E-theory Spectra

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ABSTRACT

This thesis combines the fields of functional analysis and topology.

C^*-algebras are analytic objects used in non-commutative geometry and in particular we consider an invariant of them, namely E-theory.

E-theory is a sequence of abelian groups defined in terms of homotopy classes of morphisms of C^*-algebras. It is a bivariant functor from the category where objects are C^*-algebras and arrows are $*$-homomorphisms to the category where objects are abelian groups and arrows are group homomorphisms. In particular, E-theory is a cohomology theory in its first variable and a homology theory in its second variable. We prove in the case of real graded C^*-algebras that E-theory has 8-fold periodicity.

Further we create a spectrum for E-theory. More precisely, we use the notion of quasi-topological spaces and form a quasi-spectrum, that is a sequence of based quasi-topological spaces with specific structure maps. We consider actions of the orthogonal group and we obtain a orthogonal quasi-spectrum which we prove has a smash product structure using the categorical framework. Then we obtain stable homotopy groups which give us E-theory.

Finally, we combine these ideas and a relation between E-theory and K-theory to obtain connections of the E-theory orthogonal quasi-spectrum to K-theory and K-homology orthogonal quasi-spectra.

For Stephen Paul Browne, my father who would have read every word.

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Introduction

A key aspect of this work is that we have connected operator algebras and topology.

We use complex C^*-algebras, i.e. Banach $*$-algebras A for which $||T^*T|| = ||T||^2$ for all T in A. By results of Gelfand and Naimark, any commutative C^*-algebra is isomorphic to the algebra of continuous functions on some topological space, and any C^*-algebra is a subalgebra of bounded linear operators on a Hilbert space. Additionally, we also work with real C^*-algebras [Pal84]. These have an analogous definition to complex C^*-algebras but require that the element $1 + T^*T$ is invertible for all T in A. Further it is also necessary for me to work with $\mathbb{Z}_2$-graded C^*-algebras.

In C^*-algebra theory, we are interested in studying certain invariants. We work with homotopy classes of functions between C^*-algebras. These form groups known as E-theory groups; a bivariant version of the K-theory groups. Many C^*-algebraists use the notion of K-theory and/or KK-theory but we use E-theory since it has additional properties of KK-theory and relates to K-theory.

E-theory was introduced by Higson [Hig90] as a categorical framework to include the excision property that KK-theory does not have. It was formed to aid in the study of KK-theory since it has additional structure making it easier to work with. Soon after, Connes and Higson constructed a concrete definition using almost homomorphisms [CH]. Higson did some work with Guentner [HG04] and defined E-theory for complex graded C^*-algebras. Furthermore, Guentner, Higson and Trout [GHT00], define equivariant E-theory and descent and the Baum-Connes conjecture there after.

E-theory is a bivariant functor from the category where objects are C^*-algebras and arrows are $*$-homomorphisms, to the category whose objects are abelian groups and arrows are group homomorphisms, being a cohomology theory in its first variable and a homology theory in its second variable. That is, we have a sequence of abelian groups $E^0(A, B), E^1(A, B) \ldots$ for $\mathbb{Z}_2$-graded C^*-algebras A and B.

We prove that when we consider real $\mathbb{Z}_2$-graded C^*-algebras that E-theory has 8-fold periodicity using techniques of Guentner and Higson [HG04], who prove that in the case of complex $\mathbb{Z}_2$-graded C^*-algebras we have 2-fold periodicity. That is,

Theorem A. *For real graded C^*-algebras A and B,*

$$E^n(A, B) \cong E^{n+8}(A, B).$$

As we have homotopy classes, the notion of topological spectrum is a logical choice since we can obtain a stable homotopy theory which gives resulting groups as the E-theory groups. A topological spectrum is a sequence of based topological spaces with connecting structure maps. The classes of functions that define the E-theory groups are not continuous, but satisfy a weaker property called "quasi-continuity". The appropriate structure to describe this is a quasi-topological space. A quasi-topology on a set W is a collection of maps for each compact Hausdorff space C into W, $Q(C, W)$, called quasi-continuous with additional properties. A set X with a quasi-topology is a quasi-topological space. In this thesis, we generalise the definition of a spectrum to quasi-topological spaces, that is we have a sequence of based quasi-topological spaces with structure maps. Our spaces are defined by

$$E_n(A, B) = \operatorname{Asy}(C_0(\mathbb{R})^g \widehat{\otimes} A \widehat{\otimes} \mathcal{K}, B \widehat{\otimes} \mathbb{F}_{n,0} \widehat{\otimes} \mathcal{K})$$

where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and we equip the set with a quasi-topology. We can define the structure maps too and denote this quasi-spectrum by $\mathbb{X}(A, B)$.

The notion of an orthogonal quasi-spectrum is a quasi-spectrum with an action of the orthogonal group and additional properties. This is necessary for a rich product structure called the smash product of orthogonal quasi-spectra. We prove that this structure exists in this framework and obtain the following theorem.

Theorem B. *There is a natural map of orthogonal quasi-spectra*

$$\mathbb{X}(A, B) \wedge \mathbb{X}(B, C) \to \mathbb{X}(A, C).$$

This result gives us a realisation of the E-theory product.

The thesis concludes with a connection of K-theory and K-homology orthogonal quasi-spectra using the E-theory orthogonal quasi-spectrum and the smash product as in the above theorem.

Others have done similar constructions, namely Mitchener [Mit01] constructed a symmetric spectra for K-theory and a symmetric spectra of KK-theory for C^*-categories. Joachim and Stolze [JS09] created an enrichment of KK-theory over the category of symmetric spectra.

There are many avenues for future work. The author is working in collaboration with Mitchener on extending the thesis to C^*-categories. Then in a postdoctoral position at Penn State, the author hopes to use this theory in applications to positive scalar curvature results, by generalising work of Weiss and Williams [WW95].

Chapter overview

Chapter 1
We cover key preliminaries on ungraded E-theory, some necessary topological notions, the concept of a quasi-topology and categorical definitions.
Chapter 2
Here we detail the definition of $\mathbb{Z}_2$-graded C^*-algebras and cover the generalisation of the E-theory groups in this case. We include proofs throughout and set up material for later.
Chapter 3
We consider real C^*-algebras and prove Bott periodicity for real K-theory defined in terms of E-theory and then generalise this to the case of E-theory.
Chapter 4
We prove that quasi-topological spaces satisfy various properties. Additionally we prove that the category of quasi-orthogonal sequences is a symmetric monoidal category.
Chapter 5
We define orthogonal quasi-spectrum which is a quasi-orthogonal sequence with further structure, and prove that the category of these is equivalent to the category of R-modules. This then gives a smash product structure. We construct the orthogonal quasi-spectrum for E-theory and show it has the properties required before defining a smash product structure.
Chapter 6
Finally, we combine the E-theory orthogonal quasi-spectrum with K-theory and K-homology orthogonal quasi-spectrum to connect these together.
Appendix A
We give details on functional calculus for complex, unbounded, real and graded C^*-algebras.
Appendix B
We give the definition of K-theory and detail its properties.

Contents

Chapter 1

Preliminaries

In this chapter, we detail the specific definitions and technical statements that we need to define in order to define the E-theory groups. We will cover the concept of a C^*-algebra and asymptotic morphisms as well as homotopy classes of morphisms. Some of this is taken from [WO93].

1.1 C^*-algebras

Definition 1.1.1. A *complex Banach space* is a complete normed vector space. A *Banach algebra* is a complex Banach space A with an associative multiplication operation, $A \times A \to A$, written $(a, b) \mapsto ab$ such that:

- $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$, for all $a, b, c \in A$,
- $(\lambda a)(\mu b) = (\lambda\mu)(ab)$, for all $\lambda, \mu \in \mathbb{C}$ and $a, b \in A$,

and a norm $|| - ||$ satisfying

$$||ab|| \leq ||a||||b||.$$

For a proof of the following see Theorem 2.11 in [Rud91].

Theorem 1.1.2 (Open Mapping Theorem)**.** *Let $f\colon V \to W$ be a surjective continuous linear map between Banach spaces. Then f is open.*

□

A consequence of the Open Mapping Theorem is that for a surjective linear continuous map $f\colon V \to W$ between Banach spaces, we can find a continuous linear map $g\colon W \to V$ such that $f \circ g = \mathrm{id}_W$.

Definition 1.1.3. An *involution* is a mapping $a \mapsto a^*$ satisfying:

(a) (Conjugate-linear) $(\lambda a + \mu b)^* = \overline{\lambda} a^* + \overline{\mu} b^*$ for all $a, b \in A$ and $\lambda, \mu \in \mathbb{C}$,

(b) $(a^*)^* = a$ for all $a \in A$,

(c) $(ab)^* = b^* a^*$ for all $a, b \in A$.

A C^*-*algebra* is a complex Banach algebra with an involution $*$ that satisfies the C^*-identity $||aa^*|| = ||a||^2$ for all $a \in A$.

An *algebra* A is commutative if $ab = ba$ for all $a, b \in A$. Let $C(X)$ denote the continuous functions from a locally compact Hausdorff space X to the complex numbers and $C_0(X)$ those functions of $C(X)$ that vanish in norm at infinity.

Some examples of C^*-algebras:

- Let X be a locally compact Hausdorff space, then $C(X)$ is a commutative C^*-algebra where involution is given by complex conjugation. The norm here is the standard supremum norm.
- Let H be a Hilbert space, then the set of bounded linear operators on H, $\mathcal{B}(H)$, is a C^*-algebra where involution $*$ is defined as the adjoint and with norm defined by the operator norm.

Definition 1.1.4. Let A and B be C^*-algebras. Then we call $f \colon A \to B$ a $*$-*homomorphism* if:

1. f is an algebra homomorphism, that is
   (a) $f(ka) = kf(a)$ for all $a \in A$ and $k \in \mathbb{C}$,
   (b) $f(a + b) = f(a) + f(b)$ for all $a, b \in A$,
   (c) $f(ab) = f(a)f(b)$ for all $a, b \in A$,
2. $f(a^*) = f(a)^*$.

If in addition f is bijective, then we have a C^*-isomorphism, denoted by $\cong$, and say that A and B are isomorphic as C^*-algebras.

Denote the set of $*$-homomorphisms from A to B by $\mathrm{Hom}(A, B)$.

Continuity is automatic by the above definition for a $*$-homomorphism. This is clear once you see that a $*$-homomorphism is norm-decreasing (see Theorem 2.17 in [Mur90]), and norm-decreasing implies continuous. We call a

C^*-algebra *separable*, if its underlying topological space has a countable dense subset.

For a proof of the Theorem below see Theorem 2.1.10 in [Mur90].

Theorem 1.1.5. *(Gelfand-Naimark I) Let A be a commutative C^*-algebra. Then $A \cong C_0(X)$ for some locally compact Hausdorff space X.*

$\square$

See Theorem 3.4.1 in [Mur90] for a proof of the following.

Theorem 1.1.6. *(Gelfand-Naimark II) Let A be a C^*-algebra. Then A is isomorphic to a subalgebra of $\mathcal{B}(H)$ for some Hilbert space H.*

$\square$

Definition 1.1.7. The *spectrum* of an element x in a unital C^*-algebra is the set

$$\sigma(x) = \{\lambda \in \mathbb{C} \mid x - \lambda \text{ is not invertible}\}.$$

We pass to the spectrum of an operator for later on. We need to be careful with the definition in the unbounded case, as the domain of an unbounded operator needs to be dense in our Hilbert space. We firstly consider the complement to the spectrum, namely the resolvent set. The following results are from [Kre89], namely Theorem 10.4-1, Theorem 10.4-2(page 544) and one inclusion from Theorem 7.4-2 (page 381) and proofs are consistent even with extra conditions.

Theorem 1.1.8. *Let $T\colon D(\mathcal{H}) \to \mathcal{H}$ be an unbounded self-adjoint on a complex Hilbert space $\mathcal{H}$ where $D(\mathcal{H})$ is a dense subset of $\mathcal{H}$. Then λ is in the resolvent set, if for every $x \in D(\mathcal{H})$ there exists a $C > 0$ such that*

$$||(T - \lambda I)x|| \geq C||x||.$$

$\square$

Definition 1.1.9. Consider an unbounded operator $T\colon D(\mathcal{H}) \to \mathcal{H}$ on a complex Hilbert space $\mathcal{H}$ where $D(\mathcal{H})$ is a dense subset of $\mathcal{H}$. $D(\mathcal{H})$ is not in general closed and so we define the *spectrum* of T by the set

$$\sigma(T) = \{\lambda \in \mathbb{C} \mid (T - \lambda I)^{-1} \text{ exists and is bounded}\}.$$

Theorem 1.1.10. *The spectrum of a self-adjoint unbounded operator T is real and closed.*

□

Theorem 1.1.11. *Let X be a complex Hilbert space and T an unbounded linear operator on X. Then for $a_n \neq 0$, let*

$$p(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_0.$$

Then

$$\sigma(p(T)) \subset p(\sigma(T)).$$

□

We should note that for an operator T on a real Hilbert space we have similar results. This is because we can consider the complexification of T, $T_{\mathbb{C}} = T \otimes_{\mathbb{R}} \mathbb{C}$, which is the tensor product of T and $\mathbb{C}$ over $\mathbb{R}$. Then it follows that $\sigma(T) = \sigma(T_{\mathbb{C}})$.

1.2 Asymptotic morphisms

Definition 1.2.1. Let A and B be C^*-algebras. An *asymptotic morphism* from A to B, denoted by $\varphi\colon A \dashrightarrow B$, is a family of functions $\{\varphi_t\}_{t\in[1,\infty)}$ such that:

1. the map $t \mapsto \varphi_t(x)$, from $[1,\infty)$ to B is continuous for each $x \in A$,

2. $\lim_{t\to\infty} ||\varphi_t(xy) - \varphi_t(x)\varphi_t(y)|| = 0$, for each $x, y \in A$,

3. $\lim_{t\to\infty} ||\varphi_t(x + \lambda y) - \varphi_t(x) - \lambda\varphi_t(y)|| = 0$, for each $x, y \in A$, $\lambda \in \mathbb{C}$,

4. $\lim_{t\to\infty} ||\varphi_t(x^*) - \varphi_t(x)^*|| = 0$, for each $x, y \in A$.

That is an asymptotic morphism is a family of functions which asymptotically converges to a $*$-homomorphism.

Note that we will often not refer to the family of functions that form an asymptotic morphism but if we have an asymptotic morphism φ then it is associated to the family of functions denoted φ_t.

Definition 1.2.2. Two asymptotic morphisms $\varphi, \psi \colon A \dashrightarrow B$ formed from the families of functions φ_t, ψ_t respectively are called *equivalent* if for all $a \in A$:

$$\lim_{t \to \infty} ||\varphi_t(a) - \psi_t(a)|| = 0.$$

It is easy to see that this relation is in fact an equivalence relation. Denote this equivalence by $\sim_{asy}$, and the set of equivalence classes of an asymptotic morphism φ by $\langle \varphi \rangle$. Denote the set of equivalence classes of asymptotic morphisms from A to B, by $\langle A, B \rangle$.

Definition 1.2.3. Let A be a C^*-algebra and let $T = [1, \infty)$. Then define the asymptotic algebra of A, $\mathfrak{A}A$, by:

$$\mathfrak{A}A := C_b(T, A)/C_0(T, A),$$

where $C_b(T, A)$ denotes bounded continuous functions from T to A and $C_0(T, A)$ denotes those functions of $C_b(T, A)$ which vanish in norm at infinity. Alternatively,

$$\mathfrak{A}A = \{[f] \mid f \colon [1, \infty) \to A \mid f \text{ is bounded and continuous}\},$$

and $[f]$ denotes the equivalence class of functions, where $f \sim g$ if

$$\lim_{t \to \infty} ||f(t) - g(t)|| = 0.$$

The algebra structure here is defined pointwise as expected.

Let $\widetilde{\mathfrak{A}}A$ denote the algebra of continuous bounded functions from $[1, \infty)$ to A, that is to say $C_b(T, A)$.

Denote the set of $*$-homomorphisms $f \colon A \to \mathfrak{A}B$ by $Hom(A, \mathfrak{A}B)$.

Lemma 1.2.4. *We have a map $\theta \colon \mathfrak{A}B \to \widetilde{\mathfrak{A}}B$ such that for any $*$-homomorphism $\psi \colon A \to \mathfrak{A}B$ the map $\widehat{g(\psi)} \colon A \dashrightarrow B$ defined by*

$$\widehat{g(\psi)}_t(a) = \theta\psi(a)(t),$$

is an asymptotic morphism for all $a \in A$, $t \in [1, \infty)$.

Proof. We have an obvious surjective $*$-homomorphism $\pi \colon \widetilde{\mathfrak{A}}B \to \mathfrak{A}B$ which is clearly continuous and linear. Hence by Theorem 1.1.2, we see that π is open.

Consequently we have a continuous map $\theta\colon \mathfrak{A}B \to \widetilde{\mathfrak{A}}B$, such that $\pi \circ \theta = \mathrm{id}_{\mathfrak{A}B}$. Then we can define

$$\widehat{g(\psi)_t}(a) = \theta\psi(a)(t).$$

Clearly the map $t \mapsto \widehat{g(\psi)_t}$ is continuous since θ and ψ are continuous. Let $a, b \in A$, then since π is multiplicative and linear,

$$\begin{aligned}\pi(\theta\psi(a)\theta\psi(b) - \theta\psi(ab)) &= \pi(\theta\psi(a))\pi(\theta\psi(b)) - \pi(\theta\psi(ab)) \\ &= \psi(a)\psi(b) - \psi(ab) \\ &= 0.\end{aligned}$$

Then,

$$\theta\psi(a)\theta\psi(b) - \theta\psi(ab) \in C_0([1,\infty), B),$$

so

$$||\widehat{g(\psi)_t}(a)\widehat{g(\psi)_t}(b) - \widehat{g(\psi)_t}(ab)|| = ||\theta\psi(a)(t)\theta\psi(b)(t) - \theta\psi(ab)(t)|| \to 0,$$

as $t \mapsto \infty$. The other conditions to check $\theta\psi$ defines an asymptotic morphism are checked similarly. □

We define a function $g\colon Hom(A, \mathfrak{A}B) \to \langle A, B\rangle$ using the above. So $g(\psi) = \langle\widehat{g(\psi)}\rangle$ since $g(\psi)$ is the class of the above asymptotic morphism.

The following is stated similarly in [D94] on page 3 with some justifications, but the details are given in the proof below.

Theorem 1.2.5. *There is a bijection between the set of equivalence classes of asymptotic morphisms $A \dashrightarrow B$ and the set of $*$-homomorphisms $A \to \mathfrak{A}B$.*

Proof. Define $f\colon \langle A, B\rangle \to Hom(A, \mathfrak{A}B)$ as follows. Let $\varphi\colon A \dashrightarrow B$ be an asymptotic morphism formed from the family of functions φ_t, with $\langle\varphi\rangle$ as the equivalence class of φ. Then we want a $*$-homomorphism $f(\langle\varphi\rangle) \in Hom(A, \mathfrak{A}B)$. Let $a \in A$, then $f(\langle\varphi\rangle)(a) \in \mathfrak{A}B$. First let us define $\widetilde{f}(\langle\varphi\rangle)(a) \in C_b(T, A)$ with $T = [1, \infty)$ by,

$$\widetilde{f}(\langle\varphi\rangle)(a)(t) = \varphi_t(a),$$

for all $t \in [1, \infty)$. Then define

$$f(\langle\varphi\rangle)(a) = [\widetilde{f}(\langle\varphi\rangle)(a)].$$

Now we need to check that f is well defined. Let $\varphi_0, \varphi_1\colon A \dashrightarrow B$ be equivalent asymptotic morphisms formed from the family of functions $(\varphi_0)_t$

and $(\varphi_1)_t$ respectively, then we want to show that $f(\langle\varphi_0\rangle) = f(\langle\varphi_1\rangle)$. Since $\varphi_0 \sim \varphi_1$, for all $a \in A$,

$$\lim_{t\to\infty} ||(\varphi_0)_t(a) - (\varphi_1)_t(a)|| = 0.$$

Then we have $\widetilde{f}(\langle\varphi_0\rangle)(a)(t) = (\varphi_0)_t(a)$ and $f(\langle\varphi_0\rangle)(a) = [\widetilde{f}(\langle\varphi_0\rangle)(a)]$, and similarly for φ_1. Then since $\varphi_0 \sim \varphi_1$, $\widetilde{f}(\langle\varphi_0\rangle)(a)(t) - \widetilde{f}(\langle\varphi_1\rangle)(a)(t) \to 0$ as $t \to \infty$, and hence,

$$[\widetilde{f}(\langle\varphi_0\rangle)(a)(t)] - [\widetilde{f}(\langle\varphi_0\rangle)(a)(t)] = 0.$$

Hence $f(\langle\varphi_0\rangle) = f(\langle\varphi_1\rangle)$.

Now we check that $f(\langle\varphi\rangle)$ is a $*$-homomorphism. Note that $f(\langle\varphi\rangle)(x) = 0$ means that $[\widetilde{f}(\langle\varphi\rangle)(x)] = [0]$. In our case we have that

$$\varphi_t(a^*) - \varphi_t(a)^* \to 0,$$

as $t \to \infty$, so

$$\widetilde{f}(\langle\varphi\rangle)(a^*)(t) - \widetilde{f}(\langle\varphi\rangle)(a)(t)^* \to 0,$$

as $t \to \infty$, and so

$$f(\langle\varphi\rangle)(a^*) - f(\langle\varphi\rangle)(a)^* = 0.$$

Similarly we can also prove the other requirements of a $*$-homomorphism hold.

We now define $g\colon Hom(A, \mathfrak{A}B) \to \langle A, B\rangle$ as above,

$$\widehat{g(\psi)_t}(a) = \theta\psi(a)(t).$$

where $\theta\colon \mathfrak{A}B \to \widetilde{\mathfrak{A}}B$ is such that $\widehat{g(\psi)_t}$ is a asymptotic morphism, and this is possible by Lemma 1.2.4.

Finally we need to show that $f \circ g = id$, and $g \circ f = id$.

Let $\varphi\colon A \to \mathfrak{A}B$ be a $*$-homomorphism,

$$f \circ g(\varphi) = \varphi_1,$$

then we want $\varphi = \varphi_1$. Then we have

$$f(g(\varphi))(a) = [\widetilde{f}(g(\varphi))(a)] \text{ and } \widetilde{f}(g(\varphi))(a)(t) = g(\varphi)_t(a),$$

by the above definitions. Also $\widehat{g(\varphi)_t}(a) = \theta\psi(a)(t)$, but $[\theta\psi(a)] = [\widetilde{\psi(a)}] = \varphi$, and using the relation between $g(\psi)$ and $\widehat{g(\psi)}$, we get our equivalence by the properties of our equivalence relation.

Now let $\varphi\colon A \dashrightarrow B$ be an asymptotic morphism associated to the family φ_t and denote the set of equivalence classes of φ by $\langle\varphi\rangle$. Then suppose for some ψ, $g \circ f(\langle\varphi\rangle) = \psi$. Then $g(f(\langle\varphi\rangle)(a)) = g[\widetilde{f}(\langle\varphi\rangle)(a)]$, and $\widetilde{f}(\langle\varphi\rangle)(a)(t) = g(\varphi)_t(a)$ and $\widehat{g(\psi)}t(a) = \theta\psi(a)(t)$, and similarly to the above we get that $\varphi = \psi$.

□

Definition 1.2.6. Let A and B be C^*-algebras. Then we define the object $A\widehat{\otimes}B$ to be the completion of the algebraic tensor product of A and B in the norm

$$||\sum_i a_i \otimes b_i|| = \sup_{\varphi,\psi} ||\sum_i \varphi(a_i)\psi(b_i)||$$

where $\varphi\colon A \to \mathfrak{A}C$, $\psi\colon B \to \mathfrak{A}C$ are $*$-homomorphisms for some C^*-algebra C. This is the maximal tensor product.

By Theorem 1.2.5 and the definition above we get a tensor product from asymptotic morphisms in the same manner.

1.3 Homotopy classes of asymptotic morphisms

Let A, B be C^*-algebras. Denote by IB the continuous functions from a closed interval I to B. Define the *suspension*, ΣA, by:

$$\Sigma A := \{f\colon [0,1] \to A \mid f(0) = f(1) = 0, f \text{ is continuous}\}.$$

This is a C^*-algebra itself with operations taken pointwise and its norm is the supremum norm. We can apply the suspension to a $*$-homomorphism $g\colon A \to B$ between C^*-algebras and we obtain the $*$-homomorphism $\Sigma g\colon \Sigma A \to \Sigma B$, where

$$(\Sigma g(\mu))(s) = g(\mu(s)),$$

for all $\mu \in \Sigma A, s \in [0,1]$.

Definition 1.3.1. Two $*$-homomorphisms $\varphi_0, \varphi_1\colon A \to B$ are *homotopic* if there exists a $*$-homomorphism $\varphi\colon A \to IB$ such that $\varphi(a)(0) = \varphi_0(a)$ and $\varphi(a)(1) = \varphi_1(a)$ for all $a \in A$.

The conditions in the definition above can also be written as $ev_0 \circ \varphi = \varphi_0$ and $ev_1 \circ \varphi = \varphi_1$, where ev_i denotes evaluation at i for $i = 0, 1$, that is $ev_i(f) = f(i)$ for a function f.

The notion of homotopy is an equivalence relation and the classes obtained are called homotopy classes. Denote the set of these homotopy classes of $*$-homomorphisms from A to B by $[A, B]$.

Proposition 1.3.2. *Let $\alpha, \beta\colon A \to B$ be homotopic $*$-homomorphisms, then the $*$- homomorphisms $\Sigma\alpha, \Sigma\beta\colon \Sigma A \to \Sigma B$ are homotopic.*

Proof. Let $\varphi\colon A \to IB$ be a $*$-homomorphism such that $ev_0 \circ \varphi = \alpha$ and $ev_1 \circ \varphi = \beta$. Now for $g \in \Sigma A$, $s \in [0,1]$, we can define

$$(\Sigma\alpha)(g)(s) = \alpha(g(s)) \text{ and } (\Sigma\beta)(g)(s) = \beta(g(s)).$$

Then we can define a homotopy $\psi\colon \Sigma A \to I\Sigma B$ by

$$\psi(g)(t)(s) = \varphi(g(s))(t),$$

for all $g \in \Sigma A, t \in I$ and $s \in [0,1]$. Then it suffices to check that $ev_0 \circ \psi = \Sigma\alpha$ and $ev_1 \circ \psi = \Sigma\beta$. For all $g \in \Sigma A, s \in [0,1]$ we have

$$ev_0 \circ \psi = \psi(g)(0)(s) = \varphi(g(s))(0) = \alpha(g(s)) = (\Sigma\alpha)(g)(s),$$

so $ev_0 \circ \psi = \Sigma\alpha$. Similarly we can show that $ev_1 \circ \psi = \Sigma\beta$ as required. $\square$

Definition 1.3.3. Two asymptotic morphisms $\varphi, \psi\colon A \dashrightarrow B$ with families of functions φ_t, ψ_t are *homotopic* if there exists an asymptotic morphism $\theta\colon A \dashrightarrow IB$ with family of functions θ_t such that for all $a \in A$:

$$\theta_t(a)(0) = \varphi_t(a) \text{ and } \theta_t(a)(1) = \psi_t(a).$$

Proposition 1.3.4. *Let $\varphi, \psi\colon A \dashrightarrow B$ be equivalent asymptotic morphisms. Then φ and ψ are homotopic.*

Proof. Define a homotopy $\theta\colon A \dashrightarrow IB$ by

$$\theta_t(a)(s) = (1-s)\varphi_t(a) + s\psi_t(a) = \varphi_t(a) + s(\psi_t(a) - \varphi_t(a)),$$

for all a in A, and $s \in [0,1]$.

Then we need to check that θ is an asymptotic morphism. That is check,

$$\lim_{t\to\infty} ||\theta_t(x + \lambda y) - \theta_t(x) - \lambda\theta_t(y)|| = 0,$$

for each $x, y \in A$, $\lambda \in \mathbb{C}$. For all $s \in [0,1]$,

$$\theta_t(x + \lambda y)(s) - \theta_t(x)(s) - \lambda\theta_t(y)(s),$$

equals

$$\varphi_t(x+\lambda y)+s(\psi_t(x+\lambda y)-\varphi_t(x+\lambda y))-(\varphi_t(x)+s(\psi_t(x)-\varphi_t(x)))$$
$$-\lambda(\varphi_t(y)+s(\psi_t(y)-\varphi_t(y))).$$

Since $\varphi_t(x+\lambda y)(s)-\varphi_t(x)(s)-\lambda\varphi_t(y)(s)\to 0$ as $t\to\infty$, we have

$$s(\psi_t(x+\lambda y)-\varphi_t(x+\lambda y))-s(\psi_t(x)-\varphi_t(x))-\lambda(s(\psi_t(y)-\varphi_t(y))),$$

as $t\to\infty$, and as φ and ψ are equivalent, the expression above tends to 0 as t tends to infinity. This is true for all s, so

$$\lim_{t\to\infty}||\theta_t(x+\lambda y)-\theta_t(x)-\lambda\theta_t(y)||=0.$$

The other conditions are similarly checked. □

Note that the definition of homotopy and statement above are from the paper [D94]. Let $[\![f]\!]$ denote the homotopy class of an asymptotic morphism f and if f and g are homotopic as asymptotic morphism then write $f\sim_h g$.

We now define an n-asymptotic morphism (for any natural number n) and the notion of homotopy for these, in order to form a single set of homotopy classes. Let $|-|$ denote the supremum of an ordered set. For example if $k=(t_1,t_2,\ldots,t_n)$ then $|k|=\sup_i(t_1,t_2,\ldots,t_n)$ for all $t_i\in\mathbb{R}$.

Definition 1.3.5. Let A and B be C^*-algebras. Then for any natural number n, an *n-asymptotic morphism* φ from A to B is a family of functions $\varphi_k\colon A \dashrightarrow B$ where $k=(t_1,t_2,\ldots,t_n)$ for $t_i\in[1,\infty)$ for all i, such that

1. the map $k\mapsto\varphi_k(x)$, from $[1,\infty)^n$ to B is continuous for each $x\in A$,
2. $\lim_{|k|\to\infty}||\varphi_k(xy)-\varphi_k(x)\varphi_k(y)||=0$, for each $x,y\in A$,
3. $\lim_{|k|\to\infty}||\varphi_k(x+\lambda y)-\varphi_k(x)-\lambda\varphi_k(y)||=0$, for each $x,y\in A$, $\lambda\in\mathbb{C}$,
4. $\lim_{|k|\to\infty}||\varphi_k(x^*)-\varphi_k(x)^*||=0$, for each $x,y\in A$.

Note that a 0-asymptotic morphism is a $*$-homomorphism and a 1-asymptotic morphism is just an asymptotic morphism.

Definition 1.3.6. Let $\varphi,\psi\colon A\dashrightarrow B$ be n-asymptotic morphisms with associated famililes of functions φ_k,ψ_k, where $k=(t_1,t_2,\ldots,t_n)$ for all $t_i\in[1,\infty)$ for all i. Then φ,ψ are *equivalent* if for all a in A

$$\lim_{|k|\to\infty}||\varphi_k(a)-\psi_k(a)||=0.$$

Definition 1.3.7. Let $\varphi, \psi \colon A \dashrightarrow B$ be n-asymptotic morphisms with associated families of functions φ_k, ψ_k, where $k = (t_1, t_2, \ldots, t_n)$ for all $t_i \in [1, \infty)$ for all i. Then φ, ψ are *homotopic* if there exists an n-asymptotic morphism $\theta \colon A \dashrightarrow IB$ with family of functions θ_k, such that for all $a \in A$:

$$\theta_k(a)(0) = \varphi_k(a) \text{ and } \theta_k(a)(1) = \psi_k(a).$$

Proposition 1.3.8. *Let $\varphi, \psi \colon A \dashrightarrow B$ be equivalent n-asymptotic morphisms. Then φ and ψ are homotopic.*

Proof. This proof is the same as for Proposition 1.3.4, but extended to n-asymptotic morphisms. □

Definition 1.3.9. Let A and B be C^*-algebras, then we denote by $[\![A, B]\!]_n$, the set of homotopy classes of n-asymptotic morphisms from A to B.

Note that $[\![A, B]\!]_1$ is the set of homotopy classes of asymptotic morphisms from A to B.

Definition 1.3.10. Denote by $[\![A, B]\!]$, the direct limit of the following system

$$[\![A, B]\!]_0 \to [\![A, B]\!]_1 \to [\![A, B]\!]_2 \to \cdots$$

where the linking maps $[\![A, B]\!]_n \to [\![A, B]\!]_{n+1}$ are given by

$$\alpha(\varphi)_{(t_1, t_2, \ldots, t_n, t_{n+1})} \mapsto \varphi_{(t_1, t_2, \ldots, t_n)},$$

and by taking homotopy classes.

Theorem 1.3.11. *If A is a separable C^*-algebra then the natural map in the direct limit*

$$[\![A, B]\!]_1 \to \varinjlim [\![A, B]\!]_n,$$

is a bijection. Thus $[\![A, B]\!]$ is isomorphic to the set of homotopy equivalence classes of asymptotic morphisms from A to B.

For a proof see Theorem 2.16 of [GHT00].

Henceforth we will use $[\![A, B]\!]$ for homotopy classes of asymptotic morphisms from A to B and for an asymptotic morphism $\varphi \colon A \dashrightarrow B$ denote the homotopy class by $[\![\varphi]\!]$.

It is worth noting here that it is not possible to compose asymptotic morphisms simply by composing. This is since composing two asymptotic morphisms at the same t does not guarantee an asymptotic morphism as a result. Instead we need to reparameterise one when composing. Before we describe the concrete method we require the following lemma from [CH90] (Lemma 3).

Lemma 1.3.12. *Let $\varphi\colon A \dashrightarrow B$ and $\psi\colon B \dashrightarrow C$ be asymptotic morphisms. Let A' be a dense $*$-subalgebra of A which is a countable union of compacts. Then there exists an increasing continuous function $r\colon [1,\infty) \to [1,\infty)$ such that for any increasing continuous function $s\colon [1,\infty) \to [1,\infty)$ with $s(t) \geq r(t)$ the composite*

$$\theta_t = \psi_{s(t)} \circ \varphi_t,$$

is an asymptotic morphism from $A' \dashrightarrow C$.

□

Due to this lemma and since

$$\limsup_t = ||(\psi_{s(t)} \circ \varphi_t)(a)|| \leq ||a||,$$

for all $a \in A$, the map $\theta_t\colon A' \dashrightarrow C$ is well-defined and bounded. Hence we can lift so that the function vanishes at infinity and obtain an asymptotic morphism from A to C. Henceforth we will say $r(t)$ is our reparameterisation. We should also note that the choice of reparameterisation has no effect on the homotopy classes and also composition with $*$-homomorphisms can be done in the obvious manner.

For a proof of the subsequent result see Proposition 2.12 in [GHT00].

Proposition 1.3.13. *Given asymptotic morphisms $\varphi\colon A \dashrightarrow B$ and $\psi\colon B \dashrightarrow C$ with families of functions φ_t and ψ_t, we have an associative composition law*

$$[\![A,B]\!] \times [\![B,C]\!] \to [\![A,C]\!],$$

given by

$$[\![\varphi]\!] \times [\![\psi]\!] \to [\![\psi \circ \varphi]\!].$$

Here on a representative $(\psi \circ \varphi)_t$ of $[\![\psi \circ \varphi]\!]$, we can define

$$(\psi \circ \varphi)_t = \psi_{r(t)} \circ \varphi_t.$$

□

For a proof of the next result see Proposition 2.19 in [GHT00].

Proposition 1.3.14. *For any C^*-algebra B the natural map*

$$[C_0(\mathbb{R}), B] \to [\![C_0(\mathbb{R}), B]\!],$$

defined by sending a representative φ of $[C_0(\mathbb{R}), B]$ to the constant asymptotic morphism representative in $[\![C_0(\mathbb{R}), B]\!]$ (i.e $\varphi_t = \varphi$ for all $t \in [1,\infty)$), is a bijection.

□

1.4 E-theory

Throughout this section all C^*-algebras are separable, so their underlying topological spaces each have a countable dense subset. Let $\mathcal{K}(\mathcal{H})$ denote the set of compact operators on a separable Hilbert space $\mathcal{H}$. For simplicity we will just use $\mathcal{K}$ throughout. Note that $\Sigma B \otimes \mathcal{K} = \Sigma(B \otimes \mathcal{K})$.

We start by defining two operations on the set $[\![A, \Sigma B \otimes \mathcal{K}]\!]$ which we will show are equivalent. This is in order to show that there is an operation that the set $[\![A, \Sigma B \otimes \mathcal{K}]\!]$ can be equipped with, which makes it into an abelian group.

We will use the following theorem to prove this.

The succeeding Theorem's proof can be found in [EH62] Theorem 4.17.

Theorem 1.4.1 (Eckmann-Hilton Argument)**.** *Let $+$ and $\widetilde{+}$ denote two operations on a set A with common identity element $e \in A$, such that for all a_0, a_1, a_2, a_3,*

$$(a_0 + a_1)\widetilde{+}(a_2 + a_3) = (a_0 \widetilde{+} a_2) + (a_1 \widetilde{+} a_3).$$

Then the operations are equal, and furthermore are commutative.

□

Now let us define an isomorphism $\alpha_0 \colon \mathcal{K}(\mathcal{H} \oplus \mathcal{H}) \to \mathcal{K}(\mathcal{H})$ by

$$\alpha_0(T) = UTU^*,$$

where $T \in \mathcal{K}(\mathcal{H} \oplus \mathcal{H})$ and $U \colon \mathcal{H} \oplus \mathcal{H} \to \mathcal{H}$ is a unitary operator defined by sending the elements

$$e_0 \oplus e_0, e_1 \oplus e_0, e_0 \oplus e_1, e_1 \oplus e_1, e_2 \oplus e_0, \ldots$$

to $e_0, e_1, e_2, \ldots$ where e_i form an orthonormal basis for $\mathcal{H}$.

Observe that we therefore also have isomorphisms:

$$\alpha \colon B \otimes \mathcal{K}(\mathcal{H} \oplus \mathcal{H}) \to B \otimes \mathcal{K}(\mathcal{H}), \ \ \Sigma(B \otimes \mathcal{K}(\mathcal{H} \oplus \mathcal{H})) \to \Sigma(B \otimes \mathcal{K}(\mathcal{H})).$$

Definition 1.4.2. Let A and B be C^*-algebras. Given the two asymptotic morphisms $\varphi, \psi \colon A \dashrightarrow \Sigma(B \otimes \mathcal{K}(\mathcal{H}))$ we define the operation $\widetilde{\oplus}$, $[\![\varphi]\!] \widetilde{\oplus} [\![\psi]\!]$, to be the homotopy class of the asymptotic morphism $\theta \colon A \dashrightarrow \Sigma(B \otimes \mathcal{K}(\mathcal{H}))$ given by

$$\theta_t(a) = \alpha \left[\begin{pmatrix} \varphi_t(a) & 0 \\ 0 & \psi_t(a) \end{pmatrix} \right],$$

for all $a \in A$.

The above makes sense since the C^*-algebra $B \otimes \mathcal{K}(\mathcal{H})$ is a direct limit of matrices $M_n(B)$ by the mapping

$$b \to \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}.$$

For details see [RLL00] page 102. Note that θ_t above is also the same as the homotopy class of the asymptotic morphism $(\varphi \widetilde{\oplus} \psi)_t$.

Lemma 1.4.3. *The operation $\widetilde{\oplus}$ defined in Definition 1.4.2 is well-defined.*

□

Definition 1.4.4. Given two asymptotic morphisms $\varphi, \psi \colon A \dashrightarrow \Sigma B \otimes \mathcal{K}$ we define the direct sum operation $\oplus$, $[\![\varphi]\!] \oplus [\![\psi]\!]$, to be the homotopy class of the asymptotic morphism $\theta \colon A \dashrightarrow \Sigma(B \otimes \mathcal{K})$ defined by

$$\theta_t(a)(s) = \begin{cases} \varphi_t(a)(2s), & \text{if } s \in [0, \frac{1}{2}] \\ \psi_t(a)(2s-1), & \text{if } s \in [\frac{1}{2}, 1], \end{cases}$$

for all $a \in A$ and $s \in [0, 1]$.

Note that θ_t above is also the same as the homotopy class of the asymptotic morphism $(\varphi \oplus \psi)_t$.

Proposition 1.4.5. *The group operation defined above is well-defined.*

Proof. We need to show that the direct sum operation is well-defined. It suffices to show that this holds for representative elements in the class since then it will hold for the whole class. Suppose that we have asymptotic morphisms $\varphi_0, \varphi_0', \varphi_1, \varphi_1' \colon A \dashrightarrow \Sigma B \otimes \mathcal{K}$, such that $\varphi_0 \sim_h \varphi_0'$ and $\varphi_1 \sim_h \varphi_1'$. Then we have asymptotic morphisms

$$\varphi_t \colon A \dashrightarrow I(\Sigma B \otimes \mathcal{K}) \text{ and } \psi_t \colon A \dashrightarrow I(\Sigma B \otimes \mathcal{K})$$

such that for all $a \in A, t \in [1, \infty)$, $\varphi(a)(0) = \varphi_0(a)$, $\varphi(a)(1) = \varphi_0'(a)$, $\psi(a)(0) = \varphi_1(a)$ and $\psi(a)(1) = \varphi_1'(a)$. Now we define

$$\theta \colon A \dashrightarrow I(\Sigma B \otimes \mathcal{K}) \oplus I(\Sigma B \otimes \mathcal{K}),$$

for all $a \in A$, by

$$\theta_t(a)(s) = \begin{cases} (\varphi_t \oplus \psi_t)(a)(2s), & \text{if } s \in [0, \frac{1}{2}] \\ (\varphi_t \oplus \psi_t)(a)(2s-1), & \text{if } s \in [\frac{1}{2}, 1], \end{cases}$$

and then

$$\theta \colon A \dashrightarrow I((\Sigma B \otimes \mathcal{K}) \oplus (\Sigma B \otimes \mathcal{K})).$$

Then θ is clearly an asymptotic morphism and it is clear that $\varphi_0 \oplus \varphi_1 \sim_h \varphi_0' \oplus \varphi_1'$. □

Proposition 1.4.6. *For any asymptotic morphisms $\varphi, \psi, \mu, \tau \colon A \dashrightarrow \Sigma B \otimes \mathcal{K}$ the distributive law*

$$([\![\varphi]\!] \oplus [\![\psi]\!]) \widetilde{\oplus} ([\![\mu]\!] \oplus [\![\tau]\!]) = ([\![\varphi]\!] \widetilde{\oplus} [\![\mu]\!]) \oplus ([\![\psi]\!] \widetilde{\oplus} [\![\tau]\!])$$

holds.

Proof. By the definitions of $\widetilde{\oplus}$ and $\oplus$, it suffices to check that the distributive law holds for elements of the classes, that is

$$(\varphi \oplus \psi)_t \widetilde{\oplus} (\mu \oplus \tau)_t = (\varphi \widetilde{\oplus} \mu)_t \oplus (\psi \widetilde{\oplus} \tau)_t.$$

Now we have

$$\begin{aligned} (\varphi \oplus \psi)_t(a) \widetilde{\oplus} (\mu \oplus \tau)_t(a) &= \alpha \begin{pmatrix} (\varphi \oplus \psi)_t(a) & 0 \\ 0 & (\mu \oplus \tau)_t(a) \end{pmatrix} \\ &= \alpha \begin{pmatrix} \varphi_t(a) & 0 \\ 0 & \mu_t(a) \end{pmatrix} \oplus \alpha \begin{pmatrix} \psi_t(a) & 0 \\ 0 & \tau_t(a) \end{pmatrix} \\ &= (\varphi \widetilde{\oplus} \mu)_t(a) \oplus (\psi \widetilde{\oplus} \tau)_t(a), \end{aligned}$$

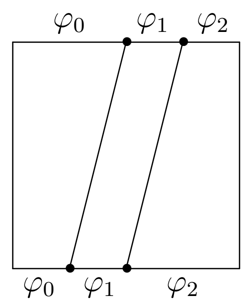
for all $a \in A$. □

It follows from the above and the Eckmann-Hilton argument that the operations $\widetilde{\oplus}$ and $\oplus$ defined on the set $[\![A, \Sigma B \otimes \mathcal{K}]\!]$ are equal.

Lemma 1.4.7. *Let A and B be C^*-algebras. The set $[\![A, \Sigma B \otimes \mathcal{K}]\!]$ is an abelian group under the direct sum operation $\oplus$ defined in Definition 1.4.4. The zero element of this group is represented by the zero asymptotic morphism.*

Proof. The group operation is well-defined by Proposition 1.4.5.

For associativity we have a homotopy $\beta\colon A \dashrightarrow I(\Sigma B \otimes \mathcal{K})$ that we first represent using the following diagram



and then by an exact formula:

$$\beta_t(a)(r)(s) = \begin{cases} (\varphi_0)_t(a)(4s/(1+r)) \text{ if } 0 \leq s \leq (r+1)/4 \\ (\varphi_1)_t(a)(4s-1-r) \text{ if } (r+1)/4 \leq s \leq (r+2)/4 \\ (\varphi_2)_t(a)(1-4\frac{(1-s)}{(2-r)}) \text{ if } (r+2)/4 \leq s \leq 1, \end{cases}$$

for all $a \in A, s \in [0,1]$ and $r \in [0,1]$.

Commutativity follows immediately from the Eckmann-Hilton argument and Proposition 1.4.6. A proof of the existence of additive inverses can be found in [Bla98], Proposition 25.4.3(c). □

Definition 1.4.8. Let A and B be C^*-algebras. Then the E-theory group is given by

$$E(A,B) = [\![\Sigma A \otimes \mathcal{K}, \Sigma B \otimes \mathcal{K}]\!].$$

In addition we have E-theory groups:

$$E^n(A,B) = [\![\Sigma A \otimes \mathcal{K}, \Sigma^{n+1} B \otimes \mathcal{K}]\!],$$

for all $n \geq 0$.

The definition of E-theory may look a bit forced but actually it is defined in this way so that we get certain desired properties. In particular the suspension of a C^*-algebra is necessary so that we obtain long exact sequences and needed for the group operation, and the compact operators are necessary for the group operation and for stability of our functor as defined later.

Lemma 1.4.9 (Functoriality)**.** *E is a bivariant functor from the category where objects are C^*-algebras and arrows are $*$-homomorphisms to the category of abelian groups and group homomorphisms. That is, it is a functor that is contravariant in its first variable and covariant in its second variable.*

Proof. By lemma 1.4.7 it is obvious that the source and target categories of the functor E are as stated in the lemma. The identity property is also clearly satisfied.

Let A, B, C and D be C^*-algebras. Let $\alpha\colon A \to B$ be a $*$-homomorphism, then we have an object $E(A, D) = [\![\Sigma A \otimes \mathcal{K}, \Sigma D \otimes \mathcal{K}]\!]$ for all A and a morphism $\alpha^*\colon E(B, D) \to E(A, D)$ for all α defined by $\alpha^*([\![x]\!]) = [\![x.\alpha]\!]$, where $(x.\alpha)_t = x_t \circ (\Sigma\alpha \otimes \mathrm{id}_{\mathcal{K}})$ for all $[\![x]\!] \in E(B, D)$. Now consider the composition of morphisms $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ on a representative x of $[\![x]\!]$,

$$\begin{aligned}(\beta \circ \alpha)^*(x_t) &= x_t \circ (\Sigma(\beta \circ \alpha) \otimes \mathrm{id}_{\mathcal{K}})\\ &= x_t \circ (\Sigma\beta \otimes \mathrm{id}_{\mathcal{K}}) \circ (\Sigma\alpha \otimes \mathrm{id}_{\mathcal{K}}) \text{ as } \Sigma \text{ is a functor}\\ &= \beta^*(x_t) \circ (\Sigma\alpha \otimes \mathrm{id}_{\mathcal{K}})\\ &= \alpha^*\beta^*(x_t)\\ &= (\alpha^* \circ \beta^*)(x_t).\end{aligned}$$

Similarly, we have for all A, $E(D, A) = [\![\Sigma D \otimes \mathcal{K}, \Sigma A \otimes \mathcal{K}]\!]$ and for all α, a morphism $\alpha_*\colon E(D, A) \to E(D, B)$ defined by $\alpha_*([\![y]\!]) = [\![\alpha.y]\!]$, where $(\alpha.y)_t = (\Sigma\alpha \otimes \mathrm{id}_{\mathcal{K}}) \circ y_t$ for all $[\![y]\!] \in E(D, A)$. Considering the composition of morphisms above and taking a representative y of $[\![y]\!] \in E(D, A)$ we see that

$$\begin{aligned}(\beta \circ \alpha)_*(y_t) &= (\Sigma(\beta \circ \alpha) \otimes \mathrm{id}_{\mathcal{K}}) \circ y_t\\ &= ((\Sigma\beta \circ \Sigma\alpha) \otimes \mathrm{id}_{\mathcal{K}}) \circ y_t\\ &= (\Sigma\beta \otimes \mathrm{id}_{\mathcal{K}}) \circ (\Sigma\alpha \otimes \mathrm{id}_{\mathcal{K}}) \circ y_t\\ &= (\Sigma\beta \otimes \mathrm{id}_{\mathcal{K}}) \circ \alpha_*(y_t)\\ &= \beta_*\alpha_*(y_t)\\ &= (\beta_* \circ \alpha_*)(y_t).\end{aligned}$$

□

Theorem 1.4.10 (Homotopy invariance)**.** *Suppose $\alpha, \beta\colon A \to B$ are homotopic $*$-homomorphisms and C is a C^*-algebra. Then the induced maps*

1. *$\alpha_*, \beta_* \colon E(C,A) \to E(C,B)$, and*

2. *$\alpha^*, \beta^* \colon E(B,C) \to E(A,C)$*

are equal.

Proof. 1. Let $f \colon \Sigma C \otimes \mathcal{K} \dashrightarrow \Sigma A \otimes \mathcal{K}$ be an asymptotic morphism representing $[\![f]\!] \in E(C,A)$, then we can define α_* and β_* by

$$\alpha_*(f_t) = (\Sigma\alpha \otimes \mathrm{id}_{\mathcal{K}}) \circ f_t = \Sigma(\alpha \otimes \mathrm{id}_{\mathcal{K}}) \circ f_t,$$
$$\beta_*(f_t) = (\Sigma\beta \otimes \mathrm{id}_{\mathcal{K}}) \circ f_t = \Sigma(\beta \otimes \mathrm{id}_{\mathcal{K}}) \circ f_t.$$

By Proposition 1.3.2, we see that $\Sigma(\alpha \otimes \mathrm{id}_{\mathcal{K}})$ and $\Sigma(\beta \otimes \mathrm{id}_{\mathcal{K}})$ are homotopic and hence α_* and β_* are equal since we are homotopic on representatives of our class.

2. Similarly to (1) we can define α^* and β^*, for an representative g of $[\![g]\!] \in E(B,C)$ by

$$\alpha^*(g_t) = g_t \circ (\Sigma\alpha \otimes id_{\mathcal{K}}),$$
$$\beta^*(g_t) = g_t \circ (\Sigma\beta \otimes id_{\mathcal{K}}).$$

Then by Proposition 1.3.2, we can clearly see that α^* and β^* are equal at the level of homotopy classes.

□

For a proof of the following see section 6 Lemma 2 and Lemma 4 in [CH].

Lemma 1.4.11 (Half exactness)**.** *Let $0 \to J \xrightarrow{\alpha} B \xrightarrow{\beta} A \to 0$ be a short exact sequence of C^*-algebras and let D be a C^*-algebra. Then*

1. *$E(D,J) \xrightarrow{\alpha_*} E(D,B) \xrightarrow{\beta_*} E(D,A)$ is exact,*

2. *$E(A,D) \xrightarrow{\beta^*} E(B,D) \xrightarrow{\alpha^*} E(J,D)$ is exact.*

□

Definition 1.4.12. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. Then a covariant functor $F \colon \mathscr{C} \to \mathscr{D}$ is called *half exact* if we have short exact sequence of objects in $\mathscr{C}$, $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ then the induced sequence $F(A) \xrightarrow{f_*} F(B) \xrightarrow{g_*} F(C)$, is exact. The functor F is called *homotopy invariant* if we have homotopic morphisms in $\mathscr{C}$, $\alpha, \beta \colon A \to B$, then the induced map $\alpha_* \colon F(A) \to F(B)$ and $\beta_* \colon F(A) \to F(B)$ are equal. If F is both half exact and homotopy invariant we call it a *homology functor.*

We obtain the following result from [WO93], Proposition 11.1.12.

Proposition 1.4.13. *Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be a short exact sequence of C^*-algebras. Let H be a homology functor from the category of C^*-algebras and $*$-homomorphisms to the category of abelian groups and group homomorphisms. Then there is a group homomorphism*

$$\delta\colon H\Sigma C \to HA,$$

such that the following sequence is exact,

$$H\Sigma A \xrightarrow{\Sigma\alpha_*} H\Sigma B \xrightarrow{\Sigma\beta_*} H\Sigma C \xrightarrow{\delta} HA \xrightarrow{\alpha_*} HB \xrightarrow{\beta_*} HC.$$

□

The same result applies in the contravariant case. For a proof of the following see Theorem 6.15 and Theorem 6.18 in [GHT00].

Theorem 1.4.14. *Let D be a C^*-algebra and let $0 \to J \xrightarrow{\alpha} B \xrightarrow{\beta} A \to 0$ be a short exact sequence of C^*-algebras. Then there is an element $\sigma \in E(\Sigma A, J)$ and long exact sequences:*

$$E(A, D) \to E(B, D) \to E(J, D) \xrightarrow{\partial^*} E(\Sigma A, D) \to \cdots,$$

and

$$\cdots \to E(D, \Sigma A) \xrightarrow{\partial_*} E(D, J) \to E(D, B) \to E(D, A),$$

in which the maps ∂^*, ∂_* are given by the E-theory product with $\sigma \in E(\Sigma A, J)$.

□

Here we can define δ^* and δ_* explicitly by,

$$\delta^*(\alpha(x)) = \alpha(\sigma(x)),$$

where α is an asymptotic morphism from $\Sigma J \otimes \mathcal{K}$ to $\Sigma D \otimes \mathcal{K}$ and $x \in \Sigma^2 A \otimes \mathcal{K}$, and

$$\delta_*(\beta(y)) = \sigma(\beta(y)),$$

where β is an asymptotic morphism from $\Sigma D \otimes \mathcal{K}$ to $\Sigma^2 A \otimes \mathcal{K}$ and $y \in \Sigma D \otimes \mathcal{K}$.

Proposition 1.4.15. *Let A, B, C and D be C^*-algebras and suppose we have a split exact sequence,*

$$0 \longrightarrow A \underset{\gamma}{\overset{\alpha}{\rightleftarrows}} B \overset{\beta}{\longrightarrow} C \longrightarrow 0,$$

then

$$0 \longrightarrow E(C,D) \overset{\beta^*}{\longrightarrow} E(B,D) \underset{\gamma^*}{\overset{\alpha^*}{\rightleftarrows}} E(A,D) \longrightarrow 0,$$

$$0 \longrightarrow E(D,A) \underset{\gamma_*}{\overset{\alpha_*}{\rightleftarrows}} E(D,B) \overset{\beta_*}{\longrightarrow} E(D,C) \longrightarrow 0,$$

are split exact.

Proof. By Theorem 1.4.14, we obtain the following long exact sequence:

$$E(C,D) \overset{\beta^*}{\longrightarrow} E(B,D) \underset{\gamma^*}{\overset{\alpha^*}{\rightleftarrows}} E(A,D) \overset{\delta^*}{\longrightarrow} E(\Sigma C,D) \longrightarrow \cdots,$$

for any separable C^*-algebra D. By exactness we have ker $\partial^* =$ im α^* but $\gamma^*\alpha^* = id_{E(B,D)}$ and by functoriality ker $\partial^* = E(A,D)$ and hence $\partial^* = 0$. So,

$$E(C,D) \overset{\beta^*}{\longrightarrow} E(B,D) \underset{\gamma^*}{\overset{\alpha^*}{\rightleftarrows}} E(A,D) \longrightarrow 0,$$

is exact. By functoriality of E-theory, lemma 2.2.11, we see β^* is surjective so

$$0 \longrightarrow E(C,D) \overset{\beta^*}{\longrightarrow} E(B,D) \underset{\gamma^*}{\overset{\alpha^*}{\rightleftarrows}} E(A,D) \overset{\partial^*}{\longrightarrow} 0,$$

is exact as required.

By Theorem 1.4.14, we obtain the following long exact sequence,

$$\cdots \longrightarrow E(D,\Sigma C) \overset{\partial_*}{\longrightarrow} E(D,A) \underset{\gamma_*}{\overset{\alpha_*}{\rightleftarrows}} E(D,B) \overset{\beta_*}{\longrightarrow} E(D,C),$$

By functoriality, α_* is injective so ker $\alpha_* = 0 =$ im ∂_*, so $\partial_* = 0$. Hence,

$$0 \longrightarrow E(D,A) \underset{\gamma_*}{\overset{\alpha_*}{\rightleftarrows}} E(D,B) \overset{\beta_*}{\longrightarrow} E(D,C).$$

Similarly β_* is surjective, we obtain

$$0 \longrightarrow E(D,A) \underset{\gamma_*}{\overset{\alpha_*}{\rightleftarrows}} E(D,B) \xrightarrow{\beta_*} E(D,C) \longrightarrow 0.$$

By functoriality we see that the sequence is clearly split exact. □

Then using a similar proof we can obtain the following result in the contravariant case.

Proposition 1.4.16. *Let A, B, C and D be C^*-algebras and suppose we have a split exact sequence,*

$$0 \longrightarrow A \xrightarrow{\alpha} B \underset{\delta}{\overset{\beta}{\rightleftarrows}} C \longrightarrow 0,$$

then

$$0 \longrightarrow E(C,D) \underset{\delta^*}{\overset{\beta^*}{\rightleftarrows}} E(B,D) \xrightarrow{\alpha^*} E(A,D) \longrightarrow 0,$$

and

$$0 \longrightarrow E(D,A) \xrightarrow{\alpha_*} E(D,B) \underset{\delta_*}{\overset{\beta_*}{\rightleftarrows}} E(D,C) \longrightarrow 0,$$

are split exact.

□

For a proof of the following Lemma, see Lemma 6.25 in [GHT00].

Lemma 1.4.17. *Let A and B be C^*-algebras, with A separable. Then*

$$E(A,B) = [\![\Sigma A, \Sigma B \otimes \mathcal{K}]\!].$$

□

Proposition 1.4.18. *Let A and B be C^*-algebras. Then*

1. $E(A \otimes \mathcal{K}, B) \cong E(A,B)$
2. $E(A, B \otimes \mathcal{K}) \cong E(A,B)$.

Proof. 1. By Lemma 1.4.17, we need to show equivalently that

$$[\![\Sigma(A \otimes \mathcal{K}), \Sigma B \otimes \mathcal{K}]\!] \cong [\![\Sigma A \otimes \mathcal{K}, \Sigma B \otimes \mathcal{K}]\!],$$

which follows if $\Sigma(A \otimes \mathcal{K}) \cong \Sigma A \otimes \mathcal{K}$. This is obvious since

$$\Sigma(A \otimes \mathcal{K}) = C_0(0,1) \otimes A \otimes \mathcal{K}.$$

2. This proof is exactly like the one above but we need to use that $\mathcal{K} \otimes \mathcal{K} \cong \mathcal{K}$.

□

A proof of the following is found in [GHT00] as proposition 6.17.

Proposition 1.4.19. *Let A and B be separable C^*-algebras. Then the suspension map*

$$\Sigma\colon E(A,B) \to E(\Sigma A, \Sigma B),$$

is an isomorphism.

□

The subsequent result is proved in section 4 in [Cun84].

Theorem 1.4.20 (Bott periodicity)**.** *Let A and B be separable C^*-algebras, then we have isomorphisms:*

$$E(A,B) \cong E(A, \Sigma^2 B) \text{ and } E(A,B) \cong E(\Sigma^2 A, B).$$

□

Proposition 1.4.21. *For separable C^*-algebras A and B, there is a natural isomorphism:*

$$E(\Sigma A, B) \cong E(A, \Sigma B).$$

Proof. This follows from the following natural isomorphisms from Theorem 1.4.20 and Proposition 1.4.19,

$$P\colon E(A,B) \to E(\Sigma^2 A, B) \text{ and } \Sigma\colon E(A,B) \to E(\Sigma A, \Sigma B).$$

□

The Theorem below is proven using Theorem 1.4.14 and Bott periodicity.

Theorem 1.4.22. *Let D be a separable C^*-algebra and let*

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0,$$

be a short exact sequence of C^*-algebras. Then there are exact sequences

$$\begin{array}{ccccc} E(D,\Sigma A) & \longrightarrow & E(D,\Sigma B) & \longrightarrow & E(D,\Sigma C) \\ \uparrow & & & & \downarrow \\ E(D,C) & \xleftarrow{\beta_*} & E(D,B) & \xleftarrow{\alpha_*} & E(D,A) \end{array},$$

and

$$\begin{array}{ccccc} E(\Sigma A,D) & \longleftarrow & E(\Sigma B,D) & \longleftarrow & E(\Sigma C,D) \\ \downarrow & & & & \uparrow \\ E(C,D) & \xrightarrow{\beta^*} & E(B,D) & \xrightarrow{\alpha^*} & E(A,D) \end{array},$$

where the boundary maps are defined by Bott periodicity and product with the E-theory class of an element $\sigma \in E(\Sigma A, J)$ associated to the short exact sequence.

□

1.5 Spectrum

Here we cover the notions of spectra and orthogonal spectra as we will need them later when we generalise their notions to quasi-topological spaces and also when combining K-theory and K-homology spectra using E-theory spectra.

Definition 1.5.1. A *spectrum* $\mathbb{X}$ is a sequence of based topological spaces $X_0, X_1, \ldots$ with structure maps $\epsilon \colon X_m \to \Omega X_{m+1}$ for $m \geq 0$.

Proposition 1.5.2. *The functors Σ_{Top} and Ω are adjoints, where Σ_{top} is the left adjoint and Ω the right adjoint.*

□

Due to the above proposition, we can also define a spectrum to be a sequence of based topological spaces with structure maps defined by

$$\sigma \colon \Sigma X_m \to X_{m+1},$$

where $\Sigma X_m = X_m \wedge S^1$.

Definition 1.5.3. An *Ω-spectrum* is a spectrum where for all natural numbers m the structure maps $\epsilon\colon X_m \to \Omega X_{m+1}$ are weak homotopy equivalences.

We will also want to use the notion of orthogonal spectra but firstly we need to recall the notions of group actions and facts about them.

Definition 1.5.4. Let G be a topological group and X a topological space. Then we call a group action $h\colon G \times X \to X$ *continuous*, written $h(g, x) = gx$ if the map h is continuous. In this case, we call X a G-space. A map of G-spaces $f\colon X \to Y$ is called *G-equivariant* if for all $g \in G$,

$$f(gx) = gf(x).$$

If the map $f\colon X \to Y$ is G-equivariant, we call it a *G-map*.

Definition 1.5.5. An *orthogonal spectrum* is a sequence of based topological spaces $X_0, X_1, \dots$ with a basepoint preserving continuous left action of $O(m)$ on each X_m for all m, and based structure maps $\sigma = \sigma_m\colon X_m \wedge S^1 \to X_{m+1}$, such that for each $m, n \geq 0$, the iterated structure map

$$\sigma_m^n\colon X_m \wedge S^n \xrightarrow{\sigma_m \wedge \mathrm{id}_{S^{n-1}}} X_{m+1} \wedge S^{n-1} \xrightarrow{\sigma_{m+1} \wedge \mathrm{id}_{S^{n-2}}} \dots \xrightarrow{\sigma_{m+n-1} \wedge \mathrm{id}_{S^1}} X_{m+n},$$

is $O(m) \times O(n)$-equivariant.

Since Σ_{Top} and Ω are adjoints, we can reformulate the structure map explicitly $\sigma\colon X_m \wedge S^1 \to X_{m+1}$ using $\epsilon\colon X_m \to \Omega X_{m+1}$ and we write

$$\sigma(x, s) = \epsilon(x)(s),$$

for all $x \in X_m$ and $s \in S^1$.

A morphism of orthogonal spectra $f\colon X \to Y$ is a collection of $O(m)$-equivariant maps $f_m\colon X_m \to Y_m$ for all m, which satisfy the following commutative diagram:

$$\begin{array}{ccc} X_m \wedge S^1 & \xrightarrow{f \wedge \mathrm{id}_{S^1}} & Y_m \wedge S^1 \\ \downarrow{\scriptstyle \sigma_m} & & \downarrow{\scriptstyle \sigma_m} \\ X_{m+1} & \xrightarrow{f_{m+1}} & Y_{m+1}. \end{array}$$

Definition 1.5.6. Let $\mathbb{X}$ be an orthogonal spectrum with spaces X_n. For each integer $k \in \mathbb{Z}$ we define the *k-th stable homotopy group* $\pi_k(\mathbb{X})$ to be the direct limit

$$\pi_k(\mathbb{X}) = \varinjlim_n \pi_{k+n} X_n,$$

under the maps $\epsilon_* \colon \pi_{k+n} X_n \to \pi_{k+n+1} X_{n+1}$ induced from the structure maps $\Omega^{k+n}\epsilon \colon \Omega^{k+n} X_n \to \Omega^{k+n+1} X_{n+1}$.

1.6 Quasi-topological spaces

This section is taken from [Spa63] and we will need quasi-topological spaces since it is not known if we can put a standard topology on the set of asymptotic morphisms.

Definition 1.6.1. A *quasi-topology* on a set X, is a collection of sets of maps from C to X for each compact Hausdorff space C, written $Q(C, X)$, called quasi-continuous and satisfying:

- any constant map $C \to X$ belongs to $Q(C, X)$,
- if $f \colon C_1 \to C_2$ is a map of compact Hausdorff spaces and $g \in Q(C_2, X)$ then $gf \in Q(C_1, X)$,
- for a disjoint union $C = C_1 \amalg C_2$ of closed compact Hausdorff spaces, a map $g \colon C \to X$ is contained in $Q(C, X)$ if and only if $g|_{C_i} \in Q(C_i, X)$ for $i = 1, 2$,
- for every $f \colon C_1 \to C_2$ surjective map of compact Hausdorff spaces, then a map $h \colon C_2 \to X$ is quasi-continuous if $h \circ f$ is quasi-continuous.

A *quasi-topological space* is a set X endowed with a quasi-topology as described above.

If X is a topological space we can obtain a quasi-topology on X by considering $Q(C, X)$ as the set of continuous maps from C to X in the topology of X.

A map of quasi-topological spaces $f \colon X \to Y$ is called *quasi-continuous* if $g \in Q(C, X)$ implies that the composite $fg \in Q(C, Y)$. Also by the definition of quasi-continuous maps, a composite of quasi-continuous maps is also quasi-continuous. A *quasi-homeomorphism* $f \colon X \to Y$ between quasi-topological spaces is a quasi-continuous bijection with a quasi-continuous inverse $g\colon Y \to X$.

As in the standard notion of topology we can define the product quasi-topology and the quotient quasi-topology. Let $X = X_1 \times X_2 \times \dots X_n$. Then define the *product quasi-topology* on X by the condition that $a\colon C \to X$ is quasi-continuous if and only if the composite

$$C \to X \xrightarrow{p_i} X_i, \text{ (where } p_i \text{ is the projection map)}$$

is in $Q(C, X_i)$ for each i.

For the quotient quasi-topology, let X be a quasi-topological space and $g\colon X \to Y$ a surjection. Then the *quotient quasi-topology* on Y is defined by the condition that $a\colon C \to Y$ in $Q(C, Y)$ if and only if there is a surjection map $g'\colon C' \to C$ of compact Hausdorff spaces and a quasi-continuous map $a'\colon C' \to X$ in $Q(C', X)$ such that

$$\begin{array}{ccc} C' & \xrightarrow{a'} & X \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow[a]{} & Y, \end{array}$$

commutes.

Using the product and quotient quasi-topology we define the quasi-topology on $X \wedge Y$. Here X and Y are quasi-topological spaces with basepoints x_0 and y_0 and we define $X \vee Y$ by

$$X \amalg Y/(x_0 \sim y_0),$$

where the equivalence class of x_0 and y_0 serve as the basepoint. Then we define the smash product of X and Y, $X \wedge Y$ by

$$X \times Y/X \vee Y.$$

A map $a\colon C \to X \wedge Y$ is quasi-continuous if and only if there exists a surjective map $g'\colon C' \to C$ of compact Hausdorff spaces and quasi-continuous maps $a'\colon C' \to X$ and $a''\colon C \to Y$ such that for every $c' \in C$,

$$ag'(c') = a'(c') \wedge a''(c').$$

Let X and Y be quasi-topological spaces. Let $F(X,Y)$ denote the set of quasi-continuous maps between X and Y. Then we can define a quasi-topology on $F(X,Y)$ by considering the evaluation map

$$E\colon F(X,Y) \times X \to Y,$$

defined by $E(f,x) = f(x)$ for all $f \in F(X,Y)$ and $x \in X$. We define the quasi topology on $F(X,Y)$ by declaring a map $a\colon C \to F(X,Y)$ to be quasi-continuous if and only if for every surjective continuous map $g\colon C' \to C$ of compact Hausdorff spaces and every quasi-continuous map $a'\colon C' \to X$, the map

$$E(ag, a')\colon C' \to Y,$$

is quasi-continuous. Due to definition of the product quasi-topology, the evaluation map is quasi-continuous.

Definition 1.6.2. Let X, Y be quasi-topological spaces and $f, g\colon X \to Y$ be quasi-continuous maps. Then a homotopy is a quasi-continuous map

$$H\colon X \to C([0,1], Y),$$

such that for all $x \in X$, $H(x)(0) = f(x)$ and $H(x)(1) = g(x)$.

From the above we can define the suspension and loop space of a quasi-topological space. The suspension is defined by

$$\Sigma_{\text{top}} X = S^1 \wedge X,$$

and the loop space is defined by

$$\Omega X = \{\mu\colon S^1 \to X \mid \mu \text{ is quasi-continuous and basepoint preserving}\}$$

and we consider the circle S^1 with the quasi-topology that comes from the topology coming from that on $\mathbb{R}^2$. That is, our quasi-continuous maps are the continuous maps from every compact Hausdorff space to S^1 in that topology.

1.7 Category Theory

We detail the definition of a symmetric monoidal category here as it will be necessary to check that we have a symmetric monoidal structure on a particular category later.

Definition 1.7.1. A *symmetric monoidal category* $(\mathscr{C}, \otimes, u)$ has the following data

- A category $\mathscr{C}$,
- A functor $\otimes \colon \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ called the tensor product,
- An object $u \in \mathrm{obj}(\mathscr{C})$ called the unit object,
- A natural isomorphism α, called the associator, with

$$\alpha_{A,B,C} \colon A \otimes (B \otimes C) \to (A \otimes B) \otimes C,$$

- A natural isomorphism γ, called the left unitor, with

$$\gamma_A \colon u \otimes A \to A,$$

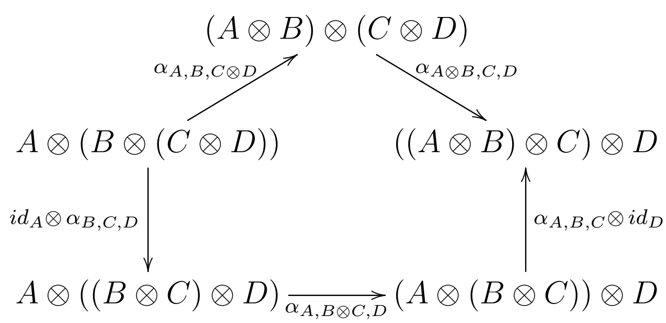
- A natural isomorphism ρ, called the right unitor, with

$$\rho_A \colon A \otimes u \to A,$$

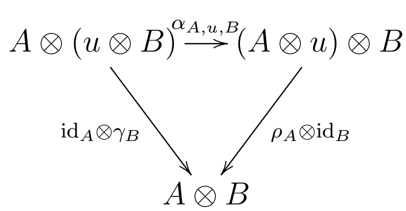
- a natural isomorphism $l_{A,B} : A \otimes B \to B \otimes A$ called the braiding

subject to the following commutative diagrams.

1.



2.



3.

$$\begin{array}{ccc} A \otimes u & \xrightarrow{\;l_{A,u}\;} & u \otimes A \\ & {\scriptstyle \rho_A}\searrow \qquad \swarrow{\scriptstyle \gamma_A} & \\ & A & \end{array}$$

4.

$$\begin{array}{ccc} A \otimes B & \xrightarrow{\;l_{A,B}\;} & B \otimes A \\ & {\scriptstyle id_{A\otimes B}}\searrow \qquad \swarrow{\scriptstyle l_{B,A}} & \\ & A \otimes B & \end{array}$$

5.

$$\begin{array}{ccc} (A \otimes B) \otimes C & \xrightarrow{\;l_{A\hat{\otimes}B,C}\;} & C \otimes (A \otimes B) \\ \downarrow{\scriptstyle \alpha^{-1}_{A,B,C}} & & \downarrow{\scriptstyle \alpha_{C,A,B}} \\ A \otimes (B \otimes C) & & (C \otimes A) \otimes B \\ \downarrow{\scriptstyle \mathrm{id}_A \otimes l_{B,C}} & & \downarrow{\scriptstyle l_{C,A} \otimes \mathrm{id}_B} \\ A \otimes (C \otimes B) & \xrightarrow{\;\alpha_{A,C,B}\;} & (A \otimes C) \otimes B \end{array} \quad ,$$

and

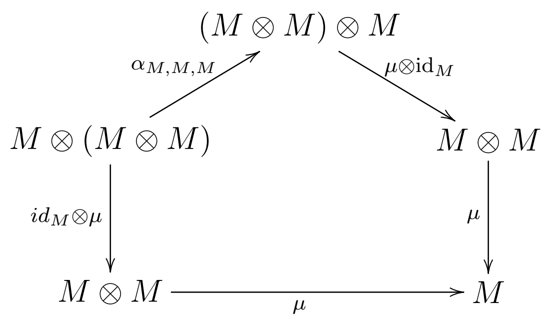
6.

$$\begin{array}{ccc} A \otimes (B \otimes C) & \xrightarrow{\;l_{A,B\hat{\otimes}C}\;} & (B \otimes C) \otimes A \\ \downarrow{\scriptstyle \alpha_{A,B,C}} & & \downarrow{\scriptstyle \alpha^{-1}_{B,C,A}} \\ (A \otimes B) \otimes C & & B \otimes (C \otimes A) \\ \downarrow{\scriptstyle l_{A,B} \otimes \mathrm{id}_C} & & \downarrow{\scriptstyle \mathrm{id}_B \otimes l_{C,A}} \\ (B \otimes A) \otimes C & \xrightarrow{\;\alpha^{-1}_{B,A,C}\;} & B \otimes (A \otimes C) \end{array} \quad ,$$

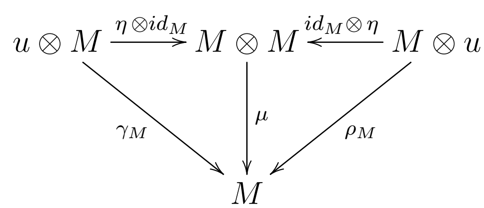
It should be noted that a *monoidal category* satisfies all the criteria as above apart from diagram (4).

Definition 1.7.2. A *commutative monoid* (M, μ, η) in a symmetric monoidal category $(\mathscr{C}, \otimes, u)$ is an object M in the $\mathscr{C}$ together with morphisms $\mu\colon M \otimes M \to M$ and $\eta\colon u \to M$ satisying the following commutative diagrams:

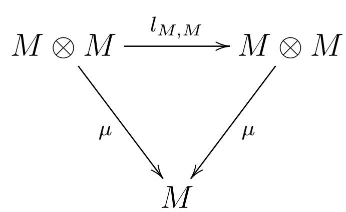
1.



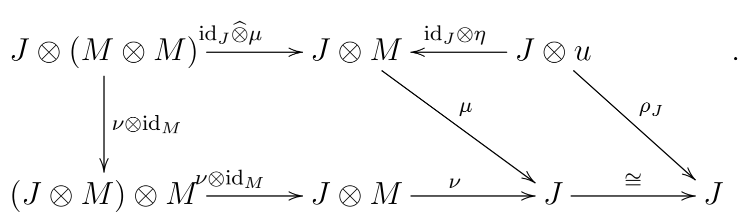
2.



3.



Definition 1.7.3. Let $(\mathscr{C}, \otimes, u)$ be a monoidal category with a monoid (M, μ, η) as defined above. Then we define a *right M-module* to be an object J with an associative morphism $\nu\colon J \otimes M \to J$ such that we have the following commutative diagram:



Let (A, ν_A) and (B, ν_B) be right modules of M. Then a *morphism of right modules of* M is a map $f\colon A \to B$ such that

$$\nu_B \circ (f \otimes \mathrm{id}_M) = f \circ \nu_A.$$

Denote the category where objects are right M-modules and where arrows are the morphisms defined above, by mod-M.

Chapter 2

Complex Graded E-theory

This chapter extends the definition of E-theory in the previous chapter to the case when we consider complex graded C^*-algebras. We start by defining the notion of gradings on a C^*-algebra alongside the definition of Clifford algebras which will play an important role for us. Then we will define the complex graded form of E-theory and state and prove the properties that it has.

2.1 Gradings and Clifford algebras

Much of this section is taken and proofs adapted from [Mit01]. To save repeating ourselves in the next chapter, where possible we include the real graded case. We will see that Clifford algebras are fundamental to the construction of the Bott map of complex graded E-theory.

Definition 2.1.1. Let A be a C^*-algebra. A *grading* on A is an automorphism $\delta\colon A \to A$ such that $\delta^2 = 1$.

Every C^*-algebra can be equipped with the trivial grading, defined as the identity map. We can also consider a grading on a C^*-algebra in terms of odd and even elements as we see below.

Definition 2.1.2. Let A be a C^*-algebra with grading δ. Then define

$$A_{\text{even}} = \{a \in A \mid \delta a = a\} \text{ and } A_{\text{odd}} = \{a \in A \mid \delta a = -a\}$$

Definition 2.1.3. Let A be a graded C^*-algebra. Then the *degree* of an element a is defined by:

$$\deg(a) = \begin{cases} 0, & \text{if } a \in A_{\text{even}} \\ 1, & \text{if } a \in A_{\text{odd}}. \end{cases}$$

Definition 2.1.4. Let A and B be graded C^*-algebras with gradings α and β respectively. Then we define the object $A\widehat{\otimes}B$ to be the completion of the algebraic tensor product of A and B in the norm

$$|| \sum_i a_i \otimes b_i || = \sup || \sum_i \varphi(a_i)\psi(b_i) ||$$

where $\varphi\colon A \dashrightarrow C$, $\psi\colon B \dashrightarrow C$ are graded asymptotic morphisms for some graded C^*-algebra C. We equip $A\widehat{\otimes}B$ with involution, multiplication and grading defined by:

1. $(a\widehat{\otimes}b)^* = (-1)^{\deg(a)\deg(b)} a^* \otimes \; b^*$
2. $(a\widehat{\otimes}b)(c\widehat{\otimes}d) = (-1)^{\deg(b)\deg(c)}(ac \otimes bd)$
3. $\gamma(a\widehat{\otimes}b) = \alpha(a) \otimes \beta(b)$

It should be noted this is the maximal tensor product.

If additionally we also extend by linearity within the definition of $A\widehat{\otimes}B$ we obtain the following result.

Proposition 2.1.5. *The object $A\widehat{\otimes}B$ is a graded C^*-algebra.*

□

Definition 2.1.6. Let p, q be natural numbers. Then define the Clifford algebra $\mathbb{F}_{p,q}$, to be the algebra over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ generated by the elements:

$$\{e_1, \ldots, e_p, f_1, \ldots, f_q\},$$

such that $e_i^2 = 1$ and $f_j^2 = -1$ and all generator anti-commute.

The following propositions are taken from [Mit01], so for the proofs see the paper.

Proposition 2.1.7. *The Clifford algebra $\mathbb{F}_{p,q}$ is a C^*-algebra with grading defined by having all generators being odd.*

□

Proposition 2.1.8. *For all p, q, r, s in the natural numbers there is an isomorphism*

$$\mathbb{F}_{p,q} \widehat{\otimes} \mathbb{F}_{r,s} \cong \mathbb{F}_{p+r,q+s}.$$

□

The following two Propositions show why Clifford algebras are really useful in this context, since they will be the reason we can define the Bott map of E-theory using them.

Proposition 2.1.9.

$$\mathbb{C}_{2,0} \cong \mathbb{C}_{1,1} \cong \mathbb{C}_{0,2}.$$

□

Proposition 2.1.10.

$$\mathbb{R}_{8,0} \cong \mathbb{R}_{4,4} \cong \mathbb{R}_{0,8}.$$

□

Proposition 2.1.11. *The Clifford algebra $\mathbb{F}_{1,1}$ and the algebra $M_2(\mathbb{F})$ are isomorphic when we have the grading:*

$$M_2(\mathbb{F})_{even} = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \;\middle|\; a, d \in \mathbb{F} \right\} \text{ and } M_2(\mathbb{F})_{odd} = \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \;\middle|\; b, c \in \mathbb{F} \right\}.$$

□

Proposition 2.1.12. *Let A be a graded C^*-algebra over the field $\mathbb{F}$. Let $M_n(A)$ denote the set of $n \times n$ matrices with entries in A. Then*

$$M_n(\mathbb{F}) \widehat{\otimes} A = M_n(A),$$

where the matrix in $M_n(A)$ is odd when every element is odd and even when every element is even.

□

Remark 2.1.13. Let $\mathcal{H}$ be a Hilbert space equipped with the orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1,$$

where $\mathcal{H}_0$ denotes the even elements and $\mathcal{H}_1$ denotes the odd elements. Then the C^*-algebra $\mathcal{K}(\mathcal{H})$ of compact operators on such a Hilbert space is graded. For this grading, we consider 2×2 matrices of operators where the diagonal matrices are even and the off diagonal ones are odd. That is we have a grading

$$\beta\colon \mathcal{K}(\mathcal{H}) \to \mathcal{K}(\mathcal{H})$$

defined by

$$\beta(T) = \begin{cases} T & \text{if } T \text{ is even} \\ -T & \text{if } T \text{ is odd.} \end{cases}$$

Note that if we have an odd element $E \in M_2(\mathcal{B}(\mathcal{H}))$ then

$$\beta(ET) = \begin{cases} ET & \text{if } T \text{ is odd} \\ -ET & \text{if } T \text{ is even,} \end{cases}$$

so $-\beta(ET) = E\beta(T)$ and similarly $-\beta(TE) = \beta(T)E$.

Then in particular with the above remark we have the following result.

Lemma 2.1.14. *We have an isomorphism*

$$\mathcal{K}(\mathcal{H})\widehat{\otimes}\mathbb{F}_{1,1} \cong M_2(\mathcal{K}(\mathcal{H})),$$

with grading

$$\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \beta(a) & -\beta(b) \\ -\beta(c) & \beta(d) \end{pmatrix},$$

for all $a, b, c, d \in \mathcal{K}(\mathcal{H})$, where β is as in Remark 2.1.13.

□

Denote $M_2(\mathcal{K}(\mathcal{H}))$ in this case by $M_2(\mathcal{K}(\mathcal{H}))^g$.

Lemma 2.1.15. *We have a set of isomorphisms*

$$\mathcal{K}(\mathcal{H})\widehat{\otimes}\mathbb{F}_{1,1} \cong M_2(\mathcal{K}(\mathcal{H}))^g \cong M_2(\mathcal{K}(\mathcal{H})) \cong \mathcal{K}(\mathcal{H}).$$

Proof. By Lemma 2.1.14 it suffices to show that $M_2(\mathcal{K}(\mathcal{H}))^g \cong M_2(\mathcal{K}(\mathcal{H}))$. The grading on $M_2(\mathcal{K}(\mathcal{H}))$ is

$$\beta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \beta(a) & \beta(b) \\ \beta(c) & \beta(d) \end{pmatrix},$$

so it suffices to check that we can define a map

$$\theta \colon M_2(\mathcal{K}(\mathcal{H})) \to M_2(\mathcal{K}(\mathcal{H}))^g,$$

which is an isomorphism. Let

$$V = \begin{pmatrix} 1 & 0 \\ 0 & E \end{pmatrix} \in M_2(\mathcal{B}(\mathcal{H})), \text{ and } E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in M_2(\mathcal{B}(\mathcal{H}))$$

and note E is odd. Notice $V^2 = 1$ and $V^* = V$. Then define $\theta(x) = VxV$, then $\theta \colon \mathcal{K}(\mathcal{H}) \to \mathcal{K}(\mathcal{H})$ is clearly a $*$-homomorphism since

$$\theta(xy) = VxyV = VxVVyV = \theta(x)\theta(y),$$

for all $x, y \in M_2(\mathcal{K}(\mathcal{H}))$. Then if $VxV = y$, then $x = VyV$ so θ is invertible.

Now we check that $\theta(\beta(x)) = \alpha(\theta(x))$.

$$\begin{aligned} \theta(\beta(x)) &= \theta \begin{pmatrix} \beta(a) & \beta(b) \\ \beta(c) & \beta(d) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} \beta(a) & \beta(b) \\ \beta(c) & \beta(d) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & E \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} \beta(a) & \beta(b)E \\ \beta(c) & \beta(d)E \end{pmatrix} \\ &= \begin{pmatrix} \beta(a) & \beta(b)E \\ E\beta(c) & E\beta(d)E \end{pmatrix} \\ &= \begin{pmatrix} \beta(a) & -\beta(bE) \\ -\beta(Ec) & \beta(EdE) \end{pmatrix} \text{ by Remark 2.1.13} \\ &= \alpha \begin{pmatrix} a & bE \\ Ec & EdE \end{pmatrix} \\ &= \alpha \begin{pmatrix} 1 & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & E \end{pmatrix} \\ &= \alpha(\theta(x)). \end{aligned}$$

□

Let $\mathcal{K}(\mathcal{H}^{\text{opp}})$ denote the compact operators on the Hilbert space $\mathcal{H}$ where the grading is reversed.

Definition 2.1.16. Let $\mathcal{S} = C_0(\mathbb{R})$, be the C^*-algebra of continuous, complex valued functions on $\mathbb{R}$ which vanish at infinity but with the following graded defined as a decomposition:

$$\mathcal{S} = C_0(\mathbb{R}) = \{\text{even functions}\} \oplus \{\text{odd functions}\}.$$

The grading automorphism is defined by $f(x) \mapsto f(-x)$. In addition define the *amplification* of a graded C^*-algebra A to be the graded tensor product $\mathcal{S}A = \mathcal{S}\widehat{\otimes}A$.

Remark 2.1.17. Let $u(x) = e^{-x^2}$ and $v(x) = xe^{-x^2}$ and notice that these are functions contained in $\mathcal{S}$ and u is even and v is odd. Now set A to be the algebra generated by u and v.

The theorem below is Theorem B page 167 in [Sim63]. It follows from the classical Stone-Weierstrass Theorem.

Theorem 2.1.18. *Let X be a locally compact Hausdorff space, and let B be a closed subalgebra of $C_0(X)$ which separates points, for each point in X contains a function which does not vanish there, and contains conjugates of the functions. Then B equals $C_0(X)$.*

□

Lemma 2.1.19. *The algebra A is dense in $\mathcal{S}$.*

Proof. This is immediate from Theorem 2.1.18 since $\mathbb{R}$ is locally compact and Hausdorff and A separates points. □

Then it follows that the elements u and v in Remark 2.1.17 generate $\mathcal{S}$.

2.2 Complex Graded E-theory

In this section we give the definition of graded E-theory and also certain properties we require when it comes to defining its spectrum.

Definition 2.2.1. Let A, B be C^*-algebras with gradings δ_A and δ_B respectively. A *graded asymptotic morphism* $\varphi\colon A \dashrightarrow B$ is an asymptotic morphism with family of functions $\{\varphi_t\}_{[1,\infty)}$which additionally satisfies the relation:-

$$\lim_{t\to\infty} ||\delta_B(\varphi_t(a)) - \varphi_t(\delta_A(a))|| = 0.$$

Denote the set of graded asymptotic morphisms from A to B by $\mathrm{Asy}_g(A, B)$.

Definition 2.2.2. Two graded asymptotic morphisms $\varphi, \psi\colon A \dashrightarrow B$ formed from the families of functions φ_t, ψ_t respectively are called *equivalent* if for all $a \in A$:

$$\lim_{t\to\infty} ||\varphi_t(a) - \psi_t(a)|| = 0.$$

Definition 2.2.3. Two graded asymptotic morphisms $\varphi, \psi\colon A \dashrightarrow B$ with families of functions φ_t, ψ_t are *graded homotopic* if there exists a graded asymptotic morphism $\theta\colon A \dashrightarrow IB$ with family of functions θ_t such that for all $a \in A$:

$$\theta_t(a)(0) = \varphi_t(a) \text{ and } \theta_t(a)(1) = \psi_t(a).$$

Denote the set of graded homotopic asymptotic morphisms from A to B by $[\![A, B]\!]_g$.

Proposition 2.2.4. *Let $\varphi, \psi\colon A \dashrightarrow B$ be equivalent graded asymptotic morphisms. Then φ and ψ are homotopic.*

Proof. As in Proposition 1.3.4 we again define a homotopy $\theta\colon A \dashrightarrow IB$ by

$$\theta_t(a)(s) = (1-s)\varphi_t(a) + s\psi_t(a) = \varphi_t(a) + s(\psi_t(a) - \varphi_t(a)),$$

which satisfies the graded property of graded asymptotic morphisms since for all $s \in [0, 1], a \in A$

$$\begin{aligned}
&\alpha_B(\theta_t(a)(s) - \theta_t(\alpha_A(a))(s) \\
&= \alpha_B[(1-s)\varphi_t(a) + s\psi_t(a)] - [(1-s)\varphi_t(\alpha_A(a)) + s\psi_t(\alpha_A(a))] \\
&= \alpha_B\varphi_t(a) - \varphi_t(\alpha_A(a)) - s(\alpha_B\varphi_t(a) - \varphi_t(\alpha_A(a))) + s(\alpha_B\psi_t(a) - \psi_t(\alpha_A(a))
\end{aligned}$$

and this tends to 0 as $t \mapsto \infty$ since φ_t and ψ_t are asymptotic morphisms. □

Now we need a notion of tensor product for our graded asymptotic morphisms so that we can define the composition of asymptotic morphisms. It is worth noting that we actually need to be pretty careful when we define a tensor product since unless our C^*-algebras are nuclear, we don't always get an asymptotic morphism when we tensor. For us to get an asymptotic morphisms we must use the maximal tensor product. In detail, let us consider two graded asymptotic morphisms $\varphi_t\colon A \dashrightarrow B$ and $\psi_t\colon C \dashrightarrow D$. We form the tensor product $\varphi_t\widehat{\otimes}\psi_t$ by taking the maximal tensor product to obtain:

$$A\widehat{\otimes}_{\max}C \dashrightarrow B\widehat{\otimes}_{\max}D,$$

which is obtained in the standard way by taking the completion and extending by linearity.

The following is Lemma 4.5 in [GHT00], so see this for a proof.

Lemma 2.2.5. *Let $\varphi\colon A_1 \dashrightarrow A_2$ and $\psi\colon B_1 \dashrightarrow B_2$ be (graded) asymptotic morphisms, then the compositions*

$$A_1\widehat{\otimes}B_1 \xrightarrow{\varphi\widehat{\otimes}1} A_2\widehat{\otimes}B_1 \xrightarrow{1\widehat{\otimes}\psi} A_2\widehat{\otimes}B_2,$$

and

$$A_1\widehat{\otimes}B_1 \xrightarrow{1\widehat{\otimes}\psi} A_1\widehat{\otimes}B_2 \xrightarrow{\varphi\widehat{\otimes}1} A_2\widehat{\otimes}B_2,$$

are equal. The 1 *symbolises the relevant identity morphism.*

□

In order to talk about compositions in the E-theory category we first need to define the $*$-homomorphism

$$\Delta\colon \mathcal{S} \to \mathcal{S}\widehat{\otimes}\mathcal{S},$$

described in [HG04]. Let $\mathcal{S}_R$ be the set of continuous functions on the interval $[-R, R]$; we have a surjection $\pi\colon \mathcal{S} \to \mathcal{S}_R$ defined by restriction. Let $X_R \in \mathcal{S}_R$ be the function $x \mapsto x$, then note that X_R is odd. Now if $f \in \mathcal{S}$, by functional calculus (see Appendix) we have an element

$$f(X_R\widehat{\otimes}1 + 1\widehat{\otimes}X_R) \in \mathcal{S}_R\widehat{\otimes}\mathcal{S}_R,$$

where 1 denotes the function 1. Then we have a graded $*$-homomorphism as follows:

Lemma 2.2.6. *There is a unique graded $*$-homomorphism $\Delta\colon \mathcal{S} \to \mathcal{S}\widehat{\otimes}\mathcal{S}$ such that*

$$(\pi\widehat{\otimes}\pi)\Delta(f) = f(X_R\widehat{\otimes}1 + 1\widehat{\otimes}X_R),$$

for every $R > 0$.

Proof. The map is clearly a graded $*$-homomorphism by the properties of functions in $\mathcal{S}$. Uniqueness follows from the fact that $\bigcup_{R>0}\mathcal{S}_R$ is dense in $\mathcal{S}$ by the Stone-Weierstrass Theorem. □

Write $\Delta_\pi(f) = (\pi\widehat{\otimes}\pi)\Delta(f)$.

Proposition 2.2.7. *For the elements $u = e^{-x^2}$ and $v(x) = xe^{-x^2}$ of $\mathcal{S}$ we have*

$$\Delta_\pi(u) = u\widehat{\otimes}u \quad \textit{and} \quad \Delta_\pi(v) = u\widehat{\otimes}v + v\widehat{\otimes}u.$$

Proof. Using the above lemma we have,

$$\begin{aligned}
\Delta_\pi(u) &= u(X_R\widehat{\otimes}1 + 1\widehat{\otimes}X_R)\\
&= e^{-(X_R\widehat{\otimes}1+1\widehat{\otimes}X_R)^2}\\
&= e^{-(X_R^2\widehat{\otimes}1+1\widehat{\otimes}X_R^2)} \text{ since } X_R \text{ is odd}\\
&= e^{-(X_R^2\widehat{\otimes}1)}e^{-(1\widehat{\otimes}X_R^2)}\\
&= e^{-X_R^2}\widehat{\otimes}e^{-X_R^2}\\
&= u\widehat{\otimes}u.
\end{aligned}$$

and

$$\begin{aligned}
\Delta_\pi(v) &= v(X_R\widehat{\otimes}1 + 1\widehat{\otimes}X_R)\\
&= (X_R\widehat{\otimes}1 + 1\widehat{\otimes}X_R)e^{-(X_R\widehat{\otimes}1+1\widehat{\otimes}X_R)^2}\\
&= (X_R\widehat{\otimes}1)e^{-(X_R\widehat{\otimes}1+1\widehat{\otimes}X_R)^2} + (1\widehat{\otimes}X_R)e^{-(X_R\widehat{\otimes}1+1\widehat{\otimes}X_R)^2}\\
&= (X_R\widehat{\otimes}1)(e^{-X_R^2}\widehat{\otimes}e^{-X_R^2}) + (1\widehat{\otimes}X_R)(e^{-X_R^2}\widehat{\otimes}e^{-X_R^2})\\
&= X_Re^{-X_R^2}\widehat{\otimes}e^{-X_R^2} + e^{-X_R^2}\widehat{\otimes}X_Re^{-X_R^2}\\
&= u\widehat{\otimes}v + v\widehat{\otimes}u,
\end{aligned}$$

as required. □

Lemma 2.2.8. *Let $\Delta' \colon \mathcal{S} \to \mathcal{S} \widehat{\otimes} \mathcal{S}$ be a graded $*$-homomorphism such that*

$$\Delta'_\pi(u) = u \widehat{\otimes} u \quad \text{and} \quad \Delta'_\pi(v) = u \widehat{\otimes} v + v \widehat{\otimes} u.$$

Then $\Delta' = \Delta$.

Proof. Let A be the algebra generated by u and v. Then by Lemma 2.1.19, $\mathcal{S}$ is generated by A. Hence Δ' is uniquely determined by $\Delta'_\pi(u)$ and $\Delta'_\pi(v)$. Then since $\Delta'_\pi(u) = \Delta_\pi(u)$ and $\Delta'_\pi(v) = \Delta_\pi(v)$, it follows that $\Delta' = \Delta$ as required. □

Now we consider compositions of asymptotic morphisms.

Let $\varphi_t \colon \mathcal{S} \widehat{\otimes} A \widehat{\otimes} \mathcal{K}(\mathcal{H}) \dashrightarrow B \widehat{\otimes} \mathcal{K}(\mathcal{H})$ and $\psi_t \colon \mathcal{S} \widehat{\otimes} B \widehat{\otimes} \mathcal{K}(\mathcal{H}) \dashrightarrow C \widehat{\otimes} \mathcal{K}(\mathcal{H})$ be asymptotic morphisms, then the composition is defined by:

$$\mathcal{S} \widehat{\otimes} A \widehat{\otimes} \mathcal{K}(\mathcal{H}) \xrightarrow{\Delta \widehat{\otimes} \mathrm{id}_{A \widehat{\otimes} \mathcal{K}(\mathcal{H})}} \mathcal{S} \widehat{\otimes} \mathcal{S} \widehat{\otimes} A \widehat{\otimes} \mathcal{K}(\mathcal{H}) \overset{\mathrm{id}_{\mathcal{S}} \widehat{\otimes} \alpha_t}{\dashrightarrow} \mathcal{S} \widehat{\otimes} B \widehat{\otimes} \mathcal{K}(\mathcal{H}) \xrightarrow{\beta_{r(t)}} C \widehat{\otimes} \mathcal{K}(\mathcal{H}),$$

where $\beta_{r(t)}$ is a reparameterisation as of Lemma 1.3.12.

Lemma 2.2.9. *Let A and B be C^*-algebras. The set $[\![\mathcal{S} \widehat{\otimes} A \widehat{\otimes} \mathcal{K}(\mathcal{H}), B \widehat{\otimes} \mathcal{K}(\mathcal{H})]\!]_g$ is an abelian group under the direct sum operation $\oplus$ defined in Definition 1.4.4. The zero element of this group is represented by the zero asymptotic morphism.*

Proof. By the previous statements in the ungraded case, it just suffices to check that we have additive inverses. Let $\varphi \colon \mathcal{S} \widehat{\otimes} A \widehat{\otimes} \mathcal{K}(\mathcal{H}) \dashrightarrow B \widehat{\otimes} \mathcal{K}(\mathcal{H})$ be a graded asymptotic morphism and define

$$\varphi_t^{\mathrm{opp}} \colon \mathcal{S} \widehat{\otimes} A \widehat{\otimes} \mathcal{K}(\mathcal{H}) \dashrightarrow B \widehat{\otimes} \mathcal{K}(\mathcal{H}^{\mathrm{opp}}),$$

to be the graded asymptotic morphism defined by $\varphi_t^{\mathrm{opp}}(x) = \varphi_t(\alpha(x))$ where α is the grading automorphism. Now let $s \geq 0$ be a fixed scalar and define

$$\Phi_t^s \colon \mathcal{S} \widehat{\otimes} \mathcal{S} \widehat{\otimes} A \widehat{\otimes} \mathcal{K}(\mathcal{H}) \dashrightarrow B \widehat{\otimes} \mathcal{K}(\mathcal{H} \oplus \mathcal{H}^{\mathrm{opp}}),$$

by

$$\Phi_t^s(f \widehat{\otimes} x) = f(s)(\varphi_t \oplus \varphi_t^{\mathrm{opp}})(x),$$

with $f \in \mathcal{S}$ and $x \in \mathcal{S} \widehat{\otimes} A \widehat{\otimes} \mathcal{K}(\mathcal{H})$. Then this is a graded asymptotic morphism and we can define a homotopy on $s \in [0, \infty)$ between $\varphi_t \oplus \varphi_t^{\mathrm{opp}}$ and the zero asymptotic morphism,

$$\mathcal{S} \widehat{\otimes} A \widehat{\otimes} \mathcal{K}(\mathcal{H}) \xrightarrow{\Delta \widehat{\otimes} \mathrm{id}_{A \widehat{\otimes} \mathcal{K}(\mathcal{H})}} \mathcal{S} \widehat{\otimes} \mathcal{S} \widehat{\otimes} A \widehat{\otimes} \mathcal{K}(\mathcal{H}) \dashrightarrow B \widehat{\otimes} \mathcal{K}(\mathcal{H} \oplus \mathcal{H}^{\mathrm{opp}}).$$

It is well defined and is clearly a graded asymptotic morphism so we need to check at the end points we obtain $\varphi_t \oplus \varphi_t^{\text{opp}}$ and 0. In order to do this it is enough to consider the functions $u = e^{-x^2}$ and $v(x) = xe^{-x^2}$ that generate $\mathcal{S}$.

For u, and $a \in A\widehat{\otimes}\mathcal{K}(\mathcal{H})$,

$$\begin{aligned}\Phi_t^s(\Delta\widehat{\otimes}\mathrm{id}_{A\widehat{\otimes}\mathcal{K}(\mathcal{H})})(u\widehat{\otimes}a) &= \Phi_t^s(\Delta(u)\widehat{\otimes}a)\\ &= \Phi_t^s(u\widehat{\otimes}u\widehat{\otimes}a)\\ &= u(s)(\varphi_t \oplus \varphi_t^{\text{opp}})(u\widehat{\otimes}a),\end{aligned}$$

and when $s = 0$, $u = e^{-0} = 1$ and so we obtain $\varphi_t \oplus \varphi_t^{\text{opp}}$ and when $s \to \infty$, we obtain 0.

Now for v,

$$\begin{aligned}\Phi_t^s(\Delta\widehat{\otimes}\mathrm{id}_{A\widehat{\otimes}\mathcal{K}(\mathcal{H})})(v\widehat{\otimes}a) &= \Phi_t^s(\Delta(v)\widehat{\otimes}a)\\ &= \Phi_t^s((u\widehat{\otimes}v + v\widehat{\otimes}u)\widehat{\otimes}a)\\ &\sim \Phi_t^s((u\widehat{\otimes}v)\widehat{\otimes}a) + \Phi_t^s((v\widehat{\otimes}u)\widehat{\otimes}a)\\ &= u(s)(\varphi_t \oplus \varphi_t^{\text{opp}})(v\widehat{\otimes}a) + v(s)(\varphi_t \oplus \varphi_t^{\text{opp}})(u\widehat{\otimes}a),\end{aligned}$$

where the equivalence is valid, since equivalent graded asymptotic morphisms are homotopic by Proposition 2.2.4. Now at $s = 0$, $v(s) = 0$ and as $s \to \infty$, $v(s) \to 0$, so the second term is equal to 0 at the end points, and hence we obtain the same endpoints as above and we are done. $\square$

Definition 2.2.10. Let A and B be C^*-algebras. Then the graded E-theory group is given by

$$E_g(A, B) = [\![\mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}(\mathcal{H}), B\widehat{\otimes}\mathcal{K}(\mathcal{H})]\!]_g.$$

In addition we have graded E-theory groups:

$$E_g^n(A, B) = [\![\mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}(\mathcal{H}), \Sigma^n B\widehat{\otimes}\mathcal{K}(\mathcal{H})]\!]_g,$$

for all $n \geq 0$. The grading on ΣB is the one coming from the grading of B, and the compact operators have grading β defined in Remark 2.1.13.

Lemma 2.2.11 (Functoriality)**.** *E_g is a bivariant functor from the category where objects are graded C^*-algebras and arrows are $*$-homomorphisms to the category where objects abelian groups and arrows are group homomorphisms. That is, it is a functor that is contravariant in its first variable and covariant in its second variable.*

Proof. By Lemma 2.2.9 it is obvious that the source and target categories of the functor E_g are as stated in the lemma. The identity property is also clearly satisfied.

Let A, B, C, D be C^*-algebras. Let $\alpha\colon A \to B$ be a graded $*$-homomorphism, then we have an object $E_g(A, D) = [\![\mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}, D\widehat{\otimes}\mathcal{K}]\!]_g$ for all A and a morphism $\alpha^*\colon E_g(B, D) \to E_g(A, D)$ for all α defined by $\alpha^*([\![x]\!]) = [\![x.\alpha]\!]$, where $(x.\alpha)_t = x_t \circ (\alpha \otimes \mathrm{id}_{\mathcal{S}\widehat{\otimes}\mathcal{K}})$ for all $[\![x]\!] \in E_g(B, D)$. Now consider the composition of morphisms $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ on a representative x of $[\![x]\!]$,

$$\begin{aligned}
(\beta \circ \alpha)^*(x_t) &= x_t \circ ((\beta \circ \alpha)\widehat{\otimes}\mathrm{id}_{\mathcal{S}\widehat{\otimes}\mathcal{K}}) \\
&= x_t \circ (\beta\widehat{\otimes}\mathrm{id}_{\mathcal{S}\widehat{\otimes}\mathcal{K}}) \circ (\alpha\widehat{\otimes}\mathrm{id}_{\mathcal{S}\widehat{\otimes}\mathcal{K}}) \\
&= \beta^*(x_t) \circ (\alpha\widehat{\otimes}\mathrm{id}_{\mathcal{S}\widehat{\otimes}\mathcal{K}}) \\
&= \alpha^*\beta^*(x_t) \\
&= (\alpha^* \circ \beta^*)(x_t).
\end{aligned}$$

Similarly, we have for all A, $E_g(D, A) = [\![\mathcal{S}\widehat{\otimes}D\widehat{\otimes}\mathcal{K}, A\widehat{\otimes}\mathcal{K}]\!]_g$ and for all $\alpha, \alpha_*\colon E_g(D, A) \to E_g(D, B)$ defined by $\alpha_*([\![y]\!]) = [\![\alpha.y]\!]$, where $(\alpha.y)_t = (\alpha \otimes \mathrm{id}_{\mathcal{K}}) \circ y_t$ for all $[\![y]\!] \in E_g(D, A)$. Considering the composition of morphisms above and taking a representative y of $[\![y]\!] \in E_g(D, A)$ we see that

$$\begin{aligned}
(\beta \circ \alpha)_*(y_t) &= ((\beta \circ \alpha) \otimes \mathrm{id}_{\mathcal{K}}) \circ y_t \\
&= ((\beta \circ \alpha) \otimes \mathrm{id}_{\mathcal{K}}) \circ y_t \\
&= (\beta \otimes \mathrm{id}_{\mathcal{K}}) \circ (\alpha \otimes \mathrm{id}_{\mathcal{K}}) \circ y_t \\
&= (\beta \otimes \mathrm{id}_{\mathcal{K}}) \circ \alpha_*(y_t) \\
&= \beta_*\alpha_*(y_t) \\
&= (\beta_* \circ \alpha_*)(y_t).
\end{aligned}$$

□

The following proposition and lemma on projections may seem out of place but they will be useful for a proof immediately after. This is since we will want to consider the compact operators and we know that they are formed of projections of rank 1. For a proof of the following proposition see [WO93], Proposition 5.26.

Proposition 2.2.12. *If p, q are projections and $||p - q|| \leq 1$ then they are homotopic.*

$\square$

Lemma 2.2.13. *Let $p, q \in \mathcal{K}(\mathcal{H})$ be rank 1 projections. Then there is a path from p to q.*

Proof. Let $[p]$ denote the homotopy class of a projection p. We know from Appendix B, that

$$\begin{aligned} K_0(\mathbb{F}) &= \{[p] - [q] \mid p, q \in M_\infty(\mathbb{F})\} \\ &= \{[p] - [q] \mid p, q \in \mathcal{K}(\mathcal{H})\}, \end{aligned}$$

since $M_\infty(\mathbb{F})$ is dense in $\mathcal{K}(\mathcal{H})$, and also since any two projections that are sufficiently close are homotopic by Proposition 2.2.12. Now we have an isomorphism

$$\theta\colon K_0(\mathbb{F}) \to \mathbb{Z},$$

defined by

$$\theta([p] - [q]) = \text{Rank}(p) - \text{Rank}(q),$$

where we recall that $\text{Rank}(p)$ denotes the rank of a projection p. Now let $p, q \in \mathcal{K}(\mathcal{H})$ be rank 1 projections. Then $\text{Rank}(p) = \text{Rank}(q)$, so $\theta([p]-[q]) = 0$, and since θ is an isomorphism, $[p] - [q] = 0$, so p and q are homotopic as required. $\square$

Lemma 2.2.14. *Let A and B be graded C^*-algebras, with A separable. Then*

$$E_g(A, B) = [\![\mathcal{S}\widehat{\otimes}A, B\widehat{\otimes}\mathcal{K}]\!]_g.$$

Proof. Let $\tau\colon \mathcal{S}\widehat{\otimes}A \to \mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}$ denote the graded $*$-homomorphism defined by

$$f\widehat{\otimes}a \mapsto f\widehat{\otimes}a\widehat{\otimes}e,$$

where $f \in \mathcal{S}$, $a \in A$ and e is the standard rank 1 projection in $\mathcal{K}$. Then define $g\colon [\![\mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}, B\widehat{\otimes}\mathcal{K}]\!]_g \to [\![\mathcal{S}\widehat{\otimes}A, B\widehat{\otimes}\mathcal{K}]\!]_g$ by

$$g(\alpha_t)(f\widehat{\otimes}a) = (\alpha_t \circ \tau)(f\widehat{\otimes}a) = \alpha_t(f\widehat{\otimes}a\widehat{\otimes}e),$$

for all $\alpha \in [\![\mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}, B\widehat{\otimes}\mathcal{K}]\!]_g$.

Then define an inverse to g, $h\colon [\![\mathcal{S}\widehat{\otimes}A, B\widehat{\otimes}\mathcal{K}]\!]_g \to [\![\mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}, B\widehat{\otimes}\mathcal{K}]\!]_g$ by

$$h(\beta_t)(f\widehat{\otimes}a\widehat{\otimes}p) = \beta_t(f\widehat{\otimes}a),$$

where $\beta_t \in [\![\mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}, B\widehat{\otimes}\mathcal{K}]\!]_g$ and p is a rank 1 projection.

Then for all $\gamma_t \in [\![\mathcal{S}\widehat{\otimes}A, B\widehat{\otimes}\mathcal{K}]\!]_g$,

$$gh(\gamma_t)(f\widehat{\otimes}a) = h(\gamma_t)(f\widehat{\otimes}a\widehat{\otimes}e) = \gamma_t(f\widehat{\otimes}a),$$

and for all $\kappa_t \in [\![\mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}, B\widehat{\otimes}\mathcal{K}]\!]_g$,

$$hg(\kappa_t)(f\widehat{\otimes}a\widehat{\otimes}p) = g(\kappa_t)(f\widehat{\otimes}a) = \kappa_t(f\widehat{\otimes}a\widehat{\otimes}e).$$

By Lemma 2.2.13 there is a path from p to e so hg is homotopic to the identity and we are done. □

Proposition 2.2.15. *Let A and B be graded C^*-algebras. Then*

1. $E_g(A\widehat{\otimes}\mathcal{K}, B) \cong E_g(A, B)$
2. $E_g(A, B\widehat{\otimes}\mathcal{K}) \cong E_g(A, B)$.

Proof. 1. This is immediate by Lemma 2.2.14.

2. Use $\mathcal{K}\widehat{\otimes}\mathcal{K} \cong \mathcal{K}$ and the proof is immediate.

□

Theorem 2.2.16 (Homotopy invariance)**.** *Suppose $\alpha, \beta\colon A \to B$ are homotopic graded $*$-homomorphisms and C is a C^*-algebra. Then the induced maps*

1. $\alpha_*, \beta_*\colon E_g(C, A) \to E_g(C, B)$, *and*
2. $\alpha^*, \beta^*\colon E_g(B, C) \to E_g(A, C)$

are equal.

Proof. 1. Let $f\colon \mathcal{S}\widehat{\otimes}C\widehat{\otimes}\mathcal{K} \dashrightarrow A\widehat{\otimes}\mathcal{K}$ be a graded asymptotic morphism representing $[\![f]\!] \in E_g(C, A)$, then we can define α_* and β_* by

$$\alpha_*(f_t) = (\alpha\widehat{\otimes}\mathrm{id}_{\mathcal{K}}) \circ f_t = (\alpha\widehat{\otimes}\mathrm{id}_{\mathcal{K}}) \circ f_t,$$

$$\beta_*(f_t) = (\beta\widehat{\otimes}\mathrm{id}_{\mathcal{K}}) \circ f_t = (\beta\widehat{\otimes}\mathrm{id}_{\mathcal{K}}) \circ f_t.$$

Since α and β are homotopic and hence α_* and β_* are equal since we are homotopic on representatives of our class.

2. Similarly to (1) we can define α^* and β^*, for an representative g of $[\![g]\!] \in E(B, C)$ by

$$\alpha^*(g_t) = g_t \circ (\alpha \widehat{\otimes} id_{\mathcal{S}\widehat{\otimes}\mathcal{K}}),$$

$$\beta^*(g_t) = g_t \circ (\beta \widehat{\otimes} id_{\mathcal{S}\widehat{\otimes}\mathcal{K}}).$$

Then again since α and β are homotopic, we can clearly see that α^* and β^* are equal at the level of homotopy classes.

□

The following definition is very similar to that of the one given in [HG04] as Definition 1.26 and the Theorem 1.14 too. The proof of Bott periodicity below is omitted since we give a proof in the case of real graded C^*-algebras in the next chapter.

Definition 2.2.17. Denote by $b \in E(\mathbb{C}, \Sigma\mathbb{C}\widehat{\otimes}\mathbb{C}_{1,0})$ the E-theory class of the $*$-homomorphism

$$\beta \colon \mathcal{S}\mathbb{C} \to \Sigma\mathbb{C}\widehat{\otimes}\mathbb{C}_{1,0}.$$

The $*$-homomorphism above is given by functional calculus and defined in explicitly for the real case in Definition 3.2.1.

Proposition 2.2.18. *The $*$-homomorphism*

$$\beta \colon \mathcal{S}\widehat{\otimes}\mathbb{C} \to \Sigma\mathbb{C}\widehat{\otimes}\mathbb{C}_{1,0},$$

induces the isomorphism:

$$E(A, \Sigma^k B\widehat{\otimes}\mathbb{C}_{n,0}) \cong E(A, \Sigma^{k+1} B\widehat{\otimes}\mathbb{C}_{n+1,0}),$$

of E-theory groups, for all positive n and k.

Proof. We consider the map

$$\beta\widehat{\otimes}\mathrm{id}_{\Sigma^k B\widehat{\otimes}\mathbb{C}_{n,0}} \colon \mathcal{S}\widehat{\otimes}B\widehat{\otimes}\mathbb{C}_{n,0} \to \Sigma^{k+1} B\widehat{\otimes}\mathbb{C}_{n+1,0},$$

which is an E-theory equivalence as β is. Consequently, we obtain the isomorphism as required. □

Call this $*$-homomorphism the Bott map.

Corollary 2.2.19. *Let A and B be graded C^*-algebras. The Bott map induces natural isomorphisms*

$$E_g^n(A, B) \cong E_g^n(A, \Sigma(B \widehat{\otimes} \mathbb{C}_{1,0})),$$

of E-theory groups.

□

To obtain the property that if we take a short exact sequence, we get a long exact sequence in graded E-theory, we need to introduce a particular asymptotic morphism that is associated to the short exact sequence. We also need some of the construction here to show that E is half-exact. For this we start by introducing the notion of approximate units as in the work by Guentner, Higson and Trout [GHT00] and the first two authors in [HG04].

Definition 2.2.20. For a separable graded C^*-algebra B and an ideal J in B, an *approximate unit* for $J \subset B$ is a norm-continuous family $\{u_t\}_{t\in[1,\infty)}$ of J, satisfying

1. $0 \leq u_t \leq 1$ for all t,
2. $\lim_{t\to\infty} ||u_t j - j|| = 0$ for all $j \in J$,
3. $\lim_{t\to\infty} ||u_t a - a u_t|| = 0$ for all $a \in A$.

Now furthermore an approximate unit as defined above always exists by Lemma 5.3 in [GHT00].

For a proof of the next result see Proposition 5.5 in [GHT00].

Proposition 2.2.21. *Let $0 \to J \to B \xrightarrow{\pi} A \to 0$ be a short exact sequence of separable graded C^*-algebras and $\{u_t\}$ an approximate unit for $J \subset B$. Then there is an asymptotic morphism $\sigma \colon \Sigma A \dashrightarrow J$ such that if we have any associated family $\{\sigma_t\}_{t\in[1,\infty)} \colon \Sigma A \to J$ and a set-theoretic section $s \colon A \to B$ of π then*

$$\sigma_t(f \widehat{\otimes} x) = f(u_t) s(x),$$

where $f \in \Sigma$ and $a \in A$.

□

Furthermore the asymptotic morphism σ is independent of the choice of section.

Definition 2.2.22. For a graded C^*-algebra A we define the *cone* of A by

$$CA = \{f : [0,1] \to A \mid f(1) = 0\}.$$

Let $\pi : B \to A$ be a $*$-homomorphism. Then the *mapping cone* of π is defined by:

$$C_\pi = \{b \oplus f \in B \oplus CA \mid \pi(b) = f(0)\}.$$

Define $*$-homomorphisms $\alpha : C_\pi \to B$ by $\alpha(b \oplus f) = b$ and $\beta : \Sigma A \to C_\pi$ by $\beta(f) = 0 \oplus f$.

By the definition of α and β above we can form a short exact sequence

$$0 \to \Sigma J \to CB \xrightarrow{\pi_1} C_\pi \to 0, \tag{2.1}$$

where we define π_1 by $\pi_1(f) = f(0) \oplus \pi(f)$. Then by Proposition 2.2.21 we obtain the associated asymptotic morphism $\sigma : \Sigma C_\pi \dashrightarrow \Sigma J$.

The proof of the following proposition follows from the short exact sequence (2.1) and by the inclusion of J in C_π.

Proposition 2.2.23. *The associated asymptotic morphism σ determines an element in $E_g(\Sigma C_\pi, \Sigma J)$ which in addition is inverse to an element $\Sigma\tau_* \in E_g(\Sigma J, \Sigma C_\pi)$.*

□

By Bott periodicity it follows that $E_g(J, A) \cong E_g(C_\pi, A)$ and $E_g(A, J) \cong E_g(A, C_\pi)$ for all C^*-algebras A.

For the following result, let $p : A \to A/J$ and then

$$C_p = \{a \oplus f \in A \oplus C(A/J) \mid p(a) = f(1)\}.$$

Lemma 2.2.24. *For a graded C^*-algebra B, the following are exact in the middle.*

1. $E_g(B, C_p) \xrightarrow{\alpha_*} E_g(B, A) \xrightarrow{p_*} E_g(B, A/J)$ *is exact,*
2. $E_g(A/J, B) \xrightarrow{p^*} E_g(A, B) \xrightarrow{\alpha^*} E_g(C_p, B)$ *is exact.*

Proof. 1. Let $\varphi_t : \mathcal{S}\widehat{\otimes}B\widehat{\otimes}\mathcal{K} \dashrightarrow A\widehat{\otimes}\mathcal{K}$, and $[\varphi_t] \in E_g(B, A)$ be such that $p_*[\varphi_t] = 0$, i.e $[\varphi_t] \in \mathrm{Ker}\ p_*$. Then $p \circ \varphi_t \sim_h 0$.

Now let $\theta\colon \mathcal{S}\widehat{\otimes}B\widehat{\otimes}\mathcal{K} \dashrightarrow A\widehat{\otimes}\mathcal{K}\widehat{\otimes}C[0,1]$ be such that $p\circ\theta_t$ is a homotopy between $p\circ\varphi_t$ and 0. Then write

$$\theta\colon \mathcal{S}\widehat{\otimes}B\widehat{\otimes}\mathcal{K} \dashrightarrow A\widehat{\otimes}C_0[0,1)\widehat{\otimes}\mathcal{K} \cong CA\widehat{\otimes}\mathcal{K},$$

and define $\psi\colon \mathcal{S}\widehat{\otimes}B\widehat{\otimes}\mathcal{K} \dashrightarrow C_p\widehat{\otimes}\mathcal{K}$ by,

$$\psi_t = \varphi_t \oplus \theta_t.$$

Then $\alpha\circ\psi_t = \varphi_t$, and $\alpha_*[\psi_t] = [\varphi_t]$ and so $[\varphi_t] \in \operatorname{Im}\alpha_*$. So $\operatorname{Ker} p_* \subseteq \operatorname{Im}\alpha_*$.

Conversely, from homological algebra we know that $\operatorname{Im}\alpha_* \subseteq \operatorname{Ker} p_*$ is equivalent to $p_*\alpha_* = 0$. Now for $x_t \in E(B, C_p)$

$$\begin{aligned} p_*\alpha_*(x_t) &= p_*(\alpha\widehat{\otimes}\mathrm{id}_{\mathcal{K}} \circ x_t) \\ &= p\widehat{\otimes}\mathrm{id}_{\mathcal{K}} \circ \alpha\widehat{\otimes}\mathrm{id}_{\mathcal{K}} \circ x_t \\ &= (p\circ\alpha)\widehat{\otimes}\mathrm{id}_{\mathcal{K}} \circ x_t \\ &= 0 \end{aligned}$$

since $p\circ\alpha$ is equal to the identity.

2. Firstly $\alpha^*\circ p^* = 0$ so $\operatorname{Im} p^* \subseteq \operatorname{Ker}\alpha^*$ by a similar method to part(1). Now we show that $\operatorname{Ker}\alpha^* \subseteq \operatorname{Im} p^*$. Let $\varphi_t\colon \mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K} \dashrightarrow B\widehat{\otimes}\mathcal{K}$ and $q\colon C_p\widehat{\otimes}\mathcal{K} \to A\widehat{\otimes}\mathcal{K}$ be the projection. Then we want $\theta_t\colon \mathcal{S}\widehat{\otimes}\Sigma A/J\widehat{\otimes}\mathcal{K} \dashrightarrow \Sigma B\widehat{\otimes}\mathcal{K}$ such that

$$[\theta_t \circ (\mathrm{id}_{\mathcal{S}} \circ \Sigma p)] = \Sigma[\varphi] \in E(\Sigma A, \Sigma B),$$

where $p\colon \mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K} \to \mathcal{S}\widehat{\otimes}A/J\widehat{\otimes}\mathcal{K}$.

Now let $\eta\colon \mathcal{S}\widehat{\otimes}C_p\widehat{\otimes}\mathcal{K} \to IB\widehat{\otimes}\mathcal{K}$ (where $I = [0,1]$) be a homotopy between $\varphi_t\circ(\mathrm{id}\widehat{\otimes}q)$ and 0. Then by symmetry of homotopy we can obtain

$$\widetilde{\eta}_t\colon \mathcal{S}\widehat{\otimes}C_p\widehat{\otimes}\mathcal{K} \to I_1B\widehat{\otimes}\mathcal{K},$$

where $I_1 = [0,1]$. Now we also have an inclusion

$$i\colon \Sigma\mathcal{S}\widehat{\otimes}A/J\widehat{\otimes}\mathcal{K} \cong \mathcal{S}\widehat{\otimes}\Sigma A/J\widehat{\otimes}\mathcal{K} \to \mathcal{S}\widehat{\otimes}C_p\widehat{\otimes}\mathcal{K},$$

defined by

$$i(g\widehat{\otimes}f\widehat{\otimes}k) = g\widehat{\otimes}(0\oplus f)\widehat{\otimes}k,$$

where $g \in \mathcal{S}$, $f\colon [-1,1] \to A/J$ and $k \in \mathcal{K}$. Then $(\mathrm{id}_{\mathcal{S}} \widehat{\otimes} q) \circ i = 0$.

Define $\theta_t\colon \mathcal{S}\widehat{\otimes}\Sigma A/J\widehat{\otimes}\mathcal{K} \dashrightarrow \Sigma B\widehat{\otimes}\mathcal{K}$ by $\theta_t = \widetilde{\eta_t} \circ i$.Then we need to show that $\theta_t \circ (\mathrm{id}_{\mathcal{S}} \circ \Sigma p)$ is homotopic to $\Sigma\varphi$. Now $\Sigma\varphi_t = \widetilde{\eta_t} \circ i \circ (\mathrm{id}_{\mathcal{S}} \circ \Sigma p)$, and so

$$||\theta_t \circ (\mathrm{id}_{\mathcal{S}} \circ \Sigma p) - (\Sigma\varphi_t)|| \to 0,$$

as $t \to \infty$ and asymptotic equivalence implies homotopic equivalence by Proposition 2.2.4, half exactness follows.

□

Then the next result follows from the isomorphisms $E_g(J, D) \cong E_g(C_\pi, D)$ and $E_g(D, J) \cong E_g(D, C_\pi)$ and Lemma 2.2.24.

Lemma 2.2.25 (Half exactness)**.** *Let $0 \to J \xrightarrow{\alpha} B \xrightarrow{\beta} A \to 0$ be a short exact sequence of graded C^*-algebras and let D be a graded C^*-algebra. Then*

1. $E_g(D, J) \xrightarrow{\alpha_*} E_g(D, B) \xrightarrow{\beta_*} E_g(D, A)$ *is exact,*

2. $E_g(A, D) \xrightarrow{\beta^*} E_g(B, D) \xrightarrow{\alpha^*} E_g(J, D)$ *is exact.*

□

Now once we see the following result, proposition 6.14 in [GHT00], we can obtain the theorem following it. Recall from chapter 1, that a functor is a homology functor if it is half exact and homotopy invariant.

The subsequent result is proven as Proposition 21.4.1 of [Bla98].

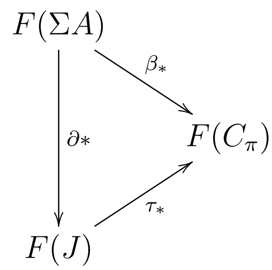
Proposition 2.2.26. *For a homology functor F from the category of graded C^*-algebras and $*$-homomorphisms to the category of abelian groups and group homomorphisms to every short exact sequence of separable graded C^*-algebras*

$$0 \to J \to B \to A \to 0,$$

we obtain a long exact sequence of abelian groups

$$\cdots \to F(\Sigma B) \to F(\Sigma A) \xrightarrow{\partial_*} F(J) \to F(B) \to F(A),$$

where the connecting map ∂_* *fits in to the commutative diagram*



$\square$

Then the following theorem is proven by using the above statements with contravariant version of the above proposition.

Theorem 2.2.27. *Let* D *be a graded* C^**-algebra and let* $0 \to J \xrightarrow{\alpha} B \xrightarrow{\beta} A \to 0$ *be a short exact sequence of graded* C^**-algebras. Then there is an element* $\sigma\colon \Sigma A \dashrightarrow J$ *and long exact sequences:*

$$E_g(A, D) \to E_g(B, D) \to E_g(J, D) \xrightarrow{\partial^*} E_g(\Sigma A, D) \to \cdots,$$

and

$$\cdots \to E_g(D, \Sigma A) \xrightarrow{\partial_*} E_g(D, J) \to E_g(D, B) \to E_g(D, A),$$

in which the maps ∂^*, ∂_* *are given by the* E*-theory product with* σ*.*

$\square$

As before we can define δ^* and δ_* explicitly by,

$$\delta^*(\alpha) = \alpha \circ (\mathrm{id}\widehat{\otimes}\sigma) \circ (\Delta\widehat{\otimes}\mathrm{id}_{\Sigma A\widehat{\otimes}\mathcal{K}}),$$

where α is in the E-theory class $E_g(J, D)$, and

$$\delta_*(\beta) = \sigma \circ (\mathrm{id}_S\widehat{\otimes}\beta) \circ (\Delta\widehat{\otimes}\mathrm{id}_{D\widehat{\otimes}\mathcal{K}}),$$

where β is in the E-theory class $E_g(D, \Sigma A)$.

Proposition 2.2.28. *Let* A, B, C *and* D *be graded* C^**-algebras and suppose we have a split exact sequence,*

$$0 \longrightarrow A \underset{\gamma}{\overset{\alpha}{\rightleftarrows}} B \xrightarrow{\beta} C \longrightarrow 0,$$

then

$$0 \longrightarrow E_g(C,D) \xrightarrow{\beta^*} E_g(B,D) \underset{\gamma^*}{\overset{\alpha^*}{\rightleftarrows}} E_g(A,D) \longrightarrow 0,$$

$$0 \longrightarrow E_g(D,A) \underset{\gamma_*}{\overset{\alpha_*}{\rightleftarrows}} E_g(D,B) \xrightarrow{\beta_*} E_g(D,C) \longrightarrow 0,$$

are split exact.

Proof. The proof is the same as in Proposition 1.4.15 where we use Theorem 2.2.27. □

Proposition 2.2.29. *Let A, B, C and D be graded C^*-algebras and suppose we have a split exact sequence,*

$$0 \longrightarrow A \xrightarrow{\alpha} B \underset{\delta}{\overset{\beta}{\rightleftarrows}} C \longrightarrow 0,$$

then

$$0 \longrightarrow E_g(C,D) \underset{\delta^*}{\overset{\beta^*}{\rightleftarrows}} E_g(B,D) \xrightarrow{\alpha^*} E_g(A,D) \longrightarrow 0,$$

and

$$0 \longrightarrow E_g(D,A) \xrightarrow{\alpha_*} E_g(D,B) \underset{\delta_*}{\overset{\beta_*}{\rightleftarrows}} E_g(D,C) \longrightarrow 0,$$

are split exact.

Proof. This proof is similar to Proposition 2.2.28. □

Proposition 2.2.30. *Let A, B be unital graded C^*-algebras. Then we have an isomorphism*

$$E_g(A,B) \cong E_g(A\widehat{\otimes}\mathbb{F}_{1,1}, B\widehat{\otimes}\mathbb{F}_{1,1}).$$

Proof. This is immediate from Lemma 2.1.14, Lemma 2.1.15 and Lemma 2.2.14. □

Corollary 2.2.31. *Let A and B be complex graded C^*-algebra. Then we have natural isomorphisms*

$$E_g^n(A,B) \cong E_g^{n+2}(A,B).$$

Proof.

$$\begin{aligned}
E^n_g(A,B) &= E^n_g(A, \Sigma^2(B \widehat{\otimes} \mathbb{C}_{1,0} \widehat{\otimes} \mathbb{C}_{1,0})), \text{ by Corollary 2.2.19,}\\
&= E^n_g(A, \Sigma^2(B \widehat{\otimes} \mathbb{C}_{1,1})), \text{ by Proposition 2.1.9,}\\
&= E^n_g(A, \Sigma^2 B), \text{ by Proposition 2.2.30,}\\
&\cong E^{n+2}_g(A,B).
\end{aligned}$$

□

The proof of the next Theorem follows from Theorem 2.2.27 and Bott periodicity.

Theorem 2.2.32. *Let D be a separable graded C^*-algebra and let*

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0,$$

be a short exact sequence of graded C^*-algebras. Then there are exact sequences

$$\begin{array}{ccccc}
E_g(D, \Sigma A) & \longrightarrow & E_g(D, \Sigma B) & \longrightarrow & E_g(D, \Sigma C) \\
\uparrow & & & & \downarrow \\
E_g(D, C) & \xleftarrow{\beta_*} & E_g(D, B) & \xleftarrow{\alpha_*} & E_g(D, A)
\end{array},$$

and

$$\begin{array}{ccccc}
E_g(\Sigma A, D) & \longleftarrow & E_g(\Sigma B, D) & \longleftarrow & E_g(\Sigma C, D) \\
\downarrow & & & & \uparrow \\
E_g(C, D) & \xrightarrow{\beta^*} & E_g(B, D) & \xrightarrow{\alpha^*} & E_g(A, D)
\end{array},$$

where the boundary maps are defined by Bott periodicity and product with the E-theory class of an element $\sigma \in E_g(\Sigma A, J)$ associated to the short exact sequence.

□

Chapter 3

Bott periodicity in the real case

We wish to pass the definition of E-theory to real C^*-algebras, so this chapter begins by giving a brief introduction to real C^*-algebras based on [Goo82] and [Pal84]. Our aim is to extend the Bott map in the previous chapter to the real case. We will start by thinking about Bott periodicity in K-theory (see Appendix B for details) since this then can be used to induce the Bott periodicity in E-theory. We will define a $*$-homomorphism

$$\beta\colon \mathcal{S} \to C_0(V, \mathrm{Cliff}(V))$$

for a finite dimensional Euclidean vector space V over $\mathbb{R}$. We will see that this gives a class in K-theory. Similarly to Guentner and Higson [HG04], we will construct an asymptotic morphism

$$\alpha_t\colon \mathcal{S}\widehat{\otimes} C_0(V, \mathrm{Cliff}(V)) \dashrightarrow \mathcal{K}(\mathcal{H}).$$

Thereafter we will prove α is asymptotically equivalent to another map before proving that the induced maps β_* and α_* are such that $\alpha_*\beta_*$ is homotopic to 1 in $K(\mathbb{R}) \cong \mathbb{Z}$. Then we will show why this is sufficient to prove Bott periodicity for E-theory.

3.1 Real C^*-algebras

Definition 3.1.1. A *real C^*-algebra* A is a real Banach algebra with involution satisfying the C^*-identity and additionally the property that $1 + x^*x$ is invertible in A for all $x \in A$.

Note that the property above that $1 + x^*x$ is invertible is equivalent to the element x^*x being positive. That is, x^*x is self adjoint and the $\sigma(x^*x) \subseteq [0, \infty)$ (For a proof see [Goo82], Proposition 13.4.).

Example 1. *Every complex C^*-algebra is a real C^*-algebra, where we restrict scalar multiplication to just real scalars.*

Example 2. *The algebra of bounded linear operators on a real Hilbert space is a real C^*-algebra with the operation of pointwise multiplication and the supremum norm.*

Example 3. *$C(X)$, the algebra of real valued functions on a compact Hausdorff space is a C^*-algebra with operation of pointwise operation and with the supremum norm.*

Here we define the *spectrum* of an element x in a real C^*-algebra by considering the set of formal expressions $x + iy$ for $x, y \in A$ which forms a complex C^*-algebra B. Then the spectrum of an element in A is the spectrum of the element in B.

3.2 The Bott map

Let Cliff(V) denote the *Clifford algebra* of a finite dimensional Euclidean real vector space V. That is, the algebra generated by elements V, that have odd grading, and such that $v = v^*$ and $v^2 = ||v||^2 \cdot 1$ for every $v \in V$. Then define a function $P \colon V \to \mathrm{Cliff}(V)$ by $P(v) = v$. Notice that $P(v)^2 = ||v||^2$. Now we require a function in $C_0(V, \mathrm{Cliff}(V))$ so using functional calculus (see Appendix A), take $f \in \mathcal{S}$ then we define the map $f(P) \colon V \to \mathrm{Cliff}(V)$ by:

$$v \mapsto f(P(v)).$$

This gives a $*$-homomorphism from $\mathcal{S}$ to $C_0(V, \mathrm{Cliff}(V))$, and hence the following definition:-

Definition 3.2.1. We define the *Bott element* b to be the class of the K-theory group $K(C_0(V, \mathrm{Cliff}(V))$ induced from the $*$-homomorphism $\beta \colon \mathcal{S} \to C_0(V, \mathrm{Cliff}(V))$ defined by $f \mapsto f(P)$.

Theorem 3.2.2. *For a finite dimensional Euclidean vector space V over $\mathbb{R}$, the Bott map induces the isomorphism of K-theory*

$$\beta_*\colon K(\mathbb{R}) \to K(\mathbb{R}\widehat{\otimes}C_0(V, \mathit{Cliff}(V))),$$

defined by the formula $\beta_*(x) = x \times b$, where $x \in K(\mathbb{R})$.

To prove this theorem we need a collection of ingredients which we now detail. We start by defining a map that induces the inverse homomorphism in K-theory but before that we require some definitions to define such a map.

Denote by $\mathcal{H}(V)$ the real Hilbert space $L^2(V, \mathrm{Cliff}(V))$ which is the Hilbert space of square-integrable $\mathrm{Cliff}(V)$-valued functions on V. Note that this is graded and the grading is the one coming from that of $\mathrm{Cliff}(V)$.

The following definition may seem irrelevant to our aim but it will be very useful for defining the operators we need to define the maps inducing Bott periodicity.

Definition 3.2.3. Let e, f be elements in V. Define linear operators

$$e, \hat{f}\colon \mathrm{Cliff}(V) \to \mathrm{Cliff}(V)$$

on a finite-dimensional Hilbert space under $\mathrm{Cliff}(V)$ by

$$e(x) = ex,$$

$$\hat{f}(x) = (-1)^{\deg(x)} xf.$$

For a vector space V, the Schwarz space Sch(V) is the topological vector space of functions $f\colon V \to \mathbb{F}$ such that f is continuously differentable and for $x \in V$, $x^\alpha \partial^\beta f(x) \to 0$ as $|x| \to \infty$ for every pair of multi-indices $\alpha, \beta \in \mathbb{N}^m$. These form a class of functions. Let $\mathrm{Sch}(V)$ denote the dense subspace of $\mathcal{H}(V)$ of Schwartz-class $\mathrm{Cliff}(V)$-valued functions.

Definition 3.2.4. Define the *Dirac operator* $D\colon \mathrm{Sch}(V) \to \mathcal{H}(V)$ by

$$(Df)(v) = \sum_1^n \hat{e}_i(\frac{\partial f}{\partial x_i}(v)),$$

where the e_i's form an orthonormal basis of V and each x_i is the corresponding coordinate in V. Here we use functional calculus to define the functions throughout the summation.

For $h \in C_0(V, \mathrm{Cliff}(V))$ let M_h denote the operator of pointwise multiplication by h on the space $\mathcal{H}(V)$. So

$$M_h(g)(v) = h(v)g(v).$$

Then for the multiplication operator we have the following result. For a proof of the this see [HG04], Lemma 1.8, since this proof is still valid when we take real finite dimensional Euclidean vector spaces.

Lemma 3.2.5. *The Dirac operator on V is formally self-adjoint. For $f \in \mathcal{S}$ and $h \in C_0(V, \mathit{Cliff}(V))$, the product $f(D)M_h$ is a compact operator on $\mathcal{H}(V)$.*

□

It will be convenient to write $h \in C_0(V, \mathrm{Cliff}(V))$ as h_t, with $t \in [1, \infty)$ and $h_t(v) = h(t^{-1}v)$.

Definition 3.2.6. Define the graded commutator of elements a, b in a real graded C^*-algebra A by

$$[a, b] = ab - (-1)^{\deg(a)\deg(b)}ba,$$

which extends by linearity to all elements in A.

Letting $V = \mathbb{R}_{n,0}$ we have the following result:

Lemma 3.2.7. *For every $f \in \mathcal{S}, h \in C_0(\mathbb{R}^n, \mathbb{R}_{n,0})$, with the finite dimensional Euclidean vector space $\mathbb{R}^n$ having Dirac operator D,*

$$\lim_{t\to\infty} ||[f(t^{-1}D), M_{h_t}]|| = 0,$$

where M_{h_t} is a bounded linear operator on $\mathcal{H}(\mathbb{R}^n)$ and $f(t^{-1}D)$ is defined using functional calculus of unbounded operators(see Appendix A).

Proof. Let $f \in C_0(\mathbb{R})\widehat{\otimes}_{\mathbb{R}}\mathbb{C}$, then f is also contained in $f \in C_0(\mathbb{R})\widehat{\otimes}_{\mathbb{C}}\mathbb{C}$. Let $f_k(x) = (x+i)^{-k}$ and $\overline{f_k}(x) = (x-i)^{-k}$ and define

$$A = \{f_k, \overline{f_k} \mid k \in \mathbb{N}_{\geq 0}\}.$$

Then A is a unital algebra closed under complex conjugation which separates points, sincs f_1 separates points. Hence by the Stone-Weierstrass Theorem, A is dense in $C_0(\mathbb{R})\widehat{\otimes}_{\mathbb{R}}\mathbb{C}$. So if

$$\lim_{t\to\infty} ||[f_k(t^{-1}D), M_{h_t}]|| = 0 \text{ and } \lim_{t\to\infty} ||[\overline{f_k}(t^{-1}D), M_{h_t}]|| = 0,$$

for all $f \in A$, then

$$\lim_{t\to\infty} ||[f(t^{-1}D), M_{h_t}]|| = 0,$$

for all $f \in C_0(\mathbb{R})\widehat{\otimes}_{\mathbb{R}}\mathbb{C}$, and so for all $f \in C_0(\mathbb{R})$. It is enough to check this for f_1, and $\overline{f_1}$ since they generate A. Let $f = \overline{f_1}$.

Then we obtain the factorisation,

$$\begin{aligned}
[f(t^{-1}D), M_{h_t}] &= [(t^{-1}D - iI)^{-1}, M_{h_t}] \\
&= (t^{-1}D - iI)^{-1}M_{h_t} \pm M_{h_t}(t^{-1}D - iI)^{-1} \\
&= (t^{-1}D - iI)^{-1}M_{h_t}(t^{-1}D - iI)(t^{-1}D - iI)^{-1} \\
&\qquad \pm (t^{-1}D - iI)^{-1}(t^{-1}D - iI)M_{h_t}(t^{-1}D - iI)^{-1} \\
&= (t^{-1}D - iI)^{-1}[M_{h_t}, (t^{-1}D - iI)](t^{-1}D - iI)^{-1} \\
&= t^{-1}(t^{-1}D - iI)^{-1}[M_{h_t}, D](t^{-1}D - iI)^{-1}.
\end{aligned}$$

Then the norm of this is bounded by $t^{-1}[M_{h_t}, D]$, but since this commutator is pointwise multiplication we obtain for $v \in \mathbb{R}^n$

$$t^{-1}[M_{h_t}, D](v) = t^{-1}\sum_{i=1}^{n} \widehat{e_i}(\frac{\partial h}{\partial t_i}(v)).$$

Then this norm tends to 0 as $t \to \infty$.

□

Now we can state the map α we require to prove Bott periodicity.

Proposition 3.2.8. *There exists a unique asymptotic morphism (up to equivalence)*

$$\alpha_t \colon \mathcal{S}\widehat{\otimes}C_0(V, \mathit{Cliff}(V)) \dashrightarrow \mathcal{K}(\mathcal{H}(V))$$

defined by

$$\alpha_t(f\widehat{\otimes}h) = f(t^{-1}D)M_{h_t},$$

for $t \geq 1$.

Proof. Firstly to prove this we consider α_t defined from the algebraic tensor product of $\mathcal{S}\widehat{\odot}C_0(V, \mathrm{Cliff}(V))$ to $\mathcal{K}(\mathcal{H}(V))$ by,

$$\alpha_t(f\widehat{\odot}h) = f(t^{-1}D)M_{h_t}.$$

Then we have a linear map, which is asymptotically $*$-linear by Lemma 3.2.7 and universality of the tensor product gives an asymptotic morphism

$$\alpha_t(f\widehat{\otimes}h) = f(t^{-1}D)M_{h_t}.$$

Also by Lemma 3.2.5, the image is contained in the compact operators. Hence we have our required asymptotic morphism. $\square$

Definition 3.2.9. On a finite dimension Euclidean real vector space we define the *Clifford operator* $C\colon \mathrm{Sch}(V) \to \mathcal{H}(V)$ by

$$(Cf)(v) = \sum_{i=1}^{n} x_i e_i(f(v)),$$

where e_i form an orthonormal basis for V, x_i are the corresponding coordinates in V. Here we use functional calculus to define the functions throughout the summation.

Lemma 3.2.10. *The composition of the Bott map and the multiplication operator given by*

$$\mathcal{S} \xrightarrow{\beta} C_0(V, \mathit{Cliff}(V)) \xrightarrow{M} \mathcal{B}(\mathcal{H}(V)),$$

is defined by

$$f \mapsto f(C),$$

where C is the Clifford operator and $f \in \mathcal{S}$.

Proof. Since the operator C is essentially self adjoint on the Hilbert space $\mathcal{H}(V)$ we can form the operator $f(C)$ in the bounded linear operators. Then

$$(M\beta(f))(g)(v) = M_{\beta(f)}(g)(v) = \beta(f)(v)g(v) = f(C)(v)g(v) = f(C)g(v).$$

$\square$

Now we need to prove that α and β satisfy the requirements that their induced maps give $\alpha_*\beta_*$ is homotopic to $1 \in K(\mathbb{R})$. In order to show this we firstly need an equivalent morphism coming from composing α and β for which we need a new operator.

Definition 3.2.11. We define an operator $B\colon \mathrm{Sch}(V) \to \mathrm{Sch}(V)$ by

$$(Bf)(v) = \sum_{i=1}^{n} x_i e_i(f(v)) + \sum_{i=1}^{n} \hat{e}_i(\frac{\partial f}{\partial x_i}(v)).$$

Call B the *Bott operator* and notice $B = C + D$.

Definition 3.2.12. Define the *number operator* $N\colon \mathrm{Cliff}(V) \to \mathrm{Cliff}(V)$ by

$$N = \sum_{i=1}^{n} \widehat{e_i} e_i,$$

where e_i's form an orthonormal basis for V.

We should note that the number operator is bounded and we can extend it to an operator on $\mathcal{H}(V)$ by composition with a map from V to $\mathrm{Cliff}(V)$. Also note that N commutes with both C and D.

Proposition 3.2.13.

$$B^2 = C^2 + D^2 + N.$$

Proof. By orthogonality of the basis of V,

$$\begin{aligned}
&B^2(f)(v)\\
&= B(Bf)(v)\\
&= \sum_{i=1}^{n}\left(x_i e_i(Bf(v)) + \hat{e}_i(\frac{\partial Bf}{\partial x_i}(v))\right)\\
&= \sum_{i=1}^{n}\left(x_i e_i \sum_{j=1}^{m}\left(x_i e_i(f(v) + \hat{e}_i(\frac{\partial f}{\partial x_i}(v))\right)\right)\\
&\qquad + \sum_{i=1}^{n}\left(\hat{e}_i \frac{\partial}{\partial x_i}\sum_{j=1}^{m}\left(x_i e_i(f(v) + \hat{e}_i(\frac{\partial f}{\partial x_i}(v))\right)\right)\\
&= \sum_{i,j=1}^{n}\left(x_i e_i x_j e_j f(v) + x_i e_i \widehat{e_j}\frac{\partial}{\partial x_j} f(v) + \widehat{e_i}\frac{\partial}{\partial x_i}(x_j e_j f)(v) + \widehat{e_i}\frac{\partial}{\partial x_i}\widehat{e_j}\frac{\partial}{\partial x_j}\right)\\
&= \sum_{i=1}^{n}\left(x_i^2 f(v) - \frac{\partial^2}{\partial {x_i}^2} f(v)\right) + A(f)(v),\\
&= C^2(f)(v) + D^2(f)(v) + A(f)(v)
\end{aligned}$$

where

$$A(f)(v) = \sum_{i=1}^{n}\sum_{j=1}^{m}\left(x_i e_i \widehat{e_j}\frac{\partial}{\partial x_j} f(v) + \widehat{e_i}\frac{\partial}{\partial x_i}(x_j e_j f)(v)\right).$$

Then for $i = j$,

$$
\begin{aligned}
A(f)(v) &= \sum_{i=1}^{n} \left(x_i e_i \widehat{e_i} \frac{\partial}{\partial x_i} f(v) + \widehat{e_i} \frac{\partial}{\partial x_i} (x_i e_i f)(v) \right) \\
&= \sum_{i=1}^{n} \left(x_i e_i \widehat{e_i} \frac{\partial}{\partial x_i} f(v) + \widehat{e_i} e_i \frac{\partial}{\partial x_i} (x_i f)(v) \right) \\
&= \sum_{i=1}^{n} \left(x_i e_i \widehat{e_i} \frac{\partial}{\partial x_i} f(v) + \widehat{e_i} e_i f(v) + \widehat{e_i} e_i x_i \frac{\partial}{\partial x_i} f(v) \right) \\
&= \sum_{i=1}^{n} \left(x_i e_i \widehat{e_i} \frac{\partial}{\partial x_i} f(v) + \widehat{e_i} e_i x_i \frac{\partial}{\partial x_i} f(v) \right) + N f(v)
\end{aligned}
$$

where the third line is established using the product rule and for $i \neq j$,

$$
A(f)(v) = \sum_{i=1}^{n} \sum_{j=1}^{m} \left(x_i e_i \widehat{e_j} \frac{\partial}{\partial x_j} f(v) + \widehat{e_i} e_j x_j \frac{\partial}{\partial x_i} f(v) \right).
$$

So

$$
\begin{aligned}
A(f)(v) &= \sum_{i=1}^{n} \sum_{j=1}^{m} \left(x_i e_i \widehat{e_j} \frac{\partial}{\partial x_j} f(v) + \widehat{e_i} e_j x_j \frac{\partial}{\partial x_i} f(v) \right) + N f(v) \\
&= N f(v),
\end{aligned}
$$

since $\widehat{e_j} e_i = -\widehat{e_j} e_i$. So as required

$$
B^2 = C^2 + D^2 + N.
$$

□

The following two results have identical proofs to the results stated in [HG04] so for proofs see Proposition 1.16 and Corollary 1.1 of that paper.

Proposition 3.2.14. *For a finite dimensional real vector space V of dimension n, consider the operator $B = C + D$. Then there exists an orthonormal basis in $Sch(V)$ for $\mathcal{H}(V)$ consisting of the eigenvectors for B^2 such that*

1. *the eigenvalues are nonnegative integers and each eigenvalue occurs with finite multiplicity,*

2. *the eigenvalue 0 occurs only once and its corresponding eigenfunction is* $e^{-\frac{1}{2}||v||^2}$.

□

Corollary 3.2.15. *For a finite dimensional real vector space V. Then the Bott operator B*

1. *is essentially self-adjoint,*

2. *has compact resolvent,*

3. *has one-dimensional kernel generated by the function* $e^{-\frac{1}{2}||v||^2}$.

□

The following two results follow from a standard result of analysis.

Proposition 3.2.16. *Let T be an unbounded self-adjoint operator. Then e^{-T^2} is bounded.*

The following is proved similarly.

Proposition 3.2.17. *For an unbounded self-adjoint operator T, Te^{-T^2} is bounded.*

□

The following follows from chapter 12 [CFKS87] and [HT92] page 114-115, and it is stated in [HG04] as Proposition 1.6.

Proposition 3.2.18 (Mehler's formula)**.** *For the Clifford operator C and the Dirac operator D defined above we have the following identities for $s > 0$*

$$e^{-s(C^2+D^2)} = e^{-\frac{1}{2}s_1C^2}e^{-s_2D^2}e^{-\frac{1}{2}s_1C^2} \quad \text{and} \quad e^{-s(C^2+D^2)} = e^{-\frac{1}{2}s_1D^2}e^{-s_2C^2}e^{-\frac{1}{2}s_1D^2}$$

with

$$s_1 = \frac{cosh(2s) - 1}{sinh(2s)} \text{ and } s_2 = \frac{sinh(2s)}{2}.$$

□

Then as in [HG04], Lemma 1.11, we have the asymptotic conditions:

Lemma 3.2.19. *For an unbounded operator X we have the following conditions:*

$$\lim_{t\to\infty} ||e^{-\frac{1}{2}s_1 X^2} - e^{-\frac{1}{2}t^{-2}X^2}|| = 0,$$

$$\lim_{t\to\infty} ||e^{-\frac{1}{2}s_2 X^2} - e^{-\frac{1}{2}t^{-2}X^2}|| = 0,$$

and

$$\lim_{t\to\infty} ||t^{-1}Xe^{-\frac{1}{2}s_1 X^2} - t^{-1}Xe^{-\frac{1}{2}t^{-2}X^2}|| = 0,$$

$$\lim_{t\to\infty} ||t^{-1}Xe^{-\frac{1}{2}s_2 X^2} - t^{-1}Xe^{-\frac{1}{2}t^{-2}X^2}|| = 0,$$

where

$$s_1 = \frac{cosh(2t^{-2}) - 1}{sinh(2t^{-2})} \text{ and } s_2 = \frac{sinh(2t^{-2})}{2}.$$

□

For a proof of the following result, see Lemma 1.12 of [HG04].

Lemma 3.2.20. *Let $f, g \in C_0(\mathbb{R})$, then*

$$\lim_{t\to\infty} ||[f(t^{-1}C), g(t^{-1}D)]|| = 0.$$

□

We can use the proof in [HG04] of Theorem 1.17 and the above statements to obtain.

Proposition 3.2.21. *For operators C and D defined before,*

$$e^{-t^{-2}B^2} \sim_{asy} e^{-t^{-2}C^2}e^{-t^{-2}D^2}.$$

□

The following is similar.

Proposition 3.2.22. *For the operators C and D,*

$$t^{-1}Be^{-t^{-2}B^2} \sim_{asy} t^{-1}(C + D)e^{-t^{-2}C^2}e^{-t^{-2}D^2}.$$

□

Theorem 3.2.23. *The composition of α_t and β given by :*

$$\mathcal{S} \xrightarrow{\Delta} \mathcal{S}\widehat{\otimes}\mathcal{S} \xrightarrow{id\widehat{\otimes}\beta} \mathcal{S}\widehat{\otimes}C_0(V, \mathit{Cliff}(V)) \overset{\alpha_t}{\dashrightarrow} \mathcal{K}(\mathcal{H}(V),$$

is asymptotically equivalent to the asymptotic morphism $\gamma_t \colon \mathcal{S} \dashrightarrow \mathcal{K}(H)$ defined by

$$\gamma_t(f) = f(t^{-1}B),$$

for all $t \geq 1$.

Proof. Here we imitate the proof of Guentner and Higson in [HG04] with additional details. Since S is generated by u and v by Lemma 2.1.19 it suffices to check that

$$\alpha(\mathrm{id}\widehat{\otimes}\beta)(\Delta(f)) \sim_{asy} \gamma_t(f),$$

for $f = u$ and $f = v$.

For $f = u$,

$$\begin{aligned}
\alpha(\mathrm{id}\widehat{\otimes}\beta)(\Delta(u)) &= \alpha(\mathrm{id}\widehat{\otimes}\beta)(u\widehat{\otimes}u) \\
&= \alpha(u\widehat{\otimes}\beta(u)) \\
&= \alpha(u\widehat{\otimes}u(C)) \\
&= u(t^{-1}D)M_{u(C)_t} \\
&= u(t^{-1}D)u(t^{-1}C) \\
&= e^{-t^{-2}C^2}e^{-t^{-2}D^2},
\end{aligned}$$

and

$$\gamma_t(u) = u(t^{-1}B) = e^{-t^{-2}B^2},$$

and these are both asymptotically equivalent by Proposition 3.2.21.

For $f = v$,

$$\begin{aligned}
\alpha(\mathrm{id}\widehat{\otimes}\beta)(\Delta(u)) &= \alpha(\mathrm{id}\widehat{\otimes}\beta)(u\widehat{\otimes}v + v\widehat{\otimes}u) \\
&= \alpha(u\widehat{\otimes}\beta(v) + v\widehat{\otimes}\beta(u)) \\
&= \alpha(u\widehat{\otimes}v(C) + v\widehat{\otimes}u(C)) \\
&\sim_{asy} \alpha(u\widehat{\otimes}v(C)) + \alpha(v\widehat{\otimes}u(C)) \\
&= u(t^{-1}D)M_{v(C)_t} + v(t^{-1}D)M_{u(C)_t} \\
&= u(t^{-1}D)v(t^{-1}C) + v(t^{-1}D)u(t^{-1}C) \\
&= e^{-t^{-2}D^2}t^{-1}Ce^{-t^{-2}C^2} + t^{-1}De^{-t^{-2}D^2}e^{-t^{-2}C^2} \\
&= t^{-1}(C+D)e^{-t^{-2}C^2}e^{-t^{-2}D^2},
\end{aligned}$$

and

$$\gamma_t(v) = v(t^{-1}B) = t^{-1}Be^{-t^{-2}B^2}.$$

Then by Proposition 3.2.22 these are asymptotically equivalent. Hence the composition of α and β is asymptotically equivalent to γ_t. □

Corollary 3.2.24. *The composition $\alpha_*\beta_*$ of the induced homomorphisms*

$$\beta_*\colon K(\mathcal{K}(\mathcal{H})) \to K(C_0(V, \mathit{Cliff}(V)))$$

and

$$\alpha_*\colon K(C_0(V, \mathit{Cliff}(V))) \to K(\mathcal{K}(\mathcal{H})),$$

is the identity homomorphism.

Proof. By Theorem 3.2.23, the composition of $\alpha_*\beta_*$ is equivalent to the asymptotic morphism γ_t, given by $\gamma_t(f) = f(t^{-1}B)$. Since f is in $\mathcal{S}$, each γ_t is a $*$-homomorphism and γ is homotopic to the $*$-homomorphism mapping f to $f(B)$. Now it suffices to define a homotopy between γ and the map $\theta\colon \mathcal{S} \to \mathcal{K}(\mathcal{H})$ defined by $\theta(f) = f(0)p$ where p is a rank 1 projection (by Corollary 3.2.15) onto the kernel of B.

Let $f = u$ or $f = v$ from Remark 2.1.17.

Consider an eigenvector w of $u(B)$ or $v(B)$ with non-zero eigenvalue, then

$$u(t^{-1}B)w \to 0 \text{ and } v(t^{-1}B)w \to 0$$

as $s \to 0$.

Now if $w \in \ker(B)$, then

$$u(t^{-1}B)w = w \text{ and } v(t^{-1}B)w = 0$$

for all s.

Finally, for an eigenvector w of $u(B)$ or $v(B)$ respectively

$$u(0)p(w) = \begin{cases} 0 & \text{if } w \notin \ker(B) \\ w & \text{if } w \in \ker(B), \end{cases}$$

and $v(0)p(w) = 0$.

By Proposition 3.2.14 and Corollary 3.2.15, $\mathcal{H}(V)$ has a basis consisting of eigenvectors of $u(B)$ and $v(B)$. Combining these, we have

$$||f(t^{-1}B) - f(0)p|| \to 0,$$

as $t \to 0$ when $f = u$ or $f = v$. Since u and v generate $\mathcal{S}$, it follows that the above holds for all $f \in \mathcal{S}$, and so we have a homotopy between γ and θ defined by

$$f \mapsto \begin{cases} f(s^{-1}B) & \text{if } s \in (0,1] \\ f(0)p & \text{if } s = 0, \end{cases}$$

where p is the projection onto the kernel of B. $\square$

Now we check that the composition $\beta_*\alpha_*$ is asymptotically equivalent to $\alpha_*\beta_*$ and consequently prove that we get the identity in this case too.

Definition 3.2.25. Define the *flip map*

$$l\colon A\widehat{\otimes}B \to B\widehat{\otimes}A,$$

by

$$l(a\widehat{\otimes}b) = (-1)^{\deg(a)\deg(b)} b\widehat{\otimes}a,$$

for all $a \in A$ and $b \in B$.

Notice that l is a $*$-isomorphism. For simplicity in the following statements let $\mathcal{C}(V) = C_0(V, \mathrm{Cliff}(V))$.

Lemma 3.2.26. *Let $\mathcal{K} = \mathcal{K}(\mathcal{H})$ for the following diagram. The diagram*

$$\begin{array}{ccccccc} \mathcal{S}\widehat{\otimes}\mathcal{C}(V) & \xrightarrow{\Delta\widehat{\otimes}id_{\mathcal{C}(V)}} & \mathcal{S}\widehat{\otimes}\mathcal{S}\widehat{\otimes}\mathcal{C}(V) & \overset{id_{\mathcal{S}}\widehat{\otimes}\alpha_t}{\dashrightarrow} & \mathcal{S}\widehat{\otimes}\mathcal{K} & \xrightarrow{\beta\widehat{\otimes}id_{\mathcal{K}}} & \mathcal{C}(V)\widehat{\otimes}\mathcal{K} \\ \downarrow{\scriptstyle \Delta\widehat{\otimes}id_{\mathcal{C}(V)}} & & & & & & \downarrow{\scriptstyle l} \\ \mathcal{S}\widehat{\otimes}\mathcal{S}\widehat{\otimes}\mathcal{C}(V) & \xrightarrow[id_{\mathcal{S}}\widehat{\otimes}\beta\widehat{\otimes}id_{\mathcal{C}(V)}]{} & \mathcal{S}\widehat{\otimes}\mathcal{C}(V)\widehat{\otimes}\mathcal{C}(V) & \xrightarrow[id_{\mathcal{S}}\widehat{\otimes}l]{} & \mathcal{S}\widehat{\otimes}\mathcal{C}(V)\widehat{\otimes}\mathcal{C}(V) & \underset{\alpha_t\widehat{\otimes}id_{\mathcal{C}(V)}}{\dashrightarrow} & \mathcal{K}\widehat{\otimes}\mathcal{C}(V), \end{array}$$

asymptotically commutes.

Proof. Since $\mathcal{S}$ is generated by the elements $u(x) = e^{-x^2}$ and $v(x) = xe^{-x^2}$ it suffices to check that the diagram asymptotically commutes for u and v in $\mathcal{S}$. Let

$$f_1 = l(\beta\widehat{\otimes}\mathrm{id}_{\mathcal{K}(\mathcal{H})})(\mathrm{id}_{\mathcal{S}}\widehat{\otimes}\alpha_t)(\Delta\widehat{\otimes}\mathrm{id}_{\mathcal{C}(V)})$$

and

$$g_1 = (\alpha_t\widehat{\otimes}\mathrm{id}_{\mathcal{C}(V)})(\mathrm{id}_{\mathcal{S}}\widehat{\otimes}l)(\mathrm{id}_{\mathcal{S}}\widehat{\otimes}\beta\widehat{\otimes}\mathrm{id}_{\mathcal{C}(V)})(\Delta\widehat{\otimes}\mathrm{id}_{\mathcal{C}(V)})$$

For u and $h \in \mathcal{C}(V)$,

$$\begin{aligned} f_1(u\widehat{\otimes}h) &= l(\beta\widehat{\otimes}\mathrm{id}_{\mathcal{K}(\mathcal{H})})(\mathrm{id}_{\mathcal{S}}\widehat{\otimes}\alpha_t)(u\widehat{\otimes}u\widehat{\otimes}h) \\ &= l(\beta\widehat{\otimes}\mathrm{id}_{\mathcal{K}(\mathcal{H})})(u\widehat{\otimes}u(t^{-1}D)M_{h_t}) \\ &= l(u(C)\widehat{\otimes}u(t^{-1}D)M_{h_t}) \\ &= u(t^{-1}D)M_{h_t}\widehat{\otimes}u(C), \end{aligned}$$

and

$$\begin{aligned} g_1(u\widehat{\otimes}h) &= (\alpha_t\widehat{\otimes}\mathrm{id}_{\mathcal{C}(V)})(\mathrm{id}_{\mathcal{S}}\widehat{\otimes}l)(\mathrm{id}_{\mathcal{S}}\widehat{\otimes}\beta\widehat{\otimes}\mathrm{id}_{\mathcal{C}(V)})(u\widehat{\otimes}u\widehat{\otimes}h) \\ &= (\alpha_t\widehat{\otimes}\mathrm{id}_{\mathcal{C}(V)})(\mathrm{id}_{\mathcal{S}}\widehat{\otimes}l)(u\widehat{\otimes}u(C)\widehat{\otimes}h) \\ &= (\alpha_t\widehat{\otimes}\mathrm{id}_{\mathcal{C}(V)})(u\widehat{\otimes}h\widehat{\otimes}u(C)) \\ &= u(t^{-1}D)M_{h_t}\widehat{\otimes}u(C). \end{aligned}$$

Hence we have an asymptotic equivalence in the case when we take u. Now consider v. We have,

$$\begin{aligned} f_1(v\widehat{\otimes}h) &= l(\beta\widehat{\otimes}\mathrm{id}_{\mathcal{K}(\mathcal{H})})(\mathrm{id}_{\mathcal{S}}\widehat{\otimes}\alpha_t)((u\widehat{\otimes}v+v\widehat{\otimes}u)\widehat{\otimes}h) \\ &\sim_{\mathrm{asy}} l(\beta\widehat{\otimes}\mathrm{id}_{\mathcal{K}(\mathcal{H})})(u\widehat{\otimes}v(t^{-1}D)M_{h_t}+v\widehat{\otimes}u(t^{-1}D)M_{h_t}) \\ &= l(u(C)\widehat{\otimes}v(t^{-1}D)M_{h_t}+v(C)\widehat{\otimes}u(t^{-1}D)M_{h_t}) \\ &= v(t^{-1}D)M_{h_t}\widehat{\otimes}u(C)+u(t^{-1}D)M_{h_t}\widehat{\otimes}v(C), \end{aligned}$$

and

$$\begin{aligned} g_1(v\widehat{\otimes}h) &= (\alpha_t\widehat{\otimes}\mathrm{id}_{\mathcal{C}(V)})(\mathrm{id}_{\mathcal{S}}\widehat{\otimes}l)(\mathrm{id}_{\mathcal{S}}\widehat{\otimes}\beta\widehat{\otimes}\mathrm{id}_{\mathcal{C}(V)})((u\widehat{\otimes}v+v\widehat{\otimes}u)\widehat{\otimes}h) \\ &= (\alpha_t\widehat{\otimes}\mathrm{id}_{\mathcal{C}(V)})(\mathrm{id}_{\mathcal{S}}\widehat{\otimes}l)(u\widehat{\otimes}v(C)\widehat{\otimes}h+v\widehat{\otimes}u(C)\widehat{\otimes}h) \\ &= (\alpha_t\widehat{\otimes}\mathrm{id}_{\mathcal{C}(V)})(u\widehat{\otimes}h\widehat{\otimes}v(C)+v\widehat{\otimes}h\widehat{\otimes}u(C)) \\ &\sim_{\mathrm{asy}} u(t^{-1}D)M_{h_t}\widehat{\otimes}v(C)+v(t^{-1}D)M_{h_t}\widehat{\otimes}u(C) \end{aligned}$$

and again we have the diagram asymptotically commuting. □

A proof of the following can be found in [HKT98], Lemma 18.

Lemma 3.2.27. *The flip map on $\mathcal{C}(V)\widehat{\otimes}\mathcal{C}(V)$ is homotopic by the graded $*$-homomorphisms to the map $h_1\widehat{\otimes}h_2 \mapsto h_1\widehat{\otimes}\iota h_2$, where ι is the automorphism on $\mathcal{C}(V)$ induced by the map $e \mapsto -e$ in V.*

$\square$

Corollary 3.2.28. *There is an isomorphism:*

$$K(\mathbb{R}) \to K(\mathcal{C}(V))$$

induced by $\beta\colon S \to \mathcal{C}(V)$.

Proof. It suffices to check that the f_1 in the proof of Lemma 3.2.26 induces an isomorphism in K-theory since by Corollary 3.2.24 we have that α_* is left inverse to β. It follows from the stability of K-theory that:

$$K(\mathcal{S}\widehat{\otimes}\mathcal{C}(V)) \xrightarrow{\alpha_*} K(\mathcal{K}(\mathcal{H})) \cong K(\mathbb{R}).$$

So we need to show:

$$\begin{array}{ccccccc}
\mathcal{S}\widehat{\otimes}\mathcal{C}(V) & \xrightarrow{\Delta\widehat{\otimes}\mathrm{id}_{\mathcal{C}(V)}} & \mathcal{S}\widehat{\otimes}\mathcal{S}\widehat{\otimes}\mathcal{C}(V) & \xrightarrow{\mathrm{id}_{\mathcal{S}}\widehat{\otimes}\alpha_t} & \mathcal{S}\widehat{\otimes}\mathcal{K}(\mathcal{H}) & \xrightarrow{\beta\widehat{\otimes}\mathrm{id}_{\mathcal{K}(\mathcal{H})}} & \mathcal{C}(V)\widehat{\otimes}\mathcal{K}(\mathcal{H}) \\
 & & & & & & \downarrow{\scriptstyle l} \\
 & & & & & & \mathcal{K}(\mathcal{H})\widehat{\otimes}\mathcal{C}(V)
\end{array}$$

induces an isomorphism on K-theory. By Lemma 3.2.26, we have an asymptotic equivalence which gives a homotopy equivalence. Further the composition in the diagram above is asymptotically equivalent to

$$g_1 = (\alpha_t\widehat{\otimes}\mathrm{id}_{\mathcal{C}(V)})(\mathrm{id}_{\mathcal{S}}\widehat{\otimes}l)(\mathrm{id}_{\mathcal{S}}\widehat{\otimes}\beta\widehat{\otimes}\mathrm{id}_{\mathcal{C}(V)})(\Delta\widehat{\otimes}\mathrm{id}_{\mathcal{C}(V)})$$

which is homotopic by Lemma 3.2.27 to the composition

$$(\alpha_t(\mathrm{id}_{\mathcal{S}}\widehat{\otimes}\beta)\Delta)\widehat{\otimes}\iota$$

and hence by Theorem 3.2.23 is homotopic to $\gamma\widehat{\otimes}\iota$, and so maps

$$f\widehat{\otimes}h \mapsto f(t^{-1}B)\widehat{\otimes}\iota(h).$$

and so induces an isomorphism in K-theory by the homotopy defined similarly to that in the proof of Corollary 3.2.24

$$f\widehat{\otimes}h \mapsto \begin{cases} f(s^{-1}B)\widehat{\otimes}\iota(h) & \text{if } s \in (0,1] \\ f(0)p\widehat{\otimes}\iota(h) & \text{if } s = 0. \end{cases}$$

$\square$

Then the proof of Theorem 3.2.2 is complete.

Observe that when $V = \mathbb{R}$ then

$$C_0(V, \mathrm{Cliff}(V)) = C_0(\mathbb{R}, \mathrm{Cliff}(\mathbb{R})) \cong C_0(\mathbb{R}) \widehat{\otimes} \mathrm{Cliff}(\mathbb{R}) \cong \Sigma \widehat{\otimes} \mathbb{R}_{1,0} \cong \Sigma \mathbb{R}_{1,0}.$$

Then we obtain

Corollary 3.2.29.

$$K(\mathbb{R}) \cong K(\Sigma \mathbb{R} \widehat{\otimes} \mathbb{R}_{1,0}).$$

□

Corollary 3.2.30. *For any real graded C^*-algebra,*

$$K(A) \cong K(\Sigma A \widehat{\otimes} \mathbb{R}_{1,0}).$$

□

Then combining this with Theorem 1.14 in [HG04], we have

Theorem 3.2.31. *For any graded C^*-algebra over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$,*

$$K_{\mathbb{F}}(A) \cong K_{\mathbb{F}}(\Sigma A \widehat{\otimes} \mathbb{F}_{1,0}),$$

where $K_{\mathbb{F}}$ denotes real or complex K-theory.

□

We finally obtain the isomorphism in E-theory.

Theorem 3.2.32. *For any graded C^*-algebra over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$,*

$$E_{\mathbb{F}}(A, B) \cong E_{\mathbb{F}}(A, \Sigma B \widehat{\otimes} \mathbb{F}_{1,0}),$$

and

$$E_{\mathbb{F}}(A, B) \cong E_{\mathbb{F}}(\Sigma A \widehat{\otimes} \mathbb{F}_{1,0}, B),$$

where $E_{\mathbb{F}}$ denotes real or complex E-theory.

Proof. The Bott map $\beta : \mathcal{S} \to \Sigma \mathbb{F}_{1,0}$ gives an invertible class, that we denote by $[\beta] \in E_{\mathbb{F}}(\mathbb{F}, \Sigma \mathbb{F} \widehat{\otimes} \mathbb{F}_{1,0})$. Also we have an invertible class

$$[\beta \widehat{\otimes} \mathrm{id}_A] \in E_{\mathbb{F}}(\mathbb{F} \widehat{\otimes} A, \Sigma \mathbb{F} \widehat{\otimes} \mathbb{F}_{1,0} \widehat{\otimes} A) = E_{\mathbb{F}}(A, \Sigma A \widehat{\otimes} \mathbb{F}_{1,0}).$$

Denote this class by $[\beta_A]$ and similarly set $[\beta_B]$ for the class $[\beta \widehat{\otimes} \mathrm{id}_B]$. Then these classes are invertible.

By the E-theory product we have a group homomorphism

$$f\colon E_{\mathbb{F}}(A,B) \to E_{\mathbb{F}}(A, \Sigma B \widehat{\otimes} \mathbb{F}_{1,0})$$

defined by

$$f(x) = x \times [\beta_B]$$

for all $x \in E_{\mathbb{F}}(A,B)$ which is invertible with inverse defined by

$$f^{-1}(y) = y \times [\beta_B]^{-1}$$

for all $y \in E_{\mathbb{F}}(A, \Sigma B \widehat{\otimes} \mathbb{F}_{1,0})$.

Similarly we define

$$g\colon E_{\mathbb{F}}(A,B) \to E_{\mathbb{F}}(\Sigma A \widehat{\otimes} \mathbb{F}_{1,0}, B)$$

by

$$g(x) = [\beta_A]^{-1} \times x$$

for all $x \in E_{\mathbb{F}}(A,B)$, and this has inverse defined by

$$g^{-1}(y) = [\beta_A] \times y,$$

for all $y \in E_{\mathbb{F}}(\Sigma A \widehat{\otimes} \mathbb{F}_{1,0}, B)$. Then the result follows.

□

Corollary 3.2.33. *Let A and B be real graded C^*-algebra. Then we have natural isomorphisms*

$$E_g^n(A,B) \cong E_g^{n+8}(A,B).$$

Proof.

$$\begin{aligned} E_g^n(A,B) &= E_g^n(A, \Sigma^8(B \widehat{\otimes} \mathbb{R}_{1,0} \widehat{\otimes} \mathbb{R}_{1,0} \widehat{\otimes} \mathbb{R}_{1,0} \widehat{\otimes} \mathbb{R}_{1,0} \widehat{\otimes} \mathbb{R}_{1,0} \widehat{\otimes} \mathbb{R}_{1,0} \widehat{\otimes} \mathbb{R}_{1,0} \widehat{\otimes} \mathbb{R}_{1,0})) \\ &= E_g^n(A, \Sigma^8(B \widehat{\otimes} \mathbb{R}_{4,4})) \text{ by Proposition 2.1.10,} \\ &= E_g^n(A, \Sigma^8 B) \text{ by Proposition 2.2.30,} \\ &\cong E_g^{n+8}(A,B) \end{aligned}$$

where the first equality follows from Corollary 2.2.19. □

Chapter 4

Quasi-topological spaces and quasi-orthogonal sequences

In this chapter we generalise the notion of orthogonal spectra to quasi-topological spaces. In order to do this, we have to define the notion of a quasi-continuous group action and further prove we have a symmetric monoidal structure on the category of quasi-orthogonal sequences which we also define.

4.1 Quasi-topological spaces

The suspension and loop space of a quasi-topological space X are defined similarly to the case of topological spaces by

$$\Sigma_{\text{top}} X = S^1 \wedge X,$$

and

$$\Omega X = \{\mu \colon S^1 \to X \mid \mu \text{ is quasi-continuous and basepoint preserving}\}$$

and we consider the circle S^1 with the quasi-topology that comes from the standard topology on $\mathbb{R}^2$. That is, our quasi-continuous maps are the continuous maps from every compact Hausdorff space to S^1 in that topology. Now we check that Σ_{top} and Ω are adjoints in the category where objects are quasi-topological spaces and arrows are quasi-continuous maps. In order to do this, we consider an abstract result and obtain it as a corollary.

Proposition 4.1.1. *Let X, Y and Z be quasi-topological spaces. Then*

$$F(X \wedge Y, Z) \text{ and } F(X, F(Y, Z)),$$

are quasi-homeomorphic.

Proof. We define $\alpha\colon F(X \wedge Y, Z) \to F(X, F(Y, Z))$ by

$$((\alpha(f))(x))(y) = f(x \wedge y),$$

where $f \in F(X \wedge Y, Z)$, $x \in X$ and $y \in Y$. Then α is quasi-continuous since f is quasi-continuous.

Define $\beta\colon F(X, F(Y, Z)) \to F(X \wedge Y, Z)$ by

$$(\beta(g))(x \wedge y) = (g(x))(y),$$

where $g \in F(X, F(Y, Z))$, $x \in X$ and $y \in Y$. So β is quasi-continous since g is quasi-continuous.

Finally α and β are inverses, so we obtain a quasi-homeomorphism. □

Corollary 4.1.2. *Σ_{top} and Ω are adjoints in the category of quasi-topological spaces. That is*

$$F(\Sigma_{top}X, Y) \text{ and } F(X, \Omega Y)$$

are quasi-homeomorphic.

Proof. This follows from Proposition 4.1.1 since

$$F(\Sigma_{\mathrm{top}}X, Y) = F(X \wedge S^1, Y),$$

and

$$F(X, \Omega Y) = F(X, F(S^1, Y)).$$

□

Let A, B be C^*-algebras. Further by work of Dardalat-Meyer [DM12] we can define a quasi topology on $\mathrm{Asy}(A, B)$, the set of asymptotic morphisms from A to B. We define the set of quasi-continuous maps from a compact Hausdorff space Y to $\mathrm{Asy}(A, B)$ to be the $\mathrm{Asy}(A, C(Y, B))$, mentioned in [DM12]. That is, more precisely we have

Definition 4.1.3. For a compact Hausdorff space Y, a map $h\colon Y \to \mathrm{Asy}(A, B)$ is quasi-continuous when for each $t \in [1, \infty)$ the map $\widetilde{h_t}(a)\colon Y \to B$ defined by

$$\widetilde{h_t}(a)(y) = h(y)_t(a),$$

is continuous.

Now we check that this is a quasi-topology.

Proposition 4.1.4. *The set of asymptotic morphisms from A to B, $Asy(A, B)$, is a quasi-topological space when equipped with the above quasi-topology.*

Proof. We must check the axioms. Let $c\colon Y \to \mathrm{Asy}(A, B)$ be constant. Then for $y \in Y$, $c(y) = f_t\colon A \dashrightarrow B$ for a fixed f. Then we need to show that c is quasi-continuous. That is to show for each map c, the map $\widetilde{c_t}(a)\colon Y \to B$ is continuous. Then we define this map for $a \in A, y \in Y$ by

$$\widetilde{c_t}(a)(y) = c(y)_t(a) = f_t(a),$$

which is continuous as a function of Y and hence c is continuous.

Now let $f\colon Y_1 \to Y_2$ be a map of compact Hausdorff spaces and let $g\colon Y_2 \to \mathrm{Asy}(A, B)$ be quasi-continuous. Then we want to show that $gf\colon Y_1 \to \mathrm{Asy}(A, B)$ is quasi-continuous. That is, we need to show that $\widetilde{gf}_t(a)\colon Y_1 \to B$ is continuous. Now f is continuous and since g is quasi-continuous, we have each $\widetilde{g_t}(a)\colon Y_2 \to B$ is continuous and

$$\widetilde{g_t}(a)(y) = g(y)_t(a).$$

Now,

$$\widetilde{gf}_t(a)(y) = \widetilde{g_t}(a)f(y),$$

which is continuous in Y so is $\widetilde{gf}_t(a)$ is continuous, yielding that gf is quasi-continuous.

Let $Y = Y_1 \amalg Y_2$ of compact Hausdorff spaces. Then we need to show that $g\colon Y \to \mathrm{Asy}(A, B)$ is quasi-continuous if and only if $g|_{Y_i}$ is quasi-continuous for $i = 1, 2$. Suppose $g\colon Y \to \mathrm{Asy}(A, B)$ is quasi-continuous. Then map $\widetilde{g_t}(a)\colon Y \to B$ defined by

$$\widetilde{g_t}(a)(y) = g(y)_t(a),$$

is continuous. Now by properties of continuous functions we know that the restriction of a continuous function is continuous. Suppose that $g|_{Y_i}$ is quasi-continuous, then $(\widetilde{g}|_{Y_i})_t(a)\colon Y_i \to B$ is continuous. Then by properties of continuous functions we know that if the restrictions are continuous then the map of the disjoint uniion will be continuous.

Finally, we need to check for every surjective map $f\colon Y_1 \to Y_2$ of compact Hausdorff spaces that $g\colon Y_2 \to \mathrm{Asy}(A,B)$ is quasi-continuous if $gf\colon Y_1 \to \mathrm{Asy}(A,B)$ is quasi-continuous. Then for a particular map f and by the above argument

$$\widetilde{gf}_t(a)(y) = \widetilde{g}_t(a)f(y),$$

is continuous. Let $f\colon Y_1 \to Y_2$ be the identity, then

$$\widetilde{gf}_t(a)(y) = \widetilde{g}_t(a)f(y) = \widetilde{g}_t(a),$$

and hence $\widetilde{g}_t(a)$ is continuous and the result follows. So we do indeed have a quasi-topology on $\mathrm{Asy}(A,B)$ defined as above. □

For a quasi-topological space X, let X_+ denote the space with a basepoint.

Proposition 4.1.5. *The quasi-topological spaces $\Omega Asy(A,B)$ and $Asy(A,\Sigma B)$ are quasi-homeomorphic.*

Proof. By the definition of a quasi-topology, we know that

$$\Omega\mathrm{Asy}(A,B) \colon = Q(S^1, \mathrm{Asy}(A,B))_+.$$

Then by the definition 4.1.3, we have that

$$Q(Y, \mathrm{Asy}(A,B))_+ = \mathrm{Asy}(A, C(Y,B)_+),$$

and then that

$$\begin{aligned}\Omega\mathrm{Asy}(A,B) &= Q(S^1, \mathrm{Asy}(A,B))_+ \\ &= \mathrm{Asy}(A, C(S^1,B)_+) \\ &= \mathrm{Asy}(A, \Sigma B).\end{aligned}$$

□

The above results hold in the case of graded asymptotic morphisms.

4.2 Group actions

The following definition makes sense since a topological group can be viewed as a quasi-topological group.

Definition 4.2.1. Let G be a topological group acting on a quasi-topological space X. Then the group action is called *quasi-continuous* if the map $G \times X \to X$ is quasi-continuous. If this is the case we say that the set X is a *quasi G-space.*

Definition 4.2.2. A map $f\colon X \to Y$ of quasi G-spaces is a *quasi G-map* if it is G-equivariant. That is, for all $g \in G$, we have

$$f(gx) = g(f(x)).$$

Now we consider basepoint preserving group actions.

Proposition 4.2.3. *Let G and H be groups. Let X be a quasi G-space, Y a quasi H-space where the group actions preserve the basepoints of both X and Y. Then there are basepoint preserving action of $G \times H$ on $X \times Y$, $X \vee Y$ and $X \wedge Y$.*

Proof. Assume the actions are left actions, as the proofs work the same for right actions.

We can define $(G \times H) \times (X \times Y) \to (X \times Y)$ by $(g, h)(x, y) \mapsto (gx, hy)$ then this is a quasi $G \times H$-action which preserves the basepoint since G and H are quasi actions and preserve the basepoint.

Now define $(G \times H) \times (X \vee Y) \to (X \vee Y)$ by $(g, h)(x \vee y) \mapsto (gx \vee hy)$ and the result follows.

Finally, define $(G \times H) \times (X \wedge Y) \to (X \wedge Y)$ by $(g, h)(x \wedge y) \mapsto (gx \wedge hy)$, and once again the result follows. □

We now need the notion of a balanced smash product.

Definition 4.2.4. Let X be a right quasi G-space and Y a left quasi G-space, then we can from the *balanced smash product* $X \wedge_G Y$, which is the quotient space $X \wedge Y/ \sim_G$ where

$$(xg \wedge y) \sim_G (x \wedge g^{-1}y) \Leftrightarrow (x \wedge y) \sim_G (xg \wedge g^{-1}y),$$

for all $g \in G$.

Let the equivalence class of $x \wedge y$ be denoted by $x \wedge_G y$. Now using these we can construct a left quasi G-space.

Let G be a topological group and H a subgroup. Then let X be a based left quasi H-space where G acts by preserving the basepoint. Let $G_+ = G \amalg \{*\}$, then we can construct the right quasi G-space denoted $G_+ \wedge_H X$ using the above equivalence classes. Additionally we can actually define a left quasi G-action on this space by the following map:

$$(f, g \wedge_H x) \mapsto fg \wedge_H x,$$

for all $f \in G$.

To prove this is well-defined action it is a formality of using the fact that H is a subgroup of G.

Let X and Y be based quasi G-spaces. Then let $Q_G(X, Y)$ denote the set of basepoint preserving quasi G-maps. Then we have the following result:

Proposition 4.2.5. *Let H be a subgroup of a group G. Let X be a left quasi H-space and Y a left quasi G-space. There there is a natural bijection*

$$Q_H(X, Y) \longleftrightarrow Q_G(G_+ \wedge_H X, Y).$$

Proof. We define $\alpha\colon Q_H(X, Y) \to Q_G(G_+ \wedge_H X, Y)$. Let $f \in Q_H(X, Y)$ and $g \wedge_H x \in G_+ \wedge_H X$, then we define

$$\alpha(f)(g \wedge_H x) = gf(x),$$

and we then need to check that α is well-defined and also G-equivariant. Since $(g \wedge_H x) \sim (gh \wedge_H h^{-1}x)$, then

$$\begin{aligned} \alpha(f)(gh \wedge_H h^{-1}x) &= ghf(h^{-1}x) = ghh^{-1}f(x) \qquad \text{since f is a quasi H-map} \\ &= gf(x) = \alpha(f)(g \wedge_H x). \end{aligned}$$

Then $\alpha(f)$ is G-equivariant since

$$\alpha(f)(g'g \wedge_H x) = g'gf(x) = g'(gf(x)) = g'\alpha(f)(g \wedge_H x),$$

for all $g' \in G$.

Now define $\beta\colon Q_G(G_+ \wedge_H X, Y) \to Q_H(X, Y)$. Let $k \in Q_G(G_+ \wedge_H X, Y)$, $x \in X$, then

$$\beta(k)(x) = k(e \wedge_H x),$$

where e denotes the identity in G. Then β is H-equivariant since

$$\begin{aligned}\beta(k)(hx) &= k(e \wedge_H hx)\\ &= k(h^{-1}h \wedge_H hx)\\ &= k(h \wedge_H x) \text{ by equivalence relations}\\ &= hk(e \wedge_H x) \text{ as } k \text{ is a } H\text{-map}\\ &= h\beta(k)x.\end{aligned}$$

Then is is clear that α and β are inverse maps, and both are natural in X and Y, so the result follows. $\square$

4.3 Quasi-Orthogonal sequences

Let $\mathscr{O}$ be the *category of finite dimensional real Euclidean inner product spaces and linear isometric isomorphisms* where we have objects to be the set

$$\text{obj}(\mathscr{O}) = \{\mathbb{R}^n \mid n = 0, 1, \ldots\}$$

and morphisms are

$$\mathscr{O}(A, B) = \begin{cases} O(n), & \text{if } A = B = \mathbb{R}^n \\ \emptyset, & \text{otherwise.} \end{cases}$$

It should be noted that this is a small category since the collection of objects is a set.

Let $\mathscr{T}$ denote the *category of quasi-topological spaces with basepoints and quasi-continuous maps*. So $\text{obj}(\mathscr{T})$ is the collection of quasi-topological spaces with basepoints and the morphisms $\mathscr{T}(X, Y)$ are the set of basepoint preserving quasi-continuous maps from X to Y.

Then we can obtain the product category $\mathscr{T} \times \mathscr{T}$ where $\text{obj}(\mathscr{T} \times \mathscr{T})$ are pairs (X, Y) of quasi-topological spaces with basepoints, and morphisms are

$$(\mathscr{T} \times \mathscr{T})((X, Y)(Z, W)) = \{(f, g) \mid f \in \mathscr{T}(X, Z), g \in \mathscr{T}(Y, W)\}.$$

Proposition 4.3.1. *The smash product* $\wedge\colon \mathscr{T} \times \mathscr{T} \to \mathscr{T}$ *of quasi-topological spaces is a functor.*

$\square$

The following definition of a quasi-orthogonal sequence is going to form part of the definition of a orthogonal quasi-spectrum.

Definition 4.3.2. Let $\mathscr{O}$ and $\mathscr{T}$ be the categories defined above. Then we define the *category of quasi orthogonal sequences* formed as the functor category $\mathscr{T}^{\mathscr{O}}$ with objects

$$\text{obj}(\mathscr{T}^{\mathscr{O}}) = \{\text{functors } X\colon \mathscr{O} \to \mathscr{T} \mid X_n := X(\mathbb{R}^n)\},$$

together with a left quasi-continuous basepoint preserving action of $O(n)$ on each X_n for all $n \geq 0$, and morphisms

$$\mathscr{T}^{\mathscr{O}}(X, Y) = \{\varphi\colon X \to Y \mid \varphi \text{ is a natural transformation}\},$$

and such that a natural transformation is formed of sets of quasi-continuous basepoint preserving maps $\varphi_n\colon X_n \to Y_n$ that are $O(n)$-equivariant for $n \geq 0$, or equivalently that the map φ_n commutes with the group action of $O(n)$ on X_n and Y_n.

A useful example of such a functor category will be the unit sequence coming from the orthogonal sequence defined below. Consider a based topological space K, then define the orthogonal sequence with n-space:

$$(G_pK)_n = \begin{cases} O(n)_+ \wedge K, & \text{if } n = p \\ \{*\}, & \text{otherwise} \end{cases}$$

Then the *unit sequence* is when we just have the topological space S^0, given by the sequence

$$G_0S^0 = \{S^0, *, *, \ldots\}.$$

We also consider quasi-biorthogonal sequences since they will help us in defining our smash product structure.

The category of *quasi-biorthogonal sequences* is defined to be the category with objects

$$\text{obj}(\mathscr{T}^{\mathscr{O}\times\mathscr{O}}) = \{X\colon \mathscr{O} \times \mathscr{O} \to \mathscr{T} \mid X \text{ is a functor } X_{m,n} := X(\mathbb{R}^m, \mathbb{R}^n)\},$$

together with a quasi-continuous basepoint preserving left-action of $O(m) \times O(n)$, and

$$\mathscr{T}^{\mathscr{O}\times\mathscr{O}}(X, Y) = \{\psi\colon X \to Y \mid \psi \text{ is a natural transformation}\},$$

formed of sets of quasi-continuous basepoint preserving maps $\psi_{m,n}\colon X_{m,n} \to Y_{m,n}$ that are $O(m) \times O(n)$-equivariant for all $m, n \geq 0$.

Using this we can define the external smash product of two quasi-orthogonal sequence X and Y.

Definition 4.3.3. Define the *external smash product* $X \overline{\wedge} Y$ to be the quasi-biorthogonal sequence given by the composition

$$\mathscr{O} \times \mathscr{O} \xrightarrow{X \times Y} \mathscr{T} \times \mathscr{T} \xrightarrow{\wedge} \mathscr{T},$$

defined by

$$(X \overline{\wedge} Y)_{m,n} = (X \overline{\wedge} Y)(\mathbb{R}^m, \mathbb{R}^n) = X(\mathbb{R}^m) \wedge Y(\mathbb{R}^n) = X_m \wedge Y_n.$$

Then by Proposition 4.2.3, the quasi-topological space $X_n \wedge Y_m$ has a quasi-$O(n) \times O(m)$-action.

For a general quasi-orthogonal sequence X we can define a quasi-biorthogonal sequence $X \circ \oplus$ by:

$$(X \circ \oplus)_{m,n} = (X \circ \oplus)(\mathbb{R}^m, \mathbb{R}^n) = X(\mathbb{R}^{m+n}) = X_{m+n}.$$

Now we can construct the tensor product of quasi-orthogonal sequences since the category $\mathscr{T}$ is complete and cocomplete.

Definition 4.3.4. For quasi orthogonal sequence X and Y we define the tensor product of X and Y to be the quasi-orthogonal sequence

$$(X \otimes Y)_n = \bigvee_{p+q=n} O(n)_+ \wedge_{O(p) \times O(q)} X_p \wedge Y_q,$$

where we define the $O(n)$-action on $(X \otimes Y)_n$ by acting on each wedge summand.

Then we can combine the external smash product and tensor product of quasi-orthogonal sequences as a natural bijection:

Proposition 4.3.5. *For quasi-orthogonal sequences X, Y and Z, there is a natural bijection*

$$\mathscr{T}^{\mathscr{O} \times \mathscr{O}}(X \overline{\wedge} Y, Z \circ \oplus) \longleftrightarrow \mathscr{T}^{\mathscr{O}}(X \otimes Y, Z).$$

Proof. Let $f \colon X \overline{\wedge} Y \to Z \circ \oplus$ be a natural transformation in the category of quasi-biorthogonal sequences. Then $f_{p,q} \colon X_p \wedge Y_q \to Z \circ \oplus$ is quasi $O(p) \times O(q)$-equivariant and then by proposition 4.2.5, this corresponds to a quasi $O(n)$-equivariant map, with $n = p + q$

$$\overline{f}_{p,q} \colon O(n)_+ \wedge_{O(p) \times O(q)} X_p \wedge Y_q \to Z_n.$$

Now fixing n and letting p and q vary, this allows us to obtain a quasi $O(n)$-equivariant map

$$\overline{f}_n = \bigvee_{p+q=n} O(n)_+ \wedge_{O(p)\times O(q)} X_p \wedge Y_q \to Z_n,$$

which is a quasi-continuous basepoint preserving $O(n)$-equivariant map in $\mathscr{T}^{\mathscr{O}}$ from $X \otimes Y$ to Z.

Now we construct a map the other way. Let $g \in \mathscr{T}^{\mathscr{O}}(X \otimes Y, Z)$. Then g is a wedge summand of basepoint preserving quasi-continuous $O(n)$-equivariant maps

$$g_n \colon \bigvee_{p+q=n} O(n)_+ \wedge_{O(p)\times O(q)} X_p \wedge Y_q \to Z_n,$$

for all $n \geq 0$. Also, we can write that $g_n = \bigvee_{p+q=n} g_{p,q}$, where

$$g_{p,q} \colon O(n)_+ \wedge_{O(p)\times O(q)} X_p \wedge Y_q \to Z_n,$$

and by proposition 4.2.5, we obtain a basepoint preserving quasi-continuous $O(p) \times O(q)$-equivariant map as required. □

Proposition 4.3.6. *The tensor product is associative, that is for quasi-orthogonal sequence X, Y and Z,*

$$X \otimes (Y \otimes Z) \cong (X \otimes Y) \otimes Z.$$

Proof.

$$\begin{aligned}
&(X \otimes (Y \otimes Z))_n \\
&\quad = \bigvee_{p+q=n} O(n)_+ \wedge_{O(p)\times O(q)} X_p \wedge (Y \otimes Z)_q \\
&\quad = \bigvee_{p+q=n} O(n)_+ \wedge_{O(p)\times O(q)} X_p \wedge \left(\bigvee_{q=r+s} O(q)_+ \wedge_{O(r)\times O(s)} Y_r \wedge Z_s \right) \\
&\quad = \bigvee_{p+r+s=n} O(n)_+ \wedge_{O(p)\times O(r+s)} X_p \wedge \big(O(r+s)_+ \wedge_{O(r)\times O(s)} Y_r \wedge Z_s \big)
\end{aligned}$$

and

$$
\begin{aligned}
&((X \otimes Y) \otimes Z)_n \\
&\quad = \bigvee_{q+s=n} O(n)_+ \wedge_{O(q)\times O(s)} (X \otimes Y)_q \wedge Z_s \\
&\quad = \bigvee_{q+s=n} O(n)_+ \wedge_{O(q)\times O(s)} \left(\bigvee_{p+r=q} O(q)_+ \wedge_{O(p)\times O(r)} X_p \wedge Y_r \right) \wedge Z_s \\
&\quad = \bigvee_{p+r+s=n} O(n)_+ \wedge_{O(p+r)\times O(s)} \big(O(p+r)_+ \wedge_{O(p)\times O(r)} X_p \wedge Y_r\big) \wedge Z_s.
\end{aligned}
$$

Then we want to show that the map

$$\alpha_{X,Y,Z} \colon X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z,$$

is an isomorphism. To do this we define two maps and prove they are inverses as follows:

$$
\begin{array}{c}
O(n)_+ \wedge_{O(p)\times O(r+s)} X_p \wedge \big(O(r+s)_+ \wedge_{O(r)\times O(s)} Y_r \wedge Z_s\big) \\
\Big\downarrow f \\
O(n)_+ \wedge_{O(p+r)\times O(s)} \big(O(p+r)_+ \wedge_{O(p)\times O(r)} X_p \wedge Y_r\big) \wedge Z_s
\end{array}
$$

by

$$
\begin{array}{c}
A \wedge_{O(p)\times O(r+s)} x \wedge \big(B \wedge_{O(r)\times O(s)} y \wedge z\big) \\
\Big\downarrow f \\
A\begin{pmatrix} I_p & 0 \\ 0 & B \end{pmatrix} \wedge_{O(p+r)\times O(s)} \big(I_{p+r} \wedge_{O(p)\times O(r)} x \wedge y\big) \wedge z
\end{array}
$$

where $A \in O(n)$, $B \in O(r+s)$, $x \in X_p$, $y \in Y_r$, $z \in Z_s$ and I_k denotes the $k \times k$ identity matrix. Then define

$$
\begin{array}{c}
O(n)_+ \wedge_{O(p+r)\times O(s)} \big(O(p+r)_+ \wedge_{O(p)\times O(r)} X_p \wedge Y_r\big) \wedge Z_s \\
\Big\downarrow g \\
O(n)_+ \wedge_{O(p)\times O(r+s)} X_p \wedge \big(O(r+s)_+ \wedge_{O(r)\times O(s)} Y_r \wedge Z_s\big)
\end{array}
$$

by

$$A \wedge_{O(p+r)\times O(s)} \big(C \wedge_{O(p)\times O(r)} x \wedge y\big) \wedge z$$

$$\Big\downarrow g$$

$$A\begin{pmatrix} C & 0 \\ 0 & I_s \end{pmatrix} \wedge_{O(p)\times O(r+s)} x \wedge \big(I_{r+s} \wedge_{O(r)\times O(s)} y \wedge z\big)$$

where $A \in O(n)$, $C \in O(p+r)$, $x \in X_p$, $y \in Y_r$, $z \in Z_s$. Since the smash product is a functor both f and g are quasi-continuous. Now we check that both f and g are well-defined. By the equivalence relations we have

$$A \wedge_{O(p)\times O(r+s)} x \wedge \big(B \wedge_{O(r)\times O(s)} y \wedge z\big)$$

$$\sim A\begin{pmatrix} I_p & 0 \\ 0 & B \end{pmatrix} \wedge_{O(p)\times O(r+s)} x \wedge \big(I_{r+s} \wedge_{O(r)\times O(s)} y \wedge z\big)$$

and then

$$\begin{aligned} & f\left(A\begin{pmatrix} I_p & 0 \\ 0 & B \end{pmatrix} \wedge_{O(p)\times O(r+s)} x \wedge \big(I_{r+s} \wedge_{O(r)\times O(s)} y \wedge z\big)\right) \\ = \;& A\begin{pmatrix} I_p & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} I_p & 0 \\ 0 & I_{r+s} \end{pmatrix} \wedge_{O(p+r)\times O(s)} \big(I_{p+r} \wedge_{O(p)\times O(r)} x \wedge y\big) \wedge z \\ = \;& A\begin{pmatrix} I_p & 0 \\ 0 & B \end{pmatrix} \wedge_{O(p+r)\times O(s)} \big(I_{p+r} \wedge_{O(p)\times O(r)} x \wedge y\big) \wedge z \\ = \;& f\big(A \wedge_{O(p)\times O(r+s)} x \wedge \big(B \wedge_{O(r)\times O(s)} y \wedge z\big)\big) \end{aligned}$$

and you can prove that g is well-defined similarly. Using this, it is clear that f and g are inverses. For completeness, we show that gf is the identity.

$$\begin{aligned} & gf(A \wedge_{O(p)\times O(r+s)} x \wedge \big(B \wedge_{O(r)\times O(s)} y \wedge z\big) \\ = \;& g\left(A\begin{pmatrix} I_p & 0 \\ 0 & B \end{pmatrix} \wedge_{O(p+r)\times O(s)} \big(I_{p+r} \wedge_{O(p)\times O(r)} x \wedge y\big) \wedge z\right) \\ = \;& A\begin{pmatrix} I_p & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} I_{p+r} & 0 \\ 0 & I_s \end{pmatrix} \wedge_{O(p)\times O(r+s)} x \wedge \big(I_{r+s} \wedge_{O(r)\times O(s)} y \wedge z\big) \\ = \;& A\begin{pmatrix} I_p & 0 \\ 0 & B \end{pmatrix} \wedge_{O(p)\times O(r+s)} x \wedge \big(I_{r+s} \wedge_{O(r)\times O(s)} y \wedge z\big) \\ \sim \;& A \wedge_{O(p)\times O(r+s)} x \wedge \big(B \wedge_{O(r)\times O(s)} y \wedge z\big) \end{aligned}$$

and so we have the identity in this case.

Now we need to extend our definitions of f and g to the wedge summand. As f and g hold for all $p+r+s=n$ we define

$$\begin{array}{c} \bigvee_{p+r+s=n} O(n)_+ \wedge_{p\times(r+s)} X_p \wedge \big(O(r+s)_+ \wedge_{O(r)\times O(s)} Y_r \wedge Z_s\big) \\ F_n \Big\downarrow \Big\uparrow G_n \\ \bigvee_{p+r+s=n} O(n)_+ \wedge_{(p+r)\times s} \big(O(p+r)_+ \wedge_{O(p)\times O(r)} X_p \wedge Y_r\big) \wedge Z_s \end{array}$$

(where $O(p)\times O(r+s)$ is denoted by $p\times(r+s)$ and $O(p+r)\times O(s)$ is denoted by $(p+r)\times s)$ by

$$F_n = \bigvee_{p+r+s=n} f_n \quad \text{and} \quad G_n = \bigvee_{p+r+s=n} g_n,$$

and then we have that the map $\alpha_{X,Y,Z}$ is an quasi-isomorphism, since our maps F_n and G_n are $O(n)$-equivariant, quasi-continuous, basepoint preserving and inverses. □

Lemma 4.3.7. *Let W, X, Y and Z be quasi-orthogonal sequences. Then the following diagram commutes:*

$$\begin{array}{ccccc} & & (W\otimes X)\otimes(Y\otimes Z) & & \\ & \nearrow^{\alpha_{W,X,Y\otimes Z}} & & \searrow^{\alpha_{W\otimes X,Y,Z}} & \\ W\otimes(X\otimes(Y\otimes Z)) & & & & ((W\otimes X)\otimes Y)\otimes Z \\ \downarrow^{id_W\otimes\alpha_{X,Y,Z}} & & & & \uparrow^{\alpha_{W,X,Y}\otimes id_Z} \\ W\otimes((X\otimes Y)\otimes Z) & & \xrightarrow{\alpha_{W,X\otimes Y,Z}} & & (W\otimes(X\otimes Y))\otimes Z \end{array}$$

□

Let

$$P_{m,n} = \begin{pmatrix} 0 & I_n \\ I_m & 0 \end{pmatrix}$$

be the matrix that when applied to a vector in $\mathbb{R}^{m+n}$ swaps the first m-entries with the last n-entries. Then note that $P_{m,n}$ is a matrix in $O(m+n)$ since $P_{m,n}^T P_{m,n} = P_{m,n} P_{m,n}^T = I_{m+n}$. Also notice that $P_{m,n}^{-1} = P_{n,m}$.

Additionally define conjugation by $P_{m,n}$ on a real $(n+m)\times(n+m)$-matrix by

$$\mathrm{conj}_{m,n}(A) = P_{m,n}AP_{m,n}^{-1} = P_{m,n}AP_{n,m}.$$

Then it is easy to see that $\mathrm{conj}_{m,n}(AB) = \mathrm{conj}_{m,n}(A)\mathrm{conj}_{m,n}(B)$.

Proposition 4.3.8. *Let X and Y be quasi-orthogonal sequences. Then we have a quasi-isomorphism*

$$L_{X,Y}\colon X\otimes Y \to Y\otimes X,$$

where $L_{X,Y}$ is the set of basepoint preserving quasi-homeomorphisms which are $O(n)$-equivariant, $l_n = \bigvee_{p+q=n}\iota_{p,q}$ with

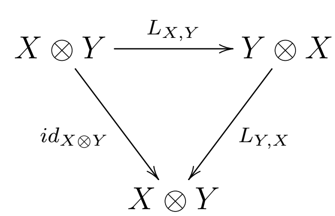
$$\iota_{p,q}\colon O(n)_+ \wedge_{O(p)\times O(q)} X_p \wedge Y_q \to O(n)_+ \wedge_{O(q)\times O(p)} Y_q \wedge X_p,$$

defined by

$$\iota_{p,q}(A \wedge_{O(p)\times O(q)} x\wedge y) = conj_{p,q}(A) \wedge_{O(q)\times O(p)} y \wedge x,$$

for all $A\in O(n)$, $x\in X_p$ and $y \in Y_q$.

Furthermore $L_{X,Y}$ satisfies the following commutative diagram



Proof. We first need to show that each $\iota_{p,q}$ is well-defined and has an inverse. For $A\in O(n)$, $B\in O(p)$, $C\in O(q)$, $x\in X_p$ and $y\in Y_q$ we have

$$A\wedge_{O(p)\times O(q)} x\wedge y \sim A\begin{pmatrix} B & 0\\ 0 & C\end{pmatrix} \wedge_{O(p)\times O(q)} Bx\wedge Cy,$$

and since

$$
\begin{aligned}
&\iota_{p,q}\left(A\begin{pmatrix}B & 0\\ 0 & C\end{pmatrix}\wedge_{O(p)\times O(q)} Bx\wedge Cy\right)\\
=\ &\mathrm{conj}_{p,q}\left(A\begin{pmatrix}B & 0\\ 0 & C\end{pmatrix}\right)\wedge_{O(q)\times O(p)} Cy\wedge Bx\\
=\ &\mathrm{conj}_{p,q}(A)\mathrm{conj}_{p,q}\left(\begin{pmatrix}B & 0\\ 0 & C\end{pmatrix}\right)\wedge_{O(q)\times O(p)} Cy\wedge Bx\\
=\ &\mathrm{conj}_{p,q}\left(A\begin{pmatrix}C & 0\\ 0 & B\end{pmatrix}\right)\wedge_{O(q)\times O(p)} Cy\wedge Bx\\
\sim\ &\mathrm{conj}_{p,q}(A)\wedge_{O(q)\times O(p)} y\wedge x\\
=\ &\iota_{p,q}\left(A\wedge_{O(p)\times O(q)} x\wedge y\right)
\end{aligned}
$$

$\iota_{p,q}$ is well-defined. Now $\iota_{p,q}^{-1}=\iota_{q,p}$ and these are clearly inverses. They are also quasi-continuous since the smash product is a functor. Now for $n=p+q$ we let $l_n=\bigvee_{p+q=n}\iota_{p,q}$ and then by taking the set of all basepoint preserving $O(n)$-equivariant quasi-homeomorphisms l_n we obtain the quasi-isomorphism

$$L_{X,Y}\colon X\otimes Y\to Y\otimes X.$$

Now the diagram commutes since for all fixed $n=p+q$ and A,x and y as above:

$$
\begin{aligned}
\iota_{q,p}\iota_{p,q}(A\wedge_{O(p)\times O(q)} x\wedge y) &= \iota_{q,p}(\mathrm{conj}_{p,q}(A)\wedge_{O(q)\times O(p)} y\wedge x)\\
&= \mathrm{conj}_{q,p}(\mathrm{conj}_{p,q}(A))\wedge_{O(p)\times O(q)} x\wedge y\\
&= A\wedge_{O(p)\times O(q)} x\wedge y = \mathrm{id}_{X\otimes Y}
\end{aligned}
$$

□

Proposition 4.3.9. *The following diagrams commute for all quasi-orthogonal* *sequences X, Y and Z.*

$$\begin{array}{ccc}
(X \otimes Y) \otimes Z & \xrightarrow{L_{X \otimes Y, Z}} & Z \otimes (X \otimes Y) \\
\downarrow{\scriptstyle \alpha^{-1}_{X,Y,Z}} & & \downarrow{\scriptstyle \alpha_{Z,X,Y}} \\
X \otimes (Y \otimes Z) & & (Z \otimes X) \otimes Y \\
\downarrow{\scriptstyle id_X \otimes L_{Y,Z}} & & \downarrow{\scriptstyle L_{Z,X} \otimes id_Y} \\
X \otimes (Z \otimes Y) & \xrightarrow{\alpha_{X,Z,Y}} & (X \otimes Z) \otimes Y
\end{array} ,$$

and

$$\begin{array}{ccc}
X \otimes (Y \otimes Z) & \xrightarrow{L_{X, Y \otimes Z}} & (Y \otimes Z) \otimes X \\
\downarrow{\scriptstyle \alpha_{X,Y,Z}} & & \downarrow{\scriptstyle \alpha^{-1}_{Y,Z,X}} \\
(X \otimes Y) \otimes Z & & Y \otimes (Z \otimes X) \\
\downarrow{\scriptstyle L_{X,Y} \otimes id_Z} & & \downarrow{\scriptstyle id_Y \otimes L_{Z,X}} \\
(Y \otimes X) \otimes Z & \xrightarrow{\alpha^{-1}_{Y,X,Z}} & Y \otimes (X \otimes Z)
\end{array} ,$$

$\square$

Now recall that G_0S^0 is the unit quasi-orthogonal sequence where

$$(G_0S^0)_n = (S^0, *, *, \ldots).$$

For a quasi-orthogonal sequence X, we want to define maps

$$G_0S^0 \otimes X \to X \text{ and } X \otimes G_0S^0 \to X,$$

such that these are quasi-isomorphisms.

Let $n = p + q$. Then for $A \in O(n), x \in X_p$ and $w \in (G_0S^0)_q$,we have an element $A \wedge_{O(p) \times O(q)} x \wedge w$ in $X \otimes G_0S^0$, we define $r_{p,q} \colon (X \otimes G_0S^0)_n \to X_n$ by

$$r_{p,q}(A \wedge_{O(p) \times O(q)} x \wedge w) = \begin{cases} x & \text{if } q = 0 \\ x_n & \text{otherwise} \end{cases}$$

where x_n is the basepoint of the quasi-topological space X_n.

Now we can also define $g_{p,q}\colon (G_0S^0 \otimes X)_n \to X_n$ to be $r_{p,q} \circ \iota_{q,p}$ Then

$$\begin{aligned} r_{p,q} \circ \iota_{q,p}(A \wedge_{O(q)\times O(p)} w \wedge x) & \\ &= r_{p,q}(\mathrm{conj}_{q,p}(A) \wedge_{O(p)\times O(q)} x \wedge w) \\ &= \begin{cases} x & \text{if } q = 0 \\ x_n & \text{otherwise} \end{cases} \end{aligned}$$

Now we can formulate the maps we require. We know that

$$O(n)_+ \wedge_{O(n)\times O(0)} X_n \wedge S^0 \cong O(n)_+ \wedge_{O(n)} X_n \to X_n,$$

we obtain a map

$$r_{p,0}\colon (X \otimes G_0S^0)_n \to X_n,$$

and similarly a map

$$l_{0,q}\colon (G_0S^0 \times X)_n \to X_n,$$

which are both quasi-homeomorphisms. Then we can vary p and q to obtain

$$\rho_X\colon X \otimes G_0S^0 \to X,$$

and

$$\gamma_X\colon G_0S^0 \times X \to X,$$

where we see that $\gamma_X = \rho_X \circ L_{G_0S^0,X}$.

Proposition 4.3.10. *The maps ρ and γ are the right unitor and the left unitor respectively in the category of quasi-orthogonal sequences wih natural transformations. That is, ρ and γ are natural isomorphisms for all quasi-orthogonal sequences X, and also for all quasi-orthogonal sequences the following diagram commutes*

$$\begin{array}{ccc} X \otimes (G_0S^0 \otimes Y) & \xrightarrow{\alpha_{X,G_0S^0,Y}} & (X \otimes G_0S^0) \otimes Y \\ & {\scriptstyle id_X \otimes \gamma_{G_0S^0 \otimes Y}} \searrow \quad \swarrow {\scriptstyle \rho_{X\otimes G_0S^0} \otimes id_Y} & \\ & X \otimes Y & \end{array}$$

□

Proposition 4.3.11. *The left and right unitor maps are compatible, in the sense that for all quasi-orthogonal sequence X the following diagram commutes*

$$\begin{array}{ccc} X \otimes G_0 S^0 & \xrightarrow{L_{X,G_0S^0}} & G_0 S^0 \otimes X \\ & {\scriptstyle \rho_X} \searrow \quad \swarrow {\scriptstyle \gamma_X} & \\ & X & \end{array}$$

□

Proposition 4.3.12. *The category of quasi-orthogonal sequences forms a symmetric monoidal category $(\mathscr{T}^{\mathscr{O}}, \otimes, G_0S^0)$ with associator $\alpha_{X,Y,Z}$, left unitor γ_X, right unitor ρ_X and braiding $L_{X,Y}$.*

The proof of this follows from Proposition 4.3.6, Lemma 4.3.7, Proposition 4.3.8, Proposition 4.3.9, Proposition 4.3.10 and Proposition 4.3.11.

Let $S = (S^0, S^1, S^2 \ldots)$ be the orthogonal sequence defined in terms of quasi-topological spaces.

Proposition 4.3.13. *The orthogonal sequence of quasi-topological spaces $S = (S^0, S^1, S^2 \ldots)$ is a commutative monoid in the symmetric monoidal category $(\mathscr{T}^{\mathscr{O}}, \otimes, G_0S^0)$.*

Proof. The map $s_{p,q}\colon S^p \wedge S^q \to S^{p+q}$ is associative since the diagram

$$\begin{array}{ccc} S^p \wedge S^q \wedge S^r & \xrightarrow{\mathrm{id}_{S^p} \wedge s_{q,r}} & S^p \wedge S^{q+r} \\ {\scriptstyle s_{p,q} \wedge \mathrm{id}_{S^r}} \downarrow & & \downarrow {\scriptstyle s_{p,q+r}} \\ S^{p+q} \wedge S^r & \xrightarrow{s_{p+q,r}} & S^{p+q+r} \end{array}$$

commutes. By Proposition 4.3.5, $s_{p,q}\colon S^p \wedge S^q \to S^{p+q}$ is equivalent to a map

$$\mu\colon S \otimes S \to S,$$

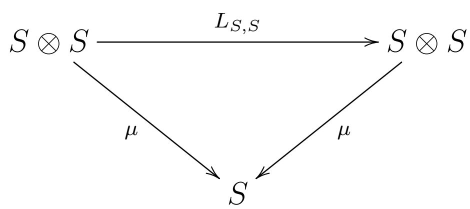
and so is associative since $s_{p,q}$ is associative.

Now we have a map $\eta_p\colon (G_0S^0)_p \to S^p$ defined to be the basepoint preserving map such that $G_0S^0 \to S^0$ is the unit quasi-homeomorphism. Then diagram (2) from Definition 1.7.2 commutes. It follows that we obtain a morphism

$$G_0S^0 \to S$$

and the diagram also commutes in this case.

Now we check that S is commutative. That is, the diagram



Let $A \wedge_{O(p)\times O(q)} x \wedge y$ be an element in $O(p+q) \wedge_{O(p)\times O(q)} S^p \wedge S^q$ then by definitions from the previous chapter yields

$$\begin{aligned} \mu(l_{S,S}(A \wedge_{O(p)\times O(q)} x \wedge y)) &= \mu(\mathrm{conj}_{p,q}(A)(y \wedge x)) \\ &= A(x \wedge y) \\ &= \mu(A \wedge_{O(p)\times O(q)} x \wedge y). \end{aligned}$$

□

Chapter 5

E-theory orthogonal quasi-spectra

This chapter brings together ideas from the previous chapter, since we will define the notion of an orthogonal quasi-spectrum which is a quasi-orthogonal sequence with added structure. We will show that we have an orthogonal quasi-spectrum representing the graded E-theory groups and thereafter show we have a smash product.

5.1 Quasi-Spectra

We begin by defining concepts we have seen before in terms of quasi-topological spaces.

A *quasi-spectrum* is a sequence of based quasi-topological spaces $X_0, X_1, \ldots$ with structure maps $\epsilon\colon X_m \to \Omega X_{m+1}$ that are quasi-continuous. An Ω-*quasi-spectrum* is a quasi-spectrum where for all natural numbers m the structure maps $\epsilon\colon X_m \to \Omega X_{m+1}$ are weak equivalences. Then we can define an orthogonal quasi-spectrum:

Definition 5.1.1. An *orthogonal quasi-spectrum* is

- a sequence of based quasi-topological spaces $X_0, X_1, \ldots$
- a basepoint preserving quasi-continuous left action of $O(m)$ on each X_m for all m, and

- a collection of based structure maps $\sigma = \sigma_m \colon X_m \wedge S^1 \to X_{m+1}$ that are quasi-continuous,

such that for each $m, n \geq 0$, the iterated map

$$\sigma_m^n \colon X_m \wedge S^n \to X_{m+1} \wedge S^{n-1} \to \ldots \to X_{m+n},$$

is quasi-continuous and $O(m) \times O(n)$-equivariant.

In the same manner, we have that a morphism of orthogonal quasi-spectrum $f \colon X \to Y$ is a collection of quasi-$O(m)$-equivariant maps $f_m \colon X_m \to Y_m$ for all m, which satisfy the following commutative diagram:

$$\begin{array}{ccc} X_m \wedge S^1 & \xrightarrow{f \wedge \mathrm{id}_{S^1}} & Y_m \wedge S^1 \\ \downarrow{\scriptstyle \sigma_m} & & \downarrow{\scriptstyle \sigma_m} \\ X_{m+1} & \xrightarrow{f_{m+1}} & Y_{m+1}, \end{array}$$

or alternatively that the following diagram commutes:

$$\begin{array}{ccc} X_m & \xrightarrow{f_m} & Y_m \\ \downarrow{\scriptstyle \epsilon_m} & & \downarrow{\scriptstyle \epsilon_m} \\ \Omega X_{m+1} & \xrightarrow{\Omega f_{m+1}} & \Omega Y_{m+1}. \end{array}$$

It is easily seen that any orthogonal spectrum is an orthogonal quasi-spectrum. By Corollary 4.1.2, the structure maps in the definition of quasi-spectrum can be defined in terms of loop spaces. Notice that an orthogonal quasi-spectrum is a quasi-orthogonal sequence with more structure.

Proposition 5.1.2. *The category of right S-modules, mod-S is naturally equivalent to the category of orthogonal quasi-spectrum.*

Proof. Consider the multiplication map $\nu \colon X \otimes S \to S$ for a right S-module X. Then by Proposition 4.3.5 we have a set of $O(m) \times O(n)$-equivariant maps

$$\nu_m^n \colon X_m \wedge S^n \to X_{m+n},$$

for $m, n \geq 0$ with unit quasi-homeomorphism ν_m^0. Now this action is associative so it follows that the structure maps are then defined by ν_m.

Conversely, consider the set of structure maps

$$\sigma_p^n \colon X_p \wedge S^p \to X_{n+p},$$

for a spectrum X and $p, n \geq 0$, with unit quasi-homeomorphism σ_p^0. Then we have a multiplicative map $\nu \colon X \otimes S \to X$ defining a right S-module. Since these constructions are inverses, we have a natural equivalence of these two categories. □

Hence we can obtain a tensor product of orthogonal quasi-spectrum since we have a tensor product in the category of right S-modules.

Definition 5.1.3. Let $\mathbb{X}$ be an orthogonal quasi-spectrum with spaces X_n. For each integer $k \in \mathbb{Z}$ we define the *k-th stable homotopy group* $\pi_k(\mathbb{X})$ to be the direct limit

$$\pi_k(\mathbb{X}) = \varinjlim_n \pi_{k+n} X_n,$$

under the maps $\epsilon_* \colon \pi_{k+n} X_n \to \pi_{k+n+1} X_{n+1}$ induced from the structure maps $\epsilon \colon \Omega^{k+n} X_n \to \Omega^{k+n+1} X_{n+1}$.

5.2 Graded E-theory Spectra

Let $\mathrm{Asy}_g(A, B)$ denote the set of graded asymptotic morphisms from A to B with the quasi-topology as defined in Definition 4.1.3.

Proposition 5.2.1. *The map of quasi-topological spaces*

$$f \colon Asy_g(A, B) \to Asy_g(D \widehat{\otimes} A, D \widehat{\otimes} B)$$

defined by

$$f(x_t) = id_D \widehat{\otimes} x_t,$$

is quasi-continuous for all $x_t \in Asy_g(A, B)$.

Proof. Since gradings follow immediately, we consider ungraded asymptotic morphisms throughout the proof. To prove a map of quasi-topological spaces is quasi-continuous, wwe need to check that for a quasi-continuous map $g : Y \to \mathrm{Asy}(A, B)$ where Y is a compact Hausdorff space, that the composition $fg \colon Y \to \mathrm{Asy}(D \widehat{\otimes} A, D \widehat{\otimes} B)$ is quasi-continuous. Suppose $g : Y \to \mathrm{Asy}(A, B)$

where Y is a compact Hausdorff space is quasi-continuous. Then by definition 4.1.3 we know that g is quasi-continuous when for each $t \in [1, \infty)$ the map $\widetilde{g}_t(a) \colon Y \to B$ defined by

$$\widetilde{g}_t(a)(y) = g(y)_t(a)$$

is continuous. Then we define $fg \colon Y \to \mathrm{Asy}(D\widehat{\otimes}A, D\widehat{\otimes}B)$ by for each $t \in [1, \infty)$

$$f(g(y)_t)(a) = \mathrm{id}_D\widehat{\otimes}g(y)_t(a) = \mathrm{id}_D\widehat{\otimes}\widetilde{g}_t(a)(y)$$

but since g is quasi-continuous and that

$$f(g(y)_t)(a) = \widetilde{fg}_t(a)(y),$$

it follows from the definition of quasi-topology on the set of asymptotic morphisms that fg is quasi-continuous. □

Definition 5.2.2. Let $\mathcal{K} = \mathcal{K}(\mathcal{H})$. Define $\mathbb{X}(A, B)$ to be the sequence of based quasi-topological spaces

$$X_m = \mathrm{Asy}_g(\mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}, B\widehat{\otimes}\mathbb{F}_{m,0}\widehat{\otimes}\mathcal{K})$$

where $m \geq 0$. Define maps $\epsilon_m \colon X_m \to \Omega X_{m+1}$:

$$\begin{array}{ccc} \mathrm{Asy}_g(\mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}, B\widehat{\otimes}\mathbb{F}_{m,0}\widehat{\otimes}\mathcal{K}) & \longrightarrow & \Omega\mathrm{Asy}_g(\mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}, B\widehat{\otimes}\mathbb{F}_{m+1,0}\widehat{\otimes}\mathcal{K}) \\ & & \Big\| \cong \\ & & \mathrm{Asy}_g(\mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}, \Sigma(B\widehat{\otimes}\mathbb{F}_{m+1,0})\widehat{\otimes}\mathcal{K}) \end{array}$$

by:

$$\epsilon(x_t) = (b\widehat{\otimes}\mathrm{id}_{B\widehat{\otimes}\mathbb{F}_{m,0}\widehat{\otimes}\mathcal{K}}) \circ (\mathrm{id}_{\mathcal{S}}\widehat{\otimes}x_t) \circ (\Delta\widehat{\otimes}\mathrm{id}_{A\widehat{\otimes}\mathcal{K}}),$$

for all $x_t \in \mathrm{Asy}_g(\mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}, B\widehat{\otimes}\mathbb{F}_{m,0}\widehat{\otimes}\mathcal{K})$ and the Bott map $b \in \mathrm{Hom}_g(\mathcal{S}, \Sigma\mathbb{F}_{1,0})$. Alternatively, we also have a map $\sigma_m \colon X_m \wedge S^1 \to X_{m+1}$ defined by

$$\sigma_m(x_t, s) = \epsilon_m(x_t)(s),$$

with $x_t \in \mathrm{Asy}_g(\mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}(\mathcal{H}), B\widehat{\otimes}\mathbb{F}_{m,0}\widehat{\otimes}\mathcal{K}(\mathcal{H}))$ and $s \in S^1$.

Definition 5.2.3. We define a quasi-continuous action of the group $O(m)$ on the space X_m as follows. First we consider the alternative definition of $\mathbb{F}_{m,0}$ as

Cliff(V) which we saw in chapter 3. Recall that for V an m-dimensional Euclidean vector space, Cliff$(V) = G(V)/\sim$ where $G(V)$ is the algebra generated by V subject to the equivalence relation $\sim$ defined by

$$v^2 = ||v||^2 \cdot 1$$

for all $v \in V$. We write ab for the product of two elements $a, b \in$ Cliff(V).

If $V = \mathbb{R}^m$, then we have a natural group action $(H, v) \mapsto Hv$ where $H \in O(m)$, $v \in V$.

Then we can define a group action of $O(m)$ on $G(V)$ by

$$H(v_1 \dots v_k) \mapsto H(v_1) \dots H(v_k) \text{ and } H(1) = 1$$

for all $H \in O(m)$. Then this gives a group action of $O(m)$ on Cliff(V) since

$$\begin{aligned} H(v^2) &= H(v)H(v) = (H(v))^2 \\ &= ||H(v)||^2 \cdot 1 = ||v|| \cdot 1 \text{ since H is orthogonal.} \end{aligned}$$

So then we get a group action

$$\lambda \colon O(m) \times \mathbb{F}_{m,0} \to \mathbb{F}_{m,0},$$

by

$$\lambda(H, (e_1, e_2, \dots e_m)) = H(e_1)H(e_2) \dots H(e_m),$$

where $H \in O(m)$, $e_1, e_2, \dots e_m$ are the generators of the algebra $\mathbb{F}_{m,0}$. Then we define

$$\lambda_* \colon O(m) \times B \widehat{\otimes} \mathbb{F}_{m,0} \widehat{\otimes} \mathcal{K}(\mathcal{H}) \to B \widehat{\otimes} \mathbb{F}_{m,0} \widehat{\otimes} \mathcal{K}(\mathcal{H})$$

by

$$\lambda_*(H, b \widehat{\otimes} x \widehat{\otimes} p) = b \widehat{\otimes} \lambda(H, x) \widehat{\otimes} p$$

with $H \in O(m)$, $b \in B$, $x \in \mathbb{F}_{m,0}$ and $p \in \mathcal{K}(\mathcal{H})$. Then we finally define a group action of $O(m)$ on X_m

$$\begin{aligned} \lambda_{**} \colon O(m) \times \operatorname{Asy}_g(\mathcal{S} \widehat{\otimes} A \widehat{\otimes} \mathcal{K}(\mathcal{H}), B \widehat{\otimes} \mathbb{F}_{m,0} \widehat{\otimes} \mathcal{K}(\mathcal{H})) \\ \longrightarrow \operatorname{Asy}_g(\mathcal{S} \widehat{\otimes} A \widehat{\otimes} \mathcal{K}(\mathcal{H}), B \widehat{\otimes} \mathbb{F}_{m,0} \widehat{\otimes} \mathcal{K}(\mathcal{H})), \end{aligned}$$

by

$$\lambda_{**}(H, \alpha_t)(x) = \lambda_*(H, \alpha_t(x)),$$

where we have $\alpha_t \in \operatorname{Asy}_g(\mathcal{S} \widehat{\otimes} A \widehat{\otimes} \mathcal{K}(\mathcal{H}), B \widehat{\otimes} \mathbb{F}_{m,0} \widehat{\otimes} \mathcal{K}(\mathcal{H}))$, $x \in \mathcal{S} \widehat{\otimes} A \widehat{\otimes} \mathcal{K}(\mathcal{H})$ and $H \in O(m)$. Then it follows that this action is $O(m)$-equivariant.

The following is true by Proposition 5.2.1.

Proposition 5.2.4. *The action in the previous definition is a basepoint preserving quasi-continuous action of $O(m)$ on X_m.*

□

Proposition 5.2.5. *The map $\epsilon_m\colon X_m \to \Omega X_{m+1}$ is quasi-continuous and hence the map $\sigma_m\colon X_m \wedge S^1 \to X_{m+1}$ is quasi-continuous.*

Proof. Since both

$$b\widehat{\otimes}\mathrm{id}_{B\widehat{\otimes}\mathbb{F}_{m,0}\widehat{\otimes}\mathcal{K}(\mathcal{H})} \quad \text{and} \quad \Delta\widehat{\otimes}\mathrm{id}_{A\widehat{\otimes}\mathcal{K}(\mathcal{H})}$$

are $*$-homomorphisms, these two maps are continuous. So it suffices to check the map $(\mathrm{id}_{\mathcal{S}}\widehat{\otimes}x_t)$ is quasi-continuous since the a composition of continuous and quasi-continuous maps yields a quasi-continuous map. By proposition 5.2.1, with $D = \mathcal{S}$ it follows that the map $(\mathrm{id}_{\mathcal{S}}\widehat{\otimes}x_t)$ is quasi-continuous. Hence the map ϵ_m is quasi-continuous. Since σ_m is defined in terms of ϵ_m it is also quasi-continuous. □

We define the iterated map $\sigma_m^n\colon X_m \wedge S^n \to X_{m+n}$ by

$$\sigma_m^n(x_t, s_1, s_2, \dots s_n) = \epsilon^n(x_t)(s_1)(s_2)\dots(s_n),$$

where $x_t \in \mathrm{Asy}_g(\mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}(\mathcal{H}), B\widehat{\otimes}\mathbb{F}_{m,0}\widehat{\otimes}\mathcal{K}(\mathcal{H}))$ and $s_1, s_2, \dots s_n$ is contained in $S^1 \wedge S^1 \wedge \dots \wedge S^1$.

Proposition 5.2.6. *The iterated map $\sigma_m^n\colon X_m \wedge S^n \to X_{m+n}$ defined above quasi-continuous and $O(m) \times O(n)$-equivariant.*

Proof. By other results it suffices to check that the map is $O(m) \times O(n)$-equivariant.

Firstly it is clear that $X_m \wedge S^n$ and X_{m+n} are quasi $O(m) \times O(n)$-spaces. Let $i\colon O(m) \times O(n) \to O(m+n)$ be the inclusion map.

Now $O(m+n)$ acts on $\mathrm{Asy}_g(\mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}(\mathcal{H}), B\widehat{\otimes}\mathbb{F}_{m+n,0}\widehat{\otimes}\mathcal{K}(\mathcal{H}))$ by

$$J(x_t) = (\mathrm{id}_B\widehat{\otimes}J\widehat{\otimes}\mathrm{id}_{\mathcal{K}(\mathcal{H})}) \circ x_t$$

for all $J \in O(m+n)$ and $x_t\colon \mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}(\mathcal{H}) \to B\widehat{\otimes}\mathbb{F}_{m+n,0}\widehat{\otimes}\mathcal{K}(\mathcal{H})$. Here J acts of $\mathbb{F}_{m+n,0}$ as defined earlier.

Then $O(m) \times O(n)$ acts on $\mathrm{Asy}_g(\mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}(\mathcal{H}), B\widehat{\otimes}\mathbb{F}_{m,0}\widehat{\otimes}\mathcal{K}(\mathcal{H})) \wedge S^n$ by,

$$(H,K)(x_t,s) = ((\mathrm{id}_B\widehat{\otimes}H\widehat{\otimes}\mathrm{id}_{\mathcal{K}(\mathcal{H})}) \circ x_t, Ks),$$

for all $H \in O(m)$, $K \in O(n)$, $x_t \in \mathrm{Asy}_g(\mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}(\mathcal{H}), B\widehat{\otimes}\mathbb{F}_{m,0}\widehat{\otimes}\mathcal{K}(\mathcal{H}))$ and $s \in S^n$. Then we need to show that for $\sigma = \sigma_m^n \colon X_m \wedge S^n \to X_{m+n}$, that

$$\sigma((H,K)(x_t,s)) = i(H,K)\sigma(x_t,s),$$

that is,

$$\sigma((Hx_t, Ks) = i(H,K)\sigma(x_t,s).$$

That is to show, by definition of σ that,

$$\epsilon(Hx_t)(Ks) = i(H,K)\epsilon(x_t)(s).$$

Let $b^n = b\widehat{\otimes}\dots\widehat{\otimes}b$ be the n-fold graded tensor product of the Bott map, $b \colon \mathcal{S} \to \Sigma\mathbb{F}_{1,0}$. Then

$$b^n = b\widehat{\otimes}\dots\widehat{\otimes}b \colon \mathcal{S}^n \to \Sigma^n\widehat{\otimes}\mathbb{F}_{n,0},$$

we have $b\widehat{\otimes}\dots\widehat{\otimes}b(\lambda)(s) \in \mathbb{F}_{n,0}$ for $\lambda \in \mathcal{S}^n$ and $s \in S^n$. Then for $K \in O(n)$,

$$(b\widehat{\otimes}\dots\widehat{\otimes}b)(\lambda)(Ks) = K(b\widehat{\otimes}\dots\widehat{\otimes}b)(\lambda)(s).$$

Then by permuting copies of Σ and extending by linearity we have an action of the orthogonal group.

Reconsidering

$$\epsilon(Hx_t)(Ks) = i(H,K)\epsilon(x_t)(s),$$

the left hand side yields

$$\epsilon(Hx_t)(Ks) = ((b^n\widehat{\otimes}\mathrm{id}_{B\widehat{\otimes}\mathbb{F}_{m,0}\widehat{\otimes}\mathcal{K}(\mathcal{H})}) \circ (\mathrm{id}_{\mathcal{S}}\widehat{\otimes}Hx_t) \circ (\Delta\widehat{\otimes}\mathrm{id}_{A\widehat{\otimes}\mathcal{K}(\mathcal{H})})(Ks)),$$

and the right hand side yields

$$i(H,K)\epsilon(x_t)(s) = i(H,K)(b\widehat{\otimes}\mathrm{id}_{B\widehat{\otimes}\mathbb{F}_{m,0}\widehat{\otimes}\mathcal{K}(\mathcal{H})}) \circ (\mathrm{id}_{\mathcal{S}}\widehat{\otimes}x_t) \circ (\Delta\widehat{\otimes}\mathrm{id}_{A\widehat{\otimes}\mathcal{K}(\mathcal{H})}).$$

Then

$$
\begin{aligned}
&\epsilon(Hx_t)(Ks)\\
&\quad = ((b^n \widehat{\otimes} \mathrm{id}_{B\widehat{\otimes}\mathbb{F}_{m,0}\widehat{\otimes}\mathcal{K}(\mathcal{H})}) \circ (\mathrm{id}_{\mathcal{S}} \widehat{\otimes} Hx_t) \circ (\Delta \widehat{\otimes} \mathrm{id}_{A\widehat{\otimes}\mathcal{K}(\mathcal{H})}))(Ks)\\
&\quad = i(H,1)((b^n \widehat{\otimes} \mathrm{id}_{B\widehat{\otimes}\mathbb{F}_{m,0}\widehat{\otimes}\mathcal{K}(\mathcal{H})}) \circ (\mathrm{id}_{\mathcal{S}} \widehat{\otimes} x_t) \circ (\Delta \widehat{\otimes} \mathrm{id}_{A\widehat{\otimes}\mathcal{K}(\mathcal{H})})(Ks))\\
&\quad = i(H,1)i(1,K)(b^n \widehat{\otimes} \mathrm{id}_{B\widehat{\otimes}\mathbb{F}_{m,0}\widehat{\otimes}\mathcal{K}(\mathcal{H})}) \circ (\mathrm{id}_{\mathcal{S}} \widehat{\otimes} x_t) \circ (\Delta \widehat{\otimes} \mathrm{id}_{A\widehat{\otimes}\mathcal{K}(\mathcal{H})})(s)\\
&\quad = i(H,K)(b^n \widehat{\otimes} \mathrm{id}_{B\widehat{\otimes}\mathbb{F}_{m,0}\widehat{\otimes}\mathcal{K}(\mathcal{H})}) \circ (\mathrm{id}_{\mathcal{S}} \widehat{\otimes} x_t) \circ (\Delta \widehat{\otimes} \mathrm{id}_{A\widehat{\otimes}\mathcal{K}(\mathcal{H})})(s)\\
&\quad = i(H,K)\epsilon(x_t)(s)
\end{aligned}
$$

Then the result follows.

□

The proof of the following result follows from the above propositions, namely Proposition 5.2.4, Proposition 5.2.5 and Proposition 5.2.6.

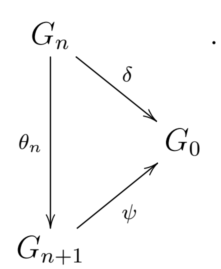
Proposition 5.2.7. *The spectrum* $\mathbb{X}(A,B)$ *is an orthogonal quasi-spectrum.*

□

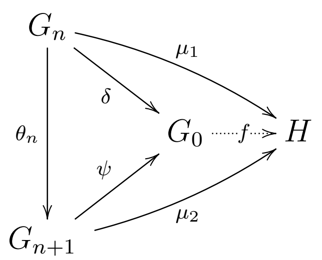
Proposition 5.2.8. *If* $G_0, G_1, G_2, \dots$ *is a sequence of groups with isomorphisms* $\theta_n\colon G_n \to G_{n+1}$ *for* $n \geq 0$*, then*

$$\varinjlim_n G_n = G_0.$$

Proof. We first need to construct a commutative diagram.



As θ_n is an isomorphism for all $n \geq 0$, we have inverses, so $\delta = (\theta_0)^{-1} \dots (\theta_{n-1})^{-1}$ and $\psi = (\theta_0)^{-1} \dots (\theta_{n-1})^{-1}(\theta_n)^{-1}$ and hence the diagram commutes. Now we check that G_0 is unique. Suppose we have a group H such that we have a group homomorphism $f\colon G_0 \to H$ which fits into the following diagram



Then define $f = \mu_1\delta^{-1}$ so our diagram commutes. Suppose that we have another group homomorphism $g\colon G_0 \to H$ fitting into the diagram. Then by commutativity we have $g\delta = \mu_1$, so $g = \mu_1\delta^{-1} = f$ so f is unique. □

Proposition 5.2.9. *The direct limit* $\varinjlim_n E_g(A, \Sigma^{k+n}B\widehat{\otimes}\mathbb{F}_{n,0})$ *is* $E_g(A, \Sigma^k B)$.

Proof. This result follows from Proposition 5.2.8 where

$$G_n = E_g(A, \Sigma^{k+n}B\widehat{\otimes}\mathbb{F}_{n,0})$$

and using Proposition 2.2.18. □

Proposition 5.2.10. *For all positive integers k,*

$$\pi_k\mathbb{X}(A, B) = E_g(A, \Sigma^k B).$$

Proof. Since $\mathbb{X}$ is an orthogonal quasi-spectrum we have that

$$\pi_k\mathbb{X}(A, B) = \varinjlim_n \pi_{k+n}E_n.$$

Then

$$\begin{aligned}
\varinjlim_n \pi_{k+n}X_n &= \varinjlim_n \pi_0\Omega^{k+n}\mathrm{Asy}_g(\mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}(\mathcal{H}), B\widehat{\otimes}\mathbb{F}_{n,0}\widehat{\otimes}\mathcal{K}(\mathcal{H}))\\
&= \varinjlim_n \pi_0\mathrm{Asy}_g(\mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}(\mathcal{H}), \Sigma^{k+n}B\widehat{\otimes}\mathbb{F}_{n,0}\widehat{\otimes}\mathcal{K}(\mathcal{H}))\\
&= \varinjlim_n [\![\mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}(\mathcal{H}), \Sigma^{k+n}B\widehat{\otimes}\mathbb{F}_{n,0}\widehat{\otimes}\mathcal{K}(\mathcal{H})]\!]_g\\
&= \varinjlim_n E_g(A, \Sigma^{k+n}B\widehat{\otimes}\mathbb{F}_{n,0})\\
&= E_g(A, \Sigma^k B) \text{ by Proposition 5.2.9.}
\end{aligned}$$

□

Proposition 5.2.11. *The orthogonal quasi-spectrum $\mathbb{X}(A,B)$ is an Ω-quasi-spectrum.*

Proof. We just need to check that the structure map $\epsilon\colon E_n \to \Omega E_{n+1}$ is a weak equivalence. That is the map $\pi_k E_n \to \pi_k \Omega E_{n+1}$ is an isomorphism for all k. Now this gives us the map:

$$E_g(A, \Sigma^k(B\widehat{\otimes}\mathbb{F}_{n,0})) \to E_g(A, \Sigma^{k+1}(B\widehat{\otimes}\mathbb{F}_{n+1,0})),$$

which is an isomorphism for all k by Corollary 2.2.19. □

Theorem 5.2.12. *Let A, B and C be graded C^*-algebras. Then there is a natural map of orthogonal quasi-spectra*

$$\mu_{m,n}\colon \mathbb{X}(A,B) \wedge \mathbb{X}(B,C) \to \mathbb{X}(A,C),$$

defined by

$$(\alpha \wedge \beta)_t \mapsto (\beta_{r(t)}\widehat{\otimes} id_{\mathbb{F}_{m,0}}) \circ (id_{\mathcal{S}}\widehat{\otimes}\alpha_t) \circ (\Delta\widehat{\otimes} id_{A\widehat{\otimes}\mathcal{K}}),$$

where $\alpha \in Asy_g(\mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}, B\widehat{\otimes}\mathbb{F}_{m,0}\widehat{\otimes}\mathcal{K})$ and $\beta \in Asy_g(\mathcal{S}\widehat{\otimes}B\widehat{\otimes}\mathcal{K}, C\widehat{\otimes}\mathbb{F}_{n,0}\widehat{\otimes}\mathcal{K})$. In addition the product is associative up to homotopy.

Proof. The product gives a natural $O(m) \times O(n)$-equivariant map:

$$\begin{aligned}\mathrm{Asy}_g(\mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}, B\widehat{\otimes}\mathbb{F}_{m,0}\widehat{\otimes}\mathcal{K}) \wedge \mathrm{Asy}_g(\mathcal{S}\widehat{\otimes}B\widehat{\otimes}\mathcal{K}, C\widehat{\otimes}\mathbb{F}_{n,0}\widehat{\otimes}\mathcal{K})\\ \longrightarrow \mathrm{Asy}_g(\mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}, C\widehat{\otimes}\mathbb{F}_{m+n,0}\widehat{\otimes}\mathcal{K}),\end{aligned}$$

given by permuting the m, n and $m+n$ copies of $\mathbb{F}_{1,0}$.

Now compatibility with the structure maps follows from the naturality of the structure maps and also since we have the following two diagrams:

$$\begin{array}{ccc}\mathbb{X}_m(A,B) \wedge \mathbb{X}_n(B,C) & \xrightarrow{\mu_{m,n}} & \mathbb{X}_{m+n}(A,C)\\ \big\downarrow{\scriptstyle \epsilon\wedge\mathrm{id}} & & \big\downarrow{\scriptstyle \epsilon}\\ \Omega\mathbb{X}_{m+1}(A,B) \wedge \mathbb{X}_n(B,C) & \xrightarrow{\mu_{m+1,n}} & \Omega\mathbb{X}_{m+n+1}(A,C),\end{array}$$

and

$$\begin{array}{ccc}
\mathbb{X}_m(A,B)\wedge\mathbb{X}_n(B,C) & \xrightarrow{\mu_{m,n}} & \mathbb{X}_{m+n}(A,C) \\
\downarrow{\scriptstyle \mathrm{id}\wedge\epsilon} & & \downarrow{\scriptstyle \epsilon} \\
\mathbb{X}_m(A,B)\wedge\Omega\mathbb{X}_{n+1}(B,C) & \xrightarrow{\mu_{m,n+1}} & \Omega\mathbb{X}_{m+n+1}(A,C),
\end{array}$$

where id denote the obvious identities, and the ϵ's denote the required structure maps. These diagrams commute since,

$$\begin{aligned}
&\mu_{m+1,n}(\epsilon\wedge\mathrm{id})(\alpha\wedge\beta)_t = \mu_{m+1,n}(\epsilon(\alpha)\wedge\beta)_t \\
&\quad= \; (\beta_{r(t)}\widehat{\otimes}\mathrm{id}_{\Sigma\mathbb{F}_{m+1,0}})\circ(\mathrm{id}_{\mathcal{S}}\widehat{\otimes}\epsilon(\alpha_t))\circ(\Delta\widehat{\otimes}\mathrm{id}_{A\widehat{\otimes}\mathcal{K}}) \\
&\quad= \; (\beta_{r(t)}\widehat{\otimes}\mathrm{id}_{\Sigma\mathbb{F}_{m+1,0}})\circ(\mathrm{id}_{\mathcal{S}}\widehat{\otimes} \\
&\qquad \Big[(b\widehat{\otimes}\mathrm{id}_{B\widehat{\otimes}\mathbb{F}_{m,0}\widehat{\otimes}\mathcal{K}})\circ(\mathrm{id}_{\mathcal{S}}\widehat{\otimes}\alpha_t)\circ(\Delta\widehat{\otimes}\mathrm{id}_{A\widehat{\otimes}\mathcal{K}})\Big])\circ(\Delta\widehat{\otimes}\mathrm{id}_{A\widehat{\otimes}\mathcal{K}}) \\
&\quad= \; (\mathrm{id}_{\Sigma\mathbb{F}_{1,0}}\widehat{\otimes}\beta_{r(t)}\widehat{\otimes}\mathrm{id}_{\mathbb{F}_{m,0}})\circ(b\widehat{\otimes}\mathrm{id}_{\mathcal{S}\widehat{\otimes}B\widehat{\otimes}\mathbb{F}_{m,0}\widehat{\otimes}\mathcal{K}})\circ(\mathrm{id}_{\mathcal{S}\widehat{\otimes}\mathcal{S}}\widehat{\otimes}\alpha_t) \\
&\qquad\qquad \circ(\Delta\widehat{\otimes}\mathrm{id}_{\mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}})\circ(\Delta\widehat{\otimes}\mathrm{id}_{A\widehat{\otimes}\mathcal{K}}) \text{ by tensor products,} \\
&\quad= \; (b\widehat{\otimes}\mathrm{id}_{C\widehat{\otimes}\mathbb{F}_{m+n,0}\widehat{\otimes}\mathcal{K}})\circ(\mathrm{id}_{\mathcal{S}}\widehat{\otimes}\beta_{r(t)}\widehat{\otimes}\mathrm{id}_{\mathbb{F}_{m,0}})\circ(\mathrm{id}_{\mathcal{S}\widehat{\otimes}\mathcal{S}}\widehat{\otimes}\alpha_t) \\
&\qquad\qquad \circ(\Delta\widehat{\otimes}\mathrm{id}_{\mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}})\circ(\Delta\widehat{\otimes}\mathrm{id}_{A\widehat{\otimes}\mathcal{K}}) \text{ by Lemma 2.2.5,} \\
&\quad= \; (b\widehat{\otimes}\mathrm{id}_{C\widehat{\otimes}\mathbb{F}_{m+n,0}\widehat{\otimes}\mathcal{K}})\circ(\mathrm{id}_{\mathcal{S}}\widehat{\otimes} \\
&\qquad \big[(\beta_{r(t)}\widehat{\otimes}\mathrm{id}_{\mathbb{F}_{m,0}})\circ(\mathrm{id}_{\mathcal{S}}\widehat{\otimes}\alpha_t)\circ(\Delta\widehat{\otimes}\mathrm{id}_{A\widehat{\otimes}\mathcal{K}})\big])\circ(\Delta\widehat{\otimes}\mathrm{id}_{A\widehat{\otimes}\mathcal{K}}) \\
&\quad= \; \epsilon((\beta_{r(t)}\widehat{\otimes}\mathrm{id}_{\mathbb{F}_{m,0}})\circ(\mathrm{id}_{\mathcal{S}}\widehat{\otimes}\alpha_t)\circ(\Delta\widehat{\otimes}\mathrm{id}_{A\widehat{\otimes}\mathcal{K}})) \\
&\quad= \; \epsilon(\mu_{m,n})(\alpha\wedge\beta)_t,
\end{aligned}$$

and

$$
\begin{aligned}
&\mu_{m,n+1}(\mathrm{id}\wedge_g \epsilon)(\alpha\wedge_g \beta)_t = \mu_{m,n+1}(\alpha\wedge\epsilon(\beta))_t \\
&= (\epsilon(\beta)_{r(t)}\widehat{\otimes}\mathrm{id}_{\mathbb{F}_{m,0}})\circ(\mathrm{id}_{\mathcal{S}}\widehat{\otimes}\alpha_t)\circ(\Delta\widehat{\otimes}\mathrm{id}_{A\widehat{\otimes}\mathcal{K}}) \\
&= \Big[(b\widehat{\otimes}\mathrm{id}_{C\widehat{\otimes}\mathbb{F}_{n,0}\widehat{\otimes}\mathcal{K}})\circ(\mathrm{id}_{\mathcal{S}}\widehat{\otimes}\beta_{r(t)}\circ(\Delta\widehat{\otimes}\mathrm{id}_{B\widehat{\otimes}\mathcal{K}})\Big]\widehat{\otimes}\mathrm{id}_{\mathbb{F}_{m,0}}) \\
&\qquad\circ(\mathrm{id}_{\mathcal{S}}\widehat{\otimes}\alpha_t)\circ(\Delta\widehat{\otimes}\mathrm{id}_{A\widehat{\otimes}\mathcal{K}}) \\
&= (b\widehat{\otimes}\mathrm{id}_{C\widehat{\otimes}\mathbb{F}_{m+n,0}\widehat{\otimes}\mathcal{K}})\circ(\mathrm{id}_{\mathcal{S}}\widehat{\otimes}\beta_{r(t)}\widehat{\otimes}\mathrm{id}_{\mathbb{F}_{m,0}})\circ(\Delta\widehat{\otimes}\mathrm{id}_{B\widehat{\otimes}\mathbb{F}_{m,0}\widehat{\otimes}\mathcal{K}}) \\
&\qquad\circ(\mathrm{id}_{\mathcal{S}}\widehat{\otimes}\alpha_t)\circ(\Delta\widehat{\otimes}\mathrm{id}_{A\widehat{\otimes}\mathcal{K}}) \\
&= (b\widehat{\otimes}\mathrm{id}_{C\widehat{\otimes}\mathbb{F}_{m+n,0}\widehat{\otimes}\mathcal{K}})\circ(\mathrm{id}_{\mathcal{S}}\widehat{\otimes}\beta_{r(t)}\widehat{\otimes}\mathrm{id}_{\mathbb{F}_{m,0}})\circ(\mathrm{id}_{\mathcal{S}\widehat{\otimes}\mathcal{S}}\widehat{\otimes}\alpha_t) \\
&\qquad\qquad\circ(\Delta\widehat{\otimes}\mathrm{id}_{\mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}})\circ(\Delta\widehat{\otimes}\mathrm{id}_{A\widehat{\otimes}\mathcal{K}}) \text{ by Lemma 2.2.5,} \\
&= (b\widehat{\otimes}\mathrm{id}_{C\widehat{\otimes}\mathbb{F}_{m+n,0}\widehat{\otimes}\mathcal{K}})\circ(\mathrm{id}_{\mathcal{S}}\widehat{\otimes} \\
&\qquad\big[(\beta_{r(t)}\widehat{\otimes}\mathrm{id}_{\mathbb{F}_{m,0}})\circ(\mathrm{id}_{\mathcal{S}}\widehat{\otimes}\alpha_t)\circ(\Delta\widehat{\otimes}\mathrm{id}_{A\widehat{\otimes}\mathcal{K}})\big])\circ(\Delta\widehat{\otimes}\mathrm{id}_{A\widehat{\otimes}\mathcal{K}}) \\
&= \epsilon((\beta_{r(t)}\widehat{\otimes}\mathrm{id}_{\mathbb{F}_{m,0}})\circ(\mathrm{id}_{\mathcal{S}}\widehat{\otimes}\alpha_t)\circ(\Delta\widehat{\otimes}\mathrm{id}_{A\widehat{\otimes}\mathcal{K}})) \\
&= \epsilon(\mu_{m,n})(\alpha\wedge\beta)_t,
\end{aligned}
$$

for all $\alpha\wedge\beta\in\mathbb{E}_m(A,B)\wedge\mathbb{E}_n(B,C)$.

Now we check that our product is associative up to homotopy.

Let $\alpha\in\mathrm{Asy}_g(\mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}, B\widehat{\otimes}\mathbb{F}_{m,0}\widehat{\otimes}\mathcal{K})$ and $\beta\in\mathrm{Asy}_g(\mathcal{S}\widehat{\otimes}B\widehat{\otimes}\mathcal{K}, C\widehat{\otimes}\mathbb{F}_{n,0}\widehat{\otimes}\mathcal{K})$ and $\gamma\in\mathrm{Asy}_g(\mathcal{S}\widehat{\otimes}C\widehat{\otimes}\mathcal{K}, D\widehat{\otimes}\mathbb{F}_{p,0}\widehat{\otimes}\mathcal{K})$. Then take the homotopy classes of these elements and we obtain E-theory groups, and we know that the E-theory product is associative.

□

Chapter 6

Connecting graded K and E-theory spectra

This chapter connects together graded K-theory and E-theory spectra. In particular we form a smash product in terms of these two spectra and consequently combine K-theory and K-homology in to a smash product.

6.1 A topology on graded $*$-homomorphisms

Let $\mathrm{Hom}_g(A, B)$ denote the set of graded $*$-homomorphisms from A to B. We equip $\mathrm{Hom}_g(A, B)$ with the compact open topology as detailed below.

Definition 6.1.1. A *basis* for a topology on a set A is a collection of subsets $\mathcal{A}$ of A such that A is a union of sets from $\mathcal{A}$ and such that if A_1, A_2 are in $\mathcal{A}$ then their intersection is a union of sets from $\mathcal{A}$.

A *subbasis* is a collection of subsets $\mathcal{B}$ of A where the set $\mathcal{A}$ of all finite intersections of sets in $\mathcal{B}$ is a basis.

Definition 6.1.2. Let A and B be graded C^*-algebras. The *compact open topology* on the set of graded $*$-homomorphisms from A to B, $\mathrm{Hom}_g(A, B)$, is generated by subsets of the following form,

$$B(K, U) = \{f \in \mathrm{Hom}_g(A, B) \mid f(K) \subset U\},$$

where K is compact in A and U is open in B. Here *generated* means that the sets defined form a subbasis for the open sets when we think of a topology. This then generates a basis for the topology.

For simplicity of notation let $\mathrm{Hom}_g(A, B)$ denote the space of graded $*$-homomorphisms from A to B equipped with the compact open topology. Denote the loop space of this space by $\Omega\mathrm{Hom}_g(A, B)$. Note that the basepoints for both of these spaces is just the zero $*$-homomorphism, which will we denote by 0.

The compact open topology is the choice for our topology since it gives us the correct path components for our loop space and it also allows us to have continuity of particular maps as we will see soon.

Now let us consider the generators of the compact open topology on the spaces $\Omega\mathrm{Hom}_g(A, B)$ and $\mathrm{Hom}_g(A, \Sigma B)$. Now we have a basis for $\mathrm{Hom}_g(A, B)$, so we just extend this for $\Omega\mathrm{Hom}_g(A, B)$, and it is not to hard to see that a basis for the loop space is the set generated by $B(K', V)$ such that $K' \subseteq [0, 1]$ compact and $V \subseteq \mathrm{Hom}_g(A, B)$ open.

Combining these, we obtain the following definition.

Definition 6.1.3. The *compact open topology* on $\Omega\mathrm{Hom}_g(A, B)$ is generated by sets of the form $B(K', B(K, U))$, where $K \subseteq A$ compact, $K' \subseteq [0, 1]$ compact and $U \subseteq B$ open. The *compact open topology* on $\mathrm{Hom}_g(A, \Sigma B)$ is generated by sets of the form $B(K, B(K', U))$ where $K \subseteq A$ compact, $K' \subseteq [0, 1]$ compact and $U \subseteq B$ open.

Before we check we have the continuity of maps in the following proof, it is worth noting that it is sufficient to check that a map is continuous under a topology by considering a subbasis. That is, to check a map of topological spaces is continuous we just need to check that continuity holds at the level of generating sets for a basis of a topology. For details, see [Sut75], Application 3.2.5.

Proposition 6.1.4. *The spaces $\Omega Hom_g(A, B)$ and $Hom_g(A, \Sigma B)$ are homeomorphic.*

Proof. We consider ungraded $*$-homomorphisms since the grading property is immediate. Define $f : \Omega\mathrm{Hom}(A, B) \to \mathrm{Hom}(A, \Sigma B)$ as follows. Let $\mu \in \Omega\mathrm{Hom}(A, B)$ based at 0, then define

$$f(\mu)(a)(s) = \mu(s)(a),$$

for all $a \in A$ and $s \in [0, 1]$.

Now define $g : \mathrm{Hom}(A, \Sigma B) \to \Omega\mathrm{Hom}(A, B)$ as follows. Let $\tau \in \mathrm{Hom}(A, \Sigma B)$, define

$$g(\tau)(s)(a) = \tau(a)(s).$$

Both f and g are well defined since μ and τ are $*$-homomorphisms.

We need to show that $f \circ g = id$, and $g \circ f = id$ where id stands for the natural identities.

Let $\varphi \in \mathrm{Hom}(A, \Sigma B)$, then for all $a \in A, s \in [0,1]$.

$$fg(\varphi)(a)(s) = g(\varphi)(s)(a) = \varphi(a)(s).$$

Similarly, let $\psi \in \Omega\mathrm{Hom}(A, B)$, then for all $a \in A, s \in [0,1]$,

$$gf(\psi)(s)(a) = f(\psi)(a)(s) = \psi(s)(a).$$

Then $f \circ g = id$ and $g \circ f = id$ as required.

Now we check that f and g are continuous. By the above discussion, it suffices to check that:

$$f^{-1}[B(K', B(K, U))] = B(K, B(K', U))$$

and

$$g^{-1}[B(K, B(K', U))] = B(K', B(K, U)),$$

for all $K \subseteq A$ compact, $K' \subseteq [0,1]$ compact and $U \subseteq B$ open.

Let $y \in B(K, B(K', U))$, then $f^{-1}(y) = \{x \mid f(x) = y\}$. Now let $x \in f^{-1}(y)$, then we know

$$f(x)(s)(a) = x(a)(s) = y(s)(a),$$

so x must be contained in $B(K', B(K, U))$, and similarly we can check the converse, so f is continuous.

Similarly we can prove that g is continuous. □

6.2 K-theory spectra

Now we can define the K-theory spectrum.

Definition 6.2.1. Let $\mathcal{K} = \mathcal{K}(\mathcal{H})$. Define $\mathbb{K}(A)$ to be the sequence of based topological spaces

$$K_n = \mathrm{Hom}_g(\mathcal{S}\widehat{\otimes}\mathbb{F}\widehat{\otimes}\mathcal{K}, A\widehat{\otimes}\mathbb{F}_{n,0}\widehat{\otimes}\mathcal{K})$$

where $m \geq 0$. Define maps $\epsilon_n \colon K_n \to \Omega K_{n+1}$:

$$\begin{array}{ccc} \mathrm{Hom}_g(\mathcal{S}\widehat{\otimes}\mathbb{F}\widehat{\otimes}\mathcal{K}, A\widehat{\otimes}\mathbb{F}_{n,0}\widehat{\otimes}\mathcal{K}) & \longrightarrow & \Omega\mathrm{Hom}_g(\mathcal{S}\widehat{\otimes}\mathbb{F}\widehat{\otimes}\mathcal{K}, A\widehat{\otimes}\mathbb{F}_{n+1,0}\widehat{\otimes}\mathcal{K}) \\ & & \| \cong \\ & & \mathrm{Hom}_g(\mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}, \Sigma(A\widehat{\otimes}\mathbb{F}_{n+1,0})\widehat{\otimes}\mathcal{K}) \end{array}$$

by:

$$\epsilon(x_t) = (b\widehat{\otimes}\mathrm{id}_{A\widehat{\otimes}\mathbb{F}_{n,0}\widehat{\otimes}\mathcal{K}}) \circ (\mathrm{id}_{\mathcal{S}}\widehat{\otimes}x_t) \circ (\Delta\widehat{\otimes}\mathrm{id}_{\mathbb{F}\widehat{\otimes}\mathcal{K}}),$$

for all $x_t \in \mathrm{Asy}_g(\mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}, B\widehat{\otimes}\mathbb{F}_{n,0}\widehat{\otimes}\mathcal{K})$ and the Bott map $b \in \mathrm{Hom}_g(\mathcal{S}, \Sigma\mathbb{F}_{1,0})$.

We now give an alternative definition for the spectrum of graded K-theory in terms of asymptotic morphisms.

Definition 6.2.2. Let $\mathcal{K} = \mathcal{K}(\mathcal{H})$. Define $\mathbb{K}'(A)$ to be the orthogonal quasi-spectrum with the sequence of based quasi-topological spaces

$$K'_n = \mathrm{Asy}_g(\mathcal{S}\widehat{\otimes}\mathbb{F}\widehat{\otimes}\mathcal{K}, A\widehat{\otimes}\mathbb{F}_{n,0}\widehat{\otimes}\mathcal{K})$$

where $n \geq 0$. The structure maps $\epsilon\colon K'_n \to \Omega K'_{n+1}$:

$$\begin{array}{ccc}
\mathrm{Asy}_g(\mathcal{S}\widehat{\otimes}\mathbb{F}\widehat{\otimes}\mathcal{K}, A\widehat{\otimes}\mathbb{F}_{n,0}\widehat{\otimes}\mathcal{K}) & \longrightarrow & \Omega\mathrm{Asy}_g(\mathcal{S}\widehat{\otimes}\mathbb{F}\widehat{\otimes}\mathcal{K}, A\widehat{\otimes}\mathbb{F}_{n+1,0}\widehat{\otimes}\mathcal{K}) \\
 & & \Big\| \cong \\
 & & \mathrm{Asy}_g(\mathcal{S}\widehat{\otimes}\mathbb{F}\widehat{\otimes}\mathcal{K}, \Sigma(A\widehat{\otimes}\mathbb{F}_{n+1,0})\widehat{\otimes}\mathcal{K})
\end{array}$$

are defined by:

$$\epsilon(x_t) = (b\widehat{\otimes}\mathrm{id}_{A\widehat{\otimes}\mathbb{F}_{n,0}\widehat{\otimes}\mathcal{K}}) \circ (\mathrm{id}_{\mathcal{S}}\widehat{\otimes}x_t) \circ (\Delta\widehat{\otimes}\mathrm{id}_{\mathbb{F}\widehat{\otimes}\mathcal{K}}),$$

for all $x_t \in \mathrm{Asy}_g(\mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}, B\widehat{\otimes}\mathbb{F}_{n,0}\widehat{\otimes}\mathcal{K})$ and the Bott map $\beta \in \mathrm{Hom}_g(\mathcal{S}, \Sigma\mathbb{F}_{1,0})$.

We now notice that Definition 6.2.1 and Definition 6.2.2 are orthogonal and orthogonal quasi-spectra for the same reason that 5.2.2 forms one and consequently the following result comes from the stable homotopy groups coming from these spectra.

Proposition 6.2.3. *The map of spectrum $f\colon \mathbb{K}(A) \to \mathbb{K}'(A)$, defined by $f(\varphi) = \varphi$ for all $\varphi \in \mathbb{K}(A)$ is a weak equivalence.*

Proof. Consider the map $f'\colon \mathrm{Hom}_g(\mathcal{S}, A) \to \mathrm{Asy}_g(\mathcal{S}, A)$ then this induces the map $f'_*\colon [\mathcal{S}, A] \to [\![\mathcal{S}, A]\!]$. Now the map,

$$\mathrm{Hom}_g(\mathcal{S}, A\widehat{\otimes}\mathbb{F}_{n+1,0}) \to \mathrm{Asy}_g(\mathcal{S}, A\widehat{\otimes}\mathbb{F}_{n+1,0}),$$

induces an isomorphism at the level of π_0, and therefore the map

$$\mathrm{Hom}_g(\mathcal{S}, A\widehat{\otimes}\mathbb{F}_{1,0}) \to \mathrm{Asy}_g(\mathcal{S}, A\widehat{\otimes}\mathbb{F}_{1,0}),$$

induces an isomorphism at the level of π_n. Therefore we have a weak equivalence. Then we can also consider the map f above and the same applies, since we obtain this map by tensoring with the suspension and the complex numbers. □

Corollary 6.2.4. *The map of spectrum $f\colon \mathbb{K}(A) \to \mathbb{K}'(A)$ has a natural inverse $g\colon \mathbb{K}' \to \mathbb{K}$ at the level of stable homotopy groups.*

□

Theorem 6.2.5. *Let A and B be C^*-algebras. Then there is a natural map of orthogonal quasi-spectra*

$$\nu'_{m,n}\colon \mathbb{K}(A) \wedge \mathbb{E}(A, B) \to \mathbb{K}'(B),$$

defined by

$$(\alpha \wedge \beta_t)_t \mapsto (\beta_t \widehat{\otimes} id_{\mathbb{F}_{m,0}}) \circ (id_{\mathcal{S}} \widehat{\otimes} \alpha) \circ (\Delta \widehat{\otimes} id_{\mathbb{F} \widehat{\otimes} \mathcal{K}(\mathcal{H})}),$$

where α is contained in $Hom_g(\mathcal{S} \widehat{\otimes} A \widehat{\otimes} \mathcal{K}(\mathcal{H}), B \widehat{\otimes} \mathbb{F}_{m,0} \widehat{\otimes} \mathcal{K}(\mathcal{H}))$ and β is contained in $Asy_g(\mathcal{S} \widehat{\otimes} B \widehat{\otimes} \mathcal{K}(\mathcal{H}), C \widehat{\otimes} \mathbb{F}_{n,0} \widehat{\otimes} \mathcal{K}(\mathcal{H}))$.

Proof. Since the composition of a $*$-homomorphism and an asymptotic morphism is an asymptotic morphism it is clear that $\alpha \wedge \beta$ is an asymptotic morphism and lies in the required spectra. □

By the above theorem and Corollary 6.2.4, we obtain

Corollary 6.2.6. *There is a natural map of spectra*

$$\nu_{m,n}\colon \mathbb{K}(A) \wedge \mathbb{E}(A, B) \to \mathbb{K}(B),$$

with the above criteria.

□

Now we finalise this section by combining the graded K-theory spectrum and K-homology spectrum noting that $\mathbb{K}_{\mathrm{hom}}(A) = \mathbb{E}(A, \mathbb{F})$.

Theorem 6.2.7. *There is a canonical map*

$$S\colon \mathbb{K}(A \widehat{\otimes} B) \wedge \mathbb{K}_{hom}(A) \to \mathbb{K}'(B)$$

of orthogonal quasi-spectra. The map S is natural in the variable B in the obvious sense and natural in the variable A, in the sense that if we have a $*$-homomorphism $f\colon A \to A'$ then we have the following commutative diagram

$$\begin{array}{ccc}
\mathbb{K}(A\widehat{\otimes}B)\wedge\mathbb{K}_{hom}(A) & \xrightarrow{S} & \mathbb{K}'(B) \\
\Big\uparrow{\scriptstyle id\wedge f^*} & & \Big\| \\
\mathbb{K}(A\widehat{\otimes}B)\wedge\mathbb{K}_{hom}(A') & & \mathbb{K}'(B) \\
\Big\downarrow{\scriptstyle f_*\wedge id} & & \Big\| \\
\mathbb{K}(A'\widehat{\otimes}B)\wedge\mathbb{K}_{hom}(A') & \xrightarrow{S} & \mathbb{K}'(B)
\end{array}$$

where f_* and f^* are defined by:

$$f_*(\alpha)(\lambda) = (f\widehat{\otimes}id_{B\widehat{\otimes}\mathbb{F}_{m,0}\widehat{\otimes}\mathcal{K}(\mathcal{H})})(\alpha(\lambda)),$$

and

$$f^*(\beta_t)(a) = \beta_t(f\widehat{\otimes}id_{\mathcal{S}\widehat{\otimes}\mathcal{K}(\mathcal{H})})(a),$$

with $\alpha \in \mathbb{K}(A\widehat{\otimes}B)$, $\beta \in \mathbb{K}_{hom}(A')$, $a \in \mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}$ and $\lambda \in \mathcal{S}\widehat{\otimes}\mathbb{F}\widehat{\otimes}\mathcal{K}$.

Proof. Writing $\mathbb{K}_{\mathrm{hom}}(A) = \mathbb{E}(A,\mathbb{F})$, we can extend the definition of S to a composition of maps, in order to obtain the following diagram:

$$\begin{array}{ccccc}
\mathbb{K}(A\widehat{\otimes}B)\wedge\mathbb{E}(A,\mathbb{F}) & \xrightarrow{id\wedge\widehat{\otimes}_B} & \mathbb{K}(A\widehat{\otimes}B)\wedge\mathbb{E}(A\widehat{\otimes}B,B) & \xrightarrow{\nu'_{m,n}} & \mathbb{K}'(B) \\
\Big\uparrow{\scriptstyle \mathrm{id}\wedge f^*} & & & & \Big\| \\
\mathbb{K}(A\widehat{\otimes}B)\wedge\mathbb{E}(A',\mathbb{F}) & & & & \mathbb{K}'(B) \\
\Big\downarrow{\scriptstyle f_*\wedge\mathrm{id}} & & & & \Big\| \\
\mathbb{K}(A'\widehat{\otimes}B)\wedge\mathbb{E}(A',\mathbb{F}) & \xrightarrow{\mathrm{id}\wedge\widehat{\otimes}_B} & \mathbb{K}(A'\widehat{\otimes}B)\wedge\mathbb{E}(A'\otimes B,B) & \xrightarrow{\nu'_{m,n}} & \mathbb{K}'(B)
\end{array}$$

$$\mathbb{K}(A\widehat{\otimes}B)\wedge\mathbb{E}(A\widehat{\otimes}B,B) \to \mathbb{K}'(B).$$

Then we have

$$
\begin{aligned}
\nu'_{m,n}&(\mathrm{id} \wedge \widehat{\otimes}_B)(\mathrm{id} \wedge f^*)(\alpha \wedge \beta_t) \\
&= \nu'_{m,n}(\mathrm{id} \wedge \widehat{\otimes}_B)(\alpha \wedge f^*(\beta_t)) \\
&= \nu'_{m,n}(\alpha \wedge f^*(\beta_t)\widehat{\otimes}\mathrm{id}_B) \\
&= (f^*(\beta_t)\widehat{\otimes}\mathrm{id}_{B\widehat{\otimes}\mathbb{F}_{m,0}}) \circ (\mathrm{id}_{\mathcal{S}}\widehat{\otimes}\alpha) \circ (\Delta\widehat{\otimes}\mathrm{id}_{\mathbb{F}\widehat{\otimes}\mathcal{K}(\mathcal{H})}) \\
&= ([\beta_t \circ (f\widehat{\otimes}\mathrm{id}_{\mathcal{S}\widehat{\otimes}\mathcal{K}(\mathcal{H})})]\widehat{\otimes}\mathrm{id}_{B\widehat{\otimes}\mathbb{F}_{m,0}}) \circ (\mathrm{id}_{\mathcal{S}}\widehat{\otimes}\alpha) \circ (\Delta\widehat{\otimes}\mathrm{id}_{\mathbb{F}\widehat{\otimes}\mathcal{K}(\mathcal{H})}) \\
&= (\beta_t\widehat{\otimes}\mathrm{id}_{B\widehat{\otimes}\mathbb{F}_{m,0}}) \circ (f\widehat{\otimes}\mathrm{id}_{\mathcal{S}\widehat{\otimes}B\widehat{\otimes}\mathbb{F}_{m,0}\widehat{\otimes}\mathcal{K}(\mathcal{H})}) \circ (\mathrm{id}_{\mathcal{S}}\widehat{\otimes}\alpha) \circ (\Delta\widehat{\otimes}\mathrm{id}_{\mathbb{F}\widehat{\otimes}\mathcal{K}(\mathcal{H})}) \\
&= (\beta_t\widehat{\otimes}\mathrm{id}_{B\widehat{\otimes}\mathbb{F}_{m,0}}) \circ (\mathrm{id}_{\mathcal{S}}\widehat{\otimes}[(f\widehat{\otimes}\mathrm{id}_{B\widehat{\otimes}\mathbb{F}_{m,0}\widehat{\otimes}\mathcal{K}(\mathcal{H})}) \circ \alpha] \circ (\Delta\widehat{\otimes}\mathrm{id}_{\mathbb{F}\widehat{\otimes}\mathcal{K}(\mathcal{H})}) \\
&= (\beta_t\widehat{\otimes}\mathrm{id}_{B\widehat{\otimes}\mathbb{F}_{m,0}}) \circ (\mathrm{id}_{\mathcal{S}}\widehat{\otimes}f_*(\alpha)) \circ (\Delta\widehat{\otimes}\mathrm{id}_{\mathbb{F}\widehat{\otimes}\mathcal{K}(\mathcal{H})}) \\
&= \upsilon'_{m,n}(f_*(\alpha) \wedge (\beta_t\widehat{\otimes}\mathrm{id}_B)) \\
&= \upsilon'_{m,n}(\mathrm{id} \wedge \widehat{\otimes}_B)(f_*(\alpha) \wedge \beta_t) \\
&= \upsilon'_{m,n}(\mathrm{id} \wedge \widehat{\otimes}_B)(f_* \wedge \mathrm{id})(\alpha \wedge \beta_t).
\end{aligned}
$$

□

Corollary 6.2.8. *There is a canonical map*

$$S\colon \mathbb{K}(A \otimes B) \wedge \mathbb{K}_{hom}(A) \to \mathbb{K}(B)$$

of orthogonal quasi-spectra. The map S is natural in the variable B in the obvious sense and natural in the variable A, in the sense that if we have a $*$-homomorphism $f\colon A \to A'$ then we have the following commutative diagram

$$
\begin{array}{ccc}
\mathbb{K}(A \otimes B) \wedge \mathbb{K}_{hom}(A) & \xrightarrow{S} & \mathbb{K}(B) \\
\uparrow{\scriptstyle id\wedge f^*} & & \| \\
\mathbb{K}(A \otimes B) \wedge \mathbb{K}_{hom}(A') & & \mathbb{K}(B) \\
\downarrow{\scriptstyle f_*\wedge id} & & \| \\
\mathbb{K}(A' \otimes B) \wedge \mathbb{K}_{hom}(A') & \xrightarrow{S} & \mathbb{K}(B).
\end{array}
$$

□

Appendices

Appendix A

Functional Calculus

Much of this appendix is taken from Section 1.5 of [Dix96] and Section 2.2 of [Mit00].

Functional Calculus is a necessity for us to define certain functions we use throughout the Bott periodicity proof.

In this appendix we give an overview of functional calculus for complex, bounded, unbounded, real and graded operators.

A.0.1 Complex bounded

Let A be a unital complex C^*-algebra and $x \in A$ be normal, that is $x^*x = xx^*$. Then we can think of the spectrum of the element x, $\sigma(x)$.

Theorem A.0.9. *There exists a unique $*$-homomorphism*

$$C(\sigma(x)) \to A,$$

written $f \mapsto f(x)$ such that

$$\sigma(f(x)) = f(\sigma(x)),$$

and

$$f(g(x)) = (f \circ g)(x),$$

if $g \in C(\sigma(x))$ and $f \in C(\sigma(g(x)))$.

□

A.0.2 Unbounded

Let H be a complex Hilbert space and $V \subseteq H$. Then consider the possibly unbounded operator $T: V \to H$.

Theorem A.0.10. *There exists a unique map*

$$C(\sigma(T)) \to \{\text{Unbounded operators on } H\},$$

written $f \mapsto f(T)$ such that the map is linear, multiplicative, $*$-preserving, satisfies

$$f(\sigma(T)) = \sigma(f(T)),$$

and $f(g(x)) = (f \circ g)(x)$ if $g \in C(\sigma(x))$ and $f \in C(\sigma(g(x)))$.

Additionally if $f \in C(\sigma(T))$ is bounded, then $f(T)$ is bounded and

$$||f(T)|| = sup\{|\lambda| \mid \lambda \in \sigma(T)\}.$$

□

A.0.3 Real

Let $C(X)$ denote the set of continuous complex-valued functions. For the real case we require a new set of functions defined by

$$C_{\mathbb{R}}(X) := \{f \in C(X) \mid f(\overline{x}) = \overline{f(x)}\}.$$

Call $f \in C_{\mathbb{R}}(X)$ a Real[1] function.

Theorem A.0.11. *Let A be a real unital C^*-algebra and let $x \in A$ be normal. There exists a unique $*$-homomorphism*

$$C_{\mathbb{R}}(\sigma(x)) \to A,$$

written $f \mapsto f(x)$ such that

$$f(\sigma(T)) = \sigma(f(T)),$$

and $f(g(x)) = (f \circ g)(x)$ if $g \in C_{\mathbb{R}}(\sigma(x))$ and $f \in C_{\mathbb{R}}(\sigma(g(x)))$.

□

[1]The capitalisation of R here is intentional.

A.0.4 Graded

Let A be a unital graded complex C^*-algebra and $x \in A$ be normal, that is $x^*x = xx^*$. Then we have the following theorem.

Theorem A.0.12. *There exists a unique $*$-homomorphism*

$$C(\sigma(x)) \to A,$$

written $f \mapsto f(x)$ such that

1. $\sigma(f(x)) = f(\sigma(x))$,
2. $f(g(x)) = (f \circ g)(x)$, *if* $g \in C(\sigma(x))$ *and* $f \in C(\sigma(g(x)))$,
3. *if x is odd then the $*$-homomorphism is graded,*
4. *if x is even, then for all $f \in C(\sigma(x))$, $f(x)$ is even.*

□

Appendix B

K-theory

Much of this appendix is taken from [Bla98], [RLL00] and [WO93]. In this Appendix we look at the notion of analytic K-theory, which is a sequence of covariant functors, $K_0, K_1, \dots K_n$, from the category where objects are C^*-algebras and arrows are $*$-homomorphisms to the category where objects are abelian groups and arrows are group homomorphisms. We will see via Bott periodicity that we only need to define K_0 and K_1, which we define in the first few subsections. We will see how we can use Bott periodicity to get the other K-theory groups and then see a relation between our two functors via the index map. Then we also use the relation of E-theory and K-theory to view K-theory differently.

Definition B.0.13. A *semigroup* is a set A with a binary operation $\circ$ such that

$$a \circ (b \circ c) = (a \circ b) \circ c,$$

for all $a, b, c \in A$. In particular, a semigroup is *abelian* if $ab = ba$, for all $a, b \in A$.

We now give a description of the construction of the Grothendieck group. We will need this later as the K_0 group is defined using this group. Let $(A, +)$ be an abelian semigroup. Let $\sim$ denote the equivalence relation on $A \times A$ defined by $(a_1, b_1) \sim (a_2, b_2)$ if there exists $z \in A$ such that $a_1 + b_2 + z = a_2 + b_1 + z$. Then the Grothendieck group $G(A)$ is defined by the quotient $A \times A/\sim$, with operation:

$$\langle a_1, b_1 \rangle + \langle a_2, b_2 \rangle = \langle a_1 + b_1, a_2 + b_2 \rangle,$$

where $\langle a, b \rangle$ denotes the equivalence class containing $\langle a, b \rangle$ in $G(S)$. Define the *Grothendieck map* for $y \in S$ by, $\gamma_s \colon S \to G(S), x \mapsto \langle x + y, y \rangle$.

Definition B.0.14. Let A be a complex C^*-algebra. A *projection*, $p \in A$, is a self-adjoint idempotent, that is to say $p = p^* = p^2$. Two projections are called *orthogonal* when $pq = 0$.

We now talk about the equivalence of projections.

Definition B.0.15. Let p, q be projections in a C^*-algebra A. Then

- p is *equivalent* to q, written $p \sim q$, if $p = v^*v$ and $q = vv^*$ for some partial isometry $v \in A$,
- p is *unitarily equivalent* to q, written $p \sim_u q$, if $p = u^*qu$ with $u \in \tilde{A}$ being unitary,
- p is *homotopic* to q, written $p \sim_h q$, if p and q are connected by a norm continuous path of projections in A.

Definition B.0.16. Define the semigroup $P_\infty(A)$, by

$$P_\infty(A) = \bigcup_{n=1}^{\infty} P_n(A),$$

where

$$P_n(A) = P(M_n(A)).$$

Define a relation $\sim_0$ on $P_\infty(A)$ for projections $p \in P_n(A)$ and $q \in P_m(A)$ by, $p \sim_0 q$ if there is an element $v \in M_{m,n}(A)$ with $p = v^*v$ and $q = vv^*$. The binary operation $\oplus$ on $P_\infty(A)$ is defined by,

$$p \oplus q = diag(p, q) = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix},$$

where $p \in P_n(A), q \in P_m(A)$.

Definition B.0.17. Define the semigroup $D(A)$ of a C^*-algebra A by,

$$D(A) = P_\infty(A)/ \sim_0 .$$

Then for each $p \in P_\infty(A)$ let $[p]_D \in P(A)$ denote the equivalence class containing p. Then addition in $P(A)$ is defined by,

$$[p]_D + [q]_D = [p \oplus q]_D.$$

Note that $(P(A), +)$ is an abelian semigroup.

We firstly have to define the functor K_0 for unital C^*-algebras and then extend it to the whole category of C^*-algebras.

Definition B.0.18. Let A be a unital C^*-algebra, and let $(D(A), +)$ be the semigroup defined in Definition B.0.17. Define $K_0(A)$ to be the Grothendieck group of $D(A)$. That is, $K_0 = G(D(A))$. Define $[.]_0 \colon D_\infty(A) \to K_0(A)$ by

$$[p]_0 = \gamma([p]_D) \in K_0(A), p \in P_\infty(A),$$

where $\gamma \colon D(A) \to K_0(A)$ is the Grothendieck map.

For a proof of the next result see [RLL00].

Proposition B.0.19. *Let A be a unital C^*-algebra. Then*

$$\begin{aligned} K_0(A) &= \{[p]_0 - [q]_0 : p, q \in P_\infty(A)\} \\ &= \{[p]_0 - [q]_0 : p, q \in P_n(A), n \in \mathbb{N}\}. \end{aligned}$$

Moreover,

- $[p + q]_0 = [p]_0 + [q]_0$ *for all projections* $p, q \in P_\infty(A)$,
- $[0_A]_0 = 0$, *where* 0_A *is the zero projection in* A,
- *if* $p, q \in P_n(A)$ *for some* n *and* $p \sim_h q \in P_n(A)$, *then* $[p]_0 = [q]_0$,
- *if* p, q *are mutually orthogonal projections in* $P_n(A)$, *then*

$$[p + q]_0 = [p]_0 + [q]_0,$$

- *for all* $p, q \in P_\infty(A)$, $[p]_0 = [q]_0$ *if and only if* $p \sim_s q$.

□

Before we move on to generalise our definition of K_0 to all C^*-algebras, we state a univeral property of K_0. For a proof of the following result see Proposition 3.1.8 in [RLL00].

Proposition B.0.20. *Suppose A is a unital C^*-algebra, G an abelian group, and that $\nu \colon P_\infty(A) \to G$ is a map satisfying:*

- $\nu(p \oplus q) = \nu(p) + \nu(q)$ *for all projections* $p, q \in P_\infty(A)$,

- $\nu(0_A) = 0$,
- *if* $p, q \in P_n(A)$ *for some* n *and* $p \sim_h q \in P_n(A)$*, then* $\nu(p) = \nu(q)$*.*

Then there is a unique group homomorphism $\alpha\colon K_0(A) \to G$ *such that the following diagram commutes:*

$$\begin{array}{ccc} P_\infty(A) & \xrightarrow{[.]_0} & K_0(A) \\ & \searrow^{\nu} & \downarrow{\scriptstyle \alpha} \\ & & G \end{array}$$

□

Now we extend our functor K_0 to the whole of the category of C^*-algebras and state the properties that this functor has.

Definition B.0.21. Let A be a C^*-algebra, not necessarily unital, and consider the associated split exact sequence

$$0 \longrightarrow A \xrightarrow{\ l\ } \widetilde{A} \underset{\lambda}{\overset{\pi}{\rightleftarrows}} \mathbb{C} \longrightarrow 0,$$

obtained by adjoining a unit to A. Define $K_0(A)$ to be the kernel of the homomophism $K_0(\pi)\colon K_0(\widetilde{A}) \to K_0(\mathbb{C})$.

Proposition B.0.22. K_0 *is a functor from the category where objects are* C^**-algebras and arrows are* $*$*-homomorphisms to the category where objects are abelian groups and arrows are group homomorphisms.*

□

Proposition B.0.23. *Let* A *and* B *be* C^**-algebras, and suppose that* $\varphi, \psi\colon A \to B$ *are homotopic* $*$*-homomorphisms, then* $K_0(\varphi) = K_0(\psi)$*.*

□

Now we talk about the standard picture of $K_0(A)$ in terms of the scalar mapping. The *Scalar mapping*, s, is defined as the composition map of λ and π in the following split exact sequence:

$$0 \longrightarrow A \xrightarrow{\ l\ } \widetilde{A} \underset{\lambda}{\overset{\pi}{\rightleftarrows}} \mathbb{C} \longrightarrow 0,$$

obtained by adjoining a unit to A. That is $s = \lambda \circ \pi\colon \widetilde{A} \to \widetilde{A}$.

Proposition B.0.24. *For a C^*-algebra A,*

$$K_0(A) = \{[p]_0 - [s(p)]_0 : p \in P_\infty(\widetilde{A})\}.$$

□

Proposition B.0.25. *For every short exact sequence of C^*-algebras:*

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0,$$

induces an exact sequence

$$K_0(A) \xrightarrow{K_0(\alpha)} K_0(B) \xrightarrow{K_0(\beta)} K_0(C).$$

□

Proposition B.0.26. *Every split exact sequence of C^*-algebras:*

$$0 \longrightarrow A \xrightarrow{\alpha} B \underset{\gamma}{\overset{\beta}{\rightleftarrows}} C \longrightarrow 0,$$

induces a split exact sequence of abelian groups

$$0 \longrightarrow K_0(A) \xrightarrow{K_0(\alpha)} K_0(B) \underset{K_0(\gamma)}{\overset{K_0(\beta)}{\rightleftarrows}} K_0(C) \longrightarrow 0.$$

□

Proposition B.0.27. *For every pair of C^*-algebras A and B, we have*

$$K_0(A \oplus B) \cong K_0(A) \oplus K_0(B).$$

□

Some examples:

- $K_0(\mathbb{C}) = \mathbb{Z}$
- $K_0(B(\mathcal{H})) = \{0\}$, where $B(\mathcal{H})$ is the set of bounded linear operators on a Hilbert space $\mathcal{H}$.

The functor K_1 also goes from the category where objects are C^*-algebras and arrows are $*$-homomorphisms to the category where objects are abelian groups and arrows are group homomorphisms just like K_0 but instead of being defined in terms of projections it is instead defined using unitaries.

Definition B.0.28. Let A be a unital C^*-algebra, and let $\mathcal{U}(A)$ denote its group of unitary elements. Set

$$\mathcal{U}_n(A) = \mathcal{U}(M_n(A)),\ \mathcal{U}_\infty(A) = \bigcup_{n=1}^{\infty} \mathcal{U}_n(A).$$

Define the binary operation $\oplus$ on $U_\infty(A)$ by,

$$u \oplus v = diag(u, v) = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix},$$

where $u \in \mathcal{U}_n(A), v \in \mathcal{U}_m(A)$.

Definition B.0.29. For each C^*-algebra A define

$$K_1(A) = \mathcal{U}_\infty(\widetilde{A})/\sim_1,$$

where $u \sim_1 v$ for all $u \in \mathcal{U}_n(A)$ and $v \in \mathcal{U}_m(A)$, if there exists a natural number $k \geq max\{m, n\}$ such that $u \oplus 1_{k-n} \sim_h v \oplus 1_{k-m}$ in $\mathcal{U}_k(A)$.

Proposition B.0.30. *Let A be a C^*-algebra. Then*

$$K_1(A) = \{[u]_1 : u \in \mathcal{U}_\infty(\widetilde{A})\}.$$

□

Note that K_1 has similar properties to K_0 but we will not restate them.
The K-theory groups are periodic with period 2.

Theorem B.0.31. *The Bott map $\beta_A \colon K_0(A) \to K_1(\Sigma A)$ is an isomorphism for every C^*-algebra A.*

□

Then we can define for $n \geq 2$,

$$K_{n+1}(A) := K_n(\Sigma A).$$

Corollary B.0.32. *For every complex C^*-algebra A and every integer $n \geq 0$,*

$$K_{n+2}(A) \cong K_n(A),$$

and for every real C^*-algebra B and every integer $n \geq 0$,

$$K_{n+8}(B) \cong K_n(B),$$

□

Equivalently we can think of the K-theory groups in an alternative way by using the relation with E-theory formulated by Guentner, Higson and Trout [GHT00]. Recall that $[A, B]$ denotes the set of homotopy classes of (graded) $*$-homomorphisms from A to B. Then we obtain the following definition from [HG04]:

Theorem B.0.33. *For a graded C^*-algebra A,*

$$K(A) = [\mathcal{S}, A \widehat{\otimes} \mathcal{K}],$$

and when A is ungraded we have

$$K(A) = [\Sigma\mathbb{C}, \Sigma A \otimes \mathcal{K}].$$

□

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