha^*$, and so $g_\alpha \in I_\alpha \otimes K_\alpha^*$. Consequently, the subgroup $I_\alpha \otimes K_\alpha^*$ contains every element of G_α of finite order, and so if G_α is periodic then $G_\alpha = I_\alpha \otimes K_\alpha^*$.

Now suppose that G_α contains an element x_α of infinite order. Suppose first that $x_\alpha \in I_\alpha \otimes K_\alpha^*$, say $x_\alpha = g_\alpha k_\alpha$. Then either g_α or k_α has infinite order, as I_α and K_α^* are of coprime order. Hence, by Lemma 6.2.5, either I_α or K_α^* contains all elements of infinite order, and so $G_\alpha = I_\alpha \otimes K_\alpha^*$. If g_α is of infinite order, then for any $m_\alpha \in K_\alpha^*$ we have that $g_\alpha m_\alpha$ has infinite order. Hence g_α^{-1} and $g_\alpha m_\alpha$, being of infinite order, are in I_α, and consequently so is $g_\alpha^{-1}(g_\alpha m_\alpha) = m_\alpha$. Thus $m_\alpha = e_\alpha$, and it follows that $G_\alpha = I_\alpha$. The case where k_α is of infinite order is proven similarly.

If no such element x_α exists, then $I_\alpha \otimes K_\alpha$ is precisely the elements of finite order as required. □

We now extend our knowledge of the final case of Lemma 6.2.21, and in particular show that each maximal subgroup is the union of its kernel subgroups.

Lemma 6.2.22. *If G_α is non-periodic and $I_\alpha \otimes K_\alpha^*$ periodic, then the inverse subsemigroup $[Y; I_\alpha \otimes K_\alpha^*, \psi_{\alpha,\beta}|_{I_\alpha \otimes K_\alpha^*}]$ of S is a HIS. Moreover, the absolute image I_α^* of G_α is trivial and $G_\alpha = \bigcup_{\beta<\alpha} K_{\alpha,\beta}$.*

Proof. The first result is immediate from Lemma 6.2.7 since $I_\alpha \otimes K_\alpha^*$, being the subgroup containing all periodic elements of G_α, is order-characteristic.

We claim that each element x_α of infinite order in G_α is contained in the kernel of some connecting morphism. For any $\beta < \alpha$ we have that $x_\alpha \psi_{\alpha,\beta}$ has finite order, say n, since $\aleph_0 \notin o(I_\alpha)$. Hence x_α^n is an element of infinite order with $x_\alpha^n \in K_{\alpha,\beta}$. The claim easily follows by homogeneity.

Now suppose that $g_\alpha \in I_\alpha^*$. Then $x_\alpha g_\alpha$ has infinite order, since otherwise $x_\alpha g_\alpha$ is an element of $I_\alpha \otimes K_\alpha^*$, and thus so is $x_\alpha = (x_\alpha g_\alpha) g_\alpha^{-1}$. By the previous claim, $x_\alpha \in K_{\alpha,\beta}$ and $x_\alpha g_\alpha \in K_{\alpha,\gamma}$ for some $\beta, \gamma < \alpha$. Hence

$$(x_\alpha g_\alpha)\psi_{\alpha,\beta\gamma} = (x_\alpha \psi_{\alpha,\beta}\psi_{\beta,\beta\gamma})(g_\alpha \psi_{\alpha,\beta\gamma}) = g_\alpha \psi_{\alpha,\beta\gamma} = e_{\beta\gamma}$$

and so $g_\alpha = e_\alpha$. Hence I_α^* is trivial as required.

Finally, if there exists $z_\alpha \notin \bigcup_{\beta<\alpha} K_{\alpha,\beta}$ then $z_\alpha \psi_{\alpha,\beta} \in I_\alpha^*$ for some $\beta < \alpha$, a contradiction, and the final result is obtained. □

We call a Clifford semigroup in which each connecting morphism is surjective a *surjective Clifford semigroup.*

Corollary 6.2.23. *Let $T = [Y; H_\alpha; \phi_{\alpha,\beta}]$ be a surjective Clifford HIS. Then the absolute kernels of T are trivial.*

Proof. Immediate from Lemma 6.2.20. □

Lemma 6.2.24. *The inverse subsemigroup $[Y; I_\alpha; \psi_{\alpha,\beta}^I]$ of S is a surjective Clifford semigroup, where $\psi_{\alpha,\beta}^I = \psi_{\alpha,\beta}|_{I_\alpha}$.*

Proof. By definition $I_\alpha\psi_{\alpha,\beta} \subseteq I_\beta$. Let $x_\beta \in I_\beta = \text{Im } \psi_{\alpha,\beta}$ (by Lemma 6.2.17). If G_α is periodic then by Lemma 6.2.21 there exist $g_\alpha \in I_\alpha$ and $k_\alpha \in K_\alpha^*$ such that $(g_\alpha k_\alpha)\psi_{\alpha,\beta} = x_\beta$. Hence $g_\alpha\psi_{\alpha,\beta} = x_\beta$ and $\psi^I_{\alpha,\beta}$ is surjective. If G_α is non-periodic, then the result is trivially true when $G_\alpha = I_\alpha$, or when $G_\alpha = K_\alpha^*$, since in this case $I_\alpha = \{e_\alpha\}$. If $\aleph_0 \notin o(I_\alpha) \cup o(K_\alpha^*)$, then as $[Y; I_\alpha \otimes K_\alpha^*; \psi_{\alpha,\beta}|_{I_\alpha \otimes K_\alpha^*}]$ forms a periodic HIS by the previous lemma, the result follows by the periodic case. □

For each $\alpha > \beta$ in Y, we let $\tau_{\alpha,\beta}$ denote the trivial morphism from K_α^* to K_β^*, and let $\tau_{\alpha,\alpha} = 1_{K_\alpha^*}$. We call a Clifford semigroup in which each connecting morphism is trivial an *image-trivial Clifford semigroup.* Note that this differs from the image-trivial normal band case, since here the images of a connecting morphism $\psi_{\alpha,\beta}$ for $\alpha > \beta$ has to be $\{e_\beta\}$.

We observe that, for each $\alpha \geq \beta$, $g_\alpha \in I_\alpha$ and $k_\alpha \in K_\alpha^*$ then

$$(g_\alpha k_\alpha)\psi_{\alpha,\beta} = (g_\alpha\psi^I_{\alpha,\beta})(k_\alpha\tau_{\alpha,\beta}).$$

Hence S has two crucial inverse subsemigroups,

$$I(S) := [Y; I_\alpha; \psi^I_{\alpha,\beta}] \text{ and } K(S) := [Y; K_\alpha^*; \tau_{\alpha,\beta}],$$

which are HIS by Lemma 6.2.7. If in addition S is periodic, then

$$S = [Y; I_\alpha \otimes K_\alpha; \psi^I_{\alpha,\beta} \otimes \tau_{\alpha,\beta}].$$

We summarise our current findings in this section as follows.

Theorem 6.2.25. *If S is a periodic Clifford HIS, then $S = [Y; I_\alpha \otimes K_\alpha^*; \psi^I_{\alpha,\beta} \otimes \tau_{\alpha,\beta}]$, where:*

(i) *Y is a homogeneous semilattice;*

(ii) *$I(S) = [Y; I_\alpha; \psi^I_{\alpha,\beta}]$ is a surjective Clifford HIS;*

(iii) *$K(S) = [Y; K_\alpha^*; \tau_{\alpha,\beta}]$ is an image-trivial Clifford HIS;*

(iv) *there exists a homogeneous group $G = I \otimes K^*$ where I and K^* are of coprime order, such that $G \cong G_\alpha$, $I_\alpha \cong I$ and $K_\alpha^* \cong K^*$ for all $\alpha \in Y$.*

A non-periodic Clifford semigroup is a HIS if and only if it is isomorphic to either a surjective Clifford HIS, or an image-trivial Clifford HIS, or a Clifford HIS with no elements of infinite order lying in the images or absolute kernels.

In the next subsection we shall prove a converse to the first result of Theorem 6.2.25. This relies on proving the stronger property of homogeneity for image-trivial Clifford semigroups.

Lemma 6.2.26. *Let $T = [Y; G_\alpha; \tau_{\alpha,\beta}]$ be an image-trivial Clifford semigroup. Then the following are equivalent:*

(i) *T is a structure-HIS;*

(ii) *T is a HIS;*

(iii) *Y is homogeneous and there exists a homogeneous group G such that $G \cong G_\alpha$ for all $\alpha \in Y$.*

Proof. (i) $\Rightarrow$ (ii) Immediate, as every structure-HIS is a HIS.

(ii) $\Rightarrow$ (iii) Immediate from Corollary 6.2.4.

(iii) $\Rightarrow$ (i) Let $A_1 = [Z'; A'_\gamma; \tau'_{\gamma,\delta}]$ and $A_2 = [Z''; A''_\eta; \tau''_{\eta,\sigma}]$ be a pair of f.g. inverse subsemigroups of T, where the maps $\tau'_{\gamma,\delta}$ and $\tau''_{\eta,\sigma}$, being restrictions of trivial morphisms, are trivial. Let $\theta = [\theta_\gamma, \pi]_{\gamma \in Z_1}$ be an isomorphism from A_1 to A_2 and let $\bar{\pi}$ be an automorphism of Y which extends π. By Lemma 4.3.7 we can extend each $\theta_\gamma : A'_\gamma \to A''_{\gamma\pi}$ to an isomorphism $\bar{\theta}_\gamma : G_\gamma \to G_{\gamma\pi}$. For each $\alpha \notin Z_1$, let $\bar{\theta}_\alpha$ be any isomorphism from G_α to $G_{\alpha\bar{\pi}}$. We claim that $\bar{\theta} = [\bar{\theta}_\alpha, \bar{\pi}]_{\alpha \in Y}$ is an automorphism of S. For any $g_\alpha \in G_\alpha$ and $\alpha > \beta$ we have

$$g_\alpha \bar{\theta}_\alpha \tau_{\alpha\bar{\pi},\beta\bar{\pi}} = e_{\beta\bar{\pi}} = e_\beta \bar{\theta}_\beta = g_\alpha \tau_{\alpha,\beta} \bar{\theta}_\beta$$

and so the diagram $[\alpha, \beta; \alpha\bar{\pi}, \beta\bar{\pi}]$ commutes, thus proving the claim. By construction, $\bar{\pi}$ extends π, and so $\bar{\theta}$ extends θ. Hence T is a structure-HIS. $\square$

Let $T = [Y; G_\alpha; \tau_{\alpha,\beta}]$ be an image-trivial Clifford semigroup such that $G_\alpha \cong G$ for each $\alpha \in Y$. Let $\theta_\alpha : G_\alpha \to G$ be an isomorphism for each $\alpha \in Y$, and define a bijection $\theta : T \to Y \times G$ by

$$g_\alpha \theta = (\alpha, g_\alpha \theta_\alpha),$$

for each $g_\alpha \in G_\alpha$, $\alpha \in Y$. Then we use θ to endow the set $Y \times G$ with a multiplication

$$(\alpha, g) * (\beta, h) = \begin{cases} (\alpha, gh) & \text{if } \alpha = \beta, \\ (\beta, h) & \text{if } \alpha > \beta, \\ (\alpha, g) & \text{if } \alpha < \beta, \\ (\alpha\beta, 1) & \text{if } \alpha \perp \beta. \end{cases}$$

We denote the resulting semigroup $(Y \times G, *)$ as $[Y; G]$. Notice that

$$[Y; G] = [Y; \bar{G}_\alpha; \bar{\tau}_{\alpha,\beta}],$$

where $\bar{G}_\alpha = \{(\alpha, g) : g \in G\}$ and $\bar{G}_\alpha \bar{\tau}_{\alpha,\beta} = \{(\beta, 1)\}$. We have thus shown that:

Lemma 6.2.27. *Let $T = [Y; G_\alpha; \tau_{\alpha,\beta}]$ be an image-trivial Clifford semigroup such that $G_\alpha \cong G$ for each $\alpha \in Y$. Then $T \cong [Y; G]$.*

6.2.1 Spined product

In the previous chapter, it was often useful to decompose a normal band as a spined product. In this section we use the spined product decomposition together with Lemma 6.2.27 to give a more succinct form of a periodic Clifford HIS semigroup $[Y; I_\alpha \otimes K_\alpha^*; \psi_{\alpha,\beta}^I \otimes \tau_{\alpha,\beta}]$.

Let $S_i = [Y; G_\alpha^{(i)}; \psi_{\alpha,\beta}^{(i)}]$ $(i = 1, 2)$ be a pair of Clifford semigroups. Then we recall that the spined product of S_1 and S_2 w.r.t to Y is

$$S_1 \bowtie S_2 = \{(a_\alpha, b_\alpha) : a_\alpha \in S_1, b_\alpha \in S_2, \alpha \in Y\}.$$

Lemma 6.2.28. *Let S_1 and S_2 be defined as above. Then $S_1 \bowtie S_2$ is isomorphic to the Clifford semigroup $S = [Y; G_\alpha^{(1)} \otimes G_\alpha^{(2)}; \psi_{\alpha,\beta}^{(1)} \otimes \psi_{\alpha,\beta}^{(2)}]$.*

Proof. The map $\phi : S_1 \bowtie S_2 \to S$ given by

$$(g_\alpha, h_\alpha)\phi = g_\alpha h_\alpha$$

for each $(g_\alpha, h_\alpha) \in G_\alpha^{(1)} \otimes G_\alpha^{(2)}$ is clearly an isomorphism. □

Note 6.2.29. *Since Clifford semigroups are completely regular, we may build isomorphisms between strong semilattices of Clifford semigroups by using Proposition 2.11.3, which we briefly recap. Let $S_i = [Y; G_\alpha^{(i)}; \psi_{\alpha,\beta}^{(i)}]$ and $S_i' = [Y'; H_{\alpha'}^{(i)}; \phi_{\alpha',\beta'}^{(i)}]$ be Clifford semigroups $(i = 1, 2)$ and consider a pair of isomorphisms*

$$\theta^{(i)} = [\theta_\alpha^{(i)}, \pi]_{\alpha \in Y} : S_i \to S_i' \quad (i = 1, 2).$$

Then the map $\theta : S_1 \bowtie S_2 \to S_1' \bowtie S_2'$ given by $(g_\alpha, h_\alpha)\theta = (g_\alpha \theta_\alpha^{(1)}, h_\alpha \theta_\alpha^{(2)})$ is an isomorphism, which we denote as $\theta^{(1)} \bowtie \theta^{(2)}$.

We now construct the Clifford semigroup analogy of corollary 5.3.2 as follows.

Corollary 6.2.30. *Let $S = S_1 \bowtie S_2$ and $S' = S_1' \bowtie S_2'$ be a pair of spined products of Clifford semigroups such that S_2 and S_2' are structure-HISs. Then $S \cong S'$ if $S_1 \cong S_1'$ and $S_2 \cong S_2'$.*

Proof. Let S have structure semilattice Y. Let $\theta^{(1)} = [\theta_\alpha^{(1)}, \pi]_{\alpha \in Y}$ be an isomorphism from S_1 to S_1' and $\theta^{(2)} = [\theta_\alpha^{(2)}, \hat{\pi}]_{\alpha \in Y}$ an isomorphism from S_2 to S_2'. Then $\pi\hat{\pi}^{-1}$ is an automorphism of Y, and so as S_2 is a structure-HIS there exists an automorphism ϕ of S_2 with induced automorphism $\pi\hat{\pi}^{-1}$. Hence $\phi\theta^{(2)} : S_2 \to S_2'$ is an isomorphism, with induced isomorphism π, and so from the note above $\theta^{(1)} \bowtie (\phi\theta^{(2)})$ is an isomorphism from S to S' as required. □

However, unlike Corollary 5.3.2, the converse of the corollary does not hold in general. Indeed, since homogeneous groups are trivially structure-HISs, counterexamples easily arise by taking S and S' to be certain homogeneous groups (for example, take both S and S' to be the infinite direct sum of a cyclic group).

If $S = [Y; I_\alpha \otimes K^*_\alpha; \psi^I_{\alpha,\beta} \otimes \tau_{\alpha,\beta}]$ is a periodic HIS then, by Lemma 6.2.28, S is isomorphic to $[Y; I_\alpha; \psi^I_{\alpha,\beta}] \bowtie [Y; K^*_\alpha; \tau_{\alpha,\beta}]$ and thus, by Corollary 6.2.30 and the structure-homogeneity of $[Y; K^*]$, to $[Y; I_\alpha; \psi^I_{\alpha,\beta}] \bowtie [Y; K^*]$, where $K^* \cong K^*_\alpha$.

We have proven the forward half of the periodic case of the following theorem.

Theorem 6.2.31. *A periodic Clifford semigroup S is a HIS if and only if there exist a homogeneous group $G = I \otimes K^*$, where I and K^* are of coprime order, and a surjective Clifford HIS $[Y; I_\alpha; \psi^I_{\alpha,\beta}]$ with $I_\alpha \cong I$ such that*

$$S \cong [Y; I_\alpha; \psi^I_{\alpha,\beta}] \bowtie [Y; K^*].$$

A non-periodic Clifford semigroup S is a HIS if and only if S is isomorphic to either a surjective Clifford HIS, or $[Y; G]$ for some homogeneous semilattice Y and group G, or a Clifford HIS with no elements of infinite order lying in the images or absolute kernels.

Proof. Let G and Y be as in the hypothesis of the theorem. Then as $[Y; K^*]$ is structure-HIS semigroup by Lemma 6.2.26, the semigroup $[Y; I_\alpha; \psi^I_{\alpha,\beta}] \bowtie [Y; K^*]$ is a HIS if and only if $[Y; I_\alpha; \psi^I_{\alpha,\beta}]$ is a HIS by Proposition 6.2.14. The non-periodic case follows immediately from Theorem 6.2.25 and Lemma 6.2.26. □

It thus suffices to consider the homogeneity of both surjective Clifford semigroups with trivial absolute kernels, and the case where there exist elements of infinite order lying outside the images and absolute kernels.

An immediate consequence of Theorem 6.2.31 is the following equivalence to a Clifford HIS being surjective.

6.2.2 A pair of classifications

We examine the case where $S = [Y; S_\alpha; \psi_{\alpha,\beta}]$ is a HIS such that there exist $\alpha > \beta$ in Y with $\psi_{\alpha,\beta}$ an isomorphism. By Lemma 5.4.4 every connecting morphism is an isomorphism, and so S is isomorphic to $Y \times G$ for some group G by Proposition 3.7.13. The following result is then immediate from Proposition 4.3.8 and Corollary 6.1.3.

Proposition 6.2.32. *Let Y be a semilattice and G be a group. Then $Y \times G$ is structure-homogeneous if and only if Y and G are homogeneous.*

This result has a number of useful consequences. First, since all surjective morphisms between finite groups are isomorphisms, the following theorem is immediate by Proposition 6.1.7 and Theorem 6.2.31.

Theorem 6.2.33. *Given a homogeneous semilattice Y and a pair of finite homogeneous groups I and K^* of coprime orders, the Clifford semigroup $(Y \times I) \bowtie [Y; K^*]$ is a HIS. Conversely, every HIS with finite maximal subgroups is isomorphic to an inverse semigroup constructed in this way.*

A complete classification of all structure-HIS Clifford semigroups can also be obtained.

Theorem 6.2.34. *A periodic Clifford semigroup S is structure-HIS if and only if there exists a homogeneous group $G = I \otimes K^*$, where I and K^* are of coprime order, such that*

$$S \cong (Y \times I) \bowtie [Y; K^*].$$

A non-periodic Clifford semigroup S is a structure-HIS if and only if S is isomorphic to either $Y \times G$ or $[Y; G]$ for some homogeneous semilattice Y and group G.

Proof. Let $S = [Y; G_\alpha; \psi_{\alpha,\beta}]$ be a structure-HIS Clifford semigroup. Suppose that there exist $\alpha > \gamma > \beta$ in Y and $a_\alpha \in G_\alpha$ such that $a_\alpha \in K_{\alpha,\beta} \setminus K_{\alpha,\gamma}$. By the homogeneity of Y there exists $\pi \in \mathrm{Aut}(Y)$ extending the unique isomorphism between $\{\alpha, \gamma\}$ and $\{\alpha, \beta\}$. Since S is structure-HIS and π extends the identity isomorphism $\{\alpha\} \to \{\alpha\}$, we extend the identity isomorphism fixing $\langle a_\alpha \rangle_I$ to an automorphism $[\theta_\alpha, \pi]_{\alpha \in Y}$ of S. Then as $[\alpha, \gamma; \alpha, \gamma]$ commutes we have

$$a_\alpha \psi_{\alpha,\gamma} \theta_\gamma = a_\alpha \theta_\alpha \psi_{\alpha,\beta} = a_\alpha \psi_{\alpha,\beta} = e_\beta$$

and so $a_\alpha \in K_{\alpha,\gamma}$, a contradiction. Hence $K_{\alpha,\beta} = K_{\alpha,\gamma}$. If $\beta, \tau < \alpha$ then we thus have

$$K_{\alpha,\beta} = K_{\alpha,\beta\tau} = K_{\alpha,\tau}$$

and so $K_{\alpha,\beta} = K^*_\alpha$ for any $\beta < \alpha$.

Suppose first that S is periodic. If S is surjective, then each $K_{\alpha,\beta}$ is trivial by Corollary 6.2.23, so that $\psi_{\alpha,\beta}$ is an isomorphism. The periodic case then follows from Theorem 6.2.31. If S is non-periodic, then the third possibility of the non-periodic case of Theorem 6.2.31 cannot hold by Lemmas 6.2.21 and 6.2.22.

Conversely, if Y is a homogeneous semilattice and G is a homogeneous group then $Y \times G$ and $[Y; G]$ are structure-HISs by Proposition 6.2.32 and Lemma 6.2.26. Hence if I and K^* are homogeneous groups of coprime order then $(Y \times I) \bowtie [Y; K^*]$ is a structure-HIS by Proposition 6.2.13. $\square$

6.2.3 Non-injective surjective Clifford semigroups

Throughout this subsection we let $S = [Y; G_\alpha; \psi_{\alpha,\beta}]$ be a surjective Clifford HIS such that each $\psi_{\alpha,\beta}$ is non-injective. Recall that the absolute kernels of S are trivial by Corollary 6.2.23. Following in line with the general case, we attempt to decompose the maximal subgroups into direct products of characteristic subgroups.

The group G_α contains two key subsets: the absolute image I_α^* and

$$T_\alpha = \{g_\alpha \in G_\alpha : \exists \beta < \alpha \text{ such that } g_\alpha \in K_{\alpha,\beta}\} = \bigcup_{\beta<\alpha} K_{\alpha,\beta}.$$

The set T_α forms a subgroup of G_α, since if $k_\alpha \in K_{\alpha,\beta}$ and $m_\alpha \in K_{\alpha,\beta'}$ then $k_\alpha m_\alpha \in K_{\alpha,\beta\beta'}$. While I_α^* may not form a subgroup, it is closed under powers, since if $g_\alpha \in I_\alpha^*$ then for each $r \in \mathbb{N}$ and $\beta \leq \alpha$,

$$o(g_\alpha) = o(g_\alpha\psi_{\alpha,\beta}) \Rightarrow o(g_\alpha^r) = o((g_\alpha\psi_{\alpha,\beta})^r) = o(g_\alpha^r\psi_{\alpha,\beta})$$

so that $g_\alpha^r \in I_\alpha^*$.

By the usual arguments we have:

Lemma 6.2.35. *For each $\alpha \in Y$, T_α is a characteristic subgroup of G_α and I_α^* is a characteristic subset of G_α with $I_\alpha^* \cap T_\alpha = \{e_\alpha\}$. Moreover, $T_\alpha \cong T_\beta$ and $\langle I_\alpha^* \rangle_I \cong \langle I_\beta^* \rangle_I$ for each $\alpha, \beta \in Y$.*

Consequently, $o(I_\alpha^*) = o(I_\beta^*)$ for each $\alpha, \beta \in Y$ and $\langle I_\alpha^* \rangle_I$ is a characteristic subgroup of G_α. Moreover, I_α^* and T_α are coprime by Corollary 6.2.8. We fix the following subsets of S.

$$A(S) := [Y; \langle I_\alpha^* \rangle_I; \psi_{\alpha,\beta}^I] \text{ and } T(S) := [Y; T_\alpha; \psi_{\alpha,\beta}^T],$$

where $\psi_{\alpha,\beta}^I = \psi_{\alpha,\beta}|_{\langle I_\alpha^* \rangle_I}$ and $\psi_{\alpha,\beta}^T = \psi_{\alpha,\beta}|_{T_\alpha}$.

Lemma 6.2.36. *For each $\alpha \in Y$, the subsets $A(S)$ and $T(S)$ of S are Clifford HISs. Moreover, if S is periodic then $A(S)$ and $T(S)$ are surjective.*

Proof. To prove that $A(S)$ and $T(S)$ are inverse subsemigroups, it suffices to show that $\psi_{\alpha,\beta}^T$ and $\psi_{\alpha,\beta}^I$ map to T_β and $\langle I_\beta^* \rangle_I$, respectively. If $k_\alpha \in T_\alpha$, say, $k_\alpha \in K_{\alpha,\gamma}$, then $k_\alpha\psi_{\alpha,\beta} \in K_{\beta,\beta\gamma} \subseteq T_\beta$. If $g_\alpha \in I_\alpha^*$ then as $o(g_\alpha) = o(g_\alpha\psi_{\alpha,\beta})$ we have that $g_\alpha\psi_{\alpha,\beta} \in I_\beta^*$ by Lemma 6.2.5, and so $\langle I_\alpha^* \rangle_I \psi_{\alpha,\beta}^I \subseteq \langle I_\beta^* \rangle_I$ as required. Hence $A(S)$ and $T(S)$ are inverse subsemigroups and, by Lemma 6.2.7, are HISs.

Finally, as G_α has trivial absolute kernel, and do $\langle I_\alpha^* \rangle_I$ and T_α. Hence as S is periodic then it follows from Theorem 6.2.31 that $A(S)$ and $T(S)$ are surjective. $\square$

Lemma 6.2.37. *If $\alpha > \beta > \gamma$ then $K_{\alpha,\beta} \subsetneq K_{\alpha,\gamma}$.*

Proof. If $K_{\alpha,\beta} = K_{\alpha,\gamma}$, then it follows by a simple application of homogeneity that $K_{\alpha,\beta} = K_{\alpha,\beta'}$ for all $\beta, \beta' < \alpha$. Hence $T_\alpha = K_{\alpha,\beta}$ is the absolute kernel of G_α which, being trivial, implies that each connecting morphism is injective, a contradiction. □

Lemma 6.2.38. *For each $\alpha \in Y$ we have $G_\alpha = T_\alpha I^*_\alpha$. Consequently, if I^*_α forms a subgroup then $G_\alpha = T_\alpha \otimes I^*_\alpha$, and if in addition G_α is non-periodic, then I^*_α is trivial, so that $G_\alpha = T_\alpha$.*

Proof. Let $a_\alpha \in G_\alpha \setminus (T_\alpha \cup I^*_\alpha)$ have finite order n. Then there exists $\beta < \alpha$ such that $a_\alpha \psi_{\alpha,\beta} = a_\beta$ has order $m < n$, say, $n = mk$. We choose β so that a_β is of minimal order, noting that $a_\beta \neq e_\beta$ as $a_\alpha \notin T_\alpha$. Then $a_\beta \in I^*_\beta$, since if $o(a_\beta \psi_{\beta,\gamma}) < m$ for some $\gamma < \beta$ then $o(a_\alpha \psi_{\alpha,\gamma}) < m$, contradicting the minimality of m. Since $o(I^*_\alpha) = o(I^*_\beta)$, it follows from Lemma 6.2.5 that a^k_α, being of order m, is in I^*_α. Moreover, $a^m_\alpha \psi_{\alpha,\beta} = a^m_\beta = e_\beta$, so that $a^m_\alpha \in T_\alpha$. Hence as T_α is characteristic by Lemma 6.2.35, T_α contains all elements of order k by Lemma 6.2.5. Since I^*_α and T_α are coprime, there exist $r, s \in \mathbb{Z}$ such that $rm + sk = 1$, and so

$$a_\alpha = a^{rm+sk}_\alpha = (a^m_\alpha)^r (a^k_\alpha)^s \in T_\alpha I^*_\alpha.$$

Now let b_α be an element of infinite order. If there exists β such that $a_\alpha \psi_{\alpha,\beta}$ has finite order n then $a^n_\alpha \in T_\alpha$ and so T_α contains all elements of infinite order. Otherwise, no such β exists, so that $b_\alpha \in I^*_\alpha$ and I^*_α contains all elements of infinite order. Hence $G_\alpha = T_\alpha I^*_\alpha$. Now suppose I^*_α forms a subgroup. Then as I^*_α and T_α are trivial intersecting characteristic, and thus normal, subgroups of G_α and $G_\alpha = T_\alpha I^*_\alpha$, it follows that $G_\alpha = T_\alpha \otimes I^*_\alpha$. If G_α is non-periodic, then we have shown that every element of G_α of infinite order lies in $T_\alpha \cup I^*_\alpha$. By a similar argument to the proof of Lemma 6.2.21 we have that G_α equals T_α or I^*_α. Since the connecting morphisms are non-injective we thus have $G_\alpha = T_\alpha$. □

Lemma 6.2.39. *The subset I^*_α is closed under prime powers. Moreover, I^*_α forms a subgroup if and only if $\langle I^*_\alpha \rangle_I$ and T_α intersect trivially.*

Proof. Suppose $p \in o(I^*_\alpha)$ for some prime p, and let $g_\alpha \in G_\alpha$ be of order p^r. Then $g^{p^{r-1}}_\alpha$ has order p, and thus is an element of I^*_α by Lemma 6.2.5. Hence $(g_\alpha \psi_{\alpha,\beta})^{p^{r-1}} = g^{p^{r-1}}_\alpha \psi_{\alpha,\beta}$ is of order p for any $\beta < \alpha$, and $(g_\alpha \psi_{\alpha,\beta})^{p^r} = e_\beta$, so that $o(g_\alpha \psi_{\alpha,\beta}) = p^r = o(g_\alpha)$. We thus have that $g_\alpha \in I^*_\alpha$, and so I^*_α is closed under prime powers

Now suppose $\langle I^*_\alpha \rangle_I \cap T_\alpha = \{e_\alpha\}$ and let $g_\alpha, h_\alpha \in I^*_\alpha$. If $(g_\alpha h^{-1}_\alpha)\psi_{\alpha,\beta}$ has finite order m for some $\beta < \alpha$ then $(g_\alpha h^{-1}_\alpha)^m \in K_{\alpha,\beta} \subseteq T_\alpha$. However as $(g_\alpha h^{-1}_\alpha)^m \in \langle I^*_\alpha \rangle_I$, this forces $(g_\alpha h^{-1}_\alpha)^m = e_\alpha$. It follows that $g_\alpha h^{-1}_\alpha$ has order m, since its order is at least the order of its image, and so $g_\alpha h^{-1}_\alpha \in I^*_\alpha$. On the other hand, if

$(g_\alpha h_\alpha^{-1})\psi_{\alpha,\beta}$ has infinite order for all $\beta < \alpha$, then $g_\alpha h_\alpha^{-1}$ is also of infinite order, and so $g_\alpha h_\alpha^{-1} \in I_\alpha^*$. The converse is immediate from Lemma 6.2.35. □

This lemma points towards a positive answer to the following question.

Open Problem 6. Are the absolute images of a surjective Clifford HIS with non-injective connecting morphisms necessarily subgroups?

Lemma 6.2.40. *If the absolute images of S form subgroups isomorphic to I^* then*

$$S \cong T(S) \bowtie (Y \times I^*),$$

where if S is non-periodic then I^* is trivial.

Proof. Suppose first that S is periodic. Since I_α^* is a subgroup, the set $[Y; I_\alpha^*; \psi_{\alpha,\beta}^I]$ is the surjective Clifford subsemigroup $A(S)$ of S by Lemma 6.2.36. By Lemma 6.2.38 we have $G_\alpha = T_\alpha \otimes I_\alpha^*$, where T_α and I_α^* are coprime, so that $\psi_{\alpha,\beta} = \psi_{\alpha,\beta}^T \otimes \psi_{\alpha,\beta}^I$ by Lemma 6.2.10. Hence $S \cong T(S) \bowtie I(S)$ by Lemma 6.2.28. By Lemma 6.2.35 we have $I_\alpha^* \cap T_\alpha = \{e_\alpha\}$, and so $I_\alpha^* \cap \operatorname{Ker} \psi_{\alpha,\beta}^I = \{e_\alpha\}$. Each connecting morphism $\psi_{\alpha,\beta}^I$ is therefore injective, and thus an isomorphism. Hence $[Y; I_\alpha; \psi_{\alpha,\beta}^I] \cong Y \times I^*$ by Proposition 3.7.13, where $I^* \cong I_\alpha^*$ for any $\alpha \in Y$. The first result then follows from Corollary 6.2.30 and the fact that $Y \times I^*$ is a structure-HIS by Proposition 4.3.8.

The final result is immediate from Lemma 6.2.38. □

We have proven the first half of the following theorem. The converse holds by Proposition 6.2.14, since the inverse semigroup $Y \times I^*$ is a structure-HIS if Y and I^* are homogeneous.

Theorem 6.2.41. *Let S be a surjective Clifford semigroup such that each absolute image forms a subgroup and the connecting morphisms are surjective but not injective. Then S is a HIS if and only if there exist a homogeneous semilattice Y, a homogeneous group $G = T \otimes I^*$ where T and I^* are of coprime order if G is periodic, or I^* is trivial otherwise, such that S is isomorphic to*

$$[Y; T_\alpha; \psi_{\alpha,\beta}^T] \bowtie (Y \times I^*)$$

where $T_\alpha \cong T$ for each $\alpha \in Y$ and $[Y; T_\alpha; \psi_{\alpha,\beta}^T]$ is a surjective Clifford HIS with T_α being the union of the kernels, none of which are equal.

In the case when the absolute images form subgroups, it consequently suffices to consider the homogeneity of a surjective Clifford semigroup $[Y; T_\alpha; \psi_{\alpha,\beta}]$, with Y and T_α homogeneous, and T_α being a (dense) union of the kernels of the connecting morphisms, none of which are equal. This leads to the following open problem.

Open Problem 7. Which homogeneous groups are a dense union of isomorphic normal subgroups?

A group is *co-Hopfian* if it is not isomorphic to a proper subgroup[1]. This is equivalent to every injective endomorphism being an automorphism [81]. Dually, a group is *Hopfian* if it is not isomorphic a proper quotient or, equivalently, if every surjective endomorphism is an automorphism. An immediate consequence of the following lemma is that T_α is both non Hopfian and non co-Hopfian:

Lemma 6.2.42. *Let $[Z; H_\alpha; \phi_{\alpha,\beta}]$ be a HIS with each connecting morphism surjective but not injective and such that $H_\alpha = \bigcup_{\beta<\alpha}$ Ker $\phi_{\alpha,\beta}$ for each $\alpha \in Z$. Then H_α is non Hopfian and non co-Hopfian, with $H_\alpha \cong$ Ker $\phi_{\alpha,\beta}$.*

Proof. For each $\alpha > \beta$, let $K_{\alpha,\beta} =$ Ker $\phi_{\alpha,\beta}$, noting that $K_{\alpha,\beta}$ is homogeneous by Lemma 6.2.18. We claim that age($K_{\alpha,\beta}$)=age(H_α), so that $K_{\alpha,\beta} \cong H_\alpha$ by Fraïssé's Theorem. Because $K_{\alpha,\beta}$ is a subgroup of H_α we have that age($K_{\alpha,\beta}$) is a subclass of age(H_α). Let $A \in$ age(H_α). Then there exists a f.g. inverse subsemigroup $A' = \langle g_{\alpha,1}, \dots, g_{\alpha,n}\rangle_I$ of H_α isomorphic to A. For each $1 \leq i \leq n$, there exists $\beta_i < \alpha$ such that $g_{\alpha,i} \in K_{\alpha,\beta_i}$. Letting $\gamma = \beta_1\beta_2\cdots\beta_n$, then $g_{\alpha,i} \in K_{\alpha,\gamma}$ for all i and so $A' \subseteq K_{\alpha,\gamma}$. Hence, as $K_{\alpha,\beta} \cong K_{\alpha,\gamma}$ by Lemma 6.2.19 we have $A' \in$ age($K_{\alpha,\beta}$). Since the age of a structure is closed under isomorphism, we have $A \in$ age($K_{\alpha,\beta}$), thus completing the proof of the claim, and so H_α is non co-Hopfian.

By Corollary 4.4.10, there exists an isomorphism $\theta : H_\alpha \to H_\beta$. The endomorphism of H_α given by $\phi_{\alpha,\beta}\theta^{-1}$ is a surjective non-automorphism, and thus H_α is non Hopfian. □

6.3 Homogeneity of commutative inverse semigroups

Given that a full classification of homogeneous abelian groups is known, it is natural to examine an extension of this to commutative inverse semigroups. As an immediate consequence of [55, Theorem 4.2.1], commutative inverse semigroups are Clifford, and as such we may use the results of the previous sections to attempt to classify commutative HIS. For consistency with earlier work, we continue with the multiplicative notation, so that the operation is denoted by juxtaposition.

By Theorem 6.2.31 it suffices to consider the homogeneity of either surjective Clifford semigroups or non-periodic Clifford semigroups with elements of infinite order not lying in the images or absolute-kernels of the maximal subgroups. We first give an overview of homogeneous abelian groups, and consider when such groups are (co-)Hopfian.

[1] Non co-Hopfian groups are also known as I-groups.

Given a prime p, the Prüfer group $\mathbb{Z}[p^\infty]$ is an abelian p-group with presentation

$$\langle g_1, g_2, g_3, \dots : g_1^p = 1, g_2^p = g_1, g_3^p = g_2, \dots \rangle_I.$$

Alternatively, $\mathbb{Z}[p^\infty]$ can be thought of as a union of a chain of cyclic p-groups of orders $p, p^2, p^3, \dots$, so that $o(\mathbb{Z}[p^\infty])$ is the set of all powers of p. Each Prüfer group is divisible, that is, for each $g \in \mathbb{Z}[p^\infty]$ and $n \in \mathbb{N}$, there exists $h \in \mathbb{Z}[p^\infty]$ such that $h^n = g$. The Prüfer groups, along with $\mathbb{Q}$, form the building blocks for all divisible abelian groups. We refer the reader to Robinson's book [82] for an in depth study of divisible groups.

By [17, Theorem 2], an abelian group is homogeneous if and only if its isomorphic to some

$$G = \begin{cases} (\bigoplus_{p\in P_1} \mathbb{Z}_{p^{m_p}}^{n_p}) \oplus (\bigoplus_{p\in P_2} \mathbb{Z}[p^\infty]^{n_p}) & \text{if } G \text{ is periodic,} \\ (\bigoplus_{p\in \mathbb{P}} \mathbb{Z}[p^\infty]^{n_p}) \oplus (\mathbb{Q}^n) & \text{otherwise,} \end{cases} \tag{6.3}$$

where P_1 and P_2 partition the set $\mathbb{P}$ of primes, $n_p, n \in \mathbb{N}^* \cup \{0\}$ and $m_p \in \mathbb{N}$. For example, if $n = \aleph_0 = n_p$ for each $p \in \mathbb{P}$ then G is the *universal abelian group*, that is the homogeneous abelian group in which every f.g. abelian group embeds.

Note that the groups $\mathbb{Z}_{p^{m_p}}, \mathbb{Z}[p^\infty]$ and $\mathbb{Q}$ are *indecomposable*, that is, they are not isomorphic to a direct sum of two non-trivial groups (again see [82]).

It follows by the work in [6] that the group G is co-Hopfian if and only if n and n_p are finite, for all $p \in \mathbb{P}$. We call G *component-wise non co-Hopfian* if $n, n_p \in \{0, \aleph_0\}$ for each p. That is, G is component-wise non co-Hopfian if and only if each of its non-trivial p-components are non co-Hopfian and $n \in \{0, \aleph_0\}$.

Let H be an abelian group with subset $A = \{h_i : i \in I\}$ for some index set I. We call A a *disjoint subset* if $\langle A \rangle_I = \bigoplus_{i\in I} \langle h_i \rangle_I$, or equivalently, if $\langle h_i \rangle_I$ and $\langle A \setminus \{h_i\} \rangle_I$ have trivial intersection for each $i \in I$. Note that if $\{g, h\}$ form a disjoint subset of H then $o(gh) = \text{lcm}\ (o(g), o(h))$, where we define $\text{lcm}(\aleph_0, n) = \aleph_0$ for all $n \in \mathbb{N}^*$.

For example, if $H = Z^n$, where $n \in \mathbb{N}^*$ and Z is either a finite cyclic p-group, a Prüfer group or $\mathbb{Q}$, then a maximal disjoint subset of H is of size n since Z is indecomposable.

6.3.1 Surjective commutative inverse semigroups

Throughout this subsection we let $S = [Y; G_\alpha; \psi_{\alpha,\beta}]$ be a surjective commutative HIS with each G_α isomorphic to the group G in (6.3) and connecting morphisms non-injective. Recall that as S is surjective, each absolute kernel is trivial by Corollary 6.2.23.

Lemma 6.3.1. *For each $\alpha \in Y$, the absolute image of G_α is a subgroup.*

Proof. By Lemma 6.2.38 the result is immediate if G_α is non-periodic, and so we assume G_α is periodic. Let $a_\alpha, b_\alpha \in I_\alpha^*$ be of orders n and m, respectively. Suppose, seeking a contradiction, that $a_\alpha b_\alpha \notin I_\alpha^*$, so that there exist $\beta < \alpha$ and $k < o(a_\alpha b_\alpha)$ such that $(a_\alpha b_\alpha)^k \in K_{\alpha,\beta} \subseteq T_\alpha$. Notice that $o((a_\alpha b_\alpha)^k)$ divides $o(a_\alpha b_\alpha)$, which in turn divides nm since G_α is abelian. This means that I_α^* and $K_{\alpha,\beta}$ are not of coprime orders, contradicting the work after Lemma 6.2.35, and so I_α^* is a subgroup. $\square$

By Theorem 6.2.41, it thus suffices to consider the case where the absolute image of each maximal subgroup is trivial, so that $G_\alpha = \bigcup_{\beta<\alpha} K_{\alpha,\beta}$ for each $\alpha \in Y$. By Lemma 6.2.42, G_α is non Hopfian and non co-Hopfian.

Lemma 6.3.2. *The group G is component-wise non co-Hopfian.*

Proof. For each $\alpha \in Y$, let $G_\alpha(p)$ denote the p-component of G_α. Then $G_\alpha(p)$ is an order-characteristic subgroup of G_α, so that the set S_p of elements of S of order some power of p forms a HIS by Lemma 6.2.7. Since S_p is periodic with trivial absolute kernel and absolute image, it follows from Theorem 6.2.31 and Theorem 6.2.41 that S_p is a surjective Clifford semigroup with each $G_\alpha(p)$ a union of kernel subgroups. In particular, $G_\alpha(p)$ is non co-Hopfian, and thus so is the p-component of G, forcing $n_p = \aleph_0$ by [6].

Suppose, seeking a contradiction, that $0 < n < \aleph_0$, so that G is non-periodic, and let $\alpha > \beta$ in Y. Pick a disjoint subset $A = \{g_{\beta,i} : 1 \le i \le n\}$ of G_β with $o(g_{\beta,i}) = \aleph_0$, and let $g_{\alpha,i}\psi_{\alpha,\beta} = g_{\beta,i}$ for each i, so clearly $\{g_{\alpha,i} : 1 \le i \le n\}$ forms a disjoint subset of G_α. Then for any $x_\alpha \in K_{\alpha,\beta}$ of infinite order we have $x_\alpha^m = g_{\alpha,1}^{m_1} \cdots g_{\alpha,n}^{m_n}$ for some large enough $m, m_i \in \mathbb{N}$, since otherwise $\{x_\alpha, g_{\alpha,i} : 1 \le i \le n\}$ forms a disjoint subset of G_α of size $n+1$. Hence

$$e_\beta = g_{\beta,1}^{m_1} \cdots g_{\beta,n}^{m_n}$$

contradicting A being a disjoint subset. Consequently, n is infinite as required. $\square$

In particular, age(G) is precisely the class of all f.g. abelian groups with elements of order from $o(G)$. We observe that if G is divisible then it is either periodic with $n_p = 0$ for each $p \in \mathbb{P}_1$ or non-periodic. Hence in both cases G is a characteristic subgroup of the universal abelian group.

Lemma 6.3.3. *The semilattice Y is the universal semilattice.*

Proof. Suppose, seeking a contradiction, that Y is a linear or semilinear order and let $\alpha > \beta$ in Y. Let $g_\alpha \in K_{\alpha,\beta}$ be of order $n \in \mathbb{N}^*$. Since the absolute kernel is trivial, there exists $\gamma < \alpha$ such that $g_\alpha \notin K_{\alpha,\gamma}$. Then $\beta \not> \gamma$, since otherwise $K_{\alpha,\beta} \subseteq K_{\alpha,\gamma}$. Hence as $Y \cdot \alpha$ forms a chain, we have $\alpha > \gamma > \beta$, so that $K_{\alpha,\gamma} \subsetneq K_{\alpha,\beta}$ by Lemma 6.2.37. Since $K_{\alpha,\beta} \cong K_{\alpha,\gamma}$ by Lemma 6.2.19 there exists an element $h_\alpha \in K_{\alpha,\gamma}$ of order n.

Suppose first that $n = p$ for some prime p. If there exists $0 < k, \ell \leq p$ such that $g_\alpha^k = h_\alpha^\ell$ then

$$g_\alpha^k \psi_{\alpha,\gamma} = h_\alpha^\ell \psi_{\alpha,\gamma} = e_\gamma$$

and so $g_\alpha^k \in K_{\alpha,\gamma}$. However $g_\alpha \psi_{\alpha,\gamma}$ has order p, and thus $g_\alpha^k = e_\alpha$, so that $k = p$. Hence $\ell = p$ as h_α is of order p, and it follows that $\langle g_\alpha, h_\alpha \rangle_I \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$. In particular, we may extend the isomorphism swapping g_α and h_α to an automorphism $\theta = [\theta_\alpha, \pi]_{\alpha \in Y}$ of S. Since Y is linear or semilinear either $\gamma \geq \gamma\pi$ or $\gamma\pi \geq \gamma$. By the commutativity of the diagram $[\alpha, \gamma; \alpha, \gamma\pi]$ we have

$$h_\alpha \psi_{\alpha,\gamma\pi} = g_\alpha \theta_\alpha \psi_{\alpha,\gamma\pi} = g_\alpha \psi_{\alpha,\gamma} \theta_\gamma \neq e_{\gamma\pi}$$

as $g_\alpha \notin K_{\alpha,\gamma}$. Hence $h_\alpha \notin K_{\alpha,\gamma\pi}$, and similarly as $h_\alpha \theta_\alpha = g_\alpha$ we attain $g_\alpha \in K_{\alpha,\gamma\pi}$. If $\gamma \geq \gamma\pi$ then $h_\alpha \in K_{\alpha,\gamma} \subseteq K_{\alpha,\gamma\pi}$, while if $\gamma\pi \geq \gamma$ then $g_\alpha \in K_{\alpha,\gamma\pi} \subseteq K_{\alpha,\gamma}$, both giving contradictions. Consequently no element of G can have prime order, and so G is torsion free with $n = \aleph_0$ by Lemma 6.3.2.

If $g_\alpha^k = h_\alpha^\ell$ for some $k, l \in \mathbb{N}$ then $g_\alpha^k \in K_{\alpha,\gamma}$, and so $(g_\alpha \psi_{\alpha,\gamma})^k = e_\gamma$, contradicting G being torsion free. Thus $\langle g_\alpha, h_\alpha \rangle_I \cong \mathbb{Z} \oplus \mathbb{Z}$, and we argue in much the same way as above to arrive at a contradiction. □

Let $\mathcal{K}(G)$ denote the age of the group G, which is a Fraïssé class by the homogeneity of G. Let $\mathcal{K}[G]$ denote the class of all f.g. commutative inverse semigroups with maximal subgroups from $\mathcal{K}(G)$. That is, $\mathcal{K}[G]$ is the class of all f.g. commutative inverse semigroups with elements of order from $o(G)$.

Proposition 6.3.4. *The class $\mathcal{K}[G]$ forms a Fraïssé class.*

Proof. Since $\mathcal{K}(G)$ is closed under substructure and direct product, so is $\mathcal{K}[G]$. It is then immediate that $\mathcal{K}[G]$ has HP and JEP. To show closure under amalgamation, we follow the construction of Imaoka in [56, Section 2]. Given $T, T' \in \mathcal{K}[G]$ with common inverse subsemigroup U, we assume w.l.o.g. that T and T' are strong semilattice of groups, say, $T = [Z; H_\alpha; \phi_{\alpha,\beta}]$ and $T' = [Z'; H'_{\alpha'}; \phi'_{\alpha',\beta'}]$. We also assume w.l.o.g. that $T \cap T' = U$. Let $1 \notin T \cup T'$, and form the semigroups T^1 and T'^1 by adjoining the identity 1, so that $1t = t1 = t$ for each $t \in T \cup T'$. We remark that this goes against the common notion of adjoining an identity if necessary, where here Imaoka *forces* an identity, even if T or T' are already monoids (forcing a zero was discussed briefly in Section 3.5).

The semigroup T^1 is a commutative Clifford semigroup, and since the maximal subgroups of T^1 are $\{1\}$ and H_α $(\alpha \in Z)$, which are members of $\mathcal{K}(G)$, we have $T^1 \in \mathcal{K}[G]$. Similarly so is $T'^1 \in \mathcal{K}[G]$. Hence $W = T^1 \times T'^1 \setminus \{(1,1)\}$, as an inverse subsemigroup of $T^1 \times T'^1$ is a member of $\mathcal{K}[G]$. Imaoka then showed that there exist a congruence ρ on W and embeddings $\theta : T \to W/\rho$ and $\theta' : T' \to W/\rho$ given by $x\theta = (x, 1)\rho$ and $x'\theta' = (1, x')\rho$ $(x \in T, x' \in T')$ such that $U\theta = U\theta' = T\theta \cap T'\theta'$.

Hence W/ρ is generated by the elements $x\theta$ and $x\theta'$, which are of orders from $o(T) \cup o(T') \subseteq o(G)$. Since G is abelian, $o(G)$ is closed under product and we thus have $o(W/\rho) \subseteq o(G)$. Consequently, W/ρ is a member of $\mathcal{K}[G]$, and AP holds.

Finally, Rédei's Theorem [77] states that every f.g. commutative semigroup is finitely presented (see also [21, Theorem 9.28]). It easily follows that the class of all f.g. commutative semigroups, and thus its subclass $\mathcal{K}[G]$, is countable. $\square$

We denote the Fraïssé limit of $\mathcal{K}[G]$ as $\mathcal{C}[G]$, noting that $\mathcal{C}[G] \cong \mathcal{C}[G']$ if and only if $G \cong G'$. We prove that $\mathcal{C}[G]$ is isomorphic to S.

Lemma 6.3.5. *Let $m, n \in o(G_\alpha)$ be such that either $m|n$ or $n = \aleph_0$. Then:*

(i) *If $\alpha > \beta$ then for every $x_\beta \in G_\beta$ of order m there exists an infinite disjoint subset of G_α of elements of order n which are the pre-image of x_β under $\psi_{\alpha,\beta}$;*

(ii) *If $\alpha > \{\beta, \gamma\} > \tau$ forms a diamond in Y, then for any $x_\gamma \in K_{\gamma,\tau}$ of order m, there exists $x_\alpha \in K_{\alpha,\beta}$ of order n such that $x_\alpha\psi_{\alpha,\gamma} = x_\gamma$.*

Proof. (i) Let $\alpha > \beta$ and $x_\beta \in G_\beta$ be of order m. We first claim that there exists $x_\alpha \in G_\alpha$ of order m with $x_\alpha\psi_{\alpha,\beta} = x_\beta$. If $m = \aleph_0$ then the result is immediate as $\psi_{\alpha,\beta}$ is surjective. Let $m = p^r$ for some prime p and $r > 0$ (the case $r = 0$ being trivial). Suppose, seeking a contradiction, that for all $\gamma < \beta$ we have $o(x_\beta\psi_{\beta,\gamma}) < p^r$. Then $x_\beta^{p^{r-1}} \in K_{\beta,\gamma}$ for all $\gamma < \beta$, contradicting the absolute kernel being trivial. Hence there exists $x_\gamma = x_\beta\psi_{\beta,\gamma}$ of order p^r. By Lemma 6.2.16 we may extend the isomorphism ϕ from $\{e_\beta\} \cup \langle x_\gamma\rangle_I$ to $\{e_\alpha\} \cup \langle x_\beta\rangle_I$, determined by $e_\beta\phi = e_\alpha$ and $x_\gamma\phi = x_\beta$, to an automorphism $[\theta_\alpha, \pi]_{\alpha\in Y}$ of S. Then the diagram $[\beta, \gamma; \alpha, \beta]$ commutes and so

$$x_\beta\theta_\beta\psi_{\alpha,\beta} = x_\beta\psi_{\beta,\gamma}\theta_\gamma = x_\gamma\theta_\gamma = x_\beta.$$

Since $x_\beta\theta_\beta \in G_\alpha$ has order p^r, the claim holds for this case. Now suppose $m = p_1^{r_1}p_2^{r_2}\cdots p_s^{r_s}$ for some primes p_i and $r_i \in \mathbb{N}$. By the Fundamental Theorem of Finite Abelian Groups, $x_\beta = x_{\beta,1}x_{\beta,2}\cdots x_{\beta,s}$ for some $x_{\beta,i} \in G_\beta$ of order $p_i^{r_i}$ and so, by the previous case, there exists $x_{\alpha,i} \in G_\alpha$ of order $p_i^{r_i}$ with $x_{\alpha,i}\psi_{\alpha,\beta} = x_{\beta,i}$ for each i. Then $x_\alpha = x_{\alpha,1}x_{\alpha,2}\cdots x_{\alpha,s}$ has order m and is such that $x_\alpha\psi_{\alpha,\beta} = x_\beta$, and the claim holds in all cases.

Notice that the set $x_\alpha K_{\alpha,\beta}$ is precisely the elements of G_α mapped to x_β. Since $K_{\alpha,\beta} \cong G_\alpha$ by Lemma 6.2.42, $K_{\alpha,\beta}$ is component-wise non co-Hopfian. By Lemma 6.3.2 there exists an infinite disjoint subset $\{g_{\alpha,i} : i \in \mathbb{N}\}$ of $K_{\alpha,\beta}$ of elements of order n. If $g_{\alpha,i}^k = x_\alpha^l$ for some $0 < k < n$ and $0 < l < m$ then $e_\beta = e_\beta^k = x_\beta^l$, a contradiction. It follows that each $x_\alpha g_{\alpha,i}$ has order n. We claim that $\{x_\alpha g_{\alpha,i} : i \in \mathbb{N}\}$ forms an infinite disjoint subset of G_α. If

$$(x_\alpha g_{\alpha,i})^k = (x_\alpha g_{\alpha,j_1})^{k_1}\cdots(x_\alpha g_{\alpha,j_t})^{k_t}$$

for some $0 < k, k_1, \dots, k_t < n$, then $g_{\alpha,i}^k g_{\alpha,j_1}^{-k_1} \cdots g_{\alpha,j_t}^{-k_t} = x_\alpha^{k_1 k_2 \cdots k_t - k}$ by commutativity, and so $e_\beta = x_\beta^{k_1 k_2 \cdots k_t - k}$. Hence $m|(k_1 k_2 \cdots k_t - k)$, so that $x_\alpha^{k_1 k_2 \cdots k_t - k} = e_\alpha$, and we thus have

$$g_{\alpha,i}^k = g_{\alpha,j_1}^{k_1} \cdots g_{\alpha,j_t}^{k_t}$$

contradicting $\{g_{\alpha,i} : i \in \mathbb{N}\}$ being disjoint, and the claim holds. The result then follows as $x_\alpha g_{\alpha,i} \in x_\alpha K_{\alpha,\beta}$.

(ii) Let $\alpha > \{\beta, \gamma\} > \tau$ form a diamond in Y and $x_\gamma \in K_{\gamma,\tau}$ be of order m. By part (i) there exists $y_\alpha \in G_\alpha$ of order n such that $y_\alpha \psi_{\alpha,\gamma} = x_\gamma$, so that $y_\alpha \in K_{\alpha,\tau}$. Let $y_\alpha \psi_{\alpha,\beta} = y_\beta$, so that $y_\beta \in K_{\beta,\tau}$. Then there exists $\beta' \in Y$ with $\beta > \beta' > \tau$ and such that $y_\beta \in K_{\beta,\beta'}$, since otherwise τ is a maximal element in the subsemilattice $\{\rho \in Y : y_\beta \in K_{\beta,\rho}\}$ of Y, which would clearly contradict homogeneity. Extend the isomorphism between

$$\begin{aligned} &\langle e_\alpha, e_{\beta'}, x_\gamma, e_\tau \rangle_I = \{e_\alpha\} \cup \{e_{\beta'}\} \cup \langle x_\gamma \rangle_I \cup \{e_\tau\} \\ \text{and} \quad &\langle e_\alpha, e_\beta, x_\gamma, e_\tau \rangle_I = \{e_\alpha\} \cup \{e_\beta\} \cup \langle x_\gamma \rangle_I \cup \{e_\tau\} \end{aligned}$$

which sends $e_{\beta'}$ to e_β and fixes all other elements to an automorphism $\theta' = [\theta'_\alpha, \pi']_{\alpha \in Y}$ of S. Then $y_\alpha \theta' \in G_\alpha$ is of order n with $y_\alpha \theta' \in K_{\alpha,\beta}$ by the commutativity of $[\alpha, \beta'; \alpha, \beta]$ (as $y_\alpha \in K_{\alpha,\beta'}$), and $y_\alpha \theta' \psi_{\alpha,\gamma} = x_\gamma$ by the commutativity of $[\alpha, \gamma; \alpha, \gamma]$. □

Lemma 6.3.6. *Let $\alpha > \{\beta, \gamma\} > \tau$ be a diamond in Y and let $x_\beta \in G_\beta$ and $x_\gamma \in G_\gamma$ be of orders $m_1, m_2 \in \mathbb{N}^*$, respectively, such that $x_\beta \psi_{\beta,\tau} = x_\tau = x_\gamma \psi_{\gamma,\tau}$. Then, for any $n \in o(G_\alpha)$ such that either $m_i|n$ ($i = 1, 2$) or $n = \aleph_0$, there exists $x_\alpha \in G_\alpha$ of order n such that $x_\alpha \psi_{\alpha,\beta} = x_\beta$ and $x_\alpha \psi_{\alpha,\gamma} = x_\gamma$.*

Proof. We may assume that $m_1 = m_2 = n$. Indeed, as Y is the universal semilattice there exist $\beta', \gamma' \in Y$ with $\alpha > \beta' > \beta$, $\alpha > \gamma' > \gamma$ and $\beta' \gamma' = \tau$ by Lemma 5.1.3. Hence by the previous lemma there exist $x_{\beta'} \in G_{\beta'}$ and $x_{\gamma'} \in G_{\gamma'}$ of order n with $x_{\beta'} \psi_{\beta',\beta} = x_\beta$ and $x_{\gamma'} \psi_{\gamma',\gamma} = x_\gamma$, and so it would suffice to consider $x_{\beta'}$ and $x_{\gamma'}$ instead.

By Lemma 6.3.5 (i) there exists $z_\alpha \in G_\alpha$ of order n such that $z_\alpha \psi_{\alpha,\beta} = x_\beta$, so that $z_\alpha \psi_{\alpha,\tau} = x_\tau$. Let $z_\alpha \psi_{\alpha,\gamma} = z_\gamma$. Then there exists γ' such that $\gamma > \gamma' > \tau$ and $o(z_\gamma \psi_{\gamma,\gamma'}) = o(x_\tau)$, else τ would be a maximal element in the set

$$\{\rho : o(z_\gamma \psi_{\gamma,\rho}) = o(x_\tau)\},$$

and thus contradict the homogeneity of S. Let $z_\gamma \psi_{\gamma,\gamma'} = z_{\gamma'}$, and pick $g_{\gamma'} \in K_{\gamma',\tau}$ of order n, noting that such an element exists as $o(G) = o(K_{\gamma',\tau})$. Arguing in much the same way as in the proof of Lemma 6.3.5 (i), the element $z_{\gamma'} g_{\gamma'}$ has order n. By Lemma 6.3.5 (ii) there exists $g_\alpha \in K_{\alpha,\beta}$ of order n with $g_\alpha \psi_{\alpha,\gamma'} = g_{\gamma'}$. Then as $(z_\alpha g_\alpha)\psi_{\alpha,\beta} = x_\beta$ has order n, it easily follows that $z_\alpha g_\alpha$ has order n, and is such that

$$(z_\alpha g_\alpha)\psi_{\alpha,\beta} = z_\alpha \psi_{\alpha,\beta} = x_\beta \quad \text{and} \quad (z_\alpha g_\alpha)\psi_{\alpha,\gamma'} = z_{\gamma'} g_{\gamma'}.$$

The map between the f.g. inverse subsemigroups $\{e_\alpha\} \cup \langle x_\beta \rangle_I \cup \langle z_{\gamma'} g_{\gamma'} \rangle_I \cup \langle x_\tau \rangle_I$ and $\{e_\alpha\} \cup \langle x_\beta \rangle_I \cup \langle x_\gamma \rangle_I \cup \langle x_\tau \rangle_I$ which sends $z_{\gamma'} g_{\gamma'}$ to x_γ and fixes all other elements, is clearly an isomorphism. Extend the isomorphism to an automorphism θ of S. Then $(z_\alpha g_\alpha)\theta \in G_\alpha$ gives the required element. $\square$

Corollary 6.3.7. *Let $\beta_1, \beta_2, \ldots, \beta_r \in Y$ be such that $\beta_i \perp \beta_j$ for each $i \neq j$, and let $x_{\beta_i} \in G_{\beta_i}$ be such that if $\gamma < \beta_i, \beta_j$, for some i, j, then $x_{\beta_i}\psi_{\beta_i,\gamma} = x_{\beta_j}\psi_{\beta_j,\gamma}$. Then for any $\alpha \in Y$ with $\alpha > \beta_i$ for all i, and any $n \in o(G_\alpha)$ with either $o(x_{\beta_i})|n$ for all i or $n = \aleph_0$, there exists an infinite disjoint subset of G_α of elements of order n which are the pre-image of each x_{β_i} under ψ_{α,β_i}.*

Proof. By Lemma 6.3.5 (i) the result holds when $r = 1$. We proceed by induction by supposing that the result holds when $r = k-1$, and letting $x_{\beta_1}, x_{\beta_2}, \ldots, x_{\beta_k}$ and $n \in o(G_\alpha)$ satisfy the conditions of the corollary. Since Y is the universal semilattice there exists $\alpha' \in Y$ with $\alpha > \alpha' > \beta_2, \ldots, \beta_k$ but $\alpha' \ngtr \beta_1$ by Lemma 5.1.3. By the induction hypothesis there exists $x_{\alpha'}$ of order n such that $x_{\alpha'}\psi_{\alpha',\beta_i} = x_{\beta_i}$ for all $2 \leq i \leq k$. Since $\alpha > \{\alpha', \beta_1\} > \alpha'\beta_1$ forms a diamond in Y, there exists $x_\alpha \in G_\alpha$ of order n such that $x_\alpha\psi_{\alpha,\alpha'} = x_{\alpha'}$ and $x_\alpha\psi_{\alpha,\beta_1} = x_{\beta_1}$ by the previous lemma. Hence $x_\alpha\psi_{\alpha,\beta_i} = x_{\beta_i}$ for all i. Let $\delta \in Y$ be such that $\alpha > \delta > \beta_i$ for each i, again noting that such an element exists by Lemma 5.1.3, and let $x_\alpha\psi_{\alpha,\delta} = x_\delta$. By Lemma 6.3.5 (i) there exists an infinite disjoint subset of G_α of elements of order n which are mapped to x_δ, and thus to each x_{β_i}. This completes the inductive step. $\square$

Proposition 6.3.8. *The age of S is $\mathcal{K}[G]$.*

Proof. Let $\mathcal{K}$ denote the age of S, noting that clearly $\mathcal{K}$ is a subclass of $\mathcal{K}[G]$. Since a 1-generated Clifford semigroup is a cyclic group, each 1-generated member of $\mathcal{K}[G]$ is a member of $\mathcal{K}(G)$, and thus of $\mathcal{K}$. Proceeding by induction, assume that every n-generated member of $\mathcal{K}[G]$ is contained in $\mathcal{K}$, for some $n \in \mathbb{N}$. Let $A = [Z; A_\alpha; \phi_{\alpha,\beta}]$ be an $n+1$-generated member of $\mathcal{K}[G]$. To avoid A trivially being a member of $\mathcal{K}(G)$ we may assume that Z is non-trivial. Let α be maximal in Z and suppose $A_\alpha = \langle a_{\alpha,1} \rangle_I \oplus \langle a_{\alpha,2} \rangle_I \oplus \cdots \oplus \langle a_{\alpha,r} \rangle_I$ is an r-generated abelian group, where each $\langle a_{\alpha,i} \rangle_I$ is a cyclic subgroup. Let A' be the inverse subsemigroup of A given by

$$A' = \begin{cases} A \setminus A_\alpha & \text{if } r = 1, \\ (A \setminus A_\alpha) \cup \langle a_{\alpha,2} \rangle_I \oplus \langle a_{\alpha,3} \rangle_I \oplus \cdots \oplus \langle a_{\alpha,r} \rangle_I & \text{if } r > 1. \end{cases}$$

Then A' is n-generated and with structure semilattice $\bar{Z}$, where $\bar{Z} = Z \setminus \{\alpha\}$ if $r = 1$, and $\bar{Z} = Z$ else. By the inductive hypothesis there exists an embedding $\theta : A' \to S$, with induced embedding $\pi : \bar{Z} \to Y$. Since Y is the universal semilattice there exists $\delta \in Y$ such that $\bar{Z}\pi \cup \{\delta\} \cong Z$ by Lemma 5.1.3, where we take $\delta = \alpha\pi$ if $r > 1$. Let α be an upper cover of $\beta_1, \dots, \beta_r$ in Z, and $a_{\alpha,1}\phi_{\alpha,\beta_i} = a_{\beta_i}$ for each i. By Corollary 6.3.7, there exists a infinite disjoint subset $\{g_{\delta,k} : k \in \mathbb{N}\}$ of G_δ with $o(g_{\delta,k}) = o(a_{\alpha,1})$ which are the pre-image of each $a_{\beta_i}\theta$ under $\psi_{\delta,\beta_i\pi}$ $(1 \leq i \leq r)$. Note that A'_α is f.g. since it is either empty or equal to $\langle a_{\alpha,2}\rangle_I \oplus \langle a_{\alpha,3}\rangle_I \oplus \cdots \oplus \langle a_{\alpha,r}\rangle_I$. On the other hand, $\bigoplus_{k\in\mathbb{N}}\langle g_{\delta,k}\rangle_I$ is infinitely generated, and it follows that there exist only finitely many $g_{\delta,k}$ with $\langle g_{\delta,k}\rangle_I \cap A'_\alpha\theta \neq \{e_\alpha\}$. Hence, for some $k \in \mathbb{N}$, we have that $\langle g_{\delta,k}\rangle_I \oplus A'_\alpha\theta$ is isomorphic to A_α, and its easily shown that the map $\theta' : A \to A'\theta \cup \langle g_{\delta,k}\rangle_I$ given by $A'\theta' = A'\theta$ and $a_{\alpha,1}\theta' = g_{\delta,k}$ is an embedding, thus completing the inductive step. □

A full classification of surjective commutative HIS is now achieved. In particular we may describe all periodic commutative HIS as follows (the non-periodic case will be considered separately in the next subsection).

Theorem 6.3.9. *Let I^*, K^* and T be periodic homogeneous abelian groups of pairwise coprime orders and T component-wise non co-Hopfian. Let Y be a homogeneous semilattice, and let U denote the universal semilattice. Then the following inverse semigroups are HIS:*

(i) $(Y \times I^*) \bowtie [Y; K^*]$;

(ii) $(U \times I^*) \bowtie \mathcal{C}[T] \bowtie [U; K^*]$;

Conversely, every periodic commutative HIS is isomorphic to an inverse semigroup constructed in this way.

Proof. Let S be a periodic commutative HIS. Then by Theorem 6.2.31 S is isomorphic to $I(S) \bowtie [Y; K^*]$, where $I(S) = [Y; I_\alpha; \psi^I_{\alpha,\beta}]$ is a surjective Clifford HIS and $I_\alpha \cong I$ is coprime to the homogeneous group K^*. By Corollary 6.2.23 the absolute kernels of $I(S)$ are trivial. If each $\psi^I_{\alpha,\beta}$ is an isomorphism, then $I(S) \cong Y \times I^*$ by Lemma 3.7.13, which is structure-HIS by Proposition 6.2.32. We then have case (i) by Corollary 6.2.30. Otherwise, as the absolute images form subgroups by Lemma 6.3.1, we have $I(S) \cong [Y; T_\alpha; \psi^T_{\alpha,\beta}] \bowtie (Y \times I^*)$ by Theorem 6.2.41, where T_α is of coprime order to I^*_α. Each group T_α is isomorphic to some component-wise non co-Hopfian group T by Lemma 6.3.2. By Propositions 6.3.4 and 6.3.8 we have $[Y; T_\alpha; \psi^T_{\alpha,\beta}] \cong \mathcal{C}[T]$, and we obtain case (ii) again by Corollary 6.2.30.

Conversely, the Clifford semigroups $Y \times I^*$ and $[Y; K^*]$ are structure-HIS by Proposition 6.2.32 and Lemmas 6.2.27 and 6.2.26. The Clifford semigroup $\mathcal{C}[T]$ is a HIS by Proposition 6.3.4. The result then follows by Proposition 6.2.14. □

6.3.2 An open case

We now consider the final case, where $S = [Y; G_\alpha; \psi_{\alpha,\beta}]$ is a commutative HIS such that each G_α is isomorphic to the group G in (6.3) and elements of infinite order are not contained in the image I_α or absolute kernel K_α^* of G_α. We observe that as $G_\alpha \neq K_\alpha^*$, the subgroup I_α is non-trivial.

By Lemma 6.2.22 the absolute images of S are trivial, so that $G_\alpha = \bigcup_{\beta<\alpha} K_{\alpha,\beta}$ and $G_\alpha \cong K_{\alpha,\beta}$ by Lemma 6.2.42. By Lemma 6.2.21, the elements of G_α of finite order form precisely the subgroup $I_\alpha \oplus K_\alpha^*$, which is clearly a characteristic subgroup. It follows by Lemma 6.2.7 that elements of S of finite order forms a HIS

$$T = [Y; I_\alpha \oplus K_\alpha^*; \psi^I_{\alpha,\beta} \bowtie \psi^K_{\alpha,\beta}]$$

where $\psi^I_{\alpha,\beta} \bowtie \psi^K_{\alpha,\beta} = \psi_{\alpha,\beta}|_{I_\alpha \oplus K_\alpha^*}$. The inverse subsemigroup $[Y; I_\alpha; \psi^I_{\alpha,\beta}]$ of T is a periodic surjective commutative HIS with trivial absolute-images and, by Corollary 6.2.23, trivial absolute kernels. Consequently, by Theorem 6.3.9, $[Y; I_\alpha; \psi^I_{\alpha,\beta}]$ is isomorphic to $\mathcal{C}[I]$, where $I \cong I_\alpha$ is component-wise non co-Hopfian, and that Y is isomorphic to universal semilattice U by Lemma 6.3.3. Hence $[Y; K_\alpha^*; \psi^K_{\alpha,\beta}]$ is isomorphic to the structure-HIS $[U; K]$ by Lemmas 6.2.26 and 6.2.27, where $K \cong K_\alpha^*$. By Corollary 6.2.30, we have that $T \cong \mathcal{C}[I] \bowtie [U; K]$, and it follows that

$$G = I \oplus K \oplus \mathbb{Q}^n \tag{6.4}$$

where $I = \bigoplus_{p\in\mathbb{P}_I} \mathbb{Z}[p^\infty]^{\aleph_0}$ and $K = \bigoplus_{p\in\mathbb{P}_K} \mathbb{Z}[p^\infty]^{n_p}$ for some $n, n_p \in \mathbb{N}^*$, where $\mathbb{P}_I$ and $\mathbb{P}_K$ are disjoint subsets of $\mathbb{P}$.

We let $\mathcal{K}^*[I; K; n]$ denote the class of all f.g. commutative inverse semigroups A with maximal subgroups in $\mathcal{K}(G)$, where G is as in (6.4), and satisfying the following properties:

1. every element of infinite order is maximal in $(A, \leq)$;

2. for each $p \in \mathbb{P}_K$, every element of order some power of p is maximal in $(T, \leq)$;

where $\leq$ is the natural order on A. In particular, if $[Z; A_\alpha; \phi_{\alpha,\beta}] \in \mathcal{K}^*[I; K; n]$ then every element of infinite order is mapped to an element of finite order by non-trivial connecting morphisms by (1) and, for each $p \in \mathbb{P}_K$, every non-trivial element of order some power of p is not contained in an image of any connecting morphism by (2), and so is contained in the absolute kernel of its maximal subgroup. Note that $\mathcal{K}[I]$ is a subclass of $\mathcal{K}^*[I; K; n]$.

Open Problem 8. For which conditions on K and n does $\mathcal{K}^*[I; K; n]$ form a Fraïssé class?

The problem we face when tackling this is open problem that the method in the proof of Proposition 6.3.4 no longer applies. For example, let $\alpha > \beta$ and $\gamma > \beta$ be a pair of chains, and let $T = \langle x_\alpha \rangle_I \cup \{e_\beta\}$ and $T' = \{e_\gamma\} \cup \{e_\beta\}$ be f.g. Clifford semigroups, with x_α of infinite order. Note that $T \cap T' = \{e_\beta\}$. Let ρ be the congruence on $W = T^1 \times T'^1 \setminus \{(1,1)\}$ as given by Imaoka. Then $(x_\alpha, 1)\rho$ and $(x_\alpha, e_\beta))\rho$ have infinite order and $(x_\alpha, 1)\rho > (x_\alpha, e_\beta)\rho$. Hence W/ρ does not satisfy (2), and is thus not a member of $\mathcal{K}^*[I; K; n]$.

We now prove that age(S) is $\mathcal{K}^*[I; K; n]$ by following the methods of the previous subsection.

Proposition 6.3.10. *Let $\beta_1, \beta_2, \ldots, \beta_r \in Y$ be such that $\beta_i \perp \beta_j$ for each $i \neq j$. Let $x_{\beta_i} \in I_{\beta_i}$ be such that if $\gamma < \beta_i, \beta_j$ then $x_{\beta_i}\psi_{\beta_i,\gamma} = x_{\beta_j}\psi_{\beta_j,\gamma}$. Then for any $\alpha \in Y$ with $\alpha > \beta_i$ for all i, and any $t \in o(G_\alpha)$ with either $o(x_{\beta_i})|t$ for all i or $t = \aleph_0$, there exists a disjoint subset of G_α of size $\aleph_0$ if t is finite, and size n otherwise, consisting of elements of order t which are the pre-image of each x_{β_i} under ψ_{α,β_i}.*

Proof. Recall that S contains the inverse subsemigroup $[Y; I_\alpha; \psi^I_{\alpha,\beta}]$ isomorphic to $\mathcal{C}[I]$, where $I \cong I_\alpha$. Hence if t is finite, then the result easily follows by Corollary 6.3.7, where the required disjoint subset of G_α is contained in I_α.

Suppose instead that $t = \aleph_0$. By the previous case there exists $x_\alpha \in I_\alpha$ of finite order with $x_\alpha\psi_{\alpha,\beta_i} = x_{\beta_i}$ for all i. Since Y is the universal semilattice, we may fix $\alpha' \in Y$ with $\alpha > \alpha' > \beta_1, \ldots, \beta_r$ by Lemma 5.1.3. Let $\{z_i : i \in \mathbb{N}\}$ be a disjoint set of size n consisting of elements of infinite order and such that $z_i \in K_{\alpha,\alpha'} \subseteq K_{\alpha,\beta_i}$. Note that such a set exists as $K_{\alpha,\alpha'} \cong G_\alpha$. Then $x_\alpha z_i$ is a disjoint set of size n consisting of elements of infinite order, and

$$(x_\alpha z_i)\psi_{\alpha,\beta_i} = x_\alpha\psi_{\alpha,\beta_i} = x_{\beta_i}$$

for each $i \in \mathbb{N}$. □

By a simple adaptation of Proposition 6.3.8 we have:

Corollary 6.3.11. *The age of S is $\mathcal{K}^*[I; K; n]$.*

Theorem 6.3.12 (Classification theorem of non-periodic commutative HIS)**.** *Let G be a homogeneous non-periodic abelian group, Y a homogeneous semilattice and U be the universal semilattice. Then the following inverse semigroups are HIS:*

(i) *$Y \times G$;*

(ii) *$[Y; G]$;*

(iii) *$\mathcal{C}[G]$, with G component-wise non co-Hopfian;*

(iv) *the Fraïssé limit of a Fraïssé class $\mathcal{K}^*[I; K^*; n]$, where $G = I \oplus K \oplus \mathbb{Q}^n$ and I is component-wise non co-Hopfian.*

Conversely, every non-periodic commutative HIS is isomorphic to an inverse semigroup constructed in this way.

Proof. By Theorem 6.2.31, A non-periodic commutative Clifford semigroup S is a HIS if and only if isomorphic to either a surjective Clifford HIS, or $[Y; G]$ for some homogeneous semilattice Y and group G, or a Clifford HIS with no elements of infinite order lying in the images or absolute kernels. By the usual argument, a surjective commutative Clifford is a HIS if and only if is isomorphic to either $Y \times G$ for some homogeneous Y and G, or a HIS with connecting morphisms non-injective, and G_α being a dense union of kernels by Theorem 6.2.41 (since the absolute image forms a subgroup by Lemma 6.3.1). By Propositions 6.3.4 and 6.3.8 the second possibility holds if and only if S isomorphic to $\mathcal{C}[G]$, where G is component-wise co-Hopfian by Lemma 6.3.2. It thus suffices to consider the case where S has no elements of infinite order lying in the images or absolute kernels. If S is a HIS then age(S) is $\mathcal{K}^*[I; K; n]$ by Corollary 6.3.11, and thus S is isomorphic to the Fraïssé limit of the Fraïssé class $\mathcal{K}^*[I; K; n]$. The converse is immediate by Fraïssé's Theorem. □

6.4 Inverse Homogeneous Semigroups

In this section we study the differences between the two concepts of homogeneity for inverse semigroups, in particular when a HIS has the stronger property of being an inverse homogeneous semigroup (HS).

Lemma 6.4.1. *Let S be a periodic inverse semigroup. Then S is a HS if and only if it is a HIS.*

Proof. Suppose S is a HIS, so that, being periodic, S is a Clifford semigroup by Theorem 6.1.6. Hence S is a HS by Lemma 4.3.2. The converse is by Lemma 6.1.1. □

Corollary 6.4.2. *A non-periodic inverse HS is a group, homogeneous as a semigroup.*

Proof. Let S be a non-periodic inverse HS. Then S is a HIS by Lemma 6.1.1, and is thus either Clifford or bisimple. If S is bisimple then there exists $x \in S$ such that $\langle x \rangle_I$ is a bicyclic semigroup. Since x and x^{-1} have infinite order and Aut(S) acts transitively on Inf(S), there exists an automorphism ϕ of S mapping x to x^{-1}. Then $x^{-1}\phi = x$ and so

$$(xx^{-1})\phi = x^{-1}x \quad \text{and} \quad (x^{-1}x)\phi = xx^{-1},$$

contradicting $xx^{-1} > x^{-1}x$. Hence S is Clifford, and is therefore completely simple by Corollary 4.5.2. However a semilattice in which every element is minimal is clearly trivial, and so S is a group. □

However the converse is not known, that is, is a homogeneous group a HS? We give a positive answer for the class of abelian groups, thus completing our study into the homogeneity of commutative inverse semigroups.

Proposition 6.4.3. *A homogeneous abelian group is a HS.*

Proof. If G is periodic then the result is clear, and so we assume G is a non-periodic abelian homogeneous group with identity 1. Let $\phi : A \to B$ be an isomorphism between f.g. subsemigroups of G, and let A_G, B_G denote the finitely generated subgroups of G generated by A and B, respectively. Since G is abelian, each element of A_G is of the form uv^{-1} for some $u, v \in A$ and so we may take the map $\hat{\phi} : A_G \to B_G$ given by

$$(uv^{-1})\hat{\phi} = (u\phi)(v\phi)^{-1}.$$

Then $\hat{\phi}$ is well-defined and injective since, for any $uv^{-1}, st^{-1} \in A_G$ we have

$$\begin{aligned}(uv^{-1})\hat{\phi} = (st^{-1})\hat{\phi} &\Leftrightarrow (u\phi)(v\phi)^{-1} = (s\phi)(t\phi)^{-1} \\ &\Leftrightarrow u\phi t\phi = s\phi v\phi \\ &\Leftrightarrow ut = sv \\ &\Leftrightarrow uv^{-1} = st^{-1}.\end{aligned}$$

If $ab^{-1} \in B_G$, then there exist $u, v \in A_G$ such that $u\phi = a$ and $v\phi = b$ since ϕ is surjective. Hence $(uv^{-1})\hat{\phi} = ab^{-1}$ and $\hat{\phi}$ is surjective. Finally,

$$\begin{aligned}(uv^{-1})\hat{\phi}(st^{-1})\hat{\phi} &= (u\phi)(v\phi)^{-1}(s\phi)(t\phi)^{-1} = (us)\phi((vt)\phi)^{-1} \\ &= (us(vt)^{-1})\hat{\phi} = ((uv^{-1})(st^{-1}))\hat{\phi}\end{aligned}$$

and $1\hat{\phi} = (uu^{-1})\hat{\phi} = (u\phi)(u\phi)^{-1} = 1$ for any $u \in A$. It follows that $\hat{\phi}$ is an isomorphism, and extends ϕ since for all $u \in A$,

$$u\hat{\phi} = u\phi(1\phi)^{-1} = u\phi.$$

Since any automorphism of G which extends $\hat{\phi}$ additionally extends ϕ, we have that G is a HS by the homogeneity of G. □

From the proposition above and Theorem 6.3.9 we obtain a complete classification of all commutative inverse HS, as either a periodic commutative HIS or a homogeneous non-periodic abelian group.

Open Problem 9. Is a non-periodic homogeneous group a HS?

We note that Open Problem 9 is simply a special case of Open Problem 3 on completely simple semigroups.

Open Problem 10. If G is a homogeneous group and $\theta : S \to T$ is an isomorphism between f.g. subsemigroups S and T of G, then can θ be extended to an isomorphism $\hat{\theta} : \langle S \rangle_I \to \langle T \rangle_I$?

The two open problems are clearly closely linked. Indeed, if G is a homogeneous group then it is a HIS, and so the isomorphism $\hat{\theta} : \langle S \rangle_I \to \langle T \rangle_I$ of Open Problem 10 can be extended to an automorphism of G, thus showing G to be a HS.

Chapter 7

Homogeneous orthogroups

Given that we now have a full description of homogeneous bands, as well as a large pool of homogeneous Clifford semigroups, the natural next step is to consider the homogeneity of orthodox completely regular semigroups, being a generalization of both bands and Clifford semigroups. We follow [72] in calling an orthodox completely regular semigroup S an *orthogroup*. If further we have that $E(S)$ is a regular band, then S is called a *regular orthogroup*.

Let S be a orthogroup with band of idempotents $E(S)$. Then as $E(S)$ is a completely regular characteristic subsemigroup of S, the homogeneity of S is inherited by $E(S)$. Furthermore, as homogeneous bands are regular bands by Proposition 5.5.2 and the classification theorem of homogeneous bands, we have the following result.

Corollary 7.0.1. *A homogeneous orthogroup is a regular orthogroup.*

A regular orthogroup in which $\mathcal{H}$ forms a congruence is called a *regular orthocryptogroup*. This is equivalent to a semigroup being a spined product of a regular band and a Clifford semigroup by [72, Lemma V.5.3]. The class of all regular orthocryptogroups forms a subvariety of the variety completely regular semigroups, defined by the identities $[xx' = x'x, x(yz)'x = xy'x'z'x]$.

A natural question then arises: is the spined product of a homogeneous band with a homogeneous Clifford semigroup necessarily homogeneous?

While we are not able to fully answer this question, we follow the usual methods from the past two chapters to obtain a generalization of Corollary 5.3.3 and Proposition 6.2.14, which allows examples of homogeneous regular orthocryptogroups to be formed *ad-lib*.

An isomorphism theorem for regular orthocryptogroups follows from [72, Proposition V.5.7], and gives a converse to Proposition 2.11.3 in the case of regular orthocryptogroups:

Proposition 7.0.2. *Let $B = \bigcup_{\alpha \in Y} B_\alpha$, $B' = \bigcup_{\alpha' \in Y'} B'_{\alpha'}$ be a pair of regular bands and $S = [Y; G_\alpha; \psi_{\alpha,\beta}]$, $S' = [Y'; G'_{\alpha'}; \psi'_{\alpha',\beta'}]$ a pair of Clifford semigroups. Let $\theta : B \to B'$ and $\phi : S \to S'$ be a pair of isomorphisms, both with induced semilattice isomorphism $\pi : Y \to Y'$. Then the map $\chi : B \bowtie S \to B' \bowtie S'$ given by*

$$(e_\alpha, g_\alpha)\chi = (e_\alpha\theta, g_\alpha\phi) \quad (e_\alpha \in B_\alpha, g_\alpha \in G_\alpha, \alpha \in Y)$$

is an isomorphism from $B \bowtie S$ to $B' \bowtie S'$, denoted by $\chi = \theta \bowtie \phi$. Conversely, every isomorphism from $B \bowtie S$ to $B' \bowtie S'$ can be constructed in this way.

Proposition 7.0.3. *Let B be a homogeneous band and S a homogeneous Clifford semigroup, both with structure semilattice Y. If either B or S are structure-homogeneous, then the regular orthocryptogroup $B \bowtie S$ is a homogeneous completely regular semigroup.*

Proof. Suppose first that B is structure-homogeneous and let A_1 and A_2 be f.g. completely regular subsemigroups of $B \bowtie S$. Since the class of regular orthocryptogroups forms a subvariety of the variety of completely regular semigroups, it follows by the usual argument that $A_i = B_i \bowtie S_i$ for some f.g. subbands B_i of B, and f.g. Clifford subsemigroups S_i of S $(i = 1, 2)$. Let A_i have structure semilattice Y_i, and χ be an isomorphism from A_1 onto A_2. Then by Proposition 7.0.2, $\chi = \theta \bowtie \phi$ for some isomorphisms $\theta : B_1 \to B_2$ and $\phi : S_1 \to S_2$, both with induced semilattice isomorphism $\pi : Y_1 \to Y_2$, say. By the homogeneity of S we may extend ϕ to an automorphism $\hat{\phi}$ of S. Let $\hat{\pi}$ be the semilattice automorphism of Y induced by $\hat{\phi}$. Since B is structure-homogeneous and $\hat{\pi}$ extends π, we may extend θ to an automorphism $\hat{\theta}$ of B with induced semilattice automorphism $\hat{\pi}$. Then $\hat{\theta} \bowtie \hat{\phi}$ is an automorphism of $B \bowtie S$ by Proposition 7.0.2, and extends $\theta \bowtie \phi$ as required. Hence $B \bowtie S$ is homogeneous.

The proof in the case of S being structure-homogeneous is argued in the same way. □

We showed after Lemma 6.1.5 that a group is homogeneous in the signature of groups L_G if and only if it is homogeneous in L_{US}. It follows that every homogeneous group is a structure-homogeneous Clifford semigroup, and so for any homogeneous group G and $n, m \in \mathbb{N}^*$, the orthogroup $S = B_{n,m} \times G$ is homogeneous by the proposition above. Moreover, by [72, Theorem III.5.2] a semigroup is a direct product of a group and a rectangular band if and only if it is an orthodox completely simple semigroup. We have thus proven the backward direction of the following corollary, the forward direction being immediate from Corollary 4.4.7.

Corollary 7.0.4. *Let $S = \mathcal{M}[G; I, \Lambda; P]$ be an orthodox completely simple semigroup. Then S is a homogeneous completely simple semigroup if and only if G is a homogeneous group.*

Proposition 7.0.3 does not give rise to all homogeneous regular orthocryptogroups, for it does not take into account the non structure-homogeneous bands, for example. However, its strength lies in our complete descriptions of both structure-homogeneous bands in Theorem 5.5.5, and structure-homogeneous Clifford semigroups in Theorem 6.2.34.

Open Problem 11. Is a homogeneous orthogroup necessarily a regular orthocryptogroup?

Open Problem 12. Is the spined product of a homogeneous band and a homogeneous Clifford semigroup (with equal structure semilattices) a homogeneous regular orthocryptogroup?

We note that by Proposition 7.0.3, an example of a negative answer to Open Problem 12 would necessarily be a spined product of a non structure-homogeneous band and a non structure-homogeneous Clifford semigroup.

7.1 Further work on homogeneity

We end this thesis by considering future directions for the work on homogeneous semigroups.

While a full classification of homogeneous bands has been given, we are interested in further understanding of the 'universal normal band' $\mathcal{B}_N$. In particular, since $\mathcal{B}_N$ is $\aleph_0$-categorical, we would like to be able to give a first order definition (that is, describe its full theory). A second direction is to follow the research in [27] by studying the automorphism group of homogeneous bands. This task should be relatively straight forward for structure-homogeneous bands, but more insightful results are likely for the image-trivial and universal cases.

The future direction to the homogeneity of inverse semigroups is more obvious, and several open problems are given in the previous chapter. Our main interest is in determining whether or not bisimple homogeneous inverse semigroups which are not groups exist. It was originally conjectured negatively, but my faith in this conjecture has wavered in time. The final open problem of that chapter is also worth highlighting, due to its sheer simplicity in its statement. It asks whether or not the properties of group homogeneity and semigroup homogeneity for a group are equivalent.

In terms of arbitrary completely regular semigroups, the first task is to determine the homogeneity of completely simple semigroups. A full classification would both determine all non-periodic completely regular homogeneous semigroups by Corollary 4.5.2, and answer Open Problem 3. We suspect that the difference between the homogeneity of a completely simple semigroup in L_S or L_{US} may depend solely on the group case. Finally, for orthodox completely regular semigroups it is hoped that by further understanding the homogeneity of Clifford semigroups, Open Problem 12 may be answered.

A research objective at the beginning of my PhD was to examine the property of *homomorphism homogeneity* (hom-hom). A structure M is hom-hom if every morphism between f.g. substructures extends to an endomorphism of M. Much like with homogeneity, the hom-hom of semigroups has only been studied in the context of semilattices in [22]. The reason for not studying hom-hom semigroups during my PhD was a positive one: the homogeneity of semigroups was a far richer field of study than expected. I hope to begin research into hom-hom semigroups soon.

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