On The Representation Theory of The Fuss-Catalan Algebras

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Abstract

Throughout this thesis, we study the representation theory of the Fuss-Catalan algebras, $\mathrm{FC}_{2,n}(a, b)$. We prove that these algebras are cellular and we define their cellular basis. In addition, we prove that they form a tower of recollement, and hence, they are quasi-hereditary. By calculating the Gram determinants of certain cell modules for the Fuss-Catalan algebras, we determine when these algebras are not semisimple. Finally, we end with defining homomorphisms between specified cell modules.

For my family

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Introduction

The Fuss-Catalan algebras was first introduced by Bisch and Jones [3] as a k-colour generalisation of the Temperley-Lieb algebra. We will denote it by $\mathrm{FC}_{k,n}(a_1, a_2, \cdots, a_k)$. They have given an explicit presentation of $\mathrm{FC}_{2,n}(a, b)$ in terms of generators and relations for the special case $k = 2$. In addition, they computed the structure of $\mathrm{FC}_{2,n}(a, b)$ when they are semisimple by using the diagram basis approach. These algebras have dimensions equal to the generalised Catalan numbers or Fuss-Catalan numbers thus they called it the Fuss-Catalan algebras.

Landau [26], generalised some results for general k that were proved in [3] for the case $k = 2$. For example, he defined a complete set of generators for $\mathrm{FC}_{k,n}(a_1, a_2, \ldots, a_k)$ [26, Theorem 4]. In addition, the isomorphism between the diagram definition and the abstract definition of the $\mathrm{FC}_{k,n}(a_1, a_2, \ldots, a_k)$ is given in [26, Theorem 6]. Francesco [14], has given new hyperbolic solutions to the Yang-Baxter equation and constructed new integrable lattice models by using the Fuss-Catalan algebras.

In this thesis we shall study the representation theory of the Fuss-Catalan algebras $\mathrm{FC}_{k,n}(a_1, a_2, \ldots, a_k)$ for the special case $k = 2$.

One motivation to study the Fuss-Catalan algebras $\mathrm{FC}_{2,n}(a, b)$ is that they satisfy the conditions of cellular algebras that have been given by Graham and Lehrer [19]. That is, they have cellular basis which introduce a filtration of $\mathrm{FC}_{k,n}$, and define a special kind of modules which is called cell modules. Furthermore, for each cell module we can define a bilinear form. One strong result of cellular algebras is that a cellular algebra is semisimple if and only if the bilinear form for every cell module is non-degenerate.

Another reason for studying these algebras is that they satisfy the axiomatic framework of towers of recollement that has been defined by Cox, Martin, Parker, and Xi [12]. This framework specifies when the algebra is semisimple and determines the simple $\mathrm{FC}_{2,n}(a,b)$-modules. In addition, the problem of finding a non-zero homomorphism between two cell modules will be reduced to the case when one of them is simple.

The main results that we have in this thesis are as following. In chapter one, we prove that $\mathrm{FC}_{2,n}(a,b)$ are cellular algebras in Theorem 1.3.14. In chapter two, Theorem 2.3.16 states that the Fuss-Catalan algebras are a tower of recollement. In chapter three, the Gram determinants for the Fuss-Catalan algebra cell modules are given in Theorem 3.4.17. Moreover, the values of a and b that make the algebras $\mathrm{FC}_{2,n}(a,b)$ not semisimple are given in Theorem 3.4.19. Finally, in chapter four, Theorem 4.1.5 and Theorem 4.1.6 give the homomorphisms between specified cell modules.

We will give a brief summary of this thesis. In chapter one, we introduce the Fuss-Catalan algebras and prove that they are cellular. We start the first section of this chapter by giving two equivalent definitions of $\mathrm{FC}_{k,n}(a_1,a_2,\dots,a_k)$, one via generators and relations and one via diagrams. In section two we summarise some known results about the Fuss-Catalan algebras when they are semisimple. In addition, definitions of the cellular algebra, cell module and the bilinear form are introduced in section three. Moreover, we prove, in Theorem 1.3.14, that our algebras $\mathrm{FC}_{2,n}(a,b)$ are cellular and we define its cellular basis. In the last section, we state some results related to set of labels for the cell modules. For example, its dimension and how to find its elements. Furthermore, we give some examples to explain the theory that we use throughout this chapter.

In chapter two, we show that the Fuss-Catalan algebras satisfy the axiomatic framework of towers of recollement introduced by Cox, Martin, Parker, and Xi [12]. The axioms A1 to A4 are proved in the first section. In the last section we study the restrictions of the cell modules so that we can give the proof for the last two axioms A5 and A6. The main theorem in this chapter is Theorem 2.3.16 which states that $\mathrm{FC}_{2,n}(a,b)$ satisfy the axioms for a tower of recollement.

In chapter three, we find the Gram determinant for $\mathrm{FC}_{2,n}(a,b)$-cell modules by using a similar technique to that used in [38, Section 4] to find the Gram determinant for the Temperley-Lieb algebra modules, and by using some of the properties of being a tower of recollement (see [12, Section 5]). This leads to finding the general form of the Gram matrices and we give the Gram determinant for some special cases of the cell modules. After that we state in Theorem 3.4.17 the general form of the Gram determinant for a large family of cell modules. We will end this chapter by our main result Theorem 3.4.19 which gives the values of a and b such that the Fuss-Catalan algebras $\mathrm{FC}_{2,n}(a, b)$ are not semisimple.

In chapter four, we work on homomorphisms between cell modules such that one of them is simple. We define morphisms from the cell modules that generated by the upper half diagrams of the generators for $\mathrm{FC}_{2,n}(a, b)$ to a cell module with certain label.

Chapter 1

Preliminaries

In this chapter we are going to give some definitions and basic concepts related to the Fuss-Catalan algebras. In addition, we prove that they are cellular algebras and define their cellular basis. Furthermore, we introduce some results on the labels of their cell modules.

1.1 The Definition of the Fuss-Catalan algebras.

Bisch and Jones [3], first defined the coloured generalisations of the Temperley-Lieb algebras which are called the Fuss-Catalan algebras. These algebras are defined in two different ways, the first as abstract algebras that defined by presentations, and the second way as diagram algebras. There is an isomorphism between these two ways of defining these algebras that has been given in [3, Theorem 4.2.14] for the special case $k = 2$. While the isomorphism for $k > 2$ is proved in [26, Theorem 6]. Here the author proved, in general, that these two algebras are isomorphic by using relation checking and basis counting.

Definition 1.1.1 ([3, p.96])**.** An (n, n)-planar diagram D is a diagram consisting of two horizontal, parallel lines with n vertices drawn on each line. The vertices are connected by strings either to vertices on the same line or to vertices on the parallel line provided that D has no crossings and strings do not leave the strip in the plane defined by the top and the bottom lines of the diagram.

Definition 1.1.2 ([3, p.97])**.** A colouring of a (kn, kn)-planar diagram D is an identical assignment of colours a_1, a_2, ..., a_k to each of the top and the bottom vertices of D such that the colouring is of the form

$$(a_1 a_2 \cdots a_k)(a_k a_{k-1} \cdots a_1) \cdots (a_1 a_2 \cdots a_k) \text{ if n is odd}$$

$$(a_1 a_2 \cdots a_k)(a_k a_{k-1} \cdots a_1) \cdots (a_k a_{k-1} \cdots a_1) \text{ if n is even.}$$

In addition, only vertices with the same colours have connecting strings.

We denote the set of all coloured planar (nk, nk)-diagrams by $\mathcal{B}_{k,n}$, so that $\mathcal{B}_{2,n}$ represents the set of all $(2n, 2n)$-diagrams.

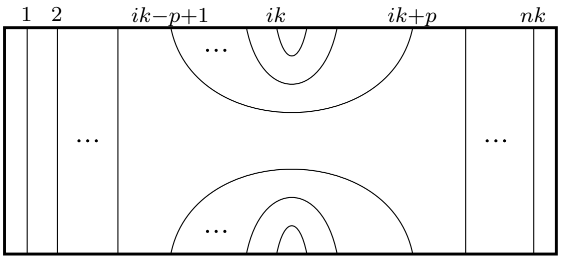
We can define the multiplication for any two diagrams in the set $\mathcal{B}_{k,n}$ to be the concatenation of two diagrams, since the top and the bottom lines of the diagrams have the same number (nk) of coloured vertices and the same colouring.
Let D_1 and D_2 are two diagrams in $\mathcal{B}_{k,n}$, then the multiplication of these two diagrams is $D_1 D_2 = (a_1)^{r_1}(a_2)^{r_2} \ldots (a_k)^{r_k} D_3$ where a_1, a_2, ..., a_k are complex numbers, D_3 is an element in $\mathcal{B}_{k,n}$ obtained by placing D_1 on the top of the D_2 and deleting the closed loops from the resulting diagram, and r_i is the number of the closed loops formed by the strings with colour a_i. The multiplication on $\mathcal{B}_{k,n}$ is not commutative in general.

Now, we are ready to introduce the diagrammatic version of the Fuss-Catalan algebras.

Definition 1.1.3 ([3, Definition 2.1.2])**.** Fix k complex numbers (the colours) $a_1, a_2, \ldots, a_k$, $k \geq 1$, and denote by $\mathrm{FC}_{k,n}(a_1, a_2, \ldots, a_k)$ the complex linear span of $\mathcal{B}_{k,n}, n \geq 1$. We set $\mathrm{FC}_{k,0} = \mathbb{C}$. Clearly, $\mathrm{FC}_{k,n}(a_1, a_2, \ldots, a_k)$ is then an associative algebra over $\mathbb{C}$ with multiplication being the multiplication of diagrams as explained above, extended linearly and respecting the distributivity law to all of $\mathrm{FC}_{k,n}(a_1, a_2, \ldots, a_k)$.

Remark 1.1.4. In this thesis we assume that the parameters $a_1, a_2, \ldots, a_k$ in the algebras $\mathrm{FC}_{k,n}$ are non-zero complex numbers.

Definition 1.1.5 ([26, Definition 11])**.** Let ${}_pe_i$, $1 \le i \le n-1, 1 \le p \le k$ denote the idempotent in $\mathrm{FC}_{k,n}$ that is $\dfrac{1}{\rho_i(p)}$ times the following diagram:



where

$$\rho_i(p) = \begin{cases} a_1a_2\cdots a_p & \text{if } i \text{ even} \\ a_ka_{k-1}\cdots a_{k-p+1} & \text{if } i \text{ odd.} \end{cases}$$

Theorem 1.1.6 ([26, Theorem 4])**.** *The set* $\{1, {}_pe_i\}$*, with* $1 \le i \le n-1$ *and* $1 \le p \le k$ *is a generating set for* $\mathrm{FC}_{k,n}$*.*

There is another definition of the Fuss-Catalan algebras, which is defined by the generators and relations.

Definition 1.1.7. [18, Definition 2.2] Let $\mathrm{FC}_{k,n}(a_1,\ldots,a_k)$ be the complex associative algebra generated by the set $\{{}_pU_i \mid 1 \le p \le k, 1 \le i \le n-1\}$ together with the identity $({}_0U_i) = ({}_0U) = 1$ subject to the relations:

$$\begin{aligned} ({}_pU_i)({}_qU_i) = ({}_qU_i)({}_pU_i) &= \rho_i(p)({}_qU_i) && \text{for } p \le q \\ ({}_pU_i)({}_qU_j) &= ({}_qU_j)({}_pU_i) && \text{for } \mid i-j \mid > 1 \text{ or} \\ & && j = i \pm 1 \text{ and } p+q \le k \\ ({}_pU_i)({}_qU_{i\pm1})({}_pU_i) &= \rho_i(k-q)({}_pU_i)({}_{k-p}U_{i\pm1}) && \text{for } p+q > k, \end{aligned}$$

for all $1 \le p \le k$, $1 \le i \le n-1$ ($n-2$ for the third relation), where $\rho_i(p)$ is as defined above.

This is called the n-th Fuss-Catalan algebra with k-colours a_1, …, $a_k \in \mathbb{C}$.

Let $\mathrm{TL}_n(a)$ be the Temperley-Lieb algebra, then under the correspondence $({}_1U_i) \leftrightarrow U_i$, $a_1 = a$, we have $\mathrm{FC}_{1,n}(a_1) = \mathrm{TL}_n(a)$. Hence, the Temperley-Lieb algebras are the first in the series of the Fuss-Catalan algebras.

Landau in his paper [26, Theorem 6], introduced the isomorphism between the two definitions of the Fuss-Catalan algebras, by showing that the map

$$\phi({}_pU_i) = \rho_i(p)_p e_i$$

is an isomorphism.

The dimension of the Fuss-Catalan algebras $\mathrm{FC}_{k,n}$ is the number of elements in the set $\mathcal{B}_{k,n}$ and it is given by the Fuss-Catalan numbers.

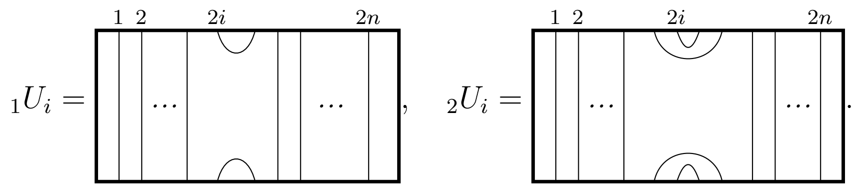
Proposition 1.1.8 ([3, Corollary 2.1.7])**.** *We have*

$$\dim \mathrm{FC}_{k,n} = \frac{1}{kn+1}\binom{(k+1)n}{n}$$

for $k \geq 1$ *and* $n \geq 1$.

Since our work is concerned with the case $k = 2$, we will denote by FC_n the algebras $\mathrm{FC}_{2,n}(a, b)$ and by $\mathcal{B}_n$ the set of all basis elements of $\mathrm{FC}_{2,n}$. Recall that throughout this thesis we will assume that the parameters a and b are non-zero complex numbers. Now, we present some definitions and examples related to the algebras FC_n.

Proposition 1.1.9 ([3, Proposition 4.1.3])**.** *The algebra* FC_n *is generated by the diagrams* 1, ${}_1U_i$ *and* ${}_2U_i$, *where* $1 \leq i \leq n-1$, *and*

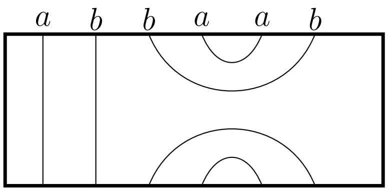


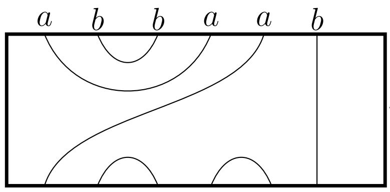
Definition 1.1.10 ([3, p.106])**.** The colouring of a basis diagram in FC_n is $a(yx)^{\frac{n}{2}-1}ya$ for n even, and $a(yx)^{\frac{n-1}{2}}b$ for n odd, where $x = aa$ and $y = bb$.

We indicate how to multiply diagrams in FC_n in this example.

Example 1.1.11. The colouring of a basis diagram in FC_3 is *abbaab*.

Let $D_1, D_2 \in \mathrm{FC}_3$, where

$D_1 =$ 

and $D_2 =$ 

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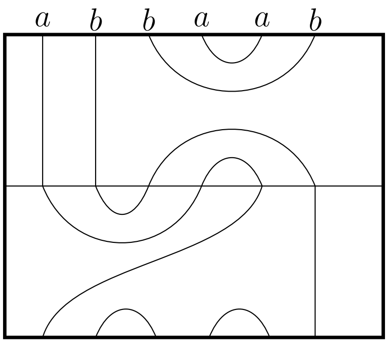
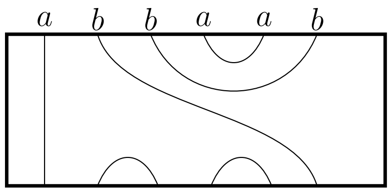
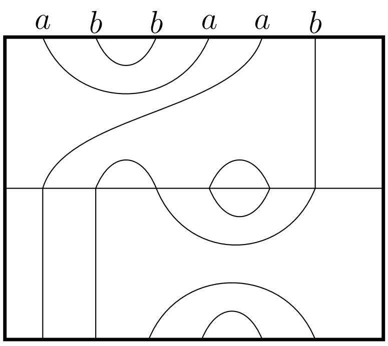
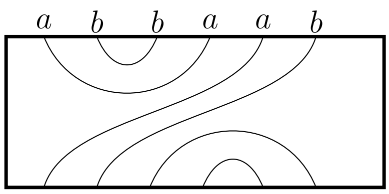
Then

$D_1D_2 =$ 

$=$ 

$D_2D_1 =$ 

$= a$ 

We can see that there are no loops in the diagram D_1D_2 while the diagram D_2D_1 has an a-loop.

1.2 The structure of $\mathrm{FC}_n(a, b)$

The Fuss-Catalan algebras over the complex field are generically semisimple [3, Lemma 2.2.1]. In this section we will review some known results when $\mathrm{FC}_n(a, b)$ are semisimple.

Definition 1.2.1 ([3, Definition 3.1.1])**.** Given a diagram $D \in \mathcal{B}_n$, a *through string* is a string connecting a vertex on the top line to a vertex on the bottom line. The number of through strings of D is denoted by $l(D)$.

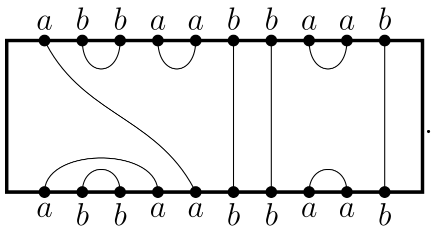
Definition 1.2.2 ([3, Definition 3.1.2])**.** To each diagram in $\mathcal{B}_n$, there is a well defined word on a and b called the *label* of the diagram, obtained by writing down the colours associated to the through strings reading from left to right.

The authors of [3] used the notation "middle pattern" instead of label in the above definition.

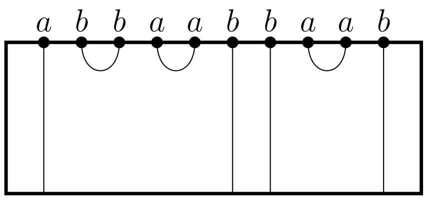
Definition 1.2.3. Let $D \in \mathcal{B}_n$, a diagram that obtained from D by removing all non-through strings and its vertices from the bottom line of D, is called the *initial part* of D, whereas a diagram that obtained from D by removing all non-through strings and its vertices from the top line of D, is called the *final part* of D.

Sometimes we call the initial part of a diagram D the upper half diagram and we call the final part of D the lower half diagram. In addition, the initial and the final parts of D have label equal the label of D, and they are uniquely determined by D.

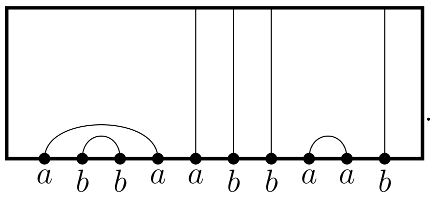
Example 1.2.4. The label of the following diagram $D \in \mathrm{FC}_5$ is $abbb = ab^3$, and $l(D) = 4$.



The initial part of D is the diagram



and the final part of D is the diagram



Lemma 1.2.5 ([3, Lemma 3.1.4])**.** *A label of a diagram $D \in \mathcal{B}_n$ is either empty (only possible if n is even) or of the form $aw(x,y)a$ if n is even and $aw(x,y)b$ if n is odd, where w is a word on $x = aa$ and $y = bb$.*

We note that not every word on a and b represents a label of a diagram. For instance, there is no label that starts with the letter b, and no label consisting of the letter b only. We will call the number of the letters in a label λ, the length of the label and we denote it by $l(\lambda)$, it is clear that, $l(D) = l(\lambda)$.

Remark 1.2.6. Let D_1, D_2 be two basis diagrams in FC_n with labels λ_1, λ_2 respectively. Let $D_3 = \beta D_1 D_2$ have a label λ_3. Since the product of two diagrams does not create new through strings, and any through strings in D_3 are obtained by connecting a through string from D_1 to a through string from D_2 we get $l(\lambda_3) \leq \min(l(\lambda_1), l(\lambda_2))$. Then we have

Definition 1.2.7 ([3, Definition 3.1.7])**.** The two-sided ideal in FC_n linearly spanned by diagrams with at most l through strings will be denoted by I_l.

By planarity, we have that the number of through strings must be even. In addition, we have

$$\mathrm{FC}_n(a,b) = I_{2n} \supset I_{2n-2} \supset \cdots \supset I_0 \supset 0.$$

Therefore, to describe the structure of $\mathrm{FC}_n(a,b)$ in the semisimple case, it is enough to find the minimal ideals for the algebras I_l/I_{l-2} for $0 \leq l \leq 2n$ where $I_{-1} = I_{-2} = \{0\}$.

Without assuming that $\mathrm{FC}_n(a,b)$ is semisimple, we have the following three lemmas.

Lemma 1.2.8 ([3, Lemma 3.1.8])**.** *Let λ be a label and let M be the initial part of some diagram in $\mathcal{B}_n$ with λ as its label. Let M^* be the reflection of M about its top line. Then $P_M = MM^*$ is a multiple of a projection, i.e. $P_M^2 = cP_M$, for some scalar $c \in \mathbb{C}$ (which is a product of a's and b's).*

Let M be the initial part of a diagram $D \in \mathrm{FC}_n$, we can define, $D'M$, the product of a diagram $D' \in \mathrm{FC}_n$ with M to be the initial part of the diagram $D'D$. That is, $D'M$ is the diagram that obtained from $D'D$ after removing all non-through strings and its vertices from the bottom line of $D'D$.

Lemma 1.2.9 ([3, p.108])**.** *The idempotent $p_M = \frac{1}{c}P_M$ has the following properties:*

(i) *p_M represents a minimal idempotent in FC_n/I_{l-2}, where the label of M has length l.*

(ii) *If M_1 and M_2 are initial parts of diagrams with distinct labels of length l, then $p_{M_1}\mathrm{FC}_n p_{M_2} = 0 \mod I_{l-2}$.*

(iii) *If M_1 and M_2 are initial parts of diagrams with the same labels λ of length l, then $M_1 = cDM_2$, where $D \in \mathcal{B}_n$ is a diagram with label λ, and c is a product of a's and b's.*

Definition 1.2.10 ([3, Definition 3.1.12])**.** If λ is a word on a and b, we define $\begin{bmatrix} n \\ \lambda \end{bmatrix}$ to be the number of distinct initial parts of the diagrams in FC_n having label λ with the convention that $\begin{bmatrix} n \\ \lambda \end{bmatrix} = 0$ if λ is not a label.

Lemma 1.2.11 ([3, Lemma 3.1.13])**.** *If λ is a label of length l and M is an initial part of a diagram with label λ, the dimension of the left ideal $p_M(I_l/I_{l-2})$ is $\begin{bmatrix} n \\ \lambda \end{bmatrix}$.*

Lemma 1.2.12 ([3, Lemma 3.1.14])**.** *Let I_l be the two sided ideal in FC_n as defined in Definition 1.2.7. Then*

$$I_l/I_{l-2} = \bigoplus_{\lambda} (I_l/I_{l-2})p_M(I_l/I_{l-2}),$$

where the sum is over all labels λ of length l and M is an initial part of a diagram with label λ (choose one M for each λ).

From the above lemmas we have the following theorem which describe the structure of the algebra $\mathrm{FC}_n(a,b)$ when it is semisimple.

Theorem 1.2.13 ([3, Theorem 3.1.15])**.** *If a and b are such that $\mathrm{FC}_n(a,b)$ is semisimple, we have*

$$\mathrm{FC}_n(a,b) \cong \bigoplus_{\lambda:\left[\begin{smallmatrix} n \\ \lambda \end{smallmatrix}\right]>0} M_{\left[\begin{smallmatrix} n \\ \lambda \end{smallmatrix}\right]}(\mathbb{C}),$$

where the sum is over all labels λ of diagrams in FC_n.

Definition 1.2.14 ([3, Definition 3.1.16])**.** Let λ be a word in a and b with $\begin{bmatrix} n \\ \lambda \end{bmatrix} > 0$ and let l be the length of λ. Let V_λ be the complex vector space of dimension $\begin{bmatrix} n \\ \lambda \end{bmatrix}$ having the distinct initial parts M of diagrams in $\mathcal{B}_n$ with label λ as a basis. Let $D \in \mathcal{B}_n$ and define a representation π_λ of $\mathrm{FC}_n(a,b)$ on V_λ by

$$\pi_\lambda(D)M = \begin{cases} DM, & \text{if } DM \text{ has } l \text{ through strings,} \\ 0, & \text{otherwise,} \end{cases}$$

for $M \in V_\lambda$ (where we replace closed loops in DM with a or b as usual).

Theorem 1.2.15 ([3, Theorem 3.1.17])**.** *If a and b are such that $\mathrm{FC}_n(a,b)$ is semisimple, then all representations π_λ as defined above are irreducible and any irreducible representation of $\mathrm{FC}_n(a,b)$ is equivalent to a π_λ.*

1.3 The $\mathrm{FC}_n(a,b)$ are cellular algebras

Cellular algebras are a class of associative algebras defined by Graham and Lehrer [19]. One main property of this kind of algebras is that they have special bases that define a set of modules called cell modules and for each cell module there is an associated bilinear form. Furthermore, the theory of cellular algebras helps us to provide the complete set of irreducible modules for a given cellular algebra by studying the quotients of the cell modules by the radical of its bilinear form. We start this section by giving some basic concepts related to the cellular algebras. Then we will prove that our algebras $\mathrm{FC}_n(a,b)$ are cellular and define their cellular bases. As well as we define their cell modules.

Definition 1.3.1 ([19, Definition 1.1])**.** Let R be a commutative ring with identity. A cellular algebra over R is an associative (unital) algebra A, together with cell datum $(\Lambda, \mathcal{W}, \mathcal{B}, *)$ where

(C1) Λ is a partially ordered set and for each $\lambda \in \Lambda$ there is a finite set $\mathcal{W}(\lambda)$ such that $\mathcal{B} = \{C^{\lambda}_{S,T} \mid \lambda \in \Lambda \text{ and } S, T \in \mathcal{W}(\lambda)\}$ is an R-basis of A.

(C2) If $\lambda \in \Lambda$ and $S, T \in \mathcal{W}(\lambda)$. Then $*$ is an R-linear anti-isomorphism of A such that $(C^{\lambda}_{S,T})^* = C^{\lambda}_{T,S}$.

(C3) If $\lambda \in \Lambda$ and $S, T \in \mathcal{W}(\lambda)$ then for any element $a \in A$ we have

$$aC^{\lambda}_{S,T} = \Big(\sum_{S' \in \mathcal{W}(\lambda)} r_a(S', S) C^{\lambda}_{S',T}\Big) + r'$$

where $r_a(S', S) \in R$ is independent of T, and r' is a linear combination of basis elements with upper label $\mu < \lambda$.

If we have a cellular algebra A, then we call the basis $\mathcal{B}$ in the above definition a *cellular basis*.

Definition 1.3.2 ([19, Definition 2.1])**.** Let A be a cellular algebra with cell datum $(\Lambda, \mathcal{W}, \mathcal{B}, *)$. For each $\lambda \in \Lambda$ define the left A-module $\Delta(\lambda)$ as a free R-module with basis $\{C_S \mid S \in \mathcal{W}(\lambda)\}$ and A-action defined by

$$aC_S = \sum_{S' \in \mathcal{W}(\lambda)} r_a(S', S) C_{S'} \qquad (a \in A,\ S \in \mathcal{W}(\lambda))$$

where $r_a(S', S) \in R$ as defined in definition 1.3.1. This module called the *cell module* of A labelled by λ.

The cell module $\Delta(\lambda)$ can be seen as a right A-module if we define the action of A on $\Delta(\lambda)$ by

$$C_S a = \sum_{S' \in \mathcal{W}(\lambda)} r_{a^*}(S', S) C_{S'}.$$

Definition 1.3.3. For each $\lambda \in \Lambda$, define A^{λ} to be the R-submodule of A with basis

$$\{C^{\mu}_{S,T} \mid \mu \in \Lambda, \mu < \lambda \text{ and } S, T \in \mathcal{W}(\mu)\}.$$

Definition 1.3.4 ([19, Definition 2.3])**.** For $\lambda \in \Lambda$, define a bilinear form

$$\Phi_\lambda : \Delta(\lambda) \times \Delta(\lambda) \to R$$

by the equation

$$C^\lambda_{S_1,T_1} C^\lambda_{S_2,T_2} \equiv \Phi_\lambda(C_{T_1}, C_{S_2}) C^\lambda_{S_1,T_2} \mod A^\lambda$$

where $S_1, S_2, T_1, T_2 \in \mathcal{W}(\lambda)$.

Proposition 1.3.5 ([19, Proposition 2.4])**.** *Let $\lambda \in \Lambda$, and $x, y \in \Delta(\lambda)$. Then*

(i) *The form Φ_λ is symmetric, i.e. we have $\Phi_\lambda(x, y) = \Phi_\lambda(y, x)$.*

(ii) *For all $a \in A$, we have $\Phi_\lambda(a^* x, y) = \Phi_\lambda(x, ay)$.*

Definition 1.3.6 ([19, Definition 3.1])**.** Let A be a cellular algebra with cell datum $(\Lambda, \mathcal{W}, \mathcal{B}, *)$. For $\lambda \in \Lambda$, define

$$\mathrm{rad}(\lambda) = \{x \in \Delta(\lambda) \mid \Phi_\lambda(x, y) = 0 \text{ for all } y \in \Delta(\lambda)\}.$$

Proposition 1.3.7 ([19, Proposition 3.2])**.** *Let $\lambda \in \Lambda$. Then*

(i) $\mathrm{rad}(\lambda)$ *is an A-submodule of* $\Delta(\lambda)$.

(ii) *If $\Phi_\lambda \neq 0$, the quotient* $\Delta(\lambda)/\mathrm{rad}(\lambda)$ *is absolutely irreducible.*

(iii) *If $\Phi_\lambda \neq 0$,* $\mathrm{rad}(\lambda)$ *is the Jacobson radical of the cell module* $\Delta(\lambda)$.

The next theorem is one of the main theorems that was introduced by Graham and Lehrer [19], to classify the irreducible cell modules for a cellular algebra A.

Theorem 1.3.8 ([19, Theorem 3.4])**.** *Let R be a field and let $(\Lambda, \mathcal{W}, \mathcal{B}, *)$ be a cell datum for the R-algebra A. For each $\lambda \in \Lambda$ define the cell module $\Delta(\lambda)$ and bilinear form Φ_λ on $\Delta(\lambda)$ as in definitions 1.3.2 and 1.3.4 respectively. Suppose $\Lambda_0 = \{\lambda \in \Lambda \mid \Phi_\lambda \neq 0\}$. Then the set $\{L_\lambda = \Delta(\lambda)/\mathrm{rad}(\lambda) \mid \lambda \in \Lambda_0\}$ is a complete set of non-isomorphic irreducible A-modules.*

Theorem 1.3.9 ([19, Theorem 3.8])**.** *Let R be a field and let $(\Lambda, \mathcal{W}, \mathcal{B}, *)$ be a cell datum for the R-algebra A. Then the following are equivalent:*

(i) *The algebra A is semisimple.*

(ii) *The non-zero cell modules of A are irreducible and pairwise inequivalent.*

(iii) *The form Φ_λ is non-degenerate, that is,* $\mathrm{rad}(\lambda) = 0$ *for each $\lambda \in \Lambda$.*

Now, we are going to identify a cell datum $(\Lambda, \mathcal{W}, \mathcal{B}, *)$ such that the algebras $\mathrm{FC}_n(a, b)$ are cellular. Let us start with this definition.

Definition 1.3.10 ([3, Definition 3.1.5])**.** If λ and λ' are words on an alphabet, we say $\lambda \leq \lambda'$ if λ is obtained from λ' by removing some (or no) letters of λ'.

For example, we have $aaabba \leq abbaabbbba$. In addition, the empty word, $\emptyset$, satisfies $\emptyset \leq \lambda$ for any word λ.

Definition 1.3.11. We define Λ to be set of all possible labels of the basis diagrams of FC_n, and $\mathcal{W}_n(\lambda)$ to be the set of all distinct initial parts of the diagrams in $\mathcal{B}_n$ having label λ.

Lemma 1.3.12. *The pair $(\Lambda, \leq)$ forms a partially ordered set.*

Proof. To see that Λ is a partially ordered, let λ_1, λ_2, $\lambda_3 \in \Lambda$. By the definition of the order $\leq$, we have $\lambda_1 \leq \lambda_1$. Now, suppose that $\lambda_1 \leq \lambda_2$ and $\lambda_2 \leq \lambda_1$. Then $l(\lambda_1) \leq l(\lambda_2)$ and $l(\lambda_2) \leq l(\lambda_1)$. So we get $l(\lambda_1) = l(\lambda_2)$. If λ_1 is not equal to λ_2 this means that we cannot get one of them from the other which is a contradiction, thus $\lambda_1 = \lambda_2$. For transitivity, suppose $\lambda_1 \leq \lambda_2$ and $\lambda_2 \leq \lambda_3$. Let S_1 and S_2 be the multi-sets of letters that we removed to get λ_1 from λ_2 and λ_2 from λ_3, respectively. Now, if we remove S_1 and S_2 from λ_3 then we get λ_1. Hence, $\lambda_1 \leq \lambda_3$. $\square$

Lemma 1.3.13. *The function $* : \mathrm{FC}_n \to \mathrm{FC}_n$ that reflects a diagram about a horizontal line in the middle of a diagram is an anti-isomorphism.*

Proof. To show that $*$ is an anti-morphism, we prove that $(D_1D_2)^* = (D_2)^*(D_1)^*$, for all $D_1, D_2 \in \mathcal{B}_n$ and extend linearly to all elements of the algebra FC_n. In fact, this holds because $(D_1D_2)^*$ is found by putting D_1 above D_2 and then reflecting them. As a result, we get the diagram D_2^* above the diagram D_1^*, which is the same as $D_2^*D_1^*$.

Since $*$ sends a basis element to a (distinct) basis element, it is injective. Using the rank-nullity theorem, we have $\dim(\mathrm{FC}_n) = \dim(\mathrm{Im}(*))$, which means that $*$ is surjective. We have shown that $*$ is bijective and an anti-morphism, so it is anti-automorphism. $\square$

Now, we will give the complete proof that the algebras $\mathrm{FC}_n(a, b)$ are cellular with cell datum $(\Lambda, \mathcal{W}_n(\lambda), \mathcal{B}_n, *)$. (We do not know of any explicit proofs in the literature).

Let Λ and $\mathcal{W}_n(\lambda)$ be as defined in Definition 1.3.11. Let S, $T \in \mathcal{W}_n(\lambda)$, and define $C^{\lambda}_{S,T}$ to be a basis diagram obtained by putting the diagram S above the diagram T after turning T upside down.

Theorem 1.3.14. *The Fuss-Catalan algebras* $\mathrm{FC}_n(a, b)$ *are cellular algebras, with cellular bases* $\mathcal{B}_n$.

Proof. (C1) We have $\mathcal{B}_n = \{C^{\lambda}_{S,T} \mid \lambda \in \Lambda \ \text{ and } \ S, T \in \mathcal{W}_n(\lambda)\}$ because each diagram D in $\mathcal{B}_n$ with label λ is a composition of the initial part with the final part of D, where the initial and the final parts of D has label λ.

(C2) By Lemma 1.3.13, we have $*$ is anti-isomorphism such that

$$(C^{\lambda}_{S,T})^* = C^{\lambda}_{T,S}.$$

(C3) Let $C^{\lambda}_{S,T}$ be a basis element of FC_n, and let $D = \sum \alpha_{S',T',\lambda'} C^{\lambda'}_{S',T'} \in \mathrm{FC}_n$ be a linear combination of basis elements of FC_n. Then

$$\begin{aligned} DC^{\lambda}_{S,T} &= \left(\sum \alpha_{S',T',\lambda'} C^{\lambda'}_{S',T'}\right) C^{\lambda}_{S,T} \\ &= \sum \alpha_{S',T',\lambda'} (C^{\lambda'}_{S',T'} C^{\lambda}_{S,T}). \end{aligned}$$

Suppose λ'' is the label of the diagram $C^{\lambda'}_{S',T'} C^{\lambda}_{S,T}$. Then, by Remark 1.2.6, we have $\lambda'' \leq \lambda$, and $\lambda'' \leq \lambda'$. Now, consider these cases:

Case I: If $\lambda' = \lambda$, and

(i) $\lambda'' = \lambda$, then the upper half diagram S' of $C^{\lambda'}_{S',T'}$ and the lower half diagram T of $C^{\lambda}_{S,T}$ are not changed in the product, but we may get some loops in the middle which give us some scalars. Thus we have $(C^{\lambda'}_{S',T'}C^{\lambda}_{S,T}) = \alpha_{T',S,\lambda}\ C^{\lambda}_{S',T}$ where $\alpha_{T',S,\lambda} \in \mathbb{C}$ depends on the half diagrams T', S and does not depend on T.

(ii) $\lambda'' < \lambda$, then $C^{\lambda'}_{S',T'}C^{\lambda}_{S,T} = \beta\ C^{\lambda''}_{S'',T''}$, where $C^{\lambda''}_{S'',T''}$ is a basis diagram obtained from the half diagrams in $\mathcal{W}(\lambda'')$, and $\beta \in \mathbb{C}$.

Case II: If $\lambda < \lambda'$, then $\lambda'' \leq \lambda$. This case is similar to Case I.

Case III: If $\lambda' < \lambda$, then $\lambda'' \leq \lambda'$. By the property of the partially ordered set, we have $\lambda'' < \lambda$, and this is similar to Case I(ii).

Case IV: If λ and λ' are incomparable. It is still true that $\lambda'' \leq \lambda$ and $\lambda'' \leq \lambda'$, and may argue in the same way as Case I and Case III.

Thus, from our cases above, we get

$$DC^{\lambda}_{S,T} = \sum \alpha_{T',S,\lambda}\ C^{\lambda}_{S',T} + \sum \beta\ C^{\lambda''}_{S'',T''},$$

where $\lambda'' < \lambda$.

Then FC_n is cellular algebra with cellular basis $\mathcal{B}_n$. □

Since FC_n are cellular algebras, then we have the following definition:

Definition 1.3.15. For each $\lambda \in \Lambda$, there is a cell module $\Delta_n(\lambda)$ with basis $\mathcal{W}_n(\lambda)$ such that for all $D \in \mathrm{FC}_n$ and $M \in \mathcal{W}_n(\lambda)$, the action is defined by

$$D \cdot M = \begin{cases} DM, & \text{if } \ l(DM) = l(M), \\ 0, & \text{otherwise.} \end{cases}$$

Consider V_λ and π_λ that defined in Definition 1.2.14. We can see that the basis diagrams for the cell module $\Delta_n(\lambda)$ are the same as the basis diagrams for the vector space V_λ. Furthermore, the action of the diagrams $D \in \mathrm{FC}_n$ on $\Delta_n(\lambda)$ is the same as the action of $\pi_\lambda(D)$ on V_λ, then we have

Proposition 1.3.16 ([4, Section 2])**.** *Suppose that $\Delta_n(\lambda)$ be a cell module with $\lambda = a^{l_1}b^{l_2}\cdots z^{l_p}$ where $z = a$ if n even and $z = b$ if n odd. The dimension of $\Delta_n(\lambda)$ is $\begin{bmatrix} n \\ \lambda \end{bmatrix}$, (defined in Definition 1.2.10), and is equal to:*

(i) *When $n = 2m$, $m \in \mathbb{N}$,*

$$\begin{bmatrix} 2m \\ \lambda \end{bmatrix} = \begin{cases} \frac{s}{3(m-r)+s}\binom{3(m-r)+s}{m-r} & \text{if } m \geq r \\ 0 & \text{if } m < r, \end{cases}$$

where $r = \frac{l-p+1}{2}$ and $s = \frac{3l-2p+4}{2}$.

(ii) *When $n = 2m+1$, $m \in \mathbb{N}$,*

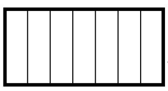
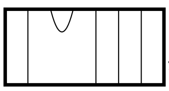
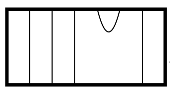
$$\begin{bmatrix} 2m+1 \\ \lambda \end{bmatrix} = \begin{cases} \frac{s}{3(m-r)+s}\binom{3(m-r)+s}{m-r} & \text{if } m \geq r \\ 0 & \text{if } m < r, \end{cases}$$

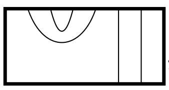
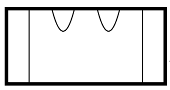
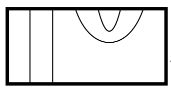
where $r = \frac{l-p}{2}$ and $s = \frac{3l-2p+4}{2}$.

We will finish this section with some examples.

Example 1.3.17. In this example we will construct the basis of FC_3 and its cell modules. Consider the algebra $\mathrm{FC}_3(a, b)$. Using Proposition 1.1.8, we have $\dim \mathrm{FC}_{2,3} = \frac{1}{7}\binom{9}{3} = 12$. The basis diagrams of FC_3 are diagrams that have six vertices on the top and the bottom lines with colouring *abbaab*.

Let

$S_1 =$ , $S_2 =$ , $S_3 =$ ,

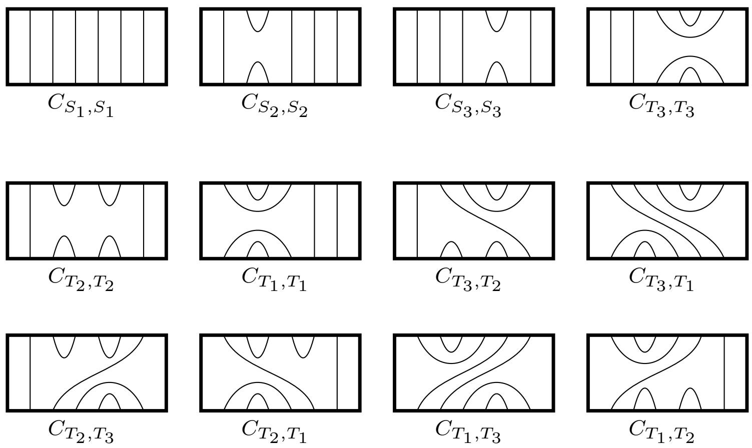
$T_1 =$ , $T_2 =$ , and $T_3 =$ .

The respective labels of these diagrams are $\lambda_1 = ab^2a^2b$, $\lambda_2 = a^3b$, $\lambda_3 = ab^3$, and of T_1, T_2 and T_3 is $\lambda_4 = ab$. For $i = 1, 2, 3$, we have $\mathcal{W}(\lambda_i) = \{S_i\}$, and $\mathcal{W}(\lambda_4) = \{T_1, T_2, T_3\}$.

Then the set of the basis elements of FC_3 is

$$\begin{aligned}\mathcal{B}_3 &= \{C^{\lambda}_{S,T} \mid \lambda \in \Lambda \text{ and } S,T \in \mathcal{W}(\lambda)\}\\ &= \{C_{S_1,S_1}, C_{S_2,S_2}, C_{S_3,S_3}, C_{T_1,T_1}, C_{T_1,T_2}, C_{T_1,T_3}, C_{T_2,T_1}, C_{T_2,T_2}, C_{T_2,T_3},\\ &\qquad C_{T_3,T_1}, C_{T_3,T_2}, C_{T_3,T_3}\}\end{aligned}$$

where the basis elements are as follows:



The set of all distinct labels is $\Lambda = \{ab, aaab, abbb, abbaab\}$.

The basis of the ideals A^{λ_1}, A^{λ_2}, A^{λ_3}, and A^{λ_4} are respectively:

$$\begin{aligned}A^{\lambda_1} &= \mathrm{Span}\{C_{S,T} \mid S,T \in \mathcal{W}(\mu) \text{ and } \mu < ab^2a^2b\}\\ &= \mathrm{Span}\{C_{S_2,S_2}, C_{S_3,S_3}, C_{T_1,T_1}, C_{T_1,T_2}, C_{T_1,T_3}, C_{T_2,T_1}, C_{T_2,T_2}, C_{T_2,T_3},\\ &\qquad C_{T_3,T_1}, C_{T_3,T_2}, C_{T_3,T_3}\}\end{aligned}$$

$$A^{\lambda_2} = \mathrm{Span}\{C_{S,T} \mid S,T \in \mathcal{W}(\mu) \text{ and } \mu < a^3b\},$$

$$A^{\lambda_3} = \mathrm{Span}\{C_{S,T} \mid S,T \in \mathcal{W}(\mu) \text{ and } \mu < ab^3\}.$$

Thus

$$\begin{aligned}A^{\lambda_2} = A^{\lambda_3} &= \mathrm{Span}\{C_{T_1,T_1}, C_{T_1,T_2}, C_{T_1,T_3}, C_{T_2,T_1}, C_{T_2,T_2},\\ &\qquad C_{T_2,T_3}, C_{T_3,T_1}, C_{T_3,T_2}, C_{T_3,T_3}\},\end{aligned}$$

$$A^{\lambda_4} = \mathrm{Span}\{C_{S,T} \mid S,T \in \mathcal{W}(\mu) \text{ and } \mu < ab\} = \emptyset.$$

Now, to find the cell modules of the algebra FC_3 we will use Definition 1.3.2. We have four cell modules:

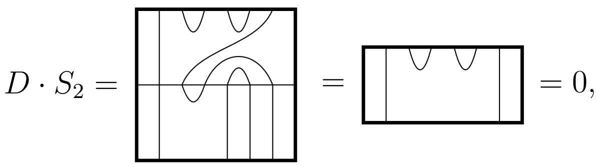
$$\Delta_3(\lambda_1) = \mathrm{Span}\{S_1\}, \quad \Delta_3(\lambda_2) = \mathrm{Span}\{S_2\}, \quad \Delta_3(\lambda_3) = \mathrm{Span}\{S_3\},$$

and

$$\Delta_3(ab) = \mathrm{Span}\{T_1, T_2, T_3\},$$

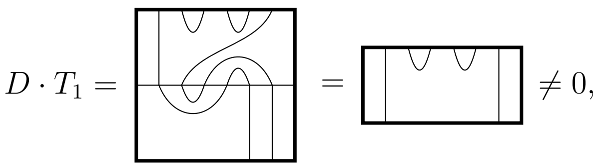
where the action is as defined in Definition 1.3.15.

To explain the action of FC_3 on the cell modules, let $D = C_{T_2,T_3} \in \mathrm{FC}_3$ and $S_2 \in \Delta_3(a^3b)$, then we have



because this diagram has two through strings and S_2 has four through strings. That is, $l(DS_2) \neq l(S_2)$.

Let $T_1 \in \Delta_3(ab)$, then



because $l(DT_1) = l(T_1)$.

Example 1.3.18. Consider the cell module $\Delta_3(ab)$ for the algebra FC_3, with the action that defined in Definition 1.3.15. We have $\Delta_3(ab) = \mathrm{Span}\{T_1, T_2, T_3\}$, where T_1, T_2, T_3 are as defined in Example 1.3.17.

Recall that the bilinear form Φ_λ is defined by the equation

$$C^\lambda_{s_1,t_1} C^\lambda_{s_2,t_2} \equiv \Phi_\lambda(C_{t_1}, C_{s_2}) C^\lambda_{s_1,t_2} \mod A^\lambda$$

where $s_1, s_2, t_1, t_2 \in \mathcal{W}(\lambda)$.

(i) For $i = 1, 2, 3$, let $C_{t_1} = C_{s_2} = T_i$ and $C_{s_1,t_2} = C_{T_i,T_i}$, then we have

$$C_{T_i,T_i}C_{T_i,T_i} = \Phi(T_i, T_i)C_{T_i,T_i}$$

Since, $C_{T_i,T_i}C_{T_i,T_i} = abC_{T_i,T_i}$ for all $i = 1, 2, 3$, then we get

$$\Phi(T_i, T_i) = ab.$$

(ii) Let $C_{t_1} = T_3$, $C_{s_2} = T_2$ and $C_{s_1,t_2} = C_{T_3,T_2}$, then we have

$$C_{T_3,T_3}C_{T_2,T_2} = \Phi(T_3, T_2)C_{T_3,T_2} = a\ C_{T_3,T_2},$$

and hence,

$$\Phi(T_3, T_2) = a.$$

By doing similar calculations we can show that

$$\Phi(T_3, T_1) = 1 \text{ and } \Phi(T_2, T_1) = b.$$

Now, we can find $\mathrm{rad}(ab)$, the radical of the cell module $\Delta_3(ab)$, which is defined by

$$\mathrm{rad}(\lambda) = \{x \in \Delta(\lambda) \mid \Phi_\lambda(x, y) = 0 \text{ for all } y \in \Delta(\lambda)\}.$$

For u, $v \in \Delta_3(ab)$, where

$$u = \alpha_1 T_1 + \alpha_2 T_2 + \alpha_3 T_3,$$
$$v = \beta_1 T_1 + \beta_2 T_2 + \beta_3 T_3,$$

suppose that u is in the radical of $\Delta_3(ab)$, thus, we have $\Phi(u, v) = 0$ for all $v \in \Delta_3(ab)$. Then we get

$$\Phi(\alpha_1 T_1 + \alpha_2 T_2 + \alpha_3 T_3,\ \beta_1 T_1 + \beta_2 T_2 + \beta_3 T_3) = 0.$$

By using the properties of the bilinear form, and substituting the values of $\Phi(T_i, T_j)$ for i, $j = 1$, 2, 3, we get

$$(\alpha_1 ab + \alpha_2 b + \alpha_3)\beta_1 + (\alpha_1 b + \alpha_2 ab + \alpha_3 a)\beta_2 + (\alpha_1 + \alpha_2 a + \alpha_3 ab)\beta_3 = 0.$$

Since u is in the radical this equation is true for all values of $\beta_1, \beta_2, \beta_3$. Thus we have the following system:

$$\begin{aligned} \alpha_1 ab + \alpha_2 b + \alpha_3 &= 0, \\ \alpha_1 b + \alpha_2 ab + \alpha_3 a &= 0, \\ \alpha_1 + \alpha_2 a + \alpha_3 ab &= 0. \end{aligned}$$

We can write this system of equations in matrix form

$$\begin{bmatrix} ab & b & 1 \\ b & ab & a \\ 1 & a & ab \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 0.$$

The values of α_1, α_2, α_3 can only be non-zero when

$$\det \begin{bmatrix} ab & a & 1 \\ a & ab & b \\ 1 & b & ab \end{bmatrix} = 0.$$

That is, when $ab(a^2 - 1)(b^2 - 1) = 0$.

Therefore, if a, $b \neq 0$, ± 1 then $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Thus we conclude that $u = 0$ and so $\mathrm{rad}(ab) = 0$. Hence, $\Delta_3(ab)$ is irreducible for all a, $b \in \mathbb{C} \setminus \{0, 1, -1\}$.

1.4 Properties of the Labels

In this section we give some properties related to $\lambda \in \Lambda(n)$.

Definition 1.4.1. Let $\lambda = a^{l_1} b^{l_2} \cdots z^{l_p}$ be a label, where $z = a$ if n is even and $z = b$ if n is odd, and $l_i > 0$ for all $1 \leq i \leq p$. Then we call p the number of *parts* of λ.

Remark 1.4.2. Let $\lambda = a^{l_1} b^{l_2} \cdots z^{l_p}$ be a label, such that $l_i > 0$ for all $1 \leq i \leq p$. If n is even then we have

(i) $z = a$ (by Lemma 1.2.5).

(ii) $\emptyset \in \Lambda(n)$ (by Lemma 1.2.5).

(iii) p is odd and $p \geq 1$, since $\lambda = a^{l_1} b^{l_2} \cdots a^{l_p}$.

(iv) $l_1 = l$ is even if $p = 1$. Otherwise, l_1 and l_p are odd, and l_i is even for all $1 < i < p$. This is from the definition of the Fuss-Catalan diagrams.

If n is odd then we have

(i) $z = b$ (by Lemma 1.2.5).

(ii) $\emptyset \notin \Lambda(n)$ (by Lemma 1.2.5).

(iii) p is even and $p \geq 2$, since $\lambda = a^{l_1} b^{l_2} a^{l_3} \cdots b^{l_p}$.

(iv) l_1 and l_p are odd, and l_i is even for all $1 < i < p$. This is from the definition of the Fuss-Catalan diagrams.

Definition 1.4.3. Let $d_n(p, l)$ be the number of the labels $\lambda = a^{l_1} b^{l_2} \cdots z^{l_p} \in \Lambda(n)$ such that λ has length l and exactly p-parts. If $\lambda = \emptyset$ then we consider that $p = 1$ and $l = 0$.

The initial values of $d_n(p, l)$ are $d_1(2, 2) = d_2(1, 2) = 1$ and $d_n(1, 0) = 1$ while $d_n(p, 0) = 0$ for all $p > 1$ and $d_n(0, l) = 0$ for all $l \geq 0$.

Proposition 1.4.4. *For $n \geq 3$, we have*

$$d_n(p, l) = \binom{\frac{l}{2}}{p-1},$$

for all $l = 2p - 2,\ 2p,\ \ldots,\ n + p - 1$, and $p = 1,\ 3,\ \ldots,\ n + 1$ if n is even and $p = 2,\ 4,\ \ldots,\ n + 1$ if n is odd.

Proof. We start by finding the lower and the upper values of p and l. The lower bound of p is 1 if n is even (when $\lambda = \emptyset$), and 2 if n is odd (when $\lambda = ab$).

The upper bound of p occurs when λ is the identity element where all strings are through strings. Then λ has the form $\lambda = (ab^2a)^{\frac{n}{2}}$ if n even, and $\lambda = (ab^2a)^{\frac{n-1}{2}}ab$ if n odd. Thus the number of parts of λ is $2(\frac{n}{2}) + 1 = n + 1$ if n even, and it is $(2(\frac{n-1}{2}) + 1) + 1 = n + 1$ if n odd.

For fixed p, the lower bound of $l(\lambda)$, where λ has exactly p-parts, happens when λ has the form $\lambda = ab^2a^2 \ldots b^2a$, (resp. $\lambda = ab^2a^2 \ldots a^2b$) if n even, (resp. if n odd). For the even and the odd case we get $l(\lambda) = 2(p-2) + 2 = 2p - 2$. From Proposition 1.3.16, to get $\begin{bmatrix} n \\ \lambda \end{bmatrix} > 0$, we have $n \geq 2r = l - p + 1$ if n is even and $n - 1 \geq 2r = l - p$ if n is odd. Then, for both cases, $l \leq n + p - 1$.

Now, to find $d_n(p, l)$, we have three cases:

Case I: If $p = 1$, then λ has the form $\lambda = \emptyset$ or $\lambda = a^l$, and for every even l such that $0 \leq l \leq n$, we have a distinct label, hence $d_n(1, l) = 1 = \binom{l/2}{0}$.

Case II: If $p = 2$, then $\lambda = a^{l_1}b^{l_2}$ where $l = l_1 + l_2$ such that l_1 and l_2 are odd. The number of distinct words $\lambda = a^{l_1}b^{l_2}$ is equivalent to finding the number of integer solutions of the equation $l = l_1 + l_2$ satisfying the conditions that l_1 and l_2 are odd. This is equivalent to finding the coefficient of x^l in E, where

$$\begin{aligned} E &= (x + x^3 + x^5 + \cdots)(x + x^3 + x^5 + \cdots) = x^2(1 + x^2 + x^4 + \cdots)^2 \\ &= x^2\frac{1}{(1-x^2)^2} = x^2\sum_{i=0}^{\infty}\binom{i+1}{1}x^{2i} = \sum_{i=0}^{\infty}\binom{i+1}{1}x^{2i+2} \end{aligned}$$

Let $l = 2i + 2$, then $i + 1 = \frac{l}{2}$, hence the coefficient of x^l in E is $\binom{l/2}{1}$. Thus, $d_n(2, l) = \binom{l/2}{1}$ for every even l such that $2 \leq l \leq n + 1$.

Case III: If $p > 2$, then $\lambda = a^{l_1}b^{l_2}\cdots z^{l_p}$ where $z = a$ if n is even and $z = b$ if n is odd. In this case, we need to find the number of integer solutions of the equation $l = l_1 + l_2 + \cdots + l_p$ where l_1 and l_p are odd and $l_i > 0$ are even for all $1 < i < p$. This is equivalent to finding the coefficient of x^l in E, where

$$\begin{aligned} E &= (x + x^3 + x^5 + \cdots)(x^2 + x^4 + x^6 + \cdots)^{p-2}(x + x^3 + x^5 + \cdots) \\ &= x^2(1 + x^2 + x^4 + \cdots)^2x^{2p-4}(1 + x^2 + x^4 + \cdots)^{p-2} \\ &= x^{2p-2}(\frac{1}{1-x^2})^p = x^{2p-2}\sum_{i=0}^{\infty}\binom{i+p-1}{p-1}x^{2i} = \sum_{i=0}^{\infty}\binom{i+p-1}{p-1}x^{2i+2p-2} \end{aligned}$$

Let $l = 2i + 2p - 2$, then $i + p - 1 = \frac{l}{2}$, hence the coefficient of x^l in E is $\binom{l/2}{p-1}$. $\square$

Proposition 1.4.5. *Let $p \geq 1$, $l \geq 2$ and $n \geq 3$ then we have*

$$d_n(p,l) = d_{n-1}(p-1, l-2) + d_{n-2}(p, l-2).$$

Proof. We have

$$\begin{aligned} d_{n-1}(p-1,l-2) + d_{n-2}(p,l-2) &= \binom{\frac{1}{2}(l-2)}{p-2} + \binom{\frac{1}{2}(l-2)}{p-1} \\ &= \binom{l/2}{p-1} = d_n(p,l). \end{aligned}$$

□

Definition 1.4.6. We define the set E_i^j to be the set of all even numbers in the interval $[i, j]$, and the set $O_{i'}^{j'}$ to be the set of all odd numbers in the interval $[i', j']$, where i, j, i', $j' \in \mathbb{Z}$.

Proposition 1.4.7. *Let $n \geq 3$. The order of the set $\Lambda(n)$ is*

$$|\Lambda(n)| = \sum_p \binom{\frac{1}{2}(n+p+1)}{p},$$

where p runs over O_1^{n+1} if n is even, and p runs over E_2^{n+1} if n is odd.

Proof. Since, $d_n(p,l)$ is the number of labels with length l and have p-parts. Thus

$$|\Lambda(n)| = \sum_p \sum_{l \in E_{2p-2}^{n+p-1}} d_n(p,l)$$

where $p \in O_1^{n+1}$ if n is even and $p \in E_2^{n+1}$ if n is odd. Let

$$R = \sum_{l \in E_{2p-2}^{n+p-1}} d_n(p,l) = \sum_{l \in E_{2p-2}^{n+p-1}} \binom{l/2}{p-1}.$$

Set $h = l/2$. When $l = 2p-2$ then $h = p-1$, when $l = n+p-1$ then $h = \frac{1}{2}(n+p-1)$. We can rewrite R as the following

$$R = \sum_{h=p-1}^{\frac{1}{2}(n+p-1)} \binom{h}{p-1}.$$

By using the fact that $\sum_{j=k}^{q} \binom{j}{k} = \binom{q+1}{k+1}$, we get $R = \binom{\frac{1}{2}(n+p+1)}{p}$. □

Proposition 1.4.8. *Consider the sets $\Lambda(n)$, $\Lambda(n-1)$ and $\Lambda(n-2)$ where $n \geq 3$. Then $|\Lambda(n)| = |\Lambda(n-1)| + |\Lambda(n-2)| + s$, where $s = 1$ if n is even and $s = 0$ if n is odd.*

Proof. If n is even, we have

$$|\Lambda(n-1)| = \sum_{p \in E_2^n} \binom{\frac{1}{2}(n+p)}{p} = \binom{\frac{n}{2}+1}{2} + \binom{\frac{n}{2}+2}{4} + \cdots + \binom{n-1}{n-2} + \binom{n}{n},$$

and

$$|\Lambda(n-2)| = \sum_{p \in O_1^{n-1}} \binom{\frac{1}{2}(n+p-1)}{p} = \binom{\frac{n}{2}}{1} + \binom{\frac{n}{2}+1}{3} + \cdots + \binom{n-1}{n-1}.$$

So, we have

$$\begin{aligned}
&|\Lambda(n-1)| + |\Lambda(n-2)| \\
&\quad = \binom{\frac{n}{2}}{1} + \left[\binom{\frac{n}{2}+1}{2} + \binom{\frac{n}{2}+1}{3}\right] + \left[\binom{\frac{n}{2}+2}{4} + \binom{\frac{n}{2}+2}{5}\right] \\
&\qquad\qquad + \cdots + \left[\binom{n-1}{n-2} + \binom{n-1}{n-1}\right] + \binom{n}{n} \\
&\quad = \binom{\frac{n}{2}}{1} + \binom{n}{n} + \binom{\frac{n}{2}+2}{3} + \binom{\frac{n}{2}+3}{5} + \cdots + \binom{n}{n-1} \\
&\quad = \binom{\frac{n}{2}+1}{1} + \sum_{p \in O_3^{n-1}} \binom{\frac{1}{2}(n+p+1)}{p} \\
&\quad = \sum_{p \in O_1^{n-1}} \binom{\frac{1}{2}(n+p+1)}{p}.
\end{aligned}$$

Hence, $1 + \sum_{p \in O_1^{n-1}} \binom{\frac{1}{2}(n+p+1)}{p} = \sum_{p \in O_1^{n+1}} \binom{\frac{1}{2}(n+p+1)}{p} = |\Lambda(n)|.$

Now, if n is odd, we have

$$|\Lambda(n-1)| = \sum_{p \in O_1^n} \binom{\frac{1}{2}(n+p)}{p}, \text{ and } |\Lambda(n-2)| = \sum_{p \in E_2^{n-1}} \binom{\frac{1}{2}(n+p-1)}{p}.$$

So, we have

$$
\begin{aligned}
|\Lambda(n-1)| + |\Lambda(n-2)| \\
&= \left[\binom{\frac{1}{2}(n+1)}{1} + \binom{\frac{1}{2}(n+1)}{2}\right] + \left[\binom{\frac{1}{2}(n+3)}{3} + \binom{\frac{1}{2}(n+3)}{4}\right] \\
&\qquad + \cdots + \left[\binom{n-1}{n-2} + \binom{n-1}{n-1}\right] + \binom{n}{n} \\
&= \binom{n}{n} + \sum_{p \in E_2^{n-1}} \binom{\frac{1}{2}(n+p+1)}{p} \\
&= \sum_{p \in E_2^{n+1}} \binom{\frac{1}{2}(n+p+1)}{p} = |\Lambda(n)|. \qquad \square
\end{aligned}
$$

Remark 1.4.9. Any basis diagram D of FC_{n-1} is a basis diagram of FC_n by adding two through strings to the right end of D and the additional strings are labelled with ab if n is odd and ba if n is even. Thus if $\lambda \in \Lambda(n-1)$ then $\lambda ab \in \Lambda(n)$ if n odd and $\lambda ba \in \Lambda(n)$ if n even. Furthermore, if D is a basis diagram of FC_{n-2} with label $\lambda \in \Lambda(n-2)$ then by adding the sub diagram

$D' =$ 

to the right end of D we get a basis diagram of FC_n with label λa^2 (resp. λb^2) if n is even (resp. odd), since D' has label a^2 (resp. b^2) if n is even (resp. odd). Let $z \in \{aa, ab, ba, bb\}$, and suppose that $\Lambda(i)z = \{\lambda z \mid \lambda \in \Lambda(i)\}$. Notice that, there is no intersection between the sets $\Lambda(n-2)a^2$ and $\Lambda(n-1)ba$. In addition, the label $\lambda = \emptyset$ does not exist in these two sets. Then by using Proposition 1.4.8, if n is even, we have that

$$\Lambda(n) = \Lambda(n-1)ba \ \cup \Lambda(n-2)aa \ \cup \{\lambda = \emptyset\}.$$

In a similar way we can show that if n is odd we have

$$\Lambda(n) = \Lambda(n-1)ab \ \cup \Lambda(n-2)bb.$$

We can find all possible labels of the cell modules, $\Delta_n(\lambda)$, inductively by using Remark 1.4.9. Alternatively, we can find them by using Proposition 1.4.8 as explained in the following example.

Example 1.4.10. Let $n = 5$. To find the elements of the set $\Lambda(5)$, we use Proposition 1.4.8. Since n is odd, then $2 \leq p \leq n+1$, and $2p-2 \leq l \leq 5+p-1$. Recall that $d_n(p,l) = \binom{l/2}{p-1}$.

When $p = 2$, the values of l are $\{2,4,6\}$. Any label with two parts has the form $\lambda = a^{l_1}b^{l_2}$ such that l_1, l_2 are odd, and $l = l_1 + l_2$. Now, when $l = 2$, we have $d_5(2,2) = 1$. That is, we have one label with length 2 and 2-parts. It is clear that $l_1 = l_2 = 1$ is the only solution. Hence, $\lambda_1 = ab$.

When $l = 4$, we have $d_5(2,4) = 2$, then we have two labels with length 4 and 2-parts. The solutions of $l_1 + l_2 = 4$ are $(l_1, l_2) = (1,3)$ and $(l_1, l_2) = (3,1)$, thus $\lambda_2 = ab^3$ and $\lambda_3 = a^3b$.

When $l = 6$, we have $d_5(2,6) = 3$, then we have three labels with length 6 and 2-parts. The solutions of $l_1 + l_2 = 6$ are $(l_1, l_2) \in \{(1,5),(3,3),(5,1)\}$, thus $\lambda_4 = ab^5$, $\lambda_5 = a^3b^3$, and $\lambda_6 = a^5b$.

Now, we move to the case $p = 4$, we have $l \in \{6,8\}$, and $\lambda = a^{l_1}b^{l_2}a^{l_3}b^{l_4}$. When $l = 6$, we have $d_5(4,6) = 1$, then we have one solution to the equation $l_1 + l_2 + l_3 + l_4 = 6$ such that l_1, l_4 are odd and l_2, l_3 are even. The only solution is $(1,2,2,1)$. Hence, $\lambda_7 = ab^2a^2b$. When $l = 8$, we have $d_5(4,8) = 4$, then we have four solutions to the equation $l_1 + l_2 + l_3 + l_4 = 8$ such that l_1, l_4 are odd and l_2, l_3 are even. The only solutions are $\{(1,2,4,1),(1,4,2,1),(1,2,2,3),(3,2,2,1)\}$. Hence,

$$\lambda_8 = ab^2a^4b, \qquad \lambda_9 = ab^4a^2b,$$
$$\lambda_{10} = ab^2a^2b^3, \qquad \lambda_{11} = a^3b^2a^2b.$$

Finally, when $p = 6$, we have $l = 10$, and $d_5(6,10) = 1$. In fact, the maximum value of p will give us the label of the identity diagram which is $\lambda_{12} = ab^2a^2b^2a^2b$. Thus we have the set $\Lambda(5)$ below.

$$\{ab, ab^3, a^3b, ab^5, a^5b, a^3b^3, ab^2a^2b, ab^2a^4b, ab^4a^2b, ab^2a^2b^3, a^3b^2a^2b, ab^2a^2b^2a^2b\}.$$

Proposition 1.4.11. *Let $x = ba$, and $y = aa$ if n is even, while $x = ab$, and $y = bb$ if n is odd.*

(i) *If $\Delta_{n-1}(\lambda)$ is one dimensional, then $\Delta_n(\lambda x)$ is one dimensional. And if $\Delta_{n-1}(\lambda)$ has dimension greater than one, then $\Delta_n(\lambda x)$ has dimension greater than one.*

(ii) *If $\Delta_{n-2}(\lambda)$ is one dimensional, then $\Delta_n(\lambda y)$ is one dimensional. And if $\Delta_{n-2}(\lambda)$ has dimension greater than one, then $\Delta_n(\lambda y)$ has dimension greater than one.*

Proof. We use Proposition 1.3.16, to count the dimension for each case.

Recall that, the dimension of $\Delta_n(\lambda)$ is

$$\begin{bmatrix} n \\ \lambda \end{bmatrix} = \begin{cases} \frac{s}{3(m-r)+s}\binom{3(m-r)+s}{m-r} & \text{if} \quad m \geq r \\ 0 & \text{if} \quad m < r \end{cases}$$

where $s = \frac{3l-2p+4}{2}$, and $r = \frac{l-p+1}{2}$ if $n = 2m$, while $r = \frac{l-p}{2}$ if $n = 2m+1$. In addition, l, p are the length and the number of parts of λ respectively. We can see that if $m = r$ then $\begin{bmatrix} n \\ \lambda \end{bmatrix} = 1$, but if $m > r$ then $\begin{bmatrix} n \\ \lambda \end{bmatrix} > 1$.

Let $r(\mu), p(\mu)$ be the values of r, p related to any label μ.

(i) Suppose $n = 2m+1$, then $n-1 = 2m$. In this case, we have

$$\begin{bmatrix} n-1 \\ \lambda \end{bmatrix} = \begin{bmatrix} 2m \\ \lambda \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} n \\ \lambda ab \end{bmatrix} = \begin{bmatrix} 2m+1 \\ \lambda ab \end{bmatrix}.$$

If $\begin{bmatrix} 2m \\ \lambda \end{bmatrix} = 1$, then $m = r(\lambda)$, and if $\begin{bmatrix} 2m \\ \lambda \end{bmatrix} > 1$, then $m > r(\lambda)$.

Now, $l(\lambda ab) = l(\lambda) + 2$ because we add only two letters to λ, moreover, we have $p(\lambda ab) = p(\lambda) + 1$ because $(n-1)$ is even, then λ ends with the letter a so that when we add ab to the end of λ we add only one new part. So,

$$\begin{aligned} r(\lambda ab) &= \frac{1}{2}\Big(l(\lambda ab) - p(\lambda ab)\Big) \\ &= \frac{1}{2}\Big(l(\lambda) + 2 - \big(p(\lambda) + 1\big)\Big) \\ &= \frac{1}{2}\Big(l(\lambda) - p(\lambda) + 1\Big) \\ &= r(\lambda). \end{aligned}$$

Therefore, when $m = r(\lambda)$, we get $m = r(\lambda ab)$, hence, $\begin{bmatrix} n \\ \lambda ab \end{bmatrix} = 1$.

In addition, when $m > r(\lambda)$, we get $m > r(\lambda ab)$, hence, $\begin{bmatrix} n \\ \lambda ab \end{bmatrix} > 1$.

By same argument we can prove the even case, $n = 2m$.

(ii) Suppose $n = 2m$, then $n - 2 = 2(m-1)$.

In this case $\begin{bmatrix} n-2 \\ \lambda \end{bmatrix} = \begin{bmatrix} 2(m-1) \\ \lambda \end{bmatrix}$ and $\begin{bmatrix} n \\ \lambda aa \end{bmatrix} = \begin{bmatrix} 2m \\ \lambda aa \end{bmatrix}$.

Now, we have $l(\lambda aa) = l(\lambda) + 2$, and $p(\lambda aa) = p(\lambda)$. This is because $n - 2$ is even which means that λ ends with the letter a, so that when we add aa to the right end of λ there is no change to the number of parts of λaa. So, we have

$$\begin{aligned} r(\lambda aa) &= \frac{1}{2}\Big(l(\lambda aa) - p(\lambda aa) + 1\Big) \\ &= \frac{1}{2}\Big(l(\lambda) - p(\lambda) + 1\Big) + 1 \\ &= r(\lambda) + 1. \end{aligned}$$

Therefore, when $\begin{bmatrix} n-2 \\ \lambda \end{bmatrix} = \begin{bmatrix} 2(m-1) \\ \lambda \end{bmatrix} = 1$, then $m - 1 = r(\lambda)$, that is, $m = r(\lambda) + 1$. This implies $m = r(\lambda aa)$, hence, $\begin{bmatrix} n \\ \lambda aa \end{bmatrix} = 1$.

Furthermore, when $\begin{bmatrix} n-2 \\ \lambda \end{bmatrix} = \begin{bmatrix} 2(m-1) \\ \lambda \end{bmatrix} > 1$, then $m - 1 > r(\lambda)$, that is, $m > r(\lambda) + 1$. This implies $m > r(\lambda aa)$, hence, $\begin{bmatrix} n \\ \lambda aa \end{bmatrix} > 1$.

We can prove the odd case, $n = 2m + 1$, in a similar way.

$\square$

Chapter 2

Towers of recollement

In 2006, Cox, Martin, Parker and Xi [12], introduced an axiomatic framework for studying the representation theory of towers of algebras, and each family of algebras that satisfy these axioms will be called a tower of recollement. To show the utility of this framework they defined a new family of algebras called contour algebras and proved that it is a tower of recollement [3, Section 2]. There are interesting algebras that are tower of recollements. For instance, the Temperley-Lieb algebras [28], cyclotomic Temperley-Lieb algebras [39], the blob algebra [32], the Partition algebra [5, 21, 29, 30], and the Brauer algebra [6, 7, 36, 37].

In this chapter we are going to discuss and apply these axioms to our algebras, the Fuss-Catalan algebras, to prove that FC_n are a tower of recollement. This axiomatic framework will help us to classify the simple FC_n-modules inductively, and to determine which of the algebras in the family are semi-simple.

2.1 Axioms for towers of recollement

Let A_n with $n \geq 0$ be a family of finite-dimensional algebras over an algebraically closed field, with idempotents e_n in A_n. Let us first introduce some definitions before we state the axioms for a tower of recollement.

Definition 2.1.1. Let Λ_n be the label set for the simple A_n-modules, and Λ^n be the label set for the simple $A_n/A_ne_nA_n$-modules.

Let e be an idempotent of an algebra A, then we have this theorem.

Theorem 2.1.2 ([20, Theorem 6.2g])**.** *Let $\{L(\lambda) \mid \lambda \in \Lambda\}$ be a full set of simple A-modules, and set $\Lambda^e = \{\lambda \in \Lambda \mid eL(\lambda) \neq 0\}$. Then $\{eL(\lambda) \mid \lambda \in \Lambda^e\}$ is a full set of simple eAe-modules. Further, the simple modules $L(\lambda)$ with $\lambda \in \Lambda \setminus \Lambda^e$ are a full set of simple A/AeA-modules.*

We apply Definition 2.1.1 to Theorem 2.1.2 by setting $A = A_n$, $e = e_n$ and $\Lambda = \Lambda_n$ then we have $\Lambda^e = \Lambda_{n-2}$ and $\Lambda^n = \Lambda_n \setminus \Lambda_{n-2}$.

Definition 2.1.3. For $m, n \in \mathbb{N}$ with $n - m \geq 0$ even, we define Λ_n^m to be the set of labels in Λ_n that first appeared in the labelling set Λ^m. If $n - m < 0$ then $\Lambda_n^m = \emptyset$.

Example 2.1.4. Consider the algebra FC_4, then the labelling sets $\Lambda_4^4 = \Lambda^4$ and Λ_4^2 are given by

$$\Lambda_4^4 = \Lambda^4 = \{ab^2a^2b^2a, a^3b^2a, ab^4a, ab^2a^3, a^4\}$$

$$\Lambda_4^2 = \Lambda^2 = \{ab^2a, a^2, \emptyset\}.$$

We will see later that the Fuss-Catalan algebras are quasi-hereditary algebras, thus we introduce this definition.

Definition 2.1.5 ([43, Definition 1.4])**.** Let A be a k-algebra. An ideal J in A is called an *heredity ideal* if J is idempotent, $J(\operatorname{rad} A)J = 0$ and J is a projective left (or, right) A-module. The algebra A is called *quasi-hereditary* provided there is a finite chain

$$0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A$$

of ideals in A such that J_j/J_{j-1} is a heredity ideal in A/J_{j-1} for all j. Such a chain is then called a *heredity chain* of the quasi-hereditary algebra A.

Now we are ready to present the axioms of a tower of recollement as stated in [12, Section 1]:

(A1) For each $n \geq 2$ we have an isomorphism $\psi_n : A_{n-2} \to e_n A_n e_n$.

We define the localisation functors

$$\mathcal{F}_n : A_n\text{-mod} \to A_{n-2}\text{-mod} \quad \text{by} \quad \mathcal{F}_n(M) = e_n M,$$

and the globalisation functors

$$\mathcal{G}_n : A_n\text{-mod} \to A_{n+2}\text{-mod} \quad \text{by} \quad \mathcal{G}_{n-2}(N) = A_n e_n \otimes_{e_n A_n e_n} N,$$

where the isomorphism in (A1) have been used in each case. Note that $\mathcal{G}_{n-2}$ is the right inverse to $\mathcal{F}_n$.

From Theorem 2.1.2 and axiom (A1), we have

$$\Lambda_n = \Lambda^n \sqcup \Lambda_{n-2}. \tag{2.1}$$

(A2) (i) The algebra $A_n/A_n e_n A_n$ is semisimple.

(ii) Set $e_{n,0} = 1$ in A_n, and for $1 \leq i \leq n/2$ define new idempotents in A_n by setting $e_{n,i} = \psi_n(e_{n-2,i-1})$. Set $A_{n,i} = A_n/(A_n e_{n,i+1} A_n)$. For each $n \geq 0$ and $0 \leq i \leq n/2$, setting $e = e_{n,i}$ and $A = A_{n,i}$, the surjective multiplication map $Ae \otimes_{eAe} eA \to AeA$ is a bijection.

By using [15, Statement 7] or ([34, Definition 3.3.1] and the remarks after it), the axiom (A2) can be replaced by

(A2') For each $n \geq 0$ the algebra A_n is quasi-hereditary, with heredity chain of the form

$$0 \subset \cdots \subset A_n e_{n,i} A_n \subset \cdots \subset A_n e_{n,0} A_n = A_n.$$

When A_n is quasi-hereditary algebra, we have that for each label $\lambda \in \Lambda_n$ a standard module $\Delta_n(\lambda)$ with simple head $L_n(\lambda)$. For $\lambda \in \Lambda_n^n$ we have $\Delta_n(\lambda) \cong L_n(\lambda)$, and that this is just the lift of a simple module for the quotient algebra $A_n/A_n e_n A_n$. From [31, Proposition 3], we have

$$G_n(\Delta_n(\lambda)) \cong \Delta_{n+2}(\lambda). \tag{2.2}$$

In similar way (see for example [17, A1]) we have

$$\mathcal{F}_n(\Delta_n(\lambda)) \cong \begin{cases} \Delta_{n-2}(\lambda) & \text{if } \lambda \in \Lambda_{n-2}, \\ 0 & \text{if } \lambda \in \Lambda^n. \end{cases} \tag{2.3}$$

(A3) For each $n \geq 0$ the algebra A_n can be identified with a subalgebra of A_{n+1}.

(A4) For all $n \geq 1$ we have $A_n e_n \cong A_{n-1}$ as a left A_{n-1}-, right A_{n-2}-bimodule.

Before we state axiom A5, we need these two definitions.

Definition 2.1.6. A Δ-filtration is defined to be a filtration of a module such that successive quotients isomorphic to $\Delta(\mu)$ for some $\mu \in \Lambda$.

Definition 2.1.7. If a module M in A_n-mod has a Δ-filtration then the set of labels λ for which $\Delta(\lambda)$ occurs in this filtration defined to be the *support* of M, and is denoted by $\text{supp}_n(M)$.

(A5) For each $\lambda \in \Lambda_n^m$ we have that $\text{res}(\Delta_n(\lambda))$ has a Δ-filtration and

$$\text{supp}\left(\text{res}(\Delta_n(\lambda))\right) \subseteq \Lambda_{n-1}^{m-1} \sqcup \Lambda_{n-1}^{m+1}.$$

(A6) For each $\lambda \in \Lambda_n^n$ there exist $\mu \in \Lambda_{n+1}^{n-1}$ such that $\lambda \in \text{supp}\left(\text{res}(\Delta_{n+1}(\mu)\right)$.

The axioms (A1) to (A6) will reduce the study of finding a non-zero homomorphism between two cell modules to the case when one is simple, as illustrated by the following theorem.

Theorem 2.1.8 ([12, Theorem 1.1])**.** (i) *For all pairs of weights $\lambda \in \Lambda_n^m$ and $\mu \in \Lambda_n^l$ we have*

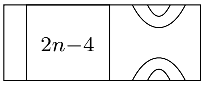
$$\text{Hom}(\Delta_n(\lambda), \Delta_n(\mu)) \cong \begin{cases} \text{Hom}(\Delta_m(\lambda), \Delta_m(\mu)) & \text{if } l \leq m, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) *Suppose that for all $n \geq 0$ and pairs of weights $\lambda \in \Lambda_n^n$ and $\mu \in \Lambda_n^{n-2}$ we have* $\text{Hom}(\Delta_n(\lambda), \Delta_n(\mu)) = 0$ *then each of the algebras A_n is semisimple.*

2.2 Proof of axioms A1 to A4

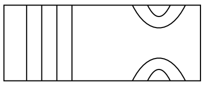
In this section we are going to prove that FC_n satisfies the axioms A1 to A4. The axioms A5 and A6 will be discussed in the next section.

Definition 2.2.1. For $n \geq 3$, we define $e_n \in \mathrm{FC}_n$ to be the diagram

$$e_n = \frac{1}{ab}$$ 

where $2n-4$ in the diagram e_n represents $2n-4$ consecutive through strings.

For example,

$$e_4 = \frac{1}{ab}$$ 

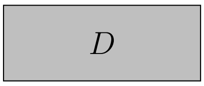
It is easy to see that e_n is an idempotent of FC_n by checking that $e_n^2 = e_n$.

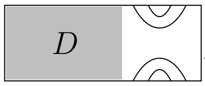
Proposition 2.2.2. *For $n \geq 3$, we have*

$$\mathrm{FC}_{n-2} \cong e_n \mathrm{FC}_n e_n$$

so FC_n satisfies axiom (A1).

Proof. For all basis element $D \in \mathrm{FC}_{n-2}$, we define $\psi_n : \mathrm{FC}_{n-2} \to e_n \mathrm{FC}_n e_n$ by

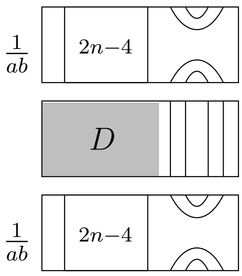


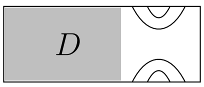
$\mapsto \frac{1}{ab}$ 

.

We will show that ψ_n is a bijection by proving that FC_{n-2} and $e_n \mathrm{FC}_n e_n$ have the same dimension.

Consider the set $\mathcal{B}_n^4 \subset \mathcal{B}_n$, the set of all basis elements of FC_n obtained by adding four consecutive through strings to the right end of each basis element of FC_{n-2}. For all $v \in \mathcal{B}_n^4$ we have

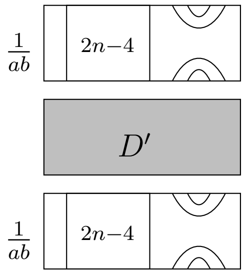
$e_n v e_n =$ 

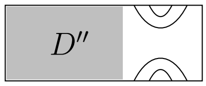
$= \frac{1}{ab}$ 

Recall that D is a basis element in FC_{n-2}, therefore,

$$\dim(e_n \mathrm{FC}_n e_n) \geq \dim(\mathrm{FC}_{n-2}).$$

On the other hand, for all $D' \in \mathcal{B}_n$ with $D' \notin \mathcal{B}_n^4$, we have

$e_n D' e_n =$ 

$= \delta$ 

where D'' is a diagram with $2n-4$ vertices on each line and $\delta \in \mathbb{C}$. We can see that there is no connection between the diagrams D'' and the four non-through strings at the right end of the diagram, . So D'' must be in $\mathcal{B}_{n-2}$, that is, D'' is a basis element in FC_{n-2}. Thus,

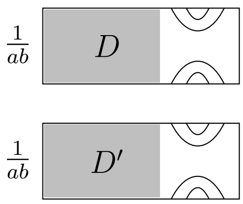
$$\dim(e_n \mathrm{FC}_n e_n) \leq \dim(\mathrm{FC}_{n-2})$$

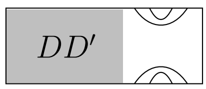
and hence,

$$\dim(e_n \mathrm{FC}_n e_n) = \dim(\mathrm{FC}_{n-2}).$$

Once we know that ψ_n is a homomorphism this will imply that ψ_n is bijective since it sends a basis element in FC_{n-2} to a distinct basis element in $e_n \mathrm{FC}_n e_n$.

Now, to show that ψ_n is a homomorphism. Let D and D' are basis elements of FC_{n-2}, then

$\psi_n(D)\psi_n(D') =$ 

$= \dfrac{1}{ab}$ 

$= \psi_n(DD').$

Therefore, ψ_n is an isomorphism. $\square$

We now move onto the proof of axiom (A2)(i). To prove that the algebra $\mathrm{FC}_n/\mathrm{FC}_n e_n \mathrm{FC}_n$ is semisimple, we recall certain results from [3] that we shall require.

Proposition 2.2.3. *Let $\Delta_n(\lambda)$ be a cell module for the algebra FC_n. Then the following are equivalent* *(i)* $\dim \Delta_n(\lambda) = 1$.

(ii) No basis diagram for $\Delta_n(\lambda)$ *has two consecutive non-through strings.*

(iii) $\Delta_{n-2}(\lambda) = 0$.

Where we mean by two consecutive non-through strings the patterns $\cup\ \cup$ *and*  .

Proof. See [3, Proposition 3.2.5]. □

Definition 2.2.4. [3, Definition 3.2.6] Let $\Delta_n(\lambda)$ be a cell module for the algebra FC_n. We say that λ is a *new label at level* n if one of the equivalent conditions of Proposition 2.2.3 are satisfied.

Remark 2.2.5. Since the dimension of $\Delta_n(\lambda)$ is one when λ is a new label at level n, then there is only one basis diagram D with label λ. That is, D is completely determined by its label. Furthermore, Definition 2.1.3 is equivalent to Definition 2.2.4, thus Λ_n^n is the set of all new label at level n, and hence, for all $\lambda \in \Lambda_n^n$ we have $\dim \Delta_n(\lambda) = 1$.

Recall that ${}_1U_i$ is the generator of FC_n that is defined in Proposition 1.1.9.

Definition 2.2.6. Define $\mathcal{J}_n = \{x \mid x = p_{j_1}p_{j_2}\cdots p_{j_h}\}$, where $|j_s - j_t| \geq 2$, $s \neq t$, $1 \leq j_1 < j_2 < \cdots < j_h \leq n-1$, and $p_i = (\frac{1}{c})_1U_i$ where $1 \leq i \leq n-1$, and $c = a$ if i even and $c = b$ if i odd.

Lemma 2.2.7. *Let* $D \in \mathrm{FC}_n$ *be a diagram with label* λ. *Then* λ *is a new label at level* n, *that is,* $\lambda \in \Lambda_n^n$, *if and only if* $D \in \mathcal{J}_n$.

Proof. See [3, Lemma 4.1.6(i)]. □

Lemma 2.2.8 ([3, Lemma 4.1.7])**.** *Let* Y_n *be the two sided ideal in* FC_n *generated by* e_n. *Then the basis for* FC_n/Y_n *is the set* $\mathcal{J}_n$

Theorem 2.2.9 ([9, Theorem 1])**.** *Suppose* F *is a field and* A *is an* F*-algebra generated by idempotents. The following are equivalent*

(i) A *is commutative,*

(ii) A *has no non-zero nilpotent elements,*

(iii) A *is* F*-isomorphic to a direct product of copies of* F.

Proposition 2.2.10. *The algebra* $\mathrm{FC}_n/(\mathrm{FC}_n e_n \mathrm{FC}_n)$ *is commutative semisimple, and so satisfies axiom (A2)(i).*

Proof. Let $Q = \mathrm{FC}_n/\mathrm{FC}_n e_n \mathrm{FC}_n$. Since $\mathrm{FC}_n e_n \mathrm{FC}_n$ is two sided ideal of FC_n generated by e_n, then by using Lemma 2.2.8, the basis elements of Q is the set $\{x \mid x = p_{j_1} p_{j_2} \cdots p_{j_h}\}$, where $|j_s - j_t| \geq 2$, $s \neq t$, $1 \leq j_1 < j_2 < \cdots < j_h \leq n-1$, and $p_i = (\frac{1}{c})_1 U_i$ where $1 \leq i \leq n-1$, and $c = a$ if i even and $c = b$ if i odd.

In addition, by Proposition 1.1.7, we have $p_i^2 = p_i$ and $p_i p_j = p_j p_i$, for all $1 \leq i, j \leq n-1$. Then Q is generated by a set of commuting idempotents. That is, Q is commutative algebra. Therefore, by Theorem 2.2.9, we have Q has no non-zero nilpotent elements. But $\mathrm{rad}(Q)$ is nilpotent, thus $\mathrm{rad}(Q) = 0$ and hence Q is semisimple. $\square$

We now turn to proving axiom (A2)(ii). Set $e_{n,0} = 1$ in FC_n, and for $1 \leq i \leq n/2$ define new idempotents in FC_n by setting $e_{n,i} = \psi_n(e_{n-2,i-1})$, as in Figure 2.1 where ψ_n is as defined in Proposition 2.2.2.

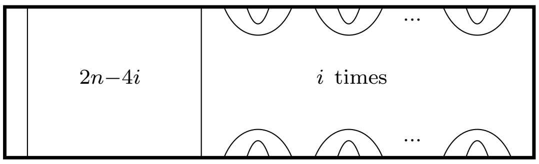
$\frac{1}{(ab)^i}$ 

Figure 2.1: The diagram $e_{n,i}$.

Recall that we say $\lambda \leq \mu$ if we can get λ from μ by removing some or no letters from μ.

Lemma 2.2.11. *Let* λ_D *be the label of a diagram* D*. For each diagram* $D \in \mathrm{FC}_n$*, we have* $\lambda_D \leq \lambda_{e_{n,i+1}}$ *if and only if* D *is in* $\mathrm{FC}_n e_{n,i+1} \mathrm{FC}_n$*.*

Proof. It is clear that if $D \in \mathrm{FC}_n e_{n,i+1} \mathrm{FC}_n$ then $\lambda_D \leq \lambda_{e_{n,i+1}}$.

On the other hand, suppose that $D \in \mathrm{FC}_n$ such that $\lambda_D \leq \lambda_{e_{n,i+1}}$. We will claim that D can be written as $u e_{n,i+1} v = cD$ for some $u, v \in \mathrm{FC}_n$ and $c \in \mathbb{C}$.

Recall that the initial part S, (resp. the final part T), of a diagram is its upper (resp. lower) half diagram which is defined in Definition 1.2.3. In addition, if S and T have label λ then $C^{\lambda}_{S,T}$ is a diagram with label λ that obtained by connecting S with T.

Let D_1, u_1 and v_1 are the initial parts of D, u and v respectively, and D_2, u_2 and v_2 are the final parts of D, u and v respectively. Set $D_1 = u_1$ and $D_2 = v_2$. Now, to determine the diagrams u and v such that $ue_{n,i+1}v$ is a scalar multiple of D we only need to find u_2 and v_1.

Since $\lambda_{e_{n,i+1}}$ is the label of the identity diagram for FC_{n-2i-2}, then there exist at least one diagram, say w, in FC_{n-2i-2} such that $\lambda_w = \lambda_D \leq \lambda_{e_{n,i+1}}$. Let w' be the initial part of w and we construct v_1 as following: let v_1 be the diagram such that the first $2n-4(i+1)$ vertices, (from the left), represented by w' and the last $4(i+1)$ vertices represented by $i+1$ consecutive nested pairs of non-through strings, moreover, we choose u_2 to be v_1^* where v_1^* is obtained by turning v_1 upside down. By this setup we have $\lambda_{v_1} = \lambda_{v_2} = \lambda_{u_1} = \lambda_{u_2} = \lambda_D$, and $e_{n,i+1}v = v$ where $v = C^{\lambda_D}_{v_1,v_2}$ and as explained in Figure 2.2.

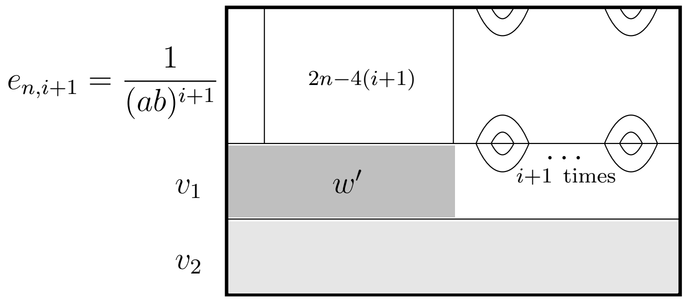


Figure 2.2: The diagram $e_{n,i+1}v$

Similarly, we can show that $ue_{n,i+1} = u$, where $u = C^{\lambda_D}_{u_1,u_2}$. Hence, we have

$$\begin{aligned} ue_{n,i+1}v &= uv \\ &= C^{\lambda_D}_{u_1,u_2} C^{\lambda_D}_{v_1,v_2} \\ &= C^{\lambda_D}_{u_1,v_1^*} C^{\lambda_D}_{v_1,v_2} \end{aligned}$$

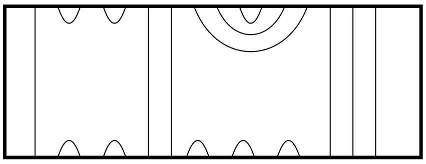
Note that $v_1^* v_1$ is a scalar multiple of the identity diagram for FC_ℓ where 2ℓ is the length of v_1. So, we get

$$\begin{aligned} ue_{n,i+1}v &= c\ C^{\lambda_D}_{u_1,v_2} \\ &= c\ C^{\lambda_D}_{D_1,D_2} = cD \end{aligned}$$

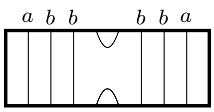
where $c \in \mathbb{C}$. □

To explain this lemma, we give this example.

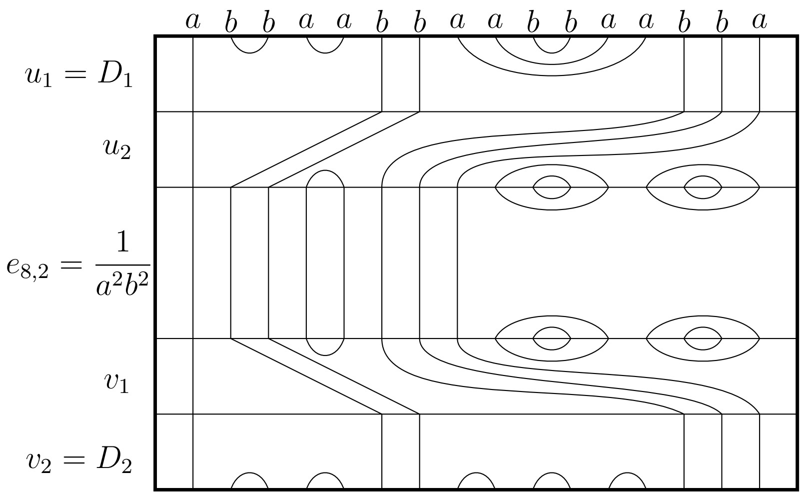
Example 2.2.12. Let $n = 8$ and $i = 1$, then $\lambda_{e_{8,2}} = ab^2a^2b^2a$. Consider the diagram $D \in \mathrm{FC}_8$, where

$D =$ 

then we have $\lambda_D = ab^4a \leq \lambda_{e_{8,2}}$. We will show that $D \in \mathrm{FC}_8 e_{8,2} \mathrm{FC}_8$ by finding two diagrams $u, v \in \mathrm{FC}_8$ such that $ue_{8,2}v = cD$ and as explained in Lemma 2.2.11. Let D_1, u_1 and v_1, (resp. D_2, u_2 and v_2), are the initial, (resp. final), parts of D, u and v respectively. Put $u_1 = D_1$ and $v_2 = D_2$. since $\lambda_D \leq \lambda_{e_{8,2}}$ then there exist a diagram $w \in \mathrm{FC}_4$ such that $\lambda_w = \lambda_D$. Since λ_D is obtained by removing some letters from $\lambda_{e_{8,2}}$, So we can find the diagram w by comparing $\lambda_D = ab^4a$ with $\lambda_{e_{8,2}} = ab^2a^2b^2a$ and then represent each letter in λ_D that not be removed from $\lambda_{e_{8,2}}$ by a through string while each letter that be removed from $\lambda_{e_{8,2}}$ by a non-through string. Then w has the form

$w =$ 

Let v_1 be the initial part of w and $u_2 = v_1^*$, then $ue_{8,2}v$ has the form



We can see that, $\quad ue_{8,2}v = a^3b^2\ D.$

Proposition 2.2.13. *For each $n \geq 1$ and $0 \leq i \leq n/2$, setting $e = e_{n,i}$ and $A = \mathrm{FC}_n/(\mathrm{FC}_n e_{n,i+1}\mathrm{FC}_n)$, the surjective multiplication map $Ae \otimes_{eAe} eA \to AeA$ is a bijection. Thus FC_n satisfies axiom (A2)(ii).*

Proof. To prove this proposition we will show that $Ae \otimes_{eAe} eA$ and AeA have same dimensions.

Define $\Gamma = \{\lambda \leq \lambda_e \mid \lambda \nleq \lambda_{e_{n,i+1}}\}$. By Lemma 2.2.11 we have, for all $D \in \mathrm{FC}_n$, that if $\lambda_D \leq \lambda_{e_{n,i+1}}$ then $D \in \mathrm{FC}_n e_{n,i+1}\mathrm{FC}_n$, and if $D' \in A$ then $\lambda_{D'} \nleq \lambda_{e_{n,i+1}}$. Therefore, Ae, eA and AeA have the diagrams with label $\lambda \in \Gamma$ as a basis. We divide the proof into three steps.

(1) We will show that diagrams with same label in Ae, (resp. eA), must have unique final, (resp. initial) part.

Recall that $\Lambda(n)$ is the label set for the basis diagrams in FC_n, and let D_1 be the initial part of $D \in eA$. Since the last $4i$ vertices of D_1 are connected to non-through strings then the label of D_1, which is the same as the label of D, is completely determined from the first $2n - 4i$ vertices of D_1. So the label for D_1 can be any label for a diagram in FC_{n-2i}, that is, the possible labels for D_1 are the set $\Lambda(n - 2i)$. From equation 2.1, we have

$$\Lambda(n-2i) = \Lambda(n-2i-2) \cup \Lambda^{n-2i}_{n-2i}.$$

Since $\lambda_{e=e_{n,i}}$ is the label of the identity diagram for FC_{n-2i} and $\lambda_{e_{n,i+1}}$ is the label of the identity diagram for FC_{n-2i-2}. Thus for all $\mu \in \Lambda(n - 2i - 2)$ we have $\mu \leq \lambda_{e_{n,i+1}}$ and hence μ is not a label for D_1. In addition, for all $\mu' \in \Lambda^{n-2i}_{n-2i}$ we have $\mu' \leq \lambda_e$ because Λ^{n-2i}_{n-2i} have labels for diagrams in FC_{n-2i}, moreover, $\mu' \nleq \lambda_{e_{n,i+1}}$ otherwise μ' must be in $\Lambda(n-2i-2)$ and this contradict to being μ' a new label at level $n-2i$. Thus $\mu' \in \Gamma$ and $\Lambda^{n-2i}_{n-2i} \subseteq \Gamma$. By the definition of Γ we have $\Gamma \subseteq \Lambda^{n-2i}_{n-2i}$. Hence, $\Gamma = \Lambda^{n-2i}_{n-2i}$. That is, the possible labels for D_1 are the set Λ^{n-2i}_{n-2i}. By Remark 2.2.5, we have, for all $\lambda \in \Lambda^{n-2i}_{n-2i}$, that $\dim \Delta_{n-2i}(\lambda) = 1$. Thus we have only one possible initial part D_1 for all diagrams $D \in eA$ with label λ. Similarly we can show that if diagrams in Ae have same label then they must have the same final part.

(2) We can read the label of a diagram from its initial part or from its final part.

So one obtains all diagrams of label $\lambda \in \Gamma$ by choosing an initial part and a final part of the same label λ and then the diagram can be uniquely determined by connecting their through strings. From (1), we have that the basis diagrams for Ae, (resp. eA), that have same label must have unique final, (resp. initial), part. Thus the basis for Ae, (resp. eA), are diagrams that obtained by connecting all possible initial, (resp. final), parts with label $\lambda \in \Gamma$ with the unique final, (resp. initial), part that have label λ. But all initial parts with label λ are precisely the basis for the cell module $\Delta_n(\lambda)$. Thus $\dim Ae = \dim eA = \sum_{\lambda \in \Gamma} d_\lambda$ where d_λ is the dimension of the cell module $\Delta_n(\lambda)$.

For each $\lambda \in \Gamma$, the diagrams that obtained by connecting all possible initial parts of the diagrams in Ae that have label λ with all possible final parts of the diagrams in eA that have label λ are basis for AeA. Then we have d_λ^2 choices, and hence, $\dim AeA = \sum_{\lambda \in \Gamma} d_\lambda^2$.

(3) Now, to find dimension of $R := Ae \otimes_{eAe} eA$, we have

$$\dim R \leq \dim(Ae)^2 = \left(\sum_{\lambda \in \Gamma} d_\lambda \right)^2 .$$

We will show that if $\lambda_{ue} \neq \lambda_{ev}$ then $ue \otimes ev = (ue)' \otimes (ev)'$ where $(ue)'$ and $(ev)'$ have same label.

From (1), we have that $\Gamma = \Lambda_{n-2i}^{n-2i}$, that is, Γ is the set of all new labels at level $n-2i$, then, by Lemma 2.2.7, Γ is the label set for the diagrams in $\mathcal{J}_{n-2i}$. By Definition 2.2.6, the diagrams in $\mathcal{J}_{n-2i}$ must have the form

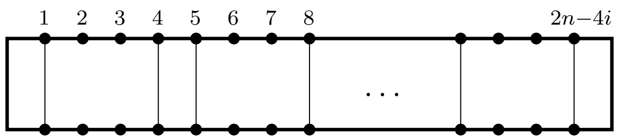
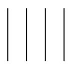
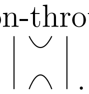


Figure 2.3: Diagram in $\mathcal{J}_{n-2i}$

where any two adjacent free vertices are either connected with through strings or connected with a non-through string, for example, the first four vertices either have the form  or . Therefore, the first $2n-4i$ vertices of the final and the initial parts for the diagrams in Ae and eA respectively have pattern similar to the pattern of the diagram in Figure 2.3.

We assign the numbers $1, 2, \ldots, 2n-4i$ to the vertices (from left to right) of the final part of ue and the initial part of ev, and let y_h be a non-through string that connect $2h$-th vertex with $(2h+1)$th-vertex where $1 < 2h < 2n-4i$.

Assume that $ue \in Ae$ and $ev \in eA$ have different label then the final part of ue and the initial part of ev have different label as well. So there is at least one non-through string in one half diagram (the final of ue or the initial of ev) that is not exist in the corresponding position in the other half diagram.

Without loss of generality, suppose that the final part of ue has a non-through string y_h whereas the $2h$-th and $(2h+1)$th-vertices in the initial part of ev are connected with through strings as explained in the following diagrams

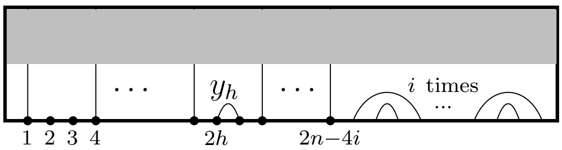


Figure 2.4: The diagram ue

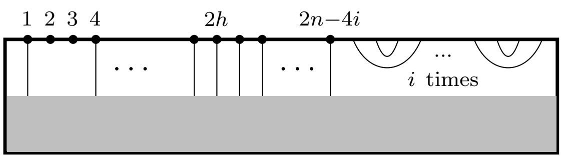


Figure 2.5: The diagram ev

Note that if a diagram $D \in \mathrm{FC}_n$ has a non-through string connected with two adjacent vertices at the bottom line of D, (for example, the diagram in Figure 2.4), then $D = cD\,{}_1U_j$ for some $1 \leq j \leq n-1$, where $c \in \{a, b\}$ and ${}_1U_j$ is the generator of FC_n which was defined in Proposition 1.1.9. In addition, $e\,{}_1U_j = {}_1U_j e$ and $e\,{}_1U_j = e^2\,{}_1U_j = e\,{}_1U_j e$ for $1 \leq j \leq n-2i-1$.

Thus we have $ue = cue\ {}_1U_h = cu\ e\ {}_1U_h\ e$, and

$$\begin{aligned} ue \otimes ev &= cu\ e\ {}_1U_h\ e \otimes ev \\ &= cu \otimes e\ {}_1U_h\ ev \\ &= cu \otimes {}_1U_h\ ev. \end{aligned}$$

Consider the diagram ${}_1U_h\ ev$

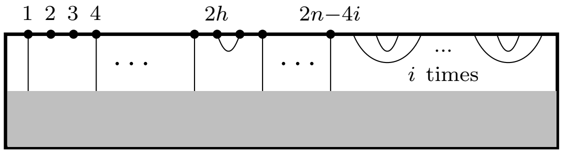


Figure 2.6: The diagram ${}_1U_h\ ev$

We see that the non-through string y_h in the final part of ue are created in the initial part of ev and in the corresponding position as in ue. Similarly, we can create any non-through string that exist in the final part of ue and does not exist in the initial part of ev and vice versa. Eventually, the final part of ue will be the same as the initial part of ev (after turning ev upside down). Hence, the two diagrams have the same label. (Sometimes, creating non-through strings in a half diagram gives us a label $\leq \lambda_{e_{n,i+1}}$ and hence the tensor product is zero).

Therefore, to give an upper bound for the dimension of R, we need only count elements of the form $ue \otimes_{eAe} ev$ with $\lambda_{ue} = \lambda_{ev}$. So, $\dim R \leq \sum_{\lambda\in\Gamma} (d_\lambda)^2$. Since the multiplication map is naturally surjective we must have that $\dim R \geq \sum_{\lambda\in\Gamma} (d_\lambda)^2$. Finally, we get $\dim R = \dim(AeA)$. $\square$

Since the algebra FC_n satisfies axiom A2, we have

Theorem 2.2.14. *The Fuss-Catalan algebras* $\mathrm{FC}_n(a, b)$ *are quasi-hereditary with heredity chain of the form*

$$0 \subset \ldots \subset \mathrm{FC}_n e_{n,i} \mathrm{FC}_n \subset \ldots \subset \mathrm{FC}_n e_{n,0} \mathrm{FC}_n = \mathrm{FC}_n$$

where $a, b \in \mathbb{C}$ *are non-zero.*

We turn now to proving axiom (A3), namely that the algebra FC_{n-1} is isomorphic to a subalgebra of FC_n for $n \geq 2$.

Remark 2.2.15. Any basis element $D \in \mathcal{B}_{n-1}$ can be viewed as a basis element $\bar{D} \in \mathcal{B}_n^2 \subset \mathcal{B}_n$ by adding two straight through strings to the right end of D with labelling ab if $(n-1)$ is even and ba if $(n-1)$ is odd. Therefore, the sets $\mathcal{B}_{n-1}$ and $\mathcal{B}_n^2$ have equal number of elements. In addition, we can see that $H_n = \mathrm{Span}(\mathcal{B}_n^2)$ is a subalgebra of FC_n.

Proposition 2.2.16. *For all $n \geq 2$, we have* $\mathrm{FC}_{n-1} \cong H_n$.

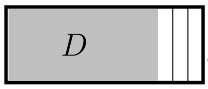
Proof. For all $D \in \mathcal{B}_{n-1}$, we define

$$\iota_n : \mathrm{FC}_{n-1} \to H_n$$

by

$$\iota_n(D) = \bar{D}.$$

As a diagram, this means that

$D \mapsto$ 

.

Recall that $H_n = \mathrm{Span}(\mathcal{B}_n^2)$, and $\mathrm{FC}_{n-1} = \mathrm{Span}(\mathcal{B}_{n-1})$. In addition, the sets $\mathcal{B}_n^2$ and $\mathcal{B}_{n-1}$ have the same number of elements, then we have

$$\dim(\mathrm{FC}_{n-1}) = \dim(H_n).$$

Since ι_n maps a basis element to a distinct basis element, it must be bijective.

Now, we need to show that ι_n is a homomorphism. For $D, D' \in \mathcal{B}_{n-1}$, we have

$\iota_n(D)\iota_n(D') =$ 

$=$ 

$= \iota_n(DD')$.

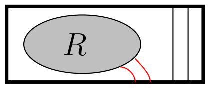
Therefore, ι_n is an isomorphism. □

We now prove axiom (A4) namely that for all $n \geq 2$, we have $\mathrm{FC}_{n-1} \cong \mathrm{FC}_n e_n$ as a left FC_{n-1}-, right FC_{n-2}-bimodule. We shall prove this module isomorphism by showing that FC_{n-1} and $\mathrm{FC}_n e_n$ have the same dimension. Then we define a bijective map between them, and show that it is a homomorphism as left FC_{n-1}-module, and right FC_{n-2}-module.

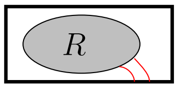
Proposition 2.2.17. *For all $n \geq 2$, we have*

$$\dim(H_n e_n) = \dim(\mathrm{FC}_{n-1}).$$

Proof. Suppose $q = \dim(\mathrm{FC}_{n-1})$. By Proposition 2.2.16, we have $\dim(H_n) = \dim(\mathrm{FC}_{n-1})$ so $\dim(H_n) = q$. We can represent a basis diagram of H_n as following

$h_R =$ 

where the subdiagram

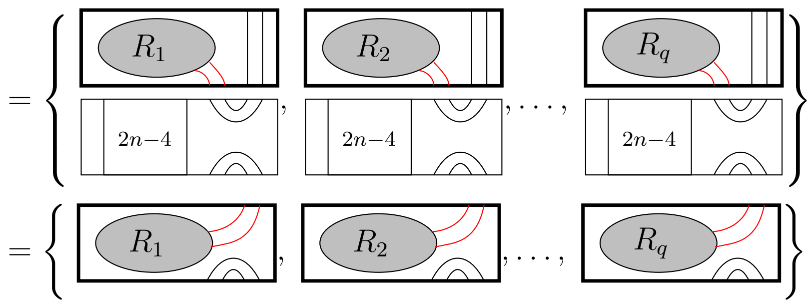


$\in \mathcal{B}_{n-1}$

and the subdiagram R has $(2n-2)$ vertices on the top line and $(2n-4)$ vertices on the bottom line.

Suppose $\mathcal{B}_n^2 = \{h_{R_i} \mid 1 \leq i \leq q\}$. Then the h_{R_i} are distinct diagrams for all $i = 1, \ldots, q$. So the R_i are distinct diagrams for all $i = 1, \ldots, q$.

Consider the multiplication of the diagrams in $\mathcal{B}_n^2$ by e_n:

$\mathcal{B}_n^2 e_n =$ 

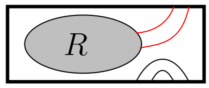
Let $v_i = h_{R_i} e_n$. To show that all the diagrams in $\mathcal{B}_n^2 e_n$ are distinct, we can ignore the last two non-through strings in the bottom line in each v_i.

Now, for all $1 \leq i \leq q$, when we multiply an element $h_{R_i} \in \mathcal{B}_n^2$ with e_n, we just move the two red strings from the bottom line to the top line and the subdiagram R_i has no changes (since it is still has $(2n-2)$ vertices on the top line and $(2n-4)$ vertices on the bottom line), that is, we do the same movement to get each diagram v_i, thus, two diagrams $v_i, v_j \in \mathcal{B}_n^2 e_n$ are equal if their subdiagrams R_i and R_j are equal. However, R_i are distinct for all $1 \leq i \leq q$, then v_i are all distinct as well. Therefore, $\dim(H_n e_n) = q$. $\square$

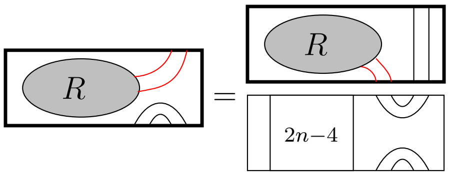
Proposition 2.2.18. *Let* $n \geq 2$. *Then* $\dim(\mathrm{FC}_n e_n) = \dim(\mathrm{FC}_{n-1})$.

Proof. Since $\mathcal{B}_n^2 e_n \subset B_n$ and by Proposition 2.2.17, we get

$$\dim(\mathrm{FC}_{n-1}) \leq \dim(\mathrm{FC}_n e_n).$$

Every basis element $v \in \mathrm{FC}_n e_n$ has the form $v =$ 

and it can be decomposed as following

$v =$ 

$= h_R e_n,$

where $h_R \in \mathcal{B}_n^2$, and R is a subdiagram with $(2n-2)$ vertices in the top line and $(2n-4)$ vertices on the bottom line. Then $\dim(\mathrm{FC}_n e_n) \leq \dim(H_n) = \dim(\mathrm{FC}_{n-1})$. Therefore, $\mathrm{FC}_n e_n$ and FC_{n-1} have the same dimension. □

Remark 2.2.19. Considering the functions ψ_n that are defined in Proposition 2.2.2 and the ι_n that are defined in Proposition 2.2.16, we have

(i) $\mathrm{FC}_n e_n$ is a left FC_{n-1}-module, where the action is defined by

$$x \cdot D = \iota_n(x) D$$

for all basis elements $x \in \mathrm{FC}_{n-1}$ and $D \in \mathrm{FC}_n e_n$.

(ii) $\mathrm{FC}_n e_n$ is a right FC_{n-2}-module, where the action is defined by

$$D \cdot y = D \psi_n(y)$$

for all basis elements $y \in \mathrm{FC}_{n-2}$ and $D \in \mathrm{FC}_n e_n$.

(iii) FC_{n-1} is a right FC_{n-2}-module, where the action is defined by

$$D \cdot y = D \iota_{n-1}(y)$$

for all basis elements $D \in \mathrm{FC}_{n-1}$ and $y \in \mathrm{FC}_{n-2}$.

Proposition 2.2.20. *For all $n \geq 2$, we have that* $\mathrm{FC}_{n-1} \cong \mathrm{FC}_n e_n$ *as a left* FC_{n-1}*-, right* FC_{n-2}*-bimodule.*

Proof. For all $D \in \mathrm{FC}_{n-1}$, define $\theta : \mathrm{FC}_{n-1} \to \mathrm{FC}_n e_n$ by

$$\theta(D) = \iota_n(D)e_n$$

Since θ maps every basis element in FC_{n-1} to a distinct basis element in $\mathrm{FC}_n e_n$ and by Proposition 2.2.18, we get θ is a bijective.

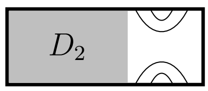
To show that θ is a homomorphism as a left FC_{n-1}-module, Suppose that D and D' are basis elements in FC_{n-1}. We have

$$\begin{aligned} \theta(D \cdot D') = \theta(DD') &= \iota_n(DD')e_n \\ &= \iota_n(D)\ \iota_n(D')e_n \\ &= \iota_n(D)\ \theta(D') \\ &= D \cdot \theta(D') \end{aligned}$$

To show that θ is a homomorphism as a right FC_{n-2}-module, suppose that D_1 is a basis element in FC_{n-1}, and D_2 is a basis element in FC_{n-2}. Then we have

$$\begin{aligned} \theta(D_1 \cdot D_2) &= \theta(D_1\ \iota_{n-1}(D_2)) \\ &= \iota_n(D_1\ \iota_{n-1}(D_2))e_n \\ &= \iota_n(D_1)\ \iota_n(\bar{D}_2)e_n \\ &= \iota_n(D_1)\ \bar{\bar{D}}_2\ e_n \\ &= \big(\iota_n(D_1)e_n\big)\ (e_n\ \bar{\bar{D}}_2) \end{aligned}$$

$$= \theta(D_1) \left(\frac{1}{ab} \right.$$



$$\left. \right)$$

$$\begin{aligned} &= \theta(D_1)\ \psi_n(D_2) \\ &= \theta(D_1) \cdot D_2 \end{aligned}$$

Hence, θ is an module isomorphism. $\square$

2.3 The Restriction of the cell module $\Delta_n(\lambda)$

In this section we are going to study the restriction of the cell modules. We need it to complete the proof of the last two axioms for the tower of recollement.

Definition 2.3.1. Let M be a basis diagram in $\Delta_n(\lambda)$ such that the last two vertices are connected to non-through strings. Then we say such a diagram has *Form 0* as depicted in Figure 2.7.

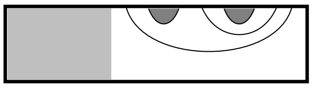


Figure 2.7: Diagram of Form 0.

Definition 2.3.2. Let M be a basis diagram in $\Delta_n(\lambda)$ such that the $2n$-th vertex (the last vertex) is connected to a through string while the $(2n-1)$-th vertex is connected to a non-through string. Then we say such diagram has *Form 1* as depicted in Figure 2.8.

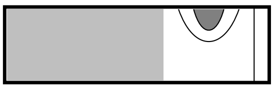


Figure 2.8: Diagram of Form 1.

Any other basis diagrams will have two through strings from the last two vertices and will be known as *Form 2*. This won't be used in the sequel however.

We will use the convention that when j_s and j_t are strings connected to vertices that the subscript indicates the labels s and t respectively on the strings, where $s, t \in \{a, b\}$. We define the following procedures.

Definition 2.3.3 (The Add-procedure)**.** Let M be a diagram with $2n-2$ vertices and label $\lambda_1 st$. If we add two vertices to M and connect the last through string j_t with the $(2n-1)$-th vertex, and connect a new through string j_s with the $2n$-th vertex (the last vertex), we get a diagram M' with $2n$ vertices and its label will be $(\lambda_1 ss)$. This process is depicted in Figure 2.9.

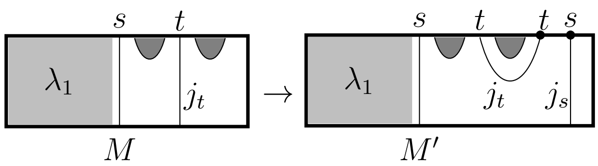


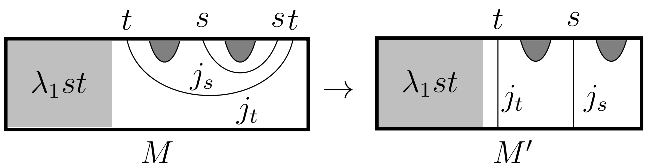
Figure 2.9: The Add-procedure.

Definition 2.3.4 (The Cut-procedure)**.** Let M be a diagram with $2n$ vertices and label $\lambda_1 st$, and let j_t be the string which is connected to the $2n$-th vertex and j_s be the string which is connected to the $(2n-1)$-th vertex. If we remove the last two vertices of M and sliding down j_s and j_t, we have

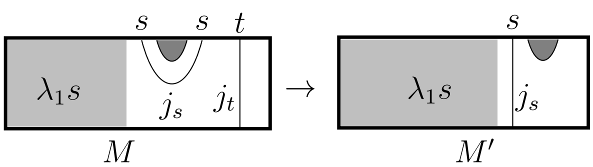
(i) If j_s and j_t are non-through strings, then we get a diagram M' with $2n-2$ vertices and its label will be $\lambda_1 stts$.

(ii) If j_s is a non-through string and j_t is a through string, then we get a diagram M' with $2n-2$ vertices and its label will be $\lambda_1 ss$.

This process is depicted in Figure 2.10.



(a) (i)



(b) (ii)

Figure 2.10: The Cut-procedure.

Let n be an odd number, then every label must end with the letter b. That is, if $\Delta_n(\lambda)$ is a cell module for FC_n, then $\lambda = \lambda_1 ab$ or $\lambda = \lambda_1 bb$. Let us first discuss the restriction of $\Delta_n(\lambda)$ when the last two letters are different and consider the restriction of $\Delta_n(\lambda_1 ab)$.

Recall that $\mathcal{W}_n(\lambda)$ is the set of basis diagrams M_λ of the cell module $\Delta_n(\lambda)$. Let $\Delta_{n-1}(\lambda_1)$ be a cell module for FC_{n-1} with basis diagrams in $\mathcal{W}_{n-1}(\lambda_1)$, and define $\mathcal{W}_n^2(\lambda_1 ab)$ to be the set of elements $\bar{M}_{\lambda_1 ab}$ obtained by adding two straight through strings to the right end of each diagram M_{λ_1} in $\mathcal{W}_{n-1}(\lambda_1)$. Then $\mathcal{W}_n^2(\lambda_1 ab)$ is a subset of $\mathcal{W}_n(\lambda_1 ab)$, and we can define the inclusion map

$$\phi_n : \Delta_{n-1}(\lambda_1) \to \mathrm{res}(\Delta_n(\lambda)) \tag{2.4}$$

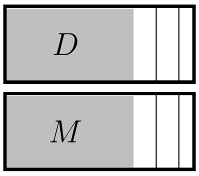
via

$$\phi_n(M_{\lambda_1}) = \bar{M}_{\lambda_1 ab}$$

and extend it by linearity to $\Delta_{n-1}(\lambda_1)$.

To show that ϕ_n is a FC_{n-1}-homomorphism, suppose $D \in \mathrm{FC}_{n-1}$ and $M = M_{\lambda_1} \in \mathcal{W}_{n-1}(\lambda_1)$ then we have

$\phi_n(D \cdot M) =$ 

$=$ 

$$= \iota_n(D)\phi_n(M)$$
$$= D \cdot \phi_n(M)$$

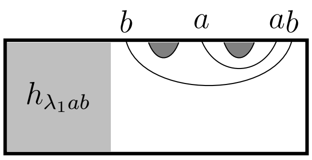
where ι_n is the inclusion map of FC_{n-1} into FC_n.

Let $Q_n(\lambda_1 ab) = \mathrm{res}(\Delta_n(\lambda_1 ab))/\Delta_{n-1}(\lambda_1)$. Now $Q_n(\lambda_1 ab)$ is the linear span of elements in

$$\{v + \Delta_{n-1}(\lambda_1) \mid v \in \mathcal{W}_n(\lambda_1 ab) \setminus \mathcal{W}_n^2(\lambda_1 ab)\}$$

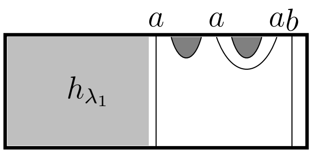
and every such element $v + \Delta_{n-1}(\lambda_1)$ has a unique coset representative in the set $\mathcal{W}_n(\lambda_1 ab) \setminus \mathcal{W}_n^2(\lambda_1 ab)$. Thus we may view basis elements of $Q_n(\lambda_1 ab)$ as diagrams.

Since for all $M_{\lambda_1 ab} \in \mathcal{W}_n^2(\lambda_1 ab)$, there exist $M_{\lambda_1} \in \Delta_{n-1}(\lambda_1)$ such that $\phi_n(M_{\lambda_1}) = M_{\lambda_1 ab}$, then every diagram $M_{\lambda_1 ab} \in Q_n(\lambda_1 ab)$ must have less than two through strings at the right end. That is, the basis diagrams of $Q_n(\lambda_1 ab)$ must be of Form 0 or Form 1. The basis diagrams for $Q_n(\lambda_1 ab)$ has the form



Form 0

and



Form 1

where $h_{\lambda_1 ab}$ and h_{λ_1} are subdiagrams with label $\lambda_1 ab$ and λ_1 respectively.

We can identify the basis of $Q_n(\lambda_1 ab)$ with the basis diagrams in the cell modules $\Delta_{n-1}(\lambda_1 abba)$ and $\Delta_{n-1}(\lambda_1 aa)$ as following:

- Let $v \in \Delta_{n-1}(\lambda_1 aa)$ be a basis diagram. Then, v has $2n-2$ vertices with label $\lambda_1 aa$. If we apply the Add-procedure to v, we get a new diagram, v' say, with $2n$ vertices and label $\lambda_1 ab$. So, $v' \in Q_n(\lambda_1 ab)$ and it has Form 1.

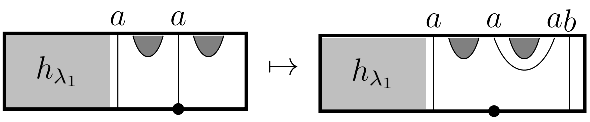
- Let $v \in Q_n(\lambda_1 ab)$ be a basis diagram. Then v has $2n$ vertices and its label is $\lambda_1 ab$. If we apply the Cut-procedure to v, we have

  (i) If v has Form 0: We get a diagram, say v', with $2n-2$ vertices and its label is $\lambda_1 abba$. Thus $v' \in \Delta_{n-1}(\lambda_1 abba)$.

  (ii) If v has Form 1: We get a diagram, say v', with $2n-2$ vertices and its label is $\lambda_1 aa$. Thus $v' \in \Delta_{n-1}(\lambda_1 aa)$.

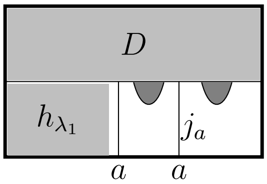
Proposition 2.3.5. *Suppose that* $Q_n(\lambda) = \mathrm{res}(\Delta_n(\lambda))/\Delta_{n-1}(\lambda_1)$ *where* $\lambda = \lambda_1 ab$. *Let* f *and* g *be the functions that represent the Add-procedure and the Cut-procedure respectively. Then* f *and* g *are* FC_{n-1}*-module homomorphisms, where* $f \in \mathrm{Hom}\big(\Delta_{n-1}(\lambda_1 aa), Q_n(\lambda)\big)$ *and* $g \in \mathrm{Hom}\big(Q_n(\lambda), \Delta_{n-1}(\lambda ba)\big)$.

Proof. (1) Define $f : \Delta_{n-1}(\lambda_1 aa) \to Q_n(\lambda_1 ab)$ by

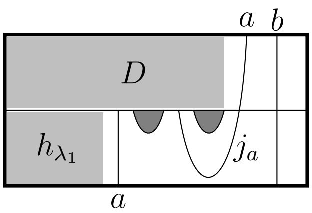
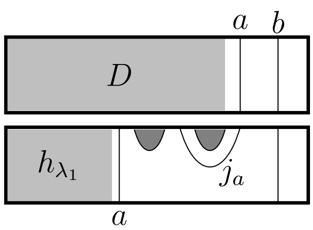


where h_{λ_1} is a subdiagram with label λ_1.

Suppose that $D \in \mathrm{FC}_{n-1}$, and $M \in \Delta_{n-1}(\lambda_1 aa)$ are basis diagrams. Consider the diagram

$D \cdot M =$ 

If j_a is a through string in $D \cdot M$ then we get

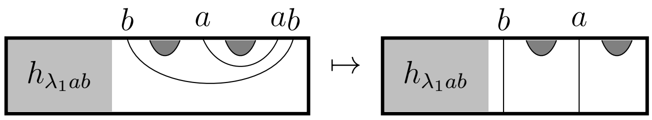
$f(D \cdot M) =$  $=$ 

So,

$$f(D \cdot M) = \iota_n(D) f(M) = D \cdot f(M)$$

If j_a is a non-through string in the diagram $D \cdot M$ then we have $l(DM) < l(M)$, that is, $D \cdot M = 0$. In addition, j_a in the diagram $D \cdot f(M) = \iota_n(D) f(M)$ will be a through string. That is, $D \cdot f(M)$ will ends with two consecutive through strings. Thus $D \cdot f(M) = 0$ because any basis diagram of $Q_n(\lambda_1 ab)$ are must be of Form 0 and Form 1. Hence, f is FC_{n-1}-module homomorphism.

(2) Let M be a basis diagram in $Q_n(\lambda_1 ab)$. Define $g : Q_n(\lambda_1 ab) \to \Delta_{n-1}(\lambda_1 abba)$ by

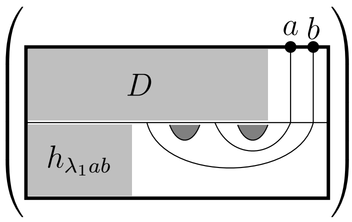
(i) 

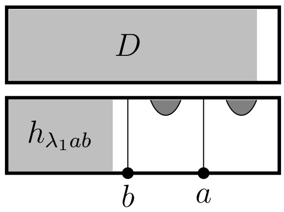
if M is of Form 0 and where $h_{\lambda_1 ab}$ is a subdiagram with label $\lambda_1 ab$.

(ii) $M \mapsto 0$ if M is of Form 1.

To prove g is an FC_{n-1}-module homomorphism, suppose that $D \in \mathrm{FC}_{n-1}$ and M is a diagram in $Q_n(\lambda_1 ab)$. We have the following cases.

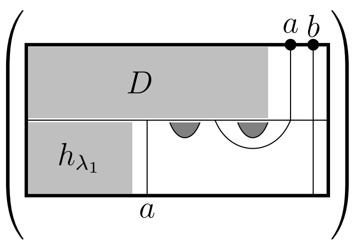
(i) M is of Form 0, then

$g(D \cdot M) = g\left(\iota_n(D)M\right) = g$ 

$=$ 

$= Dg(M)$

(ii) M is of Form 1. We have

$g(D \cdot M) = g\left(\iota_n(D)M\right) = g$ 

Recall that the basis of $Q_n(\lambda_1 ab)$ consists only of diagrams of Form 0 and Form 1. If $D \cdot M$ is a non-zero diagram, then $D \cdot M = \iota_n(D)M$ must be of Form 1, (it is not possible to be of Form 0 because it ends with one through string). Therefore, by the definition of g, we have $g(D \cdot M) = 0$. On the other hand, it is clear that $D \cdot g(M) = 0$. Thus g is a FC_{n-1}-module homomorphism. $\square$

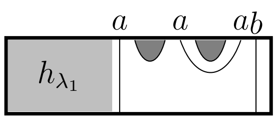
Proposition 2.3.6. *Let* $Q_n(\lambda) = \mathrm{res}(\Delta_n(\lambda))/\Delta_{n-1}(\lambda_1)$, *where* $\lambda = \lambda_1 ab$. *In addition, let* f *and* g *be as defined in Proposition 2.3.5. Then*

$$0 \to \Delta_{n-1}(\lambda_1 aa) \xrightarrow{f} Q_n(\lambda_1 ab) \xrightarrow{g} \Delta_{n-1}(\lambda_1 abba) \to 0$$

is a short exact sequence of FC_{n-1}*-modules.*

Proof. We proved that f and g are homomorphisms in Proposition 2.3.5. Since f maps every basis diagram to a basis diagram with one through string at the right end, we must have $g \circ f = 0$. Hence, $\mathrm{Im}(f) \subseteq \mathrm{Ker}(g)$.

On the other hand,

$$\mathrm{Ker}(g) = \mathrm{Span}\Bigg\{ M =$$


$$\Bigg| \; M \in Q_n(\lambda) \Bigg\}.$$

Since the diagrams of $\mathrm{Ker}(g)$ are of Form 1, then we can get these diagrams by applying the Add-procedure (which is represented by the function f) to the basis diagrams of $\Delta_{n-1}(\lambda_1 aa)$. Thus, for all $M \in \mathrm{Ker}(g)$, there exist $M' \in \Delta_{n-1}(\lambda_1 aa)$ such that $M = f(M')$. Then $\mathrm{Ker}(g) \subseteq \mathrm{Im}(f)$ and, hence, $\mathrm{Ker}(g) = \mathrm{Im}(f)$. $\square$

Putting this altogether, we may now state our first proposition about the restriction of $\Delta_n(\lambda_1 ab)$.

Proposition 2.3.7. *As an* FC_{n-1}*-module* $\Delta_n(\lambda_1 ab)$ *has a filtration*

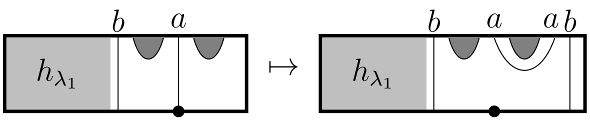
$$0 \subset V_1 \subset V_2 \subset V_3 = \Delta_n(\lambda_1 ab)$$

where $V_1 \cong \Delta_{n-1}(\lambda_1)$, $V_2/V_1 \cong \Delta_{n-1}(\lambda_1 aa)$, *and* $V_3/V_2 \cong \Delta_{n-1}(\lambda_1 abba)$.

We now turn to considering the restriction of the module $\Delta_n(\lambda_1 bb)$.
Suppose $\lambda = \lambda_1 bb$, with $\Delta_n(\lambda)$ a cell module, then $\Delta_{n-1}(\lambda_1 ba)$ is a cell module. Any basis diagram M of $\Delta_{n-1}(\lambda_1 ba)$ can be viewed as a basis diagram M' of $\Delta_n(\lambda)$ if we apply the Add-procedure to M.

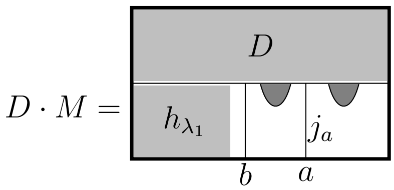
Proposition 2.3.8. *Let f and g be functions that represent the Add-procedure and the Cut-procedure on the module $\Delta_n(\lambda_1 bb)$ respectively. Then f and g are* FC_{n-1}*-module homomorphisms, where* $f \in \mathrm{Hom}\big(\Delta_{n-1}(\lambda_1 ba), \mathrm{res}(\Delta_n(\lambda_1 bb)\big)$ *and* $g \in \mathrm{Hom}\big(\mathrm{res}(\Delta_n(\lambda_1 bb), \Delta_{n-1}(\lambda_1 bbba)\big)$.

Proof. (1) Define $f : \Delta_{n-1}(\lambda_1 ba) \to \mathrm{res}(\Delta_n(\lambda_1 bb))$ via

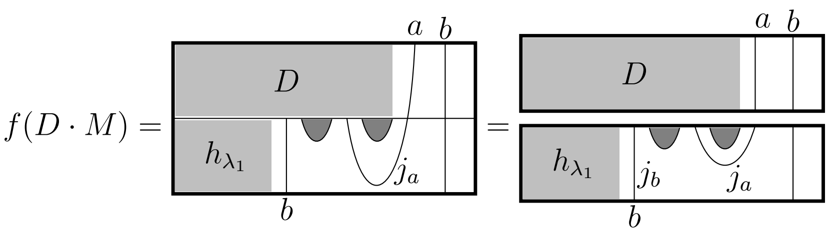


where h_{λ_1} is a subdiagram with label λ_1.

To see that f is a homomorphism, let $D \in \mathrm{FC}_{n-1}$ and $M \in \Delta_{n-1}(\lambda_1 ba)$ be basis diagrams. Then



If j_a is a through string in $D \cdot M$ then

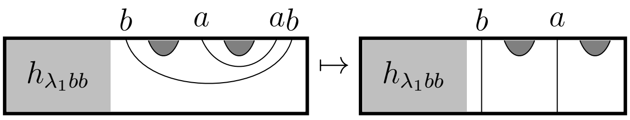


So

$$f(D \cdot M) = \iota_n(D) f(M) = D \cdot f(M).$$

If j_a is a non-through string in $D \cdot M$ then $l(D \cdot M) < l(M)$ which leads to $D \cdot M = 0$. While j_a becomes a through string in $D \cdot f(M) = \iota_n(D) f(M)$, this means that j_a is connected with an a-through string in the subdiagram h_{λ_1} but this will enforce j_b to be a non-through string in $D \cdot f(M)$ to avoid the intersection with j_a. Thus $l(D \cdot f(M)) < l(f(M))$, and $D \cdot f(M) = 0$. Then we get f is a FC_{n-1}-module homomorphism.

(2) Note that the basis diagrams of $\Delta_n(\lambda_1 bb)$ are of Form 0 and Form 1, (if there is a basis of $\Delta_n(\lambda_1 bb)$ that ends with two consecutive through strings then the last two letters of its label must be ab not bb). Let $M \in \mathrm{res}(\Delta_n(\lambda_1 bb))$ be a basis diagram, and define $g : \mathrm{res}(\Delta_n(\lambda_1 bb)) \to \Delta_{n-1}(\lambda_1 bbba)$ by

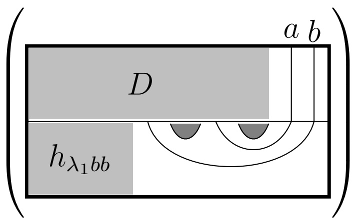
(i) 

if M is of Form 0 and where $h_{\lambda_1 bb}$ is a subdiagram with label $\lambda_1 bb$.

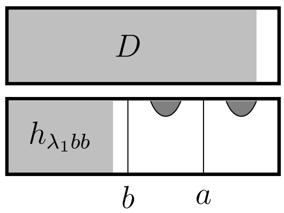
(ii) $M \mapsto 0$ if M is of Form 1.

Let $D \in \mathrm{FC}_{n-1}$ be a basis diagram. We now show that g is a homomorphism.
(i) If M is of Form 0, then

$$g(D\cdot M) = g\big(\iota_n(D)M\big) = g$$



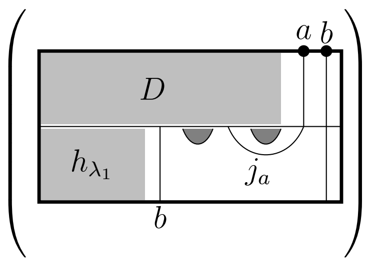
$$=$$



$$= D \cdot g(M)$$

(ii) If M is of Form 1, then $D \cdot g(M) = 0$, and

$$g(D\cdot M) = g\left(\iota_n(D)M\right) = g$$



If j_a is a through string in $D\cdot M = \iota_n(D)M$, then $D \cdot M$ has label ends with (ab) and hence, $D\cdot M = 0$. If j_a is a non-through string in $D\cdot M$, then $D\cdot M$ is of Form 1, and hence, $g(D\cdot M) = 0$. Thus g is a FC_{n-1}-module homomorphism. $\square$

Proposition 2.3.9. *Let f and g be as defined in Proposition 2.3.8. Then*

$$0 \to \Delta_{n-1}(\lambda_1 ba) \xrightarrow{f} \operatorname{res}(\Delta_n(\lambda_1 bb)) \xrightarrow{g} \Delta_{n-1}(\lambda_1 bbba) \to 0$$

is a short exact sequence of FC_{n-1}*-modules.*

Proof. By Proposition 2.3.8, f and g are FC_{n-1}-module homomorphisms.

Let M be a basis element of $\Delta_{n-1}(\lambda_1 ba)$, then, by the definition of f, the diagram $f(M)$ is of Form 1. But g sends any diagram of Form 1 to zero, thus $g \circ f = 0$, and

$$\operatorname{Im}(f) \subseteq \operatorname{Ker}(g).$$

Now, $\operatorname{Ker}(g)$ spanned by the diagrams $M \in \Delta_n(\lambda_1 bb)$ of Form 1, but any such diagrams can be obtained by applying f to the basis element $M' \in \Delta_{n-1}(\lambda_1 ba)$. That is, for all $M \in \operatorname{Ker}(g)$, there is $M' \in \Delta_{n-1}(\lambda_1 ba)$ such that $M = f(M')$. Thus

$$\operatorname{Ker}(g) \subseteq \operatorname{Im}(f).$$

Hence, we get

$$\operatorname{Ker}(g) = \operatorname{Im}(f).$$

□

Remark 2.3.10. If n is even then the general form for the labels of the cell modules $\Delta_n(\lambda)$ is either $\lambda = \lambda_1 ba$ or $\lambda = \lambda_1 aa$. All the results that we discussed when n is odd are true when n is even if we swap ab with ba and bb with aa.

Then we have

Proposition 2.3.11. *As an* FC_{n-1}*-module* $\Delta(\lambda_1 ba)$ *has a filtration*

$$0 \subset V_1 \subset V_2 \subset V_3 = \Delta(\lambda_1 ba)$$

where $V_1 \cong \Delta_{n-1}(\lambda_1)$, $V_2/V_1 \cong \Delta_{n-1}(\lambda_1 bb)$, *and* $V_3/V_2 \cong \Delta_{n-1}(\lambda_1 baab)$.

Proposition 2.3.12. *Let f and g be functions that represent the Add-procedure and the Cut-procedure on the module $\Delta_n(\lambda_1 aa)$ respectively. Then*

$$0 \to \Delta_{n-1}(\lambda_1 ab) \xrightarrow{f} \mathrm{res}(\Delta_n(\lambda_1 aa)) \xrightarrow{g} \Delta_{n-1}(\lambda_1 aaab) \to 0$$

is a short exact sequence of FC_{n-1}*-modules.*

Thus in all cases the restriction of the cell module has a Δ-filtration and thus satisfies the first part of axiom (A5).

Remark 2.3.13. Assume that $(s,t) = (a,b)$ if n even and $(s,t) = (b,a)$ if n odd. By using Proposition 1.3.16 and the same technique as that used in Proposition 1.4.11 to find the dimension of a cell module, we have

(1) If $\Delta_n(\lambda_1 st)$ is a one dimensional cell module, then

$$\mathrm{res}\,\Delta_n(\lambda_1 st) = \Delta_{n-1}(\lambda_1)$$

because the cell modules $\Delta_{n-1}(\lambda_1 ss)$ and $\Delta_{n-1}(\lambda_1 stts)$ have dimensions equal zero. In addition, we have

(i) $\Delta_{n-1}(\lambda_1)$ is a one dimensional cell module.

(ii) $\Delta_{n+1}(\lambda_1 ss)$ is a one dimensional cell module.

(iii) $\Delta_{n+1}(\lambda_1 stts)$ is a one dimensional cell module.

(2) If $\Delta_n(\lambda_1 tt)$ is a one dimensional cell module, then

$$\mathrm{res}\,\Delta_n(\lambda_1 tt) = \Delta_{n-1}(\lambda_1 ts)$$

because $\Delta_{n-1}(\lambda_1 ttts)$ has dimension equal zero. In addition, we have

(i) $\Delta_{n-1}(\lambda_1 ts)$ is a one dimensional cell module.

(ii) $\Delta_{n+1}(\lambda_1 ttts)$ is a one dimensional cell module.

Proposition 2.3.14. *For each* $\lambda \in \Lambda_n^m$ *we have*

$$\operatorname{supp}\big(\operatorname{res}(\Delta_n(\lambda))\big) \subseteq \Lambda_{n-1}^{m-1} \sqcup \Lambda_{n-1}^{m+1}.$$

Proof. Let n be odd. Then the labels λ of any cell module $\Delta_n(\lambda)$ either end with (ab) or with (bb).

Case 1: Suppose $\lambda = \lambda_1 ab$ then, by Proposition 2.3.6, we have

$$E := \operatorname{supp}\big(\operatorname{res}(\Delta_n(\lambda))\big) = \{\lambda_1, \lambda_1 aa, \lambda ba\}$$

We need to prove that every label in E is in the set $\Lambda_{n-1}^{m-1} \sqcup \Lambda_{n-1}^{m+1}$. Since $\lambda \in \Lambda_n^m$ then λ is a label of a one dimensional cell module $\Delta_m(\lambda)$ for some $m < n$. Then

(i) By Remark 2.3.13 (1)(i), λ_1 is a label of a one dimensional cell module $\Delta_{m-1}(\lambda_1)$ and hence $\lambda_1 \in \Lambda^{m-1} = \Lambda_{n-1}^{m-1}$.

(ii) By Remark 2.3.13 (1)(ii) and (iii), we have $\lambda_1 aa$ and λba are labels of one dimensional cell modules $\Delta_{m+1}(\lambda_1 aa)$ and $\Delta_{m+1}(\lambda ba)$ respectively. Hence,

$$\lambda_1 aa, \lambda ba \in \Lambda^{m+1} = \Lambda_{n-1}^{m+1}.$$

Case 2: Suppose $\lambda = \lambda_1 bb$ then, by Proposition 2.3.9, we have

$$E := \operatorname{supp}\big(\operatorname{res}(\Delta_n(\lambda))\big) = \{\lambda_1 ba, \lambda ba\}.$$

We need to prove that every label in E is in $\Lambda_{n-1}^{m-1} \sqcup \Lambda_{n-1}^{m+1}$. Since $\lambda \in \Lambda_n^m$ then λ is a label of a one dimensional cell module $\Delta_m(\lambda)$ for some $m < n$. Then

(i) By Remark 2.3.13 (2)(i), $\lambda_1 ba$ is a label of a one dimensional cell module $\Delta_{m-1}(\lambda_1 ba)$ and hence $\lambda_1 ba \in \Lambda^{m-1} = \Lambda_{n-1}^{m-1}$.

(ii) By Remark 2.3.13 (2)(ii), λba is a label of a one dimensional cell module $\Delta_{m+1}(\lambda ba)$. Hence, $\lambda ba \in \Lambda^{m+1} = \Lambda_{n-1}^{m+1}$.

We can prove this proposition for n even in a similar way as we did for n is odd. $\square$

We now prove that FC_n satisfies axiom (A6).

Proposition 2.3.15. *For each $\lambda \in \Lambda_n^n$, there exist $\mu \in \Lambda_{n+1}^{n-1}$ such that*

$$\lambda \in \operatorname{supp}\big(\operatorname{res}(\Delta_{n+1}(\mu))\big).$$

Proof. We will prove the case when n is odd, the case n is even is proved similarly.
Case 1: Suppose $\lambda = \lambda_1 ab$. Since $\lambda \in \Lambda_n^n = \Lambda^n$, then the cell module $\Delta_n(\lambda)$ is one dimensional. Suppose $\mu = \lambda_1$, then, by Remark 2.3.13 (1)(i), μ is the label for the one dimensional cell module $\Delta_{n-1}(\mu)$ and the label for the cell module $\Delta_{n+1}(\mu)$ that has dimension greater than one. Thus $\mu \in \Lambda_{n+1}^{n-1}$.

If $\lambda_1 = \lambda_2 ba$ then, by Proposition 2.3.6,

$$\operatorname{supp}\big(\operatorname{res}(\Delta_{n+1}(\mu))\big) = \{\lambda_2, \lambda_2 bb, \lambda_1 ab\}.$$

If $\lambda_1 = \lambda_2 aa$ then, by Proposition 2.3.9,

$$\operatorname{supp}\big(\operatorname{res}(\Delta_{n+1}(\mu))\big) = \{\lambda_2 ab, \lambda_1 ab\}.$$

We can see that if $\lambda_1 = \lambda_2 ba$ or $\lambda_1 = \lambda_2 aa$, we have

$$\lambda = \lambda_1 ab \in \operatorname{supp}\big(\operatorname{res}(\Delta_{n+1}(\mu))\big).$$

Case 2: Suppose $\lambda = \lambda_1 bb$. Since $\lambda \in \Lambda_n^n = \Lambda^n$, then the cell module $\Delta_n(\lambda)$ is one dimensional. Suppose $\mu = \lambda_1 ba$, then, by Remark 2.3.13 (2)(i), μ is a label for a one dimensional cell module $\Delta_{n-1}(\mu)$ and a label for a cell module $\Delta_{n+1}(\mu)$ that has dimension greater than one. Thus $\mu \in \Lambda_{n+1}^{n-1}$. By Proposition 2.3.6,

$$\operatorname{supp}\big(\operatorname{res}(\Delta_{n+1}(\mu))\big) = \{\lambda_1, \lambda_1 bb, \lambda_1 baab\}.$$

Therefore, $\lambda = \lambda_1 bb \in \operatorname{supp}\big(\operatorname{res}(\Delta_{n+1}(\mu))\big)$. $\square$

Now, we have proved that FC_n satisfies all the axioms A1 to A6 of the tower of recollement. We now have the main theorem in this chapter.

Theorem 2.3.16. *The Fuss-Catalan algebras form a tower of recollement.*

Chapter 3

The Gram determinant

In this chapter we shall define a bilinear form on each cell module and we introduce the Gram matrices related to it. In addition, we compute the determinant of the Gram matrices, in order to determine the values of a and b such that the Fuss-Catalan algebras $\mathrm{FC}_n(a, b)$ are semisimple over the complex field $\mathbb{C}$.

3.1 Definition of the Gram matrix

In chapter one we proved that FC_n are cellular algebras. Thus we can define a unique bilinear form associated to each cell module for the Fuss-Catalan algebras. Furthermore, by Theorem 1.3.9, a cell module is irreducible if its Gram matrix is non-degenerate. That is, if the determinant of the Gram matrix is non-zero.

Definition 3.1.1. Let M_1, M_2 be basis elements of a cell module $\Delta_n(\lambda)$. We define a bilinear form $\langle -, - \rangle$: $\Delta_n(\lambda) \times \Delta_n(\lambda) \to \mathbb{C}$ as follows: Turning M_2 upside down and placing it above M_1 such that each vertex in M_1 is connected with the corresponding vertex in M_2. If any through string in M_1 becomes a non-through string in this composition then $\langle M_1, M_2 \rangle = 0$, otherwise $\langle M_1, M_2 \rangle = a^{r_1} b^{r_2}$ where r_1 and r_2 are the number of a and b-loops constructed in this composition. We can extend this form bilinearly to all of $\Delta_n(\lambda)$.

Suppose that $D_1 = C^{\lambda}_{S_1,T_1}, D_2 = C^{\lambda}_{S_2,T_2} \in \mathrm{FC}_n$ where S_1, S_2 are the initial parts and T_1, T_2 are the final parts of D_1, D_2 respectively. We have that if $D_3 = D_1 D_2$ has label λ then $D_3 = c\, C^{\lambda}_{S_1,T_2}$ where $c \in \mathbb{C}$ is product of $a's$ and $b's$ that obtained from the composition of T_1 with S_2. If $D_3 = D_1D_2$ has label $\lambda' < \lambda$ then D_3 is zero in $\Delta_n(\lambda)$, (see proof of Theorem 1.3.14 case I). Now, consider the bilinear form Φ_λ that defined in Definition 1.3.4. We can see that $\Phi_\lambda(C_{T_1}, C_{S_2}) = c$ if D_3 has label λ and $\Phi_\lambda(C_{T_1}, C_{S_2}) = 0$ if D_3 has label $\lambda' < \lambda$. Therefore, the bilinear form, $\langle -, - \rangle$, that defined in Definition 3.1.1 is equivalent to Φ_λ.

Lemma 3.1.2. *For all basis diagrams M_1, $M_2 \in \Delta_n(\lambda)$ and $D \in \mathrm{FC}_n$, we have*

(i) $\langle M_1, M_2 \rangle = \langle M_2, M_1 \rangle$

(ii) $\langle M_1, DM_2 \rangle = \langle D^* M_1, M_2 \rangle$

where D^* is the reflection of D about a horizontal line which was defined in Lemma 1.3.13

Proof. We can check these relations by drawing the diagrams for each side to show that the left and the right hand sides are equal for (i) and (ii). □

Definition 3.1.3. Let $\Delta_n(\lambda)$ be a cell module with ordered basis

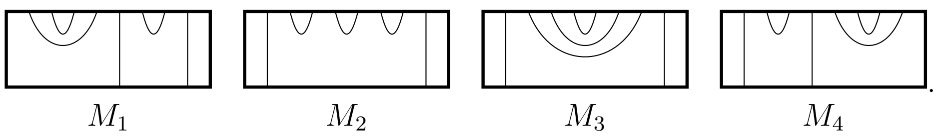
$$\{M_1, M_2, \ldots, M_r\}.$$

We define the Gram matrix, $G_n(\lambda)$, associated with $\Delta_n(\lambda)$ to be the matrix

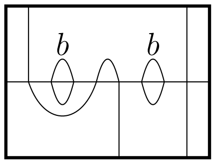
$$G_n(\lambda)_{i,j} = \langle M_i, M_j \rangle$$

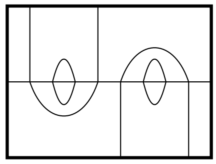
where i, $j = 1, 2, \ldots, r$.

Example 3.1.4. Consider the algebra FC_4, and the cell module $\Delta_4(aa)$. The basis of $\Delta_4(aa)$ is the set $\{M_1, M_2, M_3, M_4\}$, where



We can apply Definition 3.1.1 to find the inner product of any two basis elements of $\Delta_4(aa)$. For instance,

$\langle M_1, M_2 \rangle =$ 

$= b^2, \quad \langle M_1, M_4 \rangle =$  $= 0$

To find the Gram matrix of $\Delta_4(aa)$ we need to compute $\langle M_i, M_j \rangle$ for all i, $j = 1, \ldots, 4$, then we have

$$G_4(aa) = \begin{pmatrix} ab^2 & b^2 & b & 0 \\ b^2 & ab^2 & ab & b^2 \\ b & ab & ab^2 & b \\ 0 & b^2 & b & ab^2 \end{pmatrix}.$$

Calculating the determinant of $G_4(aa)$ we find that

$$\det G_4(aa) = a^2 b^6 (a^2 - 2)(b^2 - 1).$$

The problem of finding the zeros of determinant of the Gram matrix $G_n(\lambda)$ for general λ will be very difficult. However, the machinery of tower of recollement can give us much simpler conditions for this problem. By Theorem 2.1.8(ii), we have

Corollary 3.1.5. *The algebras* $\mathrm{FC}_n(a, b)$ *are semisimple over* $\mathbb{C}$ *if and only if the parameters* a, b *are such that*

$$\prod_{n' \leq n} \prod_{\lambda \in \Lambda_{n'}^{n'-2}} \det\left(G_{n'}(\lambda)\right) \neq 0.$$

From this corollary we deduce that in order to find the values of $a, b \in \mathbb{C}$ such that the algebras $\mathrm{FC}_n(a, b)$ are not semisimple we only need to find the determinant for the Gram matrix $G_n(\lambda)$ for all $\lambda \in \Lambda_n^{n-2}$.

3.2 The Gram matrices for the cell modules

In this section we shall use the restriction rules to give a general form of the Gram matrix, $G_n(\lambda)$, for a cell module $\Delta_n(\lambda)$, where $\lambda \in \Lambda_n^{n-2}$.

Throughout this chapter, we will use the notation (s,t) to distinguish between the even and the odd case and as following:

Definition 3.2.1. Let a, $b \in \mathbb{C}$ be the parameters of the algebra $\mathrm{FC}_n(a,b)$. We define the ordered pair (s,t) to be

$$(s,t) = \begin{cases} (a,b) \text{ if } n \text{ is even,} \\ (b,a) \text{ if } n \text{ is odd.} \end{cases}$$

Remark 3.2.2. Let $\lambda \in \Lambda_n^{n-2}$, then $\lambda \in \Lambda_{n-2}^{n-2}$ and by Remark 2.2.5 we have $\dim \Delta_{n-2}(\lambda) = 1$, and hence, the Gram matrix $G_{n-2}(\lambda)$ represents the inner product of a diagram with itself. Then $G_{n-2}(\lambda) = a^h b^{h'}$, for some integers h, h'. We know that a, b are non-zero, then $G_{n-2}(\lambda)$ is just a non-zero complex number.

Remark 3.2.3. We can use similar calculations as that used in Proposition 1.4.11 to get the following.

(1) If $\lambda = \lambda_1 ss \in \Lambda_n^{n-2}$, then

- (i) $\dim \Delta_{n-1}(\lambda_1 ssst) = 1$.
- (ii) $\dim \Delta_{n-1}(\lambda_1 st) > 1$.
- (iii) $\lambda_1 st \in \Lambda_{n-1}^{n-3}$, that is, $\dim \Delta_{n-3}(\lambda_1 st) = 1$.

2) If $\lambda = \lambda_1 st \in \Lambda_n^{n-2}$, then

- (i) $\dim \Delta_{n-1}(\lambda_1) > 1$.
- (ii) $\dim \Delta_{n-1}(\lambda_1 ss) = 1$.
- (iii) $\dim \Delta_{n-1}(\lambda_1 stts) = 1$.
- (iv) $\lambda_1 \in \Lambda_{n-1}^{n-3}$, that is, $\dim \Delta_{n-3}(\lambda_1) = 1$.

Now, suppose that $\lambda = \lambda_1 ss \in \Lambda_n^{n-2}$, and note that if the short exact sequence

$$\Delta_{n-1}(\lambda_1 st) \to \mathrm{res}(\Delta_n(\lambda_1 ss)) \to \Delta_{n-1}(\lambda_1 s^3 t)$$

is split, then there is a splitting

$$\tau : \mathrm{res}(\Delta_n(\lambda_1 ss)) \to \Delta_{n-1}(\lambda_1 st) \oplus \Delta_{n-1}(\lambda_1 s^3 t).$$

If we arrange the basis of $\Delta_n(\lambda_1 ss)$ such that those of $\Delta_{n-1}(\lambda_1 st)$ come first, we can represent τ by a matrix of the form

$$Q_{n,\lambda} = \begin{bmatrix} I & Q^1_{n,\lambda} \\ 0 & I \end{bmatrix},$$

where $Q^1_{n,\lambda}$ represents the embedding of $\Delta_{n-1}(\lambda_1 s^3 t)$ into $\mathrm{res}(\Delta_n(\lambda_1 ss))$. We define a bilinear form on $\Delta_{n-1}(\lambda_1 st) \oplus \Delta_{n-1}(\lambda_1 s^3 t)$ by

$$\langle\langle x + x', y + y' \rangle\rangle = \langle \tau^{-1}(x + x'), \tau^{-1}(y + y') \rangle_{n,\lambda_1 ss}$$

for x, $y \in \Delta_{n-1}(\lambda_1 st)$ and x', $y' \in \Delta_{n-1}(\lambda_1 s^3 t)$. This form is symmetric and invariant, thus, arguing as in [38, Lemma 4.3], we have

$$\langle\langle x + x', y + y' \rangle\rangle = t\langle x, y \rangle_{n-1,\lambda_1 st} + \alpha_{n,\lambda_1 ss} \langle x', y' \rangle_{n-1,\lambda_1 s^3 t}$$

for some $\alpha_{n,\lambda_1 ss} \in \mathbb{C}$. In matrix form this becomes

$$tG_{n-1}(\lambda_1 st) \oplus \alpha_{n,\lambda_1 ss} G_{n-1}(\lambda_1 s^3 t) = (Q^{-1}_{n,\lambda})^T G_n(\lambda_1 ss) Q^{-1}_{n,\lambda}.$$

Then

$$G_n(\lambda_1 ss) = Q^T_{n,\lambda} \begin{bmatrix} tG_{n-1}(\lambda_1 st) & 0 \\ 0 & \alpha_{n,\lambda_1 ss} G_{n-1}(\lambda_1 s^3 t) \end{bmatrix} Q_{n,\lambda}. \tag{3.1}$$

Multiplying the matrices in the right hand side of (3.1) we get

$$G_n(\lambda_1 ss) = \begin{bmatrix} G'_{11} & G'_{12} \\ G'_{21} & G'_{22} \end{bmatrix} \tag{3.2}$$

where

$$\begin{aligned} G'_{11} &= tG_{n-1}(\lambda_1 st) & G'_{12} &= tG_{n-1}(\lambda_1 st)Q^1_{n,\lambda} \\ G'_{21} &= t(Q^1_{n,\lambda})^T G_{n-1}(\lambda_1 st) & G'_{22} &= t(Q^1_{n,\lambda})^T G_{n-1}(\lambda_1 st)Q^1_{n,\lambda} + \alpha_{n,\lambda} G_{n-1}(\lambda st). \end{aligned}$$

Now, we turn to the case when $\lambda = \lambda_1 st \in \Lambda_n^{n-2}$. If the short exact sequence

$$0 \to \Delta_{n-1}(\lambda_1 ss) \to \mathrm{res}(\Delta_n(\lambda))/\Delta_{n-1}(\lambda_1) \to \Delta_{n-1}(\lambda ts) \to 0.$$

is split, then there is a splitting

$$\chi : \mathrm{res}(\Delta_n(\lambda)) \to \Delta_{n-1}(\lambda_1) \oplus \Delta_{n-1}(\lambda_1 ss) \oplus \Delta_{n-1}(\lambda ts)$$

If we arrange the basis of $\Delta_n(\lambda)$ such that those of $\Delta_{n-1}(\lambda_1)$ come first, then basis of $\Delta_{n-1}(\lambda_1 ss)$, and finally, the basis of $\Delta_{n-1}(\lambda ts)$, then we can represent χ by a matrix of the form

$$P_{n,\lambda} = \begin{bmatrix} I & P^1_{n,\lambda} & P^2_{n,\lambda} \\ 0 & I & P^3_{n,\lambda} \\ 0 & 0 & I \end{bmatrix}$$

We define a bilinear form on $\Delta_{n-1}(\lambda_1) \oplus \Delta_{n-1}(\lambda_1 ss) \oplus \Delta_{n-1}(\lambda ts)$ by

$$\langle\langle x + x' + x'', y + y' + y'' \rangle\rangle = \langle \chi^{-1}(x + x' + x''), \chi^{-1}(y + y' + y'') \rangle_{n,\lambda_1 st}$$

for x,$y \in \Delta_{n-1}(\lambda_1)$, x', $y' \in \Delta_{n-1}(\lambda_1 ss)$, x'', $y'' \in \Delta_{n-1}(\lambda ts)$.

$$\begin{aligned} &\langle\langle x + x' + x'',\ y + y' + y'' \rangle\rangle \\ &\qquad = \langle x, y \rangle_{n-1,\lambda_1} + \beta_{n,\lambda} \langle x', y' \rangle_{n-1,\lambda_1 ss} + \delta_{n,\lambda} \langle x'', y'' \rangle_{n-1,\lambda ts} \end{aligned}$$

for some $\beta_{n,\lambda}, \delta_{n,\lambda} \in \mathbb{C}$. In matrix form, we get

$$G_{n-1}(\lambda_1) \oplus \beta_{n,\lambda} G_{n-1}(\lambda_1 ss) \oplus \delta_{n,\lambda} G_{n-1}(\lambda ts) = (P_{n,\lambda}^{-1})^T G_n(\lambda_1 st) P_{n,\lambda}^{-1}.$$

Then

$$G_n(\lambda_1 st) = P_{n,\lambda}^T \begin{bmatrix} G_{n-1}(\lambda_1) & 0 & 0 \\ 0 & \beta_{n,\lambda} G_{n-1}(\lambda_1 ss) & 0 \\ 0 & 0 & \delta_{n,\lambda} G_{n-1}(\lambda ts) \end{bmatrix} P_{n,\lambda}. \tag{3.3}$$

Multiplying the matrices in the right hand side of (3.3), we get

$$G_n(\lambda_1 ab) = \begin{bmatrix} G'_{11} & G'_{12} & G'_{13} \\ G'_{21} & G'_{22} & G'_{23} \\ G'_{31} & G'_{32} & G'_{33} \end{bmatrix} \tag{3.4}$$

where

$$\begin{aligned}
G'_{11} &= G_{n-1}(\lambda_1), \\
G'_{12} &= G_{n-1}(\lambda_1) P_{n,\lambda}^1, \\
G'_{13} &= G_{n-1}(\lambda_1) P_{n,\lambda}^2, \\
G'_{21} &= (P_{n,\lambda}^1)^T G_{n-1}(\lambda_1), \\
G'_{22} &= (P_{n,\lambda}^1)^T G_{n-1}(\lambda_1) P_{n,\lambda}^1 + \beta_{n,\lambda} G_{n-1}(\lambda_1 ss), \\
G'_{23} &= (P_{n,\lambda}^1)^T G_{n-1}(\lambda_1) P_{n,\lambda}^2 + \beta_{n,\lambda} G_{n-1}(\lambda_1 ss) P_{n,\lambda}^3, \\
G'_{31} &= (P_{n,\lambda}^2)^T G_{n-1}(\lambda_1), \\
G'_{32} &= (P_{n,\lambda}^2)^T G_{n-1}(\lambda_1) P_{n,\lambda}^1 + (P_{n,\lambda}^3)^T \beta_{n,\lambda} G_{n-1}(\lambda_1 ss), \\
G'_{33} &= (P_{n,\lambda}^2)^T G_{n-1}(\lambda_1) P_{n,\lambda}^2 + (P_{n,\lambda}^3)^T \beta_{n,\lambda} G_{n-1}(\lambda_1 ss) P_{n,\lambda}^3 + \delta_{n,\lambda} G_{n-1}(\lambda ts).
\end{aligned}$$

Let $r[M]$ and $c[M]$ be number of rows and number of columns of a matrix M respectively, and we denote the dimension of M by $r \times c$. To keep the multiplication rules of matrices, the dimension of $Q_{n,\lambda}$ and $P_{n,\lambda}$ and its blocks are given in the following lemmas.

Lemma 3.2.4. *Suppose* $\lambda = \lambda_1 ss \in \Lambda_n^{n-2}$, *and consider the matrix*

$$Q_{n,\lambda} = \begin{bmatrix} I & Q^1_{n,\lambda} \\ 0 & I \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}.$$

Then we have :

$$\begin{aligned}
\dim Q_{11} &= \dim G_{n-1}(\lambda_1 st), \\
\dim Q_{22} &= \dim G_{n-1}(\lambda st) = 1 \times 1, \\
\dim Q_{21} &= r[G_{n-1}(\lambda st)] \times c[G_{n-1}(\lambda_1 st)] = 1 \times c[G_{n-1}(\lambda_1 st)], \\
\dim Q_{12} &= \dim Q^1_{n,\lambda} = r[G_{n-1}(\lambda_1 st)] \times c[G_{n-1}(\lambda st)] = r[G_{n-1}(\lambda_1 st)] \times 1.
\end{aligned}$$

Lemma 3.2.5. *Suppose* $\lambda = \lambda_1 ss \in \Lambda_n^{n-2}$, *and* $Q^1_{n,\lambda} = \begin{bmatrix} Q^{11}_{n,\lambda} \\ Q^{12}_{n,\lambda} \\ Q^{13}_{n,\lambda} \end{bmatrix}$. *Then we have*

$$\begin{aligned}
\dim Q^{11}_{n,\lambda} &= r[G_{n-2}(\lambda_1)] \times c[G_{n-2}(\lambda)] = r[G_{n-2}(\lambda_1)] \times 1, \\
\dim Q^{12}_{n,\lambda} &= r[G_{n-2}(\lambda)] \times c[G_{n-2}(\lambda)] = 1 \times 1, \\
\dim Q^{13}_{n,\lambda} &= r[G_{n-2}(\lambda_1 stts)] \times c[G_{n-2}(\lambda)] = 1 \times 1.
\end{aligned}$$

Lemma 3.2.6. *Suppose* $\lambda = \lambda_1 st \in \Lambda_n^{n-2}$, *and consider the matrix*

$$P_{n,\lambda} = \begin{bmatrix} I & P^1_{n,\lambda} & P^2_{n,\lambda} \\ 0 & I & P^3_{n,\lambda} \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}.$$

Then we have

$$\begin{aligned}
\dim P_{11} &= \dim G_{n-1}(\lambda_1), \\
\dim P_{22} &= \dim G_{n-1}(\lambda_1 ss) = 1 \times 1, \\
\dim P_{33} &= \dim G_{n-1}(\lambda ts) = 1 \times 1, \\
\dim P_{21} &= r[G_{n-1}(\lambda_1 ss)] \times c[G_{n-1}(\lambda_1)] = 1 \times c[G_{n-1}(\lambda_1)], \\
\dim P_{31} &= r[G_{n-1}(\lambda ts)] \times c[G_{n-1}(\lambda_1)] = 1 \times c[G_{n-1}(\lambda_1)],
\end{aligned}$$

$$\dim P_{32} = r[G_{n-1}(\lambda ts)] \times c[G_{n-1}(\lambda_1 ss)] = 1 \times 1,$$
$$\dim P_{12} = \dim(P^1_{n,\lambda}) = r[G_{n-1}(\lambda_1)] \times c[G_{n-1}(\lambda_1 ss)] = r[G_{n-1}(\lambda_1)] \times 1,$$
$$\dim P_{13} = \dim(P^2_{n,\lambda}) = r[G_{n-1}(\lambda_1)] \times c[G_{n-1}(\lambda ts)] = r[G_{n-1}(\lambda_1)] \times 1,$$
$$\dim P_{23} = \dim(P^3_{n,\lambda}) = r[G_{n-1}(\lambda_1 ss)] \times c[G_{n-1}(\lambda ts)] = 1 \times 1.$$

Lemma 3.2.7. *Suppose $\lambda = \lambda_1 st \in \Lambda_n^{n-2}$, and assume that $P^3_{n,\lambda} = \begin{bmatrix} P^{31}_{n,\lambda} \\ P^{32}_{n,\lambda} \end{bmatrix}$. Then we have*

$$\dim P^{31}_{n,\lambda} = \dim G_{n-2}(\lambda) = 1 \times 1,$$
$$\dim P^{32}_{n,\lambda} = r[G_{n-2}(\lambda_1 s^3 t)] \times c[G_{n-2}(\lambda)] = 1 \times 1.$$

Lemma 3.2.8. *Suppose $\lambda = \lambda_1 st \in \Lambda_n^{n-2}$ such that $\lambda_1 = \lambda_2 ts$, and assume that*

$$P^1_{n,\lambda} = \begin{bmatrix} P^{11}_{n,\lambda} \\ P^{12}_{n,\lambda} \\ P^{13}_{n,\lambda} \end{bmatrix}, \quad \textit{and} \quad P^2_{n,\lambda} = \begin{bmatrix} P^{21}_{n,\lambda} \\ P^{22}_{n,\lambda} \\ P^{23}_{n,\lambda} \end{bmatrix}$$

Then we have

$$\dim P^{11}_{n,\lambda} = \dim P^{21}_{n,\lambda} = r[G_{n-2}(\lambda_2)] \times c[G_{n-2}(\lambda)] = r[G_{n-2}(\lambda_2)] \times 1,$$
$$\dim P^{12}_{n,\lambda} = \dim P^{22}_{n,\lambda} = r[G_{n-2}(\lambda_2 tt)] \times c[G_{n-2}(\lambda)] = 1 \times 1,$$
$$\dim P^{13}_{n,\lambda} = \dim P^{23}_{n,\lambda} = \dim G_{n-2}(\lambda)] = 1 \times 1.$$

Lemma 3.2.9. *Suppose $\lambda = \lambda_1 st \in \Lambda_n^{n-2}$ such that $\lambda_1 = \lambda_2 ss$ and assume that*

$$P^1_{n,\lambda} = \begin{bmatrix} P^{11}_{n,\lambda} \\ P^{12}_{n,\lambda} \end{bmatrix}, \quad \textit{and} \quad P^2_{n,\lambda} = \begin{bmatrix} P^{21}_{n,\lambda} \\ P^{22}_{n,\lambda} \end{bmatrix}$$

Then we have

$$\dim P^{11}_{n,\lambda} = \dim P^{21}_{n,\lambda} = r[G_{n-2}(\lambda_2 st)] \times c[G_{n-2}(\lambda)] = r[G_{n-2}(\lambda_2 st)] \times 1,$$
$$\dim P^{12}_{n,\lambda} = \dim P^{22}_{n,\lambda} = \dim G_{n-2}(\lambda) = 1 \times 1$$

3.3 Another form of the Gram matrices for the cell modules

We can find another form of the Gram matrix for a cell module $\Delta_n(\lambda)$, where $\lambda \in \Lambda_n^{n-2}$, and then we can compare it with the Gram matrices that we found in the previous section. This will help us to find the values of $\alpha_{n,\lambda}$, $\beta_{n,\lambda}$, and $\delta_{n,\lambda}$, and then to find the Gram determinant for these modules.

Depending on how we restrict each cell module, we describe the general form of the basis diagrams for $\Delta_n(\lambda)$ where $\lambda \in \Lambda_n^{n-2}$. We start with the case when $\lambda = \lambda_1 ss$. In Table 3.1, we introduce the general form of the basis diagrams for $\Delta_n(\lambda)$ when $\lambda = \lambda_1 ss \in \Lambda_n^{n-2}$. We divide these basis diagrams into four subsets V_1, V_2, V_3, and V_4, such that the set of basis diagrams $\mathcal{W}_n(\lambda_1 ss) = V_1 \cup V_2 \cup V_3 \cup V_4$. For $i = 1, \ldots, 4$, the diagrams in V_i has the form v_i, where v_i are as described in Table 3.1.

When we restrict $\Delta_n(\lambda)$, we get the modules in the second row, and when we restrict the modules in the second row we get the modules in the third row. In addition, any basis diagram for the modules in row (3) must have the form of the diagrams in row (4). The diagrams in row (5) are the basis for the modules in row (2) and it is obtained by applying the appropriate restriction rule for each diagram in row (4). That is, the diagrams $m_2' = f(m_2)$, $m_3 = g(m_3')$, $m_1' = \phi_{n-1}(m_1)$, and $m_4' = \phi_{n-1}(m_4)$, where f and g are the Add and the Cut-procedures on the module $\Delta_n(\lambda_1 ss)$, and ϕ_n is the inclusion map that defined in equation 2.4. In similar way, (applying the restriction rules to the diagrams in row (5)), we can get the diagrams in row (6) which represent the basis for the cell module $\Delta_n(\lambda_1 ss)$.

Since, $\lambda_1 ss \in \Lambda_n^{n-2}$ then, by Remark 3.2.3(1)(iii), we have $\lambda_1 st \in \Lambda_{n-1}^{n-3}$. Hence, by Remark 3.2.3(2)(i), we have $\dim \Delta_{n-2}(\lambda_1) > 1$, by Remark 3.2.3(2)(ii) and (iii), we have $\dim \Delta_{n-2}(\lambda_1 ss) = \dim \Delta_{n-2}(\lambda_1 stts) = 1$.

Note that, the cardinality of V_1 is equal the dimension of $\Delta_{n-2}(\lambda_1)$ which is greater than one, whereas the cardinality of V_2, V_3 and V_4 is one because it represents the dimension of $\Delta_{n-2}(\lambda_1 ss)$, $\Delta_{n-2}(\lambda_1 stts)$ and $\Delta_{n-2}(\lambda_1 ss)$ respectively.

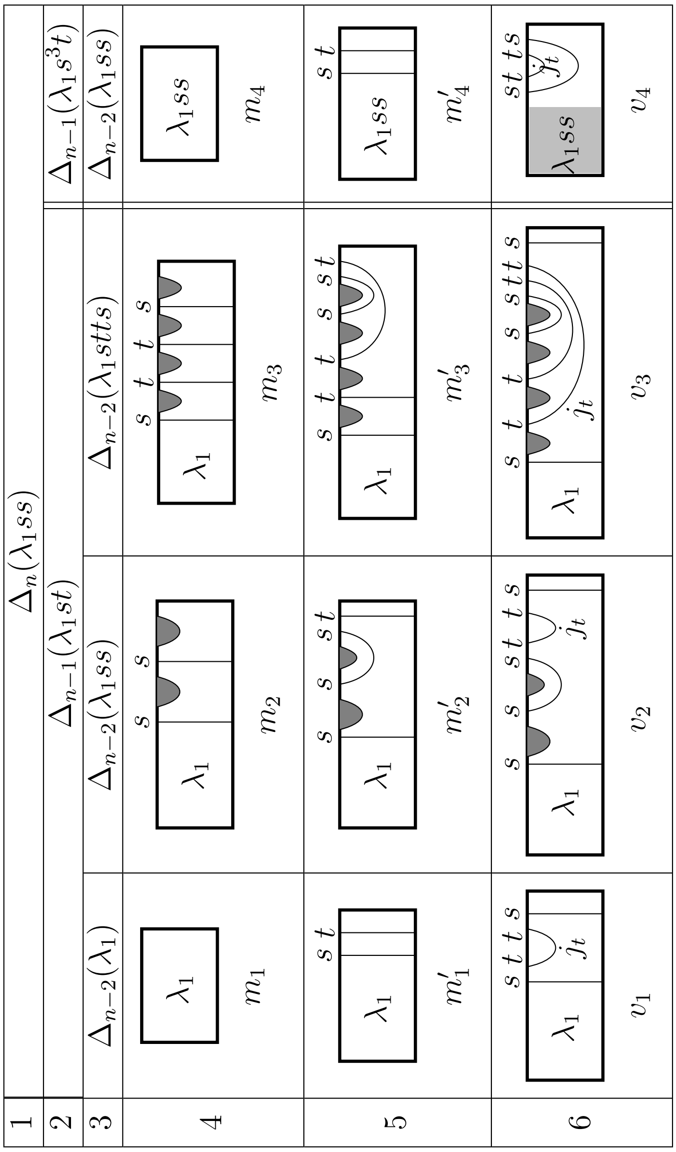


Table 3.1: Basis forms for $\Delta_n(\lambda_1 ss)$

Proposition 3.3.1. *Let $\lambda = \lambda_1 ss \in \Lambda_n^{n-2}$ and $V_1 \cup V_2 \cup V_3 \cup V_4$, be the basis of $\Delta_n(\lambda)$ such that the diagrams in V_i has the form of v_i that is described in Table 3.1. Suppose that the Gram matrix of $\Delta_n(\lambda)$ is constructed such that the basis of $\Delta_{n-1}(\lambda_1 st)$ comes first. That is, $G_n(\lambda)$ has the form*

$$\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \begin{array}{c|ccc|c} \langle -,- \rangle & V_1 & v_2 & v_3 & v_4 \\ \hline V_1 & & & & \\ v_2 & & G_{11} & & G_{12} \\ v_3 & & & & \\ \hline v_4 & & G_{21} & & G_{22} \end{array} \tag{3.5}$$

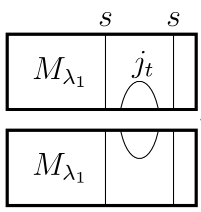
where G_{11} is the matrix of inner products of elements from $V_1 \cup V_2 \cup V_3$, G_{12} is the column vector of inner products of elements of $V_1 \cup V_2 \cup V_3$ with v_4, and $G_{22} = \langle v_4, v_4 \rangle$. Then we have

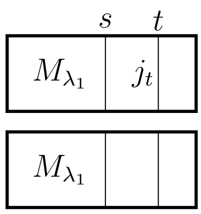
$$\text{(i) } G_{11} = tG_{n-1}(\lambda_1 st), \text{ (ii) } G_{12} = \begin{bmatrix} 0 \\ tG_{n-2}(\lambda) \\ G_{n-2}(\lambda) \end{bmatrix}, \text{ (iii) } G_{22} = stG_{n-2}(\lambda).$$

Proof. Since each basis diagram in V_1 has the form v_1 and $V_2 = \{v_2\}$, $V_3 = \{v_3\}$, $V_4 = \{v_4\}$, thus we will use the notations v_1, v_2, v_3, v_4 for the basis diagrams for $\Delta_n(\lambda_1 ss)$ and as described in Table 3.1.

Let v_i' be the diagram obtained by removing the last two vertices of v_i and sliding down the non-through string j_t. Thus v_i' is a basis diagram with $2n-2$ vertices and its label is $\lambda_1 st$, that is, v_i' is a basis diagram of $\Delta_{n-1}(\lambda_1 st)$.

(i) G_{11} represents the inner product of v_i with v_j for all i, j = 1, 2, 3. Thus we need to prove that $\langle v_i, v_j \rangle = t\langle v_i', v_j' \rangle$ for all $i, j = 1, 2, 3$. To find $\langle v_1, v_1 \rangle$, we have

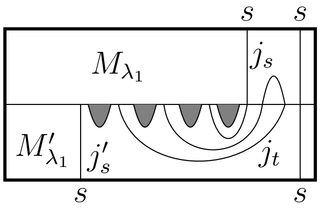
$v_1 \cdot v_1 =$ 

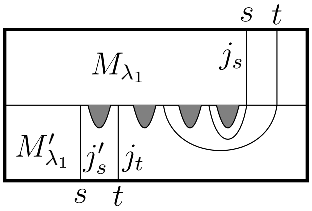
, and $v_1' \cdot v_1' =$ 

where M_{λ_1} is a diagram that has $2n-4$ vertices and its label is λ_1 . We can see that $\langle v_1, v_1 \rangle = t\langle M_{\lambda_1}, M_{\lambda_1} \rangle = t\langle v_1', v_1' \rangle$.

By the same argument, we can show that $\langle v_2, v_2\rangle = t\langle v_2', v_2'\rangle$, $\langle v_3, v_3\rangle = t\langle v_3', v_3'\rangle$ and $\langle v_1, v_2\rangle = t\langle v_1', v_2'\rangle$.

To prove that $\langle v_1, v_3\rangle = t\langle v_1', v_3'\rangle$, consider the following diagrams

$v_1 \cdot v_3 =$ 

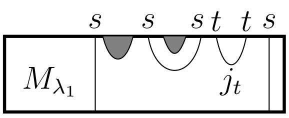
and $v_1' \cdot v_3' =$ 

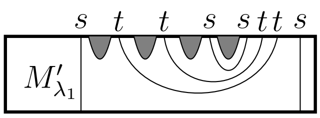
where M_{λ_1} and M'_{λ_1} are subdiagrams (their label is λ_1) of v_1 and v_3 respectively. The string j_s in v_1 becomes either a non-through string or a through string in $v_1 \cdot v_3$. We will discuss these two cases:

Case I: If j_s in $v_1 \cdot v_3$ is a non-through string, then it is a non-through string in $v_1' \cdot v_3'$ as well. We thus conclude that $\langle v_1, v_3\rangle = 0$, and $\langle v_1', v_3'\rangle = 0$.

Case II: If j_s in $v_1 \cdot v_3$ is a through string. Then j_s is either connected with j_s' or with an s-string in the subdiagram M'_{λ_1}. In both cases the string j_t must construct a t-loop in $v_1 \cdot v_3$, otherwise, j_t will cross j_s. Thus, when we remove the last two vertices of v_1 and v_3, we are cutting the t-loop while the other loops (if they exist) have no changes. Hence, the result.

We now show that $\langle v_2, v_3\rangle = t\langle v_2', v_3'\rangle$. Recall that

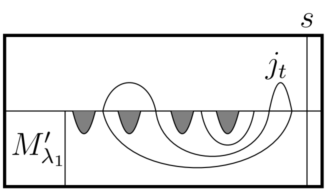
$v_2 =$ 

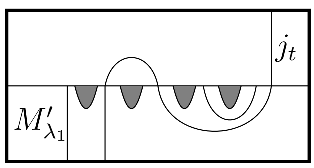
, $v_3 =$ 

.

We will discuss all the possibilities for the shape of j_t in the diagram $v_2 \cdot v_3$.

Case I: If the string j_t constructs a t-loop in the diagram $v_2 \cdot v_3$, then

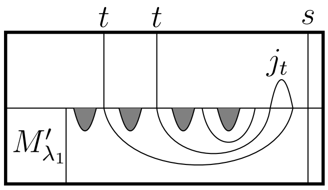
$v_2 \cdot v_3 =$ 

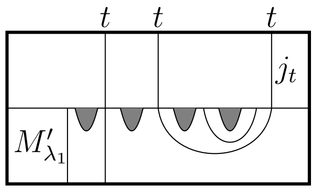
and $v_2' \cdot v_3' =$ 

The t-loop in $v_2 \cdot v_3$ becomes a through string in $v_2' \cdot v_3'$, therefore,

$$\langle v_2, v_3 \rangle = t \langle v_2', v_3' \rangle.$$

Case II: If j_t is a non-through string in $v_2 \cdot v_3$. Then

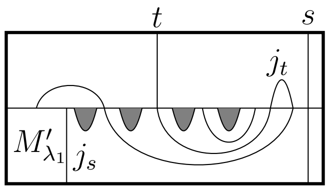
$v_2 \cdot v_3 =$ 

and $v_2' \cdot v_3' =$ 

It is clear that

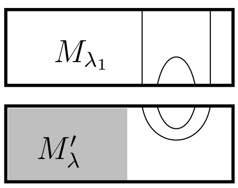
$$\langle v_2, v_3 \rangle = \langle v_2', v_3' \rangle = 0.$$

Case III: If j_t is a through string as indicated in this diagram,

$v_2 \cdot v_3 =$ 

then j_t must be connected with a t-string in the subdiagram M'_{λ_1}. This will force j_s to be a non-through string that leads to $\langle v_2, v_3 \rangle = \langle v_2', v_3' \rangle = 0$. (If j_s is a through string in $v_2 \cdot v_3$ then j_t must cross j_s).

(ii) We want to find $\langle v_i, v_4 \rangle$ for $i = 1$,2, 3. The zero submatrix in G_{12} represents the inner product of v_1 with v_4. Since,

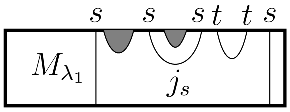
$v_1 \cdot v_4 =$ 

then

$$\langle v_1, v_4 \rangle = 0.$$

The block matrix $tG_{n-2}(\lambda_1 ss)$ in G_{12} represents the inner product of v_2 with v_4.

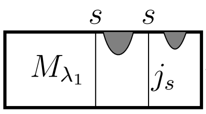
Recall that,

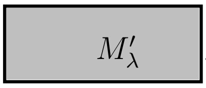
$v_2 =$ 

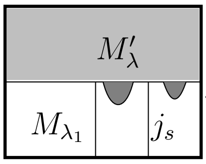
, $v_4 =$ 

.

Suppose v_2'' is the diagram obtained by removing the last four vertices of v_2 and sliding down j_s, and v_4'' is the diagram obtained by removing the last four vertices of v_4. Then v_2'' and v_4'' have $2n-4$ vertices and their label is $\lambda_1 ss$, that is, they are basis diagrams of $\Delta_{n-2}(\lambda_1 ss)$. As diagrams

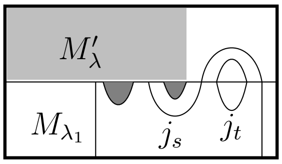
$v_2'' =$ 

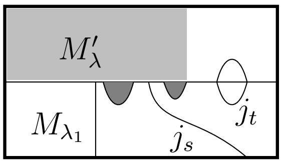
, $v_4'' =$ 

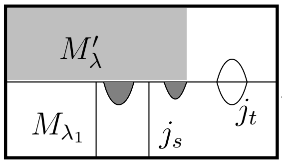
, and $v_4'' \cdot v_2'' =$ 

.

In addition, we have

$v_4 \cdot v_2 =$ 

$=$ 

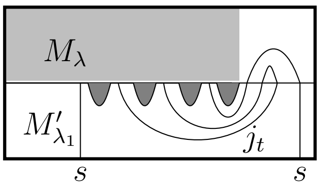
$=$ 

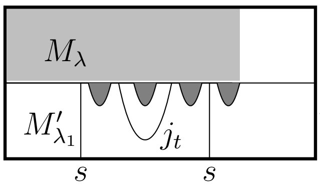
.

By comparing the last diagram with $v_4'' \cdot v_2''$, we can deduce that

$$\langle v_4, v_2 \rangle = t \langle v_4'', v_2'' \rangle.$$

We now show that $\langle v_3, v_4 \rangle = G_{n-2}(\lambda_1 ss)$. Consider the diagrams

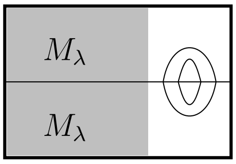
$v_4 \cdot v_3 =$ 

and $v_4'' \cdot v_3'' =$ 

.

We notice that these two diagrams have the same inner product. In addition, v_3'', v_4'' are diagrams with $2n-4$ vertices and their label is $\lambda_1 ss$, that is, v_3'', v_4'' are basis elements of $\Delta_{n-2}(\lambda_1 ss)$.

(iii) $stG_{n-2}(\lambda_1 ss)$ in G_{22} represents the inner product of v_4 with itself. Now since,

$v_4 \cdot v_4 =$ 

then

$$\langle v_4, v_4 \rangle = st \langle v_4'', v_4'' \rangle,$$

where v_4'' is as defined in (ii). $\square$

Now, we turn to the case when $\lambda = \lambda_1 st \in \Lambda_n^{n-2}$. In this case, we have two sub-cases, $\lambda_1 = \lambda_2 ss$ and $\lambda_1 = \lambda_2 ts$. By using the restriction, then the basis diagrams for $\Delta_n(\lambda_1 st)$ such that $\lambda_1 = \lambda_2 ss$ and $\lambda_1 = \lambda_2 ts$ are described in Tables 3.2 and 3.3 respectively, where the modules in the second row are the restriction of $\Delta_n(\lambda_1 st)$, the modules in the third row are the restriction of the modules in the second row. In addition, the diagrams in row(4) (resp. row(5) and row(6)), represent the basis diagrams for the modules in row(3) (resp. row(2) and row(1)).

Consider the case $\lambda = \lambda_1 st \in \Lambda_n^{n-2}$ such that $\lambda_1 = \lambda_2 ts$. Then the basis diagrams for $\Delta_n(\lambda)$ is the union of the sets W_i, $i = 1, \ldots, 5$, where W_i is the set of diagrams of form w_i that described in Table 3.3. From Remark 3.2.3, we can find that the cardinality of W_1 is greater than one, while W_i, $i = 1, \ldots, 4$ has only one diagram.

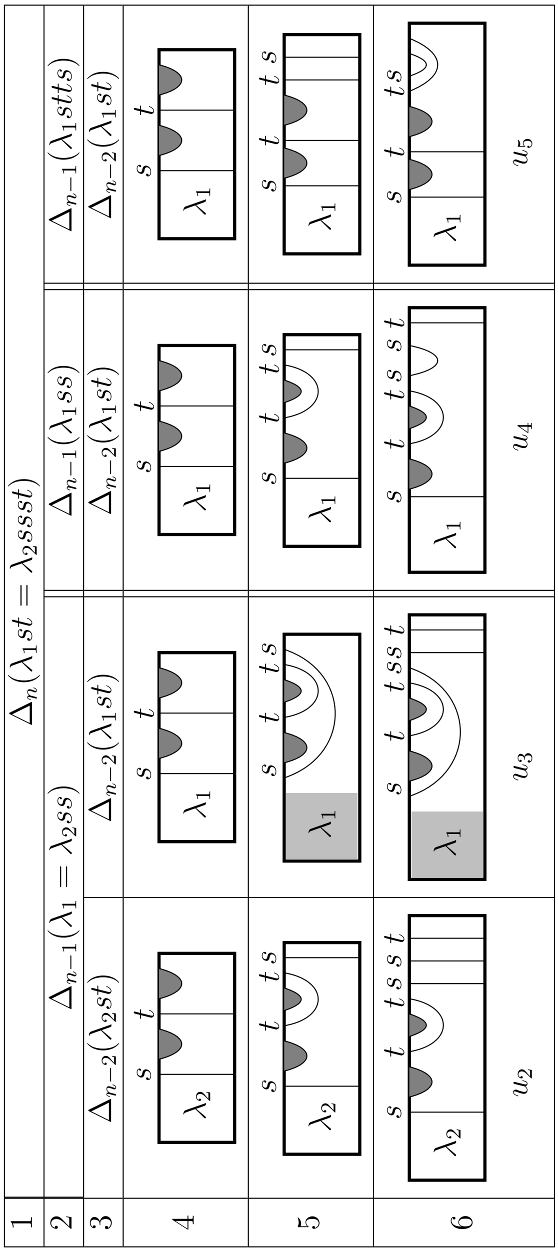


Table 3.2: Basis forms for $\Delta_n(\lambda_2 ssst)$

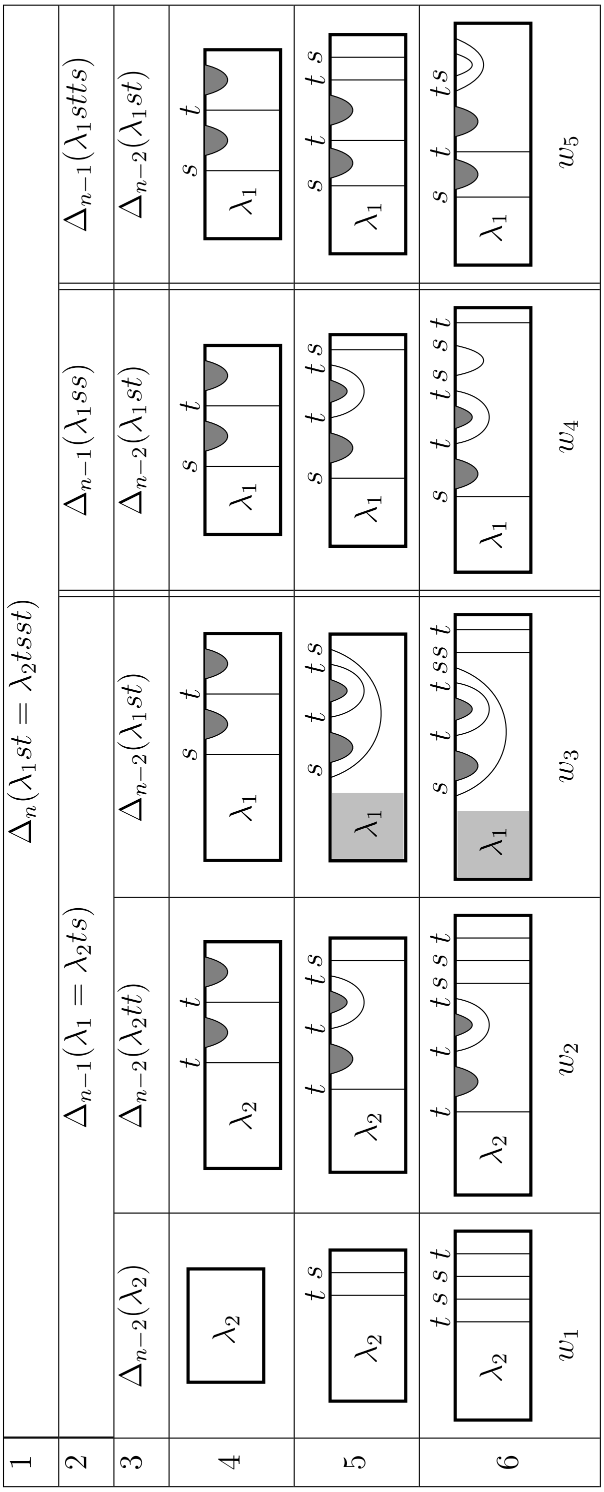


Table 3.3: Basis forms for $\Delta_n(\lambda_2 tsst)$

Proposition 3.3.2. *Let $\lambda = \lambda_1 st \in \Lambda_n^{n-2}$ with $\lambda_1 = \lambda_2 ts$, and $W_1 \cup \ldots \cup W_5$ be the basis of $\Delta_n(\lambda)$ such that the diagrams in W_i has the form of w_i that described in Table 3.3. Suppose that the Gram matrix of $\Delta_n(\lambda)$ is constructed such that the order of the basis is: the basis of $\Delta_{n-1}(\lambda_1)$, the basis of $\Delta_{n-1}(\lambda_1 ss)$, and then the basis of $\Delta_{n-1}(\lambda_1 stts)$. That is, $G_n(\lambda)$ has the form*

$$\begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix} = \begin{array}{c|ccc|c|c} \langle -,- \rangle & W_1 & w_2 & w_3 & w_4 & w_5 \\ \hline W_1 & & & & & \\ w_2 & & G_{11} & & G_{12} & G_{13} \\ w_3 & & & & & \\ \hline w_4 & & G_{21} & & G_{22} & G_{23} \\ \hline w_5 & & G_{31} & & G_{32} & G_{33} \end{array} \tag{3.6}$$

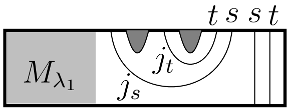
where G_{11} is the matrix of inner products of elements of $W_1 \cup W_2 \cup W_3$, the column vectors G_{12} and G_{13} are inner products of elements of $W_1 \cup W_2 \cup W_3$ with w_4 and w_5 respectively, and $G_{22} = \langle w_4, w_4 \rangle$, $G_{23} = \langle w_4, w_5 \rangle$, $G_{33} = \langle w_5, w_5 \rangle$. Then we have

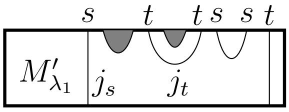
$$\text{(i)} \quad G_{11} = G_{n-1}(\lambda_1) \qquad \text{(ii)} \quad G_{12} = \begin{bmatrix} 0 \\ 0 \\ tG_{n-2}(\lambda) \end{bmatrix} \qquad \text{(iii)} \quad G_{13} = \begin{bmatrix} 0 \\ 0 \\ G_{n-2}(\lambda) \end{bmatrix}$$

$$\text{(iv)} \quad G_{22} = stG_{n-2}(\lambda) \qquad \text{(v)} \quad G_{23} = sG_{n-2}(\lambda) \qquad \text{(vi)} \quad G_{33} = stG_{n-2}(\lambda).$$

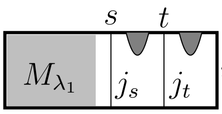
Proof. We will use the notations $w_1, \ldots, w_5$ for the basis diagrams for $\Delta_n(\lambda)$.
(i) The matrix G_{11} represents the inner product of w_i with w_j for i, $j = 1, 2, 3$. Let w_i' be the diagram obtained by removing the last two vertices of w_i. Then w_i' is a basis diagram of $\Delta_{n-1}(\lambda_1)$. Since w_i ends with at least two through strings for all $i = 1, 2, 3$, then these two through strings will not effect the value of the inner product. Hence $\langle w_i, w_j \rangle = \langle w_i', w_j' \rangle$.
(ii) The first (resp. the second) block matrix of G_{12} which is the zero matrix represents the inner product of w_1 (resp. w_2) with w_4. It is easy to check that $\langle w_1, w_4 \rangle = \langle w_2, w_4 \rangle = 0$. The submatrix $tG_{n-2}(\lambda)$ in G_{12} comes from the inner

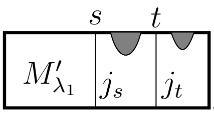
product of w_3 with w_4. Recall that

$w_3 =$ 

, and $w_4 =$ 

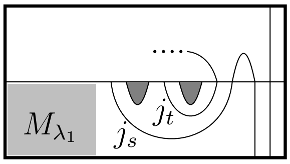
Suppose w_3'' and w_4'' are diagrams obtained by removing the last four vertices of w_3 and w_4 respectively, and sliding down every non-through string still connected to w_3 and w_4, (sliding down j_s, j_t from w_3, and j_t from w_4). Thus

$w_3'' =$ 

, and $w_4'' =$ 

.

In addition, the diagrams w_3'' and w_4'' have $2n-4$ vertices and its label is $\lambda_1 st$, therefore, they are basis diagrams of $\Delta_{n-2}(\lambda_1 st)$. Consider the diagram

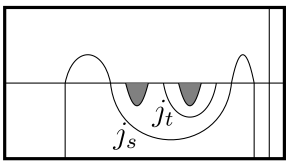
$w_4 \cdot w_3 =$ 

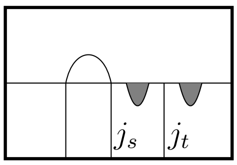
we can see that, the inner product of w_4 with w_3 is not effected when we remove the last three vertices and cut the non-through string j_s in w_3 because it will never construct loop in the diagram $w_4 \cdot w_3$. However, j_t always makes a t-loop in $w_4 \cdot w_3$. To prove this, we discuss the following cases:

Case I: If j_s in the diagram $w_4 \cdot w_3$ is connected with a vertex in w_3, then

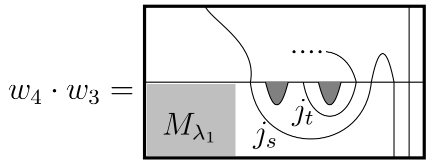
$$\langle w_3, w_4 \rangle = \langle w_3'', w_4'' \rangle = 0,$$

as explained in these diagrams

$w_4 \cdot w_3 =$ 

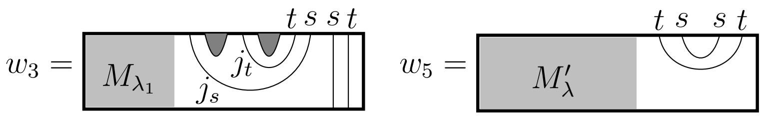
$w_4'' \cdot w_3'' =$ 

Case II: If j_s in $w_4 \cdot w_3$ is connected with a vertex in w_4, then j_t must construct a t-loop, otherwise, j_s will cross j_t as explained in this diagram

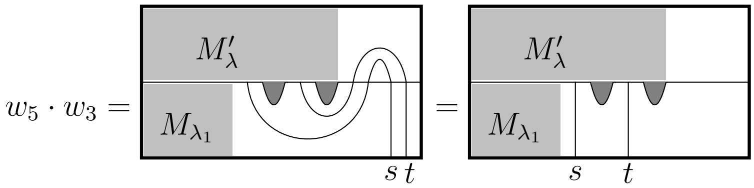


This indicate that when we remove the last four vertices of w_3 and w_4, we cut only one t-loop. Therefore, $\langle w_3, w_4\rangle = t\langle w_3'', w_4''\rangle$.

(iii) The first (resp. the second) block matrix of G_{13} which is the zero matrix is represents the inner product of w_1 (resp. w_2) with w_5. It is easy to check that $\langle w_1, w_5\rangle = \langle w_2, w_5\rangle = 0$. The submatrix $G_{n-2}(\lambda)$ in G_{13} represents the inner product of w_3 with w_5.



Suppose w_3'' is as defined in (ii), and let w_5'' be the diagram obtained by removing the last four vertices of w_5. Then w_3'' and w_5'' are basis diagrams of $\Delta_{n-2}(\lambda)$. Consider the diagram



which show that the inner product of the diagrams w_5 with w_3 has no change when we remove the last four vertices form them. Hence the result.

(iv) The first block matrix, $stG_{n-2}(\lambda)$, of G_{22} is represents the inner product of w_4 with itself. The last four vertices of the diagram $w_4 \cdot w_4$ make one s-loop and one t-loop, thus when we remove these vertices we lost the factor st.

(v) The submatrix $sG_{n-2}(\lambda)$ in G_{23} represents the inner product of w_4 with w_5. The last four vertices of of w_4 and w_5 contribute s to the inner product of them. Thus removing these vertices leads to lost the factor s.

(vi) The inner product of w_5 with itself represents the submatrix $stG_{n-2}(\lambda)$ in

G_{33}. The result follows immediately from the diagram $w_5 \cdot w_5$. □

Now, Let $\lambda = \lambda_1 st \in \Lambda_n^{n-2}$ with $\lambda_1 = \lambda_2 ss$. The basis diagrams for $\Delta_n(\lambda)$ is the union of the sets U_i, $i = 2, \ldots, 5$, (for convenience we start from 2), where U_i is the set of diagrams of form u_i that described in Table 3.2. By Remark 3.2.3, we have that U_2 has more than one diagrams, and the sets U_3, U_4 and U_5 has only one diagram.

Proposition 3.3.3. *Let $\lambda = \lambda_1 st \in \Lambda_n^{n-2}$ be such that $\lambda_1 = \lambda_2 ss$, and $U_2 \cup \ldots \cup U_5$ be the basis of $\Delta_n(\lambda)$ such that the diagrams in U_i has the form u_i that described in Table 3.2. Suppose that the Gram matrix of $\Delta_n(\lambda)$ is constructed such that the order of the basis is the basis of $\Delta_{n-1}(\lambda_1)$, the basis of $\Delta_{n-1}(\lambda_1 ss)$, and then the basis of $\Delta_{n-1}(\lambda_1 stts)$. That is, $G_n(\lambda)$ has the form*

$$\begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix} = \begin{array}{c|c|c|c} \langle -,- \rangle & U_2 \quad u_3 & u_4 & u_5 \\ \hline \begin{array}{c} U_2 \\ u_3 \end{array} & G_{11} & G_{12} & G_{13} \\ \hline u_4 & G_{21} & G_{22} & G_{23} \\ \hline u_5 & G_{31} & G_{32} & G_{33} \end{array} \tag{3.7}$$

where G_{11} is the matrix of inner products of elements of $U_2 \cup U_3$, the column vectors G_{12} and G_{13} are inner products of $U_2 \cup U_3$ with u_4 and u_5 respectively, moreover, $G_{22} = \langle u_4, u_4 \rangle$, $G_{23} = \langle u_4, u_5 \rangle$, $G_{33} = \langle u_5, u_5 \rangle$. Then we have

$$\text{(i)} \quad G_{11} = G_{n-1}(\lambda_1) \qquad \text{(ii)} \quad G_{12} = \begin{bmatrix} 0 \\ tG_{n-2}(\lambda) \end{bmatrix} \qquad \text{(iii)} \quad G_{13} = \begin{bmatrix} 0 \\ G_{n-2}(\lambda) \end{bmatrix}$$

$$\text{(iv)} \quad G_{22} = stG_{n-2}(\lambda) \qquad \text{(v)} \quad G_{23} = sG_{n-2}(\lambda) \qquad \text{(vi)} \quad G_{33} = stG_{n-2}(\lambda).$$

Proof. Consider the diagrams $u_2, \ldots, u_5$ that have been given in Table 3.2 as a basis diagrams for $\Delta_n(\lambda)$.

From Tables 3.2 and 3.3, we can notice that, for $i, j = 3, 4, 5$, we have the diagrams u_i and w_i has the same form as diagrams. Thus $\langle u_i, u_j \rangle = \langle w_i, w_j \rangle$. In addition, $\langle u_2, u_i \rangle = \langle w_2, w_i \rangle$ for all $i = 2, \ldots, 5$. Hence, the result. □

3.4 The determinant of the Gram matrices

The values of a and b are such that the algebra $\mathrm{FC}_n(a,b)$ is semisimple is determined by finding the determinant of the Gram matrices for the cell modules. We start by finding the values of $\alpha_{n,\lambda}, \beta_{n,\lambda}$, and $\delta_{n,\lambda}$. Then we introduce the general form for the Gram determinant for any cell module $\Delta_n(\lambda)$ such that $\lambda \in \Lambda_n^{n-2}$.

Lemma 3.4.1. *For $\lambda \in \Lambda_n^{n-2}$, we have $G_{n-2}(\lambda) = G_{n-1}(\lambda ts)$.*

Proof. Since, $\lambda \in \Lambda_n^{n-2}$ then by Remark 3.2.2, we have $\dim \Delta_{n-2}(\lambda) = 1$, and by Proposition 1.4.11(i), we have $\dim \Delta_{n-1}(\lambda ts) = 1$. Suppose that v is the basis diagram for $\Delta_{n-2}(\lambda)$, then v has $(2n-4)$ vertices and its label is λ. Let $\bar{v}$ be a diagram that obtained from v by adding two straight through strings (with label ts) to the right end of v. Then $\bar{v}$ is a diagram with $(2n-2)$ vertices and its label is λts. Thus $\bar{v}$ must be the basis diagram for $\Delta_{n-1}(\lambda ts)$. We can notice that $\langle v, v\rangle = \langle \bar{v}, \bar{v}\rangle$ because the last two through strings in $\bar{v}$ will not contribute to the value of the inner product. Hence, $G_{n-1}(\lambda ts) = G_{n-2}(\lambda)$. □

Proposition 3.4.2. *For $n \geq 4$, let $\lambda = \lambda_1 st \in \Lambda_n^{n-2}$ with $\lambda_1 = \lambda_2 ts$, then*

$$\beta_{n,\lambda} = \frac{s(s^2-2)}{s^2-1} \quad \textit{and} \quad \delta_{n,\lambda} = \frac{s}{t}(t^2-1).$$

Proof. To find the values of $\beta_{n,\lambda}$ and $\delta_{n,\lambda}$, we will compare the Gram matrix $G_n(\lambda_1 tsst)$ in (3.4) with the Gram matrix $G_n(\lambda_1 tsst)$ in (3.6). The block matrices $G'_{i,j}$ in (3.4) must be equal to the corresponding block matrices $G_{i,j}$ in (3.6) for $i,j = 1,2,3$. The reason is that we use the same ordered basis in both cases to construct the Gram matrix $G_n(\lambda)$. Then we have:

(1) $G'_{12} = G_{12}$. That is,

$$G_{n-1}(\lambda_1)P^1_{n,\lambda} = \begin{bmatrix} 0 \\ 0 \\ tG_{n-2}(\lambda) \end{bmatrix} \tag{3.8}$$

Since $\lambda = \lambda_1 st \in \Lambda_n^{n-2}$, then by Remark 3.2.3(2)(iv), $\lambda_1 = \lambda_2 ts \in \Lambda_{n-1}^{n-3}$. Thus $G_{n-1}(\lambda_1)$ is the matrix in (3.4) with replacing n by $n-1$, λ by λ_1, λ_1

by λ_2, and swap s with t. The matrix $P^1_{n,\lambda}$ is as defined in Lemma 3.2.8, that is, $P^1_{n,\lambda} = \begin{bmatrix} P^{11}_{n,\lambda} \\ P^{12}_{n,\lambda} \\ P^{13}_{n,\lambda} \end{bmatrix}$. Now, we multiply $G_{n-1}(\lambda_1)$ with $P^1_{n,\lambda}$ and compare it with the R.H.S. in (3.8), we get

$$0 = G_{n-2}(\lambda_2)\left(P^{11}_{n,\lambda} + P^1_{n-1,\lambda_1}P^{12}_{n,\lambda} + P^2_{n-1,\lambda_1}P^{13}_{n,\lambda}\right) \tag{3.9a}$$

$$\begin{aligned} 0 &= (P^1_{n-1,\lambda_1})^T G_{n-2}(\lambda_2)\left(P^{11}_{n,\lambda} + P^1_{n-1,\lambda_1}P^{12}_{n,\lambda} + P^2_{n-1,\lambda_1}P^{13}_{n,\lambda}\right) \\ &\quad + \beta_{n-1,\lambda_1} G_{n-2}(\lambda_2 tt)\left(P^{12}_{n,\lambda} + P^3_{n-1,\lambda_1}P^{13}_{n,\lambda}\right) \end{aligned} \tag{3.9b}$$

$$\begin{aligned} tG_{n-2}(\lambda) &= (P^2_{n-1,\lambda_1})^T G_{n-2}(\lambda_2)\left(P^{11}_{n,\lambda} + P^1_{n-1,\lambda_1}P^{12}_{n,\lambda} + P^2_{n-1,\lambda_1}P^{13}_{n,\lambda}\right) \\ &\quad + \beta_{n-1,\lambda_1}(P^3_{n-1,\lambda_1})^T G_{n-2}(\lambda_2 tt)\left(P^{12}_{n,\lambda} + P^3_{n-1,\lambda_1}P^{13}_{n,\lambda}\right) \\ &\quad + \delta_{n-1,\lambda_1} G_{n-2}(\lambda) P^{13}_{n,\lambda} \end{aligned} \tag{3.9c}$$

Substitute (3.9a) in (3.9b), we get

$$\beta_{n-1,\lambda_1} G_{n-2}(\lambda_2 tt)\left(P^{12}_{n,\lambda} + P^3_{n-1,\lambda_1}P^{13}_{n,\lambda}\right) = 0. \tag{3.10}$$

Substitute (3.9a) and (3.10) in (3.9c), we get

$$tG_{n-2}(\lambda) = \delta_{n-1,\lambda_1} G_{n-2}(\lambda) P^{13}_{n,\lambda}.$$

Since $\lambda \in \Lambda^{n-2}_n$ then, by Remark 3.2.2, $G_{n-2}(\lambda)$ is a non-zero complex number, thus

$$P^{13}_{n,\lambda} = \frac{t}{\delta_{n-1,\lambda_1}}. \tag{3.11}$$

(2) $G'_{13} = G_{13}$. That is,

$$G_{n-1}(\lambda_1) P^2_{n,\lambda} = \begin{bmatrix} 0 \\ 0 \\ G_{n-2}(\lambda) \end{bmatrix} \tag{3.12}$$

where $G_{n-1}(\lambda_1)$ is as in (1), and $P^2_{n,\lambda}$ is as defined in Lemma 3.2.8, that is

$P^2_{n,\lambda} = \begin{bmatrix} P^{21}_{n,\lambda} \\ P^{22}_{n,\lambda} \\ P^{23}_{n,\lambda} \end{bmatrix}$. If we multiply $G_{n-1}(\lambda_1)$ with $P^2_{n,\lambda}$ and compare it with the R.H.S. in (3.12), we get

$$0 = G_{n-2}(\lambda_2)\left(P^{21}_{n,\lambda} + P^1_{n-1,\lambda_1}P^{22}_{n,\lambda} + P^2_{n-1,\lambda_1}P^{23}_{n,\lambda}\right) \tag{3.13a}$$

$$\begin{aligned} 0 &= (P^1_{n-1,\lambda_1})^T G_{n-2}(\lambda_2)\left(P^{21}_{n,\lambda} + P^1_{n-1,\lambda_1}P^{22}_{n,\lambda} + P^2_{n-1,\lambda_1}P^{23}_{n,\lambda}\right) \\ &\quad + \beta_{n-1,\lambda_1}G_{n-2}(\lambda_2 tt)\left(P^{22}_{n,\lambda} + P^3_{n-1,\lambda_1}P^{23}_{n,\lambda}\right) \end{aligned} \tag{3.13b}$$

$$\begin{aligned} G_{n-2}(\lambda) &= (P^2_{n-1,\lambda_1})^T G_{n-2}(\lambda_2)\left(P^{21}_{n,\lambda} + P^1_{n-1,\lambda_1}P^{22}_{n,\lambda} + P^2_{n-1,\lambda_1}P^{23}_{n,\lambda}\right) \\ &\quad + \beta_{n-1,\lambda_1}(P^3_{n-1,\lambda_1})^T G_{n-2}(\lambda_2 tt)\left(P^{22}_{n,\lambda} + P^3_{n-1,\lambda_1}P^{23}_{n,\lambda}\right) \\ &\quad + \delta_{n-1,\lambda_1}G_{n-2}(\lambda)P^{23}_{n,\lambda} \end{aligned} \tag{3.13c}$$

Substitute (3.13a) in (3.13b), we get

$$\beta_{n-1,\lambda_1}G_{n-2}(\lambda_2 tt)\left(P^{22}_{n,\lambda} + P^3_{n-1,\lambda_1}P^{23}_{n,\lambda}\right) = 0. \tag{3.14}$$

Substitute (3.13a) and (3.14) in (3.13c), we get

$$G_{n-2}(\lambda) = \delta_{n-1,\lambda_1}G_{n-2}(\lambda)P^{23}_{n,\lambda},$$

again by Remark 3.2.2, $G_{n-2}(\lambda)$ is a non-zero complex number, thus

$$P^{23}_{n,\lambda} = \frac{1}{\delta_{n-1,\lambda_1}}. \tag{3.15}$$

(3) $G'_{22} = G_{22}$. That is,

$$stG_{n-2}(\lambda) = (P^1_{n,\lambda})^T G_{n-1}(\lambda_1)P^1_{n,\lambda} + \beta_{n,\lambda}G_{n-1}(\lambda_1 ss) \tag{3.16}$$

From (1), we have $G_{n-1}(\lambda_1)P^1_{n,\lambda} = \begin{bmatrix} 0 \\ 0 \\ tG_{n-2}(\lambda) \end{bmatrix}$, and

$$(P^1_{n,\lambda})^T = \begin{bmatrix} (P^{11}_{n,\lambda})^T & (P^{12}_{n,\lambda})^T & (P^{13}_{n,\lambda})^T \end{bmatrix}.$$

Substitute them in (3.16), we get

$$stG_{n-2}(\lambda) = \begin{bmatrix}(P^{11}_{n,\lambda})^T & (P^{12}_{n,\lambda})^T & (P^{13}_{n,\lambda})^T\end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ tG_{n-2}(\lambda) \end{bmatrix} + \beta_{n,\lambda} G_{n-1}(\lambda_1 ss)$$

$$= (P^{13}_{n,\lambda})^T tG_{n-2}(\lambda) + \beta_{n,\lambda} G_{n-1}(\lambda_1 ss). \tag{3.17}$$

Since, $\lambda = \lambda_1 st \in \Lambda_n^{n-2}$ then, by Remark 3.2.3, $\dim \Delta_{n-1}(\lambda_1 ss) = 1$, and by Remark 3.2.2, we have $\dim \Delta_{n-2}(\lambda) = 1$. Thus $G_{n-1}(\lambda_1 ss)$ and $G_{n-2}(\lambda = \lambda_1 st)$ are one by one matrices. We can argue as in the proof of Proposition 3.3.1(i) to show that

$$G_{n-1}(\lambda_1 ss) = tG_{n-2}(\lambda) \tag{3.18}$$

Thus (3.17) becomes

$$stG_{n-2}(\lambda) = (P^{13}_{n,\lambda})^T tG_{n-2}(\lambda) + \beta_{n,\lambda} tG_{n-2}(\lambda).$$

By substituting $(P^{13}_{n,\lambda})^T = \dfrac{t}{\delta_{n-1,\lambda_1}}$ (from (3.11)), and dividing by $tG_{n-2}(\lambda)$, we get

$$\beta_{n,\lambda} = s - \frac{t}{\delta_{n-1,\lambda_1}} \tag{3.19}$$

(4) $G'_{23} = G_{23}$. That is,

$$sG_{n-2}(\lambda) = (P^1_{n,\lambda})^T G_{n-1}(\lambda_1) P^2_{n,\lambda} + \beta_{n,\lambda} G_{n-1}(\lambda_1 ss) P^3_{n,\lambda} \tag{3.20}$$

Recall that, by Lemma 3.2.8, we have $(P^1_{n,\lambda})^T = \begin{bmatrix}(P^{11}_{n,\lambda})^T & (P^{12}_{n,\lambda})^T & (P^{13}_{n,\lambda})^T\end{bmatrix}$, and from (3.12) we have

$$G_{n-1}(\lambda_1) P^2_{n,\lambda} = \begin{bmatrix} 0 \\ 0 \\ G_{n-2}(\lambda) \end{bmatrix}.$$

Substitute them in (3.20), we get

$$\begin{aligned} sG_{n-2}(\lambda) &= \begin{bmatrix} (P^{11}_{n,\lambda})^T & (P^{12}_{n,\lambda})^T & (P^{13}_{n,\lambda})^T \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ G_{n-2}(\lambda) \end{bmatrix} + \beta_{n,\lambda} G_{n-1}(\lambda_1 ss) P^3_{n,\lambda} \\ &= (P^{13}_{n,\lambda})^T G_{n-2}(\lambda) + \beta_{n,\lambda} G_{n-1}(\lambda_1 ss) P^3_{n,\lambda}. \end{aligned} \tag{3.21}$$

From (3.11), we have $(P^{13}_{n,\lambda})^T = \dfrac{t}{\delta_{n-1,\lambda_1}}$, and from (3.18), we have $G_{n-1}(\lambda_1 ss) = tG_{n-2}(\lambda)$. Substituting these two values in (3.21), we get

$$sG_{n-2}(\lambda) = \frac{t}{\delta_{n-1,\lambda_1}} G_{n-2}(\lambda) + t\beta_{n,\lambda} G_{n-2}(\lambda) P^3_{n,\lambda}.$$

Dividing by $G_{n-2}(\lambda)$ and, from (3.19), substitute $\beta_{n,\lambda} = s - \dfrac{t}{\delta_{n-1,\lambda_1}}$, we get

$$P^3_{n,\lambda} = \frac{1}{t}. \tag{3.22}$$

(5) $G'_{33} = G_{33}$. That is,

$$stG_{n-2}(\lambda) = (P^2_{n,\lambda})^T G_{n-1}(\lambda_1) P^2_{n,\lambda} + (P^3_{n,\lambda})^T \beta_{n,\lambda} G_{n-1}(\lambda_1 ss) P^3_{n,\lambda} + \delta_{n,\lambda} G_{n-1}(\lambda ts) \tag{3.23}$$

By Lemma 3.4.1, we have $G_{n-1}(\lambda ts) = G_{n-2}(\lambda)$, moreover, by (3.22), we have $P^3_{n,\lambda} = \dfrac{1}{t}$, and by (3.18), we have $G_{n-1}(\lambda_1 ss) = tG_{n-2}(\lambda)$, then (3.23) becomes

$$stG_{n-2}(\lambda) = (P^2_{n,\lambda})^T G_{n-1}(\lambda_1) P^2_{n,\lambda} + \frac{1}{t}\beta_{n,\lambda} G_{n-2}(\lambda) + \delta_{n,\lambda} G_{n-2}(\lambda) \tag{3.24}$$

By (3.12), we have $G_{n-1}(\lambda_1) P^2_{n,\lambda} = \begin{bmatrix} 0 \\ 0 \\ G_{n-2}(\lambda) \end{bmatrix}$, and by Lemma 3.2.8, we have $(P^2_{n,\lambda})^T = \begin{bmatrix} (P^{21}_{n,\lambda})^T & (P^{22}_{n,\lambda})^T & (P^{23}_{n,\lambda})^T \end{bmatrix}$. Put them in (3.24), we get

$$stG_{n-2}(\lambda) = (P^{23}_{n,\lambda})^T G_{n-2}(\lambda) + \frac{1}{t}\beta_{n,\lambda} G_{n-2}(\lambda) + \delta_{n,\lambda} G_{n-2}(\lambda). \tag{3.25}$$

From (3.15), substitute $(P^{23}_{n,\lambda})^T = \dfrac{1}{\delta_{n-1,\lambda_1}}$, and from (3.19), substitute $\beta_{n,\lambda} = s - \dfrac{t}{\delta_{n-1,\lambda_1}}$, then divide by $G_{n-2}(\lambda)$, to get

$$\delta_{n,\lambda} = \frac{s}{t}(t^2 - 1)$$

where $\lambda = \lambda_1 st = \lambda_2 tsst \in \Lambda_n^{n-2}$.

Now, by Remark 3.2.3(2)(iv), we have $\lambda_1 = \lambda_2 ts \in \Lambda_{n-1}^{n-3}$, thus

$$\delta_{n-1,\lambda_1} = \frac{t}{s}(s^2 - 1),$$

(we swap s with t because λ ends with st while λ_1 ends with ts). Put δ_{n-1,λ_1} in $\beta_{n,\lambda} = s - \dfrac{t}{\delta_{n-1,\lambda_1}}$, we get

$$\beta_{n,\lambda} = \frac{s(s^2 - 2)}{(s^2 - 1)}.$$

□

Proposition 3.4.3. *For $n \geq 4$, let $\lambda = \lambda_1 st \in \Lambda_n^{n-2}$ be such that $\lambda_1 = \lambda_2 ss$. Then*

$$\beta_{n,\lambda} = s - \frac{t}{\alpha_{n-1,\lambda_1}} \quad \textit{and} \quad \delta_{n,\lambda} = \frac{s}{t}(t^2 - 1).$$

Proof. We will use the same technique that we used in the above proposition to find the values of $\beta_{n,\lambda}$ and $\delta_{n,\lambda}$. The block matrices G'_{ij} for the Gram matrix $G_n(\lambda)$ in (3.4) must be equal to the block matrices G_{ij} for the Gram matrix $G_n(\lambda)$ in (3.7) because we use the same ordered basis in both cases. Then we have:
(1) $G'_{12} = G_{12}$. That is,

$$G_{n-1}(\lambda_1)P^1_{n,\lambda} = \begin{bmatrix} 0 \\ tG_{n-2}(\lambda) \end{bmatrix} \tag{3.26}$$

Since $\lambda = \lambda_1 st \in \Lambda_n^{n-2}$, then by Remark 3.2.3(2)(iv), $\lambda_1 = \lambda_2 ss \in \Lambda_{n-1}^{n-3}$. Then $G_{n-1}(\lambda_1)$ is the matrix in (3.2) with replacing n by $n-1$, λ by λ_1, and λ_1 by λ_2.

The matrix $P^1_{n,\lambda}$ is as defined in Lemma 3.2.9, that is,

$$P^1_{n,\lambda} = \begin{bmatrix} P^{11}_{n,\lambda} \\ \\ P^{12}_{n,\lambda} \end{bmatrix}. \tag{3.27}$$

If we multiply $G_{n-1}(\lambda_1)$with the matrix $P^1_{n,\lambda}$ and compare it with the R.H.S. in (3.26), we get

$$0 = tG_{n-2}(\lambda_2 st)\left(P^{11}_{n,\lambda} + Q^1_{n-1,\lambda_1}P^{12}_{n,\lambda}\right) \tag{3.28a}$$

$$\begin{aligned} tG_{n-2}(\lambda) &= t(Q^1_{n-1,\lambda_1})^T G_{n-2}(\lambda_2 st)\left(P^{11}_{n,\lambda} + Q^1_{n-1,\lambda_1}P^{12}_{n,\lambda}\right) \\ &\quad + \alpha_{n-1,\lambda_1}G_{n-2}(\lambda)P^{12}_{n,\lambda} \end{aligned} \tag{3.28b}$$

Substituting (3.28a) into (3.28b), we get

$$tG_{n-2}(\lambda) = \alpha_{n-1,\lambda_1}G_{n-2}(\lambda)P^{12}_{n,\lambda}. \tag{3.29}$$

By Remark 3.2.2, $G_{n-2}(\lambda)$ is a non-zero complex number, so we may divide (3.29) by $G_{n-2}(\lambda)$. We have

$$P^{12}_{n,\lambda} = \frac{t}{\alpha_{n-1,\lambda_1}}. \tag{3.30}$$

(2) $G'_{13} = G_{13}$. That is,

$$G_{n-1}(\lambda_1)P^2_{n,\lambda} = \begin{bmatrix} 0 \\ G_{n-2}(\lambda) \end{bmatrix}, \tag{3.31}$$

where $G_{n-1}(\lambda_1)$ is as in (1), and $P^2_{n,\lambda}$ is as defined in Lemma 3.2.9, that is, $P^2_{n,\lambda} = \begin{bmatrix} P^{21}_{n,\lambda} \\ P^{22}_{n,\lambda} \end{bmatrix}$. If we multiply $G_{n-1}(\lambda_1)$ with the matrix $P^2_{n,\lambda}$ and compare it with the R.H.S. in (3.31), we get

$$0 = tG_{n-2}(\lambda_2 st)\left(P^{21}_{n,\lambda} + Q^1_{n-1,\lambda_1}P^{22}_{n,\lambda}\right) \tag{3.32a}$$

$$\begin{aligned} G_{n-2}(\lambda) &= t(Q^1_{n-1,\lambda_1})^T G_{n-2}(\lambda_2 st)\left(P^{21}_{n,\lambda} + Q^1_{n-1,\lambda_1}P^{22}_{n,\lambda}\right) \\ &\quad + \alpha_{n-1,\lambda_1}G_{n-2}(\lambda)P^{22}_{n,\lambda} \end{aligned} \tag{3.32b}$$

Substitute (3.32a) in (3.32b), and dividing by $G_{n-2}(\lambda)$, we get

$$P^{22}_{n,\lambda} = \frac{1}{\alpha_{n-1,\lambda_1}}. \tag{3.33}$$

(3) $G'_{22} = G_{22}$. That is,

$$stG_{n-2}(\lambda) = (P^1_{n,\lambda})^T G_{n-1}(\lambda_1) P^1_{n,\lambda} + \beta_{n,\lambda} G_{n-1}(\lambda_1 ss) \tag{3.34}$$

Substitute (3.26) and (3.27) in (3.34), we have

$$\begin{aligned} stG_{n-2}(\lambda) &= \begin{bmatrix} (P^{11}_{n,\lambda})^T & (P^{12}_{n,\lambda})^T \end{bmatrix} \begin{bmatrix} 0 \\ tG_{n-2}(\lambda) \end{bmatrix} + \beta_{n,\lambda} G_{n-1}(\lambda_1 ss) \\ &= (P^{12}_{n,\lambda})^T tG_{n-2}(\lambda) + \beta_{n,\lambda} G_{n-1}(\lambda_1 ss) \end{aligned} \tag{3.35}$$

From (3.30), we have $P^{12}_{n,\lambda} = \dfrac{t}{\alpha_{n-1,\lambda_1}}$, and by (3.18), we have $G_{n-1}(\lambda_1 ss) = tG_{n-2}(\lambda)$. Put these values in (3.35), we get

$$stG_{n-2}(\lambda) = \frac{t}{\alpha_{n-1,\lambda_1}} tG_{n-2}(\lambda) + \beta_{n,\lambda} tG_{n-2}(\lambda).$$

Dividing by $tG_{n-2}(\lambda)$, we get

$$\beta_{n,\lambda} = s - \frac{t}{\alpha_{n-1,\lambda_1}}. \tag{3.36}$$

(4) $G'_{23} = G_{23}$. That is,

$$sG_{n-2}(\lambda) = (P^1_{n,\lambda})^T G_{n-1}(\lambda_1) P^2_{n,\lambda} + \beta_{n,\lambda} G_{n-1}(\lambda_1 ss) P^3_{n,\lambda} \tag{3.37}$$

Substitute (3.27) and (3.31) in (3.37), we get

$$\begin{aligned} sG_{n-2}(\lambda) &= \begin{bmatrix} (P^{11}_{n,\lambda})^T & (P^{12}_{n,\lambda})^T \end{bmatrix} \begin{bmatrix} 0 \\ G_{n-2}(\lambda) \end{bmatrix} + \beta_{n,\lambda} G_{n-1}(\lambda_1 ss) P^3_{n,\lambda} \\ &= (P^{12}_{n,\lambda})^T G_{n-2}(\lambda) + \beta_{n,\lambda} G_{n-1}(\lambda_1 ss) P^3_{n,\lambda} \end{aligned} \tag{3.38}$$

From (3.30), we have $P^{12}_{n,\lambda} = \dfrac{t}{\alpha_{n-1,\lambda_1}}$, and from (3.18), we have

$G_{n-1}(\lambda_1 ss) = tG_{n-2}(\lambda)$. Put these values in (3.38), we get

$$sG_{n-2}(\lambda) = \frac{t}{\alpha_{n-1,\lambda_1}} G_{n-2}(\lambda) + t\beta_{n,\lambda} G_{n-2}(\lambda) P^3_{n,\lambda}.$$

From (3.36), substitute $\beta_{n,\lambda} = s - \dfrac{t}{\alpha_{n-1,\lambda_1}}$, and dividing by $G_{n-2}(\lambda)$, we get

$$P^3_{n,\lambda} = \frac{1}{t}. \tag{3.39}$$

(5) $G'_{33} = G_{33}$. That is,

$$stG_{n-2}(\lambda) = (P^2_{n,\lambda})^T G_{n-1}(\lambda_1) P^2_{n,\lambda} + (P^3_{n,\lambda})^T \beta_{n,\lambda} G_{n-1}(\lambda_1 ss) P^3_{n,\lambda} + \delta_{n,\lambda} G_{n-1}(\lambda ts) \tag{3.40}$$

From Lemma 3.4.1, we have $G_{n-1}(\lambda ts) = G_{n-2}(\lambda)$, moreover, from (3.18), we have $G_{n-1}(\lambda_1 ss) = tG_{n-2}(\lambda)$, and from (3.39), we have $P^3_{n,\lambda} = \dfrac{1}{t}$. Thus (3.40) give us

$$stG_{n-2}(\lambda) = (P^2_{n,\lambda})^T G_{n-1}(\lambda_1) P^2_{n,\lambda} + \frac{1}{t}\beta_{n,\lambda} G_{n-2}(\lambda) + \delta_{n,\lambda} G_{n-2}(\lambda) \tag{3.41}$$

Substitute $(P^2_{n,\lambda})^T = \begin{bmatrix} (P^{21}_{n,\lambda})^T & (P^{22}_{n,\lambda})^T \end{bmatrix}$ (from Lemma 3.2.9), and (3.31) into (3.41), we get

$$\begin{aligned} stG_{n-2}(\lambda) &= \begin{bmatrix} (P^{21}_{n,\lambda})^T & (P^{22}_{n,\lambda})^T \end{bmatrix} \begin{bmatrix} 0 \\ G_{n-2}(\lambda) \end{bmatrix} + \frac{1}{t}\beta_{n,\lambda} G_{n-2}(\lambda) + \delta_{n,\lambda} G_{n-2}(\lambda) \\ &= (P^{22}_{n,\lambda})^T G_{n-2}(\lambda) + \frac{1}{t}\beta_{n,\lambda} G_{n-2}(\lambda) + \delta_{n,\lambda} G_{n-2}(\lambda) \end{aligned}$$

From (3.33), substitute $P^{22}_{n,\lambda} = \dfrac{1}{\alpha_{n-1,\lambda_1}}$, and divided by $G_{n-2}(\lambda)$ we get

$$st = \frac{1}{\alpha_{n-1,\lambda_1}} + \frac{1}{t}\beta_{n,\lambda} + \delta_{n,\lambda}.$$

By (3.36), substitute $\beta_{n,\lambda} = s - \dfrac{t}{\alpha_{n-1,\lambda_1}}$, we get

$$\delta_{n,\lambda} = \frac{s}{t}(t^2 - 1). \quad \square$$

Proposition 3.4.4. *For $n \geq 4$, let $\lambda = \lambda_1 ss \in \Lambda_n^{n-2}$. Then*

$$\alpha_{n,\lambda} = st - \frac{t}{\beta_{n-1,\lambda_1 st}}.$$

Proof. The block matrices G'_{ij} in (3.2) are equals to the block matrices G_{ij} in (3.5). Then we have

(1) $G'_{12} = G_{12}$. That is,

$$G_{n-1}(\lambda_1 st) Q^1_{n,\lambda} = \frac{1}{t} \begin{bmatrix} 0 \\ tG_{n-2}(\lambda) \\ G_{n-2}(\lambda) \end{bmatrix} \tag{3.42}$$

Since, $\lambda = \lambda_1 ss \in \Lambda_n^{n-2}$ then by Remark 3.2.3(1)(iii), $\lambda_1 st \in \Lambda_{n-1}^{n-3}$. Thus $G_{n-1}(\lambda_1 st)$ is the matrix in (3.4) with replacing n by $n-1$. The matrix $Q^1_{n,\lambda}$ is as defined in Lemma 3.2.5, that is, $Q^1_{n,\lambda} = \begin{bmatrix} Q^{11}_{n,\lambda} \\ Q^{12}_{n,\lambda} \\ Q^{13}_{n,\lambda} \end{bmatrix}$. We are going to multiply $G_{n-1}(\lambda_1 st)$ with $Q^1_{n,\lambda}$ and compare it with the R.H.S. in (3.42), we get

$$0 = G_{n-2}(\lambda_1)\left(Q^{11}_{n,\lambda} + P^1_{n-1,\lambda_1 st} Q^{12}_{n,\lambda} + P^2_{n-1,\lambda_1 st} Q^{13}_{n,\lambda}\right) \tag{3.43a}$$

$$\begin{aligned} G_{n-2}(\lambda) = {} & (P^1_{n-1,\lambda_1 st})^T G_{n-2}(\lambda_1)\left(Q^{11}_{n,\lambda} + P^1_{n-1,\lambda_1 st} Q^{12}_{n,\lambda} + P^2_{n-1,\lambda_1 st} Q^{13}_{n,\lambda}\right) \\ & + \beta_{n-1,\lambda_1 st} G_{n-2}(\lambda)\left(Q^{12}_{n,\lambda} + P^3_{n-1,\lambda_1 st} Q^{13}_{n,\lambda}\right) \end{aligned} \tag{3.43b}$$

$$\begin{aligned} \frac{1}{t} G_{n-2}(\lambda) = {} & (P^2_{n-1,\lambda_1 st})^T G_{n-2}(\lambda_1)\left(Q^{11}_{n,\lambda} + P^1_{n-1,\lambda_1 st} Q^{12}_{n,\lambda} + P^2_{n-1,\lambda_1 st} Q^{13}_{n,\lambda}\right) \\ & + (P^3_{n-1,\lambda_1 st})^T \beta_{n-1,\lambda_1 st} G_{n-2}(\lambda)\left(Q^{12}_{n,\lambda} + P^3_{n-1,\lambda_1 st} Q^{13}_{n,\lambda}\right) \\ & + \delta_{n-1,\lambda_1 st} G_{n-2}(\lambda_1 stts) Q^{13}_{n,\lambda} \end{aligned} \tag{3.43c}$$

By substituting (3.43a) in (3.43b), we get

$$G_{n-2}(\lambda) = \beta_{n-1,\lambda_1 st} G_{n-2}(\lambda)\left(Q^{12}_{n,\lambda} + P^3_{n-1,\lambda_1 st} Q^{13}_{n,\lambda}\right) \tag{3.44}$$

Substitute (3.43a) and (3.44) in (3.43c), we get

$$\frac{1}{t}G_{n-2}(\lambda) = (P^3_{n-1,\lambda_1 st})^T G_{n-2}(\lambda) + \delta_{n-1,\lambda_1 st} G_{n-2}(\lambda_1 stts) Q^{13}_{n,\lambda} \tag{3.45}$$

In Propositions 3.4.2 and 3.4.3, we found that when $\lambda = \lambda_1 st \in \Lambda^{n-2}_n$ with $\lambda_1 = \lambda_2 ss$ or $\lambda_1 = \lambda_2 ts$ then we have $P^3_{n,\lambda} = \frac{1}{t}$. Recall that $\lambda_1 st \in \Lambda^{n-3}_{n-1}$, therefore, $P^3_{n-1,\lambda_1 st} = \frac{1}{t}$. Put this value in (3.45) will give us

$$\delta_{n-1,\lambda_1 st} G_{n-2}(\lambda_1 stts) Q^{13}_{n,\lambda} = 0.$$

Note that, $G_{n-2}(\lambda_1 stts)$ is a one by one matrix because $\lambda_1 st \in \Lambda^{n-3}_{n-1}$ and then by Remark 3.2.3(2)(iii), $\dim \Delta_{n-2}(\lambda_1 stts) = 1$. This means that $G_{n-2}(\lambda_1 stts)$ is a non-zero complex number, and we know that $\delta_{n-1,\lambda_1 st}$ is a non zero as well. Thus

$$Q^{13}_{n,\lambda} = 0.$$

Hence, form (3.44), we get

$$Q^{12}_{n,\lambda} = \frac{1}{\beta_{n-1,\lambda_1 st}}.$$

(2) $G'_{22} = G_{22}$. That is,

$$stG_{n-2}(\lambda) = t(Q^1_{n,\lambda})^T G_{n-1}(\lambda_1 st) Q^1_{n,\lambda} + \alpha_{n,\lambda} G_{n-1}(\lambda st) \tag{3.46}$$

Substituting $(Q^1_{n,\lambda})^T$ as defined in (1), and (3.42) in (3.46), we get

$$stG_{n-2}(\lambda) = \begin{bmatrix} (Q^{11}_{n,\lambda})^T & (Q^{12}_{n,\lambda})^T & (Q^{13}_{n,\lambda})^T \end{bmatrix} \begin{bmatrix} 0 \\ tG_{n-2}(\lambda) \\ G_{n-2}(\lambda) \end{bmatrix} + \alpha_{n,\lambda} G_{n-1}(\lambda st).$$

We know that $(Q^{12}_{n,\lambda})^T = \frac{1}{\beta_{n-1,\lambda_1 st}}$ and $(Q^{13}_{n,\lambda})^T = 0$, then we have

$$stG_{n-2}(\lambda) = \frac{t}{\beta_{n-1,\lambda_1 st}} G_{n-2}(\lambda) + \alpha_{n,\lambda} G_{n-1}(\lambda st).$$

By Lemma 3.4.1, we have $G_{n-1}(\lambda st) = G_{n-2}(\lambda)$, and dividing by $G_{n-2}(\lambda)$, we get

$$\alpha_{n,\lambda} = st - \frac{t}{\beta_{n-1,\lambda_1 st}}.$$

□

Remark 3.4.5. To find the values of $\beta_{n,\lambda}$ and $\delta_{n,\lambda}$ for $n = 3$, consider the set of labels $\Lambda(3) = \{ab^2a^2b, a^3b, ab^3, ab\}$, and $\Lambda_3^1 = \{ab\}$. From Example 1.3.18, we have $\det G_3(ab) = ab(a^2-1)(b^2-1)$. Using (3.3), we have

$$\det G_3(ab) = \det G_2(\emptyset) \times \beta_{3,ab} \det G_2(aa) \times \delta_{3,ab} \det G_2(abba).$$

It is easy to check that $\det G_2(abba) = 1$, $\det G_2(aa) = b$, and $\det G_2(\emptyset) = ab$. For convenience, suppose $\delta_{3,ab} = \frac{a}{b}(b^2-1)$, then $\beta_{3,ab} = \frac{1}{a}(a^2-1)$.

Definition 3.4.6. Let $q \in \mathbb{C}$. The *box number*, $[n]_q$, is defined by the equation

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

If $q = \pm 1$ then $[n]_q = nq^{n-1}$.

Proposition 3.4.7. *Let $a = q_a + q_a^{-1}$, $b = q_b + q_b^{-1} \in \mathbb{C}$ be the parameters of the algebra $\mathrm{FC}_n(a,b)$. Suppose $\lambda \in \Lambda_n^{n-2}$, $\mu \geq \emptyset$, $\mu' > \emptyset$, and $z \geq 1$. We have*

(i)

$$\delta_{n,\lambda} = \begin{cases} \frac{a}{b}[3]_{q_b}, & \text{if } \lambda = \mu ab \text{ (n is odd)}, \\ \frac{b}{a}[3]_{q_a}, & \text{if } \lambda = \mu' ba \text{ (n is even)}. \end{cases}$$

(ii)

$$\beta_{n,\lambda} = \begin{cases} \frac{[z+2]_{q_a}}{[z+1]_{q_a}}, & \text{if } \lambda = \mu a^z b \text{ (n is odd)}, \\ \frac{[z+2]_{q_b}}{[z+1]_{q_b}}, & \text{if } \lambda = \mu' b^z a \text{ (n is even)}. \end{cases}$$

(iii)

$$\alpha_{n,\lambda} = \begin{cases} b\frac{[z+3]_{q_a}}{[z+2]_{q_a}}, & \text{if } \lambda = \mu a^{z+1} \text{ (n is even)}, \\ a\frac{[z+3]_{q_b}}{[z+2]_{q_b}}, & \text{if } \lambda = \mu' b^{z+1} \text{ (n is odd)}. \end{cases}$$

Proof. Note that, by the definition of the box number, we have

$$a = [2]_{q_a}, \qquad a^2 - 1 = [3]_{q_a},$$
$$b = [2]_{q_b}, \qquad b^2 - 1 = [3]_{q_b}.$$

(i) This is clear from Propositions 3.4.2 and 3.4.3.

(ii) We will use induction on z to prove (ii).

When $z = 1$, this happens only when $n = 3$ and $\lambda = ab$. By Remark 3.4.5, we have $\beta_{n,ab} = \frac{1}{a}(a^2 - 1) = \frac{[3]_{q_a}}{[2]_{q_a}}$. When $z = 2$, if n is odd then $\lambda = \lambda_1 ba^2b$, where $\lambda_1 > \emptyset$. By Proposition 3.4.2, we have $\beta_{n,\lambda} = \frac{a(a^2-2)}{a^2-1} = \frac{[4]_{q_a}}{[3]_{q_a}}$. If n is even then $\lambda = \lambda_1 ab^2a$, where $\lambda_1 \geq \emptyset$. By Proposition 3.4.2, we have $\beta_{n,\lambda} = \frac{b(b^2-2)}{b^2-1} = \frac{[4]_{q_b}}{[3]_{q_b}}$.

Suppose (ii) is true for all positive integers less than z.

If n is odd then, by Proposition 3.4.3, we have

$$\beta_{n,\mu a^z b} = a - \frac{b}{\alpha_{n-1,\mu a^{z-1}}}. \tag{3.47}$$

Since n is odd, then $n-1$ is even, hence, by Proposition 3.4.4,

$$\alpha_{n-1,\mu a^{z-1}} = ab - \frac{b}{\beta_{n-2,\mu a^{z-2}b}}. \tag{3.48}$$

Applying (3.48) in (3.47), we get

$$\begin{aligned} \beta_{n,\mu a^z b} &= a - \cfrac{b}{ab - \cfrac{b}{\beta_{n-2,\mu a^{z-2}b}}} \\ &= a - \frac{\beta_{n-2,\mu a^{z-2}b}}{a\beta_{n-2,\mu a^{z-2}b} - 1}. \end{aligned} \tag{3.49}$$

By the inductive hypothesis, we have

$$\beta_{n-2,\mu a^{z-2}b} = \frac{[z]_{q_a}}{[z-1]_{q_a}} \tag{3.50}$$

Applying (3.50) in (3.49), we have (recall that $a = q_a + q_a^{-1}$)

$$\beta_{n,\mu a^z b} = a - \cfrac{\cfrac{[z]_{q_a}}{[z-1]_{q_a}}}{a\cfrac{[z]_{q_a}}{[z-1]_{q_a}} - 1}$$

$$= a - \frac{[z]_{q_a}}{a[z]_{q_a} - [z-1]_{q_a}}$$

$$= a - \frac{(q_a^z - q_a^{-z})}{(q_a + q_a^{-1})(q_a^z - q_a^{-z}) - (q_a^{z-1} - q_a^{-(z-1)})}$$

$$= a - \frac{(q_a^z - q_a^{-z})}{q_a^{z+1} - q_a^{-(z-1)} + q_a^{z-1} - q_a^{-(z+1)} - q_a^{z-1} + q_a^{-(z-1)}}$$

$$= (q_a + q_a^{-1}) - \frac{(q_a^z - q_a^{-z})}{q_a^{z+1} - q_a^{-(z+1)}}$$

$$= \frac{(q_a + q_a^{-1})(q_a^{z+1} - q_a^{-(z+1)}) - (q_a^z - q_a^{-z})}{q_a^{z+1} - q_a^{-(z+1)}}$$

$$= \frac{q_a^{z+2} - q_a^{-(z+2)}}{q_a^{z+1} - q_a^{-(z+1)}} = \frac{[z+2]_{q_a}}{[z+1]_{q_a}}$$

By the same way, we can prove that $\beta_{n,\mu' b^z a} = \dfrac{[z+2]_{q_b}}{[z+1]_{q_b}}$ if n is even.

(iii) Suppose n is even, and $\lambda = \mu a^{z+1}$. By Proposition 3.4.4, we have that $\alpha_{n,\mu a^{z+1}} = ab - \dfrac{b}{\beta_{n-1,\mu a^z b}}$. Since $\beta_{n-1,\mu a^z b} = \dfrac{[z+2]_{q_a}}{[z+1]_{q_a}}$, then we have

$$\alpha_{n,\mu a^{z+1}} = ab - \cfrac{b}{\cfrac{[z+2]_{q_a}}{[z+1]_{q_a}}}$$

$$= b\left(a - \frac{[z+1]_{q_a}}{[z+2]_{q_a}}\right)$$

$$= b\frac{[z+3]_{q_a}}{[z+2]_{q_a}}$$

By the same way, we can show that $\alpha_{n,\mu' b^{z+1}} = a\dfrac{[z+3]_{q_b}}{[z+2]_{q_b}}$ when n is odd. $\square$

Remark 3.4.8. (i) Since the values of $\alpha_{n,\lambda}$ and $\beta_{n,\lambda}$ depend only on the last part and the second to last part of the label λ respectively, we will write α_{s^i} when $\lambda = a^{l_1}b^{l_2}\cdots s^i$, $(i > 1)$, and we will write β_{t^j} when $\lambda = a^{l_1}b^{l_2}\cdots t^j s$ where $(s,t) = (a,b)$ if n is even, while $(s,t) = (b,a)$ if n is odd.
(ii) $\delta_{n,\lambda}$ has only two different values depending on whether n is even or odd thus we will use δ_a if λ ends with a. That is, $\delta_a = \frac{b}{a}[3]_{q_a}$ when n is even, and we will use δ_b if λ ends with b. That is, $\delta_b = \frac{a}{b}[3]_{q_b}$ when n is odd.

Proposition 3.4.9. *Let $\Delta_n(\lambda)$ be a cell module such that $\lambda \in \Lambda_n^{n-2}$. We have*

(i) If $\lambda = \mu t^j s^i$ with $i > 1$, then

$$\det G_n(\lambda) = a^h b^{h'} \alpha_{s^i} \det G_{n-1}(\mu t^j s^{i-1} t).$$

(ii) If $\lambda = \mu t^j s$, then

$$\det G_n(\lambda) = a^r b^{r'} \beta_{t^j} \delta_s \det G_{n-1}(\mu t^{j-1}),$$

where h, h', r, r' are integers.

Proof. (i) From (3.1), if $\lambda = \mu t^j s^i$ with $i > 1$, then

$$\det G_n(\lambda) = \det\big(tG_{n-1}(\mu t^j s^{i-1} t)\big) \det\big(\alpha_{s^i} G_{n-1}(\lambda st)\big).$$

Since $\lambda \in \Lambda_n^{n-2}$ then by Remark 3.2.3(1)(i), we have $G_{n-1}(\lambda st)$ is a one dimensional matrix and thus its determinant is $a^{h_1}b^{h_2}$ for some integers h_1, h_2. Hence,

$$\det G_n(\lambda) = t^{h_3}(a^{h_1}b^{h_2})\alpha_{s^i} \det G_{n-1}(\mu t^j s^{i-1} t)$$

where h_3 is the dimension of $\Delta_{n-1}(\mu t^j s^{i-1} t)$.
(ii) From (3.3), if $\lambda = \mu t^j s$ then

$$\det G_n(\lambda) = \det G_{n-1}(\mu t^{j-1}) \det\big(\beta_{t^j} G_{n-1}(\mu t^{j+1})\big) \det\big(\delta_s G_{n-1}(\lambda st)\big).$$

Since $\lambda \in \Lambda_n^{n-2}$ then by Remark 3.2.3(2)(ii) and (iii), we have $G_{n-1}(\mu t^{j+1})$ and $G_{n-1}(\lambda st)$ are one dimensional matrices, then the product of their determinants have the form $a^r b^{r'}$ for some integers r, r'. $\square$

We shall introduce the Gram determinant for the cell module $\Delta_n(\lambda)$ for some special forms of $\lambda \in \Lambda_n^{n-2}$.

Proposition 3.4.10. *For $n \geq 4$, the Gram determinant of the module $\Delta_n(a^{n-2})$ is $\det G_n(a^{n-2}) = a^{r_1}b^{r_2}[3]_{q_b}^{n-2/2}[n]_{q_a}$, for some integers r_1, r_2 .*

Proof. Since $a^{n-2} \in \Lambda_n^{n-2}$ then, by Proposition 3.4.9 (i), we have

$$\det G_n(a^{n-2}) = \det(bG_{n-1}(a^{n-3}b)) \det(\alpha_{a^{n-2}} G_{n-1}(a^{n-2}ab))$$

We will use the induction on n. When $n = 4$ then

$$\det G_4(a^2) = \det(bG_3(ab)) \det(\alpha_{a^2} G_3(a^2ab)).$$

From Remark 3.4.5, we have $\det G_3(ab) = ab(a^2-1)(b^2-1)$, in terms of box number $\det G_3(ab) = [2]_{q_a}[2]_{q_b}[3]_{q_a}[3]_{q_b}$. In addition, $\det G_3(a^3b) = b$, and from Proposition 3.4.7, $\alpha_{a^2} = b\dfrac{[4]_{q_a}}{[3]_{q_a}}$, we get

$$\det G_4(a^2) = ab^6[3]_{q_b}[4]_{q_a}.$$

Note that, $[3]_{q_b} = b^2 - 1$, and $[4]_{q_a} = a(a^2 - 2)$, thus it is the same result that we have in Example 3.1.4.
Suppose it is true for all integers less than n. To get the result at n, we have

$$\det G_n(a^{n-2}) = a^{i_1}b^{j_1}\alpha_{a^{n-2}} \det G_{n-1}(a^{n-3}b) \tag{3.51}$$

Since $a^{n-3}b \in \Lambda_{n-1}^{n-3}$ then

$$\det G_{n-1}(a^{n-3}b) = a^{i_2}b^{j_2}\beta_{a^{n-3}}\delta_b \det G_{n-2}(a^{n-4}) \tag{3.52}$$

Applying (3.52) in (3.51), we get

$$\det G_n(a^{n-2}) = a^{i_1+i_2}b^{j_1+j_2}\alpha_{a^{n-2}}\beta_{a^{n-3}}\delta_b \det G_{n-2}(a^{n-4})$$

By the assumption of the induction we have

$$\det G_{n-2}(a^{n-4}) = a^{r'_1} b^{r'_2} [3]_{q_b}^{n-4/2} [n-2]_{q_a}$$

for some integers r'_1, r'_2 . Thus

$$\begin{aligned}\det G_n(a^{n-2}) &= a^{(r'_1+i_1+i_2)} b^{(r'_2+j_1+j_2)} [3]_{q_b}^{n-4/2} [n-2]_{q_a} b \frac{[n]_{q_a}}{[n-1]_{q_a}} \frac{[n-1]_{q_a}}{[n-2]_{q_a}} \frac{a}{b} [3]_{q_b} \\ &= a^{r_1} b^{r_2} [3]_{q_b}^{n-2/2} [n]_{q_a}\end{aligned}$$

where $r_1 = r'_1 + i_1 + i_2 + 1$, $r_2 = r'_2 + j_1 + j_2$. □

Proposition 3.4.11. *For $n \geq 5$, the Gram determinant of the module $\Delta_n(a^{n-2}b)$ is* $\det G_n(a^{n-2}b) = a^{r_1} b^{r_2} [3]_{q_b}^{n-1/2} [n]_{q_a}$ *for some integers r_1, r_2.*

Proof. Since $a^{n-2}b \in \Lambda_n^{n-2}$, then by Proposition 3.4.9 (ii),

$$\det G_n(a^{n-2}b) = a^{i_1} b^{j_1} \beta_{a^{n-2}} \delta_b \det G_{n-1}(a^{n-3})$$

for some integers i_1, j_1. By Proposition 3.4.10, we get

$$\det G_{n-1}(a^{n-3}) = a^{i_2} b^{j_2} [3]_{q_b}^{n-3/2} [n-1]_{q_a}$$

for some integers i_2, j_2. Hence,

$$\begin{aligned}\det G_n(a^{n-2}b) &= a^{i_1+i_2} b^{j_1+j_2} [3]_{q_b}^{n-3/2} [n-1]_{q_a} \beta_{a^{n-2}} \delta_b \\ &= a^{i_1+i_2} b^{j_1+j_2} [3]_{q_b}^{n-3/2} [n-1]_{q_a} \frac{[n]_{q_a}}{[n-1]_{q_a}} \frac{a}{b} [3]_{q_b} \\ &= a^{r_1} b^{r_2} [3]_{q_b}^{n-1/2} [n]_{q_a}\end{aligned}$$

where $r_1 = i_1 + i_2 + 1$ and $r_2 = j_1 + j_2 - 1$. □

Proposition 3.4.12. *For $n \geq 5$, the Gram determinant of the module $\Delta_n(ab^{n-2})$ is* $\det G_n(ab^{n-2}) = b^{r_1} a^{r_2} [3]_{q_a}^{n-1/2} [n]_{q_b}$ *for some integers r_1, r_2.*

Proof. If we reflect any basis diagram of $\Delta_n(a^{n-2}b)$ about a vertical line and we swap a with b then we will get a basis diagram of $\Delta_n(ab^{n-2})$. We thus conclude that the Gram matrix of $\Delta_n(ab^{n-2})$ is the same as the Gram matrix of $\Delta_n(a^{n-2}b)$ after we swap a with b. Hence the result. □

Proposition 3.4.13. *For $n \geq 4$, the Gram determinant of $\Delta_n(ab^{n-2}a)$ is*

$$\det G_n(ab^{n-2}a) = a^{r_1} b^{r_2} [3]_{q_a}^{n/2} [n]_{q_b}$$

for some integers r_1, r_2.

Proof. Since $ab^{n-2}a \in \Lambda_n^{n-2}$, then by Proposition 3.4.9(ii), we have

$$\det G_n(ab^{n-2}a) = a^{i_1} b^{j_1} \beta_{b^{n-2}} \delta_a \det G_{n-1}(ab^{n-3})$$

for some integers i_1, j_1. By Proposition 3.4.12, we have

$$\det G_{n-1}(ab^{n-3}) = b^{j_2} a^{i_2} [3]_{q_a}^{n-2/2} [n-1]_{q_b}$$

for some integers j_2, i_2. Then

$$\begin{aligned}
\det G_n(ab^{n-2}a) &= a^{i_1+i_2} b^{j_1+j_2} [3]_{q_a}^{n-2/2} [n-1]_{q_b} \beta_{b^{n-2}} \delta_a \\
&= a^{i_1+i_2} b^{j_1+j_2} [3]_{q_a}^{n-2/2} [n-1]_{q_b} \frac{[n]_{q_b}}{[n-1]_{q_b}} \frac{b}{a} [3]_{q_a} \\
&= a^{r_1} b^{r_2} [3]_{q_a}^{n/2} [n]_{q_b}
\end{aligned}$$

where $r_1 = i_1 + i_2 - 1$ and $r_2 = j_1 + j_2 + 1$. □

Proposition 3.4.14. *Let $\lambda = \mu s^i t^j s \in \Lambda_n^{n-2}$. Then*

$$\det G_n(\mu s^i t^j s) = a^{r_1} b^{r_2} \; \delta_s^{j/2} \; \frac{[j+2]_{q_t}}{[3]_{q_t}} \; \det G_{n-(j-1)}(\mu s^i t)$$

where $(s,t) = (a,b)$ if n is even, and $(s,t) = (b,a)$ if n is odd.

Proof. By using Proposition 3.4.9, we get

$$\begin{aligned}
\det G_n(\mu s^i t^j s) &= a^{h_1} b^{h_2} \beta_{t^j} \delta_s \det G_{n-1}(\mu s^i t^{j-1}) \\
&= a^{h_3} b^{h_4} (\beta_{t^j} \delta_s \alpha_{t^{j-1}}) \det G_{n-2}(\mu s^i t^{j-2} s) \\
&= a^{h_5} b^{h_6} (\beta_{t^j} \delta_s \alpha_{t^{j-1}})(\beta_{t^{j-2}} \delta_s \alpha_{t^{j-3}}) \det G_{n-4}(\mu s^i t^{j-4} s).
\end{aligned}$$

Eventually we have

$$\begin{aligned}
&\det G_n(\mu s^i t^j s)\\
&= a^{h_7} b^{h_8} (\beta_{t^j}\delta_s\alpha_{t^{j-1}})(\beta_{t^{j-2}}\delta_s\alpha_{t^{j-3}})\cdots(\beta_{t^{j-(j-4)}}\delta_s\alpha_{t^{j-(j-3)}})\\
&\qquad\qquad\times \det G_{n-(j-2)}(\mu s^i t^{j-(j-2)} s)\\
&= a^{h_7} b^{h_8} (\beta_{t^j}\delta_s\alpha_{t^{j-1}})(\beta_{t^{j-2}}\delta_s\alpha_{t^{j-3}})\cdots(\beta_{t^4}\delta_s\alpha_{t^3}) \det G_{n-(j-2)}(\mu s^i t^2 s)\\
&= a^{h_9} b^{h_{10}} (\beta_{t^j}\delta_s\alpha_{t^{j-1}})(\beta_{t^{j-2}}\delta_s\alpha_{t^{j-3}})\cdots(\beta_{t^4}\delta_s\alpha_{t^3})(\beta_{t^2}\delta_s) \det G_{n-(j-1)}(\mu s^i t)
\end{aligned}$$

We can see that δ_s appeared with each β_{t^2}, β_{t^4}, ..., β_{t^j}. Thus δ_s has power equal to $j/2$. Now,

$$\begin{aligned}
&(\beta_{t^j}\alpha_{t^{j-1}})(\beta_{t^{j-2}}\alpha_{t^{j-3}})\cdots(\beta_{t^4}\alpha_{t^3})\beta_{t^2}\\
&\qquad= \frac{[j+2]_{q_t}}{[j+1]_{q_t}}\frac{[j+1]_{q_t}}{[j]_{q_t}}\frac{[j]_{q_t}}{[j-1]_{q_t}}\frac{[j-1]_{q_t}}{[j-2]_{q_t}}\cdots\frac{[6]_{q_t}}{[5]_{q_t}}\frac{[5]_{q_t}}{[4]_{q_t}}\frac{[4]_{q_t}}{[3]_{q_t}}\\
&\qquad= \frac{[j+2]_{q_t}}{[3]_{q_t}}
\end{aligned}$$

Hence, the result. □

Theorem 3.4.15. *For $n \geq 3$, if the algebras $\mathrm{FC}_n(a,b)$ are semisimple, then for each $\lambda = a^{l_1}b^{l_2}\cdots t^{l_{p-1}}s^{l_p} \in \Lambda_n^{n-2}$, we have*

$$\det G_n(\lambda) = a^{r_1}b^{r_2}\ [3]_{q_a}^{\frac{1}{2}(l_b-p+1)}\ [3]_{q_b}^{\frac{1}{2}(l_a-p+1)}\prod_{i=1}^{p}[l_i+2]_{q_c}$$

where l_a and l_b are the number of the a and b-through strings in λ respectively. In addition, $c = a$ if i is odd and $c = b$ if i is even.

Proof. We will discuss the case when n odd, that is, $\lambda = a^{l_1}b^{l_2}\cdots a^{l_{p-1}}b^{l_p} \in \Lambda_n^{n-2}$.

Set $x = l_p$, $y = l_{p-1}$, and suppose that $l_p > 1$. If we apply Theorem 3.4.9(i), we get

$$\det G_n(\lambda) = a^{i_1}b^{j_1}\alpha_{b^x}\det G_{n-1}(a^{l_1}b^{l_2}\cdots a^y b^{x-1}a).$$

Substituting $\alpha_{b^x} = \dfrac{[x+2]_{q_b}}{[x+1]_{q_b}}$ and applying Proposition 3.4.14 to evaluate $\det G_{n-1}(a^{l_1}b^{l_2}\cdots a^y b^{x-1}a)$ we get

$$\det G_n(\lambda) = a^{i_2}b^{j_2}\frac{[x+2]_{q_b}}{[x+1]_{q_b}}\delta_a^{(x-1)/2}\frac{[x+1]_{q_b}}{[3]_{q_b}}\det G_{n-(x-1)}(a^{l_1}b^{l_2}\cdots a^y b).$$

Applying Proposition 3.4.14 again we get

$$\det G_n(\lambda)$$
$$= a^{i_3}b^{j_3}\frac{[x+2]_{q_b}}{[3]_{q_b}}\ \delta_a^{(x-1)/2}\ \delta_b^{y/2}\ \frac{[y+2]_{q_a}}{[3]_{q_a}}\det G_{n-(x-1)-(y-1)}(a^{l_1}b^{l_2}\cdots b^{l_{p-2}}a)$$

If we re-apply Proposition 3.4.14 many times we will get

$$\det G_n(\lambda) = a^{i_4}b^{j_4}\delta_a^{(x-1)/2}\frac{[x+2]_{q_b}}{[3]_{q_b}}\delta_b^{y/2}\frac{[y+2]_{q_a}}{[3]_{q_a}}\delta_a^{l_{p-2}/2}\frac{[l_{p-2}+2]_{q_b}}{[3]_{q_b}}\cdots$$
$$\cdots\delta_a^{l_2/2}\frac{[l_2+2]_{q_b}}{[3]_{q_b}}\delta_b^{(l_1-1)/2}\frac{[l_1+2]_{q_a}}{[3]_{q_a}}\det G_3(ab)$$

Substitute $x = l_p$, $y = l_{p-1}$, and $\det G_3(ab) = [3]_{q_a}[3]_{q_b}[2]_{q_a}[2]_{q_b}$ we get

$$\det G_n(\lambda) = a^{i_4}b^{j_4}\delta_a^{(l_p-1)/2}\frac{[l_p+2]_{q_b}}{[3]_{q_b}}\delta_b^{l_{p-1}/2}\frac{[l_{p-1}+2]_{q_a}}{[3]_{q_a}}\delta_a^{l_{p-2}/2}\frac{[l_{p-2}+2]_{q_b}}{[3]_{q_b}}\cdots$$
$$\cdots\delta_a^{l_2/2}\frac{[l_2+2]_{q_b}}{[3]_{q_b}}\delta_b^{(l_1-1)/2}\frac{[l_1+2]_{q_a}}{[3]_{q_a}}[3]_{q_a}[3]_{q_b}$$

From the last equation we can see that for each $l_p, l_{p-2}, \ldots, l_2$ in $\det G_n(\lambda)$, we have $[3]_{q_b}$ in the denominator. As well as, for each $l_{p-1}, l_{p-3}, \ldots, l_1$ in $\det G_n(\lambda)$, we have $[3]_{q_a}$ in the denominator. Thus $[3]_{q_a}$ and $[3]_{q_b}$ in the denominator of $\det G_n(\lambda)$ has power equal to $p/2$. In addition, we can see that δ_a has power equal to

$$\tfrac{1}{2}(l_p - 1 + l_{p-2} + l_{p-4} + \cdots + l_2)$$

and δ_b has power equal to

$$\tfrac{1}{2}(l_{p-1} + l_{p-3} + \cdots + l_3 + l_1 - 1).$$

Set $l_a = l_{p-1} + l_{p-3} + \cdots + l_3 + l_1$ and $l_b = l_p + l_{p-2} + l_{p-4} + \cdots + l_2$. Then l_a and l_b is the number of a and b-through strings in λ respectively. This makes the power of δ_a is equal to $\frac{1}{2}(l_b - 1)$ and the power of δ_b is equal to $\frac{1}{2}(l_a - 1)$. Now, we have

$$\det G_n(\lambda) = a^{i_4} b^{j_4} \, \frac{\delta_a^{\frac{1}{2}(l_b-1)} \delta_b^{\frac{1}{2}(l_a-1)}}{([3]_{q_a}[3]_{q_b})^{p/2}} [3]_{q_a}[3]_{q_b} \prod_{i=1}^{p} [l_i + 2]_{q_c}$$

where $c = a$ if i odd, and $c = b$ if i even. Recall that $\delta_a = \dfrac{b}{a}[3]_{q_a}$ and $\delta_b = \dfrac{a}{b}[3]_{q_b}$. If we consider that $\dfrac{b}{a}$ and $\dfrac{a}{b}$ of δ_a and δ_b respectively, are multiplied with $a^{i_4} b^{i_4}$, then we get

$$\det G_n(\lambda) = a^{i_5} b^{j_5} \; [3]_{q_a}^{\frac{1}{2}(l_b-p+1)} [3]_{q_b}^{\frac{1}{2}(l_a-p+1)} \prod_{i=1}^{p} [l_i + 2]_{q_c}$$

By doing similar calculations, we can show that this formula is true if $l_p = 1$, or $\lambda = a^{l_1} b^{l_2} \cdots b^{l_{p-1}} a^{l_p}$. □

Note that the above theorem is true when the algebras $\mathrm{FC}_n(a, b)$ are semisimple. To generalise it to the case when $\mathrm{FC}_n(a, b)$ are not semisimple, we need this corollary.

Corollary 3.4.16 ([3, Corollary 2.2.5]). *The algebras* $\mathrm{FC}_{k,n}(a_1, \ldots, a_k)$ *are semisimple for an open dense (even a Zariski open) subset of parameters* $(a_1, \ldots, a_k) \in \mathbb{C}^k$.

This corollary says that $\mathrm{FC}_n(a, b)$ are semisimple for an open dense subset of parameters $(a, b) \in \mathbb{C}^2$. The algebras FC_n can be defined over the Laurent polynomial ring $\mathbb{Z}[a, b][a^{-1}, b^{-1}]$ and these algebras are cellular. The FC_n over $\mathbb{C}$ are obtained from the above algebras by specialising the parameters, that is, base change according to a ring homomorphism $\mathbb{Z}[a, b][a^{-1}, b^{-1}] \to \mathbb{C}$.

The Gram determinants for the FC_n over $\mathbb{C}$ are obtained by specialising the ones for the FC_n over $\mathbb{Z}[a, b][a^{-1}, b^{-1}]$. The latter are Laurent polynomials in a and b and they must be equal to the ones from Theorem 3.4.15, since they agree on a Zariski open dense subset of the parameter space $\mathbb{C}^* \times \mathbb{C}^*$. Thus we have

Theorem 3.4.17. *For each $\lambda = a^{l_1}b^{l_2}\cdots t^{l_{p-1}}s^{l_p} \in \Lambda_n^{n-2}$, and $n \geq 3$, we have*

$$\det G_n(\lambda) = a^{r_1}b^{r_2}[3]_{q_a}^{\frac{1}{2}(l_b-p+1)}[3]_{q_b}^{\frac{1}{2}(l_a-p+1)}\prod_{i=1}^{p}[l_i+2]_{q_c}$$

where l_a and l_b are the number of the a and b-through strings in λ respectively. In addition, $c = a$ if i is odd and $c = b$ if i is even.

Proposition 3.4.18. *For $n \geq 3$ and $\lambda = a^{l_1}b^{l_2}\cdots t^{l_{p-1}}s^{l_p} \in \Lambda_n^{n-2}$. If $J_\lambda = \{l_1, l_2, \ldots, l_p\}$ and $J = \bigcup_{\lambda\in\Lambda_n^{n-2}} J_\lambda$, then J is precisely the set $J = \{1, 2, \ldots, n-2\}$.*

Proof. Since $\lambda = a^{l_1}b^{l_2}\cdots t^{l_{p-1}}s^{l_p} \in \Lambda_n^{n-2}$, then $\dim \Delta_{n-2}(\lambda) = 1$. Thus, from Proposition 1.3.16, we have

$$n - 2 = l - p + 1 \tag{3.53}$$

where $l = \sum_{i=1}^{p} l_i$. First, we will show that there is no $\lambda \in \Lambda_n^{n-2}$ such that $l_i > n-2$. Consider the equation $n-2 = l-p+1$ then $l = n+p-3$. If $l_i = n-1$ for some $1 \leq i \leq p$, this means that the remaining $p-1$ parts of λ are connected to the remaining $l-(n-1) = p-2$ through strings. It is impossible, and hence, $l_i < n-1$.

Now, to show that for $n \geq 3$, $l_i = 1, 2, \ldots, n-2$, we will discuss the even and the odd case.

Let n is even, then p must be odd. Suppose $p = 3$, then, from (3.53), we get $l = n$, and λ has the form $\lambda = a^{l_1}b^{l_2}a^{l_3} \in \Lambda_n^{n-2}$. Some integer solutions for the equation $l_1 + l_2 + l_3 = n$ such that l_1, l_3 are odd and l_2 even are

$$(l_1, l_2, l_3) = \{(1, n-2, 1), (3, n-4, 1), \ldots, (n-5, 4, 1), (n-3, 2, 1)\}.$$

From the case $p = 3$, we can see that l_i take the values $\{1, 2, \ldots, n-2\}$.

Let n is odd, then p must be even. Suppose $p = 2$, then, from (3.53), we get $l = n-1$, and λ has the form $\lambda = a^{l_1}b^{l_2} \in \Lambda_n^{n-2}$. The integer solutions for the equation $l_1 + l_2 = n - 1$, such that l_1 and l_2 are odd, are that

$$(l_1, l_2) = \{(1, n-2), (3, n-4), \dots, (n-2, 1)\}$$

Suppose $p = 4$ then, from (3.53), we get $l = n+1$, and λ has the form $\lambda = a^{l_1} b^{l_2} a^{l_3} b^{l_4} \in \Lambda_n^{n-2}$. Some integer solutions for the equation $l_1 + l_2 + l_3 + l_4 = n+1$, such that l_1, l_4 are odd and l_2, l_3 are even, are that

$$(l_1, l_2, l_3, l_4) = \{(1, 2, n-3, 1), (1, 4, n-5, 1), \dots, (1, n-3, 2, 1)\}.$$

From the case $p = 2$ and $p = 4$, we can see that l_i take the values $\{1, 2, \dots, n-2\}$. $\square$

Theorem 3.4.19. *The Fuss-Catalan algebras* $\mathrm{FC}_n(a, b)$ *are semi-simple except when* $[i]_{q_a} = 0$ *or* $[i]_{q_b} = 0$ *for some* $i = 2, 3, \dots, n$, *where* $a = q_a + q_a^{-1}$ *and* $b = q_b + q_b^{-1}$.

Proof. By Theorem 3.4.17, for each $\lambda \in \Lambda_n^{n-2}$ we have

$$\det G_n(\lambda) = a^{r_1} b^{r_2} \; [3]_{q_a}^{\frac{1}{2}(l_b - p+1)} \; [3]_{q_b}^{\frac{1}{2}(l_a - p+1)} \; \prod_{i=1}^{p} [l_i + 2]_{q_c}.$$

Furthermore, from Theorem 3.4.18, for all $i = 1, 2, \dots, p$ we have that $l_i \in \{1, 2, \dots, n-2\}$. Thus for all $\lambda \in \Lambda_n^{n-2}$ the Gram matrix of $\Delta_n(\lambda)$ has zero determinant only when

$$[2]_{q_a}[2]_{q_b}[3]_{q_a}[3]_{q_b} \cdots [n]_{q_a}[n]_{q_b} = 0. \quad \square$$

Chapter 4

Homomorphisms

The homomorphisms for a special kind of cell modules for the Fuss-Catalan algebras will be introduced in this chapter. It is sufficient to find homomorphisms between cell modules such that one of them is one dimensional. Then by Theorem 2.1.8, we can reduce all the family of homomorphisms to morphisms between these modules.

4.1 Homomorphisms for specified cell modules

Let $U = \{{}_1U_1, {}_1U_2, \ldots, {}_1U_{n-1}, {}_2U_1, {}_2U_2, \ldots, {}_2U_{n-1}\}$ be the set of the generators of FC_n. For $i = 1, 2, \ldots, n-1$, suppose that v_i is the upper half diagram of ${}_1U_i$, and λ_i is the label of v_i. By counting the dimension, we have $\Delta_n(\lambda_i)$ is a one dimensional cell module with basis v_i. We are going to show that, for $n \geq 3$ and for certain values of the parameters a and b, there is a non-zero homomorphism from $\Delta_n(\lambda_i)$ to $\Delta_n(\mu_n)$, where $i = 1, 2, \ldots, n-1$ and

$$\mu_n = \begin{cases} (abba)^{m-1}ab & \text{if } n = 2m+1 \\ (abba)^{m-1} & \text{if } n = 2m. \end{cases} \tag{4.1}$$

In fact, μ_n is the label of the identity diagram for FC_{n-2}, and the first initial values of μ_n are listed in Table 4.1 .

n	μ_n
3	ab
4	ab^2a
5	ab^2a^2b
6	$ab^2a^2b^2a$
7	$ab^2a^2b^2a^2b$
8	$ab^2a^2b^2a^2b^2a$

Table 4.1: The label μ_n

Proposition 4.1.1. *Let $n \geq 3$, and μ_n be as defined in 4.1. Then we have*

$$\dim \Delta_n(\mu_n) = 2n - 3.$$

Proof. We will discuss the even and then the odd case for μ_n.

Case I: Suppose that $n = 2m$ where $m \geq 2$. Then $\mu_n = (abba)^{m-1}$. The length of μ_n is $l(\mu_n) = 4(m-1)$. We can write μ_n in the form $\mu_n = awa$, where $w = (b^2a^2)^{m-2}b^2$. Suppose that w_a is the number of a^2's in w and w_b is the number of b^2's in w. We can see that $w_a = m-2$ while $w_b = m-1$, thus the number of parts, p, of μ_n is $p = 1 + w_a + w_b + 1 = 2m-1$. Now, we use Theorem 1.3.16 to calculate the dimension of $\Delta_n(\mu_n)$. We have

$$r = m - 1 \qquad \text{and} \qquad s = 4m - 3.$$

Hence,

$$\begin{bmatrix} n \\ \mu_n \end{bmatrix} = \frac{4m-3}{4m}\binom{4m}{1} = 2n - 3.$$

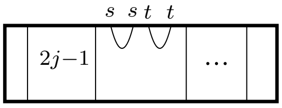
Case II: Suppose that $n = 2m+1$ where $m \geq 1$. Then $\mu_n = (abba)^{m-1}ab$. The length of μ_n is $l(\mu_n) = 4(m-1) + 2 = 4m - 2$. We can write μ_n in the form $\mu_n = awb$, where $w = (b^2a^2)^{m-1}$. Suppose that w_a is the number of a^2's in w and w_b is the number of b^2's in w. We can see that $w_a = w_b = m-1$, thus the number of parts of μ_n is $p = 1 + w_a + w_b + 1 = 2m$. By using Theorem 1.3.16 to calculate the dimension of $\Delta_n(\mu_n)$, we have

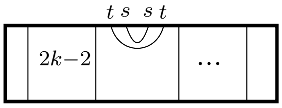
$$r = m - 1 \qquad \text{and} \qquad s = 4m - 1.$$

Hence,

$$\begin{bmatrix} n \\ \mu_n \end{bmatrix} = \frac{4m-1}{4m+1}\binom{4m+1}{1} = 2n - 3. \quad \square$$

Definition 4.1.2. For $j = 1, 2, \ldots, n-2$, and $k = 1, 2, \ldots, n-1$, we define the half diagrams M_j and N_k, as following:

$M_j =$ 

$N_k =$ 

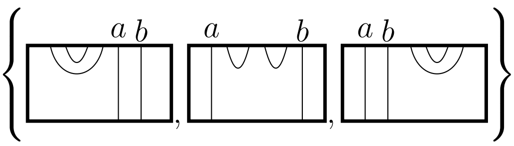
where $(s,t) = (a,b)$ if j, k are even, and $(s,t) = (b,a)$ if j, k are odd.

Note that, if we remove the two non-through strings from the diagrams M_j and N_k then we will get the identity diagram for FC_{n-2}. Hence, For $j = 1, 2, \ldots, n-2$, and $k = 1, 2, \ldots, n-1$, the diagrams M_j and N_k have label equal to μ_n.

Proposition 4.1.3. *Let $n \geq 3$ and μ_n be as defined in 4.1. The basis for $\Delta_n(\mu_n)$ is the set $\{N_1, M_1, N_2, M_2, \ldots, N_{n-2}, M_{n-2}, N_{n-1}\}$, where M_j and N_k are as defined in Definition 4.1.2 above.*

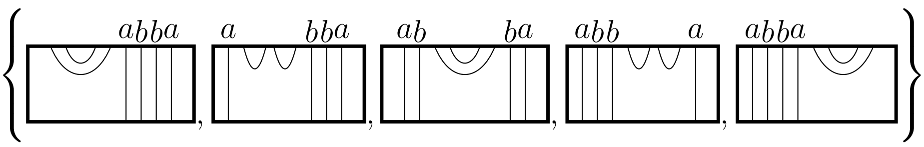
Proof. We prove this proposition by using induction on n.

When $n = 3$, then $\mu_3 = ab$, $\dim \Delta_3(ab) = 3$ and the basis elements of $\Delta_3(ab)$ are



$= \{N_1, M_1, N_2\}.$

When $n = 4$, then $\mu_4 = abba$, $\dim \Delta_4(ab^2a) = 5$ and its basis is



$$= \{N_1, M_1, N_2, M_2, N_3\}.$$

Suppose that this claim is true for all integers that are less than n. Then the basis elements of $\Delta_{n-1}(\mu_{n-1})$ form the set

$$H' = \{N'_1, M'_1, \ldots, N'_{n-3}, M'_{n-3}, N'_{n-2}\},$$

where N'_k, M'_j are the diagrams obtained by removing the last two through strings from N_k, M_j respectively.

Now, when we add two through strings to the right end of each element in H' then we get a set, H say, of diagrams with $2n$ vertices and their label is μ_n, thus these diagrams all belong to the basis of $\Delta_n(\mu_n)$. The number of elements in H is $2n-5$, and the dimension of $\Delta_n(\mu_n)$ is $2n-3$, so we miss two elements. Since every diagram in H has at least two through strings at the right end, then M_{n-2} and N_{n-1}, that have one and zero through strings at the right end respectively, are not in H. However, these diagrams have label μ_n thus they are basis diagrams for $\Delta_n(\mu_n)$. This completes the proof. $\square$

Proposition 4.1.4. *Let ${}_1U_i$, ${}_2U_i$ be the generators of* FC_n*, and consider the cell module $\Delta_n(\mu_n)$ that is spanned by the diagrams M_j, N_k. Then we have:*

$$\text{(i)}\quad {}_1U_iM_j = \begin{cases} sM_i & \text{if } j=i \\ sM_{i-1} & \text{if } j=i-1 \\ 0 & \text{otherwise.} \end{cases}$$

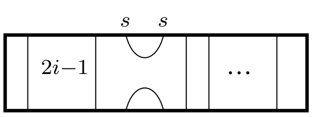
$$\text{(ii)}\quad {}_1U_iN_k = \begin{cases} sN_i & \text{if } k=i \\ M_{i-1} & \text{if } k=i-1 \\ M_i & \text{if } k=i+1 \\ 0 & \text{otherwise.} \end{cases}$$

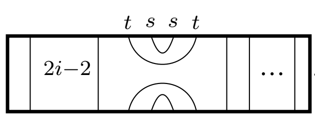
$$\text{(iii)}\quad {}_2U_iM_j = \begin{cases} sN_i & \text{if } j=i \text{ or } j=i-1 \\ 0 & \text{otherwise.} \end{cases}$$

$$\text{(iv)}\quad {}_2U_iN_k = \begin{cases} stN_i & \text{if } k=i \\ N_i & \text{if } k=i\pm 1 \\ 0 & \text{otherwise.} \end{cases}$$

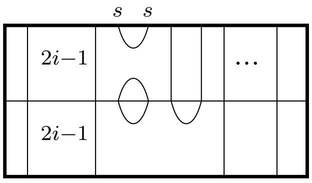
where $i,\ k = 1,\ 2,\ \ldots,\ n-1$, $j = 1,\ 2,\ \ldots,\ n-2$, and $(s,t)=(a,b)$ if i is even and $(s,t)=(b,a)$ if i is odd.

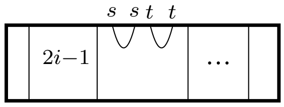
Proof. Recall that

${}_1U_i =$ 

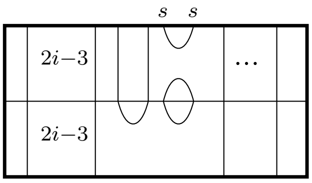
and, ${}_2U_i =$ 

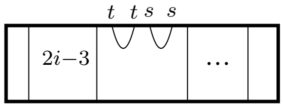
.

(i) $_1U_iM_i =$ 

$= s$ 

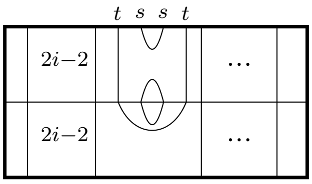
$= sM_i,$

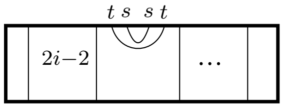
$_1U_iM_{i-1} =$ 

$= s$ 

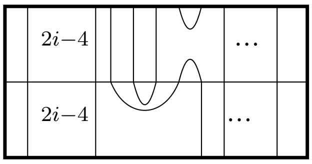
$= sM_{i-1}.$

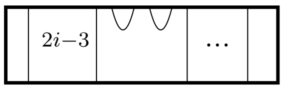
These are the only possible ways that gives a non-zero diagram in $\Delta_n(\mu_n)$, since if the two non-through strings of M_j are connected to through strings in $_1U_i$ then we get a diagram with three non through strings. That is, a diagram with length $2n-6$. Thus it is not a basis diagram of $\Delta_n(\mu_n)$ because the basis diagrams of $\Delta_n(\mu_n)$ has length equal to $2n-4$.

(ii) $_1U_iN_i =$ 

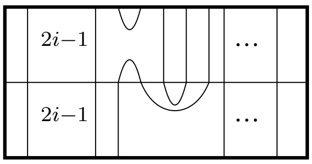
$= s$ 

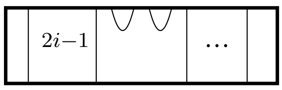
$= sN_i,$

$_1U_iN_{i-1} =$ 

$=$ 

$= M_{i-1},$

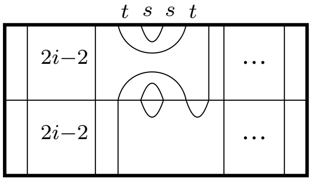
$_1U_iN_{i+1} =$ 

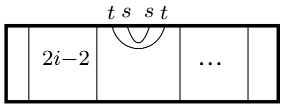
$=$ 

$= M_i.$

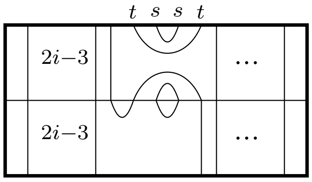
As in (i), if $i-1 > j > i+1$ then $_1U_iN_j$ will contain three non-through strings, hence it is not a basis diagram of $\Delta_n(\mu_n)$. That is,

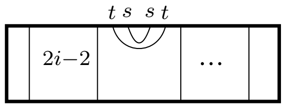
$$_1U_iN_j = 0.$$

(iii) ${}_2U_iM_i =$ 

$= s$ 

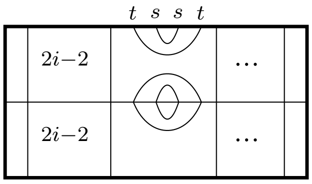
$= sN_i,$

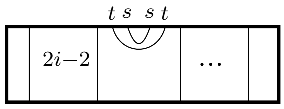
${}_2U_iM_{i-1} =$ 

$= s$ 

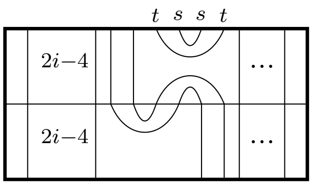
$= sN_i.$

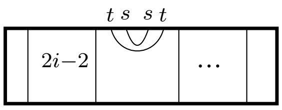
If $j \neq i, i-1$ then ${}_2U_iM_j$ is a diagram with three non through strings. Hence, it is zero in $\Delta_n(\mu_n)$.

(iv) ${}_2U_iN_i =$ 

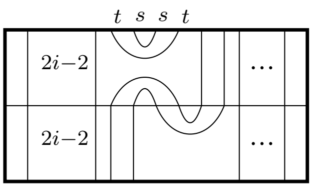
$= st$ 

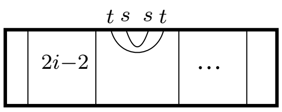
$= stN_i,$

${}_2U_iN_{i-1} =$ 

$=$ 

$= N_i,$

${}_2U_iN_{i+1} =$ 

$=$ 

$= N_i.$

Again, if $i-1 > j > i+1$ then ${}_2U_iN_j$ will give us a diagram with three non-through strings that means it is zero in the module $\Delta_n(\mu_n)$. □

Theorem 4.1.5. *Let λ_j be the label of the generators ${}_1U_j$, where $j = 2, 3, \ldots, n-2$. Let v_j be the basis element of the one dimensional cell module $\Delta_n(\lambda_j)$. Then $\theta : \Delta_n(\lambda_j) \to \Delta_n(\mu_n)$ defined by $\theta(v_j) = M_j - cN_j + M_{j-1}$ is a module homomorphism when $c = b$ and $b^2 = 2$ if j even while $c = a$ and $a^2 = 2$ if j odd.*

Proof. Let $U = \{{}_1U_1, {}_1U_2, \ldots, {}_1U_{n-1}, {}_2U_1, {}_2U_2, \ldots, {}_2U_{n-1}\}$ be the set of the generators of FC_n. In $\Delta_n(\lambda_j)$, we can see that

$$ {}_rU_i v_j = \begin{cases} sv_j & \text{if } i = j \text{ and } r = 1, \\ 0 & \text{otherwise,} \end{cases} $$

for all $i = 1, 2, \ldots, n-1$, and $j = 2, 3, \ldots, n-2$, where $s = a$ if j even and $s = b$ if j odd. To show that θ is a homomorphism, we need to prove that $\theta(uv_j) = u\theta(v_j)$ for all $u \in U$, and $j = 2, 3, \ldots, n-2$.

If $0 \neq uv_j \in \Delta_n(\lambda_j)$, then $u = {}_1U_j$, and $\theta({}_1U_j v_j) = \theta(sv_j) = s\theta(v_j)$. On the other hand, ${}_1U_j\theta(v_j) = {}_1U_j(M_j - cN_j + M_{j-1})$, by Proposition 4.1.4, we get ${}_1U_j\theta(v_j) = s(M_j - cN_j + M_{j-1}) = s\theta(v_j)$. Hence, $\theta({}_1U_j v_j) = {}_1U_j\theta(v_j)$.

Now we turn to the case when $0 = uv_j \in \Delta_n(\lambda_j)$, this happens only when $u = {}_1U_i$ with $i \neq j$ or when $u = {}_2U_i$. Let us discuss these two cases:

(i) For $i \neq j$, we have ${}_1U_i v_j = 0$. Therefore we need to show that ${}_1U_i\theta(v_j) = 0$ when $i \neq j$. We have ${}_1U_i\theta(v_j) = {}_1U_i(M_j - cN_j + M_{j-1})$. By Proposition 4.1.4, we have ${}_1U_iM_j$ is a non-zero diagram only if $i = j$ and $i = j+1$. Furthermore, ${}_1U_iN_j$ is a non-zero diagram only if $i = j$ and $i = j \pm 1$. Since $i \neq j$, thus we only need to show that ${}_1U_i\theta(v_j) = 0$ when $i = j \pm 1$. Let $i = j \pm 1$, then

$$ \begin{aligned} {}_1U_{j-1}\theta(v_j) &= {}_1U_{j-1}(M_j - cN_j + M_{j-1}) \\ {}_1U_{j+1}\theta(v_j) &= {}_1U_{j+1}(M_j - cN_j + M_{j-1}) \end{aligned} $$

If j odd, then $c = a$ and we get

$$ \begin{aligned} {}_1U_{j-1}\theta(v_j) &= 0 - aM_{j-1} + aM_{j-1} \\ {}_1U_{j+1}\theta(v_j) &= aM_j - aM_j + 0. \end{aligned} $$

If j even, then $c = b$ and we get

$$ \begin{aligned} {}_1U_{j-1}\theta(v_j) &= 0 - bM_{j-1} + bM_{j-1} \\ {}_1U_{j+1}\theta(v_j) &= bM_j - bM_j + 0. \end{aligned} $$

Hence, ${}_1U_i\theta(v_j) = 0$.

(ii) For all $i = 1, 2, \ldots, n-1$, and $j = 2, 3, \ldots, n-2$, we have ${}_2U_iv_j = 0$. Thus we need to show that ${}_2U_i\theta(v_j) = 0$ as well. We have ${}_2U_i\theta(v_j) = {}_2U_i(M_j - cN_j + M_{j-1})$, by Proposition 4.1.4, the diagram ${}_2U_iM_j$ is a non-zero only when $i = j$ and $i = j+1$. In addition, ${}_2U_iN_j$ is a non-zero diagram when $i = j$ and $i = j \pm 1$. Thus it sufficient to show ${}_2U_i\theta(v_j) = 0$ for $i = j$ and $i = j \pm 1$. Let $i = j \pm 1$ then

$$\begin{aligned} {}_2U_{j-1}\theta(v_j) &= {}_2U_{j-1}(M_j - cN_j + M_{j-1}), \\ {}_2U_{j+1}\theta(v_j) &= {}_2U_{j+1}(M_j - cN_j + M_{j-1}). \end{aligned}$$

If j odd, then $c = a$, and we get

$$\begin{aligned} {}_2U_{j-1}\theta(v_j) &= 0 - aN_{j-1} + aN_{j-1}, \\ {}_2U_{j+1}\theta(v_j) &= aN_{j+1} - aN_{j+1} + 0. \end{aligned}$$

If j even, then $c = b$, and we get

$$\begin{aligned} {}_2U_{j-1}\theta(v_j) &= 0 - bN_{j-1} + bN_{j-1}, \\ {}_2U_{j+1}\theta(v_j) &= bN_{j+1} - bN_{j+1} + 0. \end{aligned}$$

Let $i = j$, then

$${}_2U_j\theta(v_j) = {}_2U_j(M_j - cN_j + M_{j-1}).$$

If j even, then $c = b$ and

$$\begin{aligned} {}_2U_j\theta(v_j) &= aN_j - b(abN_j) + aN_j \\ &= a(2 - b^2)N_j. \end{aligned}$$

If j odd, then $c = a$ and

$$\begin{aligned} {}_2U_j\theta(v_j) &= bN_j - a(abN_j) + bN_j \\ &= b(2 - a^2)N_j. \end{aligned}$$

Since $a^2 = 2$, (resp. $b^2 = 2$) if j odd, (resp. even) then we get ${}_2U_j\theta(v_j) = 0$. $\square$

Theorem 4.1.6. *Suppose that $\Delta_n(\lambda_1)$ and $\Delta_n(\lambda_{n-1})$ are the one dimensional cell modules that generated by the initial parts v_1 and v_{n-1} of the generators ${}_1U_1$ and ${}_1U_{n-1}$ respectively. Then we have*

(i) There is a non-zero homomorphism $\theta : \Delta_n(\lambda_1) \to \Delta_n(\mu_n)$ defined by

$$\theta(v_1) = N_1 - aM_1$$

when $a^2 = 1$.

(ii) There is a non-zero homomorphism $\theta : \Delta_n(\lambda_{n-1}) \to \Delta_n(\mu_n)$ defined by

$$\theta(v_{n-1}) = M_{n-2} - cN_{n-1}$$

when $c = a$ and $a^2 = 1$ if n even while $c = b$ and $b^2 = 1$ if n odd.

Proof. Let U be the set of the generators of FC_n.

(i) For all $u \in U$ such that $u \neq {}_1U_1$, we have $uv_1 = 0$, and ${}_1U_1v_1 = bv_1$ in the cell module $\Delta_n(\lambda_1)$. In addition, $\theta({}_1U_1v_1) = \theta(bv_1) = b\theta(v_1)$, and

$${}_1U_i\theta(v_1) = {}_1U_i(N_1 - aM_1) = b(N_1 - aM_1).$$

Hence, $\theta({}_1U_iv_1) = {}_1U_i\theta(v_1)$. For ${}_1U_i \neq u \in U$, we want to show that $u\theta(v_1) = 0$. From Proposition 4.1.4, ${}_1U_iN_1$ and ${}_1U_iM_1$ are non-zero diagrams only if $i = 1$, 2. Since we discuss the case $i = 1$ then we need only to discuss the case $i = 2$. That is, we have to show that ${}_1U_2\theta(v_1) = 0$. For,

$${}_1U_2\theta(v_1) = {}_1U_2(N_1 - aM_1) = M_1 - a(aM_1) = (1 - a^2)M_1.$$

Since $a^2 = 1$, then ${}_1U_2\theta(v_1) = 0$.

Now, by Proposition 4.1.4, ${}_2U_iN_1$ and ${}_2U_iM_1$ are non-zero diagrams only if $i = 1$, 2. Therefore, we have to prove that ${}_2U_i\theta(v_1) = 0$ when $i = 1$, 2. We have

$${}_2U_1\theta(v_1) = {}_2U_1(N_1 - aM_1) = abN_1 - abN_1 = 0,$$

$${}_2U_2\theta(v_1) = {}_2U_2(N_1 - aM_1) = N_2 - a^2N_2 = 0.$$

(ii) For all $u \in U$, we are going to show that $\theta(uv_{n-1}) = u\theta(v_{n-1})$. If $u = {}_1U_{n-1}$ then ${}_1U_{n-1}v_{n-1} = sv_{n-1}$ where $s = a$ if $n-1$ even and $s = b$ if $n-1$ odd. Thus $\theta({}_1U_{n-1}v_{n-1}) = s\theta(v_{n-1})$. On the other hand,

$$
{}_1U_{n-1}\theta(v_{n-1}) = {}_1U_{n-1}(M_{n-2} - cN_{n-1}) = s(M_{n-2} - cN_{n-1}).
$$

Hence, ${}_1U_{n-1}\theta(v_{n-1}) = s\theta(v_{n-1})$.

For ${}_1U_{n-1} \neq u \in U$, we have $uv_{n-1} = 0$. Thus we need to show $u\theta(v_{n-1}) = 0$. From Proposition 4.1.4, the diagrams ${}_1U_iM_{n-2}$ and ${}_1U_iN_{n-1}$ are non-zero only if $i = n-2$, $n-1$, but we already discussed the case $i = n-1$ then we only need to discuss the case $i = n-2$. That is, we have to show that ${}_1U_{n-2}\theta(v_{n-1}) = 0$. For,

$$
{}_1U_{n-2}\theta(v_{n-1}) = {}_1U_{n-2}(M_{n-2} - cN_{n-1}).
$$

If n even, then $c = a$, and hence,

$$
{}_1U_{n-2}\theta(v_{n-1}) = aM_{n-2} - aM_{n-2}.
$$

If n odd, then $c = b$, and hence,

$$
{}_1U_{n-2}\theta(v_{n-1}) = bM_{n-2} - bM_{n-2}.
$$

Again by Proposition 4.1.4, the diagrams ${}_2U_iM_{n-2}$ and ${}_2U_iN_{n-1}$ are non-zero in $\Delta_n(\mu_n)$ only if $i = n-2$, $n-1$. We have

$$
\begin{aligned}
{}_2U_{n-2}\theta(v_{n-1}) &= {}_2U_{n-2}(M_{n-2} - cN_{n-1}) \\
{}_2U_{n-1}\theta(v_{n-1}) &= {}_2U_{n-1}(M_{n-2} - cN_{n-1}).
\end{aligned}
$$

If n even, then $c = a$, and

$$
\begin{aligned}
{}_2U_{n-2}\theta(v_{n-1}) &= aN_{n-2} - aN_{n-2} \\
{}_2U_{n-1}\theta(v_{n-1}) &= bN_{n-1} - a(abN_{n-1}) \\
&= b(1-a^2)N_{n-1}.
\end{aligned}
$$

If n odd, then $c = b$, and

$$\begin{aligned} {}_2U_{n-2}\theta(v_{n-1}) &= bN_{n-2} - bN_{n-2} \\ {}_2U_{n-1}\theta(v_{n-1}) &= aN_{n-1} - b(abN_{n-1}) \\ &= a(1-b^2)N_{n-1} \end{aligned}$$

Since $a^2 = 1$, (resp. $b^2 = 1$) if n even, (resp. odd), then ${}_2U_i\theta(v_{n-1}) = 0$. □

We recall the globalisation functor that defined in chapter two

$$\mathcal{G}_n : \mathrm{FC}_n\text{-mod} \to \mathrm{FC}_{n+2}\text{-mod}$$

via

$$\mathcal{G}_{n-2}(N) = \mathrm{FC}_n e_n \otimes_{e_n \mathrm{FC}_n e_n} N.$$

In addition, from equation 2.2, we have $\mathcal{G}_n(\Delta_n(\lambda)) \cong \Delta_{n+2}(\lambda)$. The functor $\mathcal{G}_n$ preserves all morphisms that are defined in Theorems 4.1.5 and 4.1.6.

Corollary 4.1.7. *Let λ_i be the label of the diagram generator ${}_1U_i$ of $\mathrm{FC}_n(a,b)$ where $i = 1, 2, \ldots, n-1$, and μ_n is as defined in equation 4.1. Then for all $m \geq n$ such that $m-n$ is even we have a non-zero homomorphism $\theta : \Delta_m(\lambda_i) \to \Delta_m(\mu_n)$ subject to the following values of a and b*

(i) $a^2 = 1$ *when* $i = 1$ *or* $i = n-1$ *odd.*

(ii) $b^2 = 1$ *when* $i = n-1$ *even.*

(iii) $a^2 = 2$ *when* $2 \leq i \leq n-2$ *odd.*

(iv) $b^2 = 2$ *when* $2 \leq i \leq n-2$ *even.*

Conclusion

In this thesis we introduce some results on the representation theory of the Fuss-Catalan algebras over $\mathbb{C}$. However, we think that there are areas of future research. In this thesis we stated that the Fuss-Catalan algebras are cellular, quasi-hereditary algebras and they form a tower of recollement. These results can give the opportunity to study these algebras and get new results. For example, one way is to study homomorphisms between cell modules in general. In addition, the problem of finding composition numbers. Another problem one can investigate is finding the structure of the Fuss-Catalan algebras when they are not semi-simple. Furthermore, we think that there are some results can be extended to $\mathrm{FC}_{k,n}$ for all $k > 2$, for example, the cellularity of the Fuss-Catalan algebras.

Bibliography

[1] I. Assem, D. Simson, and A. Skowronski, *Elements of the representation theory of associative algebras: Vol. 1, Techniques of representation theory*, vol. 65, Cambridge Uni. Press, Cambridge, 2006.

[2] D.J. Benson, *Representations and cohomology: Vol. 1, Basic representation theory of finite groups and associative algebras*, Cambridge Studies in Advanced Mathematics, vol. 30, Cambridge University Press, 1998.

[3] D. Bisch and V. Jones, *Algebras associated to intermediate subfactors*, Invent. Math. **128** (1997), no. 1, 89–157.

[4] ______, *A note on free composition of subfactors*, Geometry and Physics, vol. 184, Marcel Dekker, Lecture Notes in Pure and Appl. Math., (1997), 339–361.

[5] M. Bloss, *G-Colored partition algebras as centralizer algebras of wreath products*, J. Algebra. **256** (2003), 690–710.

[6] R. Brauer, *On algebras which are connected with the semisimple continuous groups*, Ann. of Math. **38** (1937), 857–872.

[7] W. Brown, *An algebra related to the orthogonal group*, Michigan Math. J. **3** (1955), 1–22.

[8] H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Uni. Press, Princeton, N. J., 1956.

[9] D. Choate, *Algebras generated by symmetric idempotents*, Bulletin of the Australian Mathematical Society **29** (1984), no. 3, 303–308.

[10] A. G. Cox, M. De Visscher, and P.P. Martin, *The blocks of the Brauer algebra in characteristic zero. representation theory*, Comm. Algebra **13** (2009), 272–308.

[11] A. G. Cox, J. J. Graham, and P. P. Martin, *The blob alegbra in positive characteristic*, J. Algebra **266** (2003), 584–635.

[12] A. G. Cox, P. P. Martin, A. E. Parker, and C. Xi, *Representation theory of towers of recollement: theory, notes, and examples*, J. Algebra **302** (2006), 340–360.

[13] C.W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*, vol. 356, American Mathematical Soc., 1966.

[14] P. Di Francesco, *New integrable lattice models from Fuss-Catalan algebras*, Nuclear Phys. B **532** (1998), no. 3, 609–634.

[15] V. Dlab and C. M. Ringel, *Quasi-hereditary algebras*, Illinois J. Math. **33** (1989), no. 2, 280–291.

[16] ———, *The module theoretical approach to quasi-hereditary algebras*, Representations of algebras and related topics (Kyoto, 1990) (H. Tachikawa and S. Brenner, eds.), London Math. Soc. Lecture Note Ser., vol. 168, Cambridge Univ. Press, Cambridge, 1992, pp. 200–224.

[17] S. Donkin, *The q–Schur Algebra*, London Math. Soc. Lecture Note Ser., vol. 253, Cambridge Univ. Press, Cambridge, 1998.

[18] J. E. Grabowski, *Fuss-Catalan recoupling theory*, Master's thesis, University of Warwick, 2002.

[19] J. J. Graham and G. I. Lehrer, *Cellular algebras*, Invent. Math. **123** (1996), 1–34.

[20] J. A. Green, *Polynomial Representations of* GL_n, Lecture Notes in Mathematics, vol. 830, Springer–Verlag, Berlin/Heidelberg/New York, 1980.

[21] R. M. Green, *Generalized Temperley-Lieb algebras and decorated tangles*, J. Knot Theory Ramifications **7** (1998), 155–171.

[22] R. M. Green, P. P. Martin, and A. E. Parker, *Towers of recollement and bases for diagram algebras: planarity and beyond*, J. Algebra **316** (2007), 392–452.

[23] J. C. Jantzen, *Representations of Algebraic Groups*, Mathematical surveys and monographs, vol. 107, AMS, 2003, second edition.

[24] S. Konig and C. Xi, *On the structure of cellular algebras*, Algebras and modules II, CMS Conf. Proc. (1996).

[25] ______, *When is a cellular algebra quasi-hereditary?*, Mathematics Annalen **315** (1999), no. 2, 281–293.

[26] Z. A. Landau, *Fuss-Catalan algebras and chains of intermediate subfactors*, Pacific J. Math. **197** (2001), no. 2, 325–367.

[27] S. MacLane, *Homology*, Springer–Verlag, Berlin/Heidelberg/New York, 1975.

[28] P. P. Martin, *Potts models and related problems in statistical mechanics*, World Scientific, Singapore, 1991.

[29] P. P. Martin, *Temperley-Lieb algebras for nonplanar statistical mechanics-the partition algebra*, J. Knot Theory Ramifications **3** (1994), 51–82.

[30] P. P. Martin and A. Elgamal, *Ramified partition algebras*, Math. Z. **246** (2004), 473–500.

[31] P. P. Martin and S. Ryom-Hansen, *Virtual algebraic Lie theory: Tilting modules and Ringel duals for blob algebras*, Proc. London Math. Soc. **89** (2004), 655–675.

[32] P. P. Martin and H. Saleur, *The blob algebra and the periodic Temperley-Lieb algebra*, Lett. Math. Phys. **30** (1994), 189–206.

[33] P. P. Martin and D. Woodcock, *On the structure of the blob algebra*, J. Algebra **225** (2000), no. 2, 957–988.

[34] S. Martin, *Schur algebras and representation theory*, Cambridge Tracts in Mathematics, vol. 112, Cambridge Uni. Press, 1993.

[35] A. Mathas, *Iwahori-hecke algebras and Schur algebras of the symmetric group*, University lecture series, vol. 15, American Mathematical Society, 1999.

[36] V. Mazorchuk, *On the strucure of Brauer semigroup and its partial analogue*, Problems in Algebra **13** (1998), 29–45.

[37] ———, *Endomorphisms of $\mathfrak{B}_n$, $\mathcal{P}\mathfrak{B}_n$ and $\mathfrak{C}_n$*, Comm. Algebra **30** (2002), 3489–3513.

[38] D. Ridout and Y. Saint-Aubin, *Standard modules, induction and the structure of the Temperley-Lieb algebra*, Adv. Theor. Math. Phys. **18** (2014), no. 5, 957–1041.

[39] H. Rui and C. Xi, *The representation theory of cyclotomic Temperley-Lieb algebras*, Comment. Math. Helv. **79** (2004), 427–450.

[40] T. A. Springer, *Linear Algebraic Groups*, Progress in Mathematics, vol. 9, Birkhäuser, Boston/Basel/Stuttgart, 1981.

[41] H. N. V. Temperley and E. H. Lieb, *Relations between the "percolation" and "colouring" problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the "percolation" problem*, Proc. Roy. Soc. London Ser. A **322** (1971), no. 1549, 251–280.

[42] B. W. Westbury, *The representation theory of the Temperley-Lieb algebras*, Math. Z. **219** (1995), no. 4, 539–565.

[43] C. Xi, *Cellular algebras*, Lecture notes from the advanced school and conference on representation theory and related topics, ICTP, Italy., 2006.