An evaluation on the gracefulness and colouring of graphs

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Abstract

In this thesis we shall introduce two interesting topics from graph theory and begin to explore what happens when we combine these together. We will be focusing on an area known as graph colouring and assessing it alongside a very unique set of graphs called graceful graphs. The two topic areas, although not mixed together often, nicely support each other in introducing various findings from each of the topics. We will start by investigating graceful graphs and determining what classes of graph can be deemed to be graceful, before introducing some of the fundamentals of graph colouring. Following this we can then begin to investigate the two topics combined and will see a whole range of results, including some fascinating less well known discoveries. Furthermore, we will introduce some different types of graph colouring based off the properties of graceful graphs. Later in the thesis there will also be a focus on tree graphs, as they have had a huge influence on research involving graceful graphs over the years. We will then conclude by investigating some results that have been formulated by combining graceful graphs with a type of graph colouring known as total colouring.

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Acknowledgements

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Declaration

I, Kymrun Dhami, declare that this thesis is a presentation of my own original work and I am the sole author. This work has not been previously presented for any other award at this, or any other, university. All the sources I have used are acknowledged in my references at the end of the thesis.

1. Introduction

Graph theory is a very interesting area of mathematics that has been written about for the last 280 years. The earliest record discussing the topic is associated with Leonhard Euler, as mentioned in [13]. He is often referred to as the father of graph theory after he famously solved what is known today as the Königsberg Bridge Problem in 1736. This involved him investigating whether it was possible to cross all the seven bridges of Königsberg, that connected four land areas together, so that each bridge was only crossed once and the person could start and finish from the same point. He proved this to be impossible by turning the problem into one using graphs and solved it using what we now consider today as graph-theoretical arguments. This led to the discovery of a special type of graph, known to us now as an Eulerian graph which we will analyse later in this section. Given this, the graphical nature of this area of mathematics has made it the ideal basis of solving many more problems over the years, as [13] explains.

In this thesis we will focus on a specialised area of graph theory known as graph colouring and combine it with a very unique type of graph called a graceful graph. The two areas, although not mixed together often, nicely support each other with introducing various findings from the topic. We also get to investigate some interesting results when concepts from the two areas are combined together and examined.

1.1. The Foundations.

We will begin by introducing the key foundations of graph theory to help develop an initial understanding of the topic. This section is based on Reinhard Diestel's 'Graph Theory' [8] throughout, as it was found to provide a useful summary of key areas within the topic.

Definition 1.1. A **graph** G consists of a finite nonempty set of points, denoted by $V(G)$, called **vertices**, and a collection of pairs of elements, $E(G)$, called **edges**. The elements of $E(G)$ are pairs of vertices.

A graph can be drawn by using dots to depict vertices and lines joining two of these dots together to denote an edge. An example of graph is shown in Figure 1, here the graph has 5 vertices and 4 edges.

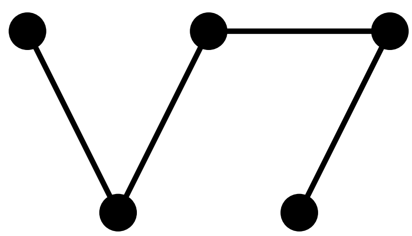

Figure 1. An example of a graph.

For further clarification, if an edge connects a vertex to itself we call this a **loop**, additionally, if two vertices are connected together by more than one edge they are known as **multiple edges**. For the entirety of this thesis we will assume that all graphs have no loops or multiple edges, these types of graphs are called **simple graphs**.

The term **incident** will be used to refer to the connection of an edge to a vertex. Whilst the term **adjacent** will be used to describe either: two vertices that are connected together by an edge; two edges that share a common vertex or, alternatively, an edge and a vertex that are incident with each other.

For a graph G we will usually have $|V(G)| = n$ and $|E(G)| = m$, unless otherwise stated. To help distinguish between the vertices of G they may be given distinct names such as v_i, where $1 \leq i \leq n$, or similar to help identify them individually. Moreover, the edges of G may be given the names e_j, where $1 \leq j \leq m$, (or something similar) to help differentiate them. However, they could also be denoted as uv, where u and v are some distinct vertices in G such that the edge uv connects the vertex u to v.

The **degree** (or **vertex degree**) of a vertex v, $deg(v)$, is equivalent to the number of edges incident with that vertex. For a graph G the largest degree in the graph, that is the largest value of $deg(v)$ for all $v \in G$, is denoted as Δ (it can also be referred to as $\Delta(G)$, especially if it is unclear which graph it relates to). A vertex with a degree equal to one is called an **end point** (or **end vertex**).

A **walk** in a graph consists of an alternating sequence of vertices and edges, beginning at a vertex called the **initial vertex** and ending at a final vertex known as the **terminal vertex**. Using the notation previously defined we can write a walk as the following sequence: v_1, e_1, v_2, e_2, $\ldots$, v_{k-1}, e_{k-1}, v_k, for some integer k, where e_i is incident with v_i and v_{i+1}. Here, v_1 denotes the initial vertex and v_k denotes the terminal vertex. A walk may also be displayed as a sequence of vertices too, such as: v_1, v_2, v_3, $\ldots$, v_k, for some value k. Moreover, the **length** of a walk is the number of edges it has, in the previous case the length would be $k-1$.

If the vertices in the walk are all distinct then it is called a **path**. Using the notation previously described, a path can be displayed either as the sequence of vertices that the path traverses (that is, the vertices crossed as we go along the path) e.g. v_1, v_2, v_3, $\ldots$, v_k for some integer k. Or, as the sequence of both the vertices and edges that are traversed, e.g. v_1, e_1, v_2, e_2, $\ldots$, v_{k-1}, e_{k-1}, v_k, for some k, where v_1 is the initial vertex and v_n is the terminal vertex.

If a walk has $v_1 = v_n$ and all edges in the walk are distinct, then we get what is known as a **cycle**. This is a closed walk, the term closed here means a path that starts and ends at the same vertex. We will denote a cycle as $\{v_1, v_2, \ldots, v_{n-1}, v_n = v_1\}$.

A **subgraph** H of a graph G is a graph whose vertices and edges lie within $V(G)$ and $E(G)$ respectively. We next will define a key concept in graph theory involving connectivity.

Definition 1.2. A graph G is **connected** if and only if there exists a path between every pair of vertices. If this was not the case then G is called **disconnected** and is said to consist of **components**, which are connected subgraphs that are maximal with respect to their vertex and edge sets. A **bridge** is the name of any edge in a connected graph which if removed increases the number of components in the graph.

We will assume that all graphs in this thesis are connected.

Considering this for a graph G we can define the **distance**, $d(u, v)$, between a vertex u and a vertex v to be the length of the shortest path between the vertices. Furthermore, we refer to the **diameter** of a graph G ($diam(G)$) as the largest distance between any two vertices in G.

1.2. **Classes of Graphs.**

We will now summarise some of the different classes of graphs that will feature in this thesis.

We have already defined what a path is, however, a path can be a graph in its own right. Denote a **path graph** by P_n, where n is the total number of vertices in the graph and $n-1$ is the number of edges, it is also connected. The vertices of P_n can be written as v_1, v_2, v_3, ..., v_n where an edge connects v_i to v_{i+1}, for $1 \leq i \leq n-1$, in the case where $i = n$, v_n is adjacent to v_{n-1}. Hence, all the vertices have a degree less than or equal to 2 (in fact only two vertices, the end points v_1 and v_n, will have a degree of value 1). Figure 2 displays the path P_6.

FIGURE 2. The path P_6.

A **star graph**, S_n, is a graph with n vertices and $n-1$ edges, such that one vertex, call this the **central vertex**, has the degree $\Delta = n-1$ whilst all other vertices are end points with degree of value 1. Figure 3 shows the graph S_9.

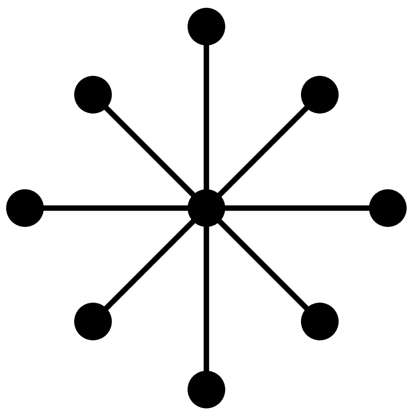

FIGURE 3. The star graph S_9.

The graph C_n is a **cycle graph** with n vertices and n edges, this is such that every vertex has degree 2 and the graph is connected. The cycle graph C_{10} can be seen in Figure 4.

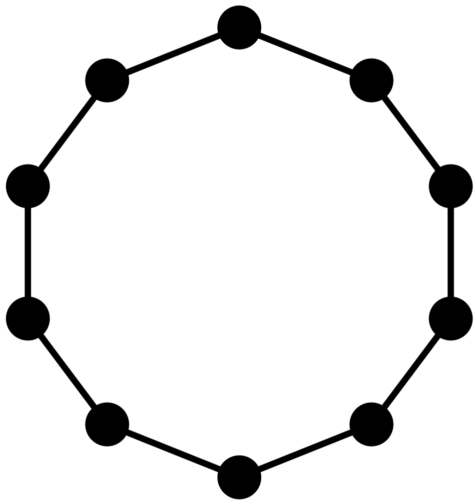

FIGURE 4. The cycle graph C_{10}.

The **complete graph** K_n is the graph with n vertices such that every pair of vertices in the graph is adjacent. This makes $|E(K_n)| = \frac{n(n-1)}{2}$. The graph K_4 is shown in Figure 5.

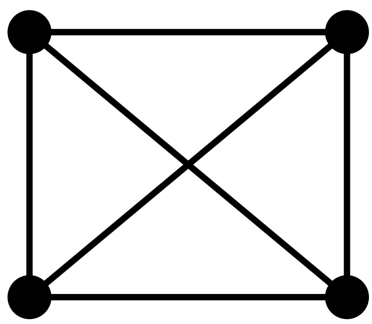

FIGURE 5. The graph K_4.

A **wheel graph**, W_n, has n vertices with $n \geq 4$. It consists of the cycle graph C_{n-1} where every vertex is adjacent to another additional single vertex, called the **central vertex**, displayed in the middle of the cycle. Hence every vertex has a degree of value 3 except the central vertex which has a degree equal to $n-1$. A diagram of the wheel graph W_9 is presented in Figure 6. The total number of edges in a wheel graph W_n is $2(n-1)$.

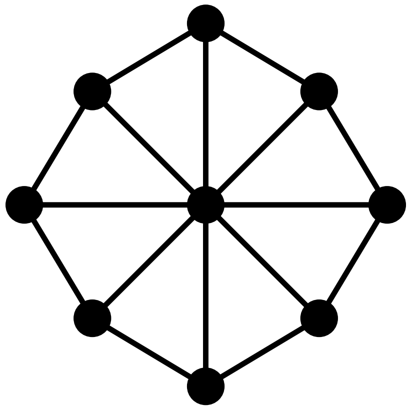

FIGURE 6. The wheel graph W_9.

A **complete bipartite graph**, $K_{a,b}$, is a graph whose vertices can be split into two sets, call these set A and set B, such that every vertex in set A is adjacent to every vertex in set B only (and vice-versa). The total number of vertices in set A is a and the number of vertices in set B is b, hence the graph $K_{a,b}$ has $a+b$ vertices in total and ab edges. Figure 7 shows the complete bipartite graph $K_{3,4}$. Note that a **bipartite graph** is also a graph whose vertices can be split into two sets, A and B, such that vertices in set A are only adjacent to a vertex in set B, and vice-versa, however unlike a complete bipartite graph, not all vertices in set A are adjacent to vertices in set B.

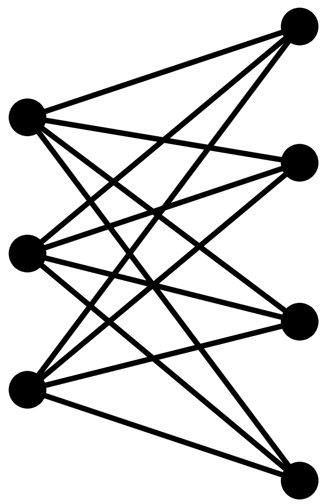

FIGURE 7. The complete bipartite graph $K_{3,4}$.

The final graph we will introduce is called an Eulerian graph (as mentioned earlier in this section). A graph G is called **Eulerian** if there exists a closed walk, that is a walk that begins and ends at the same vertex, which contains every edge in the graph exactly once. This walk is then referred to as an **Eulerian circuit** for G.

We have now been introduced to several concepts that form the basis of graph theory and have seen several of the graph classes that will feature in this thesis. With this initial understanding now grasped we can move on to introducing a very interesting new type of graph called a graceful graph.

The concept of a graceful graph was first discussed in the 1960's, as mentioned in [9], this was mainly centred around a paper published in 1967 by Alexander Rosa called 'On certain valuations of the vertices of a graph' [18]. In this paper however, Rosa refers to the idea of gracefulness as a β-valuation for a graph. To begin to understand what the concept of gracefulness is for a graph it is first useful to clarify how a graph is labelled, as done so in [9]. For a graph G a vertex labelling assigns distinct nonnegative integers to G's vertices, these are called **vertex labels**. Similarly an edge labelling of a graph G allocates distinct values called **edge labels** to each edge. Furthermore, at various points in this thesis, a set or sequence may be described as being open or closed. Here, a set of elements where all elements from the set must be included is called **closed**. On the other hand, if the set is **open** all elements do not have to be used.

Definition 2.1. A **graceful labelling** of a graph G consists of:
A) A labelling of the vertices in G from the open set of integers $\{0, \dots, m\}$, where m is the number of edges;
B) A labelling of the edges of G, where the edge label corresponds to the absolute difference of the vertex labels of the vertices that the edge is adjacent to, such that the set of edge labels is the closed set $\{1, \dots, m\}$.
G is then known as a **graceful graph**.

Example 2.2. The Petersen graph is an example of a graceful graph. A graceful labelling of this is shown below in Figure 8.

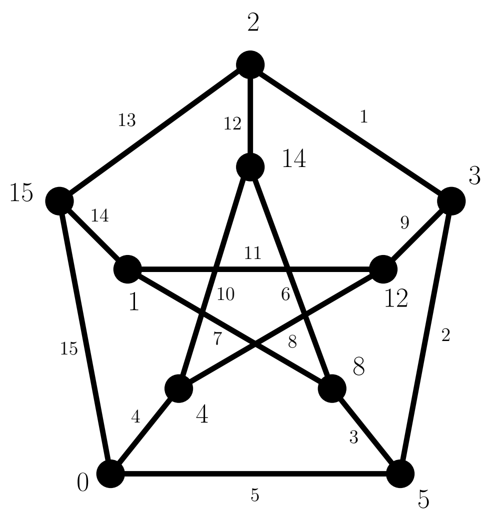

Figure 8. The gracefully labelled Petersen graph.

2.1. Properties of Graceful Graphs.

When looking at graceful graphs we are only considering simple connected graphs in this thesis. This then implies that every edge in the graph is incident with a unique pair of vertices.

An interesting property about graceful graphs is considered by Golomb in 'How to Number a Graph' [11], an adapted version of this theorem will now be presented.

Theorem 2.3. *Let G be a graph with n vertices and m edges. If G is graceful then it is possible to partition the vertices into two sets, A and B, such that the number of edges connecting set A to set B is $\frac{m}{2}$, if m is an even integer, or $\frac{(m+1)}{2}$, if m is an odd integer.*

Proof. Let G be a graph with a graceful labelling. We can partition G's vertices into two sets, one being the set of vertices with even vertex labels, A, and the other being the set with odd vertex labels, B. The m edges will be labelled from 1 to m, hence if m is even, $\frac{(m)}{2}$ of the edge labels will be odd integers or if m is odd, $\frac{(m+1)}{2}$ of the edge labels will be odd integers. Conclusively, any edge with an odd edge label must be incident with one even valued vertex label and one odd valued vertex label, since the edge labels are determined by the absolute difference of the vertices an edge is adjacent with. Therefore, in both cases, the edges labelled with odd integers represent the set of edges connecting vertices in A to B, so the theorem holds.

□

2.2. **Classes of Graceful Graphs.**

The question now arises as to how we determine whether a graph is graceful or not. There is no set way to straight off identify if any arbitrary graph is graceful, however there are many proven cases for various groups of graphs which classify whether they are graceful or not. Firstly we will consider some graphs which have been proven to be graceful.

Theorem 2.4. *All paths are graceful.* [15]

Proof. Let P_n be a graph with n vertices and m edges. Label the first vertex in the path (so a vertex of degree 1) as 0. From this point skip the next adjoining vertex in the path and label the vertex following this as 1. Continue to do this along the path, increasing the vertex label by one each time, so that alternating vertices along the path are labelled from 0 to $\frac{m}{2}$, if m is even, or 0 to $\frac{(m-1)}{2}$, if m is odd. Then starting from the second vertex in the path (adjacent to vertex 0) label this as m. From here skip the next adjoining vertex, which would now be labelled as 1, and label the following vertex as $m-1$. Continue to do this along the path, decreasing the vertex label by one each time. This newly labelled set of vertices should go from m down to $\frac{(m+2)}{2}$, if m is even, or $\frac{(m+1)}{2}$, if m is odd. Therefore P_n has $m+1$ vertices all with a distinct label from the set $\{0, \ldots, m\}$. Consequently, all edges will have a distinct label from the set.

As a result, by having the vertices of the path labelled this way the edge labels are calculated to be: m, $m-1$, $m-2$, $\ldots$, 2, 1, as you traverse along the path starting at vertex 0. This is a requirement for a graph to be graceful, hence P_n has a graceful labelling, thus proving the theorem. [15]

□

Theorem 2.5. *All star graphs, S_n, are graceful for all n.* [15]

Proof. Recall that a star graph has n vertices and $n-1$ edges. It consists of one central vertex of degree $n-1$ and has the remaining vertices, all with degree 1, connected to it. Considering this let the central vertex be labelled with the vertex label 0; therefore every other vertex in the graph

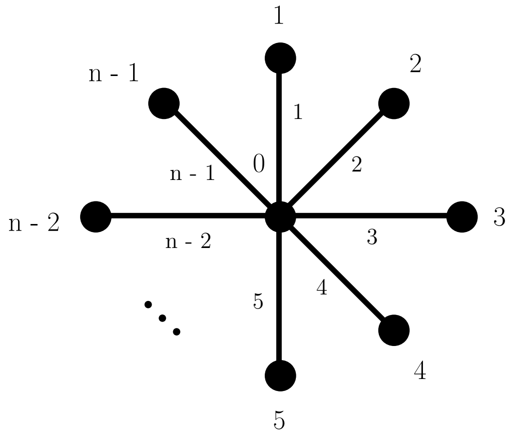

FIGURE 9. An illustration of the proof for Theorem 2.5.

may be labelled with distinct vertex labels from 1 to $n-1$. As a result the set of absolute differences along each edge, i.e. the edge labels, will be the complete set of integers from 1 to $n-1$, hence the graph is graceful for all n in S_n. An illustration of this concept is shown in Figure 9.

□

Theorem 2.6. *The complete bipartite graph $K_{a,b}$, where a and b are positive integers, is graceful for all values of a and b.* [11]

Proof. This theorem will be proven by showing a graceful labelling for $K_{a,b}$. Note that this graph has $a+b$ vertices and ab edges. Consider the two sets of vertices, A and B, consisting of a and b elements respectively. Let the vertices in set A be labelled with the integers: 0, 1, ..., $a-1$, and the vertices in set B be labelled with the integers: a, $2a$, ..., ba. This type of labelling means that every number from 1 to ab can be calculated to be a distinct edge label for the edges of the graph; this is because every edge uniquely connects a vertex in B with a vertex in A such that every individual vertex in A is adjacent to every vertex in B, and vice-versa. Hence, $K_{a,b}$ has a graceful labelling. [11]

□

Remark 2.7. The star graph S_n is in fact a complete bipartite graph, $K_{1,n}$, therefore the previous theorem provides an alternative proof to Theorem 2.5.

We shall now investigate a theorem involving wheel graphs. This theorem and proof has been taken from [10] where the formulas derived were stated without proof.

Theorem 2.8. *The graph W_n is graceful for all $n \geq 4$.*

Proof. For a wheel graph W_n, n denotes the total number of vertices in the graph. In this proof we shall let $k = n-1$, furthermore, the central vertex will always be called v_0 and the outer vertices will be represented by the closed cycle $\{v_1, v_2, \ldots, v_k\}$.

We will separate the proof into two cases, one where k is even and one where k is odd.

First we will examine the case where k is even. When $k = 4$ we can give the graph W_5 the graceful labelling shown in Figure 10.

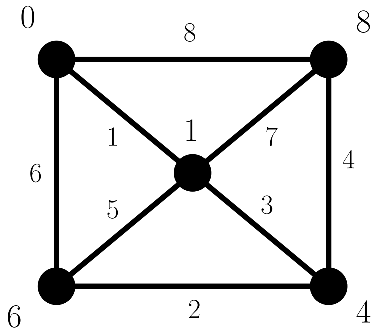

FIGURE 10. The gracefully labelled graph W_5.

For the cases where $k \geq 6$ we can get a graceful labelling of W_n using the following formula, f_e, which will allocate numbers to the vertices v_i, where $0 \leq i \leq k$:

$$f_e(v_i) = \begin{cases} 2k - i - 1 & \text{if } i = 2, 4, 6, \ldots, k-2 \\ 2 & \text{if } i = k-1 \\ i & \text{if } i = 1, 3, 5, \ldots, k-3, \text{ also if } i = 0 \\ 2k & \text{if } i = k. \end{cases} \tag{1}$$

From here the edge labels are calculated by taking the absolute difference of the vertices that edge is incident with. When $k = 8$ we can derive the graceful labelling of the graph W_9 to be as displayed in Figure 11.

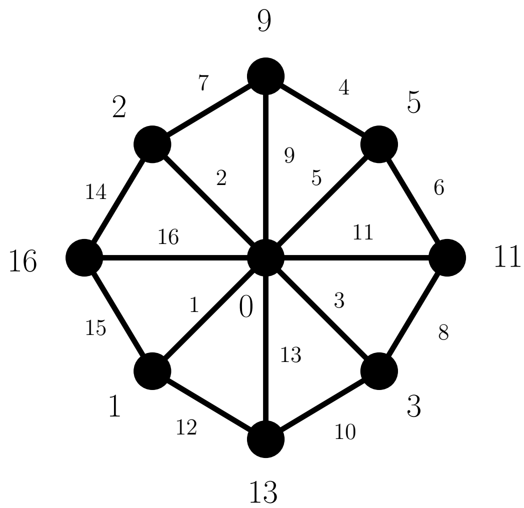

FIGURE 11. The graceful labelling of the graph W_9.

Now we will look at the case where k is odd. For $k = 3$ the wheel W_4 is also the complete graph K_4, a graceful labelling of this is shown in Figure 12.

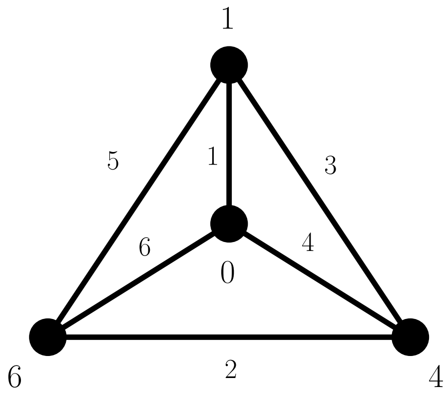

FIGURE 12. The graceful labelling of the graph $W_4 = K_4$.

When $k \geq 5$, a graceful labelling can be given to W_n using the following formula, f_o, which will allocate numbers to the vertices v_i, where $0 \leq i \leq k$:

$$f_e(v_i) = \begin{cases} 2i & \text{if } i = 0,\ 1 \text{ or } k \\ k+i & \text{if } i = 2,\ 4,\ 6,\ \ldots,\ k-1 \\ k+1-i & \text{if } i = 3,\ 5,\ 7,\ \ldots,\ k-2 \end{cases} \quad (2)$$

Again, from here the edge labels are calculated accordingly. We now have shown a way to gracefully label all wheel graphs W_n.

□

We will now begin to investigate some graphs which have been proven to not be graceful.

Theorem 2.9. *Any complete graph K_n with $n > 4$ cannot be graceful.* [11]

Proof. When $n > 4$ the graph K_n has its total number of edges $m \geq 10$. Assume K_n can have a graceful labelling. Then the vertices of the graph can be labelled using a set of values from the open set $\{0, 1, 2, \ldots, m\}$ such that all the edges of the graph can be assigned distinct labels from the closed set $\{1, 2, \ldots, m\}$.

For K_n to have an edge labelled with m, both 0 and m must be vertex labels of vertices in that graph. (This would be the case for the graph K_2 with $m = 1$.) Following this, for there to be an edge labelled with $m - 1$, either 1 or $m - 1$ must be a vertex label of a vertex in that graph. The vertex label 1 may be selected since for a graceful graph G with m edges, every vertex label v_i can be replaced by $m - v_i$ and yet all the edge labels would still remain the same, so there is no loss of generality. Hence we will proceed with the case where the vertex label 1 was chosen knowing that if $m-1$ had been chosen a similar overall outcome would have been reached.

To get an edge labelled with $m-2$ the vertex label $m-2$ must be added to a vertex in that graph, given that we already have vertex labels 0, 1 and m included. If the vertex label $m - 1$ had be chosen instead, to get the edge label $m - 2$ allocated to the edge connecting the vertices $m - 1$ and 1, there would then be two edges with the label 1, i.e. the edge incident with the vertices labelled 0 and 1 and the edge incident with the vertices labelled $m - 1$ and m, which is not allowed. Similarly, if the vertex label 2 was added to get the edge label $m - 2$ for the edge incident with m and

2, this would again result in two edges having the edge label 1, namely the edges incident with vertices 0 and 1 and vertices 1 and 2. Hence the vertex label $m-2$ must be chosen.

Now with the vertices labelled as 0, 1, $m-2$ and m we get the set of edge labels $\{1, 2, m-3, m-2, m-1, m\}$. To get an edge labelled with $m-4$ a vertex with label 4 must be added. Other choices of values for this vertex are dismissed using the same principles previously demonstrated.

We now have vertices with the labels 0, 1, 4, $m-2$ and m giving us the set of edge labels $\{1, 2, 3, 4, m-6, m-4, m-3, m-2, m-1, m\}$. Following this, there is no way to add another labelled vertex to obtain an edge with the label $m-5$. This is because all the ways to obtain the $m-5$ edge label, as a difference of the two vertices it is incident with, creates duplicate edge labels in the rest of the graph, i.e. there are no vertex label options from the set $\{2, 3, 5, m-5, m-4, m-3, m-1\}$ that can be selected. This contradicts the statement that K_n is graceful for all cases where $m-5 > 4$, i.e. when $n \geq 5$. Hence the theorem holds. [11]

□

Theorem 2.10. *C_5 is not graceful.*

Proof. For C_5 to be graceful the graph must be able to be labelled with a selection of five distinct vertex labels from the set $\{0, 1, 2, 3, 4, 5\}$. Every edge must then have a distinct edge label from the closed set $\{1, 2, 3, 4, 5\}$, where the allocated edge label is the absolute difference of the vertex labels the edge incidents with.

Assume C_5 is graceful. The only way an edge label of value 5 can feature on the graph is if the two vertices the edge incidents with have the vertex labels 0 and 5. Assign two adjacent vertices on the graph with these labels, an example of this is shown in Figure 13. Note that due to the symmetry of the graph it doesn't matter which two vertices are selected.

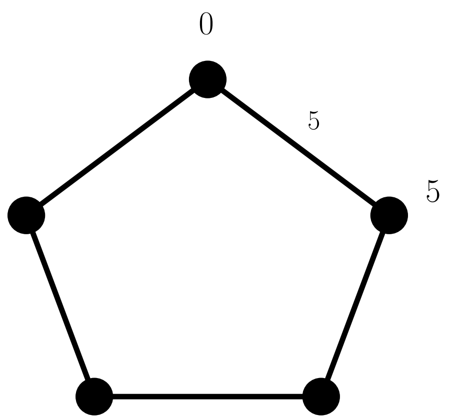

FIGURE 13. The first step attempting to gracefully label C_5.

Next we can determine that there are only two ways for the edge label 4 to be added to the graph. This is by either allowing the vertex adjacent to vertex 0 to be allocated the vertex label 4 (call this Case A) or by the vertex adjacent to vertex 5 being given the vertex label 1 (call this Case B). Now let us first investigate Case A.

Case A: Say we add a vertex label of value 4 to the vertex adjacent to vertex 0 to get an edge label of value 4. We now need to add an edge label of value 3 to the graph. Here there are again only two possible options: either allocate the vertex adjacent to vertex 4 with the vertex label 1 (call this Option 1) or allocate the vertex adjacent to vertex 5 with the vertex label 2 (call this Option 2). Let us select Option 1 and add the vertex label 1 to the graph, we are now left with one remaining vertex without a label, call this vertex w - a depiction of this is shown in Figure 14.

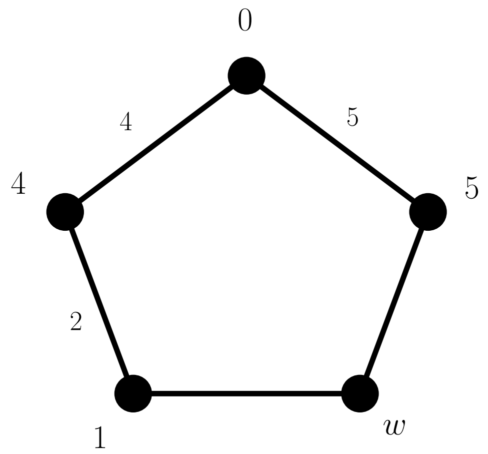

FIGURE 14. The second step attempting to gracefully label C_5.

There are now only two vertex labels left that can be given to w, label 2 or 3. However, if w was given a label of value 2 the edge label 1 can be added to the graph on the edge incident with vertices 1 and 2. But then the graph will not feature an edge label of value 2 seeing as the remaining unlabelled edge would be incident with vertices 2 and 5, for which the absolute difference is 3. On the other hand if w was given a vertex label of value 3, the vertices 1 and 5, that w is incident with, will both have the absolute difference of their vertex labels calculated to be 2. This not only means that no edge label of value 1 can be added to the graph but also that a duplication of an edge label value, in this case 2, would appear. Hence no graceful labelling can be found this way.

We will now examine Option 2. So we still add a vertex label of value 4 to the graph in Figure 13, but this time we allocate the vertex adjacent to vertex 5 with the vertex label 2 to get an edge label of 3 added to the graph. There again remains one unlabelled vertex, call this vertex x, this is shown in Figure 15.

Vertex x can now only be given a vertex label of value 1 or 3. However, similar to what occurred in Option 1, if the value 1 is selected to be this vertex label the edge label 1 can be added to the graph (between vertices 1 and 2) but not the edge label of value 2. Consequently if the vertex label 3 is given to x the edge label 1 can again be added to the graph (but this time on the edge incident with vertices 3 and 4). Nonetheless, the edge label of value 2 again cannot be added to the graph. Therefore the graph cannot be gracefully labelled this way.

Case B: We now look back at the graph in Figure 13 and add the vertex label 1 to the unlabelled vertex adjacent to vertex 5 to get an edge label of

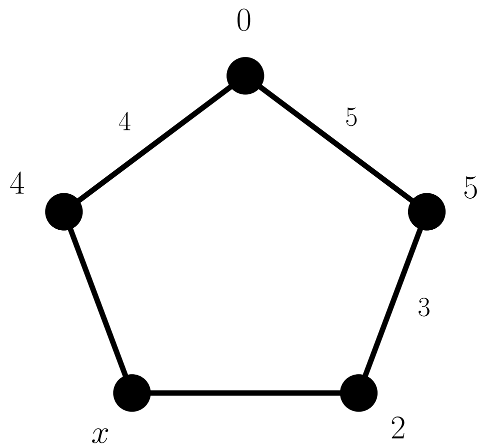

FIGURE 15. The third step attempting to gracefully label C_5.

value 4. From here we want to add the edge label of value 3 to the graph. The only two options this time is to either add the vertex label 4 to the vertex adjacent to vertex 1 (call this Option 3) or allocate the vertex adjacent to vertex 0 with vertex label 3 (call this Option 4). Say we select Option 3 and add the vertex label 4 to the graph, we are left with one unlabelled vertex again - call this y - this is shown in Figure 16.

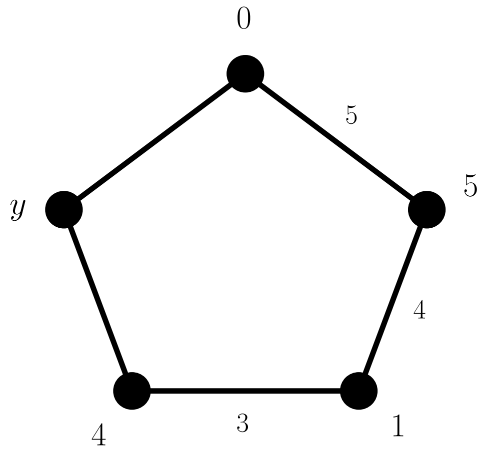

FIGURE 16. The fourth step attempting to gracefully label C_5.

From here we can see that vertex y can either be given a vertex label of value 2 or 3. Nevertheless, if the vertex label 2 was given to y the edge label 2 can be assigned to the graph on the edge between vertices 0 and 2 (and vertices 2 and 4) but the graph would not feature an edge label of 1. If the vertex label of value 3 was given to y instead, the edge incident with vertices 3 and 4 could be allocated the edge label of value 1, however no edge label of value 2 could be added to the graph (there also would be a duplication of the edge label 3 between vertices 0 and 3). Therefore no graceful labelling can be given.

Finally we will look at Option 4, this involves still adding a vertex label with value 1 to Figure 13, as done in Option 3, but now instead we add the vertex label 3 to the vertex adjacent to vertex 0 to get an edge label of value 3. Here we are left with an unlabelled vertex, call this z, which can be seen in Figure 17.

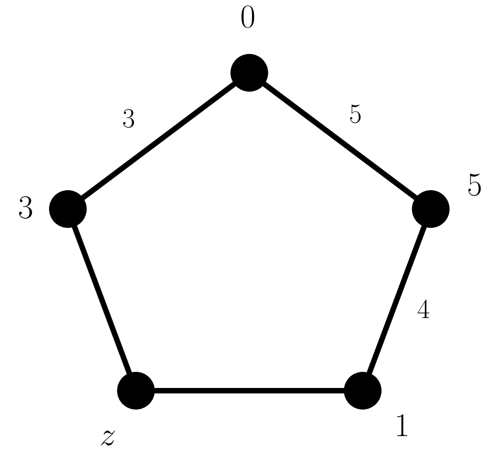

FIGURE 17. The final step attempting to gracefully label C_5.

Vertex z can now either be labelled with value 2 or 4. If the vertex label 2 was given to z then the edge label 1 would need to be added to the graph for both the edge incident with vertices 1 and 2 and the edge incident with vertices 2 and 3, this is duplication which is not allowed in a graceful graph. On the other hand, if z was given the vertex label 4 the edge label of value 1 could be assigned to the edge connecting vertices 3 and 4 but no edge label of value 2 could be added to the graph. Therefore it cannot be gracefully labelled.

As a result, every possible way of labelling an edge in the graph C_5 with the edge label 4 has been exhausted and no graceful labelling of the graph has been found. Hence, C_5 is not a graceful graph.

□

Remark 2.11. C_5 is actually one of three graphs containing 5 or fewer vertices that is not graceful, as explained in [11], the other two graphs are shown below in Figure 18. All other graphs consisting of 5 or fewer vertices can have a graceful labelling.

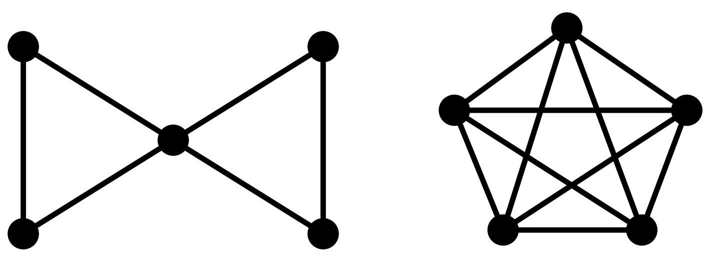

FIGURE 18. Two of three graphs containing 5 or fewer vertices that are not graceful.

Given Theorem 2.10, it is possible to determine out of all the cyclic graphs, C_n, which graphs are graceful. However, before we introduce the theorem which provides an answer to this, we will first need to examine a property for graceful graphs that are also Eulerian.

Theorem 2.12. *If G is a graceful Eulerian graph with n vertices, then $n \equiv 0 \pmod 4$ or $n \equiv 3 \pmod 4$.* [4]

Proof. Let C be an Eulerian circuit in the graph G which follows the walk: $v_0, v_1, \dots, v_{n-1}, v_n = v_0$. Let a graceful labelling of G be such that the integer a_i, where $0 \le a_i \le n$, is assigned to v_i, where $0 \le i \le n$, with $a_i = a_j$ if $v_i = v_j$. Therefore the label given to the edge incident with vertices v_{i-1} and v_i is the absolute difference of a_i and a_{i-1}, i.e. $|a_i - a_{i-1}|$. Notice that,

$$|a_i - a_{i-1}| \equiv (a_i - a_{i-1}) (\text{mod } 2) \tag{3}$$

for $1 \le i \le n$. This implies that the sum of the edge labels in G is,

$$\sum_{i=1}^{n} |a_i - a_{i-1}| \equiv \sum_{i=1}^{n} (a_i - a_{i-1}) \equiv 0 \ (\text{mod } 2), \tag{4}$$

which means the sum of the edge labels in G is even. However the sum of the edge labels is

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \tag{5}$$

so $\frac{n(n+1)}{2}$ is even. As a result, $4|n(n+1)$, which suggests $4|n$ or $4|(n+1)$, leading to $n \equiv 0$ (mod 4) or $n \equiv 3$ (mod 4). [4]

□

Theorem 2.13. *The graph C_n is graceful if and only if $n \equiv 0$ (mod 4) or $n \equiv 3$ (mod 4).* [4]

Proof. It is known that C_n is an Eulerian graph for all n, therefore by Theorem 2.12 if $n \equiv 1$ (mod 4) or $n \equiv 2$ (mod 4) then C_n is not graceful. It will now be shown that if $n \equiv 0$ (mod 4) or $n \equiv 3$ (mod 4) then C_n is graceful. Let C_n be the cycle: $\{v_1, v_2, \dots, v_n\}$.

First let $n \equiv 0$ (mod 4), we therefore can assign the vertex v_i in C_n the label a_i where,

$$a_i = \begin{cases} \frac{i-1}{2} & \text{if } i \text{ is odd} \\ n + 1 - \frac{i}{2} & \text{if } i \text{ is even and } i \le \frac{n}{2} \\ n - \frac{i}{2} & \text{if } i \text{ is even and } i > \frac{n}{2} \end{cases} \tag{6}$$

Implementing this formula will result in a graceful labelling.

Next we look at the case where $n \equiv 3$ (mod 4), we assign a vertex v_i in C_n with the label b_i where,

$$b_i = \begin{cases} n + 1 - \frac{i}{2} & \text{if } i \text{ is even} \\ \frac{i-1}{2} & \text{if } i \text{ is odd and } i < \frac{(n-1)}{2} \\ \frac{i+1}{2} & \text{if } i \text{ is odd and } i > \frac{(n-1)}{2} \end{cases} \tag{7}$$

This also gives a graceful labelling. [4]

□

We have introduced the concept of a graceful graph and started to see some classes of graphs that have been deemed to be either graceful or not graceful. The total number of graceful graphs in existence is infinite, however a published result by Sheppard [19] in 1976 has shown that there exist $n!$ graceful graphs with n edges, as explained in [9]. That being said, it is also mentioned in [9] that there is an unpublished paper by Erdős which claims most graphs are not actually graceful, making these types of graph very special thanks to their rarity.

3. Graph Colouring

We will now introduce an area of graph theory which is known as graph colouring. This topic involves colouring either a graph's vertices, edges or sometimes both and determining whether it can be coloured by a certain number of colours or, alternatively, questioning what is the minimum amount of colours needed to colour a graph when following certain rules. The following section is based on Wilson's 'Introduction to graph theory' [24].

Graph colouring has been said to have first been written about in 1852, thanks to what is known today as the Four Colour Theorem, as discussed in [22]. The problem involves determining whether every map could have its countries coloured by four colours, such that no two countries sharing a border are coloured the same colour. Here, graph theory is used by depicting the countries as vertices on a graph. The edges of this graph are allocated to connect two vertices together if the two countries represented by those vertices share a border. Consequently if every vertex in a graph that represents a map is coloured with one of four colours, such that no adjacent vertices share the same colour, then the theorem holds for that map. This type of problem involving adding colour to graphs became the basis of graph colouring.

Definition 3.1. For a graph G if the vertices can be allocated one of k colours such that no adjacent vertices have the same colour, then G is called ***k*-colourable** and is described to have a **proper vertex colouring**. (G can also be said to have a ***k*-colouring**.) The **chromatic number** of a graph G, denoted $\chi(G)$, is k if G is k-colourable but not $(k-1)$-colourable. Therefore, if a graph has n vertices the chromatic number cannot be greater than n.

Example 3.2. The Petersen graph has a chromatic number of 3, a vertex colouring is shown below in Figure 19. Note that this implies that the Petersen graph is k-colourable for all $k \geq 3$.

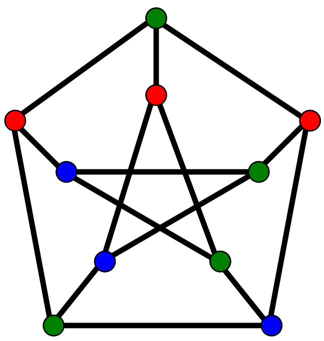

Figure 19. A vertex colouring of the Petersen graph.

We can deduce a useful result in graph colouring by knowing the highest degree of a graph.

Theorem 3.3. *If G is a simple graph with the largest vertex degree Δ, then G is $(\Delta + 1)$-colourable.*

Proof. Let G be a simple graph with n vertices. We will use induction on the number of vertices to prove Theorem 3.3. If a vertex v is deleted from the graph along with any edges incident with v the resulting graph is still simple but now with $n-1$ vertices. The largest vertex degree is at most Δ and by the initial induction hypothesis the graph is $(\Delta+1)$-colourable. Therefore when v and its adjacent edges are reinserted it can be coloured with a different colour to its at most Δ adjacent vertices, therefore making G $(\Delta+1)$-colourable.

□

Note that this then implies that for a graph G, $\chi(G) \leq (\Delta+1)$.

This property leads onto a famous result known as Brooks' theorem proved by R.L. Brooks in 1941.

Theorem 3.4. *If G is a simple connected graph which is not a complete graph and if the largest vertex degree of G is Δ (where $\Delta \geq 3$), then G is Δ-colourable.*

Proof. Let G be a graph with n vertices, using induction on the number of vertices we will prove Brooks' theorem. If G has any vertex with degree less than Δ then by using a similar method to that shown in Theorem 3.3 the proof would be complete. Since, if we let z denote a vertex in G with degree less than Δ, we can delete z and any edges incident with it from the graph to result in a new graph with $n-1$ vertices. The largest vertex degree will be at most Δ and by the induction hypothesis the graph is Δ-colourable. Hence, when z and its adjacent edges are reinserted it can be coloured in a different colour to its at most $\Delta-1$ adjacent vertices, since there are Δ choices of colour available. Therefore G is Δ-colourable, hence this would complete the proof.

Therefore, assume now G is regular of degree Δ. If a vertex v is deleted along with the edges incident with it, the graph now has $n-1$ vertices and the largest degree is still Δ. Brooks' theorem then implies this is Δ-colourable. We want to colour v with one of the Δ colours. Suppose vertices $v_1, \ldots, v_\Delta$ are adjacent to v and are arranged clockwise around it. They are coloured with distinct colours $c_1, \ldots, c_\Delta$. If these colours were not distinct then there would be an unused colour available to colour v with and we would be done; therefore we assume the colours are distinct.

Define a subgraph $H_{ij}, (i \neq j, 1 \leq i, j \leq \Delta)$ whose vertices are coloured by c_i or c_j and whose edges are those which join a vertex coloured c_i with one coloured c_j. If for some i and j the vertices v_i and v_j from $v_1, \ldots, v_\Delta$ lie in different components of the subgraph H_{ij} then all the vertices in the component containing v_i in H_{ij} can interchange colours. This is so that v_i and v_j now both have the colour c_j. Therefore v would now be able to be coloured with c_i.

Now assume for any values of i and j, the vertices v_i and v_j are connected by a path that lies entirely in H_{ij}. Denote the component containing v_i and v_j by C_{ij}. If more than one vertex adjacent to v_i is coloured c_j then there is another colour, other than c_i, that hasn't been used to colour any vertices adjacent to v_i. Therefore v_i can be recoloured with this colour and v can be coloured c_i. If this doesn't occur then every vertex in C_{ij} other than v_i

and v_j has a degree of 2. This is because if say w is the first vertex in the path from v_i to v_j with a degree greater than 2 then it can be coloured with a colour different from c_i and c_j. This contradicts the property that v_i and v_j are connected by a path lying entirely in C_{ij}. So, for any i and j, C_{ij} consists only of a path from v_i to v_j.

It can be assumed that any two paths with the property C_{ij} and C_{jl} (where $i \neq l$) - that is, C_{ij} is a path leading from v_i to v_j and similar for C_{jl} - intersect only at v_j. Otherwise, a different point of intersection, say x, could be recoloured to a colour that is not c_i, c_j, c_l contradicting that v_i and v_j are joined by a path.

Say two vertices from v_1 to v_Δ were not adjacent. We know this can occur since if we assumed otherwise then every v_i would be adjacent to v_j, for all $1 \leq i, j \leq \Delta$ where $i \neq j$, plus every v_i would also be adjacent to the vertex v. However, we know G is connected and regular of degree Δ (the degree all these vertices now have) so this would mean G would consist only of the vertices v and $v_1, \ldots, v_\Delta$ and all these vertices would be adjacent. This would then make G a complete graph, contradicting our inital assumption for the theorem, hence there can exist two vertices that are not adjacent. Now call these two vertices v_i and v_j, and let y be a vertex adjacent to v_i with the colour c_j. If C_{il}, where l is not equal j, is a path then the vertices along it can have their colours interchanged without affecting the rest of the graph's colouring. But this would lead to y becoming a vertex that is in common with the paths C_{ij} and C_{jl} which is a contradiction. Therefore this deems that Brooks' theorem holds.

□

There is no manual, systematic way to find the chromatic number of a graph other than by exhausting all colouring options for the graph using from 1 to n colours, where n is equivalent to the total number of vertices in the graph. However, there does exist a procedure, described in [5], that is known as a greedy algorithm for vertex colouring which provides a k-colouring for a graph, nonetheless the value of k may not necessarily be equivalent to the chromatic number of that graph.

Algorithm 3.5. To find a k-colouring for a graph G, where k is an integer, the following procedure (mentioned in [5]) can be used. When running this process it is necessary to number the colours that are used in consecutive order, therefore every time a new colour is introduced it gets numbered. The method follows:
1) Pick any initial vertex and colour it with colour 1.
2) Next pick an uncoloured vertex v. Colour v with the lowest numbered colour possible, such that no vertex adjacent to v also shares this colour. If all previously used colours occur on the vertices that are adjacent to v then a new colour needs to be introduced and numbered, colour v with this new colour.
3) Repeat step 2 until all vertices in the graph have been coloured.

Example 3.6. The greedy algorithm will now be applied to the graph pictured below in Figure 20, the vertices have been labelled with letters to help distinguish which one is being selected.

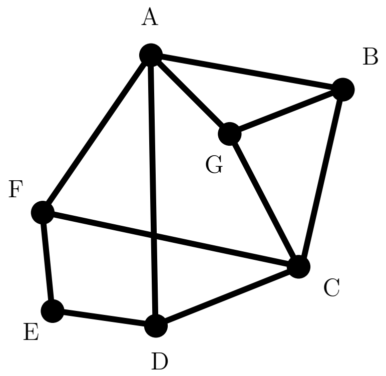

Figure 20. The graph used in Example 3.6.

The algorithm is run by first selecting the vertex labelled A at random and allocating it the colour red, which is now colour number 1. Next the vertex labelled B is chosen, as it is adjacent to vertex A, which is now coloured red, it must be coloured with a different colour - so a new choice is added, say yellow, which is now colour 2. This process continues by choosing the vertex labelled C which can be coloured with the lowest numbered colour red, as the only coloured vertex it is adjacent to is B which is yellow. Then the vertex labelled D is selected, it is connected to the coloured vertices A and C which are both red (note, it is also adjacent to the vertex labelled E but it is currently uncoloured so does not need considering) therefore D can be coloured yellow. Next the vertex labelled E is chosen and can be coloured red, then the vertex labelled F is chosen and coloured yellow (as its adjacent to three red coloured vertices). Finally the vertex labelled G is left to be coloured. However, it is adjacent to vertices A, B and C which are all coloured in either red or yellow. Therefore a new third colour needs to be introduced to colour vertex G, let this be the colour green.

We now have a complete colouring of the graph in 3 colours making it 3-colourable, as shown in Figure 21.

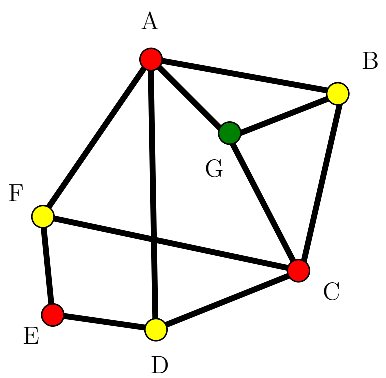

Figure 21. Vertex colouring of the graph used in Example 3.6.

In fact, the chromatic number of the graph does also happen to be 3, though note this was a coincidence and not an outcome property of the algorithm.

3.1. **Subgraphs.**

When investigating graph colouring some interesting results also arrive when examining a graph's subgraphs.

Theorem 3.7. *Let H be a subgraph of a graph G, then $\chi(H) \leq \chi(G)$.* [12]

Proof. Let the chromatic number of the graph G be denoted by k. Therefore a k-colouring of the graph G exists. Considering this it is possible to let the vertices of the subgraph H inherit their vertex colouring from G. This will then give H a colouring using k (or fewer) colours, note this does not mean k (or the number of colours used when this colouring technique is applied) is the chromatic number of H, hence $\chi(H) \leq \chi(G)$.

□

We will now give a definition involving subgraphs of a graph, as defined in [23].

Definition 3.8. If $\chi(H) < \chi(G)$ for every subgraph H of a graph G, excluding the graph G itself, then G is called **critical** (with respect to the chromatic number). If G is a critical graph with $\chi(G) = k$ then G is called ***k*-critical**.

Remark 3.9. As explained in [23], it is well known that the only 1-critical graph is K_1 and the only 2-critical graph is K_2. Furthermore all 3-critcal graphs are in fact odd cycles (that is, cycle graphs C_n where n is an odd valued integer).

It is useful to introduce some notation here for subgraphs; if a graph A is a subgraph of a graph B, with $A \neq B$, then we denote this as $B \supset A$. It is worth noting from an explanation in [23] that if the chromatic number of the graph G is k, but the graph is not k-critical, then there must exist a proper subgraph of G (that is a subgraph which is not G itself) such that its chromatic number is also k, call this subgraph G_1. From here, either G_1 is k-critical or it has a proper subgraph, call this G_2, where $\chi(G_2) = k$. Continuing this process we can obtain the following sequence for G and subgraphs of G: $G \supset G_1 \supset G_2 \supset G_3 \supset \ldots$, where $G_i \supset G_j$ denotes G_j to be a subgraph of G_i such that $i \neq j$ for some integers i and j; here every graph in the sequence has the chromatic number k. Since G is a finite graph the sequence must eventually terminate, therefore there must exist a graph G_n, where n is some integer, that is k-critical. Conclusively this shows that every graph with chromatic number k has a k-critical subgraph.

Theorem 3.10. *If a graph G is k-critical then every vertex in G has a degree of at least $k - 1$.* [27]

Proof. Let G have a vertex with a degree of at most $k-2$, call this vertex v. This means if v was removed from the graph, so we now have $G-v$, the new graph can be coloured with $k-1$ colours (since G is k-critical its subgraph $G-v$ must be able to be coloured with less than k colours). Given this, it implies that at least one of the $k-1$ colours will not have been used on the vertices that are adjacent to v, since v had a degree of $k-2$. This will then mean there is at least one colour available out of the $k-1$ colours to colour v with. In this case, G could then be coloured with $k-1$ colours which is a contradiction. Hence every vertex of a graph must have a degree of at least $k-1$ if it is to be k-critical. [27]

□

3.2. **Chromatic Polynomials.**

We can now investigate how many possible ways there are to colour a graph's vertices with a given set of colours, as described in W.D. Wallis' "A Beginners Guide to Graph Theory" [23]. When this is calculated all the different ways to colour the graph are counted if they are not identical, therefore this can include isomorphic colourings. To further understand this point consider the star graph S_3 consisting of vertices labelled x, y and z. Denote (A, B, C), where A, B and C are colours, to represent how the vertices of the graph S_n are coloured; this is such that x is coloured with A, y is coloured B and z is coloured with C. The chromatic number of S_3 is 2, so using two colours we can see that S_3 can have either the colouring (A, B, A) or (B, A, B), hence there are two possible ways to colour S_3 with two colours and in this case they happen to also be isomorphic colourings. Moreover if 3 colours were available there would be 12 possible colourings: (A, B, A), (A, B, C), (A, C, B), (A, C, A), (B, A, B), (B, A, C), (B, C, A), (B, C, B), (C, A, C), (C, A, B), (C, B, A) and (C, B, C). As we can see, not all 3 colours had to be used in a colouring.

We will now explore what a chromatic polynomial is with reference to Narsingh Deo's explanation in "Graph Theory with Applications to Engineering and Computer Science" [6].

Definition 3.11. The **chromatic polynomial**, $P_G(\lambda)$, of a graph G is the number of ways to colour the vertices of the graph with λ or fewer colours.

Let c_i denote the number of different ways to colour a graph G with exactly i different colours. We can determine that since i colours can be chosen out of λ colours in $\binom{\lambda}{i}$ different ways, there must be $c_i\binom{\lambda}{i}$ different ways to colour G with exactly i out of λ colours.

Notice that i can be any integer from 1 to n where n is the total number of vertices in G, therefore the chromatic polynomial will be the sum of all of these terms. This is such that:

$$\begin{aligned} P_G(\lambda) &= \sum_{i=1}^{n} c_i \binom{\lambda}{i} \\ &= c_1 \frac{\lambda}{1!} + c_2 \frac{\lambda(\lambda-1)}{2!} + c_3 \frac{\lambda(\lambda-1)(\lambda-2)}{3!} + \\ &\quad \ldots + c_n \frac{\lambda(\lambda-1)(\lambda-2)\ldots(\lambda-n+1)}{n!}. \end{aligned} \tag{8}$$

Note that the value of c_i needs to be determined uniquely for every graph G.

Example 3.12. We will calculate the chromatic polynomial for the graph given in Figure 22, call this graph G, the vertices of G have been labelled from v_1 to v_5 for convenience.

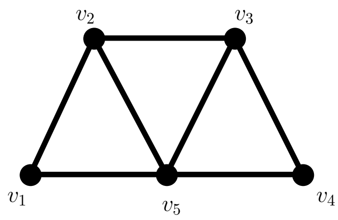

FIGURE 22. The graph used in Example 3.12.

Since the graph has five vertices, using equation (8) we can determine that the chromatic polynomial of G will be of the form:

$$P_G(\lambda) = c_1\lambda + c_2\frac{\lambda(\lambda-1)}{2!} + c_3\frac{\lambda(\lambda-1)(\lambda-2)}{3!} + c_4\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!} + c_5\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5!}. \tag{9}$$

We will now need to calculate the values of c_i for $i=$ 1, 2, 3, 4, and 5.

Analysing the graph in Figure 22 we can see that it contains a triangle (that is, three vertices all adjacent to one another) hence at least three different colours will be needed to colour the graph, such that no adjacent vertices share the same colour. Therefore we have:

$$c_1 = c_2 = 0. \tag{10}$$

Furthermore, as the graph has five vertices we can determine that,

$$c_5 = 5!. \tag{11}$$

Now, we will calculate c_3. If three colours were available to colour the graph G we can determine that for the triangle consisting of vertices v_1, v_2 and v_5 there are six possible ways for the triangle to be coloured, see Figure 23. Considering this, it can be observed that vertex v_3 can only be coloured with the same colour as vertex v_1 and similarly, v_4 must share the same colour as v_2. Hence there are only six possible ways to colour the graph with three colours, as shown in Figure 23, therefore:

$$c_3 = 6. \tag{12}$$

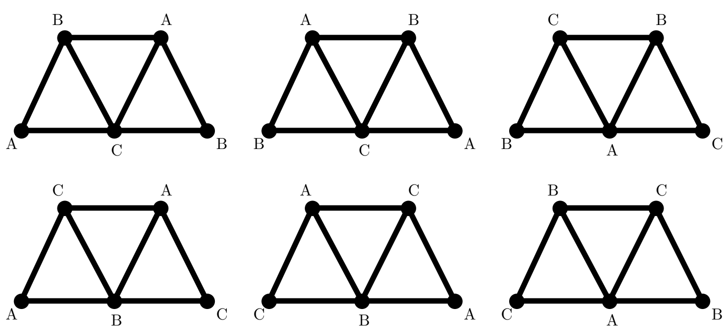

FIGURE 23. Possible colouring with three colours, denoted A, B and C, of the graph used in Example 3.12.

Next we want to calculate the value of c_4. When four colours are available there are 24 different ways to colour the triangle consisting of vertices v_1, v_2 and v_5. With this in mind it can be determined that for each case there are four possible ways to colour the remaining vertices, v_3 and v_4, since neither vertex can share the same colour as v_5 or each other and v_3 cannot be the same colour as v_2. Therefore we can determine that,

$$c_4 = 24 \times 4 = 96. \tag{13}$$

From here, we input the values of c_i into equation (8) ,

$$\begin{aligned} P_G(\lambda) =& 6\frac{\lambda(\lambda-1)(\lambda-2)}{3!} + 96\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!} \\ &+ 120\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5!}, \\ =& \lambda(\lambda-1)(\lambda-2) + 4\lambda(\lambda-1)(\lambda-2)(\lambda-3) \\ &+ \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4), \\ =& \lambda^3 - 3\lambda^2 + 2\lambda + 4\lambda^4 - 24\lambda^3 + 44\lambda^2 - 24\lambda \\ &+ \lambda^5 - 10\lambda^4 + 35\lambda^3 - 50\lambda^2 + 24\lambda, \\ =& \lambda^5 - 6\lambda^4 + 12\lambda^3 - 9\lambda^2 + 2\lambda, \end{aligned} \tag{14}$$

and hence find the chromatic polynomial for the graph G.

Remark 3.13. It is worth noting that for a graph with n vertices and no edges the chromatic polynomial would be λ^n, as every vertex can be any of the λ colours since none of them are adjacent.

3.3. **Edge Colouring.**

We will now investigate another type of colouring that can be applied to a graph's edges. Edge colouring follows similar principals to vertex colouring and raises questions such as: what the minimum number of colours required to colour a graph such that no adjacent edges share the same colour would be. We will continue to make reference to Robin Wilson's explanation in [24] throughout this section.

Definition 3.14. A graph G is called ***k*-edge-colourable** if the edges of the graph can be coloured in k colours such that no adjacent edges are the same colour. G is said to have a ***k*-edge-colouring** and is described to have a **proper edge colouring**. If G is k-edge-colourable but is not $(k-1)$-edge-colourable it is said that the **chromatic index** of G is k, denoted as $\chi'(G) = k$.

Example 3.15. The Petersen graph has a chromatic index of 4, a colouring of which is shown in Figure 24.

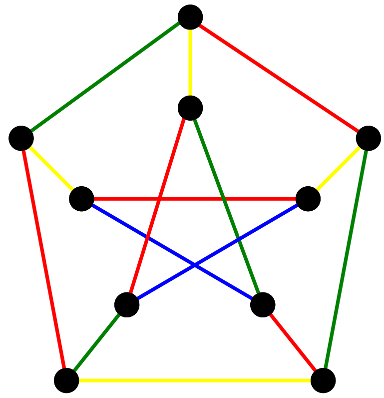

FIGURE 24. Edge colouring of the Petersen graph.

We will now introduce a key theorem in edge colouring, known as Vizing's theorem, which relates the largest degree of a graph to the chromatic index.

Theorem 3.16. *If G is a simple graph which has the largest degree denoted by Δ, then $\Delta \leq \chi'(G) \leq \Delta + 1$.*

A proof of Vizing's theorem has been omitted from this thesis but a version can be found in [20] if of interest to the reader.

Now that we have introduced some key concepts found in graph colouring we can go on to investigate what happens when we apply some of these techniques to graceful graphs.

4. Colouring Graceful Graphs

In this section we will begin to combine the concepts we learnt in Section 2 with the methods depicted in Section 3.

4.1. **Colouring Different Classes of Graceful Graphs.**

We will now investigate what happens when colour is added to some known classes of graceful graphs. Here, we will examine their chromatic numbers and indices, as well as consider some of their chromatic polynomials.

4.1.1. *Paths.*

Lemma 4.1. *Any path P_n has $\chi(P_n) = 2$ and $\chi'(P_n) = 2$.*

Proof. For a path P_n let $v_1, v_2, \ldots, v_n$ denote the consecutive vertices along the path. Colour all the vertices v_i, where i is even, with one colour and all the vertices v_i, where i is odd, with another colour (where $1 \leq i \leq n$). Hence the chromatic number of a path is always 2. Follow a similar method for the edges of the graph to show $\chi'(P_n) = 2$.

□

An illustration of the result is shown in Figure 25 for the path P_8.

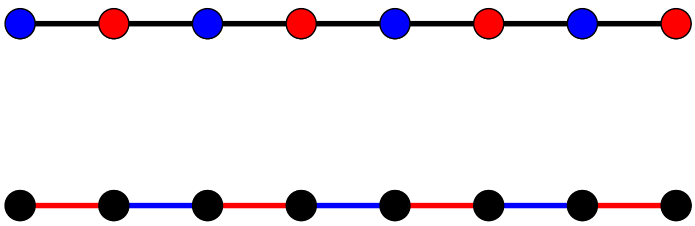

Figure 25. Vertex and edge colouring of P_8.

Theorem 4.2. *A path P_n has the chromatic polynomial, $P_{P_n}(\lambda) = \lambda(\lambda - 1)^{n-1}$, where n is the total number of vertices in the graph and λ is the total number of colours available to choose from.*

Proof. For the first vertex in P_n there is a choice of λ colours available. The next adjacent vertex in P_n can therefore be coloured with any one of the $\lambda - 1$ remaining colours. Given this, the next adjacent vertex in the path can also be assigned any colour out of $\lambda - 1$ colours; this is as the first colour selected will now be an available option again, however the colour just chosen for the adjacent vertex before this one in the path will now be unavailable. This continues until the end of the path is reached, hence $P_{P_n}(\lambda) = \lambda(\lambda - 1)^{n-1}$.

□

4.1.2. *Bipartite Graphs.*

As we have seen previously, all complete bipartite graphs, $K_{a,b}$, are graceful for all positive integers a and b. For completeness, we will examine the chromatic number and index for the class of all bipartite graphs, keeping in mind these results still apply to all graceful bipartite graphs.

Lemma 4.3. *For a bipartite graph G, $\chi(G) = 2$.* [24]

Proof. Given the definition of a bipartite graph the vertices can be split into two groups, say group A and B. This is such that for a vertex in group A any adjacent vertices are in group B and vice-versa. Therefore all the vertices in group A can be allocated one colour and all the vertices in group B can be given a different colour, so the theorem holds. [24]

□

Theorem 4.4. *If G is a bipartite graph where Δ is the largest vertex degree in G, then $\chi'(G) = \Delta$.* [24]

Proof. Induction on the number of edges of G will be used to show that if all but one of the edges are coloured with at most Δ colours, then there exists a Δ-edge-colouring for G.

Let's assume that every edge in G has been coloured except for the edge vw (the edge connecting vertices v and w). Then we can deduce that there is at least one missing colour at v and at least one missing colour at w. If a missing colour at v and w is the same colour then we can colour the edge vw with this colour. However, if this is not the case then let α denote a colour missing at v and β denote a colour missing at w. Let $H_{\alpha\beta}$ represent the connected subgraph in G which contains vertex v plus all the edges and vertices of G that can reach v by a path that consists only of edges with colours α and β. An example of this is shown in Figure 26.

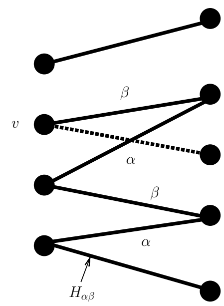

FIGURE 26. Diagram for proof of Theorem 4.4.

Since G is a bipartite graph, it is not possible for the subgraph $H_{\alpha\beta}$ to contain vertex w. Therefore the colours α and β of the edges in $H_{\alpha\beta}$ can be interchanged without affecting vertex w or the rest of the colouring of G. This would then mean the edge vw can be coloured with β, hence completing the edge colouring for the graph G. [24]

□

Remark 4.5. It is worth noting that a path P_n is a bipartite graph, hence Lemma 4.3 and Theorem 4.4 provide an additional proof to Lemma 4.1 where $\chi(P_n) = 2$ and $\chi'(P_n) = 2$.

4.1.3. *Star Graphs.*

Lemma 4.6. *Every star graph, S_n, has $\chi(S_n) = 2$.*

Proof. It is not difficult to see that the central vertex, call this v, of S_n can be coloured with one colour, whilst all the outer vertices that are adjacent to v (and have a degree of 1) can collectively be coloured with a different second colour. Hence $\chi(S_n) = 2$.

□

Lemma 4.7. *Every star graph, S_n, has the chromatic index $\chi'(S_n) = \Delta = n-1$, where Δ is the largest degree in S_n.*

Proof. Every edge in a star graph connects to the central vertex, v, and a vertex of degree 1. Hence all $n-1$ edges in the graph must be coloured differently with $n-1$ colours as they are all adjacent to each other. Therefore, $\chi'(S_n) = \Delta = deg(v) = n-1$.

□

Remark 4.8. Again, it is worth noting that all star graphs are also bipartite graphs hence the Lemma 4.3 and Theorem 4.4 could have also been applied.

4.1.4. *Cycle Graphs.*

We have seen that a cycle graph C_n is only graceful if $n \equiv 0 \pmod 4$ or 3 (mod 4). When $n \equiv 0 \pmod 4$, n is even, respectively when $n \equiv 3 \pmod 4$, n is odd. Separating cycle graphs into two categories, even and odd values of n, will be a key feature when examining the chromatic numbers and indices as they result in different answers. Again we will examine the general case for all cycle graphs, not just those that are graceful, for completion.

Lemma 4.9. *A cycle graph, C_n, has $\chi(C_n) = 2$, if n is even, or $\chi(C_n) = 3$, if n is odd.* [2]

Proof. A cycle graph with an even number of vertices is a bipartite graph, hence in relation to Lemma 4.3, $\chi(C_n) = 2$.

Now for a graph C_n where n is odd, let v_1, v_2, v_3 , . . . , $v_{2k}+1$, for some k such that $k = \frac{(n-1)}{2}$, be the vertices around C_n. This is such that v_i is adjacent to v_{i+1} and v_{i-1}, when $2 \leq i \leq 2n$, and therefore v_1 is adjacent to v_{2n+1} and v_2, similarly v_{2n+1} is adjacent to v_{2n} and v_1. If C_n could be coloured with two colours then you can allocate one colour to every v_i where i is even and the other colour to every v_i where i is odd. However, since the vertex v_{2n+1} is adjacent to v_1 they would share the same colour, hence a third colour is needed. [2]

□

Lemma 4.10. *A cycle graph, C_n, has $\chi'(C_n) = 2$, if n is even, or $\chi'(C_n) = 3$, if n is odd.*

Proof. The method for proving this lemma is very similar to the one used in the proof of Lemma 4.9. Again, a cycle graph with an even number of vertices is a bipartite graph hence, as the highest degree in the graph is of value 2, $\chi'(C_n) = 2$.

Following this, when we have C_n where n is an odd integer we let the edges around C_n be denoted as e_1, e_2, . . . , e_{2k+1}, for some integer k. If every edge e_i where i is an even number was coloured with one colour and every edge e_i where i is an odd number was coloured with another colour, again adjacent edges e_1 and e_{2n+1} would share the same colour. Therefore a third colour is needed, hence $\chi'(C_n) = 3$.

□

Theorem 4.11. *A cycle graph C_n has the chromatic polynomial, $P_{C_n}(\lambda) = (\lambda-1)^n + (-1)^n(\lambda-1)$, where n is the total number of vertices in the graph and λ is the total number of colours available to choose from.* [14]

Proof. Here we use induction based on a technique called the deletion-contraction argument (further information about it can be found in [14]). By deleting an edge from a cycle graph we get left with a path, which as we've seen previously has the chromatic polynomial $\lambda(\lambda-1)^{n-1}$. Contracting an edge in C_n leads to the graph C_{n-1}, which by the induction hypothesis has the chromatic polynomial $(\lambda-1)^{n-1}+(-1)^{n-1}(\lambda-1)$. Hence the difference is calculated to find the chromatic polynomial of C_n by subtracting the contraction result from the deletion one. Hence, $\lambda(\lambda-1)^{n-1}$ - $(\lambda-1)^{n-1}-(-1)^{n-1}(\lambda-1) = (\lambda-1)^n+(-1)^n(\lambda-1)$. [14] □

Example 4.12. The cycle graph C_{10}, has the chromatic polynomial: $P_{C_{10}}(\lambda)$ $= (\lambda-1)^{10}+(\lambda-1)$. So if say 5 colours were available to colour the graph with we can calculate that there will be $(4^{10}+4)$ different ways to provide a proper colouring to C_{10} using 5 colours.

4.1.5. *Complete Graphs.*

Lemma 4.13. *For a complete graph K_n, $\chi(K_n) = n$.*

Proof. It is very easy to see that since every vertex has degree n, as every pair of vertices in the graph are adjacent, n colours are needed. Otherwise if fewer colours were used two adjacent vertices would share the same colour. □

Theorem 4.14. *A complete graph K_n has $\chi'(K_n) = n-1$ if n is even or has $\chi'(K_n) = n$ if n is odd ($n \neq 1$).* [24]

Proof. If $n = 2$ the result is trivial, so assume $n \geq 3$.

If n is odd, the edges of K_n can be coloured with n colours. This is achieved by positioning the vertices of K_n in such a way that they form a regular n-gon (a polygon with n sides). Next the edges around the boundary of the graph (the edges that make up the shape of the n-gon) are coloured with different colours. Then each remaining edge is coloured with the colour used for the boundary that is parallel to it, hence giving the graph a n-colouring, an example of this is shown in Figure 27. K_n is not $(n-1)$-edge-colourable because it is observed that the largest number of edges that could possibly share the same colour is $\frac{(n-1)}{2}$, therefore K_n has at most $\frac{(n-1)}{2} \times \chi'(K_n)$ edges.

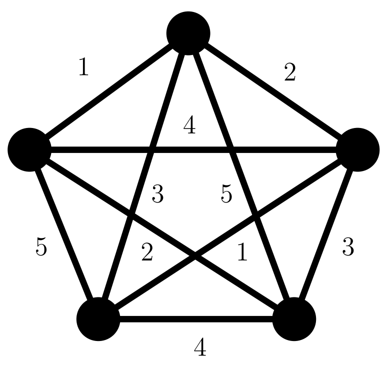

FIGURE 27. First diagram for proof of Theorem 4.14.

If n is even, it is possible to derive K_n from the complete graph K_{n-1} by joining every vertex in K_{n-1} to a single vertex v, see Figure 28. To achieve a colouring for this, colour the edges of the graph K_{n-1} as we have previously just done (for a K_n graph with an odd valued n). In doing this we will find that there is one colour missing (out of the $n-1$ colours used) from the edges incident with each vertex in K_{n-1}. These missing colours will all be different, hence the edge colouring can be completed for K_n by colouring the remaining edges attached to the single vertex v with these missing colours - as done so in Figure 28. Therefore the graph has an $n-1$-edge-colouring. [24]

□

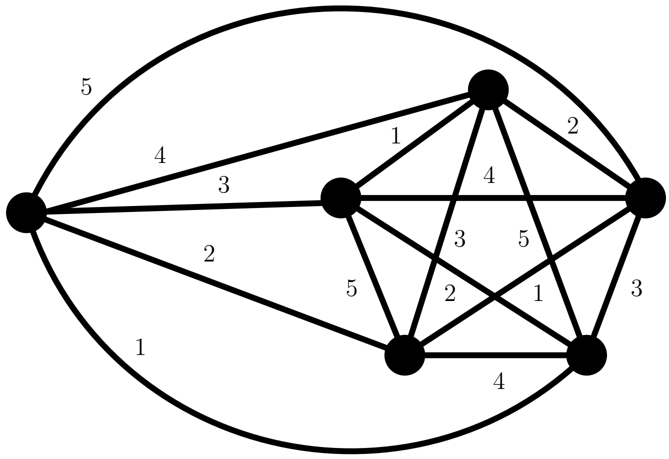

FIGURE 28. Second diagram for proof of Theorem 4.14.

Theorem 4.15. *The chromatic polynomial for the vertices of any complete graph is,* $P_{K_n}(\lambda) = \lambda(\lambda-1)(\lambda-2)\ldots(\lambda-n+1)$, *where* n *is the total number of vertices in the graph and* λ *is the total number of colours available to choose from.* [14]

Proof. Every vertex in the graph is adjacent to the other vertices so no two vertices can share the same colour. With this in mind we select an arbitrary vertex to begin with and assign it any one of the λ colours available. Next we select another vertex and random, now we can assign it a colour out of $\lambda-1$ choices. The third randomly selected vertex can then be coloured with one of $\lambda - 2$ possible colours, and so on.

□

4.1.6. *Wheel Graphs.*

Before we begin to investigate the chromatic number for wheel graphs, we first need to introduce a useful property, as mentioned in [2], for when two graphs are joined together. If a graph G is **joined** with a graph H this means every vertex in G is connected to every vertex in graph H by adding new edges (it can also be referred to as the **join** of graphs G and H), we will denote this as $G \vee H$ in this thesis.

Lemma 4.16. *The join of the graphs* G *and* H *has chromatic number,* $\chi(G \vee H) = \chi(G) + \chi(H)$. [2]

Proof. First we will show that $\chi(G \vee H) \geq \chi(G) + \chi(H)$. Since every vertex in G will be adjacent to every vertex in H, when G and H are joined, no colour used on the vertices of G can be the same as the colours used for the vertices of H. Therefore since $\chi(G)$ colours are required to colour G and $\chi(H)$ colours are required to colour H it implies $\chi(G \vee H) \geq \chi(G) + \chi(H)$.

Next we can infer that $\chi(G \vee H) \leq \chi(G) + \chi(H)$. If initially the $\chi(G)$ colours used to colour the vertices of G are different to the $\chi(H)$ colours used for H, then, when the graphs are joined, no additional colours are needed to be assigned to the vertices as all newly adjacent vertices will already be different colours.

Hence, $\chi(G \vee H) = \chi(G) + \chi(H)$.

□

Given this result we can now examine the chromatic number for wheel graphs.

Lemma 4.17. *For a wheel graph W_n (where $n \geq 4$), $\chi(W_n) = 3$ when n is odd and $\chi(W_n) = 4$ when n is even.* [2]

Proof. First we will look at the case when n is odd. The graph W_n is in fact the join of an even cycle C_{n-1} (where $n-1$ is even) and the complete graph K_1. Therefore by Lemma 4.9 and Lemma 4.13 we can deduce that for the case where n is odd, $\chi(W_n) = \chi(C_{n-1} \vee K_1) = \chi(C_{n-1}) + \chi(K_1) = 2+1 = 3$.

Next we evaluate the case where n is even. The graph W_n here is the join of an odd cycle C_{n-1} (where $n-1$ is odd) and the graph K_1. Therefore again, by Lemma 4.9 and Lemma 4.13, we can infer that for the case where n is even, $\chi(W_n) = \chi(C_{n-1} \vee K_1) = \chi(C_{n-1}) + \chi(K_1) = 3 + 1 = 4$.

□

Theorem 4.18. *A wheel graph W_n ($n \geq 4$) has the chromatic index $\chi'(W_n) = \Delta = n - 1$, where Δ is the largest vertex degree in the graph.*

Proof. We will first look at the case where n is even. The edges that incident with the centre vertex, that has a degree $\Delta = n - 1$, must all be coloured differently since they are all adjacent; hence $n - 1$ colours will be needed for this. Then, the remaining edges that form the outer cycle of the graph can be coloured by the following rule:

Colour an edge on the outer cycle of a wheel graph with the same colour as the nonadjacent edge that is directly perpendicular to it in the graph.

Again, the $n - 1$ colours will be used to colour the outer cycle, hence showing that when n is even the chromatic index is $n - 1$. An example of a colouring for the graph W_8 is shown in Figure 29.

Following this, we will now examine the case where n is odd. Let's call the centre vertex v, again all edges incident with v must be coloured differently so $\Delta = n-1$ colours are needed. Next, label the vertices on the outer cycle of the graph from v_1 to v_{n-1}, such that v_i is adjacent to vertices v_{i-1} and v_{i+1} for $2 \leq i \leq n-2$; for the case where $i = 1$ let v_1 be adjacent to v_{n-1} and v_2, similarly when $i = n-1$ let v_{n-1} be adjacent to v_{n-2} and v_1. From here the remaining edges on the outer cycle of the graph can be coloured following this rule:

Let the edge incident with both v_i and v_{i+1} be coloured with the same colour as the edge incident with both the centre vertex v and v_{i+2}.

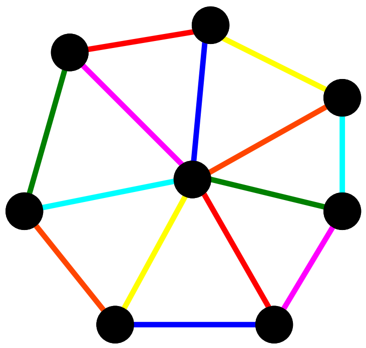

FIGURE 29. Edge colouring of graph W_8.

This will then result in the edges of the outer cycle being coloured with the $n-1$ colours already used without allowing any adjacent edges to share the same colour. Therefore the chromatic index for W_n when n is odd is also $n-1$. (See Example 4.19).

□

Example 4.19. We will provide an edge colouring to the graph W_7.

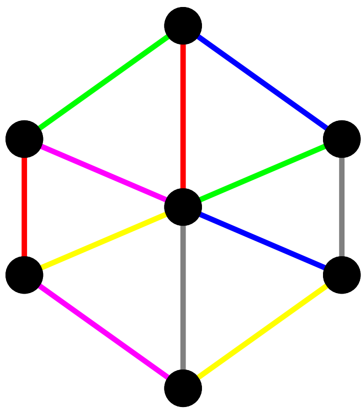

FIGURE 30. Edge colouring of graph W_7.

Theorem 4.20. *A wheel graph W_n ($n \geq 4$) has the chromatic polynomial, $P_{W_n}(\lambda) = \lambda(\lambda-2)^{n-1} + (-1)^{n-1}\lambda(\lambda-2)$, where n is the total number of vertices in the graph and λ is the total number of colours available to choose from.* [14]

Proof. Assign the central vertex any arbitrary colour, there are λ choices of colour. The $n-1$ remaining outer vertices form a cycle which has to be coloured with the remaining $\lambda-1$ colours. Using the result we determined in Theorem 4.11 we can calculate that the cycle C_{n-1} in this case can be coloured in $(\lambda-2)^{n-1} + (-1)^{n-1}(\lambda-2)$ different ways with $k-1$ colours. Therefore by multiplying the number of ways to colour the outer cycle by the number of ways to colour the centre vertex we get: $\lambda(\lambda-2)^{n-1} + (-1)^{n-1}\lambda(\lambda-2)$. [14]

□

4.2. **Generalizing the Colouring of Graceful Graphs.**

It is worth considering whether it is possible to assess if a graph is graceful or not through graph colouring. In fact we now introduce a special type of graph colouring which does exactly that.

Definition 4.21. A **rainbow vertex colouring** of a graph G with n vertices and m edges, involves assigning distinct colours from the set $\{0, \ldots, m\}$ to all the vertices of G - denote this as $f(v)$, where $v \in V(G)$. Colours are then allocated from the set $\{1, \ldots, m\}$ to the edges of G by applying the function $f'(uv) = |f(u) - f(v)|$, where uv is the edge incident with vertices u, $v \in G$. If all the edges have distinct colours then we can say that a rainbow vertex colouring has led to a **rainbow edge colouring** of G. We can then say G has a **rainbow colouring**. If $f' : E(G) \to \{1, 2, \ldots, m\}$, where every element is distinct and the set is closed, then this implies G has a graceful labelling. [26]

Example 4.22. Below we have the cycle graph C_4 which has been given a rainbow vertex colouring. Using the function defined previously this has induced a rainbow edge colouring hence depicting a graceful labelling of C_4. Note that the colour 4 is red, the colour 3 is blue, the colour 2 is green and the colour 1 is yellow.

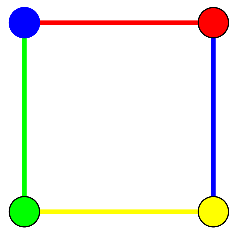

FIGURE 31. A rainbow colouring on C_4.

It is no surprise that not many theorems exist to classify properties involved with the colouring of graceful graphs overall. However below we analyse a recently proved theorem involving the existence of graceful graphs in relation to the chromatic number.

Theorem 4.23. *For any natural number n, there exists a graceful graph G such that $\chi(G) = n$.* [16]

Proof. When $n = 1$ the result is trivial, so assume $n \geq 2$. Let $v_1, \ldots, v_n$ denote the vertices in the complete graph K_n, for every i, such that $1 \leq i \leq n$. We will let 2^i be the label of v_i, note that we are purposefully exceeding the usual vertex label bounds at this stage but we will clarify this step later in the proof.

First we prove that when these vertex labels are given and the absolute difference is calculated to provide edge labels for the graph, all the edges of K_n have different labels. Suppose that v_iv_j and v_kv_l, for some i, j, k, l, are two edges with the same labels. Assume that $i > j$ and $k > l$, then we deduce that the edge labels of v_iv_j and v_kv_l are $2^i - 2^j$ and $2^k - 2^l$

respectively. Hence these must be distinct values, unless $i = k$ and $j = l$ which would mean the edges v_iv_j and v_kv_l are the same edge.

Now we note that the largest vertex label in K_n is 2^n for vertex v_n. It is obvious that for each natural number n we have $2^n > \frac{n(n-1)}{2}$. Therefore we add $2^n - \frac{n(n-1)}{2}$ new vertices to K_n by connecting them all to v_n. Call this newly formed graph G, see Figure 32. We will now show that G has a graceful labelling. For every x, $x \in \{1, \ldots, 2^n\}$ that does not exist as a label of an edge in K_n label one of the new vertices with $2^n - x$. It is clear that all the new edge labels that can then be calculated will have different values (call this Claim A) . Moreover, all the vertex labels will be from the set $\{1, \ldots, 2^n\}$ and will be distinct. This is because if two vertices had the same label then two edges (both of which would be incident with v_n) would have the same edge label, this contradicts Claim A. Hence a graceful graph exists for any arbitrary number n such that $\chi(G) = n$.

□

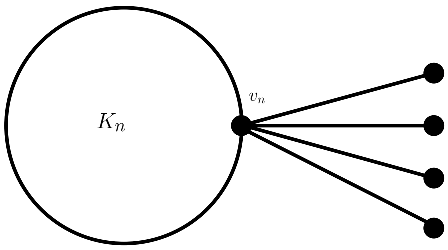

FIGURE 32. Diagram for proof of Theorem 4.23.

4.3. **Formulating a Graceful Colouring.**

Graceful graphs are formulated by the vertex labels inducing the value of edge labels. We will now examine this concept by introducing a new style of graph colouring that has been inspired by graceful labelling; here, we are looking at vertex colourings that induce edge colourings such that no adjacent vertices share the same colour and neither do any adjacent edges (however vertices may share the same colour as the edges they are incident with, and vice-versa). Note that this new colouring does not show whether a graph is graceful, like rainbow colouring, however it uses the ideas of graceful graphs to give a colouring to a graph. The entirety of this subsection is based on Zhang's explanation in 'A Kaleidoscopic View of Graph Colorings' [26].

Definition 4.24. A **graceful k-colouring** of a graph G is a vertex colouring $c : V(G) \to \{1, 2, \ldots, k\}$, where $k \geq 2$ and each integer represents a different colour, that induces an edge colouring $c' : E(G) \to \{1, \ldots, k-1\}$ defined by $c(uv) = |c(u) - c(v)|$, where u and v are vertices in G and uv is the edge connecting them. Any vertex colouring c for a graph G that is a graceful k-colouring for some integer k is called a **graceful colouring** of G. The **graceful chromatic number** of G, denoted by $\chi_g(G)$, is the minimum value of k that gives G a graceful k-colouring.

We will now list some useful results about graceful colouring. These results have been listed without proof as they go beyond the scope of the thesis and would require the introduction of a large amount of theory; the proofs can be found in [26] if of interest to the reader.

Proposition A: If G is a graph with n vertices such that $n \geq 3$ and has a diameter of at most 2, then $\chi_g(G) \geq n$.

Proposition B: If the graph H is a subgraph of a graph G, then $\chi_g(H) \leq \chi_g(G)$.

Proposition C: If G is a graph with n vertices where $n \geq 3$, then we have that $\chi_g(G) \geq \max\{\chi(G), \chi'(G)\} + 1$.

4.3.1. *Applying a graceful colouring to well known graphs.*

Theorem 4.25. *If G is a complete bipartite graph with n vertices such that $n \geq 3$, then $\chi_g(G) = n$.*

Proof. Let $G = K_{a,b}$ be the complete graph where $n = a + b$ such that the set A consists of a vertices with $A = \{u_1, u_2, \ldots, u_a\}$ and the set B consisting of b vertices with $B = \{v_1, v_2, \ldots, v_b\}$. Since the diameter of G is 2 we know by Proposition A that $\chi_g(G) \geq n$. Now consider the colouring $c : V(G) \to \{1, 2, \ldots, n\}$ where $c(u_i) = i$, for $1 \leq i \leq a$, and $c(v_j) = a + j$, for $1 \leq j \leq b$. Therefore, $c'(u_i w_j) = |a + (j - i)|$. Note that if i is fixed and $1 \leq j_1 \neq j_2 \leq b$, then $|a + (j_1 - i)| \neq |a + (j_2 - i)|$, similarly, if j is fixed $1 \leq i_1 \neq i_2 \leq a$, then $|a + (j - i_1)| \neq |a + (j - i_2)|$. Hence the edges of G have a distinct colouring such that no adjacent edges share the same colour and $\chi_g(G) = n$.

□

Before we examine the graceful chromatic number for cycle graphs we will need to introduce some notation to make formulating the proof simpler. For this we let $C_n = (v_1,\ v_2,\ \ldots,\ v_n,\ v_{n+1} = v_1)$, where $n \geq 3$. For an edge in C_n we let $e_i = v_i v_{i+1}$ for $i = 1,\ 2,\ \ldots,\ n$. Then for the colouring of the vertices, c, in C_n we let $s_c = (c(v_1), c(v_2), \ldots, c(v_n))$. And similarly for the colouring of the edges, c', of C_n we let $s_{c'} = (c'(e_1), c'(e_2), \ldots, c'(e_n))$.

Theorem 4.26. *For $n \geq 4$ we have, if $n = 5$, $\chi_g(C_5) = 5$ and if $n \neq 5$, $\chi_g(C_5) = 4$.*

Proof. Let C_n be defined as notated above and first suppose $n = 5$. Since the diameter of C_5 is 2, we know by Proposition A that $\chi_g(C_5) \geq 5$. Now define a colouring of the vertices, c, to be the following: $s_c = (1,\ 5,\ 3,\ 4,\ 2)$. This then makes the resulting edge colouring c' to be defined as $s_{c'} = (4,\ 2,\ 1,\ 2,\ 1)$. Therefore, this shows a graceful 5-colouring of C_n so $\chi_g(C_5) = 5$.

Next we examine the case where $n \neq 5$. To begin with we want to show that in this case $\chi_g(C_n) \geq 4$. Let's assume that in fact a graceful 3-colouirng of C_n exists and say $c(v_1) = 1$. Since this is a graceful colouring we must have $(c(v_2), c(v_n)) = (2, 3)$, so let's suppose $c(v_2) = 2$ and $c(v_n) = 3$. This would then mean $c(v_3) = 3$, as if it was equal to 1 then v_2 would have two edges adjacent to it with the same number hence resulting in two adjacent edges being given the same value, which isn't allowed. However, this would then lead to $c'(v_1 v_2) = c'(v_2 v_3) = 1$ which would go against it resulting in a graceful 3-colouring since again two adjacent edges share the same colour. Hence $\chi_g(C_n) \geq 4$.

Now will define graceful 4-colourings of C_n for different values of n.

$n \equiv 0 \pmod 4$. We use the following pattern to denote a graceful 4-colouring of C_n: for $n = 4$, let $s_c = (1, 2, 4, 3)$ such that $s_{c'} = (1, 2, 1, 2)$. Then for $n \geq 8$, let $s_c = (1, 2, 4, 3, \ldots, 1, 2, 4, 3)$ which implies that $s_{c'} = (1, 2, \ldots, 1, 2)$.

$n \equiv 1 \pmod 4$. We have, when $n = 9$ let $s_c = (1, 2, 4, 1, 2, 4, 1, 2, 4)$ so that $s_{c'} = (1, 2, 3, 1, 2, 3, 1, 2, 3)$. For $n \geq 13$, let $s_c = (1, 2, 4, 3, \ldots, 1, 2, 4, 3, 1, 2, 4, 1, 2, 4, 1, 2, 4)$ which means $s_{c'} = (1, 2, 1, 2, \ldots, 1, 2, 1, 2, 3, 1, 2, 3, 1, 2, 3)$.

$n \equiv 2 \pmod 4$. If $n = 6$ then $s_c = (1, 2, 4, 1, 2, 4)$ leading to $s_{c'} = (1, 2, 3, 1, 2, 3)$. Then for $n \geq 10$, let $s_c = (1, 2, 4, 3, \ldots, 1, 2, 4, 3, 1, 2, 4, 1, 2, 4)$ so that $s_{c'} = (1, 2, 1, 2, \ldots, 1, 2, 1, 2, 3, 1, 2, 3)$.

$n \equiv 3 \pmod 4$. For all cases where $n \geq 7$ let $s_c = (1, 2, 4, 3, \ldots, 1, 2, 4, 3, 1, 2, 4)$. Then $s_{c'} = (1, 2, 1, 2, \ldots, 1, 2, 1, 2, 3)$.

Hence in all cases when $n \neq 5$ there is a graceful 4-colouring and so $\chi_g(C_n) = 4$.

□

Theorem 4.27. *For a path P_n, where $n \geq 5$, $\chi_g(P_n) = 4$.*

Proof. Denote $P_n = (v_1, v_2, \ldots, v_n)$. For $n = 5$ the graceful 4-colouring, call this c^*, is such that $(c^*(v_1), c^*(v_2), c^*(v_3), c^*(v_4), c^*(v_4)) = (1, 2, 4, 1, 2)$, it is easy to see that a graceful 3-colouring is not possible for P_5, hence $\chi_g(P_5) = 4$.

For $n \geq 6$ we observe that P_n is a subgraph of C_n, therefore by using Proposition B and Theorem 4.26 we can determine that since the graceful chromatic number of C_n when $n \geq 6$ is 4 we must have that $\chi_g(P_n) \leq 4$. We will now show that P_n does not have a graceful 3-colouring. Suppose $\chi_g(P_n) = 3$ for some vertex colouring c. We can see that $c(v_3) \neq 2$ since the possible integers for $c(v_2)$ and $c(v_4)$ would result in the edges v_2v_3 and v_3v_4 having the same value, which is not allowed. Hence let $c(v_3) = 1$. Therefore $(c(v_2), v(c_4)) = (2, 3)$, so let $c(v_2) = 2$ which therefore means we must have $c(v_1) = 3$. However this results in $c'(v_1v_2) = c'(v_2v_3) = 1$, which again is not allowed. Hence no graceful 3-colouring is possible and we can conclude that $\chi_g(P_n) = 4$.

□

Theorem 4.28. *If W_n is a wheel graph with $n \geq 6$, then $\chi_g(W_n) = n$.*

Proof. The graph W_n is the join of the graphs C_{n-1} and K_1, so let $C_{n-1} = (v_1, v_2, , v_{n-1}, v_n = v_1)$ and denote the central vertex as v_0. By Proposition A we know that $\chi_g(W_n) \geq n$, hence we will show that W_n has a graceful n-colouring.

The graceful n-colouring of the graph W_n when $n = 6$, 7 and 8 means that the central vertex is coloured as 1 and the graceful n-colouring for W_7 and W_8 was obtained using the graceful $(n-1)$-colouring of W_{n-1} for $n = 6$ and 7 respectively. This was achieved by inserting a new vertex into the cycle C_{n-2} in W_{n-1} and connecting this to the central vertex, then giving this new vertex the colour n.

Now we will prove that for $n \geq 7$ there is a graceful $(n-1)$-colouring of W_{n-1}, with the central vertex, v_0, coloured as 1, where an edge xy (the edge connecting some vertex x to some vertex y) in the cycle C_{n-2} in W_{n-1} can have a new vertex, call this v, inserted into it and joined to v_0 to produce W_n. Furthermore, v can be coloured as n to produce a graceful n-colouring for the resulting graph W_n.

So suppose there exists a graceful $(n-1)$-colouring, c, of W_{n-1} for some $n \geq 7$, with the central vertex coloured with 1. We want to show that there is an edge xy in C_{n-2} where $c(x)$ and $c(y)$ satisfying the following two conditions:
i) $c(x) \neq (n+1)/2$ and $c(y) \neq (n+1)/2$,
ii) if (x', x, y, y') is a path in C_{n-2}, then $c(x) \neq c(x') + \frac{n}{2}$ and $c(y) \neq c(y') + \frac{n}{2}$.

To show this we let $C_{n-2} = (v_1, v_2, \ldots, v_{n-2}, v_{n-1} = v_1)$; the diameter of W_{n-1} is 2 so all the vertices in W_n must be assigned different colours by the vertex colouring c. Therefore, if for some i, $c(v_{i+1}) = \frac{c(v_{i+2})+n}{2}$ then $c(v_j) \neq \frac{c(v_{i+2})+n}{2}$ for all $j \neq i+1$. (Note, here the subscripts are denoted as integers modulo $n-2$.) We now consider the two cases where n is an odd integer and when n is an even integer.

Case 1
Let n be an odd integer. Suppose that for some i, $c(v_{i+2}) = \frac{c(v_{i+2}+n)}{2}$, this would mean the edge $v_i v_{i+1}$ makes the condition ii) invalid. We know that $n = 2c(v_{i+1})c(v_{i+2})$ is odd, hence we can deduce that $c(v_{i+2})$ is odd. There are $\frac{n-3}{2}$ vertices in C_{n-2} which will be assigned odd colours by c, since the central vertex is coloured with 1, therefore at most $\frac{n-3}{2}$ edges in C_{n-2} will fail condition ii). Conclusively this means there will be at least $(n-2) - \frac{n-3}{2} = \frac{n-1}{2}$ edges (which will have a value ≥ 3) in C_{n-2} that satisfy condition ii). Amongst these edges - those which satisfy condition ii) - at most two of them will fail condition i). Hence, there will be a least one edge xy in C_{n-2} for which $c(x)$ and $c(y)$ satisfy both i) and ii).

Case 2
We will now assume n is even. Suppose that for some i, $c(v_{i+2}) = \frac{c(v_{i+2}+n)}{2}$. We can see that $n = 2c(v_{i+1})c(v_{i+2})$ is even, therefore it follows that $c(v_{i+2})$ is even. There are then $\frac{n-2}{2}$ vertices in C_{n-2} which will be assigned even colours by c, since again the central vertex is coloured with 1, which means that at most $\frac{n-2}{2}$ edges in C_{n-2} will fail condition ii). Consequently there will be at least $(n-2) - \frac{n-2}{2} = \frac{n-2}{2}$ edges (which will have a value ≥ 4) in C_{n-2} that satisfy condition ii). Since the value of $\frac{n+1}{2}$ is not an integer, all of these edges will satisfy condition i). Hence, there will be a least one edge xy in C_{n-2} for which $c(x)$ and $c(y)$ satisfy both i) and ii).

$\square$

5. Trees

In this section we will introduce a new type of graph and evaluate the gracefulness and colouring of it. Trees are a unique type of graph as they always contain no cycles or multiple edges, so in theory should work very well when assessing if they are graceful or not. They have been a very central area of study when investigating graceful graphs, as we will see later. They also appear to be a very simple graph as they can feature many edges with degree 1, yet we can still develop a whole range of theorems and lemmas for them.

Definition 5.1. A **tree** is a connected graph with no cycles. If a graph is not connected but still contains no cycles then it is called a **forest**, which has components that are all trees. Any vertices in a tree with a degree of one is called a **leaf**.

We shall evaluate a very interesting theorem for trees, however we first need to introduce the following lemma.

Lemma 5.2. *If G is a connected graph with n vertices then it contains at least $n-1$ edges.* [7]

Proof. We want to show that in a connected graph G the number of vertices is at most one more than the number of edges. For the graph G let $v(G)$ be the number of vertices and $e(G)$ be the number of edges. We will be proving that for every non-negative integer m the following statement is true:

$P(m)$: Every connected graph, G, with m edges will have $v(G) \leq m+1$.

First we will look at the case where $m = 0$.

Since G has 0 edges and is a connected graph it must only have one vertex. Therefore $v(G) = 1$ and $m + 1 = 0 + 1 = 1$ so $v(G) \leq m + 1$, hence $P(0)$ holds.

Now we assume that $P(k-1)$ is true for some k.

We want to then prove that $P(k)$ holds. Here we let G be a connected graph with $k > 0$ edges and look at the two possible cases.

Case A: Say G contains a cycle, removing an edge from the cycle will lead to a new connected graph, H, with $k-1$ edges and the same number of vertices as G. Since H now has $k-1$ edges and we assumed $P(k-1)$ to be true we can infer that H must have at most $e(H) + 1$ vertices, i.e. $v(H) = (k-1) + 1 = k$. Therefore as $v(H) = v(G)$ we have that $v(G) \leq k$ and consequently $v(G) \leq k + 1$, so $v(G) \leq e(G) + 1$ as required.

Case B: If G does not contain a cycle then remove an end vertex and the edge that connects it to the rest of the graph. (We know there exists an end vertex for the graph since G is connected and would in this case be a tree, hence it must have at least one leaf.) We are now left with a connected graph F with $k-1$ edges and $v(G) - 1$ vertices. This means F has $k-1$ vertices, so applying our assumption for $P(k-1)$ to F implies that $v(F) \leq e(F) + 1$ and therefore $v(F) + 1 \leq (e(F) + 1) + 1$. However we know that $v(F) + 1 = v(G)$ and $e(G) = e(F) + 1$ - as we removed one vertex and one edge from G to get F - so therefore $v(G) \leq e(G) + 1$ as required.

Since both Case A and Case B hold we have shown that $P(k)$ is true, consequently proving Lemma 5.2. [7]

□

Theorem 5.3. *Let T be a graph with n vertices, then the following statements are equivalent:*
i) T is a tree;
ii) T contains no cycles and has $n-1$ edges;
iii) T is connected and has $n-1$ edges;
iv) T is connected and every edge is a bridge;
v) any two vertices of T are connected by exactly one path;
vi) T contains no cycles but the addition of any new edge will create exactly one cycle. [24]

Proof. If $n = 1$ then the proof for all six statements would be trivial, hence we shall assume $n \geq 2$.

i) $\Rightarrow$ *ii)*
Let T be a graph without any cycles and let e be an edge of T. If removing e leaves T connected then there is a path in T connecting the two vertices e is incident which does not consist of the edge e. Hence putting e back into the graph would give a cycle in T which is a contradiction, so e must be a bridge. Therefore, since T is a tree, removing any edge from T will disconnect the graph into two new graphs, both of which are trees. By induction, it can be determined that the number of edges in each of the two graphs is one less than the number of vertices. Therefore it can be deduced that the total number of edges in T is $n-1$.

ii) $\Rightarrow$ *iii)*
If T was disconnected then every component of T will be a connected graph with no cycles. By *ii)* this implies that the number of edges in each component is one less than the number of vertices. This would then mean that the total number of vertices in T would exceed the total number of edges by at least 2, contradicting the statement that T has $n-1$ edges.

iii) $\Rightarrow$ *iv)*
Removing any edge from T will result in a graph with n vertices and $n-2$ edges, which by Lemma 5.2 would make the graph disconnected.

iv) $\Rightarrow$ *v)*
Since T is connected, every pair of vertices is connected by at least one path. If two paths existed between a pair of vertices in T then they would be enclosed on a cycle, this contradicts the fact that every edge is a bridge.

v) $\Rightarrow$ *vi)*
If T contained a cycle then any two vertices in the graph would be connected by at least two paths, contradicting *v)*. When an edge e is added to the graph the two vertices e is incident with are already connected by a path, hence a cycle is created; only one cycle is created since to begin with there was only one path between any two vertices.

vi) $\Rightarrow$ *i)*
If T was disconnected then the addition of a new edge that joins a vertex in one component to a vertex in another component would not create a cycle,

contradicting *vi*). Hence T must be connected for vi) to hold and since it has no cycles, T is a tree.

□

5.1. **Graceful Trees.**

The graceful labelling of trees is an area of graph theory that has been studied for many years, as explained in [1]. A main focus in this area of study is on the Graceful Tree Conjecture (also known as the Ringel-Kotzig conjecture) which states that all trees are graceful. There have been lots of attempts to prove this conjecture many of which have involved proving different classes of trees to be graceful. In this section we will investigate some of these classes of trees.

Paths and star graphs are both classes of tree; as we saw previously in this thesis, Theorem 2.4 and Theorem 2.5, both of these can have graceful labelling. We will now examine a new class of trees called caterpillars. A **caterpillar** is a type of tree in which if all leaves were removed, the resulting graph would be a path.

Theorem 5.4. *All caterpillars are graceful.* [23]

Proof. Let T be a caterpillar with n vertices. Call the path that is formed from T when all the leaves are removed H. Select an endpoint of H (a vertex with degree 1) and call this x_0, then name the next adjacent vertex to x_0 in H as x_1, continue to do this along H; such that the vertices of H are given names from x_0 to x_k for some integer k. Denote X to be the set of vertices in T whose distance from x_0 is even, this includes x_0 itself. Then let Y denote the set of vertices in T which are of odd distance from x_0. Here we can note that every edge connects together two vertices, one of which is in X and one that is in Y.

Now assign the label $n-1$ to x_0. Label the neighbours of x_0 with 0, 1, 2, $\ldots$, where x_1 is the neighbour that receives the greatest label. Next assign labels $n-2$, $n-3$, $\ldots$, to the neighbours of x_1 - giving the largest label to x_2. From here we continue as follows: after x_{2i} (for some value i) receives its label assign its neighbours with increasing integer labels starting with the smallest available integer that has not yet been used as a label, do not include its neighbour x_{2i-1} as this will already have a label allocated, and give the largest label to x_{2i+1}. Then assign decreasing integer labels to the neighbours of x_{2i+1}, not including x_{2i}, starting with the largest unused integer that is smaller than n and giving the smallest value of these labels to x_{2i+2}.

This will result in all members of the set X receiving the labels: $n-1$, $n-2$, $\ldots$, $n-|X|$, whereas all member of the set Y will receive the labels 0, 1, 2, $\ldots$, $|Y|$. It can be easily observed that the graph has a graceful labelling.

□

Definition 5.5. A **spider tree** is a tree graph which has exactly one vertex with degree larger than or equal to 3, such a vertex is called the **branch point** of the tree. The paths that lead from the branch point to a leaf in the tree are referred to as **legs**. Note that if the length of a leg is of value

m, this means that m edges were crossed from the branch point at the start of the path to reach the vertex at the end of the path.

Theorem 5.6. *Let T be a spider tree with l legs. If each leg has a length of either m or $m+1$ for some $m \geq 1$, then T is graceful.* [3]

Proof. Let us assume $l \geq 3$, otherwise T would simply be a path which as we have shown previously is graceful.

First we will look at the case where l is odd. Let $l = l_0 + l_1$, where l_i is the number of legs with length $m + i$ for $i \in \{0, 1\}$. We can then calculate the number of vertices, n, in T such that $n = lm + l_1 + 1$ (this is calculated by accounting that all l legs are of length of at least m, plus l_1 of them have an extra vertex at the end, whilst noting that all the legs originate from the same single vertex, the branch point). These vertices will be given labels from the set $\{0, 1, \ldots, k\}$, where k in this case is equivalent to the number of edges in T.

Next we give names to the legs using L_1, L_2, $\ldots$, L_l, here L_1, L_2, $\ldots$, L_{l_1} are the legs of length $m+1$ and $L_{(l_1)+1}$, $L_{(l_1)+2}$, $\ldots$, L_l are the legs of length m. Let v^* denote the branch point of T and $v_{i,j}$ denote a vertex in L_i of distance j from v^*.

We now will define the following labelling function, Φ:
i) let $\Phi(v^*) = 0$,
ii) if i and j are both odd, then

$$\Phi(v_{i,j}) = k - \frac{i-1}{2} - \frac{(j-1)l}{2};$$

iii) if i and j are both even, then

$$\Phi(v_{i,j}) = k - \frac{l-1}{2} - \frac{i}{2} - \frac{(j-2)l}{2};$$

iv) if i is even and j is odd, then

$$\Phi(v_{i,j}) = \frac{i}{2} + \frac{(j-1)l}{2};$$

v) if i is odd and j is even, then

$$\Phi(v_{i,j}) = \frac{l-1}{2} + \frac{i+1}{2} + \frac{(j-1)l}{2}.$$

The Φ labelling puts a 0 label at the spider's branch point and then, by traversing along the spider's longer legs first, it give labels to the rest of the vertices. It does this by alternating between the highest and lowest remaining unused labels and spiralling away from the centre. An example of this is shown in Figure 33 where $l_0 = 2$, $l_1 = 3$ and $m = 4$.

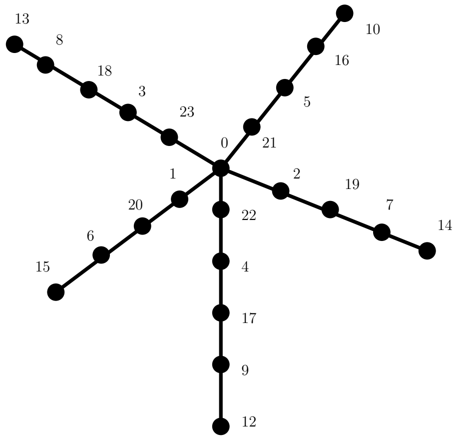

FIGURE 33. Diagram for proof of Theorem 5.6.

To compute the edge labels induced by the newly allocated vertex labels we recognise that the local maxima of Φ occurs at $v_{i,j}$ when i and j have the same parity, so when both i and j are odd or when both i and j are even. For when this is the case for i and j we have,

$$\Phi(v_{i,j}) - \Phi(v_{i,j+1}) = k - \frac{l-1}{2} - i + (1-j)l > 0, \tag{15}$$

$$\Phi(v_{i,j}) - \Phi(v_{i,j-1}) = k - \frac{l-1}{2} - i + (2-j)l > 0. \tag{16}$$

When looking for a contradiction suppose that two distinct edges share the same edge label. Consider the indices of the vertices these two edges incident with. It can be deemed that distinct pairs of indexes (i, j) and (i', j') can be chosen such that i and j have the same parity, i' and j' likewise have the same parity and an edge incident with $v_{i,j}$ shares the same label as a different edge incident with $v_{i',j'}$. Hence, this would imply one of the following three case could occur:

$$\Phi(v_{i,j}) - \Phi(v_{i,j+1}) = \Phi(v_{i',j'}) - \Phi(v_{i',j'+1}), \tag{17}$$

$$\Phi(v_{i,j}) - \Phi(v_{i,j+1}) = \Phi(v_{i',j'}) - \Phi(v_{i',j'-1}), \tag{18}$$

$$\Phi(v_{i,j}) - \Phi(v_{i,j-1}) = \Phi(v_{i',j'}) - \Phi(v_{i',j'-1}). \tag{19}$$

We will examine the case where (17) holds. By equation (15) we obtain that $i - i' + (j - j')l = 0$, however $j \neq j'$ otherwise $i = i'$ which contradicts the assumption that $(i, j) \neq (i', j')$. Therefore we can write $l = \frac{i-i'}{j'-j}$. Hence, $|i - i'| < l$ and $|j - j'| \geq 1$, and $l = |\frac{i-i'}{j'-j}| < \frac{l}{1} = l$, another contradiction. Similarly, when equations (18) and (19) hold they result in contradictions. Therefore two distinct edges cannot have the same edge labels and Φ gives a graceful labelling.

We will now look at the case where l is even. Without loss of generality, assume L_l is a leg with length m. Removing this results in a tree, call this T_0, with an odd number of legs, $l - 1$. The previous construction yields the graceful labelling Φ_0 of T_0 such that $\Phi_0(v^*) = 0$. Now we let $|V(T_0)| = k' + 1$

and define a new graceful labelling , Φ'_0 , on T_0 where $\Phi_0(v) = k' - \Phi_0(v)$ for all v.

Next we construct a new tree, T_1, by adding a new vertex, call this w_1, to T_0's branch point. Define Φ_1 on $V(T_1)$ by $\Phi_1(w_1) = 0$ and $\Phi_1(v) = \Phi'_0(v)+1$ for all $v \in V(T_0)$. Then also define Φ'_1 on T_1 by Φ_1'$(v) = k' + 1 - \Phi_1(v)$ for all v, noting that $\Phi'_1(w_1) = n' + 1$. Following this we add a new vertex w_2 to w_1 and construct the graceful labellings Φ_2 from Φ'_1 and Φ'_2 from Φ_2, and so on, until we have the full reconstruction of $L_l = w_1, w_2, \ldots, w_m$, which recovers our original graph T. This will then mean T will have a graceful labelling.

□

Following this we examine a graph known as a banana tree.

Definition 5.7. A **banana tree** $B_{z,k}$ is a tree which consists of z star graphs of the form S_k, such that one leaf in each star graph is connected to an additional distinct vertex called an **apex**.

The following theorem has had its proof adapted from [17].

Theorem 5.8. *All banana trees are graceful.*

Proof. A banana tree $B_{z,k}$ features z star graphs of the form S_k, for this proof we will simplify this notation to just S to represent a star graph within the banana tree - which we will now refer to as B. We then label each S in the graph with S^1, S^2, $\ldots$, S^z. Next, label the end vertices of S^i as $v^{i,1}$, $v^{i,2}$, $\ldots$, $v^{i,k-1}$ and the central vertex as $w^{i,0}$, where $i = 1, 2, \ldots, z$. We then label the apex with w, note that w is adjoined to every vertex $w^{i,0}$ for $i = 1, 2, \ldots, z$. We can now see that the number of vertices in B is $zk + 1$ and the number of edges is equal to $z(k-1)+k$. So we to define a labelling f for the vertices of B to be, $f : V(B) \to \{0, 1, 2, \ldots, z(k-1) + k\}$, such that:

i) $f(w^{1,0}) = 0$,
ii) $f(w^{2,0}) = 1$,
iii) $f(w^{i,0}) = f(w^{i-1,0}) + k$, $i = 3, 4, \ldots, z$,
iv) $f(v^{1,j}) = zk + 1 - j$, $j = 1, 2, \ldots, k-1$,
v) $f(w^{i,j}) = f(v^{i-1,k-1}) - 1 - j$, $i = 2, 3, \ldots, z$ and $j = 1, 2, \ldots, k-1$,
vi) $f(w) = f(w^{z,0}) + k$.

Using the above formula to allocate vertex labels to B, then using these to calculate the induced edge labels for the edges of B, will result in a graceful labelling.

□

They are many other types of trees that have been found to be graceful, here we will summarise a few which are mentioned in [1]. A symmetric tree is a graph which has a designated single vertex such that all edges are implicitly directed away from it, this is so that all vertices which are the same distance from this vertex have the same degree. In 1975 Bermond and Sotteau proved that all symmetric trees were graceful. A firecracker is a class of tree which consists of a caterpillar, such that the vertices at the end of the leaves of the caterpillar are all a central vertex of a star graph S_n $(n \geq 1)$ - note that the values of n here can be different for each leaf.

All firecrackers were proven to be graceful in 1997 by Chen, Lü and Yeh. An olive tree consists of a single vertex which is adjacent to i paths, for some integer i, here each path in the graph has a distinct length from 1 to i. Olive trees were shown to all have a graceful labelling in 1978 by Pastel and Raynaud.

5.1.1. *Trees with diameter at most 5.*

We will now look at a small collection of trees. This subsection is based on Superdock's 'The Graceful Tree Conjecture: A Class of Graceful Diameter-6 Trees' [21].

The diameter of a tree is the length of the longest path in that tree. We will show that all trees with a diameter of at most 5 are graceful.

Lemma 5.9. *All trees with a diameter of at most 4 are graceful.*

Proof. A tree with no diameter is just a single vertex so the result is trivial. A tree with a diameter of 1 will simply be a tree with one edge (and two vertices) hence again is trivial. If a tree has a diameter of 2 it is either the path P_2 or any star S_n, where n is the total number of vertices, which we have previously proven to be graceful. A tree with a diameter of 3 will be a caterpillar, if all the leaves were removed from this graph we would be left with a path consisting of a single edge and two vertices, as we have shown in Theorem 5.4 this is graceful. Finally, all trees with a diameter of 4 have been proven to be graceful, as shown in [25].

□

Theorem 5.10. *All trees of diameter 5 are graceful.*

Proof. (Sketch.) Let's look at banana tree T with an odd number of edges adjacent to the apex and with it consisting of n vertices overall. By Theorem 5.8 we know that this can have a graceful labelling by allowing the apex, call this v, to be labelled with 0 and all the k vertices, call these $v_1, \ldots, v_k$, adjacent to v to be labelled with $n, 1, n-1, 2, \ldots$.

Let T' be the new tree obtained when a new vertex which forms a leaf, call this u, is adjoined to the apex of T. The graceful labelling of T can be extended to one for T' by assigning the label $n+1$ to u. Then a graceful tree with a diameter of 5 can be formed by transferring any of the following pairs of subtrees from vertex v to vertex u: (T_{v,v_1}, T_{v,v_2}), (T_{v,v_3}, T_{v,v_4}), $\ldots$.

This concept shows how all trees of diameter 5 are determined to have a graceful labelling.

□

We will conclude this subsection by noting that although there has been a struggle when it has come to investigating the graceful labelling of trees, it has in fact been proven, as mentioned in [4], that all trees with up to 35 vertices are graceful.

5.2. **Colouring Trees.**

When tree graphs are coloured we gain some interesting results, here we will examine some of them (the first two proofs being my own) before observing a graceful colouring bound for all trees.

Theorem 5.11. *The chromatic number of any tree, T, with n vertices is $\chi(T) = 2$ (where $n \geq 2$).*

Proof. Choose any vertex in T, call this x, and colour it with colour A. Next choose any vertex adjacent to x, call this vertex y, and colour it with a second colour, colour B. We can then select any vertex adjacent to y and colour it with colour A, we know that this vertex will not be adjacent to x since that would mean T contained a cycle. Continue this colouring process, interchanging between colours A and B, until an end point is reached. From here, select any vertex, z, that is adjacent to a coloured vertex, c, and colour it with either colour A or B (such that z is a different colour to c). Again, we know z will not be adjacent to any other coloured vertices since this would once again imply T has a cycle. Carry on with this process until all vertices in the graph are coloured. Hence we can see, only two colours are needed to colour T, implying $\chi(T) = 2$.

□

Theorem 5.12. *The chromatic index of a tree, T, is $\chi'(T) = \Delta$, where Δ is the largest vertex degree in T.*

Proof. Let x be a vertex in T with degree Δ, colour all the edges adjacent to T with Δ different colours. Next select a vertex adjacent to x, call this y, this will either have degree Δ or less. If y has a degree Δ then all the uncoloured edges adjacent to it can be coloured with $\Delta - 1$ colours, this excludes the colour used for the edge adjacent to vertices x and y. If y has a degree less than Δ then the uncoloured edges can be coloured with a selection of colours from the $\Delta - 1$ colours available (again this excludes the colour used for the edge adjacent to vertices x and y). Following this we select a vertex adjacent to y, call this z, and repeat the previous process when colouring the edges adjacent to z, however this time the colour which is not available to be used from the $\Delta - 1$ colours will be the colour of the edge adjacent to vertices y and z. (The colour used for the edge connecting x and y will be avaliable to use again, since we know z cannot be adjacent to x as otherwise T would contain a cycle.) Carry on this procedure until an end point is reached, this can be done as no coloured edges with be adjacent to any newly coloured edges (excluding the one that brought us to that vertex) since this would mean T contains a cycle. Once an end point is reached, go back to a vertex adjacent to a coloured edge which does not have all of the edges incident with it coloured and repeat the method until all edges of T are coloured. Hence, T can be coloured with Δ colours.

□

Theorem 5.13. *Let T be a tree with n vertices, the chromatic polynomial of T is then: $P_T(\lambda) = \lambda(\lambda - 1)^{n-1}$.* [23]

Proof. Select any vertex in T, call this x_1. There are λ number of ways to colour x_1. Now choose any vertex adjacent to x_1 and call this x_2, there are now $\lambda - 1$ colours available to colour x_2 with (as the colour used for x_1 cannot be chosen); select a colour. Continue this process until an end point is reached and k vertices have been coloured. Now pick an uncoloured vertex adjacent to any of the k coloured vertices and call this x_{k+1}. We know that x_{k+1} is only adjacent to one coloured vertex as otherwise T would contain a cycle. Therefore if x_{k+1} is adjacent to x_i, where $1 \leq i \leq k$, it can be coloured with any of the $\lambda - 1$ colours that does not include the colour of x_i. This process is now continued until all the vertices of T are coloured.

Since there was λ choices of colour for the first vertex, x_1, and $\lambda - 1$ choices available for all of the remaining $n-1$ vertices we deduce that the chromatic polynomial is: $\lambda(\lambda-1)^{n-1}$.

□

5.2.1. *Graceful Colouring of a Tree.*

We know by Theorem 4.25 that all stars have a graceful chromatic number of value n, since they are bipartite graphs. Similarly, by Theorem 4.27 we have shown that all paths, P_n, with $n \geq 5$ have $\chi_g(P_n) = 4$. We will examine a general result which provides an upper bound for all graphs that are a tree, however, before this we will state an interesting result about caterpillars. Once again the reminder of this section will be based on Zhang's 'A Kaleidoscopic View of Graph Colorings' [26].

Theorem 5.14. *If T is a caterpillar with a maximum degree Δ, where $\Delta \geq 2$, then $\Delta + 1 \leq \chi_g(T) \leq \Delta + 2$.*

The proof of this theorem has been omitted from the thesis, however it is discussed a little further in [26] where a proof for the theorem is also referenced.

Before we investigate the final theorem of this section it is worth introducing two useful definitions and some notation. Firstly, for a vertex v in a graph G we define the **neighbourhood** of v, $N(v)$, to be the set of all vertices adjacent to v in G. Moreover we say that the **eccentricity** of a vertex v, $e(v)$, in a graph G is the maximum distance between v and any other vertex in G. Furthermore we denote that for positive integers a, b where $a \leq b$, we let $[a, b] = \{a, a+1, \dots, b\}$ and $[b] = [1, b]$. Additionally, the notation $\lceil z \rceil$ defines the smallest integer that is greater than or equal to z.

Theorem 5.15. *If T is a nontrivial tree with maximum degree Δ, then $\chi_g(T) \leq \lceil \frac{5\Delta}{3} \rceil$.*

Proof. First we define $S_1 = \left[\lceil \frac{2\Delta}{3} \rceil\right]$, $S_2 = \left[\Delta+1, \lceil \frac{5\Delta}{3} \rceil\right]$ and $S = S_1 \cup S_2$. Next we will prove a claim which will lead us towards verifying that T has a graceful colouring using the colours in S.

Claim: For every $a \in S$, there will be at least Δ distinct elements: a_1, a_2, a_3, ..., $a_\Delta \in S - \{a\}$, such that all of the Δ integers calculated to be $|a - a_1|$, $|a - a_2|$, ..., $|a - a_\Delta|$ are distinct.

We will look at proving this claim using three cases, each which correspond to the values of Δ modulo 3.

Case 1:
$\Delta \equiv 0 \pmod 3$. Let $\Delta = 3k$ for some positive integer k. This means we have $\lceil \frac{2\Delta}{3} \rceil = 2k$ and therefore $S_1 = [2k]$ and $S_2 = [3k+1, 5k]$. Let $a \in S$ and assume it is such that $a \in S_1$. Here, let $a_i = 3k + i$ for every $i = 1, 2, \dots, 2k$. Then we can see that all of $|a - a_1|$, $|a - a_2|$, ..., $|a - a_{2k}|$ are distinct and $|a - a_i| = 3k + i - a \geq 3k + i - 2k = k + i \geq k + 1$ for $1 \leq i \leq 2k$. If $a \leq k$ we then select $a_{2k+j} = a + j$ for $1 \leq j \leq k$, whereas if $a \geq k+1$ we then select $a_{2k+j} = a - j$ for $1 \leq j \leq k$. Here, $|a - a_{2k+1}|$, $|a - a_{2k+2}|$, ..., $|a - a_{3k}|$ are all distinct with $|a - a_{2k+j}| = j \leq k$. So, since $|a - a_i| \geq k+1$ for $1 \leq i \leq 2k$ and $|a - a_i| \leq k$ for $2k+1 \leq i \leq 3k$, it follows that $|a - a_1|$,

$|a-a_2|, \ldots, |a-a_{3k}|$ are distinct.

Case 2:
$\Delta \equiv 1 \pmod 3$. Let $\Delta = 3k+1$ for some positive integer k. This means we have $\lceil \frac{2\Delta}{3} \rceil = 2k+1$ and therefore $S_1 = [2k+1]$ and $S_2 = [3k+2, 5k+2]$. Let $a \in S$ and again assume it is such that $a \in S_1$. Here, let $a_i = 3k+1+i$ for every $i = 1, 2, \ldots, 2k+1$. Here we see that $|a-a_1|, |a-a_2|, \ldots, |a-a_{2k+1}|$ are all distinct and $|a-a_i| = 3k+1+i-a \geq 3k+1+i-(2k+1) = k+i \geq k+1$ for $1 \leq i \leq 2k+1$. If $a \leq k$ we then choose $a_{2k+1+j} = a+j$ for $1 \leq j \leq k$, whereas if $a \geq k+1$ we then select $a_{2k+1+j} = a-j$ for $1 \leq j \leq k$. Therefore, $|a-a_{2k+2}|, |a-a_{2k+3}|, \ldots, |a-a_{3k+1}|$ are all distinct and $|a-a_{2k+1+j}| = j \leq k$. As $|a-a_i| \geq k+1$ for $1 \leq i \leq 2k+1$ and $|a-a_i| \leq k$ for $2k+2 \leq i \leq 3k+1$, it follows that $|a-a_1|, |a-a_2|, \ldots, |a-a_{3k+1}|$ are distinct.

Case 3:
$\Delta \equiv 2 \pmod 3$. Let $\Delta = 3k+2$ for some positive integer k. This means we have $\lceil \frac{2\Delta}{3} \rceil = 2k+2$ and so $S_1 = [2k+2]$ and $S_2 = [3k+3, 5k+4]$. Principals similar to that in Case 2 can then be followed to prove this case.

Hence, the claim holds. We now need to construct a graceful colouring, c, of T using the colours in S. For $v \in V(T)$ where $deg(v) = \Delta$ and let $V_i = w \in V(T) : d(u,v) = i$ for $0 \leq i \leq e(v)$, where $e(v)$ is the eccentricity of v.

Therefore we have $V_0 = \{v\}$ and $V_1 = N(v)$. Now we let $c(v) = a$ for some $a \in S$ and from our proven claim let $a_1, a_2, a_3, \ldots, a_\Delta \in S - \{a\}$ where $|a-a_1|, |a-a_2|, \ldots, |a-a_\Delta|$ are distinct. Then colour the vertices of V_1 so that $c(w) : w \in V_1 = \{a_1, a_2, \ldots, a_\Delta\}$. Here, every vertex in $V_0 \cup V_1$ has been allocated a colour from S so that all the vertices and edges in the tree T_1 where $T_1 = T[V_0 \cup V_1]$ (that is the tree consisting of vertices from V_0 and V_1 and the edges that connect them) are properly coloured.

Next assume that the colours of vertices in the tree $T_i = T[\cup_{j=0}^{i} V_j]$, for some integer i where $1 \leq i < e(v)$, have been allocated colours from S such that all the edges and vertices are properly coloured. We now will define the colours of the vertices in V_{i+1}. Say that $w \in V_i$ where w is not an end vertex of T and suppose $deg(w) = t \leq \Delta$ and $c(w) = b \in S$. Select $b_1, b_2, b_3, \ldots, b_\Delta \in S = \{b\}$ where $|b-b_1|, |b-b_2|, \ldots, |b-b_\Delta|$ are distinct. Now let $u \in V_{i-1}$ such that $uw \in E(T)$ and assume, without loss of generality, that $b_j \neq c(u)$ and $b_j \neq 2c(w) - c(u)$, for $1 \leq j \leq t-1 \leq \Delta - 1$. Then colour the vertices in $N(w) - \{u\} \subseteq V_{i+1}$ so that $\{c(w) : w \in N(w) - \{u\}\} = \{b_1, b_2, \ldots, b_{t-1}\}$. This process is continued for every vertex in V_i, that is not an end vertex, so that the colour of each vertex in V_{i+1} gets defined. Conclusively, T has a graceful colouring using colours from the set $S \subseteq [\lceil \frac{5\Delta}{3} \rceil]$ and therefore $\chi_g(T) \leq \lceil \frac{5\Delta}{3} \rceil$.

$\square$

6. Total Colouring

In this final section we will evaluate another type of graph colouring which involves colouring both the edges and vertices of a graph. This type of colouring is called total colouring, it requires the same colour to not be shared between both adjacent edges and vertices as well as edges adjacent to vertices. Hence, the colour of the vertices influences the colour of the edges, but this differs from the previously induced way we have seen it achieved in the graceful colouring and rainbow colouring of graphs.

Definition 6.1. A **total colouring** of a graph G assigns colours to the vertices and edges of G, such that: no pair of adjacent edges or vertices share the same colour and no edge and vertex that are incident with each other are the same colour. If such a colouring for G can be achieved using k colours then G is said to be ***k*-total-colourable** and have received a ***k*-total-colouring**. The **total chromatic number**, $\chi''(G)$, for the graph G is the minimum number of colours needed to produce a total colouring of G.

Example 6.2. Below we see a total colouring of the wheel graph W_4, here the total chromatic number is 5.

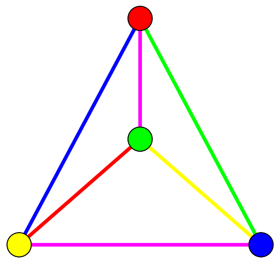

Figure 34. Total colouring of the wheel graph W_4.

In 1964, as mentioned in [20], a famous conjecture involving the total colouring of graphs was first formulated by Vizing and was officially published in 1968, today it is known as the Total Colouring Conjecture and is still unproven, it is stated below. It is also interesting to note that Behzad independently formulated the conjecture in 1965 and published it alongside Chartrand in 1967, hence there tends to be confusion towards who to accredit the conjecture to.

Conjecture 6.3 (Total Colouring Conjecture)**.** *For every graph G, $\chi''(G) \leq \Delta + 2$, where Δ is the largest vertex degree in the graph.*

6.1. **Total Colouring of Graceful Graphs.**

When initially investigating what connections lie between graceful graphs and total colouring it was found that there was not much documented on the topic. So here we begin to examine if any generalised properties for the total colouring of graceful graphs can be determined. The remainder of this section consists of my own proofs and findings.

The investigation began by first realising that for a graceful graph no two edges share the same two vertices. Therefore, as the absolute difference of the vertex labels is used to calculate the edge labels, we know that no edge label will have the same value as the vertex labels of the vertices that the edge is incident with - except in the case where the edge is adjacent to the vertex labelled 0. Furthermore, for a graceful graph with m edges the only way for the graph to receive an edge label with the value of m is for there to be an edge incident with two vertices with the vertex labels 0 and m. Considering all this we can begin to assess an independent upper bound for the total colouring of graceful graphs. We will first look at the case where the degree of the vertex labelled 0 in a graceful graph is one.

Lemma 6.4. *For any graceful graph, G, where the degree of the vertex labelled 0 is 1, $\chi''(G) \leq m + 1$ where m is the total number of edges in G.*

Proof. We know that for a graceful graph, G, other than at vertex 0, no edge has the same label value as any vertices it is incident with. Additionally, all edges and vertices have distinct labels. Using these two facts we can begin to create a total colouring for G. Firstly, let any edge or vertex labelled i, for $1 \leq i \leq m - 1$, be coloured with the colour i. Next colour the vertex m with the colour m and the edge m with the colour $m + 1$. All edges of the graph should now be coloured and the only vertex not assigned a colour is vertex 0. From here it is easy to see that since vertex 0 has a degree of 1, the only edge adjacent to it is coloured with $m + 1$ whilst the only vertex adjacent to it is coloured with m. This means vertex 0 can be coloured with any colour i, where $1 \leq i \leq m - 1$. Hence, the graceful graph receives a $(m + 1)$-total-colouring.

□

Next we look at the case where the degree of the vertex labelled 0 is greater than 1.

Lemma 6.5. *For any graceful graph, G, where the degree of the vertex labelled 0 is greater than 1, $\chi''(G) \leq m + 1$ where m is the total number of edges in G.*

Proof. We begin this proof by analysing the vertex 0 and both the edges and vertices adjacent to it. Firstly we colour the vertex 0 with the colour $m + 1$. Let us then say vertex 0 has a degree of value d. This means that there are d vertices incident with vertex 0, call these v_j for each $1 \leq j \leq d$. Now we can colour the vertices v_j with the colour of their vertex labels i, where $1 \leq i \leq m$. We next want to colour the edges that adjoin the vertices v_j to vertex 0. To do this we apply the following rule:

For an edge e_j, that is an edge connecting vertex v_j to vertex 0, colour this with the colour of vertex v_{j+1} for $1 \leq j \leq d - 1$. In the case where $j = d$, let e_d be coloured with the colour of v_1. $(*)$

From here we move onto colouring the reminder of the graph. Again, we note the fact that no edge has the same label value as any vertices it is incident with (except at vertex 0) and all vertex and edge labels are distinct. This means all the remaining vertices and edges labelled i can be coloured with colour i, where $1 \leq i \leq m-1$. (Note that no values of i previously used in the step before will be repeated and we know that the vertex labelled m must be adjacent to vertex 0 hence will already be coloured.) Following these steps will provide a valid $(m+1)$-total-colouring for the graceful graph.

□

Now we will assess what the total chromatic numbers are for different classes of graceful graphs.

6.1.1. *Paths and Stars.*

Lemma 6.6. *For a path P_n, $\chi''(P_n) = 3$.*

Proof. For a path P_n where n and m are the total number of vertices and edges in the graph respectively, denote P_n by the sequence consisting of its vertices v_i (where $1 \leq i \leq n$) and edges v_j (where $1 \leq j \leq m = n - 1$) in the order they are traversed along the path from the initial vertex v_1 to the terminal vertex v_n. This is as follow: v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_4, $\ldots$, e_{n-1}, v_n. Now if we allocate colours to the elements of the sequence we can see by inspection that in order for no adjacent elements to share the same colour, as well as no v_i to share the same colour as v_{i+1} and v_{i-1} (for $2 \leq i \leq n - 1$) and no e_j to share the same colour as e_{j+1} and e_{j-1} (for $2 \leq i \leq n - 2$), three colours are needed. Hence, v_1 can be coloured with colour 1, e_1 can be coloured with colour 2, v_2 coloured with colour 3, then e_2 can be coloured with colour 1 again and so on.

□

Theorem 6.7. *A star graph S_n has $\chi''(S_n) = \Delta + 1 = n$, where Δ is the largest degree in the graph.*

Proof. First we note that the central vertex is adjacent to all the outer vertices in the graph, hence every edge must be coloured a different colour. (Here we can straight off see that at least n colours are needed, that is, $n-1$ colours for the edges and an additional colour for the central vertex.) To achieve the total colouring of S_n we colour the central vertex (labelled 0) with colour n. We next let all the outer vertices be coloured with colour i, such that i is the value of their vertex label and $1 \leq i \leq n - 1$. Then using the same process described by the rule $(*)$ in Lemma 6.5, here vertex 0 is now defined as the central vertex of S_n, we assign colours to the edges of S_n, these will all be colours from the set i. Hence, $\chi''(S_n) = n$.

□

6.1.2. *Cycle Graphs.*

We will now analyse the total chromatic number for a cycle graph C_n, for completion we will prove this for the general case consisting of all values of n.

Theorem 6.8. *For a cycle graph, C_n, where $n \geq 3$,*

$$\chi''(C_n) = \begin{cases} 3 & \text{if } n \equiv 0 \text{ (mod 3)} \\ 5 & \text{if } n \equiv 1 \text{ (mod 3)} \\ 4 & \text{if } n \equiv 2 \text{ (mod 3).} \end{cases} \tag{20}$$

Proof. To examine this proof we will look at the three modulo individually.

Case 1: $n \equiv 0 \pmod 3$. C_n will have $n = 3k$ vertices, for some integer k, which means it also has $3k$ edges. Therefore the combined total number of vertices and edges in the graph is $6k$. If the cycle was written out as a closed sequence, similar to that seen in Lemma 6.6, it would contain $6k$ elements. Using the same requirements as those listed in Lemma 6.6 we can easily observe that a sequence of $6k$ elements can have its elements coloured with 3 colours, such that no consecutive elements share the same colour and neither do any adjacent edges or vertices that the elements represent. Furthermore each colour will be used exactly $2k$ times.

To better understand this we observe the case where $n = 9$, we have the closed sequence: v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_4, v_5, e_5, v_6, e_6, v_7, e_7, v_8, e_8, v_9, e_9. This can be coloured with colours 1, 2 and 3 in the following way: 1, 2, 3, 1,2, 3, 1, 2, 3, 1, 2, 3, 1,2, 3, 1, 2, 3.

Case 2: $n \equiv 1 \pmod 3$. C_n will have $n = 3k + 1$ vertices, for some integer k, and therefore $3k + 1$ edges. The combined total number of vertices and edges in the graph will be $6k + 2$. If the cycle was written out as a closed sequence, similar to that seen in Lemma 6.6, it would contain $6k + 2$ elements. We have seen in Case 1 that $6k$ elements can be coloured with 3 colours (following the requirements set out in Lemma 6.6) using the order: colour 1 then 2 then 3. If we apply this colouring to the first $6k$ elements in this case we are then left with 2 uncoloured elements. The first of these elements is the $(6k + 1^{th})$ element adjacent to the $6k^{th}$ coloured element (which is coloured by 3) and the $6k + 2^{th}$ uncoloured element. In graphical form, remembering this is a closed sequence, this would be a vertex adjacent to a vertex of colour 2 and another of colour 1, as well as an edge of colour 3 and an uncoloured edge. It is clear a fourth colour needs to be introduced to colour the element $6k+1$ as it is adjacent to all 3 previously used colours. When this fourth colour is applied the uncoloured edge represented by the $6k + 2^{th}$ element can only be coloured by a new fifth colour, as it will then be adjacent to all four colours, hence five colours are needed.

Case 3: $n \equiv 2 \pmod 3$. C_n will now have $n = 3k + 2$ vertices, for some integer k, and $3k + 2$ edges. The combined total number of vertices and edges in the graph will be $6k + 4$. If the cycle was written out as a closed sequence, similar to that seen in Lemma 6.6, it would contain $6k + 4$ elements. From Case 1 we again know the first $6k$ elements can be coloured using the requirements set out in Lemma 6.6 with 3 colours. The next 3 elements in the sequence can also then be coloured with these colours in the same order (that is, $6k + 1^{th}$ element coloured 1, $6k + 2^{th}$ element coloured 2 and $6k + 3^{th}$ element coloured 3). This leaves the final uncoloured element. In graphical form, remembering still this is a closed sequence, this element would be an edge adjacent to a vertex of colour 3 and another of colour 1, it will also be adjacent to two edges of colour 2. It is clear that a fourth colour needs to assigned to colour this final element (the $6k + 4^{th}$ one in the sequence) as it is adjacent to all 3 previously used colours. Hence 4 colours are needed in this case.

An example of this for the cycle C_8 would involve the element sequence denoted as: v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_4, v_5, e_5, v_6, e_6, v_7, e_7, v_8, e_8 being coloured by the following order: 1, 2, 3, 1, 2, 3, 1, 2, 3, 1, 2, 3, 1, 2, 3, 4. The proof is now complete.

□

6.1.3. *Wheel Graphs.*

Theorem 6.9. *All wheel graphs, W_n, with n vertices ($n \geq 4$) have $\chi''(W_n) = \Delta + 1 = n$, where Δ is the largest vertex degree in W_n.*

Proof. We can recognise that a subgraph of W_n is S_n. Therefore to begin this proof, apply the total colouring derived in Theorem 6.7 to the subgraph S_n in W_n, recalling also the procedure used in Lemma 6.5 ($*$). This results in a colour being allocated to all the vertices and inner edges of W_n using $\Delta + 1$ colours. Next we must assign colours to the outer edges of W_n. To achieve this we colour the edge connecting vertex v_j to v_{j+1} with the colour of vertex v_{j+2}, for $1 \leq j \leq j - 2$. In the case where $j = j - 1$ we colour the edge connecting v_{j-1} to v_j with the colour of vertex v_1. Then for the case where $j = d$ (in this case $d = \Delta = n - 1$) we colour the edge connecting v_j to v_1 with the colour of vertex v_2. The result will be a $(\Delta + 1)$-total-colouring for W_n.

□

6.1.4. *Complete Bipartite Graphs.*

Theorem 6.10. *A complete bipartite graph, $K_{a,b}$, has $\chi''(K_{a,b}) = \Delta + 2$, where Δ is the largest degree in the graph.*

Proof. Given that the chromatic number of a bipartite graph is 2 (as the vertices of the graph can be split into two sets, A and B, such that vertices in set A are only adjacent to vertices in set B), we begin by colouring all the vertices in set A with what we will for now refer to as colour A and all vertices in set B with the colour B. Next we recognise that since every vertex in set A is adjacent to every vertex in set B no edges in the graph can be coloured with colour A or B. Furthermore we observe that $\Delta(K_{a,b}) = max\,\{a, b\} = k$, for some integer k, hence at least k colours are needed to colour the edges of the graph. Note that from this point on the colours 1 to k do not consist of the colours A or B. Considering this, for the set where the vertices have degree $\Delta = k$, label them as $v_{1,1}$, $v_{1,2}$, $\ldots$, $v_{1,s}$, where s is the integer a or b (note that $s \neq k$ unless $a = b$). For the case where $a = b$, hence every vertex in $K_{a,b}$ has degree k, either set A or B can be chosen to be labelled in this way. Following this, label the second set of vertices as $v_{2,1}$, $v_{2,2}$, $\ldots$, $v_{2,k}$. We will now begin to assign the edges colours starting with those incident with $v_{1,1}$. Here, let $v_{1,1}v_{2,x}$ denote the edge connecting vertex $v_{1,1}$ with vertex $v_{2,x}$, where $1 \leq x \leq k$. Then for each value of x we assign the colour x to the edge $v_{1,1}v_{2,x}$. Next we look at the edges incident with $v_{1,2}$, here we assign the colour $x + 1$ to the edge $v_{1,2}v_{2,x}$, letting colour $k + 1 = 1$. From this we can formulate the following rule:

For the edge $v_{1,y}v_{2,x}$, where $1 \leq y \leq s$ and $1 \leq x \leq k$, assign the colour $x + (y - 1)$ to the edges, noting that whenever the colour is $k + (y - 1)$ let this be equivalent to the colour $y - 1$.

Following this process will result in a proper edge colouring for $K_{a,b}$ using k colours. Therefore, by noting that two further colours (colours A and B) were used for the vertices of the graph to give a proper vertex colouring, when coloured in this way, both the proper edge and vertex colourings result in a proper total colouring. Hence $\chi''(K_{a,b}) = \Delta + 2$.

□

7. Conclusion

Throughout this thesis we have introduced many exciting results found in graph theory, when focusing on graceful graphs and graph colouring. We have also begun to produce some new findings when combining these two topic areas together. This was achieved by grasping a fundamental knowledge on some key concepts found in graph theory before moving on to define what a graceful graph was and properties of it. This meant we were able to evaluate different classes of graph and determine whether or not they were graceful. Here, it was interesting to note that although the majority of graphs have been said to not be graceful, as mentioned in [9], when considering the set of graphs with 5 or fewer vertices only three of these do not have a graceful labelling, as [11] explains. It would interesting to evaluate this idea and see why this is the case, for graphs with 5 or fewer vertices, as well as investigate if any further sets of graphs have similar properties to this collection, hence nearly all can consequently be deemed to be graceful too. When initially considering this it may be worth noting all the possible vertex label combinations available for a pair of vertices from the set $\{0, 1, 2, 3, 4, 5\}$, then noting the amount of possible combinations that can lead to a graceful labelling. This could then be narrowed down further by implementing properties for graceful graphs that we have determined in this thesis, such as the vertices labelled 0 and 5 must be adjacent to get a graceful labelling. Therefore, it could be predicted that we have this property for graphs with 5 or fewer vertices since there are many ways available to calculate an appropriate edge labelling to make the graph graceful.

Following this, we introduced the popular area of graph theory known as graph colouring and explored some of the main results found in the topic. This provided an excellent basis for us to begin an evaluation on the colouring of graceful graphs. It would be fascinating to further examine this and see whether any links between the chromatic number of a graph and the properties of a graceful graph can be found, as this appears to be relatively unexplored. It was found that there was not much published material displaying any findings in this combined area, nonetheless we were able to see a really interesting result in Theorem 4.23, taken from [16], which could be a starting point for finding new results. We also portrayed two new types of graph colouring that use the principals of graceful graphs when allocating colours, this was rainbow colouring and graceful colouring. The fact that a graph with a rainbow colouring is also a graceful graph is an incredible finding and would have been intriguing to research further, especially considering the current amount of available information on that topic is sparse - which may just be due to it not being examined to great lengths of detail yet, hence provides a good scope for further study.

We went on to evaluate tree graphs, a type of graph that has had a massive influence on the topic of graceful graphs over the years. It would be interesting to see if any techniques from graph colouring could be used to help towards proving the Graceful Tree Conjecture, especially if this could be determined from say, a rainbow colouring. Finally, we concluded the thesis by investigating a final type of graph colouring called total colouring. Here, every class of graceful graph examined was seen to abide to the

Total Colouring Conjecture (which we would expect as no counter example to the conjecture yet exists). However, it would fascinating to investigate further if any link between total colouring, in particular the total chromatic number, and all graceful graphs can be found. This is especially considering if techniques such as those implemented in Lemma 6.4 and 6.5, that were determined specifically from the properties of graceful graphs, are used and developed further.

Overall we have investigated many interesting results regarding graceful graphs and the colouring of graphs. With this evaluation now complete it may provide a starting point for further study into the combined topic area, as we have focused on introducing key material which can form a good basis and initial understanding of the area.

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