Differentiating L-functions

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May 2017

Abstract. The Riemann zeta function is well known due to its link to prime numbers. The Riemann Xi function is related to the zeta function, and is commonly used due to its nicer analytic properties (such as its lack of a pole and its Fourier transform).

The work within this thesis was inspired by Haseo Ki's result, which showed that, under repeated differentiation and suitable scaling, the Riemann Xi function tends to the cosine function.

We prove a similar result for the Selberg Class of L-functions, albeit with different scalings.

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Introduction

Primes are the building block of integers, and from that all numbers. Therefore, it naturally makes sense to consider how many primes there are in an interval. The function

$$\pi(x) = \sum_{p \leq x} 1$$

where p are the prime numbers, counts the number of primes up to height x. Riemann [38] used what is now known as 'his' zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

and linked the density of primes to the zeros of the zeta function.

Since then, work has been done on many aspects of the zeta function, as well as the more general L-functions. Some of the results about the Riemann zeta function and its zeros are explained in chapter 1, as well as the analogous results about L-functions.

At the end of chapter 1 we give Ki's [27] proof of the result that

$$\lim_{n \to \infty} A_n \Xi^{(2n)}(C_n z) = \cos(z).$$

This result inspired my own research, which is extending this result to the Selberg Class of L-functions. This is explained and worked through in chapter 2, as is a discussion of the sequences and error terms.

Chapter 3 is focused on the computer work required to get plots of the derivatives of the Riemann Xi function. Since there are two error terms, there isn't a one-size-fits-all method of generating suitable plots, and in total three methods are used.

Acknowledgments

Firstly, a great many thanks are owed to my supervisor Chris Hughes, who spent so much time answering all my questions (both meaningful and utterly inane)! The bubble of understanding around you was invaluable (no matter how much you deny it).

Secondly, to the rest of my TAP for keeping the pair of us on some sort of track!

Thanks are also owed to my friends and family. There are too many to list here, but especially Evie Moon, Tom, Nikita and Sami.

And finally, but perhaps the most important, to maths itself. All my love and devotion have been poured into this epic for you; I hope this isn't the end.

Declaration

I declare that this thesis is a presentation of original work and I am the sole author. This work has not previously been presented for an award at this, or any other, University. All sources are acknowledged as References.

Notation and Key Results

$$
\begin{aligned}
s &= \sigma + it \qquad \sigma, t \in \mathbb{R} \\
\zeta(s) &= \sum_{n} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}, \qquad \sigma > 1 \\
&= 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s) \\
&= \chi(s)\zeta(1-s) \\
\chi(s) &= 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \\
&= \pi^{s-1/2} \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \\
\vartheta(t) &= \sqrt{\chi\left(\frac{1}{2} - it\right)} \\
&= -\frac{t}{2}\log(\pi) + \mathrm{Im}(\log\Gamma\left(\frac{1}{4} + \frac{it}{2}\right)) \\
Z(t) &= e^{i\vartheta(t)}\zeta\left(\frac{1}{2} + it\right) \qquad \in \mathbb{R} \\
\zeta^{(k)}(s) &= (-1)^k \sum_{n=2}^{\infty} \frac{\log^k(n)}{n^s} \qquad \sigma > 1 \\
\zeta^{(k)}(1-s) &= (-1)^k \sum_{m=0}^{k} \binom{k}{m} \left(e^{sz} z^{k-m} + e^{s\overline{z}} (\overline{z})^{k-m}\right) (\Gamma(s)\zeta(s))^{(m)} \qquad z = -\log(2\pi) - \frac{i\pi}{2} \\
N(T) &= \frac{T}{2\pi} \log\left(\frac{T}{2\pi e}\right) + S(T) + \frac{7}{8} + \mathcal{O}\left(\frac{1}{T}\right) \\
S(T) &= \frac{1}{\pi} \arg\left(\zeta\left(\frac{1}{2} + iT\right)\right) \\
S_1(T) &= \int_0^T S(u)\mathrm{d}u \\
\xi(s) &= \frac{1}{2} s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) \\
&= \xi(1-s) \\
\Xi(t) &= \xi\left(\frac{1}{2} + it\right) \\
&= \Xi(-t)
\end{aligned}
$$

As well as the Riemann zeta function, the Selberg Class of L-functions is also used. These formulas are introduced in chapter 2.

$$\begin{aligned}
F(s) &= \sum_{n=1}^{\infty} \frac{a_n}{n^s} \qquad a_1 = 1, a_n = \mathcal{O}(1) \\
&= \prod_{p \text{ prime}} P(p, s) \\
\Phi(s) &= \varepsilon Q^s \prod_{j=1}^{k} \Gamma(\lambda_j s + \mu_j) F(s) \\
&= \overline{\Phi(1 - \overline{s})} \\
\xi_F(s) &= s^m (1-s)^m \lambda_1 \cdots \lambda_m \Phi(s) \\
&= \varepsilon Q^s \prod_{l=1}^{m} \lambda_l s(1-s)\Gamma(\lambda_l s) \prod_{j=m+1}^{k} \Gamma(\lambda_j s + \mu_j) F(s) \\
\Xi_F(z) &= \xi_F\left(\frac{1}{2} + iz\right) \\
\mathrm{Li}(x) &= \int_2^x \frac{1}{\log(t)} \mathrm{d}t
\end{aligned}$$

Chapter 1

1. Prime Numbers

1.1. **Riemann's Paper.** In 1859, Riemann wrote a paper [38] about the distribution and density of prime numbers. His aim was to create a formula for

$$\pi(x) = \sum_{p \leq x} 1$$

i.e. the number of primes less than x, using the relationship Euler discovered 90 years earlier [13]

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}, \quad \mathrm{Re}(s) > 1,$$

where $s = \sigma + it$, and worked with it to get

$$\begin{aligned} \log(\zeta(s)) &= -\sum_{p \text{ prime}} \log(1-p^{-s}) \quad \mathrm{Re}(s) > 1 \\ &= \sum_{p \text{ prime}} \sum_{n=1}^{\infty} \frac{1}{n} p^{-ns}. \end{aligned}$$

Since

$$p^{-ns} = s \int_{p^n}^{\infty} x^{-s-1} \mathrm{d}x$$

we have that

$$\frac{\log(\zeta(s))}{s} = \sum_{p \text{ prime}} \sum_{n=1}^{\infty} \frac{1}{n} \int_{p^n}^{\infty} x^{-s-1} \mathrm{d}x.$$

This is absolutely convergent for any s with $\mathrm{Re}(s) > 1$, so we can change the order to summation to give

$$\begin{aligned}\frac{\log(\zeta(s))}{s} &= \sum_{n=1}^{\infty} \frac{1}{n} \sum_{p \text{ prime}} \int_{p^n}^{\infty} x^{-s-1} \mathrm{d}x \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \int_{1}^{\infty} (\sum_{\substack{p \\ p^n < x}} 1) x^{-s-1} \mathrm{d}x \\ &= \int_{1}^{\infty} (\sum_{p^n < x} \frac{1}{n}) x^{-s-1} \mathrm{d}x \\ &= \int_{1}^{\infty} J(x) x^{-s-1} \mathrm{d}x \\ &= \int_{0}^{\infty} J(e^u) e^{-su} \mathrm{d}u,\end{aligned}$$

where

$$J(x) = \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \frac{1}{4}\pi(x^{1/4}) + \cdots$$

counts primes and prime powers.

The integral is a Laplace transform in t (that is, the imaginary part of s) so the inverse formula gives

$$\begin{aligned}J(e^u) &= \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{\log(\zeta(s))}{s} e^{us} \mathrm{d}s \qquad \sigma > 1 \\ J(x) &= \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{\log(\zeta(s))}{s} x^{s} \mathrm{d}s.\end{aligned}$$

Rather than trying to use the product or sum representation of the zeta function, which can be problematic as it only converges for $\sigma > 1$, it is more useful to use

$$\begin{aligned}\xi(s) &= \frac{s}{2}(s-1)\Gamma\left(\frac{s}{2}\right)\pi^{-s/2}\zeta(s) \\ &= (s-1)\Gamma\left(\frac{s}{2}+1\right)\pi^{-s/2}\zeta(s)\end{aligned}$$

and the Hadamard product [45]

$$\xi(s) = \frac{1}{2} e^{Bs} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho},$$

as together these define the zeta function in terms of its zeros, poles and other behaviour. The variable in the product over *rho* are the zeros of the zeta function. In this instance, we have that [11]

$$B = -\sum_{\rho} \frac{1}{\rho}$$

so that we can simplify the product formula to

$$\xi(s) = \frac{1}{2} \prod_{\rho} \left(1 - \frac{s}{\rho}\right).$$

Therefore, we have that

$$\begin{aligned}
\log(\zeta(s)) = \frac{s}{2}\log(\pi) - \log(s-1) - \log\left(\Gamma\left(\frac{s}{2}+1\right)\right) \\
+ \sum_{\operatorname{Im}(\rho)>0} \left(\log\left(1-\frac{s}{\rho}\right) + \log\left(1-\frac{s}{1-\rho}\right)\right) \\
+ \log\left(\frac{1}{2}\right).
\end{aligned}$$

Directly making this substitution leads to divergent integrals, and so first the integral must be integrated by parts to give

$$J(x) = -\frac{1}{2\pi i}\frac{1}{\log(x)} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{\log(\zeta(s))}{s}\right] x^s \mathrm{d}s.$$

This then becomes

$$\begin{aligned}
J(x) = &-\frac{1}{4\pi i}\frac{1}{\log(x)} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\mathrm{d}}{\mathrm{d}s}\left(\log(\pi)\right) x^s \mathrm{d}s \\
&+ \frac{1}{2\pi i}\frac{1}{\log(x)} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\log(s-1)}{s}\right) x^s \mathrm{d}s \\
&+ \frac{1}{2\pi i}\frac{1}{\log(x)} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\log(\Gamma(s/2+1))}{s}\right) x^s \mathrm{d}s \\
&- \sum_{\operatorname{Im}(\rho)>0} \frac{1}{2\pi i}\frac{1}{\log(x)} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\log(1-s/\rho)+\log(1-s/(1-\rho))}{s}\right) x^s \mathrm{d}s \\
&+ \frac{1}{2\pi i}\frac{\log(2)}{\log(x)} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s}\right) x^s \mathrm{d}s.
\end{aligned}$$

Each of these integrals can be calculated, and together they give

$$J(x) = \operatorname{Li}(x) - \sum_{\operatorname{Im}(\rho)>0} \left[\operatorname{Li}(x^{\rho}) + \operatorname{Li}(x^{1-\rho})\right] + \int_x^{\infty} \frac{1}{(t^2-1)t\log(t)}\mathrm{d}t + \log\left(\frac{1}{2}\right).$$

In order to return this to what Riemann was looking for, a formula for the prime counting function, Möbius inversion formula must be used, which leads to

$$\pi(x) = \sum_n \mu(n) \frac{1}{n} J(x^{1/n}).$$

Therefore, if we use the approximation

$$J(x) \approx \mathrm{Li}(x)$$

a better approximation for $\pi(x)$ is

$$\pi(x) \approx \mathrm{Li}(x) - \frac{1}{2}\,\mathrm{Li}(x^{1/2}).$$

1.2. **The Prime Number Theorem.** The statement that

$$\pi(x) \sim \mathrm{Li}(x)$$

is known as the Prime Number Theorem. Currently the best known estimate for this number is [16]

$$\pi(x) = \mathrm{Li}(x) + \mathcal{O}\left(x e^{-\frac{A \log(x)^{3/5}}{\log\log(x)^{1/5}}} \right).$$

Under the assumption of the Riemann Hypothesis, this error term can be reduced and simplified to [12]

$$\pi(x) = \mathrm{Li}(x) + \mathcal{O}\left(\sqrt{x}\log(x)\right),$$

which is the best possible error term.

For small values of x, we have that $\pi(x) < \mathrm{Li}(x)$, as we can see from fig 1 of both $\pi(x)$ and $\mathrm{Li}(x)$.

However, Littlewood [31] showed that the function $\pi(x) - \mathrm{Li}(x)$ changes sign infinitely often, although he did not give or propose an upper bound for the first change.

Riemann's formula, that

$$\pi(x) \approx \mathrm{Li}(x) - \frac{1}{2}\,\mathrm{Li}(x^{1/2}),$$

shows that the approximation $\mathrm{Li}(x)$ is usually much larger than the prime counting function. The error term of the above sum is dependent on the location of the zeros of the Riemann zeta function, and it is this sum which causes the change of sign. It has been shown that [39]

$$\pi(x) > \mathrm{Li}(x)$$

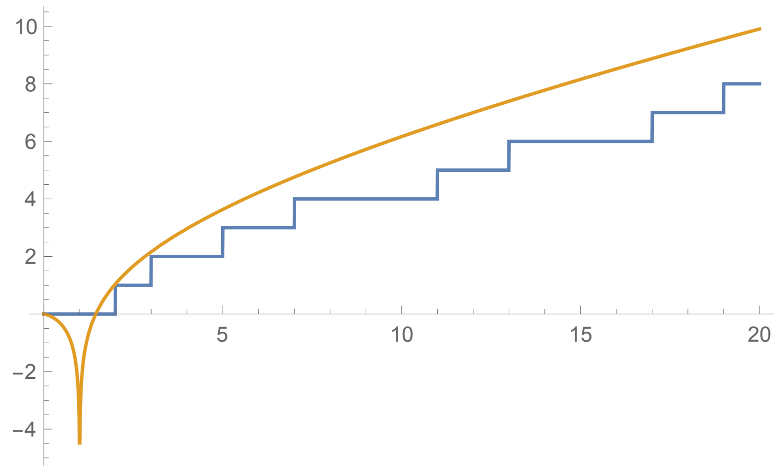


FIGURE 1. A plot of $\pi(x)$ and $\mathrm{Li}(x)$

happens approximately 0.0000026% of the time.

Skewes [41] was able to find an upper bound of

$$x < 10^{10^{10^{1000}}},$$

for the first sign change of $\pi(x) - \mathrm{Li}(x)$ by showing that this is an upper bound assuming a particular result about the real part of the first set number of zeros, and also assuming the negation of the same result. However, he only had a rough estimate for the location of the first few zeros, and therefore his work can easily be improved upon.

Using Riemann's result

$$J(x) = \mathrm{Li}(x) - \sum_{\mathrm{Im}(\rho)>0} \left[\mathrm{Li}(x^{\rho}) + \mathrm{Li}(x^{1-\rho})\right] + \int_x^{\infty} \frac{1}{(t^2-1)t\log(t)} \mathrm{d}t + \log\left(\frac{1}{2}\right),$$

and the location of the first two million zeros to a much higher precision than was available to Skewes, the best known upper bound for the first change in sign of $\pi(x) - \mathrm{Li}(x)$so far [42] is

$$x < 1.397166161527 \times 10^{316}.$$

This result is numerically calculated using the location of the first two million non-trivial zeros to a high precision.

Although the formula for $J(x)$ is too inefficient to be used to calculate $J(x)$ to a suitable accuracy, a variation has been used [7] to calculate $\pi(10^{25})$, using the the zeros with imaginary part less than 10^{11}. This result required a runtime of 40,000 CPU hours.

Certainly, there are better methods of calculating prime numbers than calculating $\pi(x)$ accurately enough to spot a jump where prime numbers occur.

Prime numbers are used in many areas of maths, sciences and computing, and so understanding their behaviour is very useful. Understanding the zeta function and the behaviour of its zeros is one way to do this.

2. The Riemann Zeta Function

2.1. **Properties of the Zeta Function.** Euler proved his result by sieving the primes and prime factors from the summation formula. Beginning with (assuming $\mathbb{R}(s) > 1$)

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots$$

so

$$\frac{1}{2^s}\zeta(s) = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \cdots$$

which leads to

$$\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \cdots$$

where we can see that all the even numbers on the right have been removed. Repeating this for 3 gives

$$\frac{1}{3^s}\left(1 - \frac{1}{2^s}\right)\zeta(s) = \frac{1}{3^s} + \frac{1}{9^s} + \frac{1}{15^s} + \frac{1}{21^s} + \cdots$$

so that

$$\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \cdots .$$

Repeating this process for every prime number leads to

$$\prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)\zeta(s) = 1$$

which can be rearranged to give

$$\zeta(s) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}$$

so we have that

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}, \quad \mathrm{Re}(s) > 1.$$

However, the series diverges $\sigma \leq 1$ and the product for $\sigma < 1$ and at $s = 1$ [45], because of the pole of the function at $s = 1$. Therefore, an analytic continuation must be found in order to continue studying the function. The functional equation

$$\Gamma\left(\frac{s}{2}\right)\zeta(s)\pi^{-s/2} = \Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)\pi^{(s-1)/2},$$

or, more elegantly,

$$\zeta(s) = \Gamma(1-s)2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\zeta(1-s),$$

can be proved in a number of ways. One way is by showing that the left hand side of the first equation,

$$\Gamma\left(\frac{s}{2}\right)\zeta(s)\pi^{-s/2},$$

remains unchanged if s is replaced with $1-s$. Beginning with (for $\mathrm{Re}(s) > 0$),

$$\Gamma\left(\frac{s}{2}\right) = \int_0^{\infty} x^{s/2-1}e^{-x}\mathrm{d}x,$$

so

$$\begin{aligned}\frac{\Gamma\left(\frac{s}{2}\right)}{n^s\pi^{s/2}} &= \int_0^{\infty}\left(\frac{x}{\pi n^2}\right)^{s/2-1} e^{-x}\frac{\mathrm{d}x}{\pi n^2} \\ &= \int_0^{\infty} u^{s/2-1}e^{-n^2\pi u}\mathrm{d}u,\end{aligned}$$

using the substitution

$$\frac{x}{\pi n^2} = u.$$

Summing over the natural numbers to introduce the whole zeta function, (now under the assumption that $\mathrm{Re}(s) > 1$), the function becomes

$$\begin{aligned}\Gamma\left(\frac{s}{2}\right)\zeta(s)\pi^{-s/2} &= \sum_{n=1}^{\infty}\int_0^{\infty} x^{s/2-1}e^{-n^2\pi x}\mathrm{d}x \\ &= \int_0^{\infty} x^{s/2-1}\sum_{n=1}^{\infty} e^{-n^2\pi x}\mathrm{d}x \\ &= \left[\int_0^1 + \int_1^{\infty}\right] x^{s/2-1}\sum_{n=1}^{\infty} e^{-n^2\pi x}\mathrm{d}x.\end{aligned}$$

Using the Poisson summation result that

$$\sum_{n=-\infty}^{\infty} e^{-n^2\pi x} = \frac{1}{\sqrt{x}}\sum_{n=-\infty}^{\infty} e^{-n^2\pi/x},$$

which, when rearranged gives

$$\sum_{n=1}^{\infty} e^{-n^2\pi x} = \frac{1}{\sqrt{x}}\sum_{n=1}^{\infty} e^{-n^2\pi/x} + \frac{1}{2\sqrt{x}} - \frac{1}{2},$$

so that, on inserting this result in the first integral but not the second, the function becomes

$$\begin{aligned}\Gamma\left(\frac{s}{2}\right)\zeta(s)\pi^{-s/2} = \int_0^1 x^{s/2-1}\left(\frac{1}{\sqrt{x}}\sum_{n=1}^{\infty} e^{-n^2\pi/x} + \frac{1}{2\sqrt{x}} - \frac{1}{2}\right)\mathrm{d}x \\ + \int_1^{\infty} x^{s/2-1}\sum_{n=1}^{\infty} e^{-n^2\pi x}\mathrm{d}x.\end{aligned}$$

Working on the first integral, we have that

$$\begin{aligned}&\int_0^1 x^{s/2-1}\left(\frac{1}{\sqrt{x}}\sum_{n=1}^{\infty} e^{-n^2\pi/x} + \frac{1}{2\sqrt{x}} - \frac{1}{2}\right)\mathrm{d}x \\ &= \int_1^{\infty} u^{-1-s/2}\left(\sqrt{u}\sum_{n=1}^{\infty} e^{-n^2\pi u} + \frac{\sqrt{u}}{2} - \frac{1}{2}\right)\mathrm{d}u \\ &= \int_1^{\infty} u^{-1/2-s/2}\sum_{n=1}^{\infty} e^{-n^2\pi u}\mathrm{d}u + \frac{1}{s(s-1)},\end{aligned}$$

so that

$$\Gamma\left(\frac{s}{2}\right)\zeta(s)\pi^{-s/2} = \frac{1}{s(s-1)} + \int_1^{\infty}\left(u^{-1/2-s/2} + u^{s/2-1}\right)\sum_{n=1}^{\infty} e^{-n^2\pi u}\mathrm{d}u.$$

The integral on the right hand side of this is convergent and remains the same if we replace s with $1-s$ for any value s, and therefore, so too must the left hand side. Therefore, we have that

$$\Gamma\left(\frac{s}{2}\right)\zeta(s)\pi^{-s/2}=\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)\pi^{s/2-1/2},$$

which, after rearranging and using the result that

$$\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}=2^{1-s}\pi^{-1/2}\cos\left(\frac{\pi s}{2}\right)\Gamma(s),$$

gives

$$\zeta(s)=2^{s}\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s),$$

or

$$\zeta(s)=\chi(s)\zeta(1-s)$$

the functional equation for the Riemann zeta function.

We can see that the pole of the zeta function at $s=1$ is canceled by a zero of the cosine function, so it is not repeated at $s=0$. The cosine function has zeros at the odd natural numbers, leading to zeros of the zeta function at the negative even integers. These are called the trivial zeros because they are so easily found and are unimportant to results about the density of zeros. The only other interesting behaviour is the critical strip, in the region $0<\sigma<1$. Riemann mentioned in passing that he expects all the zeros in the critical strip to be on the critical line $\sigma=1/2$, and this statement has become the famous Riemann Hypothesis, and is still unproven (although generally believed to be true).

The result

$$\zeta(s)=\overline{\zeta(\overline{s})}$$

shows that the zeros obey reflective properties, so that, given a zero at point ρ_n, the point $\overline{\rho_n}$ is also a zero. If ρ_n is not on the critical line, then the functional equation shows that the distinct points $1-\rho_n$ and $1-\overline{\rho_n}$ are both zero as well.

As previously mentioned, Riemann used the function

$$\xi(s)=\frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

because it is an entire function containing just the non-trivial zeros. It also satisfies an elegant functional equation, that of

$$\xi(s) = \xi(1-s).$$

The location of the non-trivial zeros remains the same in the xi function, so under the assumption of RH, all the zeros of the xi function are on the critical line. So far, the best result towards this claim is that more than 41% [6] of the zeros lie on it. All recent improvements to this result have been fairly small, and folklore suggests that new maths will be required to improve this result significantly.

2.2. **Density of Zeros.**

It is easier to use

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

to calculate the density of zeros in the critical strip because of the functional equation

$$\xi\left(\frac{1}{2}+x+it\right) = \overline{\xi\left(\frac{1}{2}-x+it\right)}, \qquad x,t \in \mathbb{R}$$

and because the pole and trivial zeros have been removed.The formula

$$\frac{1}{2\pi i}\oint_R \frac{\xi'(s)}{\xi(s)}\mathrm{d}s$$

counts the number of zeros inside the region R,(since the xi function doesn't have any poles). This is the same as $1/2\pi$ times the change in argument of the xi function around the contour R. Therefore, by constructing R to be the box with corners at $2, 2+iT, -1+iT, -1$, (see fig 2) we can define $N(T)$ to be the number of zeros of the zeta function inside the critical strip from the origin to height T (assuming that there are no zeros with imaginary part equal to T).

Since the xi function is real and positive along the real axis, the argument can be assumed to be 0 along it. The functional equation shows that the change in argument along the line from 2 to $1/2+iT$ is the same as the change in argument from $1/2+iT$ to -1, and therefore

$$N(T) = \frac{1}{\pi} * \{\text{change in argument of } \xi(s) \text{ along the lines from 2 to } 2+iT \text{ to } 1/2+iT\}.$$

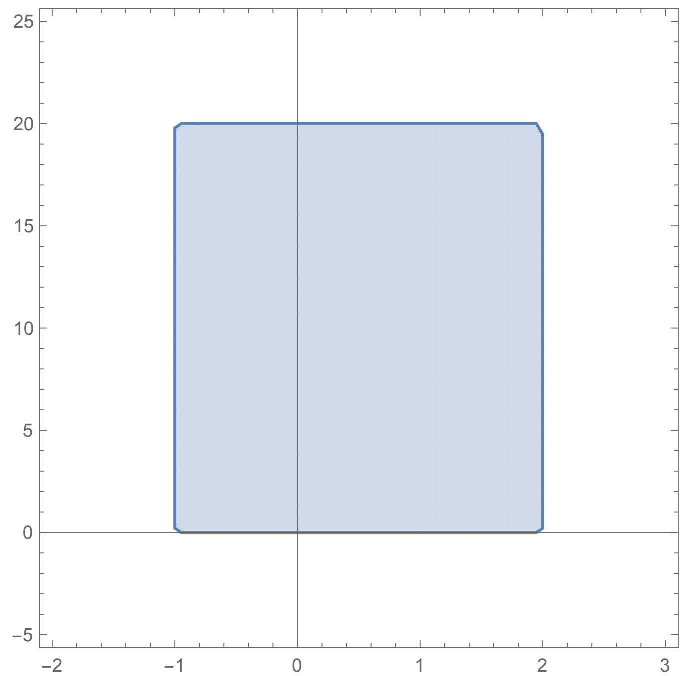


FIGURE 2. The box used to calculate $N(20)$

Calculating the change in argument of each of the terms can be done separately, and then added together. For $s, (s-1)$ and $\pi^{-s/2}$, the change in the argument is $\frac{\pi}{2} + \mathcal{O}(\frac{1}{T})$, $\frac{\pi}{2} + \mathcal{O}(\frac{1}{T})$ and $-\frac{T}{2}\log(\pi)$ respectively. In order to calculate the change in argument of the Gamma function, Stirling's formula,

$$\Gamma\left(\frac{s}{2}\right) \approx \sqrt{\frac{4\pi}{s}}\left(\frac{s}{2e}\right)^{s/2}\left(1 + \mathcal{O}\left(\frac{1}{s}\right)\right),$$

is used. The change in argument can then be approximated as

$$\begin{aligned} -\frac{\pi}{8} + \mathcal{O}\left(\frac{1}{T}\right) + \frac{T}{2}\log\left(\frac{T}{2e}\right) + \mathcal{O}\left(\frac{1}{T}\right) \\ = \frac{T}{2}\log\left(\frac{T}{2e}\right) - \frac{\pi}{8} + \mathcal{O}\left(\frac{1}{T}\right) \end{aligned}$$

For now, the change in argument of the zeta function will be called $\pi S(T)$. It is $\mathcal{O}(\log(T))$, and will be discussed in more detail later on. Combining everything, we have that

$$N(T) = \frac{1}{\pi}\left(\frac{T}{2}\log\left(\frac{T}{2e}\right) - \frac{T}{2}\log(\pi) + \pi S(T) + \frac{7\pi}{8} + \mathcal{O}\left(\frac{1}{T}\right)\right)$$
$$= \frac{T}{2\pi}\log\left(\frac{T}{2\pi e}\right) + S(T) + \frac{7}{8} + \mathcal{O}\left(\frac{1}{T}\right).$$

i.e. there are more zeros in an interval of any given size the further away from the origin you go. This means that trivially, the lim inf of the gap between successive zeros is 0. In order to make this calculation more meaningful, the zeros can be scaled such that $\gamma_n' = \gamma_n \log(\gamma_n)/2\pi$, so the rescaled average spacing is 1. After rescaling, assuming RH, the lim inf and lim sup can be improved [37] to 0.5154 and 2.7327 respectively, and it is conjectured [34] that these are 0 and infinity. Under the assumption of the Generalised Riemann Hypothesis(that all the non-trivial zeros of all Dirichlet L-functions lie on the critical line), it has been shown [15] that the lim sup is great than 3.072.

2.3. **Pair Correlation.** As well as studying the lim sup and lim inf of the gap between consecutive zeros, it is also possible to study the general distribution of $\gamma - \gamma'$, where γ, γ' are the imaginary parts of (not necessarily consecutive) zeros of the zeta function.

Montgomery originally conjectured [34] that for $\alpha < \beta$ fixed,

$$\sum_{\substack{0<\gamma,\gamma'<T \\ \frac{2\pi\alpha}{\log(T)} \leq \gamma-\gamma' \leq \frac{2\pi\beta}{\log(T)}}} 1 \sim \left(\int_\alpha^\beta \left(1 - \left(\frac{\sin(\pi u)}{\pi u}\right)^2\right) \mathrm{d}u + \delta(\alpha,\beta)\right) \frac{T\log(T)}{2\pi}$$

where

$$\delta(\alpha,\beta) = \begin{cases} 1 & 0 \in [\alpha,\beta] \\ 0 & \text{otherwise.} \end{cases}$$

This is equivalent to the statement that the pair correlation function for the Riemann zeta function is

$$1 - \left(\frac{\sin(\pi u)}{\pi u}\right)^2 + \delta(u).$$

A quick way of thinking about this is: given a zero at the point $1/2 + iT$, go a distance $2\pi u/\log(T)$ away, and study a small interval of size $2\pi L/\log(T)$. The probability of finding a zero in that interval is then L times the pair correlation function evaluated at u. The function $2\pi/\log(T)$ is used here to scale the zeros so that the density is constant.

It was noted that this suggested pair correlation function is the same as that used for random complex Hermitian and unitary matrices as the size of the matrix tends to infinity. Therefore, this suggests that there exists an as-yet undiscovered linear operator[3] with eigenvalues which match the non-trivial zeros of the zeta function.

In 1987 Odlyzko [36] used the first 10^5 zeros as well as the 10^{12}th and the following 10^5 zeros to numerically study Montgomery's pair correlation. The reason these two groups of zeros was chosen was so any differences in behaviour could be seen and studied to see if it is reasonable that the pair correlation conjecture is true.

As well as plotting pictures which give a rather qualitative view of the zeros compared to the expected distribution based on the GUE, they were also quantitatively studied. The general view from Odlyzko's paper is that the gaps between consecutive zeros is fairly consistent with the gaps predicted by the GUE (although it must be remembered that the GUE statistics are from $\lim n \to \infty$ matrices, whereas we can only see the gaps between zeros relatively close to the origin). Additionally, it was noted that the tails of the distribution of the gaps between consecutive and next-but-one zeros may indicate that long range behaviour may not satisfy the pair correlation conjecture. However, it is important to note that the range of zeros used here is too small to draw any meaningful conclusions, and computational work can only be used as an indicator rather than final answer.

2.4. **Derivatives of the Riemann Zeta Function.** As well as the properties of the zeta function, the properties of derivatives of the zeta function are of interest. Using the functional equation, we have that [1]

$$(-1)^k \zeta^{(k)}(1-s) = \sum_{m=0}^{k} \binom{k}{m} \left(e^{sz} z^{k-m} + e^{s\overline{z}} (\overline{z})^{k-m} \right) \left(\Gamma(s)\zeta(s) \right)^{(m)},$$

where

$$z = -\log(2\pi) - \frac{i\pi}{2}.$$

Zeros of the derivatives of the zeta function are not bound to the critical line in the same way as for the zeta function, and so the density of zeros for the first derivative of the zeta function depends upon a horizontal component. It has been shown unconditionally that [28] asymptotically as $T \to \infty$ and for σ bounded by

$$\frac{\log\log(T)^2}{\log(T)^{1/3}} \leq 2\sigma - 1 \leq \frac{1}{20\log\log(T)},$$

we have that

$$\begin{aligned} N_1(T,\sigma) &= \sum_{\substack{\beta'>\sigma \\ 0\leq\gamma'<T}} 1 \\ &\sim \frac{T}{2\pi(\sigma - 1/2)} \end{aligned}$$

where $\beta' + i\gamma'$ are the zeros of the derivative of the zeta function. This function is bounded to the right of the critical line $\sigma = 1/2$.

To the left of the critical line, things get more interesting. Speiser [43] showed that RH is equivalent to there being no zeros of the first derivatives of the zeta function to the left of the critical line. This work was extended by Levinson and Montgomery, who proved [29] that there exists an infinite sequence of points T_j such that

$$N^-(T_j) = N_1^-(T_j)$$

where $N^-(T)$ is the number of zeros of the zeta function to the left of the critical line up to height T, and $N_1^-(T)$ is the number of zeros of the first derivative of the zeta function to the left of the critical line. The proof of this result assumes that

$$N^-(T) \leq \frac{T}{2}.$$

If this inequality does not hold, the best result which has so far been found is

$$N_1^-(T) = N^-(T) + \mathcal{O}(\log(T)).$$

Also of interest is the number of zeros of higher derivatives of the zeta function. It has been shown that [2]

$$\begin{aligned} N_k(T) &= \frac{T}{2\pi}\log(T) - \left(\frac{1+\log(4\pi)}{2\pi}\right)T + \mathcal{O}(\log T) \\ &= \frac{T}{2\pi}\log\left(\frac{T}{2\pi e}\right) - \frac{T\log(2)}{2\pi} + \mathcal{O}(\log T) \end{aligned}$$

where $N_k(T)$ is the number of zeros of the kth derivative of the zeta function with positive imaginary part less than T for large k. This is found by calculating a contour integral around all the non-trivial zeros of the kth derivative of the zeta function. The zero free region is calculated [44] using a different formula for the derivatives, namely

$$\zeta^{(k)}(s) = (-1)^k \sum_{n=2}^{\infty} \frac{\log^k(n)}{n^s}.$$

Using this formula, we get that

$$|\zeta^{(k)}(s)| \geq \frac{\log^k(2)}{2^\sigma} - \sum_{n=3}^{\infty} \frac{\log^k(n)}{n^\sigma}.$$

Setting

$$f(x) = \frac{\log^k(x)}{x^\sigma}$$

we have that

$$f'(x) = (k - \sigma \log(x)) \frac{\log^{k-1}(x)}{x^{\sigma+1}},$$

which will be negative for $x > 2$, provided $\sigma > k/\log(2)$. Therefore

$$\sum_{n=3}^{\infty} \frac{\log^k(n)}{n^\sigma} < \int_2^\infty \frac{\log^k(x)}{x^\sigma} \mathrm{d}x.$$

We can use integration by parts to calculate the integral, giving the recursive formula

$$\begin{aligned} I_k &= \int_2^\infty \frac{\log^k(x)}{x^\sigma} \mathrm{d}x \\ &= \left. \frac{\log^k(x) x^{1-\sigma}}{1-\sigma} \right|_2^\infty + \frac{k}{\sigma - 1} \int_2^\infty \frac{\log^{k-1}(x)}{x^\sigma} \mathrm{d}x \\ &= \frac{\log^k(2) 2^{1-\sigma}}{\sigma - 1} + \frac{k}{\sigma - 1} I_{k-1}. \end{aligned}$$

This, combined with the result that

$$I_0 = \frac{2^{1-\sigma}}{\sigma - 1}$$

means that we can write

$$I_k = \frac{2^{1-\sigma} k!}{(\sigma - 1)^{k+1}} \sum_{j=0}^{k} \frac{\log^j(2)(\sigma - 1)^j}{j!},$$

and therefore

$$|\zeta^{(k)}(s)| \geq \frac{\log^k(2)}{2^\sigma} - \frac{2^{1-\sigma}k!}{(\sigma-1)^{k+1}} \sum_{j=0}^{k} \frac{\log^j(2)(\sigma-1)^j}{j!}.$$

We can then bound the RHS away from zero

$$\frac{\log^k(2)}{2^\sigma} - \frac{2^{1-\sigma}k!}{(\sigma-1)^{k+1}} \sum_{j=0}^{k} \frac{\log^j(2)(\sigma-1)^j}{j!} > 0$$

which can be rearranged to give

$$\frac{\log^k(2)}{2^\sigma} > \frac{2^{1-\sigma}k!}{(\sigma-1)^{k+1}} \sum_{j=0}^{k} \frac{\log^j(2)(\sigma-1)^j}{j!}.$$

Finding the values of σ_0 for which this inequality holds bounds the zeros of the derivatives of the zeta function for $\sigma > \sigma_0$, where σ_0 depends on the k, the number of derivatives taken.

Setting

$$z = \log(2)(\sigma - 1)$$

the previous inequality can be written as

$$\frac{z^{k+1}}{2\log(2)k!} > \sum_{j=0}^{k} \frac{z^j}{j!}.$$

Approximating the sum on the right as

$$\sum_{j=0}^{k} \frac{z^j}{j!} \leq \frac{z^k}{k!} + \frac{kz^{k-1}}{(k-1)!},$$

so the inequality will definitely hold if it can be shown that

$$\frac{z^{k+1}}{2\log(2)k!} > \frac{z^k}{k!} + \frac{kz^{k-1}}{(k-1)!}$$

or, equivalently

$$\frac{z^2}{2\log(2)} > z + k^2,$$

which holds for

$$z > \log(2)\left(1 + \sqrt{\frac{1+2k^2}{\log(2)}}\right).$$

This can be simplified, for $k \geq 3$, to

$$z \geq \log(2)\left(1 + \frac{7k}{8}\right),$$

so, remembering that

$$z = \log(2)(\sigma - 1),$$

we have that

$$\sigma \geq 2 + \frac{7k}{8}.$$

Therefore, we have that there aren't any zeros of the kth derivative (where $k \geq 3$) to the right of the line $\sigma > 7k/4 + 2$. Using the same method and the functional equation for the zeta function, we also have that there is a sequence r_k, such that all the zeros such that $|s| > r_k$ and $\sigma < -\varepsilon$ satisfies $|t| < \varepsilon$.

It is conjectured [44] that

$$N(T) = N_k(T) + \left[\frac{T\log(2)}{2\pi}\right] \pm 1,$$

which, given that

$$S(T) = \mathcal{O}(\log(T))$$

appears not implausible. However, more work into understand $S(T)$ is probably needed before such a result is conclusively proven.

2.5. **S(T).** As mentioned previously,

$$S(T) = \frac{1}{\pi}\arg\left(\zeta\left(\frac{1}{2} + iT\right)\right)$$

or, if there is a zero of zeta at $1/2 + iT$,

$$S(T) = \frac{1}{2\pi}\lim_{\epsilon \to 0}\arg\left(\zeta\left(\frac{1}{2} + i(T+\epsilon)\right)\right) + \arg\left(\zeta\left(\frac{1}{2} + i(T-\epsilon)\right)\right)$$

where the argument is found by considering the increment of the arguement of the zeta function along the straight lines from 2 to $2 + iT$ to $1/2 + iT$. We can also consider $S(T)$ to be the error term

$$N(T) - \frac{1}{\pi}\vartheta(T) - 1,$$

where

$$\vartheta(T) = \mathrm{Im}(\log\left(\Gamma\left(\frac{1}{4} + \frac{iT}{2}\right)\right)) - \frac{T}{2}\log(\pi)$$

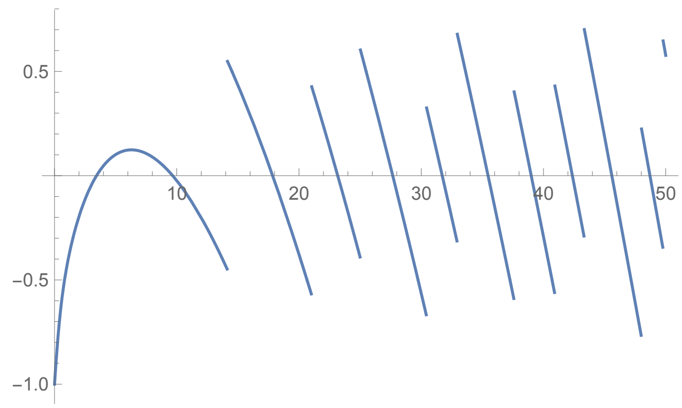


Figure 3. S(t)

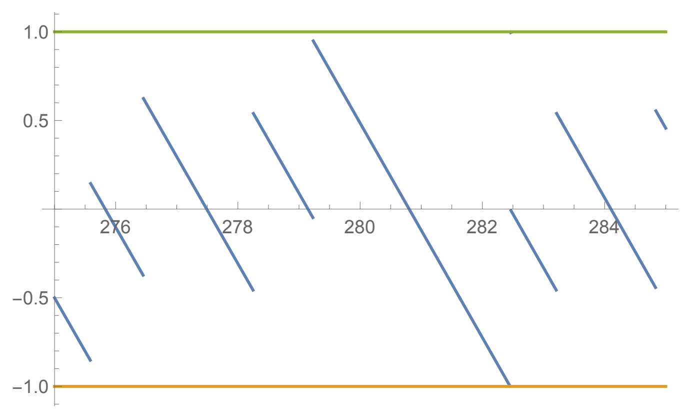


Figure 4. S(t) for larger t

which shows that $S(T)$ is a decreasing function for $T > 8$, except where a zero occurs, where the function is discontinuous and increases by the multiplicity of the zero. The function $\vartheta(t)$ is examined in more detail later on.

In fig 3, $S(t)$ is bounded by $|S(t)| < 1$. However, this doesn't always hold, and fig 4 shows that at $t \approx 282$ this inequality fails to hold. Although $S(t)$ is unbounded, it grows very slowly, and currently the largest value which has been found is $S(T) \approx 3.3455$ at $T \approx 7 \times 10^{27}$ [4].

It has been shown that [45]

$$\int_0^T S(u)\mathrm{d}u \ll \log(T),$$

so that

$$\int_{T}^{T+\log^2(T)} S(u)\mathrm{d}u \ll \log(T)$$

from which it follows that $S(t)$ changes sign infinitely often. This can most easily be seen by considering the opposite— assume that for all $t > t_0$, $S(t) > \epsilon$. However, then the integral would be

$$\int_{T}^{T+\log^2(T)} S(u)\mathrm{d}u = \mathcal{O}(\log^2(T)).$$

This argument is the same for the case that $S(t) < -\epsilon$ for all $t > t_0$.

Since

$$N(T) = \frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + \frac{7}{8} + S(T) + \mathcal{O}\left(\frac{1}{T}\right)$$

we have that

(1)

$$\begin{aligned} N(T+H) - N(T) &= \frac{H}{2\pi}\log\left(\frac{T}{2\pi}\right) + S(T+H) - S(T) + \mathcal{O}\left(\frac{1+H^2}{T^2}\right) \\ &= \frac{H}{2\pi}\log\left(\frac{T}{2\pi}\right) + S(T+H) - S(T) + \mathcal{O}\left(\frac{1}{T}\right) \end{aligned}$$

for $0 < H < \sqrt{T}$.

It has been shown, under the assumption of RH that [18]

$$|N(T+H) - N(T) - \frac{H}{2\pi}\log\left(\frac{T}{2\pi}\right)| \le \left(\frac{1}{2} + o(1)\right)\frac{\log(T)}{\log\log(T)},$$

which, together with 1 means that

$$|S(T+H) - S(T)| \le \left(\frac{1}{2} + o(1)\right)\frac{\log(T)}{\log\log(T)}.$$

A different technique was used to show that [8]

$$|S(T)| \le \left(\frac{1}{4} + o(1)\right)\frac{\log(T)}{\log\log(T)}.$$

This result shows that the number of zeros can vary from the expected value more the larger T gets.

2.6. $\mathbf{S_1(T)}$. In the previous section, we mentioned the integral of $S(T)$. Calling this

$$S_1(T) = \int_0^T S(u)\mathrm{d}u,$$

which we looked at briefly in the previous section. In order to calculate this, the function

$$\log(\zeta(s))$$

is integrated around the contour connecting the point $1/2$, A, $A+iT$ and $1/2+iT$, where A is a suitably large real number, which will tend to ∞ later. The contour integral,

$$\int_{1/2}^{A}\log(\zeta(\sigma))\mathrm{d}\sigma + i\int_{0}^{T}\log(\zeta(A+iu))\mathrm{d}u - \int_{1/2}^{A}\log(\zeta(\sigma+iT))\mathrm{d}\sigma - i\int_{0}^{T}\log(\zeta(1/2+iu))\mathrm{d}u,$$

is imaginary, since the sum of integrals will be $2\pi i$*the residue of the integrand, which will be the sum of zeros of the zeta function to the right of the critical line, which must be imaginary. Therefore, the real part of the sum of integrals is zero, and explicitly calculating this can lead to a result for $S_1(T)$. Calculating some of these integrals directly leads to divergent results, and so the integrand must be rewritten, using the result that

$$\log(\zeta(s)) = \sum_{n=2}^{\infty}\frac{\Lambda(n)}{\log(n)}n^{-s}$$

for $\sigma > 1$.

Starting with the first and third integrals, we have that

$$\mathrm{Re}\left[\int_{1/2}^{A}\log(\zeta(\sigma))\mathrm{d}\sigma\right] = \int_{1/2}^{A}\log|\zeta(\sigma)|\mathrm{d}\sigma$$

and

$$-\,\mathrm{Re}\left[\int_{1/2}^{A}\log(\zeta(\sigma+iT))\mathrm{d}\sigma\right] = -\int_{1/2}^{A}\log|\zeta(\sigma+iT)|\mathrm{d}\sigma$$

For the second integral,

$$i\int_{0}^{T}\log(\zeta(A+iu))\mathrm{d}u$$

the inequality

$$|\log(\zeta(s))| = |\sum_{n=2}^{\infty}\frac{\Lambda(n)}{\log(n)}n^{-s}| < \sum_{n=2}^{\infty}\frac{1}{n^{\sigma}} < 2^{1-\sigma}$$

holds for σ suitably large, so therefore the integral can be written as

$$\begin{aligned}\operatorname{Re}\left[\int_0^T i\log(\zeta(A+iu))\mathrm{d}u\right] &\leq \left|\int_0^T \log(\zeta(A+iu))\mathrm{d}u\right| \\ &< \int_0^T |\log(\zeta(A+iu))|\mathrm{d}u \\ &< \int_0^T 2^{1-A}\mathrm{d}u \\ &= T2^{1-A}\end{aligned}$$

The final integral is

$$\begin{aligned}\operatorname{Re}\left[-\int_0^T i\log\left(\zeta\left(\frac{1}{2}+iu\right)\right)\mathrm{d}u\right] &= \operatorname{Im}\left[\int_0^T \log\left(\zeta\left(\frac{1}{2}+iu\right)\right)\mathrm{d}u\right] \\ &= \int_0^T \arg\left(\zeta\left(\frac{1}{2}+iu\right)\right)\mathrm{d}u \\ &= \int_0^T \pi S(u)\mathrm{d}u \\ &= \pi S_1(T).\end{aligned}$$

Therefore, we have that

$$\pi S_1(T) = \int_{1/2}^A \log|\zeta(\sigma+iT)|\mathrm{d}\sigma - \int_{1/2}^A \log|\zeta(\sigma)|\mathrm{d}\sigma + \mathcal{O}\left(T2^{1-A}\right),$$

and by taking the limit as $A\to\infty$, this becomes

$$S_1(T) + \frac{1}{\pi}\int_{1/2}^\infty \log|\zeta(\sigma)|\mathrm{d}\sigma = \frac{1}{\pi}\int_{1/2}^\infty \log|\zeta(\sigma+iT)|\mathrm{d}\sigma.$$

The integral on the left is a constant (and is integrable despite the pole at $\sigma = 1$), and so doesn't need any more work doing to it. The integral on the right can be split up to give

$$S_1(T) = \frac{1}{\pi}\int_{1/2}^2 \log|\zeta(\sigma+iT)|\mathrm{d}\sigma + \frac{1}{\pi}\int_2^\infty \log|\zeta(\sigma+iT)|\mathrm{d}\sigma + C,$$

which is split up in this way as [45]

$$\log|\zeta(s)| = \sum_{|t-\gamma|<1} \log|s-\rho| + \mathcal{O}(\log(t))$$

uniformly for $-1 \leq \sigma \leq 2$, and $\rho = \beta + i\gamma$ are the non-trivial zeros. Therefore,

$$S_1(T) = \frac{1}{\pi} \sum_{|T-\gamma|<1} \int_{1/2}^{2} \log|\sigma + iT - \rho| \mathrm{d}\sigma + \mathcal{O}(\log(T)).$$

as the second integral can be approximated using

$$|\log(\zeta(s))| < 2^{1-\sigma}$$

as above. The integral can be bounded from below as

$$\begin{aligned}\int_{1/2}^{2} \log((\sigma-\beta)^2 + (T-\gamma)^2)\mathrm{d}\sigma &> \int_{1/2}^{2} \log|\sigma-\beta|\mathrm{d}\sigma \\ &> C\end{aligned}$$

and from above as

$$\int_{1/2}^{2} \log((\sigma-\beta)^2 + (T-\gamma)^2)\mathrm{d}\sigma < \frac{3}{2}\log\left(\left(\frac{3}{2}\right)^2 + 1\right)$$

without assuming RH. Therefore, we have that

$$S_1(T) = \mathcal{O}(\log(T)).$$

Using much more work, and under the assumption of RH, it is possible to solidify this result [8] to give

$$-\left(\frac{\pi}{24} + o(1)\right)\frac{\log(T)}{(\log\log T)^2} \leq S_1(T) \leq \left(\frac{\pi}{48} + o(1)\right)\frac{\log(T)}{(\log\log T)^2}$$

This is another way to show that $S(T)$ has an infinite number of sign changes, since if $S(T)$ had no sign changes beyond some point, because of the behaviour of this function, it would be expected that $S_1(T)$ would grow linearly, which it doesn't.

3. Variant Functions

Dealing with the zeta function directly, with its poles and trivial zeros, is difficult. Instead, Riemann created the xi function, and many others have been used to either showcase or hide different aspects of the zeta function.

3.1. **Z(t).**

The most basic change to the zeta function is to make the critical line the real axis, and make the function map the reals to the reals. Let

$$Z(t) = e^{i\vartheta(t)}\zeta\left(\frac{1}{2} + it\right)$$

where $\vartheta(t)$ is the negative of the argument of the zeta function along the critical line. By considering the functional equation,

$$\zeta(s) = \chi(s)\zeta(1-s)$$

which can be rearranged to give

$$\sqrt{\chi(1-s)}\zeta(s) = \sqrt{\chi(s)}\zeta(1-s)$$

which is real along the critical line, and

$$\left|\chi\left(\frac{1}{2} - it\right)\right| = 1$$

we therefore have that

$$\vartheta(t) = \sqrt{\chi\left(\frac{1}{2} - it\right)}.$$

It is important to note that the trivial zeros, which are at $t = i(2n + 1/2)$ and the pole at $t = -i/2$ remain in this function. Rearranging χ, we have that

$$\chi\left(\frac{1}{2} - it\right) = \pi^{-it}\frac{\Gamma\left(\frac{1}{4} + \frac{it}{2}\right)}{\Gamma\left(\frac{1}{4} - \frac{it}{2}\right)}.$$

Since

$$\Gamma(s) = \overline{\Gamma(\overline{s})},$$

this means that

$$\sqrt{\chi\left(\frac{1}{2} - it\right)} = \pi^{-it/2}e^{i\,\mathrm{Im}\log(\Gamma(1/4+it/2))}$$

so

$$\begin{aligned} Z(t) &= \pi^{-it/2}e^{i\,\mathrm{Im}\log(\Gamma(1/4+it/2))}\zeta\left(\frac{1}{2} + it\right) \\ &= e^{i\vartheta(t)}\zeta\left(\frac{1}{2} + it\right), \end{aligned}$$

where

$$\vartheta(t) = -\frac{t}{2}\log(\pi) + \mathrm{Im}(\log\Gamma\left(\frac{1}{4} + \frac{it}{2}\right))$$

Using Stirling's formula for the Gamma function,

$$\Gamma(z) = \sqrt{\frac{2\pi}{z}}\left(\frac{z}{e}\right)^z\left(1 + \mathcal{O}\left(\frac{1}{z}\right)\right),$$

we have that

$$\vartheta(t) = \frac{t}{2}\log\left(\frac{t}{2\pi}\right) - \frac{t}{2} - \frac{\pi}{8} + \mathcal{O}\left(\frac{1}{t}\right).$$

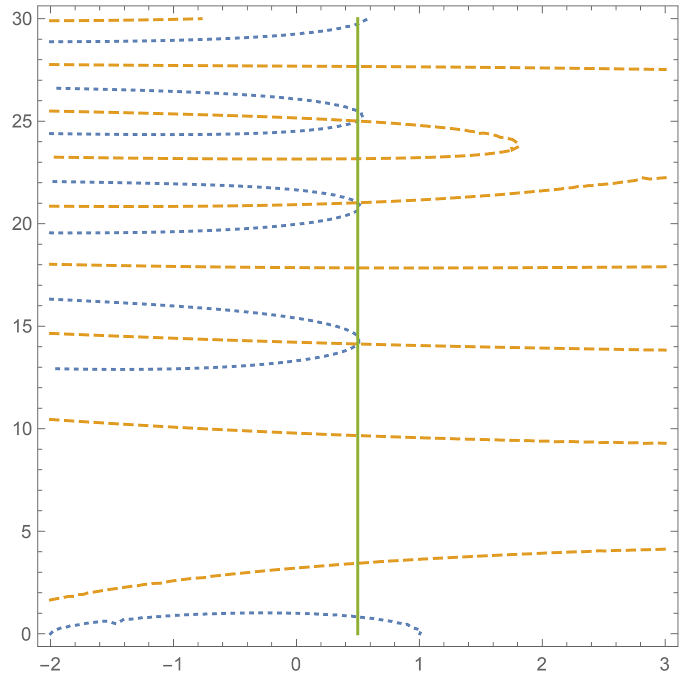


FIGURE 5. The lines where the real(blue, dotted) and imaginary (orange, dashed) parts of the Riemann zeta function are zero

It is possible to calculate this error term more precisely (see e.g. [25]), which allows us to estimate the derivative of this function without worrying about the problems normally raised from attempting the differentiating an error term of this form. An estimation of the derivative of this function yields

$$\vartheta'(t) \sim \frac{1}{2}\log\left(\frac{t}{2\pi}\right),$$

which is positive for $t > 10$, meaning that zeros of $\cos(\vartheta(t))$ and $\sin(\vartheta(t))$ will alternate for $t > 10$. Since the real part of the zeta function is positive more often than it is negative, on average the function $Z(t)$ changes sign an odd number of times between successive points where $\sin(\vartheta(t)) = 0$, meaning that there must be an odd number of zeros of $Z(t)$, and consequently the same number of the zeta function too. Although it is not known exactly the proportion of times this happens, for the first one and half billion intervals [48], 72.8% contain an odd number of zeros of the zeta function.

In order for

$$\sin(\vartheta(t)) = 0$$

we must have that

$$\vartheta(t) = n\pi,$$

and these points are called Gram Points, after the mathematician Gram who first studied this behaviour. [19] The idea that there is one zero of the zeta function between two successive Gram Points is called Gram's Law, although it fails infinitely often. The first failure occurs between the 125th and 126th Gram Point, and the missing zero is found in the next interval, not far from the Gram Point. This is the point where $S(t) < -1$ for the first time.

In the first one and a half billion Gram Intervals [48], 13.8% have no zeros, 72.6% have one zero, 13.4% have two zeros, 0.2% have three zeros, and just 33 intervals have four zeros. This work also suggests that the number of intervals with just one zero is decreasing, while the number of other intervals is increasing, i.e. the number of zeros in a given interval becomes more irregular.

It is conjectured that the limiting behaviour of Gram Intervals with 0, 1, and 2 zeros in [47] are 17%, 66.1% and 16.7% respectively. This is because of the similarities between the location of the zeros and the spacings of the eigenvalues of Random Matrices.

Differentiating evens out the zeros, and so it makes sense to consider the behaviour of the zeros and the function in general under repeated differentiation. However, the trivial zeros and the pole are still in the function, and they introduce some unwanted behaviour near the origin. Therefore, a different function is introduced later on which does not contain these problem points, and that is then differentiated instead. My thesis generalises this result to the entirety of the Selberg Class of L-functions.

However, further away from the origin, the expected behaviour does occur, and it has been shown that [33] under the assumption of RH, for $t > t_k$, there is exactly one zero of $Z^{(k+1)}(t)$ between any two successive zeros of $Z^{(k)}(t)$.

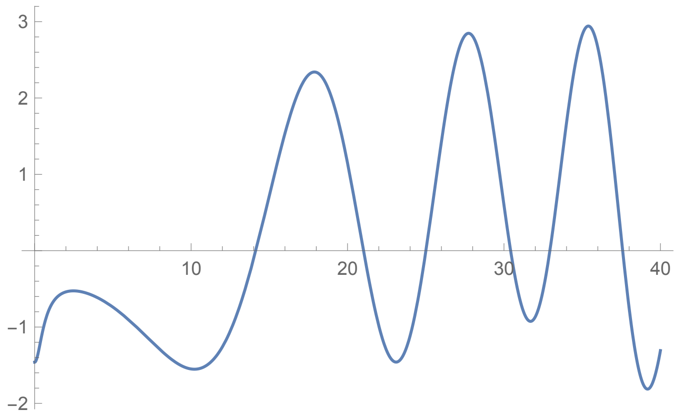


FIGURE 6. Z(t)

3.2. **The Riemann Xi-function.** Using the zeta function to calculate the location of the prime numbers is awkward due to the trivial zeros at the negative even integers and the pole at $s = 1$. By removing these, and rescaling, we create the function

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

which now only has the non-trivial zeros, and satisfies the functional equation

$$\xi(s) = \xi(1-s),$$

and

$$\xi(s) = \overline{\xi(\overline{s})}$$

so that it is real along the critical line. Defining a_k to be the proportion of zeros of the kth derivative of the xi function on the critical line, it has been shown that [9] $a_0 \geq 0.3658$, $a_1 \geq 0.8137$, $a_2 \geq 0.9584$, $a_3 \geq 0.9873$, $a_4 \geq 0.9948$, $a_5 \geq 0.997$, and, as $k \to \infty$, $a_k = 1 + \mathcal{O}(k^{-2})$. These results are not optimal, and it is interesting to note that different methods don't always immediately give the same result. These results are calculated by showing that

$$\xi^{(k)}(s) = Q_k(s) + (-1)^k\overline{Q_k(1-\overline{s})}$$

where $Q_k(s)$ is defined as a complex sum over k terms, and can be seen in [9]. Along the critical line, this becomes

$$\xi^{(k)}\left(\frac{1}{2}+it\right) = Q_k\left(\frac{1}{2}+it\right) + (-1)^k\overline{Q_k\left(\frac{1}{2}+it\right)}.$$

This shows that after an even(odd) number of derivatives, the real part of the function is even(odd), and the imaginary part os odd(even). Therefore, in order for the to be a zero along the critical line, we must have that

$$\arg\left(Q_k\left(\frac{1}{2}+it\right)\right) \equiv \frac{(k+1)\pi}{2} \mod (\pi).$$

It is interesting to see from these results that there is potential for zeros with higher multiplicity off the critical line.

Since the xi function is real along the critical line, it makes sense to turn it so the critical line becomes the real axis, i.e.

$$\Xi(z) = \xi\left(\frac{1}{2}+iz\right),$$

where for all the zeros of the Xi function, the absolute value of the imaginary part must be less than 1/2, and the Riemann Hypothesis is equivalent to the statement that all the zeros of the Xi function are real. This is now the (rotated) critical strip. Since it is explicitly a function of a complex variable, z is used rather than t.

The growth of the Xi function along the real axis can be used for some interesting results. Using

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

and the Stirling formula

$$\Gamma(z) \approx \sqrt{\frac{2\pi}{z}}\left(\frac{z}{e}\right)^z$$

we have that the Xi function is

$$\Xi(t) \approx \frac{1}{2}\left(t^2-\frac{1}{4}\right)\pi^{-1/4-it/2}\sqrt{\frac{\pi}{1/2+it}}\left(\frac{1/2+it}{2e}\right)^{1/4+it/2}\zeta\left(\frac{1}{2}+it\right),$$

which, assuming that $t \in \mathbb{R}$, can be written as

$$\begin{aligned}\Xi(t) &= \mathcal{O}\left(t^{2-1/2+1/4}e^{-\pi t/4}t^{\varepsilon}\right)\\ &= \mathcal{O}\left(e^{-\pi t/4}t^{7/4+\varepsilon}\right)\end{aligned}$$

since the Lindelöf Hypothesis [45] states that for any $\varepsilon > 0$,

$$|\zeta\left(\frac{1}{2}+it\right)| = \mathcal{O}(t^{\varepsilon})$$

as $t \to \infty$. Therefore, we have that the Xi function decays exponentially fast along the real axis, which means it can be written as a Fourier transform. This is discussed later on.

3.3. **Approximate Functional Equations.** The functional equation

$$\zeta(s) = \chi(s)\zeta(1-s)$$

means that the zeta function can be calculated by Dirichlet series for $\sigma < 0$ and $\sigma > 1$, but can't be used to calculate the zeta function in the critical strip $0 \leq \sigma \leq 1$ since the original formula for the zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

only holds for $\sigma > 1$.

Instead, approximate functional equations (so-called since they include finite sums of n^{-s}, and because they mostly include the $\chi(s)$ term used in the functional equation) are used to calculate the zeta function to a high degree of accuracy. The most basic approximate functional equation is [24]

$$\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} + \frac{x^{1-s}}{s-1} + \mathcal{O}(x^{-\sigma})$$

where $x \geq |t|/\pi$, although this is not used often due to the large number of terms needed in the sum.

The approximate functional equation more commonly used is [22]

$$\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} + \chi(s) \sum_{n \leq y} n^{s-1} + \mathcal{O}(x^{-\sigma}) + \mathcal{O}(t^{1/2-\sigma} y^{\sigma-1}) \tag{2}$$

where $t = 2\pi xy$ and

$$\chi(s) = 2^s \pi^{s-1} \Gamma(1-s) \sin\left(\frac{\pi s}{2}\right).$$

This result is proved by first showing that

$$\zeta(s) + \frac{x^{1-s}}{1-s} - \sum_{n \leq x} n^{-s} = 2^s \pi^{s-1} \sum_{n=1}^{\infty} \int_{2\pi n x}^{\infty} u^{-s} \cos(u) \mathrm{d}u$$

and then splitting the sum on the RHS into 5 separate sums and bounding the integral in each case separately.

A similar method can be used to calculate the approximate functional equation

$$\zeta^2(s) = \sum_{n\le x} \frac{d(n)}{n^s} + \chi(s)\sum_{n\le y}\frac{d(n)}{n^{1-s}} + \mathcal{O}\left(x^{1/2-\sigma}\left(\frac{x+y}{t}\right)^{1/4}\log(t)\right)$$

where now we have that

$$xy = \left(\frac{t}{2\pi}\right)^2.$$

3.4. **AFE for Z(t).** The function $Z(t)$ is more commonly used to computationally find zeros of the zeta function along the critical line, since the (simple) zeros correspond to changes of sign of $Z(t)$.

Beginning with the result [24]

$$\zeta(s) = \chi(s)\zeta(1-s)$$

which means that

$$\chi(s)\chi(1-s) = 1.$$

Therefore, we can write

$$\zeta(s)\chi^{1/2}(1-s) = \zeta(1-s)\chi^{1/2}(s)$$

which shows that along the critical line, this function is real. Therefore

$$\begin{aligned} Z(t) &= \zeta\left(\frac{1}{2}+it\right)\chi^{-1/2}\left(\frac{1}{2}+it\right) \\ &= \zeta\left(\frac{1}{2}+it\right)\chi^{1/2}\left(\frac{1}{2}-it\right) \end{aligned}$$

Using the approximate functional equation 2 along the critical line, we have that the error terms will be minimised by setting $x = y$, and so setting

$$x = y = \sqrt{\frac{t}{2\pi}}$$

gives us

$$\zeta\left(\frac{1}{2}+it\right) = \sum_{n\le\sqrt{\frac{t}{2\pi}}}\frac{1}{n^{1/2+it}} + \chi\left(\frac{1}{2}+it\right)\sum_{n\le\sqrt{\frac{t}{2\pi}}}\frac{1}{n^{1/2-it}} + \mathcal{O}(t^{-1/4}).$$

We can then use this to generate a formula for $Z(t)$, namely

$$\begin{aligned} Z(t) &= \chi^{1/2}\left(\frac{1}{2}-it\right)\zeta\left(\frac{1}{2}+it\right) \\ &= \chi^{1/2}\left(\frac{1}{2}-it\right)\sum_{n\leq\sqrt{\frac{t}{2\pi}}}\frac{1}{n^{1/2+it}} + \chi^{1/2}\left(\frac{1}{2}-it\right)\chi\left(\frac{1}{2}+it\right)\sum_{n\leq\sqrt{\frac{t}{2\pi}}}\frac{1}{n^{1/2-it}} \\ &\qquad + \mathcal{O}(t^{-1/4}). \end{aligned}$$

Since we have that

$$\chi(s)\chi(1-s) = 1$$

we must also have that

$$\chi^{1/2}\left(\frac{1}{2}-it\right)\chi\left(\frac{1}{2}+it\right) = \chi^{1/2}\left(\frac{1}{2}+it\right)$$

so our equation becomes

$$Z(t) = \chi^{1/2}\left(\frac{1}{2}-it\right)\sum_{n\leq\sqrt{\frac{t}{2\pi}}}\frac{1}{n^{1/2+it}} + \chi^{1/2}\left(\frac{1}{2}+it\right)\sum_{n\leq\sqrt{\frac{t}{2\pi}}}\frac{1}{n^{1/2-it}} + \mathcal{O}(t^{-1/4}).$$

Remembering also that

$$Z(t) = e^{i\vartheta(t)}\zeta\left(\frac{1}{2}+it\right)$$

so that

$$\chi^{1/2}\left(\frac{1}{2}-it\right) = e^{i\vartheta(t)}$$

and

$$\chi^{1/2}\left(\frac{1}{2}+it\right) = e^{-i\vartheta(t)}$$

the function then becomes

$$\begin{aligned} Z(t) &= \sum_{n\leq\sqrt{\frac{t}{2\pi}}}\frac{e^{i\vartheta(t)}}{n^{1/2+it}} + \sum_{n\leq\sqrt{\frac{t}{2\pi}}}\frac{e^{-i\vartheta(t)}}{n^{1/2-it}} + \mathcal{O}(t^{-1/4}) \\ &= \sum_{n\leq\sqrt{\frac{t}{2\pi}}}\frac{e^{i(\vartheta(t)-t\log(n))}}{\sqrt{n}} + \sum_{n\leq\sqrt{\frac{t}{2\pi}}}\frac{e^{-i(\vartheta(t)-t\log(n))}}{\sqrt{n}} + \mathcal{O}(t^{-1/4}) \\ &= 2\sum_{n\leq\sqrt{\frac{t}{2\pi}}}\frac{\cos(\vartheta(t)-t\log(n))}{\sqrt{n}} + \mathcal{O}(t^{-1/4}). \end{aligned}$$

In order to use $Z(t)$ to check RH, it suffices to show that the number of sign changes of Z match the expected number of zeros by calculating $N(T)$. So far, all the zeros of zeta are simple and lie on the critical line. However, this method will fall down for repeated roots or zeros off the critical line- this method cannot positively identify a repeated root rather than a pair of zeros off the line, and since it is possible to miss pairs of zeros, you cannot be sure that there aren't zeros you haven't found rather than zeros off the critical line. Turing's method [5] uses the located zeros and the formula

$$S(T) = N(T) - \frac{\vartheta(T)}{\pi} - 1$$

to find where $S(T)$ remains away from 0 for long periods of time, which suggests that zeros have been missed.

4. L-functions

The Riemann zeta function is just one example of a group of functions called L-functions. Beginning with a Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \qquad a_n = \mathcal{O}(1)$$

Without any loss of generality, it can be assumed that $a_1 = 1$, and the abscissa of convergence the line $\sigma = 1$. There are potentially a finite number of poles along this line, modified so they all lie on the point $s = 1$. The analytic continuation to the whole complex plane is then called an L-function. [40]

There are several different commonly used methods of constructing the sequence a_n, and these produce different families of L-functions. Dirichlet characters, Maass forms and elliptic curves are three different ways of constructing L-series and then L-functions.

4.1. **Prime Number Theorem extension.**

The natural extension of the PNT is Dirichlet's Theorem [26], which concerns itself with primes in arithmetic progressions. Given an arithmetic progression $a(\text{mod } q)$, where a and q are co-prime, how large is the first prime, commonly denoted $P(a, q)$?

It has been conjectured that [20]

$$\max_a P(a, q) \gg \phi(q) \log^2(q),$$

where ϕ is Euler's totient function, i.e. for any given q, there is an arithmetic progression with a surprisingly large number of composite numbers before the first prime. In the other direction, it has been proven that [30]

$$P(a, q) \ll q^L$$

for some absolute constant L, called Linnik's constant. Much work has been done to reduce this constant, currently the best bound for it is [46]

$$L \leq 5.$$

Assuming GRH, this result can be improved to give [23]

$$P(a, q) \ll \phi(q)^2 \log^2(q).$$

4.2. **Riemann Hypothesis variations.** Just as the zeta function has the Riemann Hypothesis, so too are there variations for different L-functions. The Grand or Generalised Riemann Hypothesis(GRH) [26] either refers to L-functions created with Dirichlet characters— Dirichlet L-functions- or all L—functions which have an Euler product and functional equation. The Euler product of L-functions is of the form

$$\prod_{p \text{ prime}} (1 - \alpha_1(p) p^{-s})^{-1} \cdots (1 - \alpha_d(p) p^{-s})^{-1}$$

where d is the degree of the L-function. The functional equation is

$$\begin{aligned} \Lambda(f, s) &= q^{s/2} \gamma_F(s) F(s) \\ &= \Lambda(\overline{f}, 1 - s) \end{aligned} \tag{3}$$

where f defines the construction of the sequence a_n. For all currently known L-functions

$$\gamma_F(s) = \pi^{ds/2} \prod_{j=1}^{d} \Gamma\left(\frac{s + \mu_j}{2}\right),$$

where $q \in \mathbb{R}$ and $\mu_j \in \mathbb{C}$. The requirement for an Euler product and gamma factor removes L-functions which are constructed by adding or subtracting L-functions from each other, which can generate L-functions with zeros clearly off the critical line.

4.3. **Density of zeros of L-functions.** Results about the density of zeros can also be applied to L-functions with an Euler product and a functional equation. Because the functional equation involves the conjugate L-function, counting the zeros in the same method as the zeta function case calculates

$$N(T, f) := N'(T, f) + N'(T, \overline{f})$$

where N' is the number of zeros of the L-function in the critical strip with $0 < \gamma \leq T$, and the conjugate L-function is the L-function where every complex number has been replaced with its complex conjugate. The final result is that [26]

$$N(T, f) = \frac{T}{\pi} \log\left(\frac{qT^d}{(2\pi e)^d}\right) + \mathcal{O}(\log(\mathfrak{q}(f, s)))$$

where q is the constant from the previously defined functional equation 3 and d is the degree of the L-function. The $\mathfrak{q}(f, s)$ in the error term is unimportant for this work.

Because this is calculating the zeros of the L-function as well as the zeros of the conjugate of the L-function, this result is in agreement with the calculated number of zeros of the zeta function ($q = d = 1$)

$$N(T) = \frac{T}{2\pi} \log\left(\frac{T}{2\pi e}\right) + \mathcal{O}(\log(T)).$$

5. Differentiation

5.1. **Smoothing out Zero Gaps.**

Differentiation smooths out functions, including the gaps between zeros, and Farmer and Rhoades [14] showed that

Theorem 1. *Let f be a real, entire function of order 1 containing just real zeros. If the density of zeros is constant, then*

$$Ae^{Bx} f^{(n)}(Cx + D) \to \cos(x)$$

as $n \to \infty$, where A, B, C and D may depend on n but not x.

This is shown by showing that differentiating a function with all real zeros causes the infimum to increase, and the supremum to decrease. This result requires the zeros to be roughly regularly spaced out, with constant density, otherwise the infimum or supremum is vacuously 0 or infinity respectively, and this result won't hold.

In the case that the zeros are not all real, the result that the infimum will always increase doesn't hold either. Instead, it has been shown [10] that if a real, entire function of order less than 2 has a finite number of nonreal zeros, after a finite number of derivatives, all zeros will be real, and then stay real when more derivatives are taken.

5.2. **Differentiating the Riemann Xi function.** Farmer and Rhoade's work cannot be used to study the derivatives of the Xi function, because the count of zeros

$$\frac{z}{2\pi}\log\left(\frac{z}{2\pi e}\right)+\mathcal{O}(\log(z)),$$

is not of the required form. Instead, using a different technique, Ki [27] showed that

$$\lim_{n\to\infty} A_n\Xi^{(2n)}(C_n z)=\cos(z).$$

This result is obtained by showing that the Xi-function can be written as a quasi-Fourier transform, namely

$$\Xi(z)=\int_{-\infty}^{\infty}\varphi(t)e^{itz/2}\mathrm{d}t.$$

This can be easily differentiated with respect to z. Then, the integral can be split up into the main term and some error terms, all of which are suitably small.

Rather than calculate φ using the inverse Fourier transform result, it is easier to create the required integral by using the integral representation of the Gamma function

$$\begin{aligned}\Gamma(x)&=\int_0^{\infty}e^{-t}t^{x-1}\mathrm{d}t\\&=\int_{-\infty}^{\infty}e^{-e^t}e^{tx}\mathrm{d}t,\end{aligned}$$

so that

$$\begin{aligned}\frac{s}{2}(s-1)\Gamma\left(\frac{s}{2}\right)&=2\Gamma\left(\frac{s}{2}+2\right)-3\Gamma\left(\frac{s}{2}+1\right)\\&=\int_{-\infty}^{\infty}e^{-e^u}e^{us/2}\left(2e^{2u}-3e^u\right)\mathrm{d}u.\end{aligned}$$

Remembering that

$$\xi(s)=\frac{s}{2}(s-1)\Gamma\left(\frac{s}{2}\right)\pi^{-s/2}\zeta(s),$$

so that

$$\begin{aligned}\xi(s) &= \pi^{-s/2}\zeta(s)\int_{-\infty}^{\infty} e^{-e^u} e^{us/2}\left(2e^{2u}-3e^u\right)\mathrm{d}u\\ &= \sum_{n=1}^{\infty}\int_{-\infty}^{\infty} e^{-e^u}\left(\frac{e^u}{n^2\pi}\right)^{s/2}\left(2e^{2u}-3e^u\right)\mathrm{d}u\\ &= \sum_{n=1}^{\infty}\int_{-\infty}^{\infty} e^{-\pi n^2 e^x} e^{xs/2}\left(2\pi^2 n^4 e^{2x}-3\pi n^2 e^x\right)\mathrm{d}x\end{aligned}$$

and therefore

$$\begin{aligned}\Xi(z) &= \int_{-\infty}^{\infty}\sum_{n=1}^{\infty} e^{-\pi n^2 e^x}\left(2\pi^2 n^4 e^{9x/4}-3\pi n^2 e^{5x/4}\right)e^{ixz/2}\mathrm{d}x\\ &= \int_{-\infty}^{\infty}\varphi(x)e^{ixz/2}\mathrm{d}x,\end{aligned}$$

where

$$\begin{aligned}\varphi(x) &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^x}\left(2\pi^2 n^4 e^{9x/4}-3\pi n^2 e^{5x/4}\right)\\ &= Ae^{-ae^x}e^{bx}(1+\mathcal{O}(e^{-x}))\end{aligned}$$

for $a=\pi$ and $b=9/4$.

Since the Xi-function is even

$$\Xi(z)=\Xi(-z),$$

we must have that

$$\varphi(x)=\varphi(-x)$$

and therefore

$$\Xi(z)=\int_0^{\infty}\varphi(x)\left(e^{ixz/2}+e^{-ixz/2}\right)\mathrm{d}x.$$

Ki then showed that there exist constants A_n and C_n such that

$$\lim_{n\to\infty} A_n f^{(2n)}(C_n z)=\frac{1}{2}e^{iz},$$

where

$$f(z)=\int_0^{\infty}\varphi(x)e^{ixz/2}\mathrm{d}x,$$

so

$$\Xi(z)=f(z)+f(-z)$$

by differentiating and then rescaling so that the maximum occurs around 1. The integral is then split into the main part around 1 and three error terms.

Differentiating and then rescaling gives

$$f^{(2n)}(z) = \int_0^\infty \varphi(x)(-1)^n \left(\frac{x}{2}\right)^{2n} e^{ixz/2}\mathrm{d}x.$$

Rescaling with a positive constant C_n gives

$$\begin{aligned} f^{(2n)}(C_n z) &= (-1)^n 2^{-2n} \int_0^\infty \varphi(x) x^{2n} e^{iC_n xz/2}\mathrm{d}x \\ &= (-1)^n w_n^{2n+1} 2^{-2n} \int_0^\infty \varphi(w_n x) x^{2n} e^{ixz}\mathrm{d}x \\ &= A(-1)^n w_n^{2n+1} 2^{-2n} \int_0^\infty \exp\left[-ae^{w_n x} + bw_n x + 2n\log(x)\right] \\ &\qquad \left(1 + \mathcal{O}(e^{-w_n x})\right) e^{ixz}\mathrm{d}x \end{aligned}$$

where

$$w_n = \frac{2}{C_n}.$$

We can set the main part of the integral to occur around 1 by ensuring

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[-ae^{w_n x} + bw_n x + 2n\log(x)\right]\Big|_{x=1} = 0$$

by choosing w_n such that

$$aw_n e^{w_n} = bw_n + 2n.$$

For larger n, this can be approximated by

$$w_n \approx \log\left(\frac{2n}{a}\right) - \log\log\left(\frac{2n}{a}\right).$$

This means that the integrand has its maximum at 1, and so we can write

$$f^{(2n)}(C_n z) = A(-1)^n w_n^{2n+1} 2^{-2n}\left[I_m + I_\epsilon\right]$$

where

$$I_m = \int_{1-u_n}^{1+u_n} \exp\left[-ae^{w_n x} + bw_n x + 2n\log(x)\right]\left(1 + \mathcal{O}(e^{-w_n x})\right) e^{ixz}\mathrm{d}x$$

and

$$\begin{aligned} I_\epsilon = \left[\int_0^{1-u_n} + \int_{1+u_n}^2 + \int_2^\infty\right] &\exp\left[-ae^{w_n x} + bw_n x + 2n\log(x)\right] \\ &\left(1 + \mathcal{O}(e^{-w_n x})\right) e^{ixz}\mathrm{d}x \end{aligned}$$

where u_n will be defined later, after approximating the integrand and seeing for what values of x the approximation is allowed. We can change the range of integration by using

$$
\begin{aligned}
I_m &= \int_{1-u_n}^{1+u_n} \exp\left[-ae^{w_n x} + bw_n x + 2n\log(x)\right]\left(1+\mathcal{O}(e^{-w_n x})\right)e^{ixz}\mathrm{d}x \\
&= e^{iz}\int_{-u_n}^{u_n} \exp\left[-ae^{w_n(1+x)} + bw_n(1+x) + 2n\log(1+x)\right]\left(1+\mathcal{O}(e^{-w_n x})\right)e^{ixz}\mathrm{d}x
\end{aligned}
$$

and so the part of the integrand we are interested in is

$$
\begin{aligned}
&\exp\left[-ae^{w_n(1+x)} + bw_n(1+x) + 2n\log(1+x)\right] \\
&= \exp\left[-ae^{w_n}(1+w_n x+\frac{w_n^2x^2}{2}+\mathcal{O}(w_n^3x^3)) + bw_n(x+1) + 2n(x-\frac{x^2}{2}+\mathcal{O}(x^3))\right] \\
&= \exp\left[-ae^{w_n} + bw_n - x^2(ae^{w_n}\frac{w_n^2}{2}+n) + \mathcal{O}(x^3n + x^3w_n^3e^{w_n})\right] \\
&= \exp\left[-ae^{w_n} + bw_n - x^2(nw_n+b\frac{w_n^2}{2}+n) + \mathcal{O}(x^3n + x^3w_n^3e^{w_n})\right].
\end{aligned}
$$

The largest (in n) of the x^2 terms is

$$x^2nw_n,$$

as $w_n \sim \log(n)$. Comparing this to the largest of the error terms,

$$x^3w_n^3e^{w_n},$$

we therefore require

$$xw_n^2 \to 0.$$

Ki sets

$$u_n = \frac{1}{n^\theta}, \qquad \theta < \frac{1}{2},$$

which is a smaller value than necessary, but for the purpose of showing a bound, rather than minimising the error term doesn't matter. Therefore, we have that

$$
\begin{aligned}
I_m &= e^{-ae^{w_n}}e^{bw_n}e^{iz}\int_{-u_n}^{u_n} e^{-x^2(nw_n+b\frac{w_n^2}{2}+n)}(1+\mathcal{O}(w_n^2x^2))\mathrm{d}x \\
&= e^{-ae^{w_n}}e^{bw_n}e^{iz}u_n\int_{-1}^{1} e^{-x^2u_n^2(nw_n+b\frac{w_n^2}{2}+n)}(1+\mathcal{O}(w_n^2u_n^2x^2))\mathrm{d}x \\
&= e^{-ae^{w_n}}e^{bw_n}e^{iz}\sqrt{\frac{\pi}{(nw_n+bw_n^2/2+n)}}\left(1+\mathcal{O}\left(\frac{w_n^2}{nw_n+bw_n^2/2+n}\right)\right)
\end{aligned}
$$

The first two error terms

$$\left[\int_0^{1-u_n} + \int_{1+u_n}^{2}\right]\exp\left[-ae^{w_n x} + bw_n x + 2n\log(x)\right]\left(1+\mathcal{O}(e^{-w_n x})\right)e^{ixz}\mathrm{d}x$$

can be bounded using the same method as the main integral. The error term can be ignored since we are only interested in the leading behaviour, so we are only interested in the complex part (which will be dealt with later) and the exponential part

$$\exp\left[-ae^{w_n x} + bw_n x + 2n\log(x)\right].$$

The maximum of this part is at 1 (by construction), and so we have that

$$\exp\left[-ae^{w_n x} + bw_n x + 2n\log(x)\right] \leq e^{-ae^{w_n}} e^{bw_n} e^{-w_n n u_n^2}.$$

The complex part can be bound by

$$|e^{ixz}| < e^{|2z|}$$

and so the two error terms are bounded by

$$2e^{-ae^{w_n}} e^{bw_n} e^{-nw_n u_n^2} e^{2|z|}.$$

This is suitably small when compared to the main itnegral term.

The final error term

$$\int_2^\infty \exp\left[-ae^{w_n x} + bw_n x + 2n\log(x)\right] e^{ixz}\mathrm{d}x,$$

requires rearranging in a different way. Recalling that

$$aw_n e^{w_n} = bw_n + 2n$$

so that

$$\begin{aligned} ae^{w_n x} &= a\frac{w_n}{w_n} e^{w_n} e^{w_n(x-1)} \\ &= \frac{1}{w_n} e^{w_n(x-1)}\left(b + \frac{2n}{w_n}\right). \end{aligned}$$

Therefore, the exponential term of the integrand can be rearranged to give

$$\begin{aligned} &- ae^{w_n x} + bw_n x + 2n\log(x) \\ &\qquad = -be^{w_n(x-1)} - \frac{2n}{w_n} e^{w_n(x-1)} + bw_n x + 2n\log(x). \end{aligned}$$

The largest term of this integrand is

$$-\frac{2n}{w_n} e^{w_n(x-1)},$$

and so

$$\int_2^\infty \exp\left[-ae^{w_n x}+ \ bw_n x + 2n\log(x)\right] e^{ixz} \mathrm{d}x$$

$$\approx \int_2^\infty \exp\left[-\frac{2n}{w_n} e^{w_n(x-1)}\right] \mathrm{d}x$$

$$= \frac{1}{w_n} \int_{e^{w_n}}^\infty \exp\left[-\frac{2n}{w_n} u\right] \frac{\mathrm{d}u}{u}$$

$$< \frac{1}{w_n} \int_{e^{w_n}}^\infty \exp\left[-\frac{2n}{w_n} u\right] \mathrm{d}u$$

$$= \frac{1}{2n} e^{-\frac{2n}{w_n} e^{w_n}}.$$

Therefore, we have that

$$I_m + I_\epsilon = \sqrt{\pi} e^{-ae^{w_n}} e^{bw_n} e^{iz} \left(nw_n + \frac{bw_n^2}{2} + n\right)^{-1/2} \left(1 + \mathcal{O}\left(\frac{w_n^2}{nw_n + bw_n^2/2 + n}\right)\right) + \mathcal{O}(e^{-ae^{w_n}} e^{bw_n} e^{-nw_n u_n^2}) + \mathcal{O}(\frac{1}{n} e^{\frac{-2n}{w_n} e^{w_n}})$$

$$= \sqrt{\pi} e^{-ae^{w_n}} e^{bw_n} e^{iz} \left(nw_n + \frac{bw_n^2}{2} + n\right)^{-1/2} \Bigg[1 + \mathcal{O}(\frac{w_n^2}{nw_n + bw_n^2/2 + n}) + \mathcal{O}(e^{-nw_n u_n^2} \left(nw_n + bw_n^2/2 + n\right)^{1/2}) + \mathcal{O}(\frac{1}{n} e^{\frac{-2n}{w_n} e^{w_n}} e^{ae^{w_n}} e^{-bw_n} \left(2nw_n + \frac{bw_n^2}{2} + n\right)^{1/2})\Bigg].$$

The largest of these error terms comes is the

$$\frac{w_n^2}{nw_n + bw_n^2/2 + n} \sim \frac{w_n}{n}$$

and so we have that

$$I + m + I_\epsilon = \sqrt{\pi} e^{-ae^{w_n}} e^{bw_n} e^{iz} \left(nw_n + \frac{bw_n^2}{2} + n\right)^{-1/2} \left(1 + \mathcal{O}\left(\frac{w_n}{n}\right)\right).$$

Remembering that

$$f^{(2n)}(C_n z) = A(-1)^n w_n^{2n+1} 2^{-2n} [I_m + I_\epsilon]$$

so that

$$f^{(2n)}(C_n z) = A(-1)^n w_n^{2n+1} 2^{-2n} \sqrt{\pi} e^{-ae^{w_n}} e^{bw_n} e^{iz} \left(nw_n + \frac{bw_n^2}{2} + n \right)^{-1/2} \left(1 + \mathcal{O}\left(\frac{w_n}{n}\right)\right)$$

Therefore, in order that

$$\lim_{n\to\infty} A_n f^{(2n)}(C_n z) = e^{iz}$$

we have that

$$A_n = \frac{2}{A\sqrt{\pi}}(-1)^n 2^{2n-1} w_n^{-2n-1} e^{ae^{w_n}} e^{-bw_n} \left(nw_n + \frac{bw_n^2}{2} + n \right)^{1/2},$$

so

$$\begin{aligned} A_n \Xi^{(2n)}(C_n z) &= \frac{1}{2} A_n (f^{(2n)}(C_n z) + f^{(2n)}(-C_n z)) \\ &= \frac{1}{2}(e^{iz} + e^{-iz})(1 + \mathcal{O}\left(\frac{w_n}{n}\right)) \\ &= \cos(z)(1 + \mathcal{O}\left(\frac{w_n}{n}\right)) \end{aligned}$$

as required.

Ki's method works because a single Gamma function can be written as a single integral. This work can be used for the Hecke L-function case, since although it has two Gamma functions, the duplication formula reduces it to just one integral. Since more Gamma functions means more integrals, extending Ki's work in this way is unfeasible, given how the area of integration must be split up. Instead, in the second half of this thesis I focus on extending Ki's work by using the Fourier convolution theorem to create a single integral from an arbitrary number of Gamma functions.

Chapter 2

6. Introduction

In this chapter Ki's result on differentiating Riemann's Xi-function is extended to the Selberg Class of L-functions, showing that, given an L-function from the Selberg class, there exists sequences A_n, C_n and a constant Θ such that, uniformly on compact subsets of $\mathbb{C}$,

$$\lim_{n\to\infty} A_n \frac{\mathrm{d}^{2\mathrm{n}}}{\mathrm{d}z^{2n}} \Xi_F(C_n(z-\Theta)) = \cos(z+\theta),$$

where $\Xi_F(z)$ is related to the Selberg Class of L-functions in the same way that the the Riemann Xi-function is related to the Riemann zeta-function. The constants Λ, M and θ and the sequences C_n and A_n will be discussed in section 10. This is analogous to Ki's [27] work, which we discussed previously. Ki's method of using the integral representation of the Gamma function also holds for Hecke L-functions, since the Ξ_F-function related to these L-functions can also be written with a single Gamma function. However, the Selberg Class of L-functions generally includes a product of disparate Gamma functions, which cannot be simplified down to one by the multiplication or duplication formulas of the Gamma function.

In section 7 the Selberg Class of L-functions is discussed, and an explanation given for the particular choice of function used for this work.

Section 8 is dedicated to showing that the equation

$$\Xi_F(z) = \varepsilon Q^{1/2+iz} \prod_{l=1}^{m} \lambda_l(-\frac{1}{4} - z^2)\Gamma(i\lambda_l z + \frac{\lambda_l}{2})$$
$$\prod_{j=m+1}^{k} \Gamma\left(i\lambda_j z + \mu_j + \frac{\lambda_j}{2}\right) F(\frac{1}{2} + iz)$$

can be written as a quasi-Fourier transform

$$\Xi_F(z) = B\varepsilon \int_{-\infty}^{\infty} \varphi(x) e^{i(\Lambda z+\Theta)x} \mathrm{d}x$$

This is done by showing that the product of Gamma functions can be reduced asymptotically to one using Fourier convolution. Then the other parts of the function are introduced.

Section 9 uses the result from section 8

$$\Xi_F(z) = B\varepsilon \int_{-\infty}^{\infty} \varphi(x) e^{i(\Lambda z + M)x} \mathrm{d}x,$$

to show that

$$\lim_{n\to\infty} A_n \Xi_F^{(2n)}(C_n(z-M)) = \cos(z+\theta)$$

where $\theta = \arg(B)$, using similar methods to those used by Ki in his paper.

In section 10, the sequences A_n and C_n, the constants Λ, M and θ, as well as the error term are studied to see how quickly the zeros even out. This is extended in chapter 3, where derivatives of the Ξ-function are plotted using a variety of techniques so that they can be practically studied, rather than just discussing them in a theoretical context.

7. The Selberg Class of L-functions

In 1989, Selberg [40] gave a talk at the Amalfi Conference on Number Theory, about L-functions which satisfy a number of axioms. Beginning with a Dirichlet series, which is of the form

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad s = \sigma + it,$$

where a_n is a complex series. The first term a_1 must be 1, and for any $\varepsilon > 0$, $a_n = \mathcal{O}(n^{\varepsilon})$. This causes $F(s)$ to converge absolutely for $\mathrm{Re}(s) = \sigma > 1$.

The second axiom states that $(s-1)^m F(s)$ is an entire function, i.e. the only pole of the L-series occurs at $s = 1$ and is of order m, where $m \in \mathbb{N}_0$.

Axiom three states that there exists a functional equation

$$\Phi(s) = \gamma_F(s)F(s) = \overline{\Phi}(1-s)$$

where

$$\gamma_F(s) = \varepsilon Q^s \prod_{j=1}^{k} \Gamma(\lambda_j s + \mu_j).$$

where $|\varepsilon| = 1$, $Q, \lambda_j, \mathrm{Re}(\mu_j) \geq 0$. The degree of the L-function is

$$d = 2\sum_{j=1}^{k} \lambda_j,$$

and so far, using the Duplication formula for the Gamma functions, we can always take

$$\lambda_j = \frac{1}{2},$$

so the degree is equal to k. This functional equation still contains the pole of the L-function at $s = 1$, and so to cancel it, there must be a corresponding pole at $s = 0$ of order m of the $\gamma_F(s)$, so we can write it as

$$\gamma_F(s) = \varepsilon Q^s \prod_{l=1}^{m} \Gamma(\lambda_l s) \prod_{j=m+1}^{k} \Gamma(\lambda_j s + \mu_j).$$

The functional equation shows that $\Phi(1/2+it)$ is real, and contains just the non-trivial zeros and the poles.

The last axiom states that

$$\log(F(s)) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$$

where $b_n = 0$ unless n is a prime or prime power. This is equivalent to $F(s)$ having an Euler equation of the form

$$F(s) = \prod_{p \text{ prime}} P(p, s),$$

although this is unimportant for this work. It is suggested that the degree of the polynomial $P(p, s)$ is the degree of the L-function, although this has not yet been proven.

The functional equation created for these axioms still contains the pole at $s = 1$, and so we will work with a variant function

$$\xi_F(s) = \varepsilon Q^s \prod_{l=1}^{m} \lambda_l s(s-1)\Gamma(\lambda_l s) \prod_{j=m+1}^{k} \Gamma(\lambda_j s + \mu_j) F(s).$$

This function is analogous to the Riemann xi-function, including the factors of λ_l which correspond to the factor of $1/2$ in the Riemann xi-function, kept there for historical reasons. This satisfies the functional equation

$$\begin{aligned} \xi_F(s) &= \overline{\xi}_F(1-s) \\ &= \overline{\xi_F(1-\overline{s})}, \end{aligned}$$

from which it follows that along the Critical Line,

$$\xi_F\left(\frac{1}{2}+iz\right)=\overline{\xi_F\left(\frac{1}{2}+iz\right)},$$

which shows that the function is real when $z\in\mathbb{R}$.

The similarities to the Riemann Ξ-function can continue by setting

$$\Xi_F(z)=\xi_F\left(\frac{1}{2}+iz\right).$$

To replicate Ki's work, it is necessary to find a functional equation which shows how the function changes if z is replaced with $-z$. In order to find this, the function needs to be studied more carefully. Beginning with

$$\begin{aligned}F(\overline{s})&=\sum_{n=1}^{\infty}\frac{a_n}{n^{\overline{s}}}\\&=\overline{\sum_{n=1}^{\infty}\frac{\overline{a_n}}{n^{s}}}\end{aligned}$$

and

$$\Gamma(\lambda_j\overline{s}+\mu_j)=\overline{\Gamma(\lambda_j s+\overline{\mu_j})},$$

since $\lambda_j\in\mathbb{R}$. Therefore, the functional equation can be written as

$$\xi_{\overline{F}}(s)=\overline{\varepsilon}Q^s\prod_{l=1}^{m}\lambda_l s(s-1)\Gamma(\lambda_l s)\prod_{j=m+1}^{k}\Gamma(\lambda_j s+\overline{\mu_j})\overline{F}(s)$$

where

$$\overline{F}(s)=\sum_{n=1}^{\infty}\frac{\overline{a_n}}{n^s}.$$

This means that we then have

$$\xi_F\left(\frac{1}{2}-iz\right)=\xi_{\overline{F}}\left(\frac{1}{2}+iz\right)$$

and therefore

$$\Xi_F(-z)=\Xi_{\overline{F}}(z)$$

where $\overline{F}$ represents the complex conjugate of the function, i.e. where all the complex constants (μ_j,a_n,ε) have been replaced by their complex conjugates. This means that calculating the derivative of this function in section 9 will be much quicker, as only half the work needs doing.

The individual μ_j terms in the functional equation are complex. It was once thought [26] that these terms were either real or came in complex conjugate pairs, however most degree 3 L-functions on the LMFDB website [32] do not satisfy this. It is more likely that

$$\sum_{j=1}^{k} \mu_j$$

is real. This is constant under operations such as the multiplication formula, and currently all known L-functions satisfy this condition.

In this section, it has been shown that the Ξ_F-function is an entire function which is real when $z \in \mathbb{R}$ and decays exponentially fast (due to the Gamma functions). Therefore, this function can be expressed as a Fourier transform, which makes differentiating it much more tractable. The next section is devoted to showing the Fourier transform.

8. Proving the integral representation of the Xi function

In this section, the focus will be on showing that:

Theorem 2. *The Ξ_F-function corresponding to F, an element of the Selberg Class of L-functions, can be written as an integral in a similar way to that of the Riemann Xi-function*

$$\Xi_F(z) = B \int_{-\infty}^{\infty} \varphi(t) e^{it\Lambda z} \mathrm{d}t,$$

where

$$\varphi(t) = e^{-ae^t} e^{b't}(1 + \mathcal{O}(e^{-t}))$$

where

$$a = \Lambda Q^{-1/\Lambda} \prod_{j=1}^{k} \lambda_j^{-\lambda_j/\Lambda}$$

$$\Lambda = \sum_{j=1}^{k} \lambda_j$$

$$b' = M + 2m + \frac{\Lambda}{2} - \frac{k-1}{2}$$

$$M = \sum_{j=m+1}^{k} \mu_j$$

Remark 2.1. It is possible for $\mathrm{Im}(M) \neq 0$, which means that Ki's work cannot be immediately applied to this result. However, that will be resolved in the next section.

8.1. **Fourier transform results.** Standard Fourier transform results state that the Fourier transform of a function $f(t)$ (assuming it exists), can be written as

$$\mathcal{F}(w) = \mathcal{F}[f(t)](w) = \int_{-\infty}^{\infty} f(t)e^{-iwt}\mathrm{d}t$$

and the inverse of this gives

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}(w)e^{iwt}\mathrm{d}w,$$

where f is continuous. The other important general result is the Convolution theorem, which states that

$$\begin{aligned}\mathcal{F}[f(x)g(x)](w) &= \mathcal{F}[f(x)](t) * \mathcal{F}[g(x)](w-t)\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}[f(x)](t)\mathcal{F}[g(x)](w-t)\mathrm{d}t.\end{aligned}$$

8.2. **Finding the Fourier transform of the Xi function.** Since the Selberg Class xi function is defined as

$$\begin{aligned}\xi_F(z) &= \varepsilon Q^s \prod_{l=1}^{m} \lambda_l s(s-1)\Gamma(\lambda_l s) \prod_{j=m+1}^{k} \Gamma(\lambda_j s + \mu_j)F(s)\\ &= \varepsilon Q^s \prod_{l=1}^{m} \frac{1}{\lambda_l}\left(\Gamma(\lambda_l s + 2) - (\lambda_l + 1)\Gamma(\lambda_l s + 1)\right) \prod_{j=m+1}^{k} \Gamma(\lambda_j s + \mu_j)F(s)\end{aligned}$$

with the variation

$$\Xi_F(z) = \xi_F\left(\frac{1}{2} + iz\right)$$

the product of Gamma functions ensures that the function decays suitably fast for its Fourier transform to exist. Rather than starting from

$$\Xi_F(z) = \int_{-\infty}^{\infty} \varphi(t)e^{\Lambda itz}\mathrm{d}t$$

we will instead begin with the Fourier transform mentioned above,

$$\Xi_F(z) = \int_{-\infty}^{\infty} \phi(t)e^{itz}\mathrm{d}t,$$

where $\phi(t)$ is defined as a function with an error term as $t \to \infty$, and then later modify the integral to create a suitable $\varphi(t)$. It is important to note that these two functions— ϕ and φ are slightly, albeit importantly, different from each other.

This error term is acceptable due to how the integral is calculated. Therefore, it is possible to write

$$\phi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Xi_F(z) e^{-itz} \mathrm{d}z.$$

This becomes

$$\begin{aligned}\phi(t) = \frac{\varepsilon}{2\pi}\int_{-\infty}^{\infty} Q^{1/2+iz} \prod_{l=1}^{m} \frac{1}{\lambda_l}\Bigg(\Gamma(i\lambda_l z + 2 + \frac{\lambda_l}{2}) \\ - (\lambda_l + 1)\Gamma(i\lambda_l z + 1 + \frac{\lambda_l}{2})\Bigg) \\ \prod_{j=m+1}^{k} \Gamma(i\lambda_j z + \mu_j + \frac{\lambda_j}{2}) \sum_n \frac{a(n)}{n^{1/2+iz}} e^{-itz} \mathrm{d}z.\end{aligned}$$

Shifting the integration contour to a region where the sum is convergent, we can then change the order of integration and summation and shift back to give

$$\begin{aligned}\phi(t) = \sum_n \frac{\varepsilon Q^{1/2} a(n)}{2\pi n^{1/2}} \int_{-\infty}^{\infty} \prod_{l=1}^{m} \frac{1}{\lambda_l}\Bigg(\Gamma(i\lambda_l z + 2 + \frac{\lambda_l}{2}) \\ - (\lambda_l + 1)\Gamma(i\lambda_l z + 1 + \frac{\lambda_l}{2})\Bigg) \\ \prod_{j=m+1}^{k} \Gamma(i\lambda_j z + \mu_j + \frac{\lambda_j}{2}) \left(\frac{e^t n}{Q}\right)^{-iz} \mathrm{d}z.\end{aligned}$$

Therefore, defining $T = t + \log(n/Q)$, we are interested in

$$\begin{aligned}\phi(T) = \sum_n \frac{\varepsilon Q^{1/2} a(n)}{2\pi n^{1/2}} \int_{-\infty}^{\infty} \prod_{l=1}^{m} \frac{1}{\lambda_l}\Bigg(\Gamma(i\lambda_l z + 2 + \frac{\lambda_l}{2}) \\ - (\lambda_l + 1)\Gamma(i\lambda_l z + 1 + \frac{\lambda_l}{2})\Bigg) \\ \prod_{j=m+1}^{k} \Gamma(i\lambda_j z + \mu_j + \frac{\lambda_j}{2}) e^{-iTz} \mathrm{d}z\end{aligned}$$

for large T.

In the rest of this section, we will use the integral representation of the Gamma functions to calculate the quasi-Fourier convolution of them. This reduces the number of integrals, eventually to one integral which is how we get the following result.

Theorem 3. *Let*

$$\phi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Xi_F(z)e^{-itz}\mathrm{d}z$$

as previously defined. By calculating the Fourier convolution of the Gamma functions, it is possible to show that

$$\phi(t) = \varepsilon B e^{\frac{t(M+2m+\Lambda/2-(k-1)/2)}{\Lambda}} \exp\left[-\Lambda Q^{-1/\Lambda}\prod_{j=1}^{k}\lambda_j^{-\lambda_j/\Lambda}e^{t/\Lambda}\right]\left(1+\mathcal{O}\left(e^{-t/\Lambda}\right)\right),$$

where

$$\Lambda = \sum_{j=1}^{k}\lambda_j$$

and

$$M = \sum_{j=m+1}^{k}\mu_j$$

and the other variables are as previously defined. We also have that $B \in \mathbb{R}$.

This result is first proven in the instance where $k = 2$. Then after this, a third Gamma function is introduced, and it is proven that the convolution of a third Gamma function with the original two Gamma functions looks the same (up to suitable constants) as the convolution of two Gamma functions, thereby generalising it to any number of Gamma functions.

8.3. **The Fourier transform of one Gamma function.** In order to use the convolution theorem, we first need to prove that

Lemma 4.

$$\mathcal{F}[\Gamma(i\lambda_1 z + A)](T) = \frac{2\pi}{\lambda_1}e^{-e^{T/\lambda_1}}e^{\frac{AT}{\lambda_1}}.$$

Proof. Recall that

$$\Gamma(z) = \int_0^{\infty} e^{-x}x^{z-1}\mathrm{d}x$$

so that

$$\begin{aligned}\Gamma(i\lambda_j z + A) &= \int_0^\infty e^{-x} x^{i\lambda_j z + A - 1} \mathrm{d}x \\ &= \int_{-\infty}^\infty e^{-e^u} e^{u(i\lambda_j z + A)} \mathrm{d}u \\ &= \frac{1}{\lambda_j} \int_{-\infty}^\infty e^{-e^{T/\lambda_j}} e^{AT/\lambda_j} e^{iTz} \mathrm{d}T.\end{aligned}$$

Since this is the inverse Fourier transform (positive complex exponential term), we need to have a fraction of $1/2\pi$ out the front of the integral, to give

$$\Gamma(i\lambda_j z + A) = \frac{2\pi}{\lambda_j} \frac{1}{2\pi} \int_{-\infty}^\infty e^{-e^{T/\lambda_j}} e^{AT/\lambda_j} e^{iTz} \mathrm{d}T$$

as required. □

8.4. **The Fourier transform of two Gamma functions.**

Theorem 5. *The Fourier convolution of two Gamma functions can be written as*

$$\begin{aligned}&\mathcal{F}[\Gamma(i\lambda_1 z + A)\Gamma(i\lambda_2 z + B)](T) \\ &\qquad = C_2 \exp\left[-e^{\frac{T}{\Lambda}} \Lambda \lambda_1^{\frac{-\lambda_1}{\Lambda}} \lambda_2^{\frac{-\lambda_2}{\Lambda}}\right] \\ &\qquad\qquad e^{\frac{T}{\Lambda}(A+B-1/2)} \left(1 + \mathcal{O}\left(e^{\frac{-T}{\Lambda}}\right)\right)\end{aligned}$$

where

$$\Lambda = \lambda_1 + \lambda_2$$

and

$$C_2 = (2\pi)^{3/2} \lambda_1^{\frac{\lambda_2(A-1/2) - B\lambda_1}{\Lambda}} \lambda_2^{\frac{\lambda_1(B-1/2) - A\lambda_2}{\Lambda}} \Lambda^{-1/2}.$$

Proof. Recalling that

$$\begin{aligned}\mathcal{F}[f(x)g(x)](T) &= \Big(\mathcal{F}[f(x)] * \mathcal{F}[g(x)]\Big)(T) \\ &= \frac{1}{2\pi} \int_{-\infty}^\infty \mathcal{F}[f(x)](w) \mathcal{F}[g(x)](T - w) \mathrm{d}w\end{aligned}$$

so that

$$(*)\quad \mathcal{F}[\Gamma(i\lambda_1 z + A)\Gamma(i\lambda_2 z + B)](T) =$$

$$= \frac{2\pi}{\lambda_1\lambda_2}\int_{-\infty}^{\infty} e^{-(e^{v/\lambda_1}+e^{(T-v)/\lambda_2})} e^{Av/\lambda_1} e^{B(T-v)/\lambda_2} \mathrm{d}v$$

$$= 2\pi e^{\frac{BT}{\lambda_2}} \int_{-\infty}^{\infty} e^{-(e^{\lambda_2 w}+e^{T/\lambda_2-\lambda_1 w})} e^{w(A\lambda_2 - B\lambda_1)} \mathrm{d}w.$$

For large values of T, the maximum of the integrand occurs very near the point at which

$$\frac{\mathrm{d}}{\mathrm{d}w}\left(e^{\lambda_2 w} + e^{T/\lambda_2 - \lambda_1 w}\right) = 0.$$

This occurs at

$$e^{w(\lambda_1+\lambda_2)} = \frac{\lambda_1}{\lambda_2} e^{T/\lambda_2}$$

$$e^{w} = \left(\frac{\lambda_1}{\lambda_2} e^{T/\lambda_2}\right)^{\frac{1}{\lambda_1+\lambda_2}}$$

Therefore, the substitution

$$e^{w} = (1+v)\left(\frac{\lambda_1}{\lambda_2} e^{T/\lambda_2}\right)^{\frac{1}{\lambda_1+\lambda_2}}$$

will put the maximum of the integrand at the origin of the integral, to give

$$(*)\quad = 2\pi e^{\frac{BT}{\lambda_2}} \int_{-1}^{\infty} \exp\left[- e^{\frac{T}{\lambda_1+\lambda_2}} \left(\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{\lambda_2}{\lambda_1+\lambda_2}} (1+v)^{\lambda_2} + \left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{\lambda_1}{\lambda_1+\lambda_2}} (1+v)^{-\lambda_1} \right)\right] \left(\frac{\lambda_1}{\lambda_2} e^{\frac{T}{\lambda_2}}\right)^{\frac{A\lambda_2}{\lambda_1+\lambda_2}} (1+v)^{A\lambda_2} \left(\frac{\lambda_1}{\lambda_2} e^{\frac{T}{\lambda_2}}\right)^{\frac{-B\lambda_1}{\lambda_1+\lambda_2}} (1+v)^{-B\lambda_1} \frac{\mathrm{d}v}{1+v}$$

$$= 2\pi e^{T\left(\frac{B}{\lambda_2} + \frac{A}{\lambda_1+\lambda_2} - \frac{B\lambda_1}{\lambda_2(\lambda_1+\lambda_2)}\right)} \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{A\lambda_2 - B\lambda_1}{\lambda_1+\lambda_2}} \int_{-1}^{\infty} \exp\left[- e^{\frac{T}{\lambda_1+\lambda_2}} \left(\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{\lambda_2}{\lambda_1+\lambda_2}} (1+v)^{\lambda_2} + \left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{\lambda_1}{\lambda_1+\lambda_2}} (1+v)^{-\lambda_1} \right)\right] (1+v)^{A\lambda_2 - B\lambda_1 - 1} \mathrm{d}v$$

$$= 2\pi e^{T\left(\frac{A}{\lambda_1+\lambda_2}+\frac{B}{\lambda_1+\lambda_2}\right)} \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{A\lambda_2-B\lambda_1}{\lambda_1+\lambda_2}}$$

$$\int_{-1}^{\infty} \exp\left[- e^{\frac{T}{\lambda_1+\lambda_2}} \left(\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{\lambda_2}{\lambda_1+\lambda_2}} (1+v)^{\lambda_2} + \left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{\lambda_1}{\lambda_1+\lambda_2}} (1+v)^{-\lambda_1} \right)\right] (1+v)^{A\lambda_2-B\lambda_1-1} \mathrm{d}v$$

Setting $\lambda_1+\lambda_2=\Lambda$, we want the main integral (around the origin) to have integration limits of $\pm\exp[-T/3\Lambda]$, in order to allow some approximations of the integrand. Therefore, we have

$$\text{(M)} \quad I_M = \int_{-e^{-T/3\Lambda}}^{e^{-T/3\Lambda}} \exp\left[- e^{\frac{T}{\Lambda}} \left(\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{\lambda_2}{\Lambda}} (1+v)^{\lambda_2} + \left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{\lambda_1}{\Lambda}} (1+v)^{-\lambda_1} \right)\right] (1+v)^{A\lambda_2-B\lambda_1-1} \mathrm{d}v.$$

and

$$\text{(E)} \quad I_E = \int_{-1}^{-e^{-T/3\Lambda}} + \int_{e^{-T/3\Lambda}}^{\infty} \exp\left[- e^{\frac{T}{\Lambda}} \left(\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{\lambda_2}{\Lambda}} (1+v)^{\lambda_2} + \left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{\lambda_1}{\Lambda}} (1+v)^{-\lambda_1} \right)\right] (1+v)^{A\lambda_2-B\lambda_1-1} \mathrm{d}v.$$

Dealing with the main integral (M) first, we have that

Lemma 6.

$$I_M = \exp\left[-e^{\frac{T}{\Lambda}}\Lambda\lambda_1^{\frac{-\lambda_1}{\Lambda}}\lambda_2^{\frac{-\lambda_2}{\Lambda}}\right]\sqrt{2\pi}\Lambda^{-1/2}\lambda_1^{\frac{-\lambda_2}{2\Lambda}}\lambda_2^{\frac{-\lambda_1}{2\Lambda}}e^{\frac{-T}{2\Lambda}}(1+\mathcal{O}(e^{\frac{-T}{\Lambda}})).$$

Proof. Beginning with Taylor series, we have that

$$(1+v)^{\lambda_2} = 1+\lambda_2 v+\lambda_2(\lambda_2-1)\frac{v^2}{2}+\lambda_2(\lambda_2-1)(\lambda_2-2)\frac{v^3}{6}\mathcal{O}(v^4)$$

and

$$(1+v)^{-\lambda_1} = 1-\lambda_1 v+\lambda_1(\lambda_1+1)\frac{v^2}{2}-\lambda_1(\lambda_1+1)(\lambda_1+2)\frac{v^3}{6}+\mathcal{O}(v^4)$$

so that the exponential part of the integrand in (M)

$$\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{\lambda_2}{\Lambda}}(1+v)^{\lambda_2}+\left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{\lambda_1}{\Lambda}}(1+v)^{-\lambda_1}$$

can be written as

$$\Bigg[\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{\lambda_2}{\Lambda}}+\left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{\lambda_1}{\Lambda}}+\frac{v^2}{2}\lambda_1^{\frac{\lambda_2}{\Lambda}}\lambda_2^{\frac{\lambda_1}{\Lambda}}\Lambda$$

$$+\frac{v^3}{6}\lambda_1^{\frac{\lambda_2}{\Lambda}}\lambda_2^{\frac{\lambda_1}{\Lambda}}\left[(\lambda_2-1)(\lambda_2-2)-(\lambda_1+1)(\lambda_1+2)\right]+\mathcal{O}(v^4)\Bigg]$$

$$=\left[\Lambda\lambda_1^{\frac{-\lambda_1}{\Lambda}}\lambda_2^{\frac{-\lambda_2}{\Lambda}}+\frac{v^2}{2}\lambda_1^{\frac{\lambda_2}{\Lambda}}\lambda_2^{\frac{\lambda_1}{\Lambda}}\Lambda+\alpha v^3+\mathcal{O}(v^4)\right].$$

The rest of this integrand can be easily expanded to give

$$(1+v)^{A\lambda_2-B\lambda_1-1}=1+(A\lambda_2-B\lambda_1-1)v+\mathcal{O}(v^2).$$

Therefore, the integral (M) can be written as

$$I_M=\exp\left[-e^{\frac{T}{\Lambda}}\Lambda\lambda_1^{\frac{-\lambda_1}{\Lambda}}\lambda_2^{\frac{-\lambda_2}{\Lambda}}\right]$$

$$\int_{-e^{-T/3\Lambda}}^{e^{-T/3\Lambda}}\exp\left[-e^{\frac{T}{\Lambda}}\left(\frac{v^2}{2}\lambda_1^{\frac{\lambda_2}{\Lambda}}\lambda_2^{\frac{\lambda_1}{\Lambda}}\Lambda+\alpha v^3+\mathcal{O}(v^4)\right)\right]$$

$$\left(1+Av+\mathcal{O}(v^2)\right)\mathrm{d}v$$

$$=\exp\left[-e^{\frac{T}{\Lambda}}\Lambda\lambda_1^{\frac{-\lambda_1}{\Lambda}}\lambda_2^{\frac{-\lambda_2}{\Lambda}}\right]$$

$$\int_{-e^{-T/3\Lambda}}^{e^{-T/3\Lambda}}\exp\left[-\frac{1}{2}\lambda_1^{\frac{\lambda_2}{\Lambda}}\lambda_2^{\frac{\lambda_1}{\Lambda}}\Lambda e^{\frac{T}{\Lambda}}v^2\right]$$

$$(1+e^{\frac{T}{\Lambda}}\alpha v^3+\mathcal{O}(e^{\frac{T}{\Lambda}}v^4)+\mathcal{O}(e^{\frac{2T}{\Lambda}}v^6))(1+Av+\mathcal{O}(v^2))\mathrm{d}v,$$

where the approximation

$$e^{-e^{T/\Lambda}(\alpha v^3+\mathcal{O}(v^4))}=1+e^{\frac{T}{\Lambda}}\alpha v^3+\mathcal{O}(e^{\frac{T}{\Lambda}}v^4)+\mathcal{O}(e^{\frac{2T}{\Lambda}}v^6)$$

is valid within this range of integration. We can now increase the range of integration to whole real axis which introduces an error term, to give

$$= \exp\left[-e^{\frac{T}{\Lambda}}\Lambda\lambda_1^{\frac{-\lambda_1}{\Lambda}}\lambda_2^{\frac{-\lambda_2}{\Lambda}}\right]$$

$$\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\lambda_1^{\frac{\lambda_2}{\Lambda}}\lambda_2^{\frac{\lambda_1}{\Lambda}}\Lambda e^{\frac{T}{\Lambda}}v^2\right]$$

$$(1+e^{\frac{T}{\Lambda}}\alpha v^3 + \mathcal{O}(e^{\frac{T}{\Lambda}}v^4))(1+Av+\mathcal{O}(v^2))\mathrm{d}v$$

$$+\mathcal{O}(e^{-\frac{1}{2}\lambda_1^{\frac{\lambda_2}{\Lambda}}\lambda_2^{\frac{\lambda_1}{\Lambda}}\Lambda e^{T/3\Lambda}}).$$

Due to the range of integration for this function, any integral which includes odd powers of v will vanish. Therefore, the main error terms from this integral will be those from the v^2 and $e^{T/\Lambda}v^4$ terms. The main integral is

$$\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\lambda_1^{\frac{\lambda_2}{\Lambda}}\lambda_2^{\frac{\lambda_1}{\Lambda}}\Lambda e^{\frac{T}{\Lambda}}v^2\right]\mathrm{d}v = \sqrt{2\pi}\Lambda^{-1/2}\lambda_1^{\frac{-\lambda_2}{2\Lambda}}\lambda_2^{\frac{-\lambda_1}{2\Lambda}}e^{\frac{-T}{2\Lambda}},$$

and the three main error terms have the same order of magnitude since

$$\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\lambda_1^{\frac{\lambda_2}{\Lambda}}\lambda_2^{\frac{\lambda_1}{\Lambda}}\Lambda e^{\frac{T}{\Lambda}}v^2\right]v^2\mathrm{d}v = \mathcal{O}(e^{\frac{-3T}{2\Lambda}})$$

is the same as

$$\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\lambda_1^{\frac{\lambda_2}{\Lambda}}\lambda_2^{\frac{\lambda_1}{\Lambda}}\Lambda e^{\frac{T}{\Lambda}}v^2\right]e^{\frac{T}{\Lambda}}v^4\mathrm{d}v = \mathcal{O}(e^{\frac{-3T}{2\Lambda}})$$

and also

$$\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\lambda_1^{\frac{\lambda_2}{\Lambda}}\lambda_2^{\frac{\lambda_1}{\Lambda}}\Lambda e^{\frac{T}{\Lambda}}v^2\right]e^{\frac{2T}{\Lambda}}v^6\mathrm{d}v = \mathcal{O}(e^{\frac{-3T}{2\Lambda}}).$$

Therefore, for large T,

$$I_M = \exp\left[-e^{\frac{T}{\Lambda}}\Lambda\lambda_1^{\frac{-\lambda_1}{\Lambda}}\lambda_2^{\frac{-\lambda_2}{\Lambda}}\right]\sqrt{2\pi}\Lambda^{-1/2}\lambda_1^{\frac{-\lambda_2}{2\Lambda}}\lambda_2^{\frac{-\lambda_1}{2\Lambda}}e^{\frac{-T}{2\Lambda}}(1+\mathcal{O}(e^{\frac{-T}{\Lambda}}))$$

as required. □

We now need to calculate the error integrals (E), which are both smaller than the error term from the main integral.

Lemma 7. *We have that the two integrals in this error term are both or roughly the same size, that is*

$$I_E = \mathcal{O}\left(e^{\frac{-T}{\Lambda}}e^{-e^{T/\Lambda}}\left((1+e^{\frac{-T}{3\Lambda}})^{1-\lambda_2}+(1-e^{\frac{-T}{3\Lambda}})^{\lambda_1-1}\right)\right)$$

Proof. The first of these two error integrals needs to be rearranged to be in the same form as the second. Then they can be evaluated using the same techniques. Beginning with

$$\int_{-1}^{-e^{-T/3\Lambda}} \exp\left[- e^{\frac{T}{\Lambda}}\left(\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{\lambda_2}{\Lambda}} (1+v)^{\lambda_2} + \left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{\lambda_1}{\Lambda}} (1+v)^{-\lambda_1} \right)\right] (1+v)^{A\lambda_2 - B\lambda_1 - 1} \mathrm{d}v$$

$$= \int_{0}^{1-e^{-T/3\Lambda}} \exp\left[- e^{\frac{T}{\Lambda}}\left(\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{\lambda_2}{\Lambda}} v^{\lambda_2} + \left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{\lambda_1}{\Lambda}} v^{-\lambda_1} \right)\right] v^{A\lambda_2 - B\lambda_1 - 1} \mathrm{d}v$$

$$= \int_{(1-e^{-T/3\Lambda})^{-1}}^{\infty} \exp\left[- e^{\frac{T}{\Lambda}}\left(\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{\lambda_2}{\Lambda}} v^{-\lambda_2} + \left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{\lambda_1}{\Lambda}} v^{\lambda_1} \right)\right] v^{B\lambda_1 - A\lambda_2 - 1} \mathrm{d}v.$$

We can now approximate this integrand by

$$\approx \int_{(1-e^{-T/3\Lambda})^{-1}}^{\infty} \exp\left[- e^{\frac{T}{\Lambda}} \left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{\lambda_1}{\Lambda}} v^{\lambda_1} \right] \mathrm{d}v$$

since we must always have that $\lambda_1, \lambda_2 > 0$. This integral can then be approximated to give

$$\mathcal{O}\left(e^{\frac{-T}{\Lambda}} (1 - e^{\frac{-T}{3\Lambda}})^{\lambda_1 - 1} e^{-e^{T/\Lambda}} \right).$$

The second error integral, that of

$$\int_{e^{-T/3\Lambda}}^{\infty} \exp\left[- e^{\frac{T}{\Lambda}}\left(\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{\lambda_2}{\Lambda}} (1+v)^{\lambda_2} + \left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{\lambda_1}{\Lambda}} (1+v)^{-\lambda_1} \right)\right] (1+v)^{A\lambda_2 - B\lambda_1 - 1} \mathrm{d}v$$

can be approximated in the same way to give

$$= \int_{1+e^{-T/3\Lambda}}^{\infty} \exp\left[- e^{\frac{T}{\Lambda}}\left(\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{\lambda_2}{\Lambda}} v^{\lambda_2} + \left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{\lambda_1}{\Lambda}} v^{-\lambda_1} \right)\right] v^{A\lambda_2 - B\lambda_1 - 1} \mathrm{d}v$$

$$\approx \int_{1+e^{-T/3\Lambda}}^{\infty} \exp\left[-e^{\frac{T}{\Lambda}}\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{\lambda_2}{\Lambda}} v^{\lambda_2}\right] \mathrm{d}v$$
$$= \mathcal{O}\left(e^{\frac{-T}{\Lambda}}(1+e^{\frac{-T}{3\Lambda}})^{1-\lambda_2} e^{-e^{T/\Lambda}}\right)$$

□

The error integrals (E) are smaller than the error term from (M), and therefore

$$I_M + I_E = \exp\left[-e^{\frac{T}{\Lambda}}\Lambda\lambda_1^{\frac{-\lambda_1}{\Lambda}}\lambda_2^{\frac{-\lambda_2}{\Lambda}}\right]\sqrt{2\pi}\Lambda^{-1/2}\lambda_1^{\frac{-\lambda_2}{2\Lambda}}\lambda_2^{\frac{-\lambda_1}{2\Lambda}} e^{\frac{-T}{2\Lambda}}(1+\mathcal{O}(e^{\frac{-T}{\Lambda}})).$$

Recalling that

$$\mathcal{F}[\Gamma(i\lambda_1 z + A)\Gamma(i\lambda_2 z + B)](T) =$$
$$= 2\pi e^{\frac{T(A+B)}{\Lambda}}\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{A\lambda_2 - B\lambda_1}{\Lambda}}(I_M + I_E)$$

$$= (2\pi)^{3/2}\Lambda^{-1/2}\lambda_1^{\frac{(A-1/2)\lambda_2 - B\lambda_1}{\Lambda}}\lambda_2\frac{(B-1/2)\lambda_1 - A\lambda_2}{\Lambda}$$
$$e^{\frac{T(A+B-1/2)}{\Lambda}}\exp\left[-e^{\frac{T}{\Lambda}}\Lambda\lambda_1^{\frac{-\lambda_1}{\Lambda}}\lambda_2^{\frac{-\lambda_2}{\Lambda}}\right](1+\mathcal{O}(e^{\frac{-T}{\Lambda}}))$$
$$= C_2\exp\left[-e^{\frac{T}{\Lambda}}\Lambda\lambda_1^{\frac{-\lambda_1}{\Lambda}}\lambda_2^{\frac{-\lambda_2}{\Lambda}}\right]e^{\frac{T}{\Lambda}(A+B-1/2)}\left(1+\mathcal{O}\left(e^{\frac{-T}{\Lambda}}\right)\right)$$

as required.

□

8.5. **Three Gamma functions.** The situation with three Gamma functions is what extends this result to any number of Gamma functions. By showing that the Fourier transform of three Gamma functions has the same important properties as the Fourier transform of two Gamma functions, we can extrapolate that the Fourier transform of any number of Gamma functions looks the same.

Theorem 8. *The Fourier transform of three Gamma functions can be written as*

$$F[\Gamma(i\lambda_1 z + A)\Gamma(i\lambda_2 z + B)\Gamma(i\lambda_3 z + C)](T)$$
$$= C_3 e^{\frac{T(A+B+C-1)}{\Lambda}}$$
$$\exp\left[-\Lambda\lambda_1^{\frac{-\lambda_1}{\Lambda}}\lambda_2^{\frac{-\lambda_2}{\Lambda}}\lambda_3^{\frac{-\lambda_3}{\Lambda}}e^{\frac{T}{\Lambda}}\right]$$

where

$$\Lambda = \lambda_1 + \lambda_2 + \lambda_3.$$

Proof. This can be treated as the Fourier transform of a product, and so

$$F[\Gamma(i\lambda_1 z + A)\Gamma(i\lambda_2 z + B)\Gamma(i\lambda_3 z + C)](T) = \\ = \Big(F[\Gamma(i\lambda_1 z + A)\Gamma(i\lambda_2 z + B)] * F[\Gamma(i\lambda_3 z + C)]\Big)(T)$$

We have already shown that

$$F[\Gamma(i\lambda_1 z + A)\Gamma(i\lambda_2 z + B)](T) \\ = C_2 \exp\left[-e^{\frac{T}{\lambda_1+\lambda_2}}(\lambda_1+\lambda_2)\lambda_1^{\frac{-\lambda_1}{\lambda_1+\lambda_2}}\lambda_2^{\frac{-\lambda_2}{\lambda_1+\lambda_2}}\right] \\ e^{\frac{T}{\lambda_1+\lambda_2}(A+B-1/2)}\left(1+\mathcal{O}\left(e^{\frac{-T}{\lambda_1+\lambda_2}}\right)\right)$$

so we need to calculate the integral

$$\frac{C_2 2\pi}{\lambda_3}\int_{-\infty}^{\infty}\exp\left[-e^{\frac{w}{\lambda_3}} - e^{\frac{T-w}{\lambda_1+\lambda_2}}(\lambda_1+\lambda_2)\lambda_1^{\frac{-\lambda_1}{\lambda_1+\lambda_2}}\lambda_2^{\frac{-\lambda_2}{\lambda_1+\lambda_2}}\right] \\ e^{\frac{Cw}{\lambda_3}+\frac{T-w}{\lambda_1+\lambda_2}(A+B-1/2)}\left(1+\mathcal{O}(e^{-\frac{T-w}{\lambda_1+\lambda_2}})\right)\mathrm{d}w.$$

Making the change of variable

$$w \to \lambda_3(\lambda_1+\lambda_2)w$$

gives

$$= C_2 2\pi(\lambda_1+\lambda_2)e^{\frac{T}{\lambda_1+\lambda_2}(A+B-1/2)} \\ \int_{-\infty}^{\infty}\exp\left[-e^{(\lambda_1+\lambda_2)w} - e^{\frac{T}{\lambda_1+\lambda_2}}e^{-\lambda_3 w}(\lambda_1+\lambda_2)\lambda_1^{\frac{-\lambda_1}{\lambda_1+\lambda_2}}\lambda_2^{\frac{-\lambda_2}{\lambda_1+\lambda_2}}\right] \\ e^{w(C(\lambda_1+\lambda_2)-\lambda_3(A+B-1/2))}\left(1+\mathcal{O}\left(e^{\frac{-T}{(\lambda_1+\lambda_2})+\lambda_3 w}\right)\right)\mathrm{d}w.$$

This is calculated in the same way as the previous Fourier convolution, by finding the change of variable which puts the maximum at the origin, and then splitting the integral up into the main and error integrals.

Setting

$$D = (\lambda_1+\lambda_2)\lambda_1^{\frac{-\lambda_1}{\lambda_1+\lambda_2}}\lambda_2^{\frac{-\lambda_2}{\lambda_1+\lambda_2}}$$

in order to make the calculation easier to follow, we can calculate the maximum of the integrand. This occurs at

$$\frac{\mathrm{d}}{\mathrm{d}w} e^{(\lambda_1+\lambda_2)w} + De^{\frac{T}{\lambda_1+\lambda_2}} e^{-\lambda_3 w} = 0$$

$$e^{(\Lambda)w} = e^{\frac{T}{\lambda_1+\lambda_2}} \frac{\lambda_3 D}{\lambda_1+\lambda_2}.$$

Reinserting the formula for D, we can rewrite this as

$$e^{(\Lambda)w} = \left(\frac{\lambda_3}{\lambda_1}\right)^{\frac{\lambda_1}{\lambda_1+\lambda_2}} \left(\frac{\lambda_3}{\lambda_2}\right)^{\frac{\lambda_2}{\lambda_1+\lambda_2}} e^{\frac{T}{\lambda_1+\lambda_2}}$$

so the maximum of the integrand occurs at

$$e^w = \left(\left(\frac{\lambda_3}{\lambda_1}\right)^{\frac{\lambda_1}{\lambda_1+\lambda_2}} \left(\frac{\lambda_3}{\lambda_2}\right)^{\frac{\lambda_2}{\lambda_1+\lambda_2}} e^{\frac{T}{\lambda_1+\lambda_2}} \right)^{\frac{1}{\Lambda}}$$

and the most suitable change of variable is therefore

$$e^w = (1+v) \left(\left(\frac{\lambda_3}{\lambda_1}\right)^{\frac{\lambda_1}{\lambda_1+\lambda_2}} \left(\frac{\lambda_3}{\lambda_2}\right)^{\frac{\lambda_2}{\lambda_1+\lambda_2}} e^{\frac{T}{\lambda_1+\lambda_2}} \right)^{\frac{1}{\Lambda}}.$$

The integrand then becomes

$$e^{(\lambda_1+\lambda_2)w} = (1+v)^{\lambda_1+\lambda_2} \lambda_3 \lambda_1^{\frac{-\lambda_1}{\Lambda}} \lambda_2^{\frac{-\lambda_2}{\Lambda}} \lambda_3^{\frac{-\lambda_3}{\Lambda}} e^{\frac{T}{\Lambda}}$$

and

$$\begin{aligned} &e^{\frac{T}{\lambda_1+\lambda_2}} e^{-\lambda_3 w} (\lambda_2+\lambda_2) \lambda_1^{\frac{-\lambda_1}{\lambda_1+\lambda_2}} \lambda_2^{\frac{-\lambda_2}{\lambda_1+\lambda_2}} \\ &\qquad = (1+v)^{-\lambda_3} e^{\frac{T}{\Lambda}} (\lambda_1+\lambda_2) \lambda_1^{\frac{-\lambda_1}{\Lambda}} \lambda_2^{\frac{-\lambda_2}{\Lambda}} \lambda_3^{\frac{-\lambda_3}{\Lambda}}. \end{aligned}$$

Summing these together, we have that the top line of the integral is

$$\begin{aligned} &e^{(\lambda_1+\lambda_2)z} + e^{\frac{T}{\lambda_1+\lambda_2}} De^{-\lambda_3 z} \\ &\qquad = e^{\frac{T}{\Lambda}} \lambda_1^{\frac{-\lambda_1}{\Lambda}} \lambda_2^{\frac{-\lambda_2}{\Lambda}} \lambda_3^{\frac{-\lambda_3}{\Lambda}} \\ &\qquad\qquad \left(\lambda_3 (1+v)^{\lambda_1+\lambda_2} + (\lambda_1+\lambda_2)(1+v)^{-\lambda_3} \right). \end{aligned}$$

The other part of the integrand is

$$e^{w(C(\lambda_1+\lambda_2)-\lambda_3(A+B-1/2))} \left(1 + \mathcal{O}\left(e^{\frac{-T}{(\lambda_1+\lambda_2)}) + \lambda_3 w}\right)\right) \mathrm{d}w,$$

and the same change of variables makes this

$$C'e^{\frac{T(C(\lambda_1+\lambda_2)-\lambda_3(A+B-1/2))}{(\lambda_1+\lambda_2)(\Lambda)}}(1+v)^{(C(\lambda_1+\lambda_2)-\lambda_3(A+B-1/2))}\left(1+\mathcal{O}\left((1+v)^{\lambda_3}e^{\frac{-T}{\Lambda}}\right)\right)\frac{\mathrm{d}v}{1+v}.$$

Therefore, the integral can be written as

$$C_2 2\pi(\lambda_1+\lambda_2)C'e^{\frac{T(C(\lambda_1+\lambda_2)-\lambda_3(A+B-1/2))}{(\lambda_1+\lambda_2)(\Lambda)}}e^{\frac{A+B-1/2}{\lambda_1+\lambda_2}}\int_{-1}^{\infty}\exp\Big[-e^{\frac{T}{\Lambda}}\lambda_1^{\frac{-\lambda_1}{\Lambda}}\lambda_2^{\frac{-\lambda_2}{\Lambda}}\lambda_3^{\frac{-\lambda_3}{\Lambda}}\left(\lambda_3(1+v)^{\lambda_1+\lambda_2}+(\lambda_1+\lambda_2)(1+v)^{-\lambda_3}\right)\Big](1+v)^{\beta}\left(1+\mathcal{O}\left(e^{\frac{-T}{\Lambda}}(1+v)^{\lambda_3}\right)\right)\mathrm{d}v$$

$$= C_2 2\pi(\lambda_1+\lambda_2)C'e^{\frac{T}{\Lambda}(A+B+C-1/2)}\int_{-1}^{\infty}\exp\Big[-e^{\frac{T}{\Lambda}}\lambda_1^{\frac{-\lambda_1}{\Lambda}}\lambda_2^{\frac{-\lambda_2}{\Lambda}}\lambda_3^{\frac{-\lambda_3}{\Lambda}}\left(\lambda_3(1+v)^{\lambda_1+\lambda_2}+(\lambda_1+\lambda_2)(1+v)^{-\lambda_3}\right)\Big](1+v)^{\beta}\left(1+\mathcal{O}\left(e^{\frac{-T}{\Lambda}}(1+v)^{\lambda_3}\right)\right)\mathrm{d}v$$

where $\beta = C(\lambda_1+\lambda_2)-\lambda_3(A+B-1/2)-1$ and

$$C' = \lambda_3^{\frac{C(\lambda_1+\lambda_2)-\lambda_3(A+B-1/2)}{\Lambda}}\lambda_1^{\frac{-\lambda_1 C}{\Lambda}+\frac{\lambda_1\lambda_3(A+B-1/2)}{(\lambda_1+\lambda_2)(\Lambda)}}\lambda_2^{\frac{-\lambda_2 C}{\Lambda}+\frac{\lambda_2\lambda_3(A+B-1/2)}{(\lambda_1+\lambda_2)(\Lambda)}}.$$

This integral is then split up in the same way as the previous one, to give

$$\text{(M3)}\quad I_M = \int_{-e^{-T/3\Lambda}}^{e^{-T/3\Lambda}}\exp\Big[-e^{\frac{T}{\Lambda}}\lambda_1^{\frac{-\lambda_1}{\Lambda}}\lambda_2^{\frac{-\lambda_2}{\Lambda}}\lambda_3^{\frac{-\lambda_3}{\Lambda}}\left(\lambda_3(1+v)^{\lambda_1+\lambda_2}+(\lambda_1+\lambda_2)(1+v)^{-\lambda_1}\right)\Big](1+v)^{\beta}\mathrm{d}v.$$

and

$$\text{(E3)}\quad I_E = \int_{-1}^{-e^{-T/3\Lambda}} + \int_{e^{-T/3\Lambda}}^{\infty} \exp\Bigg[- e^{\frac{T}{\Lambda}} \lambda_1^{\frac{-\lambda_1}{\Lambda}} \lambda_2^{\frac{-\lambda_2}{\Lambda}} \lambda_3^{\frac{-\lambda_3}{\Lambda}} \Big(\lambda_3(1+v)^{\lambda_1+\lambda_2} + (\lambda_1+\lambda_2)(1+v)^{-\lambda_3}\Big)\Bigg](1+v)^{\beta}\mathrm{d}v.$$

Dealing with the main integral (M3) first, we have that

Lemma 9.

$$I_M = \exp\left[-e^{\frac{T}{\Lambda}}\Lambda\lambda_1^{\frac{-\lambda_1}{\Lambda}}\lambda_2^{\frac{-\lambda_2}{\Lambda}}\lambda_3^{\frac{-\lambda_3}{\Lambda}}\right]\sqrt{\frac{\pi}{a}}\lambda_1^{\frac{\lambda_1}{2\Lambda}}\lambda_2^{\frac{\lambda_2}{2\Lambda}}\lambda_3^{\frac{\lambda_3}{2\Lambda}}e^{\frac{-T}{2\Lambda}}(1+\mathcal{O}(e^{\frac{-T}{\Lambda}}))$$

where

$$a = \frac{1}{2}\left[(\lambda_1+\lambda_2)(\lambda_1+\lambda_2-1)+\lambda_3(\lambda_3+1)\right],$$

is the coefficient of the quadratic terms of the exponential part of the integrand.

Proof. Beginning with Taylor series, we have that

$$(1+v)^{\lambda_1+\lambda_2} = 1 + (\lambda_1+\lambda_2)v + (\lambda_1+\lambda_2)(\lambda_1+\lambda_2-1)\frac{v^2}{2} + (\lambda_1+\lambda_2)(\lambda_1+\lambda_2-1)(\lambda_1+\lambda_2-2)\frac{v^3}{6}\mathcal{O}(v^4)$$

and

$$(1+v)^{-\lambda_3} = 1 - \lambda_3 v + \lambda_3(\lambda_3+1)\frac{v^2}{2} - \lambda_3(\lambda_3+1)(\lambda_3+2)\frac{v^3}{6} + \mathcal{O}(v^4)$$

so that the exponential part of the integrand in (M3)

$$\lambda_3(1+v)^{\lambda_1+\lambda_2} + (\lambda_1+\lambda_2)(1+v)^{\lambda_3}$$

can be written as

$$\begin{aligned}\Lambda &+ \frac{v^2}{2}\left[(\lambda_1+\lambda_2)(\lambda_1+\lambda_2-1)+\lambda_3(\lambda_3+1)\right] \\ &+ \frac{v^3}{6}\left[(\lambda_1+\lambda_2)(\lambda_1+\lambda_2-1)(\lambda_1+\lambda_2-2) = \lambda_3(\lambda_3+1)(\lambda_3+2)\right] + \mathcal{O}(v^4) \\ &= \Lambda + av^2 + bv^3 + \mathcal{O}(v^4).\end{aligned}$$

The rest of this integrand can be easily expanded to give

$$(1+v)^{\beta} = 1 + \beta v + \mathcal{O}(v^2).$$

Therefore, the integral (M3) can be written as

$$\begin{aligned} I_M = \exp\left[-e^{\frac{T}{\Lambda}}\Lambda\lambda_1^{\frac{-\lambda_1}{\Lambda}}\lambda_2^{\frac{-\lambda_2}{\Lambda}}\lambda_3^{\frac{-\lambda_3}{\Lambda}}\right] \\ \int_{-e^{-T/3\Lambda}}^{e^{-T/3\Lambda}} \exp\left[-\lambda_1^{\frac{-\lambda_1}{\Lambda}}\lambda_2^{\frac{-\lambda_2}{\Lambda}}\lambda_3^{\frac{-\lambda_3}{\Lambda}} e^{\frac{T}{\Lambda}}\left(av^2 + bv^3 + \mathcal{O}(v^4)\right)\right] \\ \left(1 + \beta v + \mathcal{O}(v^2)\right) \mathrm{d}v \end{aligned}$$

$$\begin{aligned} = \exp\left[-e^{\frac{T}{\Lambda}}\Lambda\lambda_1^{\frac{-\lambda_1}{\Lambda}}\lambda_2^{\frac{-\lambda_2}{\Lambda}}\lambda_3^{\frac{-\lambda_3}{\Lambda}}\right] \\ \int_{-e^{-T/3\Lambda}}^{e^{-T/3\Lambda}} \exp\left[-\lambda_1^{\frac{-\lambda_1}{\Lambda}}\lambda_2^{\frac{-\lambda_2}{\Lambda}}\lambda_3^{\frac{-\lambda_3}{\Lambda}} e^{\frac{T}{\Lambda}} av^2\right] \\ (1 + e^{\frac{T}{\Lambda}} bv^3 + \mathcal{O}(e^{\frac{T}{\Lambda}} v^4) + \mathcal{O}(e^{\frac{2T}{\Lambda}} v^6))(1 + \beta v + \mathcal{O}(v^2))\mathrm{d}v, \end{aligned}$$

where the approximation

$$e^{-e^{T/\Lambda}(bv^3 + \mathcal{O}(v^4))} = 1 + e^{\frac{T}{\Lambda}} bv^3 + \mathcal{O}(e^{\frac{T}{\Lambda}} v^4) + \mathcal{O}(e^{\frac{2T}{\Lambda}} v^6)$$

is again valid within this range of integration. We can now increase the range of integration to whole real axis which introduces an error term, to give

$$\begin{aligned} = \exp\left[-e^{\frac{T}{\Lambda}}\Lambda\lambda_1^{\frac{-\lambda_1}{\Lambda}}\lambda_2^{\frac{-\lambda_2}{\Lambda}}\lambda_3^{\frac{-\lambda_3}{\Lambda}}\right] \\ \int_{-\infty}^{\infty} \exp\left[-\lambda_1^{\frac{-\lambda_1}{\Lambda}}\lambda_2^{\frac{-\lambda_2}{\Lambda}}\lambda_3^{\frac{-\lambda_3}{\Lambda}} e^{\frac{T}{\Lambda}} av^2\right] \\ (1 + e^{\frac{T}{\Lambda}} bv^3 + \mathcal{O}(e^{\frac{T}{\Lambda}} v^4))(1 + \beta v + \mathcal{O}(v^2))\mathrm{d}v \\ + \mathcal{O}(e^{-ae^{T/3\Lambda}}). \end{aligned}$$

Due to the range of integration for this function, any integral which includes odd powers of v will vanish. Therefore, the main error terms from this integral will be those from the v^2 and $e^{T/\Lambda}v^4$ terms. The main integral is

$$\int_{-\infty}^{\infty} \exp\left[-\lambda_1^{\frac{-\lambda_1}{\Lambda}}\lambda_2^{\frac{-\lambda_2}{\Lambda}}\lambda_3^{\frac{-\lambda_3}{\Lambda}} ae^{\frac{T}{\Lambda}} v^2\right] \mathrm{d}v = \sqrt{\frac{\pi}{\lambda_1^{\frac{-\lambda_1}{\Lambda}}\lambda_2^{\frac{-\lambda_2}{\Lambda}}\lambda_3^{\frac{-\lambda_3}{\Lambda}} a}}\, e^{\frac{-T}{2\Lambda}}$$

$$= \exp\left[-e^{\frac{T}{\Lambda}}\Lambda\lambda_1^{\frac{-\lambda_1}{\Lambda}}\lambda_2^{\frac{-\lambda_2}{\Lambda}}\lambda_3^{\frac{-\lambda_3}{\Lambda}}\right]\sqrt{\frac{\pi}{a}}\lambda_1^{\frac{\lambda_1}{2\Lambda}}\lambda_2^{\frac{\lambda_2}{2\Lambda}}\lambda_3^{\frac{\lambda_3}{2\Lambda}}e^{\frac{-T}{2\Lambda}}.$$

The three main error terms have the same order of magnitude since

$$\int_{-\infty}^{\infty}\exp\left[-\lambda_1^{\frac{-\lambda_1}{\Lambda}}\lambda_2^{\frac{-\lambda_2}{\Lambda}}\lambda_3^{\frac{-\lambda_3}{\Lambda}}ae^{\frac{T}{\Lambda}}v^2\right]v^2\mathrm{d}v = \mathcal{O}(e^{\frac{-3T}{2\Lambda}})$$

is the same as

$$\int_{-\infty}^{\infty}\exp\left[-\lambda_1^{\frac{-\lambda_1}{\Lambda}}\lambda_2^{\frac{-\lambda_2}{\Lambda}}\lambda_3^{\frac{-\lambda_3}{\Lambda}}ae^{\frac{T}{\Lambda}}v^2\right]e^{\frac{T}{\Lambda}}v^4\mathrm{d}v = \mathcal{O}(e^{\frac{-3T}{2\Lambda}})$$

and

$$\int_{-\infty}^{\infty}\exp\left[-\lambda_1^{\frac{-\lambda_1}{\Lambda}}\lambda_2^{\frac{-\lambda_2}{\Lambda}}\lambda_3^{\frac{-\lambda_3}{\Lambda}}ae^{\frac{T}{\Lambda}}v^2\right]e^{\frac{2T}{\Lambda}}v^6\mathrm{d}v = \mathcal{O}(e^{\frac{-3T}{2\Lambda}}).$$

Therefore we have that, for large T,

$$I_M = \exp\left[-e^{\frac{T}{\Lambda}}\Lambda\lambda_1^{\frac{-\lambda_1}{\Lambda}}\lambda_2^{\frac{-\lambda_2}{\Lambda}}\lambda_3^{\frac{-\lambda_3}{\Lambda}}\right]\sqrt{\frac{\pi}{a}}\lambda_1^{\frac{\lambda_1}{2\Lambda}}\lambda_2^{\frac{\lambda_2}{2\Lambda}}\lambda_3^{\frac{\lambda_3}{2\Lambda}}e^{\frac{-T}{2\Lambda}}(1+\mathcal{O}(e^{\frac{-T}{\Lambda}}))$$

where

$$a = \frac{1}{2}\left[(\lambda_1+\lambda_2)(\lambda_1+\lambda_2-1)+\lambda_3(\lambda_3+1)\right]$$

as required. □

We now need to calculate the error integrals (E3), which are both smaller than the error term from the main integral.

Lemma 10. *We have that the two integrals in this error term are both or roughly the same size, that is*

$$I_E = \mathcal{O}\left(e^{\frac{-T}{\Lambda}}e^{-e^{T/\Lambda}}((1-e^{\frac{-T}{3\Lambda}})^{\lambda_3-1}+(1+e^{\frac{-T}{3\Lambda}})^{1-\lambda_1-\lambda_2})\right)$$

Proof. As with the previous calculation of these error integrals, the first needs to be rearranged to be in the same form as the second. Beginning with

$$\int_{-1}^{-e^{-T/3\Lambda}}\exp\Bigg[-e^{\frac{T}{\Lambda}}\lambda_1^{\frac{-\lambda_1}{\Lambda}}\lambda_2^{\frac{-\lambda_2}{\Lambda}}\lambda_3^{\frac{-\lambda_3}{\Lambda}}\Big(\lambda_3(1+v)^{\lambda_1+\lambda_2}$$
$$+(\lambda_1+\lambda_2)(1+v)^{-\lambda_3}\Big)\Bigg](1+v)^{\beta}\mathrm{d}v$$

$$= \int_0^{1-e^{-T/3\Lambda}} \exp\left[-e^{\frac{T}{\Lambda}} \lambda_1^{\frac{-\lambda_1}{\Lambda}} \lambda_2^{\frac{-\lambda_2}{\Lambda}} \lambda_3^{\frac{-\lambda_3}{\Lambda}} \Big(\lambda_3 v^{\lambda_1+\lambda_2} + (\lambda_1+\lambda_2) v^{-\lambda_3} \Big) \right] v^{\beta} \mathrm{d}v$$

$$= \int_{(1-e^{-T/3\Lambda})^{-1}}^{\infty} \exp\left[-e^{\frac{T}{\Lambda}} \lambda_1^{\frac{-\lambda_1}{\Lambda}} \lambda_2^{\frac{-\lambda_2}{\Lambda}} \lambda_3^{\frac{-\lambda_3}{\Lambda}} \Big(\lambda_3 v^{-\lambda_1-\lambda_2} + (\lambda_1+\lambda_2) v^{\lambda_3} \Big) \right] v^{\beta-1} \mathrm{d}v.$$

We can now approximate this integrand by

$$\approx \int_{(1-e^{-T/3\Lambda})^{-1}}^{\infty} \exp\left[-e^{\frac{T}{\Lambda}} \lambda_1^{\frac{-\lambda_1}{\Lambda}} \lambda_2^{\frac{-\lambda_2}{\Lambda}} \lambda_3^{\frac{-\lambda_3}{\Lambda}} (\lambda_1+\lambda_2) v^{\lambda_3} \right] \mathrm{d}v$$

since we must always have that $\lambda_1, \lambda_2 > 0$. This integral can then be approximated to give

$$\mathcal{O}\left(e^{\frac{-T}{\Lambda}} (1 - e^{\frac{-T}{3\Lambda}})^{\lambda_3 - 1} e^{-e^{T/\Lambda}} \right).$$

The second error integral, that of

$$\int_{e^{-T/3\Lambda}}^{\infty} \exp\left[-e^{\frac{T}{\Lambda}} \lambda_1^{\frac{-\lambda_1}{\Lambda}} \lambda_2^{\frac{-\lambda_2}{\Lambda}} \lambda_3^{\frac{-\lambda_3}{\Lambda}} \Big(\lambda_3 (1+v)^{\lambda_1+\lambda_2} + (\lambda_1+\lambda_2)(1+v)^{-\lambda_3} \Big) \right] (1+v)^{\beta} \mathrm{d}v$$

can be approximated in the same way to give

$$= \int_{1+e^{-T/3\Lambda}}^{\infty} \exp\left[-e^{\frac{T}{\Lambda}} \lambda_1^{\frac{-\lambda_1}{\Lambda}} \lambda_2^{\frac{-\lambda_2}{\Lambda}} \lambda_3^{\frac{-\lambda_3}{\Lambda}} \Big(\lambda_3 v^{\lambda_1+\lambda_2} + (\lambda_1+\lambda_2) v^{-\lambda_3} \Big) \right] v^{\beta} \mathrm{d}v$$

$$\approx \int_{1+e^{-T/3\Lambda}}^{\infty} \exp\left[-e^{\frac{T}{\Lambda}} \lambda_1^{\frac{-\lambda_1}{\Lambda}} \lambda_2^{\frac{-\lambda_2}{\Lambda}} \lambda_3^{\frac{-\lambda_3}{\Lambda}} \lambda_3 v^{\lambda_1+\lambda_2} \right] \mathrm{d}v$$
$$= \mathcal{O}\left(e^{\frac{-T}{\Lambda}} (1 + e^{\frac{-T}{3\Lambda}})^{1-\lambda_1-\lambda_2} e^{-e^{T/\Lambda}} \right).$$

□

These errors are the same, and are both smaller than the main error term introduced from the main integral. Therefore, we have that

$$I_M + I_E = C'' \exp\left[-e^{\frac{T}{\Lambda}} \Lambda \lambda_1^{\frac{-\lambda_1}{\Lambda}} \lambda_2^{\frac{-\lambda_2}{\Lambda}} \lambda_3^{\frac{-\lambda_3}{\Lambda}}\right] e^{\frac{-T}{2\Lambda}} (1 + \mathcal{O}(e^{\frac{-T}{\Lambda}}))$$

where

$$\Lambda = \lambda_1 + \lambda_2 + \lambda_3.$$

Recalling that the Fourier transform of three Gamma functions can be written as

$$C_2 2\pi(\lambda_1 + \lambda_2) C' e^{\frac{T}{\Lambda}(A+B+C-1/2)} \left[I_M + I_\varepsilon + I_E\right],$$

we therefore have that

$$\begin{aligned} F[\Gamma(i\lambda_1 z + A)\Gamma(i\lambda_2 z + B)\Gamma(i\lambda_3 z + C)](T) \\ = C_2 2\pi(\lambda_1 + \lambda_2) C' C'' e^{\frac{T(A+B+C-1)}{\Lambda}} \\ \exp\left[-\Lambda \lambda_1^{\frac{-\lambda_1}{\Lambda}} \lambda_2^{\frac{-\lambda_2}{\Lambda}} \lambda_3^{\frac{-\lambda_3}{\Lambda}} e^{\frac{T}{\Lambda}}\right] \end{aligned}$$

as required.

Combining the constants out the front, we get that

$$\begin{aligned} C_2 2\pi(\lambda_1 + \lambda_2) C' C'' = \\ \frac{\sqrt{2\pi}^5}{\sqrt{\lambda_1}\sqrt{\lambda_2}} \lambda_1^{\frac{\lambda_2(A_1/2) - B\lambda_1}{\lambda_1+\lambda_2}} \lambda_2^{\frac{\lambda_1(B-1/2) - A\lambda_2}{\lambda_1\lambda_2}} \lambda_3^{\frac{C(\lambda_1+\lambda_2) - \lambda_3(A+B-1/2)}{\Lambda}} \\ \lambda_1^{\frac{-\lambda_1 C}{\Lambda} + \frac{\lambda_1\lambda_3(A+B-1/2)}{(\lambda_1+\lambda_2)\Lambda}} \lambda_2^{\frac{-\lambda_2 C}{\Lambda} + \frac{\lambda_2\lambda_3(A+B-1/2)}{(\lambda_1+\lambda_2)\Lambda}} \\ \frac{\sqrt{2\pi}}{\sqrt{\Lambda}\sqrt{\lambda_3}} \lambda_1^{\frac{\lambda_1}{2\Lambda}} \lambda_2^{\frac{\lambda_2}{2\Lambda}} \lambda_3^{\frac{\lambda_3}{2\Lambda}}. \end{aligned}$$

After rearranging, this becomes

$$\begin{aligned} \frac{(2\pi)^3}{\sqrt{\lambda_1\lambda_2\lambda_3}\sqrt{\Lambda}} \\ \lambda_1^{\frac{-\lambda_1(B+C-1)}{\Lambda} + \frac{A(\lambda_2+\lambda_3)}{\Lambda}} \lambda_2^{\frac{-\lambda_2(A+C-1)}{\Lambda} + \frac{B(\lambda_1+\lambda_3)}{\Lambda}} \lambda_3^{\frac{-\lambda_3(A+B-1)}{\Lambda} + \frac{C(\lambda_1+\lambda_2)}{\Lambda}}. \end{aligned}$$

□

It can also be clearly seen how this generalises to more Gamma functions. Setting

$$\Lambda = \sum_j \lambda_j$$

and

$$M = \sum_j \mu_j$$

we therefore have that

$$\mathcal{F}\left[\prod_{j=1}^{k} \Gamma\left(i\lambda_j z + \mu_j + \frac{\lambda_j}{2}\right)\right](T) = C_k e^{\frac{T(M+\Lambda/2-(k-1)/2))}{\Lambda}}$$

$$\exp\left[-\Lambda \prod_{j=1}^{k} \lambda_j^{-\lambda_j/\Lambda} e^{T/\Lambda}\right]\left(1 + \mathcal{O}\left(e^{-T/\Lambda}\right)\right).$$

where

$$C_k = \frac{(2\pi)^{3(k-1)/2}}{\sqrt{\Lambda}} \prod_{j=1}^{k} \lambda_j^{-\frac{1}{2} - \frac{\lambda_j(M-\mu_j-(k-1)/2)}{\Lambda} + \frac{\mu_j(\Lambda-\lambda_j)}{\Lambda}}$$

Recalling that

$$F(T) = \sum_n \frac{\varepsilon Q^{1/2} a(n)}{2\pi n^{1/2}}$$

$$\mathcal{F}\left[\prod_{l=1}^{m} \frac{1}{\lambda_l}\left(\Gamma\left(i\lambda_l z + 2 + \frac{\lambda_l}{2}\right) - (\lambda_l + 1)\Gamma\left(i\lambda_l z + 1 + \frac{\lambda_l}{2}\right)\right)\right.$$

$$\left.\prod_{j=m+1}^{k} \Gamma\left(i\lambda_j z + \mu_j + \frac{\lambda_j}{2}\right)\right](T)$$

$$= C'' \sum_n \frac{\varepsilon a(n)}{n^{1/2}} \sum_{q=0}^{m} C'_q e^{\frac{T(M+m+q+\Lambda/2-(k-1)/2)}{\Lambda}} \exp\left[-\Lambda \prod_{j=1}^{k} \lambda_j^{-\lambda_j/\Lambda} e^{T/\Lambda}\right]$$

$$\left(1 + \mathcal{O}\left(e^{-T/\Lambda}\right)\right)$$

where

$$M = \sum_{j=m+1}^{k} \mu_j$$

and

$$\Lambda = \sum_{j=1}^{k} \lambda_j.$$

At the beginning of this work, we set

$$T = t + \log\left(\frac{n}{Q}\right),$$

which becomes

$$e^T = \frac{Q}{n}e^t$$

and so returning to this again, our equation becomes

$$\begin{aligned}\phi(t) \\ &= C'' \sum_n \frac{\varepsilon a(n)}{n^{1/2}} \sum_{q=0}^{m} C'_q \left(\frac{n}{Q}\right)^{\frac{M+m+q+\Lambda/2-(k-1)/2}{\Lambda}} e^{\frac{t(M+m+q+\Lambda/2-(k-1)/2)}{\Lambda}} \\ &\qquad \exp\left[-\Lambda\left(\frac{n}{Q}\right)^{1/\Lambda} \prod_{j=1}^{k} \lambda_j^{-\lambda_j/\Lambda} e^{t/\Lambda}\right]\left(1+\mathcal{O}\left(e^{-t/\Lambda}\right)\right).\end{aligned}$$

The leading order behaviour of this function is $n = 1$ and $q = m$, giving

$$\begin{aligned}\phi(t) = \\ \varepsilon C''' e^{\frac{t(M+2m+\Lambda/2-(k-1)/2)}{\Lambda}} \exp\left[-\Lambda Q^{-1/\Lambda} \prod_{j=1}^{k} \lambda_j^{-\lambda_j/\Lambda} e^{t/\Lambda}\right]\left(1+\mathcal{O}\left(e^{-t/\Lambda}\right)\right)\end{aligned}$$

as required.

Recall that

$$\Xi_F(z) = \int_{-\infty}^{\infty} \phi(t) e^{izt} \mathrm{d}z.$$

We can rescale the variable of integration to make the integrand the same format as the integrand for the Riemann Xi function, i.e.

$$\Xi_F(z) = \frac{1}{\Lambda} \int_{-\infty}^{\infty} \phi\left(\Lambda t\right) e^{\Lambda itz} \mathrm{d}t$$

where

$$\frac{1}{\Lambda}\varphi(\Lambda t) = Be^{-ae^t} e^{b't}(1 + \mathcal{O}(e^{-t}))$$

with

$$a = \Lambda Q^{-1/\Lambda} \prod_{j=1}^{k} \lambda_j^{-\lambda_j/\Lambda}$$

and

$$b' = M + 2m + \frac{\Lambda}{2} - \frac{k-1}{2}. \tag{4}$$

This is now in the same format as the Riemann Xi format, and so a similar result can be obtained.

9. Differentiating the function

In the previous section, it was shown that

$$\Xi_F(z) = \frac{1}{\Lambda}\int_{-\infty}^{\infty} \phi(\Lambda x) e^{i\Lambda x z} \mathrm{d}x$$

where

$$\frac{1}{\Lambda}\phi(\Lambda x) = A e^{-ae^x} e^{b'x}(1 + \mathcal{O}(e^{-x})). \tag{5}$$

In this section, it will be shown that this function behaves like the Riemann Ξ-function under repeated differentiation, so that

$$\lim_{n\to\infty} A_n \Xi_F^{(2n)}(C_n z - \frac{\Theta}{\Lambda}) = \cos(z + \theta)$$

where

$$A_n, C_n, \Theta, \theta \in \mathbb{R}.$$

Note that Θ and θ have yet to be defined.

However, in 5, it is possible that $b' \in \mathbb{C} \setminus \mathbb{R}$, and so it needs to be split it up into the real and imaginary parts, $b' = b + i\Theta$, so the Selberg Ξ_F-function can be written as

$$\begin{aligned}
\Xi_F(z) &= A\int_{-\infty}^{\infty} \frac{1}{\Lambda} e^{-ae^x} e^{b'x}(1 + \mathcal{O}(e^{-x}))e^{i\Lambda x z}\mathrm{d}x \\
&= \frac{A}{\Lambda}\int_{-\infty}^{\infty} e^{-}ae^x e^{(b+i\Theta)x}(1 + \mathcal{O}(e^{-x}))e^{i\Lambda x z}\mathrm{d}x \\
&= \frac{A}{\Lambda}\int_{-\infty}^{\infty} e^{-ae^x} e^{bx}(1 + \mathcal{O}(e^{-x}))e^{ix(\Lambda z + \Theta)}\mathrm{d}x \\
&= \frac{A}{\Lambda}\int_{-\infty}^{\infty} \varphi(x) e^{ix(\Lambda z + \Theta)}\mathrm{d}x
\end{aligned}$$

where

$$\varphi(x) = e^{-ae^x} e^{bx}(1 + \mathcal{O}(e^{-x}))$$

with $a, b \in \mathbb{R}$, as required for Ki's work. Remembering that 4

$$b' = M + 2m + \frac{\Lambda}{2} - \frac{k-1}{2}$$

the only way it can have a non-zero imaginary part is if M is complex. Given that $M = \sum \mu_j$, so far for all known L-functions, this is real. However, this has not yet been proven for all L-functions, and so the possibility remains that is may be complex.

Rescaling z, the function becomes

$$\Xi_F\left(z-\frac{\Theta}{\Lambda}\right)=\frac{A}{\Lambda}\int_{-\infty}^{\infty}\varphi(x)e^{i\Lambda xz}\mathrm{d}x.$$

The functional equation

$$\Xi_F\left(z-\frac{\Theta}{\Lambda}\right)=\Xi_{\overline{F}}\left(-z+\frac{\Theta}{\Lambda}\right)$$

means that we can replace z with $-z$ in the integral, provided that we take the complex conjugate of all the other terms. Therefore, we have that

$$\begin{aligned}\Xi_F\left(z-\frac{\Theta}{\Lambda}\right)&=B\int_{-\infty}^{\infty}\varphi(x)e^{i\Lambda xz}\mathrm{d}x\\&=B\int_{-\infty}^{\infty}\varphi(-x)e^{-i\Lambda xz}\mathrm{d}x.\\&=\overline{B}\int_{-\infty}\varphi(-x)e^{i\Lambda xz}\mathrm{d}x,\end{aligned}$$

which means that we have

$$B\varphi(x)=\overline{B}\varphi(-x)$$

and we can write the integral as

$$\begin{aligned}\Xi_F\left(z-\frac{\Theta}{\Lambda}\right)&=\int_{-\infty}^{\infty}B\varphi(x)e^{i\Lambda xz}\mathrm{d}x\\&=\int_{-\infty}^{0}B\varphi(x)e^{i\Lambda xz}\mathrm{d}x+\int_{0}^{\infty}B\varphi(x)e^{i\Lambda xz}\mathrm{d}x\\&=\int_{0}^{\infty}B\varphi(-x)e^{-i\Lambda xz}\mathrm{d}x+\int_{0}^{\infty}B\varphi(x)e^{i\Lambda xz}\mathrm{d}x\\&=\int_{0}^{\infty}\overline{B}\varphi(x)e^{-i\Lambda xz}\mathrm{d}x+\int_{0}^{\infty}B\varphi(x)e^{i\Lambda xz}\mathrm{d}x\\&=\int_{0}^{\infty}\varphi(x)\left(Be^{i\Lambda xz}+\overline{B}e^{-i\Lambda xz}\right)\mathrm{d}x.\end{aligned}$$

Now it is possible to show that under repeated differentiation(which in this case due to the Fourier transform, is the same as multiplying by $i\Lambda x$), the integral smooths out.

Theorem 11. *There exist sequences A_n and C_n such that*

$$\lim_{n\to\infty} A_n \int_0^\infty \varphi(x)(C_n ix)^{2n} e^{iC_n xz} \mathrm{d}x = e^{iz+\theta}.$$

where

$$A_n = \frac{\sqrt{n}(-1)^n e^{ae^{w_n}} e^{-bw_n} \Lambda^{2n}}{\mid B \mid w_n^{2n} \sqrt{\pi} 2},$$

$$w_n = \frac{1}{\Lambda c_n} \approx \log\left(\frac{2n}{a}\right) - \log\log\left(\frac{2n}{a}\right)$$

and

$$\theta = \arg(B).$$

Proof. Beginning by defining

$$\Xi_F\left(z - \frac{\Theta}{\Lambda}\right) = |B| \left(e^{i\theta} f(z) + e^{-i\theta} f(-z)\right)$$

where

$$f(z) = \int_0^\infty \varphi(x) e^{i\Lambda xz} \mathrm{d}x,$$

so that

$$f^{(2n)}(z) = (-1)^n \int_0^\infty \varphi(x) \Lambda^{2n} x^{2n} e^{i\Lambda xz} \mathrm{d}x$$

and rescaling so that

$$\begin{aligned} f^{(2n)}(C_n z) &= (-1)^n \Lambda^{2n} \int_0^\infty \varphi(x) x^{2n} e^{iC_n \Lambda xz} \mathrm{d}x \\ &= (-1)^n w_n \Lambda^{2n} \int_0^\infty \varphi(w_n x)(w_n x)^{2n} e^{ixz} \mathrm{d}x, \end{aligned}$$

where

$$w_n = \frac{1}{\Lambda C_n}.$$

The main part of the integrand can be written as

$$\varphi(w_n x) x^{2n} = \exp\Big[- ae^{w_n x} + bw_n x + 2n\log(x)\Big] \left(1 + \mathcal{O}(e^{-w_n x})\right).$$

Excluding the error term, the maximum of this will occur when

$$\frac{\mathrm{d}}{\mathrm{d}x}\Bigg(- ae^{w_n x} + bw_n x + 2n\log(x)\Bigg) = 0.$$

Setting the maximum of this to occur at 1 means that

$$aw_n e^{w_n} = bw_n + 2n,$$

and it is this equation which sets w_n and C_n. For large n, we have that

$$w_n \approx \log\left(\frac{2n}{a}\right) - \log\log\left(\frac{2n}{a}\right).$$

The main part of the integral now occurs around 1, so the integral can be split up into the main term and error terms

$$\begin{aligned} f^{(2n)}(C_n z) &= (-1)^n w_n^{2n+1} \Lambda^{2n} \int_0^\infty \varphi(w_n x) x^{2n} e^{ixz} \mathrm{d}x \\ &= (-1)^n w_n^{2n+1} \Lambda^{2n} \Big[\int_0^{1-u_n} + \int_{1-u_n}^{1+u_n} + \int_{1+u_n}^\infty \Big] \varphi(w_n x) x^{2n} e^{ixz} \mathrm{d}x, \end{aligned}$$

where u_n will be defined later. Each of these integrals now needs to be considered separately.

9.1. **Main term.**

Lemma 12. *The main integral can be calculated to give*

$$\int_{1-u_n}^{1+u_n} \varphi(w_n x) x^{2n} e^{ixz} \mathrm{d}x = \frac{\sqrt{\pi} e^{iz} e^{b w_n - a e^{w_n}}}{\sqrt{n} w_n} \left(1 + \mathcal{O}(w_n^{-3})\right)$$

where the error term is suitably small since $w_n \to \infty$.

Proof. Beginning by rescaling the integrand the integral becomes

$$I_m = e^{iz} \int_{-u_n}^{u_n} \varphi(w_n(1+x))(1+x)^{2n} e^{ixz} \mathrm{d}x$$

The main part of this integrand can be written as

$$\begin{aligned} &\varphi(w_n(1+x))(1+x)^{2n} = \\ &\qquad \exp\Big[-a e^{w_n(1+x)} + b w_n(1+x) + 2n \log(1+x) \Big] \\ &\qquad\qquad \left(1 + \mathcal{O}(e^{-w_n(1+x)})\right). \end{aligned}$$

The main part of this can be rearranged using

$$a w_n e^{w_n} = b w_n + 2n$$

and various Taylor series to give

$$\begin{aligned}
-ae^{w_n(1+x)} &+ bw_n(1+x) + 2n\log(1+x) \\
&= -ae^{w_n}e^{w_n x} + bw_n + bw_n x + 2n(x + \mathcal{O}(x^2)) \\
&= -ae^{w_n}(1 + w_n x + \frac{w_n^2 x^2}{2} + \mathcal{O}(w_n^3 x^3)) + bw_n + bw_n x \\
&\qquad + 2nx + \mathcal{O}(nx^2) \\
&= -ae^{w_n} + bw_n - aw_n x e^{w_n} + bw_n x + 2nx - \frac{aw_n^2 x^2}{2} e^{w_n} \\
&\qquad + \mathcal{O}(nx^2) + \mathcal{O}(w_n^3 x^3 e^{w_n}) \\
&= -ae^{w_n} + bw_n - x^2(nw_n + \frac{bw_n^2}{2}) + \mathcal{O}(nx^2) + \mathcal{O}(w_n^3 x^3 e^{w_n}) \\
&= -ae^{w_n} + bw_n - nw_n x^2 + \mathcal{O}(w_n^2 x^2) + \mathcal{O}(nx^2) + \mathcal{O}(w_n^3 x^3 e^{w_n}).
\end{aligned}$$

Recalling that

$$w_n \approx \log(n)$$

the largest of these error terms is (for suitable small x)

$$\mathcal{O}(nx^2)$$

and so for the error term to tend to 0, we need

$$x = o(n^{-1/2}).$$

Therefore, it makes sense to maximise the range of integration by setting

$$u_n = \frac{1}{n^{1/2}},$$

and so the integral becomes

$$I_M = e^{iz}e^{bw_n - ae^{w_n}} \int_{-1/n^{1/2}}^{1/n^{1/2}} e^{-nw_n^2 x^2 + \mathcal{O}(nx^2)} e^{ixz}(1 + \mathcal{O}(e^{-w_n(x+1)}))\mathrm{d}x.$$

We can rewrite

$$e^{ixz} = 1 + \mathcal{O}(x^2)$$

which is smaller than the

$$e^{\mathcal{O}(nx^2)} = 1 + \mathcal{O}(nx^2)$$

and so our equation becomes

$$
\begin{aligned}
I_M &= e^{iz} e^{bw_n - ae^{w_n}} \int_{-1/n^{1/2}}^{1/n^{1/2}} e^{-nw_n^2 x^2} \left(1 + \mathcal{O}(nx^2)\right) \mathrm{d}x \\
&= e^{iz} e^{bw_n - ae^{w_n}} n^{-1/2} \int_{-1}^{1} e^{-w_n^2 x^2} (1 + \mathcal{O}(x^2)) \mathrm{d}x \\
&= e^{iz} e^{bw_n - ae^{w_n}} n^{-1/2} \left(\frac{\sqrt{\pi}}{w_n} + \mathcal{O}(w_n^{-11}) + \mathcal{O}(w_n^{-3}) \right) \\
&= e^{iz} e^{bw_n - ae^{w_n}} n^{-1/2} \frac{\sqrt{\pi}}{w_n} \left(1 + \mathcal{O}(w_n^{-3})\right)
\end{aligned}
$$

as required. □

9.2. **Error terms.** The error terms

$$\left[\int_0^{1-1/n^{1/2}} + \int_{1+1/n^{1/2}}^{\infty} \right] \varphi(w_n x) x^{2n} e^{ixz} \mathrm{d}x$$

still need to be calculated. The integrals

$$\left[\int_0^{1-1/n^{1/2}} + \int_{1+1/n^{1/2}}^{2} \right] \varphi(w_n x) x^{2n} e^{ixz} \mathrm{d}x$$

are calculated in the same manner, and so are treated together, and then finally the integral

$$\int_2^{\infty} \varphi(w_n x) x^{2n} e^{ixz} \mathrm{d}x$$

is calculated separately.

Lemma 13. *The first two error integrals can be calculated to give*

$$
\begin{aligned}
\left[\int_0^{1-1/n^{1/2}} + \int_{1+1/n^{1/2}}^{2} \right] \varphi(w_n x) x^{2n} e^{ixz} \mathrm{d}x &= \mathcal{O}\left(e^{bw_n - ae^{w_n} - w_n} \right) \\
&= \frac{e^{bw_n - ae^{w_n}}}{\sqrt{n} w_n} \mathcal{O}(\sqrt{n} w_n e^{-w_n}).
\end{aligned}
$$

Since only the leading order behaviour is of interest here, the error terms associated with $\varphi(x)$ can be ignored.

Proof. Since the function $\varphi(w_n x) x^{2n}$ has only the one maximum at $x = 1$, this part of the integrand will be largest the closest to $x = 1$ it can get. The approximation

$$\varphi(w_n(1 + x))(1 + x)^{2n} \approx \exp[-ae^{w_n} + bw_n - nw_n x^2]$$

which was previously found can be used here. The rest of the integrand can be approximated by $|e^{ixz}| \leq e^{|2z|}$ in this range. Therefore

$$\begin{aligned}
&\left|\int_{1+1/n^{1/2}}^{2} \varphi(w_n x)x^{2n}e^{ixz}\mathrm{d}x\right| \\
&\leq \int_{1+1/n^{1/2}}^{2} \varphi(w_n(1+1/n^{1/2}))(1+1/n^{1/2})^{2n}e^{|2z|}\mathrm{d}x \\
&= e^{|2z|}\exp[-ae^{w_n} + bw_n - w_n + \mathcal{O}(1)] \\
&= \frac{e^{bw_n - ae^{w_n}}}{\sqrt{n}w_n}\mathcal{O}(\sqrt{n}w_n e^{-w_n})
\end{aligned}$$

as required. The other integral mentioned is calculated in exactly the same way, and so is not shown here. Since $e^{w_n} \sim n$, this error term is suitably small. □

Lemma 14. *The final error term is*

$$\begin{aligned}
\int_{2}^{\infty} \varphi(w_n x)x^{2n}e^{ixz}\mathrm{d}x &= \mathcal{O}\left(\frac{e^{-ae^{2w_n}}}{w_n}\right) \\
&= \frac{e^{bw_n - ae^{w_n}}}{\sqrt{n}w_n}\mathcal{O}(\exp[-ae^{2w_n} + ae^{w_n} - bw_n]\sqrt{n}).
\end{aligned}$$

Proof. The expansion

$$\varphi(1+x)(1+x)^{2n} = \exp[-ae^{w_n} + bw_n - nw_n x^2 + \mathcal{O}(nx^2)]$$

cannot be used here as the error term tends to infinity. Instead, the integral is

$$\begin{aligned}
&\int_{2}^{\infty} \varphi(w_n x)x^{2n}e^{ixz}\mathrm{d}x \\
&= \int_{2}^{\infty} e^{-ae^{w_n x} + bw_n x + 2n\log(x) + ixz}\mathrm{d}x.
\end{aligned}$$

The behaviour of this integral is most influenced by the $-ae^{w_n x}$ term, and so, since the result is only needed as an error term, the integral can be written as

$$
\begin{aligned}
&= \mathcal{O}\left[\int_2^\infty e^{-ae^{w_n x}}\mathrm{d}x\right] \\
&= \mathcal{O}\left[\frac{1}{w_n}\int_{e^{2w_n}}^\infty \frac{e^{-ax}}{x}\mathrm{d}x\right] \\
&\le \mathcal{O}\left[\frac{1}{w_n}\int_{e^{2w_n}}^\infty e^{-ax}\mathrm{d}x\right] \\
&= \mathcal{O}\left[\frac{e^{-ae^{2w_n}}}{w_n}\right].
\end{aligned}
$$

Comparing this to the main term, this is

$$
= \frac{e^{bw_n - ae^{w_n}}}{\sqrt{n}w_n}\mathcal{O}(e^{-ae^{2w_n}+ae^{w_n}-bw_n}\sqrt{n})
$$

as required. □

9.3. **The complete integral.** We can now combine all the terms, to give

$$
\begin{aligned}
&\left[\int_0^{1-u_n} + \int_{1-u_n}^{1+u_n} + \int_{1+u_n}^\infty\right]\varphi(w_n x)x^{2n}e^{ixz}\mathrm{d}x \\
&= \frac{\sqrt{\pi}e^{iz}e^{bw_n - ae^{w_n}}}{\sqrt{n}w_n}\Bigg(1 + \mathcal{O}(w_n^{-3}) + \mathcal{O}(\sqrt{n}w_n e^{-w_n}) \\
&\qquad + \mathcal{O}\left(e^{-ae^{2w_n}+ae^{w_n}-bw_n}\sqrt{n}\right)\Bigg) \\
&= \frac{\sqrt{\pi}e^{iz}e^{bw_n - ae^{w_n}}}{\sqrt{n}w_n}\left(1 + \mathcal{O}(w_n^{-3})\right),
\end{aligned}
$$

and, remembering that

$$
f^{(2n)}(C_n z) = (-1)^n w_n^{2n+1}\Lambda^{2n}\int_0^\infty \varphi(w_n x)x^{2n}e^{ixz}\mathrm{d}x
$$

the function can now be written as

$$
f^{(2n)}(C_n z) = \frac{(-1)^n w_n^{2n}\sqrt{\pi}e^{iz}e^{bw_n - ae^{w_n}}\Lambda^{2n}}{\sqrt{n}}\left(1 + \mathcal{O}(w_n^{-3})\right).
$$

Remembering also that

$$
\Xi_F\left(z - \frac{\Theta}{\Lambda}\right) = \mid B \mid (e^{i\theta}f(z) + e^{-i\theta}f(-z))
$$

shows that

$$\Xi_F^{(2n)}\left(C_n(z-\frac{\Theta}{\Lambda})\right) = \frac{\mid B \mid (-1)^n w_n^{2n}\sqrt{\pi}e^{bw_n - ae^{w_n}}\Lambda^{2n}}{\sqrt{n}}$$
$$(e^{i(z+\theta)} + e^{-i(z+\theta)})(1+\mathcal{O}(w_n^{-3}).$$

and so setting the sequence

$$A_n = \frac{\sqrt{n}e^{ae^{w_n} - bw_n}(-1)^n}{2 \mid B \mid w_n^{2n}\sqrt{\pi}\Lambda^{2n}}$$

means that

$$A_n\Xi_F^{(2n)}\left(C_n(z-\frac{\Theta}{\Lambda})\right) = \cos(z+\theta)(1+\mathcal{O}(w_n^{-3}))$$

and so

$$\lim_{n\to\infty} A_n\Xi_F^{(2n)}\left(C_n(z-\frac{\Theta}{\Lambda})\right) = \cos(z+\theta)$$

where

$$\Theta = \mathrm{Im}(\sum_j \mu_j) = \mathrm{Im}(M)$$

and

$$\theta = \arg(B)$$

as required. □

10. Analysis of error terms and series

In this section, the sequences A_n and C_n are studied, as well as the error term w_n^{-3}.

The following chapter is dedicated to discussing how the plots are generated, but it is important to mention them here. All the plots are of $4n$ derivatives, so that they look similar to each other ($f(0) > 0$). Additionally, all the plots are of derivatives of the Riemann Xi-function. This is because trying to calculate the Xi-functions for L-functions proved impossible with the computing power I had. Up to 100 derivatives, the plots are generated by approximating derivatives using the function evaluated at points. This means that the plots are of $\Xi^{(4n)}(z)$. The differences are mostly in the y-axis, but it is also important to note that the zeros are still moving towards the origin for these plots.

For more than 100 derivatives, a different method of approximating derivatives is used, one which uses the integral representation. This means that the plots here do include the C_n, and are plotting

$$\frac{\Xi^{(4n)}(C_n z)2^{4n}}{w_n^{4n+1}},$$

which does scale the zeros, but doesn't scale the y-axis completely. However, these plots can be used to see how the error term shrinks.

The sequence C_n determines the behaviour of the zeros, other than them becoming more evenly spaced out. Because the density of zeros of L-functions is an increasing function, repeatedly differentiating the function causes the zeros to move towards the origin (while still being evenly spaced out). The sequence

$$C_n = \frac{1}{\Lambda w_n} \approx \frac{1}{\Lambda \log(n)}$$

shows that the zeros move very slowly towards the origin. In the interval $(0, 20)$, there is one zero of the Riemann Xi-function, compared with 8 zeros of the 100th derivative.The time taken to calculate plots of higher numbers of derivatives using this method became too time-consuming, and so I had to find an alternative method, which does not show this behaviour.

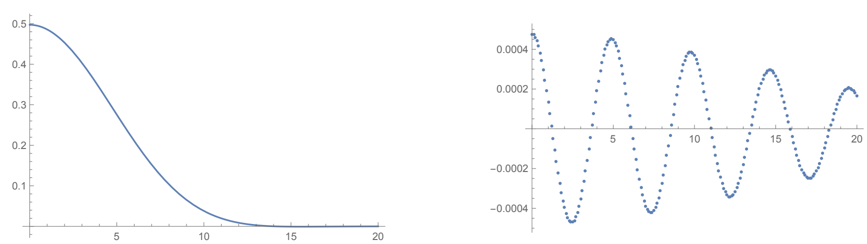


FIGURE 7. A comparison of the Riemann Xi function at 0 and 100 derivatives (no scaling)

Looking at these plots, it is also possible to see that the error term (w_n^{-3}) is also slowly decaying, although at a faster rater than the movement of zeros towards the origin.

A plot of the Xi-function after 1,000 derivatives looks very similar to one of 100 derivatives

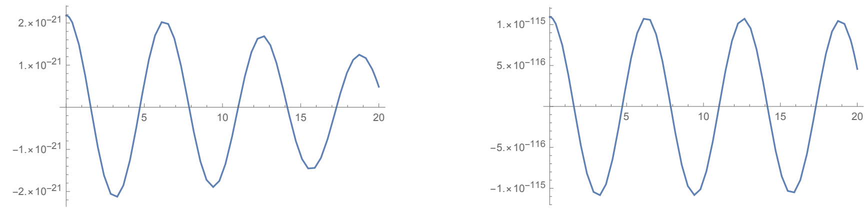


FIGURE 8. A comparison of 100 and 1,000 derivatives of the Riemann Xi function (with scaling)

and the main difference here is that the decay of the function is much less noticeable. Comparing the 1,000 derivatives with the cosine function

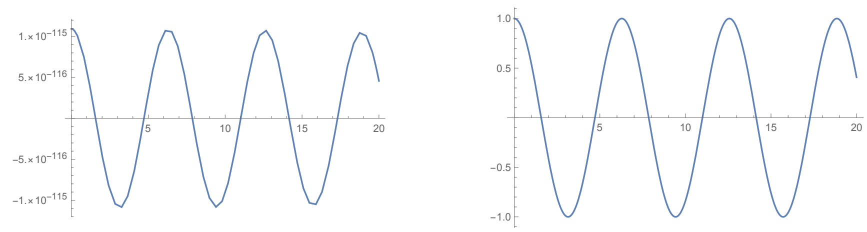


FIGURE 9. A comparison of 1,000 derivatives of the scaled Riemann Xi function and the cosine function

we can see that the differences are very minimal.

The A_n term dictates how large the derivatives of Ξ and Ξ_F can get. Recalling that

$$\begin{aligned} A_n &= \frac{\sqrt{n} e^{ae^{w_n} - bw_n} (-1)^n}{2|B| w_n^{2n} \sqrt{\pi} \Lambda^{2n}} \\ &\approx \frac{(-1)^n}{2\sqrt{\pi}|B|} \exp\left[ae^{w_n} - bw_n - 2n\log(w_n) + \frac{1}{2}\log(n) - 2n\log(\Lambda)\right]. \end{aligned}$$

This eventually tends to 0 since the largest term here is the $-2n\log(w_n)$. Therefore, the size of the derivatives of $\Xi(C_n z)$ and $\Xi_F(C_n(z - \Theta/\Lambda))$ grows as the number of derivatives taken increases. This is difficult to see in practice, since the first 100 derivatives are of $\Xi(z)$, without the C_n term.

Higher derivatives are calculated in a different way, and so plots of more than 100 derivatives are of

$$\frac{\Xi^{(4n)}(C_n z)}{w_n^{4n} \Lambda^{4n}}.$$

This will (excepting scaling of the y axis) tend to the cosine function due to the C_n term in the Xi function. However, the scaling of this function is not A_n, and so the scaling is not of the right form to really examine the scaling A_n in any practical way.

The methods used to calulate these plots are discussed in more detail in the following chapter; a brief overview is provided here to understand the limitations of the plots shown.

Chapter 3

11. Introduction

Pictures are a useful way to visualise maths. However, differentiating the Ξ-function and the Ξ_F-functions directly is difficult, due to the exponential decay. Using Mathematica directly to plot derivatives only works for the first few derivatives, before inaccuracies start appearing. Explicitly telling Mathematica to work to a higher accuracy by using finite differences works for more derivatives, until the errors caused by subtracting very similar numbers starts to play too large a part.

An alternative is to realise that rather than differentiating, it is possible to calculate the derivatives using the same method used to calculate the result analytically; by differentiating the Fourier Transform and then calculating the integral. This can be done using the trapezium rule, or alternatively, DENIM(Double Exponential Numerical Integration Method). This is a more accurate method of approximating integrals by rescaling the integrand to reduce the error term significantly. This method will only work for larger numbers of derivatives, as the error term associated with the $\varphi(u)$ plays a roll here and must be suitably small.

All the pictures used in this section are of $4n$ derivatives. This is so that all the pictures look similar with $f(0) > 0$, so that they can be reasonably compared to each other. All plots generated are of derivatives of the Riemann Xi function, as the accuracy needed to plot derivatives of Ξ_F functions required more computer memory than was available to me.

It is also important to note what is being plotted here. In the previous chapter, it was shown that

$$\lim_{n\to\infty} A_n \Xi^{(2n)} \left(C_n(z - \frac{\Theta}{\Lambda}) \right) = \cos(z + \theta)$$

and so, on one level it would make sense to plot

$$A_{2n}\Xi^{(4n)}(C_n(z - \frac{\Theta}{\Lambda})).$$

However, it is more natural to use Mathematica to plot

$$\Xi^{(4n)}(z)$$

up to 100 derivatives. This is because the approximations of the derivatives use the function to be differentiated— in this case $\Xi(z)$. Beyond 100 derivatives, the method changes to approximate the integral

$$\int_0^\infty \varphi(w_{2n}t)t^{4n}(e^{itz}+e^{-itz})\mathrm{d}t$$

and this is an approximation to

$$\frac{\sqrt{n}e^{ae^{w_n}-bw_n}}{2|B|}\Xi^{(4n)}(C_n z).$$

Because of the error term in

$$\varphi(w_n t) = e^{-ae^{w_n t}}e^{bw_n t}(1+\mathcal{O}(w_n^{-3}))$$

the DENIM method can only be used for large numbers of derivatives. The main difference of including the C_n term means that comparing plots of less than and more than 100 derivatives can be misleading. However, being aware of this mitigates the effect. The other difference, that the scaling of the plots is much less obvious as that only scales the vertical axis and can be ignored.

12. Differentiating using Mathematica

Using Mathematica's inbuilt differentiate command is problematic due to the Ξ-function being defined as a product. This means that the derivatives of it are a sum of similar terms. This has two main issues— firstly that Mathematica thinks that the derivatives aren't real(see fig 10). This problem can be solved by plotting the real part of the derivatives.

The second problem is that beyond the eighth derivative, the real part of sums of similar terms becomes inaccurate due to rounding errors. It can easily be seen that fig 12 which is of the plot of the twelfth derivative of the Ξ-function has errors near the origin, and these errors become more intrusive the more derivatives are taken.

These problems mean that an alternative method of plotting the derivatives must be found for higher derivatives, one where the accuracy can be explicitly controlled to deal with rounding errors.

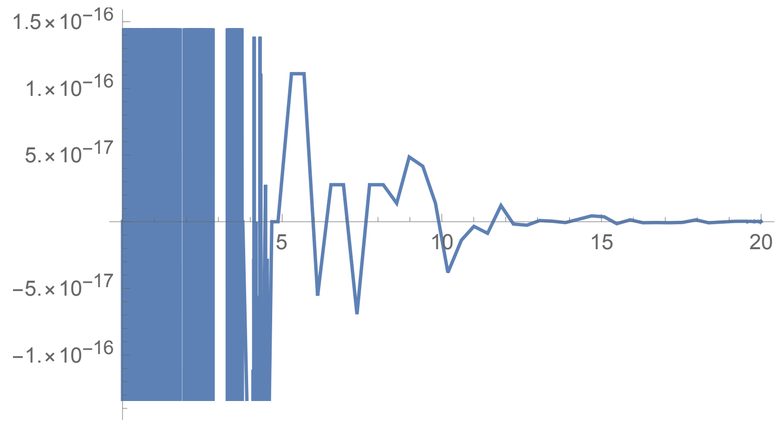


FIGURE 10. A plot of what Mathematica thinks the imaginary part of the fourth dervative of Xi is

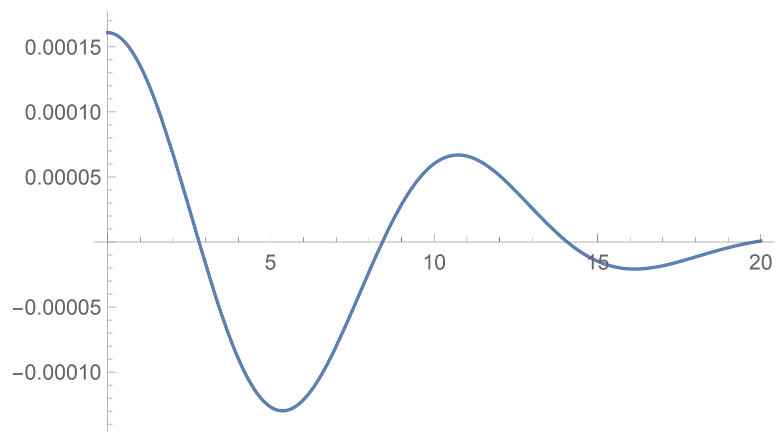


FIGURE 11. A plot of the real part of the eighth derivative of Xi as calculated by Mathematica

13. Numerical Approximation

If a function cannot be differentiated analytically, one alternative is to numerically approximate it [17]. The commonly mentioned

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

can easily be extended to

$$\left.\frac{\mathrm{d}^m f(x)}{\mathrm{d}x^m}\right|_{x=x_0} \approx \sum_{v=0}^{n} \delta_{n,v}^m f(\alpha_v)$$

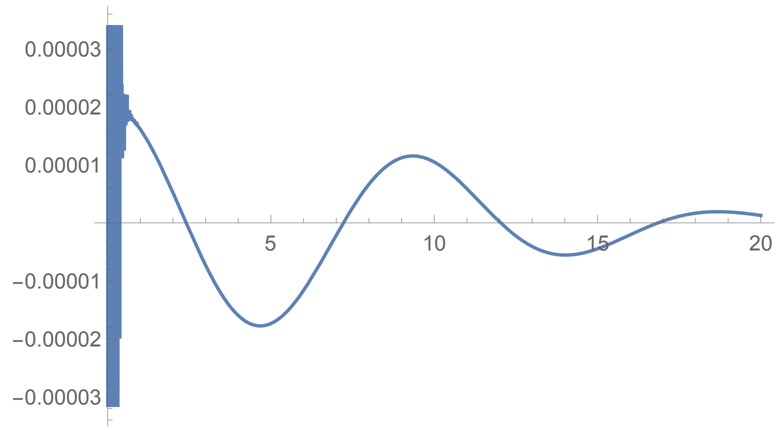


FIGURE 12. A plot of the real part of the 12th derivative of Xi as calculated by Mathematica

where α_v are distinct and $n > m$. Calculating the recursive formula for $\delta_{n,v}$ starts with defining

$$w_n(x) = \prod_{k=0}^{n} (x - \alpha_k)$$

and

$$F_{n,v}(x) = \frac{w_n(x)}{w_n'(\alpha_v)(x - \alpha_v)}.$$

Since

$$F_{n,v}(\alpha_v) = 1$$

and

$$F_{n,v}(\alpha_k) = 0, \qquad k \neq v$$

this can be used as a Lagrange multiplier, so

$$p(x) = \sum_{v=0}^{n} F_{n,v}(x) f(\alpha_v).$$

where f is any function and p is the polynomial approximation to f. Differentiating both sides then gives

$$\left.\frac{\mathrm{d}^m p(x)}{\mathrm{d}x^m}\right|_{x=x_0} = \sum_{v=0}^{n} \left.\frac{\mathrm{d}^m F_{n,v}(x)}{\mathrm{d}x^m}\right|_{x=x_0} f(\alpha_v)$$

and therefore

$$\delta_{n,v}^m = \left.\frac{\mathrm{d}^m F_{n,v}(x)}{\mathrm{d}x^m}\right|_{x=x_0},$$

and the dependence of the RHS for x_0 will be entirely dependent on the $f(\alpha_v)$ term. Using Taylor series gives

$$F_{n,v}(x) = \sum_{m=0}^{n} \frac{\delta_{n,v}^m}{m!} x^m$$

and so a recursive formula for $\delta_{n,v}^m$ can be found from using the original definition for $F_{n,v}(x)$ to find a recursive formula for that. Remembering that

$$\begin{aligned} w_n(x) &= \prod_{k=0}^{n} (x - \alpha_k) \\ &= (x - \alpha_n) w_{n-1}(x) \end{aligned}$$

so therefore

$$w_n'(x) = w_{n-1}(x) + (x - \alpha_n) w_{n-1}'(x),$$

and, for $v \neq n$

$$w_n'(\alpha_v) = (\alpha_v - \alpha_n) w_{n-1}'(\alpha_v).$$

Therefore,

$$\begin{aligned} F_{n,v}(x) &= \frac{w_n(x)}{w_n'(\alpha_v)(x - \alpha_v)} \\ &= \frac{(x - \alpha_n) w_{n-1}(x)}{(\alpha_v - \alpha_n) w_{n-1}'(\alpha_v)(x - \alpha_v)}, \qquad n \neq v. \end{aligned}$$

For $v = n$, using

$$w_n'(\alpha_n) = w_{n-1}(\alpha_n)$$

and

$$w_{n-1}(x) = (x - \alpha_{n-1}) w_{n-2}(x),$$

we get that

$$\begin{aligned} F_{n,n}(x) &= \frac{w_n(x)}{w_n'(\alpha_n)(x - \alpha_n)} \\ &= \frac{w_{n-1}(x)}{w_{n-1}(\alpha_n)} \\ &= \frac{(x - \alpha_{n-1}) w_{n-2}(x) w_{n-2}(\alpha_{n-1})}{w_{n-1}(\alpha_n) w_{n-2}(\alpha_{n-1})} \\ &= \frac{(x - \alpha_{n-1}) w_{n-2}(\alpha_{n-1})}{w_{n-1}(\alpha_n)} \frac{w_{n-2}(x)}{w_{n-2}(\alpha_{n-1})} \\ &= \frac{(x - \alpha_{n-1}) w_{n-2}(\alpha_{n-1})}{w_{n-1}(\alpha_n)} F_{n-1,n-1}(x). \end{aligned}$$

Using these two results for F, along with the Taylor series means that it is possible to find a recursive formula for $\delta^m_{n,v}$. Firstly for $v \neq n$, the functions are

$$\begin{aligned} F_{n,v}(x) &= \frac{x-\alpha_n}{\alpha_v-\alpha_n} F_{n-1,v}(x) \\ &= \sum_{m=0}^{n} \frac{\delta^m_{n,v}}{m!} x^m. \end{aligned}$$

Therefore

$$\sum_{m=0}^{n} \frac{\delta^m_{n,v}}{m!} x^m = \frac{1}{\alpha_v-\alpha_n} \sum_{m=1}^{n} \frac{\delta^{m-1}_{n-1,v}}{(m-1)!} x^m - \frac{\alpha_n}{\alpha_v-\alpha_n} \sum_{m=0}^{n-1} \frac{\delta^m_{n-1,v}}{m!} x^m,$$

which is the same as

$$\delta^m_{n,v} = \frac{1}{\alpha_n-\alpha_v} (\alpha_n \delta^m_{n-1,v} - m\delta^{m-1}_{n-1,v}).$$

In the case $v = n$, we have

$$\begin{aligned} F_{n,n}(x) &= \frac{(x-\alpha_{n-1}) w_{n-2}(\alpha_{n-1})}{w_{n-1}(\alpha_n)} F_{n-1,n-1}(x) \\ &= \sum_{m=0}^{n} \frac{\delta^m_{n,n}}{m!} x^m, \end{aligned}$$

which, upon rearranging becomes

$$\sum_{m=0}^{n} \frac{\delta^m_{n,n}}{m!} x^m = \frac{w_{n-2}(\alpha_{n-1})}{w_{n-1}(\alpha_n)} \left[\sum_{m=1}^{n} \frac{\delta^m_{n-1,n-1}}{(m-1)!} x^m - \alpha_{n-1} \sum_{m=0}^{n-1} \frac{\delta^m_{n-1,n-1}}{m!} x^m \right]$$

so

$$\delta^m_{n,n} = \frac{w_{n-2}(\alpha_{n-1})}{w_{n-1}(\alpha_n)} \left[m\delta^{m-1}_{n-1,n-1} - \alpha_{n-1}\delta^m_{n-1,n-1} \right].$$

Therefore, it is easy to see that these recursive formulas depend upon M, N, and α_v, where M is the maximum number of derivatives to be calculated, and N the maximum value of n used in the sum. We must have that $N > M$.

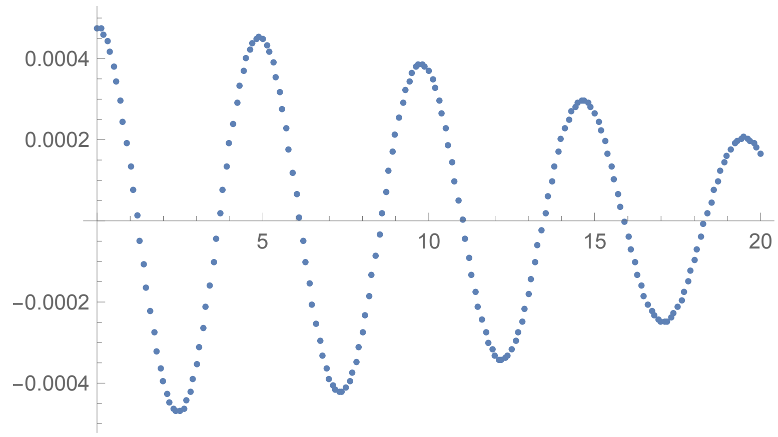


FIGURE 13. A plot of the $100th$ derivative of Xi without zero scaling

The maximum number of derivatives to be calculated must be agreed upon ahead of time, or save the calculated $\delta_{n,v}$ to insert into the formula later on. However, the size of N must be decided at the start, and this necessarily limits the choice for M.

These values of $\delta_{n,v}^{m}$ are the best possible, and naturally, the more terms which are used will make this more accurate. However, this must be balanced against the error terms which arise from subtracting very similar numbers from each other. I used this method to approximate the first 100 derivatives for the Riemann Xi-function.

By choosing which points to plot ahead of time, the term $(x - \alpha_n)$ can be simplified in the calculations, thereby making the terms easier to calculate and save, as they are constants rather than variables which depend on x. In the particular case I was using them for, this also meant that the zeta function only needed to be calculated to a high degree at points, rather than being treated as a function.

By setting the α_v to depend upon x_0, the $\delta_{n,v}$ are all constants, which makes using this as a method of plotting derivatives much easier. In order to do this, it makes sense to have the $x_0 = \delta_{n,0}$, and then alternate so that

$$\delta_{n,2v} = x_0 - v\epsilon$$

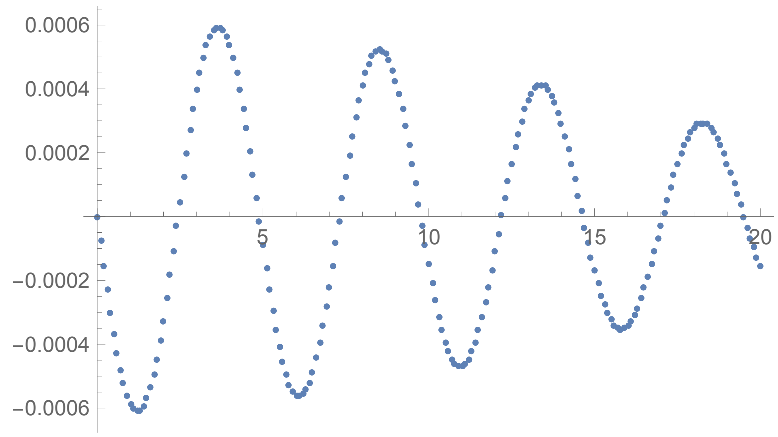


FIGURE 14. A plot of the $101st$ derivative of Xi without zero scaling

and

$$\delta_{n,2v+1} = x_0 + (v+1)\epsilon$$

where ϵ is the desired gap between points.

This method starts to take an unreasonable amount of time to calculate the $\delta_{n,v}^m$ terms for $m > 100$, and so I sought an alternative. Rather than differentiating the function, as I had been doing for $m < 100$, I used the integral calculated in chapter 2 to generate the plots.

All the plots used in chapter 2 are of $4n$ derivatives of the Xi function, so they all look similar enough to compare to each other. Therefore, in order to check that the plots are of $4n$ derivatives, rather than some other approximation to the cosine function, we include here the approximation to the $101st$ derivative.

14. DENIM

14.1. **General Method.** DENIM(Double Exponential Numerical Integration Method) [35] [21] is a method which allows integrals to be approximated more accurately than the trapezium method. This is done by converting the integrand so it is $\mathcal{O}(e^{-e^x})$, and therefore decays very rapidly.

In order to use this method, the integral to be evaluated must be of the form

$$\int_{-1}^{s} f(x)\mathrm{d}x$$

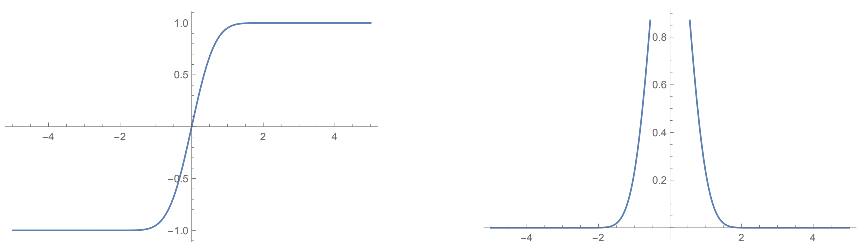
where $-1 < s < 1$ and $f(x)$ must be analytic in the open interval $(-1, 1)$ and must decay exponentially as $|x| \to \infty$. Many different changes of variable can be used, which can affect the final error term. The transform [35]

$$x = \phi(t) = \tanh\left(\frac{\pi}{2}\sinh t\right)$$

is commonly used as the derivative

$$\phi'(t) = \frac{\pi \cosh t}{2\cosh^2(\pi \sinh t/2)} = \mathcal{O}\left(\exp\left(-\frac{\pi(1-\varepsilon)}{2} m \exp|t|\right)\right).$$

Plotting $\phi(t)$ and $\phi'(t)$ (fig 15) illustrates that the derivative decays double exponentially fast, which is the key requirement of this method.



(A) $\phi(t)$ (B) $\phi'(t)$

Figure 15. $\phi(t)$ and $\phi'(t)$

The integral then becomes

$$\int_{-1}^{s} f(x)\mathrm{d}x = \int_{-\infty}^{\tau} f(\phi(t))\phi'(t)\mathrm{d}t = \int_{-\infty}^{\tau} u(t)\mathrm{d}t,$$

where

$$\tau = \phi^{-1}(s).$$

It is possible to create an infinite series expansion for $u(z)$ [21] under the following conditions:

(1) $u(z)$ is analytic in some interval $|y| < d$
(2) The integral $\int_{-\infty}^{\infty} |u(x \pm i(d-\varepsilon)|\mathrm{d}x$ is bounded in ε
(3) The integral $\int_{-d+\varepsilon}^{d-\varepsilon} |u(x+iy)|\mathrm{d}y$ is bounded in x

These conditions are needed as the series comes from calculating the contour integral

$$\frac{1}{2\pi i}\int_{C_{n,\varepsilon}} \frac{u(z)\mathrm{d}z}{(z-t)\sin(\pi z/h)}$$

where the contour $C_{n,\varepsilon}$ is a rectangle with corners at $x = \pm(n+1/2)h, y = \pm(d-\varepsilon)$, so that $u(z)$ is analytic within the whole rectangle (condition 1).

This integrand and contour are chosen so that the conditions listed above mean that the integral is bounded. The denominator is chosen so that the residue theorem produces an infinite sum linking $u(t)$, which is the desired result with $u(kh)$, which means that when the function $u(t)$ is integrated, the right hand side does not include the integral of u.

Beginning with the residue theorem, the singularities occur at $z = t$, and at $z = kh$, where $-n \le k \le n$ is an integer. The residues for these singularities are $u(t)/\sin(\pi t/h)$ and $(-1)^k u(kh)h/\pi(kh-t)$ respectively. Using these, we have that

$$\frac{1}{2\pi i}\int_{C_{n,\varepsilon}} \frac{u(z)\mathrm{d}z}{(z-t)\sin(\pi z/h)} = \frac{u(t)}{\sin(\pi t/h)} + \sum_{k=-n}^{n} \frac{(-1)^k u(kh)h}{\pi(kh-t)}$$

which can be rearranged to give

$$\begin{aligned}
u(t) &= \sum_{k=-n}^{n} \frac{(-1)^k u(kh)h\sin(\pi t/h)}{\pi(t-kh)} + \frac{\sin(\pi t/h)}{2\pi i}\int_{C_{n,\varepsilon}} \frac{u(z)\mathrm{d}z}{(z-t)\sin(\pi z/h)} \\
&= \sum_{k=-n}^{n} u(kh)\frac{(-1)^k \sin(\pi t/h)}{\pi t/h - \pi k} + \frac{\sin(\pi t/h)}{2\pi i}\int_{C_{n,\varepsilon}} \frac{u(z)\mathrm{d}z}{(z-t)\sin(\pi z/h)} \\
&= \sum_{k=-n}^{n} u(kh)\frac{\sin(\pi t/h - \pi k)}{\pi t/h - \pi k} + \frac{\sin(\pi t/h)}{2\pi i}\int_{C_{n,\varepsilon}} \frac{u(z)\mathrm{d}z}{(z-t)\sin(\pi z/h)} \\
&= \sum_{k=-n}^{n} u(kh)\,\mathrm{sinc}\left(\frac{\pi t}{h} - \pi k\right) + \frac{\sin(\pi t/h)}{2\pi i}\int_{C_{n,\varepsilon}} \frac{u(z)\mathrm{d}z}{(z-t)\sin(\pi z/h)}
\end{aligned}$$

where

$$\mathrm{sinc}(x) = \frac{\sin(x)}{x}$$

and

$$(-1)^k \sin(\pi t/h) = \sin(\pi t/h - k\pi)$$

is an extension of

$$-\sin(x) = \sin(x-\pi).$$

The third condition specified earlier states that

$$\int_{-d+\varepsilon}^{d-\varepsilon} |u(x+iy)| \mathrm{d}y$$

is bounded. Therefore, the two vertical integrals must tend to 0 as n (and consequently x) $\to \infty$. The two horizontal integrals can be written as

$$I_{\pm} = \int_{-\infty}^{\infty} \frac{u(x \pm i(d-\varepsilon))\mathrm{d}x}{(x - t \pm i(d-\varepsilon))\sin(\pi(x \pm i(d-\varepsilon))/h)}$$

and therefore we have that

$$u(t) = \sum_{k=-n}^{n} u(kh)\,\mathrm{sinc}\left(\frac{\pi t}{h} - \pi k\right) + \frac{\sin(\pi t/h)}{2\pi i}\left(I_{-} - I_{+}\right).$$

The error terms here are the two integrals I_{-} and I_{+}. These are suitably small, from the conditions imposed upon u, but it becomes easier to deal with them after integrating over the interval $(-\infty, \tau)$ to give

$$\int_{-\infty}^{\tau} u(t)\mathrm{d}t = \sum_{k=-n}^{n} u(kh) \int_{-\infty}^{\tau} \mathrm{sinc}\left(\frac{\pi t}{h} - \pi k\right)\mathrm{d}t + \frac{1}{2\pi i}\int_{-\infty}^{\tau} \sin\left(\frac{\pi t}{h}\right)(I_{-} - I_{+})\mathrm{d}t.$$

For the first integral, we have that

$$\begin{aligned}\int_{-\infty}^{\tau} \mathrm{sinc}\left(\frac{\pi t}{h} - \pi k\right)\mathrm{d}t &= \int_{-\infty}^{\tau} \frac{\sin(\pi t/h - \pi k)}{\pi t/h - \pi k}\mathrm{d}t \\ &= \frac{h}{\pi}\int_{-\infty}^{0} \frac{\sin(x)}{x}\mathrm{d}x + \frac{h}{\pi}\int_{0}^{\pi\tau/h - \pi k} \frac{\sin(x)}{x}\mathrm{d}x \\ &= \frac{h}{2} + \frac{h}{\pi}\,\mathrm{Si}\left(\frac{\pi\tau}{h} - \pi k\right).\end{aligned}$$

The error integrals $I_{\pm}$ are bounded, and so the order of integration can be swapped around, to give

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{u(x \pm i(d-\varepsilon))}{\sin(\pi(x \pm i(d-\varepsilon))/h)} \int_{-\infty}^{\tau} \frac{\sin(\pi t/h)}{x - t \pm i(d-\varepsilon)}\mathrm{d}t\mathrm{d}x.$$

The inner integral can be approximated by

$$\begin{aligned}\int_{-\infty}^{\tau} \frac{\sin(\pi t/h)}{x - t \pm i(d-\varepsilon)}\mathrm{d}t &= \frac{h}{\pi}\int_{-\infty}^{\pi\tau/h} \frac{\sin(y)}{x - hy/\pi \pm i(d-\varepsilon)}\mathrm{d}y \\ &= \mathcal{O}(h),\end{aligned}$$

and the outer, which is now independent of the inner, can be approxuimated by

$$\left| \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{u(x \pm i(d-\varepsilon))\mathrm{d}x}{\sin(\pi(x \pm i(d-\varepsilon))/h)} \right|.$$

The denominator of this integrand can be approximated by

$$|\sin\left(\frac{\pi}{h}(x \pm i(d-\varepsilon))\right)| > \sinh\left(\frac{\pi(d-\varepsilon)}{h}\right) \geq \frac{1}{2}e^{\pi(d-\varepsilon)/h}$$

so we have that this integral can be approximated by

$$\frac{e^{-\pi(d-\varepsilon)/h}}{\pi i} \int_{-\infty}^{\infty} u(x \pm i(d-\varepsilon))\mathrm{d}x.$$

Remembering that this integral must be bounded (condition 2), we therefore have that the error term is

$$\mathcal{O}(he^{-\pi d/h})$$

and the equation is

$$\int_{-\infty}^{\tau} u(t)\mathrm{d}t = h \sum_{k=-\infty}^{\infty} u(kh)\left(\frac{1}{2} + \frac{1}{\pi}\,\mathrm{Si}\left(\frac{\pi\tau}{h} - \pi k\right)\right) + \mathcal{O}(he^{-\pi d/h}).$$

Since it is impossible to calculate an infinite sum, it must be truncated at some point N. The error associated with truncating at this point is

$$\mathcal{O}(he^{-\alpha Nh})$$

where α is a constant such that

$$f(x) = \mathcal{O}(e^{-\alpha|x|}) \text{ as } |x| \to \infty.$$

Balancing the error terms gives the optimal place to truncate the sum is at

$$N = \frac{\pi d}{\alpha h^2}$$

so that

$$\int_{-\infty}^{\tau} u(t)\mathrm{d}t = h \sum_{k=-N}^{N} u(kh)\left(\frac{1}{2} + \frac{1}{\pi}\,\mathrm{Si}\left(\frac{\pi\tau}{h} - \pi k\right)\right) + \mathcal{O}(he^{-\alpha Nh}).$$

Remembering then that

$$\int_{-1}^{s} f(x)\mathrm{d}x = \int_{-\infty}^{\tau} u(t)\mathrm{d}t$$

where

$$u(t) = f(\phi(t))\phi'(t)$$

so we therefore have that

$$\int_{-1}^{s} f(x)\mathrm{d}x = h\sum_{k=-N}^{N} f(\phi(kh))\phi'(kh)\left(\frac{1}{2}+\frac{1}{\pi}\operatorname{Si}\left(\frac{\pi\phi(s)}{h}-\pi k\right)\right) + \mathcal{O}(he^{-\alpha Nh})$$

with

$$\phi(t) = \tanh\left(\frac{\pi}{2}\sinh t\right),$$
$$\phi'(t) = \frac{\pi}{2}(1-\tanh^2(\frac{\pi}{2}\sinh t))\cosh(t),$$

and

$$N = \frac{\pi d}{\alpha h^2}$$

14.2. **Modifying this method.** This method as standard cannot be used for the integral

$$\int_0^\infty \varphi(w_n t)t^{2n}e^{ixt}\mathrm{d}t$$

due to the limits, and the fact that the integrand already has double exponential decay. However, modifying the previous work does still allow the integral of the function to be written as a sum of terms. Defining

$$u(t) = \varphi(w_n t)t^{2n}e^{ixt},$$

where

$$\varphi(w_n t) = e^{-ae^{w_n t}}e^{bw_n t}$$

means that the integral

$$\frac{1}{2\pi i}\int_{C_{n,\varepsilon}} \frac{u(z)\mathrm{d}z}{(z-t)\sin(\pi z/h)}$$

can be calculated in the same manner. In order for the conditions on the integrals of u to hold, it is required that $d = \pi/2$. Therefore, as before

$$u(t) = \sum_{k=-\infty}^{\infty} u(kh)\operatorname{sinc}\left(\frac{t}{h}-k\right) + \frac{\sin(\pi t/h)}{2\pi i}(I_- - I_+),$$

which, upon integrating, becomes

$$\int_0^\infty u(t)\mathrm{d}t = \sum_{k=-\infty}^{\infty} u(kh)\int_0^\infty \operatorname{sinc}\left(\frac{t}{h}-k\right)\mathrm{d}t + \frac{1}{2\pi i}\int_0^\infty \sin\left(\frac{\pi t}{h}\right)(I_- - I_+)\mathrm{d}t.$$

The second integral is bound in the same way as for the general DENIM method, and so won't be recreated here. The first integral is dealt with in a similar way, but the result is significantly different.

$$\begin{aligned}\int_0^\infty \operatorname{sinc}\left(\frac{t}{h}-k\right)\mathrm{d}t &= \int_0^\infty \frac{\sin(\pi t/h-\pi k)}{\pi t/h-\pi k}\mathrm{d}t\\ &= \frac{h}{\pi}\int_{-\pi k}^\infty \frac{\sin(x)}{x}\mathrm{d}x\\ &= \frac{h}{\pi}\int_{-\pi k}^0 \frac{\sin(x)}{x}\mathrm{d}x + \frac{h}{\pi}\int_0^\infty \frac{\sin(x)}{x}\mathrm{d}x\\ &= \frac{h}{\pi}\operatorname{Si}(\pi k)+\frac{h}{2}\end{aligned}$$

Combining this all into the integral gives

$$\int_0^\infty u(t)\mathrm{d}t = h\sum_{k=-\infty}^{\infty} u(kh)\left(\frac{1}{2}+\frac{1}{\pi}\operatorname{Si}(\pi k)\right)+\mathcal{O}(he^{-\pi^2/2h}).$$

This needs to be truncated in the same was as before, and the optimum value of N remains the same, so the function can be approximated by

$$\int_0^\infty u(t)\mathrm{d}t = h\sum_{k=-N}^{N} u(kh)\left(\frac{1}{2}+\frac{1}{\pi}\operatorname{Si}(\pi k)\right)+\mathcal{O}(he^{-\pi^2/2h})$$

where

$$N=\frac{\pi^2}{2bw_nh^2}$$

since, in this example,

$$\begin{aligned}d &= \frac{\pi}{2}\\ \alpha &= bw_n\\ u(t) &= \varphi(w_nt)t^{2n}e^{izt}+\varphi(-w_nt)t^{2n}e^{-izt}.\end{aligned}$$

We can use the result that

$$u(t)=u(-t)$$

so that

$$\begin{aligned}\int_0^\infty u(t)\mathrm{d}t \\ &\approx h \sum_{k=-N}^{N} u(kh) \left(\frac{1}{2} + \frac{1}{\pi} \operatorname{Si}(\pi k) \right) \\ = h \sum_{k=1}^{N} u(kh) \left(\frac{1}{2} + \frac{1}{\pi} \operatorname{Si}(\pi k) \right) + h \sum_{k=1}^{N} u(kh) \left(\frac{1}{2} - \operatorname{Si}(\pi k) \right) + \frac{h}{2} u(0) \\ &= \frac{h}{2} u(0) + h \sum_{k=1}^{N} u(kh).\end{aligned}$$

This is the trapezium formula, which is unsurprising since the integrand already has double exponential decay. However, it is important to note that this is only the case because of the reflection property $u(t) = u(-t)$, and in general, even for functions which decay double exponentially, the sin integral term would still be involved.

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