Inhomogeneous Diophantine Approximation, M_0 sets and projections of fractals

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ABSTRACT

The thesis deals with two main subjects, one being metric Diophantine approximation and the other Fractal Geometry.

As far as the first subject is concerned, the results presented lie in the setup of inhomogeneous Diophantine approximation. The following is shown: suppose $\mathcal{A} = (q_n)_{n=1}^{\infty} \subseteq \mathbb{N}$ is a lacunary sequence and μ is a probability measure with Fourier transform of a prescribed logarithmic decay rate. Then for any $\gamma \in \mathbb{R}$ and any decreasing approximating function $\psi : \mathbb{N} \to \mathbb{R}^+$, the set $W_{\mathcal{A}}(\gamma, \psi) = \{x \in [0, 1) : \|q_n x - \gamma\| \leq \psi(q_n)$ for i.m. $n \in \mathbb{N}\}$ satisfies a Khintchine-type law with respect to the measure μ. This result builds on the work of Pollington and Velani in [51]. It is also shown that $W(\gamma, \psi)$ is a Salem set, generalising a result of Kaufman in [38].

Regarding Fractal Geometry, we present a refinement of Marstrand's famous projection theorem for arbitrary dimension functions. We state an analogue of Kaufman's result on the dimension of the set of angles of exceptional projections and discuss on the necessity of the conditions imposed in our main theorem.

Contents

Preface

Chapter 1 is an introduction to well known basic facts and results which are used in subsequent parts of the thesis. Proofs are included for most of the results presented. The main areas covered are continued fractions, metric Diophantine approximation and connections of fractal geometry with Diophantine approximation.

In Chapter 2 we formulate and prove a Khintchine-type law on inhomogeneous Diophantine approximation with restricted denominators. The size of the set of well approximable numbers is given with respect to a probability measure the Fourier transform of which has a prescribed logarithmic decay rate. The denominators are restricted to a lacunary sequence.

In Chapter 3 we show that the set of inhomogeneously ψ-well approximable numbers, where ψ is an arbitrary approximating function, is a Salem set. That is, its Hausdorff dimension equals its Fourier dimension. This generalises the result proved by Kaufman for the homogeneous case.

Finally in Chapter 4 we present a refinement of Marstrand's projection theorem to arbitrary dimension functions. This refinement allows us to discriminate, for example, between projections of sets of dimension zero. We also show that the conditions imposed in the theorem are in some sense necessary.

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I also feel grateful to all of my colleagues and peers who contributed to creating this wonderful atmosphere in the Maths Department.

Declaration

I declare that this thesis is a presentation of original work and I am the sole author. This work has not previously been presented for an award at this, or any other, University. All sources are acknowledged as References.

Chapter 2 is joint work with A.D. Pollington, S. Velani and E. Zorin, and is part of a paper in preparation. Chapter 4 is essentially the paper *Marstrand's Theorem Revisited: Projecting Sets of Dimension Zero,* with V. Beresnevich, K. Falconer, S. Velani, with an *Appendix: The Gap of Uncertainty* with D. Simmons and H. Yu (arXiv:1703.08554).

Chapter 1

Introduction

1.1 Continued Fractions

We begin this introductory chapter by presenting some of the basic facts and properties of the expansion of reals into continued fractions. Let

$$x = [a_0; a_1, a_2, \ldots] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}$$

be the continued fraction expansion of some real number x. Here $a_0 \in \mathbb{Z}$ and $a_1, a_2, \ldots \in \mathbb{N}$. The integers $(a_n)_{n=1}^{\infty}$ are known as the partial quotients of x. Note that when x is rational the continued fraction terminates.

The n-th convergent of an irrational $x \in \mathbb{R}$ is defined to be the rational number

$$\frac{p_n}{q_n} = [a_0; a_1, \ldots, a_n].$$

The sequences $(p_n)_{n=1}^{\infty}$, $(q_n)_{n=1}^{\infty}$ are called the sequences of numerators and denominators associated to the continued fraction expansion of x, respectively. They satisfy the following recursive relations

$$\begin{cases} p_{n+1} = a_{n+1}p_n + p_{n-1} \\ p_{-1} = 1,\, p_0 = a_0 \end{cases} \qquad \begin{cases} q_{n+1} = a_{n+1}q_n + q_{n-1} \\ q_{-1} = 0,\, q_0 = 1 \end{cases} \tag{1.1}$$

which may also be written in matrix form as

$$\begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix}. \tag{1.2}$$

Taking determinants in (1.2) gives

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1}, \tag{1.3}$$

and this implies that $(p_n, q_n) = 1$ for all $n = 1, 2, \ldots$, i.e. the convergents are all in reduced form.

The convergents of x also satisfy the inequalities

$$\frac{1}{q_n(q_{n+1} + q_n)} < \left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}}, \qquad n = 1, 2, \ldots \tag{1.4}$$

There are several properties of real numbers which are reflected in the corresponding continued fraction expansion. The continued fraction $[a_1, a_2, \ldots]$ is called *eventually periodic* if there exist integers $k, d \geq 1$ such that $(a_{k+1}, \ldots, a_{k+d}) = (a_{k+nd+1}, \ldots, a_{k+(n+1)d})$ for all $n = 1, 2, \ldots$

Proposition 1.1. *Let $x \in \mathbb{R}$. Then x is a quadratic irrational if and only if the continued fraction expansion of x is eventually periodic.*

For example, the golden ratio $\frac{1}{2}(1+\sqrt{5})$ and $\sqrt{2}$ have the well-known continued fraction expansions

$$\frac{1+\sqrt{5}}{2} = [1;\, 1,\, 1,\, \ldots]$$

and

$$\sqrt{2} = [1;\, 2,\, 2\, \ldots].$$

For more details on continued fractions we refer to [40], [21, Chapter 3], [54].

1.2 Dirichlet and Khintchine's Theorems

1.2.1 Dirichlet's Theorem

The fundamental result in Diophantine approximation is Dirichlet's theorem.

Theorem 1.2 (Dirichlet)**.** *Let $x \in \mathbb{R}$ and $N \in \mathbb{N}$ be a positive integer. There exist $p \in \mathbb{Z}$ and a positive integer $1 \leq q \leq N$ such that*

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{qN}\,.$$

The proof is an easy consequence of the pigeonhole principle, see for example [30, Chapter 2]. In what follows, we use the notation

$$\|x\| := \min\{|x-k| : k \in \mathbb{Z}\}$$

for the distance of the real number x to the nearest integer. Then Dirichlet's theorem implies that for any real number x and for any integer $N \geq 1$, there is some $q \geq 1$ with $q \leq N$ and

$$\|qx\| \leq \frac{1}{N}.$$

An immediate corollary to Dirichlet's theorem is the following.

Corollary 1.3. *Let $x \in \mathbb{R}\setminus\mathbb{Q}$. There exist infinitely many rational numbers $\frac{p}{q} \in \mathbb{Q}$ such that*

$$\left|x - \frac{p}{q}\right| \leq \frac{1}{q^2}.$$

So Dirichlet's theorem gives a rate of approximation which is attained by all real numbers. Actually this uniform rate of approximation can be improved in the following sense.

Theorem 1.4 (Hurwitz)**.** *Let $x \in \mathbb{R}\setminus\mathbb{Q}$. There exist infinitely many rational numbers $\frac{p}{q} \in \mathbb{Q}$ such that*

$$\left|x - \frac{p}{q}\right| \leq \frac{1}{\sqrt{5}\,q^2}.$$

The rate of approximation described in Hurwitz's theorem is actually the best possible which is achieved by all irrational numbers, as the following proposition implies. For the proof see [40].

Proposition 1.5. *Let $\phi = [1; 1, 1, \ldots] = \frac{1}{2}(1+\sqrt{5})$ be the golden ratio. For any $\varepsilon > 0$, there exist at most finitely many rational numbers $p/q \in \mathbb{Q}$ such that*

$$\left|\phi - \frac{p}{q}\right| \leq \frac{1}{(\sqrt{5}+\varepsilon)q^2}.$$

In view of the previous statement, it makes sense to define as *badly approximable* those real numbers for which the aforementioned rate of approximation is actually the best possible. In what follows, we restrict to numbers in the unit interval $[0, 1)$ since all properties of reals with respect to Diophantine approximation we study are invariant under translation by integers.

Definition 1. A number $x \in [0, 1)$ is called badly approximable if there exists a constant $c = c(x) > 0$ such that

$$q\|qx\| \geq c, \qquad \text{for all } q = 1, 2, \ldots$$

The set of badly approximable numbers is denoted by **Bad**.

By Proposition 1.5 it follows that the set **Bad** is non-empty. The fact that **Bad** is uncountable may be deduced from Khintchine's characterisation of badly approximable numbers, see [40].

Proposition 1.6. *Let $x \in [0,1)$ have the continued fraction expansion $x = [a_1, a_2, \ldots]$. Then $x \in$ **Bad** if and only if the sequence $(a_n)_{n=1}^{\infty}$ of partial quotients is bounded.*

This proposition implies, for example, that all quadratic irrationals are in **Bad**. It remains an open problem to determine whether there are any other algebraic irrationals of degrees greater than 2 which are in **Bad**. It is conjectured that the answer to this question is negative.

For each $N = 1, 2, \ldots$ define the set

$$\mathrm{F}_N = \{x \in [0,1) : x = [a_1, a_2, \ldots] \text{ with } a_i \leq N,\, i = 1, 2, \ldots\}. \tag{1.5}$$

Then Proposition 1.6 implies

$$\mathbf{Bad} = \bigcup_{N=1}^{\infty} \mathrm{F}_N.$$

1.2.2 Khintchine's Theorem

Dirichlet's theorem states that all irrational numbers admit approximations by rationals p/q of the rate of $1/q^2$. It is natural to ask what is the size of the set of numbers which have approximations of various rates. In what follows, we write $\mathbb{R}^+$ for the set $(0, \infty)$.

Given a function $\psi : \mathbb{N} \to \mathbb{R}^+$, define the set

$$W(\psi) = \{x \in [0,1) : \|qx\| \leq \psi(q) \quad \text{for infinitely many } \ q \in \mathbb{N}\}$$

of *ψ-well approximable numbers*. The function ψ determines the rate of approximation by rationals, and will be referred to as the approximating function. Also for any $\tau \geq 1$ we define $W(\tau)$ to be simply the set $W(\psi)$, when $\psi(q) = q^{-\tau}$; that is,

$$W(\tau) = \left\{x \in [0,1) : \|qx\| \leq \frac{1}{q^\tau} \quad \text{for infinitely many } \ q \in \mathbb{N}\right\}. \tag{1.6}$$

Khintchine's theorem gives the size of the set $W(\psi)$ in terms of one-dimensional Lebesgue measure. In what follows, $|\cdot|$ stands for the Lebesgue measure on the real line.

Theorem 1.7 (Khintchine)**.** *Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a decreasing function. Then*

$$|W(\psi)| = \begin{cases} 0, & \text{if } \sum\limits_{q=1}^{\infty} \psi(q) < \infty \\ 1, & \text{if } \sum\limits_{q=1}^{\infty} \psi(q) = \infty. \end{cases}$$

Before we proceed to more details on the proof of the theorem, let us study the set $W(\psi)$ more closely. Given a sequence $(A_q)_{q=1}^{\infty}$ of subsets of an arbitrary set X (not necessarily a subset of $\mathbb{R}$), we may define the set $\limsup\limits_{q\to\infty} A_q$ by

$$\begin{aligned} \limsup_{q\to\infty} A_q &= \bigcap_{q=1}^{\infty}\bigcup_{i=q}^{\infty} A_i \\ &= \{x \in X : x \in A_q \text{ for infinitely many } q \in \mathbb{N}\} . \end{aligned}$$

Using this definition, the set $W(\psi)$ can be written as $W(\psi) = \limsup\limits_{q\to\infty} A_q(\psi)$, where we define

$$\begin{aligned} A_q(\psi) &= \{x \in [0,1) : \|qx\| \le \psi(q)\} \\ &= \bigcup_{p=0}^{q} B\left(\frac{p}{q}, \frac{\psi(q)}{q}\right) \cap [0,1) . \end{aligned}$$

Here we define $B(x,r) = \{y \in \mathbb{R} : |y-x| < r\}$ for each $x \in \mathbb{R}$ and $r > 0$. So each set $A_q(\psi)$ has Lebesgue measure $|A_q(\psi)| \le 2\psi(q)$, and this explains the appearance of the series $\sum\limits_{q=1}^{\infty} \psi(q)$ in Khintchine's theorem.

The difficult part in the proof of this theorem and many other similar results we will encounter is the divergence case. The convergence case is relatively easy, as it follows directly from the Borel-Cantelli Lemma.

Lemma 1.8 (Borel-Cantelli)**.** *Let $(X, \mathcal{B}, \mu)$ be a probabilty space and $(A_n)_{n=1}^{\infty} \subseteq \mathcal{B}$ be a sequence of subsets of X. If $\sum\limits_{n=1}^{\infty} \mu(A_n) < \infty$ then*

$$\mu\left(\limsup_{n\to\infty} A_n\right) = 0.$$

Since $W(\psi) = \limsup A_q(\psi)$ and each set $A_q(\psi)$ has Lebesgue measure $|A_q(\psi)| \le 2\psi(q)$, in the convergence case we deduce that

$$\sum_{q=1}^{\infty} |A_q(\psi)| < \infty$$

and $|W(\psi)| = 0$ by the Borel-Cantelli Lemma. Observe that the assumption of monotonicity of ψ is not necessary for the convergence case of Khintchine's theorem.

Khintchine's original proof [40] is under the additional assumption that the function $q \mapsto q\psi(q)$ is monotonic, and uses the apparatus of continued fractions. Cassels [12] proved the theorem under the assumption of monotonicity of ψ. As we explain later in subsection 1.2.4, the assumption of monotonicity cannot be removed. Schmidt [56] proved a quantitative version of the theorem. For an alternative proof related to the theory of *ubiquitous systems* we refer to [4].

Equipped with Khintchine's theorem, we can easily deduce that $|\mathbf{Bad}| = 0$.

Proposition 1.9. *The set* **Bad** *has zero Lebesgue measure.*

Proof. Consider the function $\psi(q) = 1/(q \log q)$, $q \geq 2$. Then

$$\mathbf{Bad} \subseteq [0, 1) \setminus W(\psi).$$

Khintchine's theorem implies $|W(\psi)| = 1$, hence $|\mathbf{Bad}| = 0$. □

1.2.3 A proof of Khintchine's Theorem

Here we give a proof of Khintchine's theorem which makes implicit use of the apparatus of ubiquitous systems. The proof relies on the notion of quasi-independence on average and the following zero-one law due to Cassels [13].

Theorem 1.10 (Cassels' Zero-One Law)**.** *Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be any function. Then the Lebesgue measure of the set $W(\psi)$ satisfies*

$$|W(\psi)| = 0 \qquad or \qquad |W(\psi)| = 1.$$

In view of Cassels' zero-one law, in order to prove the divergence case for Khintchine's theorem it is sufficient to show that $|W(\psi)| > 0$. Hence we want to show that the Lebesgue measure of a certain lim-sup set is positive. With this in mind, the following partial converse to the Borel-Cantelli Lemma is useful, see [30, Lemma 2.3].

Lemma 1.11. *Let $(X, \mathcal{B}, \mu)$ be a probabilty space and $(A_n)_{n=1}^{\infty} \subseteq \mathcal{B}$ be a sequence of* *subsets of* X*. If* $\sum_{n=1}^{\infty} \mu(A_n) = \infty$ *then*

$$\mu\left(\limsup_{n\to\infty} A_n\right) \geq \limsup_{N\to\infty} \frac{\left(\sum_{n=1}^{N} \mu(A_n)\right)^2}{\sum_{n=1}^{N}\sum_{m=1}^{N} \mu(A_n \cap A_m)}.$$

Let us give the following definition.

Definition 1.12. Let $(X, \mathcal{B}, \mu)$ be a probability space. The subsets $(A_n)_{n=1}^{\infty}$ of X are called *quasi-independent on average* if there exists a constant $C > 0$ such that

$$\sum_{n=1}^{N}\sum_{m=1}^{N} \mu(A_n \cap A_m) \leq C\left(\sum_{n=1}^{N} \mu(A_n)\right)^2, \quad N = 1, 2, \ldots \tag{1.7}$$

So the divergence Borel-Cantelli Lemma (Lemma 1.11) implies that whenever the sets $(A_n)_{n=1}^{\infty}$ are quasi-independent on average, the set $\limsup_{n\to\infty} A_n$ has positive Lebesgue measure. More precisely, whenever (1.7) is satisfied, then

$$\left|\limsup_{n\to\infty} A_n\right| \geq \frac{1}{C}.$$

We proceed with the divergence case of Kintchine's theorem. Consider the sets

$$B_n(\psi) = \bigcup_{q=2^{n-1}}^{2^n-1} \bigcup_{\substack{p=1\\(p,q)=1}}^{q} B\left(\frac{p}{q}, \frac{\psi(2^n)}{2^n}\right), \qquad n = 1, 2, \ldots \tag{1.8}$$

Clearly $\limsup_{n\to\infty} B_n(\psi) \subseteq W(\psi)$ and it suffices to show that $\left|\limsup_{n\to\infty} B_n(\psi)\right| > 0$. In view of the previous remarks, it suffices to show that the sequence $(B_n(\psi))_{n=1}^{\infty}$ is pairwise quasi-independent. Notice that without loss of generality we may assume that

$$\psi(q) < \frac{1}{2q}, \qquad q = 1, 2, \ldots$$

Indeed, if this is not the case, define

$$\Psi(q) = \min\left\{\frac{1}{2q}, \psi(q)\right\}, \qquad q = 1, 2, \ldots$$

and observe that $\sum_{q=1}^{\infty} \Psi(q) = \infty$ and $W(\Psi) \subseteq W(\psi)$. Now we estimate the measure of the sets $B_n(\psi)$. If $2^{n-1} \leq q_1 < q_2 < 2^n$ and $0 \leq p_1 \leq q_1$, $0 \leq p_2 \leq q_2$ then

$$\left|\frac{p_1}{q_1} - \frac{p_2}{q_2}\right| \geq \frac{1}{q_1 q_2} > \frac{1}{2^{2n}} > \frac{2\psi(2^n)}{2^n}.$$

Hence the intervals in (1.8) are pairwise disjoint and each set $B_n(\psi)$ has Lebesgue measure

$$|B_n(\psi)| = \sum_{q=2^{n-1}}^{2^n-1} \sum_{\substack{p=1 \\ (p,q)=1}}^{q} 2\frac{\psi(2^n)}{2^n} = 2\frac{\psi(2^n)}{2^n} \sum_{q=2^{n-1}}^{2^n-1} \phi(q) \asymp 2^n \psi(2^n).$$

Here we have used the fact that Euler's totient function satisfies

$$\sum_{q=1}^{N} \phi(q) \;=\; \frac{3}{\pi^2} N^2 + O(N \log N), \qquad N \to \infty.$$

Next we estimate the measure of the intersections of the form $B_m(\psi) \cap B_n(\psi)$. Let $m < n$ and fix one of the intervals $B\left(\frac{p}{q}, \frac{\psi(2^m)}{2^m}\right)$ which comprise $B_m(\psi)$. This interval overlaps with intervals of the form $B\left(\frac{p}{q}, \frac{\psi(2^n)}{2^n}\right)$, the centers of which have distance at least $1/2^{2n}$. Thus the number of intervals of the form $B\left(\frac{p}{n}, \frac{\psi(2^n)}{2^n}\right)$ contained in an interval of the form $B\left(\frac{p}{m}, \frac{\psi(2^m)}{2^m}\right)$ is at most

$$\frac{2\psi(2^m)/2^m}{1/2^{2n}} + 2 = 2\frac{\psi(2^m)}{2^m} 2^{2n} + 2.$$

Each of these intervals of the form $B\left(\frac{p}{q}, \frac{\psi(2^n)}{2^n}\right)$ has Lebesgue measure equal to $\psi(2^n)/2^n$. Also the set $B_m(\psi)$ consists of

$$\sum_{2^{m-1}}^{2^m} \phi(q)$$

intervals in total. Thus for $m < n$,

$$\begin{aligned} |B_m(\psi) \cap B_n(\psi)| \;&\leq\; \left(2\frac{\psi(2^m)}{2^m} 2^{2n} + 2\right) \cdot \frac{2\psi(2^n)}{2^n} \sum_{q=2^{n-1}}^{2^n-1} \phi(q) \\ &\ll\; \left(\frac{\psi(2^m)}{2^m} 2^{2n} + 1\right) \frac{\psi(2^n)}{2^n} 2^{2m}. \end{aligned}$$

So for N large enough we get

$$\begin{aligned}
\sum_{m=1}^{N}\sum_{n=1}^{N}|B_m(\psi)\cap B_n(\psi)| &\ll \sum_{m=1}^{N}\sum_{n=1}^{N}\left(\frac{\psi(2^m)}{2^m}2^{2n}+1\right)\frac{\psi(2^n)}{2^n}2^{2m} \\
&\ll \sum_{m=1}^{N}\sum_{n=1}^{N}\frac{\psi(2^m)}{2^m}2^{2n}\frac{\psi(2^n)}{2^n}2^{2m}+\sum_{m=1}^{N}\sum_{n=1}^{N}\frac{\psi(2^n)}{2^n}2^{2m} \\
&\ll \left(\sum_{n=1}^{N}2^n\psi(2^n)\right)^2+\sum_{\substack{1\le m\le N\\ m<n\le N}}\sum\frac{\psi(2^n)}{2^n}2^{2m} \\
&\ll \left(\sum_{n=1}^{N}|B_n(\psi)|\right)^2+\sum_{m=1}^{N}\left(2^m\psi(2^m)\sum_{m<n\le N}\frac{1}{2^{n-m}}\right) \\
&\ll \left(\sum_{n=1}^{N}|B_n(\psi)|\right)^2+\sum_{n=1}^{N}|B_n(\psi)| \\
&\ll \left(\sum_{n=1}^{N}|B_n(\psi)|\right)^2.
\end{aligned}$$

The last estimate follows because

$$\sum_{m=1}^{\infty}|B_m(\psi)| = \sum_{m=1}^{\infty}2^m\psi(2^m) = \infty$$

by the assumption of monotonicity of ψ and Cauchy's condensation test. Thus we have shown that

$$\sum_{m=1}^{N}\sum_{n=1}^{N}|B_m(\psi)\cap B_n(\psi)| \ll \left(\sum_{m=1}^{N}|B_m(\psi)|\right)^2, \quad N\to\infty.$$

Hence the sets $(B_n(\psi))_{n=1}^{\infty}$ are quasi-independent on average and Lemma 1.11 implies that

$$|W(\psi)| \ge \left|\limsup_{n\to\infty}B_n(\psi)\right| > 0\,.$$

In view of Theorem 1.10 we have that $|W(\psi)|=1$.

1.2.4 The Duffin-Schaeffer counterexample

Here we explain why the assumption of monotonicity in the divergence case in Khintchine's theorem cannot be removed.

Duffin and Schaeffer [19] constructed an approximating function $\theta : \mathbb{N} \to \mathbb{R}^+$ which is not monotonic and for which

$$\sum_{q=1}^{\infty} \theta(q) = \infty \quad \text{and} \quad |W(\theta)| = 0$$

hold simultaneously. The definition of the function θ is as follows. Let $(N_k)_{k=1}^{\infty}$ be an increasing sequence of integers such that the following properties are satisfied:

(i) N_k is squarefree for $k = 1, 2, \ldots$

(ii) $(N_k, N_l) = 1$ for $k \neq l$ (i.e. the integers N_k are pairwise coprime), and

(iii) for all $k = 1, 2, \ldots$
$$\prod_{p \,|\, N_k} \left(1 + \frac{1}{p}\right) > 1 + 2^k.$$

Note that the choice of the sequence $(N_k)_{k=1}^{\infty}$ is possible because the product $\prod_p (1+p^{-1})$ diverges to infinity. The function $\theta : \mathbb{N} \to \mathbb{R}^+$ is defined by

$$\theta(q) = \begin{cases} \dfrac{q}{2^{k+1} N_k}, & \text{if } q > 1 \text{ and } q \,|\, N_k \\ 0, & \text{otherwise .} \end{cases}$$

Observe that for any $k = 1, 2, \ldots$ we get

$$\begin{aligned} \sum_{\substack{q \,|\, N_k \\ q>1}} \theta(q) &= \sum_{\substack{q \,|\, N_k \\ q>1}} \frac{q}{2^{k+1} N_k} \\ &\geq \frac{1}{2^{k+2} N_k} \prod_{p \,|\, N_k} (1+p) \\ &= \frac{1}{2^{k+2}} \prod_{p \,|\, N_k} \left(1 + \frac{1}{p}\right) \end{aligned}$$

$$
\begin{aligned}
&> \frac{1+2^k}{2^{k+2}} \\
&> \frac{1}{4},
\end{aligned}
$$

hence

$$\sum_{q=1}^{\infty} \theta(q) \;=\; \sum_{k=1}^{\infty} \sum_{\substack{q \mid N_k \\ q>1}} \theta(q) \;=\; \infty.$$

In order to show that the set $W(\theta)$ has zero Lebesgue measure, we need to examine the sets

$$A_q(\theta) \;=\; \bigcup_{p=0}^{q-1} B\left(\frac{p}{q}, \frac{\theta(q)}{q}\right), \qquad q = 1, 2, \ldots .$$

When $q > 1$ is not a divisor of any of the integers N_k, then $A_q(\theta)$ is empty. Fix some $k \in \mathbb{N}$; then for any divisor $q > 1$ of N_k the set $A_q(\theta)$ is a union of intervals centered at points of the form p/q, $0 \le p < q$ and with radii equal to

$$\frac{\theta(q)}{q} \;=\; \frac{1}{2^{k+1} N_k} \;=\; \frac{\theta(N_k)}{N_k}.$$

Since each rational of the form p/q, $0 \le p < q$ can be written as p'/N_k, $0 \le p' < N_k$, the intervals comprising the sets $A_q(\theta)$ are among the intervals comprising $A_{N_k}(\theta)$. Thus

$$A_{N_k}(\theta) \;=\; \bigcup_{q \mid N_k} A_q(\theta)$$

and

$$W(\theta) \;=\; \limsup_{k\to\infty} A_{N_k}(\theta). \tag{1.9}$$

Now

$$|A_{N_k}(\theta)| \;=\; 2\theta(N_k) \;=\; \frac{1}{2^k}, \qquad k = 1, 2, \ldots$$

hence $|W(\theta)| = 0$ by (1.9) and the Borel-Cantelli Lemma.

1.2.5 The Duffin-Schaeffer Conjecture

In their paper [19], Duffin and Schaeffer not only showed that the assumption of monotonicity is necessary in Khintchine's theorem, but they also formulated a statement regarding arbitrary functions. This statement remains open to date and is known as the *Duffin-Schaeffer Conjecture*.

Given an arbitrary function $\psi : \mathbb{N} \to \mathbb{R}^+$, define the set

$$W'(\psi) = \left\{ x \in [0,1) : \left| x - \frac{p}{q} \right| < \frac{\psi(q)}{q} \text{ for inf. many } p/q \in \mathbb{Q} \text{ with } (p,q) = 1 \right\}.$$

Intuitively, the set $W'(\psi)$ consists of numbers x which admit the same rate of approximation as the elements of $W(\psi)$, but with the restriction that the corresponding rationals are in reduced form. Trivially, $W'(\psi) \subseteq W(\psi)$.

Just like in Khintchine's theorem, the set $W'(\psi)$ can be written as $W'(\psi) = \limsup_{q\to\infty} A'_q(\psi)$, where we define

$$A'_q(\psi) = \bigcup_{\substack{p=1 \\ (p,q)=1}}^{q} B\left(\frac{p}{q}, \frac{\psi(q)}{q}\right).$$

Each of the sets $A'_q(\psi)$ has Lebesgue measure $|A'_q(\psi)| \leq \phi(q)\dfrac{\psi(q)}{q}$, and once more the Borel-Cantelli Lemma implies that $|W'(\psi)| = 0$ whenever $\sum_{q=1}^{\infty} \phi(q)\frac{\psi(q)}{q} < \infty$. Here $\phi(q)$ is Euler's totient function. The Duffin-Schaeffer conjecture is the statement that $|W'(\psi)| = 1$ whenever the aforementioned series diverges.

Conjecture (Duffin & Schaeffer). Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a function. Then

$$|W'(\psi)| = \begin{cases} 0, & \text{if } \sum_{q=1}^{\infty} \phi(q)\dfrac{\psi(q)}{q} < \infty, \\ 1, & \text{if } \sum_{q=1}^{\infty} \phi(q)\dfrac{\psi(q)}{q} = \infty. \end{cases}$$

Remark 1. When ψ is a decreasing function, the Duffin-Schaeffer conjecture and Khintchine's theorem coincide as statements. To see this, first observe that $W'(\psi) = W(\psi)$. Indeed, trivially $W'(\psi) \subseteq W(\psi)$ and for the converse suppose $x \in W(\psi)$. There exist infinitely many $p/q \in \mathbb{Q}$ such that

$$\left| x - \frac{p}{q} \right| \leq \frac{\psi(q)}{q}.$$

Write each of them as $p/q = p'/q' \in \mathbb{Q}$ with $(p', q') = 1$. The assumption of monotonicity of ψ yields

$$\left|x - \frac{p'}{q'}\right| = \left|x - \frac{p}{q}\right| \leq \frac{\psi(q)}{q} \leq \frac{\psi(q')}{q'},$$

so $x \in W'(\psi)$ as well. Furthermore the series

$$\sum_{q=1}^{\infty} \psi(q) \qquad \text{and} \qquad \sum_{q=1}^{\infty} \phi(q)\frac{\psi(q)}{q}$$

appearing in Khintchine's theorem and the Duffin-Schaeffer Conjecture, respectively, either both converge or both diverge. Clearly if the first series converges, then so does the second. Also for any positive integer N large enough,

$$\begin{aligned}
\sum_{q=1}^{2^N} \phi(q)\frac{\psi(q)}{q} &> \sum_{n=0}^{N-1} \sum_{q=2^n+1}^{2^{n+1}} \phi(q)\frac{\psi(q)}{q} \\
&\geq \sum_{n=0}^{N-1} \sum_{q=2^n+1}^{2^{n+1}} \phi(q)\frac{\psi(2^n)}{2^n} \\
&= \sum_{n=0}^{N-1} \frac{\psi(2^n)}{2^n} \sum_{q=2^n+1}^{2^{n+1}} \phi(q) \\
&\asymp \sum_{n=0}^{N-1} \frac{\psi(2^n)}{2^n} 2^{2n} \\
&= \sum_{n=1}^{N-1} 2^n \psi(2^n),
\end{aligned}$$

hence the divergence of the first series implies the divergence of the second series by Cauchy's condensation test.

Remark 2. Duffin and Schaeffer's function θ is not a counterexample to their conjecture. To see this, observe that

$$\sum_{q=1}^{\infty} \phi(q)\frac{\theta(q)}{q} = \sum_{k=1}^{\infty} \sum_{\substack{q \mid N_k \\ q>1}} \phi(q)\frac{\theta(q)}{q}$$

$$
\begin{aligned}
&= \sum_{k=1}^{\infty} \sum_{\substack{q \mid N_k \\ q>1}} \phi(q) \frac{1}{2^{k+1} N_k} \\
&= \sum_{k=1}^{\infty} \frac{1}{2^{k+1} N_k} \sum_{\substack{q \mid N_k \\ q>1}} \phi(q) \\
&= \sum_{k=1}^{\infty} \frac{1}{2^{k+1} N_k} (N_k - 1) \\
&< \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} \\
&< \infty \,.
\end{aligned}
$$

Regarding the Duffin-Schaeffer Conjecture, Gallagher [26] showed that for the divergence case it is sufficient to prove that $|W'(\psi)| > 0$. The following result is a natural analogue of the aforementioned Cassels' zero-one law.

Theorem 1.13 (Gallagher's Zero-One Law)**.** *Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be any function. Then the Lebesgue measure of the set $W'(\psi)$ satisfies*

$$|W'(\psi)| = 0 \qquad \text{or} \qquad |W'(\psi)| = 1.$$

For several partial results related to the Duffin-Schaeffer Conjecture we refer to [1, 5, 22, 31, 50, 62]. It is also worth mentioning the following partial result proved by Duffin and Schaeffer in their attempt to prove their conjecture.

Theorem 1.14 (Duffin, Schaeffer)**.** *Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be any function. If*

$$\limsup_{N \to \infty} \left(\sum_{q=1}^{N} \phi(q) \frac{\psi(q)}{q} \right) \left(\sum_{q=1}^{N} \psi(q) \right)^{-1} > 0 \,,$$

then $|W'(\psi)| = 1$.

For the proof we refer to [30, Chapter 2].

1.3 Inhomogeneous Diophantine Approximation

The object of Diophantine approximation, which is finding rational approximations to a real number x, can be equivalently stated as follows: given $x \in \mathbb{R}$, how close does the sequence $qx \pmod 1$, $q \geq 1$ approximate the point 0? It is natural to study an "inhomogeneous" generalisation of this question: given any $\gamma \in \mathbb{R}$ and an irrational $x \in \mathbb{R}$, how close does the sequence qx, $q \geq 1$ approximate γ mod 1?

In the setup of inhomogeneous Diophantine Approximation, several variants and generalisations of the aforementioned results for the homogeneous case ($\gamma = 0$) have been established. We mention just a few of them, starting with a Theorem due to Chebyshev with the proof presented in [33].

Theorem 1.15 (Chebyshev). *Let $x \in \mathbb{R}$ be an irrational and $\gamma \in \mathbb{R}$ be any real number. Then*

$$\|qx - \gamma\| < \frac{3}{q} \qquad \textit{for inf. many } q = 1, 2, \ldots$$

Proof. Let $(P_n/Q_n)_{n=1}^{\infty}$ be the convergents of x. By (1.4), for each $n = 1, 2, \ldots$ we can write

$$x = \frac{P_n}{Q_n} + \frac{\delta_n}{Q_n^2}, \qquad 0 < |\delta_n| < 1. \tag{1.10}$$

For each $n = 1, 2, \ldots$ there exists $t_n \in \mathbb{Z}$ such that $|Q_n\gamma - t_n| = \|Q_n\gamma\| \leq \frac{1}{2}$, hence

$$\gamma = \frac{t_n}{Q_n} + \frac{\varepsilon_n}{2Q_n}, \qquad |\varepsilon_n| \leq 1.$$

Since $(P_n, Q_n) = 1$ there exist $p_n, q_n \in \mathbb{N}$ such that

$$\frac{Q_n}{2} \leq q_n < \frac{3Q_n}{2} \qquad \text{and} \qquad P_n q_n - Q_n p_n = t_n. \tag{1.11}$$

Thus for all n,

$$\begin{aligned}
\|q_n x - \gamma\| &\leq |q_n x - p_n - \gamma| \\
&\overset{(1.10)}{=} \left| q_n \left(\frac{P_n}{Q_n} + \frac{\delta_n}{Q_n^2} \right) - p_n - \frac{t_n}{Q_n} - \frac{\varepsilon_n}{2Q_n} \right| \\
&\overset{(1.11)}{\leq} \frac{q_n}{Q_n^2} + \frac{1}{2Q_n} \\
&\overset{(1.11)}{<} \frac{3}{q_n}.
\end{aligned}$$

□

Khintchine [33] improved the constant appearing in the previous theorem.

Theorem 1.16 (Khintchine)**.** *Let $x \in \mathbb{R}$ be an irrational, $\gamma \in \mathbb{R}$ be a real number and $\varepsilon > 0$. Then*

$$\|qx - \gamma\| < \frac{1+\varepsilon}{\sqrt{5}\,q} \qquad \text{for inf. many } q = 1, 2, \ldots$$

Theorem 1.16 is an obvious generalisation of Hurwitz's result (Theorem 1.4). Observe that the assumption that x is irrational is necessary, otherwise the approximation rate to γ may not be achieved.

In the inhomogeneous setup, it is also natural to ask if a variant of Khintchine's metric theorem is true. Namely, if $\gamma \in \mathbb{R}$ and $\psi : \mathbb{N} \to \mathbb{R}$ is a function, we would like to know the Lebesgue measure of the set

$$W(\gamma, \psi) = \{x \in [0,1) : \|qx - \gamma\| < \psi(q) \quad \text{for infinitely many } \; q \in \mathbb{N}\}.$$

Szüss [59] proved that an inhomogeneous version of Khintchine's theorem is true under the same assumptions.

Theorem 1.17. *Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a decreasing function. Then*

$$|W(\gamma, \psi)| = \begin{cases} 0, & \text{if } \sum_{q=1}^{\infty} \psi(q) < \infty \\ 1, & \text{if } \sum_{q=1}^{\infty} \psi(q) = \infty. \end{cases}$$

Again, the assumption of monotonicity of ψ is required only for the divergence case.

The set **Bad** of badly approximable numbers admits an inhomogeneous analogue.

Definition 1.18. Let $\gamma \in \mathbb{R}$. Define the set

$$\mathbf{Bad}_\gamma = \left\{ x \in [0,1) \, : \, \inf_{q \in \mathbb{N}} q\|qx - \gamma\| > 0 \right\}.$$

The elements of $\mathbf{Bad}_\gamma$ are called *inhmogeneously badly approximable numbers* with respect to γ.

Remark 3. In view of the inhomogeneous Khintchine theorem, we can deduce that the Lebesgue measure of $\mathbf{Bad}_\gamma$ is zero in the same way we derived the same conclusion for **Bad**.

1.4 The Hausdorff theory of Metric Diophantine approximation

As we have seen, Khintchine's theorem implies that the sets $W(\tau)$, $\tau > 1$ all have zero Lebesgue measure. However it does not allow any further discrimination between them, even though we know for example that $W(3)$ is in some sense "bigger" than $W(100)$. The theory of Hausdorff dimension helps us overcome this problem.

1.4.1 Hausdorff measure and dimension

Let us start with the definition of a dimension function.

Definition 1.19. A *dimension function* (or *gauge function*) is a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ which is increasing and continuous with $f(r) \to 0$ as $r \to 0$.

Definition 1.20. Let A be a non–empty subset of $\mathbb{R}^n$. For $\rho > 0$, let

$$\mathcal{H}^f_\rho(A) \;=\; \inf\left\{\sum_{i=1}^{\infty} f(|U_i|) : A \subseteq \bigcup_{i=1}^{\infty} U_i,\; |U_i| \le \rho \text{ for all } i = 1, 2\ldots\right\},$$

where $|U|$ denotes the diameter of the set U and the infimum is taken over countable covers $(U_i)_{i=1}^{\infty}$ of A by sets of diameter at most ρ. The *Hausdorff f-measure* of A is defined by

$$\mathcal{H}^f(A) = \lim_{\rho \to 0} \mathcal{H}^f_\rho(A)\,.$$

When $f(r) = r^s$ for some $s > 0$, the measure $\mathcal{H}^f$ is the usual *s-dimensional Hausdorff measure $\mathcal{H}^s$*.

Remark 4. When $0 < \rho_1 < \rho_2$, $\mathcal{H}^f_{\rho_1}(A) \ge \mathcal{H}^f_{\rho_2}(A)$ so in the definition of $\mathcal{H}^f$ we actually have

$$\mathcal{H}^f(A) = \lim_{\rho \to 0} \mathcal{H}^f_\rho(A) = \sup_{\rho > 0} \mathcal{H}^f(A)\,.$$

Remark 5. According to the previous definition, the value of the f-Hausdorff measure $\mathcal{H}^f$ only depends on $f(r)$ for $r \in [0, r_0)$ where r_0 is arbitrarily small. Thus we can define the Hausdorff measure $\mathcal{H}^f$ for any function f which is continuous and increasing on any interval of the form $[0, r_0)$.

Definition 1.21. The *Hausdorff dimension* $\dim A$ of a set $A \subseteq \mathbb{R}^n$ is defined by

$$\begin{aligned} \dim A &= \inf\{s : \mathcal{H}^s(A) = 0\} \\ &= \sup\{s : \mathcal{H}^s(A) = \infty\}\,. \end{aligned}$$

Let $A \subseteq \mathbb{R}^n$ be a set. According to the definition,

$$\mathcal{H}^s(A) \;=\; \begin{cases} 0, & \text{if } s > \dim A \\ \\ \infty, & \text{if } s < \dim A. \end{cases}$$

Furthermore, if $s = \dim A$ any of the statements $\mathcal{H}^s(A) = 0$, $\mathcal{H}^s(A) = \infty$ or $0 < \mathcal{H}^s(A) < \infty$ might hold. Sets for which the latter is true are called *s-sets*.

Defining Hausdorff measures for general dimension functions allows a more precise notion of dimension than just a numerical value. For example, a set A may have Hausdorff dimension s but $\mathcal{H}^s(A) = 0$. However, it may be that $0 < \mathcal{H}^f(A) < \infty$ where, say $f(r) = r^s \log(1/r)$, in which case we think of A having dimension 'logarithmically smaller' than s. Introducing a partial order $\prec$ on the set of dimension functions by

$$f \prec g \quad \text{if} \quad \lim_{r\to 0} \frac{g(r)}{f(r)} = 0,$$

which implies that $\mathcal{H}^g(A) = 0$ whenever $\mathcal{H}^f(A) < \infty$, allows a much finer notion of dimension, see [55]. It is also worth noting that there are sets $A \subseteq \mathbb{R}^n$ for which there is no dimension function f such that $0 < \mathcal{H}^f(A) < \infty$, see [16].

A variant of the Hausdorff measure which we use in Chapter 4 is the centred Hausdorff measure. Even though it is not essentially different, it appears to be more suitable to use in some applications. For its definition we consider covers by a countable collection of balls $(B(x_i, r_i))_{i=1}^{\infty}$ of radii $r_i \le \rho$ with centres in A.

Definition 1.22. For $\rho > 0$ we set

$$\mathcal{H}^f_{C,\rho}(A) \;=\; \inf\left\{ \sum_{i=1}^{\infty} f(r_i) : A \subseteq \bigcup_{i=1}^{\infty} B(x_i, r_i),\; x_i \in A,\; r_i \le \rho,\; i = 1, 2, \ldots \right\}.$$

We define the *centred Hausdorff f-measure* of A by

$$\mathcal{H}^f_C(A) = \lim_{\rho \to 0} \mathcal{H}^f_{C,\rho}(A)\;.$$

The standard and centered Hausdorff measures are equivalent, in the sense that for all $A \subseteq \mathbb{R}^n$

$$\mathcal{H}^f_C(A) \;\le\; \mathcal{H}^f(A) \;\le\; m_n\, \mathcal{H}^f_C(A), \tag{1.12}$$

where m_n depends only on n. This follows easily from the definitions, noting that every set U that intersects A is contained in a ball with centre in A and diameter $|U|$, and that every ball $B \subseteq \mathbb{R}^n$ of radius r is contained in a finite number m_n of balls of radius $\frac{1}{2}r$, that is diameter r; in particular $m_2 = 7$.

It follows from (1.12) that we get the same value for Hausdorff dimension if we replace $\mathcal{H}^s$ by $\mathcal{H}^s_C$ in this definition. For further discussion of Hausdorff measures and dimensions, see [23, 44, 55].

1.4.2 An explicit example: the Cantor set

In the current subsection we give a specific example of the calculation of the Hausdorff dimension of a specific set, namely the middle-third Cantor set K. We follow Falconer [24].

The set K is defined as follows. First we divide the unit interval $[0,1]$ into three subintervals of equal lengths and remove the middle one, so we obtain the set $F_1 = [0,1/3] \cup [2/3,1]$. We repeat the same procedure with each of the intervals of F_1, so we obtain the set $F_2 = [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1]$. Inductively we obtain the sets $(F_n)_{n=1}^{\infty}$, where each F_n is the union of 2^n intervals of length 3^{-n}. We define

$$K = \bigcap_{n=1}^{\infty} F_n \,.$$

We explain why $\dim K = s$, where $s = \dfrac{\log 2}{\log 3}$. For each n, F_n is a cover of K consisting of 2^n intervals of length 3^{-n}, hence

$$\mathcal{H}^s_{3^{-n}}(K) \;\leq\; 2^n 3^{-ns} \;=\; 1\,.$$

Letting $n \to \infty$ we get $\mathcal{H}^s(K) \leq 1$. Thus $\dim K \leq s$.

for the converse inequality, let $(B_n)_{n=1}^{\infty}$ be any cover of K consisting of intervals. Since K is compact, we may assume we have a finite cover $(B_n)_{n=1}^{N}$. Also we may assume the endpoints of each B_n are endpoints of the intervals comprising the sets F_k, $k \geq 1$. For each $n = 1, \ldots, N$ let I_n and J_n be the largest such intervals. Then B_n consists of three consecutive intervals; I_n, an interval L_n in the comlpement of K, and J_n.

Since J_n has length greater or equal to the maximum length of I_n, J_n we have

$$|L_n| \;\geq\; \frac{1}{2}\left(|I_n| + |J_n|\right)\,. \tag{1.13}$$

Thus

$$|B_n|^s \;=\; (|I_n| + |L_n| + |J_n|)^s$$

$$
\begin{aligned}
&\overset{(1.13)}{\geq} && \left(\frac{3}{2}\right)^s (|I_n| + |J_n|)^s \\
&= && 2\left(\frac{|I_n| + |J_n|}{2}\right)^s \\
&\geq && |I_n|^s + |J_n|^s.
\end{aligned}
$$

Observe that for each interval I appearing in the definition of the sets F_k, $k \geq 1$, hence for the intervals I_n, J_n as well, if I', I'' are the two intervals of the "next level" which form I, then

$$|I|^s = |I'|^s + |I''|^s .$$

The upshot is, in the estimates for $|B_n|^s$ we can replace $|I_n|^s$ and $|J_n|^s$ with the corresponding sums taken over all subintervals of the same common level. Thus if k_0 denotes the maximum level of all intervals $I_n, J_n, 1 \leq n \leq N$ and F_{k_0} is the union of the intervals $A_j, 1 \leq j \leq 2^j$ then

$$
\begin{aligned}
\sum_{n=1}^{N} |B_n|^s &\geq \sum_{n=1}^{N} (|I_n|^s + |J_n|^s) \\
&\geq \sum_{j=1}^{2^{k_0}} |I_j|^s \\
&= 2^{k_0} 3^{-k_0 s} \\
&= 1.
\end{aligned}
$$

Thus $\mathcal{H}^s(K) \geq 1$ and $\dim K \geq s$. Also note that we have shown something much stronger than $\dim K = s$; we have proved that $\mathcal{H}^s(K) = 1$.

Remark 6. The fact that the Hausdorff dimension of K is equal to $\log 2/\log 3$ and the fact that K was constructed by dividing intervals into 3 pieces and keeping 2 of them at each stage are not unrelated at all. Using the same methods we can show that if we construct a set K' by dividing intervals into b subintervals and selecting a of them at each stage, the set will have Hausdorff dimension equal to $\log a/\log b$. See [23] for more details.

In many cases, a lower bound for the Hausdorff dimension of a set can be found by the properties of the probability measures supported on the set.

Theorem 1.23 (Mass Distribution Principle)**.** *Let μ be a probability measure supported* *on a set $F \subseteq \mathbb{R}^n$. If there exist constants $c, s, r_0 > 0$ such that*

$$\mu(B) \le c|B|^s \tag{1.14}$$

for any ball B of diameter $|B| \le r_0$, then $\dim F \ge s$.

Proof. Let $(B_n)_{n=1}^{\infty}$ be a cover of the set F consisting of balls of diameters at most r_0. Then

$$1 = \mu(B) \le \sum_{n=1}^{\infty} \mu(B_n) \le c \sum_{n=1}^{\infty} |B_n|^s.$$

Taking the infimum over all such possible covers, we obtain $c\mathcal{H}^s_{r_0}(A) \ge 1$. Thus

$$\mathcal{H}^s(A) = \sup_{r>0} \mathcal{H}^s_r(A) > \frac{1}{c} > 0$$

and $\dim A \ge s$, as required. □

Remark 7. If f is a dimension function and the probability measure μ is such that

$$\mu(B) \ \le \ f(|B|)$$

for any ball B of sufficiently small diameter, we say μ satisfies the *Hölder condition* with respect to the dimension function f. In particular, when (1.14) holds, we say μ satisfies the Holder condition with exponent s.

Remark 8. The Mass Distribution Principle can be generalised to arbitrary dimension functions. In particular, if μ is supported on $A \subseteq \mathbb{R}^n$ and satisfies the Hölder condition with respect to the dimension function f, then $\mathcal{H}^f(A) > 0$.

The Mass Distribution Principle can be used to calculate the lower bound for the Hausdorff dimension of the Cantor set K in the previous example. Indeed, for $n = 1, 2, \dots$ let μ_n be the probability measure assigning mass equal to $\frac{1}{2^n}$ uniformly on each of the intervals of F_n, and let μ be the weak limit of $(\mu_n)_{n=1}^{\infty}$.

If $I \subseteq [0, 1)$ is an interval with $I \cap K \ne \emptyset$, let $n \ge 1$ be such that $3^{-(n+1)} \le |I| \le 3^{-n}$. Then I intersects at most one of the intervals of E_n, hence

$$\mu(I) \ \le \ \frac{1}{2^n} \ = \ \frac{1}{3^{ns}} \ = \ \frac{3^s}{3^{(n+1)s}} \ \le \ 2|I|^s \, .$$

Now the Mass Distribution Principle implies that $\mathcal{H}^s(K) \ge \dfrac{1}{2}$ and $\dim K \ge s$.

1.4.3 The Hausdorff theory of well approximable numbers

Jarník [35] and Besicovitch [8] independently calculated the Hausdorff dimension of the sets $W(\tau)$ defined in (1.6).

Theorem 1.24 (Jarník-Besicovitch). *Let $\tau \geq 1$. The Hausdorff dimension of the set $W(\tau)$ is*

$$\dim W(\tau) = \frac{2}{\tau+1}.$$

The Jarník-Besicovitch Theorem tells us the Hausdorff dimension of the set $W(\tau)$ but it gives no information for the value of the corresponding Hausdorff measure at the critical exponent $s = 2/(\tau+1)$. This was found later by Jarník [36]. We mention that Jarník proved his theorem under more assumptions on the functions involved, which were shown to be unnecessary in [4].

Theorem 1.25 (Jarník). *Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be decreasing and let f be a dimension function such that $f(r)/r$ is decreasing and $f(r)/r \to \infty$ as $r \to 0$. Then*

$$\mathcal{H}^f\left(W(\psi)\right) = \begin{cases} 0, & \text{if } \sum\limits_{q=1}^{\infty} qf(\psi(q)/q) < \infty \\ \infty, & \text{if } \sum\limits_{q=1}^{\infty} qf(\psi(q)/q) = \infty. \end{cases}$$

Observe that when the function f is such that $f(r)/r \to \infty$ as $r \to 0$, then $\mathcal{H}^f([0,1)) = \infty$. Thus Khintchine's and Jarník's theorems can be unified into a single theorem, referred to as the Khintchine-Jarník theorem.

Theorem 1.26 (Khintchine-Jarník). *Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be decreasing and let f be a dimension function such that $f(r)/r$ is decreasing. Then*

$$\mathcal{H}^f\left(W(\psi)\right) = \begin{cases} 0, & \text{if } \sum\limits_{q=1}^{\infty} qf(\psi(q)/q) < \infty \\ \mathcal{H}^f([0,1)), & \text{if } \sum\limits_{q=1}^{\infty} qf(\psi(q)/q) = \infty. \end{cases}$$

As an immediate corollary we obtain the value of the Hausdorff measure of the set $W(\tau)$ at the crucial exponent $s = 2/(1+\tau)$.

Corollary 2. For any $\tau > 1$, we have

$$\mathcal{H}^{2/(1+\tau)}(W(\tau)) = \infty.$$

Theorem 1.26 gives the f-Hausdorff measure of the set $W(\psi)$ for arbitrary approximating function ψ. The Hausdorff dimension of this set is related to the lower order of $1/\psi$ at infinity.

Definition 1.27. Let $f : \mathbb{N} \to (0, \infty)$ be a function. We define the *lower logarithmic order of f at infinity* to be the number

$$\lambda(f) = \liminf_{q \to \infty} \frac{\log f(q)}{\log q} \, .$$

Dodson [18] calculated the Hausdorff dimension of $W(\psi)$ and its higher-dimensional analogues. Levesley [41] generalised the result to the inhomogeneous case. Here we state explicitly only the one-dimensional result, which gives the Hausdorff dimension of the set $W(\gamma, \psi)$.

Theorem 1.28. *Let $\psi : \mathbb{N} \to (0, \infty)$ be any function. If $\lambda = \lambda(1/\psi)$ is the lower logarithmic order of $1/\psi$, then*

$$\dim W(\gamma, \psi) \; = \; \begin{cases} 1, & \text{if } \lambda < 1 \\ \dfrac{2}{1+\lambda}, & \text{if } \lambda \geq 1. \end{cases}$$

1.4.4 The Hausdorff theory of badly approximable numbers

It turns out that we can calculate the precise value of the Hausdorff dimension of many sets which appear in the theory of Diophantine Approximation.

We have already seen that **Bad** is minimal in terms of Lebesgue measure. However, according to the following theorem due to Jarník [34], **Bad** is maximal in terms of Hausdorff dimension. We present the proof given in [6].

Theorem 1.29 (Jarník)**.** *The Hausdorff dimension of* **Bad** *is full in the unit interval; that is,*

$$\dim \mathbf{Bad} = 1.$$

Proof. Let $R \geq 4$ be an integer and $0 < \delta < \frac{1}{2}$. For $n = 1, 2, \ldots$ define

$$Q_n \; = \; \left\{ \frac{p}{q} \in \mathbb{Q} \, : \, (p,q) = 1 \quad \text{and} \quad R^{\frac{n-3}{2}} \leq q < R^{\frac{n-2}{2}} \right\} .$$

Clearly the sets Q_n, $n \geq 1$ form a partition of $\mathbb{Q}$. For each rational $p/q \in \mathbb{Q}$ with $(p,q) = 1$ define the interval

$$\Delta\left(\frac{p}{q}\right) = \left\{x \in [0,1) : \left|x - \frac{p}{q}\right| < \frac{\delta}{R^n}\right\}.$$

We construct inductively a sequence $(E_n)_{n=0}^{\infty}$ of sets with the following properties:

(i) $E_0 \supseteq E_1 \supseteq E_2 \supseteq \dots$

(ii) For $n = 1, 2, \dots$ the set E_n is the union of $(R-2)^n$ intervals of length R^{-n}.

(iii)For $n = 1, 2, \dots$ $E_n \cap \Delta(p/q) = \emptyset$ for all $p/q \in Q_n$.

Set $E_0 = [0,1]$. Assume for some $n \geq 1$ we have defined the sets $E_1, \dots E_{n-1}$ such that the desired properties hold. Observe that for any distinct points p_1/q_1, $p_2/q_2 \in Q_n$,

$$\left|\frac{p_1}{q_1} - \frac{p_2}{q_2}\right| = \frac{|p_1q_2 - p_2q_1|}{q_1q_2} > \frac{1}{R^{n-2}}. \tag{1.15}$$

Now we partition E_{n-1} into consecutive subintervals of length R^{-n}. Let I be any of these subintervals.

Claim: I intersects at most one interval of the form $\Delta(p/q)$, $p/q \in Q_n$.
Proof of Claim: For each $p/q \in Q_n$,

$$\left|\Delta\left(\frac{p}{q}\right)\right| = \frac{2\delta}{R^n} < \frac{1}{R^n}.$$

Also by (1.15) intervals $\Delta(p/q)$, $p/q \in Q_n$ have their centers at distance at least $R^{-(n-2)}$, hence the Claim follows.

In view of the Claim, if we divide each of the $(R-2)^{n-1}$ intervals of E_{n-1} into R consecutive subintervals of length R^{-n} we can choose $R-2$ of them so that the resulting set E_n does not intersect any of the $\Delta(p/q)$, $p/q \in Q_n$ and hence property (iii) is satisfied. This completes the construction of the sequence $(E_n)_{n=1}^{\infty}$. Finally define

$$K_R = \bigcap_{n=1}^{\infty} E_n.$$

We show that $K_R \subseteq$ **Bad**. Indeed, let $x \in K_R$ and $p/q \in \mathbb{Q}$. Let $n \in \mathbb{N}$ be the unique positive integer such that $p/q \in Q_n$. Then $x \in E_n \subseteq \Delta(p/q)^c$, so

$$\left|x - \frac{p}{q}\right| \geq \frac{\delta}{R^n} \geq \frac{\delta}{R^3q^2}.$$

Thus for each integer $R \geq 4$, the set **Bad** contains a subset K_R, the Hausdorff dimension of which is $\dim K_R = \dfrac{\log(R-2)}{\log R}$ (see Remark 6 and the relevant subsection). Hence

$$\dim \mathbf{Bad} \geq \limsup_{R\to\infty} \frac{\log(R-2)}{\log R} = 1\,.$$

□

Jarník [34] also gave estimates for the Hausdorff dimension of the sets F_N, $N \geq 1$ defined in (1.5).

Theorem 1.30 (Jarník)**.** *For $N > 8$, the Hausdorff dimension of the set F_N is*

$$1 - \frac{4}{N\log 2} \leq \dim \mathrm{F}_N \leq 1 - \frac{1}{8N\log N}\,.$$

It has been shown that the set of inhomogeneously badly approximable points is of full Hausdorff dimension, as well.

Theorem 1.31. *Let $\gamma \in \mathbb{R}$. The set of inhomogeneously badly approximable points with respect to γ has full Hausdorff dimension; that is,*

$$\dim \mathbf{Bad}_\gamma = 1.$$

For the proof of this theorem we refer to [7]. The result proved there is much stronger, since it is actually shown that $\dim(\mathbf{Bad}\cap\mathbf{Bad}_\gamma) = 1$.

Remark 9. The results regarding the Hausdorff dimension of the aforementioned sets of badly approximable numbers can be obtained via the theory of *Schmidt games* introduced in [57]. Using this theory it has been shown that if $(\gamma_n)_{n=1}^\infty$ is a sequence of reals, then

$$\dim\left(\bigcap_{n=1}^{\infty} \mathbf{Bad}_{\gamma_n}\right) = 1.$$

We refer the reader to [7] and [20] for more information.

1.5 Diophantine Approximation on higher dimensions

In this subsection we give a brief description of the higher-dimensional analogues of the notions and results we have already presented in the one-dimensional case. The one-dimensional results admit generalisations in two different setups in higher dimensions, namely the setup of simultaneous Diophantine approximation and that of dual Diophantine approximation. Here we only present results relevant only to the former.

First, let us begin with some useful notation. As in the one-dimensional case, we restrict our attention to the unit cube $[0,1)^n$. The generic element $\mathbf{x} \in [0,1)^n$ is a vector $\mathbf{x} = (x_1, \ldots, x_n)$. For such an $\mathbf{x} \in [0,1)^n$, we seek rational approximates of the form $\mathbf{p}/q \in \mathbb{Q}^n$. For $\mathbf{x} \in [0,1)^n$ we define

$$\|\mathbf{x}\| = \max\{ \|x_1\|, \ldots, \|x_n\| \}.$$

Dirichlet's theorem admits the following generalisation, which is also proved using the pigeonhole principle. Alternatively it is an easy consequence of Minkowski's convex body theorem from the geometry of numbers.

Theorem 1.32 (Dirichlet's Theorem in $\mathbb{R}^n$)**.** *Let $\mathbf{x} \in \mathbb{R}^n$ and $N \in \mathbb{N}$ be a positive integer. There exists a positive integer $1 \leq q \leq N$ such that*

$$\|q\mathbf{x}\| \leq \frac{1}{N^{1/n}}.$$

Corollary 1.33. *Let $\mathbf{x} \in \mathbb{R}^n$. There exist infinitely many positive integers $q = 1, 2, \ldots$ such that*

$$\|q\mathbf{x}\| \leq \frac{1}{q^{1/n}}.$$

If $\psi : \mathbb{N} \to \mathbb{R}^+$ is an approximating function, we define the set $W_n(\psi)$ of *simultaneously ψ-well approximable* points to be

$$W_n(\psi) = \{\mathbf{x} \in [0,1)^n : \|q\mathbf{x}\| \leq \psi(q) \text{ for infinitely many } q \in \mathbb{N}\}.$$

A generalised version of Khintchine's theorem gives the n-dimensional Lebesgue measure of $W_n(\psi)$.

Theorem 1.34 (Khintchine's Theorem in $\mathbb{R}^n$)**.** *Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a decreasing function. Then*

$$|W_n(\psi)| = \begin{cases} 0, & \text{if } \sum_{q=1}^{\infty} \psi(q)^n < \infty \\ 1, & \text{if } \sum_{q=1}^{\infty} \psi(q)^n = \infty. \end{cases}$$

Remark 10. As we have seen, in the one-dimensional case the assumption of monotonicity cannot be removed. However Gallagher [28] has proved that for $n \geq 2$ the conclusion in Khintchine's theorem is true without the assumption of monotonicity.

The Khintchine-Jarník theorem can also be generalised to the multiple dimensions setup and to the inhomogeneous setup. If $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_n) \in \mathbb{R}^n$ and $\psi : \mathbb{N} \to \mathbb{R}^n$ is an approximating function, define the set

$$W_n(\boldsymbol{\gamma}, \psi) = \{\mathbf{x} \in [0,1)^n : \|q\mathbf{x} - \boldsymbol{\gamma}\| \leq \psi(q) \quad \text{for infinitely many } q \in \mathbb{N}\}.$$

The inhomogeneous Khintchine-Jarník theorem is the following statement. For details see [6, 3, 4].

Theorem 1.35. *Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be decreasing and let f be a dimension function such that $f(r)/r^n$ is decreasing. Then*

$$\mathcal{H}^f\left(W_n(\boldsymbol{\gamma}, \psi)\right) = \begin{cases} 0, & \text{if } \sum\limits_{q=1}^{\infty} q^n f(\psi(q)/q)^n < \infty \\ \mathcal{H}^f([0,1)^n), & \text{if } \sum\limits_{q=1}^{\infty} q^n f(\psi(q)/q)^n = \infty. \end{cases}$$

Let us write $W_n(\boldsymbol{\gamma}, \tau)$ for the set $W_n(\boldsymbol{\gamma}, \psi)$ when $\psi(q) = q^{-\tau}$. The Hausdorff dimension of the set $W_n(\boldsymbol{\gamma}, \tau)$ follows as a direct corollary of the previous theorem.

Corollary 1.36. *Let $\tau > \frac{1}{n}$ and $\boldsymbol{\gamma} \in \mathbb{R}^n$. The Hausdorff dimension of the set $W_n(\boldsymbol{\gamma}, \tau)$ is*

$$\dim W_n(\boldsymbol{\gamma}, \tau) = \frac{1+n}{1+\tau}.$$

1.6 Diophantine Approximation with Restricted Denominators

In the field of Diophantine Approximation, one can consider the problem of approximating real numbers by rationals with denominators in a fixed subset of positive integers. If $\mathcal{A} \subseteq \mathbb{N}$ is a subset of the positive integers and $\psi : \mathbb{N} \to \mathbb{R}^+$ is an approximating function, define the set

$$W_{\mathcal{A}}(\psi) = \{x \in [0,1) : \|qx\| \leq \psi(q) \text{ for infinitely many } q \in \mathcal{A}\}. \tag{1.16}$$

For the specific choice $\psi(q) = q^{-\tau}$ of the approximating function, the above set is denoted by $W_{\mathcal{A}}(\tau)$.

A metric result for sets of the form $W_{\mathcal{A}}(\psi)$ can be proved when the denominators are restricted to the set of prime numbers. The proof is a simple application of Theorem 1.14, see [30, p. 27].

Theorem 1.37. *Let $\mathcal{P} \subseteq \mathbb{N}$ be the set of primes, and $\psi : \mathbb{N} \to \mathbb{R}^+$ be a function. Then*

$$|W_{\mathcal{P}}(\psi)| = \begin{cases} 0, & \text{if } \sum\limits_{p \in \mathcal{P}} \psi(p) < \infty \\ 1, & \text{if } \sum\limits_{p \in \mathcal{P}} \psi(p) = \infty. \end{cases}$$

As we shall see later in Chapter 2, a case of special interest is when the sequence $\mathcal{A}$ is lacunary. In that case, we can prove a Khintchine-type metrical result for the set $W_{\mathcal{A}}(\psi)$. The proof we present here follows to a large extent the theory developed in [4].

Let us first give the definition of a lacunary sequence.

Definition 1.38. A sequence $(q_n)_{n=1}^{\infty}$ of positive integers is called *lacunary* if there exists a constant $K > 1$ such that

$$\frac{q_{n+1}}{q_n} \geq K, \qquad n = 1, 2, \dots \tag{1.17}$$

Theorem 1.39. *Let $\mathcal{A} = (q_n)_{n=1}^{\infty}$ be a lacunary sequence of positive integers and $\psi : \mathbb{N} \to \mathbb{R}^+$ be a function. If $W_{\mathcal{A}}(\psi)$ is the set defined in* (1.16)*, then its Lebesgue measure is*

$$|W_{\mathcal{A}}(\psi)| = \begin{cases} 0, & \text{if } \sum\limits_{n=1}^{\infty} \psi(q_n) < \infty \\ 1, & \text{if } \sum\limits_{n=1}^{\infty} \psi(q_n) = \infty. \end{cases}$$

Proof. Define the sets

$$A_n = \bigcup_{p=0}^{q_n - 1} B\left(\frac{p}{q_n}, \frac{\psi(q_n)}{q_n}\right), \qquad n \geq 1.$$

Then clearly $W_{\mathcal{A}}(\psi) = \limsup\limits_{n\to\infty} A_n$ and the convergence case follows from the Borel-Cantelli Lemma. For the divergence case, it suffices to show that $(A_n)_{n=1}^{\infty}$ is a sequence of quasi-independent on average sets. Since the sequence $(q_n)_{n=1}^{\infty}$ is lacunary, there exists a constant $K > 1$ such that (1.17) holds.

Let $m < n$. Each of the intervals of A_m has the form $B(p/q_m, \psi(q_m)/q_m)$ and contains at most

$$\frac{2\psi(q_m)/q_m}{1/q_n} + 2 = \frac{2\psi(q_m)}{q_m} q_n + 2$$

points of the form p/q_n, so

$$\left| B\left(\frac{p}{q_m}, \frac{\psi(q_m)}{q_m}\right) \cap A_n \right| \leq \left(\frac{2\psi(q_m)}{q_m} q_n + 2\right) \frac{2\psi(q_n)}{q_n}$$

and

$$|A_m \cap A_n| \leq \left(\frac{2\psi(q_m)}{q_m} q_n + 2\right) \frac{2\psi(q_n)}{q_n} q_m .$$

Hence for all N large enough,

$$\begin{aligned}
\sum_{n=1}^{N} \sum_{m=1}^{N} |A_m \cap A_n| &\leq 2 \sum_{\substack{1\leq m\leq N \\ m<n\leq N}} \sum |A_m \cap A_n| \\
&\leq 2 \sum_{\substack{1\leq m\leq N \\ m<n\leq N}} \sum 2\frac{\psi(q_m)}{q_m} q_n \cdot \frac{2\psi(q_n)}{q_n} q_m + 2 \sum_{\substack{1\leq m\leq N \\ m<n\leq N}} \sum 4\frac{\psi(q_n)}{q_n} q_m \\
&\leq 2 \sum_{\substack{1\leq m\leq N \\ m<n\leq N}} \sum |A_m||A_n| + 2 \sum_{\substack{1\leq m\leq N \\ m<n\leq N}} \sum 4\psi(q_m) \frac{q_m}{q_n} \\
&\ll \sum_{m=1}^{N} \left(|A_m| \sum_{n=m+1}^{N} |A_n| \right) + \sum_{m=1}^{N} \left(\psi(q_m) \sum_{n=m+1}^{N} \frac{1}{K^{n-m}} \right) \\
&\ll \left(\sum_{n=1}^{N} |A_n| \right)^2 + \sum_{n=1}^{N} |A_n| \\
&\ll \left(\sum_{n=1}^{N} |A_n| \right)^2 .
\end{aligned}$$

The last estimate is because

$$\sum_{n=1}^{\infty} |A_n| = \sum_{n=1}^{\infty} 2\psi(q_n) = \infty .$$

We have actually shown that for all N,

$$\sum_{n=1}^{N} \sum_{m=1}^{N} |A_m \cap A_n| \leq C \left(\sum_{n=1}^{N} |A_n| \right)^2 ,$$

where $C > 0$ is an absolute constant. So by the divergence Borel-Cantelli Lemma (Lemma 1.11),

$$|W_{\mathcal{A}}(\psi)| = \left|\limsup_{n\to\infty} A_n\right| > 0.$$

The fact that $|W_{\mathcal{A}}(\psi)| = 1$ follows by Cassels' zero-one law (Theorem 1.10), because $|W_{\mathcal{A}}(\psi)| = |W(\widetilde{\psi})|$, where $\widetilde{\psi} : \mathbb{N} \to \mathbb{R}^+$ is defined by

$$\widetilde{\psi}(n) = \begin{cases} \psi(n), & \text{if } n \in \mathcal{A} \\ 0, & \text{if } n \notin \mathcal{A}. \end{cases}$$

□

Regarding the Hausdorff theory of restricted Diophantine approximation, Borosh and Fraenkel [11] have calculated the Hausdorff dimension of the sets $W_{\mathcal{A}}(\tau)$.

Theorem 1.40 (Borosh, Fraenkel). *If $\mathcal{A} \subseteq \mathbb{N}$ is a set of positive integers, then the set $W_{\mathcal{A}}(\tau)$ has Hausdorff dimension*

$$\dim W_{\mathcal{A}}(\tau) \;=\; \min\left\{\frac{1+\nu(\mathcal{A})}{1+\tau}\,,\,1\right\}\,,$$

where $\nu(\mathcal{A})$ is defined by

$$\nu(\mathcal{A}) = \inf\left\{\eta > 0 \;:\; \sum_{q\in\mathcal{A}} \frac{1}{q^{\eta}} < \infty\right\}.$$

For more details on Diophantine approximation with restricted denominators we refer to [30].

1.7 Potential theory and Hausdorff dimension

We have already discussed several ways to calculate the Hausdorff dimension of a given set. In addition to geometric methods, potential theory provides another powerful tool to estimate the Hausdorff dimension of a given set. We will make use of potential theoretic notions and results in Chapter 4.

1.7.1 Fourier transforms of Probability Measures

Given a probability measure μ supported on a subset of $\mathbb{R}^n$, the *Fourier transform* of μ is the function $\widehat{\mu} : \mathbb{R}^n \to \mathbb{C}$ defined by

$$\widehat{\mu}(t) \;=\; \int e^{-2\pi i t\cdot x}\mathrm{d}\mu(x), \qquad t \in \mathbb{R}^n\,.$$

When the measure μ is supported on a subset of the real line, the Fourier transform is simply the function

$$\widehat{\mu}(t) \;=\; \int e^{-2\pi i t x} \mathrm{d}\mu(x), \qquad t \in \mathbb{R}\,.$$

For example, if λ denotes the restriction of the Lebesgue measure on the unit interval $[0,1]$, direct calculation shows that

$$\widehat{\lambda}(t) \;=\; \int_0^1 e^{-2\pi i t x} \mathrm{d}x = \begin{cases} 1, & \text{if } t = 0, \\ \dfrac{1 - e^{-2\pi i t}}{2\pi i t}, & \text{if } t \neq 0. \end{cases}$$

The properties of the Fourier transform $\widehat{\mu}$ reflect many of the properties of the measure μ. The following fundamental inversion theorem asserts that distinct measures have distinct Fourier transforms.

Theorem 1.41. *Let μ be a probability measure supported on a subset of $\mathbb{R}$ and $a, b \in \mathbb{R}$ be points with $\mu(\{a\}) = \mu(\{b\}) = 0$. Then*

$$\mu\left((a,b]\right) \;=\; \lim_{T\to\infty} \int_{-T}^{T} \frac{e^{-2\pi i t a} - e^{-2\pi i t b}}{it} \widehat{\mu}(t)\mathrm{d}t.$$

We refer to [9] for the proof and for more details on the Fourier transform and connections with probability theory.

Remark 11. In the setup of Diophantine Approximation we restrict our attention to subsets of $[0,1)$, which can be identified with the unit circle $\mathbb{T}$. For a probability measure μ supported on $\mathbb{T}$, it is common to refer to its *Fourier coefficients* defined by

$$\widehat{\mu}(n) = \int e^{-2\pi i n t} \mathrm{d}\mu(t), \qquad n \in \mathbb{Z}$$

instead of the Fourier transform. Clearly the two definitions coincide for integral values of the argument. In what follows we consider all probability measures as measures on the real line and follow the former definition.

1.7.2 Energies and Capacities

This subsection is a brief introduction to the notion of s-energy of a measure and s-capacity of a set and their connection with Hausdorff dimension.

Definition 1.42. Let $s \geq 0$. If μ is a probability measure supported on a subset of $\mathbb{R}^n$, we define the *s-potential* at $x \in \mathbb{R}^n$ due to μ to be

$$\phi_s(x) = \int \frac{\mathrm{d}\mu(y)}{|x-y|^s}.$$

The s-*energy* of μ is defined to be

$$I_s(\mu) = \int \phi_s(x)\mathrm{d}\mu(x) = \int\int \frac{\mathrm{d}\mu(x)\mathrm{d}\mu(y)}{|x-y|^s}.$$

Definition 1.43. Let $s \geq 0$. Let $A \subseteq R^n$ be a Borel set. The s-capacity of A is defined to be

$$C_s(A) = \sup\left\{\frac{1}{I_s(\mu)} : \mu \in M_1^+(A)\right\}.$$

Here $M_1^+(A)$ denotes the set of positive probability measures supported on the Borel set $A \subseteq \mathbb{R}$. The actual connection between Hausdorff dimension and capacities is given by the following theorem.

Theorem 1.44. *Let $A \subseteq \mathbb{R}^n$. Then*

$$\dim A = \sup\{s > 0 : C_s(A) > 0\} = \inf\{s > 0 : C_s(A) = 0\}.$$

We do not give a complete proof for the relation of s-capacity with Hausdorff dimension, as this will be done later in the text, when the generalised energy and capacity are introduced. Also see [23, Chapter 4] and [44, Chapter 8]. Instead we present the clear heuristic explantion given in [44].

If μ is a probability measure supported in a Borel set $A \subseteq \mathbb{R}^n$ and $s, t, c > 0$ consider the conditions

$$\mu\left(B(x,r)\right) \leq cr^s, \quad x \in \mathbb{R}^n,\, r > 0 \tag{1.18}$$

and

$$I_t(\mu) < \infty. \tag{1.19}$$

We already know by the Mass Distribution Principle (Theorem 1.23) that if μ satisfies (1.18), then A has Hausdorff dimension $\dim A \geq s$. Let us see how (1.18) and (1.19) are related. Observe that

$$\begin{aligned}\phi_t(x) &= \int \frac{\mathrm{d}\mu(y)}{|x-y|^t} \\ &= \int_0^\infty \mu\left(\{y : |x-y|^{-t} \geq u\}\right)\mathrm{d}u\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty \mu\left(B(x,u^{-1/t})\right) \mathrm{d}u \\
&= t\int_0^\infty \frac{\mu\left(B(x,r)\right)}{r^{t+1}} \mathrm{d}r\,.
\end{aligned}
$$

Thus if (1.18) holds, then (1.19) is satisfied for any $t < s$.

Conversely, if (1.19) holds, there exists some $M > 0$ such that the set

$$A_1 = \{x \,:\, \phi_t(x) \le M\}$$

has $\mu(A_1) > 0$. Let ν be the restriction of μ on A_1. Then for any $x \in A_1$, $r > 0$ we have

$$
\begin{aligned}
\nu\left(B(x,r)\right) &= \int_{B(x,r)} \mathrm{d}\mu(y) \\
&\le r^t \int_{B(x,r)} \frac{\mathrm{d}\mu(y)}{|x-y|^t} \\
&\le Mr^t,
\end{aligned}
$$

Also for any $x \in A$ and $r > 0$ such that $B(x,r) \cap A_1 \neq \emptyset$, there exists $z \in A_1$ with $B(x,r) \subseteq B(z,2r)$, so

$$\nu\left(B(x,r)\right) \le \nu\left(B(z,2r)\right) \le 2^t M r^t$$

and the measure ν satisfies (1.18) with $s = t$.

1.7.3 Frostman's Lemma for Fourier transforms

The result connecting the Fourier transform of a probability measure μ with the Hausdorff dimension of its support set is known as Frostman's Lemma. For completeness, we provide a proof of the theorem taken from [23, Chapter 4]. In order to do so, we briefly describe the necessary properties of Fourier transforms.

If μ is a probability measure supported in a subset of $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a function in $L_1(\mu)$, we define the convolution of f with μ to be the real-valued function

$$(f * \mu)(x) = \int f(x-y)\mathrm{d}\mu(y), \qquad x \in \mathbb{R}^n\,.$$

Then the Convolution Theorem states that if f and μ are as previously, the Fourier transform of $f * \mu$ satisfies

$$\widehat{(f * \mu)}(x) = \widehat{f}(x)\widehat{\mu}(x), \qquad x \in \mathbb{R}^n$$

where $\widehat{f}(x)$ denotes the Fourier transform of the function f. We are now in a position to present Frostman's Lemma for Fourier transforms.

Theorem 1.45. *Let μ be a probability measure supported on a set $F \subseteq \mathbb{R}^n$. If there exist constants $c, \eta > 0$ such that*

$$|\widehat{\mu}(x)| \leq c\,|x|^{-\eta/2}, \qquad |x| \neq 0 \tag{1.20}$$

then $\dim_H F \geq \min\{n, \eta\}$.

Proof. For any $s > 0$, consider the function $f(x) = |x|^{-s}$. Then the s-potential due to the measure μ is by definition

$$\phi_s(x) = \int \frac{\mathrm{d}\mu(y)}{|x-y|^s} = (f * \mu)(x).$$

By the Convolution Theorem,

$$\begin{aligned} \widehat{\phi_s}(x) &= \widehat{f}(x)\widehat{\mu}(x) \\ &= C(n,s)\frac{1}{|x|^{n-s}}\widehat{\mu}(x). \end{aligned}$$

Thus

$$\begin{aligned} I_s(\mu) &= \int \phi_s(x)\mathrm{d}\mu(x) \\ &= \int \widehat{\phi}_s(t)\overline{\widehat{\mu}(t)}\mathrm{d}\mu \qquad \text{(by Parseval's Theorem)} \\ &= C(n,s)\int |t|^{s-n}|\widehat{\mu}(t)|^2\mathrm{d}\mu(t). \end{aligned}$$

This implies that whenever μ satisfies (1.20) then for any $s < \eta$ the s-energy of μ is finite, and hence $\dim F \geq \eta$ by Theorem 1.44 provided $\eta \leq n$. $\square$

Frostman's Lemma for Fourier transforms naturally gives rise to the following definition:

Definition 1.46. If $A \subseteq \mathbb{R}^n$ is a Borel set, the *Fourier dimension* of A is defined as

$$\dim_F A = \sup\left\{\eta \in [0,n] \,:\, \exists \mu \in M_1^+(A) \text{ s.t. } \widehat{\mu}(t) = O\left(|t|^{-\eta/2}\right),\ |t| \to \infty\right\}.$$

Theorem 1.45 implies that for any $A \subseteq \mathbb{R}^n$, we have $\dim_F A \le \dim A$. There exist subsets of $\mathbb{R}$ for which the aforementioned relation is true with equality.

Definition 1.47. A Borel set $A \subseteq \mathbb{R}^n$ is called a *Salem set* if its Fourier dimension is equal to its Hausdorff dimension, i.e.

$$\dim_F A = \dim A.$$

Trivial examples of Salem sets are intervals $[a,b] \subseteq \mathbb{R}$ as well as finite sets $X \subseteq \mathbb{R}^n$. Kaufman [39] showed that for $\tau > 1$ the set $W(\tau)$ of τ-well approximable numbers is a Salem set. We will show in Chapter 3 that the set $W(\gamma,\psi)$ of inhomogeneous ψ-well approximable numbers is also a Salem set.

Chapter 2

Inhomogeneous Diophantine Approximation on M_0 sets with restricted denominators

2.1 Introduction and main result

In the present chapter we formulate and prove a Khintchine-type theorem within the framework of inhomogeneous Diophantine approximation. Here the denominators q of the "shifted" rational approximates $(p+\gamma)/q$ form a lacunary sequence and the "size" of the set of well approximable numbers is measured in terms of probability measures μ with sufficiently rapid Fourier transform decay. Subsets of the real line which support a probability measure with Fourier transform tending to zero are referred to as M_0 sets. Results in this chapter are joint work with A.D. Pollington, S. Velani and E. Zorin.

2.1.1 Motivation

A long-standing open problem in Diophantine approximation dating back to the 1930s is Littlewood's Conjecture.

Littlewood's Conjecture : For all $\alpha, \beta \in [0, 1)$,

$$\liminf_{q\to\infty} q\|q\alpha\|\|q\beta\| = 0. \tag{2.1}$$

Observe that (2.1) is trivially true if α is not an element of **Bad**. In view of this observation, Pollington & Velani [51] consider the following basic question: Given $\alpha \in$ **Bad**, are there $\beta \in$ **Bad** such that the pair (α, β) satisfies condition (2.1)?

Their answer to this question is positive. To be more specific, they prove the following statement:

Theorem 2.1 (Pollington, Velani)**.** *Given $\alpha \in$ **Bad**, there exists a subset $\mathbb{G}(\alpha) \subseteq$ **Bad** with $\dim \mathbb{G}(\alpha) = 1$ such that for any $\beta \in \mathbb{G}(\alpha)$,*

$$q\,||q\alpha||\,||q\beta|| \;\leq\; \frac{1}{\log q} \qquad \text{for infinitely many } q = 1, 2 \ldots .$$

A key fact for the proof of Theorem 2.1 is the following result proved by Kaufman [37] and later improved by Queffelec & Ramaré [52].

Theorem 2.2 (Kaufman, Queffelec & Ramaré)**.** *Let $N \geq 2$. For any $\frac{1}{2} < \delta < \dim F_N$, the set F_N supports a probability measure $\mu = \mu(N, \delta)$ with the following properties:*

(i) $\mu(J) \leq c_1 |J|^\delta$ for any interval J of sufficiently small length, and

(ii) $|\widehat{\mu}(t)| \leq c_2 (1 + |t|)^{-\eta}$ for all $t \in \mathbb{R}$, where $\eta > 0$ is a constant.

Here $c_1, c_2 > 0$ are absolute constants.

The proof of Theorem 2.1 in [51] involves the following steps: The set $\mathbb{G}(\alpha)$ is defined as the union of the sets

$$\mathbb{G}_N(\alpha) = \{x \in F_N : \|q_n x\| \leq 1/\log q_n \ \text{ for infinitely many } n \in \mathbb{N}\},$$

where $(q_n)_{n=1}^\infty$ is the sequence of denominators of α. This implies that

$$q_n \|q_n x\| \;\leq\; 1, \qquad n = 1, 2, \ldots$$

hence for any $\beta \in \mathbb{G}(\alpha)$ we have

$$q_n \|q_n \alpha\| \|q_n \beta\| \leq \frac{1}{\log q_n}, \qquad n = 1, 2, \ldots .$$

This clearly shows the connection of the set $\mathbb{G}(\alpha)$ with the statement in Theorem 2.1. It is then shown that

$$\mu(\mathbb{G}_N(\alpha)) > 0$$

for any of the aforementioned measures μ supported on F_N. Then, fact (i) together with the Mass Distribution Principle implies that the set $\mathbb{G}_N(\alpha)$ has Hausdorff dimension equal to that of F_N, and thus $\dim \mathbb{G}(\alpha) = 1$.

Theorem 2.1 naturally opens up some directions for further research. To be more specific, the following questions arise naturally.

- Can we prove a similar statement in the setup of inhomogeneous Diophantine Approximation?

- Can we prove a full measure statement rather than just a positive measure statement?

- If $\alpha \in \mathbf{Bad}$, the corresponding sequence $(q_n)_{n=1}^{\infty}$ of denominators satisfies the inequality
$$K_1 \leq \frac{q_{n+1}}{q_n} \leq K_2, \qquad n = 1, 2, \ldots \tag{2.2}$$
where $K_2 > K_1 > 1$ are constants. Can this assumption on the sequence $(q_n)_{n=1}^{\infty}$ be relaxed?

- Can we show a result for arbitrary approximating functions $\psi : \mathbb{N} \to \mathbb{R}^+$, rather than just for the specific choice $\psi(q) = \dfrac{1}{\log q}$?

- Can we prove a result valid for arbitrary probability measures with sufficient assumptions on the decay rate of their Fourier transform?

These are the problems we intend to answer, at least partially, in the current chapter.

2.1.2 The main result

The following theorem is the main result of this chapter.

Theorem 2.3. *Let $\gamma \in \mathbb{R}$, $\psi : \mathbb{N} \to \mathbb{R}^+$ be a decreasing function, $\mathcal{A} = (q_n)_{n=1}^{\infty} \subseteq \mathbb{N}$ be a lacunary sequence and μ be a Borel probability measure on $[0,1)$. Assume that*
$$\widehat{\mu}(t) = O\left(\frac{1}{(\log |t|)^A}\right), \qquad |t| \to \infty \tag{2.3}$$
where $A > 5$ is a constant. Consider the set
$$W_{\mathcal{A}}(\gamma; \psi) = \left\{x \in [0,1) : \|q_n x - \gamma\| \leq \psi(q_n) \text{ for infinitely many } n \in \mathbb{N}\right\}.$$
Then
$$\mu(W_{\mathcal{A}}(\gamma;\psi)) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} \psi(q_n) < \infty \\ 1, & \text{if } \sum_{n=1}^{\infty} \psi(q_n) = \infty. \end{cases}$$

In the case of convergence we are able to prove the following stronger statement for general sequences.

Theorem 2.4. *Let $\gamma \in \mathbb{R}$, $\psi : \mathbb{N} \to \mathbb{R}^+$ be a decreasing function, $\mathcal{A} = (q_n)_{n=1}^{\infty} \subseteq \mathbb{N}$ be a sequence of integers and μ be a Borel probability measure on $[0,1)$. Assume that*

$$\sum_{n=1}^{+\infty} \sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} \frac{|\widehat{\mu}(kq_n)|}{|k|} < \infty. \tag{2.4}$$

Then

$$\mu(W_{\mathcal{A}}(\gamma;\psi)) = 0 \quad \text{if} \quad \sum_{n=1}^{\infty} \psi(q_n) < \infty. \tag{2.5}$$

It is easily verified that Theorem 2.4 implies the convergence case of Theorem 2.3. Indeed, if $\mathcal{A} = (q_n)_{n=1}^{\infty}$ is lacunary and the probability measure μ satisfies (2.3), then

$$\begin{aligned}
\sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} \frac{|\widehat{\mu}(kq_n)|}{|k|} &\ll \sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} \frac{1}{|k|(\log|kq_n|)^A} \\
&\ll \frac{1}{(\log q_n)^{A-2}} \\
&\ll \frac{1}{n^{A-2}}, \quad n \to \infty \qquad \text{(by the lacunarity of } (q_n)_{n=1}^{\infty} \text{)}
\end{aligned}$$

hence (2.4) is satisfied.

Remark 12. Theorem 2.3 is a Khintchine-type result for denominators restricted to a lacunary sequence. We would like to know if similar results are valid for other sequences of denominators with slower growth rate. Consider for example the sequence

$$\begin{aligned}
\mathcal{A}_1 &:= \{2^a 3^b \,:\, a, b \geq 0\} \\
&= \{q_1 < q_2 < \dots\}.
\end{aligned}$$

It can be shown that $\lim_{n\to\infty} \frac{q_{n+1}}{q_n} = 1$, hence $\mathcal{A}_1 = (q_n)_{n=1}^{\infty}$ is not lacunary. It is an open problem whether we can prove a statement like Theorem 2.3 for the set $W_{\mathcal{A}_1}(\gamma,\psi)$, maybe with more restrictions on the probability measure μ.

2.2 Proof of Theorems

Let $\gamma \in \mathbb{R}$, $\psi : \mathbb{N} \to \mathbb{R}^+$ and $\mathcal{A} = (q_n)_{n=1}^{\infty}$ be as in Theorem 2.3. For $n = 1, 2, \dots$ define the set

$$E_n = \{x \in [0,1) : \|q_n x - \gamma\| \leq \psi(q_n)\}. \tag{2.6}$$

Then the set $W_{\mathcal{A}}(\gamma;\psi)$ can be expressed as

$$W_{\mathcal{A}}(\gamma;\psi) \;=\; \limsup_{n\to\infty} E_n.$$

2.2.1 The convergence case

Taking into account the previous remarks, we estimate the measure of the sets $(E_n)_{n=1}^{\infty}$.

Lemma 2.5. *Let $\psi:\mathbb{N}\to\mathbb{R}^+$ be a decreasing function. For all $q\in\mathbb{N}$ consider the set $E_q^{\gamma}(\psi)=\{x\in[0,1):\|qx-\gamma\|\le\psi(q)\}$. Suppose μ is a Borel probability measure supported on $[0,1)$ such that*

$$\widehat{\mu}(t) \;=\; O\left(\frac{1}{(\log|t|)^A}\right), \qquad |t|\to\infty \tag{2.7}$$

where $A>5$ is a constant. Then

$$\mu(E_q^{\gamma}(\psi)) \;=\; 2\psi(q)+O\left(\frac{1}{(\log q)^{\Gamma}}\right), \qquad q\to\infty$$

where $\Gamma>4$ is a constant.

Proof. Fix $q\in\mathbb{N}$ and set $\delta=\dfrac{\psi(q)}{q}>0$. Consider the function

$$\chi_\delta:[0,1)\to\mathbb{R}, \qquad \chi_\delta(x)=\begin{cases}1, & \text{if } \|x\|\le\delta\\ 0, & \text{if } \|x\|>\delta\end{cases}$$

and its continuous approximation

$$\chi_{\delta,\varepsilon}^{+}:[0,1)\to\mathbb{R}, \qquad \chi_{\delta,\varepsilon}^{+}(x)=\begin{cases}1, & \text{if } \|x\|\le\delta\\ 1+\dfrac{1}{\varepsilon}(\delta-\|x\|), & \text{if } \delta<\|x\|\le\delta+\varepsilon\\ 0, & \text{if } \|x\|>\delta+\varepsilon\end{cases}$$

where $0<\varepsilon\le\delta$. Also consider for each $q\in\mathbb{N}$ the function

$$W_{q,\gamma}^{+} \;=\; \sum_{p=0}^{q-1}\delta_{\frac{p+\gamma}{q}} * \chi_{\delta,\varepsilon}^{+}, \tag{2.8}$$

where δ_x denotes the Dirac function at $x\in\mathbb{R}$. This definition implies

$$W_{q,\gamma}^{+}(x)=\sum_{p=0}^{q-1}\chi_{\delta,\varepsilon}^{+}\left(x-\frac{p+\gamma}{q}\right),$$

hence

$$\mu(E_q^\gamma(\psi)) \leq \int_0^1 W_{q,\gamma}^+(x)\mathrm{d}\mu(x). \tag{2.9}$$

Regarding the Fourier coefficients of the functions $\chi_{\delta,\varepsilon}^+$, calculation yields

$$\widehat{\chi}_{\delta,\varepsilon}^+(k) = \begin{cases} 2\delta+\varepsilon, & \text{if } k=0, \\ \dfrac{\cos(2\pi k\delta)-\cos(2\pi k(\delta+\varepsilon))}{2\pi^2k^2\varepsilon}, & \text{if } k\neq 0 \end{cases}$$

By (2.8),

$$\widehat{W}_{q,\gamma}^+(k) = \sum_{p=0}^{q-1} \widehat{\delta}_{\frac{p+\gamma}{q}}(k)\widehat{\chi}_{\delta,\varepsilon}^+(k),$$

so for $k\neq 0$ we have

$$\widehat{W}_{q,\gamma}^+(k) = \begin{cases} \exp\left(-\dfrac{2\pi i k\gamma}{q}\right)\dfrac{q\left(\cos(2\pi k\delta)-\cos(2\pi k(\delta+\varepsilon))\right)}{2\pi^2k^2\varepsilon}, & \text{if } q\mid k \\ 0, & \text{if } q\nmid k \end{cases} \tag{2.10}$$

and

$$\widehat{W}_{q,\gamma}^+(0) = 2\psi(q)+q\varepsilon.$$

Since $\sum\limits_{k=-\infty}^{+\infty}\left|\widehat{W}_{q,\gamma}^+(k)\right| < \infty$,

$$W_{q,\gamma}^+(x) = \sum_{k=-\infty}^{+\infty} \widehat{W}_{q,\gamma}^+(k)e^{2\pi kix}$$

uniformly for all $0\leq x<1$. Integration with respect to μ gives

$$\begin{aligned}
\int_0^1 W^+_{q,\gamma}(x)\mathrm{d}\mu(x) &= \sum_{k=-\infty}^{+\infty} \widehat{W}^+_{q,\gamma}(k)\widehat{\mu}(-k) \\
&= 2\psi(q) + q\varepsilon + \sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} \widehat{W}^+_{q,\gamma}(k)\widehat{\mu}(-k) \\
&\overset{(2.10)}{=} 2\psi(q) + q\varepsilon + \sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} \widehat{W}^+_{q,\gamma}(kq)\widehat{\mu}(-kq).
\end{aligned}$$

Write $A = 5 + 2\eta$, where $\eta > 0$. The sum in the third term is

$$\begin{aligned}
\sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} \widehat{W}^+_{q,\gamma}(kq)\widehat{\mu}(-kq) &= \sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} e^{-2\pi ik\gamma}\frac{\cos(2\pi kq\delta) - \cos(2\pi kq(\delta+\varepsilon))}{2\pi^2k^2q\varepsilon}\widehat{\mu}(-kq) \\
&\ll \sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} \frac{1}{|k| \ \log^A(|kq|)} \\
&\ll \sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} \frac{1}{|k| \ \log^{1+\eta}(1+|k|)} \cdot \frac{1}{(\log q)^{4+\eta}} \\
&\ll \frac{1}{(\log q)^{4+\eta}}.
\end{aligned}$$

If $C > 0$ is the implicit constant in the previous estimates, letting $\varepsilon \to 0$ we obtain

$$\mu(E_q^\gamma(\psi)) \leq 2\psi(q) \ + \ \frac{C}{(\log q)^{4+\eta}}. \tag{2.11}$$

Now we introduce the lower approximating functions

$$\chi^-_{\delta,\varepsilon}(x) = \begin{cases} 1, & \text{if } \|x\| \leq \delta - \varepsilon \\ 1 + \dfrac{1}{\varepsilon}(\delta - \|x\|), & \text{if } \delta - \varepsilon < \|x\| \leq \delta, \\ 0, & \text{if } \|x\| > \delta \end{cases}$$

and

$$W^-_{q,\gamma} = \sum_{p=0}^{q-1} \delta_{p/q} * \chi^-_{\delta,\varepsilon}.$$

Their Fourier coefficients are

$$\widehat{\chi^-_{\delta,\varepsilon}}(k) \;=\; \begin{cases} 2\delta - \varepsilon, & \text{if } k = 0, \\ \dfrac{\cos(2\pi k(\delta-\varepsilon)) - \cos(2\pi k\delta)}{2\pi^2 k^2 \varepsilon}, & \text{if } k \neq 0 \end{cases}$$

and for $k \neq 0$,

$$\widehat{W^-_{q,\gamma}}(k) \;=\; \begin{cases} \exp\left(-\dfrac{2\pi i k\gamma}{q}\right) \dfrac{q\,(\cos(2\pi k(\delta-\varepsilon)) - \cos(2\pi k\delta))}{2\pi^2 k^2 \varepsilon}, & \text{if } q \mid k \\ 0, & \text{if } q \nmid k \end{cases}$$

while

$$\widehat{W^-_{q,\gamma}}(0) \;=\; 2\psi(q) - q\varepsilon\,.$$

Using the same arguments as previously, we obtain

$$\mu(E^\gamma_q(\psi)) \geq 2\psi(q) - \frac{C}{(\log q)^{4+\eta}}\,. \tag{2.12}$$

Combining (2.11) and (2.12) gives the required result. □

Now the convergence case of Theorem 2.3 follows easily from the Borel-Cantelli Lemma (Lemma 1.8). If $(E_n)_{n=1}^\infty$ is the sequence of sets defined in (2.6), we have shown that

$$\begin{aligned} \mu(E_n) \;&=\; 2\psi(q_n) \,+\, O\left(\frac{1}{(\log q_n)^\Gamma}\right) \\ &=\; 2\psi(q_n) \,+\, O\left(\frac{1}{n^\Gamma}\right), \quad n \to \infty\,. \end{aligned}$$

The last estimate in the error term is due to the lacunarity of the sequence $(q_n)_{n=1}^\infty$. Thus the convergence of $\sum_{n=1}^{\infty} \psi(q_n)$ in the hypothesis implies $\sum_{n=1}^{\infty} \mu(E_n) < \infty$ and finally $\mu(W_{\mathcal{A}}(\gamma;\psi)) = 0$.

The arguments used here can be utilised to prove Theorem 2.4, as well.

Proof of Theorem 2.4

Let $\chi^+_{\delta,\varepsilon}$, $W^+_{q,\gamma}$ be the upper approximation functions used before. By (2.9), (2.10) and the estimates following them, for all $n = 1, 2\ldots$ we obtain

$$\begin{aligned}\mu(E^\gamma_{q_n}(\psi)) &\leq 2\psi(q_n) + q_n\varepsilon + \sum_{\substack{k=-\infty\\k\neq 0}}^{+\infty} e^{-2\pi ik\gamma}\frac{\cos(2\pi kq_n\delta) - \cos(2\pi kq_n(\delta+\varepsilon))}{2\pi^2k^2q_n\varepsilon}\widehat{\mu}(-kq_n)\\ &= 2\psi(q_n) + q_n\varepsilon + O\left(\sum_{\substack{k=-\infty\\k\neq 0}}^{+\infty}\frac{|\widehat{\mu}(kq_n)|}{|k|}\right).\end{aligned}$$

Since this is true for all $\varepsilon > 0$, letting $\varepsilon \to 0$ we obtain

$$\mu(E^\gamma_{q_n}(\psi)) \leq 2\psi(q_n) + \frac{1}{\pi}\sum_{\substack{k=-\infty\\k\neq 0}}^{+\infty}\frac{|\widehat{\mu}(kq_n)|}{|k|}.$$

Thus the conclusion of Theorem 2.4 follows by (2.4) and the Borel-Cantelli Lemma.

2.2.2 The divergence case

In the case $\sum_{n=1}^{\infty}\psi(q_n) = \infty$, the previous estimates imply that $\sum_{n=1}^{\infty}\mu(E_n) = \infty$. Since we aim to show that $\mu\left(\limsup_{n\to\infty} E_n\right) = 1$, the divergence Borel-Cantelli Lemma (Lemma 1.11) will be useful. However, as explained in Section 1.2.3, the divergence Borel-Cantelli Lemma can only show that the set $\limsup_{n\to\infty} E_n$ has positive Lebesgue measure. In order to prove the desired full-measure result, more tools from measure theory are needed.

Proposition 2.6. *Let μ be a Borel probability measure supported on $[0,1)$ and $A \subseteq [0,1)$ be a Borel set. Assume there exist constants $c, l_0 > 0$ such that for any interval $B \subseteq [0,1)$ with length $|B| < l_0$*

$$\mu(A\cap B) \geq c\mu(B)\,.$$

Then $\mu(A) = 1$.

This is [4, Lemma 7] adapted to the case when the underlying metric (or topological) space is a subspace of the real line.

The following definition enables us to use Lemma 1.11 together with Proposition 2.6.

Definition 2.7. Let μ be any Borel measure on $[0,1)$. A sequence $(A_n)_{n=1}^{\infty}$ of subsets of $[0,1)$ is called *locally pairwise quasi-independent on average* if there exists a constant $C>0$ such that for any interval $B \subseteq [0,1)$ of sufficiently small length with $\mu(B)>0$,

$$\sum_{n=1}^{N}\sum_{m=1}^{N}\mu(A_n\cap A_m\cap B)\leq \frac{C}{\mu(B)}\left(\sum_{n=1}^{N}\mu(A_n\cap B)\right)^2, \quad N=1,2,\dots \tag{2.13}$$

Lemma 1.11 implies that if the Borel sets $(A_n)_{n=1}^{\infty}\subseteq[0,1)$ are locally pairwise quasi-independent on average and $\sum_{n=1}^{\infty}\mu(E_n)=\infty$, then

$$\mu\left(B\cap\limsup_{n\to\infty}A_n\right)\geq\frac{1}{C}\,\mu(B)$$

for any interval $B\subseteq[0,1)$ of sufficiently small length. In turn, this fact combined with Proposition 2.6 implies that $\mu\left(\limsup_{n\to\infty}A_n\right)=1$. Hence for the proof of Theorem 2.3 it suffices to show that the sets $(E_n)_{n=1}^{\infty}$ are locally pairwise quasi-independent on average.

For any interval $B\subseteq[0,1)$ define the probability measure μ_B to be the normalised restriction of μ on B, that is,

$$\mu_B(A)\;=\;\frac{1}{\mu(B)}\mu(A\cap B)\quad\text{for all Borel subsets } A\subseteq[0,1). \tag{2.14}$$

Then showing that the sets $(E_n)_{n=1}^{\infty}$ are locally pairwise quasi-independent is equivalent to proving that there is some $C>0$ such that

$$\sum_{n=1}^{N}\sum_{m=1}^{N}\mu_B(E_n\cap E_m)\leq C\left(\sum_{n=1}^{N}\mu_B(E_n)\right)^2, \quad N=1,2,\dots \tag{2.15}$$

for any interval $B\subseteq[0,1)$.

Remark 13. Suppose that instead of (2.15) we are able to show that there is a constant $C>0$ so the following holds: for any interval $B\subseteq[0,1)$, there exists a subsequence $(E_{k_n})_{n=1}^{\infty}$, possibly depending on B, such that

$$\sum_{n=1}^{N}\sum_{m=1}^{N}\mu_B(E_{k_n}\cap E_{k_m})\le C\left(\sum_{n=1}^{N}\mu_B(E_{k_n})\right)^2, \quad N=1,2,\ldots \tag{2.16}$$

Then

$$\mu\left(\limsup_{n\to\infty} E_n\cap B\right) \;\ge\; \mu\left(\limsup_{n\to\infty} E_{k_n}\cap B\right) \;\ge\; \frac{1}{C}\,\mu(B)$$

and the requested result remains true. The upshot is that in order to prove the divergence case of Theorem 2.3 it suffices to prove (2.16), as long as the constant $C>0$ does not depend on the choice of the interval B or the subsequence $(E_{n_k})_{k=1}^{\infty}$.

Properties of the measure μ_B.

Proposition 2.8. *Let μ be a Borel probability measure on $[0,1)$ such that*

$$\widehat{\mu}(t)=O\left(\frac{1}{(\log|t|)^A}\right), \qquad |t|\to\infty$$

for some constant $A>1$. If $B\subseteq[0,1)$ is an interval with $\mu(B)>0$ and μ_B is the probability measure defined by (2.14), then

$$\widehat{\mu}_B(t) \;=\; \frac{1}{\mu(B)}\,O\left(\frac{1}{(\log|t|)^{A-1}}\right), \qquad |t|\to\infty\,. \tag{2.17}$$

We present the proof of the proposition decomposed into several steps:

Lemma 2.9. *Let $(\mu_n)_{n=1}^{\infty}$ be a sequence of probability measures supported on $[0,1)$. Suppose $f:\mathbb{R}\to(0,\infty)$ is a function such that $|\widehat{\mu}_n(t)|\le f(t),\ \ t\in\mathbb{R}$. If $(\mu_n)_{n=1}^{\infty}$ converges weakly to some measure μ, then also*

$$|\widehat{\mu}(t)|\le f(t), \qquad \textit{for all}\ \ t\in\mathbb{R}.$$

Proof. For all $t\in\mathbb{R}$,

$$\begin{aligned}
|\widehat{\mu}(t)| &= \left|\int e^{-2\pi ixt}\mathrm{d}\mu(x)\right| \\
&= \lim_{n\to\infty}\left|\int e^{-2\pi ixt}\mathrm{d}\mu_n(x)\right| \quad \text{(by the hypothesis)} \\
&= \lim_{n\to\infty}|\widehat{\mu}_n(t)| \\
&\le f(t).
\end{aligned}$$

$\square$

Next, for all $\varepsilon > 0$ sufficiently small consider the function

$$\chi_{B,\varepsilon} : [0,1) \to \mathbb{R}, \qquad \chi_{B,\varepsilon}(x) = \begin{cases} 1, & \text{if } \alpha + \varepsilon \le x \le \beta - \varepsilon \\ (x-\alpha)/\varepsilon, & \text{if } \alpha < x < \alpha + \varepsilon \\ (\beta - x)/\varepsilon, & \text{if } \beta - \varepsilon < x < \beta \\ 0, & \text{otherwise,} \end{cases}$$

where α, β are the endpoints of B. Also consider the measures $\mu_{B,\varepsilon}$ defined by the relation

$$\int_0^1 g(x) \mathrm{d}\mu_{B,\varepsilon}(x) = \int_0^1 g(x) \chi_{B,\varepsilon}(x) \mathrm{d}\mu_B(x)$$

for any continuous function $g : [0,1) \to \mathbb{R}$.

Lemma 2.10. *The family of measures $(\mu_{B,\varepsilon})_{\varepsilon>0}$ converges weakly to μ_B as $\varepsilon \to 0$.*

Proof. Let $f : [0,1) \to \mathbb{R}$ be bounded and continuous. For any $\varepsilon > 0$,

$$\begin{aligned} \int_0^1 f(x) \mathrm{d}\mu_{B,\varepsilon}(x) &= \int_{\alpha+\varepsilon}^{\beta-\varepsilon} f(x) \mathrm{d}\mu_B(x) + \int_\alpha^{\alpha+\varepsilon} f(x)\chi_{B,\varepsilon}(x) \mathrm{d}\mu_B(x) \\ &\quad + \int_{\beta-\varepsilon}^{\beta} f(x)\chi_{B,\varepsilon} \mathrm{d}\mu_B(x), \end{aligned}$$

so

$$\begin{aligned} \left| \int_0^1 f(x) \mathrm{d}\mu_B(x) - \int_0^1 f(x) \mathrm{d}\mu_{B,\varepsilon}(x) \right| &= \Big| \int_\alpha^{\alpha+\varepsilon} (1 - \chi_{B,\varepsilon}(x)) f(x) \mathrm{d}\mu_B(x) \\ &\quad + \int_{\beta-\varepsilon}^{\beta} (1 - \chi_{B,\varepsilon}(x)) f(x) \mathrm{d}\mu_B(x) \Big| \\ &\le \left| \int_\alpha^{\alpha+\varepsilon} (1 - \chi_{B,\varepsilon}(x)) f(x) \mathrm{d}\mu_B(x) \right| \\ &\quad + \left| \int_{\beta-\varepsilon}^{\beta} (1 - \chi_{B,\varepsilon}(x)) f(x) \mathrm{d}\mu_B(x) \right| \\ &\le \frac{1}{\mu(B)} \|f\|_\infty \, \mu\left((\alpha, \alpha+\varepsilon) \cup (\beta - \varepsilon, \beta)\right) \\ &\to 0, \quad \text{as } \varepsilon \to 0. \quad \square \end{aligned}$$

Proof. (of Proposition 2.8) According to the previous, it suffices to show that the

Fourier transforms of all measures $\mu_{B,\varepsilon}$ have the requested logarithmic decay rate as in (2.17).

Fix some $\varepsilon > 0$. The Fourier coefficients of the function $\chi_{B,\varepsilon}$ are

$$\widehat{\chi}_{B,\varepsilon}(k) = \begin{cases} \beta - \alpha - \varepsilon, & \text{if } k = 0, \\ \dfrac{(e^{2\pi i k\beta} - e^{2\pi i k(\beta-\varepsilon)}) - (e^{-2\pi i k(\alpha+\varepsilon)} - e^{-2\pi i k\alpha})}{(2\pi i k)^2 \varepsilon}, & \text{if } k \neq 0 \end{cases}$$

This implies that

$$\widehat{\chi}_{B,\varepsilon}(k) \ll \frac{1}{|k|^2 \varepsilon}, \qquad |k| \to \infty \tag{2.18}$$

so $\sum_{k=-\infty}^{+\infty} |\widehat{\chi}_{B,\varepsilon}(k)| < \infty$ and the Fourier series

$$\chi_{B,\varepsilon}(t) = \sum_{k=-\infty}^{+\infty} \widehat{\chi}_{B,\varepsilon}(k) e^{2\pi i k t} \tag{2.19}$$

converges uniformly to $\chi_{B,\varepsilon}$. Using the Mean Value Theorem we get the alternative estimate

$$\widehat{\chi}_{B,\varepsilon}(k) \ll \frac{1}{|k|}, \qquad |k| \to \infty \tag{2.20}$$

for the Fourier coefficients of $\chi_{B,\varepsilon}$, which is uniform for all $\varepsilon > 0$. Now by definition

$$\begin{aligned} \widehat{\mu}_{B,\varepsilon}(t) &= \int_0^1 e^{-2\pi i t x} \mathrm{d}\mu_{B,\varepsilon}(x) \\ &= \frac{1}{\mu(B)} \int_0^1 e^{-2\pi i t x} \chi_{B,\varepsilon}(x) \mathrm{d}\mu(x) \\ &= \frac{1}{\mu(B)} \int_0^1 e^{-2\pi i t x} \sum_{k=-\infty}^{+\infty} \widehat{\chi}_{B,\varepsilon}(k) e^{2\pi i k x} \mathrm{d}\mu(x) \\ &= \frac{1}{\mu(B)} \sum_{k=-\infty}^{+\infty} \widehat{\chi}_{B,\varepsilon}(k) \int_0^1 e^{-2\pi i (t-k) x} \mathrm{d}\mu(x) \\ &= \frac{1}{\mu(B)} \sum_{k=-\infty}^{+\infty} \widehat{\chi}_{B,\varepsilon}(k) \widehat{\mu}(t-k). \end{aligned}$$

Decompose the sum in the right hand side into the terms

$$A_1 = \sum_{\substack{k=-\infty \\ |t-k|\geq 2|t|}}^{+\infty} \widehat{\chi}_{B,\varepsilon}(k)\widehat{\mu}(t-k),$$

$$A_2 = \sum_{\substack{k=-\infty \\ \frac{1}{2}|t|<|t-k|<2|t|}}^{+\infty} \widehat{\chi}_{B,\varepsilon}(k)\widehat{\mu}(t-k),$$

$$A_3 = \sum_{\substack{k=-\infty \\ |t-k|\leq \frac{1}{2}|t|}}^{+\infty} \widehat{\chi}_{B,\varepsilon}(k)\widehat{\mu}(t-k).$$

Since $k=0$ gives the term

$$\widehat{\chi}_{B,\varepsilon}(0)\widehat{\mu}(t) \ll \frac{1}{(\log|t|)^A}$$

and values of k with $|k-t|<2$ give terms

$$\widehat{\chi}_{B,\varepsilon}(k)\widehat{\mu}(t-k) \ll \frac{1}{|t|},$$

these terms can be ignored in our estimates. Also we only need to deal with the case $t>0$; the other one is similar due to symmetry.

Regarding the first sum, the condition of summation $|k-t|\geq 2|t|$ implies that $k\geq 3t$ or $k\leq -t$, hence

$$\frac{2}{3} \leq 1-\frac{t}{k} \leq 2.$$

So

$$|t-k| = |k|\left|1-\frac{t}{k}\right| \asymp |k|, \qquad |k|\to\infty$$

and

$$\begin{aligned}\hat{\chi}_{B,\varepsilon}(k)\hat{\mu}(t-k) &\ll \frac{1}{|k|\,(\log|t-k|)^A} \\ &\ll \frac{1}{|k|\,(\log|k|)^A}, \qquad |k|\to\infty.\end{aligned}$$

The condition of summation also implies that $|k| \geq |t|$, and we estimate

$$\begin{aligned}
A_1 &= \sum_{\substack{k=-\infty \\ |t-k|\geq 2|t|}}^{+\infty} \widehat{\chi}_{B,\varepsilon}(k)\widehat{\mu}(t-k) \\
&\leq \sum_{|k|\geq|t|} \widehat{\chi}_{B,\varepsilon}(k)\widehat{\mu}(t-k) \\
&\ll \sum_{|k|\geq|t|} \frac{1}{|k|(\log|k|)^A} \\
&\ll \frac{1}{(\log|t|)^{A-1}}.
\end{aligned}$$

Regarding the second sum, the condition of summation implies $|k| \leq 3|t|$, hence

$$\begin{aligned}
A_2 &= \sum_{\substack{k=-\infty \\ \frac{1}{2}|t|\leq|t-k|\leq 2|t|}}^{+\infty} \widehat{\chi}_{B,\varepsilon}(k)\widehat{\mu}(t-k) \\
&\ll \sum_{\substack{k=-\infty \\ \frac{1}{2}|t|\leq|t-k|\leq 2|t|}}^{+\infty} \frac{1}{|k|}\cdot\frac{1}{(\log|t-k|)^A} \\
&\ll \sum_{\substack{k=-\infty \\ \frac{1}{2}|t|\leq|t-k|\leq 2|t|}}^{+\infty} \frac{1}{|k|}\cdot\frac{1}{(\log|t|)^A} \\
&\ll \frac{1}{(\log|t|)^A}\sum_{|m|\leq 3|t|}\frac{1}{|k|} \\
&\ll \frac{1}{(\log|t|)^{A-1}}.
\end{aligned}$$

Finally the condition $|t-k| \leq \frac{1}{2}|t|$ in the third sum implies that

$$\frac{1}{2}|t| \leq |k| \leq \frac{3}{2}|t|.$$

So

$$\begin{aligned} A_3 &= \sum_{\substack{k=-\infty \\ |t-k|\le\frac{1}{2}|t|}}^{+\infty} \widehat{\chi}_{B,\varepsilon}(k)\widehat{\mu}(t-k) \\ &\ll \sum_{\substack{k=-\infty \\ 2\le|t-k|<\frac{1}{2}|t|}}^{+\infty} \frac{1}{|k|}\cdot\frac{1}{(\log|t-k|)^A} \\ &\ll \frac{1}{|t|}\sum_{2\le k\le|t|}\frac{1}{(\log|k|)^A} \\ &\ll \frac{1}{(\log|t|)^{A-1}}, \end{aligned}$$

where the last estimate follows if we compare the sum with the corresponding integral. Combining all previous cases yields the required decay rate for the measures $\mu_{B,\varepsilon}$. The same is true for μ_B by Lemma 2.9, since the implicit constant is independent of $\varepsilon > 0$. □

Finally, we establish the following analogue of Lemma 2.5 for the measure μ_B.

Lemma 2.11. *Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a decreasing function. Let μ be a Borel probability measure supported on $[0, 1)$ such that*

$$\widehat{\mu}(t) = O\left(\frac{1}{(\log|t|)^A}\right), \qquad |t| \to \infty$$

where $A > 5$ is a constant. Define the sets $E_q^\gamma(\psi) = \{x \in [0,1) : \|qx-\gamma\| \le \psi(q)\}$, $q = 1, 2, \ldots$ Then

$$\mu_B(E_q^\gamma(\psi)) = 2\psi(q) + \frac{1}{\mu(B)}O\left(\frac{1}{(\log q)^\Gamma}\right), \qquad q \to \infty$$

where $\Gamma > 3$ is a constant.

Proof. By Proposition 2.8, the Fourier transform of the measure μ_B satisfies

$$\widehat{\mu}_B(t) = \frac{1}{\mu(B)}\,O\left(\frac{1}{(\log|t|)^{A-1}}\right), \qquad |t| \to \infty\,.$$

Consider the approximating functions $\chi_{\delta,\varepsilon}^+$, $\chi_{\delta,\varepsilon}^-$ and $W_{q,\gamma}^+$, $W_{q,\gamma}^-$ as in the proof of Lemma 2.5. Combining the same arguments as in the proof of Lemma 2.5 together with Proposition 2.8, we get

$$\mu_B(E_q^\gamma(\psi)) \le 2\psi(q) + \frac{1}{\mu(B)} \cdot \frac{C_1}{(\log q)^\Gamma}$$

and

$$\mu_B(E_q^\gamma(\psi)) \ge 2\psi(q) - \frac{1}{\mu(B)} \cdot \frac{C_1}{(\log q)^\Gamma}$$

where $\Gamma > 3$ is a constant. The constant $C_1 > 0$ only depends on A. Combining the two inequalities we get the required result. □

Completing the proof of Theorem 2.3

From now on, in the definition of the functions $W_{q,\gamma}^+$ we consider the implicit constants to be equal to

$$\delta = \varepsilon = \frac{\psi(q)}{q}\cdot$$

Let $K > 1$ be an absolute constant such that

$$\frac{q_{n+1}}{q_n} \ge K, \qquad n = 1, 2, \dots \tag{2.21}$$

holds. Also write $\Gamma = 3 + 2\eta$, $\eta > 0$ for the constant $\Gamma > 3$ in Lemma 2.11. If we restrict to values of n such that $(\log q_n)^\eta \mu(B) \ge 1$ (see Remark 13), by Lemma 2.11 we have the estimate

$$\mu_B(E_n) = 2\psi(q_n) + O\left(\frac{1}{(\log q_n)^{3+\eta}}\right), \qquad n \to \infty.$$

Hence we may assume without loss of generality that

$$\mu_B(E_n) = 2\psi(q_n) + O\left(\frac{1}{n^\Gamma}\right), \qquad n \to \infty \tag{2.22}$$

with $\Gamma > 3$. Set $W_{m,n}^+ = W_{q_m,\gamma}^+ W_{q_n,\gamma}^+$, so

$$\begin{aligned}\mu_B(E_m \cap E_n) &\le \int_0^1 W_{q_m,\gamma}^+(x) W_{q_n,\gamma}^+(x) \mathrm{d}\mu_B(x) \\ &= \int_0^1 W_{m,n}^+(x) \mathrm{d}\mu_B(x)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=-\infty}^{+\infty} \widehat{W}^{+}_{m,n}(k)\widehat{\mu}_B(-k) \\
&= \widehat{W}^{+}_{m,n}(0) + \sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} \widehat{W}^{+}_{m,n}(k)\widehat{\mu}_B(-k). \qquad (2.23)
\end{aligned}
$$

Now

$$
\begin{aligned}
\widehat{W}^{+}_{m,n}(0) &= \int_0^1 W^{+}_{q_m,\gamma}(x)W^{+}_{q_n,\gamma}(x)\mathrm{d}x \\
&\leq \left|(2E_m)\cap(2E_n)\right|,
\end{aligned}
$$

where

$$
2E_n = \bigcup_{p=0}^{q_n-1} B\left(\frac{p+\gamma}{q_n}, \frac{2\psi(q_n)}{q_n}\right), \qquad n\in\mathbb{N}
$$

is the set which consists of the intervals of E_n dilated by a factor of 2. Under the assumption $m<n$, each interval of the form $B\left(\frac{p+\gamma}{q_m}, \frac{2\psi(q_m)}{q_m}\right)$ contains at most

$$
\frac{2\psi(q_m)/q_m}{1/q_n} + 2 = \frac{2\psi(q_m)q_n}{q_m} + 2
$$

numbers of the form $\frac{p+\gamma}{q_n}$, $0\leq p<q_n$ hence

$$
\begin{aligned}
\sum\sum_{1\leq m<n\leq N} \widehat{W}^{+}_{m,n}(0) &\leq \sum\sum_{1\leq m<n\leq N} \left|(2E_m)\cap(2E_n)\right| \\
&\leq \sum\sum_{1\leq m<n\leq N} \left(\frac{2\psi(q_n)q_m}{q_m}+2\right)\frac{4\psi(q_n)}{q_n}q_m \\
&\ll \sum\sum_{1\leq m<n\leq N} \left(\frac{\psi(q_m)q_n\psi(q_n)q_m}{q_mq_n} + \frac{\psi(q_n)q_m}{q_n}\right) \\
&= \sum\sum_{1\leq m<n\leq N} \psi(q_m)\psi(q_n) + \sum\sum_{1\leq m<n\leq N} \frac{\psi(q_n)q_m}{q_n}
\end{aligned}
$$

$$
\begin{aligned}
&\ll \sum\sum_{1\le m<n\le N} \psi(q_m)\psi(q_n) + \sum\sum_{1\le m<n\le N} \frac{\psi(q_n)}{K^{n-m}} \\
&\ll \left(\sum_{1\le n\le N} \psi(q_n)\right)^2 + \sum\sum_{1\le m<n\le N} \frac{\psi(q_m)}{K^{n-m}} \\
&\ll \left(\sum_{1\le n\le N} \psi(q_n)\right)^2 + \sum_{1\le n\le N} \psi(q_n) \\
&\ll \left(\sum_{1\le n\le N} \psi(q_n)\right)^2 \\
&\overset{(2.22)}{\ll} \left(\sum_{1\le n\le N} \mu_B(E_n)\right)^2 .
\end{aligned}
$$

Now we focus our attention on the second term of (2.23). This is

$$
\begin{aligned}
S_{m,n} &= \sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} \widehat{W}^+_{m,n}(k)\widehat{\mu}_B(-k) \\
&= \sum_{\substack{s=-\infty \\ sq_m+tq_n\neq 0}}^{\infty} \sum_{t=-\infty}^{\infty} \widehat{W}^+_{q_m,\gamma}(sq_m)\widehat{W}^+_{q_n,\gamma}(tq_n)\widehat{\mu}_B\left(-(sq_m+tq_n)\right) \\
&=: S_1(m,n) + S_2(m,n) + S_3(m,n),
\end{aligned}
$$

where

$$
\begin{aligned}
S_1(m,n) &= \sum_{\substack{t=-\infty \\ t\neq 0}}^{+\infty} \widehat{W}^+_{q_m,\gamma}(0)\widehat{W}^+_{q_n,\gamma}(tq_n)\widehat{\mu}_B(-tq_n) \\
&= 3\psi(q_m) \sum_{\substack{t=-\infty \\ t\neq 0}}^{+\infty} \widehat{W}^+_{q_n,\gamma}(tq_n)\widehat{\mu}_B(-tq_n),
\end{aligned}
$$

$$\begin{aligned} S_2(m,n) &= \sum_{\substack{s=-\infty \\ s\neq 0}}^{+\infty} \widehat{W}^+_{q_n,\gamma}(0)\widehat{W}^+_{q_m,\gamma}(sq_m)\widehat{\mu}_B(-sq_m) \\ &= 3\psi(q_n)\sum_{\substack{s=-\infty \\ s\neq 0}}^{+\infty} \widehat{W}^+_{q_m,\gamma}(sq_m)\widehat{\mu}_B(-sq_m), \\ S_3(m,n) &= \sum\sum_{\substack{s,t\in\mathbb{Z}\setminus\{0\} \\ sq_m+tq_n\neq 0}} \widehat{W}^+_{q_m,\gamma}(sq_m)\widehat{W}^+_{q_n,\gamma}(tq_n)\widehat{\mu}_B\left(-(sq_m+tq_n)\right). \end{aligned}$$

By (2.10) we get the estimate

$$\widehat{W}^+_{q_n,\gamma}(tq_n) \ll \frac{1}{|t|}, \qquad |t|\to\infty \tag{2.24}$$

which is uniform for all $n\in\mathbb{N}$. Also Proposition 2.8 together with the argument leading to (2.22) we may assume

$$\widehat{\mu}_B(tq_n) \ll \frac{1}{(\log|tq_n|)^{\Delta}}, \qquad |t|\to\infty \tag{2.25}$$

for some constant $\Delta > 4$. Thus, if we write $\Delta = 4+2\theta$, with $\theta > 0$, we obtain

$$\begin{aligned} S_1(m,n) &\overset{(2.24),(2.25)}{\ll} \psi(q_m)\sum_{\substack{t=-\infty \\ t\neq 0}}^{+\infty} \frac{1}{|t|(\log|tq_n|)^{\Delta}} \\ &\leq \frac{\psi(q_m)}{(\log q_n)^{3+\theta}}\sum_{\substack{t=-\infty \\ t\neq 0}}^{+\infty} \frac{1}{|t|(\log|t|)^{1+\theta}} \\ &\ll \frac{\psi(q_m)}{n^{3+\theta}} \end{aligned}$$

and

$$\begin{aligned}
\sum\sum_{1\le m<n\le N} S_1(m,n) &\ll \sum\sum_{1\le m<n\le N} \frac{\psi(q_m)}{n^{3+\theta}} \\
&\asymp \sum_{1\le m\le N} \psi(q_m) \\
&\overset{(2.22)}{\ll} \left(\sum_{n=1}^{N} \mu_B(E_n)\right)^2 .
\end{aligned}$$

Similarly,

$$\sum\sum_{1\le m<n\le N} S_2(m,n) \ll \left(\sum_{n=1}^{N} \mu_B(E_n)\right)^2 .$$

Now decompose the third sum as

$$\begin{aligned}
S_3(m,n) &= \sum\sum_{\substack{s,t\in\mathbb{Z}\setminus\{0\}\\ sq_m+tq_n\neq 0}} \widehat{W}^+_{q_m,\gamma}(sq_m)\widehat{W}^+_{q_n,\gamma}(tq_n)\widehat{\mu}_B\left(-(sq_m+tq_n)\right) \\
&= \sum\sum_{\substack{s,t\in\mathbb{Z}\setminus\{0\},st>0\\ sq_m+tq_n\neq 0}} \widehat{W}^+_{q_m,\gamma}(sq_m)\widehat{W}^+_{q_n,\gamma}(tq_n)\widehat{\mu}_B\left(-(sq_m+tq_n)\right) \\
&\quad + \sum\sum_{\substack{s,t\in\mathbb{Z}\setminus\{0\},st<0\\ sq_m+tq_n\neq 0}} \widehat{W}^+_{q_m,\gamma}(sq_m)\widehat{W}^+_{q_n,\gamma}(tq_n)\widehat{\mu}_B\left(-(sq_m+tq_n)\right) \\
&= S_4(m,n)+S_5(m,n)\,,
\end{aligned}$$

where

$$\begin{aligned}
S_4(m,n) &:= \sum\sum_{\substack{s,t\in\mathbb{Z}\setminus\{0\},st>0\\ sq_m+tq_n\neq 0}} \widehat{W}^+_{q_m,\gamma}(sq_m)\widehat{W}^+_{q_n,\gamma}(tq_n)\widehat{\mu}_B\left(-(sq_m+tq_n)\right)\,, \\
S_5(m,n) &:= \sum\sum_{\substack{s,t\in\mathbb{Z}\setminus\{0\},st<0\\ sq_m+tq_n\neq 0}} \widehat{W}^+_{q_m,\gamma}(sq_m)\widehat{W}^+_{q_n,\gamma}(tq_n)\widehat{\mu}_B\left(-(sq_m+tq_n)\right).
\end{aligned}$$

Observe that (2.10) gives the alternative estimates

$$\widehat{W}^+_{q_n,\gamma}(tq_n) \ll \psi(q_n), \qquad n \to \infty \tag{2.26}$$

and

$$\widehat{W}^+_{q_n,\gamma}(tq_n) \ll \frac{1}{|t|^2\psi(q_n)}, \qquad |t| \to \infty. \tag{2.27}$$

Regarding $S_4(m,n)$, observe that when s,t have the same sign then

$$\begin{aligned} |sq_m + tq_n| &\geq 2\sqrt{|sq_m tq_n|} \;\Rightarrow \\ \frac{1}{\log|sq_m+tq_n|} &\ll \frac{1}{\log|sq_m tq_n|} \end{aligned}$$

hence

$$\begin{aligned} S_4(m,n) &= \sum_{\substack{s,t\in\mathbb{Z}\setminus\{0\},st>0 \\ sq_m+tq_n\neq 0}}\sum \widehat{W}^+_{q_m,\gamma}(sq_m)\widehat{W}^+_{q_n,\gamma}(tq_n)\widehat{\mu}_B\left(-(sq_m+tq_n)\right) \\ &\overset{(2.24)}{\ll} \sum_{s=1}^{\infty}\sum_{t=1}^{\infty}\frac{1}{st\left(\log|st|\right)^{2+\theta}}\cdot\frac{1}{\left(\log|q_nq_m|\right)^{2+\theta}} \\ &\ll \frac{1}{(m+n)^{2+\theta}} \end{aligned}$$

and

$$\begin{aligned} \sum_{1\leq m<n\leq N}\sum S_4(m,n) &\ll 1 \\ &\ll \left(\sum_{n=1}^{N}\mu_B(E_n)\right)^2. \end{aligned}$$

Now write

$$
\begin{aligned}
S_5(m,n) &\ll \sum_{\substack{s=1 \\ sq_m+tq_n\neq 0}}^{\infty}\sum_{t=1}^{\infty} \widehat{W}^{+}_{q_m,\gamma}(sq_m)\widehat{W}^{+}_{q_n,\gamma}(tq_n)\cdot\frac{1}{(\log|sq_m-tq_n|)^{\Delta}} \\
&= \sum_{\substack{s=1 \\ |sq_m-tq_n|\geq q_n/2}}^{\infty}\sum_{t=1}^{\infty} \widehat{W}^{+}_{q_m,\gamma}(sq_m)\widehat{W}^{+}_{q_n,\gamma}(tq_n)\cdot\frac{1}{(\log|sq_m-tq_n|)^{\Delta}} \\
&\quad + \sum_{\substack{s=1 \\ |sq_m-tq_n|< q_n/2}}^{\infty}\sum_{t=1}^{\infty} \widehat{W}^{+}_{q_m,\gamma}(sq_m)\widehat{W}^{+}_{q_n,\gamma}(tq_n)\cdot\frac{1}{(\log|sq_m-tq_n|)^{\Delta}} \\
&= S_6(m,n)+S_7(m,n),
\end{aligned}
$$

where

$$
S_6(m,n) := \sum_{\substack{s=1 \\ |sq_m-tq_n|\geq q_n/2}}^{\infty}\sum_{t=1}^{\infty} \widehat{W}^{+}_{q_m,\gamma}(sq_m)\widehat{W}^{+}_{q_n,\gamma}(tq_n)\cdot\frac{1}{(\log|sq_m-tq_n|)^{\Delta}}
$$

and

$$
S_7(m,n) := \sum_{\substack{s=1 \\ |sq_m-tq_n|< q_n/2}}^{\infty}\sum_{t=1}^{\infty} \widehat{W}^{+}_{q_m,\gamma}(sq_m)\widehat{W}^{+}_{q_n,\gamma}(tq_n)\cdot\frac{1}{(\log|sq_m-tq_n|)^{\Delta}}\,.
$$

Clearly

$$
\begin{aligned}
S_6(m,n) &\overset{(2.27)}{\ll} \sum_{s=1}^{\infty}\sum_{t=1}^{\infty}\frac{1}{s^2t^2\psi(q_m)\psi(q_n)\,(\log q_n)^{\Delta}} \\
&\ll \frac{1}{\psi(q_m)\psi(q_n)n^{\Delta}}.
\end{aligned}
$$

Now since the series $\sum\limits_{n=1}^{\infty}\psi(q_n)$ diverges but $\sum\limits_{n=1}^{\infty}n^{-\Delta/4}<\infty$, passing to a further subsequence if necessary (see Remark 13), we may assume without loss of generality that

$$
\psi(q_n)\gg\frac{1}{n^{\Delta/4}}, \qquad n\to\infty.
$$

Thus

$$S_6(m,n) \le \frac{1}{(mn)^{\Delta/4}}$$

and

$$\begin{aligned}
\sum_{1\le m<n\le N}\sum S_6(m,n) &\ll 1 \\
&\ll \left(\sum_{n=1}^{N}\mu_B(E_n)\right)^2.
\end{aligned}$$

Regarding the sum $S_7(m,n)$, observe that in each term the value of s determines at most one value of t. Now decompose $S_7(m,n)$ further into

$$\begin{aligned}
S_7(m,n) &= \sum_{\substack{s=1 \\ q_m/2\le|sq_m-tq_n|<q_n/2}}^{\infty}\sum_{t=1}^{\infty} \widehat{W}^{+}_{q_m,\gamma}(sq_m)\widehat{W}^{+}_{q_n,\gamma}(tq_n)\left(\log|sq_m-tq_n|\right)^{-\Delta/2} \\
&+ \sum_{\substack{s=1 \\ |sq_m-tq_n|<q_m/2 \\ s\le 1/\psi(q_m)}}^{\infty}\sum_{t=1}^{\infty} \widehat{W}^{+}_{q_m,\gamma}(sq_m)\widehat{W}^{+}_{q_n,\gamma}(tq_n)\left(\log|sq_m-tq_n|\right)^{-\Delta/2} \\
&+ \sum_{\substack{s=1 \\ |sq_m-tq_n|<q_m/2 \\ s>1/\psi(q_m)}}^{\infty}\sum_{t=1}^{\infty} \widehat{W}^{+}_{q_m,\gamma}(sq_m)\widehat{W}^{+}_{q_n,\gamma}(tq_n)\left(\log|sq_m-tq_n|\right)^{-\Delta/2} \\
&=: S_8(m,n)+S_9(m,n)+S_{10}(m,n).
\end{aligned}$$

Now easily

$$\begin{aligned}
S_8(m,n) &\overset{(2.27)}{\ll} \sum_{s=1}^{\infty}\frac{1}{s^2\psi(q_m)}\psi(q_n)\left(\log q_m\right)^{-\Delta/2} \\
&\ll \psi(q_n)\frac{1}{m^{3\Delta/4}},
\end{aligned}$$

hence this term gives

$$\begin{aligned}
\sum\sum_{1\le m<n\le N} S_8(m,n) &\ll \sum_{n=1}^{N}\psi(q_n)\\
&\ll \left(\sum_{n=1}^{N}\mu_B(E_n)\right)^2.
\end{aligned}$$

Regarding $S_9(m,n)$ and $S_{10}(m,n)$ observe that the condition of summation implies

$$\begin{aligned}
|sq_m - tq_n| &\le \frac{q_m}{2} \quad\Rightarrow\\
-\frac{q_m}{2} \le sq_m - tq_n &\le \frac{q_m}{2} \quad\Rightarrow\\
-\frac{1}{2} \le s - \frac{tq_n}{q_m} &\le \frac{1}{2},
\end{aligned}$$

hence each t determines a unique value of s, which is trivially

$$s = s_t \asymp \frac{q_n}{q_m}t.$$

Thus

$$\begin{aligned}
S_9(m,n) &\overset{(2.26)}{\ll} \sum\sum_{t<\frac{q_m}{q_n\psi(q_m)}} \psi(q_m)\psi(q_n)\\
&\ll \frac{q_m}{q_n}\psi(q_n)\\
&\ll \frac{q_m}{q_n}\psi(q_m),
\end{aligned}$$

which leads to

$$\begin{aligned}
\sum\sum_{1\le m<n\le N} S_9(m,n) &\ll \sum_{n=1}^{N}\psi(q_n)\\
&\ll \left(\sum_{n=1}^{N}\mu_B(E_n)\right)^2.
\end{aligned}$$

Finally

$$\begin{aligned}
S_{10}(m,n) &\ll \sum_{\substack{s>1/\psi(q_m)\\ t>1/\psi(q_n)}}\sum \widehat{W}^{+}_{q_m,\gamma}(sq_m)\widehat{W}^{+}_{q_n,\gamma}(tq_n)\left(\log|sq_m-tq_n|\right)^{-\Delta/2} \\
&\quad + \sum_{\substack{s>1/\psi(q_m)\\ t\le 1/\psi(q_n)}}\sum \widehat{W}^{+}_{q_m,\gamma}(sq_m)\widehat{W}^{+}_{q_n,\gamma}(tq_n)\left(\log|sq_m-tq_n|\right)^{-\Delta/2} \\
&\overset{(2.26),(2.27)}{\ll} \sum_{\substack{s>1/\psi(q_m)\\ t>1/\psi(q_n)}}\sum \frac{1}{s^2\psi(q_m)}\cdot\frac{1}{t^2\psi(q_n)} + \sum_{\substack{s>1/\psi(q_m)\\ t\le 1/\psi(q_n)}}\sum \left(\frac{q_m}{q_n t}\right)^2\cdot\frac{\psi(q_n)}{\psi(q_m)} \\
&\ll \sum_{t>1/\psi(q_n)} \frac{q_m^2}{q_n^2 t^4}\cdot\frac{1}{\psi(q_m)\psi(q_n)} + \sum_{t>\frac{q_m}{q_n\psi(q_m)}} \frac{q_m^2}{q_n^2 t^2}\cdot\frac{\psi(q_n)}{\psi(q_m)} \\
&\ll \left(\frac{q_m}{q_n}\right)^2 \frac{\psi(q_n)^3}{\psi(q_n)\psi(q_m)} + \left(\frac{q_m}{q_n}\right)^2 \frac{\psi(q_n)}{\psi(q_m)}\frac{q_n\psi(q_m)}{q_m} \\
&\ll \left(\frac{q_m}{q_n}\right)^2 \psi(q_m) + \frac{q_m}{q_n}\psi(q_m),
\end{aligned}$$

which implies

$$\begin{aligned}
\sum_{1\le m<n\le N}\sum S_{10}(m,n) &\ll \sum_{1\le m<n\le N}\sum \frac{\psi(q_m)}{K^{2(n-m)}} + \sum_{1\le m<n\le N}\sum \frac{\psi(q_m)}{K^{n-m}} \\
&\ll \sum_{n=1}^{N}\psi(q_n) \\
&\ll \left(\sum_{n=1}^{N}\mu_B(E_n)\right)^2 .
\end{aligned}$$

Combining all previous cases, we get the desired result.

Chapter 3

The Fourier dimension of the set $W(\gamma, \psi)$

3.1 Introduction and main result

Let us recall the definition of inhomogeneously well approximable numbers with respect to a given approximating function. If $\gamma \in \mathbb{R}$ and $\psi : \mathbb{N} \to \mathbb{R}^+$ is a function, the set $W(\gamma, \psi)$ of *inhomogeneously ψ-well approximable numbers* with respect to γ is defined as

$$W(\gamma, \psi) = \{x \in [0, 1) : \|qx - \gamma\| \leq \psi(q) \text{ for infinitely many } q \in \mathbb{N}\}.$$

As we have already seen in the introductory chapter, the Hausdorff dimension of the set $W(\gamma, \psi)$ is related to the decay rate of the approximating function ψ. If $f : \mathbb{N} \to (0, \infty)$ is a function, the lower logarithmic order of f at infinity is defined as

$$\lambda(f) = \liminf_{q \to \infty} \frac{\log f(q)}{\log q}.$$

Let us recall the theorem of Levesley [41] which gives the Hausdorff dimension of the set $W(\gamma, \psi)$. The theorem actually refers to the Hausdorff dimension of higher dimension analogues of $W(\gamma, \psi)$, but here we only present its one-dimensional version.

Theorem 3.1. *Let $\gamma \in \mathbb{R}$ and $\psi : \mathbb{N} \to \mathbb{R}^+$ be a function. If $\lambda = \lambda(1/\psi)$ is the lower logarithmic order of $1/\psi$, then*

$$\dim W(\gamma, \psi) = \begin{cases} 1, & \text{if } \lambda < 1 \\ \dfrac{2}{1+\lambda}, & \text{if } \lambda \geq 1. \end{cases}$$

We show that for a decreasing approximation function $\psi : \mathbb{N} \to \mathbb{R}$ the set $W(\gamma,\psi)$ of inhomogeneously ψ-well approximable numbers is a Salem set, generalizing the result of Kaufman in [39]. For the definition of Salem sets, see Chapter 1. We follow Bluhm's method of proof in [10].

Theorem 3.2. *Let $\gamma \in \mathbb{R}$ and $\psi : \mathbb{N} \to \mathbb{R}^+$ be a decreasing function. Let λ be the lower logarithmic order of $1/\psi$. For any $\varepsilon > 0$, the set $W(\gamma,\psi)$ supports a probability measure $\mu = \mu(\psi,\varepsilon)$ the Fourier transform of which satisfies*

$$\widehat{\mu}(x) = o(\log|x|)\ |x|^{-(1-5\varepsilon)/(1+\lambda+\varepsilon)}, \qquad |x| \to \infty. \tag{3.1}$$

Before we proceed to the proof of Theorem 3.2, let us see how it yields the requested information for the set $W(\gamma,\psi)$.

Corollary 3.3. *Let $\gamma \in \mathbb{R}$ and $\psi : \mathbb{N} \to \mathbb{R}^+$ be a decreasing function. The set $W(\gamma,\psi)$ is a Salem set.*

Proof. Let ε, $\varepsilon' > 0$. By Theorem 3.2 the set $W(\gamma,\psi)$ supports a probability measure μ with the property

$$\widehat{\mu}(x) \ll |x|^{-\frac{1-5\varepsilon}{1+\lambda+\varepsilon}+\varepsilon'}, \qquad |x| \to \infty\,.$$

Now this implies that the Fourier dimension of $W(\gamma,\psi)$ is

$$\dim_F W_\gamma(\psi) \ \geq\ \frac{2-10\varepsilon}{1+\lambda+\varepsilon} - 2\varepsilon',$$

so letting ε, $\varepsilon' \to 0$ we obtain the desired lower bound. □

3.2 Proof of the main result

3.2.1 Notation

For the proof of Theorem 3.2, fix some $\varepsilon > 0$. By the definition of λ we have

$$q^{\lambda-\varepsilon} < \frac{1}{\psi(q)} < q^{\lambda+\varepsilon} \qquad \text{for inf. many } q \in \mathbb{N}. \tag{3.2}$$

First let us begin with some notation. For all $M \in \mathbb{N}$ let

$$\mathbf{P}_M = \{p \text{ prime } : M \leq p \leq 2M\}.$$

We are going to choose a sequence $(M_k)_{k=1}^{\infty}$ of positive integers which satisfies

$$M_1 < 2M_1 < M_2 < 2M_2 < \dots \tag{3.3}$$

$$|\mathbf{P}_{M_k}| \geq \frac{M_k}{2\log M_k}, \qquad k = 1, 2, \ldots \tag{3.4}$$

and

$$(2M_k)^{\lambda-\varepsilon} < \frac{1}{\psi(2M_k)} < (2M_k)^{\lambda+\varepsilon}, \qquad k = 1, 2, \ldots \tag{3.5}$$

The choice of a sequence $(M_k)_{k=1}^{\infty}$ satisfying (3.4) is possible by the Prime Number Theorem, while the fulfilment of assumption (3.5) is possible because of (3.2).

The measure μ to be constructed will be supported in the set

$$S_\psi := \bigcap_{k=1}^{+\infty} \bigcup_{p \in \mathbf{P}_{M_k}} \{x \in [0,1) \, : \, \|px - \gamma\| \leq \psi(p)\} \, .$$

From now on, whenever we refer to a positive integer $M \geq 1$, we shall implicitly mean a term of the aforementioned sequence $(M_k)_{k=1}^{\infty}$. For each $M \in \mathbb{N}$ such that

$$R = R_M := \psi(2M) < \frac{1}{2}$$

define a function

$$F_M : \mathbb{R} \to \mathbb{R}, \quad F_M(x) = \begin{cases} \dfrac{15}{16R^5}(R^2 - x^2)^2, & \|x\| \leq R \\ 0, & \|x\| > R. \end{cases}$$

Clearly F_M is a continuous $1-$periodic function. Consider the Fourier series expansion

$$F_M(x) = \sum_{m=-\infty}^{+\infty} a_m^{(M)} e^{2\pi i m x}. \tag{3.6}$$

The Fourier coefficients of F_M are the numbers

$$a_m^{(M)} = \int_0^1 F_M(t) e^{-2\pi i m t} \mathrm{d}t, \qquad m = 0, \pm 1, \pm 2, \ldots$$

Observe that

$$\begin{aligned} a_0^{(M)} &= 1, \\ |a_m^{(M)}| &\le 1, \\ |a_m^{(M)}| &\le \frac{1}{m^2R^2}, \qquad m \in \mathbb{Z}\backslash\{0\}, \end{aligned} \tag{3.7}$$

where the last estimate follows by partial integration.

Finally define

$$g_M(x) = \frac{1}{|\mathbf{P}_M|}\sum_{p\in\mathbf{P}_M} F_M(px-\gamma) = \frac{1}{|\mathbf{P}_M|}\sum_{m=-\infty}^{+\infty}\sum_{p\in\mathbf{P}_M} a_m^{(M)} e^{2\pi i m(px-\gamma)},$$

which is clearly a 1-periodic function with $\hat{g}_M(0) = 1$. The connection of the functions g_M with the result to be shown becomes clear by the following:

Proposition 3.4. *If $x \in \mathbb{R}$ is such that $g_M(x) > 0$, there exists a prime $p \in \mathbf{P}_M$ such that $\|px-\gamma\| \le \psi(p)$.*

Proof. Since the functions F_M are non-negative, there exists $p \in \mathbf{P}_M$ such that $F_M(px-\gamma) > 0$, and the definition of F_M implies that

$$\|px-\gamma\| \le R = \psi(2M) < \psi(p).$$

□

3.2.2 Some auxiliary lemmas

In what follows we use the function

$$\theta(x) = \begin{cases} 1, & \text{if } |x| < x_{\lambda,\varepsilon} \\ (\log|x|)\,|x|^{-\frac{1-5\varepsilon}{1+\lambda+\varepsilon}} & \text{if } |x| \ge x_{\lambda,\varepsilon}\,, \end{cases}$$

where

$$x_{\lambda,\varepsilon} := 2^{1+\lambda+\varepsilon} e^{\frac{1+\lambda+\varepsilon}{1-5\varepsilon}}\,.$$

We are only interested in large values of $\theta(x)$, as it is clearly related to the requested decay rate mentioned in the main result.

Lemma 3.5. *Let $k \in \mathbb{N}$, $M \in \mathbb{N}$. Then*

$$\#\{p \in \mathbf{P}_M : p \,|\, k\} \le \frac{\log|k|}{\log M}\,. \tag{3.8}$$

Proof. By the canonical factorisation of k as a product of prime powers we obtain

$$\begin{aligned}\log|k| &\geq \sum_{p\,|\,k}\log p\\ &\geq \sum_{\substack{p\,|\,k\\ p\in\mathbf{P}_M}}\log p\\ &\geq (\log M)\cdot\#\{p\in\mathbf{P}_M : p\,|\,k\}\,.\end{aligned}$$

□

Lemma 3.6. *For all $k\in\mathbb{Z}\setminus\{0\}$ and for M large enough, the Fourier coefficients of g_M satisfy*

$$|\widehat{g}_M(k)| \leq \frac{2\log|k|}{M} \tag{3.9}$$

and

$$|\widehat{g}_M(k)| \leq \frac{8M\log|k|}{R^2|k|^2}. \tag{3.10}$$

Proof. The Fourier coefficients of g_M are

$$\begin{aligned}\widehat{g}_M(k) &= \int_0^1 g_M(x)e^{-2\pi ikx}\mathrm{d}x\\ &= \int_0^1 e^{-2\pi ikx}\frac{1}{|\mathbf{P}_M|}\sum_{m=-\infty}^{+\infty}\sum_{p\in\mathbf{P}_M}a_m^{(M)}e^{2\pi im(px-\gamma)}\mathrm{d}x\\ &= \frac{1}{|\mathbf{P}_M|}\sum_{m=-\infty}^{+\infty}\sum_{p\in\mathbf{P}_M}a_m^{(M)}e^{-2\pi im\gamma}\int_0^1 e^{2pi(mp-k)x}\mathrm{d}x.\end{aligned}$$

Note that we are allowed to interchange the order of summation and integration since (3.7) implies that the Fourier series (3.6) converges uniformly. Thus

$$|\widehat{g}_M(k)| \leq \frac{1}{|\mathbf{P}_M|}\left(\max_{\substack{mp=k\\ p\in\mathbf{P}_M}}|a_m^{(M)}|\right)\#\{(p,m)\in\mathbf{P}_M\times\mathbb{Z}\,:\,k=mp\}. \tag{3.11}$$

Observe that each prime $p\in\mathbf{P}_M$ determines a unique $m\in\mathbb{Z}$ such that $k=mp$, hence also a unique pair $(p,m)\in\mathbf{P}_M\times\mathbb{Z}$ such that $k=mp$. Thus (3.4) and Lemma 3.5 yield

$$\begin{aligned}|\widehat{g}_M(k)| &\le \frac{2\log M}{M}\#\{p\in\mathbf{P}_M \,:\, p\,|\,k\}\\ &\le \frac{2\log|k|}{M},\end{aligned}$$

so (3.9) is proved. Alternatively, note that when $k = mp$ for some $p \in \mathbf{P}_M$, then

$$|m| = \frac{|k|}{p} \ge \frac{|k|}{2M}.$$

Thus (3.7) implies

$$|a_m^{(M)}| \le \frac{4M^2}{|k|^2R^2},$$

which together with (3.4), (3.8) and (3.11) give (3.10). □

Lemma 3.7. *There exist constants $M_1 > 0,\ A = A(\psi,\varepsilon) > 0$ such that for all $M \ge M_1$,*

$$\begin{aligned}|\widehat{g}_M(k)| &\le A\frac{\log M}{M^{1-4\varepsilon}}, \qquad \textit{for all}\ \ k\neq 0\\ |\widehat{g}_M(k)| &\le A\frac{\log|k|}{|k|^{\frac{1-4\varepsilon}{1+\lambda-\varepsilon}}}, \qquad |k| > MR^{-1}.\end{aligned}$$

Proof. Consider the cases:

CASE I: $1 \le |k| \le MR^{-1}$. Using (3.9) and then (3.5) we obtain

$$\begin{aligned}|\widehat{g}_M(k)| &\le \frac{2}{M}(\log(M) - \log R)\\ &= \frac{2}{M}(\log(M) - \log\psi(2M))\\ &\le \frac{2}{M}(\log M + (\lambda+\varepsilon)(\log 2 + \log M))\\ &\le 2(1+2\lambda+2\varepsilon)\frac{\log M}{M}\\ &\le 2(1+2\lambda+2\varepsilon)\frac{\log M}{M^{1-4\varepsilon}}.\end{aligned}$$

CASE II: $|k| > MR^{-1}$. Then by (3.5),

$$|k| \ge 2^{\lambda-\varepsilon}M^{1+\lambda-\varepsilon}. \tag{3.12}$$

So (3.10) yields

$$\begin{aligned}
|\widehat{g}_M(k)| &\leq 8M(2M)^{2(\lambda+\varepsilon)} \cdot \frac{\log|k|}{|k|^2} \\
&= 2^{3+2\lambda+2\varepsilon} M^{1+2(\lambda+\varepsilon)} \cdot \frac{\log|k|}{|k|^2} \\
&\leq 2^{3+2\lambda+2\varepsilon} \cdot 2^{-\frac{1+2\lambda+2\varepsilon}{1+\lambda+\varepsilon}(\lambda-\varepsilon)} \cdot |k|^{\frac{1+2\lambda+2\varepsilon}{1+\lambda-\varepsilon}} \cdot \frac{\log|k|}{|k|^2} \\
&= 2^{\frac{3+4\lambda}{1+\lambda-\varepsilon}} \frac{\log|k|}{|k|^{\frac{1-4\varepsilon}{1+\lambda-\varepsilon}}}.
\end{aligned}$$

Now we use (3.12) together with the fact that for $a > 0$ the function $x \mapsto (\log|x|)|x|^{-a}$ is finally decreasing. We obtain

$$\begin{aligned}
|\widehat{g}_M(k)| &\leq 2^{\frac{3+4\lambda}{1+\lambda-\varepsilon}} \frac{\log\left(2^{\lambda-\varepsilon} M^{1+\lambda-\varepsilon}\right)}{\left(2^{\lambda-\varepsilon} M^{1+\lambda-\varepsilon}\right)^{\frac{1-4\varepsilon}{1+\lambda-\varepsilon}}} \\
&\leq (1+\lambda-\varepsilon) 2^{4^{\frac{1+\lambda+\varepsilon(\lambda-\varepsilon)}{1+\lambda-\varepsilon}}} \frac{\log M}{M^{1-4\varepsilon}},
\end{aligned}$$

hence the desired statement is true with

$$A = \max\left\{ 2(1+2\lambda+2\varepsilon),\ 2^{\frac{3+4\lambda}{1+\lambda-\varepsilon}},\ (1+\lambda-\varepsilon)2^{4^{\frac{1+\lambda+\varepsilon(1-\varepsilon)}{1+\lambda-\varepsilon}}} \right\}.$$

∎

In what follows, we write $C_c^2(\mathbb{R})$ for the set of compactly supported functions on $\mathbb{R}$ with continuous second derivative.

Lemma 3.8. *Let $M_1 > 0$ be as in Lemma 3.7. For every $\phi \in C_c^2(\mathbb{R})$ and $M \geq M_1$ there exists a constant $B = B(\phi, \psi, \varepsilon) > 0$ such that*

$$|(\widehat{\phi g_M})(x) - \widehat{\phi}(x)| \leq B \frac{\log M}{M^{1-4\varepsilon}} \qquad \textit{for all } x \in \mathbb{R}.$$

Proof. Since $\phi \in C_c^2(\mathbb{R})$ its Fourier transform satisfies

$$\widehat{\phi}(x) \ll \frac{1}{(1+|x|)^2}, \qquad |x| \to \infty.$$

Since $\psi, g_M \in C_c^2(\mathbb{R})$, by the convolution theorem we obtain

$$(\widehat{\phi g_M})(x) = \sum_{k=-\infty}^{+\infty} \widehat{g}_M(k)\widehat{\phi}(x-k), \qquad x \in \mathbb{R}.$$

Hence for all $x \in \mathbb{R}$,

$$\begin{aligned}
\left|(\widehat{\phi g_M})(x) - \widehat{\phi}(x)\right| &= \left|\sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} \widehat{g}_M(k)\widehat{\phi}(x-k)\right| \\
&\leq \sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} |\widehat{g}_M(k)\widehat{\phi}(x-k)| \\
&\ll \sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} \frac{\log M}{M^{1-4\varepsilon}} \cdot \frac{1}{(1+|x-k|)^2} \\
&\ll \frac{\log M}{M^{1-4\varepsilon}}.
\end{aligned}$$

□

Lemma 3.9. *For every $\phi \in C_c^2(\mathbb{R})$ and $\delta > 0$ there exists a positive integer $M_0 = M_0(\phi, \delta, \varepsilon)$ such that*

$$|(\widehat{\phi g_M})(x) - \widehat{\phi}(x)| \leq \delta\theta(x) \quad \textit{for all } x \in \mathbb{R}$$

for all $M \geq M_0$.

Proof. Once again consider two cases:

CASE I: $|x| < 2MR^{-1}$. Then $x \leq (2M)^{1+\lambda+\varepsilon}$ and by Lemma 3.8 we have

$$\begin{aligned}
\left|(\widehat{\phi g_M})(x) - \widehat{\phi}(x)\right| &\ll \frac{\log M}{M^{1-4\varepsilon}} \\
&\ll \frac{1}{M^{\varepsilon}} \cdot \frac{\log M^{1-5\varepsilon}}{M^{1-5\varepsilon}}
\end{aligned}$$

$$
\begin{aligned}
&\ll \frac{1}{M^{\varepsilon}} \cdot \frac{\log\left(2^{-(1-5\varepsilon)}\,|x|^{\frac{1-5\varepsilon}{1+\lambda+\varepsilon}}\right)}{|x|^{\frac{1-5\varepsilon}{1+\lambda+\varepsilon}}} \\
&\ll \frac{1}{M^{\varepsilon}} \cdot \frac{\log|x|}{|x|^{\frac{1-5\varepsilon}{1+\lambda+\varepsilon}}}\,.
\end{aligned}
$$

Choosing M to be sufficiently large, the right hand side can become less than $\delta\theta(x)$.
CASE II: $|x| \geq 2MR^{-1}$. Then $|x| \geq (2M)^{1+\lambda-\varepsilon}$ and as in the proof of Lemma 3.8,

$$
\begin{aligned}
\left|\widehat{(\phi g_M)}(x) - \widehat{\phi}(x)\right| &\ll \sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} |\widehat{g}_M(k)|\,(1+|x-k|)^{-2} \\
&\ll \sum_{\substack{k\neq 0 \\ |x-k|\geq \frac{1}{2}|x|}} |\widehat{g}_M(k)|\,(1+|x-k|)^{-2} \\
&\quad + \sum_{\substack{k\neq 0 \\ |x-k|< \frac{1}{2}|x|}} |\widehat{g}_M(k)|\,(1+|x-k|)^{-2}\,,
\end{aligned}
$$

where the first term is

$$
\sum_{\substack{k\neq 0 \\ |x-k|\geq \frac{1}{2}|x|}} |\widehat{g}_M(k)|\,(1+|x-k|)^{-2} \ll |\widehat{g}_M(0)| \sum_{k\geq \frac{1}{2}|x|} \frac{1}{|k|^2} \ll \frac{1}{|x|}
$$

and the second is

$$
\begin{aligned}
\sum_{\substack{k\neq 0 \\ |x-k|< \frac{1}{2}|x|}} |\widehat{g}_M(k)|\,(1+|x-k|)^{-2} &\ll \sum_{|x-k|<\frac{|x|}{2}} |\widehat{g}_M(k)|\,(1+|x-k|)^{-2} \\
&\ll \sup_{|k|>\frac{|x|}{2}} |\widehat{g}_M(k)| \\
&\ll \sup_{|k|>\frac{|x|}{2}} \frac{\log|k|}{|k|^{\frac{1-4\varepsilon}{1+\lambda-\varepsilon}}} \qquad (\text{ by } (3.10)\ ) \\
&\ll \frac{\log|x|}{|x|^{\frac{1-4\varepsilon}{1+\lambda+\varepsilon}}}
\end{aligned}
$$

$$\ll \quad (2M)^{-\frac{1+\lambda-\varepsilon}{1+\lambda+\varepsilon}\varepsilon} \frac{\log|x|}{|x|^{\frac{1-5\varepsilon}{1+\lambda+\varepsilon}}}$$

so they are both less than $\delta\theta(x)$ for M large enough. $\square$

3.2.3 Finishing the Proof of Theorem 3.2

We use Lemma 3.9 to construct the requested measure μ. Choose a function $\phi_0 \in C_c^2(\mathbb{R})$ such that

$$\int \phi_0(x)\mathrm{d}x = 1, \quad \phi_0|_{(0,1)} > 0 \quad \text{and} \quad \phi_0|_{[0,1]^c} = 0.$$

Pick an arbitrary $0 < \delta < \frac{1}{2}$. Using Lemma 3.9 we can find a sequence $(M_k)_{k=1}^{\infty} \subseteq \mathbb{N}$ satisfying (3.3), (3.4) and (3.5) where

$$\begin{aligned} M_1 &= M_1\left(\phi_0, \delta 2^{-1}, \varepsilon\right), \\ M_2 &= M_2\left(\phi_0 g_{M_1}, \delta 2^{-2}, \varepsilon\right), \\ &\vdots \\ M_k &= M_k\left(\phi_0 g_{M_1} g_{M_2} \dots g_{M_{k-1}}, \delta 2^{-k}, \varepsilon\right), \quad k \in \mathbb{N}. \end{aligned}$$

We point out that the dependence of the integers $M_k, k = 1, 2, \dots$ on the corresponding parameters is in accordance with Lemma 3.9. Also define a sequence of functions $(G_k)_{k=0}^{\infty}$ by

$$\begin{aligned} G_0 &= 1, \\ G_k &= g_{M_1} g_{M_2} \cdots g_{M_k}, \quad k = 1, 2, \dots \end{aligned}$$

According to Lemma 3.9, for all $k \in \mathbb{N}$ we have

$$|(\widehat{\phi_0 G_{k+1}})(x) - (\widehat{\psi_0 G_k})(x)| \leq \delta 2^{-(k+1)}\theta(x), \quad \text{for all} \quad x \in \mathbb{R}. \tag{3.13}$$

If λ denotes the Lebesgue measure on $[0,1]$, the measures $(\mu_k)_{k=1}^{\infty}$ defined by $\mu_k = \phi_0 G_k \lambda$, $k = 1, 2, \dots$ have Fourier transforms

$$\begin{aligned} \widehat{\mu}_k(x) &= \int e^{-2\pi i t x}\mathrm{d}\mu_k(x) \\ &= \int e^{-2\pi i t x}\phi_0(x)G_k(x)\mathrm{d}\lambda(x) \\ &= (\widehat{\phi_0 G_k})(x), \quad x \in \mathbb{R}. \end{aligned}$$

Now (3.13) shows that $(\widehat{\mu}_k)_{k=1}^{\infty}$ is a $\| \, \|_\infty$-Cauchy sequence, hence there exists a constant $c = c(\psi, \varepsilon) > 0$ and a probability measure μ supported on the unit interval such that the sequence $(c\mu_k)_{k=1}^{\infty}$ converges weakly to μ.

The following proposition essentially proves Theorem 3.2.

Proposition 3.10. *The measure μ is supported on $W(\gamma, \psi)$ and has Fourier transform*

$$\widehat{\mu}(x) = O(\theta(x)), \qquad |x| \to \infty .$$

Proof. The fact that the support of μ lies within $W(\gamma, \psi)$ follows directly from the definition of the functions $(G_k)_{k=0}^{\infty}$ and the measure μ. Also observe that since $\widehat{\mu}_k(x) = (\widehat{\phi_0 G_k})(x)$ and ϕG_k is a C^2-function, we have

$$\widehat{\mu}_k(x) \le \frac{C_0}{(1+|x|)^2}, \qquad \text{for all } x \in \mathbb{R} .$$

for all $k = 1, 2, \ldots$. Now for all $k, p \in \mathbb{N}$ we have

$$\begin{aligned} |\widehat{\mu}(x)| &= \left| \widehat{\mu}(x) - \widehat{\mu}_{k+p}(x) + \sum_{j=1}^{p} (\widehat{\mu}_{k+j}(x) - \widehat{\mu}_{k+j-1}(x)) + \widehat{\mu}_k(x) \right| \\ &\le |\widehat{\mu}(x) - \widehat{\mu}_{k+p}(x)| + \sum_{j=1}^{p} |\widehat{\mu}_{k+j}(x) - \widehat{\mu}_{k+j-1}(x)| + |\widehat{\mu}_k(x)| \\ &\le \|\widehat{\mu} - \widehat{\mu}_{k+p}\|_\infty + \frac{\delta}{2^k}\theta(x) + \frac{C_0}{(1+|x|)^2} \qquad (\text{by } (3.13)) . \end{aligned}$$

Letting $p \to \infty$ we obtain the requested bound. $\square$

Chapter 4

A refinement of Marstrand's projection Theorem

4.1 Motivation

4.1.1 Marstrand's Theorem

Given $0 \leq \theta < \pi$, let L_θ denote the line through the origin of $\mathbb{R}^2$ that forms an angle θ with the horizontal axis. Let proj_θ denote orthogonal projection onto the line L_θ and $\dim A$ denote the Hausdorff dimension of a set $A \subseteq \mathbb{R}^2$. Then proj_θ is a Lipschitz mapping; indeed for all θ,

$$|\mathrm{proj}_\theta x - \mathrm{proj}_\theta y| \leq |x - y| \qquad \text{for all} \ \ x, y \in \mathbb{R}^2 . \tag{4.1}$$

This together with the trivial fact that $\mathrm{proj}_\theta A$ is a subset of a line, implies

$$\dim \mathrm{proj}_\theta A \leq \min\{1, \dim A\} \ , \tag{4.2}$$

see for example [23, Proposition 3.3]. The famous projection theorem of Marstrand [43], dating back to 1954, states that equality holds in (4.2) for almost almost all directions θ with respect to Lebesgue measure. Equivalently, the exceptional values of $\theta \in [0, \pi)$ for which the inequality (4.2) is strict, form a set of one-dimensional Lebegue measure zero.

Theorem 4.1 (Marstrand)**.** *Let $A \subseteq \mathbb{R}^2$ be a Borel set.*

(i) If $\dim A \leq 1$ then $\dim \mathrm{proj}_\theta A = \dim A$ for almost all $\theta \in [0, \pi)$.

(ii) If $\dim A > 1$ then $|\mathrm{proj}_\theta A| > 0$ for almost all $\theta \in [0, \pi)$.

Observe that the measure conclusion of (ii) is significantly stronger than the corresponding dimension statement; it trivially implies that $\dim \mathrm{proj}_\theta A = 1$ for almost all $\theta \in [0, \pi)$.

Marstrand's proof depends heavily on delicate and, in places, complicated geometric and measure theoretic arguments. Subsequently, Kaufman [39] gave a slick, two page, proof that made natural use of the potential theoretic characterization of Hausdorff dimension and Fourier transform methods.

4.1.2 The motivating example

Before moving onto our main result, Theorem 4.4, which is an analogue of Marstrand's Theorem for a general class of dimension functions, let us recall the definition of an explicit class of sets that has motivated our work and which illustrates and clarifies the need for statements such as Theorem 4.6.

Recall that if $\psi : \mathbb{N} \to \mathbb{R}^+$ is a decreasing function, a point $(y_1, \ldots, y_k) \in \mathbb{R}^n$ is called *simultaneously ψ–well approximable* if there are infinitely many $q \in \mathbb{N}$ and $(p_1, \ldots, p_k) \in \mathbb{Z}^n$ such that

$$\left| y_i - \frac{p_i}{q} \right| < \frac{\psi(q)}{q}, \qquad 1 \leq i \leq n \,. \tag{4.3}$$

The set of simultaneously ψ-well approximable points in $[0,1)^n$ will be denoted by $W_n(\psi)$. As we have already seen, the Khintchine-Jarník theorem (Theorem 1.35) provides a criterion for the 'size' of $W_n(\psi)$ in terms of Hausdorff measures $\mathcal{H}^f$.

For all $\tau > 0$, let ψ_τ be the approximating function given by $\psi_\tau(q) = q \exp(-q^\tau)$. Then by definition, when $n = 1$ the corresponding set $W_n(\psi_\tau)$ is a subset of the set of Liouville numbers, which is well-known to be of Hausdorff dimension zero. In fact $\dim W_n(\psi_\tau) = 0$ for all positive integers n. To see this, note that for any dimension function $f_s(r) = r^s$,

$$\sum_{q=1}^{\infty} q^n \, f_s\left(\psi_\tau(q)/q\right) \;=\; \sum_{q=1}^{\infty} \exp\big(-(s\, q^\tau - n \log q)\big) \;<\; \infty$$

for all $\tau > 0$ and $n \in \mathbb{N}$. Hence it follows from the Khintchine-Jarník Theorem and the definition of Hausdorff dimension that $\dim W_n(\psi_\tau) = 0$ for all $\tau > 0$ and $n \geq 1$. The upshot of this is that by (4.2), for all $\theta \in [0, \pi)$

$$\dim\big(\mathrm{proj}_\theta \, W_2(\psi_\tau)\big) = 0$$

and Marstrand's Theorem is not particularly informative. The problem is that the dimension functions f_s given by $f_s(r) = r^s$ are not delicate enough to differentiate between sets of dimension zero. Instead, for $s > 0$ consider the logarithmic dimension function f_s given by $f_s(r) = (-\log r)^{-s}$ for $0 < r < 1$. Then, for $\tau > 0$ and $n \geq 1$, it is easily verified that

$$\sum_{q=1}^{\infty} q^n f_s\left(\psi_\tau(q)/q\right) \;=\; \sum_{q=1}^{\infty} q^{-(\tau s-n)} \quad \begin{cases} <\infty, & \text{if} \quad s>s_0 \\ =\infty, & \text{if} \quad s<s_0 \end{cases} ,$$

where

$$s_0 = \frac{n+1}{\tau} \, .$$

It then follows from the Khintchine-Jarník Theorem and the definition of logarithmic Hausdorff dimension (Definition 4.5 in the following Section) that $\dim_{\log} W_n(\psi_\tau) = s_0$ for all $\tau > 0$ and $n \geq 1$. In turn Theorem 4.6 which follows will imply the non-trivial statement that for almost all $\theta \in [0, \pi)$,

$$\dim_{\log}\left(\mathrm{proj}_\theta\ W_n(\psi_\tau)\right) = s_0 \, .$$

4.2 The main result

The results of this subsection are joint work with V. Beresnevich, K. Falconer and S. Velani. In order to state our main theorem we first need to introduce the notion of a doubling function.

4.2.1 The doubling condition

Definition 4.2. A dimension function f is said to be *doubling* if there exist constants $c > 1$ and $r_0 > 0$ such that

$$f(2r) \leq cf(r) \qquad \text{for all } \ 0 < r < r_0 \, . \tag{4.4}$$

The number c in the above definition is called a *doubling constant.* Note that if f is given by $f(r) = r^s$ $(s > 0)$ then $f(2r) = 2^s f(r)$ and so $c = 2^s$ is a doubling constant for f.

We state an equivalent form of the doubling condition (4.4).

Lemma 4.3. *Let f be a dimension function. Then f is doubling if and only if there exist constants $s > 0$, $\kappa > 0$ and $r_1 > 0$ such that*

$$f(r\lambda) \,\geq\, \kappa\, \lambda^s f(r) \qquad \text{for all} \quad 0 < \lambda < 1 \quad \text{and} \quad 0 < r < r_1 \, . \tag{4.5}$$

Moreover, if f has a doubling constant $c > 1$ then (4.5) *holds with $\kappa = c^{-1}$ and $s = \log_2 c$.*

Proof. Suppose f is doubling with constant $c > 1$. For each positive integer n, applying (4.4) n times gives

$$f(r) \;\leq\; c^n f\left(\frac{1}{2^n}\, r\right) \qquad \text{for all} \;\; r < 2^n r_0 .$$

Set $s = \log_2 c > 0$. For each $0 < \lambda < 1$, let $m \geq 0$ be the unique integer such that

$$2^m \leq \frac{1}{\lambda} < 2^{m+1} .$$

Then

$$\begin{aligned} f(\lambda r) \;&\geq\; f\left(\frac{1}{2^{m+1}}\, r\right) \\ &\geq\; \frac{1}{c^{m+1}}\, f(r) \\ &\geq\; \frac{1}{c}\, \lambda^s f(r) . \end{aligned}$$

For the converse implication simply put $\lambda = \frac{1}{2}$ in (4.5). □

4.2.2 The main result

We are now in the position to state our main result.

Theorem 4.4. *Let $A \subseteq \mathbb{R}^2$ be a Borel set.*

(i) Let f be a dimension function. Then $\mathcal{H}^f(\mathrm{proj}_\theta A) \leq \mathcal{H}^f(A)$ for all $\theta \in [0, \pi)$. In particular if $\mathcal{H}^f(A) = 0$ then $\mathcal{H}^f(\mathrm{proj}_\theta A) = 0$ for all $\theta \in [0, \pi)$.

(ii) Let f be a dimension function such that $\mathcal{H}^f(A) > 0$. Suppose g is a dimension function that is doubling with constant $c < 2$ and such that

$$-\int_0^1 f(r)\; \mathrm{d}\!\left(\frac{1}{g(r)}\right) < \infty . \tag{4.6}$$

Then, $\mathcal{H}^g(\mathrm{proj}_\theta A) = \mathcal{H}^g(A) = \infty$ for almost all $\theta \in [0, \pi)$.

Several remarks are in order.

Remark 14. Part (i) of Theorem 4.4 is an immediate consequence of the Lipschitz condition (4.1) and the definition of $\mathcal{H}^f$, see [23, Proposition 3.1] where the case of $f(r) = r^s$ is given. Thus the main substance of the theorem is part (ii) when $\mathcal{H}^f(A) > 0$.

Remark 15. The conclusion of (ii) remains true if the range of integration in (4.6) is an interval $[0, r_0]$ for any $r_0 > 0$. Moreover, if g is differentiable, or at least differentiable except at finitely many points, then

$$-\int_0^1 f(r)\ \mathrm{d}\left(\frac{1}{g(r)}\right) = \int_0^1 f(r)\frac{g'(r)}{g^2(r)}\ \mathrm{d}r\ . \tag{4.7}$$

In particular, if f and g are dimension functions satisfying (4.6) then

$$\lim_{r\to 0}\frac{f(r)}{g(r)} = 0. \tag{4.8}$$

For suppose not. Then there exists $a > 0$ and a decreasing sequence $(r_n)_{n=1}^{\infty}$ tending to 0 such that

$$\frac{f(r_n)}{g(r_n)} \geq a \qquad \text{for all } n = 1, 2, \ldots$$

Let $r_n' > r_n$ be the least number such that $g(r_n') = 2g(r_n)$; such an r_n' exists by continuity and monotonicity of g provided that the sequence is chosen taking r_1 sufficiently small. Then

$$\begin{aligned}\int_{r_n}^{r_n'} f(r)\frac{g'(r)}{g^2(r)}\ \mathrm{d}r \quad &\geq \quad \int_{r_n}^{r_n'}\frac{f(r_n)}{2g(r_n)}\frac{g'(r)}{g(r)}\ \mathrm{d}r \\ &\geq \quad \frac{1}{2}a\log\frac{g(r_n')}{g(r_n)} \\ &= \quad \frac{1}{2}a\log 2.\end{aligned}$$

Since $0 < r_n < r_n' \to 0$, the integrals in (4.7) and (4.6) cannot be finite.

Remark 16. It is easily verified that if f and g are dimension functions satisfying (4.8) and $\mathcal{H}^f(A) > 0$ then $\mathcal{H}^g(A) = \infty$. Thus, the main substance of part (ii) of Theorem 4.4 is the statement that $\mathcal{H}^g(\mathrm{proj}_\theta A) \geq \mathcal{H}^g(A)$ for almost all $\theta \in [0, \pi)$. For further relations between measures with respect to different gauge functions, see [55, Section 4].

Remark 17. Regarding the dimension function g, the condition that $c < 2$ on the doubling constant is necessary. To see this, we derive the dimension aspect of Marstrand's Theorem from our result. With this in mind, assume without loss of generality that $\dim A > 0$ and let s_1, s_2 be arbitrary real numbers satisfying $0 < s_1 < s_2 < \dim A$. Now let g and f be dimension functions given by $g(r) = r^{s_1}$ and $f(r) = r^{s_2}$. It follows from the definition of Hausdorff dimension that $\mathcal{H}^{s_1}(A) = \mathcal{H}^{s_2}(A) = \infty$. Also it is easily checked that condition (4.6) is satisfied and thus, modulo the condition on the doubling constant, part (ii) of Theorem 4.4 implies that $\mathcal{H}^{s_1}(\mathrm{proj}_\theta A) = \infty$ for almost all $\theta \in [0, \pi)$. In turn, it follows (from the definition of Hausdorff dimension) that

$$\dim \operatorname{proj}_\theta A \geq s_1 \tag{4.9}$$

for almost all $\theta \in [0, \pi)$. The application of Theorem 4.4 is legitimate as long as the doubling constant $c = 2^{s_1}$ associated with g satisfies $s_1 < 1$. Now with reference to (4.9) this restriction on s_1 makes perfect sense since $\dim \operatorname{proj}_\theta A \leq 1$ regardless of the size of A. By continuity, we can replace s_1 in (4.9) by $\dim A$. The complementary upper bound can easily be deduced via part (i) of Theorem 4.4 but inequality (4.2) gives it directly.

Remark 18. Even if $\mathcal{H}^f(A) = \infty$, the conclusion of part (ii) of Theorem 4.4 is not in general valid for the dimension function f. Indeed, if f is given by $f(r) = r$ so that $\mathcal{H}^f$ is simply 1-dimension Lebesgue measure, it is known [23, Section 6.4] that there are sets A for which $\mathcal{H}^f(A) > 0$ but $\mathcal{H}^f(\operatorname{proj}_\theta A) = 0$ for almost all $\theta \in [0, \pi)$.

As alluded to in Remark 19, in §4.4 we will investigate the size of the set of exceptional angles θ for which the conclusion of part (ii) of Theorem 4.4 fails. In short, by replacing the integral convergence condition (4.6) by a suitable rate of convergence condition we are able to conclude that the exceptional set of $\theta \in [0, \pi)$ for which $\mathcal{H}^g(\operatorname{proj}_\theta A) < \infty$ is of $\mathcal{H}^f$-measure 0, see Theorem 4.12 for the precise statement.

The logarithmic dimension result

One consequence of Theorem 4.4, our main result, is the following analogue of Marstrand's Theorem for logarithmic Hausdorff dimension.

In terms of dimension theory, when we are confronted with sets of Hausdorff dimension 0 it is natural to change the usual 'r^s-scale' in the definition of Hausdorff dimension to a logarithmic scale. For $s > 0$, let f_s be the dimension function given by

$$f_s(r) = (-\log r)^{-s}, \tag{4.10}$$

for $0 < r < \frac{1}{2}$. As explained in Remark 5, the fact that we have restricted to sufficiently small values of $r > 0$ does not affect the Hausdorff measures defined by the functions f_s. In what follows, whenever we refer to dimension functions with a logarithmic factor in their definition, it is implied that these functions are studied in an appropriately chosen interval of the form $[0, r_0)$.

Definition 4.5. Let $A \subseteq \mathbb{R}^n$. The *logarithmic Hausdorff dimension* $\dim_{\log} A$ of A is defined by

$$\dim_{\log} A = \inf\left\{s : \mathcal{H}^{f_s}(A) = 0\right\} = \sup\left\{s : \mathcal{H}^{f_s}(A) = \infty\right\}, \tag{4.11}$$

where the functions f_s are as in (4.10).

It is easily verified that if $\dim A > 0$ then $\dim_{\log} A = \infty$, precisely as one would expect.

Armed with Theorem 4.4 it is straightforward to prove the following.

Theorem 4.6. *Let $A \subseteq \mathbb{R}^2$ be a Borel set. Then*

(i) $\dim_{\log} \mathrm{proj}_\theta A \le \dim_{\log} A$ *for all* $\theta \in [0, \pi)$,

(ii) $\dim_{\log} \mathrm{proj}_\theta A = \dim_{\log} A$ *for Lebesgue-almost all* $\theta \in [0, \pi)$.

Proof. Part (i) is immediate from Theorem 4.4(i) and (4.11).

For part (ii), without loss of generality, assume that $\dim_{\log} A > 0$ and let s_1, s_2 be real numbers satisfying

$$0 < s_1 < s_2 < \dim_{\log} A.$$

Consider the dimension functions given by $g(r) = (-\log r)^{-s_1}$ and $f(r) = (-\log r)^{-s_2}$. It follows from (4.11) that $\mathcal{H}^g(A) = \mathcal{H}^f(A) = \infty$. It is easily verified that condition (4.6) is satisfied and that g is doubling with constant $c < 2$. Thus, Theorem 4.4(ii) implies that $\mathcal{H}^g(\mathrm{proj}_\theta A) = \infty$ for almost all $\theta \in [0, \pi)$. In turn, it follows from (4.11) that for almost all $\theta \in [0, \pi)$, $\dim_{\log} \mathrm{proj}_\theta A \ge s_1$ and thus $\dim_{\log} \mathrm{proj}_\theta A \ge \dim_{\log} A$.
□ □

Remark 19. By considering the size of sets of exceptional angles, see §4.4, we are further able to conclude that

$$\dim_{\log} \{\theta \in [0, \pi) : \dim_{\log} \mathrm{proj}_\theta A < \dim_{\log} A\} \le \dim_{\log} A\,. \tag{4.12}$$

Of course, the interesting case is when $\dim_{\log} A$ is finite. Then, by definition $\dim A = 0$ and so (4.12) is significantly stronger than Theorem 4.6.

Explicitly exposing the gap of uncertainty

With reference to our motivating example, for all $\tau > 0$ let $\psi_\tau : \mathbb{N} \to \mathbb{R}^+$ be the approximating function given by

$$\psi_\tau(q) = \frac{1}{q^{\tau-1}(\log q)^\tau}, \qquad q \ge 2.$$

It follows, via the Khintchine-Jarník Theorem and the definition of Hausdorff dimension, that for all $\tau \ge \dfrac{1+n}{n}$,

$$\delta = \delta(\tau) := \dim W_n(\psi_\tau) = \frac{n+1}{\tau}\,\cdot$$

In fact, the Khintchine-Jarník Theorem implies a much finer conclusion. Fix $\tau > (1+n)/n$ and consider the family of dimension functions $(f_{\delta,s})_{s>0}$ given by

$$f_{\delta,s}(r) = r^{\delta}\left(-\frac{1}{\tau}\log r\right)^{s}.$$

It is easily verified that

$$\sum_{q=2}^{\infty} q^n f_{\delta,s}\left(\psi_\tau(q)/q\right) \asymp \sum_{q=2}^{\infty} \frac{1}{q(\log q)^{1+n-s}},$$

in the sense that the series either both converge or diverge, and so the Khintchine-Jarník Theorem implies that

$$\mathcal{H}^{f_{\delta,s}}\left(W_k(\psi_\tau)\right) = \begin{cases} 0 & \text{if} \quad s < n\,, \\ \infty & \text{if} \quad s \geq n\,. \end{cases}$$

Loosely speaking, the set $W_n(\psi_\tau)$ has "$\delta(\tau)$-logarithmic dimension" equal to n.

Now let $n = 2$ and with reference to Theorem 4.4, put $f = f_{\delta,2}$ and $g = f_{\delta,s}$. Suppose that $\tau > 3$ so that $\delta(\tau) < 1$. This ensures that g is doubling with constant $c < 2$. Theorem 4.4 then implies that for almost all $\theta \in [0, \pi)$,

$$\mathcal{H}^{f_{\delta,s}}\left(\operatorname{proj}_\theta W_2(\psi_\tau)\right) = \begin{cases} 0 & \text{if} \quad s < 2\,, \\ \infty & \text{if} \quad s > 3\,. \end{cases}$$

Of course, part (i) of Theorem 4.4 implies that the zero measure statement associated with $s < 2$ is true for all $\theta \in [0, \pi)$. Regarding the application of part (ii), we need $s > 3$ in order for the integral convergence condition (4.6) to be satisfied. Thus the latter gives rise to a gap of uncertainty; namely $s \in (2, 3)$ in the specific example under consideration. We suspect that the infinity measure statement for $s > 3$ is actually true for $s > 2$.

Problem: Show that $\mathcal{H}^{f_{\delta,s}}\left(\operatorname{proj}_\theta W_2(\psi_\tau)\right) = \infty$ if $s > 2$.

The fact of the matter is that it is highly unlikely that any set $W_2(\psi)$ of simultaneously ψ–approximable points will have the necessary 'dense rotational' structure that underpins the construction of the sets associated with Theorem 4.14 which follows later.

4.3 Proof of main result

Our proof of Theorem 4.4 will follow Kaufman's potential theoretic proof [39] of Mastrand's Theorem. We adapt the proof that he gave for the specific functions $f(r) = r^s$ $(s > 0)$ to general dimension functions.

4.3.1 Energies and capacities

We first generalise the standard notions of s-energy and s-capacity of a measure which have been defined in the introductory chapter, see for example [23, Section 4.3] and [44, Chapter 8]. In what follows, f is a dimension function.

Definition 4.7. The *f-energy* of $\mu \in \mathcal{M}_1(A)$ is defined as

$$I_f(\mu) = \iint \frac{\mathrm{d}\mu(x)\mathrm{d}\mu(y)}{f(|x-y|)} .$$

Alternatively, we could have defined the f-energy via the *f-potential* at a point $x \in \mathbb{R}^2$, that is

$$\phi_f(x) := \int \frac{\mathrm{d}\mu(y)}{f(|x-y|)} \quad \text{and so} \quad I_f(\mu) = \int \phi_f(x)\ \mathrm{d}\mu(x).$$

Definition 4.8. The *f-capacity* of a Borel set $A \subseteq \mathbb{R}^2$ is defined as

$$C_f(A) = \sup\left\{\frac{1}{I_f(\mu)} : \mu \in \mathcal{M}_1(A)\right\}$$

with the interpretation that $C_f(\emptyset) = 0$.

Naturally, when f is given by $f(r) = r^s$ $(s > 0)$ we recover the familiar notions of s-energy and s-capacity.

We now establish the connection between the Hausdorff measure $\mathcal{H}^f(A)$ and the capacity $C_f(A)$ of a set A with respect to a general dimension function f. These results stated below have a long history: apart from notational differences they appear as Theorems 1 and 2 in [60], though versions for the dimension functions of the form $f(r) = r^s$ date back to the 1930s. The paper [60] discusses the historical development to increasingly general dimension functions and includes further references. Proofs for dimension functions $f(r) = r^s$ may be found in several more recent accounts of fractal geometry, for example [23],[44]. Even for general dimension functions the proofs are relatively short, so for the sake of completeness we include these proofs.

Proposition 4.9. *Let $A \subseteq \mathbb{R}^2$ be a Borel set and f be a dimension function. If $\mathcal{H}^f(A) < \infty$ then $C_f(A) = 0$.*

Proof. Assume $C_f(A) > 0$. By definition, the set A supports a Radon probability measure μ such that $I_f(\mu) < \infty$. Thus

$$\int \frac{\mathrm{d}\mu(y)}{f(|x-y|)} < \infty \quad \text{for } \mu\text{-almost all } x \in A.$$

For such $x \in A$,

$$\lim_{r\to 0} \int\limits_{B(x,r)} \frac{\mathrm{d}\mu(y)}{f(|x-y|)} = 0\,.$$

By Egorov's theorem, for all $\varepsilon > 0$ there exist $\delta > 0$ and a Borel set $K \subseteq A$ such that $\mu(K) > \frac{1}{2}$ and

$$\begin{aligned} \mu(B(x,r)) &\le f(r) \int\limits_{B(x,r)} \frac{\mathrm{d}\mu(y)}{f(|x-y|)} \\ &\le \varepsilon f(r), \qquad \text{for all } x \in K \text{ and } 0 < r \le \delta. \end{aligned}$$

Now let $\big(B(x_i,r_i)\big)_{i=1}^{\infty}$ be a cover of K by balls with $x_i \in K$ and $r_i \le \delta$ such that

$$\sum_{i=1}^{\infty} f(r_i) \;<\; \mathcal{H}^f_C(K) + 1$$

Then

$$\frac{1}{2} < \;\mu(K) \le \sum_{i=1}^{\infty} \mu(B(x_i,r_i)) \le \varepsilon \sum_{i=1}^{\infty} f(r_i) \le \varepsilon \left(\mathcal{H}^f_C(A) + 1\right),$$

where $\mathcal{H}^f_C$ is centred Hausdorff measure. Since $\varepsilon > 0$ can be made arbitrarily small, we conclude that $\mathcal{H}^f(A) = \mathcal{H}^f_C(A) = \infty$, using (1.12). This contradicts our hypothesis that $\mathcal{H}^f(A)$ is finite. □

The following statement is referred to as Frostman's lemma, and can be viewed as a partial converse to the Mass Distribution Principle, Theorem 1.23 in Chapter 1. Throughout, given a Borel set $A \subseteq \mathbb{R}^2$ we denote by $\mathcal{M}_1(A)$ the set of Radon probability measures μ with compact support in A.

Theorem 4.10 (Frostman's Lemma). *Let $A \subseteq \mathbb{R}^2$ be a Borel set and f be a dimension function. Then $\mathcal{H}^f(A) > 0$ if and only if there exist a measure $\mu \in \mathcal{M}_1(A)$ and a constant $c_1 > 0$ such that*

$$\mu(B(x,r)) \le c_1 f(r) \qquad \text{for all } \; x \in \mathbb{R}^2 \; \text{ and } \; r > 0\,.$$

Two very different proofs for the case where $f(r) = r^s$ $(s > 0)$ are given in [44, Theorem 8.8], where it is explicitly pointed out that both proofs are valid for general dimension functions. Alternatively, for the harder of the implications, namely that suitable measures exist, the result in Rogers [55, Theorem 57], that for general dimension functions there exists a compact subset A' of A with $0 < \mathcal{H}^f(A') < \infty$, followed by a density argument akin to [23, Proposition 4.11], also gives the conclusion.

Proposition 4.11. *Let $A \subseteq \mathbb{R}^2$ be a Borel set and let f and g be dimension functions satisfying the integral convergence condition* (4.6)*. If $\mathcal{H}^f(A) > 0$ then $C_g(A) > 0$.*

Proof. By Frostman's lemma, Theorem 4.10, the Borel set A supports a Radon probability measure μ such that

$$\mu\left(B(x,r)\right) \le c_1\, f(r) \qquad \text{for all} \quad r>0 \quad \text{and} \quad x\in\mathbb{R}^2 \tag{4.13}$$

for some constant $c_1>0$. Fix $x\in\mathbb{R}^2$ and let

$$m(r)=\mu\left(B(x,r)\right).$$

Using (4.13) and that $\mu(\mathbb{R}^2)=1$ and integrating by parts,

$$\begin{aligned}
\int \frac{\mathrm{d}\mu(y)}{g(|x-y|)} &= \int\limits_{|x-y|\le 1} \frac{\mathrm{d}\mu(y)}{g(|x-y|)} \;+\; \int\limits_{|x-y|>1} \frac{\mathrm{d}\mu(y)}{g(|x-y|)} \\
&\le \int_0^1 \frac{1}{g(r)}\,\mathrm{d}m(r) \;+\; \frac{\mu(\mathbb{R}^2)}{g(1)} \\
&= \left[\frac{m(r)}{g(r)}\right]_0^1 \;-\; \int_0^1 m(r)\,\mathrm{d}\left(\frac{1}{g(r)}\right) \;+\; \frac{\mu(\mathbb{R}^2)}{g(1)} \\
&\le \frac{m(1)}{g(1)} \;-\; \lim_{r\to 0+}\frac{m(r)}{g(r)} \;-\; \int_0^1 f(r)\,\mathrm{d}\left(\frac{1}{g(r)}\right) \;+\; \frac{\mu(\mathbb{R}^2)}{g(1)} \\
&< \infty,
\end{aligned}$$

noting that

$$\frac{m(r)}{g(r)} \le c_1\frac{f(r)}{g(r)} \to 0$$

by (4.8). This bound is uniform for all $x\in\mathbb{R}$ and so

$$I_g(\mu)=\iint \frac{\mathrm{d}\mu(x)\mathrm{d}\mu(y)}{g(|x-y|)} \;<\; \infty$$

giving $C_g(A)>0$ by Definition 4.8. □

Remark 20. Fix $0<\delta<1$ and consider the family of dimension functions $(f_{\delta,s})_{s>0}$ given by

$$f_{\delta,s}(r)=r^{\delta}(-\log r)^{s},$$

to within constants the same as those considered in §4.2.2. Let $A\subseteq\mathbb{R}^2$ be a Borel set and $\alpha>0$. Then, by Propositions 4.9 and 4.11,

(i) if $s \leq \alpha$ and $\mathcal{H}^{f_{\delta,\alpha}}(A) < \infty$ then $C_{f_{\delta,s}}(A) = 0$,

(ii) if $s > \alpha + 1$ and $\mathcal{H}^{f_{\delta,\alpha}}(A) > 0$ then $C_{f_{\delta,s}}(A) > 0$.

The upshot is that if $\alpha < s \leq \alpha+1$, condition (4.6) is not satisfied and the propositions provide no information. The main aim of the paper [60] is to show that this "gap of uncertainty" is genuine – it really exists! So for example, by [60, Theorem 3], if f and g are dimension functions not satisfying condition (4.6), then there exist Borel sets A with $0 < \mathcal{H}^f(A) < \infty$ but $C_g(A) = 0$.

4.3.2 Proof of Theorem 4.4

(i) As pointed out in Remark 14, this is a trivial consequence of the definition of the Hausdorff measures that projection is a Lipschitz mapping.

(ii) From Remark 16, $\mathcal{H}^g(A) = \infty$. Thus it suffices to show that $\mathcal{H}^g(\mathrm{proj}_\theta A) = \infty$ for almost all $\theta \in [0, \pi)$.

Since $\mathcal{H}^g(A) > 0$, it follows via Proposition 4.11 and the definition of capacity, that A supports a Radon probability measure μ such that $I_g(\mu) < \infty$. For each $\theta \in [0, \pi)$, projecting μ onto the line L_θ gives a measure μ_θ supported on $\mathrm{proj}_\theta A$ defined by the requirement that $\mu_\theta(K) = \mu(\mathrm{proj}_\theta^{-1}(K))$ for each Borel set $K \subseteq L_\theta$. For each $x \in \mathbb{R}^2$, let $\phi(x)$ denote the angle that x (viewed as a vector) forms with the horizontal axis. Then, by Lemma 4.3 and using the fact that g is doubling with constant $c < 2$, it follows that

$$\begin{aligned}
\int_0^\pi I_g(\mu_\theta)\mathrm{d}\theta &= \int_0^\pi \iint \frac{\mathrm{d}\mu_\theta(x)\mathrm{d}\mu_\theta(y)}{g(|x-y|)}\mathrm{d}\theta \\
&= \int_0^\pi \iint \frac{\mathrm{d}\mu(x)\mathrm{d}\mu(y)}{g(|\mathrm{proj}_\theta x - \mathrm{proj}_\theta y|)}\mathrm{d}\theta \\
&\leq \iint \left(\int_0^\pi \frac{c}{g(|x-y|)|\cos(\phi(x-y)-\theta)|^s}\mathrm{d}\theta\right)\mathrm{d}\mu(x)\mathrm{d}\mu(y) \\
&\leq c_1 \iint \frac{\mathrm{d}\mu(x)\mathrm{d}\mu(y)}{g(|x-y|)} \qquad \text{(because } s = \log_2 c < 1\text{)} \\
&= c_1 I_g(\mu) \\
&< \infty\,.
\end{aligned}$$

This implies that $I_g(\mu_\theta) < \infty$ for almost all $\theta \in [0, \pi)$. From the definition of capacity, $C_g(\mathrm{proj}_\theta A) > 0$ for such θ, so by Proposition 4.9, $\mathcal{H}^g(\mathrm{proj}_\theta A) = \infty$ for almost all $\theta \in [0, \pi)$. □

4.4 Exceptional projections

Marstrand's Theorem trivially implies that the set of exceptional angles

$$E(A) := \{\theta \in [0, \pi) \,:\, \dim \mathrm{proj}_\theta A < \dim A\} \ ,$$

is a set of (one-dimensional) Lebesgue measure zero. Kaufman also showed [39] that

$$\dim\, E(A) \ \leq \ \min\{1, \dim A\} \tag{4.14}$$

(see also Remark 21 below). Clearly, when $\dim A < 1$, this bound on the size of the set of exceptional angles is significantly stronger than the measure zero statement of Marstrand's Theorem. It is natural to attempt to extend Theorem 4.4 in a similar fashion. With this in mind, let $E_g(A)$ denote the exceptional set of $\theta \in [0, \pi)$ for which the conclusion of part (ii) of Theorem 4.4 fails; that is

$$E_g(A) := \{\theta \in [0, \pi) : \mathcal{H}^g(\mathrm{proj}_\theta A) < \infty\} \ . \tag{4.15}$$

By replacing the integral convergence condition (4.6) by a rate of convergence condition we are able to establish the following strengthening of Theorem 4.4. It is easily verified that condition (4.16) below implies condition (4.6) of Theorem 4.4.

Theorem 4.12. *Let $A \subseteq \mathbb{R}^2$ be a Borel set. Let f be a dimension function such that $\mathcal{H}^f(A) > 0$ and let g be a dimension function that is doubling. Suppose that there exist constants t_0 and $c_2 > 0$ such that*

$$-\int_0^1 f(r)\ \mathrm{d}\!\left(\frac{1}{g(tr)}\right) \ < \ c_2\,\frac{1}{g(t)} \qquad \text{for all} \quad 0 < t < t_0\,. \tag{4.16}$$

Then, $\mathcal{H}^f(E_g(A)) = 0$.

Proof. In view of Proposition 4.9,

$$E_g(A) \subseteq E_* := \{\theta \in [0, \pi) : C_g(\mathrm{proj}_\theta A) = 0\} \ .$$

Thus, it suffices to show that $\mathcal{H}^f(E_*) = 0$. Suppose this is not the case. Then $\mathcal{H}^f(E_*) > 0$ and by Theorem 4.10 the set E_* supports a probability measure $\nu \in \mathcal{M}_1(E_*)$ such that

$$\nu\,(B(x, r)) \ \leq \ c_1\, f(r) \qquad \text{for all} \ \ x \in \mathbb{R}^2, \quad r > 0$$

where $c_1 > 0$ is an absolute constant. On the other hand, since $\mathcal{H}^f(A) > 0$ and the fact that condition (4.16) implies condition (4.6), it follows via Proposition 4.11 and the definition of capacity, that A supports a probability measure $\mu \in \mathcal{M}_1(A)$ such that

$$I_g(\mu) < \infty. \tag{4.17}$$

For each $\theta \in [0, \pi)$, let μ_θ be the projection of μ onto the line L_θ supported on $\mathrm{proj}_\theta A$, so that $\mu_\theta(K) = \mu(\mathrm{proj}_\theta^{-1}(K))$ for any Borel set $K \subseteq L_\theta$ as in the proof of Theorem 4.4. Let us assume for the moment that

$$\int_{E_*} I_g(\mu_\theta)\mathrm{d}\nu(\theta) < \infty\,. \tag{4.18}$$

This implies that $I_g(\mu_\theta) < \infty$ for ν-almost all $\theta \in E_*$. By the definition of capacity, $C_g(\mathrm{proj}_\theta A) > 0$ for such θ, contradicting that $C_g(\mathrm{proj}_\theta A) = 0$ if $\theta \in E_*$. This completes the proof of the theorem modulo establishing (4.18).

To establish (4.18), we first observe that for all $x \in \mathbb{R}^2 \setminus \{0\}$ and $d > 0$, the set

$$\{\theta \in [0,\pi) : |\mathrm{proj}_\theta x| \le d\}$$

is a union of at most two intervals each of diameter at most $\pi\, d/|x|$. The upshot is that

$$\nu\left(\{\theta \in [0,\pi) : |\mathrm{proj}_\theta x| \le d\}\right) \;\le\; 2\,c_1 f\Big(\pi\frac{d}{|x|}\Big)\,.$$

This, together with the fact that g is doubling, implies that

$$\begin{aligned}
\int_{E_*} \frac{1}{g(|\mathrm{proj}_\theta x|)}\,\mathrm{d}\nu(\theta) \;&=\; \int_0^\infty \nu\left(\left\{\theta : \frac{1}{g(|\mathrm{proj}_\theta x|)} \ge r\right\}\right)\mathrm{d}r \\
&=\; \int_0^{1/g(|x|)} \nu\left(\left\{\theta : \frac{1}{g(|\mathrm{proj}_\theta x|)} \ge r\right\}\right)\mathrm{d}r \\
&\qquad +\; \int_{1/g(|x|)}^\infty \nu\left(\left\{\theta : \frac{1}{g(|\mathrm{proj}_\theta x|)} \ge r\right\}\right)\mathrm{d}r \\
&\le\; \frac{1}{g(|x|)} \;+\; \int_{1/g(|x|)}^\infty 2\,c_1 f\left(\frac{\pi}{|x|}\,g^{-1}\left(\frac{1}{r}\right)\right)\mathrm{d}r \\
&\le\; \frac{1}{g(|x|)} \;-\; 2c_1 \int_0^\pi f(u)\,\mathrm{d}\left(\frac{1}{g\left(\frac{1}{\pi}|x|u\right)}\right) \\
&=\; \frac{1}{g(|x|)} \;+\; 2c_1 \int_0^1 f(u)\,\mathrm{d}\left(\frac{-1}{g\left(\frac{1}{\pi}|x|u\right)}\right) \\
&\qquad +\; 2c_1 \int_1^\pi f(u)\,\mathrm{d}\left(\frac{-1}{g\left(\frac{1}{\pi}|x|u\right)}\right)
\end{aligned}$$

$$\begin{aligned} &\overset{(4.16)}{\leq} \quad \frac{1}{g(|x|)} \;+\; 2c_1c_2\,\frac{1}{g\left(\frac{1}{\pi}|x|\right)} \\ &\qquad +\; 2c_1\,f(\pi)\,\left(\frac{1}{g\left(\frac{1}{\pi}|x|\right)} - \frac{1}{g(|x|)}\right) \\ &\leq \quad c_3\,\frac{1}{g(|x|)} \end{aligned}$$

for some c_3 and for all $x \neq 0$ with $|x| < t_0$. Hence, using Fubini's theorem,

$$\begin{aligned} \int_{E_*} I_g(\mu_\theta)\,\mathrm{d}\nu(\theta) &= \int_{E_*}\iint \frac{\mathrm{d}\mu_\theta(x)\,\mathrm{d}\mu_\theta(y)}{g(|x-y|)}\,\mathrm{d}\nu(\theta) \\ &= \int_{E_*}\iint \frac{\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)}{g(|\mathrm{proj}_\theta x - \mathrm{proj}_\theta y|)}\,\mathrm{d}\nu(\theta) \\ &= \iint\int_{E_*} \frac{\mathrm{d}\nu(\theta)}{g(|\mathrm{proj}_\theta(x-y)|)}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) \\ &\leq c_3\iint \frac{\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)}{g(|x-y|)} \\ &< \infty \end{aligned}$$

by (4.17). This establishes (4.18) and completes the proof. □

Remark 21. The above proof of Theorem 4.12 is based on that of the special case (4.14) presented in [45, Theorem 5.1] which in derives from [39]. Indeed, it is easy to deduce (4.14) from Theorem 4.12. To see that this, without loss of generality assume that $0 < \dim A < 1$ and let s_1, s_2 be real numbers satisfying $0 < s_1 < s_2 < \dim A$. Let

$$E(A, s_1) := \{\theta \in [0,\pi) : \dim \mathrm{proj}_\theta A < s_1\}.$$

Let g and f be the dimension functions given by $g(r) = r^{s_1}$ and $f(r) = r^{s_2}$. It follows that $\mathcal{H}^{s_2}(A) = \infty$ and that $\mathcal{H}^{s_1}\big(\mathrm{proj}_\theta A\big) = 0$ for all $\theta \in E(A, s_1)$. Thus

$$E(A, s_1) \subseteq E_g(A)\,,$$

with $E_g(A)$ as in (4.15). Clearly, the function g is doubling and it is easily checked that f and g satisfy condition (4.16). Theorem 4.12 implies that $\mathcal{H}^{s_2}(E_g(A)) = 0$ and so $\dim\big(E_{\log}(A, s_1)\big) \leq s_2$, and (4.14) follows on taking s_1, s_2 arbitrarily close to $\dim A$.

Armed with Theorem 4.12 it is straightforward to prove (4.12) which we formally state as a corollary.

Corollary 3. Let $A \subseteq \mathbb{R}^2$ be a Borel set. Then, $\dim_{\log} E_{\log}(A) \leq \dim_{\log} A$ where

$$E_{\log}(A) := \{\theta \in [0, \pi) \,:\, \dim_{\log} \mathrm{proj}_\theta(A) < \dim_{\log} A\}.$$

Proof. Without loss of generality, assume that $0 < \dim_{\log} A < \infty$ and let s_1, s_2 be real numbers satisfying $0 < s_1 < s_2 < \dim_{\log} A$. Let

$$E_{\log}(A, s_1) := \{\theta \in [0, \pi) : \dim_{\log} \mathrm{proj}_\theta(A) < s_1\}.$$

As in the proof of Theorem 4.6, let g and f be the dimension functions $g(r) = (-\log r)^{-s_1}$ and $f(r) := (-\log r)^{-s_2}$. Then $\mathcal{H}^f(A) = \infty$ and $\mathcal{H}^g\big(\mathrm{proj}_\theta(A)\big) = 0$ for all $\theta \in E_{\log}(A, s_1)$ so $E_{\log}(A, s_1) \subseteq E_g(A)$. Clearly g is doubling. Assume for the moment that f and g satisfy condition (4.16). Then Theorem 4.12 implies that $\mathcal{H}^f(E_g(A)) = 0$ so from the definition of logarithmic Hausdorff dimension (4.11),

$$\dim_{\log}\big(E_{\log}(A, s_1)\big) \leq s_2\,.$$

The conclusion now follows on taking s_1, s_2 arbitrarily close to $\dim_{\log} A$.

It remains to verify (4.16). For all sufficiently small $t > 0$,

$$\begin{aligned} -\int_0^1 f(r)\, \mathrm{d}\Big(\frac{1}{g(tr)}\Big) &= \Big[\frac{f(r)}{g(tr)}\Big]_1^0 + \int_0^1 \frac{\mathrm{d}f(r)}{g(tr)} \\ &= -\frac{f(1)}{g(t)} + \int_0^t \frac{\mathrm{d}f(r)}{g(tr)} + \int_t^1 \frac{\mathrm{d}f(r)}{g(tr)}\,. \end{aligned}$$

For the first integral, $r \leq t$ implies that

$$\frac{1}{g(tr)} \leq 2^{s_1}(-\log r)^{s_1} = 2^{s_1}\frac{1}{g(r)}$$

and hence it follows that

$$\begin{aligned} \int_0^t \frac{\mathrm{d}f(r)}{g(tr)} &\leq 2^{s_1}\int_0^t \frac{\mathrm{d}f(r)}{g(r)} \\ &\leq 2^{s_1}\int_0^1 \frac{\mathrm{d}f(r)}{g(r)} \\ &= \frac{2^{s_1} s_2(s_2 - s_1)}{(\log 2)^{s_2 - s_1}}. \end{aligned}$$

For the second integral, $r \geq t$ implies that

$$\frac{1}{g(tr)} < 2^{s_1}\frac{1}{g(t)}$$

and hence it follows that

$$\int_t^1 \frac{\mathrm{d}f(r)}{g(tr)} < 2^{s_1}\int_0^1 \frac{\mathrm{d}f(r)}{g(t)} = 2^{s_1}\frac{f(1)}{g(t)}\,.$$

On combining these estimates, we obtain that

$$\begin{aligned} -\int_0^1 f(r)\,\mathrm{d}\left(\frac{1}{g(tr)}\right) &= (2^{s_1}-1)f(1)\frac{1}{g(t)} + \frac{2^{s_1}s_2(s_2-s_1)}{(\log 2)^{s_2-s_1}} \\ &\leq c_2\frac{1}{g(t)} \end{aligned}$$

for some constant c_2, as desired.

$\square$

4.5 Final comments

Apart from working in higher dimensions, there are several other directions in which one could attempt to strengthen/generalize the main theorem. We concentrate on just a few of them.

The gap of uncertainty. Theorem 4.14 in the following section shows that we can not in general replace condition (4.6) by (4.8) in Theorem 4.4. Thus there is a genuine gap of uncertainty associated with Theorem 4.4. It would be highly desirable to know whether or not condition (4.6) is really necessary. Namely, if f and g are dimension functions such that (4.6) is not satisfied, then does there exist a set $A \subseteq \mathbb{R}^2$ such that $\mathcal{H}^f(A) > 0$ but $\mathcal{H}^g(\mathrm{proj}_\theta A) = 0$ for almost all $0 \leq \theta < \pi$? Theorem 4.14 provides sufficient conditions on f and g for the existence of such a set A.

Brownian paths. Brownian motion sample paths, see [23, Chapter 16] for a general introduction, illustrate the sort of situation that can arise for projections of sets in $\mathbb{R}^3$, and which perhaps may occur in $\mathbb{R}^2$, though there is no direct analogue. Let $B[0,1] \subseteq \mathbb{R}^3$ be (random) Brownian motion path over the unit time interval. Then, almost surely, the Hausdorff dimension of $B[0,1]$ is logarithmically smaller than 2, more precisely $0 < \mathcal{H}^f(B[0,1]) < \infty$ where f is the dimension function $f(r) = r^2\log\log(1/r)$ (for small r), see [15]. However, the projection $\mathrm{proj}_P(B[0,1])$ of $B[0,1]$ onto any given plane P has exactly the same distribution as a Brownian motion in the plane, which is almost surely of Hausdorff dimension 2, or precisely, $0 < \mathcal{H}^g(\mathrm{proj}_P(B[0,1])) < \infty$ where g is the dimension function $g(r) = r^2\log(1/r)\log\log\log(1/r)$, see [61]. This example, where the exact dimension functions of a set and of almost all its projections onto a plane can be identified, illustrates the sort of change in exact dimension that may occur under projection.

Sets with no exceptional projections. The dimension result (4.14) for the set of exceptional projections has been extended in various ways – see [25, 45] and references within. We highlight a result concerning sets A for which there are no exceptional projections; that is, sets A for which $E(A) = \emptyset$.

Theorem (Peres–Shmerkin). Let $A \subseteq \mathbb{R}^2$ be a self-similar set with dense rotations. Then

$$\dim \mathrm{proj}_\theta A = \min\{\dim A, 1\} \quad \textit{for all} \quad \theta \in [0, \pi). \tag{4.19}$$

This theorem was proved by Peres and Shmerkin [47] and subsequently generalized by Hochman and Shmerkin [32]. Now suppose A is self-similar set with dense rotations and f and g are dimension functions as in Theorem 4.4. It is natural to ask whether or not the conclusion of part (ii) of Theorem 4.4 is actually valid for all θ rather than just almost all $\theta \in [0, \pi)$.

Lengths of projections. It is natural to seek a finer version of part (ii) of Marstrand's theorem which gives a criterion for almost all projections of a set to have positive length. One aspect of this was investigated by Peres and Solomyak [48], who considered dimension functions f such that $f(r)/r^2$ is decreasing for $r > 0$ (a condition that holds in virtually all cases of interest). They showed that $|\mathrm{proj}_\theta A| > 0$ for almost all $\theta \in [0, \pi)$ for all Borel sets A satisfying $\mathcal{H}^f(A) > 0$ if and only if $\int_0^1 r^{-2} f(r) \mathrm{d}r < \infty$.

4.6 On the gap of uncertainty

As we have explicitly demonstrated in §4.2.2, the integral convergence condition (4.6) gives rise to a gap of uncertainty. It is natural to ask whether this condition is really necessary. Namely, if f and g are dimension functions such that (4.6) is not satisfied, then does there exist a set $A \subseteq \mathbb{R}^2$ such that $\mathcal{H}^f(A) > 0$ but $\mathcal{H}^g(\mathrm{proj}_\theta A) = 0$ for almost all $0 \le \theta < \pi$?

In joint work with David Simmons and Han Yu, in this section we partially answer this question by providing sufficient conditions on f and g for the existence of such a set A. Our construction of this set is a generalization of a construction of Martin and Mattila [42], in which they proved that for every $0 < s \le 1$ there exists a set $A \subseteq \mathbb{R}^2$ such that $\mathcal{H}^s(A) > 0$ but $\mathcal{H}^s(\mathrm{proj}_\theta A) = 0$ for all $0 \le \theta < \pi$. By making a careful quantitative analysis of their construction, we are able to improve their result and establish Theorem 4.14.

For the statement of this result we need the notion of a codoubling dimension function, which is similar to the notion of a doubling dimension function.

Definition 4.13. Let f be a dimension function. Then f is called *codoubling with exponent s* if there exist constants $\kappa > 0$ and $r_1 > 0$ such that

$$f(\lambda r) \le \kappa\, \lambda^s f(r) \qquad \text{for all } \ 0 < \lambda < 1 \ \text{ and } \ 0 < r < r_1\,. \tag{4.20}$$

Remark 22. Just as in the case of doubling functions, we could have defined a codoubling function to be a dimension function f for which there exist constants $c > 1$ and $r_0 > 0$ such that

$$f(2r) \ge c\, f(r) \qquad \text{for all } \ 0 < r < r_0\,, \tag{4.21}$$

and then show that this condition is equivalent to (4.20) for some appropriate choice of the constants. However, in what follows we shall only use condition (4.20).

The theorem we prove in this section is the following.

Theorem 4.14. *Let f, g be dimension functions such that f is doubling with exponent $s_1 \le 1$ and codoubling with exponent $s_2 > 0$, and such that*

$$g(r) \le M\ f\left(r \log \frac{1}{r}\right) \qquad \textit{for all } \ 0 < r < r_0 \tag{4.22}$$

where $r_0 > 0$ and $M > 0$ are constants. Then there exists a set $A \subseteq \mathbb{R}^2$ with $0 < \mathcal{H}^f(A) < \infty$ but $\mathcal{H}^g(\mathrm{proj}_\theta A) = 0$ for all $0 \le \theta < \pi$.

Remark 23. The assumption that the function f is doubling with exponent $s \le 1$ restricts our attention to subsets $A \subseteq \mathbb{R}^2$ of Hausdorff dimension at most 1, which is expected given the nature of the problem (cf. Remark 17).

Remark 24. The growth condition (4.22) can be replaced by any condition of the form

$$g(r) \ll f\left(r \log(r^{-1}) \log_3(r^{-1}) \log_4(r^{-1}) \cdots \log_p(r^{-1})\right) \tag{4.23}$$

where $p \ge 3$ is a positive integer and $r > 0$ is sufficiently small – see Remark 27 below. Here we write $\log_2 t = \log\log t$, $\log_3 t = \log\log\log t$, etc .

Remark 25. Under the assumptions of Theorem 4.14, the integral

$$\int_0^1 \frac{\mathrm{d}f(r)}{g(r)} \;=\; \frac{f(1)}{g(1)} \;-\; \int_0^1 f(r)\ \mathrm{d}\!\left(\frac{1}{g(r)}\right)$$

diverges, hence the integral convergence condition (4.6) is not satisfied and in turn Theorem 4.4 is not violated. To see this, observe that since f is doubling with exponent $s_1 \le 1$, we have

$$f\left(r \log \frac{1}{r}\right) \ll f(r)\, \log \frac{1}{r} \qquad r \to 0^+.$$

Now since f is codoubling with exponent $s_2 > 0$, we have

$$f(r) \ll r^{s_2}$$

and thus

$$\log \frac{1}{r} \ll \log \frac{1}{f(r)} \qquad (r \to 0^+).$$

Together the above estimates yield

$$f\left(r \log \frac{1}{r}\right) \ll f(r) \log \frac{1}{f(r)} \cdot$$

It then follows that for any g satisfying condition (4.22), we have

$$\begin{aligned}
\int_0^1 \frac{\mathrm{d}f(r)}{g(r)} &\gg \int_0^1 \frac{\mathrm{d}f(r)}{f\left(r \log \frac{1}{r}\right)} \\
&\gg \int_0^1 \frac{\mathrm{d}f(r)}{f(r) \log \frac{1}{f(r)}} \\
&= \int_0^{f^{-1}(1)} \frac{\mathrm{d}x}{x \log \frac{1}{x}} \\
&= \infty .
\end{aligned}$$

4.6.1 Construction of an f-set

Given a dimension function f, a set $A \subseteq \mathbb{R}^n$ is called an *f-set* if $0 < \mathcal{H}^f(A) < \infty$. Here we present the construction of an f-set $A \subseteq \mathbb{R}^2$ for a given function f, which is similar to the one presented by Martin and Mattila in [42, Section 5.3] for dimension functions of the form $r \mapsto r^s$. In the next section, we will show that by choosing the parameters of the construction appropriately, the resulting f-set A will satisfy $\mathcal{H}^g(\mathrm{proj}_\theta A) = 0$ for all $0 \leq \theta < \pi$.

Throughout, $(r_k)_{k=0}^{\infty}$ is a decreasing sequence of positive real numbers tending to 0, $(N_k)_{k=1}^{\infty}$ is a sequence of positive integers ≥ 2, and $(\theta_k)_{k=1}^{\infty}$ is a sequence of angles $0 \leq \theta_k < \pi,\ k \geq 1$. The sequences $(r_k)_{k=0}^{\infty}$ and $(N_k)_{k=1}^{\infty}$ will be assumed to satisfy the inequalities

$$a \leq N_1 \cdots N_k f(r_k) \leq 2a \tag{4.24}$$

and

$$N_{k+1} r_{k+1} < r_k \tag{4.25}$$

for all $k \geq 0$, for some constant $a > 0$.

Let A_0 be the closed disc of radius r_0 centered at the origin. In the first step, inside A_0 we consider N_1 subdiscs of radius r_1, denoted $C_1, \dots, C_{N_1}$ and defined as follows: their centers are equally spaced, lying on the diameter of A_0 which forms angle θ_1 (measured counterclockwise) with the horizontal axis, and the boundaries of first and last subdisc are tangent to the boundary of A_0. Condition (4.25) guarantees that these subdiscs are disjoint. Let $d_1 = \theta_1$, and set

$$A_1 = \bigcup_{i=1}^{N_1} C_i.$$

Now inductively assume that for some $k \geq 1$ we have defined the discs $C_{i_1 \dots i_k},\ 1 \leq i_j \leq N_j,\ 1 \leq j \leq k$, each of radius r_k.

At the $(k+1)$st step, inside each disc $C_{i_1 \dots i_k}$ we consider N_{k+1} subdiscs $C_{i_1 \dots i_k 1}, \dots, C_{i_1 \dots i_k N_{k+1}}$, each of radius r_{k+1}, defined as follows: their centers are equally spaced along the diameter of $C_{i_1 \dots i_k}$ which forms angle θ_{k+1} with the line containing the centers of the discs of the kth step, and the boundaries of the first and last subdiscs are tangent to the boundary of $C_{i_1 \dots i_k}$. Again, condition (4.25) guarantees that these subdiscs are disjoint. Let $d_{k+1} \equiv \theta_1 + \dots + \theta_{k+1} \pmod{\pi}$, so that for any disc $C_{i_1 \dots i_k}$ of A_k, d_{k+1} is the angle between the diameter of $C_{i_1 \dots i_k}$ used to define the subdiscs of $C_{i_1 \dots i_k}$ and the horizontal axis. Set

$$A_{k+1} = \bigcup_{\substack{1 \leq i_j \leq N_j \\ (j=1,\dots,k+1)}} C_{i_1 \dots i_{k+1}}.$$

We complete the construction by setting

$$A = \bigcap_{k=1}^{\infty} A_k.$$

We show that under certain conditions on f and appropriate choices of the sequences $(r_k)_{k=0}^{\infty}$ and $(N_k)_{k=1}^{\infty}$, the set A is an f-set.

Proposition 4.15. *Let f be a dimension function which is doubling with exponent $s \leq 1$, and let $(r_k)_{k=0}^{\infty}$ be a sequence satisfying the inequalities*

$$f(r_{k+1}) < \frac{1}{4} f(r_k) \tag{4.26}$$

and

$$\frac{f(r_{k+1})}{r_{k+1}} > 3 \frac{f(r_k)}{r_k} \tag{4.27}$$

for all $k \geq 0$. Let $(\theta_k)_{k=1}^{\infty}$ be any sequence of real numbers. Then the parameter $a > 0$ and the sequence $(N_k)_{k=1}^{\infty}$ can be chosen so as to satisfy (4.24) and (4.25) for all $k \geq 0$. The resulting set $A \subseteq \mathbb{R}^2$ constructed as above is an f-set.

Proof. Let $a = f(r_0)$, so that (4.24) automatically holds when $k = 0$. Now inductively assume that for some $k \geq 0$ we have chosen $N_1, \ldots, N_k \geq 2$ such that (4.24) holds. Since

$$\begin{aligned}
\frac{2a}{N_1 \cdots N_k f(r_{k+1})} - \frac{a}{N_1 \cdots N_k f(r_{k+1})} \quad &= \quad a \frac{f(r_k)}{f(r_{k+1})} \frac{1}{N_1 \cdots N_k \; f(r_k)} \\
&\overset{(4.24)}{\geq} \quad \frac{1}{2} \frac{f(r_k)}{f(r_{k+1})} \\
&\overset{(4.26)}{>} \quad 2,
\end{aligned}$$

the interval

$$\left[\frac{2a}{N_1 \cdots N_k f(r_{k+1})}, \frac{a}{N_1 \cdots N_k f(r_{k+1})} \right]$$

contains a positive integer $N_{k+1} \geq 2$. Thus, the inequality

$$a \;\leq\; N_1 \cdots N_k N_{k+1} \, f(r_{k+1}) \;\leq\; 2a \tag{4.28}$$

is satisfied. This completes the inductive step, thus demonstrating that the sequence $(N_k)_{k=1}^{\infty}$ can be chosen so that (4.24) holds for all $k \geq 0$.

To demonstrate (4.25), we note that

$$N_{k+1} \overset{(4.28)}{\leq} \frac{2a}{N_1 \cdots N_k f(r_{k+1})} \overset{(4.24)}{\leq} 2 \cdot \frac{f(r_k)}{f(r_{k+1})} \overset{(4.27)}{<} \frac{2}{3} \cdot \frac{r_k}{r_{k+1}} \tag{4.29}$$

and in particular $N_{k+1} r_{k+1} < r_k$.

For each $k \in \mathbb{N}$, the set A_k is a cover of A consisting of discs of radius r_k. The number of balls in this cover is $N_1 \cdots N_k$, hence for $k \in \mathbb{N}$ we have

$$\mathcal{H}^f_{r_k}(A) \;\leq\; N_1 \cdots N_k f(r_k) \;\leq\; 2a$$

and thus

$$\mathcal{H}^f(A) \;=\; \sup_{k>0} \mathcal{H}^f_{r_k}(A) \;\leq\; 2a \;<\; \infty.$$

Now consider the probability measure μ supported on A which is defined by assigning each of the discs $C_{i_1\dots i_k}$ of A_k the same measure, i.e. by setting

$$\mu(C_{i_1\dots i_k}) = \frac{1}{N_1\cdots N_k}\cdot$$

We claim that for all $x \in A$ and $r > 0$ small enough,

$$\mu\left(B(x,r)\right) \leq Cf(r) \tag{4.30}$$

for some constant $C > 0$. By the Mass Distribution Principle, this will imply that $\mathcal{H}^f(A) > 0$ and thus complete the proof.

Whenever r, t are sufficiently small and $r < t$, since f is doubling with exponent $s \leq 1$, for some constant $\kappa > 0$ we have

$$\begin{aligned} \frac{f(r)}{r} &\geq \kappa\,\frac{1}{r}\left(\frac{r}{t}\right)^s f(t) \\ &= \kappa\left(\frac{t}{r}\right)^{1-s}\frac{f(t)}{t} \\ &\geq \kappa\,\frac{f(t)}{t}, \end{aligned}$$

which gives

$$\frac{f(t)}{t} \leq \kappa^{-1}\frac{f(r)}{r}\cdot \tag{4.31}$$

Now fix $x \in A$ and $r > 0$, and let $k \in \mathbb{N}$ be maximal such that $B(x,r) \cap A$ is contained in only one disc of A_k. Consider the following cases:

Case 1: $r < r_k$. Let s_{k+1} be the common distance between any two consecutive subdiscs of A_{k+1}. Then by subdividing the appropriate diameter of $C_{i_1\dots i_k}$ into intervals consisting of its intersections with discs $C_{i_1\dots i_k j}$ as well as the gaps between them, we find that

$$2N_{k+1}r_{k+1} + (N_{k+1}-1)s_{k+1} = 2r_k. \tag{4.32}$$

Now in any sequence of n consecutive subdiscs of A_{k+1}, the distance between the first and last subdiscs in this sequence is

$$(n-1)s_{k+1} + (n-2)2r_{k+1} > (n-2)(s_{k+1} + 2r_{k+1}).$$

Since the diameter of $B(x,r)$ is $2r$, if n is the number of subdiscs of A_{k+1} that intersect $B(x,r)$, then the distance given above must be less than $2r$. It follows that

$$n \leq 2 + \frac{2r}{2r_{k+1} + s_{k+1}}\cdot \tag{4.33}$$

On the other hand, we have

$$\begin{aligned} N_{k+1}(s_{k+1}-r_{k+1}) &> (N_{k+1}-1)s_{k+1}-N_{k+1}r_{k+1} \\ &\overset{(4.32)}{=} 2r_k - 3N_{k+1}r_{k+1} \\ &\overset{(4.29)}{>} 0, \end{aligned}$$

which implies that

$$s_{k+1} > r_{k+1}\,. \tag{4.34}$$

On the other hand, by the maximality of k, $B(x,r)$ intersects at least 2 discs of A_{k+1}, including the disc containing x, and thus it follows that $r > s_{k+1}$. This together with (4.34) implies that $r > r_{k+1}$ and so

$$\frac{2r}{2r_{k+1}+s_{k+1}} \geq \frac{2}{3}\,.$$

Hence,

$$\begin{aligned} n &\overset{(4.33)}{\leq} 4\left(\frac{2r}{2r_{k+1}+s_{k+1}}\right) \\ &\overset{(4.32)}{=} 8r\left(2r_{k+1}+2\frac{r_k-N_{k+1}r_{k+1}}{N_{k+1}-1}\right)^{-1} \\ &\leq 4r\left(r_{k+1}+\frac{r_k-N_{k+1}r_{k+1}}{N_{k+1}}\right)^{-1} \\ &= 4\frac{N_{k+1}}{r_k}r \\ &\overset{(4.29)}{\leq} 8\frac{f(r_k)}{f(r_{k+1})}\cdot\frac{r}{r_k} \\ &\overset{(4.31)}{\leq} \frac{8}{\kappa}\,\frac{f(r)}{f(r_{k+1})}\,. \end{aligned}$$

Since each subdisc of A_{k+1} has measure $\dfrac{1}{N_1\cdots N_{k+1}}$, it follows that

$$\mu\left(B(x,r)\right) \leq \frac{8}{\kappa}\,\frac{f(r)}{f(r_{k+1})}\cdot\frac{1}{N_1\cdots N_{k+1}} \overset{(4.28)}{\leq} \frac{8}{a\kappa}\,f(r).$$

Case 2: $r \geq r_k$. Let $C_{i_1\ldots i_k}$ be the unique disc of A_k intersecting $B(x,r)$, which exists by the definition of k. Then

$$\begin{aligned}\mu(B(x,r)) &\leq \mu(C_{i_1\ldots i_k}) \\ &= \frac{1}{N_1\cdots N_k} \\ &\overset{(4.24)}{\leq} \frac{1}{a}f(r_k) \\ &\leq \frac{1}{a}f(r),\end{aligned}$$

where in the last inequality, we have used the fact that f is increasing.

Thus in either case, (4.30) holds with $C = \max\left\{\frac{8}{a\kappa}, \frac{1}{a}\right\} > 0$. □

Remark 26. Note that Proposition 4.15 applies to any possible sequence of angles $(\theta_k)_{k=1}^\infty$, indicating that varying the sequence of angles may cause the quantity $\mathcal{H}^f(A)$ to change slightly but will not affect the fact that it is finite and positive. The role of the sequence $(\theta_k)_{k=1}^\infty$ will become apparent in the next section.

4.6.2 Proof of Theorem 4.14

We show that if g satisfies the growth condition (4.22) relative to f, the sequences $(r_k)_{k=0}^\infty$, $(N_k)_{k=1}^\infty$ and $(\theta_k)_{k=1}^\infty$ in the aforementioned construction can be suitably selected so that the corresponding f-set $A \subseteq \mathbb{R}^2$ satisfies $\mathcal{H}^g(\mathrm{proj}_\theta A) = 0$ for all $0 \leq \theta < \pi$.

First, we claim that the sequence $(r'_k)_{k=k_0}^\infty$ defined by the formula

$$r'_k = (k \log k \log\log k)^{-k} \tag{4.35}$$

satisfies (4.26) and (4.27) for all sufficiently large k. To prove this, we first observe that

$$\frac{r'_{k+1}}{r'_k} \asymp \frac{1}{k\log k\log\log k} \to 0 \qquad \text{as} \qquad k\to\infty. \tag{4.36}$$

On the other hand, by the doubling and codoubling hypotheses imposed on f, there exist constants $\kappa_1, \kappa_2 > 0$ such that

$$\kappa_1\lambda^{s_1}f(r) \;\leq\; f(\lambda r) \;\leq\; \kappa_2\lambda^{s_2}f(r)$$

for all $0 < \lambda < 1$ and $r > 0$ sufficiently small. Since $r'_k > r'_{k+1}$ for all k sufficiently large, we have that

$$f(r'_{k+1}) \leq \kappa_2 \left(\frac{r'_{k+1}}{r'_k}\right)^{s_2} f(r_k)$$

and

$$\begin{aligned} \frac{f(r'_{k+1})}{r'_{k+1}} &\geq \kappa_1 \frac{1}{r'_{k+1}} \left(\frac{r'_{k+1}}{r'_k}\right)^{s_1} f(r'_k) \\ &= \kappa_1 \left(\frac{r'_{k+1}}{r'_k}\right)^{s_1-1} \frac{f(r'_k)}{r'_k}. \end{aligned}$$

Thus by (4.36), the inequalities $(4.26)_{r_k=r'_k}$ and $(4.27)_{r_k=r'_k}$ are satisfied for all k large enough. Let $k_1 \geq k_0$ be chosen so that $(4.26)_{r_k=r'_k}$ and $(4.27)_{r_k=r'_k}$ are satisfied for all $k \geq k_1$.

Now consider the sequence $(r_k)_{k=0}^{\infty}$ defined by the formula

$$r_k = r'_{k+k_1},$$

and note that (4.26) and (4.27) are satisfied for all $k \geq 0$. Thus by Proposition 4.15, we can choose a sequence $(N_k)_{k=1}^{\infty}$ such that (4.24) and (4.25) hold for all $k \geq 0$. Also note that by (4.36) we have that

$$\frac{r_{k+1}}{r_k} \asymp \frac{1}{k \log k \log\log k}. \tag{4.37}$$

Let the sequence of angles be defined by

$$\theta_{k+1} = \frac{r_{k+1}}{r_k}, \qquad k \geq 0.$$

Then

$$\sum_{k=1}^{\infty} \theta_k = \infty.$$

Take an arbitrary $0 \leq \theta < \pi$. Let d_θ denote the direction perpendicular to L_θ, i.e. the direction of projection, $d_\theta \equiv \theta + \pi/2 \pmod{\pi}$. Since the series $\sum_{k=1}^{\infty} \theta_k$ diverges, there are infinitely many $k \in \mathbb{N}$ such that d_θ lies between d_k and d_{k+1}. For each of these values of k, the angle between d_θ and d_{k+1} is at most θ_{k+1}, and thus for each disc $C_{i_1 \ldots i_k}$ of A_k, the distances from the centers of all subdisc $C_{i_1 \ldots i_k j}$ of $C_{i_1 \ldots i_k}$ from the diameter of $C_{i_1 \ldots i_k}$ in the direction d_θ are at most

$$r_k \sin\theta_{k+1} \;\leq\; r_k \theta_{k+1} \;=\; r_{k+1}.$$

This is because all of these centers lie on the diameter of $C_{i_1 \ldots i_k}$ in the direction d_{k+1}.

This means that within each disc of A_k, when we project the union of the subdiscs of A_{k+1} onto L_θ we get an interval of length at most $4r_{k+1}$, which we can think of as the union of at most 4 intervals of length at most r_{k+1}. The number of such intervals is equal to the number of discs of A_k, that is,

$$N_1 \cdots N_k \leq \frac{2a}{f(r_k)}\,.$$

We have shown that for infinitely many values of k there is a cover of $\mathrm{proj}_\theta A$ which consists of $4N_1 \cdots N_k$ intervals of length at most r_{k+1}, hence for such k we obtain that

$$\begin{aligned}
\mathcal{H}^g_{r_{k+1}}(\mathrm{proj}_\theta A) &\leq 8a\frac{g(r_{k+1})}{f(r_k)} \\
&\leq 8aM\frac{1}{f(r_k)} f\left(r_{k+1}\log(r_{k+1}^{-1})\log\log(r_{k+1}^{-1})\right). \qquad (4.38)
\end{aligned}$$

Now (4.35) implies that

$$\begin{aligned}
\log(r_k^{-1}) &\asymp \log(r_k'^{-1}) \\
&\asymp k\log(k\log k\log\log k) \\
&\asymp k\log k,
\end{aligned}$$

and

$$\begin{aligned}
\log\log(r_k^{-1}) &\asymp \log\log(r_k'^{-1}) \\
&\asymp \log(k\log k) \\
&\asymp \log k
\end{aligned}$$

as $k \to \infty$. Combining this estimate with (4.37), (4.38), and the fact that f is codoubling shows that

$$\mathcal{H}^g_{r_{k+1}}(\mathrm{proj}_\theta A) \leq \frac{M_1}{(\log k)^{s_2}}\,,$$

where $M_1 > 0$ is some absolute constant. This implies that

$$\mathcal{H}^g(\mathrm{proj}_\theta A) = \lim_{k\to\infty} \mathcal{H}^g_{r_{k+1}}(\mathrm{proj}_\theta A) = 0$$

and thereby completes the proof of Theorem 4.14.

Remark 27. As mentioned in Remark 24, the growth condition (4.22) in Theorem 4.14

can be replaced by any condition of the form (4.23). The corresponding set in that case is constructed using the sequence $(r'_k)_{k \geq k_0}$ defined by the formula

$$r'_k = \left(k \log k \log_2 k \cdots \log_p k\right)^{-k}.$$

The proof is nearly identical.

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