ON THE FUZZY CONCEPT COMPLEX

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Abstract

Every relation between posets gives rise to an adjunction, known as a Galois connection, between the corresponding power sets. Formal concept analysis (FCA) studies the fixed points of these adjunctions, which can be interpreted as latent "concepts" [20], [19]. In [47] Pavlovic defines a generalisation of posets he calls *proximity sets* (or *proxets*), which are equivalent to the generalised metric spaces of Lawvere [37], and introduces a form of *quantitative concept analysis* (QCA) which provides a different viewpoint from other approaches to fuzzy concept analysis (for a survey see [4]).

The *nucleus* of a fuzzy relation between proxets is defined in terms of the fixed points of a naturally arising adjunction based on the given relation, generalising the Galois connections of formal concept analysis. By giving the unit interval $[0,1]$ an appropriate category structure it can be shown that proxets are simply $[0,1]$-enriched categories and the nuclues of a proximity relation between proxets is a generalisation of the notion of the Isbell completion of an enriched category.

We prove that the sets of fixed points of an adjunction arising from a fuzzy relation can be given the structure of complete idempotent semimodules and show that they are isomorphic to tropical convex hulls of point configurations in tropical projective space, in which addition and scalar multiplication are replaced with pointwise minima and addition, respectively. We show that some the results of Develin and Sturmfels on tropical convex sets [13] can be applied to give the nucleus of a proximity relation the structure of a cell complex, which we term the *fuzzy concept complex*. We provide a formula for counting cells of a given dimension in generic situations.

We conclude with some thoughts on computing the fuzzy concept complex using ideas from Ardila and Develin's work on tropical oriented matroids [1].

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Chapter 1

Introduction

One aim of this thesis is to provide some additional tools for studying concepts in quantitative (or "fuzzy") concept analysis. This is a generalisation of the study of formal concept analysis, which we now briefly describe.

1.1 Formal concept analysis

The aim of Formal Concept Analysis (FCA) is to extract latent concepts, in the form of a *concept lattice*, from a formal context that describes the relationship between two collections. A *formal context* is a triple $K = (X, Y, M)$, where X and Y are sets, normally interpreted as a collection of *objects* and a collection of *attributes* that these objects may satisfy, respectively, and $M\colon X \times Y \to \{0, 1\}$ is a relation encoding information about which attributes are satisfied by each object and, conversely, which objects satisfy each attribute. That is, given an object $x \in X$ and an attribute $y \in Y$ we have

$$M(x, y) = \begin{cases} 1 & \text{if } x \text{ has the attribute } y, \\ 0 & \text{otherwise.} \end{cases}$$

If $X = \{x_1, x_2, \ldots, x_r\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ are finite sets, such a relation can be written as an $(r \times n)$-matrix, whose (i, j)th entry is equal to $M(x_i, y_j)$ for all $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, n\}$. We will abuse notation and call this matrix M too, so that $m_{ij} = M(x_i, y_j)$.

There are obvious functions $M^\bullet\colon X \to \mathcal{P}(Y)$ and $M_\bullet\colon Y \to \mathcal{P}(X)$. Each object $x \in X$ gives rise to a subset $M^\bullet(x) \subseteq Y$ consisting of those attributes satisfied by x. Similarly, each $y \in Y$ gives rise to a subset $M_\bullet(x) \subseteq Y$ consisting of those objects satisfying x.

This correspondence extends to a Galois connection between the powersets of X and Y. Given any subset of objects $A \subseteq X$ one can find the set $A^\uparrow = M^*(A) \subseteq Y$

	male	English	beard	retired
Albert	✓	✓		
Betty		✓		
Charles	✓		✓	✓
Doris				✓
Eric	✓	✓	✓	

Table 1.1: Some people and their attributes

consisting of those attributes satisfied by *all* $a \in A$ and given subset of attributes $B \subseteq Y$ one can find the set $B^{\downarrow} = M_*(B) \subseteq X$ consisting of those objects satisfying *all* $b \in B$. These subsets are the intersections of the subsets corresponding to individual elements:

$$M^*(A) = \bigcap_{a \in A} M^{\bullet}(a) \qquad \text{and} \qquad M_*(B) = \bigcap_{b \in B} M_{\bullet}(b). \tag{1.1}$$

On the one hand, $M^*(A)$ can be viewed as the intersection of the subsets $M^{\bullet}(a)$ corresponding to the elements $a \in A$. From a slightly more sophisticated perspective, which will be useful later, it can be viewed as the intersections of all the subsets $M^{\bullet}(x)$ corresponding to each element $x \in X$, where only those for which $x \in A$ are counted; in other words, the inclusion of $M^{\bullet}(x)$ in the intersection is "weighted" by the truth value of the statement "$x \in A$".

Given a formal context as above, a *formal concept* is a pair (A, B), consisting of subsets $A \subseteq X$ and $B \subseteq Y$, such that $A^{\uparrow} = B$ and $A = B^{\downarrow}$. Each concept can be looked at in two ways: *extrinsically* in terms of its objects, i.e. the subset A, or *intrinsically* in terms of its attributes, i.e. the subset B. This is best illustrated in an example.

Example 1.1.1. For this example let X be the set of people

$$\{\text{Albert, Betty, Charles, Doris, Eric}\}$$

and let Y be the set

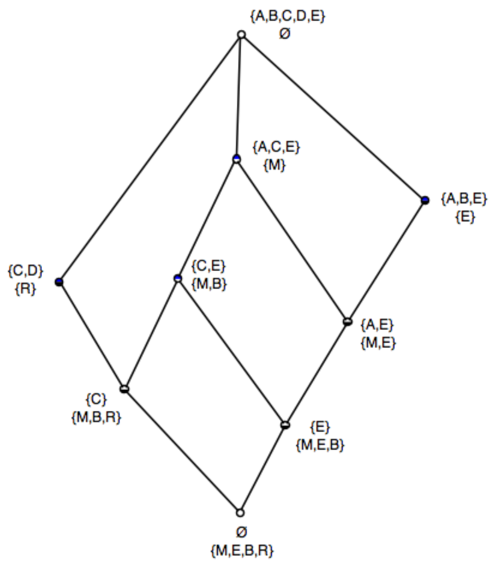
$$\{\text{being male, being English, having a beard, being retired}\}$$

consisting of some of the possible attributes they may have. Which people have which attributes is shown in Table 1.1.

The fact that the powersets $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ are in fact partially ordered sets (with respect to inclusion) allows formal concepts to be organised into a *formal concept lattice*. Given concepts $C = (A, B)$ and $C' = (A', B')$, we say that $C \leqslant C'$

whenever $A \subseteq A'$ *and* $B' \subseteq B$. (Note the opposite ordering on $\mathcal{P}(Y)$; this is due to the Galois connection $M^* \dashv M_*$ being antitone.) Such lattices can be conveniently illustrated as graphs, the vertices of which are simply the formal concepts; the edges do not have any direct meaning other than to indicate the ordering of the concepts.

Here is the concept lattice for the relation defined by the table. Each concept is indicated by a node marked with the corresponding subsets of X and Y that describe the concept extrinsically or intrinsically, respectively. For instance, the node on the far left of the diagram, labelled "$\{C, D\}\{R\}$" corresponds to the concept consisting of (from an extrinsic perspective) Charles and Doris, or equivalently (from an intrinsic perspective) those people who are retired.



In this thesis we study the structure that arises when we consider relations between sets that are not restricted to Boolean values, but instead can describe the relationship between objects in a "fuzzier" way.

1.2 Outline of contents

This thesis is inspired predominantly by two research papers. The first of these is Pavlovic's "Quantitative Concept Analysis" [47], which generalises the idea of formal concept analysis to a quantitative setting. The *nucleus* of a relation is defined as a quantitative alternative to traditional qualitative concept lattices. The second paper is Develin and Sturmfels' "Tropical Convexity" [13] in which it is shown that the tropical convex hull of a set of points in tropical projective space admits a natural cell complex structure. A third paper, Cohen, Gaubert, and

Quadrat's "Duality and Separation Theorems in Idempotent Semimodules" [9] was also influential.

By adapting the definitions of tropical geometry we are able to show that the nucleus of the first paper can be treating in a similar way to the tropical convex hulls of the second paper and that the nucleus admits a cell complex structure. This may have applications in fuzzy concept analysis and it allows one to classify fuzzy concepts in terms of their type, dimension and neighbouring cells.

In Chapter 2 we provide relevant background material on proxets, as introduced in [47], and explain how they can be thought of as enriched categories. We also provide further background material on profunctors and the Isbell completion of an enriched category in preparation for defining proximity relations between proxets (which are a special case of profunctors) and the nucleus of such a proximity relation (which is a special case of a more genereal construction related to the Isbell completion).

In Chapter 3 we define the nucleus of a proximity relation between proxets. We restrict our attention to finite discrete proxets so that proximity relations can be thought of as matrices. We show that $[0,1]^k$ obtains a natural semimodule structure and show that the nucleus of a proximity relation is isomorphic to a submodule of this semimodule. Using this we show that the nucleus of a proximity relation can be expressed geometrically as a span of its rows or columns.

In Chapter 4 we provide a brief introduction to some key ideas from tropical geometry and describe a correspondence between the aforementioned spans and tropical convex hulls of point configurations in tropical projective space. We translate some definitions and results from [13] into the context of these spans. We show that points in the nucleus can be classified according to their "type" and that this classification provides a cell complex structure for the nucleus. We provide a formula for counting k-cells in this cell complex by translating another result from [13].

In Chapter 5 we briefly describe how to compute certain cells in the nucleus of a fuzzy relation and discuss tropical oriented matroids with reference to [1] (see also [23, 24]).

Appendix A provides some background on enriched category theory which may be of particular use in relation to the material on general profunctors and Isbell completions in Chapter 2.

1.3 Acknowledgements

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Chapter 2

Background

This chapter is roughly divided into three parts. In the first part we provide some of the background material needed to define fuzzy relations between proxets and the nuclei of such relations. We have assumed a basic knowledge of category theory, but some definitions and results from the theory of enriched categories can be found in Appendix A, which we will refer to throughout this chapter.

We then move on to talk about profunctors in general enriched categories. Profunctors are a powerful generalisation of functors, relations, and bimodules, amongst other things. Indeed, fuzzy relations between proxets are enriched profunctors.

In the final part we discuss the Isbell completion (also known as the "reflexive completion" [2]). The nucleus of a fuzzy relation is a special case of a generalisation of the Isbell completion of an enriched category.

2.1 Proximity sets

In his paper "Quantitative Concept Analysis" [47], Pavlovic introduced objects he calls *proximity sets* (also called *proxets*), to provide a natural setting for studying fuzzy concepts. Although there are many ways to extend formal concept analysis to a fuzzy (i.e. quantitative) setting — surveys can be found in [3, 4] — proxets are particularly effective. They turn out to be equivalent to the generalised metric spaces introduced by Lawvere [37] and have many nice properties. Definitions and results about proxets can equivalently be stated in terms of generalised metric spaces. See, for example, [48].

Proxets can be seen as categories enriched over the closed interval $[0,1]$ with an appropriate monoidal category structure. We will explain in some detail how this is done. However, the approach we take is to state the basic definitions regarding proxets initially in elementary terms, with no direct reference to enriched categories, before showing that proxets are $[0,1]$-categories. In a number of cases our conventions and notation differ from that of the original.

Definition 2.1.1. A *proximity set* (or *proxet* for short) is a set X together with a function $(-,-)_X \colon X \times X \to [0,1]$, so that any two elements $x, y \in X$ have a *proximity* $(x,y)_X \in [0,1]$, such that the following conditions are satisfied for all $x, y, z \in X$:

$$\begin{aligned}(x,x)_X &= 1,\\ (x,y)_X \cdot (y,z)_X &\leqslant (x,z)_X.\end{aligned}$$

We typically abuse notation and refer to a proxet simply by the name of its underlying set, leaving its proximity operation implicit.

Definition 2.1.2. A proxet X is said to be *extensional* if it satisfies the additional property

$$(x,y)_X = 1 \text{ and } (y,x)_X = 1 \implies x = y, \tag{2.1}$$

for all $x, y \in X$. Otherwise we say that X is *intensional*.

Definition 2.1.3. A proxet X is *discrete* if $(x,y)_X = 0$ for all $x, y \in X$ with $x \neq y$.

Here are some examples of proxets.

Example 2.1.4 (Discrete proxets)**.**

Any set X can be thought of as a discrete proxet by setting, for $x, y \in X$,

$$(x,y)_X = \begin{cases} 1 & \text{if } x = y,\\ 0 & \text{otherwise.}\end{cases}$$

It is trivial to check that the conditions of Definition 2.1.1 are satisfied.

Example 2.1.5 (Preorders)**.**

More generally, the above extends to any preorder $(P, \preceq)$, by setting

$$(x,y)_P = \begin{cases} 1 & \text{if } x \preceq y,\\ 0 & \text{otherwise.}\end{cases}$$

The conditions of Definition 2.1.1 follow immediately from the fact that $\preceq$ is reflexive and transitive.

Example 2.1.6.

The set $[0,1]$ itself can be given the structure of a proxet by defining

$$(x,y)_{[0,1]} = x\backslash y = \begin{cases} y/x & \text{if } y < x, \\ 1 & \text{otherwise.} \end{cases} \tag{2.2}$$

This operation, called *truncated division*, will be extremely important in later chapters. Note that $y\backslash z$ is equal to the largest number in $[0,1]$ that, when multiplied by x, is no larger than y, i.e.

$$x\backslash y = \sup\{w \in [0,1] \mid x \cdot w \leqslant y\}. \tag{2.3}$$

This gives us the defining condition

$$x \cdot y \leqslant z \quad \text{if and only if} \quad x \leqslant y\backslash z \tag{2.4}$$

for all $x, y, z \in [0,1]$.

This means that truncated division is a *residuation* in the sense of [9], as discussed in Section 3.2, where further properties of residuations, and truncated division in particular, are given. One such property is that

$$x(x\backslash y) \leqslant y$$

for all $x, y \in [0,1]$. This follows by straightforward application of (2.4) to the statement $x\backslash y \leqslant x\backslash y$. Another particularly useful property is the following.

Lemma 2.1.7. *Let* $x, y, z \in [0,1]$. *Then*

$$x\backslash(y\backslash z) = (x \cdot y)\backslash z = y\backslash(x\backslash z). \tag{2.5}$$

Proof. Using (2.3) we have

$$\begin{aligned} x\backslash(y\backslash z) &= \sup\{w \in [0,1] \mid x \cdot w \leqslant y\backslash z\} \\ &= \sup\{w \in [0,1] \mid x \cdot w \cdot y \leqslant z\} && \text{(by (2.4))} \\ &= \sup\{w \in [0,1] \mid (x \cdot y) \cdot w \leqslant z\} \\ &= (x \cdot y)\backslash z. \end{aligned}$$

The second equation follows from commutativity of multiplication in $[0,1]$. ◻

The above result will be proved more generally in the context of residuations in idempotent semimodules in the next chapter. More properties will be given when they are needed.

We still need to check that this definition satisfies the conditions of Definition 2.1.1. For any $x \in [0,1]$ it is clear that $x\backslash x = 1$, since $x \geqslant x$. For the second condition, let $x, y, z \in [0,1]$. Then, by the above-stated property of truncated division,

$$x \cdot (x\backslash y) \cdot (y\backslash z) \leqslant y \cdot (y\backslash z) \leqslant z,$$

and then applying (2.4) gives the required condition.

Definition 2.1.8. A *morphism of proximity sets* (or *proximity map*) is a function $f\colon X \to Y$ between the underlying sets of two proxets, such that for all $x, y \in X$

$$(x, y)_X \leqslant (f(x), f(y))_Y.$$

Proximity maps can be composed to produce proximity maps.

Lemma 2.1.9. *Let X, Y, Z be proxets and let $f\colon X \to Y$ and $g\colon Y \to Z$ be proximity maps. Then $g \circ f\colon X \to Z$ is a proximity map.*

Proof. Let $x, y \in X$. Then

$$\begin{aligned}((g \circ f)(x), (g \circ f)(y))_Z &= (g(f(x)), g(f(y)))_Z \\ &\leqslant (f(x), f(y))_Y \qquad \text{(since } g \text{ is a proximity map)} \\ &\leqslant (x, y)_X. \qquad \text{(since } f \text{ is a proximity map)}\end{aligned}$$

Hence $g \circ f$ is a proximity map. □

The category of proxets and proximity maps is denoted Prox.

Definition 2.1.10. Let X and Y be proximity sets. A proximity map $f\colon X \to Y$ is an *isomorphism* of proximity sets (or a *proximity isomorphism*) if there exists a proximity map $g\colon Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

Corollary 2.1.11. *Let $f\colon X \to Y$ be a proximity isomorphism. Then for all $x, x' \in X$ we have $(x, x')_X = (f(x), f(x'))_Y$.*

Proof. Let $x, x' \in X$. Then

$$\begin{aligned}(x, x')_X &\leqslant (f(x), f(x'))_Y \\ &\leqslant (g(f(x)), g(f(x')))_X \\ &= (x, x')_X\end{aligned}$$

so $(x, x')_X = (f(x), f(x'))_Y$ as required. □

2.1.1 Proxets as enriched categories

We now provide an interpretation of proxets as enriched categories. We refer the reader to Appendix A for an introduction to enriched categories, including the definitions of enriched categories, enriched functors, enriched natural transformations, and enriched functor categories, which are mathematical objects that can be thought of as representing the collection of all enriched natural transformations between enriched functors. We will attempt to explain many relevant points in the main text, only referring to Appendix A when necessary.

We must first provide the category to enrich over.

Proposition 2.1.12. *The closed unit interval* $[0,1]$ *can be given the structure of a closed symmetric monoidal category.*

Proof. Firstly, the objects of $[0,1]$ are, unsurprisingly, the elements of the closed unit interval, i.e. $\mathrm{Ob}[0,1] = [0,1]$. There is a morphism in $[0,1]$ from x to y precisely when $x \leqslant y$. In particular, this guarantees a morphism $1_x\colon x \to x$ for all $x \in X$. It is easy to see that this gives $[0,1]$ the structure of a category; transitivity of $\leqslant$ guarantees that composition is associative, while for each $x \in [0,1]$ the morphism 1_x acts as an identity.

The monoidal product on $[0,1]$ is defined to be ordinary multiplication, i.e. $x \otimes y = x \cdot y$ for all $x, y \in [0,1]$. The unit $\mathbb{1}$ for this product is the object $1 \in [0,1]$, since $x \cdot 1 = 1 \cdot x = x$ for all $x \in [0,1]$. The monoidal product thus defined is clearly symmetric, since $x \cdot y = y \cdot x$ for all $x \in [0,1]$.

To see that $[0,1]$ is a closed monoidal category, we must show that for each $y \in [0,1]$ the order-preserving map $x \mapsto x \cdot y$ has a right adjoint. By (2.4), this adjoint is easily seen to the map $z \mapsto x \backslash z$, where $\backslash$ is the truncated division operation defined in (2.2). ◻

It is then easy to see that proxets are equivalent to $[0,1]$-categories. Each proxet X has an underlying set, corresponding to the objects of a $[0,1]$-category $\mathcal{X}$ and vice versa. For each $x, y \in X$, the proximity $(x,y)_X$ is the hom-object $\mathcal{X}(x,y) \in [0,1]$. The reflexivity and transitivity axioms of X are precisely the identity and associativity axioms for composition in $\mathcal{X}$.

We will now therefore consider proxets as if they were defined as $[0,1]$-categories, we will think of proximity maps as $[0,1]$-functors, and so on. This will allow to apply results from Appendix A to proxets.

Extensional proxets are $[0,1]$-categories in which no two distinct objects are isomorphic. In other words, extensional proxets are *skeletal* $[0,1]$-categories.

Before making any further definitions, we highlight another important equivalence of categories. This provides a slightly different way to think of proxets.

Proposition 2.1.13. *The set* $\overline{\mathbb{R}_+} := [0,\infty]$ *can be given the structure of a closed symmetric monoidal category.*

Proof. The objects of the category $\overline{\mathbb{R}_+}$ are non-negative real numbers together with infinity, and there is a morphism $x \to y$ precisely when $x \geqslant y$ (note that this is the other way round to how we defined morphisms in $[0,1]$). Addition acts as the monoidal product and 0 is the monoidal unit. The monoidal product is clearly symmetric. The right adjoint to $x \mapsto x + y$ is given by $z \mapsto z \neg y = \max\{z - y, 0\}$. This is called *truncated subtraction.* ◻

A category X enriched in $\overline{\mathbb{R}_+}$ has, for each pair of objects x and y, a number $X(x,y)$, which can be thought of as the distance from x to y. For $x, y, z \in X$, composition is, by the definition of morphisms in $\overline{\mathbb{R}_+}$, the inequality

$$X(y,z) + X(x,y) \geqslant X(x,z), \tag{2.6}$$

i.e. the familiar *triangle inequality* from the classical study of metric spaces; the 'identity morphism' for $x \in X$ is the inequality

$$0 \geqslant X(x,x), \tag{2.7}$$

which, since the right-hand side is always non-negative, means that $X(x,x) = 0$ for every $x \in X$.

Definition 2.1.14. An $\overline{\mathbb{R}_+}$-category X is called a *generalised metric space*, after Lawvere introduced the notion in his excellent paper [37]. There are a number of differences between such spaces and classical metric spaces:

- distances in X may be infinite;
- X may be non-symmetric, i.e. it is possible that $X(x,y) \neq X(y,x)$;
- distinct points in X may be zero distance apart, i.e. having $X(x,y) = 0$ does not necessarily imply that $x = y$.

Definition 2.1.15. If X and Y are generalised metric spaces, we define an $\overline{\mathbb{R}_+}$-functor $f\colon X \to Y$ to be a map of sets such that, for $x_1, x_2 \in X$,

$$X(x_1,x_2) \geqslant Y(f(x_1), f(x_2)).$$

Such a map is called a *distance non-increasing map*, or a *short map* for short.

The category of generalised metric spaces and distance non-increasing maps is denoted GMet.

Proposition 2.1.16. *The category* GMet *is equivalent to the category* Prox.

Proof. Any generalised metric space X gives rise to a proxet ΦX. Given such a space X and points $x, y \in X$, let $d_X(x,y)$ denote the distance from x to y in X. Given any $b > 1$, we can then define the proximity from x to y as

$$(x,y)_{\Phi X} := b^{-d_X(x,y)}.$$

In particular, if X allows infinite distances and $d_X(x,y) = \infty$ then $(x,y)_{\Phi X} = 0$.

Conversely, any proxet X gives rise to a generalised metric space ΨX by defining

$$d_{\Psi X}(x,y) := -\log_b((x,y)_X),$$

where we use the convention that $\log_b 0 = -\infty$.

If $f\colon X \to Y$ is a distance non-increasing map between generalised metric spaces, i.e. such that

$$d_X(x, x') \geqslant d_Y(f(x), f(x'))$$

for all $x, x' \in X$, then $\Phi f\colon \Phi X \to \Phi Y$, defined simply as $\Phi f(x) := f(x)$ is a proximity map, since

$$(x, x')_{\Phi X} = b^{-d_X(x,x')} \leqslant b^{-d_Y(f(x),f(x'))} = (\Phi f(x), \Phi f(x'))_{\Phi Y},$$

so $\Phi\colon \mathsf{GMet} \to \mathsf{Prox}$ is a functor. We can show that $\Psi\colon \mathsf{Prox} \to \mathsf{GMet}$ is a functor in the same way and it is clear that $\Phi \circ \Psi = \mathrm{id}_{\mathsf{Prox}}$ and $\Psi \circ \Phi = \mathrm{id}_{\mathsf{GMet}}$, so Φ and Ψ constitute an equivalence (in fact an isomorphism) of categories. □

Since proxets are simply $[0, 1]$-categories, the following definitions can be arrived at by specialising the general definitions for enriched categories.

Definition 2.1.17. Let X be a proxet. A *subproxet* of X is a sub-$[0, 1]$-category of X, i.e. a proxet whose underlying set is a subset $Y \subset X$, with proximities given by

$$(x, y)_Y := (x, y)_X$$

for all $x, y \in Y$.

Definition 2.1.18. Let X be a proxet. The *opposite* proxet of X is the opposite $[0, 1]$-category of X, i.e. the proxet X^{op} whose underlying set is equal to that of X, but with proximities given by

$$(x, y)_{X^{\mathrm{op}}} := (y, x)_X,$$

for all $x, y \in X$.

To define products of proxets in the same way — directly from the definition for general enriched categories — we first need to see what the categorical product of two objects in Prox is.

Proposition 2.1.19. *Let X and Y be objects of Prox. The categorical product of X and Y is the proxet $X \times Y$ whose underlying set is the set of pairs (x, y) with $x \in X$ and $y \in Y$, with proximities given by*

$$((x, y), (x', y'))_{X\times Y} := \min\{(x, x')_X, (y, y')_Y\},$$

for all $x, x' \in X$ and $y, y' \in Y$.

Proof. The proxet $X \times Y$ as defined clearly has projections $\pi_X \colon X \times Y \to X$ and $\pi_Y \colon X \times Y \to Y$, defined by $\pi_X(x, y) = x$ and $\pi_Y(x, y) = y$. These can be seen to be proximity maps, since if $x, x' \in X$ and $y, y' \in Y$ we see

$$\begin{aligned}((x, y), (x', y'))_{X \times Y} &= \min\{(x, x')_X, (y, y')_Y\} \\ &\leqslant (x, x')_X \\ &= (\pi_X(x, y), \pi_X(x', y'))_X\end{aligned}$$

and, similarly,

$$\begin{aligned}((x, y), (x', y'))_{X \times Y} &= \min\{(x, x')_X, (y, y')_Y\} \\ &\leqslant (y, y')_Y \\ &= (\pi_Y(x, y), \pi_Y(x', y'))_Y.\end{aligned}$$

Given any other proxet Z with projections $f \colon Z \to X$ and $g \colon Z \to Y$ we can define a proximity map $h \colon Z \to X \times Y$ by $h(z) = (f(z), g(z))$ for $z \in Z$ so that $\pi_X \circ h = f$ and $\pi_Y \circ h = g$. This is a proximity map: for all $z, z' \in Z$,

$$(z, z')_Z \leqslant (f(z), f(z'))_X,$$

since f is a proximity map, and

$$(z, z')_Z \leqslant (g(z), g(z'))_Y,$$

since g is a proximity map, hence

$$(z, z')_Z \leqslant \min\{(f(z), f(z'))_X, (g(z), g(z'))_Y\} = (h(z), h(z'))_{X \times Y}$$

and h is clearly the unique map with this property. This shows that $X \times Y$ is the categorical product of X and Y. □

Definition 2.1.20. Let X and Y be proxets. The *product* of X and Y is the proxet $X \times Y$, defined in Proposition 2.1.19

The definition of the power of a proxet, which can be thought of as the proxet of all proximity maps between two proxets, can also be read off from the general definition of an enriched functor category. See Section A.2 for information about enriched functor categories and ends and coends in general. In the case of proxets ends in $[0, 1]$ appear as infima (see Example A.2.4), so we obtain the following definition:

Definition 2.1.21. Let X and Y be proxets. The *power* proxet Y^X is the proxet whose underlying set is the set $\mathsf{Prox}(X, Y)$ of proximity maps from X to Y, with proximities given by

$$(f, g)_{Y^X} := \inf_{x \in X}\{(f(x), g(x))_Y\},$$

where $f, g \colon X \to Y$ are proximity maps.

Let $*$ be the unique proxet with a single point. The existence of products and powers suggests the following statement.

Proposition 2.1.22. $(\mathsf{Prox}, \times, *)$ *is a closed monoidal category.*

Proof. It is clear that $(\mathsf{Prox}, \times, *)$ is a monoidal category. To see that it is closed, we show that there are natural correspondences of proximity maps

$$\mathsf{Prox}(X, Y) \times \mathsf{Prox}(X, Z) \cong \mathsf{Prox}(X, Y \times Z). \tag{2.8}$$

$$\mathsf{Prox}(X \times Y, Z) \cong \mathsf{Prox}(X, Z^Y). \tag{2.9}$$

In the first isomorphism, a pair of proximity maps $f\colon X \to Y$ and $g\colon X \to Z$ is sent to the proximity map $(f, g)\colon X \to Y \times Z$ given by $(f, g)(x) = (f(x), g(x))$ for $x \in X$. Conversely a proximity map $h\colon X \to Y \times Z$ is sent to the pair of proximity maps $\pi_Y \circ h\colon X \to Y$ and $\pi_Z \circ h\colon X \to Z$, where π_Y and π_Z are the projections from the product $Y \times Z$.

Since f and g are proximity maps, we have

$$(x, x')_X \leqslant (f(x), f(x'))_Y \qquad \text{and} \qquad (x, x')_X \leqslant (g(x), g(x'))_Z,$$

for all $x, x' \in X$. Thus,

$$\begin{aligned}(x, x')_X &\leqslant \min\{(f(x), f(x'))_Y (g(x), g(x'))_Z\} \\ &= ((f(x), g(x)), (f(x'), g(x')))_{X\times Y},\end{aligned}$$

so (f, g) is a proximity map. If h is a proximity map, we see that $\pi_Y \circ h$ and $\pi_Z \circ h$ are proximity maps, since compositions of proximity maps are proximity maps.

In the second isomorphism, $f\colon X \times Y \to Z$ to the proximity map $\hat{f}\colon X \to Z^Y$ defined by $\hat{f}(x)(y) = f(x, y)$ for $x \in X$ and $y \in Y$. Conversely, a proximity map $F\colon X \to Z^Y$ is send to the proximity map $\widetilde{F}\colon X \times Y \to Z$ defined by $\widetilde{F}(x, y) = F(x)(y)$ for $x \in X$ and $y \in Y$.

This shows that $- \times Y$ is left adjoint to $(-)^Y$, which proves the result. □

2.1.2 Fuzzy subsets

In Proposition 2.1.12 we showed that the category $[0, 1]$ is a closed monoidal category. This is what allowed us to define a proxet structure on $[0, 1]$ itself in a natural way, as we did in Example 2.1.6. We can therefore consider $[0, 1]$-functors, i.e. proximity maps, out of or into $[0, 1]$ itself.

For a general closed monoidal category $\mathcal{V}$, a $\mathcal{V}$-functor $P\colon C^{\mathrm{op}} \to \mathcal{V}$ is called a $\mathcal{V}$*-presheaf* (or simply a *presheaf*), while a $\mathcal{V}$-functor $Q\colon C \to \mathcal{V}$ is called a $\mathcal{V}$*-copresheaf* (or just a *copresheaf*). See Appendix A for more details. (Note that the terminology “copresheaf” is not entirely standard. Many authors use the terms “covariant presheaf” or simply “functor” instead.)

In the context of proxets, when $\mathcal{V} = [0,1]$, this leads us to the following definition.

Definition 2.1.23. Let X be a proxet. A *lower subset* of X is a proximity map $A\colon X^{\mathrm{op}} \to [0,1]$, i.e. a function $A\colon X \to [0,1]$ such that for all $x, y \in X$

$$(x,y)_X \cdot A(y) \leqslant A(x). \tag{2.10}$$

An *upper subset* of X is a proximity map $B\colon X \to [0,1]$, i.e. a function $B\colon X \to [0,1]$ such that for all $x, y \in X$

$$B(x) \cdot (x,y)_X \leqslant B(y). \tag{2.11}$$

It is clear from Definition 2.1.21 that the sets $X^{\Downarrow} = [0,1]^{X^{\mathrm{op}}}$ and $X^{\Uparrow} = \left([0,1]^X\right)^{\mathrm{op}}$ of all lower subsets of X and of all upper subsets of X form proxets with proximities given by

$$\begin{aligned} (A,A')_{X^{\Downarrow}} &:= \inf_{x\in X}\{A(x)\backslash A'(x)\}, \\ (B,B')_{X^{\Uparrow}} &:= \inf_{x\in X}\{B'(x)\backslash B(x)\}, \end{aligned}$$

whenever $A, A' \in X^{\Downarrow}$ and $B, B' \in X^{\Uparrow}$.

Lower subsets and upper subsets are collectively referred to as *fuzzy subsets*. In some contexts, particularly when X or Y are finite sets, we may sometimes refer lower and upper subsets as lower and upper *vectors*, respectively. Lower subsets are $[0,1]$-presheaves, while upper subsets are $[0,1]$-copresheaves. Fuzzy subsets generalise the downward- and upward-closed subsets of preorders discussed in Example A.1.15

Fuzzy subsets are useful in the study of fuzzy concept analysis as they will represent the fuzzy concepts appearing in the fuzzy concept complex generated by a fuzzy relation between proxets. Fuzzy relations between proxets are a specific example of enriched *profunctors*, so before defining them we take a brief excursion to discuss profunctors in general $\mathcal{V}$-categories.

2.2 Profunctors

Ordinary (unenriched) profunctors can be thought of as relations between categories. The notion of a profunctor is a generalisation of the notion of a functor. This follows from the fact that the Yoneda embedding $y\colon C \to \widehat{C} = [C^{\mathrm{op}}, \mathsf{Set}]$ is full and faithful. In an enriched setting, profunctors turn out to provide various generalised notions of relations, including, as we shall see, an appropriate notion of a fuzzy relation between proxets.

2.2.1 Definition and examples

In the following, $\mathcal{V}$ is always taken to be a closed symmetric monoidal category, unless otherwise stated.

Definition 2.2.1. Let $\mathcal{C}$ and $\mathcal{D}$ be $\mathcal{V}$-categories. A *profunctor* from $\mathcal{C}$ to $\mathcal{D}$, written $F\colon \mathcal{C} \rightsquigarrow \mathcal{D}$, is a $\mathcal{V}$-functor $F\colon \mathcal{D}^{\mathrm{op}} \otimes \mathcal{C} \to \mathcal{V}$.

There are differing conventions regarding the definition. There are essentially two choices to be made: firstly, which "variable" should be contravariant and which covariant; secondly, which order should they be written in? Taking a profunctor $\mathcal{C} \rightsquigarrow \mathcal{D}$ to be contravariant in its codomain $\mathcal{D}$ and with that argument written first is most consistent with the usual notation for hom-functors.

Note, however, that when we define fuzzy relations between proxets as $[0,1]$-profunctors we will use a slightly different convention, in order to be more consistent with [47].

Lawvere [37] gives an alternative definition in terms of "actions":

Definition 2.2.2 (Alternative definition)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be $\mathcal{V}$-categories. A *profunctor* $F\colon \mathcal{C} \rightsquigarrow \mathcal{D}$ consists of a family of objects $F(D,C)$ in $\mathcal{V}$, indexed by the objects of $\mathcal{D}$ and $\mathcal{C}$, together with morphisms

$$\begin{aligned} \lambda\colon \mathcal{D}(D',D) \otimes F(D,C) &\to F(D',C) \\ \rho\colon F(D,C) \otimes \mathcal{C}(C,C') &\to F(D,C') \end{aligned}$$

that behave as *actions* in the sense that the obvious associativity and unitality axioms (including mixed associativity) hold.

Remark. The two definitions given are equivalent. To see this, recall that $\mathcal{V}$-functoriality for $F\colon \mathcal{D}^{\mathrm{op}} \otimes \mathcal{C} \to \mathcal{V}$ yields morphisms

$$\mathcal{D}(D',D) \otimes \mathcal{C}(C,C') \to [F(D,C), F(D',C')]$$

for all $C, C' \in \mathcal{C}$ and $D, D' \in \mathcal{D}$. The actions λ and ρ are the adjuncts, under the hom–tensor adjunction, of the composites

$$\mathcal{D}(D',D) \xrightarrow{(\mathrm{id}\otimes 1_C)\circ r^{-1}} \mathcal{D}(D',D) \otimes \mathcal{C}(C,C) \xrightarrow{F} [F(D,C), F(D',C)]$$

$$\mathcal{C}(C,C') \xrightarrow{(1_D\otimes \mathrm{id})\circ l^{-1}} \mathcal{D}(D,D) \otimes \mathcal{C}(C,C') \xrightarrow{F} [F(D,C), F(D,C')]$$

Example 2.2.3. Profunctors generalise $\mathcal{V}$-functors. Since the Yoneda embedding is full and faithful, a $\mathcal{V}$-functor $\mathcal{C} \to \widehat{\mathcal{D}} = [\mathcal{D}^{\mathrm{op}}, \mathcal{V}]$ is a generalisation of a $\mathcal{V}$-functor $\mathcal{C} \to \mathcal{D}$. The hom–tensor adjunction gives a natural correspondence between $\mathcal{V}$-functors $F\colon \mathcal{D}^{\mathrm{op}} \otimes \mathcal{C} \to \mathcal{V}$ and $\mathcal{V}$-functors $\widehat{F}\colon \mathcal{C} \to \widehat{\mathcal{D}}$.

Given any $\mathcal{V}$-functor $F: \mathcal{C} \to \mathcal{D}$, there are two canonical ways to produce a profunctor from F.

$$F_*: \mathcal{C} \rightsquigarrow \mathcal{D}: (D, C) \mapsto \mathcal{D}(D, FC) \tag{2.12}$$
$$F^*: \mathcal{D} \rightsquigarrow \mathcal{C}: (C, D) \mapsto \mathcal{D}(FC, D). \tag{2.13}$$

The profunctor F_* is sometimes referred to as "F considered as a profunctor" and F^* is its "right adjoint profunctor" (see 2.2.12 below), for example in [6].

We now give some examples of how profunctors manifest themselves in some of the enriched categories defined in Appendix A.

Example 2.2.4 (Hom-functors)**.** For a $\mathcal{V}$-category $\mathcal{C}$, the hom-functor $\mathrm{Hom}_{\mathcal{C}} = \mathcal{C}(-,-): \mathcal{C}^{\mathrm{op}} \otimes \mathcal{C} \to \mathcal{V}$ is a profunctor from $\mathcal{C}$ to itself. When $\mathcal{V}$ is Set, $\mathrm{Hom}_{\mathcal{C}}$ sends a pair of objects X and Y to the set $\mathcal{C}(X, Y)$ of morphisms $X \to Y$ and sends a pairs of morphisms $f: X' \to X$ and $g: Y \to Y'$ to the function $g \circ - \circ f: \mathcal{C}(X, Y) \to \mathcal{C}(X', Y')$ which maps $h: X \to Y$ to $g \circ h \circ f$.

In fact, as we will see, $\mathrm{Hom}_{\mathcal{C}}$ acts as an identity under profunctor composition and is thus the identity on $\mathcal{C}$ in the category $\mathcal{V}$-Prof defined below. When it clear that we are referring to this profunctor, and not to the identity $\mathcal{V}$-functor $\mathrm{id}_{\mathcal{C}}: \mathcal{C} \to \mathcal{C}$, we will sometimes use the notation $\mathrm{id}_{\mathcal{C}}$.

Example 2.2.5 (Bimodules)**.** Let R and S be rings, i.e. one-object Ab-categories. A profunctor $M: R \rightsquigarrow S$ is an Ab-functor $S^{\mathrm{op}} \otimes R \to \mathsf{Ab}$. This gives an Abelian group $M := M(*, *)$ and a morphism $S^{\mathrm{op}} \otimes R \to \mathrm{End}(M)$ which, under the hom–tensor adjunction corresponds to a morphism $S^{\mathrm{op}} \otimes M \otimes R \to M$ that sends $s \otimes m \otimes r$ to $s \cdot m \cdot r$. Functoriality says that this acts as an action:

$$s_2 \cdot (s_1 \cdot m \cdot r_1) \cdot r_2 = (s_2 s_1) \cdot m \cdot (r_1 r_2)$$

Thus M is an R-S-bimodule (i.e. simultaneously a left R-module and a right S-module).

Example 2.2.6 ("Categorified matrices")**.** Let X and Y be sets, considered as discrete categories. A profunctor $F: X \rightsquigarrow Y$ is a functor $F: Y^{\mathrm{op}} \times X \to \mathsf{Set}$. Since there are no non-identity arrows in $Y^{\mathrm{op}} \times X$, this is just a function assigning to each pair (y, x) a set $F(y, x)$. In other words, F is just a bundle of sets over the product $Y \times X$. Such an object can be thought of as a sort of "categorified matrix" where each entry is now a set, rather than simply a number.

Example 2.2.7 (Relations)**.** Let X and Y be posets, i.e. 2-categories. A profunctor $R: X \rightsquigarrow Y$ is a 2-functor $Y^{\mathrm{op}} \times X \to 2$, which assigns to each pair $(y, x) \in Y^{\mathrm{op}} \times X$ a "truth value" indicating whether or not x and y are related by R. Such maps correspond to relations $R \subset Y \times X$ with the following transitivity property ensured by functoriality: if $x \leqslant x'$ and $R(y, x)$ is true (x is R-*related* to y) then also $R(y, x')$, and similarly in the other variable. In particular, if X and Y are sets, i.e. posets with trivial ordering, a profunctor $X \rightsquigarrow Y$ is just an ordinary relation between X and Y.

The previous example allows us to think of profunctors as "relations between $\mathcal{V}$-categories" where the "truth values" of R-relatedness are objects of $\mathcal{V}$. This is particularly relevant when considering fuzzy relations as we shall see in the next chapter.

2.2.2 The Bicategory $\mathcal{V}$-Prof

As seen in Example 2.2.3, $\mathcal{V}$-profunctors can be seen as generalised $\mathcal{V}$-functors. However, because the domains and codomains of profunctors $F\colon \mathcal{C} \rightsquigarrow \mathcal{D}$ and $G\colon \mathcal{D} \rightsquigarrow \mathcal{E}$ do not match up (when considered as $\mathcal{V}$-functors), composing them is not as straightforward as it is for $\mathcal{V}$-functors. In this subsection we will show how composition of $\mathcal{V}$-profunctors can nonetheless be defined and how this can be used to define a bicategory of $\mathcal{V}$-profunctors.

Definition 2.2.8 (Composition of profunctors)**.** Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be $\mathcal{V}$-categories and let $F\colon \mathcal{C} \rightsquigarrow \mathcal{D}$ and $G\colon \mathcal{D} \rightsquigarrow \mathcal{E}$ be profunctors. The composite $G \circ F\colon \mathcal{C} \rightsquigarrow \mathcal{E}$ is defined, for $C \in \mathcal{C}$ and $E \in \mathcal{E}$, via the following coend:

$$(G \circ F)(E, C) := \int^{D \in \mathcal{D}} G(E, D) \otimes F(D, C). \tag{2.14}$$

Analogously to (A.3), $(G \circ F)(E, C)$ can be written as the coequaliser of the two morphisms

$$\coprod_{D_1, D_2 \in \mathcal{D}} G(E, D_1) \otimes \mathcal{D}(D_1, D_2) \otimes F(D_2, C) \rightrightarrows \coprod_{D \in \mathcal{D}} G(E, D) \otimes F(D, C), \tag{2.15}$$

the top arrow being induced by the action

$$\mathrm{id} \otimes \rho\colon G(E, D_1) \otimes \mathcal{D}(D_1, D_2) \otimes F(D_2, C) \to G(E, D_1) \otimes F(D_1, C)$$

and the bottom arrow induced similarly by $\lambda \otimes \mathrm{id}$. This is a many-object analogue of the quotienting out by the equality of two actions that takes place when we form the tensor product of bimodules, as we describe in more detail below. See also [51].

Example 2.2.9 (Tensor product of bimodules)**.** Consider an R-S-bimodule M and an S-T-bimodule N over S. We form the tensor product $M \otimes_S N$ by quotienting out $M \otimes N$ by the equivalence relation generated by $(m \cdot s) \otimes n \sim m \otimes (s \cdot n)$. This is the one object specialisation of the vastly more general definition 2.2.8.

Example 2.2.10. If X, Y and Z are posets and $R\colon X \rightsquigarrow Y$ and $S\colon Y \rightsquigarrow Z$ are relations, the composite $S \circ R$ corresponds to the subset

$$\{(z,x) \in Z \times X \mid \exists y \in Y \text{ such that } R(y,x) \wedge S(z,y)\},$$

which agrees with the usual definition of composition of relations.

We refer to [33] for a proof of the following important result

Proposition 2.2.11. *$\mathcal{V}$-categories, enriched profunctors and natural transformations form a bicategory $\mathcal{V}$-*Prof.

As in any 2-category, we can now talk about adjunctions in $\mathcal{V}$-Prof.

Definition 2.2.12. Let $\mathcal{C}$ and $\mathcal{D}$ be $\mathcal{V}$-categories and let $P\colon \mathcal{C} \rightsquigarrow \mathcal{D}$ and $Q\colon \mathcal{D} \rightsquigarrow \mathcal{C}$ be profunctors. We say that P is *left adjoint* to Q (and, conversely, that Q is *right adjoint* to P) if there are natural transformations $\eta\colon \mathrm{id}_{\mathcal{C}} \Rightarrow Q \circ P$ and $\varepsilon\colon P \circ Q \Rightarrow \mathrm{id}_{\mathcal{D}}$. We write $P \dashv Q$, as for an any adjunction. Here, $Q \circ P$ and $P \circ Q$ are, of course, composites as defined in 2.2.8 and $\mathrm{id}_{\mathcal{C}}$ and $\mathrm{id}_{\mathcal{D}}$ are the identity profunctors.

2.2.3 Fuzzy relations between proxets

Fuzzy relations between proxets can now be defined as $[0,1]$-profunctors. The following definition is of fundamental importance in this thesis.

Fuzzy relations, as the name suggests, provide a way of describing how data in two proxets is related in a looser way than that of Formal Concept Analysis as described in the introduction, since they essentially allow for two objects to be only partially related. In this setting, instead of a concept lattice arising from a relation between sets we study a certain proxet arising from a fuzzy relation between proxets which we call the *nucleus*, which is a specific example of the generalised Isbell completion of a profunctor, to be defined in the next section. We will later show that for a fuzzy relation between finite discrete proxets (i.e. ordinary sets) the nucleus can be given the structure of a cell complex.

Definition 2.2.13. Let X and Y be proxets. A *fuzzy relation* (or *proximity relation*) $M\colon X \rightsquigarrow Y$ between X and Y is a $[0,1]$-profunctor from X to Y. In other words, a fuzzy relation $M\colon X \rightsquigarrow Y$ is a proximity map $M\colon X^{\mathrm{op}} \times Y \to [0,1]$, i.e. a function $M\colon X \times Y \to [0,1]$ such that for all $x, x' \in X$, $y, y' \in Y$

$$(x',x)_X \cdot M(x,y) \cdot (y,y')_Y \leqslant M(x',y'),$$

where $M(x,y)$ denotes the image of (x,y) under M.

As stated below Definition 2.2.1, note that we are using a slightly different convention to that given for general $\mathcal{V}$-profunctors in order to be more consistent with [47]. Using a different convention doesn't change any important properties of profunctors.

Proximity relations can be composed in the following way, following Definition 2.2.8 (and noting the change in convention).

Definition 2.2.14. Let X, Y, and Z be proxets and let $M\colon X \rightsquigarrow Y$ and $N\colon Y \rightsquigarrow Z$ be proximity relations. The *composite* of M and N is the proximity relation $N \circ M\colon X \rightsquigarrow Z$ defined by

$$(N \circ M)(x,z) := \sup_{y \in Y}\{M(x,y) \cdot N(y,z)\},$$

for $x \in X$ and $z \in Z$.

More information about fuzzy relations can be found in [47], in which they are referred to as "proximity matrices". For example, Pavlovic defines the notions of the "dual" of a proximity matrix and a "connection" between proximity matrices and shows that every proximity matrix forms a connection with its dual.

2.3 The Isbell completion

The Isbell completion of a $\mathcal{V}$-category C simultaneously generalises two superficially non-categorical constructions, namely the Dedekind–MacNeille completion of a poset and the tight span of a metric space. We will describe each of these in detail before introducing the general definition of the Isbell completion of a $\mathcal{V}$-category and showing that this reduces to the motivating examples in the cases where $\mathcal{V}$ is 2 or $\overline{\mathbb{R}_+}$, respectively.

Example 2.3.1 (Dedekind–MacNeille completion of a poset)**.** Let $(X, \leqslant)$ be a poset. Recall that a subset $D \subseteq X$ is called *downward-closed* or *descending* if, whenever $x \in X$ and $x' \leqslant x$, then $x' \in X$ and similarly, a subset $U \subseteq X$ is called *upward-closed* or *ascending* if, whenever $x \in X$ and $x \leqslant x'$, then $x' \in X$.

Given a downward-closed subset $D \subseteq X$, we can form an upward-closed subset

$$D^u := \left\{x \in X \mid x' \leqslant x \text{ for all } x' \in D\right\};$$

conversely, given an upward-closed subset $U \subseteq X$, we can form a downward-closed subset

$$U^d := \left\{x \in X \mid x \leqslant x' \text{ for all } x' \in U\right\}.$$

Note that, by definition, $D \subseteq (D^u)^d$ and $U \subseteq (U^d)^u$.

The *Dedekind–MacNeille completion* of X, written $\mathrm{DM}(X)$, is defined to be the set of downward-closed subsets $D \subseteq X$ such that $D = (D^u)^d$; equivalently, $\mathrm{DM}(X)$ is the set of upward-closed subsets $U \subseteq X$ such that $U = (U^d)^u$. An alternative, but equivalent, definition is

$$\mathrm{DM}(X) := \left\{(D, U) \mid D^u = U \text{ and } D = U^d\right\}.$$

This set has a natural ordering given by setting $(D_1, U_1) \leqslant (D_2, U_2)$ if and only if $D_1 \subseteq D_2$ (or, equivalently, if $U_1 \supseteq U_2$).

As an example, let X be the set $\mathbb{Q}$ of rational numbers with the usual ordering. For each real number $x \in \mathbb{R}$ there is a downward-closed subset $D_x := \{x' \in \mathbb{R} \mid x' \leqslant x\}$ and every downward-closed subset is of this form. It is straightforward to check that $(D_x^u)^d = D_x$, so we have a bijection $\mathrm{DM}(\mathbb{Q}) \cong \mathbb{R}$. Moreover, this is an order-isomorphism: $x \leqslant y$ if and only if $(D_x, D_x^u) \leqslant (D_y, D_y^u)$ for all $x, y \in \mathbb{R}$. This example is known as the completion of $\mathbb{Q}$ by Dedekind cuts. Furthermore, there is a full and faithful embedding $\mathbb{Q} \hookrightarrow \mathrm{DM}(\mathbb{Q})$ given by sending each $x \in \mathbb{Q}$ to the downward-closed subset D_x as defined above.

Note that not every downward-closed subset $D \subset X$ satisfies $(D^u)^d = D$. For example, if X is the real line and $D = \{x \in \mathbb{R} \mid x < 0\}$, then we find that $(D^u)^d = \{x \in \mathbb{R} \mid x \leqslant 0\}$, which is a proper superset of D.

Example 2.3.2 (Tight span of a metric space)**.** Let X be a generalised metric space, i.e. a category enriched in $\overline{\mathbb{R}_+}$ as defined in Definition 2.1.14. A presheaf on X is an enriched functor $f\colon X^{\mathrm{op}} \to \overline{\mathbb{R}_+}$, i.e. a function $f\colon X \to [0, \infty]$ satisfying

$$X(x_1, x_2) \geqslant f(x_1) - f(x_2)$$

for all $x_1, x_2 \in X$. Write $L(X)$ for the space of all such functions, with distances given by the sup metric. It follows that $f(x) \geqslant \sup_{x' \in X}(f(x') - X(x', x))$ for all $x \in X$ for each $f \in L(X)$. This is in fact an equality, since the supremum is attained when $x' = x$.

On the other hand, given such a function f we can define

$$l(f)(x) := \sup_{x' \in X}(X(x', x) - f(x')).$$

The function $l(f)$ satisfies

$$\begin{aligned} l(f)(x_2) - l(f)(x_1) &= \sup_{x' \in X}(X(x', x_2) - f(x')) - \sup_{x' \in X}(X(x', x_1) - f(x')) \\ &\leqslant \sup_{x' \in X}(X(x', x_2) - f(x') - X(x', x_1) + f(x')) \\ &\leqslant \sup_{x' \in X}(X(x', x_2) - X(x', x_1)) \\ &\leqslant \sup_{x' \in X}(X(x_1, x_2)) \\ &\leqslant X(x_1, x_2), \end{aligned}$$

and is thus a copresheaf. Define the *tight span* of X to be the subspace of $L(X)$ consisting of those functions for which this is an equality, i.e. for which

$$f(x) = \sup_{x' \in X}(X(x', x) - f(x')).$$

In [60], Willerton has shown that Isbell completions of generalised metric spaces are complete if and only if they admit a certain type of "semi-tropical" module structure.

We now turn to the general case. Let $\mathcal{V}$ be a symmetric monoidal closed category and let C be a $\mathcal{V}$-category. There is a $\mathcal{V}$-adjunction between the $\mathcal{V}$-category of presheaves on C and the opposite of the $\mathcal{V}$-category of copresheaves on C

$$\begin{tikzcd} {[C^{\mathrm{op}}, \mathcal{V}]} \arrow[r, shift left, "L"] \arrow[r, phantom, "\perp"] & {[C, \mathcal{V}]^{\mathrm{op}}} \arrow[l, shift left, "R"] \end{tikzcd}$$

where the $\mathcal{V}$-functors L and R are given by

$$\begin{aligned} L(P)&\colon C \mapsto [C^{\mathrm{op}}, \mathcal{V}](P, C(-, C)) \\ R(Q)&\colon C \mapsto [C, \mathcal{V}](Q, C(C, -)) \end{aligned}$$

This is known as the *Isbell adjunction*. The *Isbell completion* of C, denoted $I(C)$, is defined to be the full sub-$\mathcal{V}$-category of $[C^{\mathrm{op}}, \mathcal{V}]$ consisting of those presheaves that are *Isbell self-dual*, i.e. those presheaves for which the unit of the Isbell adjunction is an isomorphism:

$$\mathrm{Ob}\, I(C) := \{P\colon C^{\mathrm{op}} \to \mathcal{V} \mid \eta_P\colon \mathbb{1} \to [C^{\mathrm{op}}, \mathcal{V}](P, RL(P)) \text{ is an iso}\}\,.$$

Another way of saying this is to say that $I(C)$ consists of the fixed points of the monad RL induced by the adjunction. Equivalently, $I(C)$ can be defined as the fixed points of the induced comonad LR. Fullness simply means that the hom-object for two presheaves in the Isbell completion is the same as the hom-object when these presheaves are interpreted as objects of $[C^{\mathrm{op}}, \mathcal{V}]$.

There is also a third way to describe the Isbell completion. Define the objects of a $\mathcal{V}$-category $\mathrm{Dual}(L, R)$ to be quadruples (P, Q, α, β), with $P\colon C^{\mathrm{op}} \to \mathcal{V}$, $Q\colon C \to \mathcal{V}$, $\alpha\colon \mathbb{1} \Rightarrow [C^{\mathrm{op}}, \mathcal{V}](P, RQ)$, $\beta\colon \mathbb{1} \Rightarrow [C, \mathcal{V}](LP, Q)$ such that α and β are isomorphisms and are mutually adjoint under the adjunction $L \dashv R$. In the case when $\mathcal{V} = \mathsf{Set}$, the hom-objects are pairs of morphisms (f, g) such that the squares

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "\alpha"'] & P' \arrow[d, "\alpha'"] \\ RQ \arrow[r, "Rg"'] & RQ' \end{tikzcd} \qquad \begin{tikzcd} LP \arrow[r, "Lf"] \arrow[d, "\beta"'] & LP' \arrow[d, "\beta'"] \\ Q \arrow[r, "g"'] & Q' \end{tikzcd} \tag{2.16}$$

commute. In the enriched case, this can be written as an equaliser.

Given an object (P, Q, α, β) in $\mathrm{Dual}(L, R)$, one finds that P is an object of $\mathrm{Fix}(RL)$. Conversely, each P in $\mathrm{Fix}(RL)$ corresponds to the tuple $(P, LP, \eta_P, \mathrm{id}_{LP})$ in $\mathrm{Dual}(L, R)$.

Any profunctor between $\mathcal{V}$-categories gives rise to a $\mathcal{V}$-category which generalises the Isbell completion.

Definition 2.3.3. Let $\mathcal{V}$ be a complete and cocomplete monoidal category and let $\mathcal{C}$ and $\mathcal{D}$ be $\mathcal{V}$-categories. Let $M\colon \mathcal{C} \rightsquigarrow \mathcal{D}$ be a $\mathcal{V}$-profunctor. The *generalised Isbell completion* is the sub-$\mathcal{V}$-category $\mathrm{Nuc}\, M \subset [\mathcal{C}^{\mathrm{op}}, \mathcal{V}] \times [\mathcal{D}, \mathcal{V}]^{\mathrm{op}}$ consisting of the objects

$$\{(P, Q, \alpha, \beta) \mid \alpha\colon P \cong M_*Q, \beta\colon M^*P \cong Q, \alpha \text{ and } \beta \text{ natural, mutually adjoint}\},$$

where $M^* \dashv M_*$ are defined as follows:

$$M^*P(D) = \int_{C \in \mathcal{C}} \mathcal{V}(P(C), M(C, D)),$$
$$M_*Q(C) = \int_{D \in \mathcal{D}} \mathcal{V}(Q(D), M(C, D)).$$

In the next chapter we will study a special case of the generalised Isbell completion for a profunctor, namely the nucleus of a fuzzy relation between proxets. Since proxets can be thought of $[0,1]$-categories and fuzzy relations as $[0,1]$-profunctors, the nucleus of a fuzzy relation between proxets will turn out to be a sub-$[0,1]$-category, i.e. a subproxet, of $X^{\Downarrow} \times Y^{\Uparrow}$.

The nucleus of a fuzzy relation between proxets can be thought of as a generalisation of the concept lattice generated by a relation between sets, as described in the introduction. Elements of the nucleus can be thought of as fuzzy concepts. In Chapter 4 we will show that for a fuzzy relation between finite discrete proxets (i.e. ordinary finite sets) the nucleus actually has the structure of a cell complex and these concepts can be classified in ways which are not found in classical concept analysis.

Chapter 3

The nucleus of a fuzzy relation

In the previous chapter we introduced proxets, upper and lower subsets, and fuzzy relations. These ideas were introduced in [47] and are specific examples of enriched categories. In this chapter we move on to define the *nucleus* of a fuzzy relation, focusing particularly on fuzzy relations between finite sets.

After introducing some of the theory of idempotent semirings and their semimodules, following Cohen et al. [9], we show that the nucleus of a fuzzy relation M is in bijection with certain sets, denoted $\mathrm{Fix}^{\Downarrow}(M)$ and $\mathrm{Fix}^{\Uparrow}(M)$, that can be regarded as submodules of a complete idempotent semimodule over $[0,1]$ and study the structure of these submodules. Because of their smaller dimensions, these submodules are easier to deal with than the nucleus.

Much of what is done in this chapter can be done in considerably more generality. See, for example, [52, 55–58]. We restrict our attention to fuzzy relations between finite sets as it is in this case that we can adapt ideas and results from tropical geometry over the idempotent semiring $\mathbb{T} = (\mathbb{R} \cup \{\infty\}, \min, +)$ to show that the sets $\mathrm{Fix}^{\Downarrow}(M)$ and $\mathrm{Fix}^{\Uparrow}(M)$ can be naturally given the structure of a cell complex in addition to their structure as proxets and $[0,1]$-semimodules as we will do in the next chapter. While nucleus-like structures have been studied in various contexts — as proxets, as more general enriched categories, as semimodules — this restricted definition allows us to study several structures at once.

3.1 Basic definitions

Recall from Definition 2.1.23 that $X^{\Downarrow}$ is the proxet of lower subsets of X and $Y^{\Uparrow}$ is the proxet of upper subsets of Y, i.e.

$$X^{\Downarrow} = \{A\colon X \to [0,1] \mid (x,x')_X \cdot A(x') \leqslant A(x)\ \forall x,x' \in X\},$$
$$Y^{\Uparrow} = \{B\colon Y \to [0,1] \mid B(y) \cdot (y,y')_Y \leqslant B(y')\ \forall y,y' \in Y\}.$$

Definition 3.1.1. Let X and Y be proxets and let $M\colon X \rightsquigarrow Y$ be a fuzzy relation. Then the proximity maps

$$M^*\colon X^{\Downarrow} \to Y^{\Uparrow} \quad \text{and} \quad M_*\colon Y^{\Uparrow} \to X^{\Downarrow}$$

are defined by setting, for $A \in X^{\Downarrow}$ and $B \in Y^{\Uparrow}$,

$$(M^*A)(y) = \inf_{x\in X} \{A(x)\backslash M(x,y)\}\,, \tag{3.1}$$

$$(M_*B)(x) = \inf_{y\in Y} \{B(y)\backslash M(x,y)\}\,. \tag{3.2}$$

When it is clear which proximity relation M we are using, we will sometimes use the shorthand notation $A^{\uparrow}$ and $B^{\downarrow}$ to indicate the upper and lower subsets M^*A and M_*B, respectively.

Proposition 3.1.2. *Given $M\colon X \rightsquigarrow Y$ as above, the proximity maps M^* and M_* form an adjunction $M^* \dashv M_*$, in the sense that*

$$(M^*A, B)_{Y^{\Uparrow}} = (A, M_*B)_{X^{\Downarrow}}, \tag{3.3}$$

whenever $A \in X^{\Downarrow}$ and $B \in Y^{\Uparrow}$.

Proof. Let $A \in X^{\Downarrow}$ and $B \in Y^{\Uparrow}$. Then we can simply compute

$$\begin{aligned}
(M^*A, B)_{Y^{\Uparrow}} &= \inf_{y\in Y} \{B(y)\backslash(M^*A)(y)\} && \text{(by definition of proximities in } Y^{\Uparrow}\text{)}\\
&= \inf_{y\in Y} \left\{B(y)\backslash \inf_{x\in X} \{A(x)\backslash M(x,y)\}\right\} && \text{(by definition of } M^*A\text{)}\\
&= \inf_{x\in X} \inf_{y\in Y} \{B(y)\backslash(A(x)\backslash M(x,y))\}\\
&= \inf_{x\in X} \inf_{y\in Y} \{(A(x)\cdot B(y))\backslash M(x,y)\} && \text{(by (2.5))}\\
&= \inf_{x\in X} \inf_{y\in Y} \{A(x)\backslash(B(y)\backslash M(x,y))\} && \text{(by (2.5) again)}\\
&= \inf_{x\in X} \left\{A(x)\backslash \inf_{y\in Y} \{B(y)\backslash M(x,y)\}\right\}\\
&= \inf_{x\in X} \{A(x)\backslash(M_*B)(x)\} && \text{(by definition of } M_*B\text{)}\\
&= (A, M_*B)_{X^{\Downarrow}}. && \square
\end{aligned}$$

The above results hold for all proxets X and Y. However, in order to prove the main results of the next section we must restrict our attention to *extensional* proxets (cf. Definition 2.1.2)

Lemma 3.1.3. *Let X and Y be extensional proxets and let $f\colon X \to Y$ and $g\colon Y \to X$ be proximity maps such that f is left adjoint to g, in the sense of Proposition 3.1.2, i.e. such that $(f(x), y)_Y = (x, f(y))_X$ for all $x \in X$ and $y \in Y$. Then*

$$fgf = f \qquad \textit{and} \qquad gfg = g.$$

Proof. We prove only the first equation in the first statement as the second is entirely analogous. For $y \in Y$ it follows from the fact that $(g(y), g(y))_X = 1$ that $(fg(y), y)_Y = 1$, via the adjunction. So, in particular, when $y = f(x)$ for some $x \in X$ we have

$$(fgf(x), f(x))_Y = 1.$$

Similarly, for each $x \in X$ it follows from the fact that $(f(x), f(x))_Y = 1$ that $(x, gf(x))_X = 1$. By applying the proximity map $f : X \to Y$ we see that

$$(f(x), fgf(x))_Y = 1.$$

Since Y was assumed to be extensional, it follows that $f(x) = fgf(x)$ for each $x \in X$ and hence $f = fgf$. $\square$

Proposition 3.1.2 and Lemma 3.1.3 immediately give the following result.

Corollary 3.1.4. *Let X and Y be extensional proxets. Given any fuzzy relation $M : X \rightsquigarrow Y$, the proximity maps M^* and M_* satisfy*

$$M^* M_* M^* = M^* \qquad \textit{and} \qquad M_* M^* M_* = M_*. \tag{3.4}$$

Definition 3.1.5. Let X and Y be extensional proxets and let $M : X \rightsquigarrow Y$ be a fuzzy relation between X and Y. Define the following subproxets of $X^{\Downarrow}$ and $Y^{\Uparrow}$, respectively:

$$\mathrm{Fix}^{\Downarrow}(M) = \left\{A \in X^{\Downarrow} \mid M_* M^* A = A\right\} \subseteq X^{\Downarrow}, \tag{3.5}$$
$$\mathrm{Fix}^{\Uparrow}(M) = \left\{B \in Y^{\Uparrow} \mid M^* M_* B = B\right\} \subseteq Y^{\Uparrow}. \tag{3.6}$$

Proposition 3.1.6. *Let X and Y be extensional proxets and let $M : X \rightsquigarrow Y$ be a fuzzy relation between X and Y. Then*

$$\mathrm{Fix}^{\Downarrow}(M) = \operatorname{im} M_* \qquad \textit{and} \qquad \mathrm{Fix}^{\Uparrow}(M) = \operatorname{im} M^*.$$

Proof. Let $A \in \mathrm{Fix}^{\Downarrow}(M)$. Then $A = M_* M^* A$, so clearly $A \in \operatorname{im} M_*$. Conversely, suppose $A \in \operatorname{im} M_*$, i.e. $A = M_* B$ for some $B \in Y^{\Uparrow}$. Then

$$M_* M^* A = M_* M^* M_* B = M_* B = A,$$

by Lemma 3.1.3. Hence $A \in \mathrm{Fix}^{\Downarrow}(M)$.

The second statement is proved in the same way. $\square$

The following definition is very important.

Definition 3.1.7. Let X and Y be extensional proxets and let $M\colon X \rightsquigarrow Y$ be a fuzzy relation between X and Y. We define the *nucleus* of M to be the proxet whose underlying set is

$$\mathrm{Nuc}(M) = \left\{(A,B) \in X^{\Downarrow} \times Y^{\Uparrow} \mid M^{*}A = B, A = M_{*}B\right\}, \tag{3.7}$$

with proximities given by

$$((A,B),(A',B'))_{\mathrm{Nuc}(M)} = (A,A')_{X^{\Downarrow}} = (B,B')_{Y^{\Uparrow}},$$

where the last equality follows immediately from Proposition 3.1.2.

The nucleus of a fuzzy relation $M\colon X \rightsquigarrow Y$ between proxets is the generalised Isbell completion of M regarded as a $[0,1]$-profunctor. In this sense it is a generalisation of the Dedekind–MacNeille completion of a poset.

Proposition 3.1.8. *There are isomorphisms of proxets*

$$\mathrm{Nuc}(M) \cong \mathrm{Fix}^{\Downarrow}(M) \cong \mathrm{Fix}^{\Uparrow}(M).$$

Proof. Let $(A,B) \in \mathrm{Nuc}(M)$. Then it is immediate that $A \in \mathrm{Fix}^{\Downarrow}(M)$ and $B \in \mathrm{Fix}^{\Uparrow}(M)$. Conversely, given $A \in \mathrm{Fix}^{\Downarrow}(M)$, we find that $(A,A^{\uparrow}) \in \mathrm{Nuc}(M)$ and, similarly, given $B \in \mathrm{Fix}^{\Uparrow}(M)$, we find that $(B^{\downarrow},B) \in \mathrm{Nuc}(M)$. Moreover, by Definition 3.1.7, the maps $(A,B) \mapsto A$, $(A,B) \mapsto B$, $A \mapsto (A,A^{\uparrow})$, and $B \mapsto (B^{\downarrow},B)$ are all easily seen to be proximity maps, giving the desired result. $\square$

3.1.1 Nuclei of finite discrete proxets

In this section we restrict our attention to fuzzy relations between finite discrete proxets, i.e. ordinary finite sets. See Definitions 2.1.2 and 2.1.3. Since the proximities of elements in a discrete proxet are only non-zero for identical elements, such proxets are automatically extensional. Under this assumption we are able to provide a geometric description of the sets $\mathrm{Fix}^{\Uparrow}(M)$, $\mathrm{Fix}^{\Downarrow}(M)$, and $\mathrm{Nuc}(M)$ for a fuzzy relation $M\colon X \rightsquigarrow Y$.

Let $X = \{x_1,\dots,x_r\}$ and $Y = \{y_1,\dots,y_n\}$ be finite sets considered as discrete proxets. Since non-equal elements in a discrete proxet must have zero proximity, it is clear that upper and lower subsets of such a proxet can both be represented simply as tuples, since the conditions of equations (2.10) and (2.11) hold vacuously.

Nonetheless, it will be helpful to distinguish between lower and upper subsets in our notation. Lower subsets of X will be represented by column vectors

$$v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_r \end{pmatrix},$$

while upper subsets of Y will be represented by row vectors

$$w = \begin{pmatrix} w_1 & w_2 & \cdots & w_n \end{pmatrix},$$

where all entries are in $[0,1]$.

That is, for finite discrete proxets X and Y with $|X| = r$ and $|Y| = n$ we have:

$$X^{\Downarrow} = [0,1]^{r,1}, \tag{3.8}$$

$$Y^{\Uparrow} = [0,1]^{1,n}. \tag{3.9}$$

A fuzzy relation $M\colon X \rightsquigarrow Y$ can be written as an $(r \times n)$-matrix

$$M = \begin{pmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & \ddots & \vdots \\ m_{r1} & \cdots & m_{rn} \end{pmatrix},$$

all of whose entries are in $[0,1]$.

The proximity morphisms $M^*\colon X^{\Downarrow} \to Y^{\Uparrow}$ and $M_*\colon Y^{\Uparrow} \to X^{\Downarrow}$ reduce in this context to the definitions

$$M^*(v)_j = \min_{1\leqslant i\leqslant r} \{v_i \backslash m_{ij}\} \in [0,1]^n, \tag{3.10}$$

$$M_*(w)_i = \min_{1\leqslant j\leqslant n} \{w_j \backslash m_{ij}\} \in [0,1]^r, \tag{3.11}$$

where $v \in [0,1]^r$ and $w \in [0,1]^n$.

Conversely, any matrix in $[0,1]^{r,n}$ can be regarded as a fuzzy relation between finite sets. Given such a matrix M, the underlying sets of the proxets $\mathrm{Fix}^{\Downarrow}(M)$ and $\mathrm{Fix}^{\Uparrow}(M)$ will be subsets of $[0,1]^r$ and $[0,1]^n$, respectively. We will show that $\mathrm{Col}(M) \subseteq \mathrm{Fix}^{\Downarrow}(M) \subseteq [0,1]^r$ and $\mathrm{Row}(M) \subseteq \mathrm{Fix}^{\Uparrow}(M) \subseteq [0,1]^n$. The nucleus, $\mathrm{Nuc}(M)$, is naturally a subset of $[0,1]^r \times [0,1]^n$.

Lemma 3.1.9. *Let X and Y be finite discrete proxets with $|X| = r$ and $|Y| = n$ and let $M\colon X \rightsquigarrow Y$ be a fuzzy relation. The nucleus of M is given by*

$$\mathrm{Nuc}(M) = \left\{ (v,w) \in [0,1]^{r,1} \times [0,1]^{1,n} \;\middle|\; \begin{array}{l} \forall i \ (v_i = \min\limits_{1\leqslant j\leqslant n} \{w_j \backslash m_{ij}\}) \\ \forall j \ (w_j = \min\limits_{1\leqslant i\leqslant r} \{v_i \backslash m_{ij}\}) \end{array} \right\}. \tag{3.12}$$

Proof. By Definition 3.1.1, $v = M_*(w)$ if and only if $v_i = \min_{1\leqslant j\leqslant n}\{w_j \backslash m_{ij}\}$ for all $i \in [r]$. Similarly, $w = M^*(v)$ if and only if $w_j = \min_{1\leqslant i\leqslant r}\{v_i \backslash m_{ij}\}$ for all $j \in [n]$. The result follows by consideration of Definition 3.1.7. □

3.2 Complete idempotent semirings and semimodules

In [9], Cohen et al. consider semimodules over complete idempotent semirings. We show that $[0,1]$ can be given the structure of a complete idempotent semiring and that the spaces $[0,1]^r$ and $[0,1]^n$ can be considered as complete semimodules over the semiring $[0,1]$ and that for a given fuzzy relation M between finite sets, the sets $\mathrm{Fix}^{\Downarrow}(M)$ and $\mathrm{Fix}^{\Uparrow}(M)$ can be considered as (finitely-generated) sub-semimodules of these semimodules.

First, for convenience we given some fundamental definitions here. The interested reader should consult [9] for further details about idempotent semimodules.

Definition 3.2.1. A *semiring* $R = (R, \oplus, \odot, 0, 1)$ is a set R equipped with two binary operations $\oplus\colon R \times R \to R$ (called *addition*) and $\odot\colon R \times R \to R$ (called *multiplication*) such that:

1. $(R, \oplus, 0)$ is a commutative monoid, i.e.

$$a \oplus (b \oplus c) = (a \oplus b) \oplus c,$$
$$a \oplus 0 = a,$$
$$a \oplus b = b \oplus a,$$

   for all $a, b, c \in R$ and $0 \in R$ is called the *additive identity* for R;

2. $(R, \odot, 1)$ is a monoid, i.e.

$$a \odot (b \odot c) = (a \odot b) \odot c,$$
$$a \odot 1 = a,$$
$$1 \odot a = a,$$

   for all $a, b, c \in R$ and $1 \in R$ is called the *multiplicative identity* for R;

3. multiplication distributes over addition on both sides, i.e.

$$a \odot (b \oplus c) = a \odot b \oplus a \odot c,$$
$$(a \oplus b) \odot c = a \odot c \oplus b \odot c,$$

   for all $a, b, c \in R$;

4. multiplication by 0 annihilates elements of R, i.e.

$$a \odot 0 = 0 \odot a = 0, \tag{3.13}$$

   for all $a \in R$.

Note that although condition in (3.13) is automatically true for rings, it does not follow from the other semiring axioms, so it is necessary to include it in the definition.

Homomorphisms of semirings are defined in the obvious way.

Definition 3.2.2. A *homomorphism of semirings* $f\colon R \to S$ is a function that is simultaneously a monoid homomorphism from $(R, \oplus_R, 0_R)$ to $(S, \oplus_S, 0_S)$ and a monoid homomorphism from $(R, \odot_R, 1_R)$ to $(S, \odot_S, 1_S)$, i.e.

$$f(a \oplus_R b) = f(a) \oplus_S f(b), \tag{3.14}$$

$$f(a \odot_R b) = f(a) \odot_S f(b), \tag{3.15}$$

$$f(0_R) = 0_S, \tag{3.16}$$

$$f(1_R) = 1_S, \tag{3.17}$$

for all $a, b \in R$.

We are primarily interested in semirings with the following additional properties. We follow the presentation of Cohen et al. [9] fairly closely, although we work in less generality and omit some details.

Definition 3.2.3. A semiring $(R, \oplus, \odot)$ is *commutative* if $(R, \odot)$ is a commutative monoid, i.e. if $a \odot b = b \odot a$ for all $a, b \in R$.

Definition 3.2.4. A semiring $(R, \oplus, \odot)$ is *idempotent* if $a \oplus a = a$ for all $a \in R$.

Recall that an ordered set X is said to be *complete* if any subset $Y \subseteq X$ has a supremum $\sup Y \in X$. Note that the existence of suprema guarantees the existence of infima, since for any subset $Y \subseteq X$, we have

$$\inf Y = \sup\{x \in X \mid x \leqslant y \ \forall y \in Y\}. \tag{3.18}$$

Any idempotent commutative monoid $(X, \oplus, 0)$ — in particular, any idempotent semiring — can be given a *natural order*, by defining $x \leqslant y \iff x \oplus y = y$ for $x, y \in X$. With respect to this order $x \oplus y = \sup\{x, y\}$ for all $x, y \in X$ and $0 \in X$ is the bottom element.

This allows us to make the following definition.

Definition 3.2.5. A semiring $R = (R, \oplus, \odot)$ is *complete* if it is complete with respect to the natural order on R (i.e. if each subset $S \subseteq R$ has a supremum $\sup S \in R$) and for all $S \subseteq R$ and $b \in R$ we have

$$\sup\{a \odot b \mid b \in S\} = a \odot (\sup S).$$

Our primary example of a complete commutative idempotent semiring will be $[0, 1]$ together with the operations of taking maxima and ordinary multiplication.

Proposition 3.2.6. *The set* $[0,1]$*, together with the operations*

$$a \oplus b = \max\{a, b\} \qquad \text{and} \qquad a \odot b = a \cdot b, \tag{3.19}$$

for $a, b \in [0,1]$*, is a complete commutative idempotent semiring with additive and multiplicative identities given by* 0 *and* 1*, respectively.*

Proof. The axioms for a semiring can be easily checked. Let $a, b, c \in [0,1]$.

1. It is clear that $\max\{a, \max\{b, c\}\} = \max\{\max\{a, b\}, c\}$, that $\max\{a, 0\} = a$ and that $\max\{a, b\} = \max\{b, a\}$, so $([0,1], \max, 0)$ is a commutative monoid.

2. That $([0,1], \cdot, 1)$ is a monoid follows immediately from the associativity and unitality of ordinary multiplication of real numbers.

3. To demonstrate distributivity of $\odot$ over $\oplus$, we check

$$\begin{aligned} a \odot (b \oplus c) &= a \cdot \max\{b, c\} \\ &= \max\{a \cdot b, a \cdot c\} && \text{(since } a \geqslant 0) \\ &= (a \odot b) \oplus (a \odot c) \end{aligned}$$

   and

$$\begin{aligned} (a \oplus b) \oplus c &= \max\{a, b\} \cdot c \\ &= \max\{a \cdot c, b \cdot c\} && \text{(since } c \geqslant 0) \\ &= (a \odot c) \oplus (b \odot c) \end{aligned}$$

4. Finally, $a \cdot 0 = 0 \cdot a = 0$, so the annihilation property is satisfied.

The additive identity is 0, since $\max\{a, 0\} = a$ for all $a \in [0,1]$, while the multiplicative identity is 1. Since $a \cdot b = b \cdot a$ and $\max\{a, a\} = a$ for all real numbers a, b it is clear that $[0,1]$ is commutative and idempotent.

The natural order on $[0,1]$ given by $a \leqslant b \iff \max\{a, b\} = b$ is simply the standard order on $\mathbb{R}$. As a closed bounded subset of $\mathbb{R}$, $[0,1]$ is complete with respect to this order and for all $S \subseteq [0,1]$ and $a \in [0,1]$ we clearly have $\sup\{a \cdot b \mid b \in S\} = a \cdot (\sup S)$. □

Note that $[0,1]$ is only a *semi*ring and not a ring, since given $a \in [0,1]$ there does not, in general, exist any $b \in [0,1]$ for which $\max\{a, b\} = 0$, unless $a = 0$. Nor is $[0,1]$ a semi*field*, since unless $a = 1$ there is no $b \in [0,1]$ with $a \cdot b = 1$.

The semiring structure defined above is not the only complete commutative idempotent semiring structure that can be defined on $[0,1]$. Alternatively we could take, for example, $a \oplus b = \min\{a, b\}$ and $a \odot b = a \cdot b$.

Definition 3.2.7. Let R be a commutative idempotent semiring. An R*-semimodule* is a commutative monoid $X = (X, \oplus_X, 0_X)$ equipped with an *action* $R \times X \to X$, where the image of $(a, x) \in R \times X$ is written $a \cdot x$, such that:

$$(a \odot b) \cdot x = a \cdot (b \cdot x), \tag{3.20}$$

$$a \cdot (x \oplus y) = (a \cdot x) \oplus (a \cdot y), \tag{3.21}$$

$$(a \oplus b) \cdot x = (a \cdot x) \oplus (b \cdot x), \tag{3.22}$$

$$0_R \cdot x = 0_X, \tag{3.23}$$

$$1_R \cdot x = x, \tag{3.24}$$

for all $a, b \in R$ and $x, y \in X$.

Definition 3.2.8. An R-semimodule $(X, \oplus_X, 0_X)$ is *idempotent* if the monoid operation $\oplus_X$ is idempotent, i.e. if for all $x, y \in X$ we have $x \oplus_X x = x$.

When R is a non-commutative semiring, one must define *left* R*-semimodules* and *right* R*-semimodules* separately. An R*-bisemimodule* is then a set equipped with both a left R-semimodule structure and a right R-semimodule structure, such that the left and right actions commute. However, since the particular semiring that we will be principally dealing with happens to be commutative, left and right semimodules coincide, i.e. all our semimodules are bisemimodules.

The following two observations are straightforward.

Lemma 3.2.9. *If* R *is a commutative idempotent semiring and* X *is an* R*-semimodule, then* X *is idempotent.*

Proof. Let $x \in X$. Then, since R is idempotent,

$$x = 1_R \cdot x = (1_R \oplus 1_R) \cdot x = (1_R \cdot x) \oplus (1_R \cdot x) = x \oplus x. \qquad \square$$

Lemma 3.2.10. *If* R *is a commutative semiring and* X *is an* R*-semimodule, then* $a \cdot 0_X = 0_X$ *for all* $a \in R$.

Proof. Let $a \in R$ and $x \in X$. Then

$$a \cdot 0_X = a \cdot (0_R \odot x) = (a \odot 0_R) \cdot x = 0_R \cdot x = 0_X. \qquad \square$$

Definition 3.2.11. If R is a complete commutative idempotent semiring, an R-semimodule X is said to be *complete* if it is complete with respect to the natural order (i.e. if each subset $Y \subseteq X$ has a supremum $\sup Y \in X$) and if whenever $a \in R, x \in X, S \subseteq R, Y \subseteq X$, we have

$$\sup\{a \cdot y \mid y \in Y\} = a \cdot (\sup Y), \tag{3.25}$$

$$\sup\{a \cdot x \mid a \in S\} = (\sup S) \cdot x. \tag{3.26}$$

Definition 3.2.12. Let R be a complete commutative idempotent semiring, and let X be an R-semimodule. A submonoid $Y \subseteq X$ is an *R-subsemimodule* of X if $a \cdot y \in Y$ for all $a \in R$ and $y \in Y$.

For brevity, we also refer to R-subsemimodules as R-submodules.

Definition 3.2.13. Let R be a commutative complete idempotent semiring. A *free complete R-semimodule* is of the form R^I for some set I: the elements of R^I are functions $f: I \to R$, with addition and action defined pointwise as $(f \oplus g)(i) := f(i) \oplus g(i)$ and $(a \cdot f)(i) := a \cdot f(i)$ for all $f, g \in R^I$, $a \in R$, and $i \in I$.

If I is a finite set, say $I = [r]$ for some $r \in \mathbb{N}$, we use the notation R^r for R^I and write elements of R^r as vectors $v = (v_1, \ldots, v_r)$, where $v_i = v(i)$.

Example 3.2.14. Consider the complete commutative idempotent semiring $[0,1] = ([0,1], \max, \cdot)$. Let $r \in \mathbb{N}$. The set $[0,1]^r$ can be given the structure of a free complete $[0,1]$-semimodule, with

$$(v \oplus v')_i = \max\left\{v_i, v'_i\right\}, \tag{3.27}$$

and

$$(\lambda \cdot v)_i = \lambda \cdot v_i. \tag{3.28}$$

for all $v, v' \in [0,1]^r$ and $\lambda \in [0,1]$. Note that the natural order on $[0,1]^r$ is defined as $v \leqslant v' \iff \max\{v, v'\} = v'$, so $v \leqslant v'$ if and only if $v_i \leqslant v'_i$ for all $i \in [r]$. Thus $[0,1]^r$ is only partially ordered; v and v' will not necessarily be comparable.

The following definition will ultimately allow us to think of the spaces $[0,1]^{r,1}$ and $[0,1]^{1,n}$ as complete idempotent $[0,1]$-semimodules in the appropriate way.

Definition 3.2.15. A map $f: X \to Y$ between ordered sets is *residuated* if there exists a map $g: Y \to X$ such that

$$f(x) \leqslant y \quad \text{if and only if} \quad x \leqslant g(y), \tag{3.29}$$

for all $x \in X$ and $y \in Y$. The map g is called the *residuation* of f.

For more information on residuation theory, see [7, 25].

If R is a complete commutative idempotent semiring, the residuation of the map $b \mapsto a \odot b$ (multiplication by a) is the map $c \mapsto c \oslash a$, where

$$c \oslash a := \sup\{b \in R \mid a \odot b \leqslant c\}, \tag{3.30}$$

which is guaranteed to exist, since R is complete. Therefore, by definition,

$$a \odot b \leqslant c \quad \text{if and only if} \quad b \leqslant c \oslash a \tag{3.31}$$

for all $a, b, c \in R$.

Example 3.2.16. When $R = ([0,1], \max, \cdot)$, this operation is simply the truncated division operated defined earlier in (2.2), since for $a, b \in [0,1]$:

$$\begin{aligned} c \oslash a &= \sup\{b \in [0,1] \mid a \cdot b \leqslant c\} \\ &= \begin{cases} \sup\left\{b \in [0,1] \mid b \leqslant \frac{c}{a}\right\} & \text{if } a \neq 0, \\ 1 & \text{if } a = 0, \end{cases} \\ &= \begin{cases} \frac{c}{a} & \text{if } a > c, \\ 1 & \text{otherwise}, \end{cases} \\ &= a \backslash c. \end{aligned}$$

This definition can also be extended to complete idempotent semimodules. If R is a complete commutative idempotent semiring and X is a complete idempotent R-semimodule we can define for $a \in R$ and $x \in X$:

$$x \oslash a := \sup\{y \in X \mid a \cdot y \leqslant x\}. \tag{3.32}$$

Thus, by definition,

$$a \cdot y \leqslant x \quad \text{if and only if} \quad y \leqslant x \oslash a. \tag{3.33}$$

Some useful properties follow from (3.33).

Lemma 3.2.17. *Let R be a complete commutative idempotent semiring and let X be a complete idempotent R-semimodule. Let $a, b \in R$, $x \in X$. Then*

$$a \cdot (x \oslash a) \leqslant x, \tag{3.34}$$

$$(a \cdot x) \oslash a \geqslant x, \tag{3.35}$$

$$(x \oslash a) \oslash b = x \oslash (a \odot b) = (x \oslash b) \oslash a. \tag{3.36}$$

Proof. Equations (3.34) and (3.35) are immediate from (3.33). To prove the first equation in (3.36), let $a, b \in R$, $x \in X$. Then

$$\begin{aligned} x \oslash (a \odot b) \leqslant (x \oslash a) \oslash b &\iff b \cdot (x \oslash (a \odot b)) \leqslant x \oslash a && \text{(by (3.33))} \\ &\iff a \cdot (b \cdot (x \oslash (a \odot b))) \leqslant x \\ &\iff (a \odot b) \cdot (x \oslash (a \odot b)) \leqslant x && \text{(by (3.20))} \\ &\iff x \oslash (a \odot b) \leqslant x \oslash (a \odot b), \end{aligned}$$

but the last statement is a tautology. Similarly,

$$\begin{aligned} (x \oslash a) \oslash b \leqslant x \oslash (a \odot b) &\iff (a \odot b) \cdot ((x \oslash a) \oslash b) \leqslant x && \text{(by (3.33))} \\ &\iff a \cdot (b \cdot ((x \oslash a) \oslash b)) \leqslant x && \text{(by (3.20))} \\ &\iff b \cdot ((x \oslash a) \oslash b) \leqslant x \oslash a \\ &\iff (x \oslash a) \oslash b \leqslant (x \oslash a) \oslash b, \end{aligned}$$

where, again, the last statement is a tautology. Hence $(x \oslash a) \oslash b = x \oslash (a \odot b)$. The second equation in (3.36) is proved in exactly the same way. $\square$

Example 3.2.18. When R is the complete idempotent semiring $([0,1], \max, \cdot)$ and X is the complete free idempotent semimodule $[0,1]^r$ we have

$$v \oslash \lambda = \lambda \backslash v = \sup\{w \in [0,1]^r \mid \lambda \cdot w \leqslant v\},$$

where $v \in [0,1]^r$ and $\lambda \in [0,1]$.

Recall that the natural order on $[0,1]^r$ is defined so that $v \leqslant v'$ if and only if $\max\{v, v'\} = v'$, i.e. $v \leqslant v'$ if and only if $v_i \leqslant v'_i$ for all $i \in [r]$. This means that $\lambda \backslash v$ is such that for each $i \in [r]$ we have $(\lambda \backslash v)_i = \sup\{w_i \in [0,1] \mid \lambda \cdot w_i \leqslant v_i\}$. That is, $\lambda \backslash v \in [0,1]^r$ has coordinates $\lambda \backslash v_i$ for each $i \in [r]$.

Given any ordered set X, one can consider the set X^{op} with the same elements as X but the opposite order, i.e. $x \leqslant^{\mathrm{op}} y$ in X^{op} if and only if $y \leqslant x$, for all $x, y \in X$. If $(X, \oplus, 0_X)$ is a monoid then $(X^{\mathrm{op}}, \boxplus, 0_{X^{\mathrm{op}}})$ is a monoid, where $x \boxplus y = \min\{x, y\}$ (with respect to X) and $0_{X^{\mathrm{op}}}$ is the top element of X^{op}.

Definition 3.2.19. Let R be a complete commutative idempotent semiring and let X be a complete R-semimodule. The *opposite semimodule* of X is the R-semimodule X^{op} with the same underlying set but with addition

$$x \boxplus y = \min\{x, y\}, \tag{3.37}$$

where min is with respect to the natural order $\leqslant$ on X (*not* $\leqslant^{\mathrm{op}}$), and with action

$$a \boxdot x = x \oslash a, \tag{3.38}$$

for $x, y \in X$ and $a \in R$.

It is straightforward to check that X^{op} really is a semimodule. Crucially, it follows from (3.36) that for all $x \in X^{\mathrm{op}}$ and $a, b \in [0,1]$

$$(a \odot b) \boxdot x = a \boxdot (b \boxdot x).$$

For $x, y \in X^{\mathrm{op}}$ and $a, b \in [0,1]$ we can apply (3.32) to see

$$\begin{aligned}
a \boxdot (x \boxplus y) &= \min\{x, y\} \oslash a \\
&= \sup\{z \in X \mid a \cdot z \leqslant \min\{x, y\}\} \\
&= \min\{\sup\{z \in X \mid a \cdot z \leqslant x\}, \sup\{z \in X \mid a \cdot z \leqslant y\}\} \\
&= \min\{x \oslash a, y \oslash a\} \\
&= (a \boxdot x) \boxplus (a \boxdot y)
\end{aligned}$$

and, similarly,

$$\begin{aligned}
(a \oplus b) \boxdot x &= x \oslash \max\{a, b\} \\
&= \sup\{z \in X \mid \max\{a, b\} \cdot z \leqslant x\} \\
&= \min\{\sup\{z \in X \mid a \cdot z \leqslant x\}, \sup\{z \in X \mid b \cdot z \leqslant x\}\} \\
&= \min\{x \oslash a, x \oslash b\} \\
&= (a \boxdot x) \boxplus (b \boxdot x).
\end{aligned}$$

Recalling that the additive and multiplicative identities for the semiring $[0,1]$ are 0 and 1, respectively, and remembering that the additive identity for X^{op} is $\underline{1}$, we also check that

$$0 \boxdot x = x \oslash 0 = \sup\{z \in X \mid 0 \cdot z \leqslant x\} = \underline{1}$$

and

$$1 \boxdot x = x \oslash 1 = \sup\{z \in X \mid 1 \cdot z \leqslant x\} = x$$

for all $x \in X^{\mathrm{op}}$.

Moreover, it follows from the properties of residuation that X^{op} is a complete R-semimodule. See [9] for a proof.

Note that (3.37) defines addition in the opposite semimodule of a general semimodule X in terms of the natural order on X. It is not an analogue of (3.27); that role is filled by (3.39).

The next example is of fundamental importance to this thesis.

Example 3.2.20. When R is the complete commutative idempotent semiring $([0,1], \max, \cdot)$, the opposite semimodule of the free complete semimodule $[0,1]^r$ (for some $r \in \mathbb{N}$) is the semimodule with underlying set $[0,1]^r$, with addition

$$(v \boxplus v')_i = \min\{v_i, v'_i\}, \tag{3.39}$$

and action

$$(\lambda \boxdot v)_i = \lambda \backslash v_i, \tag{3.40}$$

for $v, v' \in [0,1]^r$, $\lambda \in [0,1]$, and $i \in [r]$.

Note that the vector $\underline{1} = (1, \ldots, 1) \in [0,1]^r$ is the additive identity element in this semimodule: $v \boxplus \underline{1} = \underline{1} \boxplus v = v$ for all $v \in [0,1]^r$.

This complete idempotent $[0,1]$-semimodule and its submodules will be of primary importance in later chapters. For convenience we may refer to this semimodule as the *fuzzy semimodule* $[0,1]^r$, due to its importance of its submodules $\mathrm{Fix}^{\Downarrow}(M)$ and $\mathrm{Fix}^{\Uparrow}(M)$ for a fuzzy relation M. More often, however, when we simply refer to "the semimodule $[0,1]^r$", we mean this semimodule.

Note that, by (3.36),

$$\lambda \boxdot (\mu \boxdot v) = (\lambda \mu) \boxdot v, \tag{3.41}$$

for all $\lambda, \mu \in [0,1]$ and $v \in [0,1]^r$.

For vectors $v, v' \in [0,1]^k$ we refer to the vector $v \boxplus v'$ as the *fuzzy sum* of v and v' and we refer to the vector $\lambda \boxdot v$ as the *λ-scale of v*. We will sometimes refer to the operations $\boxplus$ and $\boxdot$ as the *box operations.*

3.3 Fuzzy spans

In this section we make some observations about the structure of the semimodule $[0,1]^r$ with the structure defined in Example 3.2.20. This section will serve as a prelude to the discussion of tropical polytopes in the next chapter.

Definition 3.3.1. Let V be a finite subset of $[0,1]^r$, where $r \in \mathbb{N}$. A *fuzzy linear combination* of elements of V is a vector

$$\boxed{+}_{v \in V} \lambda_v \boxdot v = \min_{v \in V} \{\lambda_v \backslash v\}$$

where $\lambda_v \in [0,1]$ for all $v \in V$.

Note that $0 \boxdot v = (1, \ldots, 1)$ for any $v \in [0,1]^r$, where $(1, \ldots, 1)$ is the identity of the semimodule $[0,1]^r$ defined above.

Definition 3.3.2. Let V be a finite subset of $[0,1]^r$, where $r \in \mathbb{N}$. The *fuzzy span* of V is the set of all fuzzy linear combinations of elements of V:

$$\operatorname{span}_{\boxplus\boxdot}(V) = \left\{ \boxed{+}_{v \in V} \lambda_v \boxdot v \;\middle|\; \lambda_v \in [0,1] \text{ for all } v \in V \right\} \tag{3.42}$$

Fuzzy spans are $[0,1]$-submodules of the $[0,1]$-semimodule $[0,1]^r$ defined in Example 3.2.20.

Proposition 3.3.3. *Let V be a finite subset of $[0,1]^r$, where r is a natural number. Then $\operatorname{span}_{\boxplus\boxdot}(V) \subset [0,1]^r$ is a complete $[0,1]$-submodule of $[0,1]^r$ with the operations $\boxplus$ and $\boxdot$.*

Proof. We need to check that $\operatorname{span}_{\boxplus\boxdot}(V)$ is a submonoid of $[0,1]^r$ and that for all $\alpha \in [0,1]$ and all $x \in \operatorname{span}_{\boxplus\boxdot}(V)$ we have $\alpha \boxdot x \in \operatorname{span}_{\boxplus\boxdot}(V)$. It is clear that $\operatorname{span}_{\boxplus\boxdot}(V)$ is a monoid. To see that it is a submonoid of $[0,1]^r$, consider $x, y \in \operatorname{span}_{\boxplus\boxdot}(V)$. Writing

$$x = \boxed{+}_{v \in V} \lambda_v \boxdot v, \qquad y = \boxed{+}_{v \in V} \mu_v \boxdot v,$$

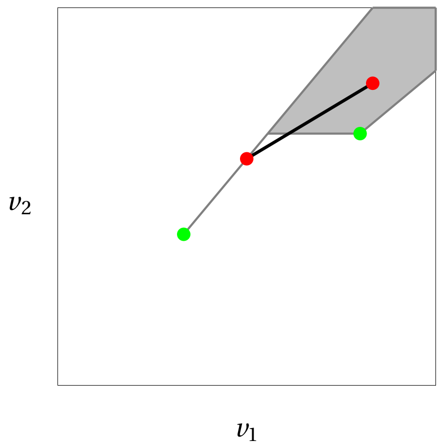


Figure 3.1: Fuzzy spans are not convex in general. The area in grey, consisting of a closed region and a line segment, is the fuzzy span of the two green points. The two red points are contained in the fuzzy span of the green points but the affine line segment connecting them is not.

we see that

$$x \boxplus y = \bigboxplus_{v \in V} (\lambda_v \boxplus \mu_v) \boxdot v \in \mathrm{span}_{\boxplus\boxdot}(V).$$

Similarly,

$$\alpha \boxdot x = \bigboxplus_{v \in V} (\alpha \lambda_v) \boxdot v \in \mathrm{span}_{\boxplus\boxdot}(V).$$

Thus $\mathrm{span}_{\boxplus\boxdot}(V)$ is a submodule of $[0,1]^r$.

To see that $\mathrm{span}_{\boxplus\boxdot}(V)$ is complete, note that for any subset $X \subseteq \mathrm{span}_{\boxplus\boxdot}(V)$, the point $\bigboxplus_{x \in X} x$ is in $\mathrm{span}_{\boxplus\boxdot}(V)$ (if $X = \varnothing$, the empty sum is interpreted as the point $\underline{1} \in [0,1]^r$ while if X is infinite we can interpret this sum as the point whose ith coordinate is given by $\inf_{x \in X}\{x_i\}$ for each i). Since the natural order on $\mathrm{span}_{\boxplus\boxdot}(V)$ (inherited from $[0,1]^r$) is such that $x \leqslant y$ (in $\mathrm{span}_{\boxplus\boxdot}(V)$) if and only if $x_i \geqslant y_i$ (in the usual sense, in $[0,1]$) for all $i \in [r]$, this point is a supremum for X. It is easy to check that the conditions in (3.25) also hold. □

It is clear that the fuzzy span of a finite set of points in $[0,1]^r$ is not generally convex in the ordinary (affine) sense, i.e. given a finite set $V \subset [0,1]^r$ it is easy to find points $x, y \in \mathrm{span}_{\boxplus\boxdot}(V)$ such that the affine straight line connecting x and y is not contained in $\mathrm{span}_{\boxplus\boxdot}(V)$. See Figure 3.1.

Instead, we make the following definition.

Definition 3.3.4. Let V be a finite subset of $[0,1]^r$, where $r \in \mathbb{N}$. Let $x, y \in$

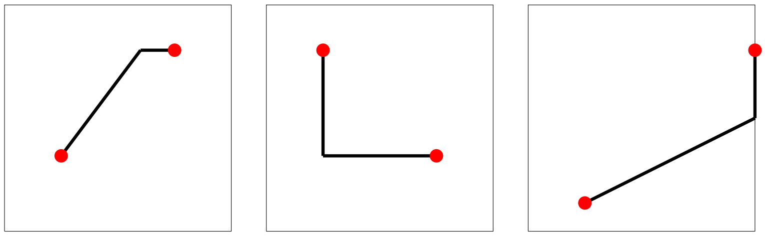


Figure 3.2: Examples of fuzzy line segments between pairs of points in $[0,1]^2$. In two dimensions every fuzzy line segment is of one of these forms.

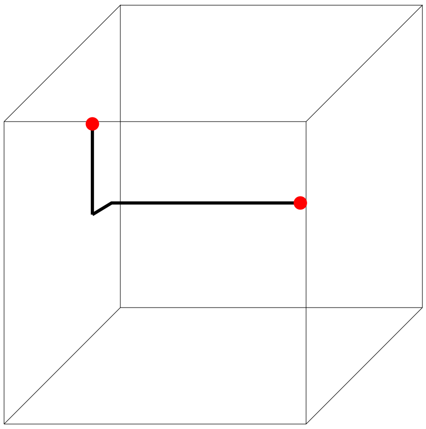


Figure 3.3: The fuzzy line segment between two points in $[0,1]^3$.

$\mathrm{span}_{\boxplus\boxdot}(V)$. The *fuzzy line segment* between x and y is the set

$$[x,y]_{\boxplus\boxdot} = \{(\lambda \boxdot x) \boxplus y \mid \lambda \in [0,1]\} \cup \{x \boxplus (\mu \boxdot y) \mid \mu \in [0,1]\}. \tag{3.43}$$

The fuzzy line segments between some pairs of points in $[0,1]^2$ and $[0,1]^3$ are shown in Figures 3.2 and 3.3. We will see in the next chapter that fuzzy line segments look like projectivisations of tropical line segments. We will study λ-*scales* of points, which are related to fuzzy line segments, in Chapter 4.

Definition 3.3.5. A subset $X \subseteq [0,1]^r$ is *fuzzy convex* if it contains the fuzzy line segment between every pair of its points, i.e. if for all $x, y \in X$ we have $[x,y]_{\boxplus\boxdot} \subset X$. The smallest fuzzy convex set containing X is called the *fuzzy convex hull* of X and is written $\mathrm{conv}_{\boxplus\boxdot}(X)$.

It seems clear from Definition 3.3.2 that for a set $V \subset [0,1]^r$, the fuzzy span of V, $\mathrm{span}_{\boxplus\boxdot}(V)$, is fuzzy convex. Here is the proof.

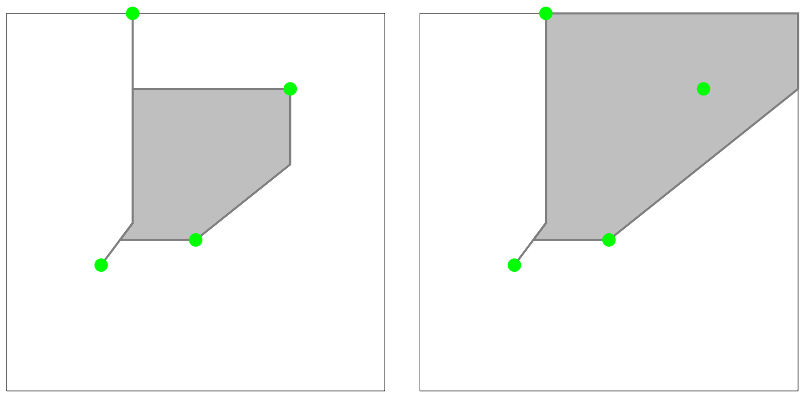


Figure 3.4: The diagram on the left shows the fuzzy convex hull of the four green points. The diagram on the right show the fuzzy span of the same points. The fuzzy span is a fuzzy convex set and contains the fuzzy convex hull.

Lemma 3.3.6. *Let V be a finite subset of $[0,1]^r$ for $r \in \mathbb{N}$. Then the fuzzy span* $\operatorname{span}_{\boxplus\boxdot}(V)$ *is fuzzy convex.*

Proof. Let x and y be two fuzzy linear combinations in $\operatorname{span}_{\boxplus\boxdot}(V)$, given by

$$x = \mathop{\boxplus}\limits_{v \in V} \lambda_v \boxdot v, \qquad y = \mathop{\boxplus}\limits_{v \in V} \mu_v \boxdot v.$$

Then for $\alpha, \beta \in [0,1]$,

$$(\alpha \boxdot x) \boxplus y = \mathop{\boxplus}\limits_{v \in V} (\alpha\lambda_v) \boxdot v \boxplus \mathop{\boxplus}\limits_{v \in V} (\mu_v) \boxdot v \in \operatorname{span}_{\boxplus\boxdot}(V)$$

and

$$x \boxplus (\beta \boxdot y) = \mathop{\boxplus}\limits_{v \in V} (\alpha\lambda_v) \boxdot v \boxplus \mathop{\boxplus}\limits_{v \in V} (\beta\mu_v) \boxdot v \in \operatorname{span}_{\boxplus\boxdot}(V),$$

so $[x,y]_{\boxplus\boxdot} \subseteq \operatorname{span}_{\boxplus\boxdot}(V)$. Hence $\operatorname{span}_{\boxplus\boxdot}(V)$ is fuzzy convex. □

In general, the fuzzy span is *not* equal to the fuzzy convex hull of V, since, for example, $\underline{1} = (1,\ldots,1) \in \operatorname{span}_{\boxplus\boxdot}(V)$ for any set $V \subset [0,1]^r$, even if $\underline{1} \notin V$. This is because $0 \boxdot v = \underline{1}$ for any $v \in V$. If $\underline{1} \notin V$, it cannot lie on the fuzzy line segment between any two points in $\operatorname{span}_{\boxplus\boxdot}(V)$. Figure 3.4 shows the fuzzy convex hull of a finite set $V \subset [0,1]^r$ next to its fuzzy span.

The proof of the following proposition is similar to [13, Proposition 4].

Proposition 3.3.7. *Let V be a finite subset of $[0,1]^r$, where r is any natural number and let $\underline{1} = (1,\ldots,1) \in [0,1]^r$ be the point with every coordinate equal to 1. Then*

$$\operatorname{span}_{\boxplus\boxdot}(V) = \operatorname{conv}_{\boxplus\boxdot}(V \cup \{\underline{1}\}).$$

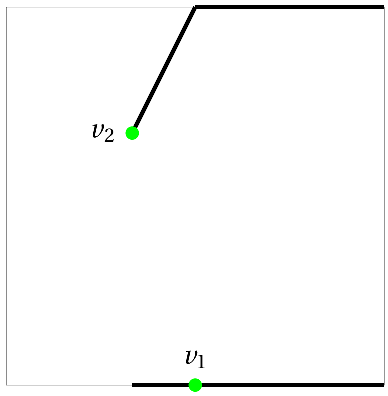


Figure 3.5: The fuzzy span of a set of points in $[0,1]^2$ need not be connected. In particular, no point with second coordinate y for $0 < y < \frac{2}{3}$ is contained in $\mathrm{span}_{\boxplus\boxdot}(\{v_1, v_2\})$.

Proof. Lemma 3.3.6 shows that the fuzzy span of V, $\mathrm{span}_{\boxplus\boxdot}(V)$, contains the fuzzy convex hull of V, $\mathrm{conv}_{\boxplus\boxdot}(V)$. If $x \in \mathrm{span}_{\boxplus\boxdot}(V)$ each fuzzy line segment from x to $\underline{1}$ consists of points of the form $\lambda \boxdot x$ for $\lambda \in [0,1]$ and these are clearly in $\mathrm{span}_{\boxplus\boxdot}(V)$. Hence $\mathrm{conv}_{\boxplus\boxdot}(V \cup \{\underline{1}\}) \subseteq \mathrm{span}_{\boxplus\boxdot}(V)$.

To show the converse, we proceed by induction on the number of points in V. If V has just one point, say x, it is clear that $\mathrm{span}_{\boxplus\boxdot}(\{x\})$ and $\mathrm{conv}_{\boxplus\boxdot}(\{x\} \cup \{\underline{1}\})$ coincide, both consisting solely of the fuzzy line segment $[x, \underline{1}]_{\boxplus\boxdot}$. If $V = \{v_1, \ldots, v_n\}$ with $n > 1$, consider $x = \boxplus_{j=1}^{n} \lambda_j \boxdot v_j \in \mathrm{span}_{\boxplus\boxdot}(V)$. Then we can write $x = \lambda_1 \boxdot v_1 \boxplus \left(\boxplus_{j=2}^{n} \lambda_j \boxdot v_j \right)$, where the bracketed term is in $\mathrm{conv}_{\boxplus\boxdot}(V \cup \{\underline{1}\})$ by the induction hypothesis. But this means that x lies on the fuzzy line segment between this bracketed point and v_1 and is therefore contained in the convex hull $\mathrm{conv}_{\boxplus\boxdot}(V \cup \{\underline{1}\})$. Thus $\mathrm{span}_{\boxplus\boxdot}(V) \subseteq \mathrm{conv}_{\boxplus\boxdot}(V \cup \{\underline{1}\})$. $\square$

Example 3.3.8. Whenever any vectors in V have entries equal to 0, it is possible for $\mathrm{span}_{\boxplus\boxdot}(V)$ to be disconnected. For example, consider

$$V = \left\{ \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix} \right\} \subset [0,1]^2.$$

The fuzzy span of V, $\mathrm{span}_{\boxplus\boxdot}(V)$, is illustrated in Figure 3.5.

Note that $\mathrm{span}_{\boxplus\boxdot}(V)$ is not connected, since for all $\lambda \neq 0$ we have $\lambda \boxdot \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda \backslash (1/2) \\ 0 \end{pmatrix}$ and so it is impossible to obtain any point with non-zero second coordinate by scaling $\begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$ unless $\lambda = 0$, in which case $\lambda \boxdot \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} = \binom{1}{1}$. Thus the set $\{\binom{1}{y} \mid 0 < y < 1\}$ is not contained in $\mathrm{span}_{\boxplus\boxdot}(V)$ as one might naively expect by merely extrapolating from examples of more familiar connected fuzzy spans.

3.3.1 The nucleus of a fuzzy relation as a fuzzy span

We are now in a position to link all of this back to the nucleus of a fuzzy relation.

Theorem 3.3.9. *Let $M \in [0,1]^{r,n}$ be a matrix, considered as a fuzzy relation between finite sets. Let* $\mathrm{Row}(M)$ *and* $\mathrm{Col}(M)$ *denote the sets of rows and columns of M, respectively. Then*

$$\mathrm{Fix}^{\Downarrow}(M) = \mathrm{span}_{\boxplus\boxdot}(\mathrm{Col}(M)), \tag{3.44}$$

$$\mathrm{Fix}^{\Uparrow}(M) = \mathrm{span}_{\boxplus\boxdot}(\mathrm{Row}(M)). \tag{3.45}$$

Proof. We prove the first equation; the second is analogous. Write $m_{\bullet j}$ for the jth column of M, i.e. $(m_{\bullet j})_i := m_{ij}$, so that $\mathrm{Col}(M) = \{m_{\bullet j} \mid j \in [n]\}$. By Proposition 3.1.6, $\mathrm{Fix}^{\Downarrow}(M) = \mathrm{im}\, M_*$. Therefore, let $v \in \mathrm{im}\, M_*$ and let $w \in Y^{\Downarrow}$ be such that $v = M_*(w)$. Then for each $i \in [r]$

$$\begin{aligned} v_i &= M_*(w)_i \\ &= \min_{1\leqslant j\leqslant n} \{w_j \backslash m_{ij}\} \\ &= \left(\boxplus_{j=1}^{n} w_j \boxdot (m_{\bullet j}) \right)_i \end{aligned}$$

so

$$v = \boxplus_{j=1}^{n} w_j \boxdot m_{\bullet j} \in \mathrm{span}_{\boxplus\boxdot}(\mathrm{Col}(M)).$$

Conversely, any $x = \boxplus_{j=1}^{n} \lambda_j \boxdot (m_{\bullet j})$ can be obtained as $M_*(\lambda)$, where $\lambda = (\lambda_1, \dots, \lambda_n) \in [0,1]^n$. □

The following corollary to Theorem 3.3.9 is then immediate from Proposition 3.1.8.

Corollary 3.3.10. *Let $M \in [0,1]^{r,n}$ be a matrix, considered as a fuzzy relation between finite sets. Then there are proxet isomorphisms*

$$\mathrm{Nuc}(M) \cong \mathrm{span}_{\boxplus\boxdot}(\mathrm{Col}(M)) \cong \mathrm{span}_{\boxplus\boxdot}(\mathrm{Row}(M)).$$

An explicit isomorphism between the second and third proxets is provided by the proximity maps $M^*\colon \mathrm{span}_{\boxplus\boxdot}(\mathrm{Col}(M)) \to \mathrm{span}_{\boxplus\boxdot}(\mathrm{Row}(M))$ and $M_*\colon \mathrm{span}_{\boxplus\boxdot}(\mathrm{Row}(M)) \to \mathrm{span}_{\boxplus\boxdot}(\mathrm{Col}(M))$.

We illustrate these results with some examples.

Example 3.3.11. Consider the matrix

$$M = \begin{pmatrix} 1/8 & 1/3 & 1/2 \\ 1/7 & 2/3 & 1/4 \end{pmatrix},$$

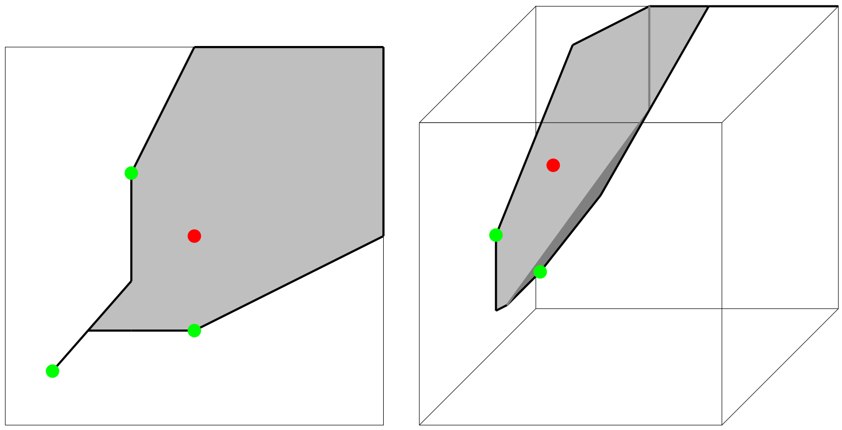


Figure 3.6: The set $\mathrm{Fix}^{\Downarrow}(M)$ is equal to the fuzzy span of three points in $[0,1]^2$ corresponding to the columns of M, while $\mathrm{Fix}^{\Uparrow}(M)$ is equal to the fuzzy span of two points in $[0,1]^3$ corresponding to the rows of M. The spanning points are highlighted in green. The points highlighted in red correspond to each other via the bijection in Corollary 3.3.10.

thought of as a fuzzy relation between finite sets. The fuzzy spans of the columns and rows of M are illustrated in Figure 3.6. By Theorem 3.3.9, these sets are equal to $\mathrm{Fix}^{\Downarrow}(M)$ and $\mathrm{Fix}^{\Uparrow}(M)$, respectively.

Any point $v \in \mathrm{Fix}^{\Downarrow}(M)$ corresponds to a unique point in $\mathrm{Fix}^{\Uparrow}(M)$, given by $w = M^*(v)$. One such example is given by the (column) vector $v = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \in \mathrm{Fix}^{\Downarrow}(M)$, which corresponds to the (row) vector $w = \begin{pmatrix} 1/4 & 2/3 & 1/2 \end{pmatrix} \in \mathrm{Fix}^{\Uparrow}(M)$.

Example 3.3.12. The bijection between $\mathrm{span}_{\boxplus\boxdot}(\mathrm{Row}(M))$ and $\mathrm{span}_{\boxplus\boxdot}(\mathrm{Col}(M))$ can be seen most clearly when M is a square matrix, i.e. when $r = n$. We give a particularly simple 2-dimensional example, which hints at the fact that the correspondence between $\mathrm{Fix}^{\Downarrow}(M)$ and $\mathrm{Fix}^{\Uparrow}(M)$ is actually an isomorphism of cell complexes, as we will see in Theorem 4.4.23.

Consider the matrix

$$M = \begin{pmatrix} 1/4 & 1/2 \\ 1/8 & 1/2 \end{pmatrix}.$$

The fuzzy spans of the columns and rows of M, i.e. the sets $\mathrm{Fix}^{\Downarrow}(M)$ and $\mathrm{Fix}^{\Uparrow}(M)$, are shown in Figure 3.7.

Note that the disconnectedness of the fuzzy span in Example 3.3.8 is consistent with Theorem 3.3.9, in particular the fact that $\mathrm{Fix}^{\Downarrow}(M) = \mathrm{span}_{\boxplus\boxdot}(\mathrm{Col}(M))$. It is easy to check that for $0 < y < 1$, we have $M_* \begin{pmatrix} 1 \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1/3 \end{pmatrix}$ but $M^* \begin{pmatrix} 0 & 1/3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 1 \\ y \end{pmatrix}$, so $\begin{pmatrix} 1 \\ y \end{pmatrix} \notin \mathrm{Fix}^{\Downarrow}(M)$, as required.

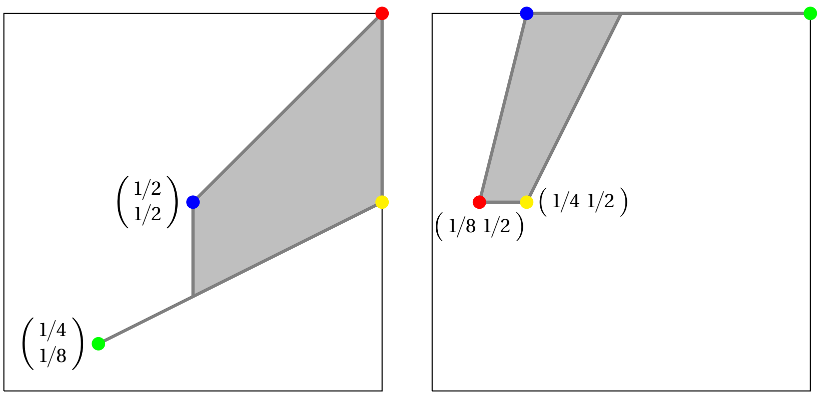


Figure 3.7: The fuzzy spans of the columns and rows of the matrix $M = \begin{pmatrix} 1/4 & 1/2 \\ 1/8 & 1/2 \end{pmatrix}$, i.e. the sets $\mathrm{Fix}^{\Downarrow}(M)$ and $\mathrm{Fix}^{\Uparrow}(M)$. Points that correspond to each other under the bijections M^* and M_* are marked in the same colour.

The discontinuity inherent in the operation of truncated division, as illustrated in Example 3.3.8, means that we focus mainly on fuzzy relations whose matrices contain only non-zero components. We will be able to prove further results related to the nuclei of fuzzy relations, under such conditions, by drawing analogies with tropical linear algebra, which we consider in the next chapter.

Chapter 4

The nucleus as a cell complex

In the previous chapter we showed that the nucleus of a fuzzy relation between finite sets (i.e. finite discrete proxets) is isomorphic to the span of its columns (or rows) with respect to the operations of "fuzzy addition" (pointwise minimum) and "fuzzy scalar multiplication" (truncated division of each coordinate by a constant):

$$(x \boxplus y)_i = \min\{x_i, y_i\} \quad \text{and} \quad (\lambda \boxdot x)_i = \lambda \backslash x_i,$$

for $x, y \in \mathrm{Col}(M)$ and $\lambda \in [0, 1]$.

In this chapter we will show that, under certain conditions, these spans can in fact be seen as tropical convex hulls of certain related (finite) sets of points in tropical projective space, i.e. as tropical polytopes. As tropical polytopes have a natural cell decomposition [13, Theorem 15], we are therefore able to describe a cell decomposition of the nucleus of a fuzzy relation between finite sets.

We begin with a summary of the important definitions and results from tropical linear algebra, focusing on the definition of the tropical convex hull of a finite point configuration in tropical projective space. We then prove that the sets $\mathrm{Fix}^{\Downarrow}(M)$ and $\mathrm{Fix}^{\Uparrow}(M)$ of downward- and upward-fixed vectors of a fuzzy relation between finite sets are isomorphic to the tropical convex hulls of certain points in tropical projective space.

Knowing that $\mathrm{Fix}^{\Downarrow}(M)$ and $\mathrm{Fix}^{\Uparrow}(M)$ are tropical polytopes allows us to transfer many definitions and results from tropical mathematics into the context of nuclei of fuzzy relations. In particular, we are able to define the *type* of a fuzzy concept $(v, w) \in \mathrm{Nuc}(M)$. The tropical convex hull of a finite set of points in tropical projective space inherits a natural cell complex structure where cells are labelled by these types. Develin and Sturmfels [13, Theorem 1] showed that, up to combinatorial type, tropical complexes arising from an arrangement of r points in $(n-1)$-dimensional tropical projective space are in natural bijection with regular subdivisions of the product of simplices $\Delta_{r-1} \times \Delta_{n-1}$.

Notation. In this thesis we frequently consider finite sets of points in finite-dimensional spaces. Dimensions will usually be indexed by a variable $i \in \{1, \ldots, r\}$, while points will usually be indexed by a variable $j \in \{1, \ldots, n\}$. This is different to the conventions of some other authors. When working in tropical projective space, we label the additional coordinate with a zero, so that $i \in \{0, \ldots, r\}$. Later, when we add an additional point to a configuration of n points, we also label it with a zero, so that the $j \in \{0, \ldots, n\}$. It is convenient to make use of the shorthand notations $[k] = \{1, \ldots, k\}$ and $\overline{[k]} = \{0, \ldots, k\}$.

4.1 Tropical mathematics

In this section we introduce the basics of tropical mathematics. Our primary reference is the seminal paper [13] of Develin and Sturmfels. Other introductions can be found in [42, 50, 54].

We begin with an important definition.

Definition 4.1.1. The *tropical semiring* $\mathbb{T} = (\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ consists of the set of real numbers together with the operations of *tropical addition* and *tropical multiplication*, given by

$$x \oplus y := \min\{x, y\} \quad \text{and} \quad x \odot y := x + y, \tag{4.1}$$

respectively, for $x, y \in \mathbb{R}$.

Some authors define tropical addition in terms of maxima, i.e. for $x, y \in \mathbb{T}$, $x \oplus y = \max\{x, y\}$. Since $\max\{x, y\} = -\min\{-x, -y\}$ for all $x, y \in \mathbb{R}$, geometry over the semiring $(\mathbb{R}, \max, +)$ is equivalent to geometry over the semiring $(\mathbb{R}, \min, +)$. Nonetheless, this is the first of several varying conventions within the tropical literature that a reader should be aware of.

It is not difficult to show that $\mathbb{T}$ is a commutative semiring. The additive identity for $\mathbb{T}$ is ∞ and the multiplicative identity is 0, since $\min\{x, \infty\} = x$ and $x + 0 = x$ for all $x \in \mathbb{R} \cup \{\infty\}$.

Tropical addition and tropical multiplication can be extended pointwise to $\mathbb{R}^r$ in the obvious way. For $x = (x_1, \ldots, x_r)$, $y = (y_1, \ldots, y_r) \in \mathbb{R}^r$, $\lambda \in \mathbb{R}$ we define

$$x \oplus y = (x_1 \oplus y_1, \ldots, x_r \oplus y_r) \qquad \lambda \odot x = (\lambda \odot x_1, \ldots, \lambda \odot x_r).$$

Tropical operations arise naturally in the context of addition and multiplication of power series under the map that sends each power series $P = \sum a_i t^i \in \mathbb{R}[[t]]$ to its degree, i.e. the exponent of its leading term. Provided the leading terms do not cancel, $\deg(P + Q) = \max\{\deg P, \deg Q\}$ and $\deg(P \cdot Q) =$

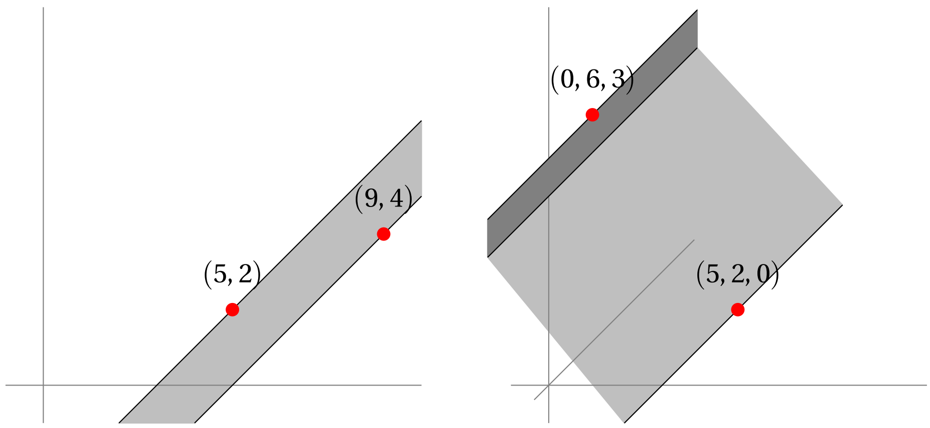


Figure 4.1: Tropical line segments in $\mathbb{R}^2$ and $\mathbb{R}^3$. The (unprojectivised) tropical line segment between two points in $\mathbb{R}^2$ consists of two (infinite) lines and the (infinite) planar section bounded by them (shaded in grey). In $\mathbb{R}^3$ the (unprojectivised) tropical line segment between two points consists of two such (infinite) planar sections intersecting in an (infinite) line as well as the (infinite) lines bounding each of these planar sections.

$\deg P + \deg Q$. In fact there are strong connections between tropical geometry and generalised power series known as Puiseux series, which have the form

$$P = \sum_{i \geqslant k} a_i t^{i/n}$$

for some fixed $k \in \mathbb{Z}$ and $n \in \mathbb{N}$. See, for example, [14, 32, 44].

Definition 4.1.2. The *tropical line segment* between two points $x, y \in \mathbb{R}^r$ is defined to be the set of *all* tropical linear combinations of x and y:

$$[x, y]_{\oplus\odot} := \{\lambda \odot x \oplus \mu \odot y \mid \lambda, \mu \in \mathbb{R}\}. \tag{4.2}$$

Note that there is no restriction on the parameters λ and μ in Definition 4.1.2; in particular, we do not require $\lambda + \mu = 1$ as in the affine case.

Examples of tropical line segments in $\mathbb{R}^2$ and $\mathbb{R}^3$ are illustrated in Figure 4.1. In this form, tropical line segments do not look very much like what one might expect line segments to look like.

Definition 4.1.3. A set $X \subseteq \mathbb{R}^r$ is said to be *tropically convex* if contains the tropical line segment $[x, y]_{\oplus\odot}$ between each pair of points $x, y \in X$.

It is easy to see that tropically convex sets are closed under tropical multiplication by scalars, i.e. if $X \subseteq \mathbb{R}^r$ is tropically convex and $x \in X$, then $\lambda \odot x \in X$ for all $\lambda \in \mathbb{R}$. We can therefore simplify things by identifying each x with all of its tropical scalings $\lambda \odot x$.

Definition 4.1.4. For r a natural number, r-dimensional *tropical projective space* $\mathbb{TP}^r$ is defined to be the quotient space $\mathbb{R}^{r+1}/\sim$, where $\sim$ is the equivalence relation $(x_0, \ldots, x_r) \sim \lambda \odot (x_0, \ldots, x_r)$ for all $\lambda \in \mathbb{R}$.

Note that points in $\mathbb{TP}^r$ have $r+1$ coordinates, which we index from zero. Some authors, e.g. Joswig in [31], call this space *tropical affine space* and use the notation $\mathbb{TA}^r$.

For each point in $x \in \mathbb{TP}^r$ there is a unique vector $c(x) \in \mathbb{R}^{r+1}$ with non-negative coordinates such that at least one coordinate is zero. The coordinates of this vector are sometimes called the *canonical coordinates* of x. However, since there is a unique representative of the coset $\mathbb{R} \odot x$ with zeroth coordinate zero, it is often more useful to consider the image of a point in $\mathbb{TP}^r$ under the natural projection $c_0 \colon \mathbb{TP}^r \to \mathbb{R}^r$ given by eliminating the zeroth coordinate, i.e.

$$c_0 \colon (x_0, \ldots, x_r) \mapsto (x_1 - x_0, \ldots, x_r - x_0).$$

This projection is easily seen to be a bijection. We call the image of this projection the *projectivisation* of x.

Note that instead of eliminating the zeroth coordinate, some authors choose to eliminate a different coordinate, typically the last. This is another varying convention to be aware of.

Example 4.1.5. The point $x = (-2, 1, -3) \in \mathbb{TP}^2$ has canonical coordinates $(1, 4, 0)$. We can also represent x by the vector $c_0(x) = (3, -1) \in \mathbb{R}^2$. This is shown, together with the (projectivisation of the) tropical line segment between $(0, 1, 2)$ and $(0, 8, 6)$ in $\mathbb{TP}^2$, in Figure 4.2.

Note that projectivised tropical line segments in $\mathbb{TP}^2$ look a lot like the fuzzy line segments we defined in the previous chapter. To see why this is, consider two points $x = (x_0, x_1, x_2)$ and $y = (y_0, y_1, y_2)$ in $\mathbb{TP}^2$. Without loss of generality we may assume $x_0 = y_0 = 0$. The tropical line segment between x and y is the set of points of the form

$$\begin{aligned}\lambda \odot x \oplus \mu \odot y &= (\lambda \oplus \mu, \lambda \odot x_1 \oplus \mu \odot y_1, \lambda \odot x_2 \oplus \mu \odot y_2) \\ &= \begin{cases} (0, x_1 \oplus (\mu - \lambda) \odot y_1, x_2 \oplus (\mu - \lambda) \odot y_2) & \text{if } \lambda < \mu \\ (0, (\lambda - \mu) \odot x_1 \oplus y_1, (\lambda - \mu) \odot x_2 \oplus y_2) & \text{if } \lambda > \mu, \end{cases}\end{aligned}$$

so, eliminating the zeroth coordinate using the projection c_0 defined above, we see

$$[x, y]_{\oplus\odot} \cong \{(\lambda - \mu) \odot x \oplus y \mid \lambda, \mu \in \mathbb{R}\} \cup \{x \oplus (\mu - \lambda) \odot y \mid \lambda, \mu \in \mathbb{R}\} \subset \mathbb{R}^2,$$

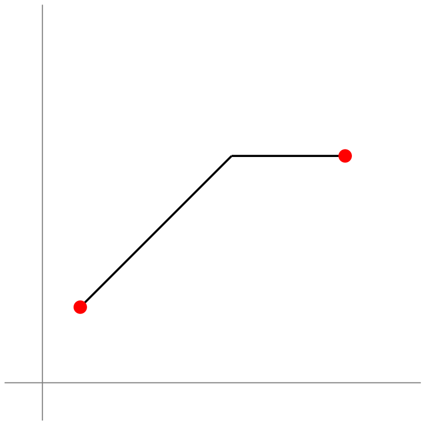


Figure 4.2: A (projectivised) tropical line segment in $\mathbb{TP}^2$. Tropical line segments in $\mathbb{TP}^r$ are equal to the concatenation of at most r ordinary line segments, each of which is parallel to a vector with coordinates in $\{0,1\}$ (see [13, Proposition 3]).

where the symbol $\cong$ indicates the projectivisation bijection. Comparing this formula with (3.43):

$$[x,y]_{\boxplus\boxdot} = \{(\lambda \boxdot x) \boxplus y \mid \lambda \in [0,1]\} \cup \{x \boxplus (\mu \boxdot y) \mid \mu \in [0,1]\},$$

makes the correspondence clear.

The relationship between projectivised tropical line segments and fuzzy line segments is harder to see in higher dimensions.

Given a set of points in $\mathbb{TP}^r$ we can find the smallest tropically convex set containing them.

Definition 4.1.6. The *tropical convex hull* of a set of points $V = \{v_1, \ldots, v_n\} \subseteq \mathbb{TP}^r$ is defined to be the smallest tropically convex set containing V and is denoted $\mathrm{conv}_{\oplus\odot}(V)$. A subset $X \subseteq \mathbb{TP}^r$ is called a *tropical polytope* if it is the tropical convex hull of a finite set of points.

In [13, Proposition 4], the tropical convex hull of V is shown to be equal to the set of all tropical linear combinations of points in V.

Example 4.1.7. An example of a tropical polytope in $\mathbb{TP}^2$ is shown in Figure 4.3. This example, which is the tropical convex hull of the points $(0,0,2)$, $(0,2,0)$, $(0,1,-2) \in \mathbb{TP}^2$ is taken from [13]. As with all tropical polytopes, it is a finite union of closed, bounded, (classically) convex polyhedra (in this case polygons), namely the pentagon and the closed line segment from $(0,0,0)$ to $(0,0,2)$.

The tropical analogues of hyperplanes are particularly useful examples of tropically convex sets [13, Proposition 6].

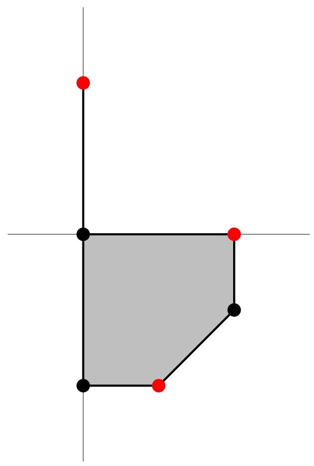


Figure 4.3: A tropical polytope in $\mathbb{TP}^2$. This is the tropical convex hull of the points highlighted in red. It is the finite union of closed, bounded, convex polygons.

Definition 4.1.8. A *tropical hyperplane* H_a in $\mathbb{TP}^r$ is the set of points for which a tropical linear form

$$\bigoplus_{k=0}^{r} a_k \odot x_k = a_0 \odot x_0 \oplus \cdots \oplus a_r \odot x_r \tag{4.3}$$

is attained at least twice, i.e. $x \in H_a$ if and only there exist distinct coordinates $i \neq j$ for which

$$a_i + x_i = a_j + x_j = \min_{0 \leqslant k \leqslant r} \{a_k + x_k\}.$$

The point $(-a_0, -a_1, \ldots, -a_r)$ for which this minimum is attained in every coordinate is called the *apex* of H_a.

Each tropical hyperplane divides $\mathbb{TP}^r$ into $r+1$ sectors. We write

$$H[i] = \{x \in \mathbb{TP}^r \mid a_i + x_i = \min_{0 \leqslant k \leqslant r} \{a_k + x_k\}\}$$

for the closed sector consisting of those points $x \in \mathbb{TP}^r$ for which the minimum in (4.3) is attained by $a_i + x_i$. All tropical hyperplanes are translates of each other.

Example 4.1.9. The tropical hyperplane corresponding the the tropical linear form $1 \odot x_0 \oplus (-2) \odot x_1 \oplus 0 \odot x_2$ is the set of points $(x_0, x_1, x_2) \in \mathbb{R}^3$ such that

$$\begin{aligned} & 1 + x_0 = (-2) + x_1 \leqslant 0 + x_2, \\ \text{or} \quad & 1 + x_0 = 0 + x_2 \leqslant (-2) + x_1, \\ \text{or} \quad & (-2) + x_1 = 0 + x_2 \leqslant 1 + x_0. \end{aligned}$$

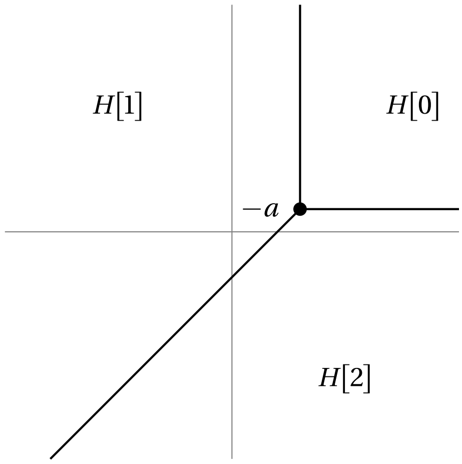


Figure 4.4: The (projectivisation of the) tropical hyperplane in $\mathbb{TP}^2$ defined by the tropical linear form $1 \odot x_0 \oplus (-2) \odot x_1 \oplus 0 \odot x_2$. The three lines are the sets of points for which the minimum encoded in the tropical linear form is attained twice, with the apex $-a = (-1, 2, 0)$ (represented by the point $(3, 1) \in \mathbb{R}^2$) being the point at which the minimum is attained three times. The sectors $H[0]$, $H[1]$, $H[2]$ are indicated.

In its unprojectivised form, this hyperplane consists of three sections of ordinary planes in $\mathbb{R}^3$ meeting in a line. The projectivised form of this hyperplane in $\mathbb{TP}^2$ is illustrated in Figure 4.4.

It will be particularly useful to work with "inverted" tropical hyperplanes, defined in terms of maxima rather than minima. Explicitly, given a tropical linear form $a_0 \odot x_0 \oplus \cdots \oplus a_r \odot x_r$, a point $x \in \mathbb{TP}^r$ is contained in the *inverted tropical hyperplane* H_a^{inv} if and only if there exist distinct coordinates $i \neq j$ for which

$$a_i + x_i = a_j + x_j = \max_{0 \leqslant k \leqslant r} \{a_k + x_k\}.$$

It is clear that $x \in H_a^{\mathrm{inv}}$ if and only if $-x \in H_{-a}$. The ith sector of an inverted tropical hyperplane H_a^{inv} is the closed set

$$H_a^{\mathrm{inv}}[i] = \{x \in \mathbb{TP}^r \mid a_i + x_i = \max_{0 \leqslant k \leqslant r} \{a_k + x_k\}\}.$$

Much work has been done on tropical convex hulls and their natural cell complex structures. See, for example, [14, 15, 28, 30, 31]. We aim to show that under certain conditions nuclei of fuzzy relations between finite sets are in one-to-one correspondence with tropical convex hulls of point configurations in tropical projective space and thus inherit a natural cell complex structure.

4.2 The nucleus as a tropical convex hull

In this section we provide a bijection between the set $\mathrm{Fix}^{\Downarrow}(M)$ of lower vectors of a fuzzy relation given by an $(r \times n)$-matrix with entries in $(0, 1]$ and to the set of points of a certain tropical convex hull of $n + 1$ vectors in $\mathbb{TP}^r$. Similarly, there is a bijection between the set $\mathrm{Fix}^{\Uparrow}(M)$ of upper vectors of M and a certain tropical convex hull of $r + 1$ points in $\mathbb{TP}^n$. It is important that all the entries of M are strictly positive, in which case we say that such a fuzzy relation M is *non-vanishing*. These bijections respect the semimodule structure in a manageable way, so it is easy to move between fuzzy spans of vectors in $[0, 1]$-space and tropical convex hulls in tropical projective space.

Definitions and results about tropical polytopes can then be translated directly to corresponding statements about fuzzy spans. In particular, since each tropical polytope has a natural cell decomposition [13, Theorem 15] we are able to show that $\mathrm{Fix}^{\Downarrow}(M)$ and $\mathrm{Fix}^{\Uparrow}(M)$ also have such a structure, allowing us to classify concepts in $\mathrm{Fix}^{\Downarrow}(M)$ or $\mathrm{Fix}^{\Uparrow}(M)$ according to their *type*, a piece of combinatorial data from which the dimension of the cell in which a concept lies can be read off. Concepts in a classical concept lattice can all be thought of as constituting cells of dimension zero. The ability to classify fuzzy concepts may have applications to fuzzy concept analysis as it allows different fuzzy concepts to be thought of in qualitatively different ways.

We begin by describing a way to move between $[0, 1]$-space and tropical projective space. Write $\mathbb{R}_+$ for the set of strictly positive real numbers. For any base $b > 0$, the function $x \mapsto \log_b(x)\colon \mathbb{R}_+ \to \mathbb{R}$ is a bijection, with inverse given by $y \mapsto \exp_b(y) := b^y$. When we do not wish to specify the base we abbreviate these functions to log and exp. This should be taken to mean that the base b can be taken as desired, not that we are necessarily using natural logarithms and exponentials (i.e. with $b = e$). For any $b > 1$, both log and exp are order-preserving functions, i.e. if $x \leqslant y$ then $\log(x) \leqslant \log(y)$ and $\exp(x) \leqslant \exp(y)$.

Since log and exp interchange addition and multiplication, i.e.

$$\log(xy) = \log(x) + \log(y) \quad \text{and} \quad \exp(x + y) = \exp(x)\exp(y),$$

these functions are obvious candidates for moving from the essentially multiplicative realm of fuzzy spans to the additive realm of tropical mathematics, which has been widely studied. To this end, define a map $\phi\colon (0, 1]^r \to \mathbb{TP}^r$ as follows. For $x \in (0, 1]^r$ define $\phi(x)$ to be the equivalence class in $\mathbb{TP}^r$ of the point with coordinates $(0, \log(x_1), \ldots, \log(x_r))$, i.e.

$$\phi(x)_i := \begin{cases} 0 & \text{if } i = 0, \\ \log(x_i) & \text{if } i > 0. \end{cases} \tag{4.4}$$

Note that $(0,1]^r$ is a $[0,1]$-submodule of the $[0,1]$-semimodule $[0,1]^r$ from which it inherits the operations $\boxplus$ and $\boxdot$ defined in (3.39) and (3.40): for $x, x' \in (0,1]^r$ and $\lambda \in [0,1]$

$$(x \boxplus x')_i = \min\{x_i, x'_i\} \quad \text{and} \quad (\lambda \boxdot x)_i = \lambda \backslash x_i \tag{4.5}$$

for each $i \in [r]$, where $\backslash$ is the truncated division operation defined in (2.2). We can see that $(0,1]^r$ really is a submodule as for any $\lambda \in [0,1]$ and $x \in (0,1]^r$ we have $\lambda \boxdot x \in (0,1]^r$ since $\lambda \backslash x_i \geqslant x_i$ for each i by definition.

When X is a subset of $(0,1]^r$, write $\phi(X) = \{\phi(x) \mid x \in X\}$. Points in the image of ϕ are those points $(y_0, y_1, \ldots, y_r) \in \mathbb{TP}^r$ for which $y_i - y_0 \leqslant 0$ for all $i \in [r]$, and this property is invariant under tropical scalar mutliplication. Write $\mathbb{TP}^r_{\leqslant 0}$ for this set, a subspace of $\mathbb{TP}^r$. Restricting to $\mathbb{TP}^r_{\leqslant 0}$ is not limiting; because $\mathbb{TP}^r$ is homogeneous, every configuration of points in $\mathbb{TP}^r$ that does not have this property can be seen to be equivalent to one that does by simply changing the zeroth coordinate of each point, and vice versa. This equivalence is analagous to the equivalence of geometry in the whole Cartesian plane and its first quadrant.

Lemma 4.2.1. *The map ϕ defined in* (4.4) *is a bijection onto its image, with inverse given by $\psi \colon \mathbb{TP}^r_{\leqslant 0} \to (0,1]^r$, where we define $\psi(y)_i := \exp(y_i - y_0)$ for $y = (y_0, y_1, \ldots, y_r) \in \mathbb{TP}^r$. Moreover, ϕ respects the semimodule structure of $(0,1]^r$ and $\mathbb{TP}^r$ in the following way: for all $x, x' \in (0,1]$ and $\lambda \in (0,1]$,*

$$\phi(x \boxplus x') = \phi(x) \oplus \phi(x') \quad \textit{and} \quad \phi(\lambda \boxdot x) = (-\log(\lambda)) \odot \phi(x) \oplus \underline{0}, \tag{4.6}$$

where $\underline{0} \in \mathbb{TP}^r$ is the point with all coordinates equal to zero.

Proof. We first check that the map ψ is well-defined. If y and y' represent the same point in $\mathbb{TP}^r$ there is some $\lambda \in \mathbb{R}$ such that $y'_i = \lambda + y_i$ for each $i \in \overline{[r]}$. Then for all $i \in [r]$ we have $y'_i - y'_0 = \lambda + y_i - (\lambda + y_0) = y_i - y_0$, so ψ is well-defined. Note also that since $y \in \mathbb{TP}^r_{\leqslant 0}$ we have $\psi(y) \in (0,1]^r$ as required. It is then easy to see that ϕ and ψ are mutually inverse. For $x \in (0,1]^r$ and $i \in [r]$ we see that $\psi(\phi(x))_i = \exp(\log(x_i) - 0) = x_i$. For $y \in \mathbb{TP}^r$ we have $\phi(\psi(y))_i = \log(\exp(y_i - y_0)) = y_i - y_0$ for all $i \in [r]$. Adding y_0 to each coordinate shows that this is in the same equivalence class of $\mathbb{TP}^r$ as y.

For $x, x' \in (0,1]^r$ and $i \in [r]$, we see

$$\begin{aligned}
\phi(x \boxplus x')_i &= \log((x \boxplus x')_i) \\
&= \log(\min\{x_i, x'_i\}) \\
&= \min\{\log(x_i), \log(x'_i)\} \\
&= \phi(x)_i \oplus \phi(x')_i \\
&= (\phi(x) \oplus \phi(x'))_i,
\end{aligned}$$

since log is order-preserving. For $i = 0$ both sides are equal to 0.

Given a constant $\lambda \in (0,1]$ and $i \in [r]$, we see

$$\begin{aligned}
\phi(\lambda \boxdot x)_i &= \log((\lambda \boxdot x)_i) \\
&= \log(\min\{x_i/\lambda, 1\}) \\
&= \min\{\log(x_i/\lambda), 0\} \\
&= \min\{-\log(\lambda) + \log(x_i), 0\} \\
&= \min\{(-\log(\lambda)) \odot \phi(x)_i, 0\} \\
&= ((-\log(\lambda)) \odot \phi(x) \oplus \underline{0})_i
\end{aligned}$$

For $i = 0$, both sides are again equal to 0. □

We are now able to prove the most important result in this section.

Theorem 4.2.2. *Let $V \subseteq (0,1]^n$ be a finite set of vectors. The map ϕ defined in* (4.4) *restricts to a bijection of sets (respecting the semimodule structures of $(0,1]^r$ and $\mathbb{TP}^r$ as descried in* (4.6)*)*

$$\phi\colon \operatorname{span}_{\boxplus\boxdot}(V) \cong \operatorname{span}_{\oplus\odot}(\phi(V) \cup \{\underline{0}\}), \tag{4.7}$$

where $\underline{0} \in \mathbb{TP}^r$ is the point with all coordinates equal to zero.

Proof. Let $x \in \operatorname{span}_{\boxplus\boxdot}(V)$. Then we can write

$$x = \boxplus_{v \in V} \lambda_v \boxdot v \tag{4.8}$$

for some constants $\lambda_v \in (0,1]$ depending on v. By Lemma 4.2.1, we see that

$$\phi(x) = \bigoplus_{v \in V} (-\log \lambda_v) \odot \phi(v) \oplus \underline{0}.$$

Hence $\phi(x) \in \operatorname{span}_{\oplus\odot}(\phi(V) \cup \{\underline{0}\})$.

Conversely, let $y \in \operatorname{span}_{\oplus\odot}(\phi(V) \cup \{\underline{0}\})$. Then y can be written as

$$y = \bigoplus_{v \in V} \mu_v \odot \phi(v) \oplus \mu_0 \odot \underline{0},$$

for some constants $\mu_0, \mu_v \in (0,1]$. Let $\mu_{\min} = \min\{\min_{v \in V}\{\mu_v\}, \mu_0\}$, so $y_0 = \mu_{\min}$ and for each $i \in [r]$,

$$y_i = \min_{v \in V}\{\min\{\mu_v + \log(v_i), \mu_0\}\}. \tag{4.9}$$

We claim that y lies in the same equivalence class of $\mathbb{TP}^r$ as $\phi(z)$, where z is the fuzzy linear combination

$$z = \boxplus_{v \in V} \exp(\mu_{\min} - \mu_v) \boxdot v.$$

Lemma 4.2.1 gives

$$\begin{aligned}\phi(z) &= \bigoplus_{v\in V}(-\log(\exp(\mu_{\min}-\mu_v)))\odot\phi(v)\oplus\underline{0}\\ &= \bigoplus_{v\in V}(\mu_v-\mu_{\min})\odot\phi(v)\oplus\underline{0}.\end{aligned}$$

Hence

$$\begin{aligned}\mu_{\min}\odot\phi(z) &= \bigoplus_{v\in V}\mu_v\odot\phi(v)\oplus\mu_{\min}\odot\underline{0}\\ &= \bigoplus_{v\in V}(\mu_v\odot\phi(v)\oplus\mu_v\odot\underline{0})\oplus\mu_0\odot\underline{0},\end{aligned}$$

by definition of $\mu_{\min}$. But since, for each $v\in V$, $\phi(v)_i\leqslant 0$ for all $i\in\overline{[r]}$, we have $\mu_v\odot\phi(v)\oplus\mu_v\odot\underline{0}=\mu_v\odot\phi(v)$ for each v and so the right-hand side is equal to y. Thus $y=\mu_{\min}\odot\phi(z)$, so $y\sim\phi(z)$ in $\mathbb{TP}^r$. □

Theorem 4.2.2 shows that ϕ preserves the convexity of sets in $(0,1]^r$. The following is essentially a restatement of the previous theorem.

Corollary 4.2.3. *The bijection $\phi\colon (0,1]^r\to\mathbb{TP}^r$ defined in (4.4) sends fuzzy spans in $(0,1]^r$ to tropical convex hulls in $\mathbb{TP}^r$. More precisely, the fuzzy span of n points $\{v_1,\dots,v_n\}$ in $(0,1]^r$ is in bijective correspondence with the tropical convex hull of $n+1$ points $\{\phi(v_1),\dots,\phi(v_n),\underline{0}\}$ in $\mathbb{TP}^r$.*

This correspondence is illustrated in Figure 4.5.

4.3 Tropical hyperplane arrangements and types

In the previous section we showed that the fuzzy span of n vectors in $(0,1]^r$ is in bijective correspondence with the tropical convex hull of $n+1$ points in $\mathbb{TP}^r$. This allows us to translate many definitions and results into statements about the nuclei of fuzzy relations between finite sets.

We begin with the definition of the *type* of a point with respect to a configuration of points. The definition we give is actually the transpose of the definition given in [13], although the two are equivalent. Our definition is closer to that given by Ardila and Develin in [1].

Definition 4.3.1. Let $V=\{v_0,\dots,v_n\}$ be a finite set of points in $\mathbb{TP}^r$ and let x be any point in $\mathbb{TP}^r$. The *type* of $x\in\mathbb{TP}^r$ with respect to V, denoted $\operatorname{type}_V(x)$, is the ordered $(n+1)$-tuple $S=(S_0,\dots,S_n)$ of subsets $S_j\subseteq\overline{[r]}$ such that $i\in S_j$ if and only if

$$v_{ji}-x_i=\min_{0\leqslant k\leqslant r}\{v_{jk}-x_k\},$$

where v_{ji} is the ith coordinate of the vector v_j.

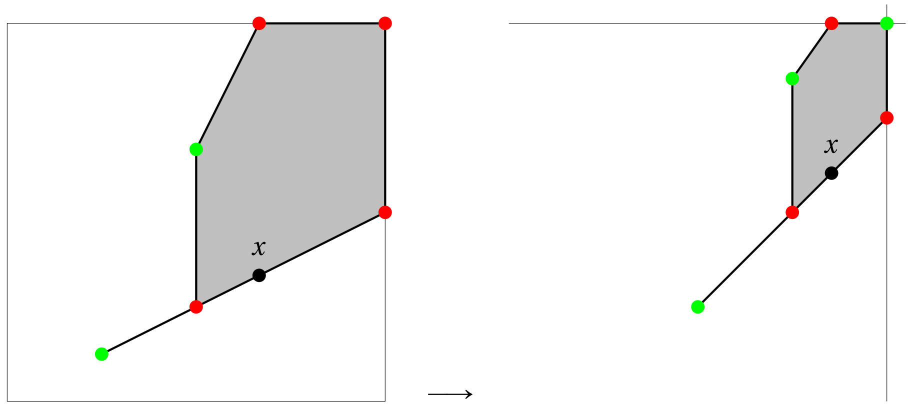


Figure 4.5: The bijective correspondence between fuzzy spans and tropical polytopes. Note that the fuzzy span on the left only has two generating vertices, while the tropical polytope on the right is the tropical convex hull of three points, namely the images under ϕ of the generating vertices of the left-hand diagram and the zero vector.

Definition 4.3.2. Let $V = \{v_0, \ldots, v_n\}$ be a finite set of points in $\mathbb{TP}^r$. A *candidate (r,n)-type* is any $(n+1)$-tuple $S = (S_0, \ldots, S_n)$ of subsets $S_j \subseteq \overline{[r]}$. A candidate type S is a type if there exists $x \in \mathbb{TP}^r$ such that $\text{type}_V(x) = S$.

The transpose of Definition 4.3.1 defines a type to be an $(r+1)$-tuple $S^\top = (S_0^\top, \ldots, S_r^\top)$ of subsets $S_i \subseteq \overline{[n]}$ such that $j \in S_i^\top$ if and only if $i \in S_j$. Whether to use S or $S^\top$ is another varying convention between authors to be aware of.

Note that for a genuine type S, each minimum must be attained at least once so each set S_j must be non-empty. In general, however, it is not necessary that each $i \in \overline{[r]}$ appears in some S_j. When this holds we make the following definition.

Definition 4.3.3. A type $S = (S_0, \ldots, S_n)$ of a point $x \in \mathbb{TP}^r$ is *full* if, for each $i \in \overline{[r]}$, there exists some $j \in \overline{[n]}$ such that $i \in S_j$.

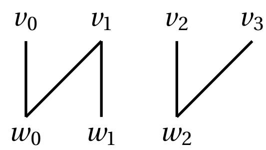
Abusing notation slightly, a type S can be conveniently represented as a matrix $S \in \{0,1\}^{r+1,n+1}$, where $S_{ij} = 1$ if and only if $i \in S_j$. By definition a type S is full if and only if every row of this matrix is non-zero or, equivalently, if every column of the matrix corresponding to the transpose type $S^\top$ is non-zero. For types of points with respect to configurations of points in tropical projective space we do not need distinct notions of “row full” and “column full”.

It is also sometimes useful to represent S by the undirected bipartite graph for which this matrix is the adjacency matrix.

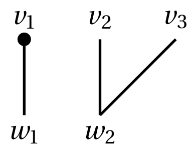
Example 4.3.4. The type $S = (\{0,1\}, \{1\}, \{2,3\})$ can be represented by the matrix

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

or by the undirected bipartite graph



or by the marked graph



where a vertex v_i is marked if $i \in S_0$ and w_j is marked if $0 \in S_j$.

Remark. Here is a convenient way to compute the type of a point $x \in \mathbb{TP}^r$ with respect to a point configuration $V = \{v_0, \ldots, v_n\} \subset \mathbb{TP}^r$. The type of x encodes the position of x relative to the n inverted tropical hyperplanes centred at the vertices v_j. Translating Definition 4.3.1, if $S = \text{type}_V(x)$ then $i \in S_j$ if and only x lies in the ith sector of the jth inverted hyperplane H_j^{inv}, whose apex is v_j. This is illustrated in Figure 4.6

.

We now make an important definition.

Definition 4.3.5. Let $V = \{v_0, \ldots, v_n\}$ be a finite set of points in $\mathbb{TP}^r$. Let S be a candidate (r, n)-type and define $C_V(S)$ to be the set

$$C_V(S) = \{x \in \mathbb{TP}^r \mid S \subseteq \text{type}_V(x)\}, \tag{4.10}$$

where a candidate (r, n)-type A is contained in another candidate (r, n)-type B if and only if $A_j \subseteq B_j$ for each $j \in \overline{[n]}$. For reasons we will explain shortly, we refer to $C_V(S)$ as the closed *cell* corresponding to the candidate type S.

The set $C_V(S)$ is shown to be an ordinary convex polyhedron in [13, Lemma 10] and that it is bounded precisely when S is full, in the sense of Definition 4.3.3, in [13, Corollary 12]. (Note that the second of these results is stated slightly differently in [13] due because their definition of "type" is the transpose of ours.)

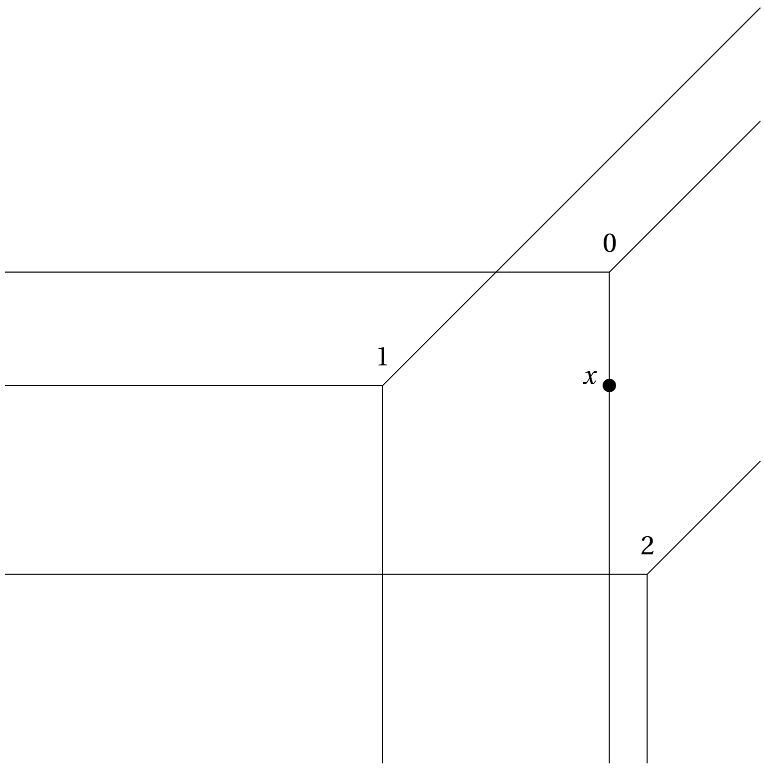


Figure 4.6: Three inverted tropical hyperplanes in $\mathbb{TP}^2$. Moving anticlockwise from the bottom left, the three sectors of each inverted tropical hyperplane are oriented in the order 0, 1, 2. The lines indicate the boundaries between these sectors; a point lying on such a boundary is considered to lie in all of the sectors it bounds. The type of x can be seen to be $(\{0,1\},\{1\},\{2\})$, since x lies in sectors 0 and 1 of H_0^{inv}, in sector 1 of H_1^{inv}, and in sector 2 of H_2^{inv}. Note that every point must lie in one or more sectors for each inverted tropical hyperplane.

It is then shown in [13, Theorem 15] that the set of all cells $C_V(S)$, where S ranges over all possible types, provides a cell decomposition of $\mathbb{TP}^r$ and that the tropical convex hull of the points $\{v_0, \ldots, v_n\}$ consists of the bounded cells, i.e. the cells $C_V(S)$ for those types S that are full.

Write $\mathcal{C}_V$ for the collection of all sets $C_V(S)$ for candidate (r, n)-types S. What it means for $\mathcal{C}_V$ to be a "cell decomposition" of $\mathbb{TP}^r$ can be summed up by the following results, for which full proofs can be found in [13, Lemma 10–Theorem 15]:

1. Every point $x \in \mathbb{TP}^r$ lies in some cell $C_V(S)$;
2. Each cell $C_V(S)$ is a convex polyhedron;
3. The faces of $C_V(S)$ are precisely those cells $C_V(T)$ for which $S \subseteq T$;
4. Every cell $C_V(S)$ for a candidate (r, n)-type S is equal to $C_V(T)$ for some actual (r, n)-type T, i.e. with T such that $T = \mathrm{type}_V(x)$ for some $x \in \mathbb{TP}^r$;
5. The intersection of two cells $C_V(S)$ and $C_V(T)$ in $\mathcal{C}_V$ is a cell in $\mathcal{C}_V$ and is a face of both $C_V(S)$ and $C_V(T)$.

The dimension of each polyhedron $C_V(S)$ can be calculated using the following definition.

Definition 4.3.6. Given a candidate (r, n)-type $S = (S_0, \ldots, S_n)$ (not necessarily the type of any point $x \in \mathbb{TP}^r$), let G_S be the undirected graph with vertex set $\overline{[r]}$, where for $i_1, i_2 \in \overline{[r]}$, there is an edge in G_S between i_1 and i_2 if and only if there exists $j \in \overline{[n]}$ such that $i_1 \in S_j$ and $i_2 \in S_j$.

A proof of the following result can be found in [13, Proposition 17].

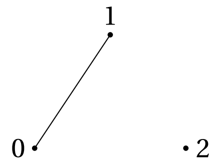
Proposition 4.3.7. *Let S be a candidate (r, n)-type. The dimension of $C_V(S)$ is equal to one less than the number of connected components of G_S.*

If $C_V(S)$ has dimension k we call the relative interior of $C_V(S)$ a *k-cell*. Thus, 0-cells are points, 1-cells are open line segments, 2-cells are the relative interiors of closed convex polygons, etc.

Example 4.3.8. Consider the point x in Figure 4.6. The type of x with respect to the given arrangement of tropical hyperplanes can be represented as the matrix

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since $r = 2$, the graph G_S has 3 vertices. There is an (undirected) edge between 0 and 1 since $S_{00} = S_{10} = 1$, as shown



Hence G_S has two connected components and x lies in a cell of dimension 1, i.e. a line segment.

4.4 The cell complex structure of the nucleus

In this section we apply the results of the previous section to the specific case of point configurations obtained from fuzzy relations between finite sets. It was shown in the previous chapter that the nucleus of a fuzzy relation $M: X \rightsquigarrow Y$ between finite sets is isomorphic as a proxet to its projections into $[0,1]^r$ or $[0,1]^n$, i.e.

$$\mathrm{Nuc}(M) \cong \mathrm{Fix}^{\Downarrow}(M) \cong \mathrm{Fix}^{\Uparrow}(M),$$

and the underlying sets of these proxets are equal to fuzzy spans of the columns and rows of M, respectively. It follows from Theorem 4.2.2 that when M is such that $M(x, y) \neq 0$ for all $x \in X$, $y \in Y$, i.e. if the matrix of M contains no zeros, then $\mathrm{Fix}^{\Downarrow}(M)$ is isomorphic (as a set) to a tropical convex hull of $n+1$ points in $\mathbb{TP}^r$, namely the images of the n columns of M under the map ϕ (defined in (4.4)), together with the zero vector $\underline{0} \in \mathbb{TP}^r$; similarly, $\mathrm{Fix}^{\Uparrow}(M)$ is isomorphic (as a set) to the tropical convex hull of the images of the r rows of M under ϕ, together with $\underline{0} \in \mathbb{TP}^n$.

The upshot of this is that if M is a fuzzy relation whose matrix contains no zeros, there is a natural cell complex structure on $\mathrm{Fix}^{\Downarrow}(M)$ and $\mathrm{Fix}^{\Uparrow}(M)$. Since we are only concerned with such fuzzy relations, we give them a name.

Definition 4.4.1. A fuzzy relation $M: X \rightsquigarrow Y$ is said to be *non-vanishing* if $M(x, y) \neq 0$ for all $x \in X$ and all $y \in Y$, i.e. if the matrix of M has no zero entries.

Imposing a non-vanishing condition on a fuzzy relation may appear to be quite a limiting restriction, but in practice it is not. It is clear that from any fuzzy relation $M: X \rightsquigarrow Y$ we can define a non-vanishing fuzzy relation $\widetilde{M}: X \rightsquigarrow Y$ by picking $\varepsilon > 0$ sufficiently small and setting

$$\widetilde{M}(x, y) = \begin{cases} \varepsilon & \text{if } M(x, y) = 0, \\ M(x, y) & \text{otherwise.} \end{cases}$$

for all $x \in X$, $y \in Y$. Although there is a clear qualitative difference between the nuclei of M and $\widetilde{M}$, from the perspective of fuzzy concept analysis this modification can be justified in practice by arguing that no statement is ever completely false, i.e. there will be some context in which it could be considered true.

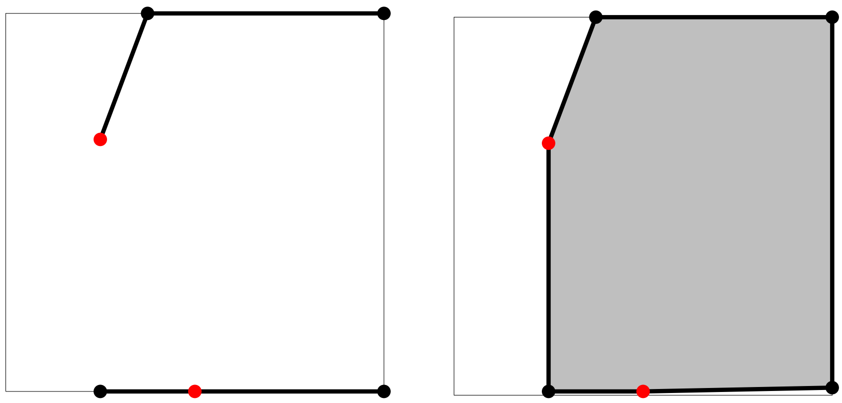


Figure 4.7: The diagram on the left is $\mathrm{Fix}^{\Downarrow}(M)$ for a fuzzy relation that is not non-vanishing. The diagram on the right is $\mathrm{Fix}^{\Downarrow}(\widetilde{M})$, where $\widetilde{M}$ is a non-vanishing fuzzy relation obtained from M by changing all zeros in the matrix of M.

Example 4.4.2. Let M be the fuzzy relation with matrix

$$M = \begin{pmatrix} 1/4 & 1/2 \\ 2/3 & 0 \end{pmatrix}.$$

Since $m_{22} = 0$, M is clearly not non-vanishing. Let $\widetilde{M}$ be the non-vanishing fuzzy relation with matrix

$$M = \begin{pmatrix} 1/4 & 1/2 \\ 2/3 & \varepsilon \end{pmatrix},$$

where $\varepsilon > 0$ is small. The $\mathrm{Fix}^{\Downarrow}(M)$ and $\mathrm{Fix}^{\Downarrow}(\widetilde{M})$ are shown in Figure 4.7. Note that these are qualitatively very different; $\mathrm{Fix}^{\Downarrow}(M)$ is not connected and consists only of 0- and 1-cells, while $\mathrm{Fix}^{\Downarrow}(\widetilde{M})$, whose points are in bijection with those of a tropical polytope, is necessarily connected and includes a 2-cell, as well as additional 1-cells.

Given sets X and Y with $|X| = r$ and $|Y| = n$ and a fuzzy relation $M : X \rightsquigarrow Y$ we have seen that the nucleus of M is the proxet with underlying set

$$\mathrm{Nuc}(M) = \{(v, w) \in [0,1]^r \times [0,1]^n \mid M^*(v) = w \text{ and } v = M_*(w)\}. \tag{4.11}$$

If M is non-vanishing it is possible to express the set on the right-hand side of (4.11) in a different way. First, we note that the nuclei of non-vanishing fuzzy relations cannot contain any points with any zero coordinates.

Lemma 4.4.3. *Let $M : X \rightsquigarrow Y$ be a fuzzy relation between finite sets with $|X| = r$ and $|Y| = n$ and let $(v, w) \in \mathrm{Nuc}(M)$. If M is non-vanishing then $v_i > 0$ for all $i \in [r]$ and $w_j > 0$ for all $j \in [n]$, i.e. $(v, w) \in (0,1]^r \times (0,1]^n$.*

Proof. If $(v, w) \in \mathrm{Nuc}(M)$ then $v \in \mathrm{Fix}^{\Downarrow}(M) = \mathrm{span}_{\boxplus\boxdot} \mathrm{Col}(M)$ and $w \in \mathrm{Fix}^{\Uparrow}(M) = \mathrm{span}_{\boxplus\boxdot} \mathrm{Row}(M)$. Since M is non-vanishing, $\mathrm{Col}(M)$ and $\mathrm{Row}(M)$ both contain no elements with any coordinates equal to zero, and hence neither can their fuzzy spans, since $(\lambda \boxdot x)_i = \lambda \backslash x_i \geqslant x_i$ for all $\lambda \in [0,1]$. Hence $v_i > 0$ and $w_j > 0$ for all $i \in [r]$ and all $j \in [n]$. □

From this we obtain the following characterisation of points in the nucleus of a non-vanishing relation between finite sets.

Lemma 4.4.4. *Let $(v, w) \in (0,1]^r \times (0,1]^n$ and let $M \in (0,1]^{r,n}$ be the matrix of a non-vanishing fuzzy relation between finite sets. Then $v = M_*(w)$ and $M^*(v) = w$ if and only if:*

$$\begin{gathered} \forall i \; \forall j \; v_i \cdot w_j \leqslant m_{ij}, \\ \forall i \; v_i = 1 \textit{ or } \exists j : \; v_i \cdot w_j = m_{ij}, \\ \forall j \; w_j = 1 \textit{ or } \exists i : \; v_i \cdot w_j = m_{ij}. \end{gathered}$$

Proof. Suppose $v = M_*(w)$. By the definition of M_*, this means that, for all $i \in [r]$, we have $v_i = \min_{1 \leqslant j \leqslant n}\{w_j \backslash m_{ij}\}$, which means that for each $j \in [n]$ we have $v_i \leqslant w_j \backslash m_{ij}$ and hence $v_i w_j \leqslant m_{ij}$. By finiteness, there must exist $l \in [n]$ such that $v_i = w_l \backslash m_{il}$. If $w_l < m_{il}$ then $v_i = 1$; otherwise $v_i = m_{il}/w_l$, so $v_i w_l = m_{il}$. In the same way, $w = M^*(v)$ implies that, for all $j \in [n]$, we have $w_j = 1$ or there exists $k \in [r]$ such that $v_k w_j = m_{kj}$.

Conversely, suppose $(v, w) \in (0,1]^r \times (0,1]^n$ satisfies the displayed conditions. Fix $i \in [r]$. Firstly, by the definition of truncated division, $v_i w_j \leqslant m_{ij}$ implies $v_i \leqslant w_j \backslash m_{ij}$ for each $j \in [n]$. Hence $v_i \leqslant \min_{1 \leqslant j \leqslant n}\{w_j \backslash m_{ij}\}$. Now, either $v_i = 1$ or there exists some $l \in [n]$ such that $v_i w_l = m_{il}$. In the first case we clearly have $v_i \geqslant \min_{1 \leqslant j \leqslant n}\{w_j \backslash m_{ij}\}$; otherwise suppose $v_i < \min_{1 \leqslant j \leqslant n}\{w_j \backslash m_{ij}\}$. In that case, for each $j \in [n]$ we must have $v_i < w_j \backslash m_{ij}$, and since $w_j > 0$, by the previous lemma, mutliplying by w_j gives $v_i w_j < w_j(w_j \backslash m_{ij}) \leqslant m_{ij}$ for all $j \in [n]$, contradicting the fact that there exists some j with $v_i w_j = m_{ij}$. Hence $v_i \geqslant \min_{1 \leqslant j \leqslant n}\{w_j \backslash m_{ij}\}$ and thus $v = M_*(w)$.

The equality $w = M^*(v)$ is proved similarly. □

Thus, for a non-vanishing fuzzy relation, $\mathrm{Nuc}(M)$ is equal to the set

$$\left\{ (v, w) \in (0,1]^r \times (0,1]^n \;\middle|\; \begin{gathered} \forall i \; \forall j \; v_i \cdot w_j \leqslant m_{ij} \\ \forall i \; v_i = 1 \text{ or } \exists j : \; v_i \cdot w_j = m_{ij} \\ \forall j \; w_j = 1 \text{ or } \exists i : \; v_i \cdot w_j = m_{ij} \end{gathered} \right\}. \tag{4.12}$$

Remark. Note that it is necessary that M be non-vanishing for $\mathrm{Nuc}(M)$ to be written as the set in (4.12). Consider the fuzzy relation with matrix

$$M = \begin{pmatrix} 1/4 & 1/2 \\ 2/3 & 0 \end{pmatrix},$$

as in Example 4.4.2, let $v = \binom{1/2}{1/2}$ and let $w = M^*(v) = \left(1/2\ 0\right)$. Then $M_*(w) = \binom{1/2}{1} \neq v$, so $(v, w) \notin \mathrm{Nuc}(M)$, even though $v_1 w_1 = m_{11}$ and $v_2 w_2 = m_{22}$.

We can introduce some notation to simplify the set in (4.12).

For $v \in [0,1]^r$, let $\overline{v} \in [0,1]^{r+1}$ be the vector defined by setting $\overline{v}_i = 1$ if $i = 0$ and $\overline{v}_i = v_i$ otherwise. For $w \in [0,1]^n$, define $\overline{w} \in [0,1]^{n+1}$ similarly. Note that we index these "extended" vectors from zero. For a matrix $M \in [0,1]^{r,n}$, let $\overline{M} \in [0,1]^{r+1,n+1}$ be the matrix with $\overline{m}_{ij} = 1$ if $i = 0$ or $j = 0$ and $\overline{m}_{ij} = m_{ij}$ otherwise. With this notation, the set in (4.12) becomes

$$\mathrm{Nuc}(M) = \left\{ (v, w) \in (0,1]^r \times (0,1]^n \;\middle|\; \begin{array}{c} \forall i\ \forall j\ \overline{v}_i \cdot \overline{w}_j \leqslant \overline{m}_{ij} \\ \forall i\ \exists j:\ \overline{v}_i \cdot \overline{w}_j = \overline{m}_{ij} \\ \forall j\ \exists i:\ \overline{v}_i \cdot \overline{w}_j = \overline{m}_{ij} \end{array} \right\}. \tag{4.13}$$

Definition 4.4.5. Let $M\colon X \rightsquigarrow Y$ be a non-vanishing fuzzy relation between finite sets, where $|X| = r$ and $|Y| = n$. Let $v \in \mathrm{Fix}^{\Downarrow}(M)$. Define the *type* of v with respect to M, written $\mathrm{type}_M(v)$, to be the type of $\phi(v)$ with respect to the tropical hyperplane arrangement given by $\phi(\mathrm{Col}(\overline{M})) = \phi(\mathrm{Col}(M)) \cup \{\underline{0}\}$, i.e.

$$\mathrm{type}_M(v) := \mathrm{type}_{\phi(\mathrm{Col}(\overline{M}))}(\phi(v)). \tag{4.14}$$

The following proposition is essentially a translation of Definition 4.4.5.

Proposition 4.4.6. *Let $M\colon X \rightsquigarrow Y$ be a non-vanishing fuzzy relation between finite sets, where $|X| = r$ and $|Y| = n$. Let $v \in (0,1]^r$, let $w = M^* v$, and let $S = \mathrm{type}_M(v)$. Then $i \in S_j$ if and only if $\overline{v}_i \cdot \overline{w}_j = \overline{m}_{ij}$.*

Proof. For $j \in \overline{[n]}$, write $\overline{m}_{\bullet j}$ for the jth column of $\overline{M}$, whose ith coordinate is $\overline{m}_{ij}$. Note that for any $v \in (0,1]^r$ and $i \in \overline{[r]}$, we have $\phi(v)_i = \log(\overline{v}_i)$. It therefore follows from Definition 4.3.1 that $i \in S_j$ if and only if

$$\log(\overline{m}_{ij}) - \log(\overline{v}_i) = \min_{0 \leqslant k \leqslant r} \{\log(\overline{m}_{kj}) - \log(\overline{v}_k)\}.$$

Using the properties of logarithms, it follows that $i \in S_j$ if and only if

$$\log\left(\frac{\overline{m}_{ij}}{\overline{v}_i}\right) = \min_{0 \leqslant k \leqslant r} \left\{\log\left(\frac{\overline{m}_{kj}}{\overline{v}_k}\right)\right\} \tag{4.15}$$

$$= \log\left(\min_{0 \leqslant k \leqslant r} \left\{\frac{\overline{m}_{kj}}{\overline{v}_k}\right\}\right). \tag{4.16}$$

Note that $\overline{v}_k \neq 0$ for all $k \in \overline{[r]}$ because of the condition that the matrix of M has no zero entries, so division by $\overline{v}_k$ is safe.

Exponentiating both sides of (4.15) gives $i \in S_j$ if and only if

$$\begin{aligned}
\frac{\overline{m}_{ij}}{\overline{v}_i} &= \min_{0 \leqslant k \leqslant r} \left\{ \frac{\overline{m}_{kj}}{\overline{v}_k} \right\} \\
&= \min \left\{ \min_{1 \leqslant k \leqslant r} \left\{ \frac{\overline{m}_{kj}}{\overline{v}_k} \right\}, 1 \right\} \\
&= \min_{1 \leqslant k \leqslant r} \left\{ \min \left\{ \frac{\overline{m}_{kj}}{\overline{v}_k}, 1 \right\} \right\} \\
&= \min_{1 \leqslant k \leqslant r} \left\{ \overline{v}_k \backslash \overline{m}_{kj} \right\} \\
&= \begin{cases} 1 & \text{if } j = 0 \\ \min_{1 \leqslant k \leqslant r} \{ v_k \backslash m_{kj} \} & \text{otherwise} \end{cases} \\
&= \overline{w}_j,
\end{aligned}$$

hence $i \in S_j$ if and only if $\overline{v}_i \cdot \overline{w}_j = \overline{m}_{ij}$. □

We can define the type of $w \in \mathrm{Fix}^{\Uparrow}(M)$ in a similar way.

Proposition 4.4.7. *Let $M \colon X \rightsquigarrow Y$ be a non-vanishing fuzzy relation, where $|X| = r$ and $|Y| = n$. Let $w \in (0,1]^n$ and let $S = \mathrm{type}_M(v)$, where $v = M_*(w)$. Then*

$$\mathrm{type}_{\phi(\mathrm{Row}(\overline{M}))}(\phi(w)) = S^{\top}.$$

Proof. Following exactly the same steps as in Proposition 4.4.6, we find that if $T = \mathrm{type}_{\phi(\mathrm{Row}(\overline{M}))}(\phi(w))$ then $j \in T_i$ if and only if $\overline{v}_i \cdot \overline{w}_j = \overline{m}_{ij}$. But this is true if and only if $i \in S_j$. Thus $T = S^{\top}$. □

Proposition 4.4.7 shows that if $(v, w) \in \mathrm{Nuc}(M)$, the types of v and w are transposes of each other. In other words, they contain exactly the same information. We therefore make the following definition.

Definition 4.4.8. Let $M \colon X \rightsquigarrow Y$ be a non-vanishing fuzzy relation between finite sets, where $|X| = r$ and $|Y| = n$. Let $p = (v, w) \in \mathrm{Nuc}(M)$. The *type* of p (with respect to M) is the $((r+1) \times (n+1))$-matrix S with $S_{ij} = 1$ if and only if $\overline{v}_i \cdot \overline{w}_j = \overline{m}_{ij}$.

One way to think about types is as solution sets to systems of equations and inequalities. It is clear from (4.13) that points in $\mathrm{Nuc}(M)$ satisfy $\overline{v}_i \cdot \overline{w}_j \leqslant \overline{m}_{ij}$ for all $i \in \overline{[r]}$ and $j \in \overline{[n]}$, while a certain number of these must be equalities. The types of points in $\mathrm{Nuc}(M)$ encode which inequalities are strict and which are in fact equalities.

If $M \colon X \rightsquigarrow Y$ is a non-vanishing fuzzy relation between finite sets with $|X| = r$ and $|Y| = n$ we know that $\mathrm{Fix}^{\Downarrow}(M)$ and $\mathrm{Fix}^{\Uparrow}(M)$ are in bijection with the underlying sets of tropical polytopes in $\mathbb{TP}^r$ and $\mathbb{TP}^n$, respectively. Since the spanning vertices of $\mathrm{Fix}^{\Downarrow}(M)$ and $\mathrm{Fix}^{\Uparrow}(M)$ are mapped to spanning vertices of these tropical polytopes (cf. Theorem 4.2.2) we can use Definition 4.4.5 to define natural cell decompositions of $(0,1]^r$ and $(0,1]^n$. The following definitions will allow us to make this explicit.

Definition 4.4.9. Let $M\colon X \rightsquigarrow Y$ be a non-vanishing fuzzy relation between finite sets, where $|X| = r$ and $|Y| = n$. Let S be a candidate (r,n)-type. Define the following sets.

$$C_M^{\mathrm{col}}(S) = \{v \in (0,1]^{r,1} \mid S \subseteq \mathrm{type}_M(v)\}, \tag{4.17}$$

$$C_M^{\mathrm{row}}(S) = \{w \in (0,1]^{1,n} \mid S^\top \subseteq \mathrm{type}_M(w)\}. \tag{4.18}$$

Remark. Note that $C_M^{\mathrm{col}}(S) = \psi(C_{\phi(\mathrm{Col}\,\overline{M})}(S))$ and $C_M^{\mathrm{row}}(S) = \psi(C_{\phi(\mathrm{Row}\,\overline{M})}(S))$, where ϕ and its inverse ψ are defined in Theorem 4.2.2. That is, cells in $(0,1]^r$ and $(0,1]^n$ can be defined in terms of cells in $\mathbb{TP}^r$ and $\mathbb{TP}^n$ and many results from [13] can be carried over.

We can compute the dimension of these sets using Proposition 4.3.7.

Proposition 4.4.10. *Let $M\colon X \rightsquigarrow Y$ be a non-vanishing fuzzy relation between finite sets, where $|X| = r$ and $|Y| = n$. Let S be a candidate (r,n)-type. The dimension of the cell $C_M^{\mathrm{col}}(S)$ is equal to the dimension of the cell $C_M^{\mathrm{row}}(S)$ and this is one less than the number of connected components of G_S (cf. Definition 4.3.6).*

This means that the dimension of cells is entirely determined by its type. The following result is adapted from [13, Lemma 10].

Proposition 4.4.11. *Let S be a candidate (r,n)-type and let $C_M^{\mathrm{col}}(S)$ and $C_M^{\mathrm{row}}(S)$ be defined as above. Then $C_M^{\mathrm{col}}(S)$ and $C_M^{\mathrm{row}}(S)$ can be expressed in terms of inequalities as*

$$C_M^{\mathrm{col}}(S) = \left\{ v \in (0,1]^r \;\middle|\; \frac{\overline{v}_k}{\overline{v}_i} \leqslant \frac{\overline{m}_{kj}}{\overline{m}_{ij}} \text{ for all } i,k \in \overline{[r]} \text{ such that } S_{ij} = 1 \right\}, \tag{4.19}$$

$$C_M^{\mathrm{row}}(S) = \left\{ w \in (0,1]^n \;\middle|\; \frac{\overline{w}_l}{\overline{w}_j} \leqslant \frac{\overline{m}_{il}}{\overline{m}_{ij}} \text{ for all } j,l \in \overline{[n]} \text{ such that } S_{ij} = 1 \right\}. \tag{4.20}$$

Proof. Let $v \in (0,1]^r$ and let $w = M^*(v) \in (0,1]^n$. Let $T = \mathrm{type}_M(v)$. It follows from Proposition 4.4.7 that $\mathrm{type}_M(w) = T^\top$.

Suppose $v \in C_M^{\mathrm{col}}(S)$. Then $S \subseteq T$. So, for all $i,k \in \overline{[r]}$, $j \in \overline{[n]}$ such that $S_{ij} = 1$ we have $T_{ij} = 1$, so $\overline{v}_i \cdot \overline{w}_j = \overline{m}_{ij}$, which implies that

$$\frac{\overline{v}_k}{\overline{v}_i} \leqslant \frac{\overline{m}_{kj}}{\overline{m}_{ij}}, \tag{4.21}$$

since all components of $\overline{v}$ and $\overline{m}$ are positive. Hence v is in the right-hand side of $C_M^{\mathrm{col}}(S)$.

Conversely, suppose that for all $i \in \overline{[r]}$ and $j \in \overline{[n]}$ with $S_{ij} = 1$ and for all $k \in \overline{[r]}$ the inequality (4.21) holds. Then $\overline{m}_{ij}/\overline{v}_i \leqslant \overline{m}_{kj}/\overline{v}_k$ for all such i, j, k. This implies $\overline{v}_i \cdot \overline{w}_j = \overline{m}_{ij}$, since $\overline{w}_j = \overline{m}_{kj}/\overline{v}_k$ (cf. the proof of Proposition 4.4.6) so $T_{ij} = 1$. Hence $S \subseteq T$ and $v \in C_M^{\mathrm{col}}(S)$.

The proof of the second equation is almost the same. Note that $S^{\top}_{ji} = 1$ if and only if $S_{ij} = 1$. □

Proposition 4.4.11 implies the following.

Corollary 4.4.12. *The sets $C_M^{\mathrm{col}}(S)$ and $C_M^{\mathrm{row}}(S)$ are closed bounded polyhedra in $(0,1]^r$ and $(0,1]^n$, respectively, that are convex in the ordinary sense.*

We give an example of how these inequalities pick out such a polyhedron in $(0,1]^r$.

Example 4.4.13. Once again, consider the matrix

$$M = \begin{pmatrix} 1/2 & 1/3 \\ 1/4 & 2/3 \end{pmatrix},$$

thought of as a fuzzy relation between finite sets. Consider the candidate type

$$S = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

By Proposition 4.4.11, we find that $C_M^{\mathrm{col}}(S)$ is defined as

$$\begin{aligned} C_M^{\mathrm{col}}(S) = \Bigg\{ v \in (0,1]^2 \;\Bigg|\; & \frac{\overline{v}_1}{\overline{v}_0} \leqslant \frac{\overline{m}_{10}}{\overline{m}_{00}},\ \frac{\overline{v}_2}{\overline{v}_0} \leqslant \frac{\overline{m}_{20}}{\overline{m}_{00}},\ \frac{\overline{v}_1}{\overline{v}_0} \leqslant \frac{\overline{m}_{11}}{\overline{m}_{01}},\ \frac{\overline{v}_2}{\overline{v}_0} \leqslant \frac{\overline{m}_{21}}{\overline{m}_{01}}, \\ & \frac{\overline{v}_0}{\overline{v}_1} \leqslant \frac{\overline{m}_{01}}{\overline{m}_{11}},\ \frac{\overline{v}_2}{\overline{v}_1} \leqslant \frac{\overline{m}_{21}}{\overline{m}_{11}},\ \frac{\overline{v}_0}{\overline{v}_1} \leqslant \frac{\overline{m}_{02}}{\overline{m}_{12}},\ \frac{\overline{v}_2}{\overline{v}_1} \leqslant \frac{\overline{m}_{22}}{\overline{m}_{12}} \Bigg\} \end{aligned}$$

Rearranging gives

$$\begin{aligned} C_M^{\mathrm{col}}(S) = \{ v \in (0,1]^2 \mid\ & v_1 \leqslant 1,\ v_2 \leqslant 1,\ v_1 \leqslant 1/2,\ v_2 \leqslant 1/4, \\ & v_1 \geqslant 1/2,\ v_2 \leqslant v_1/2,\ v_1 \geqslant 1/3,\ v_2 \leqslant 2v_1 \}. \end{aligned}$$

The cell $C_M^{\mathrm{col}}(S)$ is shown in Figure 4.8. Note that this cell does not lie in $\mathrm{Fix}^{\Downarrow}(M)$. In fact, we will see that whether or not a cell lies in $\mathrm{Fix}^{\Downarrow}(M)$ can be read off from its type.

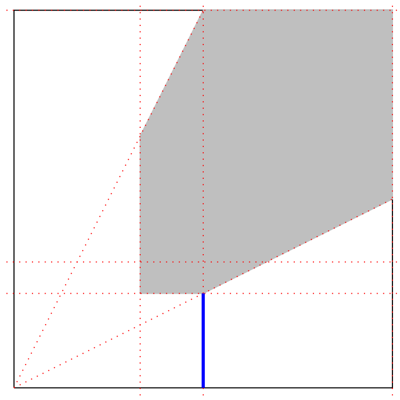


Figure 4.8: A cell $C_M^{\mathrm{col}}(S)$ bounded by inequalities. The cell is highlighted in blue. Note that it is not contained in the set $\mathrm{Fix}^{\Downarrow}(M)$, shaded in grey.

In order to state a later result we need to make the following distinction.

Definition 4.4.14. Let S be a candidate (r,n)-type. We say that S is *column full* if S is full in the sense of Definition 4.3.3, i.e. if for each $i \in \overline{[r]}$ there exists $j \in \overline{[n]}$ such that $i \in S_j$. We say that S is *row full* if $S^\top$ is full, i.e. if for each $j \in \overline{[n]}$ there exists $i \in \overline{[n]}$ such that $j \in S_i^\top$ or, equivalently, such that $i \in S_j$.

For types of points in the nucleus of a fuzzy relation we have the following result.

Lemma 4.4.15. *Let* $S = \mathrm{type}_M(v)$ *for some* $v \in \mathrm{Fix}^{\Downarrow}(M)$. *Then* S *is column full and row full.*

Proof. Since S is a type, we must have $S_j \neq \varnothing$ for each $j \in \overline{[n]}$, i.e. for each $j \in \overline{[n]}$ there exists $i \in \overline{[r]}$ such that $i \in S_j$. Hence S is row full and $S^\top$ is column full. But by Proposition 4.4.7, $S^\top$ is the type of $w = M^*(v)$, and we must therefore have for each $i \in \overline{[r]}$ some $j \in \overline{[n]}$ such that $j \in S_i^\top$, i.e. $S^\top$ must be row full. Hence S is column full. $\square$

An exactly similar result holds for types of points in $\mathrm{Fix}^{\Uparrow}(M)$. We simply say that the type of a point in $\mathrm{Fix}^{\Downarrow}(M)$ or $\mathrm{Fix}^{\Uparrow}(M)$ is *full* since such a type is both column full and row full.

It is shown in [13, Corollary 12] that in the tropical setting a polyhedron $C_V(S)$ is bounded if and only if S is full, i.e. if for each $i \in \overline{[r]}$ there exists $j \in \overline{[n]}$ with $i \in S_j$. Since $(0,1]^r$ is a bounded subset of $\mathbb{R}^r$, every subset $X \subset (0,1]^r$ is necessarily bounded, so we need a slightly subtler notion.

Definition 4.4.16. Let $X \subset (0,1]^r$. We say that X is *well-bounded* if it does not contain any points with any coordinate arbitrarily close to 0, i.e. if there exists $\varepsilon > 0$ such that $X \subset (\varepsilon, 1]^r$.

It is easy to see that a subset $X \subset (0,1]^r$ is well-bounded if and only if the set $\phi(X) \subset \mathbb{TP}^r$ is bounded. The following result follows from [13, Corollary 12].

Corollary 4.4.17. *A cell $C_M^{\mathrm{col}}(S)$ or $C_M^{\mathrm{row}}(S)$ is well-bounded if and only if S is full.*

As mentioned in the remark above, the fact that $C_M^{\mathrm{col}}(S)$ and $C_M^{\mathrm{row}}(S)$ can be defined in terms of cells in tropical projecive space means that results from [13] can easily be carried over. The most important of these are summed up the following theorem.

Theorem 4.4.18. *Let $M \colon X \rightsquigarrow Y$ be a non-vanishing fuzzy relation between finite sets, with $|X| = r$ and $|Y| = n$. The collection of convex polyhedra $C_M^{\mathrm{col}}(S)$, where S ranges over all types, defines a cell decomposition of $(0,1]^r$. That is*

1. *Every $v \in (0,1]^r$ lies in some cell $C_M^{\mathrm{col}}(S)$;*
2. *Each cell $C_M^{\mathrm{col}}(S)$ is a convex polyhedron;*
3. *The faces of $C_M^{\mathrm{col}}(S)$ are precisely those cells $C_M^{\mathrm{col}}(T)$ for which $S \subseteq T$;*
4. *Every cell $C_M^{\mathrm{col}}(S)$ for a candidate (r,n)-type S is equal to $C_M^{\mathrm{col}}(T)$ for some actual (r,n)-type T;*
5. *The intersection of $C_M^{\mathrm{col}}(S)$ and $C_M^{\mathrm{col}}(T)$ is a face of both $C_M^{\mathrm{col}}(S)$ and $C_M^{\mathrm{col}}(T)$.*

The set $\mathrm{Fix}^{\Downarrow}(M)$ is equal to the union of all well-bounded cells $C_M^{\mathrm{col}}(S)$, i.e. those $C_M^{\mathrm{col}}(S)$ for which S is (column) full. This gives $\mathrm{Fix}^{\Downarrow}(M)$ the structure of a cell complex.

Corresponding results hold for the sets $C_M^{\mathrm{row}}(S)$, giving a cell decomposition of $(0,1]^n$ and a cell complex structure on $\mathrm{Fix}^{\Uparrow}(M)$.

Definition 4.4.19. As for general tropical hyperplane arrangements, for a candidate (r,n)-type S we can define the set of points in $(0,1]^r \times (0,1]^n$ whose type contains S as

$$C_M(S) = \{(v,w) \in (0,1]^r \times (0,1]^n \mid S \subseteq \mathrm{type}_M(v,w)\}, \tag{4.22}$$

where we say that a matrix $A \in \{0,1\}^{r+1,n+1}$ is contained in $B \in \{0,1\}^{r+1,n+1}$ if $a_{ij} \leqslant b_{ij}$ for all $i \in \overline{[r]}$ and $j \in \overline{[n]}$.

A direct cell complex structure on $\mathrm{Nuc}\, M$ can also be obtained by adapting the unbounded polyhedron $\mathcal{P}_V$ described in the proof of [13, Lemma 22].

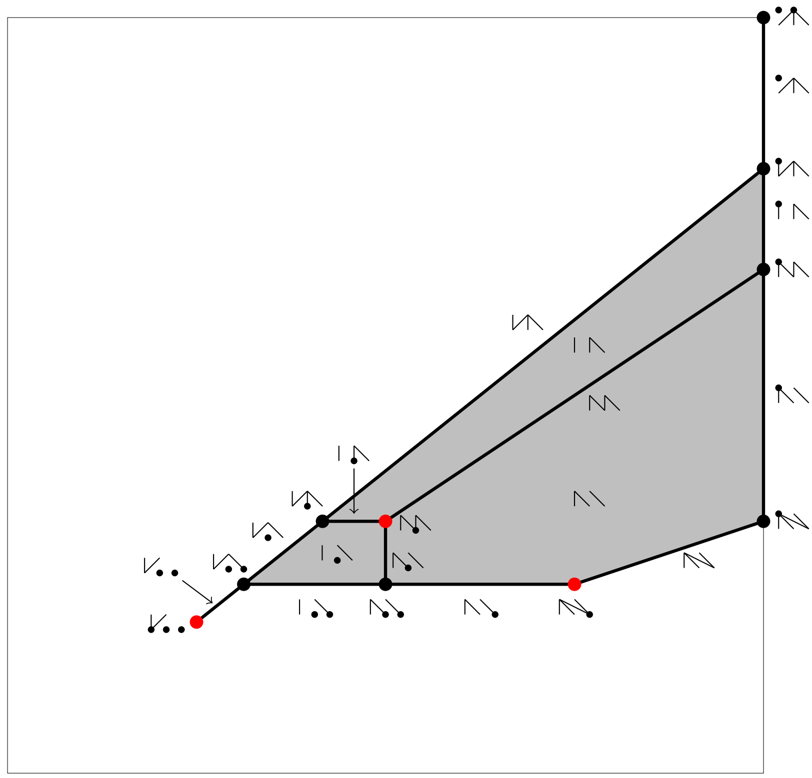


Figure 4.9: The cell decomposition of the set $\mathrm{Fix}^{\Downarrow}(M)$ for the matrix $\begin{pmatrix} 1/4 & 1/2 & 3/4 \\ 1/5 & 1/3 & 1/4 \end{pmatrix}$ in $[0,1]^r$. Types are shown in the form of marked bipartite graphs (cf. Example 4.3.4). The three spanning vectors appear as 0-cells, marked in red.

The next example illustrates the cell complex structure of $\mathrm{Fix}^{\Downarrow}(M)$.

Example 4.4.20. Consider the matrix

$$M = \begin{pmatrix} 1/4 & 1/2 & 3/4 \\ 1/5 & 1/3 & 1/4 \end{pmatrix},$$

regarded as a fuzzy relation between finite sets. The set $\mathrm{Fix}^{\Downarrow}(M)$ is shown in Figure 4.9. The types are labelled with their corresponding marked bipartite graphs (cf. Example 4.3.4).

As another example to help illustrate the additional structure that the cell decomposition provides, consider the sets $\mathrm{Fix}^{\Downarrow}(M)$ and $\mathrm{Fix}^{\Uparrow}(M)$ of Figure 3.6, depicted in Figure 4.10 with their cell complex structure.

Definition 4.4.21. Two fuzzy relations are said to have the same *combinatorial type* if their nuclei have the same types up to relabelling.

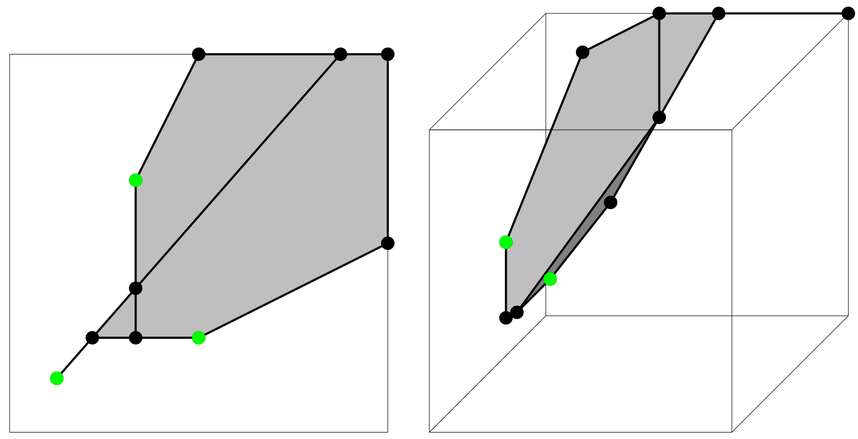


Figure 4.10: The sets $\mathrm{Fix}^{\Downarrow}(M)$ and $\mathrm{Fix}^{\Uparrow}(M)$ of Figure 3.6 as cell complexes. 0-cells are represented by dots (the green dots are the spanning 0-cells), 1-cells by thick line segments, and 2-cells by shaded grey regions.

We can combine Theorem 4.4.18 with [13, Theorem 1] to get the following result.

Theorem 4.4.22. *There is a natural bijection between combinatorial types of non-vanishing fuzzy relations $M\colon X \rightsquigarrow Y$, where $|X| = r$ and $|Y| = n$ and regular subdivisions of the product of simplices $\Delta_r \times \Delta_n$.*

Applying Proposition 4.4.7 and Theorem 4.4.18 together with [13, Theorem 23] then gives an explicit correspondence between $\mathrm{Fix}^{\Downarrow}(M)$ and $\mathrm{Fix}^{\Uparrow}(M)$ in which each cell $C_M^{\mathrm{col}}(S)$ is sent to the cell $C_M^{\mathrm{row}}(S)$ and vice versa. This correspondence is an isomorphism of cell complexes in the sense that k-cell are mapped to k-cells and the faces of a given cell are mapped to faces of the image of that cell.

Theorem 4.4.23. *Let $M\colon X \rightsquigarrow Y$ be a non-vanishing fuzzy relation between finite sets, where $|X| = r$ and $|Y| = n$. Then the restrictions of the maps $M^*\colon (0,1]^r \to (0,1]^n$ and $M_*\colon (0,1]^n \to (0,1]^r$ provide an explicit isomorphism of cell complexes between $\mathrm{Fix}^{\Downarrow}(M)$ and $\mathrm{Fix}^{\Uparrow}(M)$.*

4.4.1 Generic nuclei

In order to transfer some further results, we need to translate what it means for the columns or rows of a fuzzy relation to be in "general position".

In the tropical setting we have the following definition:

Definition 4.4.24. The *tropical determinant* of a $(k \times k)$-matrix M is the tropical polynomial

$$\det_{\oplus\odot}(M) := \bigoplus_{\sigma\in\mathrm{Sym}_k}\left(\bigodot_{i=1}^{k} m_{i\sigma(i)}\right),$$

where Sym_k is the symmetric group of permutations of k elements. A matrix M is said to be *tropically singular* if this minimum is attained at least twice.

For more information on the tropical determinant, see [11, 53].

Definition 4.4.25. A matrix $M \in (0,1]^{r,n}$ considered as a fuzzy relation between finite sets is said to be *generic* if no $(k\times k)$-submatrix of $\phi(M)$ is tropically singular. We say that a fuzzy relation between finite sets is *generic* if its matrix is generic. If M is generic we say that the rows and columns of M are in *general position*.

Example 4.4.26. Let $M = \begin{pmatrix} 1/2 & 1/3 \\ 1/4 & 2/3 \end{pmatrix}$. Then $\phi(M) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -\log_2 3 \\ 0 & -2 & 1-\log_2 3 \end{pmatrix}$, so

$$\begin{aligned}\det_{\oplus\odot}(\phi(M)) &= (\phi(m)_{11}\odot\phi(m)_{22}\odot\phi(m)_{33})\oplus(\phi(m)_{12}\odot\phi(m)_{21}\odot\phi(m)_{33})\\ &\oplus(\phi(m)_{13}\odot\phi(m)_{22}\odot\phi(m)_{31})\oplus(\phi(m)_{11}\odot\phi(m)_{23}\odot\phi(m)_{32})\\ &\oplus(\phi(m)_{12}\odot\phi(m)_{23}\odot\phi(m)_{31})\oplus(\phi(m)_{13}\odot\phi(m)_{21}\odot\phi(m)_{32})\\ &=(0\odot -1\odot 1-\log_2 3)\oplus(0\odot 0\odot 1-\log_2 3)\oplus(0\odot -1\odot 0)\\ &\oplus(0\odot -\log_2 3\odot -2)\oplus(0\odot -\log_2 3\odot 0)\oplus(0\odot 0\odot -2)\\ &=(-\log_2 3)\oplus(1-\log_2 3)\oplus(-1)\oplus(-2-\log_2 3)\\ &\oplus(-\log_2 3)\oplus(-2)\\ &=-2-\log_2 3.\end{aligned}$$

Since this minimum is attained only once (for the transposition $\sigma = (23)$), the matrix M is *not* singular. Thus the vectors $\begin{pmatrix} 1/2 \\ 1/4 \end{pmatrix}$ and $\begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}$ are in general position in $[0,1]^2$.

It can be shown that a matrix M is generic if none of its columns can be obtained from the others via the box operations. This follows from a corresponding statement about tropical determinants; see [53].

The main result of [13] is the theorem that there is a bijection between tropical complexes generated by r points in $\mathbb{TP}^{n-1}$ and regular triangulations of the product of simplices $\Delta^r \times \Delta^n$. By the same argument it can be shown that nuclear complexes of $(r\times n)$-matrices in $[0,1]^r \times [0,1]^n$ are in bijection with regular triangulations of the higher-dimensional product of simplices $\Delta^{r+1}\times\Delta^{n+1}$. The extra dimensions arise from the variables $\overline{v}_0$ and $\overline{w}_0$.

The next result is a translation of [13, Corollary 25]. We spell out some points in the proof which were only very briefly explained in the original.

Theorem 4.4.27. *Let $M\colon X \rightsquigarrow Y$ be a non-vanishing fuzzy relation between finite sets, where $|X| = r$ and $|Y| = n$. Then the number of k-cells in the cell decomposition of* $\mathrm{Nuc}(M)$ *is equal to the multinomial coefficient*

$$\binom{r+n-k}{r-k,\ n-k,\ k} = \frac{(r+n-k)!}{(r-k)!\cdot(n-k)!\cdot k!}.$$

Proof. This follows from [13, Corollary 25]. By Theorem 4.4.18 and the results leading up to it, the cell complex structure on $\mathrm{Nuc}(M)$ is equivalent to the cell complex structure on a tropical complex of $n+1$ points in $\mathbb{TP}^r$. The number of k-cells can then be counted for a specific example, since all generic tropical complexes of $n+1$ points in $\mathbb{TP}^r$ have the same number of k-cells. This is essentially because tropical complexes correspond to triangulations of $\Delta_r \times \Delta_n$ with the k-cells of the tropical complex corresponding to interior faces of codimension k in the triangulation. The polytope $\Delta_r \times \Delta_n$ is known to be equidecomposable, meaning every triangulation has the same number of cells of each dimension, i.e. every triangulation has the same f-vector (a sequence of integers listing the number of faces of each dimension). See the original proof for further details on this point.

All of this holds for the spaces $\mathrm{Fix}^{\Downarrow}(M)$ and $\mathrm{Fix}^{\Uparrow}(M)$, since these are isomorphic to tropical convex hulls. In other words, we can pick any generic fuzzy span $X = \mathrm{span}_{\boxplus\boxdot}\{x_1,\dots,x_n\}$ of n points in $(0,1]^r$ and compute the number of k-cells in X, knowing by the argument above that it doesn't matter which fuzzy span we pick.

We therefore choose the fuzzy span of the columns of the $(r \times n)$-matrix

$$M = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \cdots \\ \frac{1}{4} & \frac{1}{16} & \frac{1}{64} & \cdots \\ \frac{1}{8} & \frac{1}{64} & \frac{1}{512} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where each entry is given by $m_{ij} = 2^{-ij}$. This is depicted in Figure 4.11. M is generic, since it is not possible to obtain any column of M from the others via the box operations.

Let $v \in \mathrm{span}_{\boxplus\boxdot}(\mathrm{Col}(M))$. Let $j < l$. We claim that the sets S_j and S_l are intervals and intersect in at most one point, in which case the intersection is the largest element of S_j and the smallest element of S_l. To see this, let $i \in S_j$ and $k \in S_l$. The first of these statements means that $v_i \cdot w_j = \overline{m}_{ij}$, which implies, since all components are positive, that

$$\frac{\overline{m}_{ij}}{\overline{v}_i} \leqslant \frac{\overline{m}_{kj}}{\overline{v}_k}.$$

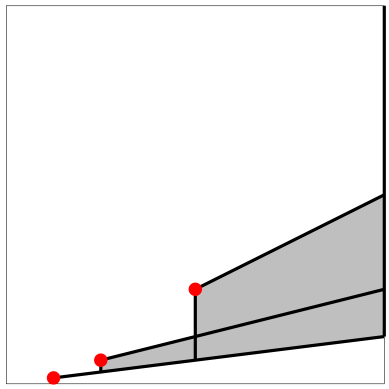


Figure 4.11: A generic fuzzy span.

Similarly, $k \in S_l$ implies

$$\frac{\overline{m}_{kl}}{\overline{v}_k} \leqslant \frac{\overline{m}_{il}}{\overline{v}_i}.$$

Multiplying these two inequalities together and then multiplying through by $v_i v_k$ implies

$$\overline{m}_{ij} \cdot \overline{m}_{kl} \leqslant \overline{m}_{il} \cdot \overline{m}_{kj}.$$

Since $\overline{m}_{ij} = 2^{-ij}$ for all $i \in \overline{[r]}$ and $j \in \overline{[n]}$ and $j < l$ this implies $i \leqslant k$. Each S_j is non-empty and, since the types of points in $\mathrm{span}_{\boxplus\boxdot}(\mathrm{Col}(M))$ are full, each i lies in some j. This means that the sets S_j must be intervals. It follows that the number of types with k degrees of freedom is equal to the number of ways of covering $n+1$ points with $r+1$ intervals such that $S_l \cap S_{l+1} \neq \varnothing$ for exactly k values of l.

Such a covering can be represented compactly by duplicating those points that are contained in two or more intervals, marking overlapping intervals with horizontal lines, and disjoint intervals with vertical lines. For example, the type

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

has $S_0 = \{0, 1\}$, $S_1 = \{1\}$, $S_2 = \{2, 3\}$ and corresponds to the following covering of $\{0, 1, 2, 3\}$:

$$\bullet \quad \bullet - \bullet \mid \bullet \quad \bullet$$

In general, there are $r + n - k$ spaces between points in such a diagram, k of which must be filled with vertical lines and $r - k$ of which must be filled with horizontal lines, so it is clear that the number of possible diagrams is equal to the stated multinomial coefficient. (In the example above, $r = 2$, $n = 3$, and $k = 1$, so by hypothesis S is one 1-cell out of $\frac{2+3-1!}{(2-1)!(3-1)!1!} = 12$ in total.)

To prove the theorem, we must show that not only does every point in $\mathrm{span}_{\boxplus\boxdot}(\mathrm{Col}(M))$ have a type of this form, but that every type of this form is in fact the type of a point in $\mathrm{span}_{\boxplus\boxdot}(\mathrm{Col}(M))$.

Given a type $S = (S_0, \ldots, S_n)$ with these properties, which for convenience we will refer to as the *covering properties*, i.e. such that

1. each S_j is an interval,
2. if $i \in S_j$, $k \in S_l$ and $j < l$ then $i \leqslant k$,
3. $\max(S_j) \leqslant \min(S_{j+1})$,
4. each $i \in \overline{[r]}$ is in some S_j,

we define a vector that lies in $\mathrm{span}_{\boxplus\boxdot}(\mathrm{Col}(M))$ and has the desired type.

Let $c_0 = 0$ and for each $i \in [r]$, let $c_i = \frac{1}{2}(\max(S^\top_{i-1}) + \min(S^\top_i))$. Then for each $i \in \overline{[r]}$ set

$$v_i = 2^{-C_i}, \tag{4.23}$$

where $S^\top$ is the transpose of S and C_k is the kth partial sum of the c_i, i.e. $C_k = c_0 + \cdots + c_k$. Note that by the third covering property above, the sequence $c_0, c_1, c_2, \ldots$ is increasing and each $c_i \geqslant 1$. This means the partial sums are also increasing.

We claim that $\mathrm{type}_M(v) = S$. To prove this we must show that when $w = M^*(v)$ we have $\overline{v}_i \cdot \overline{w}_j = \overline{m}_{ij}$ precisely when $i \in S_j$.

Substituting in (4.23) we see that $w = M^*(v)$ has coordinates

$$\begin{aligned}
\overline{w}_j &= \min_{0\leqslant k\leqslant r} \{\overline{v}_k \backslash \overline{m}_{kj}\} \\
&= \min_{0\leqslant k\leqslant r} \{2^{-C_k} \backslash 2^{-kj}\} \\
&= \min_{0\leqslant k\leqslant r} \{2^{C_k - kj}\}
\end{aligned}$$

and so it follows that we need to show that

$$2^{C_i - ij} = \min_{0\leqslant k\leqslant r} \{2^{C_k - kj}\} \quad \text{if and only if } i \in S_j.$$

This further reduces to showing that

$$C_i - ij = \min_{0\leqslant k\leqslant r} \{C_k - kj\} \quad \text{if and only if } i \in S_j.$$

The result follows from the following lemma. □

Lemma 4.4.28. *Let $S = (S_0, \ldots, S_n)$ be a type satisfying the covering properties listed above and define $c_0 = 0$, $c_i := \frac{1}{2}(\max(S_{i-1}^\top) + \min(S_i^\top))$ for $i \in [r]$. Then*

$$\sum_{l=0}^{i} c_l - ij = \min_{0 \leqslant k \leqslant r} \left\{ \sum_{l=0}^{k} c_l - kj \right\} \tag{4.24}$$

if and only if $i \in S_j$.

Proof. Write $T = S^\top$. It is straightforward to check that if $S = (S_0, \ldots, S_n)$ satisfies the covering properties above, then so does T, once subscripts have been changed appropriately.

Suppose that $i \in S_j$, so $j \in T_i$. It follows that $\min(T_i) \leqslant j$ and $\max(T_i) \geqslant j$, hence

$$\begin{aligned} \min(S_l) \leqslant \min(T_i) \leqslant j \qquad & \text{for all } l \leqslant i, \\ \max(S_l) \geqslant \max(T_i) \geqslant j \qquad & \text{for all } l \geqslant i. \end{aligned}$$

First, let $k \leqslant i$. Then

$$\begin{aligned} \sum_{l=0}^{i} c_l &= \sum_{l=0}^{k} c_l + \sum_{l=k+1}^{i} c_l \\ &\leqslant \sum_{l=0}^{k} c_l + (i-k)j \end{aligned}$$

since $c_l \leqslant \min(T_l) \leqslant j$ for all $l \leqslant i$. Hence, for $k \leqslant i$,

$$\sum_{l=0}^{i} c_l - ij \leqslant \sum_{l=0}^{k} c_l - kj.$$

Similarly, for $k \geqslant i$ we have

$$\begin{aligned} \sum_{l=0}^{k} c_l &= \sum_{l=0}^{i} c_l + \sum_{l=i+1}^{k} c_l \\ &= \sum_{l=0}^{i} c_l + \sum_{l=i}^{k-1} c_{l+1} \\ &\geqslant \sum_{l=0}^{i} c_l + (k-i)j \end{aligned}$$

since $c_{l+1} \geqslant \max(T_l) \geqslant j$ for all $l \geqslant i$. Hence, for $k \geqslant i$,

$$\sum_{l=0}^{i} c_l - ij \leqslant \sum_{l=0}^{k} c_l - kj.$$

This completes the proof that if $i \in S_j$ then

$$\sum_{l=0}^{i} c_l - ij = \min_{0 \leqslant k \leqslant r} \left\{ \sum_{l=0}^{k} c_l - kj \right\}.$$

To prove the converse, assume that equation (4.24) holds. We will show that $j \in T_i$. Since T_i is an interval, it is sufficient to prove that $\min(T_i) \leqslant j$ and $\max(T_i) \geqslant j$.

If $i = 0$ then $\min(T_i) = 0 \leqslant j$. Otherwise it follows immediately from equation (4.24) that for $k < i$

$$\sum_{l=k+1}^{i} c_l \leqslant (i-k)j.$$

In particular, for $k = i-1$ this reduces to $c_i \leqslant j$, i.e. $\frac{1}{2}(\max(T_{i-1}) + \min(T_i)) \leqslant j$. By the third and fourth covering properties of T, either $\max(T_{i-1}) = \min(T_i)$, in which case $\min(T_i) \leqslant j$ as required, or $\max(T_{i-1}) = \min(T_i) - 1$, in which case $\min(T_i) \leqslant j + \frac{1}{2}$, but this also implies $\min(T_i) \leqslant j$ since $\min(T_i)$ must be an integer.

If $i = r$ then $\max(T_i) = n \geqslant j$. Otherwise, in a similar way to above, for $k > i$ we have, again from equation (4.24), that

$$\sum_{l=i+1}^{k} c_l \geqslant (k-i)j.$$

For $k = i+1$ this reduces to $c_{i+1} = \frac{1}{2}(\max(T_i) + \min(T_{i+1})) \geqslant j$. By a similar argument to the above it follows that $\max(T_i) \geqslant j$. Hence $j \in T_i$, i.e. $i \in S_j$. □

Chapter 5

Computing the nucleus

In this chapter we discuss some methods for computing the fuzzy concept complex. We use some ideas from [1] and [29].

We have seen that the nucleus of a non-vanishing fuzzy relation between finite sets is equal to the union of closed cells of the form $C_M(S)$, where S ranges over all (r, n)-types that are full. Therefore, one approach to computing the nucleus of a fuzzy relation between finite sets would be the brute force approach of attempting to solve the systems of equations and inequalities described by each full candidate (r, n)-type. Unsurprisingly, this is not an effective method of computation.

We can be much more efficient with our search if we know some cells that are guaranteed to appear. In the first section of this chapter we prove the existence of certain 0-cells that must lie in $\mathrm{Fix}^{\Downarrow}(M)$ for a given non-vanishing fuzzy relation between finite sets.

5.1 Fuzzy sums and fuzzy scales

In this section we introduce some fundamental operations on points in the fuzzy concept complex and prove the existence of certain guaranteed 0-cells.

Definition 5.1.1. In Chapter 2 we saw how the space $[0,1]^{r,1}$ can be considered as a module over the semiring $([0,1], \min, \cdot)$, where for $v, v' \in [0,1]^{r,1}$ and $\lambda \in [0,1]$ the operations $\boxplus$ and $\boxdot$ are given by

$$(v \boxplus v')_i = \min\left\{v_i, v'_i\right\} \tag{5.1}$$

$$(\lambda \boxdot v)_i = \lambda \backslash v_i \tag{5.2}$$

for $i \in [r]$. We refer to the vector $v \boxplus v'$ as the *sum* of v and v' and we refer to the vector $\lambda \boxdot v$ as the *λ-scale of v*.

The sum and scaling operations are illustrated in Figure 5.1.

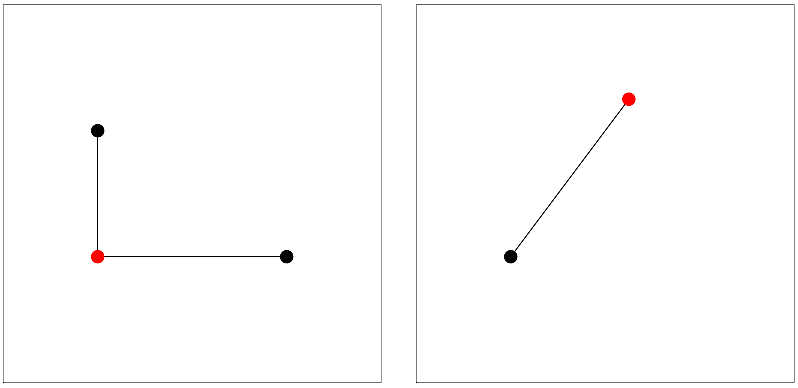


Figure 5.1: The fuzzy sum of two points and the λ-scale of a point (here $\lambda = 4/9$). The line segments are for illustrative purposes only.

For a given fuzzy relation M, Theorem 3.3.9 shows that the spaces $\mathrm{Fix}^{\Downarrow}(M) \cong \mathrm{Fix}^{\Uparrow}(M) \cong \mathrm{Nuc}(M)$ are closed under these operations. However, taking sums and scales of points does not generally preserve the dimension of the cells they lie in. For example, if $v, v' \in \mathrm{Fix}^{\Downarrow}(M)$ are 0-cells it is not generally the case that $v \boxplus v'$ or $\lambda \boxdot v$ are 0-cells. The next very simple example illustrates this point.

Example 5.1.2. Consider the matrix

$$M = \begin{pmatrix} 1/2 & 1/4 \\ 1/4 & 2/3 \end{pmatrix}$$

It is easy to check that $v = \binom{1}{1/2}$ and $v' = \binom{3/8}{1}$ are both 0-cells in $\mathrm{Fix}^{\Downarrow}(M)$. However, their sum $v \boxplus v' = \begin{pmatrix} 3/8 \\ 1/2 \end{pmatrix}$ lies in a 2-cell. See Figure 5.2.

Lemma 5.1.3. *Let $M \in (0,1]^{r,n}$ be the matrix of a fuzzy relation between finite sets. Let $v \in \mathrm{Fix}^{\Downarrow}(M)$ and let $S = \mathrm{type}_M(v)$. Then v is a 0-cell if and only if, for all $i, i' \in \overline{[r]}$ there exist $j_1, j_2, \ldots, j_N \in \overline{[n]}$ (not necessarily distinct) and $k_1, k_2, \ldots, k_{N-1} \in \overline{[r]}$ (also not necessarily distinct) such that $i, k_1 \in S_{j_1}, k_1, k_2 \in S_{j_2}, \ldots, k_{N-1}, i' \in S_{j_N}$.*

Proof. Recall that the dimension of an (r, n)-type S is equal to one less than the number of connected components of the graph G_S, where G_S has vertex set $\overline{[r]}$ and an edge between i and i' if and only if there exists $j \in \overline{[n]}$ such that $i \in S_j$ and $i' \in S_j$. The stated condition is satisfied precisely when G_S is connected, in which case $\dim(S) = 1 - 1 = 0$. $\square$

First we show that columns of M are 0-cells.

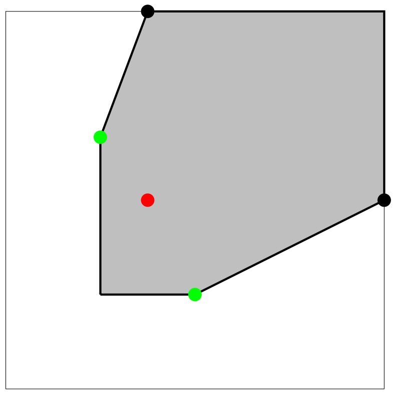


Figure 5.2: The fuzzy sum of two 0-cells in $\mathrm{Fix}^{\Downarrow}(M)$ is not necessarily a 0-cell. The cell complex illustrated is the generated by the two vectors marked in green, corresponding to the columns of a fuzzy relation. This cell complex consists of six 0-cells, six 1-cells and one 2-cell (shaded in grey). The fuzzy sum of the two black 0-cells, marked in red, is not a 0-cell, but instead lies in the 2-cell.

Proposition 5.1.4. *Let $M \in (0,1]^{r,n}$ be the matrix of a fuzzy relation between finite sets. For $l \in \overline{[n]}$, let $c^l = \overline{m}_{\bullet l} \in \mathrm{Col}(\overline{M})$ be the lth column of $\overline{M}$. Then c^l is a 0-cell.*

Proof. First, note that for $j \in [n]$ we have $(M^*(c^l))_j = \min_{1\leqslant i\leqslant r}\{m_{il}\backslash m_{ij}\}$ so, in particular, $(M^*(c^l))_l = 1$. So for all $i \in \overline{[r]}$ we have $\overline{(c^l)}_i \cdot \overline{M^*(c^l)}_l = (c^l)_i = \overline{m}_{il}$. If $S = \mathrm{type}_M(c^l)$, we therefore have $i \in S_l$ for all $i \in \overline{[r]}$. In other words, the lth column of the matrix representation of S consists entirely of 1s.

The result follows from Lemma 5.1.3. □

More generally, fuzzy sums of columns of M are 0-cells.

Proposition 5.1.5. *Let $M \in (0,1]^{r,n}$ be the matrix of a fuzzy relation between finite sets. Let $L \subseteq \overline{[n]}$ be a (finite) subset of the column indices of M, and let $c^L = \boxplus_{l\in L} c^l$. Then c^L is a 0-cell.*

Proof. For each $l \in L$, by definition $c^L_i \leqslant c^l_i = m_{il}$ for all $i \in [r]$. Thus for all $l \in L$,

$$(M^*(c^L))_l = \min_{1\leqslant i\leqslant r}\{c^L_i\backslash m_{il}\} = 1.$$

Hence $\overline{(c^L)}_0 \cdot \overline{M^*(c^L)}_l = 1 = \overline{m}_{00}$, so $0 \in S_l$ for all $l \in L$. In other words, if $S = \mathrm{type}_M(c^L)$, the top row of the matrix representation of S has a 1 in each column in L.

Now, as L is finite, for each $i \in [r]$, there exists $l \in L$ such that $c^L_i = c^l_i = m_{il}$ and so $c^L_i \cdot M^*(c^L)_l = m_{il}$, so $i \in S_l$. It follows from Lemma 5.1.3 that c^L is a 0-cell. □

Definition 5.1.6. Let $v, v' \in (0,1]^r$ be such that $v_i \leqslant v'_i$ for all $i \in [r]$. Let

$$\lambda = \min_{1 \leqslant i \leqslant r} \left\{ \frac{v'_i}{v_i} \right\} \tag{5.3}$$

and let k be a value of i for which this minimum is attained, i.e. k is such that $\lambda = v'_k / v_k$. Note that there is not necessarily a unique k with this property. The *(1-)scale of v towards v'* is the vector $\mathrm{Sc}(v, v')$ with coordinates

$$\mathrm{Sc}(v, v')_i = \lambda v_i = \left(\frac{v'_k}{v_k} \right) v_i \tag{5.4}$$

for $i \in [r]$. When v' is the unique vector with $v'_i = 1$ for all $i \in [r]$, we abbreviate $\mathrm{Sc}(v, v')$ to $\mathrm{Sc}(v)$, and simply call it the *(1-)scale of v*.

Scaling v towards v' once results in a vector whose coordinates agree with v' in coordinate k, i.e. $\mathrm{Sc}(v, v')_k = v'_k$. Note, however, that if this is already the case, i.e. if there is any $i \in [r]$ for which $v_i = v'_i$, then $\lambda = 1$ and $\mathrm{Sc}(v, v') = v$. Because of this fact, in order to continue scaling v towards v' we cannot simply define the 2-scale of v towards v' recursively as $\mathrm{Sc}^2(v, v') = \mathrm{Sc}(\mathrm{Sc}(v, v'), v')$, since $\mathrm{Sc}(v, v')_k = v'_k$ by construction. Instead, we wish to ignore those coordinates which have already been fully scaled up.

Define $\lambda_1, \ldots, \lambda_r$ and $k_1, \ldots, k_r$ such that

$$\begin{aligned}
\lambda_1 &= \frac{v'_{k_1}}{v_{k_1}} = \min_{1 \leqslant i \leqslant r} \left\{ \frac{v'_i}{v_i} \right\} \\
\lambda_2 &= \frac{v'_{k_2}}{v_{k_2}} = \min_{1 \leqslant i \leqslant r} \left\{ \frac{v'_i}{v_i} \;\middle|\; i \neq k_1 \right\} \\
&\;\;\vdots \\
\lambda_r &= \frac{v'_{k_r}}{v_{k_r}} = \min_{1 \leqslant i \leqslant r} \left\{ \frac{v'_i}{v_i} \;\middle|\; i \notin \{k_1, \ldots, k_{r-1}\} \right\}.
\end{aligned}$$

This gives an ordering of $[r]$ based on how close v is to v' in each coordinate. We have $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_r$ by construction. To repeatedly scale one vector towards another we keep scaling in this order, keeping coordinates fixed once they have been scaled fully.

We can now make the following definition.

Definition 5.1.7. Let $v, v' \in [0,1]^r$ be such that $v_i \leqslant v'_i$ for all $i \in [r]$ and let $R \in \overline{[r]}$. The *R-scale of v towards v'* is the vector $\mathrm{Sc}^R(v, v') \in [0,1]^r$ with

$$
\begin{aligned}
\mathrm{Sc}^0(v, v') &= v, \\
\mathrm{Sc}^R(v, v')_i &= \begin{cases} v'_i & \text{if } i \in \{k_1, \ldots, k_R\} \\ \mu_R \, \mathrm{Sc}^{R-1}(v, v')_i & \text{otherwise} \end{cases}
\end{aligned}
$$

where

$$
\mu_R = \min_{1 \leqslant i \leqslant r} \left\{ \frac{v'_i}{\mathrm{Sc}^{R-1}(v, v')_i} \;\middle|\; i \notin \{k_1, \ldots, k_{R-1}\} \right\}. \tag{5.5}
$$

Again, when v' is the unique vector with $v'_i = 1$ for all $i \in [r]$, we abbreviate $\mathrm{Sc}^R(v, v')$ to $\mathrm{Sc}^R(v)$, and simply call it the *R-scale of v*.

Lemma 5.1.8. *Let $R \in [r]$. The R-scale of v towards v' has coordinates*

$$
\mathrm{Sc}^R(v, v')_i = \begin{cases} v'_i & \text{if } i \in \{k_1, \ldots, k_R\} \\ \lambda_R v_i & \text{otherwise,} \end{cases} \tag{5.6}
$$

for $i \in [r]$.

Proof. For $R = 1$, we have, by Definition 5.1.7, that $\mathrm{Sc}^1(v, v')_{k_1} = v'_{k_1}$ and that $\mathrm{Sc}^1(v, v')_i = \mu_1 v = \lambda_1 v$ for $i \neq k_1$ as required. Again by Definition 5.1.7, we find that $\mathrm{Sc}^R(v, v')_i = v'_i$ for $i \in \{k_1, \ldots, k_R\}$. Suppose that $\mathrm{Sc}^{R-1}(v, v')$ satisfies (5.6). By (5.5),

$$
\begin{aligned}
\mu_R &= \min_{1 \leqslant i \leqslant r} \left\{ \frac{v'_i}{\mathrm{Sc}^{R-1}(v, v')_i} \;\middle|\; i \notin \{k_1, \ldots, k_{R-1}\} \right\} \\
&= \min_{1 \leqslant i \leqslant r} \left\{ \frac{v'_i}{\lambda_{R-1} v_i} \;\middle|\; i \notin \{k_1, \ldots, k_{R-1}\} \right\} \qquad \text{(by the induction hypothesis)} \\
&= \frac{1}{\lambda_{R-1}} \frac{v'_{k_R}}{v_{k_R}} \\
&= \frac{\lambda_R}{\lambda_{R-1}},
\end{aligned}
$$

so

$$
\mathrm{Sc}^R(v, v')_i = \mu_R \, \mathrm{Sc}^{R-1}(v, v')_i = \frac{\lambda_R}{\lambda_{R-1}} \lambda_{R-1} v_{k_R} = \lambda_R v_i,
$$

for all $i \notin \{k_1, \ldots, k_R\}$, as required.

The result follows by induction. □

Proposition 5.1.9. *Let $M \in (0,1]^{r,n}$ be the matrix of a fuzzy relation between finite sets and let $l, l' \in \overline{[n]}$ be column indices such that $m_{il} \leqslant m_{il'}$ for all $i \in [r]$. Then $\mathrm{Sc}^R(c^l, c^{l'})$ is a 0-cell for all $R \in \overline{[r]}$.*

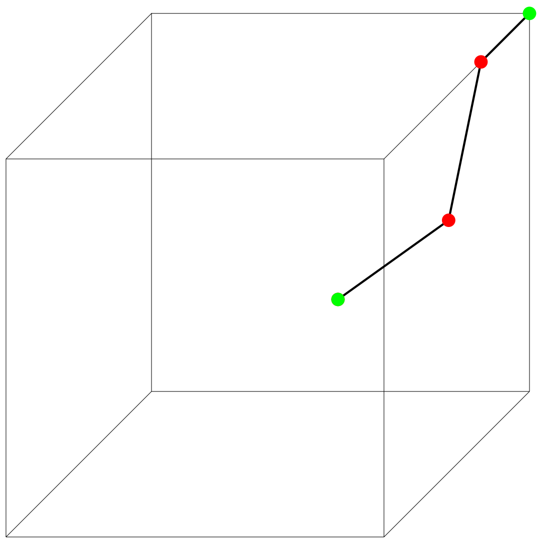


Figure 5.3: Scaling one 0-cell towards another. The line segments are for illustrative purposes only.

Proof. First note that, by Lemma 5.1.8,

$$\mathrm{Sc}^R(c^l, c^{l'})_k = c_k^{l'} = m_{kl'}$$

for all $k \in \{k_1, \ldots, k_R\}$ and

$$\mathrm{Sc}^R(c^l, c^{l'})_i = \lambda_R c_i^l = \left(\frac{c_{k_R}^{l'}}{c_{k_R}^l}\right) c_i^l = \left(\frac{m_{k_R l'}}{m_{k_R l}}\right) m_{il}$$

otherwise. The condition that $m_{il} \leqslant m_{il'}$ together with the fact that $\lambda \leqslant 1$ implies that $\lambda_R m_{il} \leqslant m_{il'}$ for all $i \in [r]$, so $(\lambda_R m_{il}) \backslash m_{il'} = 1$ for all $i \in [r]$. Thus, by definition of M^*,

$$M^*(\mathrm{Sc}^R(c^l, c^{l'}))_{l'} = \min_{1 \leqslant i \leqslant r} \{(\lambda_R m_{il}) \backslash m_{il'}\} = 1.$$

Hence

$$\overline{\mathrm{Sc}^R(c^l, c^{l'})_0} \cdot M^*(\mathrm{Sc}^R(c^l, c^{l'}))_{l'} = 1 = \overline{m}_{0l'}$$

so $0 \in S_{l'}$ and, for $k \in \{k_1, \ldots, k_R\}$,

$$\mathrm{Sc}^R(c^l, c^{l'})_k \cdot M^*(\mathrm{Sc}^R(c^l, c^{l'}))_{l'} = m_{kl'}$$

so $k \in S_{l'}$ for each $k \in \{k_1, \ldots, k_R\}$, where $S = \mathrm{type}_M(\mathrm{Sc}^R(c^l, c^{l'}))$.

Because $\lambda_R \leqslant 1$ we also have $m_{il} \leqslant \lambda_R m_{il}$ for all $i \in [r]$, so looking at the lth coordinate we see

$$
\begin{aligned}
M^*(\mathrm{Sc}^R(c^l, c^{l'}))_l &= \min_{1 \leqslant i \leqslant r} \{(\lambda_R m_{il}) \backslash m_{il}\} \\
&= \min_{1 \leqslant i \leqslant r} \left\{ \frac{m_{il}}{\lambda_R m_{il}} \right\} \qquad \text{(by definition of truncated division)} \\
&= \frac{1}{\lambda_R} \\
&= \frac{m_{k_R l}}{m_{k_R l'}}.
\end{aligned}
$$

Hence

$$
\mathrm{Sc}^R(c^l, c^{l'})_{k_R} \cdot M^*(\mathrm{Sc}^R(c^l, c^{l'}))_l = m_{k_R l'} \cdot \frac{m_{k_R l}}{m_{k_R l'}} = m_{k_R l}
$$

and, for $i \in [r] \backslash \{k_1, \ldots, k_R\}$,

$$
\mathrm{Sc}^R(c^l, c^{l'})_i \cdot M^*(\mathrm{Sc}^R(c^l, c^{l'}))_l = m_{il} \left(\frac{m_{k_R l'}}{m_{k_R l}} \right) \left(\frac{m_{k_R l}}{m_{k_R l'}} \right) = m_{il}
$$

so $k_R \in S_l$ and $i \in S_l$, for all $i \in [r] \backslash \{k_1, \ldots, k_R\}$.

Taking $N = 2$, $j_1 = l$, $j_2 = l'$ and $k = k_R$ in Lemma 5.1.3 it follows that $\mathrm{Sc}^R(c^l, c^{l'})$ is a 0-cell. □

Proposition 5.1.10. *Let $M \in (0,1]^{r,n}$ be the matrix of a fuzzy relation between finite sets and let $l, l' \in \overline{[n]}$ be column indices of M. Then $\mathrm{Sc}^R(c^l \boxplus c^{l'}, c^l)$ is a 0-cell for all $R \in \overline{[r]}$.*

Proof. For each $i \in [r]$, either $(c^l \boxplus c^{l'})_i = m_{il}$ or $(c^l \boxplus c^{l'})_i = m_{il'}$, depending on whether $m_{il} \leqslant m_{il'}$ or vice versa. For a given $i \in [r]$ we find, via Lemma 5.1.8, that

$$
\mathrm{Sc}^R(c^l \boxplus c^{l'}, c^l)_i = m_{il} \leqslant m_{il'}
$$

in the first case, and

$$
\mathrm{Sc}^R(c^l \boxplus c^{l'}, c^l)_i = \frac{m_{k_R l}}{m_{k_R l'}} m_{il'} \leqslant m_{il}
$$

in the second case. It follows that

$$
M^*(\mathrm{Sc}^R(c^l \boxplus c^{l'}, c^l))_l = \min_{1 \leqslant i \leqslant r} \{m_{il} \backslash m_{il}\} = 1
$$

and

$$
M^*(\mathrm{Sc}^R(c^l \boxplus c^{l'}, c^l))_{l'} = \min_{1 \leqslant i \leqslant r} \left\{ \left(\frac{m_{k_R l}}{m_{k_R l'}} m_{il'} \right) \backslash m_{il'} \right\} = \frac{m_{k_R l'}}{m_{k_R l}}.
$$

So, if $m_{il} \leqslant m_{il'}$, then

$$\mathrm{Sc}^R(c^l \boxplus c^{l'}, c^l)_i \cdot M^*(\mathrm{Sc}^R(c^l \boxplus c^{l'}, c^l))_l = m_{il} \cdot 1 = m_{il},$$

and if $m_{il'} \leqslant m_{il}$, then

$$\mathrm{Sc}^R(c^l \boxplus c^{l'}, c^l)_i \cdot M^*(\mathrm{Sc}^R(c^l \boxplus c^{l'}, c^l))_{l'} = m_{il'} \frac{m_{k_R l}}{m_{k_R l'}} \cdot \frac{m_{k_R l'}}{m_{k_R l}} = m_{il'}.$$

Hence, if $S = \mathrm{type}_M(\mathrm{Sc}^R(c^l \boxplus c^{l'}, c^l))$, we have shown that for all $i \in [r]$, either $i \in S_l$ or $i \in S_{l'}$ (possibly both).

Next, note that $\mathrm{Sc}^R(c^l \boxplus c^{l'}, c^l)_{k_R} = m_{k_R l}$ by definition. Hence,

$$\mathrm{Sc}^R(c^l \boxplus c^{l'}, c^l)_{k_R} \cdot M^*(\mathrm{Sc}^R(c^l \boxplus c^{l'}, c^l))_l = m_{k_R l} \cdot 1 = m_{k_R l}$$

and

$$\mathrm{Sc}^R(c^l \boxplus c^{l'}, c^l)_{k_R} \cdot M^*(\mathrm{Sc}^R(c^l \boxplus c^{l'}, c^l))_{l'} = m_{k_R l} \cdot \frac{m_{k_R l'}}{m_{k_R l}} = m_{k_R l'}.$$

This means that $k_R \in S_l$ and $k_R \in S_{l'}$.

Finally, we note that $\overline{\mathrm{Sc}^R(c^l \boxplus c^{l'}, c^l)_0} \cdot M^*(\mathrm{Sc}^R(c^l \boxplus c^{l'}, c^l))_l = 1 = \overline{m}_{00}$, so $0 \in S_l$.

Taking $N = 2$, $j_1 = l$, $j_2 = l'$ and $k = k_R$ in Lemma 5.1.3 it follows that $\mathrm{Sc}^R(c^l \boxplus c^{l'}, c^l)$ is a 0-cell. □

Example 5.1.11. Note that if $L \subset \overline{[n]}$ and $l' \in L$ it is generally *not* the case that the point $\mathrm{Sc}^R(\boxplus_{l \in L} c^l, c^{l'})$ is a 0-cell if $|L| > 2$. For example, consider the matrix

$$M = \begin{pmatrix} 1/4 & 1/2 & 2/3 \\ 4/5 & 1/3 & 1/5 \\ 3/4 & 1/4 & 2/3 \\ 1 & 1/2 & 5/6 \end{pmatrix}.$$

The sum of all three columns of M is the column vector

$$c = \begin{pmatrix} 1/4 \\ 1/5 \\ 1/4 \\ 1/2 \end{pmatrix}$$

and the first non-trivial scale of this towards $c^1 \in \mathrm{Col}(M)$ is obtained by multiplying all but the first coordinate by $\lambda = 2$:

$$\mathrm{Sc}^1(c, c^1) = \begin{pmatrix} 1/4 \\ 2/5 \\ 1/2 \\ 1 \end{pmatrix}.$$

This vector has dual

$$M^*(\mathrm{Sc}^1(c, c^1)) = \begin{pmatrix}1 & 1/2 & 1/2\end{pmatrix};$$

its type matrix can then be seen to be

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

from which it is clear that $\mathrm{Sc}^1(c, c^1)$ is not a 0-cell. (In fact, in this case this is a 1-cell.)

Based on numerical experimentation, the correct generalisation of Proposition 5.1.10 appears to be the following.

Conjecture 5.1.12. *Let $M \in (0,1]^{r,n}$ be the matrix of a fuzzy relation between sets. Let $L \subseteq \overline{[n]}$ be column indices and let $l' \in L$. Then for all $R \in [r]$, the vector*

$$\mathrm{Sc}^R\left(\boxplus_{l\in L} c^l, \boxplus_{l\in L\setminus\{l'\}} c^l\right)$$

is a 0-cell.

Since the cell complex structure of $\mathrm{Fix}^{\Uparrow}(M)$ is isomorphic to the cell complex structure of $\mathrm{Fix}^{\Downarrow}(M)$ and $\mathrm{Row}(M) = \mathrm{Col}(M^\top)$, all of the above statements are true for row vectors in $\mathrm{Fix}^{\Uparrow}(M)$. This also allows us to compute additional 0-cells in $\mathrm{Fix}^{\Downarrow}(M)$ using M_*. Note that $\mathrm{Fix}^{\Downarrow}(M) = \mathrm{Fix}^{\Uparrow}(M^\top)$ and $\mathrm{Fix}^{\Downarrow}(M^\top) = \mathrm{Fix}^{\Uparrow}(M)$.

Lemma 5.1.13. *If $w \in (0,1]^n$ is a 0-cell in $\mathrm{Fix}^{\Uparrow}(M)$ then $M_*(w) \in (0,1]^r$ is a 0-cell in $\mathrm{Fix}^{\Downarrow}(M)$.*

Proof. This follows directly from Theorem 4.4.23. □

Lemma 5.1.14. *If $v \in \mathrm{Fix}^{\Downarrow}(M)$, then the dimension of the cell in $\mathrm{Fix}^{\Downarrow}(M)$ containing v is equal to the dimension of the cell in $\mathrm{Fix}^{\Uparrow}(M)$ containing $M^*(v)$.*

Proof. Since $\mathrm{type}_M(v) = \mathrm{type}_M(M^*(v))^\top$ and the dimension of a cell is entirely determined by the type data, it follows immediately that the dimensions agree. □

We have the following corollaries. We prove the first of these explicitly; the others can be proved in exactly the same way.

Corollary 5.1.15. *Let $M \in (0,1]^{r,n}$ be the matrix of a fuzzy relation between finite sets. For $k \in \overline{[r]}$, let $d^k = \overline{m}_{k\bullet} \in \mathrm{Row}(\overline{M})$ be the kth row of $\overline{M}$. Then $M_*(d^k)$ is a 0-cell in* $\mathrm{Fix}^{\Downarrow}(M)$.

Proof. Since $\mathrm{Row}(M) = \mathrm{Col}(M^\top)$, we have $d^k \in \mathrm{Col}(\overline{M^\top})$. Thus, by Proposition 5.1.4, d^k is a 0-cell in $\mathrm{Fix}^{\Downarrow}(M^\top) = \mathrm{Fix}^{\Uparrow}(M)$. Then, by Lemma 5.1.13, $M_*(d^k)$ is a 0-cell in $\mathrm{Fix}^{\Downarrow}(M)$. □

Corollary 5.1.16. *Let $M \in (0,1]^{r,n}$ be the matrix of a fuzzy relation between finite sets. Let $K \subseteq \overline{[r]}$ be a (finite) subset of the row indices of M, and let $d^K = \boxplus_{k\in K} d^k$. Then $M_*(d^K)$ is a 0-cell in* $\mathrm{Fix}^{\Downarrow}(M)$.

Corollary 5.1.17. *Let $M \in (0,1]^{r,n}$ be the matrix of a fuzzy relation between finite sets and let $k, k' \in \overline{[r]}$ be row indices such that $m_{kj} \leqslant m_{k'j}$ for all $j \in [n]$. Then $M_*(\mathrm{Sc}^N(d^k, d^{k'})) \in \mathrm{Fix}^{\Downarrow}(M)$ is a 0-cell for all $N \in \overline{[n]}$.*

Corollary 5.1.18. *Let $M \in (0,1]^{r,n}$ be the matrix of a fuzzy relation between finite sets and let $k, k' \in \overline{[r]}$ be row indices of M. Then $M_*(\mathrm{Sc}^N(c^l \boxplus c^{l'}, c^l)) \in \mathrm{Fix}^{\Downarrow}(M)$ is a 0-cell for all $N \in \overline{[n]}$.*

5.2 A heuristic for computing the skeleton of $\mathrm{Fix}^{\Downarrow}(M)$

The author has produced a *Sage* program for computing the 1-skeleton of the nucleus, based on a heuristic whereby we start with certain 0-cells that are known to lie in the nucleus, and then calculate a list of potential types of neighbouring 1-cells. We can use some of the guaranteed 0-cells described in the previous section, for instance. This provides a much smaller list of candidate types for which to then attempt to find a solution to the corresponding system of equations and inequalities. Once we have determined which of these candidate types are actually types of 1-cells, by successfully solving the system of equations and inequalities they describe, we calculate a list of potential neighbouring 0-cells, which we then attempt to solve. We repeat this process until we reach an iteration in which no new cells are discovered. Since the nucleus of a non-vanishing fuzzy relation is always connected, it is therefore reasonable to assume that all 0- and 1-cells in the nucleus have been found.

We refrain from referring to this approach as an algorithm as it has not been proven to be successful in every case, although its effectiveness, at least in low dimensions, is backed up by empirical evidence.

The program can be used to produce graphical representations of the nucleus, as in Figure 5.4.

The difficulty with the heuristic described above is computing the neighbouring cells. One way to do this is to try adding or removing equations until the rank

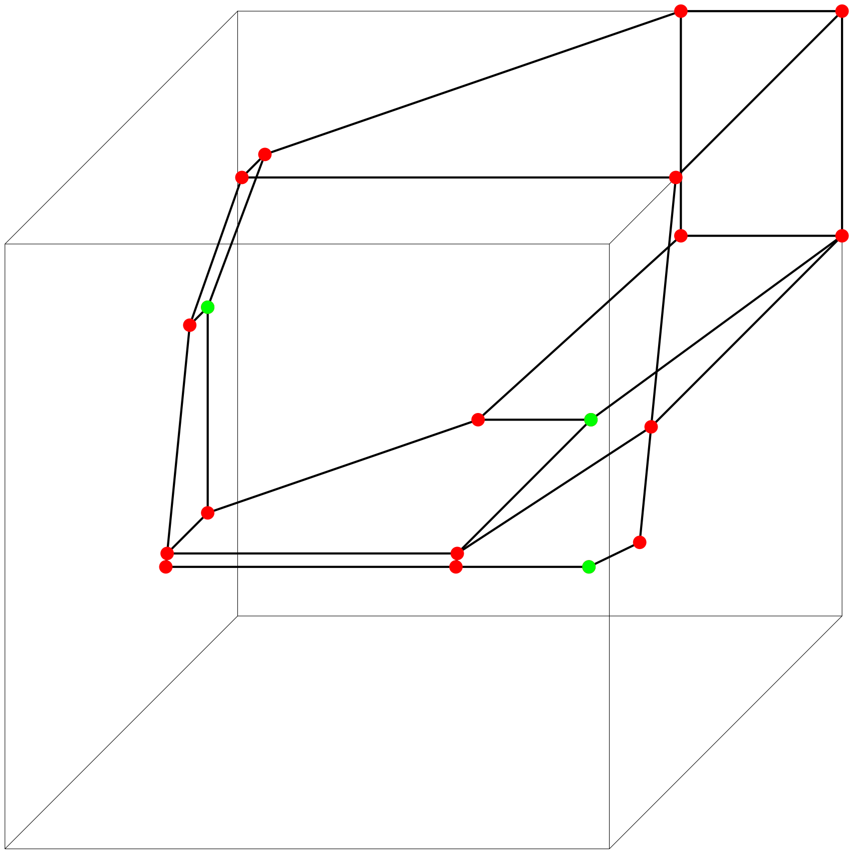


Figure 5.4: The 1-skeleton of the fuzzy concept complex of a fuzzy relation. The points corresponding to the columns of the matrix of the relation are highlighted in green.

of the system changes in the appropriate direction, but this is in inefficient. Tropical oriented matroids allow a much more direct computation of neighbouring types, as described in the next section.

5.3 Tropical oriented matroids

Matroids were introduced by Whitney [59] as a way of unifying and generalising ideas about linear independence arising in linear algebra and graph theory. A brief introduction can be found in [46].

Oriented matroids allow one to describe arrangements of ordinary hyperplanes. They consist of covectors which encode which side of each hyperplane a given point lies, in much the same way that the type of a point in tropical space encodes in which sector of each tropical hyperplane the point lies. The correspondence between oriented matroids and hyperplane arrangements is not one-to-one: every arrangement of hyperplanes gives rise to an oriented matroid, but not every oriented matroid is obtained from an arrangement of hyperplanes. However, every oriented matroid does correspond to an arrangement of *pseudo*-hyperplanes. Oriented matroids have been used in a wide range of areas, including linear programming.

In [1], Ardila and Develin introduced *tropical oriented matroids* to describe arrangements of tropical hyperplanes. Again, tropical oriented matroids do not correspond bijectively with tropical hyperplane arrangements, but they do correspond bijectively with tropical pseudohyperplane arrangements. Topological representation theorems to that effect have been proved in [23, 24].

We restate some definitions from [1], translated to use our own conventions.

Definition 5.3.1. A *type* with parameters (r, n), or an (r, n)*-type* is an n-tuple of subsets of $[r]$.

Note that this is different to how we defined an (r, n)-type in the previous chapter, where it was an $(n + 1)$-tuple of subsets of $\overline{[r]}$.

Definition 5.3.2. Let A and B be (r, n)-types. The *comparability graph* $\mathrm{CG}_{A,B}$ has vertex set $[r]$ with an edge between i_1 and i_2 whenever $i_1 \in A_j$ and $i_2 \in B_j$ for some $j \in [n]$. The edge $\{i_1, i_2\}$ is undirected if $i_1, i_2 \in A_j \cap B_j$ and directed $i_1 \to i_2$ otherwise.

The comparability graph of two types is a mathematical object called a *semi-directed graph*, or *semidigraph*. This just means it has some directed edges and some undirected edges.

Definition 5.3.3. An *ordered partition* of a finite set X is an ordered tuple $P = (P_1, \ldots, P_d)$ of subsets of X such that for each $x \in X$ there is exactly one index $k \in [d]$ such that $x \in P_k$.

Definition 5.3.4. Given an (r, n)-type $A = (A_1, \dots, A_r)$ and an ordered partition $P = (P_1, \dots, P_d)$ of $[r]$, the *refinement* of A with respect to P is the (r, n)-type A_P with

$$(A_P)_j = A_j \cap P_{m(j)}, \tag{5.7}$$

where $m(j)$ is the largest index in $[d]$ for which $A_j \cap P_{m(j)} \neq \varnothing$. A refinement A_P is *total* if each of its entries is a singleton set.

Example 5.3.5. Let $r = n = 3$. Let $A = (\{1, 2\}, \{1\}, \{3\})$ be a type and let $P = (\{2, 3\}, \{1\})$ be an ordered partition of $[n] = \{1, 2, 3\}$. Then we find

$$\begin{aligned}
A_1 \cap P_1 = \{2\} \neq \varnothing, \qquad & A_1 \cap P_2 = \{1\} \neq \varnothing && \implies m(1) = 2,\\
A_2 \cap P_1 = \varnothing, \qquad & A_2 \cap P_2 = \{1\} \neq \varnothing && \implies m(2) = 2,\\
A_3 \cap P_1 = \{3\} \neq \varnothing, \qquad & A_3 \cap P_2 = \varnothing && \implies m(3) = 1.
\end{aligned}$$

So $A_P = (A_1 \cap P_2, A_2 \cap P_2, A_3 \cap P_1) = (\{1\}, \{1\}, \{3\})$. Since each entry is a singleton, A_P is a total refinement.

Each ordered partition of $[r]$ determines a vector in $\mathbb{R}^r$. Given an ordered partition $P = (P_1, \dots, P_d)$ define $f \colon [r] \to [n]$ by taking $f(i)$ to be the unique $k \in [d]$ such that $i \in P_k$. Write $f(P)$ for the vector $(f(1), \dots, f(r)) \in \mathbb{R}^r$. By applying the usual isomorphism $\mathbb{R}^d \cong \mathbb{TP}^{d-1}$ this also gives a vector in $\mathbb{TP}^{d-1}$.

Example 5.3.6. Consider the oriented partition $P = (\{2, 3\}, \{1\})$ of the previous example. It is easy to see that $f(1) = 2$, while $f(2) = f(3) = 1$. Hence P corresponds to the vector $f(P) = (2, 1, 1) \in \mathbb{R}^3$. This projectivises to the vector $(0, -1, -1) \in \mathbb{TP}^2$, which can be represented by the vector $(-1, -1) \in \mathbb{R}^2$.

The upshot of this is that given any type we can immediately compute all neighbouring types of higher dimension by simply evaluating the refinement of our type with respect to each possible ordered partition. This essentially tells us, for each ordered partition P, what type we hit next if we move infinitessimally in the direction of the vector $f(P)$ described above.

Definition 5.3.7. A *tropical oriented matroid* M with parameters (r, n) consists of a set of (r, n)-types subject to the following four conditions.

- **Boundary**. For each $i \in [r]$, the type $i = (\{i\}, \dots, \{i\})$ is in M.
- **Elimination**. If A and B are types in M and $j \in [n]$, then there is a type C in M with $C_j = A_j \cup B_j$ and $C_k \in \{A_k, B_k, A_k \cup B_k\}$ for all $k \in [n]$.
- **Comparability**. If A and B are types in M, the comparability graph $\mathrm{CG}_{A,B}$ is acyclic.

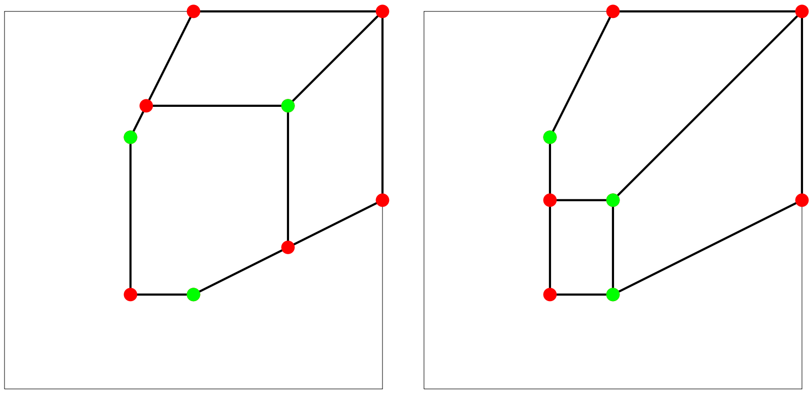


Figure 5.5: The skeleta of two fuzzy spans with the same underlying set but different cell complex structures.

- **Surrounding**. If A is a type in M and P is any ordered partition of $[r]$, then the refinement A_P is also in M.

Ardila and Develin prove the following fundamental, if unsurprising, result in [1, Theorem 3.8].

Theorem 5.3.8. *The collection of types in a tropical hyperplane arrangement constitutes a tropical oriented matroid.*

They also showed the following.

Theorem 5.3.9. *A tropical oriented matroid with parameters (n, d) is determined by its vertices.*

This means that for a fuzzy relation M between finite sets, the fuzzy spans $\mathrm{Fix}^{\Downarrow}(M)$ and $\mathrm{Fix}^{\Uparrow}(M)$ are also determined by their vertices. It is clear that just the underlying set is not sufficient to distinguish two fuzzy spans. An example is shown in Figure 5.5.

Figure 5.6 shows two skeleta of fuzzy spans with the same underlying set and the same 0-cells, but different cell complex structures. However, this does not contradict Theorem 5.3.9, since the parameters of these two spans are not the same: the left-hand diagram has an additional generating 0-cell, i.e. it corresponds to a fuzzy relation whose matrix representation has an additional column. If we were to look at $\mathrm{Fix}^{\Uparrow}(M)$ rather than $\mathrm{Fix}^{\Downarrow}(M)$ the underlying sets may not even agree.

It may be possible to use these ideas to turn the heuristic described in Section 5.2 into an algorithm. This is an area for future research. .

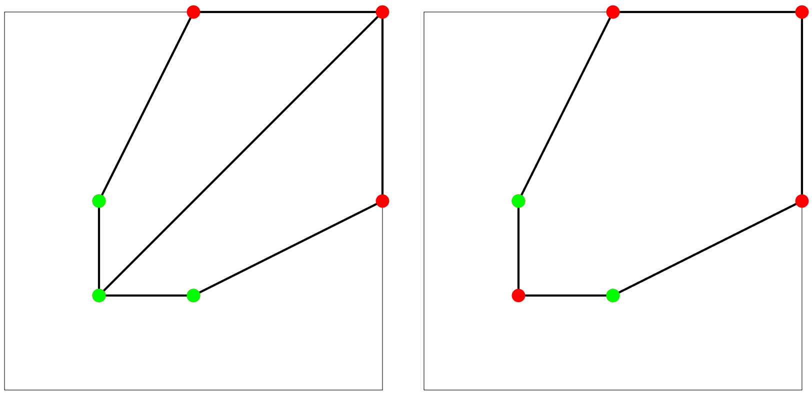


Figure 5.6: Two skeleta of fuzzy spans with the same 0-cells and the same underlying set but different cell complex structures overall.

Appendix A

Basic enriched category theory

A.1 Enriched categories

The theory of enriched categories is motivated by the observation that in many applications of ordinary category theory the hom-sets are not simply sets, but more complex objects. For instance, many categories used in homological algebra, such as categories of modules, have extra structure: one is able to add morphisms as well as compose them. In linear algebra the definition of the dual space of a vector space W as the space of linear functionals from W into the base field relies on the fact that the collection of linear maps between two vector spaces is itself a vector space.

The general idea of enriched category theory is to modify the *definition* of a category by replacing the hom-*sets* with objects of some other 'enriching category' $\mathcal{V}$. We stress at this point that we do *not* start with an underlying category and 'enrich' it; instead, an 'enriched category' is a distinct concept, which generalises many of the properties of an ordinary category.

Naturally, the first examples of possible enriching categories that come to mind are *concrete* categories of 'sets with structure', such as modules over some ring. In a category enriched in Banach spaces, for example, the hom-sets are Banach spaces and one is able to talk about the limit of a sequence of morphisms between two objects. It is important, however, to note that *non-concrete* enrichment is also possible. We will find that objects such as metric spaces and posets can be described as $\mathcal{V}$-categories for suitable (non-concrete) choice of $\mathcal{V}$. All we require is that $\mathcal{V}$ has enough structure that the result of our 'enrichment' still looks something like a category. In particular, we must still be able to define composition of morphisms.

In an ordinary category C, composition of two morphisms is defined if the domain of the first is the codomain of the second: given morphisms $f\colon A \to B$ and $g\colon B \to C$ we can form the composite morphism $g \circ f\colon A \to C$. In other words, composition constitutes a function on hom-sets,

$$C(B,C) \times C(A,B) \to C(A,C),$$

sending the pair (g, f) to its composite $g \circ f$. One key property of the category of sets that makes this definition of composition possible is the existence of a product for any two objects. In defining the identity morphism on an object X we also use the fact that a morphism from the terminal set to $C(X,X)$ specifies a morphism from X to itself.

In general we will take our enriching category to be a *monoidal category*. This is a category $\mathcal{V}_0$ with a monoidal product $\otimes\colon \mathcal{V}_0 \times \mathcal{V}_0 \to \mathcal{V}_0$ and a unit object $\mathbb{1}$, satisfying associativity and identity axioms. We write $\mathcal{V} = (\mathcal{V}_0, \otimes, \mathbb{1})$ for a general monoidal category and we say that $\mathcal{V}$ is *symmetric* if $X \otimes Y \cong Y \otimes X$ for all X, Y in $\mathcal{V}$. Note that, in particular, any category with finite products is monoidal under its categorical product with the terminal object as unit.

We are now ready to define an *enriched category*, first defined by Eilenberg and Kelly in [16], though we give a slightly more general definition. The definition is entirely analogous to the definition of an ordinary category, but with all references to the category of sets replaced by references to $\mathcal{V}$. The standard reference for enriched category theory is Kelly's book [33], where most of these results can be found. Another good source is [8].

Definition A.1.1. Let $\mathcal{V} = (\mathcal{V}_0, \otimes, \mathbb{1})$ be a symmetric monoidal category. A *category enriched in* $\mathcal{V}$, called a $\mathcal{V}$*-category* for short, consists of the following data:

- a set $\operatorname{Ob} C$ of *objects*;
- for all $X, Y \in \operatorname{Ob} C$, an object $C(X,Y)$ in $\mathcal{V}$, called the *hom-object* of morphisms from X to Y;
- for all $X, Y, Z \in \operatorname{Ob} C$, a morphism in $\mathcal{V}$

  $$\mu_{X,Y,Z}\colon C(Y,Z) \otimes C(X,Y) \to C(X,Z),$$

  called *composition*;
- for each $X \in \operatorname{Ob} C$, a morphism $1_X\colon \mathbb{1} \to C(X,X)$, the *identity on* X;

such that the following conditions hold:

- composition is associative, i.e., for all $W, X, Y, Z \in \operatorname{Ob} C$, the following diagram commutes:

$$\begin{tikzcd}
(\mathcal{C}(Y,Z)\otimes\mathcal{C}(X,Y))\otimes\mathcal{C}(W,X) \arrow[r, "\cong"] \arrow[dd, "\mu_{X,Y,Z}\otimes\mathrm{id}"'] & \mathcal{C}(Y,Z)\otimes(\mathcal{C}(X,Y)\otimes\mathcal{C}(W,X)) \arrow[d, "\mathrm{id}\otimes\mu_{W,X,Y}"] \\
 & \mathcal{C}(Y,Z)\otimes\mathcal{C}(W,Y) \arrow[d, "\mu_{W,Y,Z}"] \\
\mathcal{C}(X,Z)\otimes\mathcal{C}(W,X) \arrow[r, "\mu_{W,X,Z}"'] & \mathcal{C}(W,Z);
\end{tikzcd}$$

- the identities act as units for composition, i.e., for all $X, Y \in \operatorname{Ob}\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{C}(X,Y) \arrow[r, "\cong"] \arrow[d, "\cong"'] \arrow[ddrr, "\mathrm{id}"] & \mathcal{C}(X,Y)\otimes\mathbb{1} \arrow[r, "\mathrm{id}\otimes 1_X"] & \mathcal{C}(X,Y)\otimes\mathcal{C}(X,X) \arrow[dd, "\mu_{X,X,Y}"] \\
\mathbb{1}\otimes\mathcal{C}(X,Y) \arrow[d, "1_Y\otimes\mathrm{id}"'] & & \\
\mathcal{C}(Y,Y)\otimes\mathcal{C}(X,Y) \arrow[rr, "\mu_{X,Y,Y}"'] & & \mathcal{C}(X,Y).
\end{tikzcd}$$

The associativity and unitality conditions above are direct analogues of the associativity and unitality conditions for an ordinary category. Normally these are phrased as certain equalities between composites of morphisms. However, if our enriching category is non-concrete there may not be any morphisms, i.e. the hom-objects may not have elements, and so in general everything must be phrased in terms of the hom-objects themselves and the composition morphisms.

A.1.1 Examples of enriched categories

Defintion A.1.1 is quite general; choosing a different enriching category can significantly affect what a $\mathcal{V}$-category looks like. In some cases the result is a category with some extra structure, such as the facility to add morphisms to each other; in other cases it might be something that, on the surface, does not look much like a category at all. We give a range of examples below.

First of all, the following should not be a surprise.

Example A.1.2 (Ordinary categories)**.** An ordinary (locally small) category is a Set-category. The Cartesian product $\times$ plays the role of the monoidal product with any terminal set $\{*\}$ acting as the monoidal unit. The functions $1_X : \{*\} \to \mathcal{C}(X,X)$ pick out the identity morphisms $\mathrm{id}_X \in \mathcal{C}(X,X)$.

This example makes it clear that the concept of an enriched category generalises the concept of an ordinary category. Our first real example is a useful concrete enrichment.

Example A.1.3 (Pre-additive categories and linear categories)**.** For a ring R, the category R-Mod of R-modules and R-linear transformations forms a monoidal category whose monoidal product is the usual tensor product of modules and whose monoidal unit is R. In particular, $(\mathsf{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$ and $(\mathsf{Vect}, \otimes_k, k)$, where k is any field, are monoidal categories; Ab- and Vect-categories are called, respectively, *pre-additive categories* and *linear categories*. Pre-additive categories that satisfy some particular further axioms are called *Abelian categories* and are especially useful in homological algebra; see [17].

Another example, well known to category theorists, is the following.

Example A.1.4 (2-categories)**.** The category Cat of small categories is a monoidal category with respect to the Cartesian product. The terminal category 1 is the monoidal unit. A category $\mathcal{C}$ enriched in Cat is called a *strict 2-category*. Such an object has, for each pair of objects X and Y, a category of morphisms $\mathcal{C}(X, Y)$; in other words, the hom-objects themselves have objects and morphisms. Thus a strict 2-category has three types of component: objects (*0-cells*), morphisms (*1-cells*) between objects, and *2-cells* between 1-cells.

The prototypical example of a strict 2-category is Cat itself, with categories, functors, and natural transformations taking the roles of 0-cells, 1-cells, and 2-cells, respectively. The functoriality of composition amounts to what is sometimes known as the *middle four interchange law* for natural transformations: given functors $F, F', F'' \colon \mathcal{C} \to \mathcal{D}$ and $G, G', G'' \colon \mathcal{D} \to \mathcal{E}$ and natural transformations $\alpha \colon F \Rightarrow F'$, $\alpha' \colon F' \Rightarrow F''$, $\beta \colon G \Rightarrow G'$, $\beta' \colon G' \Rightarrow G''$ we have

$$(\beta' \circ \alpha') \cdot (\beta \circ \alpha) = (\beta' \cdot \beta) \circ (\alpha' \cdot \alpha),$$

where $\cdot$ indicates vertical composition of natural transformations.

In fact, for each enriching category $\mathcal{V}$, the collection of $\mathcal{V}$-categories, $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations (to be defined presently) forms a (strict) 2-category, which we denote $\mathcal{V}$-Cat.

Weaker than a strict 2-category is the notion of a *bicategory*, introduced by Bénabou in [5]. In a bicategory, composition is not associative 'on the nose', but only up to (coherent) isomorphism. Similarly, the identities are only weakly unital with respect to composition. In practice, bicategories arise much more frequently than strict 2-categories and in many contexts the term '2-category' will mean a bicategory rather than a strict one. A key example of a bicategory is $\mathcal{V}$-Prof, the bicategory of $\mathcal{V}$-categories, $\mathcal{V}$- profunctors, and $\mathcal{V}$-natural transformations between $\mathcal{V}$-profunctors, which we define in section 2.2. we omit any further details of bicategories, instead referring the reader to [5], [36] or [39].

Example A.1.5 (Preorders)**.** The category $\underline{2}$ with just two objects $\{\mathsf{true}, \mathsf{false}\}$ and one non-identity arrow $\mathsf{false} \to \mathsf{true}$ is a category of truth values, where morphisms correspond to entailment of propositions. Logical conjunction ('and') gives $\underline{2}$ a monoidal structure. The monoidal unit is true, since $p \wedge \mathsf{true} = p$ for any proposition p. Thus a $\underline{2}$-category A (not to be confused with a 2-category above) has, for each pair of objects a and b, an object $A(a, b)$ in $\underline{2}$, either true or false, specifying whether or not a and b are related.

The final example we give of a $\mathcal{V}$-category is one that will particularly useful later on when we discuss completions of enriched categories.

Definition A.1.6. For any symmetric monoidal category $\mathcal{V}$, the *trivial* or *unit* *$\mathcal{V}$-category,* denoted $\mathcal{I}$, has just one object $*$ with $\mathcal{I}(*, *) = \mathbb{1}$. Composition and identities are given by the canonical isomorphisms $\mathbb{1} \otimes \mathbb{1} \cong \mathbb{1}$ and $\mathbb{1} \cong \mathbb{1}$, respectively.

When $\mathcal{V}$ is Set, the trivial $\mathcal{V}$-category is the terminal category, with one object and a single identity morphism. When $\mathcal{V}$ is $\underline{2}$ it can be thought of the one-element poset.

We introduce $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations as the appropriate generalisations of functors and natural transformations to an enriched setting.

A.1.2 Enriched functors and enriched natural transformations

The notions of functor and natural transformation generalise straightforwardly to an enriched setting.

Definition A.1.7. Let $\mathcal{C}$ and $\mathcal{D}$ be $\mathcal{V}$-categories. A *$\mathcal{V}$-functor* $F\colon \mathcal{C} \to \mathcal{D}$ consists of a function $F\colon \operatorname{Ob}\mathcal{C} \to \operatorname{Ob}\mathcal{D}$ together with, for all $X, Y \in \operatorname{Ob}\mathcal{C}$, morphisms in $\mathcal{V}$

$$F_{X,Y}\colon \mathcal{C}(X, Y) \to \mathcal{D}(FX, FY),$$

such that the following diagrams commute:

$$\begin{tikzcd}
\mathcal{C}(Y,Z) \otimes \mathcal{C}(X,Y) \arrow[r, "F\otimes F"] \arrow[d, "\mu"'] & \mathcal{D}(FY,FZ) \otimes \mathcal{D}(FX,FY) \arrow[d, "\mu"] \\
\mathcal{C}(X,Z) \arrow[r, "F"'] & \mathcal{D}(FX,FZ),
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{1} \arrow[r, "1_X"] \arrow[dr, "1_{FX}"'] & \mathcal{C}(X,X) \arrow[d, "F"] \\
 & \mathcal{D}(FX,FX)
\end{tikzcd}$$

When $\mathcal{V}$ is concrete, these diagrams correspond to the ordinary functoriality conditions that $F(g \circ f) = Fg \circ Ff$ and $F(\mathrm{id}_X) = \mathrm{id}_{FX}$.

It is obvious that Set-functors are ordinary functors, so we turn our attention to enriched functors for the other examples of enriched categories given in the previous section.

Example A.1.8 (Additive functors and linear functors)**.** An Ab-functor between pre-additive categories C and $\mathcal{D}$ is a functor F between the underlying categories satisfying

$$F(f+g) = F(f) + F(g)$$

for all morphisms $f, g \in C(X,Y)$. Such a functor is called *additive*. Similarly, a Vect-functor is called a *linear functor*.

Example A.1.9 (Strict 2-functors)**.** Enriched functors between strict 2-categories, called *strict 2-functors*, send 0-cells to 0-cells, 1-cells to 1-cells and 2-cells to 2-cells in a way that is compatible with the structure of the 2-categories involved and are, by themselves, rather unremarkable. More interestingly, however, strict 2-categories and strict 2-functors form a monoidal category $2\text{-}\mathsf{Cat}$, so it is possible to define a *strict 3-category* as a $2\text{-}\mathsf{Cat}$-category and, more generally, a *strict n-category* as an $(n-1)\text{-}\mathsf{Cat}$-category.

Example A.1.10 (Order-preserving maps)**.** Let A and B be preorders, i.e. $\underline{2}$-categories. A $\underline{2}$-functor $f\colon A \to B$ is a map of sets such that, for $a, a' \in A$,

$$A(a,a') \vdash B(f(a), f(a')).$$

In other words,

$$a \leqslant_A a' \implies f(a) \leqslant_B f(a'),$$

so f is an order-preserving map.

We now turn to enriched natural transformations. In ordinary category theory, a natural transformation between two functors consists of a family of morphisms in the codomain category that satisfy a 'naturality condition', which just amounts to the commutativity of certain squares for every possible morphism in the domain. In a general enriched setting we cannot necessarily talk about individual morphisms, since general hom-objects might not have elements, so as usual we rephrase the definition in terms of hom-objects and composition.

Definition A.1.11. Let $F, G\colon C \to \mathcal{D}$ be $\mathcal{V}$-functors between $\mathcal{V}$-categories. A *$\mathcal{V}$-natural transformation* $\theta\colon F \Rightarrow G$ consists of a collection of morphisms $\theta_X\colon \mathbb{1} \to \mathcal{D}(FX, GX)$ in $\mathcal{V}$, indexed by the objects of C, such that for all $X, Y \in C$ the following diagram commutes, where $r\colon C(X,Y) \otimes \mathbb{1} \to C(X,Y)$ and $l\colon \mathbb{1} \otimes$

$\mathcal{C}(X,Y) \to \mathcal{C}(X,Y)$ are canonical isomorphisms:

$$\begin{tikzcd}
 & \mathcal{C}(X,Y)\otimes \mathbb{1} \arrow[r, "G\otimes\theta_X"] & \mathcal{D}(GX,GY)\otimes\mathcal{D}(FX,GX) \arrow[dr, "\mu"] & \\
\mathcal{C}(X,Y) \arrow[ur, "r^{-1}"] \arrow[dr, "l^{-1}"'] & & & \mathcal{D}(FX,GY). \\
 & \mathbb{1}\otimes\mathcal{C}(X,Y) \arrow[r, "\theta_Y\otimes F"'] & \mathcal{D}(FY,GY)\otimes\mathcal{D}(FX,FY) \arrow[ur, "\mu"'] &
\end{tikzcd}$$

When $\mathcal{V}$ is concrete, $\mathcal{V}$-natural transformations are not significantly different to ordinary natural transformations. For preorders we get something quite different.

Example A.1.12. If $f, g\colon A \to B$ are order-preserving maps, a $\underline{2}$-natural transformation from f to g consists of an A-indexed family of morphisms in $\underline{2}$

$$\mathsf{true} \vdash B(f(a), g(a)),$$

i.e. the statement $f(a) \leqslant g(a)$, for each $a \in A$.

Enriched natural transformations allow us to talk about the category of $\mathcal{V}$-functors between $\mathcal{V}$-categories $\mathcal{C}$ and $\mathcal{D}$. What we really want, though, is to make this a *$\mathcal{V}$-category*. This will have to wait until Section A.2 when we introduce ends.

A.1.3 Closed categories

One interesting property of the category of sets is that there is a one-to-one correspondence between two-variable functions $f\colon X \times Y \to Z$ and functions $\tilde{f}\colon X \to Z^Y$ into the 'function set' of functions from Y to Z. Given such an f we can define, for each x, a function $\tilde{f}(x) = f_x\colon Y \to Z$ by setting $f_x(y) := f(x,y)$, a process known as 'currying'; conversely, we can piece together such functions f_x to define f. Thus there is a bijection

$$\mathrm{Hom}(X \times Y, Z) \cong \mathrm{Hom}(X, Z^Y);$$

furthermore, this is natural in X and Z. In other words, for each set Y, the functor $-\times Y\colon \mathsf{Set} \to \mathsf{Set}$ is left adjoint to the functor $(-)^Y\colon \mathsf{Set} \to \mathsf{Set}$.

In general we say that a symmetric monoidal category $\mathcal{V}$ is *closed* if, for every $Y \in \mathrm{Ob}\,\mathcal{V}$, the $\mathcal{V}$-functor $-\otimes Y\colon \mathcal{V} \to \mathcal{V}$ has a right adjoint, which we denote by $[Y,-]\colon \mathcal{V} \to \mathcal{V}$. We call $[Y,Z]$ the *internal hom* from Y to Z. In particular, if a category is closed with respect to its categorical product we say it is *Cartesian closed*.

We have already discussed the closedness of Set. It turns out that the monoidal category $\underline{2}$ is also closed.

Example A.1.13. Given propositions p, q, r, it is a theorem of logic that

$$(p \wedge q) \vdash r \qquad \text{if and only if} \qquad p \vdash (q \Rightarrow r).$$

Thus the internal hom in $\underline{2}$ is given by implication.

For any closed monoidal category $\mathcal{V}$, the adjunction $- \otimes Y \dashv [Y, -]$ is known as the *hom–tensor adjunction.* Its unit and counit have components

$$\eta_X \colon X \to [Y, X \otimes Y], \qquad \varepsilon_X \colon [Y, X] \otimes Y \to X,$$

the latter being known as *evaluation* and sometimes denoted ev. The adjunction also gives an internal composition functor $[Y, Z] \otimes [X, Y] \to [X, Z]$ as the adjunct of the composite

$$[Y, Z] \otimes [X, Y] \otimes X \xrightarrow{\mathrm{id} \otimes \mathrm{ev}} [Y, Z] \otimes Y \xrightarrow{\mathrm{ev}} Z. \tag{A.1}$$

When $\mathcal{V}$ is closed it is possible to 'think of $\mathcal{V}$ itself as a $\mathcal{V}$-category'. More precisely, we can define a $\mathcal{V}$-category $\tilde{\mathcal{V}}$ with $\operatorname{Ob} \tilde{\mathcal{V}} := \operatorname{Ob} \mathcal{V}$ and $\tilde{\mathcal{V}}(X, Y) := [X, Y]$. For brevity we will often refer to $\tilde{\mathcal{V}}$ simply as $\mathcal{V}$. This allows us to talk about $\mathcal{V}$-functors into $\mathcal{V}$.

A $\mathcal{V}$-functor $P \colon C^{\mathrm{op}} \to \mathcal{V}$ is called a *presheaf* on C, while a $\mathcal{V}$-functor $Q \colon C \to \mathcal{V}$ is called a *copresheaf.* Here C^{op} is the $\mathcal{V}$-category whose objects are the same as those of C, with $C^{\mathrm{op}}(X, Y) := C(Y, X)$.

Example A.1.14. A one-object Ab-category R is a ring. A presheaf $M \colon R^{\mathrm{op}} \to \mathsf{Ab}$ turns out to be a right R-module, while a copresheaf is a left R-module. Similarly a one-object Vect-category is an algebra and a presheaf is an algebra representation.

Example A.1.15 (Upward-closed and downward-closed sets)**.** A copresheaf on a preorder A, i.e. a $\underline{2}$-functor $U \colon A \to \underline{2}$, gives a truth value $U(a)$ for each $a \in A$, which can be interpreted as the statement "$x \in U$", together with the entailment

$$A(a, a') \vdash (U(a) \Rightarrow U(a')),$$

which means if $a \in U$ and $a \leqslant a'$ then $a' \in U$. Thus U is an *upward-closed* or *ascending* set.

Similarly, a presheaf on A is a *downward-closed* or *descending* subset.

A.2 Ends and coends

An end is a special type of limit for $\mathcal{V}$-functors of mixed variance. We will restrict our attention to $\mathcal{V}$-functors of the form $C^{\mathrm{op}} \otimes C \to \mathcal{D}$. Recall that an ordinary limit of an ordinary functor $F \colon C \to \mathcal{D}$ consists of an object $\lim F$ in $\mathcal{D}$ together with a universal natural transformation from $\Delta \lim F$ to F. In order to define ends in a similar way, we first need to introduce a special type of naturality between functors of mixed variance, which we call *extraordinary naturality* or *extranaturality*. In fact, we only describe extranatural transformations to or from a constant functor. This is just one specific instance of the more general concept of *dinaturality*, but we do not need the full generality here.

Definition A.2.1. Let $F\colon \mathcal{C}^{\mathrm{op}} \otimes \mathcal{C} \to \mathcal{D}$ be a bifunctor and let $K \in \mathcal{D}$. An *extranatural transformation* $\alpha\colon K \Rightarrow F$ consists of a family of maps $\alpha_X\colon \mathbb{1} \to \mathcal{D}(K, F(X,X))$, indexed by the objects of $\mathcal{C}$, such that each diagram of the following form commutes:

$$\begin{tikzcd}
\mathcal{C}(X,Y) \arrow[r, "F(X,-)"] \arrow[d, "F(-,Y)"'] & \mathcal{D}(F(X,X), F(X,Y)) \arrow[d, "\mathcal{D}(\alpha_X, \mathrm{id})"] \\
\mathcal{D}(F(Y,Y), F(X,Y)) \arrow[r, "\mathcal{D}(\alpha_Y, \mathrm{id})"'] & \mathcal{D}(K, F(X,Y))
\end{tikzcd} \tag{A.2}$$

Such a transformation is called a *wedge* from K to F. An extranatural transformation $\beta\colon F \Rightarrow K$ is defined dually and is called a *cowedge*.

The $\mathcal{V}$-functor $\mathcal{D}(\alpha_X, \mathrm{id})$ is the composite

$$\mu \circ (\mathrm{id} \otimes \alpha_X) \circ r^{-1}\colon \mathcal{D}(F(X,X), F(X,Y)) \to \mathcal{D}(K, F(X,Y)),$$

corresponding, in the case of ordinary Set-categories, to the functor "precompose with α_X". Thus, in this unenriched setting, diagram (A.2) reduces to the commutativity of each diagram

$$\begin{tikzcd}
K \arrow[r, "\alpha_X"] \arrow[d, "\alpha_Y"'] & F(X,X) \arrow[d, "F(X,f)"] \\
F(Y,Y) \arrow[r, "F(f,Y)"'] & F(X,Y).
\end{tikzcd}$$

We can now define ends of functors valued in our enriching category $\mathcal{V}$. Ends of arbitrary functors can be defined in terms of weighted limits.

Definition A.2.2. Let $F\colon \mathcal{C}^{\mathrm{op}} \otimes \mathcal{C} \to \mathcal{V}$ be a $\mathcal{V}$-functor. An *end* of F consists of an object $K = \int_{C \in \mathcal{C}} F(C,C)$ of $\mathcal{V}$ together with a wedge $\lambda\colon K \Rightarrow F$ that is *universal* in the sense that, whenever $\alpha\colon K' \Rightarrow F$ is another wedge, there is a unique $f\colon K' \to K$ such that $\alpha_X = \lambda_X \circ f$ for every $X \in \mathcal{C}$.

Dually, a *coend* of F consists of an object $L = \int^{C \in \mathcal{C}} F(C,C)$ and a universal cowedge from F to L.

It turns out that the extranaturality condition of (A.2) reduces, in the case of $\mathcal{V}$-valued functors, to the equality of certain composites obtained as adjuncts of $F(X,-)$ and $F(-,Y)$, indicated by ρ and σ respectively. For brevity the details are omitted but can be found in [33]. The upshot is that we can write the end of F as the following equaliser

$$\textstyle\int_{C\in\mathcal{C}} F(C,C) \xrightarrow{\lambda} \prod_{C\in\mathcal{C}} F(C,C) \overset{\rho}{\underset{\sigma}{\rightrightarrows}} \prod_{X,Y\in\mathcal{C}}[\mathcal{C}(X,Y),F(X,Y)]. \tag{A.3}$$

The maps ρ and σ can be thought of as 'actions'. With this interpretation the end is the subobject of the product consisting of those objects for which the actions coincide.

In a similar way, the coend of F can be written as a coequaliser of morphisms between coproducts

$$\textstyle\coprod_{X,Y\in\mathcal{C}} \mathcal{C}(X,Y)\otimes F(X,Y) \rightrightarrows \coprod_{C\in\mathcal{C}} F(C,C) \longrightarrow \int^{C\in\mathcal{C}} F(C,C), \tag{A.4}$$

and represents the 'quotienting out' of the coproduct by identifying the results of the actions.

Example A.2.3. Given a preorder A and an order-preserving map $f\colon A^{\mathrm{op}}\otimes A\to\underline{2}$, the end of f is given by the universal quantification $(\forall a\in A)f(a,a)$, since, for each $a'\in A$,

$$(\forall a\in A)f(a,a) \implies f(a',a')$$

and this is optimum. Similarly, coends in $\underline{2}$ correspond to existential quantification.

Example A.2.4. Given a proxet X and a proximity map $f\colon X^{\mathrm{op}}\otimes X\to[0,1]$, the end of f is given by infimum $\inf_{x\in X} f(x,x)$, since, for each $x'\in X$,

$$\inf_{x\in X} f(x,x) \leqslant f(x',x'),$$

i.e. it is a lower bound for f, and, by definition, it is the greatest such lower bound and thus constitutes a universal wedge. Similarly, coends in $[0,1]$ correspond to suprema.

A.2.1 Enriched functor categories

In section A.1.2 we defined $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations for a given symmetric monoidal closed category $\mathcal{V}$ and observed that together with $\mathcal{V}$-categories these form a 2-category $\mathcal{V}$-Cat. We now show that, in keeping with the general enriched feel of our discussion, we can define a $\mathcal{V}$-object of $\mathcal{V}$-natural transformations between two $\mathcal{V}$-functors. The following observation is useful.

Lemma A.2.5. *A family of morphisms $\alpha_X : \mathbb{1} \to \mathcal{D}(FX, GX)$ constitutes the components of a $\mathcal{V}$-natural transformation $\alpha : F \Rightarrow G$ precisely when it forms an extranatural transformation $\alpha : \mathbb{1} \to \mathcal{D}(F-, G-)$.*

By definition, a morphism $\lambda : \mathbb{1} \to \int_{C \in \mathcal{C}} \mathcal{D}(FC, GC)$ corresponds to an extranatural transformation $\lambda : \mathbb{1} \Rightarrow \mathcal{D}(F-, G-)$, i.e., by the lemma, a $\mathcal{V}$-natural transformation $\lambda : F \Rightarrow G$. Thus the *set of elements* of the end, i.e. the image of the end under the forgetful functor $V = \mathcal{V}_0(\mathbb{1}, -) : \mathcal{V} \to \mathsf{Set}$, consists of the $\mathcal{V}$-natural transformations from F to G. We make the following definition.

Definition A.2.6. Let $\mathcal{C}$ and $\mathcal{D}$ be $\mathcal{V}$-categories with $\mathcal{C}$ small. The *enriched functor category* $[\mathcal{C}, \mathcal{D}]$ is the $\mathcal{V}$-category whose objects are $\mathcal{V}$-functors from $\mathcal{C}$ to $\mathcal{D}$ and, given $\mathcal{V}$-functors $F, G : \mathcal{C} \to \mathcal{D}$, whose hom-objects are

$$[\mathcal{C}, \mathcal{D}](F, G) := \int_{C \in \mathcal{C}} \mathcal{D}(FC, GC). \tag{A.5}$$

Details of the composition and identities in $[\mathcal{C}, \mathcal{D}]$ are given in [33].

For more information about enriched functor categories, see also [10].

We also have an enriched version of the Yoneda Lemma.

Lemma A.2.7 (Yoneda)**.** *Let $\mathcal{C}$ be a $\mathcal{V}$-category and let $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{V}$ be a $\mathcal{V}$-functor. Then there is an isomorphism in $\mathcal{V}$*

$$[\mathcal{C}^{\mathrm{op}}, \mathcal{V}](F, \mathcal{C}(-, C)) \cong F(C), \tag{A.6}$$

natural in C.

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